EE456 – Digital Communications

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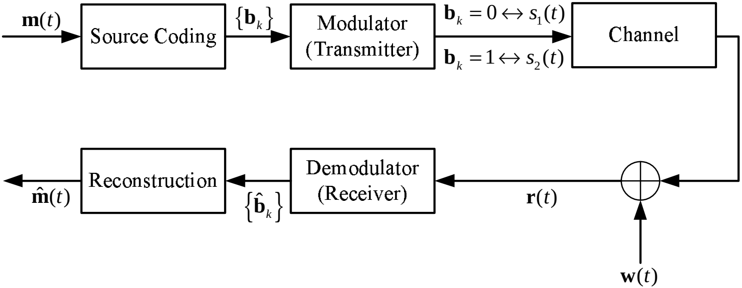
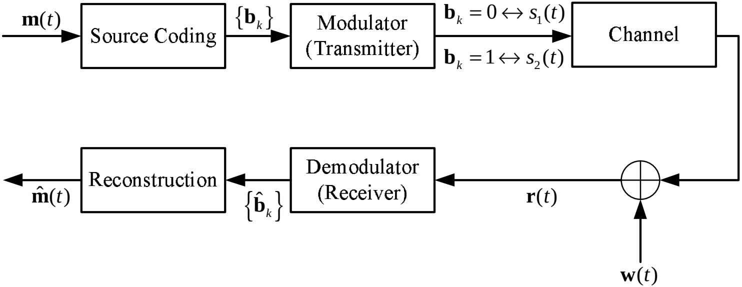
Block Diagram of Binary Communication Systems



- Bits in two different time slots are *statistically independent*.
- *a priori* probabilities: $P[\mathbf{b}_k = 0] = P_1$, $P[\mathbf{b}_k = 1] = P_2$.
- Signals $s_1(t)$ and $s_2(t)$ have a duration of T_b seconds and finite energies: $E_1 = \int_0^{T_b} s_1^2(t)\mathrm{d}t$, $E_2 = \int_0^{T_b} s_2^2(t)\mathrm{d}t$.
- Noise $\mathbf{w}(t)$ is stationary *Gaussian*, zero-mean *white* noise with two-sided power spectral density of $N_0/2$ (watts/Hz):

$$E\{\mathbf{w}(t)\} = 0,\ E\{\mathbf{w}(t)\mathbf{w}(t+\tau)\} = \frac{N_0}{2}\delta(\tau),\ \mathbf{w}(t) \sim \mathcal{N}\left(0, \frac{N_0}{2}\right).$$

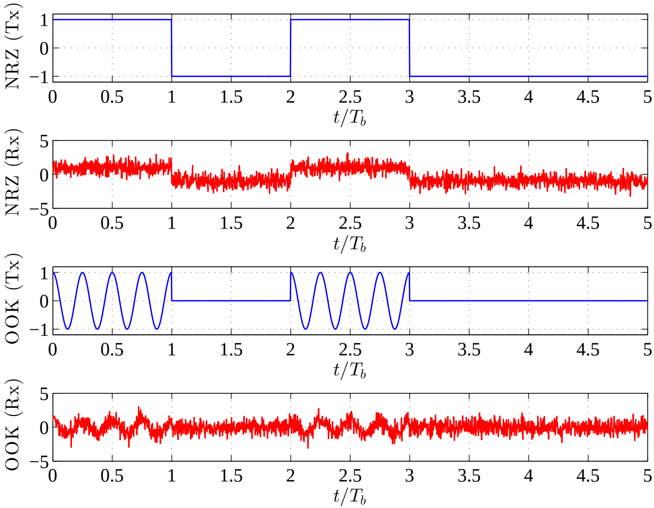


- Received signal over $[(k-1)T_b, kT_b]$:

$$\mathbf{r}(t) = s_i(t-(k-1)T_b) + \mathbf{w}(t), \quad (k-1)T_b \le t \le kT_b.$$

- Objective is to design a receiver (or demodulator) such that *the probability of making an error is minimized*.
- Shall reduce the problem from the observation of a time waveform to that of observing a set of numbers (which are random variables).

Can you Identify the Signal Sets $\{s_1(t), s_2(t)\}$?



Geometric Representation of Signals $s_1(t)$ and $s_2(t)$ (I)

- Wish to represent two arbitrary signals $s_1(t)$ and $s_2(t)$ as *linear combinations* of two *orthonormal* basis functions $\phi_1(t)$ and $\phi_2(t)$.
- $\phi_1(t)$ and $\phi_2(t)$ form a set of *orthonormal* basis functions if and only if: $\phi_1(t)$ and $\phi_2(t)$ are orthonormal if:

$$\int_0^{T_b} \phi_1(t)\phi_2(t)\mathrm{d}t = 0 \text{ (\textit{ortho}gonality)},$$
$$\int_0^{T_b} \phi_1^2(t)\mathrm{d}t = \int_0^{T_b} \phi_2^2(t)\mathrm{d}t = 1 \text{ (\textit{normal}ized to have unit energy)}.$$

- If $\{\phi_1(t), \phi_2(t)\}$ can be found, the representations are

$$\begin{aligned} s_1(t) &= s_{11}\phi_1(t) + s_{12}\phi_2(t), \\ s_2(t) &= s_{21}\phi_1(t) + s_{22}\phi_2(t). \end{aligned}$$

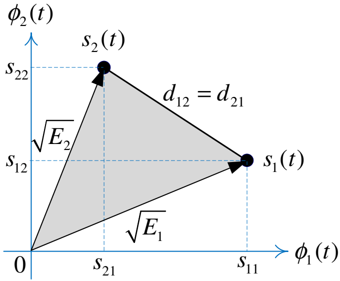
- It can be checked that the coefficients s_{ij} can be calculated as follows:

$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t, \quad i, j \in \{1, 2\},$$

- An important question is: Given the signal set $s_1(t)$ and $s_2(t)$, can one always find an orthonormal basis functions to represent $\{s_1(t), s_2(t)\}$ exactly? If the answer is YES, is the set of orthonormal basis functions UNIQUE?

Geometric Representation of Signals $s_1(t)$ and $s_2(t)$ (II)

Provided that $\phi_1(t)$ and $\phi_2(t)$ can be found, the signals (which are waveforms) can be represented as *vectors* in a vector space (or signal space) spanned (i.e., defined) by the orthonormal basis set $\{\phi_1(t), \phi_2(t)\}$.



$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t),$$
$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t),$$
$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t,\ \ i, j \in \{1, 2\},$$
$$E_i = \int_0^{T_b} s_i^2(t)\mathrm{d}t = s_{i1}^2 + s_{i2}^2,\ \ i \in \{1, 2\},$$
$$d_{12} = d_{21} = \sqrt{\int_0^{T_b} [s_2(t) - s_1(t)]^2 \mathrm{d}t}$$

- $\int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t$ is the projection of signal $s_i(t)$ onto basis function $\phi_j(t)$.
- The length of a signal vector equals to the square root of its energy.
- It is always possible to find *orthonormal* basis functions $\phi_1(t)$ and $\phi_2(t)$ to represent $s_1(t)$ and $s_2(t)$ exactly. In fact, there are infinite number of choices!

Gram-Schmidt Procedure

Gram-Schmidt (G-S) procedure is one method to find a set of orthonormal basis functions for a given arbitrary set of waveforms.

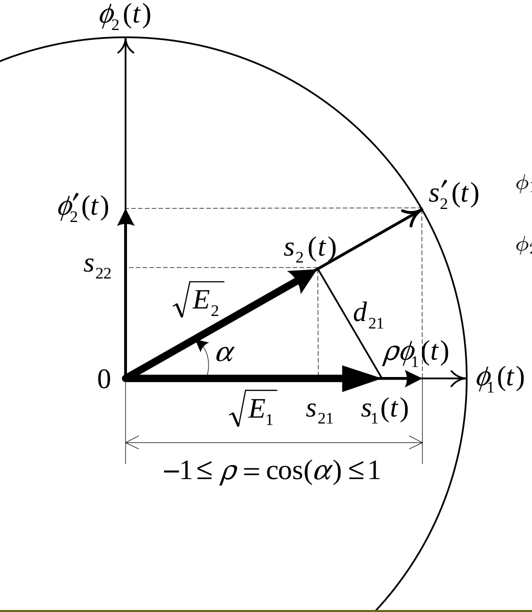
1. Let $\phi_1(t) \equiv \frac{s_1(t)}{\sqrt{E_1}}$. Note that $s_{11} = \sqrt{E_1}$ and $s_{12} = 0$.
2. Project $s_2^{'}(t) = \frac{s_2(t)}{\sqrt{E_2}}$ onto $\phi_1(t)$ to obtain the *correlation coefficient*:

$$\rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}}\phi_1(t)\mathrm{d}t = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t)s_2(t)\mathrm{d}t.$$

3. Subtract $\rho\phi_1(t)$ from $s_2^{'}(t)$ to obtain $\phi_2^{'}(t) = \frac{s_2(t)}{\sqrt{E_2}} - \rho\phi_1(t)$.
4. Finally, normalize $\phi_2^{'}(t)$ to obtain:

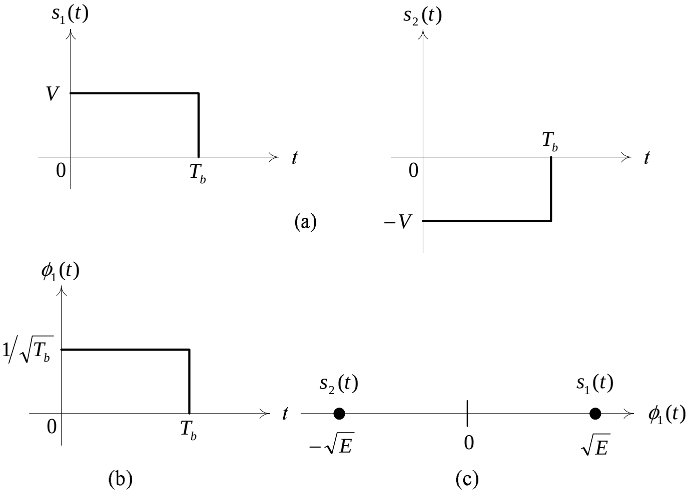
$$\begin{aligned}\phi_2(t) &= \frac{\phi_2^{'}(t)}{\sqrt{\int_0^{T_b}\left[\phi_2^{'}(t)\right]^2\mathrm{d}t}} = \frac{\phi_2^{'}(t)}{\sqrt{1-\rho^2}}\\ &= \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right].\end{aligned}$$

