ELEC264: Signals And Systems

Topic 5:Discrete-Time Fourier Transform (DTFT)

- **Introduction**
- DT Fourier Transform
- Sufficient condition for the DTFT
- DT Fourier Transform of Periodic Signals
- DTFT and LTI systems: Frequency response
- Properties of DT Fourier Transform
- Summary
- Appendix: Transition from DT Fourier Series to DT Fourier Transform
- Appendix: Relations among Fourier Methods

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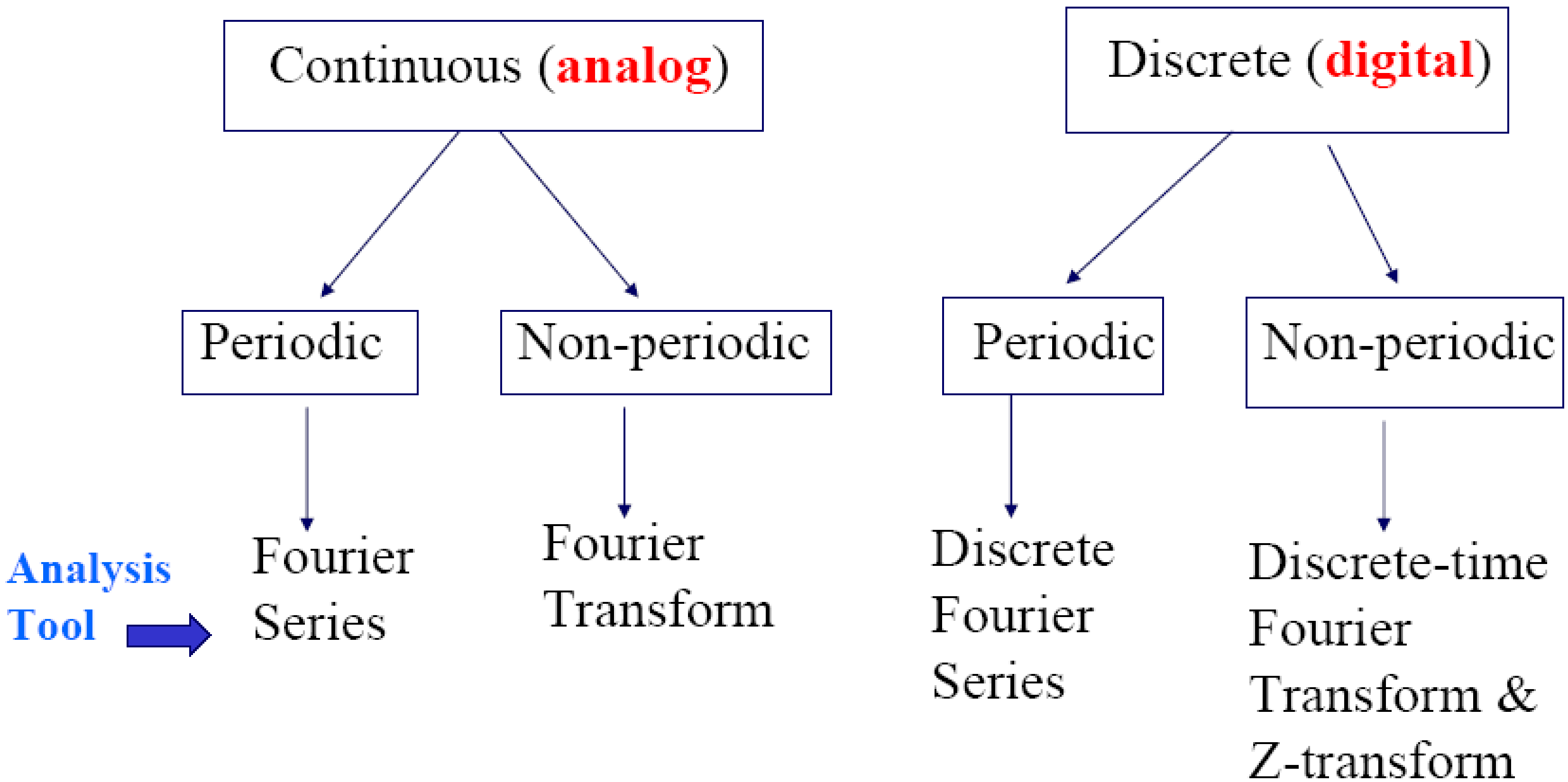
Figures and examples in these course slides are taken from the following sources:

- **A. Oppenheim, A.S. Willsky and S.H. Nawab, Signals and Systems, 2nd Edition, Prentice-Hall, 1997**
- **M.J. Roberts, Signals and Systems, McGraw Hill, 2004**
- **J. McClellan, R. Schafer, M. Yoder, Signal Processing First, Prentice Hall, 2003**

Fourier representation

- A Fourier function is unique, i.e., no two same signals in time give the same function in frequency
- The DT Fourier Series is a good analysis tool for systems with periodic excitation but **cannot** represent an aperiodic DT signal **for all time**
- The DT Fourier Transform can represent an **aperiodic discrete-time** signal **for all time**
  - Its development follows exactly the same as that of the Fourier transform for continuous-time aperiodic signals

Overview of Frequency Analysis Methods



Overview of Fourier Analysis Methods

	Periodic in Time Discrete in Frequency	Aperiodic in Time Continuous in Frequency
Continuous in Time Aperiodic in Frequency	$\otimes$ CT Fourier Series : CT - $P_T \Rightarrow$ DT $a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$ $\otimes$ CT Inverse Fourier Series : DT $\Rightarrow$ CT - P_T $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\otimes$ CT Fourier Transform: CT $\Rightarrow$ CT $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $\otimes$ Inverse CT Fourier Transform: CT $\Rightarrow$ CT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
Discrete in Time Periodic in Frequency	$\otimes$ DT Fourier Series DT - $P_N \Rightarrow$ DT - P_N $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_0 kn}$ $\otimes$ Inverse DT Fourier Series DT - $P_N \Rightarrow$ DT - P_N $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\omega_0 kn}$	$\otimes$ DT Fourier Transform: DT $\Rightarrow$ CT + $P_{2\pi}$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $\otimes$ Inverse DT Fourier Transform: CT + $P_{2\pi} \Rightarrow$ DT $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

Overview of Fourier symbols

	Variable	Period	Continuous Frequency	Discrete Frequency
DT *x[n]*	*n*	*N*	ω	*k* $\omega_k = 2\pi k / N$
CT *x(t)*	*t*	*T*	Ω	*k* $\omega_k = 2\pi k / T$

