Haberlesme Sistemlerine Giris (ELE 361) 30 Ekim 2017
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Lecture Notes
Chapter 8 Digital Modulation Methods in an Additive White Gaussian Noise Channel

- AWGN: Simplest model for channel impairments
- Advantages of Digital Transmission:
- Digital Transmission: Binary data is mapped to analog waveforms
- A finite number of analog waveforms are used
- Reason of using analog waveforms:
- Will first consider baseband transmission
- In reality communications occur in a pass band , away from $f = 0$
- Binary data is mapped to the
  - Phase of the carrier signal
  - Frequency of the carrier signal
  - Both phase and amplitude of the carrier
- In this chapter we concentrate on the transmission of a single symbol.

8.1 Geometric Representation of Signal Waveforms

- Suppose we want to send block of 6 bits.
- There are bit combinations
- In digital transmission we assign each bit sequence to a different analog signal.

- So $\boxed{}$ different analog signals are needed.
- What should the receiver do?
  - Compare the received signal to these 64 signals?
  - There is a simpler way
  - Most of the time the 64 signals can be written as a combination of 1 or 2 base signals.
  - Then the receiver calculates the projection of the received signal on these 2 signals.
  - Much simpler and systematic.
  - Now we will learn to determine minimal a set of base signals corresponding to a set of signals.
- In order to send k bits, we need $M = 2^k$ different analog signals.$s_m(t), 1 \leq m \leq M$ $\boxed{}$
- We can express these M signals as linear combinations of N orthonormal signals $\psi_n(t)$ $\boxed{}$
- What is orthonormal? orthogonal: $\boxed{}$ and normal $\boxed{}$
- There are infinite possible choices of orthonormal basis sets
- Systematic way: Gram-Schmidt Orthonormalization

Gram-Schmidt Orthonormalization Procedure

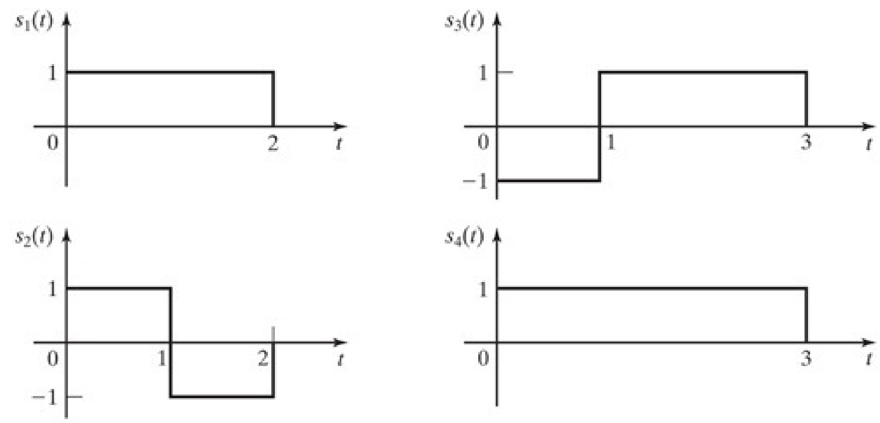
- Begin with $s_1(t)$ and **normalize** it to find $\psi_1(t)$ $\boxed{}$
- Take $s_2(t)$ and
  - compute its projection on $\psi_1(t)$ $\boxed{}$
  - compute $s_2(t) - c_{21}\psi_1(t)$ to yield $\boxed{}$
  - Normalize $d_2(t)$ to find $\psi_2(t) =$ $\boxed{}$
- ....

- Take $s_k(t)$ and
  - compute its projection on $\psi_1(t), \ldots, \psi_{k-1}(t)$
  - compute $s_k(t) - \sum_{i=1}^{k-1} c_{ki}\psi_i(t)$ to yield
  - Normalize $d_k(t)$ to find $\psi_k(t) =$
- If at any step $d_k(t)$ then no new $\psi(t)$

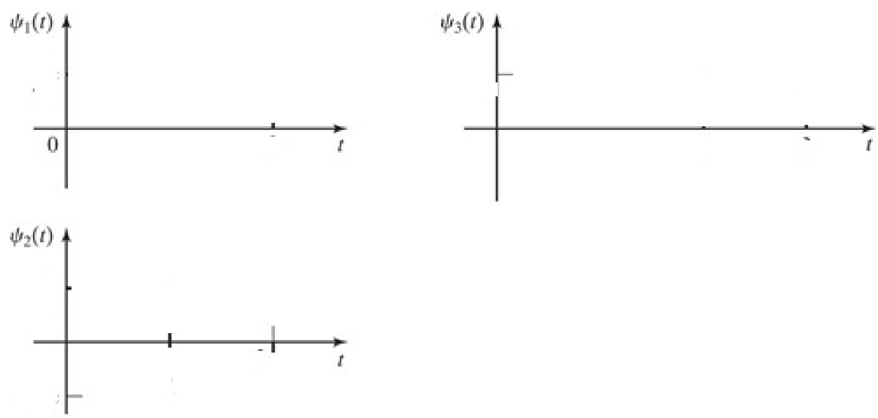
Example 8.1.1: Apply Gram-Schmit procedure to these signals.
Solution:

- Expressing $s_m(t)$ as a vector in the ψ space.
- Find the projections of $s_m(t)$ on each $\psi_n(t)$:
- So $s_m(t) =$
- Energy of $s_m(t)$: $\mathcal{E}_m =$
- Vector:

Example 8.1.2: Determine the vector representations of $s_m(t)$ in the previous example.
Solution:



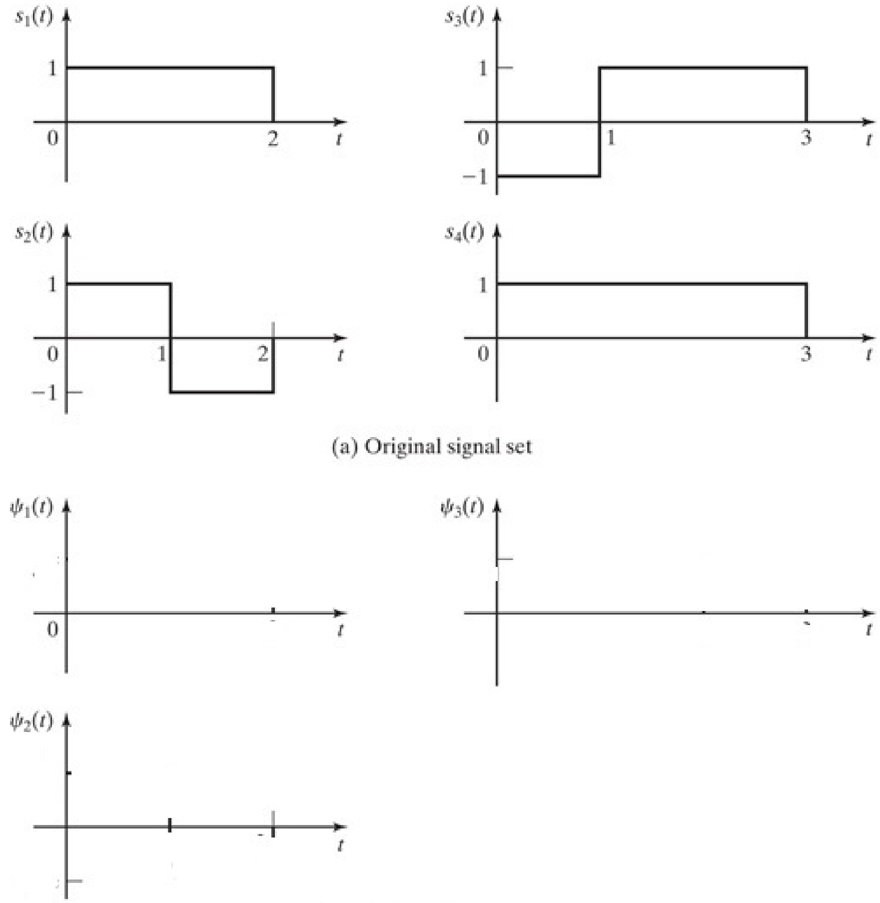
(a) Original signal set



(b) Orthonormal waveforms

A set of signals and their orthogonalization

Most of the time a convenient orthonormal basis set can be found without using Gram-Schmidt method.



(a) Original signal set

(b) Orthonormal waveforms

A set of signals and their orthogonalization

8.4 M-Ary Digital Modulation

Binary modulation: Transmit one bit at a time. Two signal waveforms are required

- Binary antipodal and binary orthogonal are two main binary modulation schemes.
- Others are special cases of these two.
- 1 bit/symbol
- Some examples

M-ary modulation: Use M different waveforms. Transmit $k = \log_2 M$ bits at a time.

$$
\begin{aligned}
R_s &= \frac{1}{T} \\
R_b &= kR_s = \\
T_b &=
\end{aligned}
$$

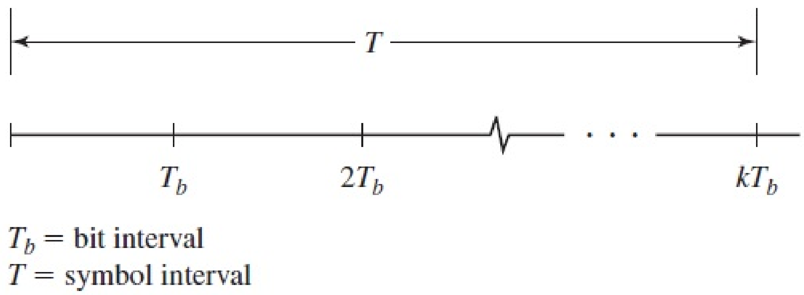
8.4.1 Optimal Receiver for M-ary Signals in AWGN

$$r(t) = s_m(t) + n(t), 0 \leq t \leq T, m = 1, 2, ..., M$$

$n(t)$ is

Signal Demodulator:

Receiver is divided into two such as



M-ary bit and symbol intervals

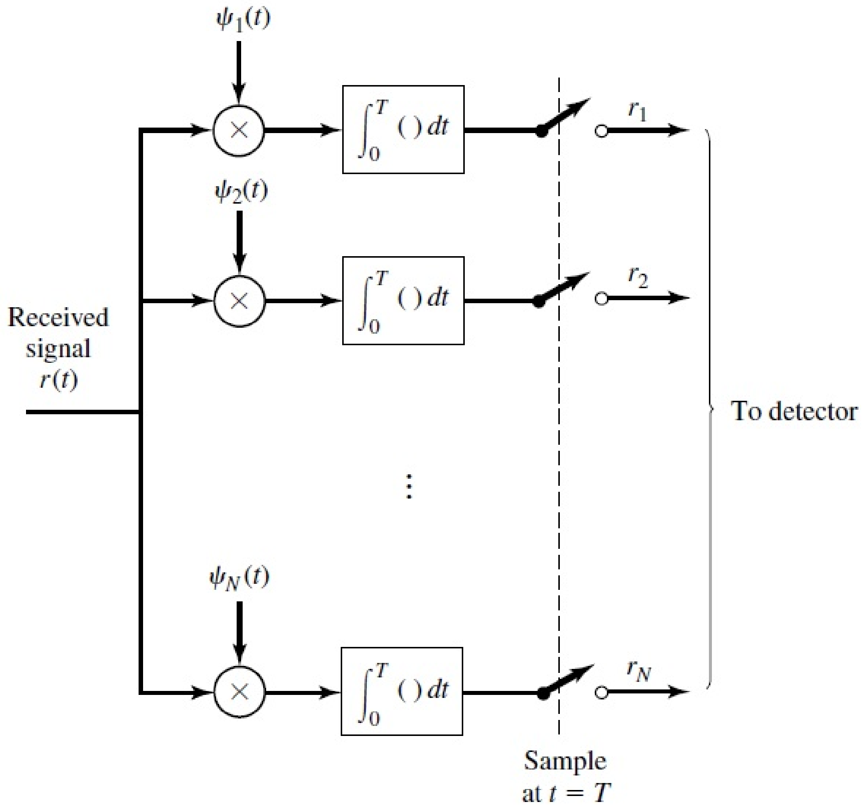
1. Demodulator $\boxed{}$
2. Detector $\boxed{}$

M-ary signal waveform can be represented as $\boxed{}$

$$s_m(t) = \sum_{k=1}^{N} s_{mk}\Psi(t), 0 \leq t \leq T, m = 1, ..., M$$

Signal can be represented as a vector $\boxed{}$

Demodulator: CORRELATOR. $\boxed{}$



Correlation type demodulator

$$
\begin{aligned}
\int_0^T r(t)\Psi_k(t) &= \boxed{} \\
&= \boxed{} \\
y_k &= \boxed{} \\
\mathbf{y} &= \mathbf{s}_m + \mathbf{n} \\
r(t) &= \boxed{} \\
&= \boxed{} \\
n'(t) &= \boxed{}
\end{aligned}
$$

$n'(t)$ is orthogonal to every dimension of the signal waveform (s_{mk}), so it is irrelevant.