Gram-Schmidt Procedure: Summary



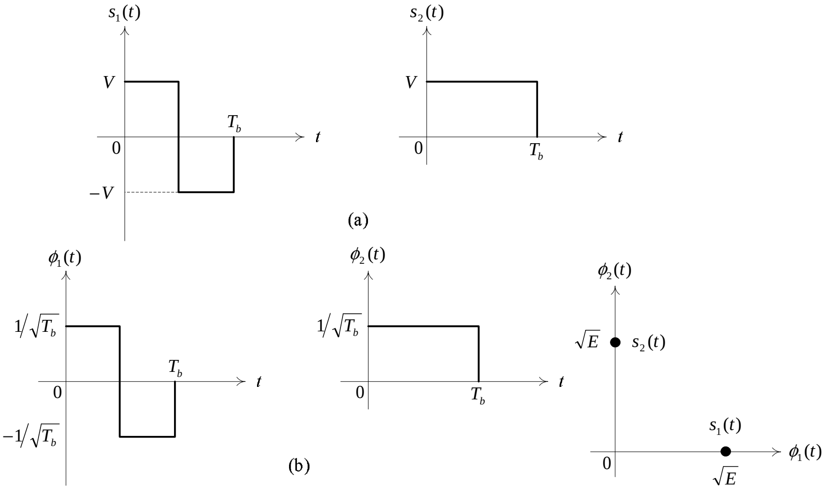
$$
\begin{aligned}
\phi_1(t) &= \frac{s_1(t)}{\sqrt{E_1}}, \\
\phi_2(t) &= \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right], \\
s_{21} &= \int_0^{T_b} s_2(t)\phi_1(t)\mathrm{d}t = \rho\sqrt{E_2}, \\
s_{22} &= \left(\sqrt{1-\rho^2}\right)\sqrt{E_2}, \\
d_{21} &= \sqrt{\int_0^{T_b}[s_2(t)-s_1(t)]^2\mathrm{d}t} \\
&= E_1 - 2\rho\sqrt{E_1E_2} + E_2.
\end{aligned}
$$

Example 1

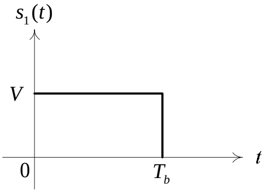


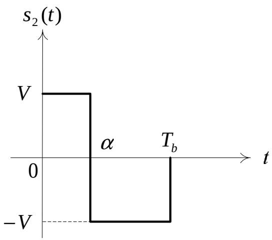
(a) Signal set. (b) Orthonormal function. (c) Signal space representation.

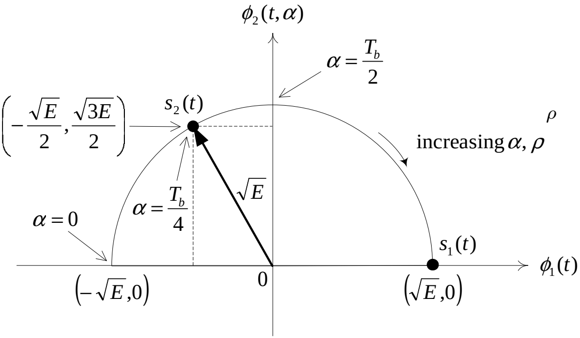
Example 2


$s_1(t)$
V
T_b
t
0
$-V$
$s_2(t)$
V
0
T_b
t
(a)
$\phi_1(t)$
$1/\sqrt{T_b}$
T_b
t
0
$-1/\sqrt{T_b}$
$\phi_2(t)$
$1/\sqrt{T_b}$
0
T_b
t
(b)
$\phi_2(t)$
$\sqrt{E}$
$s_2(t)$
$s_1(t)$
0
$\phi_1(t)$
$\sqrt{E}$

Example 3

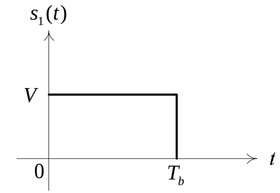






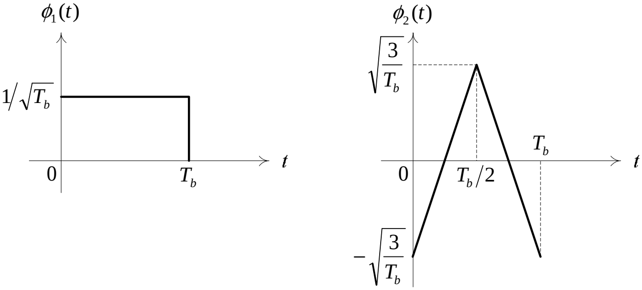
$$
\begin{aligned}
\rho &= \frac{1}{E}\int_0^{T_b} s_2(t)s_1(t)\mathrm{d}t \\
&= \frac{1}{V^2T_b}\left[V^2\alpha - V^2(T_b-\alpha)\right] \\
&= \frac{2\alpha}{T_b} - 1
\end{aligned}
$$

Example 4



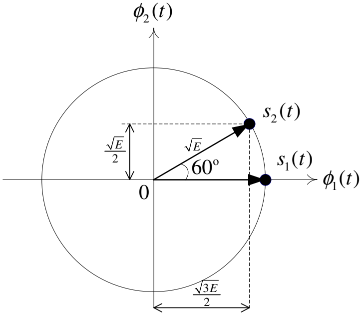
$s_1(t)$
V
0
T_b
t
$s_2(t)$
$\sqrt{3}V$
0
$T_b/2$
T_b
t

(a)



$\phi_1(t)$
$1/\sqrt{T_b}$
0
T_b
t
$\phi_2(t)$
$\sqrt{\frac{3}{T_b}}$
0
$T_b/2$
T_b
t
$-\sqrt{\frac{3}{T_b}}$

(b)



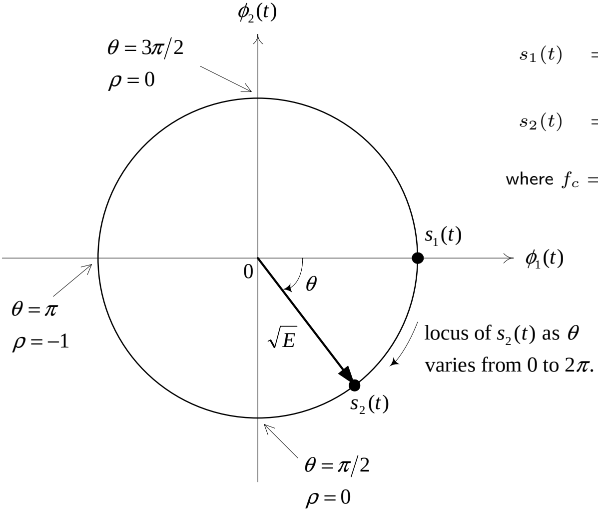
$$\rho = \frac{1}{E}\int_0^{T_b} s_2(t)s_1(t)\mathrm{d}t = \frac{2}{E}\int_0^{T_b/2}\left(\frac{2\sqrt{3}}{T_b}Vt\right)V\mathrm{d}t = \frac{\sqrt{3}}{2},$$

$$\phi_2(t) = \frac{1}{(1-\frac{3}{4})^{\frac{1}{2}}}\left[\frac{s_2(t)}{\sqrt{E}} - \rho\frac{s_1(t)}{\sqrt{E}}\right] = \frac{2}{\sqrt{E}}\left[s_2(t) - \frac{\sqrt{3}}{2}s_1(t)\right],$$

$$s_{21} = \frac{\sqrt{3}}{2}\sqrt{E}, \quad s_{22} = \frac{1}{2}\sqrt{E}.$$

$$d_{21} = \left[\int_0^{T_b}[s_2(t)-s_1(t)]^2\mathrm{d}t\right]^{\frac{1}{2}} = \sqrt{\left(2-\sqrt{3}\right)E}.$$

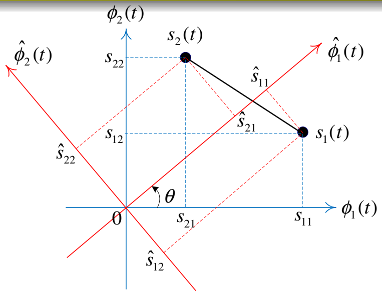
Example 5



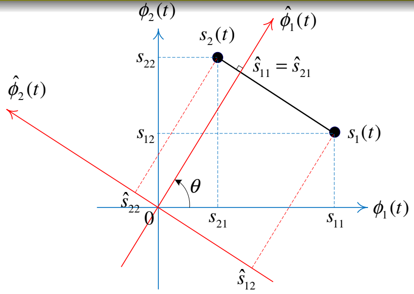
$$
\begin{aligned}
s_1(t) &= \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), \\
s_2(t) &= \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \theta).
\end{aligned}
$$

where $f_c = \frac{k}{2T_b}$, k an integer.

Obtaining Different Basis Sets by Rotation



(a) Arbitrary rotation



(b) Rotation to make $\hat{s}_{11} = \hat{s}_{21}$

$$\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}.$$

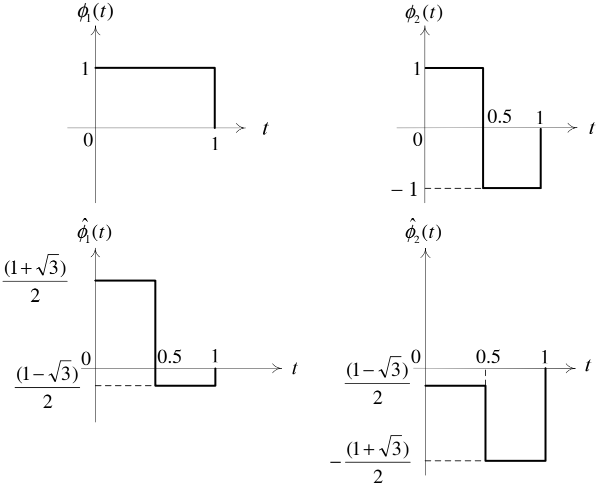
(a) Show that, regardless of the angle θ, the set $\{\hat{\phi}_1(t), \hat{\phi}_2(t)\}$ is also an orthonormal basis set.