- **DT-FT:** Discrete in time; Aperiodic in time; Continous in Frequency; Periodic in Frequency
- **DT-FS:** Discrete in time; Periodic in time; Discrete in Frequency; Periodic in Frequency
- **CT-FS:** Continuous in time; Periodic in time; Discrete in Frequency; Aperiodic in Frequency
- **CT-FT:** Continuous in time; Aperiodic in time; Continous in Frequency; Aperiodic in Frequency

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DT Fourier Transform

- DT Fourier transform and the inverse FT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- FT describes which frequencies are present in the original function

- The original signal can be recovered from knowing the Fourier transform, and vice versa

- The function $X(e^{j\omega})$ is periodic in ω with period 2π
  - (The function $e^{j\omega}$ is periodic with N=2π)

- Forms : $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $\qquad X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$

DT Fourier Transform

- DT signal representations:
  - A sum of scaled, delayed impulse

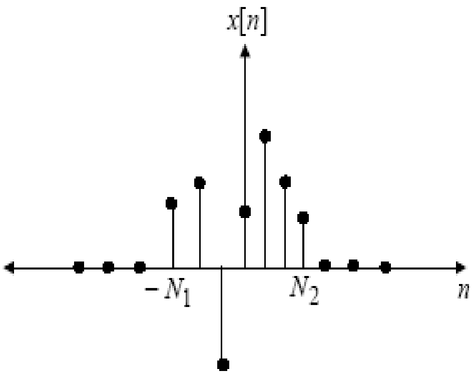
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

  - A linear combination of weighted sinusoidal signals

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

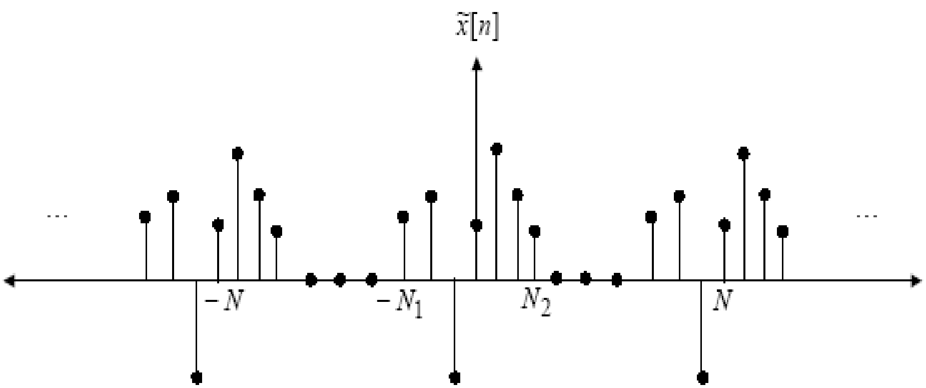
DT Fourier Transform: Derivation

- Let *x[n]* be the aperiodic DT signal
- We construct a periodic signal $\tilde{x}[n]$ for which *x[n]* is one period
  - $\tilde{x}[n]$ is comprised of infinite number of replicas of *x[n]*
  - Each replica is centered at an integer multiple of N
  - N is the period of $\tilde{x}[n]$



- Consider the following figure which illustrates an example of x[n] and the construction of $\tilde{x}[n]$
- Clearly, x[n] is defined between $-N_1$ and N_2
- Consequently, N has to be chosen such that $N > N_1 + N_2 + 1$ so that adjacent replicas do not overlap
- Clearly, as we let

as desired $N \to \infty,\ \tilde{x}[n] = x[n]$



DT Fourier Transform: Derivation

- Let us now examine the FS representation of $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{<N>} a_k e^{jk(2\pi/N)n} \qquad \text{where} \qquad a_k = \frac{1}{N}\sum_{<N>} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

- Since x[n] is defined between $-N_1$ and N_2

  ➔ a_k in the above expression simplifies to

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk(2\pi/N)n} \\ &= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n} \end{aligned}$$

$\omega = 2\pi/N$

DT Fourier Transform: Derivation

- Now defining the function $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$
- We can see that the coefficients a_k are related to $X(e^{j\omega})$ as $a_k = \frac{1}{N}X(e^{jkw_0})$
- where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain
- Therefore
$$\begin{aligned}\tilde{x}[n] &= \sum_{<N>} \frac{1}{N}X(e^{jkw_0})e^{jkw_0 n} \\ &= \frac{1}{2\pi}\sum_{<N>} X(e^{jkw_0})e^{jkw_0 n}w_0\end{aligned}$$
- As N increases ω_0 decreases, and as $N \to \infty$ the above equation becomes an integral

DT Fourier Transform: Derivation

- One important observation here is that the function $X(e^{j\omega})$ is periodic in ω with period 2π
  - Therefore, as N $\longrightarrow$ ∞, $\tilde{x}[n] = x[n]$
- (Note: the function $e^{j\omega}$ is periodic with N=2π)
- This leads us to the DT-FT pair of equations

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \qquad \text{Synthesis equation} \\
X(e^{jw}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \qquad \text{Analysis equation}
\end{aligned}
$$

DT Fourier Transform: Examples

Let $x[n] = \delta[n] \rightarrow X(e^{jw}) = 1$

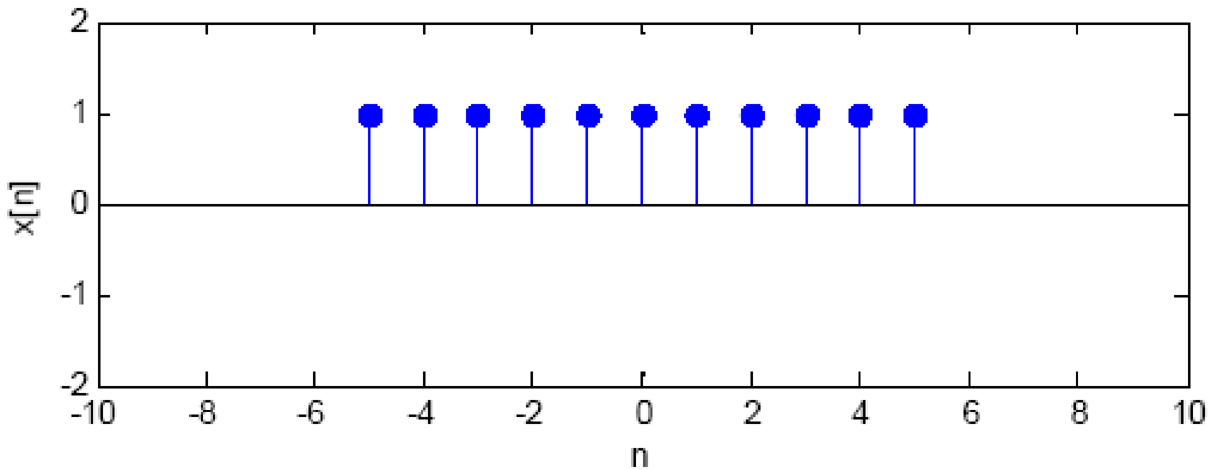
Let $x[n] = 1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$

