ELEC264: Signals And Systems

Topic 4: Continuous-Time Fourier Transform (CTFT)

- **Introduction to Fourier Transform**
- Fourier transform of CT **aperiodic** signals
- CT Fourier transform examples
- Convergence of the CT Fourier Transform
- Convergence examples
- Properties of CT Fourier Transform
- CT Fourier transform of **periodic** signals
- Summary
- Appendix: Transition from CT Fourier Series to CT Fourier Transform
- Appendix: Applications

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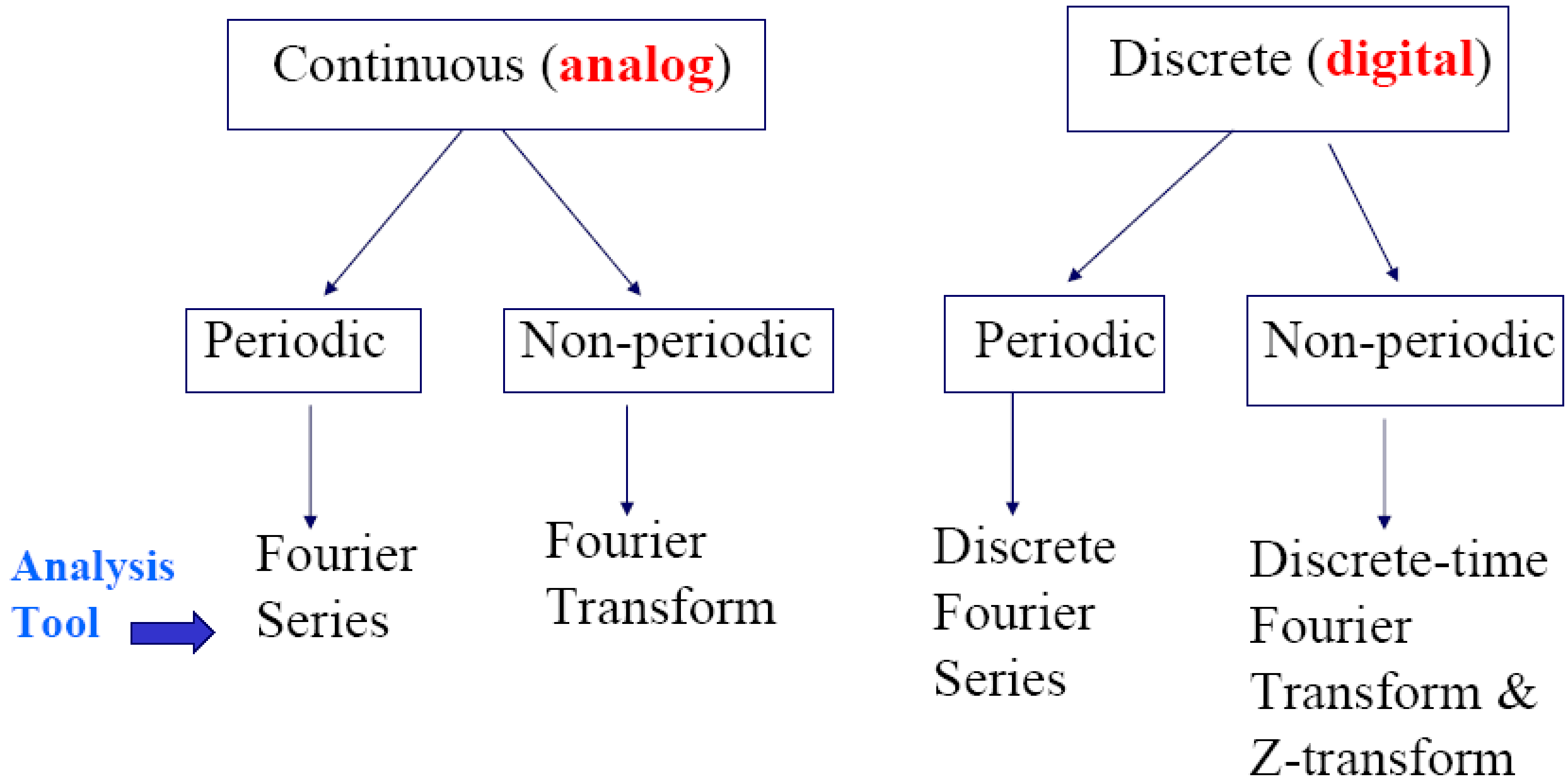
Figures and examples in these course slides are taken from the following sources:

- **A. Oppenheim, A.S. Willsky and S.H. Nawab, Signals and Systems, 2nd Edition, Prentice-Hall, 1997**
- **M.J. Roberts, Signals and Systems, McGraw Hill, 2004**
- **J. McClellan, R. Schafer, M. Yoder, Signal Processing First, Prentice Hall, 2003**

Fourier representations



A Fourier representation is <u>unique</u>, i.e., no two same signals in time domain give the same function in frequency domain

Fourier Series versus Fourier Transform

- Fourier Series (FS): a discrete representation of a **periodic** signal as a linear combination of complex exponentials
  - The CT Fourier Series cannot represent an aperiodic signal for all time

- Fourier Transform (FT): a continuous representation of a **not periodic** signal as a linear combination of complex exponentials
  - The CT Fourier transform represents an aperiodic signal for all time
- A not-periodic signal can be viewed as a periodic signal with an **infinite** period

Fourier Series versus Fourier Transform

- FS of periodic CT signals:

$$X[k]=\frac{1}{T}\int_0^T x(t)\cdot e^{-jk\omega_0 t}dt \;\; ; \;\; x(t)=\sum_{k=-\infty}^{\infty} X[k]\cdot e^{jk\omega_0 t} \;\; ; \omega_0 = 2\pi / T$$

- As the period increases T↑, ω_0↓
  ➔The harmonically related components ***kw₀*** become closer in frequency

- As T becomes infinite
  ➔the frequency components *form a continuum* and the FS sum becomes an integral

Fourier Series versus Fourier Transform

$\otimes$ CT Fourier Series: CTime - $\text{Per}_T \Rightarrow$ DFreq

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

$\otimes$ CT Inverse Fourier Series: DFreq $\Rightarrow$ CTime - Per_T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\otimes$ CT Fourier Transform: CTime $\Rightarrow$ CFreq

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$\otimes$ Inverse CT Fourier Transform: CFreq $\Rightarrow$ CTime

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

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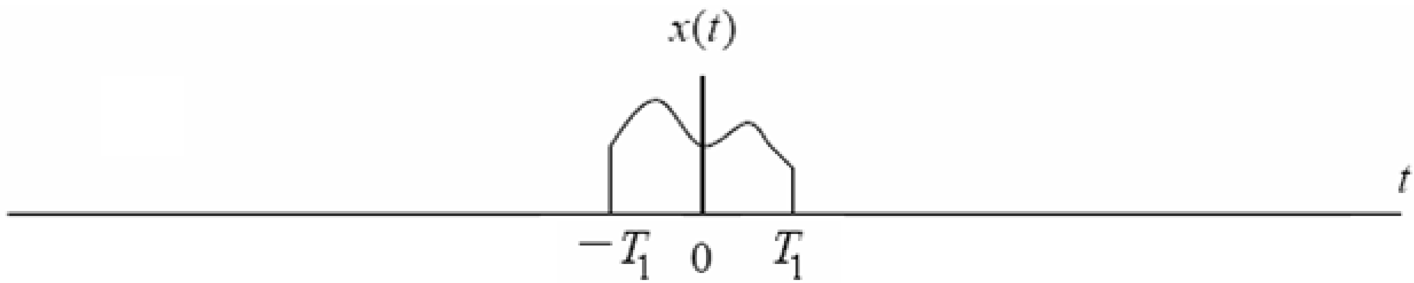
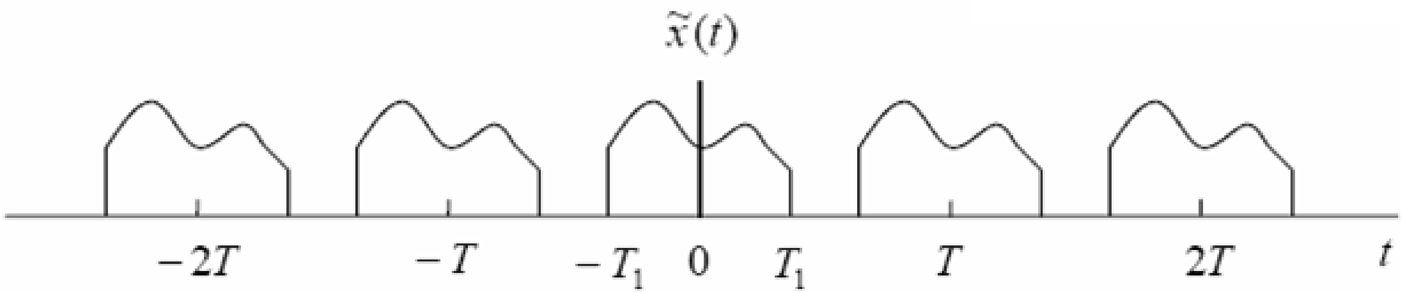
Fourier Transform of CT aperiodic signals

- Consider the CT aperiodic signal given below:



One can construct a periodic signal equal to $x(t)$ over the interval $-T_1 \le t \le T_1$ as follows:



Fourier Transform of CT aperiodic signals

- FS gives:

$$\boxed{\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

$T \rightarrow \infty$

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt, \quad \omega_0 = \frac{2\pi}{T}$$

- Define: $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

- This means that:

$$a_k = X(jk\omega_0),$$

T

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$T = \dfrac{2\pi}{\omega_0}$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

With $\omega_0 = \dfrac{2\pi}{T} : T \rightarrow \infty, \ \omega_0 \rightarrow 0 \Rightarrow \Sigma \rightarrow \int$

Fourier Transform of CT aperiodic signals

- As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$
  - ➔ ω_0 approaches zero
  - ➔ uncountable number of harmonics $k\omega_0$
  - ➔ Integral instead of

$$\text{Fourier Synthesis (inverse transform)}: x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$\text{Fourier analysis (forward transform)}: X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

CT Fourier transform for aperiodic signals

- Forward $\boxed{f \text{ form}}$ Inverse

$$X(f) = \mathrm{FT}(x(t)) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt \qquad x(t) = \mathrm{FT}^{-1}(X(f)) = \int_{-\infty}^{\infty} X(f) e^{+j2\pi ft}\, df$$

- Forward $\boxed{\omega \text{ form}}$ Inverse

$$X(j\omega) = \mathrm{FT}(x(t)) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \qquad x(t) = \mathrm{FT}^{-1}(X(j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t}\, d\omega$$

- Commonly-used notation:

$$x(t) \xleftrightarrow{\mathrm{F}} X(f) \quad \text{or} \quad x(t) \xleftrightarrow{\mathrm{F}} X(j\omega)$$

CT Fourier Transform of aperiodic signal

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

- The CT FT expresses
  - a finite-amplitude **aperiodic signal** x(t)
  - as **a linear combination** (integral) of
    - an infinite continuum of **weighted**, **complex sinusoids**, each of which is **unlimited in time**
- *Time limited* means “having non-zero values only for a finite time”
- x(t) is, in general, time-limited but must not be

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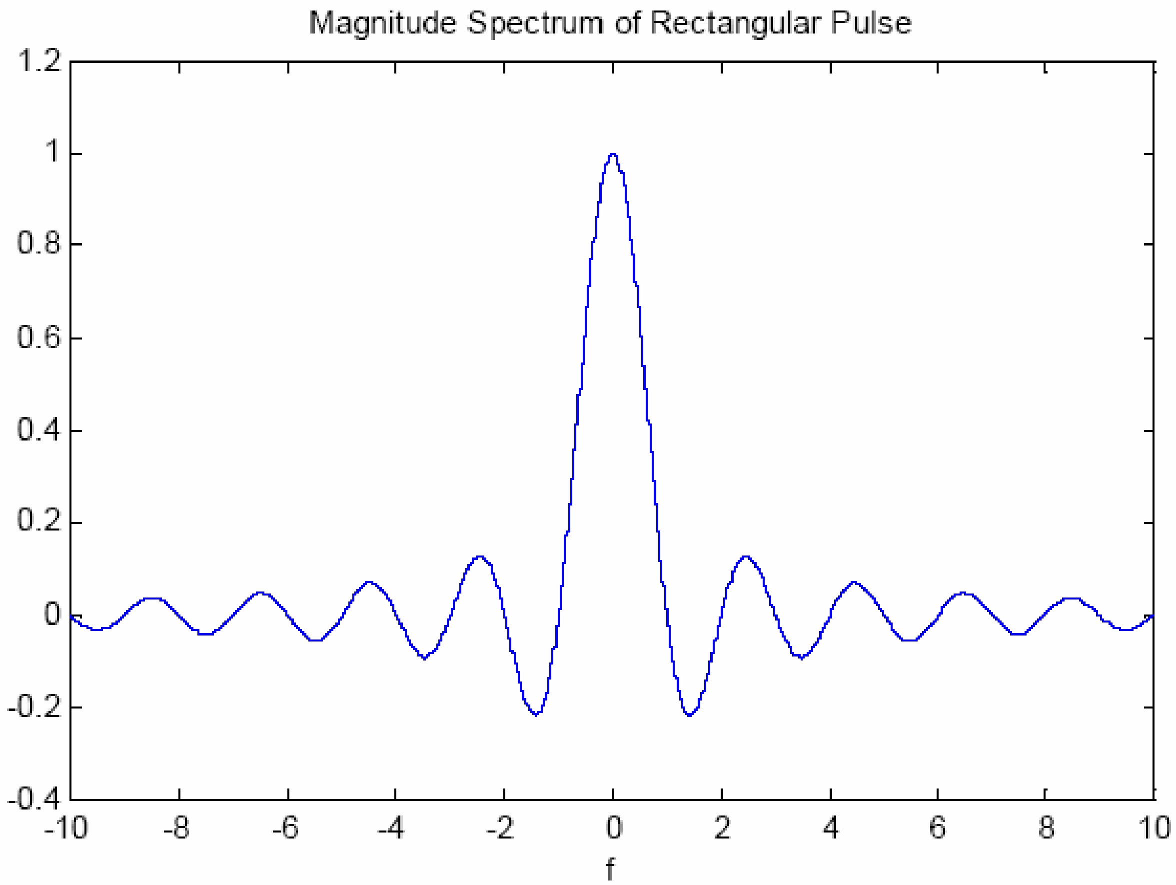
CTFT: Rectangular Pulse Function

$$x(t)=\begin{cases}1 & -T/2<t<T/2\\ 0 & \text{otherwise}\end{cases} \Leftrightarrow X(j\omega)=2\frac{\sin(\omega T)}{\omega}$$ $\boxed{\textit{sinc function}}$



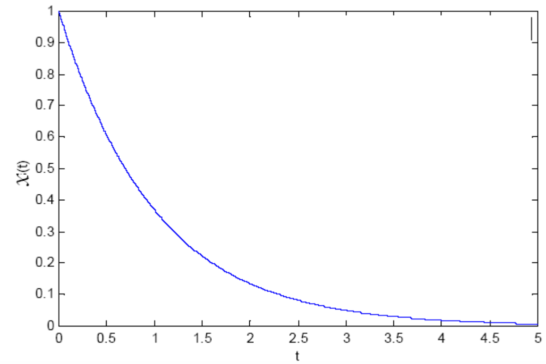
- The peaks of the FT magnitude drops slowly This is because the pulse function has sharp transition, which contributes to very high frequency in the signal

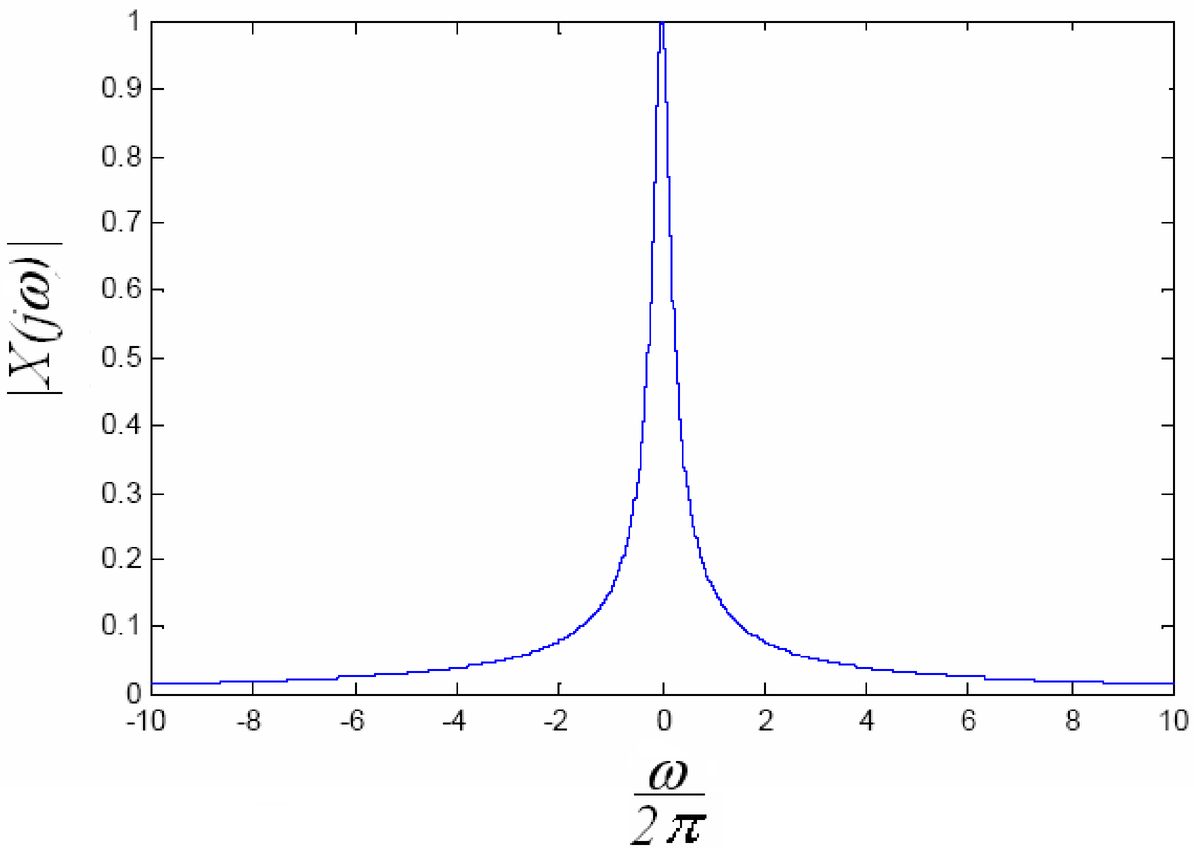
CTFT: Exponential Decay (Right-sided)

$$x(t) = e^{-at}u(t) \qquad (a > 0)$$

$$\mathcal{F}[e^{-at}u(t)] \;:\; \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$: \; \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$





The FT magnitude drops much faster than for the pulse function. This is because the exponential decay function does not has sharp transition

TABLE
Some Selected Fourier Transform Pairs

$x(t)$	$X(\omega)$
1. 1	$2\pi\delta(\omega)$
2. $u(t)$	$\pi\delta(\omega) + \dfrac{1}{j\omega}$
3. $\delta(t)$	1
4. $\delta(t - t_0)$	$\exp[-j\omega t_0]$
5. $\text{rect}(t/\tau)$	$\tau \operatorname{sinc} \dfrac{\omega\tau}{2\pi} = \dfrac{2 \sin \omega\tau/2}{\omega}$
6. $\dfrac{\omega_B}{\pi} \operatorname{sinc} \dfrac{\omega_B t}{\pi} = \dfrac{\sin \omega_B t}{\pi t}$	$\text{rect}(\omega/2\omega_B)$
7. $\operatorname{sgn} t$	$\dfrac{2}{j\omega}$
8. $\exp[j\omega_0 t]$	$2\pi\delta(\omega - \omega_0)$
9. $\displaystyle\sum_{n=-\infty}^{\infty} a_n \exp[jn\omega_0 t]$	$2\pi \displaystyle\sum_{n=-\infty}^{\infty} a_n \delta(\omega - n\omega_0)$
10. $\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
11. $\sin \omega_0 t$	$\dfrac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$\delta(t) = \frac{du(t)}{dt}.$$

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Convergence of CT Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

- Dirichlet's sufficient conditions for the convergence of Fourier transform are similar to the conditions for the CT-FS:

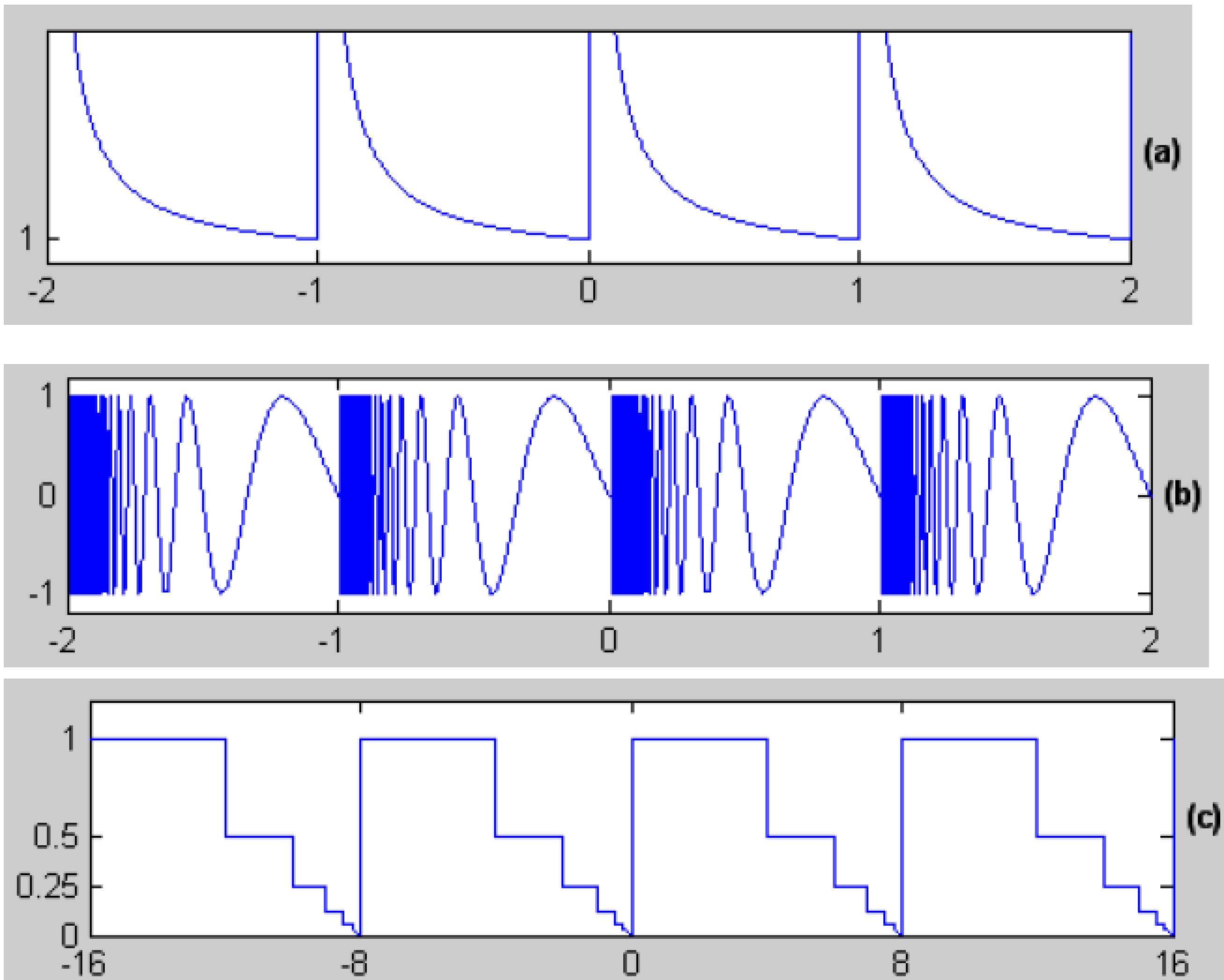
a) *x(t)* must be absolutely integrable

$$\int_{-\infty}^{+\infty} |x(t)|\,dt < \infty$$

b) *x(t)* must have a finite number of maxima and minima within any finite interval

c) *x(t)* must have a finite number of discontinuities, all of finite size, within any finite interval

Convergence of CT Fourier Transform



Convergence of CT Fourier Transform: Example 1

- Find the Fourier transform of the following signal:

$$\boxed{x(t) = e^{-at}u(t), \quad \mathrm{Re}\{a\} > 0}$$

Solution: $x(t)$ meets all three Dirichlet conditions. Using $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ we will have:

$$X(j\omega) = \int_{0}^{+\infty} e^{-at}e^{-j\omega t}dt$$

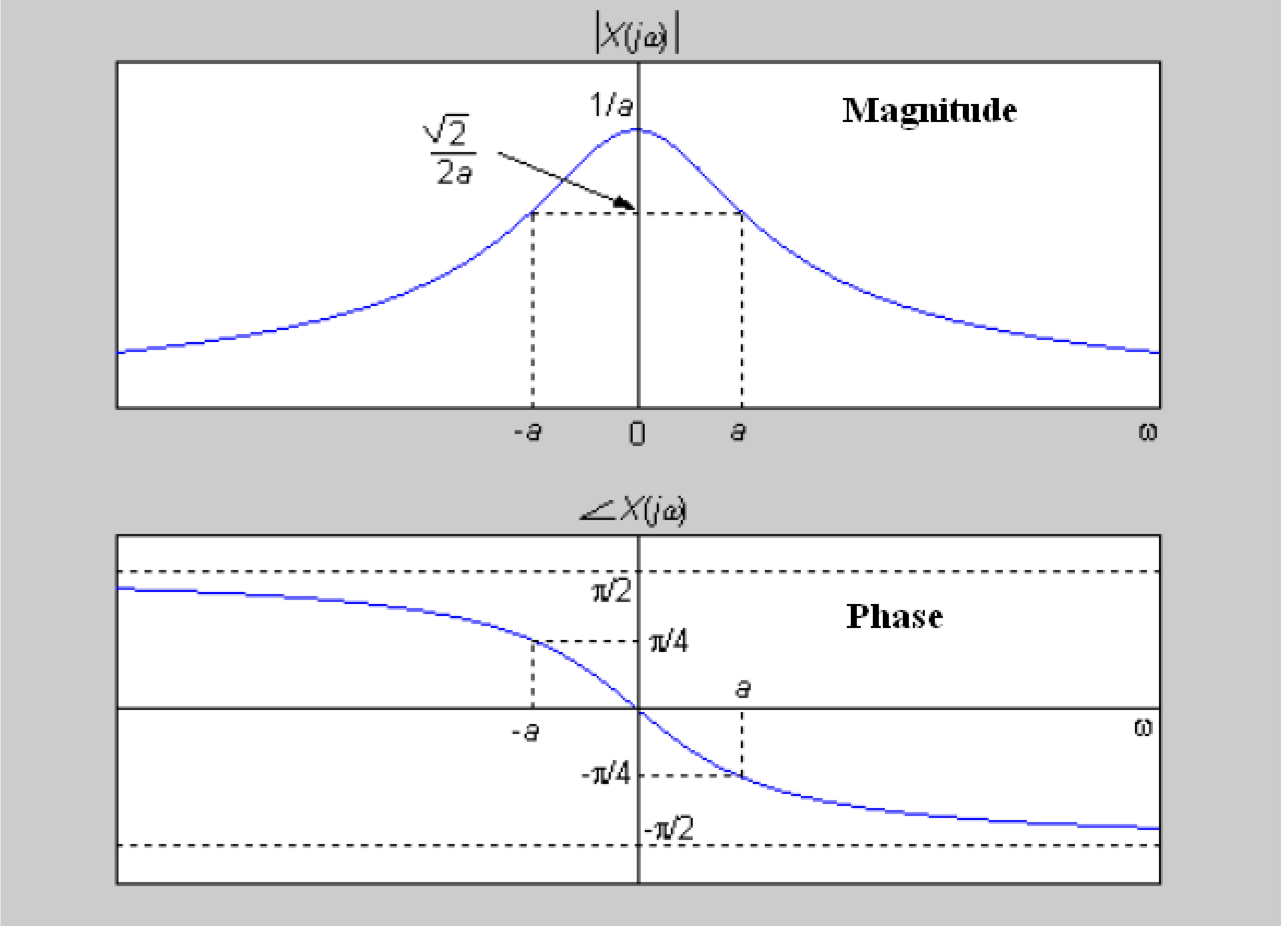
$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega}, \quad \mathrm{Re}\{a\} > 0$$

Like the Fourier series coefficients, the Fourier transform is a complex function of frequency in general and must be represented in two separate figures

Convergence of CT Fourier Transform: Example 1

The magnitude and phase of $X(j\omega)$ for a real value of a for this example are



Convergence of CT Fourier Transform: Example 2

- Find the Fourier transform of the following signal:

$$\boxed{x(t) = e^{-a|t|}u(t), \quad \mathrm{Re}\{a\} > 0}$$

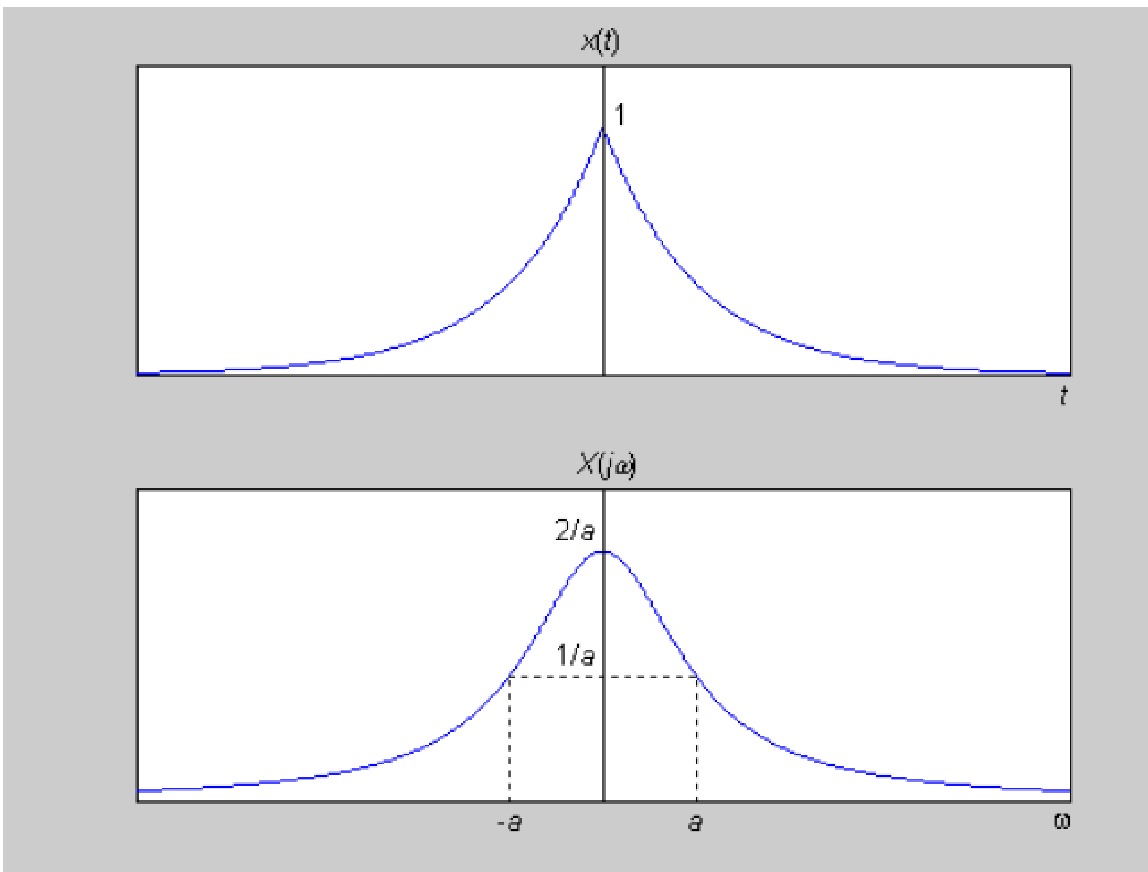
Solution: $x(t)$ meets all three Dirichlet conditions. We know that $|t| := \begin{cases} -t & t < 0 \\ t & t > 0 \end{cases}$

So, using $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ we will have:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t} dt \\ &= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{+\infty} e^{-at} e^{-j\omega t} dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2}, \quad \mathrm{Re}\{a\} > 0 \end{aligned}$$

Convergence of CT Fourier Transform: Example 2

- The Fourier transform for this example is real at all frequencies
- The time signal and its Fourier transform are



Convergence of CT Fourier Transform: Example 3

- Properties of unit impulse:

$$\delta(t) = \frac{du(t)}{dt}. \qquad \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

- Find the Fourier transform of the unit impulse signal

Solution: Using $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ and the sifting property of the impulse we will have:

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}dt = 1$$

Convergence of CT Fourier Transform: Example 4

- Find the Fourier transform of the following rectangular pulse signal

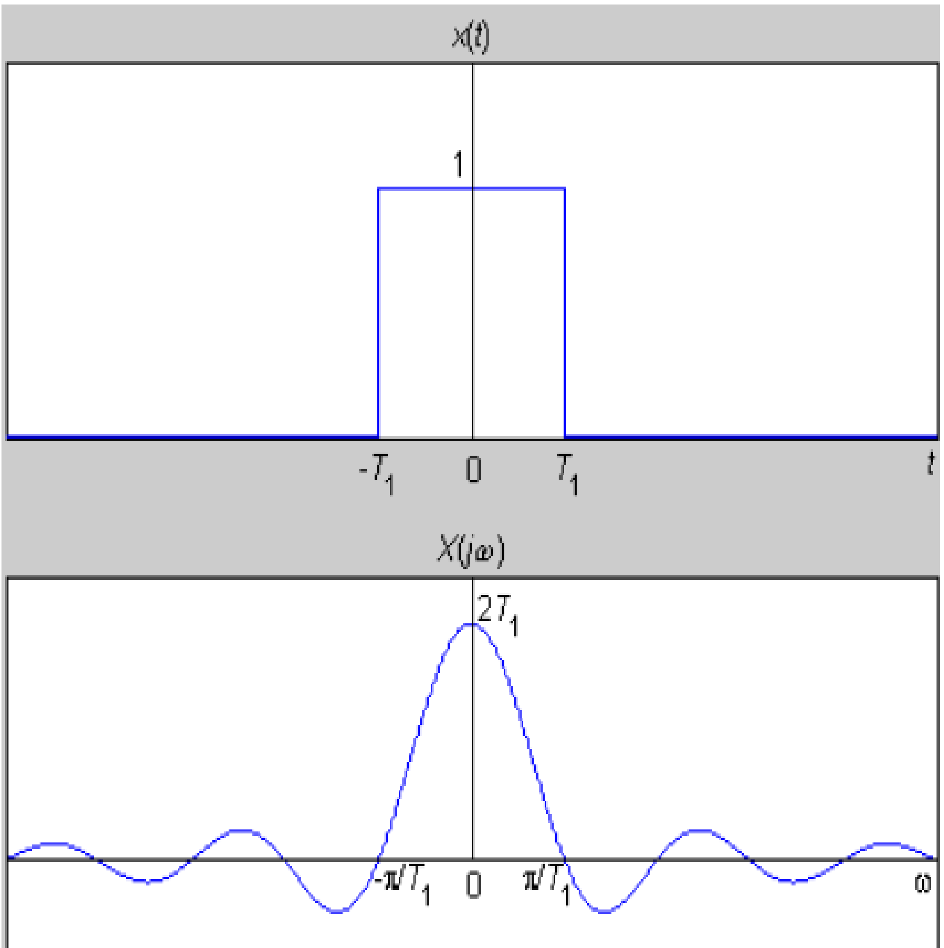
$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}.$$

➔**discontinuities at t=T_1 and t=-T_1**

Solution: Using $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ we will have:

$$X(j\omega) = \int_{-T_1}^{+T_1} e^{-j\omega t} dt = 2\frac{\sin(\omega T_1)}{\omega}.$$

Convergence of CT Fourier Transform: Example 4



- ➔Reducing the width of x(t) will have an **opposite** effect on X(jω)
- ➔Using the inverse Fourier transform, we get **x_{rec}(t) which is equal to x(t)** at all points except discontinuities ($t=T_1$ & $t=-T_1$), where the inverse Fourier transform is equal to the average of the values of x(t) on both sides of the discontinuity

Convergence of CT Fourier Transform: Example 5

- Find a time signal $x(t)$ whose Fourier transform is given by:

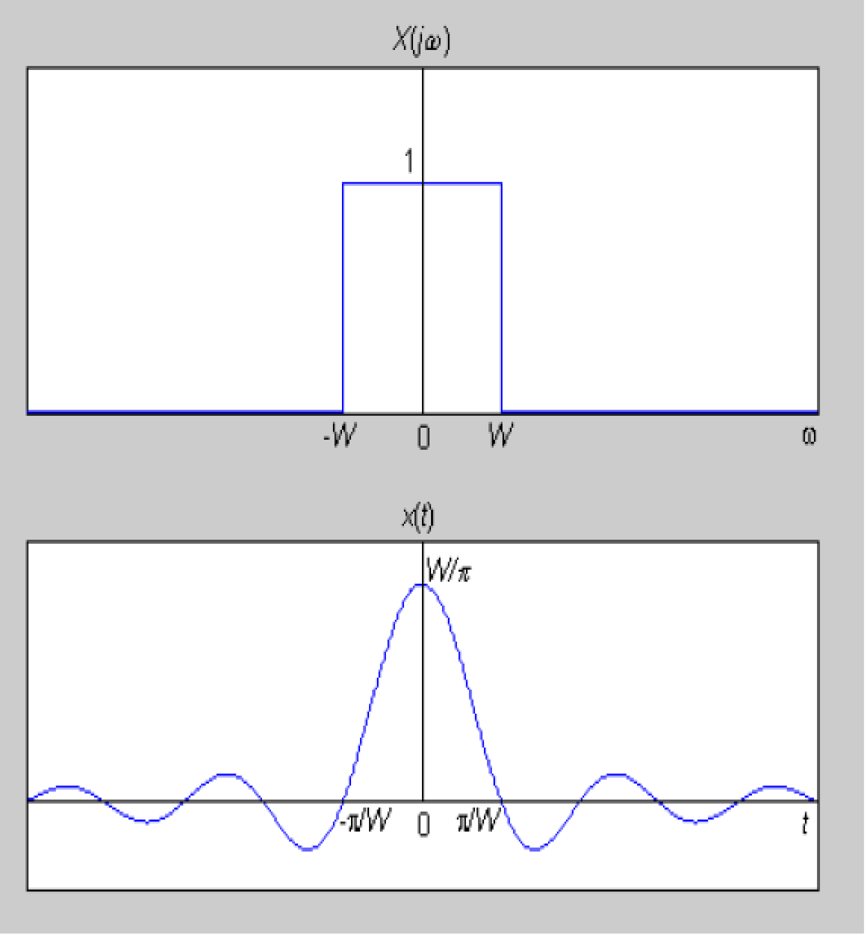
$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

Solution: Using $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$ we will have:

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin(Wt)}{\pi t}$$

The Fourier transform $X(j\omega)$ and the corresponding time signal $x(t)$ are shown in the next slide

Convergence of CT Fourier Transform: Example 5



- This example shows the <u>reveres effect</u> in the time and frequency domains <u>in terms of the width</u> of the time signal and the corresponding FT

Convergence of CT Fourier Transform: Example 5

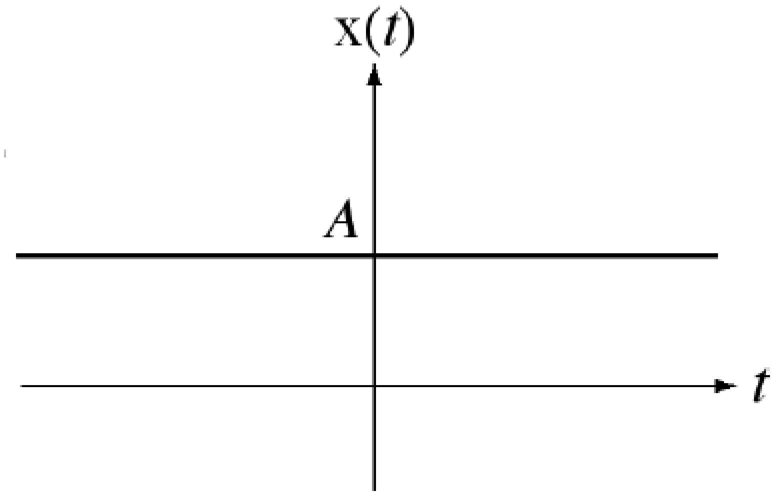
- Functions of the form $f(x)=\frac{\sin(ax)}{bx}$ are very frequently used in signals and systems. In general, the function $\frac{\sin(\pi x)}{\pi x}$ in referred to as sinc function. So, the time signal of Example 5.. can be expressed in terms of the $\boxed{\text{sinc function}}$ as follows:

$$x(t)=\frac{\sin(Wt)}{\pi t}=\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

Convergence of the CT-FT: Generalization

- Let $x(t) = A$
- Then from the definition of the CT Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} A e^{-j2\pi ft} dt = A \int_{-\infty}^{\infty} e^{-j2\pi ft} dt$$

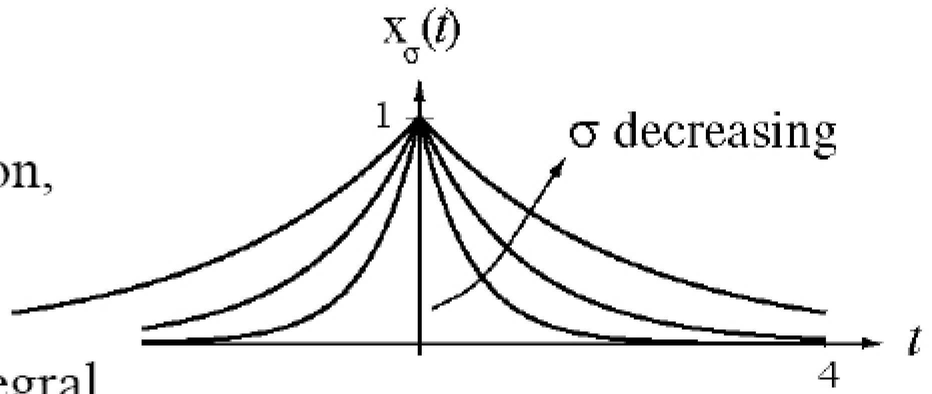


- This integral does not converge so, strictly speaking, the CT Fourier Transform does not exist

Convergence of the CTFT: Generalization

- But consider a similar function,

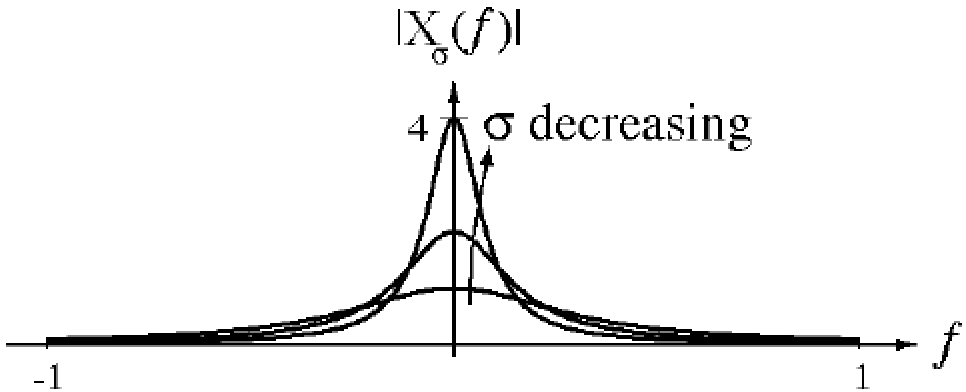
$$x_\sigma(t) = Ae^{-\sigma|t|} \quad , \quad \sigma > 0$$