(b) What are the values of θ that make $\hat{\phi}_1(t)$ perpendicular to the line joining $s_1(t)$ to $s_2(t)$? For these values of θ, mathematically show that the components of $s_1(t)$ and $s_2(t)$ along $\hat{\phi}_1(t)$, namely $\hat{s}_{11}$ and $\hat{s}_{21}$, are identical.

Remark: **Rotating counter-clockwise for positive θ and clock-wise for negative θ.**

Example: Rotating $\{\phi_1(t), \phi_2(t)\}$ by $\theta = 60^\circ$ to Obtain $\{\hat{\phi}_1(t), \hat{\phi}_2(t)\}$

$$\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \Rightarrow \begin{cases} \hat{\phi}_1(t) = & \cos\theta \times \phi_1(t) + \sin\theta \times \phi_2(t) \\ \hat{\phi}_2(t) = & -\sin\theta \times \phi_1(t) + \cos\theta \times \phi_2(t) \end{cases}$$

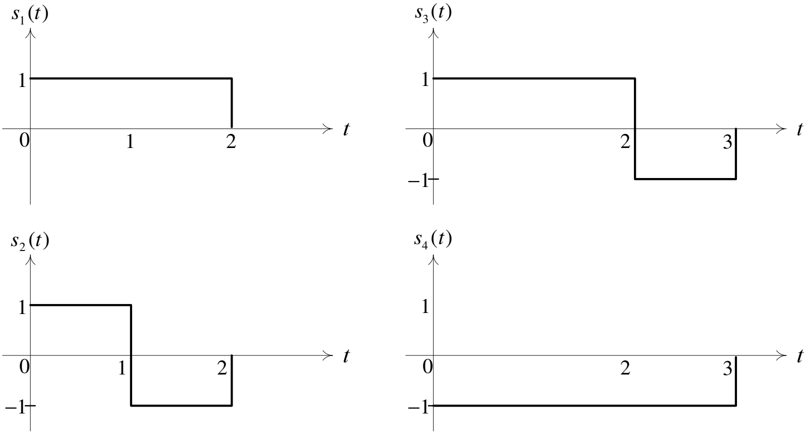


Gram-Schmidt Procedure for M Waveforms $\{s_i(t)\}_{i=1}^{M}$

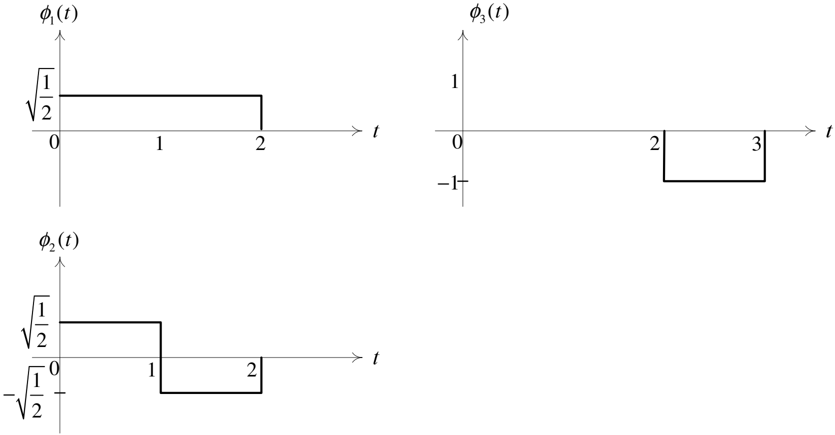
$$
\begin{aligned}
\phi_1(t) &= \frac{s_1(t)}{\sqrt{\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t}}, \\
\phi_i(t) &= \frac{\phi_i^{'}(t)}{\sqrt{\int_{-\infty}^{\infty} \left[\phi_i^{'}(t)\right]^2 \mathrm{d}t}}, \quad i = 2, 3, \ldots, N, \\
\phi_i^{'}(t) &= \frac{s_i(t)}{\sqrt{E_i}} - \sum_{j=1}^{i-1} \rho_{ij}\phi_j(t), \\
\rho_{ij} &= \int_{-\infty}^{\infty} \frac{s_i(t)}{\sqrt{E_i}} \phi_j(t)\mathrm{d}t, \quad j = 1, 2, \ldots, i-1.
\end{aligned}
$$

If the waveforms $\{s_i(t)\}_{i=1}^{M}$ form a *linearly independent set*, then $N = M$. Otherwise $N < M$.

Example: Find a Basis Set for the Following $M = 4$ Waveforms

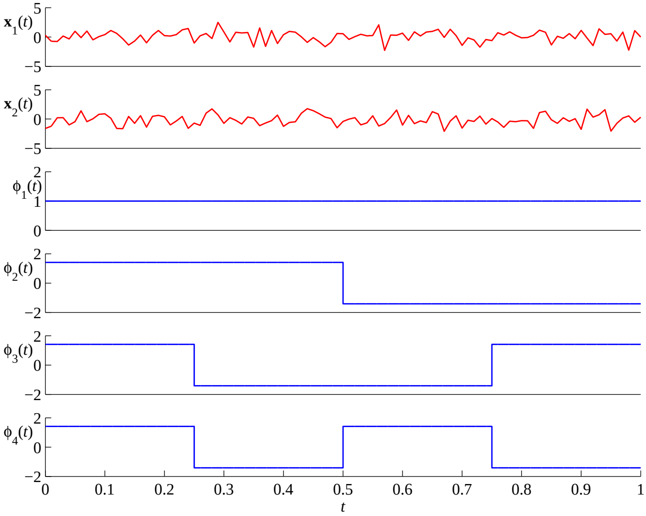


Answer Found by Applying the G-S Procedure



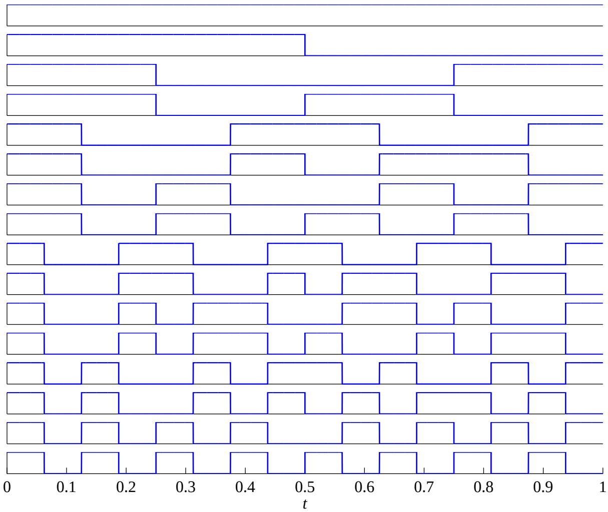
$$\begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \end{bmatrix}$$

Representation of Noise with Walsh Functions



Exact representation of noise using 4 Walsh functions is not possible.

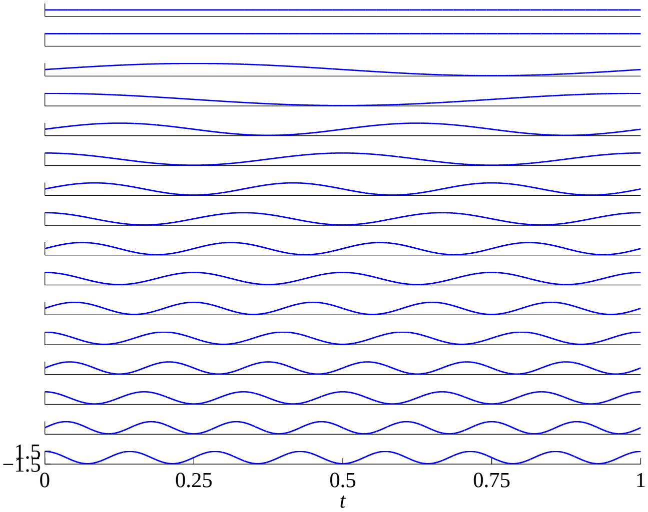
The First 16 Walsh Functions



<u>Exact</u> representation might be possible by using many more Walsh functions.

The First 16 Sine and Cosine Functions

Can also use sine and cosine functions (Fourier representation).



Representation of Noise

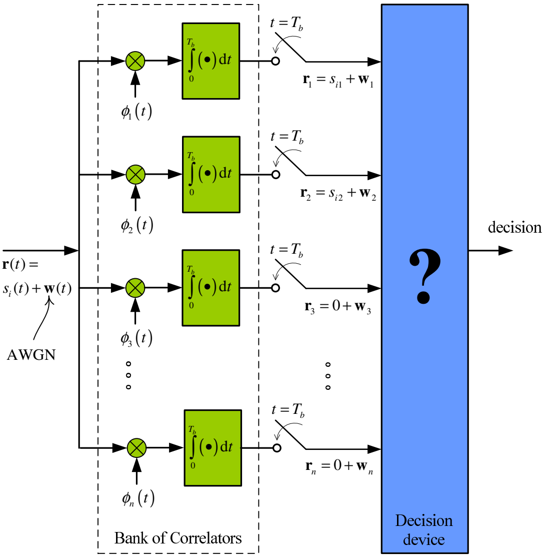
- To represent noise $\mathbf{w}(t)$, need to use a *complete* set of orthonormal functions:

$$\mathbf{w}(t) = \sum_{i=1}^{\infty} \mathbf{w}_i \phi_i(t), \quad \text{where} \quad \mathbf{w}_i = \int_0^{T_b} \mathbf{w}(t)\phi_i(t)\mathrm{d}t.$$

- The coefficients $\mathbf{w}_i$'s are *random variables* and understanding their statistical properties is imperative in developing the optimum receiver.
- Of course, the statistical properties of random variables $\mathbf{w}_i$'s depend on the statistical properties of the noise $\mathbf{w}(t)$, which is a *random process*.
- In communications, a major source of noise is *thermal noise*, which is modelled as *Additive White Gaussian Noise (AWGN)*:
  - **White**: The *power spectral density (PSD)* is a constant (i.e., flat) over all frequencies.
  - **Gaussian**: The *probability density function (pdf)* of the noise amplitude at any given time follows a Gaussian distribution.
- When $\mathbf{w}(t)$ is modelled as AWGN, the projection of $\mathbf{w}(t)$ on each basis function, $\mathbf{w}_i = \int_0^{T_b} \mathbf{w}(t)\phi_i(t)\mathrm{d}t$, is a Gaussian random variable (this can be proved).
- For zero-mean and white noise $\mathbf{w}(t)$, $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3, \ldots$ are zero-mean and uncorrelated random variables:
  1. $E\{\mathbf{w}_i\} = E\left\{\int_0^{T_b} \mathbf{w}(t)\phi_i(t)\mathrm{d}t\right\} = \int_0^{T_b} E\{\mathbf{w}(t)\}\phi_i(t)\mathrm{d}t = 0.$
  2. $E\{\mathbf{w}_i\mathbf{w}_j\} = E\left\{\int_0^{T_b} \mathrm{d}\lambda \mathbf{w}(\lambda)\phi_i(\lambda) \int_0^{T_b} \mathrm{d}\tau \mathbf{w}(\tau)\phi_j(\tau)\right\} = \begin{cases} \frac{N_0}{2}, & i = j \\ 0, & i \neq j \end{cases}.$
- Since $\mathbf{w}(t)$ is not only zero-mean and white, but also Gaussian $\Rightarrow \{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ are Gaussian and *statistically independent*!!!