$$
\begin{aligned}
E[n_k] &= \\
E[n_k n_m] &= \\
&= \\
&=
\end{aligned}
$$

So, N noise components are uncorrelated and independent. Distribution is

Correlator output:

$$y_k = s_{mk} + n_k, k = 1, ..., N$$

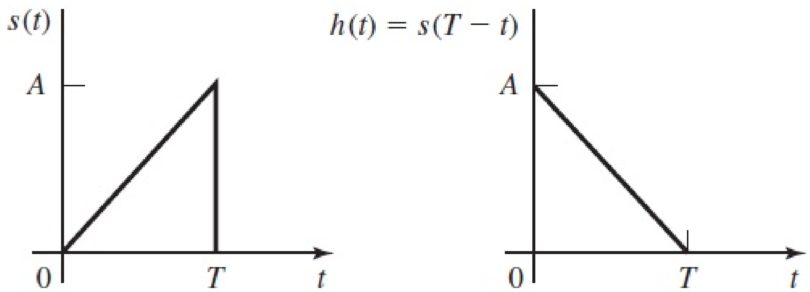
Distribution of y_k is:

y_k is also independent of $n'(t)$

Another type of demodulator: MATCHED FILTER

$$
\begin{aligned}
r(t) &= s_m(t) + n(t), 0 \leq t \leq T, m = 1, ..., M \\
&= \\
h_k(t) &= \Psi_k(T - t), 0 \leq t \leq T, k = 1, .., N \\
y_k(t) &= \int_0^t r(\tau) h_k(t - \tau) d\tau \\
&= \\
\texttt{sample at t=T} \quad &= \\
&= \quad \texttt{same as correlation type}
\end{aligned}
$$

Matched Filter: Impulse Response matched to the signal waveform
Signal $s(t)$: matched filter $h(t) = s(T - t)$



(a) Signal $s(t)$ (b) Impulse response of filter matched to $s(t)$

Matched Filter example

$$s(t) * s(T - t) = \int_0^t s(\tau)s(T - t + \tau)$$

Shetch:

Maximizes the output SNR. Output SNR depends on the energy of the matched filter but not on the signal waveform itself.

In the frequency domain:

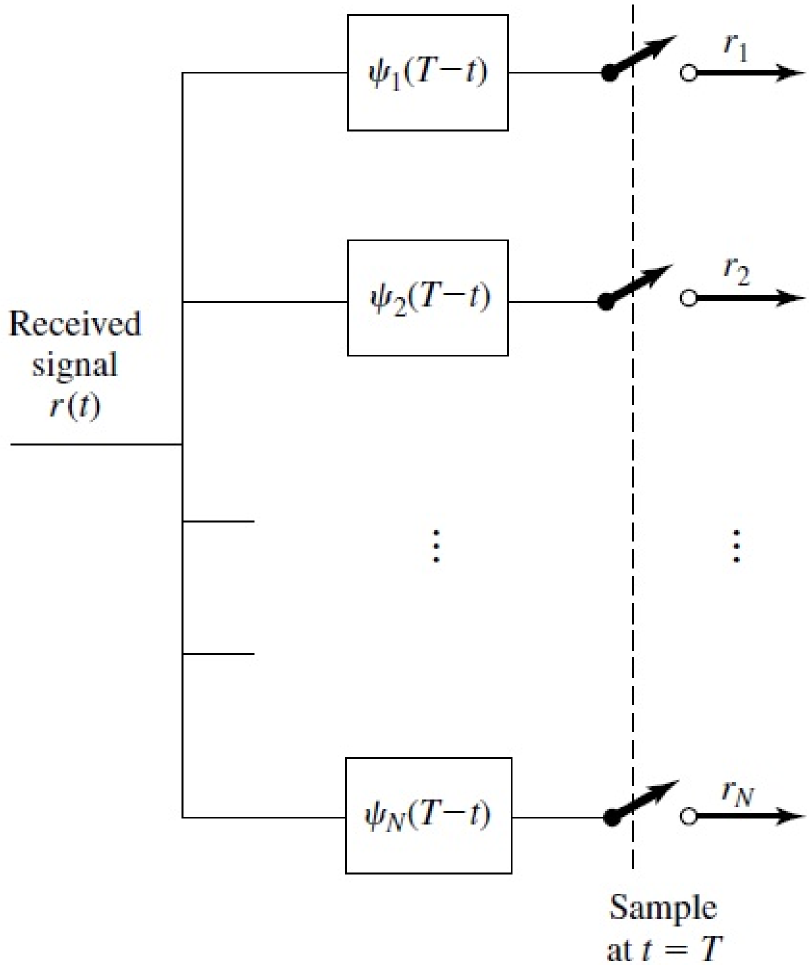
$$
\begin{aligned}
h(t) &= s(T-t) \\
s(t) &\to S(f) \\
s(-t) &= S^*(f) \\
s(T-t) &= \boxed{} \\
Y(f) &= S(f)H(f) = \boxed{} \\
y(t) &= \boxed{} \\
y(T) &= \boxed{} \\
\texttt{output signal power} &= \boxed{}
\end{aligned}
$$

What about the noise component?

$$
\begin{aligned}
\texttt{output noise power} &= \boxed{} \\
&= \boxed{} \\
&= \boxed{} \\
SNR &= \boxed{}
\end{aligned}
$$

Example 8.4.1: Consider 4-PAM. Determine the PDF of the received signal at the output of the correlator and sketch its PDF.

Solution:



Matched Filter type demodulator

Example 8.4.1: Consider M-4 orthogonal signaling. Determine the PDF of the received signal at the output of the correlator and sketch its PDF.