- Its CT Fourier Transform integral

$$X_\sigma(f) = \int_{-\infty}^{\infty} Ae^{-\sigma|t|} e^{-j2\pi ft}\, dt$$

does converge



Convergence of the CTFT: Generalization

- Carrying out the integral, $X_\sigma(f) = A\dfrac{2\sigma}{\sigma^2 + (2\pi f)^2}$
- Now let σ approach zero
- If $f \neq 0$ then $\lim_{\sigma \to 0} A\dfrac{2\sigma}{\sigma^2 + (2\pi f)^2} = 0$. The area under this function is

$$\text{Area} = A\int_{-\infty}^{\infty} \frac{2\sigma}{\sigma^2 + (2\pi f)^2}\,df$$

which is equal to *A*, independent of the value of σ

- So, in the limit as σ approaches zero, the CT Fourier Transform has an area of *A* and is zero unless *f* = 0
  - This exactly defines an impulse of strength, *A*. Therefore

$$\boxed{A \xleftrightarrow{\mathcal{F}} A\delta(f)}$$

Convergence of the CTFT: Generalization

By a similar process it can be shown that

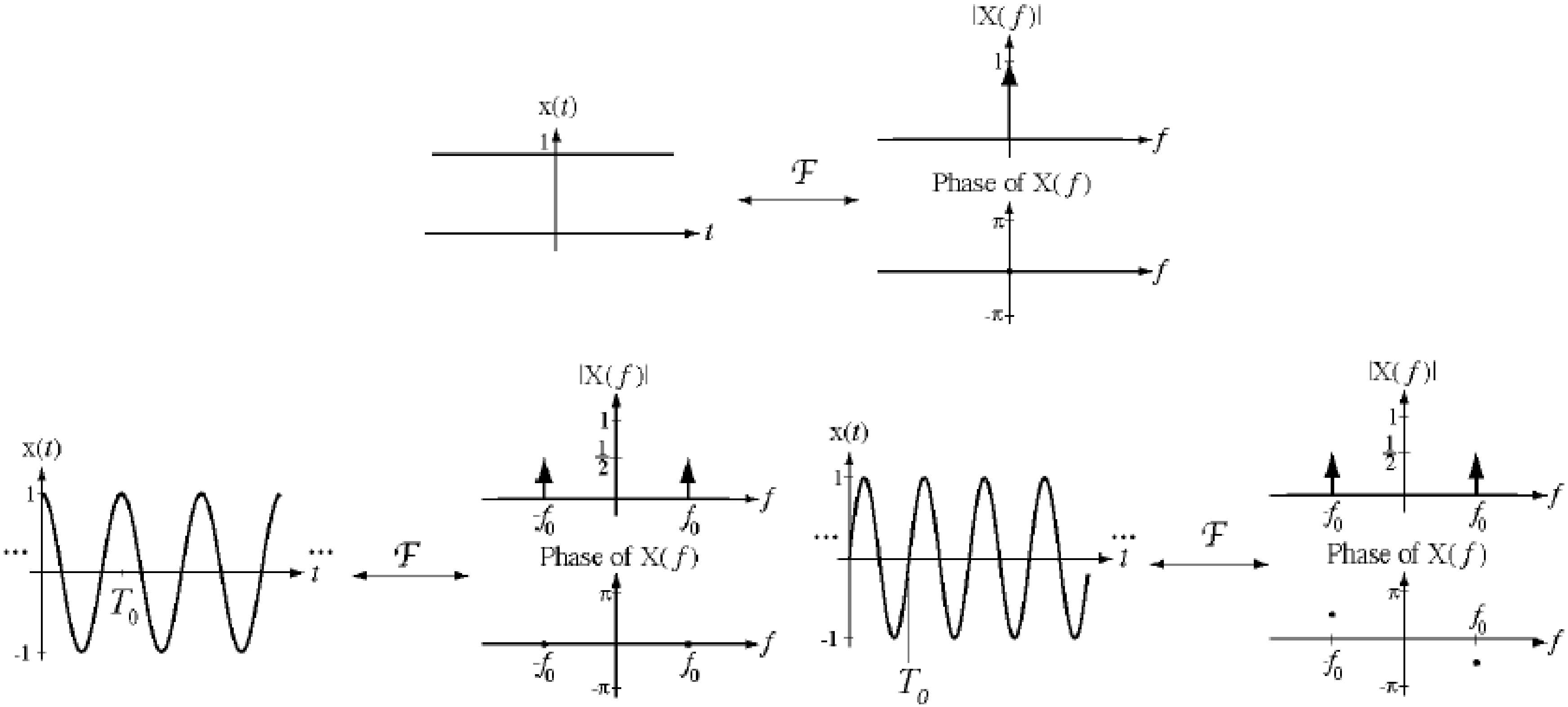
$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

and

$$\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{j}{2}\left[\delta(f + f_0) - \delta(f - f_0)\right]$$

These CT Fourier transforms which involve impulses are called *generalized* Fourier transforms

Convergence of the CTFT: Generalization



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Properties of the CT Fourier Transform

- The properties are useful in determining the Fourier transform or inverse Fourier transform
- They help to represent a given signal in term of operations (e.g., convolution, differentiation, shift) on another signal for which the Fourier transform is known
- Operations on {x(t)} $\Leftrightarrow$ Operations on {X(jω)}
- Help find analytical solutions to Fourier transform problems of complex signals
- Example:

$$FT\{\, y(t) = a^t u(t-5)\} \;\rightarrow delay\, and\; multiplication$$

Properties of the CT Fourier Transform

- The properties of the CT Fourier transform are very similar to those of the CT Fourier series
- Consider two signals x(t) and y(t) with Fourier transforms X(jω) and Y(jω), respectively (or X(f) and Y(f))
- The following properties can easily been shown using

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

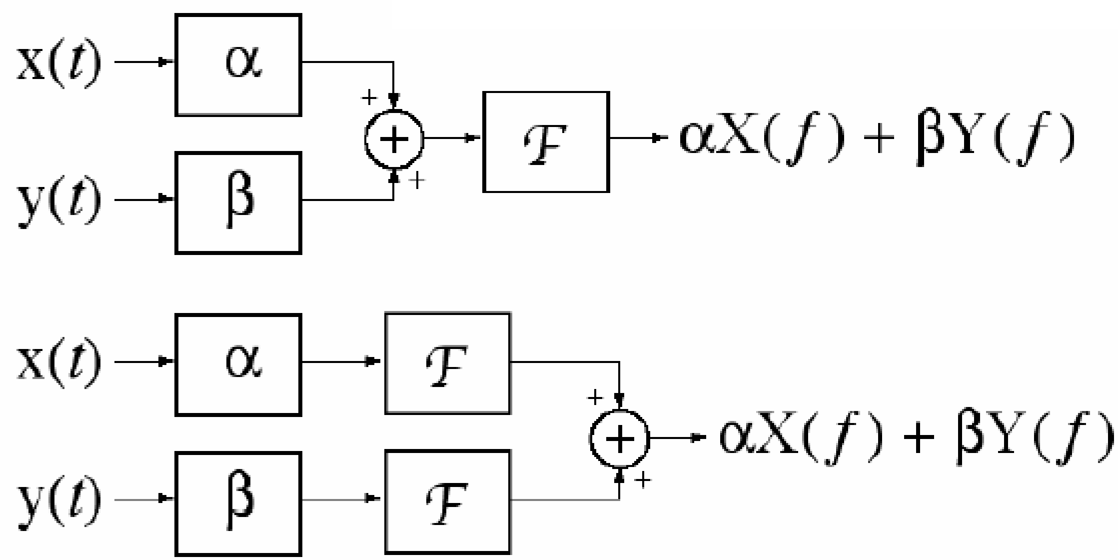
Properties of the CTFT: Linearity

- Linearity:

$$x(t) \xleftrightarrow{\text{FT}} X(j\omega), \quad y(t) \xleftrightarrow{\text{FT}} Y(j\omega)$$

$$\Rightarrow \quad \alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \xleftrightarrow{\text{FT}} \alpha\,\mathrm{X}(f) + \beta\,\mathrm{Y}(f)$$

$$\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \xleftrightarrow{\text{FT}} \alpha\,\mathrm{X}(j\omega) + \beta\,\mathrm{Y}(j\omega)$$



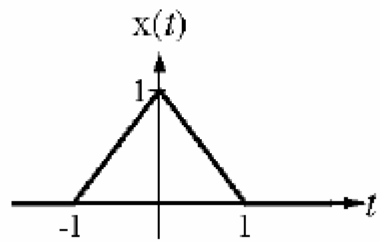
Properties of the CTFT: Time shift

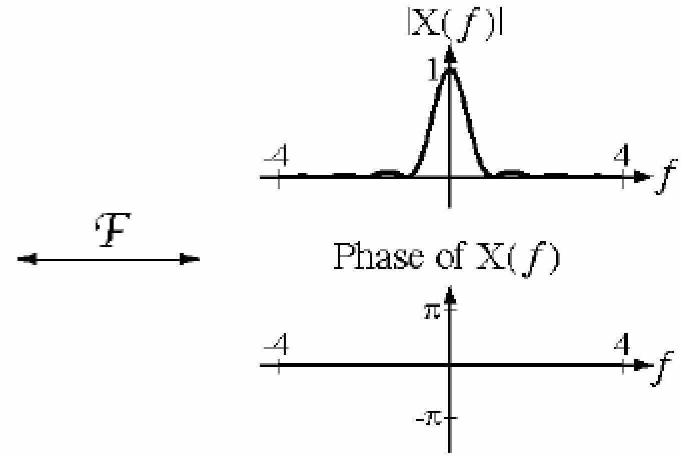
- Time Shifting

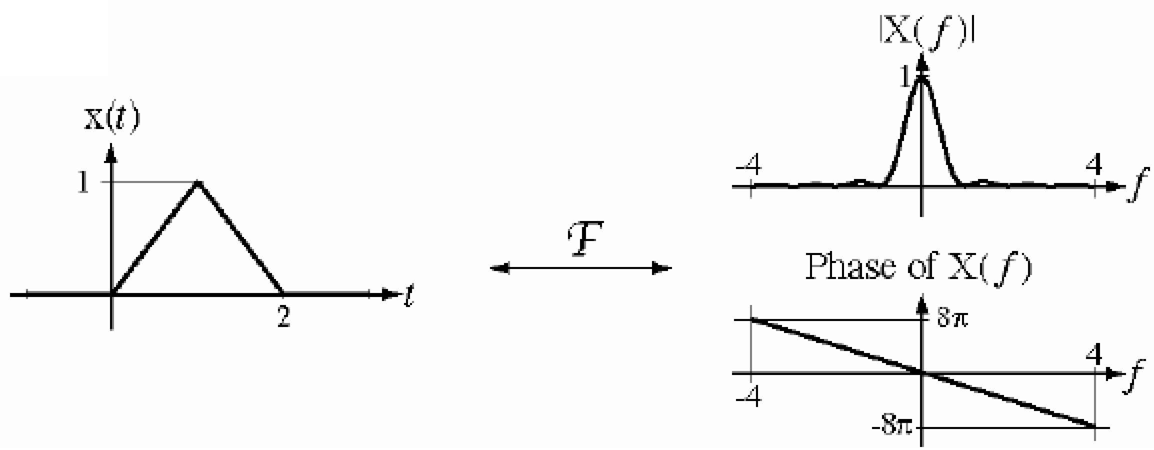
$$x(t) \xleftrightarrow{\text{FT}} X(j\omega)$$

$$\Rightarrow \mathrm{x}(t-t_0) \xleftrightarrow{\text{FT}} \mathrm{X}(f)e^{-j2\pi f t_0}$$

$$\mathrm{x}(t-t_0) \xleftrightarrow{\text{FT}} \mathrm{X}(j\omega)e^{-j\omega t_0}$$





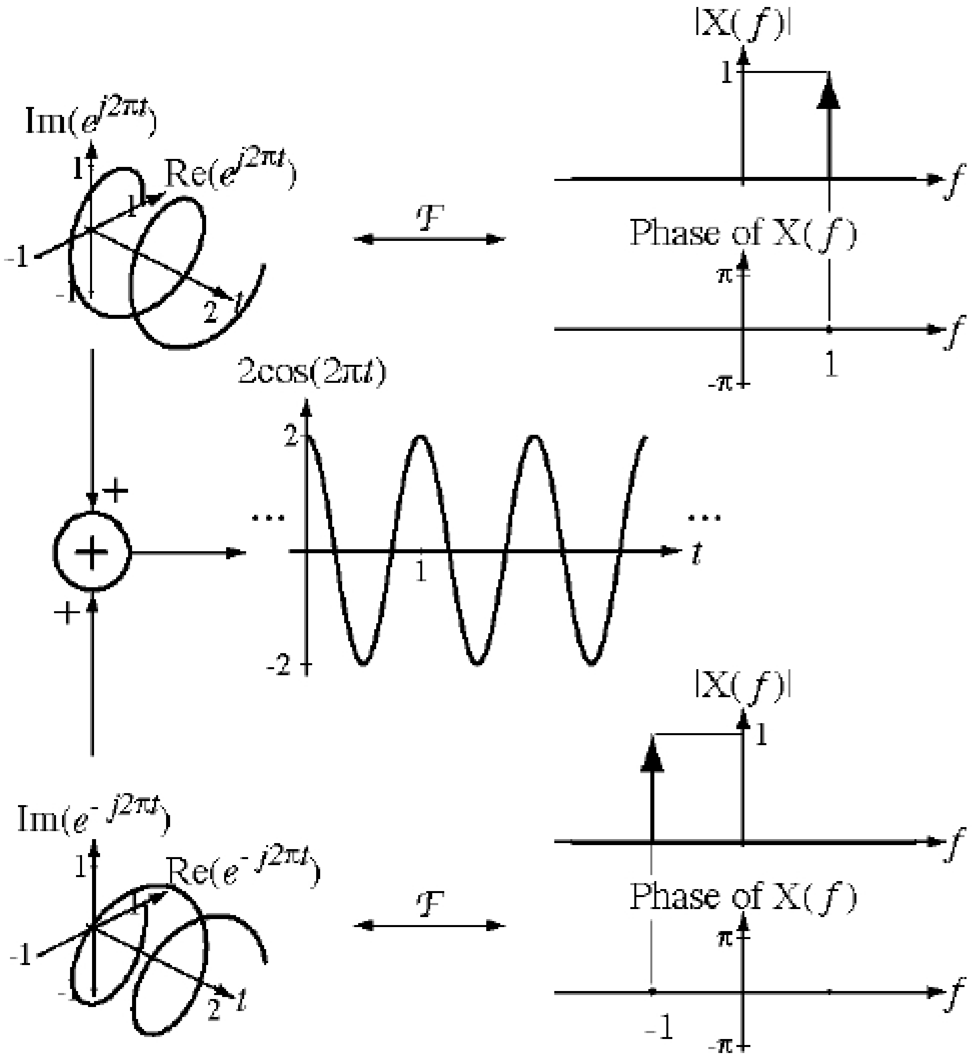


Properties of the CTFT: Freq. shift

- Frequency Shifting

$$x(t)e^{+j2\pi f_0 t} \overset{F}{\longleftrightarrow} X(f - f_0)$$

$$x(t)e^{+j\omega_0 t} \overset{F}{\longleftrightarrow} X(\omega - \omega_0)$$



Frequency shift property: Example

- **Example** $2\pi\delta(\omega-\omega_0)$

From the shift in frequency property, we will have:

$$1 \xleftrightarrow{\text{FT}} 2\pi\delta(\omega) \Rightarrow e^{j\omega_0 t} \xleftrightarrow{\text{FT}} 2\pi\delta(\omega-\omega_0)$$

Properties of the CTFT

- Time reversal:

$$x(t) \xleftrightarrow{FT} X(j\omega)$$
$$\Rightarrow x(-t) \xleftrightarrow{FT} X(-j\omega)$$

- Time Scaling

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$
$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

- Frequency Scaling

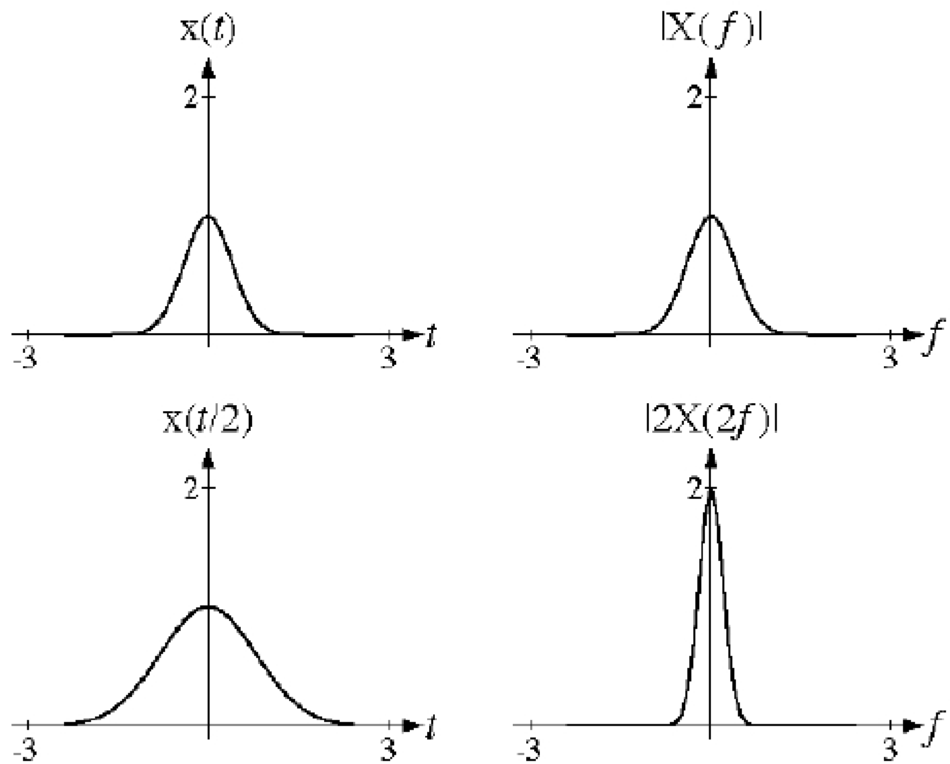
$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \xleftrightarrow{F} X(af)$$
$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \xleftrightarrow{F} X(ja\omega)$$

Properties of the CTFT

- The time and frequency scaling properties indicate that if a signal is expanded in one domain it is compressed in the other domain ➔ This is also called the “uncertainty principle” of Fourier analysis

$$e^{-\pi t^2} \xleftrightarrow{\mathrm{F}} e^{-\pi f^2}$$

$$e^{-\pi\left(\frac{t}{2}\right)^2} \xleftrightarrow{\mathrm{F}} 2e^{-\pi(2f)^2}$$



Properties of the CTFT: Scaling **Proofs**

- Time and frequency scaling: For any nonzero real number a we have:

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\Rightarrow x(at) \xleftrightarrow{FT} \frac{1}{|a|} X(\frac{j\omega}{a})$$

  - To prove this, we can write: $x(at) \xleftrightarrow{FT} \int^{+\infty} x(at) e^{-j\omega t} dt$

    Now substituting at with τ we will have:

$$x(at) \xleftrightarrow{FT} \begin{cases} \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a < 0 \end{cases}$$

Properties of the CTFT: Scaling **Proofs**

- Scaling property

Let a be any real constant that is not zero and let $z(t) = x(at)$

Then the CT Fourier Transform of $z(t)$ is

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t}\, dt.$$

Make the change of variable $\lambda = at$ and $d\lambda = a\, dt$

Then, if $a > 0$,

$$Z(f) = \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f(\lambda/a)} \frac{d\lambda}{a} = \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi(f/a)\lambda}\, d\lambda = \frac{1}{a} X\left(\frac{f}{a}\right)$$

and if $a < 0$,

$$Z(f) = \int_{\infty}^{-\infty} x(\lambda) e^{-j2\pi f(\lambda/a)} \frac{d\lambda}{a} = -\frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi(f/a)\lambda}\, d\lambda = -\frac{1}{a} X\left(\frac{f}{a}\right)$$

Therefore, in either case,

$$Z(f) = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Properties of the CTFT: Time Shifting & Scaling: **Example**

If $x(t) = 10\sin(t)$, then find the CT Fourier Transform of

a. $x(t)$ b. $x(t-2)$ c. $x(2(t-1))$ d. $x(2t-1)$

■ Solution

a. Using the linearity property and looking up the transform of the general sine f

$$\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{j}{2}[\delta(f+f_0) - \delta(f-f_0)]$$

$$\sin(t) \xleftrightarrow{\mathcal{F}} \frac{j}{2}\left[\delta\left(f+\frac{1}{2\pi}\right) - \delta\left(f-\frac{1}{2\pi}\right)\right]$$

$$10\sin(t) \xleftrightarrow{\mathcal{F}} j5\left[\delta\left(f+\frac{1}{2\pi}\right) - \delta\left(f-\frac{1}{2\pi}\right)\right]$$

or, in the radian-frequency form,

$$10\sin(t) \xleftrightarrow{\mathcal{F}} j10\pi[\delta(\omega+1) - \delta(\omega-1)].$$

b. Using the result of part (a)

$$10\sin(t) \xleftrightarrow{\mathcal{F}} j5\left[\delta\left(f+\frac{1}{2\pi}\right) - \delta\left(f-\frac{1}{2\pi}\right)\right]$$

and the time-shifting property,

$$10\sin(t-2) \xleftrightarrow{\mathcal{F}} j5\left[\delta\left(f+\frac{1}{2\pi}\right) - \delta\left(f-\frac{1}{2\pi}\right)\right]e^{-j4\pi f}$$

or

$$10\sin(t-2) \xleftrightarrow{\mathcal{F}} j10\pi[\delta(\omega+1) - \delta(\omega-1)]e^{-j2\omega}$$

Properties of the CTFT: Time Shifting & Scaling Example

c. From part (a),

$$10\sin(t) \xleftrightarrow{\mathcal{F}} j5\left[\delta\left(f+\frac{1}{2\pi}\right)-\delta\left(f-\frac{1}{2\pi}\right)\right].$$