Need to Review Probability Theory & Random Processes – Chapter 3 Slides

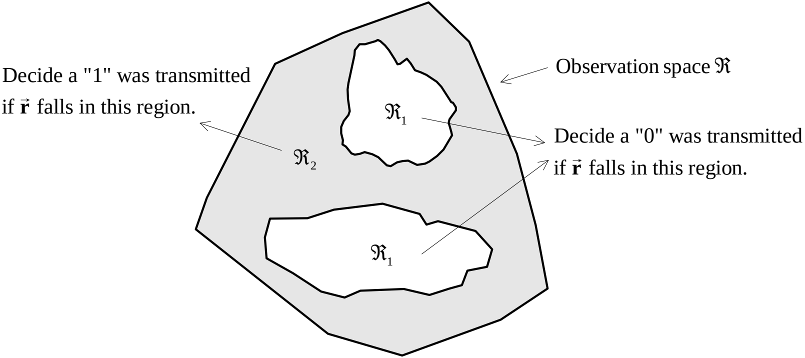
Observing a waveform ⇒ Observing a set of numbers



- Choose $\phi_1(t)$ and $\phi_2(t)$ so that they can be used to represent the two signals $s_1(t)$ and $s_2(t)$ exactly. The remaining orthonormal basis functions are simply chosen to complete the set in order to represent noise exactly.
- The decision can be based on the observations $r_1, r_2, r_3, r_4, \ldots$.
- Note that $\mathbf{r}_j$, for $j = 3, 4, 5, \ldots$, does not depend on which signal ($s_1(t)$ or $s_2(t)$) was transmitted.

Optimum Receiver

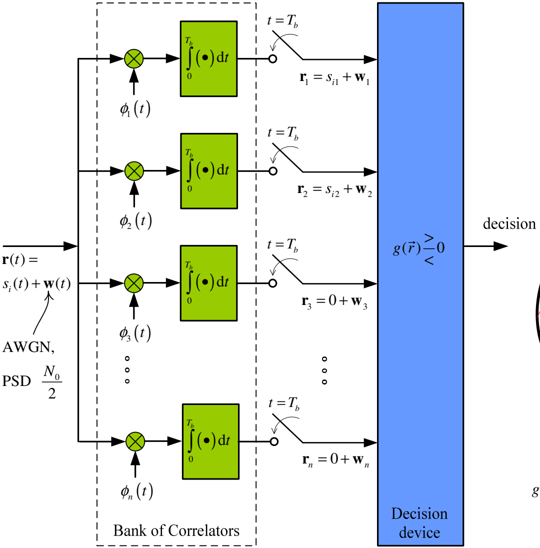
- The criterion is *to minimize the bit error probability*.
- Consider only the first n terms (n can be very very large), $\vec{r} = \{r_1, r_2, \ldots, r_n\}$ $\Rightarrow$ Need to partition the n-dimensional observation space into *two decision regions*, $\Re_1$ and $\Re_2$.



$$
\begin{aligned}
P[\text{error}] = & P[(\text{“0” decided and “1” transmitted}) \text{ or} \\
& (\text{“1” decided and “0” transmitted})]. \\
= \quad & P[0_D, 1_T] + P[1_D, 0_T] \\
= \quad & P[0_D|1_T]P[1_T] + P[1_D|0_T]P[0_T] \\
= \quad & P[\vec{\mathbf{r}} \in \Re_1|1_T]P_2 + P[\vec{\mathbf{r}} \in \Re_2|0_T]P_1 \\
= \quad & P_2 \int_{\Re_1} f(\vec{r}|1_T)\mathrm{d}\vec{r} + P_1 \int_{\Re_2} f(\vec{r}|0_T)\mathrm{d}\vec{r} \\
= \quad & P_2 \int_{\Re-\Re_2} f(\vec{r}|1_T)\mathrm{d}\vec{r} + P_1 \int_{\Re_2} f(\vec{r}|0_T)\mathrm{d}\vec{r} \\
= \quad & P_2 \int_{\Re} f(\vec{r}|1_T)\mathrm{d}\vec{r} + \int_{\Re_2} [P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T)]\mathrm{d}\vec{r} \\
= \quad & P_2 + \int_{\Re_2} [P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T)]\, \mathrm{d}\vec{r} \\
= \quad & P_1 - \int_{\Re_1} [P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T)]\, \mathrm{d}\vec{r}.
\end{aligned}
$$

- The minimum error probability decision rule is

$$
\begin{cases}
P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T) \geq 0 & \Rightarrow \quad \text{decide “0” } (0_D) \\
P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T) < 0 & \Rightarrow \quad \text{decide “1” } (1_D)
\end{cases}.
$$

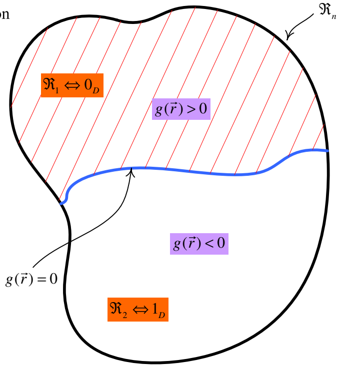


Pr[error] =

$$P_2 + \int_{\Re_2} \underbrace{[P_1 f(\vec{r}\,|\,0_T) - P_2 f(\vec{r}\,|\,1_T)]}_{g(\vec{r})} \mathrm{d}\vec{r}$$

$$= P_1 + \int_{\Re_1} \underbrace{[P_2 f(\vec{r}\,|\,1_T) - P_1 f(\vec{r}\,|\,0_T)]}_{-g(\vec{r})} \mathrm{d}\vec{r}$$

$\Rightarrow$ Optimal decision rule: $g(\vec{r}) \underset{1_D}{\overset{0_D}{\gtrless}} 0$



- Equivalently,

$$\frac{f(\vec{r}|1_T)}{f(\vec{r}|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}. \tag{1}$$

- The expression $\frac{f(\vec{r}|1_T)}{f(\vec{r}|0_T)}$ is called the *likelihood ratio*.
- The decision rule in (1) was derived without specifying any statistical properties of the noise process $\mathbf{w}(t)$.
- Simplified decision rule when the noise $\mathbf{w}(t)$ is zero-mean, white and Gaussian:

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2 + N_0 \ln\left(\frac{P_1}{P_2}\right).$$

- For the special case of $P_1 = P_2$ (signals are equally likely):

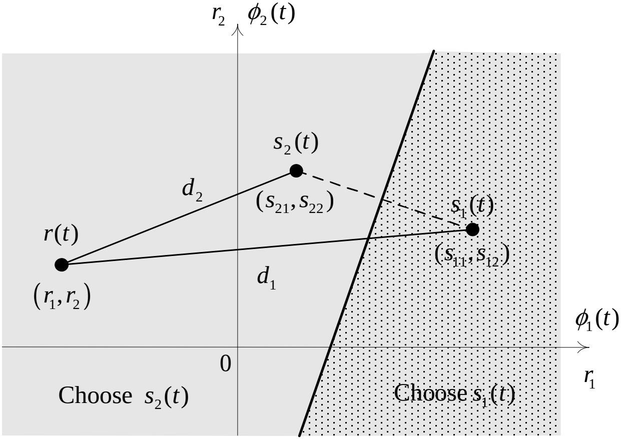
$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2.$$

$\Rightarrow$ *minimum-distance* receiver!

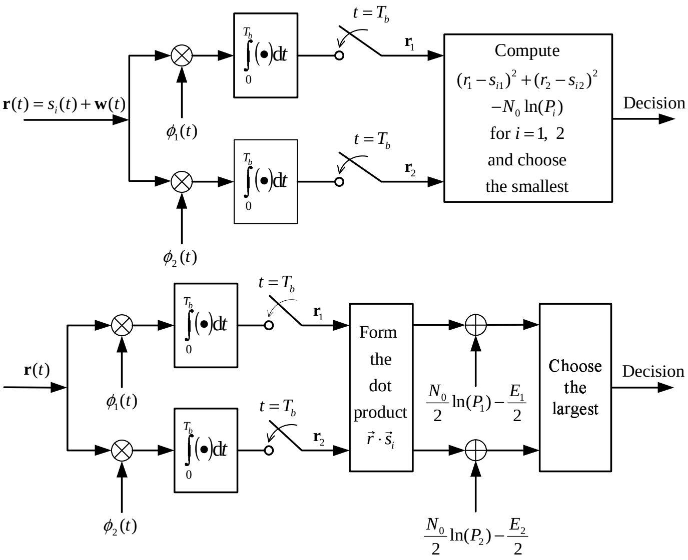
Minimum-Distance Receiver

$$\underbrace{(r_1 - s_{11})^2 + (r_2 - s_{12})^2}_{d_1^2} \underset{0_D}{\overset{1_D}{\gtrless}} \underbrace{(r_1 - s_{21})^2 + (r_2 - s_{22})^2}_{d_2^2}$$