Examples of orthogonal signaling:

1. Pulse position modulation

2. Frequency Shift Keying

Solution:

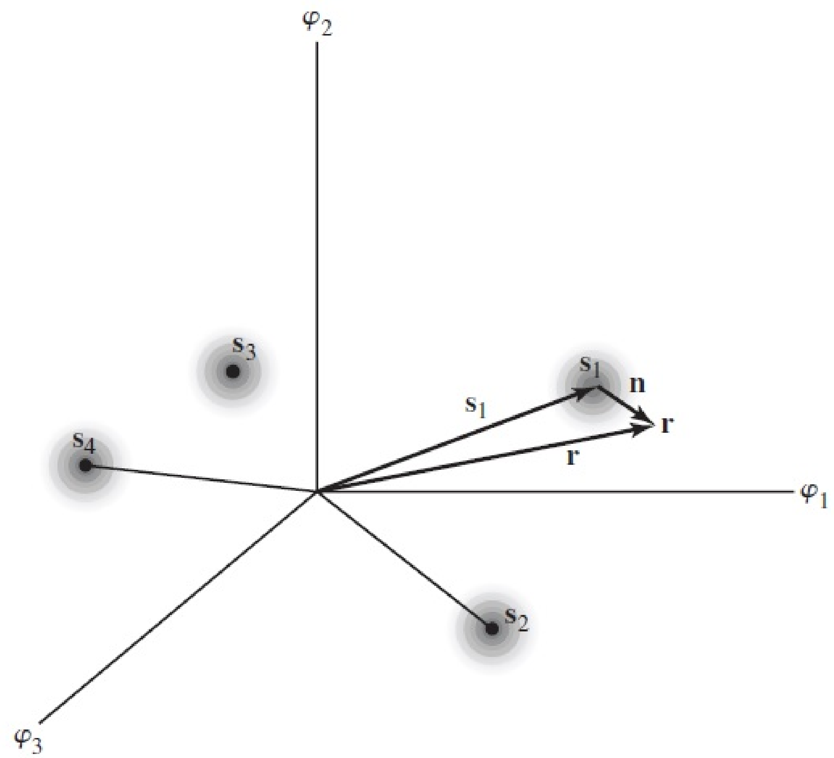
Optimum Detector

The problem is::

$$\arg \max_{m=1,\ldots,M} P(\texttt{signal m was transmitted}|\mathbf{r})$$

"Find the most likely transmitted waveform, given the demodulator output"

This is *Maximum A Posteriori Probability (MAP) detector*



Signal constellation, noise cloud and received vector (M=4, N=3)

$$P(\mathbf{s}_m|\mathbf{y}) = \frac{f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)}{f(\mathbf{y})}$$

$$\texttt{where} f(\mathbf{y}) = \sum_{m=1}^{M} f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)$$

so the problem is $\max\limits_m \frac{f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)}{f(\mathbf{y})}$

simplification $=$

For equiprobable signals, MAP rule becomes Maximum Likelihood detection.

$$f(\mathbf{y}|\mathbf{s}_m) = \prod_{k=1}^{N} \frac{1}{\sqrt{\pi N_o}} e^{(y_k - s_{mk})^2/N_o}$$

take logarithm: $\ln f(\mathbf{y}|\mathbf{s}_m) =$

$$\max_m P(\mathbf{s}_m|\mathbf{y}) = \min_m$$

$$D(\mathbf{y}, \mathbf{s}_m) =$$

$D(\mathbf{y}, \mathbf{s}_m)$ is the

Example 8.4.3: Consider binary PAM. Signal points are $s_1 = -s_2 = \sqrt{\epsilon_b}$. Prior probabilities are unequal (i.e. $P(s_1) \neq P(s_2)$) Determine the metric.

Solution:

MAP detection minimizes the probability of error.

8.4.1a Probability of error for binary PAM

Output of demodulator is

Assume equiprobable signaling

Assume $s_1(t)$ was transmitted. Then demodulator output is

$$
\begin{aligned}
P(error) &= \\
\texttt{Due to} &= \texttt{symmetry} \\
&= \boxed{} \\
&= \boxed{} \\
\texttt{Standardize} & \quad \texttt{the Gaussian noise} \\
&= \\
&= \boxed{} \\
\texttt{Write in terms} & \quad \texttt{of the distance}
\end{aligned}
$$

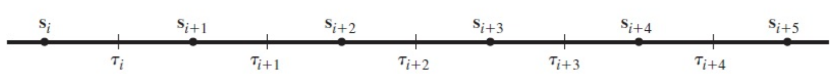
8.4.2 A Union Bound on the Probability of Error

For binary equiprobable signaling (which is)

8.5 M-ary Pulse Amplitude Modulation

M-ary PAM is used to transmit $\boxed{}$ bits per symbol.

Single dimension $\boxed{}$



M-ary PAM

$$
\begin{aligned}
s_m(t) &= s_m \Psi(t), 0 \le t \le T, m = 1, 2, ..., M \\
s_m &= (2m - 1 - M)d \\
Average symbol energy \epsilon_s &= \boxed{} \\
Average bit energy \epsilon_b &= \boxed{}
\end{aligned}
$$

Example 8.5.1: Sketch the transmitted waveform for 4-PAM and the bit sequence 011001010101
Solution:

8.5.1 Carrier Modulated PAM for bandpass channels (M-ary ASK)

A regular baseband PAM signal is multiplied with a cosine to make it bandpass

This is like a

The occupied bandwidth is

$$
\begin{aligned}
u_m(t) &= s_m(t)\cos(2\pi f_c t), m = 1, ..., M \\
u_m(t) &= s_m\sqrt{2}\Psi(t)\cos(2\pi f_c t), m = 1, ..., M \\
s_m &= (2m - 1 - M)d \\
sU_m(f) &= \\
\epsilon_m &= \int_{-\infty}^{\infty} u_m^2(t)dt \\
&= \\
&=
\end{aligned}
$$

8.5.2 Demodulation and Detection of Bandpass PAM

The received signal is correlated with

The result is

The detector decides according to

8.5.3 Probability of Error for M-PAM

For baseband PAM, if $s_m(t)$ is transmitted, the demodulator output is:

Error event is related to the instantaneous noise being greater than d.

Sketch:

$$
\begin{aligned}
y &= s_m + n \\
P_M &= \frac{M-2}{M} P(|y - s_m| > d) + \frac{2}{M} P(y - s_m > d) \\
&= \\
&= \\
&= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2d^2}{N_o}}\right)
\end{aligned}
$$