Using the time-scaling property,

$$10\sin(2t) \xleftrightarrow{\mathcal{F}} j\frac{5}{2}\left[\delta\left(\frac{f}{2}+\frac{1}{2\pi}\right)-\delta\left(\frac{f}{2}-\frac{1}{2\pi}\right)\right].$$

Then, using the time-shifting property,

$$10\sin(2(t-1)) \xleftrightarrow{\mathcal{F}} j\frac{5}{2}\left[\delta\left(\frac{f}{2}+\frac{1}{2\pi}\right)-\delta\left(\frac{f}{2}-\frac{1}{2\pi}\right)\right]e^{-j2\pi f}$$

Finally, using the scaling property of the impulse function

$$\boxed{\delta(\alpha x)=\frac{\delta(x)}{|\alpha|}}$$

$$10\sin(2(t-1)) \xleftrightarrow{\mathcal{F}} j5\left[\delta\left(f+\frac{1}{\pi}\right)-\delta\left(f-\frac{1}{\pi}\right)\right]e^{-j2\pi f}$$

or $$10\sin(2(t-1)) \xleftrightarrow{\mathcal{F}} j5\left[\delta\left(f+\frac{1}{\pi}\right)e^{j2}-\delta\left(f-\frac{1}{\pi}\right)e^{-j2}\right]$$

or $$10\sin(2(t-1)) \xleftrightarrow{\mathcal{F}} j10\pi[\delta(\omega+2)e^{j2}-\delta(\omega-2)e^{-j2}]$$

Properties of the CTFT: average power

- Parseval's Theorem or Parseval's relation:

$$\int_{-\infty}^{\infty} |\mathrm{x}(t)|^2 dt = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2 df$$

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

- Use Parseval's relation to find the value of the following integral:

$$\int_{-\infty}^{+\infty} \frac{\sin^2(t)}{\pi^2 t^2} dt$$

Solution:

$$\frac{\sin(t)}{\pi t} \xleftrightarrow{\text{FT}} X(j\omega) = \begin{cases} 1 & |\omega| < 1 \\ 0 & |\omega| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\sin^2(t)}{\pi^2 t^2} dt = \frac{2}{2\pi} = \frac{1}{\pi}$$

Properties of the CTFT: Conjugation

- Conjugation and conjugate symmetry: $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$

$$\Rightarrow x^*(t) \xleftrightarrow{\text{FT}} X^*(-j\omega)$$

  - For real signals we have $x(t) = x^*(t)$, which results in the following conjugate symmetry: $X(-j\omega) = X^*(j\omega)$
  - This implies that for real signals the real part and the magnitude of $X(j\omega)$ are even functions of ω while the imaginary part and phase of $X(j\omega)$ are odd functions of ω. In other words:

$$\mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$$
$$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$$
$$|X(j\omega)| = |X(-j\omega)|$$
$$\angle X(j\omega) = -\angle X(-j\omega)$$

Properties of the CTFT: Conjugation

- Conjugation and conjugate symmetry: $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$

$$\Rightarrow x^*(t) \xleftrightarrow{\text{FT}} X^*(-j\omega)$$

  - For real and even signals we have $x(t) = x^*(t)$ and $x(t) = x(-t)$, which result in real and even Fourier transforms:

$$X(j\omega) = X^*(j\omega), \quad X(-j\omega) = X(j\omega)$$

    This property can be verified with Examples 5.5, 5.7, 6.1, 6.2

  - For real and odd signals we have $x(t) = x^*(t)$ and $x(t) = -x(-t)$, which result in purely imaginary and odd Fourier transforms:

$$X(j\omega) = -X^*(j\omega), \quad X(-j\omega) = -X(j\omega)$$

Properties of the CTFT: Convolution & Multiplication

- Multiplication-Convolution Duality

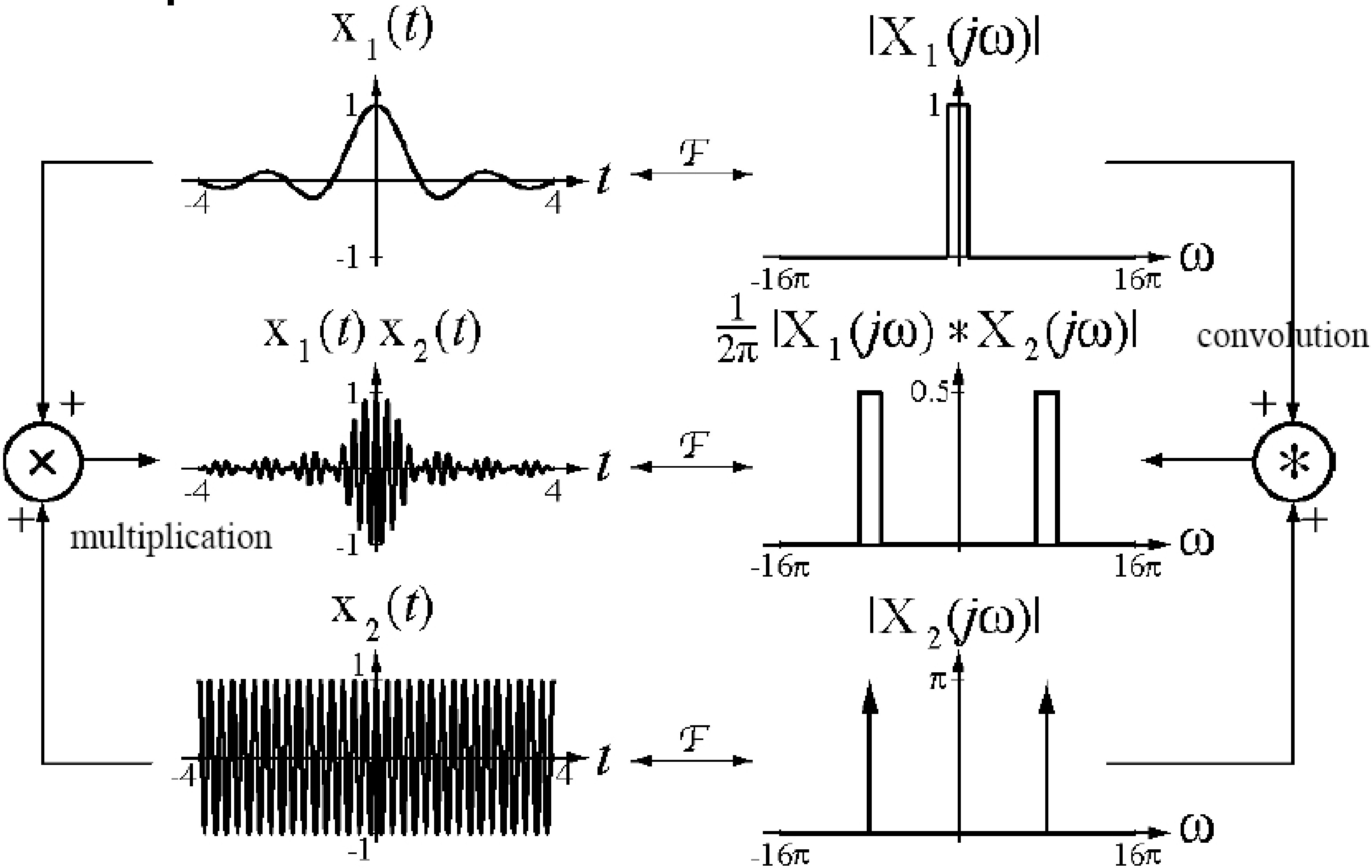
$$x(t) * y(t) \xleftrightarrow{F} X(f)Y(f)$$

$$x(t) * y(t) \xleftrightarrow{F} X(j\omega)Y(j\omega)$$

$$\boxed{\begin{aligned} x(t)y(t) &\xleftrightarrow{F} X(f) * Y(f) \\ x(t)y(t) &\xleftrightarrow{F} \frac{1}{2\pi} X(j\omega) * Y(j\omega) \end{aligned}}$$

- x(t)y(t)
  - Multiplication of x(t) and y(t)
  - Modulation of x(t) by y(t)
- Example: x(t)cos(w_0t)
  - w_0 is much higher than the max. frequency of x(t)

Properties of the CTFT: Convolution & Multiplication



Properties of the CTFT: Modulation Property Example

- Modulation

$$x(t)e^{j\omega_0 t} \leftrightarrow X(j(\omega-\omega_0))$$

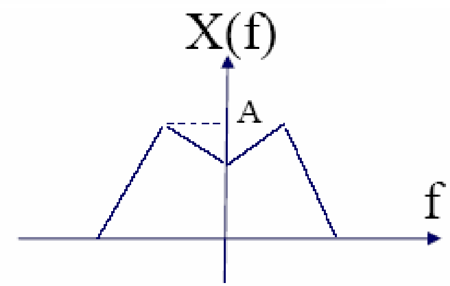
$$x(t)\cos(2\pi f_0 t) \xleftrightarrow{F} \frac{1}{2}\left[X(f-f_0)+X(f+f_0)\right]$$

$$x(t)\cos(\omega_0 t) \xleftrightarrow{F} \frac{1}{2}\left[X(j(\omega-\omega_0))+X(j(\omega+\omega_0))\right]$$

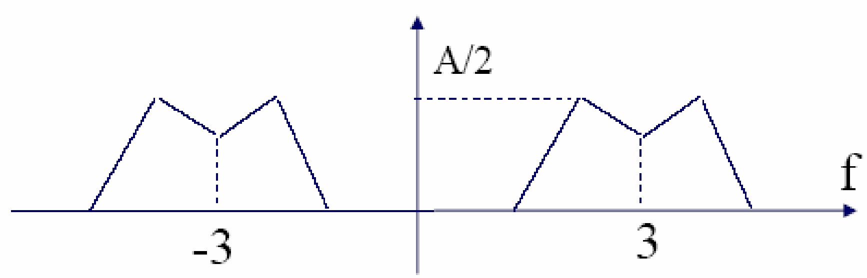
$$x(t)\sin(\omega_0 t) \leftrightarrow \frac{1}{2j}(X(j(\omega-\omega_0))-X(j(\omega+\omega_0)))$$

- Suppose that

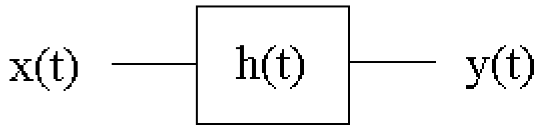
$$x(t) \xleftrightarrow{F}$$



- Sketch the Fourier transform of $x(t)\cos(6\pi t)$



Properties of the CTFT: Convolution



- h(t) is the impulse response of the LTI system
- Convolution property:

$$y(t) = h(t) * x(t) \Rightarrow \boxed{Y(j\omega) = H(j\omega)X(j\omega)}$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

- $H(j\omega)$ is called the frequency response of the system
- One can design a filter with a desired frequency response $H(j\omega)$ to pass or stop certain frequency components of the input signal

Properties of the CTFT: Convolution Property Example

- An LTI system has an impulse response

$$h(t) = \exp[-at]u(t)$$

and output

$$y(t) = [\exp[-bt] - \exp[-ct]]u(t)$$

Using the convolution property, we find that the transform of the input is

$$\begin{aligned} X(\omega) &= \frac{Y(\omega)}{H(\omega)} \\ &= \frac{(c-b)(j\omega + a)}{(j\omega + b)(j\omega + c)} \\ &= \frac{D}{j\omega + b} + \frac{E}{j\omega + c} \end{aligned}$$

where

$$D = a - b \qquad \text{and} \qquad E = c - a$$

Therefore,

$$x(t) = [(a-b)\exp[-bt] + (c-a)\exp[-ct]]u(t)$$

Properties of the CTFT: Multiplication-Convolution Duality Proof

Let the convolution of $x(t)$ and $y(t)$ be $z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau$.

The CT Fourier Transform of $z(t)$ is $Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi f t}\, dt$

or $$Z(f) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau \right] e^{-j2\pi f t}\, dt$$

Reversing the order of integration $$Z(f) = \int_{-\infty}^{\infty} x(\tau) \underbrace{\left[\int_{-\infty}^{\infty} y(t-\tau) e^{-j2\pi f t}\, dt \right]}_{\mathcal{F}[y(t-\tau)]} d\tau.$$

Then, using the time-shifting property $$Z(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} Y(f)\, d\tau$$

Since $Y(f)$ is not a function of τ, $$Z(f) = Y(f) \underbrace{\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau}\, d\tau}_{\mathcal{F}[x(\tau)]}$$

and finally $Z(f) = X(f) Y(f)$

Properties of the CTFT: Differentiation & Integration

- Differentiation and integration:

$$x(t) \xleftrightarrow{\text{FT}} X(j\omega)$$

$$\Rightarrow \quad \frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\text{F}} j2\pi f\, \mathrm{X}(f)$$

k^{th} derivative :

$$\frac{d^k}{dt^k} x(t) \overset{F}{\leftrightarrow} (j\omega)^k X(j\omega)$$

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\text{F}} j\omega\, \mathrm{X}(j\omega)$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\text{FT}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

- The term $\pi X(0)\delta(\omega)$ in the integration property reflects the DC or average value of the signal

Properties of the CTFT: Integration Property Example 1

- Recall: Relation unit step and impulse:

$$\delta(t) = \frac{du(t)}{dt}. \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{0}^{\infty} \delta(t-\sigma)d\sigma,$$

- Find the Fourier transform of the unit step signal

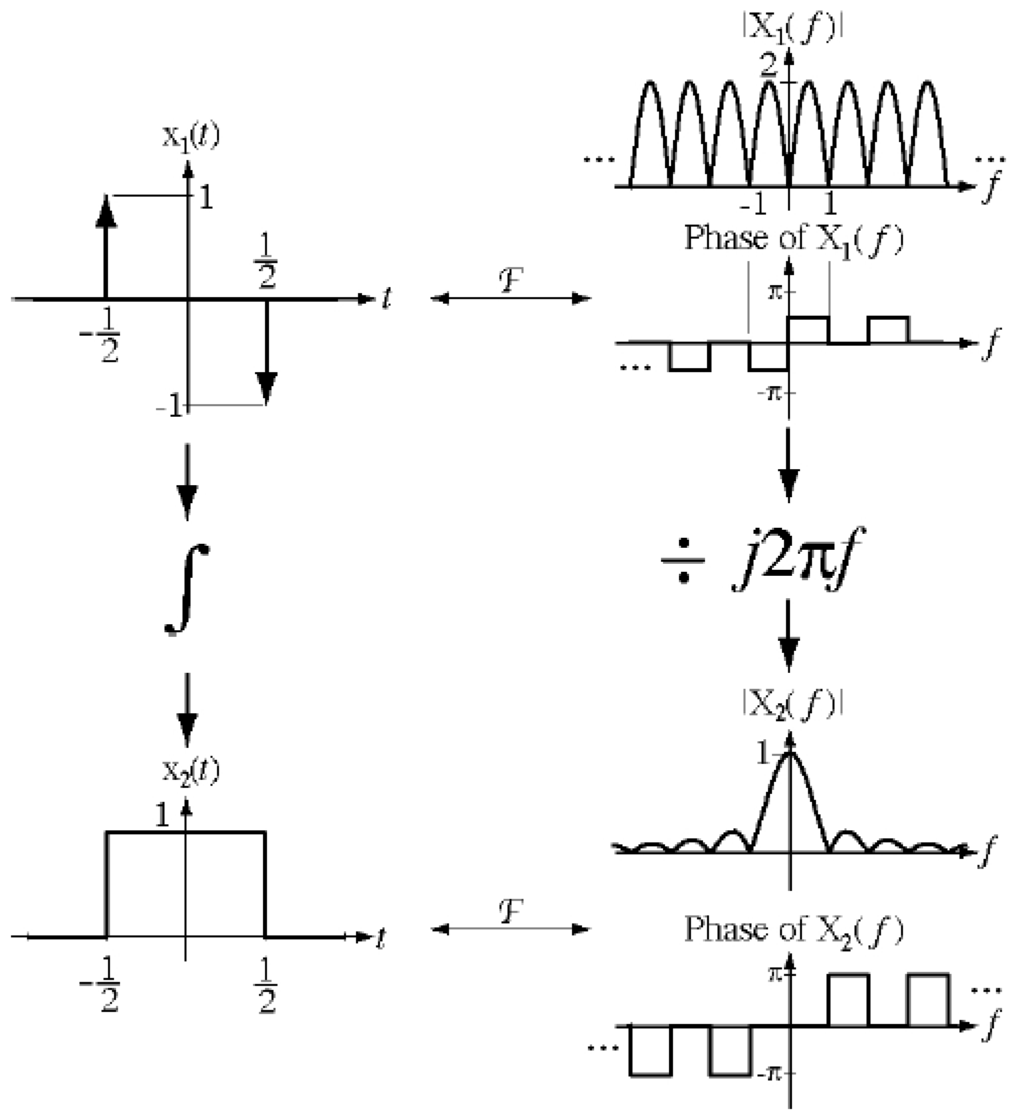
  Solution: We know that the Fourier transform of a unit impulse signal $g(t) = \delta(t)$ is equal to $G(j\omega) = 1$. This means that:

$$u(t)\int_{-\infty}^{t} \delta(\tau)d\tau \xleftrightarrow{\text{FT}} \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Properties of the CTFT: Integration Property Example 2

$$\int_{-\infty}^{t} x(\lambda)d\lambda \xleftrightarrow{F} \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

$$\int_{-\infty}^{t} x(\lambda)d\lambda \xleftrightarrow{F} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$



Properties of the CTFT: Differentiation Property Example 1

$\boxed{\text{sgnt(t)=2u(t)-1}}$

- Consider the unit-step function $u(t) = \frac{1}{2} + \left[u(t) - \frac{1}{2}\right] = \frac{1}{2} + \frac{1}{2}\operatorname{sgn} t$

- The first term has $\pi\delta(\omega)$ as its transform. Although $\operatorname{sgn} t$ does not have a derivative in the regular sense, we defined the derivatives of discontinuous signals in terms of the delta function

- As a consequence, $$\frac{d}{dt}\left\{\frac{1}{2}\operatorname{sgn} t\right\} = \delta(t)$$

  Since $\operatorname{sgn} t$ has a zero dc component (it is an odd signal), applying $\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathrm{F}} j\omega\,\mathrm{X}(j\omega)$

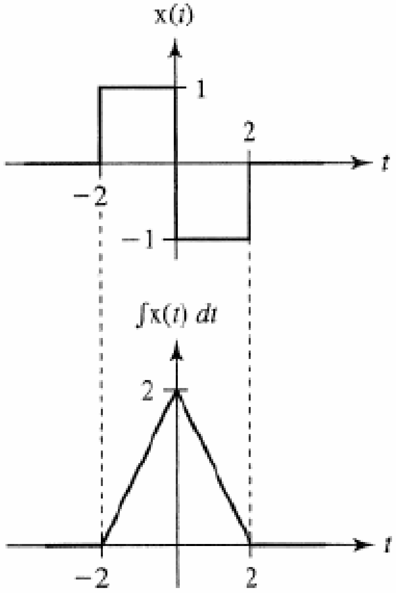
  $$j\omega\,\mathcal{F}\left\{\frac{1}{2}\operatorname{sgn} t\right\}=1 \quad \text{or} \quad \mathcal{F}\left\{\frac{1}{2}\operatorname{sgn} t\right\} = \frac{1}{j\omega}$$

- By the linearity of the Fourier transform, we obtain

  $$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

- Therefore, the Fourier transform of the unit-step function contains an impulse at $\omega = 0$ corresponding to the average value of 1/2. It also has all the high-frequency components of the signum function, reduced by one-half

Properties of the CTFT: Differentiation Property Example 2



x(t) and its integral.