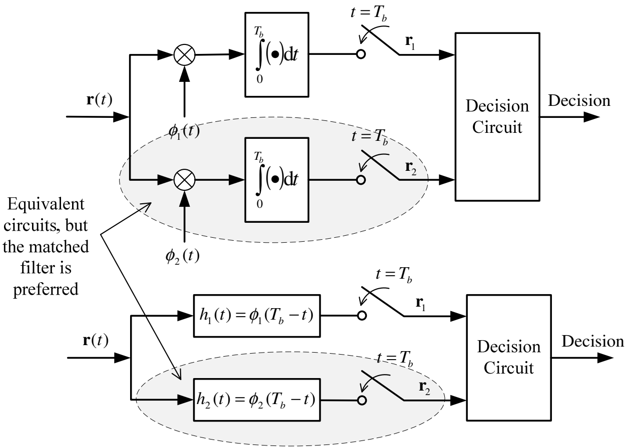
$$d_1^2 \underset{0_D}{\overset{1_D}{\gtrless}} d_2^2$$



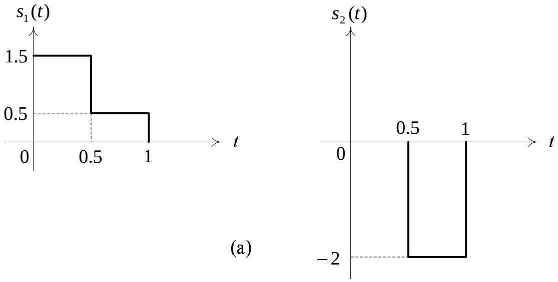
Correlation Receiver Implementation



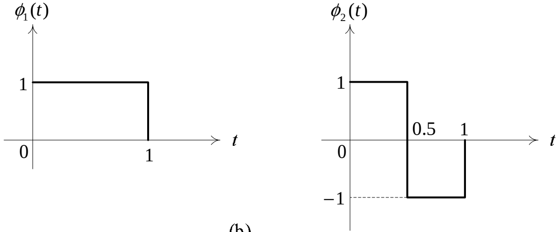
Receiver Implementation using Matched Filters



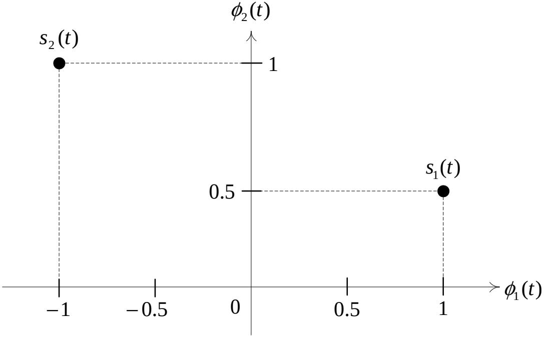
Example 5.6



(a)

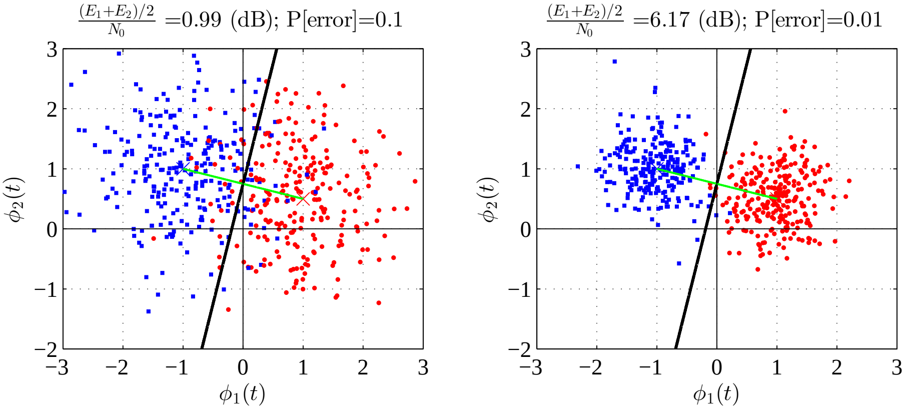


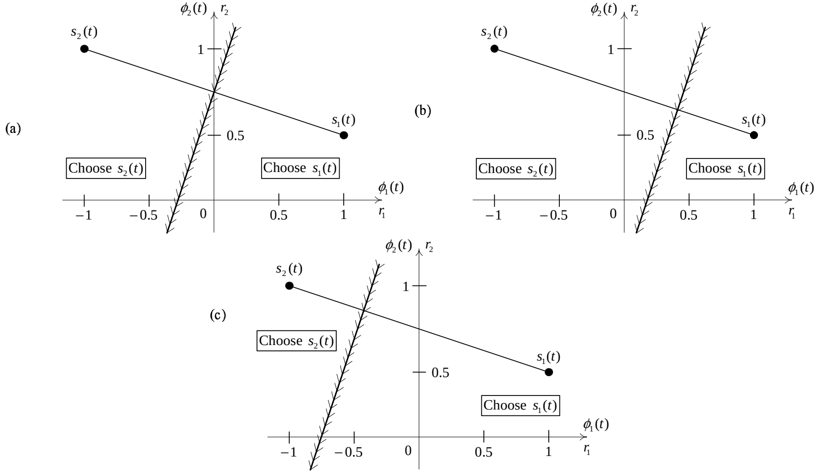
(b)



$$
\begin{aligned}
s_1(t) &= \phi_1(t) + \frac{1}{2}\phi_2(t), \\
s_2(t) &= -\phi_1(t) + \phi_2(t).
\end{aligned}
$$

For each value of the signal-to-noise ratio (SNR), Matlab simulation was conducted for transmitting/receiving 500 equally-likely bits.

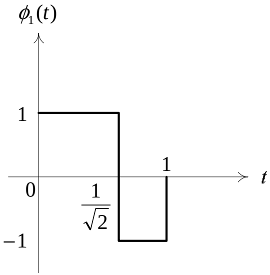


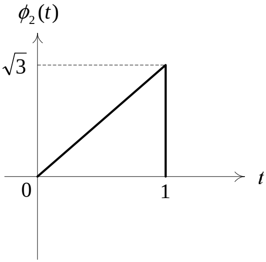


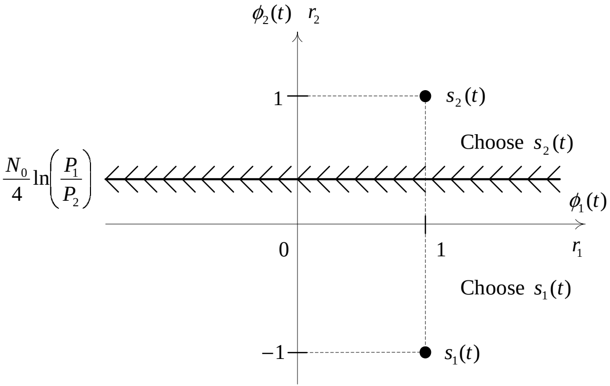
(a) $P_1 = P_2 = 0.5$, (b) $P_1 = 0.25,\ P_2 = 0.75$. (c) $P_1 = 0.75,\ P_2 = 0.25$.

Example 5.7

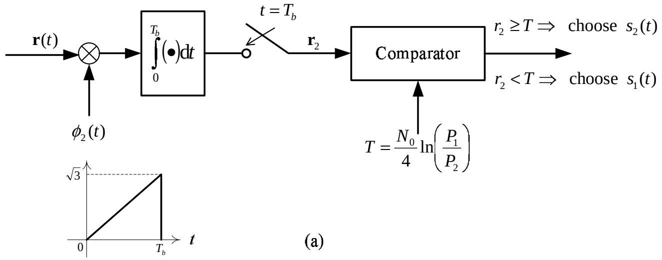
$$s_2(t) = \phi_1(t) + \phi_2(t),$$
$$s_1(t) = \phi_1(t) - \phi_2(t).$$





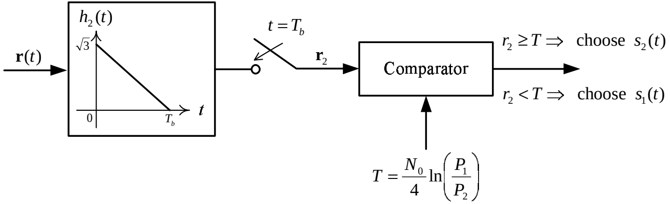


$\phi_2(t)$ r_2
1
$s_2(t)$
Choose $s_2(t)$
$\frac{N_0}{4}\ln\left(\frac{P_1}{P_2}\right)$
$\phi_1(t)$
0
1
r_1
Choose $s_1(t)$
-1
$s_1(t)$



$\mathbf{r}(t)$
$\int_0^{T_b} (\bullet) \mathrm{d}t$
$\phi_2(t)$
$t = T_b$
$\mathbf{r}_2$
Comparator
$r_2 \geq T \Rightarrow$ choose $s_2(t)$
$r_2 < T \Rightarrow$ choose $s_1(t)$
$T = \frac{N_0}{4} \ln\left(\frac{P_1}{P_2}\right)$
$\sqrt{3}$
0
T_b
t

(a)

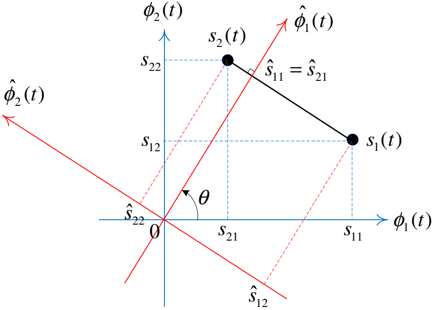


$\mathbf{r}(t)$
$h_2(t)$
$\sqrt{3}$
0
T_b
t
$t = T_b$
$\mathbf{r}_2$
Comparator
$r_2 \geq T \Rightarrow$ choose $s_2(t)$
$r_2 < T \Rightarrow$ choose $s_1(t)$
$T = \frac{N_0}{4} \ln\left(\frac{P_1}{P_2}\right)$

(b)

Implementation with One Correlator/Matched Filter

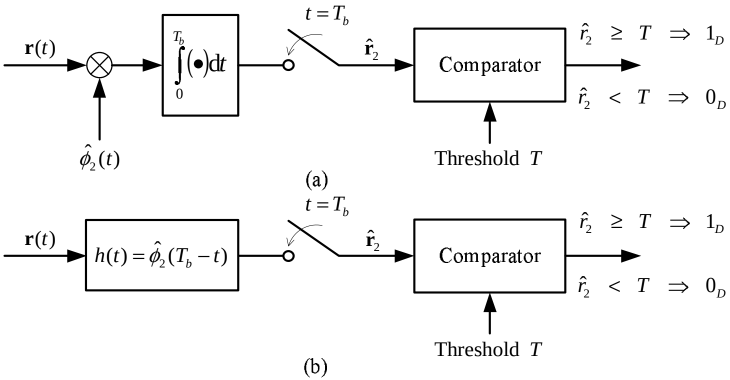
Always possible by choosing $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ such that one of the two basis functions is *perpendicular* to the line joining the two signals.