The periodic impulse train

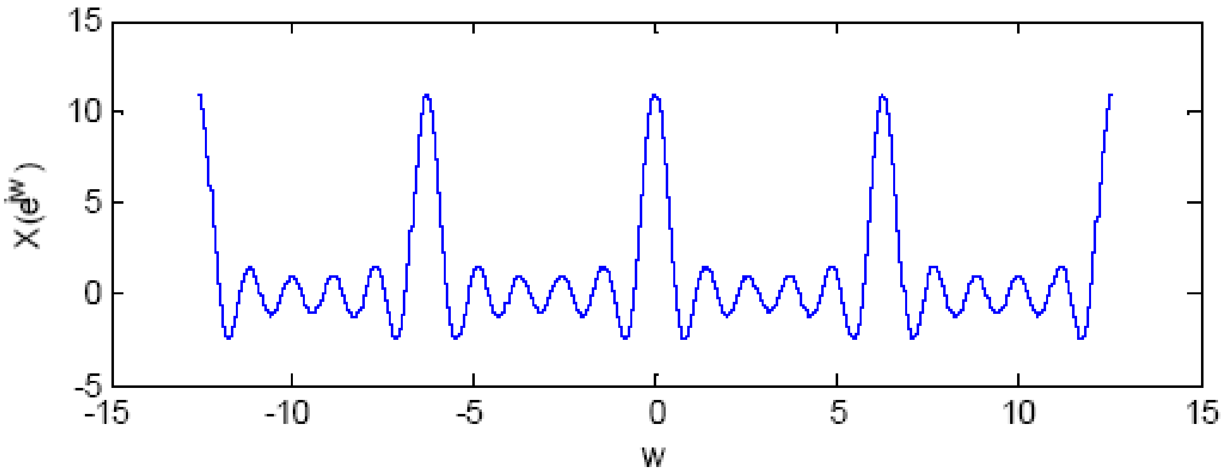
Let $x[n] = a^n u[n] \quad |a| < 1 \Leftrightarrow X(e^{j\omega}) = \dfrac{1}{1 - ae^{-j\omega}}$

DT Fourier Transform: Examples

$$x[n] = \begin{cases} 1, & |n| \leq 5 \\ 0, & \text{e.w.} \end{cases} \longrightarrow \quad X(e^{jw}) = \sum_{n=-5}^{5} e^{-jwn} = \frac{\sin[w(11/2)]}{\sin[w/2]}$$



Aperiodic



Periodic

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1-ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

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Sufficient condition for DTFT

- Condition for the convergence of the infinite sum

$$|X(e^{j\omega})| = \left|\sum_{-\infty}^{\infty} x[n] e^{-j\omega n}\right|$$

$$\leq \sum_{-\infty}^{\infty} |x[n]||e^{-j\omega n}| \quad \leq \sum_{-\infty}^{\infty} |x[n]| < \infty$$

➔If x[n] is absolutely summable, its FT exists (sufficient condition)

Example: Exponential sequence

$$x[n] = a^n u[n] \qquad |a| < 1: \quad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$a = 1: \quad X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$$

$$|a| > 1: \quad \text{DTFT does not exist}$$

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FT of Periodic DT Signals

- Consider the continuous time signal $x(t) = e^{jw_0 t}$
  - This signal is periodic
  - Furthermore, the Fourier series of this signal is just an impulse of weight one centered at $\omega = \omega_0$
- Now consider this signal $x[n] = e^{jw_0 n}$
  - It is also periodic and there is one impulse per period
  - However, the separation between adjacent impulses is 2π
  - In particular, the DT Fourier Transform for this signal is

$$X(e^{jw}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(w - w_0 - 2\pi l)$$

- **DTFT of a periodic signal with period N**

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - \omega_k); \quad \omega_k = \frac{2\pi k}{N}$$

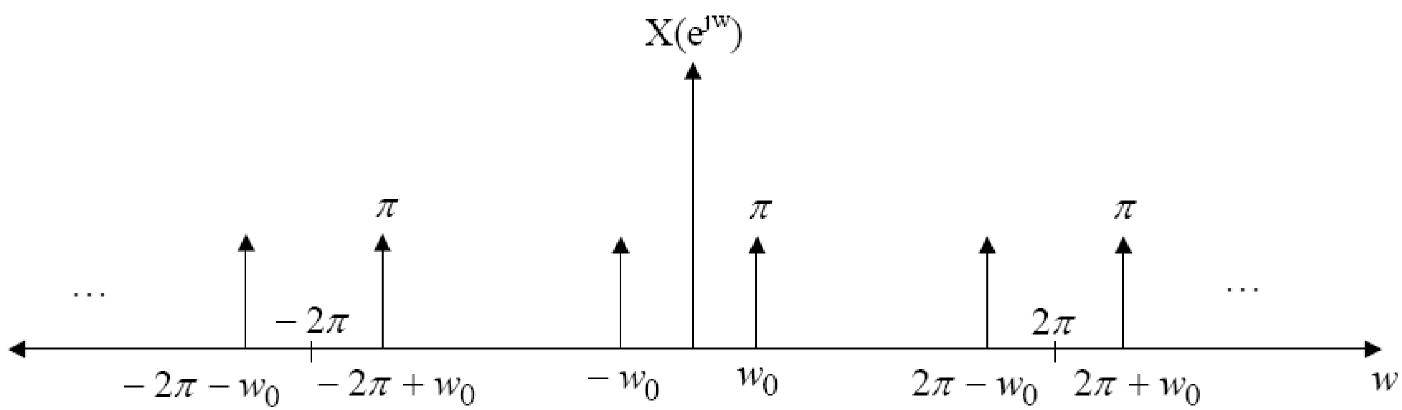
DTFT: Periodic signal

- Let $x[n] = \cos w_0 n \qquad \text{with } w_0 = \frac{2\pi}{5}$
- The signal can be expressed as $x[n] = \frac{1}{2}\left(e^{jw_0 n} + e^{-jw_0 n}\right)$
- We can immediately write

$$X(e^{jw}) = \sum_{l=-\infty}^{\infty} \pi\delta(w - \frac{2\pi}{5} - 2\pi l) + \sum_{l=-\infty}^{\infty} \pi\delta(w + \frac{2\pi}{5} - 2\pi l)$$