Distance parameter d can be related to the average symbol energy ϵ_{av} as,

$$P_M =$$

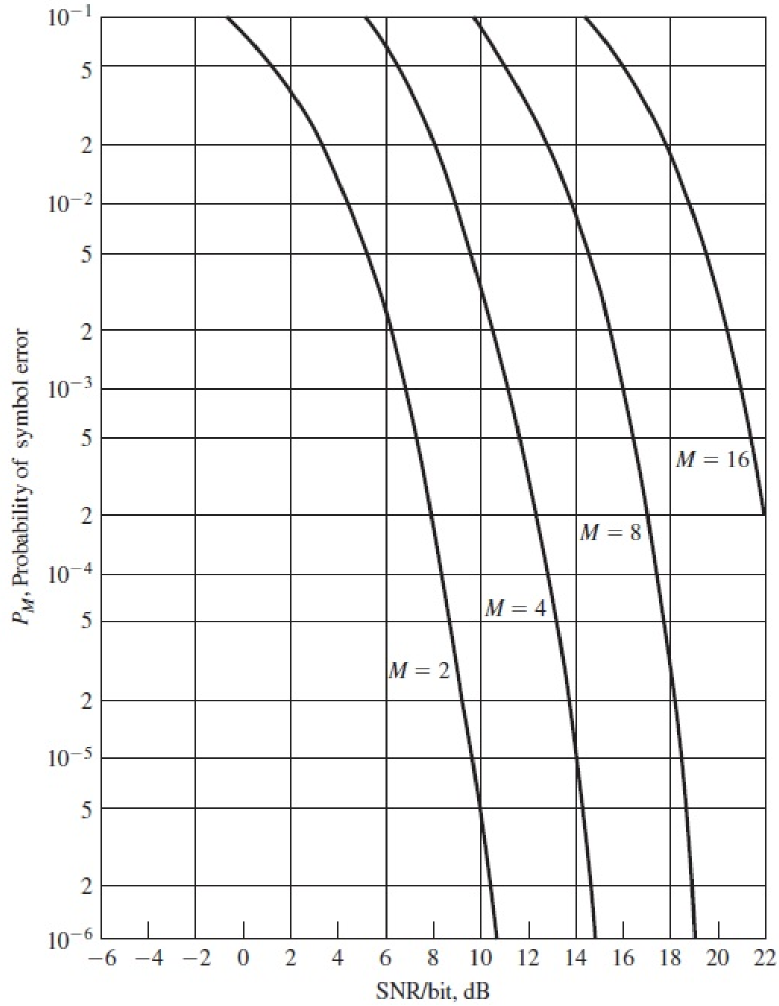
In terms of the average bit energy ϵ_{bav}

$$P_M =$$

Example 8.5.3: Using the figure below, determine the approximate SNR/bit required to achieve a SER of $P_M = 10^{-6}$ for M=2,4,8.

Phase Shift Keying

Each analog waveform has equal energy. Equally-spaced on a circle around the origin of the constellation diagram.

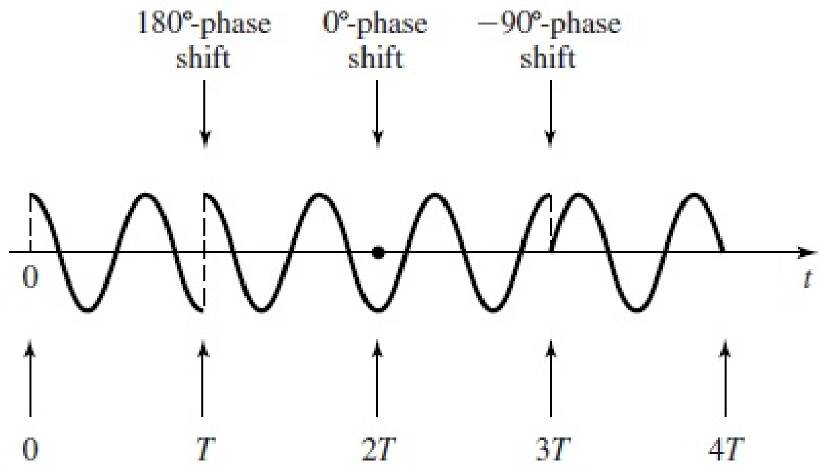


Bit error rate for PAM: The horizontal axis is $\frac{\epsilon_{bav}}{N_o}$ (in dB). Every extra 1 bit/symbol requires extra 4dB SNR. For better spectral efficiency we need more transit power.

Special case: For rectangular pulse

$$
\begin{aligned}
\texttt{Special case} &: \texttt{For rectangular pulse} \\
u_m(t) &= \sqrt{\frac{2\epsilon_s}{T}} \cos(2\pi f_c t + \frac{2\pi m}{M}), 0 \leq t \leq T m = 1, ..., M \\
\epsilon_{bav} &= \frac{\epsilon_{av}}{\log_2 M} \\
General case &: \texttt{Pulse} g_T(t) \\
u_m(t) &= g_T(t) \cos(2\pi f_c t + \frac{2\pi m}{M}), 0 \leq t \leq T m = 1, ..., M \\
\texttt{expanding the cosine} u_m(t) &= \boxed{}
\end{aligned}
$$

Sketch the modulator:



Example of a 4-PSK

8.6.1 Geometric representation of M-PSK Signals

For general pulse $g_T(t)$

Sketch the constellation for M=2,4,8

$$\mathbf{s}_m =$$

$$\Psi_1(t) =$$

$$\Psi_2(t) =$$

Minimum Euclidian distance $d_{min} =$

d_{min} plays an important role in determining the error probability

Example 8.6.1: For M=8 PSK determine how many dB the transmitted energy must be increased to achieve the same d_{min} as M=4.

Solution:

8.6.2 Demodulation and Detection of PSK Signals

$$
\begin{aligned}
r(t) &= u_m(t) + n(t) m = 0, 1..., M-1 \\
&= \boxed{} \\
n(t) &= n_c(t)\cos(2\pi f_c t) + n_s(t)\sin(2\pi f_c t) \\
\texttt{Correlated with}\Psi_1(t) &= n_c(t)\cos(2\pi f_c t) \\
\texttt{and}\Psi_2(t) &= n_c(t)\cos(2\pi f_c t) \\
\texttt{Output vector } \mathbf{y} &= \mathbf{s}_m + \mathbf{n} \\
&= \boxed{} \\
\texttt{by definition} n_c &= \boxed{} \\
n_s &= \boxed{} \\
E[n_c^2] = E[n_s^2] &= \boxed{}
\end{aligned}
$$

Optimal detector: Compute the phase of the detector output ($\boxed{}$) and find the signal $\mathbf{s}_m$ whose phase is closest to $\boxed{}$

8.6.3 Probability of Error for Phase Coherent PSK

Let $s_0 = (\sqrt{\epsilon_s}, 0)$ be transmitted

Demodulator output is $(y_1, y_2) = (\sqrt{\epsilon_s} + n_c, n_s)$. Then the phase of $\mathbf{y}$ is checked. Error probability is about the phase being $\theta > \pi/M$ or $\theta < -\pi/M$. The pdf of Θ is complicated, and exact error probability calculation requires numerical integration.