The optimum receiver is still the minimum-distance receiver. However the terms $(\hat{r}_1 - \hat{s}_{11})^2$ and $(\hat{r}_1 - \hat{s}_{21})^2$ are the same on both sides of the comparison and hence can be removed. This means that one does not need to compute $\hat{r}_1$!

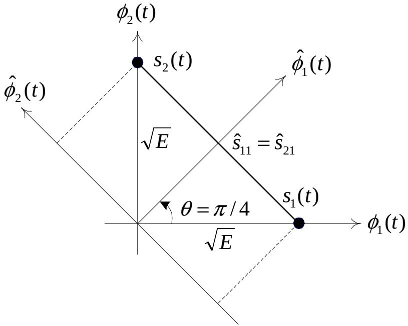
$$\underbrace{(\hat{r}_1 - \hat{s}_{11})^2 + (\hat{r}_2 - \hat{s}_{12})^2}_{d_1^2} \mathop{\gtrless}_{0_D}^{1_D} \underbrace{(\hat{r}_1 - \hat{s}_{21})^2 + (\hat{r}_2 - \hat{s}_{22})^2}_{d_2^2} \Leftrightarrow (\hat{r}_2 - \hat{s}_{12})^2 \mathop{\gtrless}_{0_D}^{1_D} (\hat{r}_2 - \hat{s}_{22})^2$$

$$\hat{r}_2 \mathop{\gtrless}_{0_D}^{1_D} \underbrace{\frac{\hat{s}_{22} + \hat{s}_{12}}{2}}_{\text{midpoint of two signals}} + \underbrace{\left(\frac{N_0/2}{\hat{s}_{22} - \hat{s}_{12}}\right) \ln\left(\frac{P_1}{P_2}\right)}_{\text{equal to 0 if } P_1 = P_2} \equiv T.$$

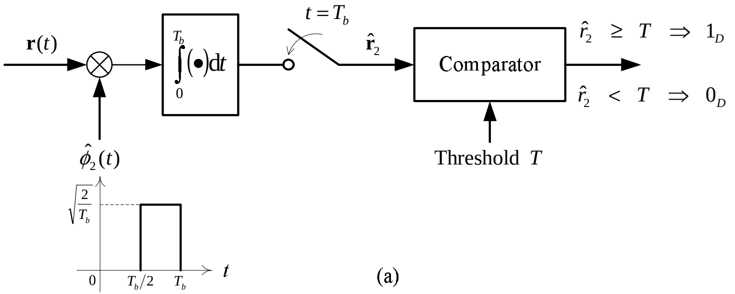


$$\hat{\phi}_2(t) = \frac{s_2(t) - s_1(t)}{(E_2 - 2\rho\sqrt{E_1 E_2} + E_1)^{\frac{1}{2}}}, \; T \equiv \frac{\hat{s}_{22} + \hat{s}_{12}}{2} + \left(\frac{N_0/2}{\hat{s}_{22} - \hat{s}_{12}}\right) \ln\left(\frac{P_1}{P_2}\right).$$

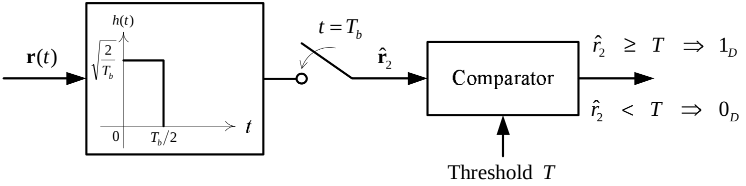
Example 5.8



$$
\begin{aligned}
\hat{\phi}_1(t) &= \frac{1}{\sqrt{2}}[\phi_1(t) + \phi_2(t)], \\
\hat{\phi}_2(t) &= \frac{1}{\sqrt{2}}[-\phi_1(t) + \phi_2(t)].
\end{aligned}
$$


r(t)
T_b
$\int_0^{T_b} (\bullet) dt$
$t = T_b$
$\hat{\mathbf{r}}_2$
Comparator
$\hat{r}_2 \geq T \Rightarrow 1_D$
$\hat{r}_2 < T \Rightarrow 0_D$
Threshold T
$\hat{\phi}_2(t)$
$\sqrt{\frac{2}{T_b}}$
0
$T_b/2$
T_b
t

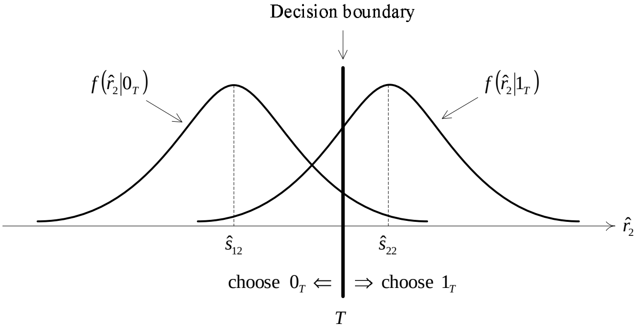
(a)


r(t)
$h(t)$
$\sqrt{\frac{2}{T_b}}$
0
$T_b/2$
t
$t = T_b$
$\hat{\mathbf{r}}_2$
Comparator
$\hat{r}_2 \geq T \Rightarrow 1_D$
$\hat{r}_2 < T \Rightarrow 0_D$
Threshold T

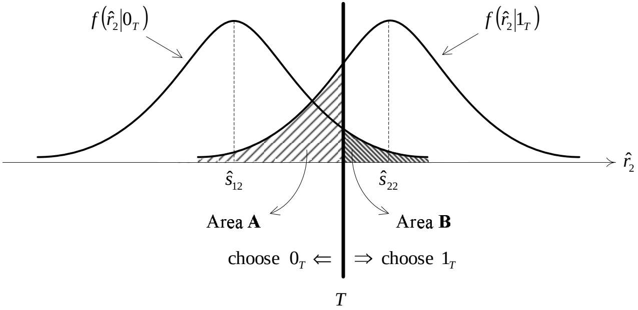
(b)

Receiver Performance

To detect $\mathbf{b}_k$, compare $\hat{\mathbf{r}}_2 = \int_{(k-1)T_b}^{kT_b} \mathbf{r}(t)\hat{\phi}_2(t)\mathrm{d}t$ to the threshold $T = \frac{\hat{s}_{12}+\hat{s}_{22}}{2} + \frac{N_0}{2(\hat{s}_{22}-\hat{s}_{12})} \ln\left(\frac{P_1}{P_2}\right)$.

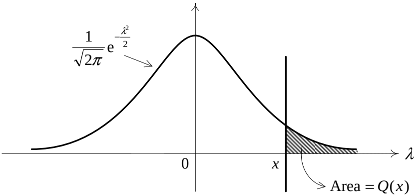


$$
\begin{aligned}
P[\text{error}] &= P[(0 \text{ transmitted and } 1 \text{ decided}) \text{ or } (1 \text{ transmitted and } 0 \text{ decided})] \\
&= P[(0_T, 1_D) \text{ or } (1_T, 0_D)].
\end{aligned}
$$

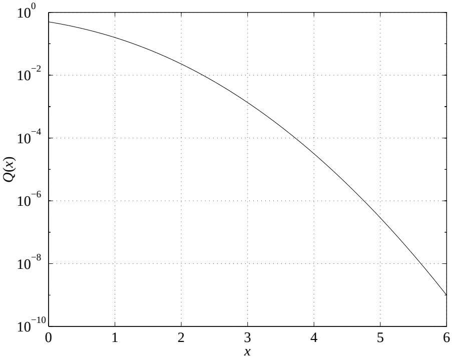


$$
\begin{aligned}
P[\text{error}] &= P[0_T, 1_D] + P[1_T, 0_D] = P[1_D|0_T]P[0_T] + P[0_D|1_T]P[1_T] \\
&= P_1 \underbrace{\int_T^{\infty} f(\hat{r}_2|0_T)\mathrm{d}\hat{r}_2}_{\text{Area }\mathbf{B}} + P_2 \underbrace{\int_{-\infty}^{T} f(\hat{r}_2|1_T)\mathrm{d}\hat{r}_2}_{\text{Area }\mathbf{A}} \\
&= P_1 Q\left(\frac{T - \hat{s}_{12}}{\sqrt{N_0/2}}\right) + P_2\left[1 - Q\left(\frac{T - \hat{s}_{22}}{\sqrt{N_0/2}}\right)\right].
\end{aligned}
$$

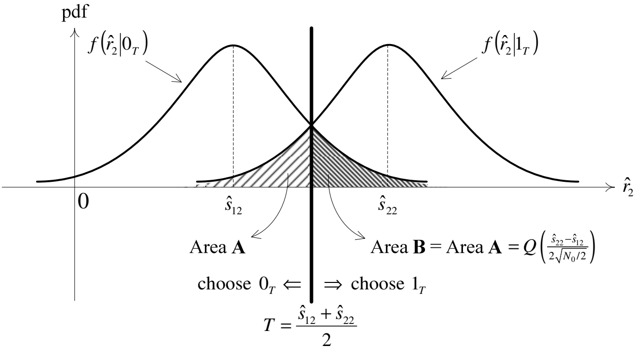
Q-function



$$Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{\lambda^2}{2}\right) \mathrm{d}\lambda.$$



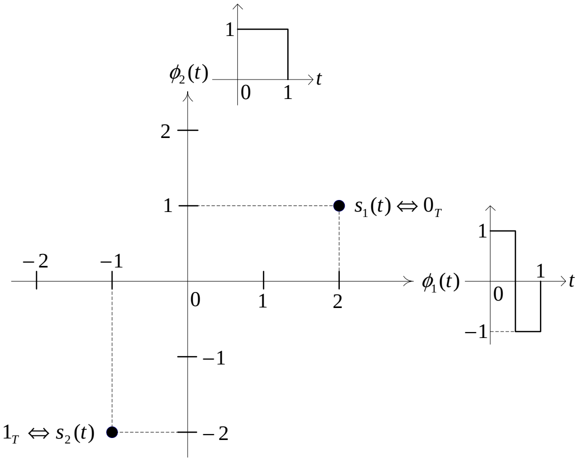
Performance when $P_1 = P_2$



$$P[\text{error}] = Q\left(\frac{\hat{s}_{22} - \hat{s}_{12}}{2\sqrt{N_0/2}}\right) = Q\left(\frac{\text{distance between the signals}}{2 \times \text{noise RMS value}}\right).$$

- Probability of error decreases as either the two signals become more dissimilar (increasing the distances between them) or the noise power becomes less.
- To maximize the distance between the two signals one chooses them so that they are placed 180° from each other $\Rightarrow s_2(t) = -s_1(t)$, i.e., *antipodal signaling*.
- The error probability does *not* depend on the signal shapes but only on the distance between them.