- Equivalently $X(e^{jw}) = \pi\delta(w - \frac{2\pi}{5}) + \pi\delta(w + \frac{2\pi}{5}) \qquad -\pi \leq w < \pi$

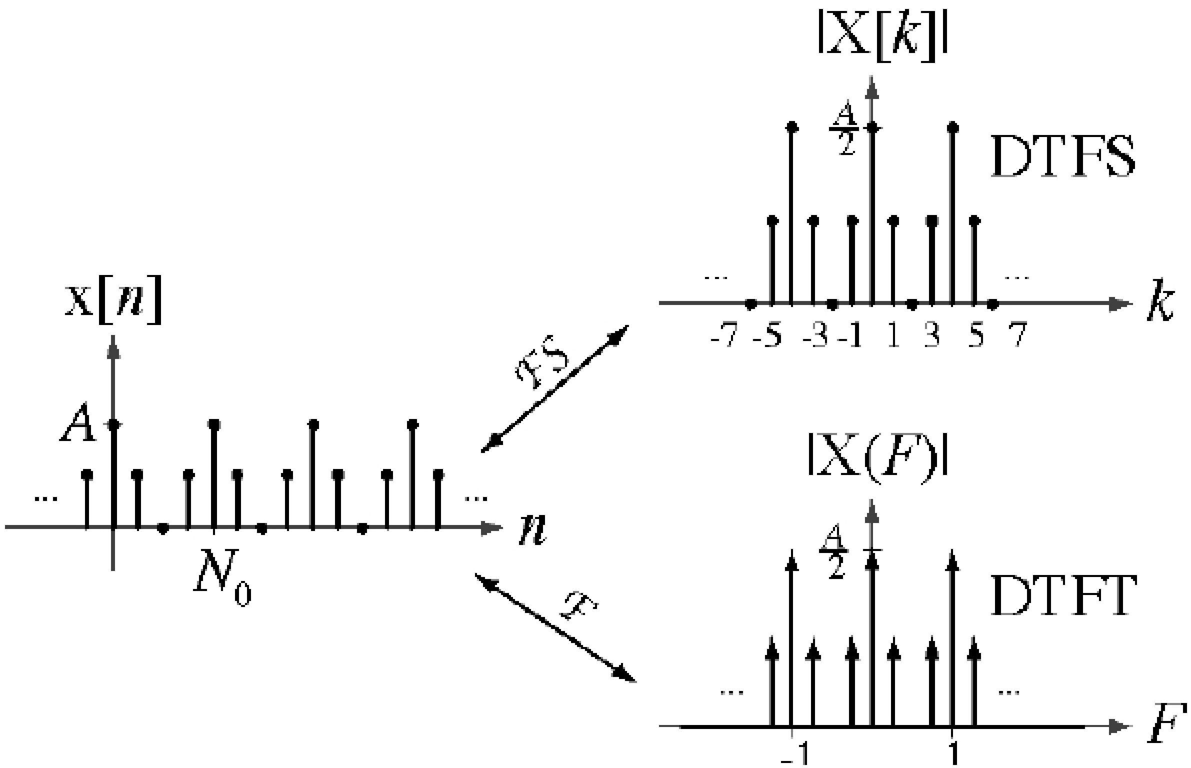
  period 2π



DT FT of periodic signals

FS vs. FT

$$X(F) = \sum_{k=-\infty}^{\infty} X[k]\delta(F - kF_0)$$



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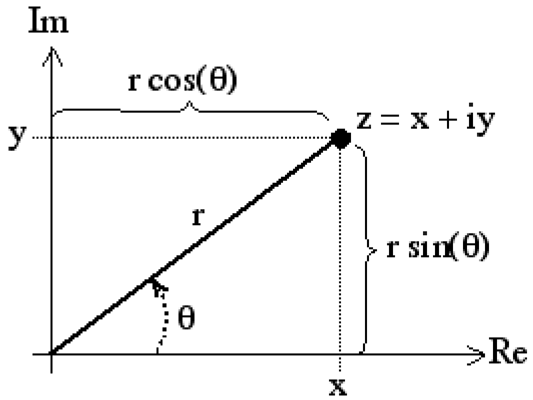
Complex numbers



$* \text{Cartesian representation}: z = x + jy$

Magnitude of z $\boxed{r = |z| = \sqrt{x^2 + y^2}}$

It is the distance of a point z from the origin

Phase (argument) of z $\boxed{\theta = \angle z = \tan^{-1}\frac{y}{x}}$

θ is the angle to the real positive axis

θ can change by any multiple of 2π and still give the same angle

(radians not degrees are being used)

$* \text{Polar representation}: \; z = |z| e^{j\theta} = |z| \cos\theta + j |z| \sin\theta$

∘ Complex Conjugate:

$z^* = x - jy \quad ; (z + z^*) \; and \; (zz^*) \; are \; real$

Properties of the DTFT

- Periodicity

$$X(e^{j(w+2\pi)}) = X(e^{jw})$$

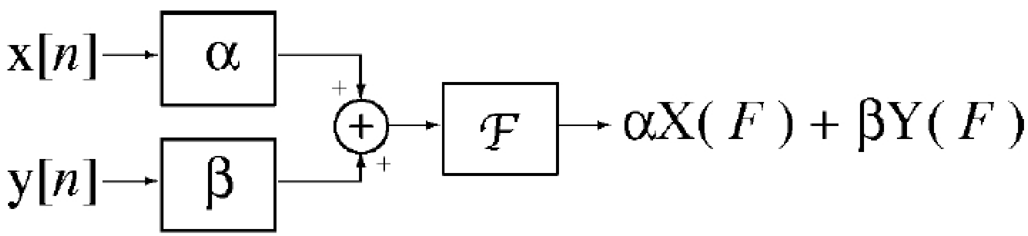
- The function $e^{j\omega}$ is periodic with $N=2\pi$