Find the CT Fourier Transform of $x(t) = \mathrm{rect}((t+1)/2) - \mathrm{rect}((t-1)/2)$ using the differentiation property of the CT Fourier Transform and the CT Fourier Transform of the triangle function

■ **Solution**

The function $x(t)$ is the derivative of a triangle function centered at zero with a half-width of two,

$$x(t) = \frac{d}{dt}\left(2\,\mathrm{tri}\left(\frac{t}{2}\right)\right)$$

Using the scaling property,

$$2\,\mathrm{tri}\left(\frac{t}{2}\right) \xleftrightarrow{\mathcal{F}} 4\,\mathrm{sinc}^2(2f)$$

Then, using the differentiation property,

$$x(t) \xleftrightarrow{\mathcal{F}} j8\pi f\,\mathrm{sinc}^2(2f)$$

2nd method:

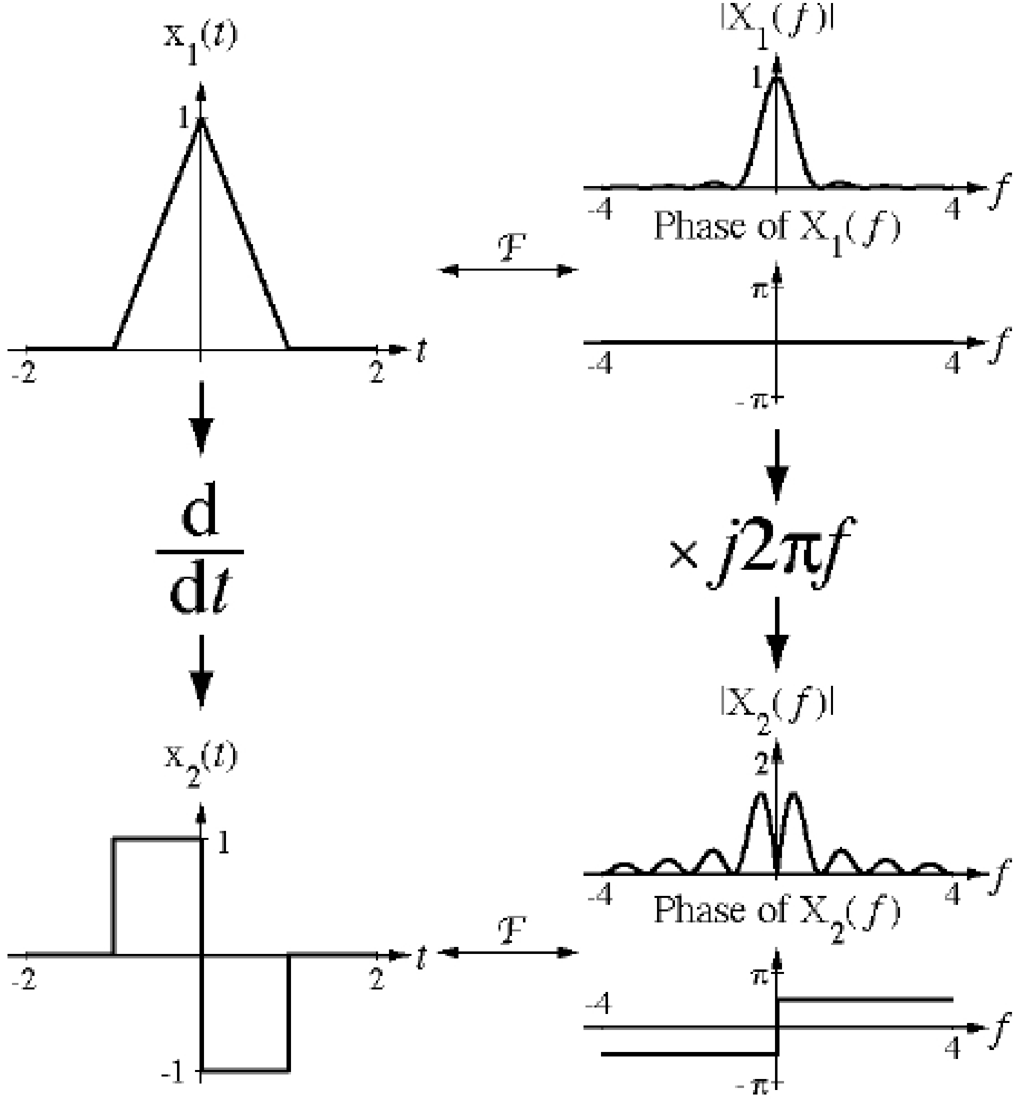
$$rect(t/2) \leftrightarrow 2\sin c(2f)$$

$$rect((t+1)/2) \leftrightarrow 2e^{j2\pi f}\sin c(2f)$$

$$rect((t-1)/2) \leftrightarrow 2e^{-j2\pi f}\sin c(2f)$$

$$x(t) \leftrightarrow j4\sin c(2f)\sin(2\pi f)$$

Properties of the CTFT: Differentiation Property Example 2



Proof of Integration Property using convolution property

$$\int_{-\infty}^{t} x(\lambda)d\lambda \xleftrightarrow{F} \frac{X(f)}{j2\pi f}+\frac{1}{2}X(0)\delta(f)$$

$$\int_{-\infty}^{t} x(\lambda)d\lambda \xleftrightarrow{F} \frac{X(j\omega)}{j\omega}+\pi X(0)\delta(\omega)$$

- If a signal $x(t)$ is convolved with a unit step,

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)\,d\tau,$$

proving that the convolution of a signal with a unit step is the same as the cumulative integral of the signal. Now we can prove the integration property of the CTFT. Using the convolution property and the equivalence property of an impulse,

$$\mathcal{F}(x(t) * u(t)) = X(f)\left[\frac{1}{j2\pi f}+\frac{1}{2}\,\delta(f)\right] = \frac{X(f)}{j2\pi f}+\frac{1}{2}X(0)\,\delta(f)$$

- Then, finally, the integration property of the CTFT is

$$\boxed{\int_{-\infty}^{t} x(\lambda)\,d\lambda \xleftrightarrow{\mathcal{F}} \frac{X(f)}{j2\pi f}+\frac{1}{2}X(0)\,\delta(f)}$$

or

$$\boxed{\int_{-\infty}^{t} x(\lambda)\,d\lambda \xleftrightarrow{\mathcal{F}} \frac{X(j\omega)}{j\omega}+\pi X(0)\,\delta(\omega)}.$$

Properties of the CTFT: Systems Characterized by linear constant-coefficient **differential equations** (LCCDE)

- Systems characterized by linear constant coefficients differential equations:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

We know that:

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{FT}} j\omega X(j\omega)$$

$$\frac{d^2 x(t)}{dt^2} \xleftrightarrow{\text{FT}} (j\omega)^2 X(j\omega)$$

$$\vdots$$

$$\frac{d^k x(t)}{dt^k} \xleftrightarrow{\text{FT}} (j\omega)^k X(j\omega)$$

This results in:

$$Y(j\omega)[\sum_{k=0}^{N} a_k (j\omega)^k] = X(j\omega)[\sum_{k=0}^{M} b_k (j\omega)^k]$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Properties of the CTFT: Systems Characterized by linear constant-coefficient **differential equations**

- $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- $$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

- $$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Properties of the CTFT: Systems Characterized by LCCDE Example 1

- $$\frac{dy(t)}{dt} + ay(t) = x(t) \quad a>0$$
- Find the impulse response $h(t)$ of this system
- $$H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} = \frac{1}{j\omega + a}$$
- $$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}\,d\omega = e^{-at}u(t)$$

Properties of the CTFT: Systems Characterized by LCCDE Example 2a

- Find the frequency response of a system whose input $x(t)$ and output $y(t)$ are related through the following differential equation:

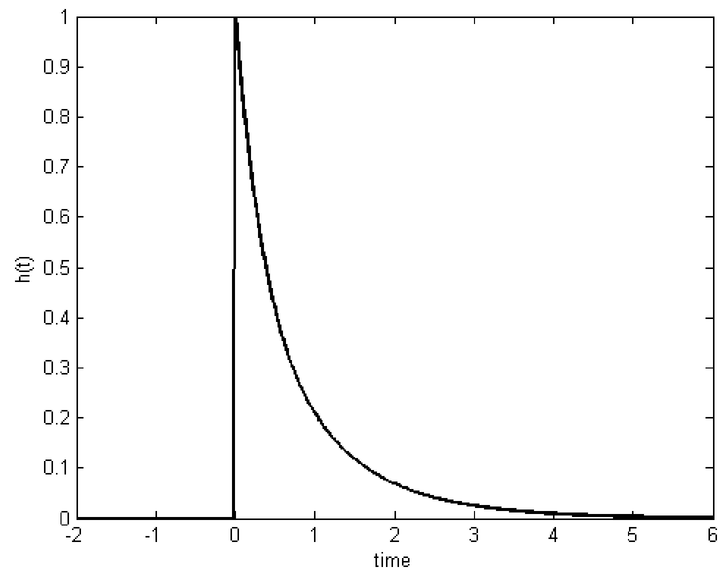
$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Solution: We have: $H(j\omega) = \dfrac{(j\omega)+2}{(j\omega)^2 + 4(j\omega) + 3}$

Properties of the CTFT: Systems Characterized by LCCDE Example 2b

- $$\frac{d^2 y(t)}{dt^2} + 4\frac{d\,y(t)}{dt} + 3y(t) = \frac{d\,x(t)}{dt} + 2x(t)$$
- $$H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} = \frac{(j\omega)+2}{(j\omega)^2 + 4(j\omega) + 3} = \frac{\frac{1}{2}}{j\omega+1} + \frac{\frac{1}{2}}{j\omega+3}$$
- Find the impulse response:

$$h(t) = [\tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-3t}]u(t)$$



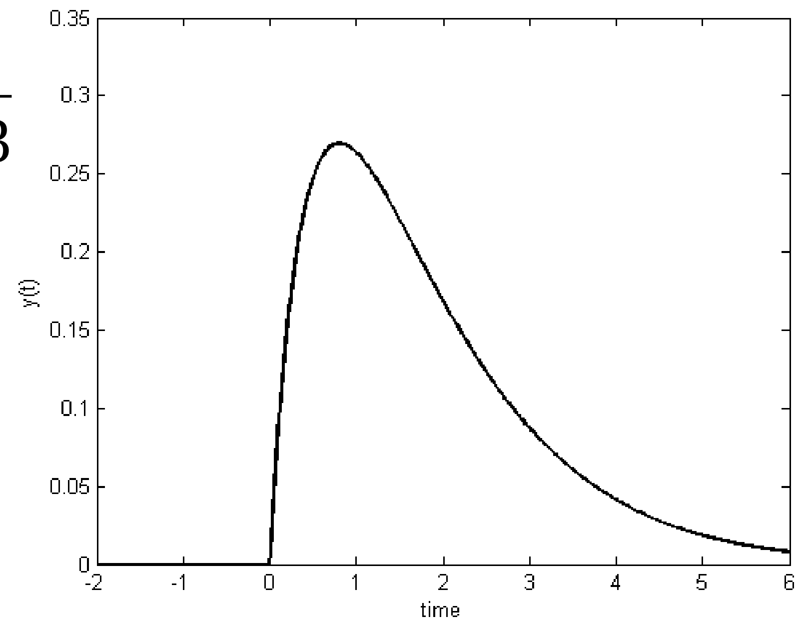
Properties of the CTFT: Systems Characterized by LCCDE Example 2c

- Let $x(t) = e^{-t}u(t) \overset{FT}{\Leftrightarrow} \dfrac{1}{j\omega+1}$; Find $y(t)$

- $$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega+2}{(j\omega+1)(j\omega+3)}\right]\left[\frac{1}{j\omega+1}\right] = \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}$$

- $$Y(j\omega) == \frac{\frac{1}{4}}{j\omega+1} + \frac{\frac{1}{2}}{(j\omega+1)^2} - \frac{\frac{1}{4}}{j\omega+3}$$

- $$y(t) = [\tfrac{1}{4}e^{-t} + \tfrac{1}{2}te^{-t} - \tfrac{1}{4}e^{-3t}]u(t)$$



Properties of the CTFT: Duality property

- Duality

$$\mathrm{X}(t) \xleftrightarrow{\mathcal{F}} \mathrm{x}(-f) \quad \text{and} \quad \mathrm{X}(-t) \xleftrightarrow{\mathcal{F}} \mathrm{x}(f)$$

$$\mathrm{X}(jt) \xleftrightarrow{\mathcal{F}} 2\pi\,\mathrm{x}(-\omega) \quad \text{and} \quad \mathrm{X}(-jt) \xleftrightarrow{\mathcal{F}} 2\pi\,\mathrm{x}(\omega)$$

- Duality Property

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

We prove this equation by replacing t with $-t$ in to get

$$2\pi x(-t) = \int_{\omega=-\infty}^{\infty} X(\omega)\exp[-j\omega t]\,d\omega$$

$$= \int_{\tau=-\infty}^{\infty} X(\tau)\exp[-j\tau t]\,d\tau$$

since ω is just a dummy variable for integration

Now replacing t by ω and τ by t gives

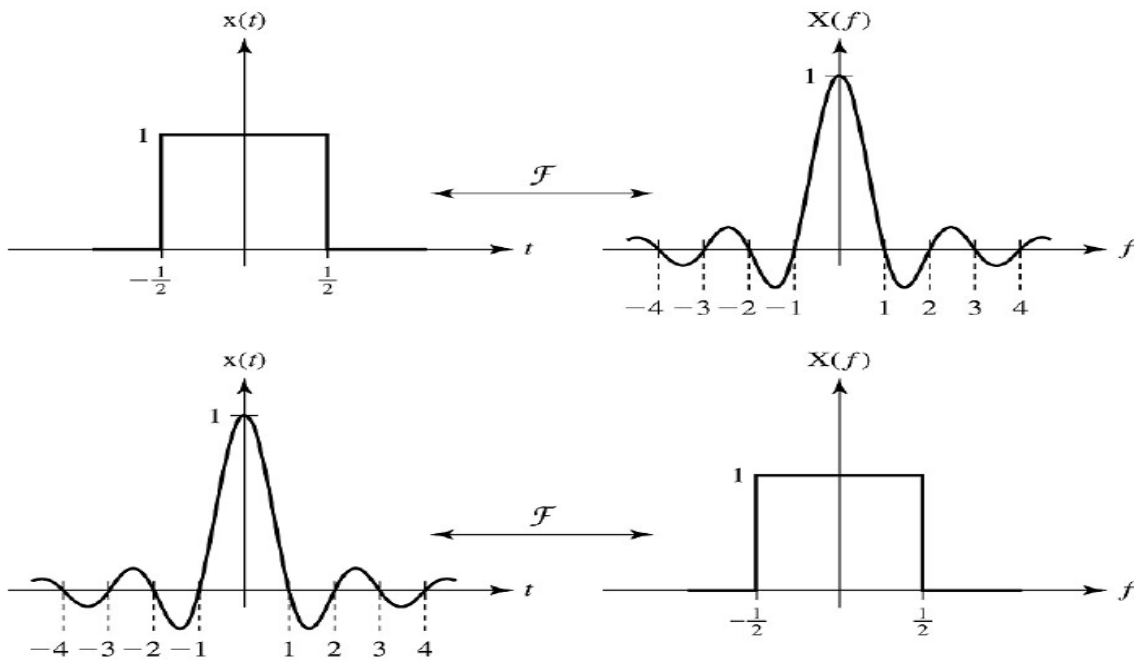
$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Properties of the CTFT: Duality property

Duality: We have seen in that:

$$x(t)=\frac{\sin(Wt)}{\pi t}\xleftrightarrow{\text{FT}}X(j\omega)=\begin{cases}1 & |\omega|<W\\ 0 & |\omega|>W\end{cases}$$

$$\Rightarrow x(t)=\begin{cases}1 & |t|<T_1\\ 0 & |t|>T_1\end{cases}\xleftrightarrow{\text{FT}}X(j\omega)=2\frac{\sin(\omega T_1)}{\omega}$$



Properties of the CTFT: Duality Property Example

- Consider the signal $$x(t) = \mathrm{Sa}\,\frac{\omega_B t}{2} = \mathrm{sinc}\,\frac{\omega_B t}{2\pi}$$

$$\mathcal{F}\left\{\mathrm{Sa}\,\frac{\omega_B t}{2}\right\} = \int_{-\infty}^{\infty} \mathrm{Sa}\,\frac{\omega_B t}{2} \exp[-j\omega t]\,dt$$

- This is a very difficult integral to evaluate directly. However, using

$$\mathrm{rect}\,(t/\tau) \leftrightarrow \tau\,\mathrm{Sa}\,\frac{\omega\tau}{2}$$

Then according to $X(t) \leftrightarrow 2\pi x(-\omega)$

$$\mathcal{F}\left\{\mathrm{Sa}\,\frac{\omega_B t}{2}\right\} = \frac{2\pi}{\omega_B}\mathrm{rect}\,(-\omega/\omega_B) = \frac{2\pi}{\omega_B}\mathrm{rect}\,(\omega/\omega_B)$$

because the rectangular pulse is an even signal. Note that the transform $X(\omega)$ is zero outside the range $-\omega_B/2 \leq \omega \leq \omega_B/2$, but that the signal $x(t)$ is not time limited

- Signals with Fourier transforms that vanish outside a given frequency band are called band-limited signals (signals with no spectral content above a certain maximum frequency, in this case, $\omega_B/2$.).
- It can be shown that time limiting and frequency limiting are mutually exclusive phenomena: i.e., a time-limited signal $x(t)$ always has a Fourier transform that is not band limited
- On the other hand, if $X(\omega)$ is band limited, then the corresponding time signal is never time limited.

Properties of the CTFT: relation between duality & other properties

- From duality, one can find the differentiation in frequency and shift in frequency properties of the Fourier transform as follows:

$$-jtx(t) \xleftrightarrow{\text{FT}} \frac{dX(j\omega)}{d\omega}$$

$$\Rightarrow e^{j\omega_0 t}x(t) \xleftrightarrow{\text{FT}} X(j(\omega-\omega_0))$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\text{FT}} \frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$$

Properties of the CTFT:
Differentiation in frequency Example

- **Example** $te^{-at}u(t)$

From the differentiation in frequency property, we will have:

$$e^{-at}u(t) \xleftrightarrow{\text{FT}} \frac{1}{j\omega + a}, \quad \operatorname{Re}\{a\} > 0 \Rightarrow te^{-at}u(t) \xleftrightarrow{\text{FT}} \frac{1}{(j\omega + a)^2}, \quad \operatorname{Re}\{a\} > 0$$

Properties of the CTFT: Total area-Integral

Total-Area Integral

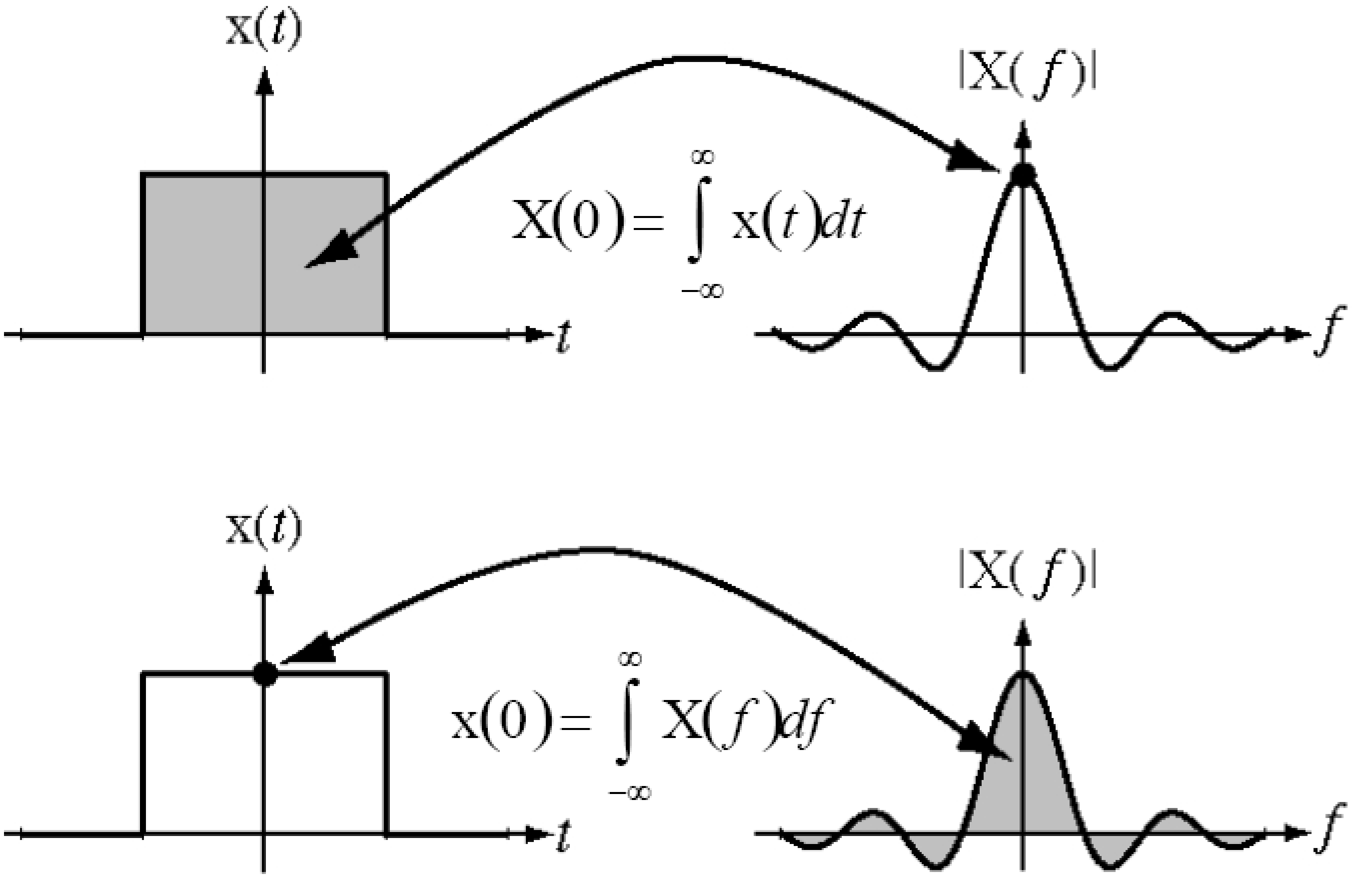
$$X(0) = \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \right]_{f \to 0} = \int_{-\infty}^{\infty} x(t) dt$$

$$x(0) = \left[\int_{-\infty}^{\infty} X(f) e^{+j2\pi ft} df \right]_{t \to 0} = \int_{-\infty}^{\infty} X(f) df$$

$$X(0) = \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right]_{\omega \to 0} = \int_{-\infty}^{\infty} x(t) dt$$

$$x(0) = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t} d\omega \right]_{t \to 0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) d\omega$$

Properties of the CTFT: Area Property Illustration



Properties of the CTFT: Area Property Example 1

- Compute the total area under the function $x(t) = \sin c(t)$

  $$\text{Area}=\int_{-\infty}^{\infty} \text{sinc(t)dt}=\int_{-\infty}^{\infty} \frac{\sin(\pi t)}{\pi t} dt = X(0)$$

  where $x(t) \leftrightarrow X(f)$

- We know that $\sin c(t) \leftrightarrow rect(f)$

- So, X(0)=1, and therefore $\text{Area}=\int_{-\infty}^{\infty} \frac{\sin(\pi t)}{\pi t}=1$

Properties of the CTFT: Area Property Example 2

Find the total area under the function $x(t) = 10\,\mathrm{sinc}\left(\frac{t+4}{7}\right)$

Solution

Ordinarily we would try to directly integrate the function over all t,

$$\text{Area} = \int_{-\infty}^{\infty} x(t)\,dt = \int_{-\infty}^{\infty} 10\,\mathrm{sinc}\left(\frac{t+4}{7}\right)dt = \int_{-\infty}^{\infty} 10\frac{\sin(\pi(t+4)/7)}{\pi(t+4)/7}\,dt.$$

This integral is a variant of a type of mathematical function called a *sine integral* defined by

$$\mathrm{Si}(z) = \int_{0}^{z} \frac{\sin(t)}{t}\,dt$$

The sine integral is related to a more general function, the *exponential integral*

The sine integral and exponential integral can be found tabulated in mathematical tables books, and MATLAB has a built-in function to evaluate the exponential integral

However, a plunge into the sine integral is not necessary to solve this problem

We can use $X(0) = \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt\right]_{f\to 0} = \int_{-\infty}^{\infty} x(t)\,dt$

First we find the CT Fourier Transform of $x(t)$ $\quad X(f) = 70\,\mathrm{rect}(7f)\,e^{j8\pi f}$