Example 5.9



(a) Determine and sketch the two signals $s_1(t)$ and $s_2(t)$.

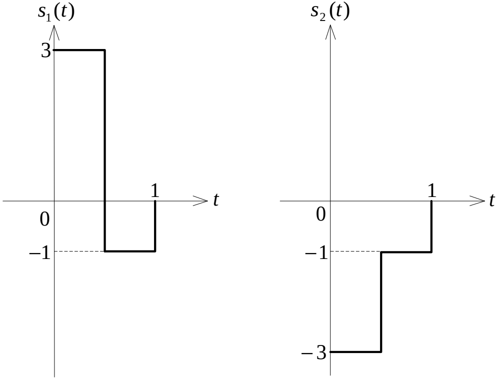
(b) The two signals $s_1(t)$ and $s_2(t)$ are used for the transmission of equally likely bits 0 and 1, respectively, over an additive white Gaussian noise (AWGN) channel. Clearly draw the decision boundary and the decision regions of the optimum receiver. Write the expression for the optimum decision rule.

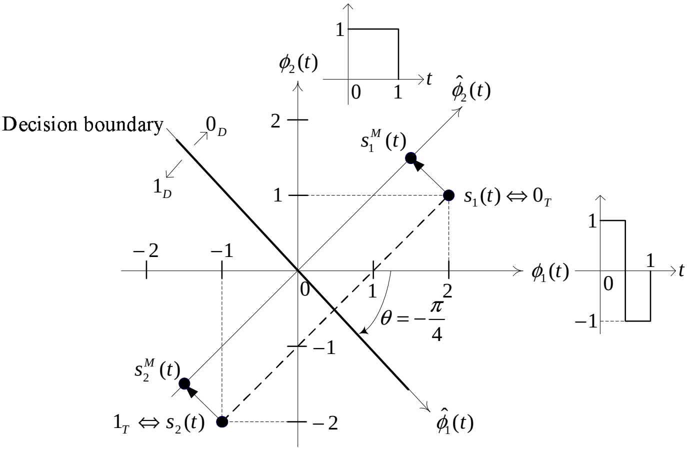
(c) Find and sketch the two orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ such that the optimum receiver can be implemented using only the projection $\hat{\mathbf{r}}_2$ of the received signal $\mathbf{r}(t)$ onto the basis function $\hat{\phi}_2(t)$. Draw the block diagram of such a receiver that uses a matched filter.

(d) Consider now the following argument put forth by your classmate. She reasons that since the component of the signals along $\hat{\phi}_1(t)$ is not useful at the receiver in determining which bit was transmitted, one should not even transmit this component of the signal. Thus she modifies the transmitted signal as follows:

$$\begin{aligned} s_1^{(\mathrm{M})}(t) &= s_1(t) - \Big(\text{component of } s_1(t) \text{ along } \hat{\phi}_1(t)\Big) \\ s_2^{(\mathrm{M})}(t) &= s_2(t) - \Big(\text{component of } s_2(t) \text{ along } \hat{\phi}_1(t)\Big) \end{aligned}$$

Clearly identify the locations of $s_1^{(\mathrm{M})}(t)$ and $s_2^{(\mathrm{M})}(t)$ in the signal space diagram. What is the average energy of this signal set? Compare it to the average energy of the original set. Comment.

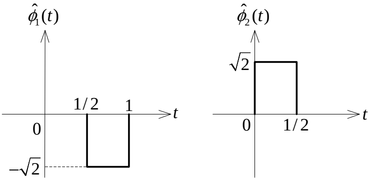

$s_1(t)$
3
1
t
0
−1
$s_2(t)$
1
t
0
−1
−3

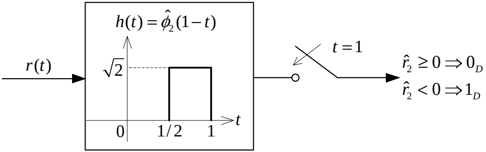


1
0
1
t
$\phi_2(t)$
$\hat{\phi}_2(t)$
Decision boundary
0_D
1_D
2
1
$s_1^M(t)$
$s_1(t) \Leftrightarrow 0_T$
-2
-1
0
1
2
$\phi_1(t)$
$\theta = -\frac{\pi}{4}$
-1
-2
$s_2^M(t)$
$1_T \Leftrightarrow s_2(t)$
$\hat{\phi}_1(t)$
1
0
1
t
-1

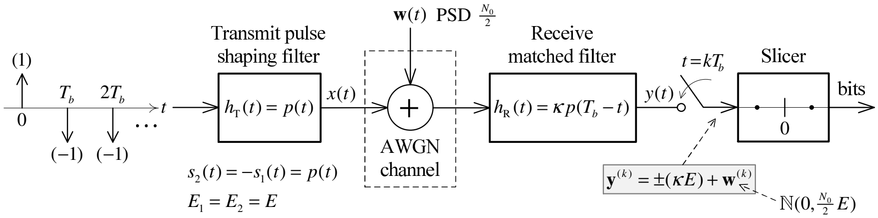
$$\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}.$$

$$\hat{\phi}_1(t) = \frac{1}{\sqrt{2}}[\phi_1(t) - \phi_2(t)], \quad \hat{\phi}_2(t) = \frac{1}{\sqrt{2}}[\phi_1(t) + \phi_2(t)].$$





Antipodal Signalling



- The pulse shaping filter $h_{\mathrm{T}} = p(t)$ defines the *power spectrum density* of the transmitted signal, which can be shown to be proportional to $|P(f)|^2$.
- The error performance, $P[\text{error}]$ only depends on the energy E of $p(t)$ and noise PSD level N_0. Specifically, the distance between $s_1(t)$ and $s_2(t)$ is $2\sqrt{E}$ (you should show this for yourself, algebraically or geometrically). Therefore

$$P[\text{error}] = Q\left(\sqrt{\frac{2E}{N_0}}\right).$$

- For antipodal signalling, the optimum decisions are performed by comparing the samples of the matched filter's output (sampled at exactly integer multiples of the bit duration) with a threshold 0. **Of course such an optimum decision rule does not change if the impulse response of the matched filter is scaled by a positive constant.**

- Scaling the matched filter's impulse response $h_{\mathrm{R}}(t)$ does not change the receiver performance because it scales both signal and noise components by the same factor, leaving the signal-to-noise ratio (SNR) of the decision variable unchanged!
- In the above block diagram, $h_{\mathrm{R}}(t) = \kappa p(T_b - t)$. We have been using $\kappa = 1/\sqrt{E}$ in order to represent the signals on the signal space diagram (which would be at $\pm\sqrt{E}$) and to conclude that the variance of the noise component is exactly $N_0/2$.
- For an arbitrary scaling factor κ, the signal component becomes $\pm\kappa E$, while the variance of the noise component is $\frac{N_0}{2}\kappa^2 E$. Thus, the SNR is

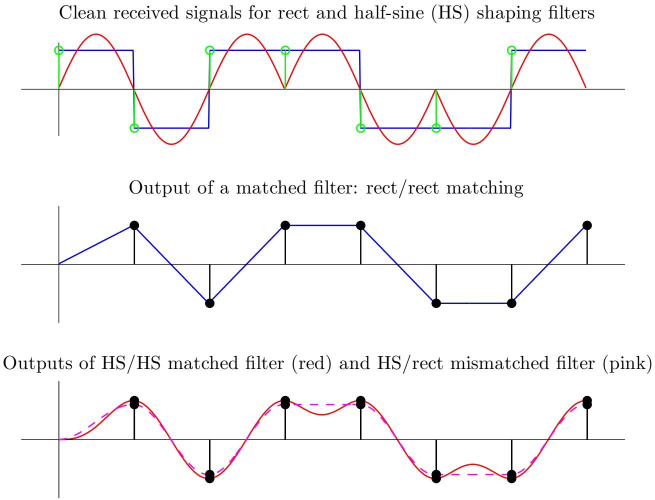
$$\mathrm{SNR} = \frac{\text{Signal power}}{\text{Noise power}} = \frac{(\pm\kappa E)^2}{\frac{N_0}{2}\kappa^2 E} = \frac{2E}{N_0}, \quad \text{(indepedent of } \kappa\text{!)}$$

- In terms of the SNR, the error performance of antipodal signalling is

$$\boxed{P[\text{error}] = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\text{SNR}}\right)}$$

- In fact, it can be proved that the receive filter that maximizes the SNR of the decision variable must be the *matched* filter. It is important to emphasize that the *matching* property here concerns the shapes of the impulse responses of the transmit and receive filters.

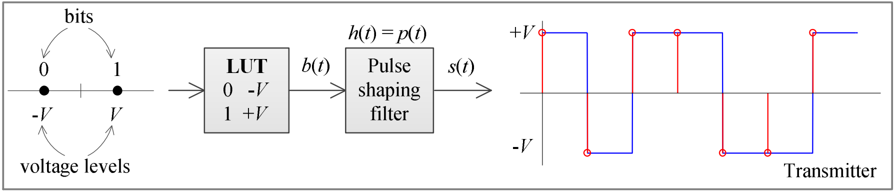
Outputs of the Matched/Mismatched Filters (No-Noise Scenario)

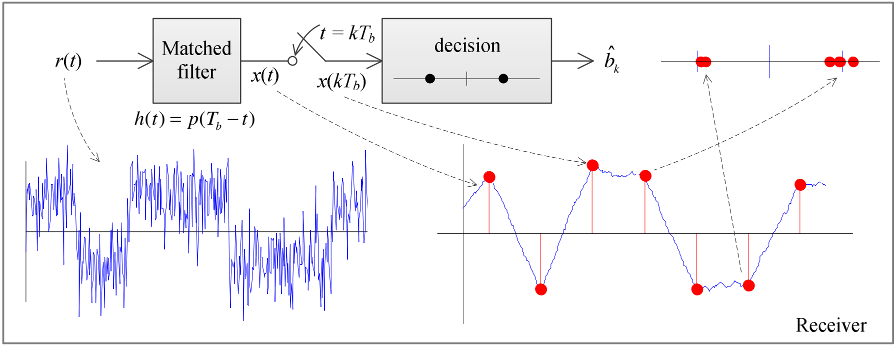


When the matched filter is used, sampling at exact multiples of the bit duration maximizes the power of the signal component in the decision variable, hence maximizing the SNR. A timing error (imperfect sampling) would reduce the power of the signal component, hence reducing the SNR, hence degrading the performance, i.e., increasing $P\,[\text{error}]$.