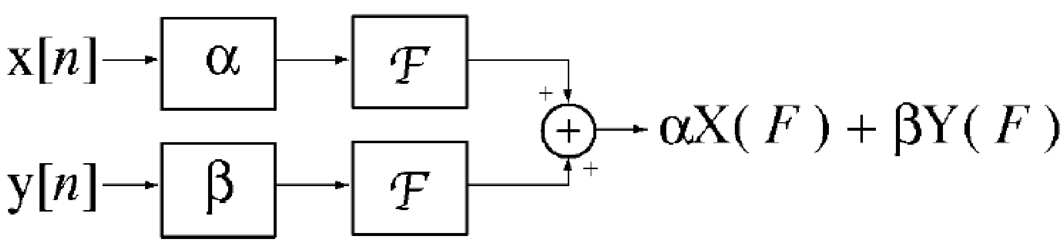
Properties of the DTFT

- Linearity: If $x_1[n] \overset{F}{\longleftrightarrow} X_1(e^{jw})$
  $x_2[n] \overset{F}{\longleftrightarrow} X_2(e^{jw})$

  Then $\Rightarrow \alpha x_1[n] + \beta x_2[n] \overset{F}{\longleftrightarrow} \alpha X_1(e^{jw}) + \beta X_2(e^{jw})$

$$\alpha \mathrm{x}[n] + \beta \mathrm{y}[n] \overset{F}{\longleftrightarrow} \alpha \mathrm{X}(F) + \beta \mathrm{Y}(F)$$



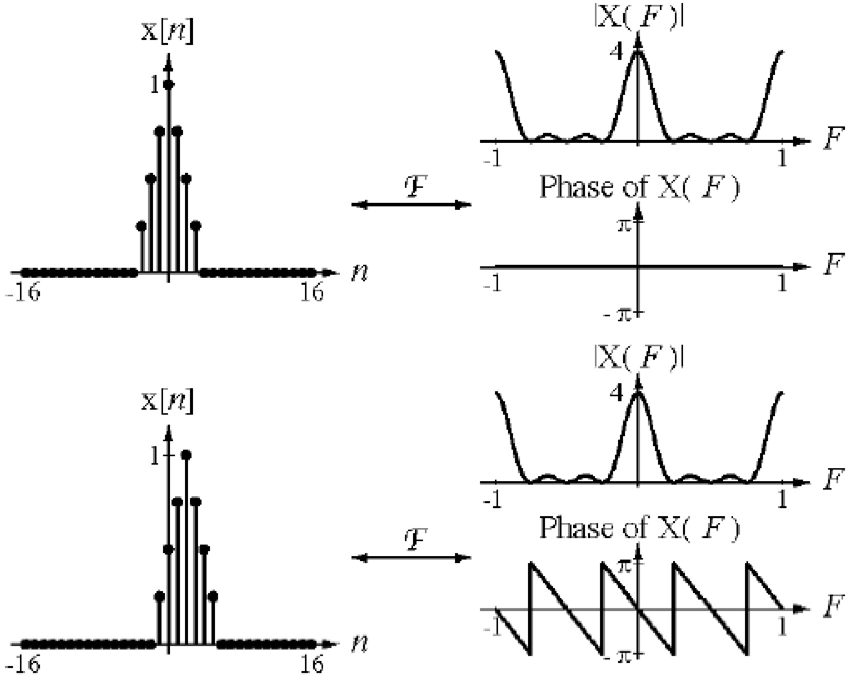


Properties of the DTFT

- Time-Shifting: If $x[n] \xleftrightarrow{F} X(e^{jw})$

  Then $x[n-n_0] \xleftrightarrow{F} e^{-jwn_0}X(e^{jw})$

$$\mathrm{x}[n-n_0] \xleftrightarrow{\mathrm{F}} e^{-j2\pi F n_0}\,\mathrm{X}(F)$$



Example: Time shift

- Determining the DTFT of

$$x[n] = a^n u[n-5]$$

- Solution

$$x_1[n] = a^n u[n] \overset{F}{\longleftrightarrow} X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$x_2[n] = x_1[n-5] \qquad (\text{i.e.} = a^{n-5}u[n-5])$$

$$X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega}) = \frac{e^{-j5\omega}}{1-ae^{-j\omega}}$$

$$x[n] = a^5 x_2[n] \qquad (\text{i.e.} = a^n u[n-5])$$

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1-ae^{-j\omega}}$$

Properties of the DTFT

- Frequency Shifting: If $x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$

  Then

$$e^{-jw_0 n}x[n] \overset{F}{\longleftrightarrow} X(e^{j(w-w_0)})$$

$$e^{j2\pi F_0 n}\,\mathrm{x}[n] \xleftrightarrow{\mathrm{F}} \mathrm{X}(F - F_0)$$

Properties of the DTFT

- Conjugation and Conjugate Symmetry

$$x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$$
$$x^*[n] \overset{F}{\longleftrightarrow} X^*(e^{-jw})$$

  - For real-valued signals,

$$x^*[n] = x[n] \Rightarrow X(e^{jw}) = X^*(e^{-jw})$$

  - For real-valued and even signals,
    the Fourier transform is real and even

  - For real-valued and odd signals,
    the Fourier transform is purely imaginary and odd

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\|X(e^{j\omega})\| = \|X(e^{-j\omega})\|$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Symmetry properties of the DTFT

x[n]=$x_e[n]$+ $x_o[n]$

conjugate-symmetric sequence: $x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x^*{}_e[-n]$

conjugate-antisymmetric sequence: $x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = -x^*{}_o[-n]$

even sequence is conjugate-symmetric : $x_e[n] = x_e[-n]$

odd sequence is conjugate-antisymmetric : $x_o[n] = -x_o[-n]$

According to Table 2.1(property 5&6)

$$x_e[n] \quad \overset{\mathcal{F}}{\rightarrow} \quad X_R(e^{jw}) = \mathcal{R}e\{X(e^{jw})\}$$

$$x_o[n] \quad \overset{\mathcal{F}}{\rightarrow} \quad jX_I(e^{jw}) = jIm\{X(e^{jw})\}$$