For M=2: equivalent to 2PAM (antipodal signaling) $p_2 = Q(\sqrt{\frac{2\epsilon_b}{N_o}})$

For M=4: We have two 2PSK signals (one in the horizontal and the other in the vertical axis of the constellation)

$$
\begin{aligned}
P_4 &= 1-(1-P_2)^2 \\
&= 1-\left[1-Q\left(\sqrt{\frac{2\epsilon_b}{N_o}}\right)\right]^2 \\
&= \\
\texttt{for high SNR} &=
\end{aligned}
$$

For general M: An approximate approach:
Sketch the constellation and calculate the minimum distance:

For high SNR: Each point in the constellation can be mixed with only the nearest neighbors:

8.7 Quadrature Amplitude Modulated Digital Signals

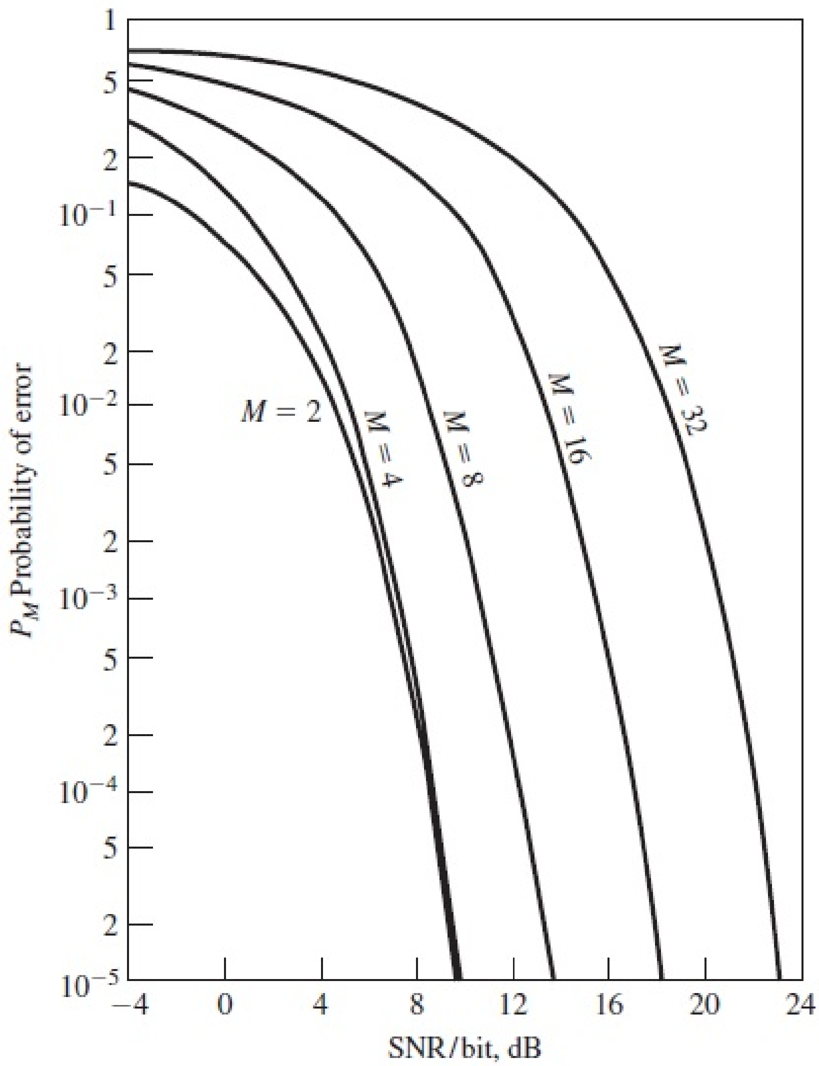
For M-PSK analog waveforms have the same energy

In the constellation diagram

In M-QAM we don't have this constraint.

Like PSK

$$
\begin{aligned}
u_m(t) &= Amcg_T(t)\cos(2\pi f_c t) - A_{ms}g_T(t)\sin(2\pi f_c t), m = 1, ..., M \\
u_{mn}(t) &= A_m g_T(t)\cos(2\pi f_c t + \theta_n), m = 1, ..., M_1, n = 1, ..., M_2 \\
M_1 &= 2^{k_1} \\
M_2 &= 2^{k_2} \\
M &= M_1 + M_2 =
\end{aligned}
$$



Bit error rate for PSK: The horizontal axis is $\frac{\epsilon_{bav}}{N_o}$ (in dB). Every extra 1 bit/symbol requires extra 4dB SNR. For better spectral efficiency we need more transit power.