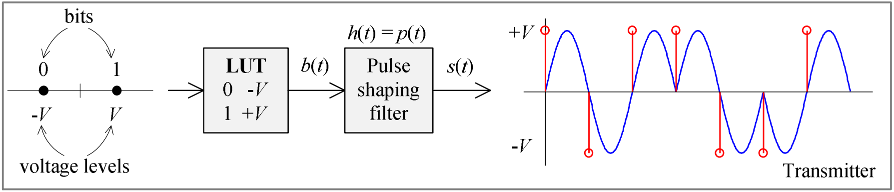
When the receive filter is not matched to the transmit filter, the power of the signal component and the SNR are not maximized, even under perfect sampling!

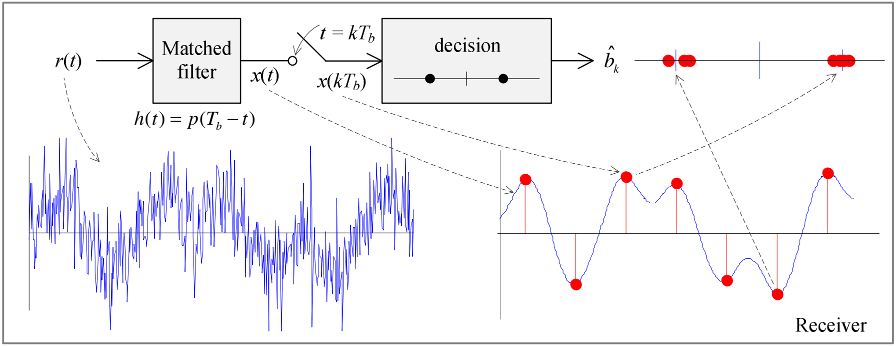
Antipodal Baseband Signalling with Rectangular Pulse Shaping





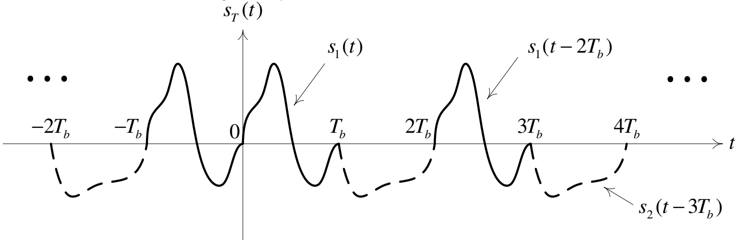
Antipodal Baseband Signalling with Half-Sine Pulse Shaping





PSD Derivation of Arbitrary Binary Modulation

- Applicable to *any* binary modulation with *arbitrary* a priori probabilities, but restricted to *statistically independent* bits.



$$\mathbf{s}_T(t) = \sum_{k=-\infty}^{\infty} \mathbf{g}_k(t), \quad \mathbf{g}_k(t) = \begin{cases} s_1(t-kT_b), & \text{with probability } P_1 \\ s_2(t-kT_b), & \text{with probability } P_2 \end{cases}.$$

The derivation on the next slide shows that:

$$S_{\mathbf{s}_T}(f) = \frac{P_1P_2}{T_b}|S_1(f) - S_2(f)|^2 + \sum_{n=-\infty}^{\infty} \left| \frac{P_1S_1\left(\frac{n}{T_b}\right) + P_2S_2\left(\frac{n}{T_b}\right)}{T_b} \right|^2 \delta\left(f - \frac{n}{T_b}\right).$$

$$\mathbf{s}_T(t) = \underbrace{E\{\mathbf{s}_T(t)\}}_{\text{DC}} + \underbrace{\mathbf{s}_T(t) - E\{\mathbf{s}_T(t)\}}_{\text{AC}} = v(t) + \mathbf{q}(t)$$

$$v(t) = E\{\mathbf{s}_T(t)\} = \sum_{k=-\infty}^{\infty} [P_1 s_1(t-kT_b) + P_2 s_2(t-kT_b)]$$

$$S_v(f) = \sum_{n=-\infty}^{\infty} |D_n|^2 \delta\left(f - \frac{n}{T_b}\right), \; D_n = \frac{1}{T_b}\left[P_1 S_1\left(\frac{n}{T_b}\right) + P_2 S_2\left(\frac{n}{T_b}\right)\right],$$

$$S_v(f) = \sum_{n=-\infty}^{\infty} \left|\frac{P_1 S_1\left(\frac{n}{T_b}\right) + P_2 S_2\left(\frac{n}{T_b}\right)}{T_b}\right|^2 \delta\left(f - \frac{n}{T_b}\right).$$

To calculate $S_{\mathbf{q}}(f)$, apply the basic definition of PSD:

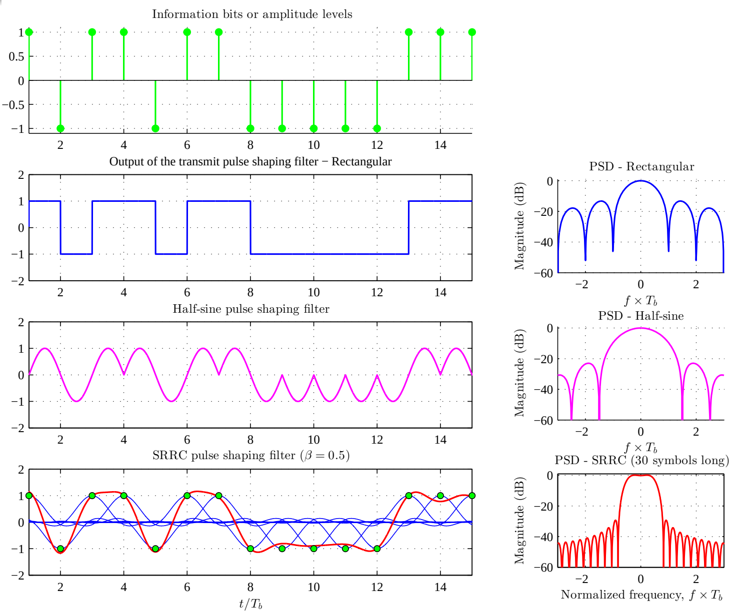
$$S_{\mathbf{q}}(f) = \lim_{T\to\infty} \frac{E\{|\mathbf{G}_T(f)|^2\}}{T} = \cdots = \frac{P_1 P_2}{T_b}|S_1(f) - S_2(f)|^2.$$

$$S_{\mathbf{s}_T}(f) = \frac{P_1 P_2}{T_b}|S_1(f) - S_2(f)|^2 + \sum_{n=-\infty}^{\infty} \left|\frac{P_1 S_1\left(\frac{n}{T_b}\right) + P_2 S_2\left(\frac{n}{T_b}\right)}{T_b}\right|^2 \delta\left(f - \frac{n}{T_b}\right).$$

For the special, but important case of antipodal signalling, $s_2(t) = -s_1(t) = p(t)$, and equally likely bits, $P_1 = P_2 = 0.5$, the PSD of the transmitted signal is solely determined by the Fourier transform of $p(t)$:

$$\boxed{S_{\mathbf{s}_T}(f) = \frac{|P(f)|^2}{T_b}}$$

Baseband Message Signals with Different Pulse Shaping Filters



Building A Binary (Antipodal) Comm. System in Labs #4 and #5

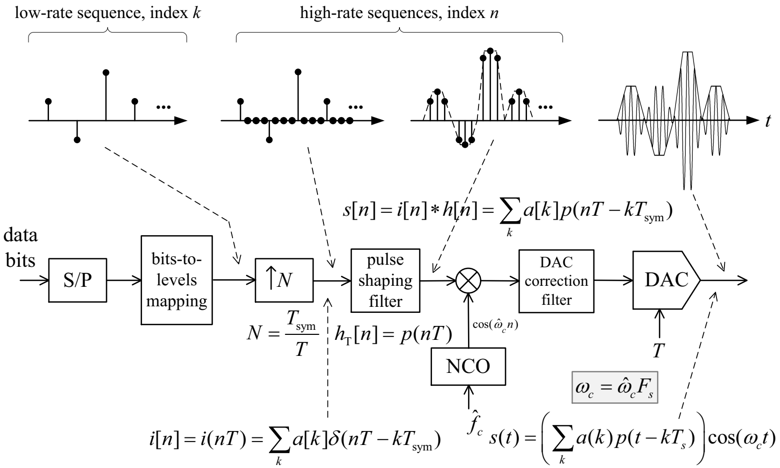


Figure 1: Block diagram of the transmitter.

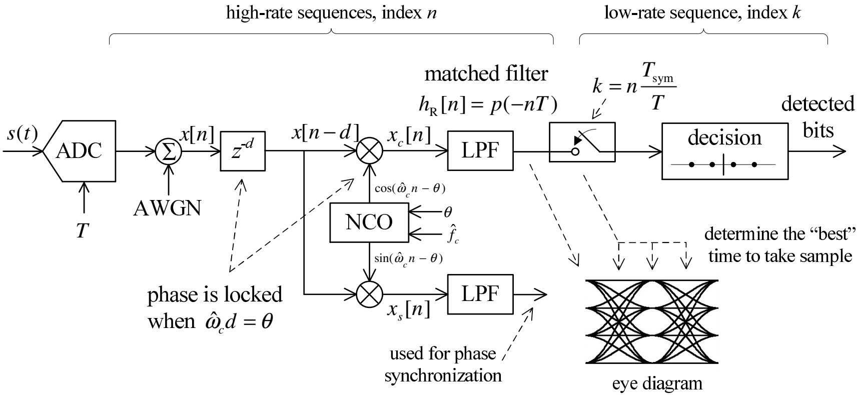


Figure 2: Block diagram of the receiver.