Symmetry properties of the DTFT
Duality property

$X(e^{jw}) = X_e(e^{jw}) + X_o(e^{jw})$

conjugate-symmetric FT: $X_e(e^{jw}) = \frac{1}{2}(X(e^{jw}) + X^*(e^{-jw}))$

conjugate-antisymmetric FT: $X_o(e^{jw}) = \frac{1}{2}(X(e^{jw}) - X^*(e^{-jw}))$

conjugate-symmetric Function : $X_e(e^{jw}) = X^*{}_e(e^{-jw})$

conjugate-antisymmetric Function : $X_o(e^{jw}) = -X^*{}_o(e^{-jw})$

According to Table 2.1(property 3&4)

$$\mathcal{R}e\{x[n]\} \quad \xrightarrow{\mathcal{F}} \quad X_e(e^{jw})$$

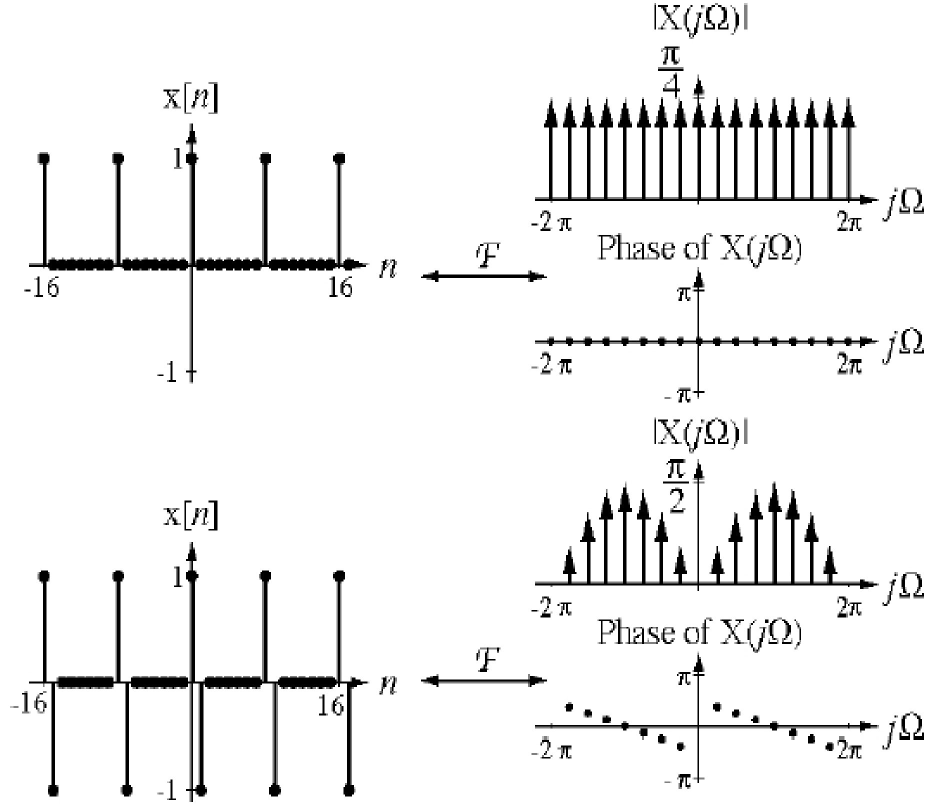
$$jIm\{x[n]\} \quad \xrightarrow{\mathcal{F}} \quad X_o(e^{jw})$$

Properties of the DTFT

- Differencing

$$x[n] - x[n-1] \overset{F}{\longleftrightarrow} \left(1 - e^{-jw}\right) X(e^{jw})$$

$$\mathrm{x}[n] - \mathrm{x}[n-1] \overset{\mathrm{F}}{\longleftrightarrow} \left(1 - e^{-j2\pi F}\right) \mathrm{X}(F)$$



Properties of the DT FT

Accumulation :

$$\sum_{m=-\infty}^{n} x[m] \leftrightarrow \frac{1}{1\text{-}e^{-j\omega}} X(e^{-j\omega}) + \pi X(e^{-j0}) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)$$

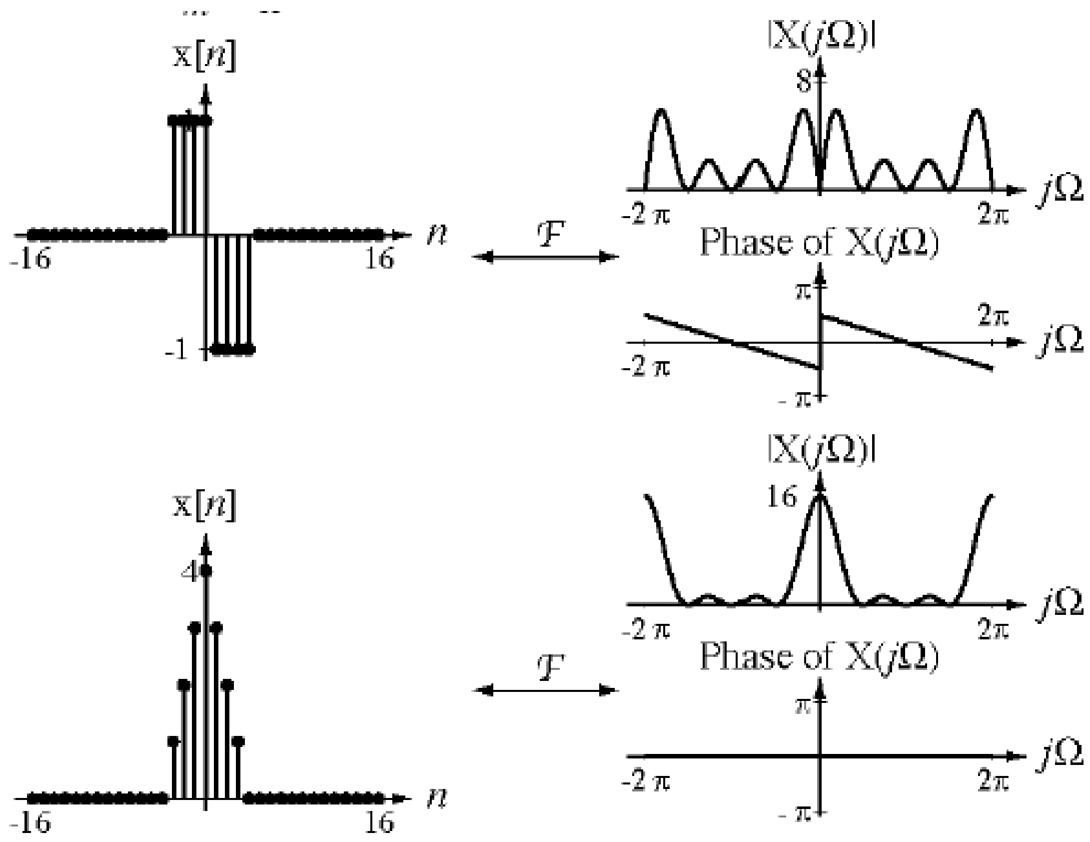
$$\sum_{m=-\infty}^{n} x[m] \leftrightarrow \frac{1}{1\text{-}e^{-j2\pi f}} X(f) + \frac{1}{2} X(0) comb(f)$$

where the impulse train on the right-hand side reflects the average value (or dc component) that may result from the summation