Then $\text{Area} = X(0) = 70$

Properties of the CTFT: Periodic signals

- Transforms of Periodic Signals

$$x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{-j2\pi(kf_F)t} \xleftrightarrow{\mathcal{F}} X(f)=\sum_{k=-\infty}^{\infty} X[k]\delta(f-kf_0)$$

$$x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{-j(k\omega_F)t} \xleftrightarrow{\mathcal{F}} X(j\omega)=2\pi\sum_{k=-\infty}^{\infty} X[k]\delta(\omega-k\omega_0)$$

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- Introduction to Fourier Transform
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Fourier transform for periodic signals

- Integral Definition of an Impulse $\int_{-\infty}^{\infty} e^{-j2\pi tf} df = \delta(t)$
- Consider a signal x(t) whose Fourier transform is given by

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

- Using the inverse Fourier transform, we will have:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t} \quad \text{(sifting property of the impulse)}$$

- This implies that the time signal corresponding to the following Fourier transform

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$ is given by $$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Fourier transform for periodic signals

- Note that $$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$
  is the Fourier series representation of periodic signals
  - *Fourier transform of a periodic signal is a train of impulses with the area of the impulse at the frequency $k\omega_0$ equal to the k^{th} coefficient of the Fourier series representation a_k times 2π*

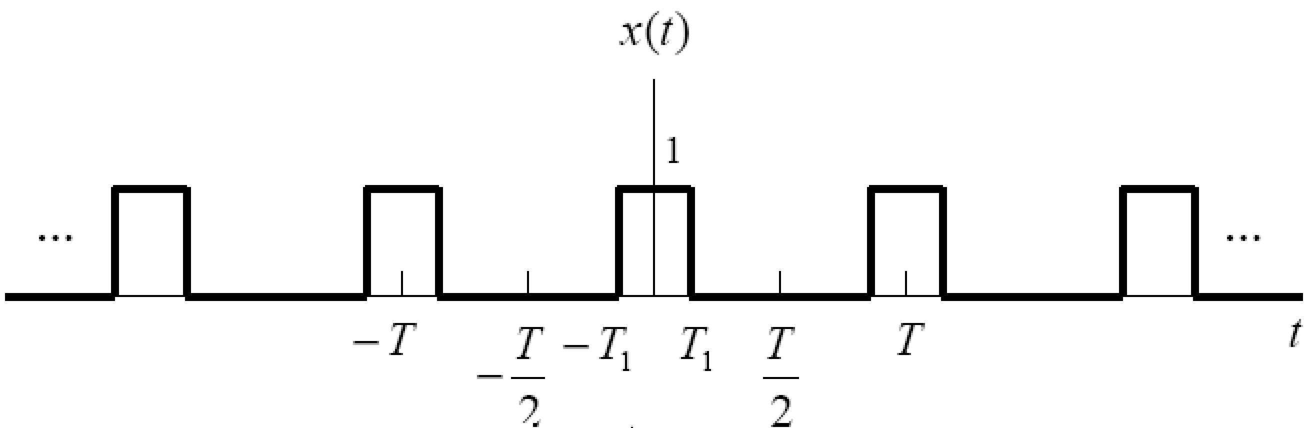
Transforms of Periodic signals

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{-j2\pi(kf_F)t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(f - kf_0)$$

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{-j(k\omega_F)t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(\omega - k\omega_0)$$

Fourier transform for periodic signals: Example 1

- Find the Fourier transform of the following square wave:



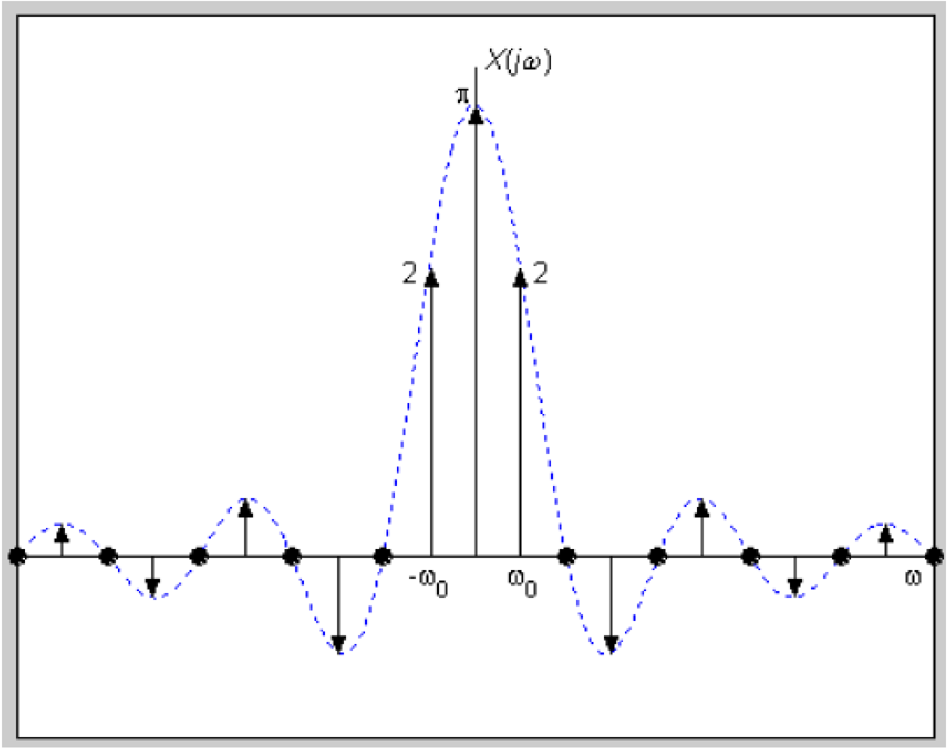
Solution: Using equation $X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$ we will have:

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \boxed{\frac{2\sin(k\omega_0 T_1)}{k}} \delta(\omega - k\omega_0)$$

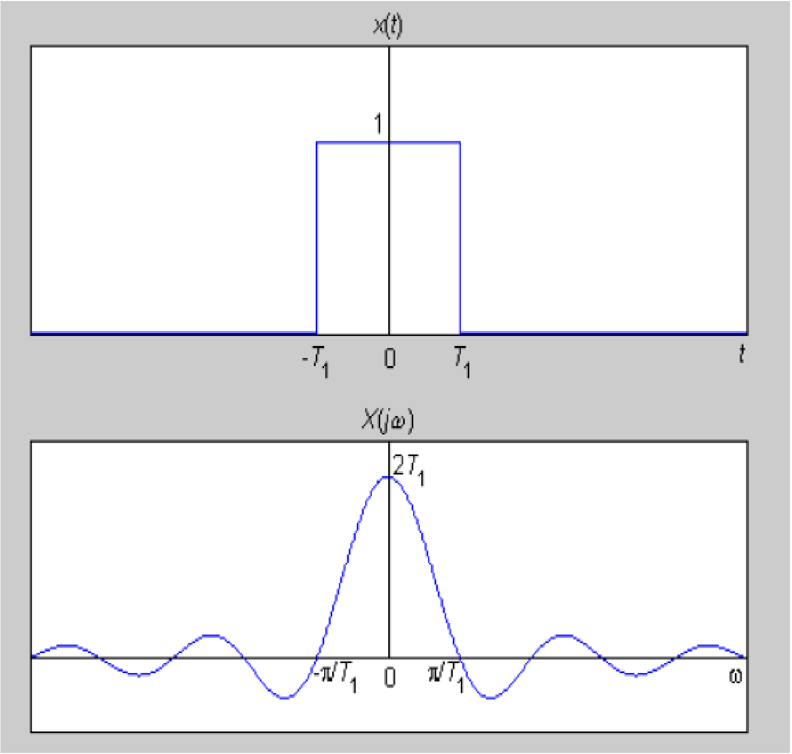
Fourier transform for periodic signals: Example 1

The Fourier transform $X(j\omega)=\sum_{k=-\infty}^{+\infty}\frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)$

is sketched in the following figure for $T=4T_1$



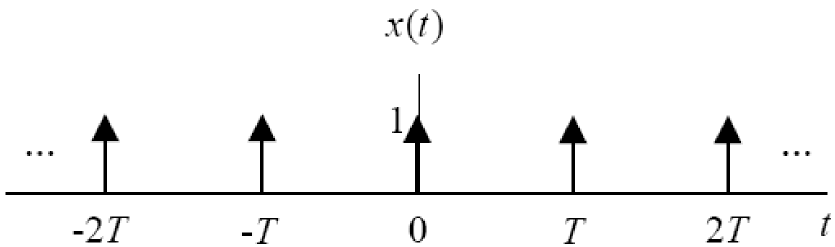
Recall: FT of square signal



Fourier transform for periodic signals: Example 2

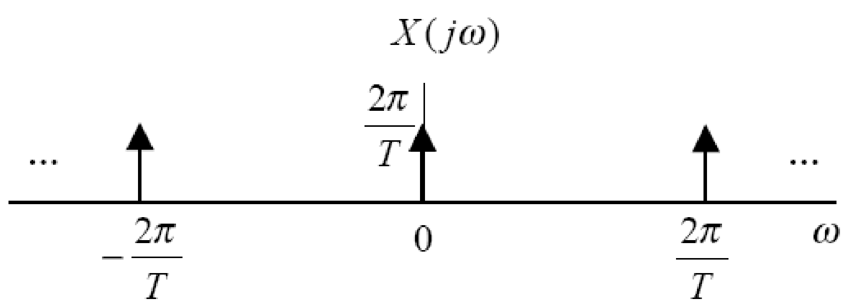
- Find the Fourier transform of the following periodic impulse train:

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$



Solution: We know that the Fourier series coefficients of a periodic impulse train is given by $a_k = \frac{1}{T}$ for all k. This results in the following Fourier transform for $x(t)$:

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0), \quad \omega_0 = \frac{2\pi}{T}$$



➔FT of periodic time impulse train is another impulse train (in frequency)

Fourier Transform of Periodic Signals: Example 2 (details)

- Consider the periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad = \text{comb(t)}$$

which has period T

- To find the Fourier transform, we first have to compute the Fourier series coefficients

The Fourier-series coefficients are

$$c_n = \frac{1}{T}\int_{\langle T\rangle} x(t)\exp\left[-\frac{j2\pi nt}{T}\right]dt = \frac{1}{T}$$

since $x(t) = \delta(t)$ in any interval of length T

- Thus, the impulse train has the Fourier-series representation

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\exp\left[\frac{j2\pi nt}{T}\right]$$

- We find that the Fourier transform of the impulse train is

$$X(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right)$$

- That is, the Fourier transformation of a sequence of impulses in the time domain yields a sequence of impulses in the frequency domain.

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CTFT: Summary

- Fourier transform represents a signal (aperiodic in general) as a sum of infinitely many complex exponentials, with the frequency varying continuously in $(-\infty, \infty)$

  $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t}\,dt, \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

- Know how to calculate the Fourier transform of simple functions
- Know Fourier transforms of special functions:

  $\delta(t), u(t), e^{-\alpha t}u(t), e^{j\omega_0 t}$, $\cos\omega_0 t, \sin\omega_0 t$, rectangular pulse, sinc,

- Know how to calculate the inverse transform directly for simple functions
- Properties of Fourier transform (shifting, differentiation, etc.)
- Fourier transform of aperiodic signals

  Understand the meaning of the inverse Fourier trainsform

  Sketch the spectrum of simple functions

  Know how to interpret a spectrum

Summary of CTFT Properties

Linearity	$\alpha x(t)+\beta y(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha X(f)+\beta Y(f)$
	$\alpha x(t)+\beta y(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha X(j\omega)+\beta Y(j\omega)$
Time shifting	$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(f)e^{-j2\pi f t_0}$
	$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)e^{-j\omega t_0}$
Frequency shifting	$x(t)e^{+j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(f-f_0)$
	$x(t)e^{+j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X[j(\omega-\omega_0)]$
Time scaling	$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
	$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{\lvert a\rvert}X\left(j\frac{\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert a\rvert}x\left(\frac{t}{a}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(af)$
	$\frac{1}{\lvert a\rvert}x\left(\frac{t}{a}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(ja\omega)$
Transform of a conjugate	$x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-f)$

Summary of CTFT Properties

- Multiplication–convolution duality

$$x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(f)Y(f)$$

$$x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(j\omega)Y(j\omega)$$

$$x(t)y(t) \xleftrightarrow{\mathcal{F}} X(f) * Y(f)$$

$$x(t)y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

Differentiation

$$\frac{d}{dt}(x(t)) \xleftrightarrow{\mathcal{F}} j2\pi f X(f)$$

$$\frac{d}{dt}(x(t)) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)$$

Modulation

$$x(t)\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}[X(f - f_0) + X(f + f_0)]$$

$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}[X(j(\omega - \omega_0)) + X(j(\omega + \omega_0))]$$

Transforms of periodic signals

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{-j2\pi(kf_F)t} \xleftrightarrow{\mathcal{F}} X(f) = \sum_{k=-\infty}^{\infty} X[k]\delta(f - kf_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{-j(k\omega_F)t} \xleftrightarrow{\mathcal{F}} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

CTFT: Summary of Pairs

TABLE
Some Selected Fourier Transform Pairs

$x(t)$	$X(\omega)$
1. 1	$2\pi\delta(\omega)$
2. $u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
3. $\delta(t)$	1
4. $\delta(t - t_0)$	$\exp[-j\omega t_0]$
5. $\text{rect}(t/\tau)$	$\tau \,\text{sinc}\,\frac{\omega\tau}{2\pi} = \frac{2\sin\omega\tau/2}{\omega}$
6. $\frac{\omega_B}{\pi}\,\text{sinc}\,\frac{\omega_B t}{\pi} = \frac{\sin\omega_B t}{\pi t}$	$\text{rect}(\omega/2\omega_B)$
7. $\text{sgn}\, t$	$\frac{2}{j\omega}$
8. $\exp[j\omega_0 t]$	$2\pi\delta(\omega - \omega_0)$
9. $\sum_{n=-\infty}^{\infty} a_n \exp[jn\omega_0 t]$	$2\pi \sum_{n=-\infty}^{\infty} a_n \delta(\omega - n\omega_0)$
10. $\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
11. $\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

CTFT: Summary of Pairs

12.	$(\cos\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$
13.	$(\sin\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$
14.	$\cos\omega_0 t \,\mathrm{rect}(t/\tau)$	$\tau\,\mathrm{sinc}\,\frac{(\omega-\omega_0)\tau}{2\pi}$
15.	$\exp[-at]u(t),\quad \mathrm{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
16.	$t\exp[-at]u(t),\quad \mathrm{Re}\{a\}>0$	$\left(\frac{1}{a+j\omega}\right)^2$
17.	$\frac{t^{n-1}}{(n-1)!}\exp[-at]u(t),\quad \mathrm{Re}\{a\}>0$	$\frac{1}{(a+j\omega)^n}$
18.	$\exp[-a\lvert t\rvert],\quad a>0$	$\frac{2a}{a^2+\omega^2}$
19.	$\lvert t\rvert\exp[-a\lvert t\rvert],\quad \mathrm{Re}\{a\}>0$	$\frac{4aj\omega}{a^2+\omega^2}$

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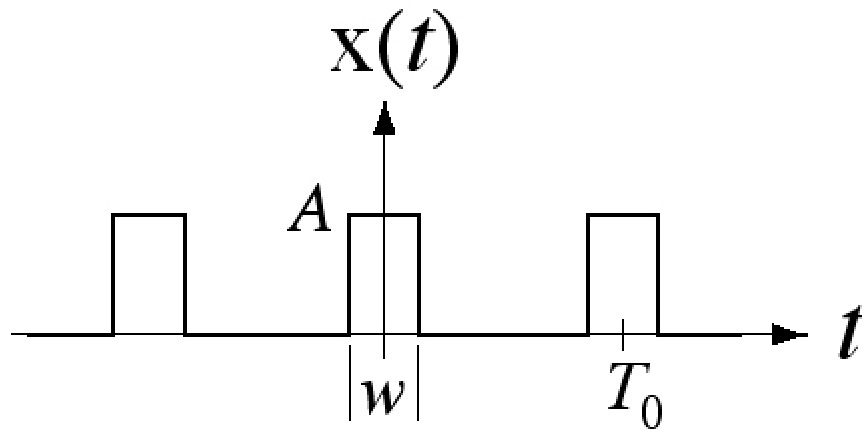
Transition: CT Fourier Series to CT Fourier Transform

- Consider a periodic pulse-train signal, x(t), with duty cycle, $\dfrac{w}{T_0}$

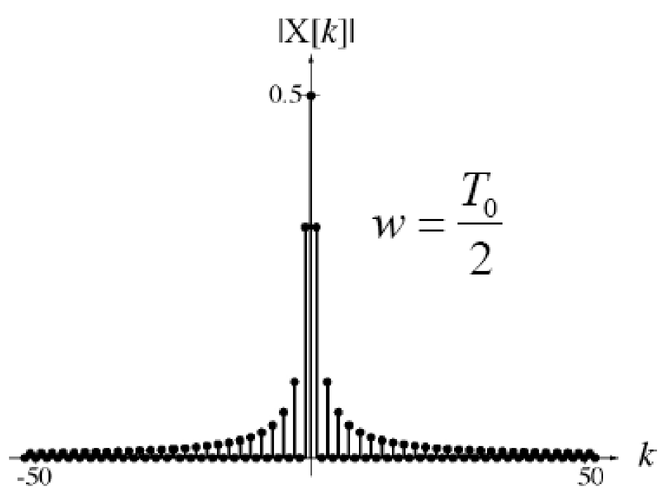


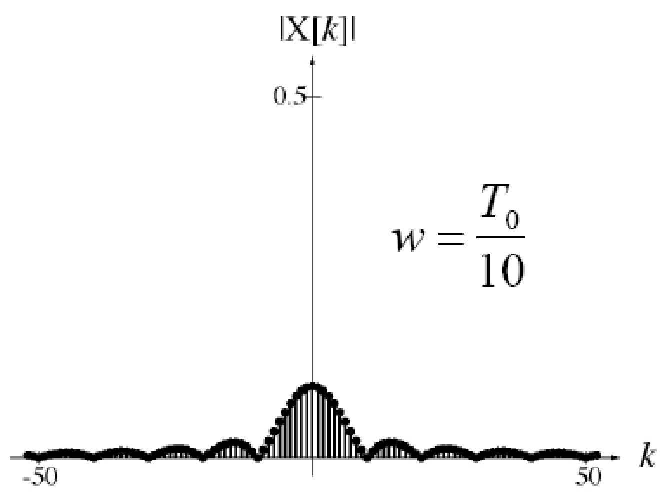
Its CTFS harmonic function is $\mathrm{X}[k] = \dfrac{Aw}{T_0}\operatorname{sinc}\left(\dfrac{kw}{T_0}\right)$

- As the period, T_0 , is increased, holding w constant, the duty cycle is decreased. When the period becomes infinite (and the duty cycle becomes zero) x(t) is no longer periodic

Transition: CT Fourier Series to CT Fourier Transform

- Below are plots of the magnitude of X[*k*] for 50% and 10% duty cycles
- As the period increases the sinc function widens and its magnitude falls
- As the period approaches infinity, the CT Fourier Series harmonic function becomes an infinitely-wide sinc function with zero amplitude (since X(k) is divided by T_o)



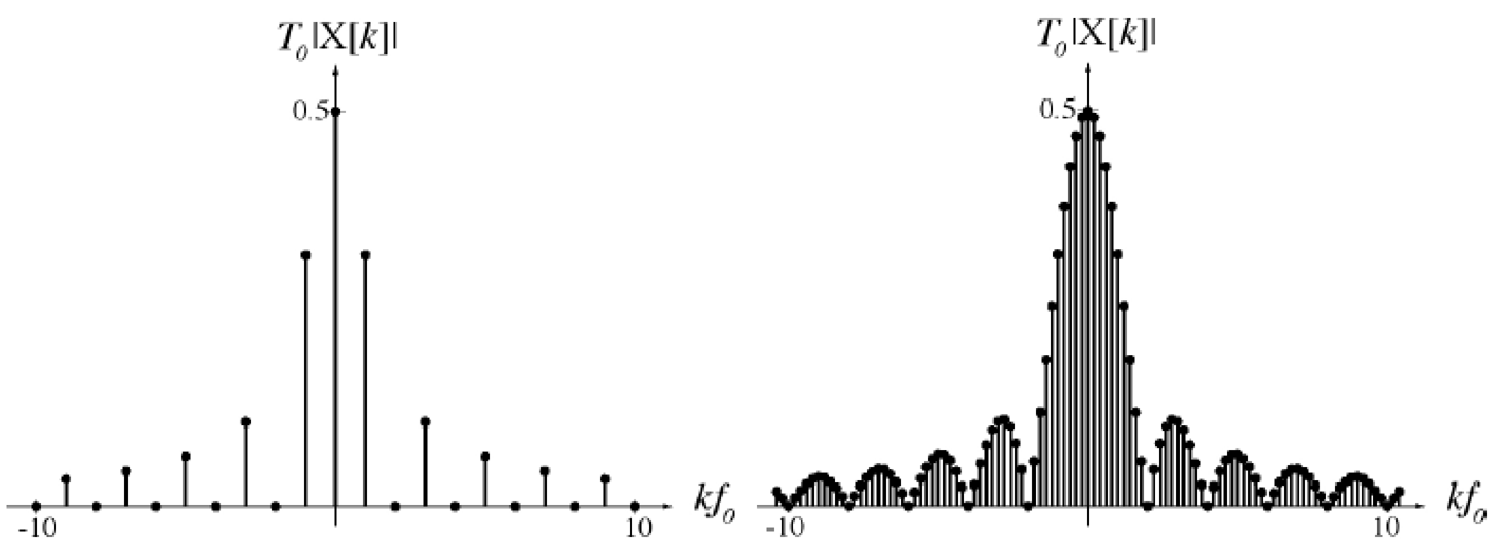


Transition: CT Fourier Series to CT Fourier Transform

- This infinity-and-zero problem can be solved by normalizing the CT Fourier Series harmonic function
- Define a new "modified" CT Fourier Series harmonic function

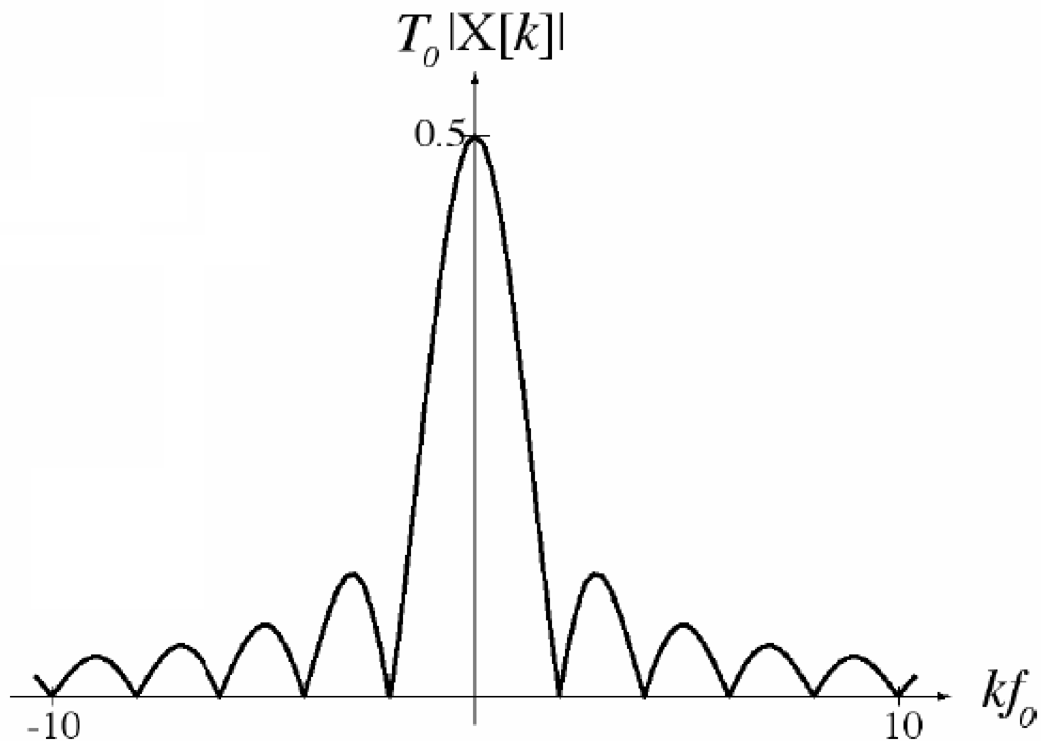
$$T_0 \, \mathrm{X}[k] = Aw \operatorname{sinc}\left(w\left(kf_0\right)\right)$$

and graph it versus kf_0 instead of versus k.