8.7.1 Geometric Representation of QAM Signals

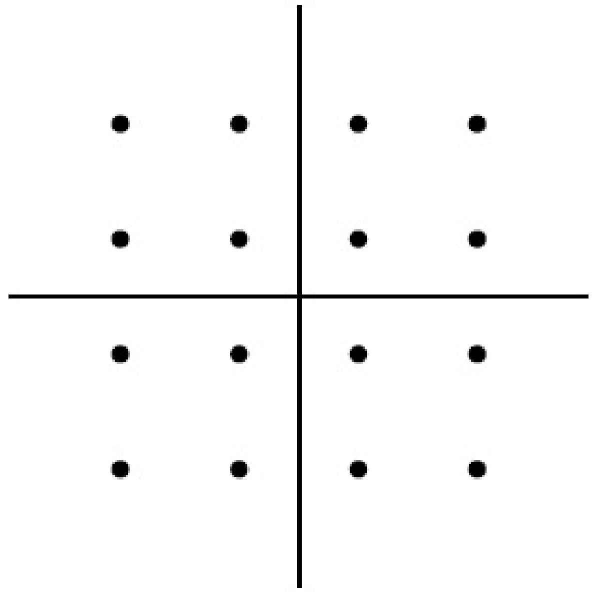
$$\Psi_1(t) =$$

$$\Psi_2(t) =$$

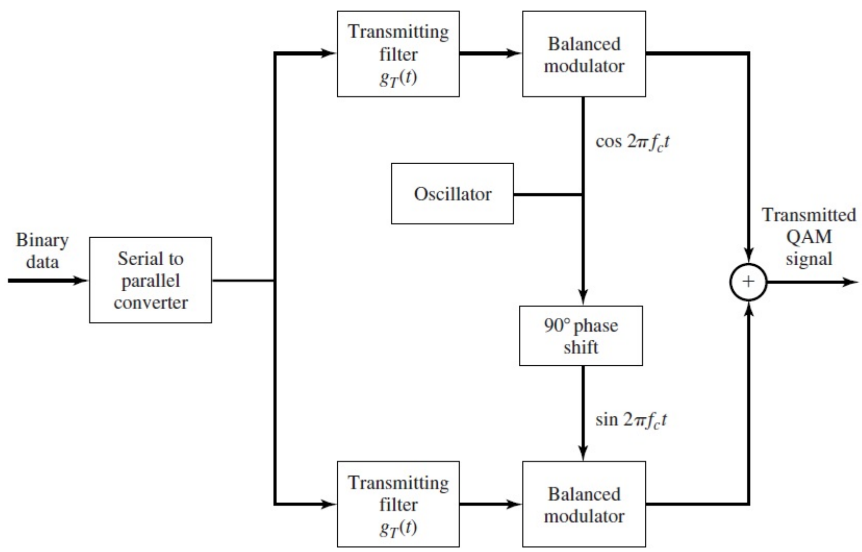
$$\mathbf{s}_m =$$

$$\epsilon_s = \epsilon_{av} =$$

$$d =$$

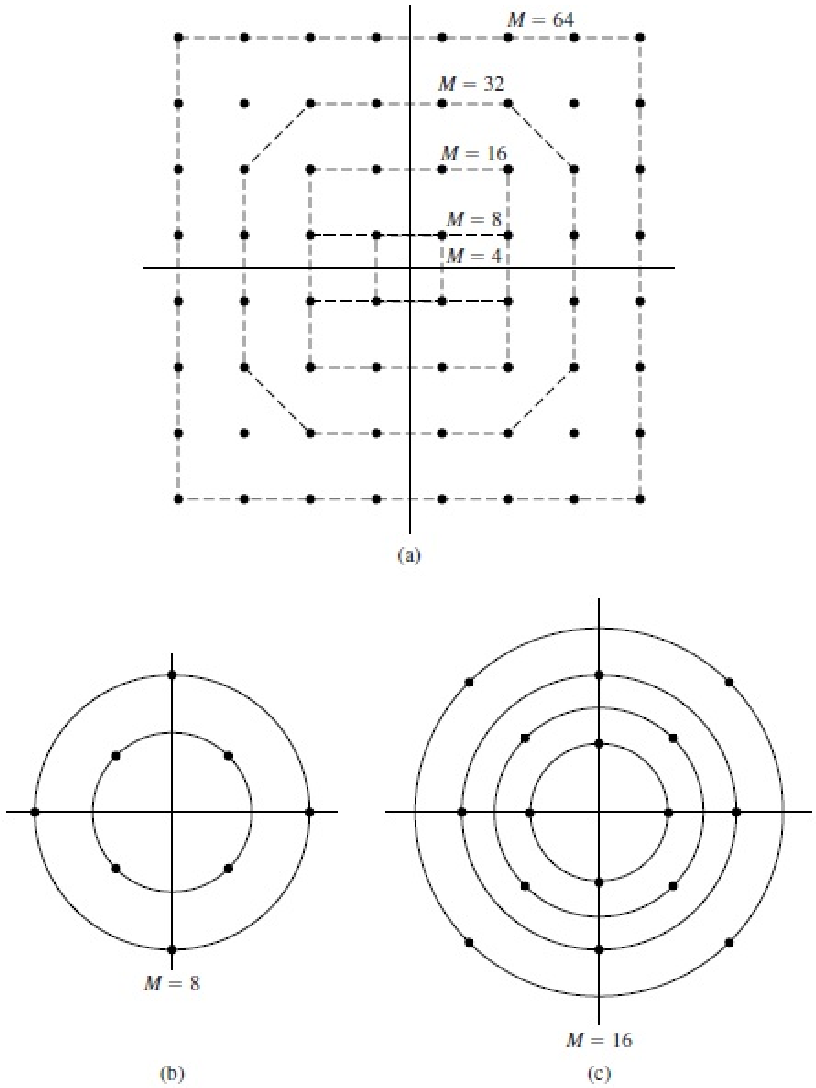


16QAM Constellation. It's like modulating the two quadrature carriers by 4PAM

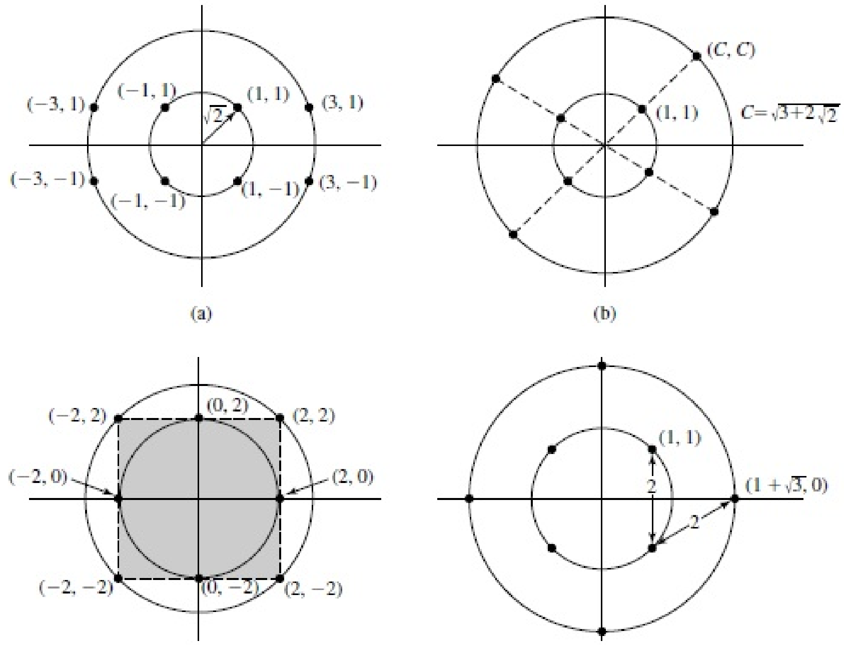


Functional block diagram of QAM

Example 8.7.1: Determine the average energy of the below QAM contellations
Solution:



Example QAM constellations



Example 8.7.1

8.7.2 Demodulation and Detection of QAM Signals

$$
\begin{aligned}
r(t) &= Amcg_T(t)\cos(2\pi f_c t) - A_{ms}g_T(t)\sin(2\pi f_c t) + n(t) \\
\texttt{Cross correlated} &= \texttt{by}\Psi_1(t)\texttt{and}\Psi_2(t) \\
\mathbf{y} &= \mathbf{s}_m + \mathbf{n} \\
&= \boxed{\quad} \\
D(\mathbf{y}, \mathbf{s}_m) &= \boxed{\quad}
\end{aligned}
$$

Optimum detector selects the signal corresponding to the smallest value of $D(\mathbf{y}, \mathbf{s}_m)$