Train of impulses $\quad comb(\omega) = \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

Properties of the DT FT

- Accumulation



Properties of the DT FT

- Time Reversal: If

$$x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$$

Then

$$x[-n] \overset{F}{\longleftrightarrow} X(e^{-jw})$$

Properties of the DT FT

- Time Expansion:

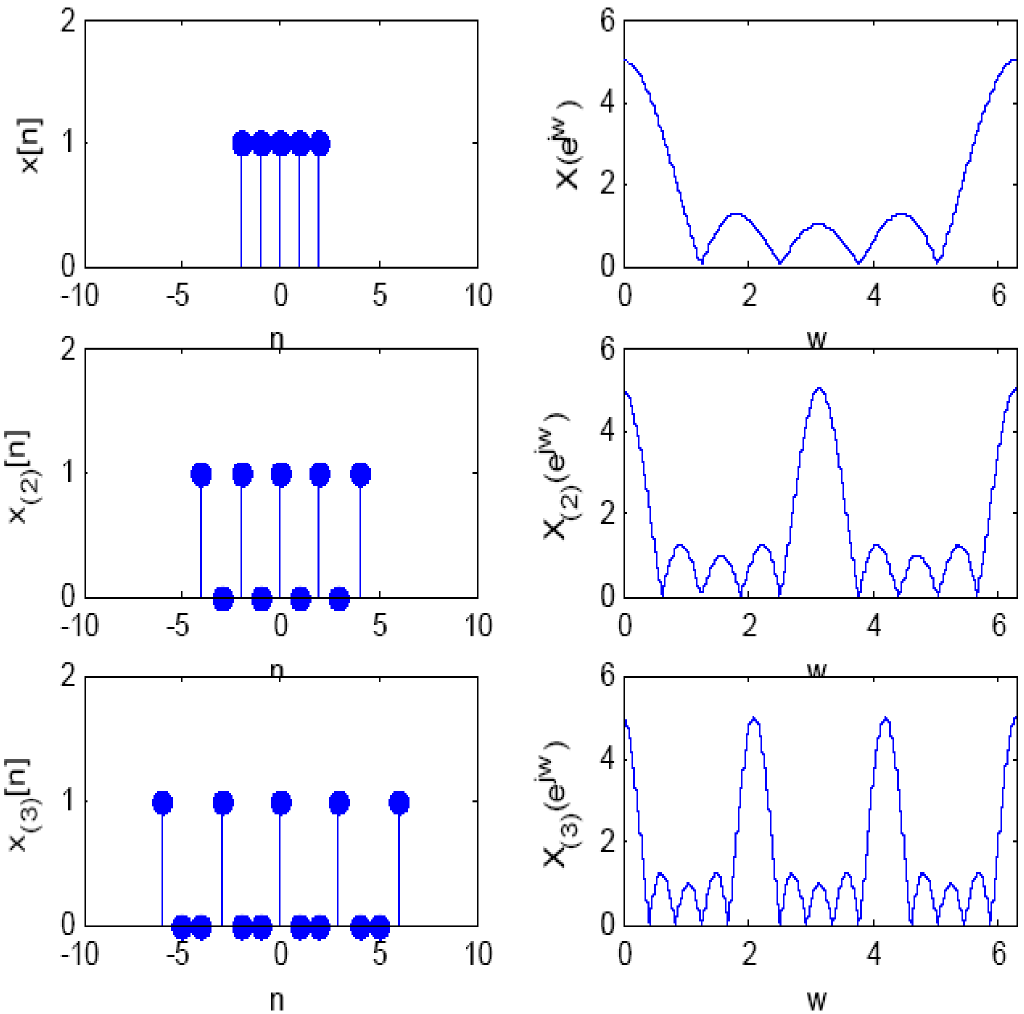
  Let k be a positive integer

  Define

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{otherwise} \end{cases}$$

Now if $x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$

then $x_{(k)}[n] \overset{F}{\longleftrightarrow} X(e^{jkw})$



Properties of the DT FT

- Differentiation in Frequency: If

$$x[n] \quad \overset{F}{\longleftrightarrow} \quad X(e^{jw})$$

then

$$nx[n] \quad \overset{F}{\longleftrightarrow} \quad j\frac{dX(e^{jw})}{dw}$$

Properties of the DT FT

- Parseval's Relation

$$\sum_{n=-\infty}^{n} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} \left|X\left(e^{jw}\right)\right|^2 dw$$

$$\sum_{n=-\infty}^{\infty} |\mathrm{x}[n]|^2 = \int_1 |\mathrm{X}(F)|^2 dF$$

The signal energy is proportional to the integral of the squared magnitude of the DTFT of the signal over one period.

Properties of the DT FT

Multiplication & Convolution duality :

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi} X(e^{j\omega}) * Y(e^{j\omega})$$

$$x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$$

It follows : for an LTI system with impulse response h[n]:

$$y[n] = h[n] * x[n] \leftrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Properties of the DT FT