Transition: CT Fourier Series to CT Fourier Transform

- In the limit as the period approaches infinity, the modified CT Fourier Series harmonic function approaches a function of continuous frequency $f\,(kf_0)$



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Applications of frequency- domain representation

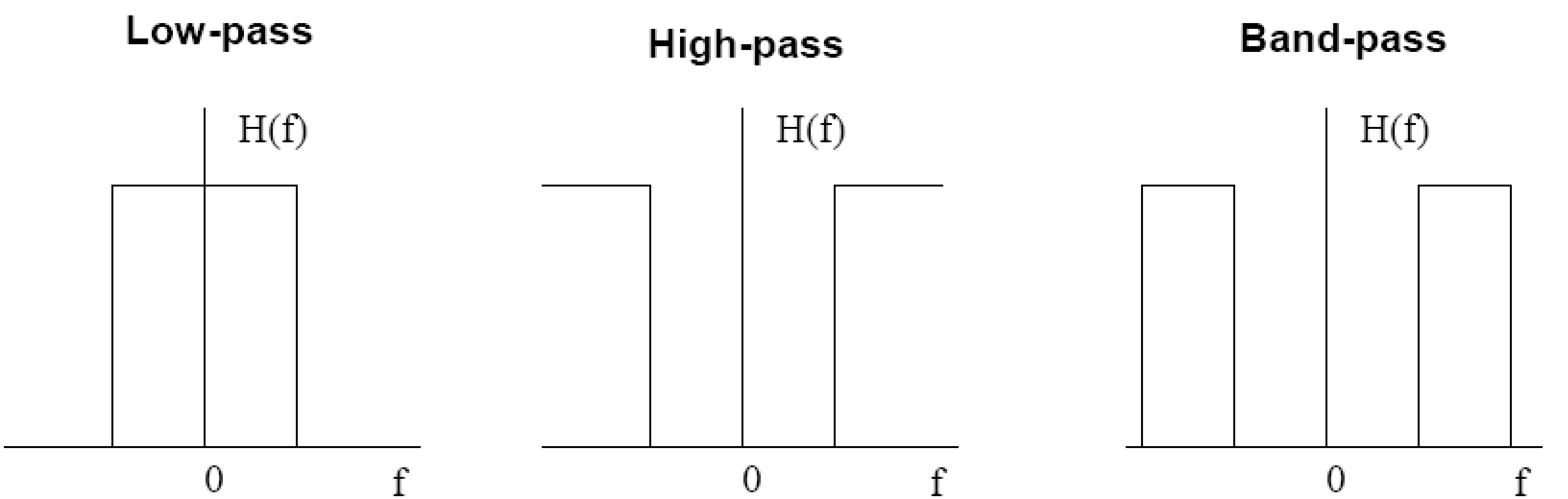
- Clearly shows the frequency composition a signal
- Can change the magnitude of any frequency component arbitrarily by a **filtering operation**
  - Lowpass -> smoothing, noise removal
  - Highpass -> edge/transition detection
  - High emphasis -> edge enhancement

➔Processing of **speech and music signals**

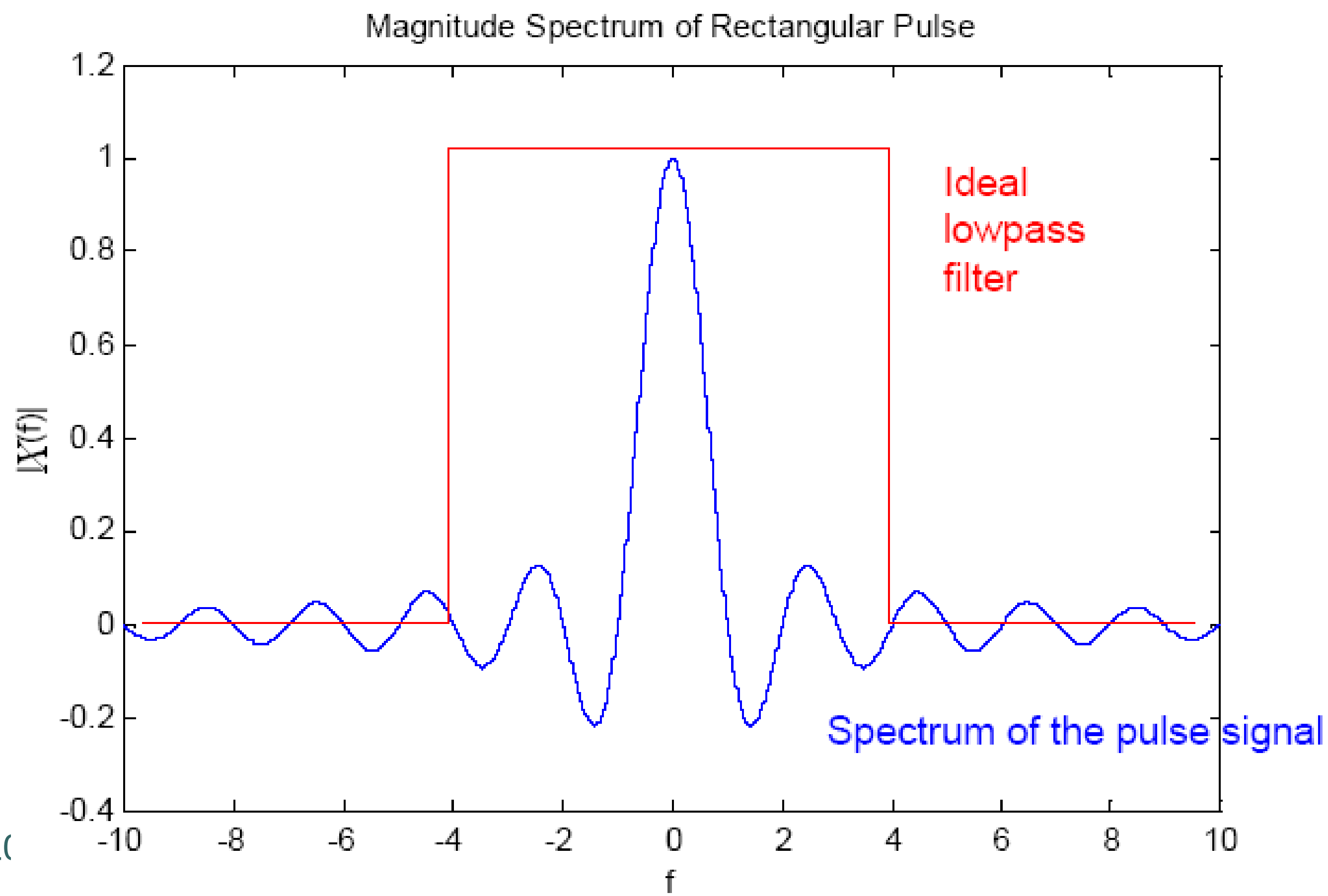
- Can shift the central frequency by modulation
  - A core technique for **communication**, which uses modulation to multiplex many signals into a *single* composite signal, to be carried over the same physical medium

Typical Filters

- Lowpass -> smoothing, noise removal
- Highpass -> edge/transition detection
- Bandpass -> Retain only a certain frequency range



Low Pass Filtering (Remove high freq, make signal smoother)



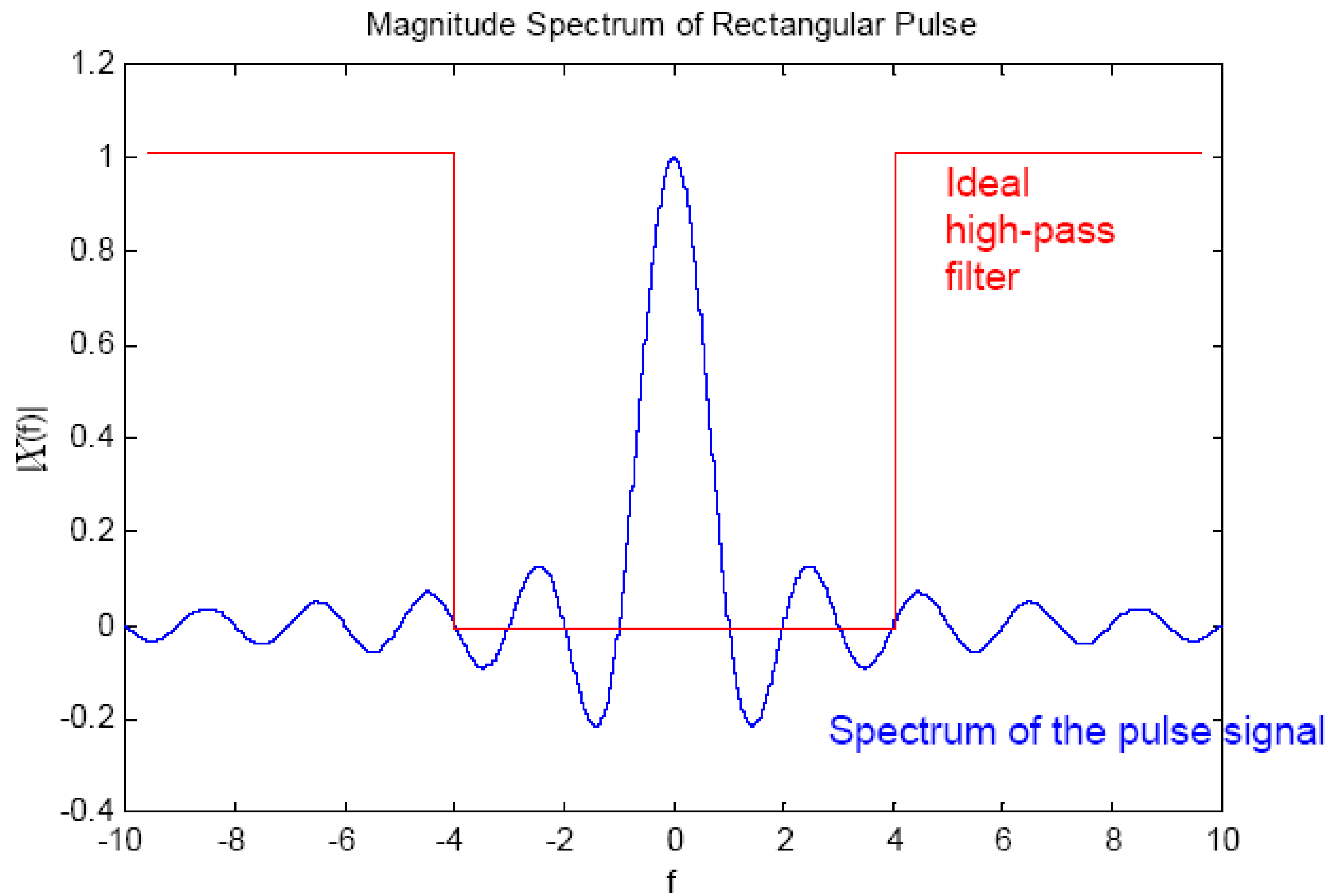
Filtering is done by a simple multiplication:

$Y(f) = X(f)\ H(f)$

H(f) is designed to magnify or reduce the magnitude (and possibly change phase) of the original signal at different frequencies.

A pulse signal after low pass filtering (left) will have rounded corners.

High Pass Filtering (remove low freq, detect edges)



Filtering in Temporal Domain (Convolution)

- Convolution theorem

$$X(f)H(f) \Leftrightarrow x(t) * h(t)$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

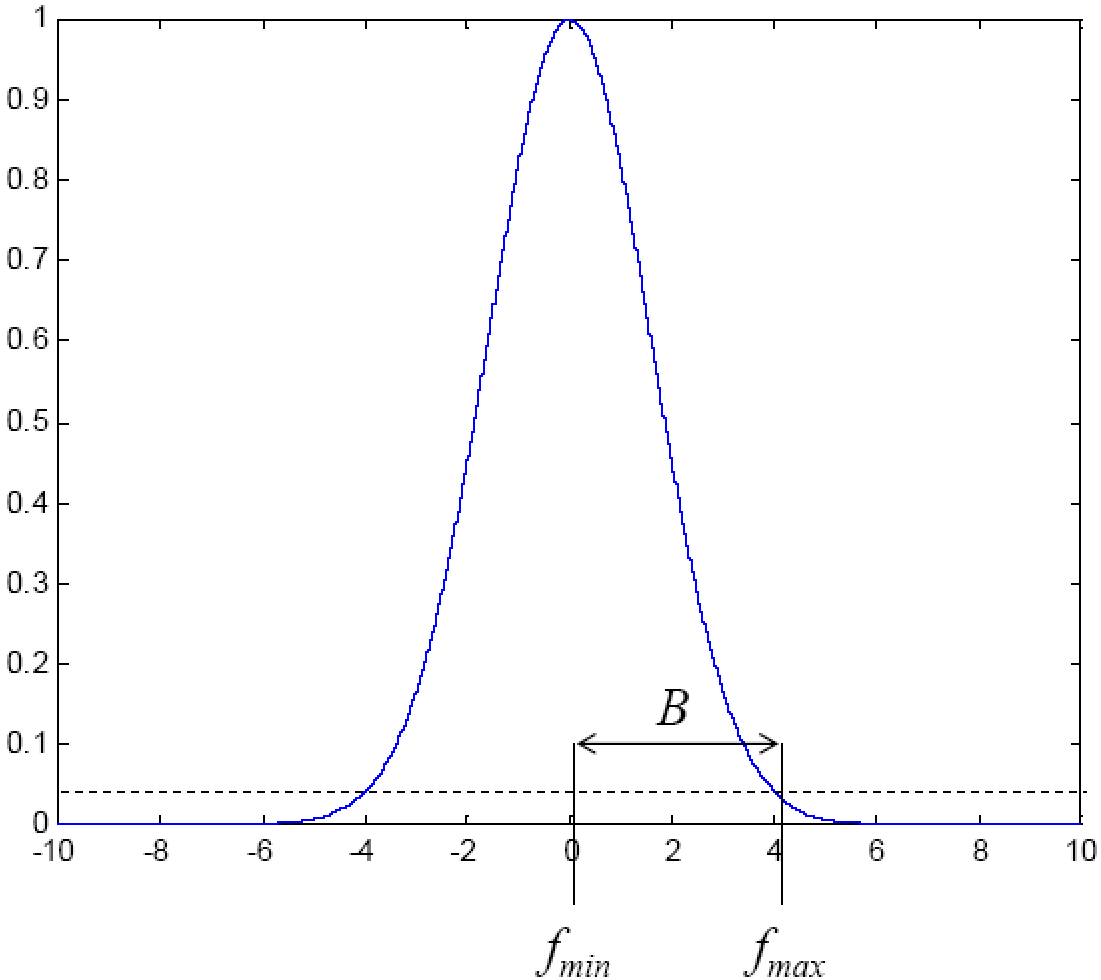
- Interpretation of convolution operation
  - replacing each pixel by a weighted sum of its neighbors
  - Low-pass: the weights sum = weighted average
  - High-pass: the weighted sum = left neighbors –right neighbors

Communication: Signal Bandwidth

- Bandwidth of a signal is a critical feature when dealing with the transmission of this signal
- A communication channel usually operates only at certain frequency range (called channel bandwidth)
  - The signal will be severely attenuated if it contains frequencies outside the range of the channel bandwidth
  - To carry a signal in a channel, the signal needed to be modulated from its baseband to the channel bandwidth
  - Multiple narrowband signals may be multiplexed to use a single wideband channel

Signal Bandwidth:

Highest frequency estimation in a signal: Find the shortest interval between peak and valleys

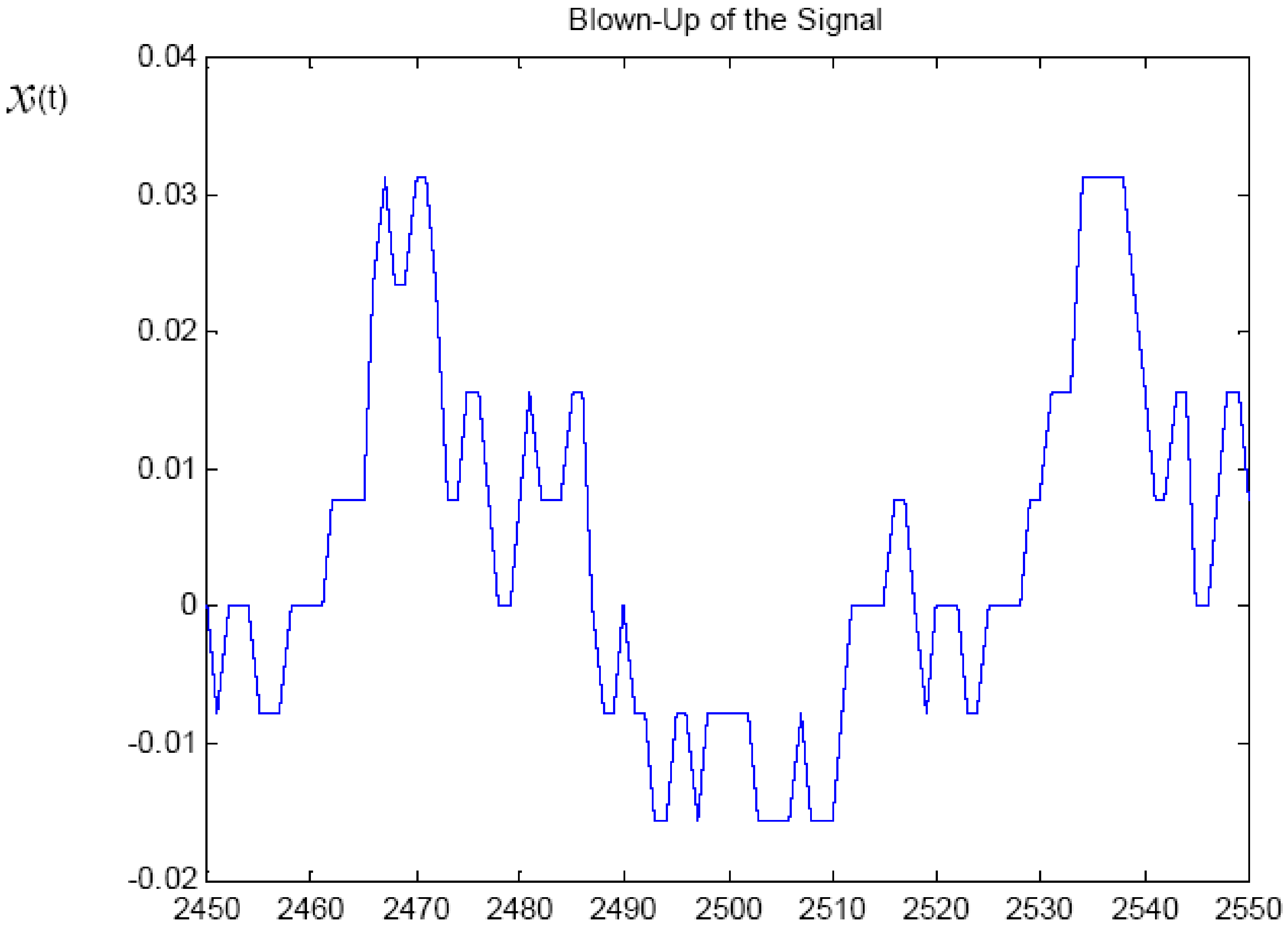


- f_{min} (f_{ma}): lowest (highest) frequency where the FT magnitude is above a threshold
- Bandwidth:

  $B=f_{max}-f_{min}$

- The threshold is often chosen with respect to the peak magnitude, expressed in dB
- dB=10 log10(ratio)
- 10 dB below peak = 1/10 of the peak value
- 3 dB below=1/2 of the peak