8.7.3 Probability of Error for QAM

`1st 4QAM contellation` $\epsilon_{av} = \epsilon_s \quad = \boxed{}$

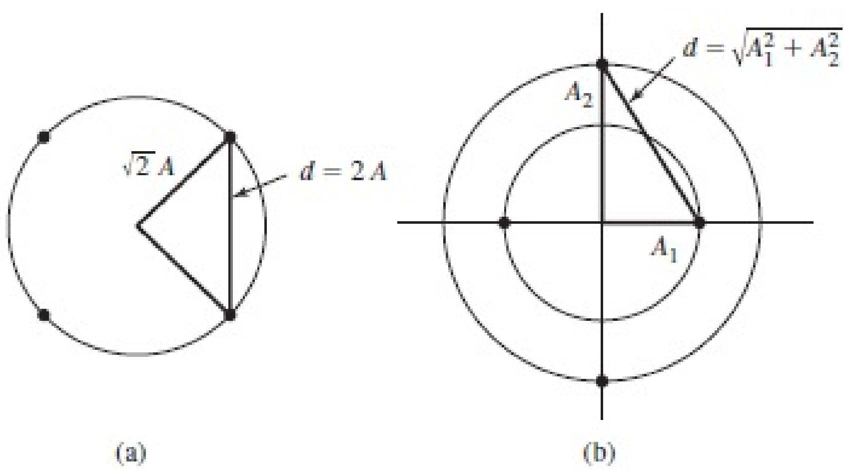
`2nd 4QAM contellation` $\epsilon_{av} = \epsilon_s \quad = \boxed{}$

`1st 8QAM contellation` $\epsilon_{av} = \epsilon_s \quad = \boxed{}$

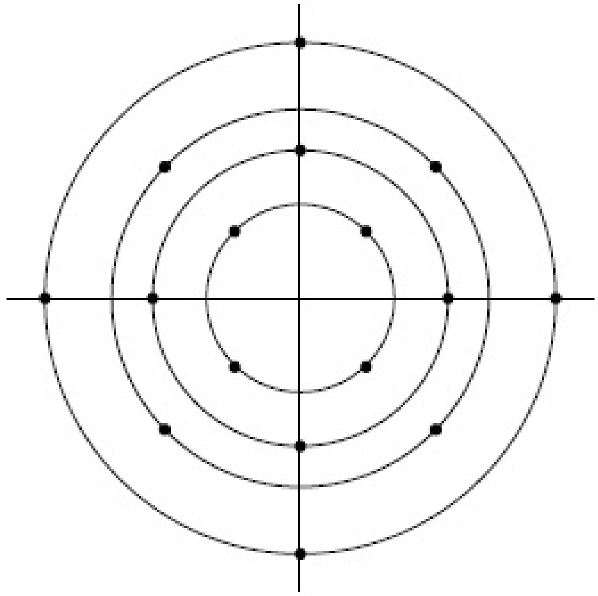
`2nd 8QAM contellation` $\epsilon_{av} = \epsilon_s \quad = \boxed{}$

`3rd 8QAM contellation` $\epsilon_{av} = \epsilon_s \quad = \boxed{}$

`4th 8QAM contellation` $\epsilon_{av} = \epsilon_s \quad = \boxed{}$



Two 4-point QAM Constellations



$M = 16$

A circular 16QAM constellation. Actually rectangular QAM constellations are preferred, as they are much easier to generate (two PAM signals).

$$
\begin{aligned}
\texttt{When k is even} P_M^{QAM} &= 1 - (1 - P_{\sqrt{M}}^{PAM}) \\
&= \boxed{} \\
\texttt{When k is odd} P_M^{QAM} &= \boxed{} \\
&= \boxed{} \\
\texttt{Compare with MPSK} : P_M^{PSK} &= \boxed{} \\
\texttt{Ratio of the arguments} R &= \boxed{}
\end{aligned}
$$

Comment: $\boxed{}$

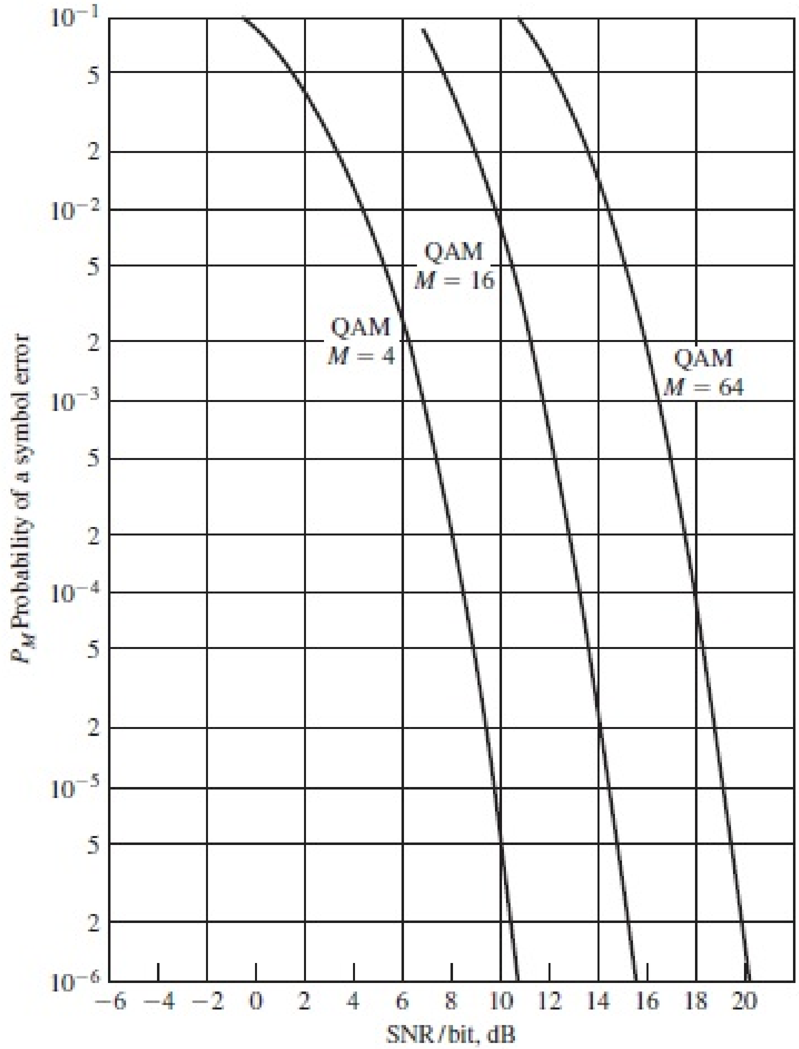


Figure 7.62 Probability of a symbol error for QAM.

QAM symbol error rate graphs. Determine the required increase in transmit energy for each extra bit/symbol

End-of-chapter problems: 8.1-8.40