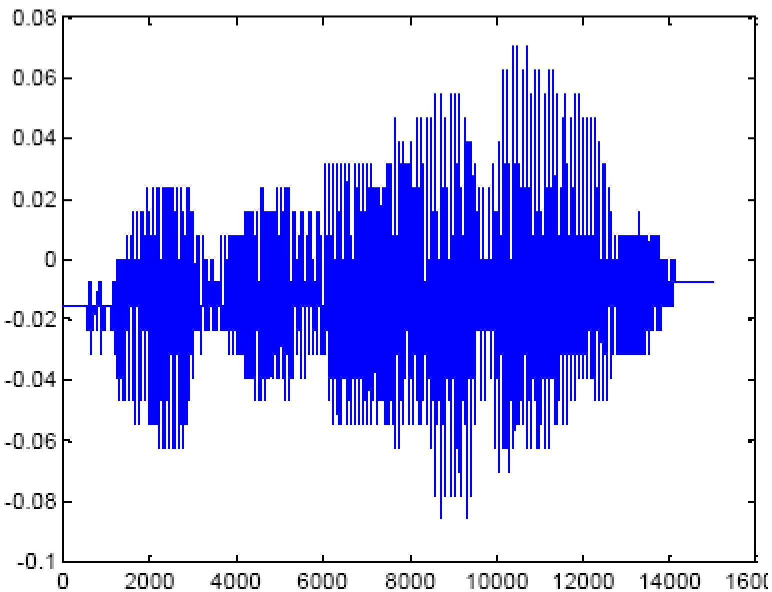
Estimation of Maximum Frequency

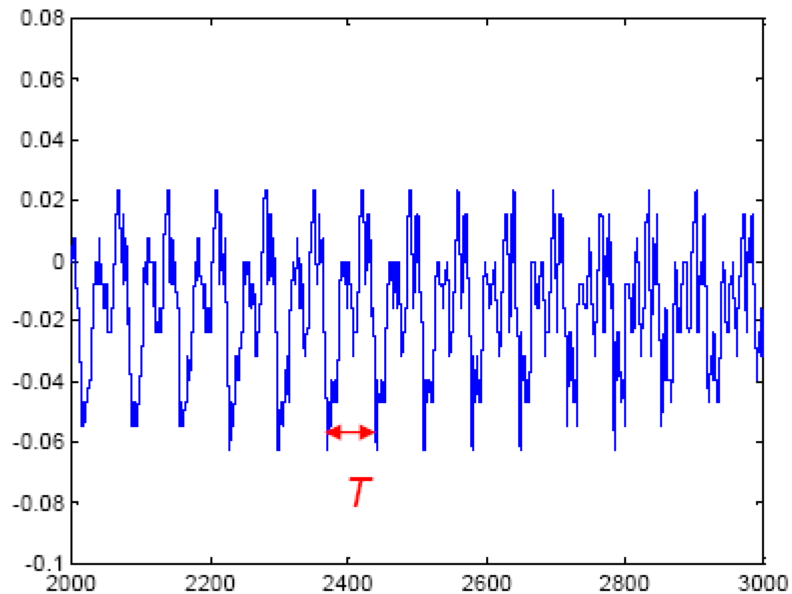


Processing Speech & Music Signals

- Typical speech and music waveforms are semi-periodic
  - The fundamental period is called pitch period
  - The fundamental frequency (f_0)
- Spectral content
  - Within each short segment, a speech or music signal can be decomposed into a pure sinusoidal component with frequency f_0, and additional harmonic components with frequencies that are multiples of f_0.
  - The maximum frequency is usually several multiples of the fundamental frequency
  - Speech has a frequency span up to 4 KHz
  - Audio has a much wider spectrum, up to 22KHz

Sample Speech Waveform 1





Entire waveform

Blown-up of a section.

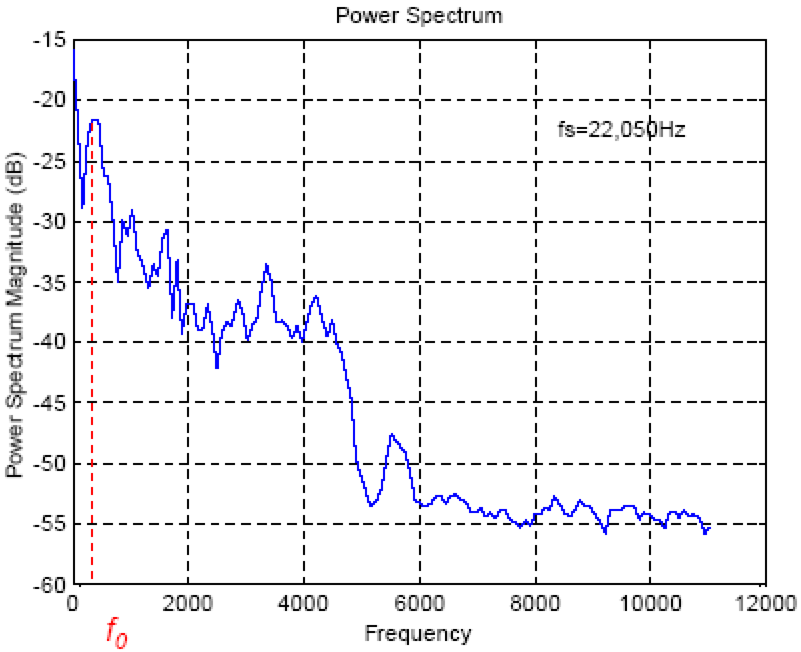
Signal within each short time interval is periodic. The period *T* is called "pitch".
The pitch depends on the vowel being spoken, changes in time. T~70 samples in this ex.

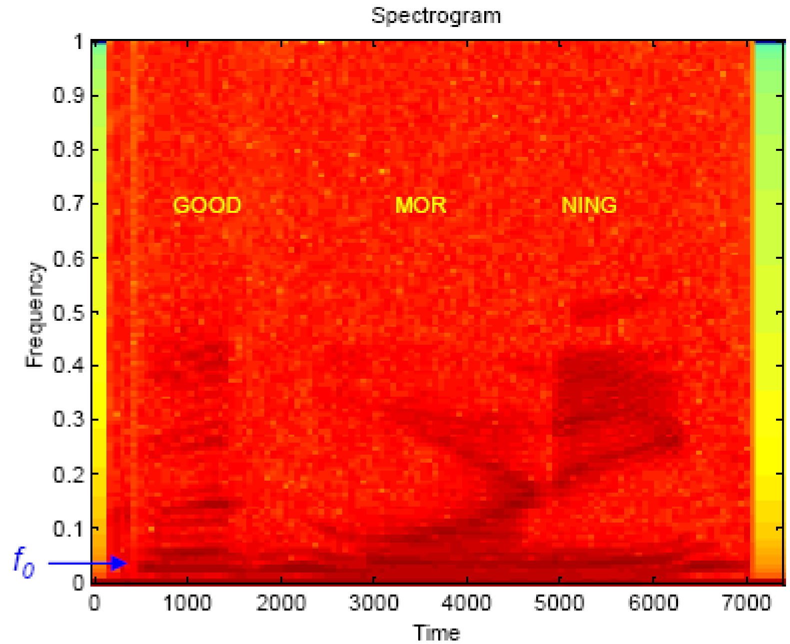
$f_0=1/T$ is the fundamental frequency (also known as formant frequency). $f_0=1/70fs=315$ *Hz.*
$k*f_0$ (k=integers) are the harmonic frequencies.

Numerical Calculation of CT FT

- The original signal is digitized, and then a Fast Fourier Transform (FFT) algorithm is applied, which yields samples of the FT at equally spaced intervals
- For a signal that is very long, e.g. a speech signal or a music piece, spectrogram is used.
  - Fourier transforms over successive overlapping short intervals

Sample Speech Spectrogram 1





» figure;
» psd(x,256,fs);

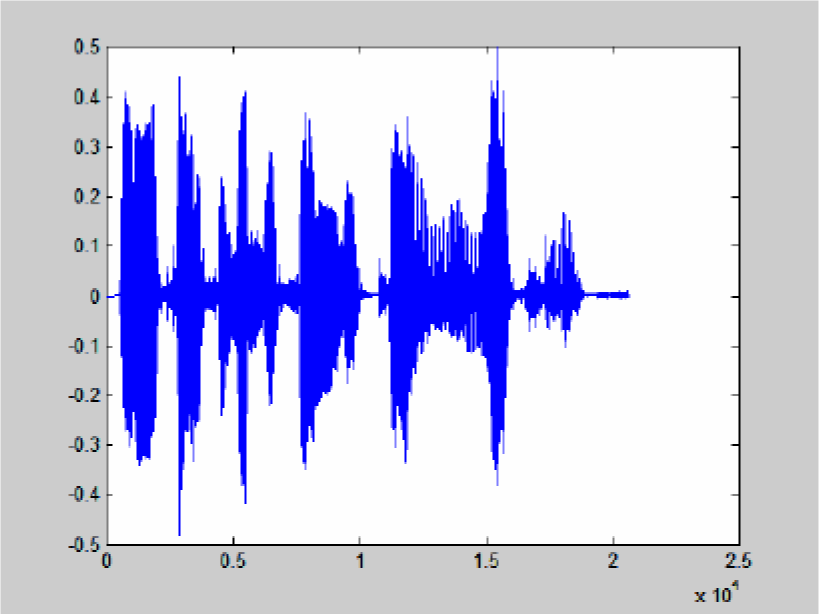
» figure;
» specgram(x,256,fs);

Signal power drops sharply at about 4KHz

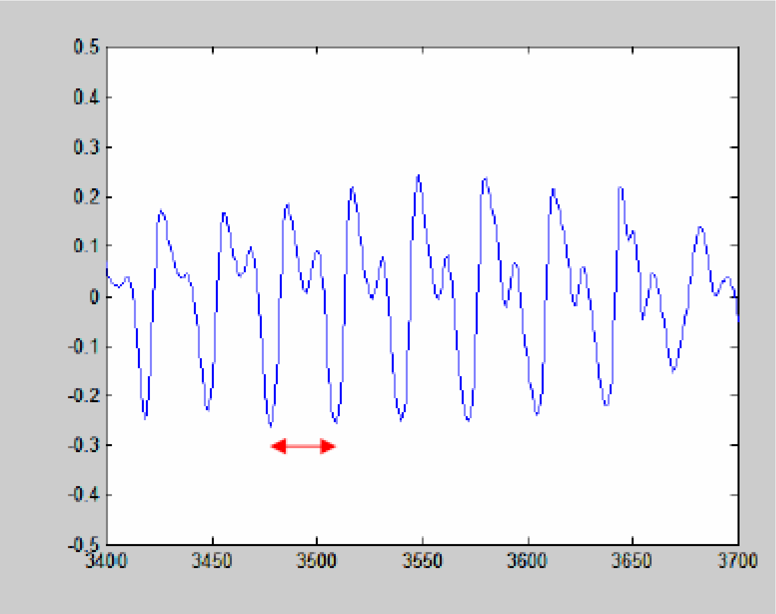
Line spectra at multiple of f0,
maximum frequency about 4 KHz

What determines the maximum freq?

Sample Speech Waveform 2



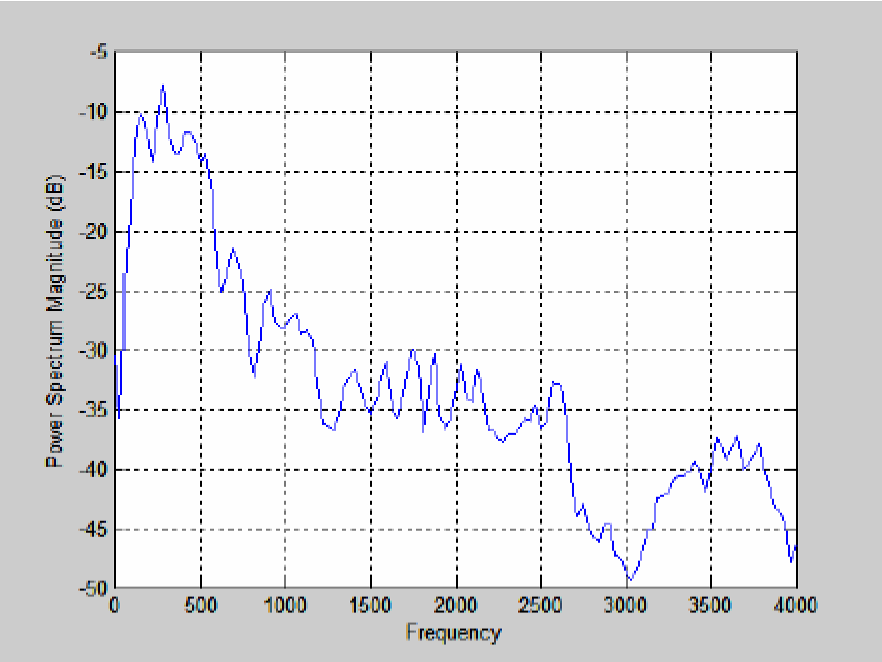
Entire waveform



Blown-up of a section.

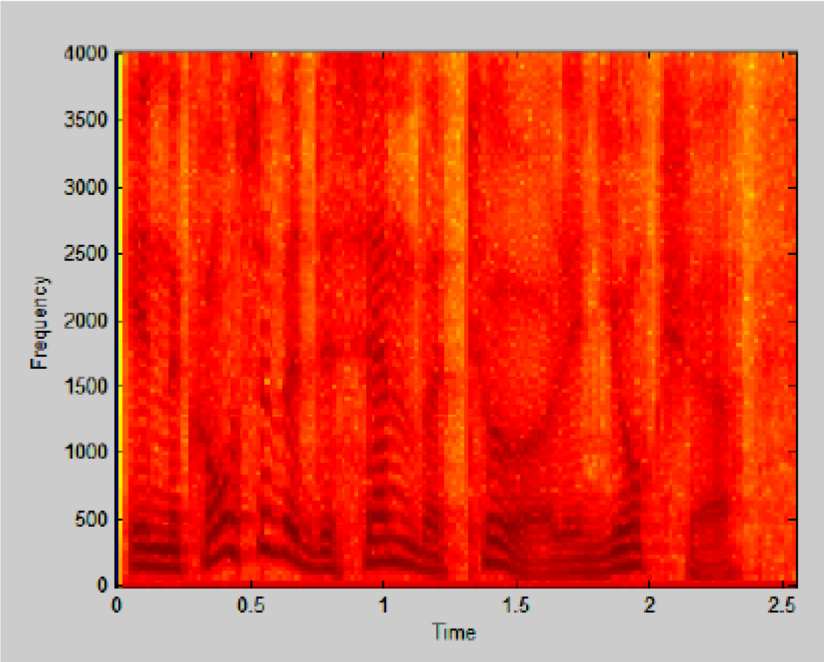
"In the course of a December tour in Yorkshire"

Speech Spectrogram 2




```
» figure;
» psd(x,256,fs);
```

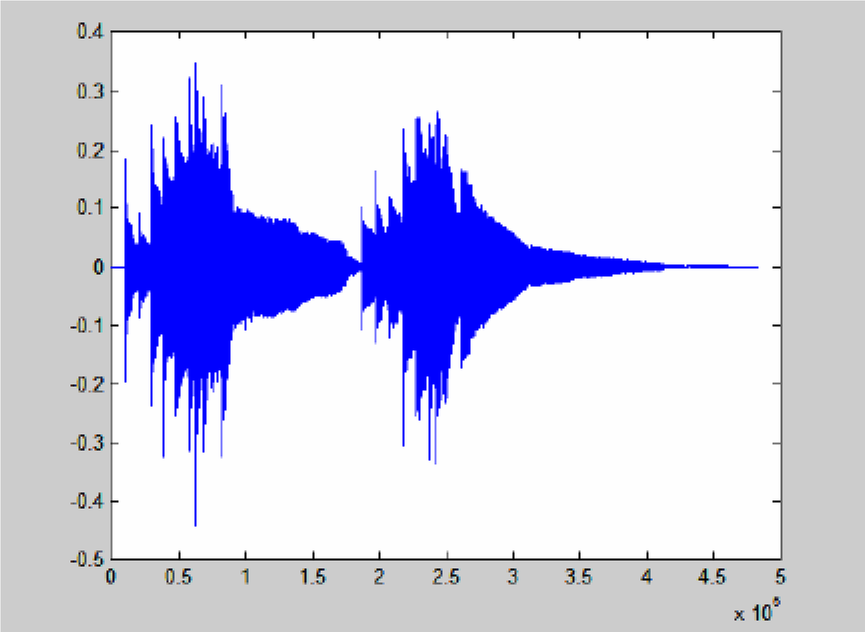
Signal power drops sharply at about 4KHz

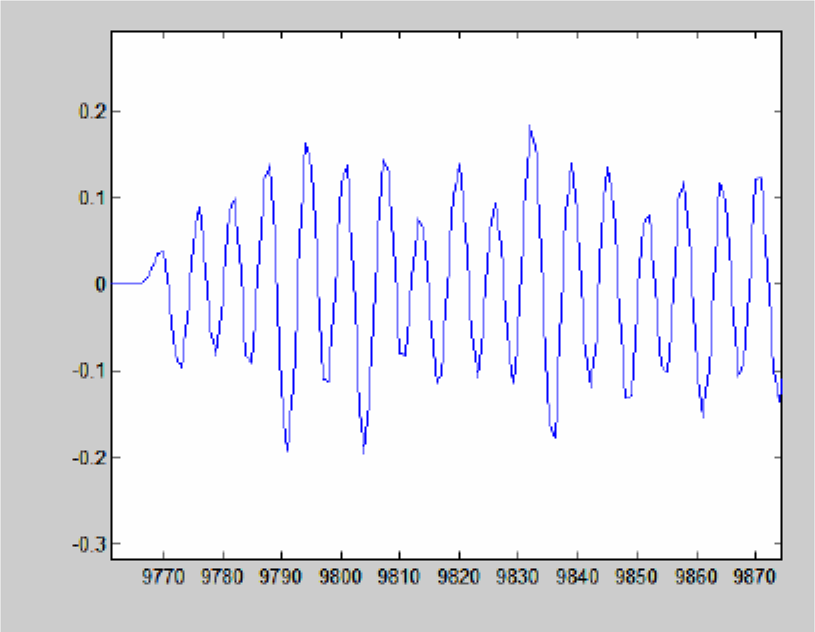



```
» figure;
» specgram(x,256,fs);
```

Line spectra at multiple of f0,
maximum frequency about 4 KHz

Sample Music Waveform





Entire waveform

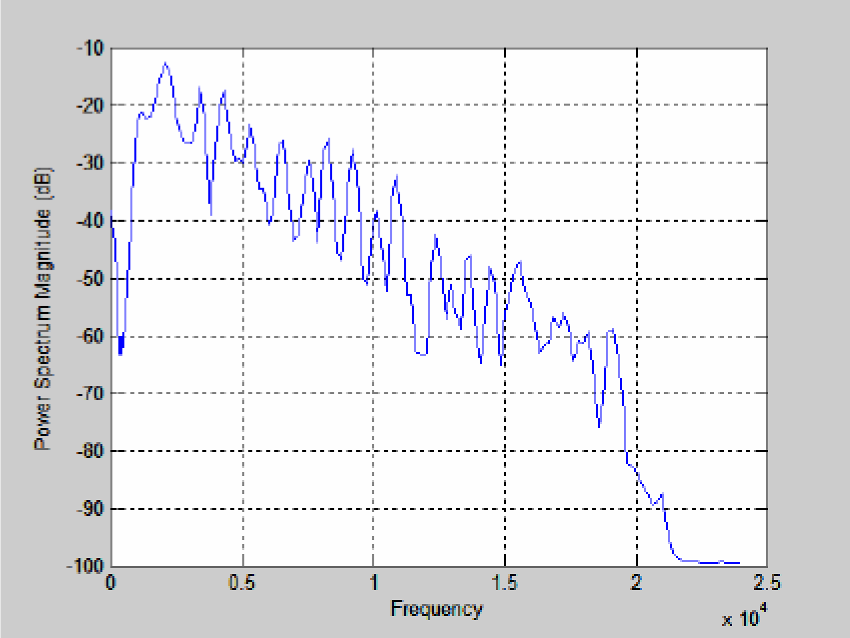
» [y,fs]=wavread('sc01_L.wav');
» sound(y,fs);
» figure; plot(y);

Blown-up of a section

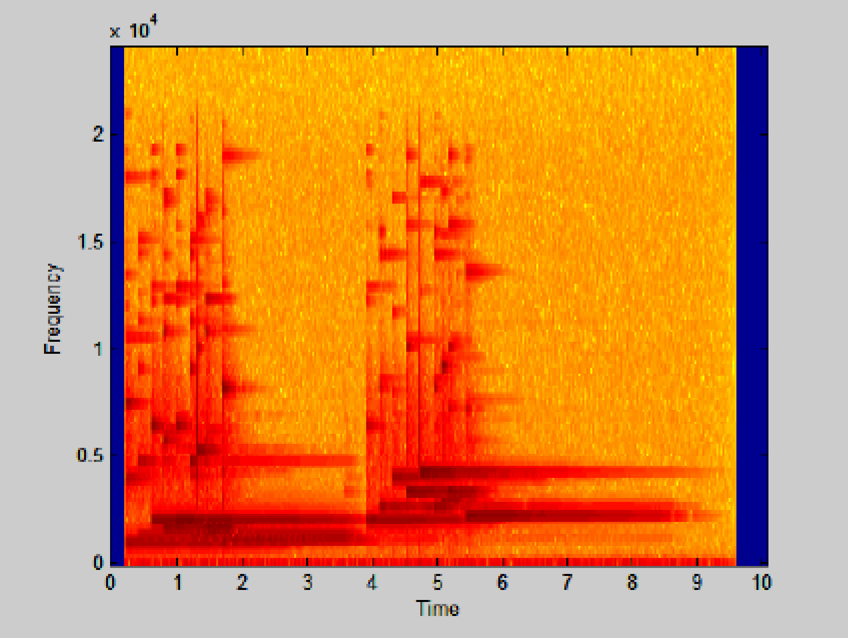
» v=axis;
» axis([1.1e4,1.2e4,-.2,.2])

Music typically has more periodic structure than speech
Structure depends on the note being played

Sample Music Spectrogram



» figure; » psd(y,256,fs);



» figure; » specgram(y,256,fs);

Signal power drops gradually in the entire frequency range

Line spectra are more stationary, Frequencies above 4 KHz, more than 20KHz in this ex.