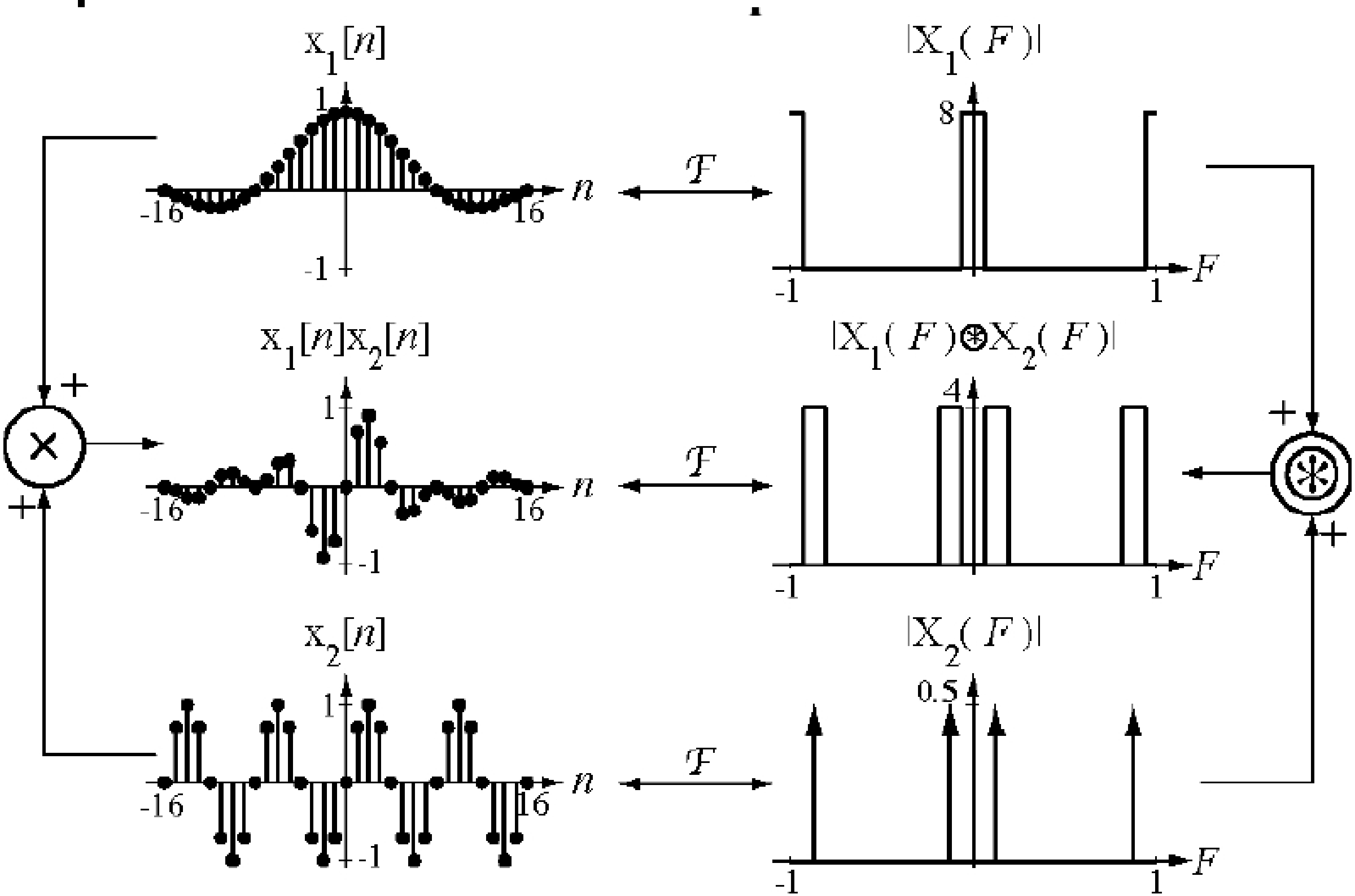
- Multiplication: Let

$$y[n] = x_1[n] \cdot x_2[n]$$

then

$$Y(e^{jw}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(w-\theta)}) d\theta$$

Properties of the DT FT



Properties of the DT FT: Difference equation

- DT LTI Systems are characterized by Linear Constant-Coefficient Difference Equations

$$y[n] - ay[n-1] = x[n]$$

- A general linear constant-coefficient difference equation for an LTI system with input x[n] and output y[n] is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Now applying the FT to both sides of the above equation, we have

$$\sum_{k=0}^{N} a_k e^{-jkw} Y(e^{jw}) = \sum_{k=0}^{M} b_k e^{-jkw} X(e^{jw})$$

- But we know that the input and the output are related to each other through the impulse response of the system, denoted by h[n], i.e.,

$$y[n] = x[n] * h[n]$$

Properties of the DT FT : Difference equation

- Applying the convolution property

$$Y(e^{jw}) = X(e^{jw})H(e^{jw}) \quad \text{Or} \quad H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}}$$

➔ if one is given a difference equation corresponding to some system, the <u>FT of the impulse response</u> of the system can found <u>directly from the difference equation</u> by applying the Fourier transform

- FT of the <u>impulse response</u> = <u>Frequency response</u>
- Inverse FT of the frequency response = Impulse response

Properties of the DT FT: Example

- With |a| < 1 , consider the causal LTI system that is characterized by the difference equation

$$y[n] - ay[n-1] = x[n]$$

- The frequency response of the system is

$$H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

- From tables (or by applying inverse FT), we get

$$h[n] = a^n u[n]$$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

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Frequency response of LTI systems

- If input is complex exponentials

$$x[n] = e^{j\omega n} \Rightarrow$$

$$y[n] = T\{e^{j\omega n}\} = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = \Big(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}\Big)\, e^{j\omega n}$$

- Define $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \Rightarrow y[n] = H(e^{j\omega}) e^{j\omega n}$
  - *(h[n] & H():* Frequency and impulse responses are a FT pair)

$$\Rightarrow \text{if } x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \rightarrow y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) \cdot H(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

Frequency response

- The frequency response of discrete-time LTI systems is always a periodic function of the frequency variable w with period 2π

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}e^{-j2\pi n} = H(e^{j\omega})$$

➔Only specify over the interval $-\pi < \omega \leq \pi$

➔The ‘low frequencies’ are close to 0

➔The ‘high frequencies’ are close to $\pm\pi$

- Frequency response is generally complex

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

$$= |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

➔describes changes to x[n] in magnitude and phase

Frequency response: Example

- Frequency response of the ideal delay system

  - $\text{Ideal delay}: \ y[n] = x[n - n_d] \ \rightarrow \ h[n] = \delta[n - n_d]$

$$\rightarrow H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}$$

  - $H_R(e^{j\omega}) = \cos(\omega n_d), \qquad H_I(e^{j\omega}) = -\sin(\omega n_d)$
  - $|H(e^{j\omega})| = 1, \qquad \angle H(e^{j\omega}) = -\omega n_d$

Frequency response

- Response of LTI – systems:

$x[n] = \delta[n] \quad \rightarrow \quad y[n] = h[n]$; with $h[n]$ the impulse response

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \;\rightarrow\; y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Convolution theorem:

If input $X(e^{j\omega}) \rightarrow$ Response: $X(e^{j\omega}) \cdot H(e^{j\omega})$;

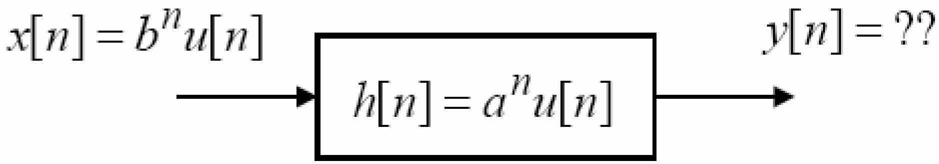
with $H(e^{j\omega})$ the frequency response

$\Rightarrow$ Frequency & impulse responses are a FT pair

Frequency response: Example

- Let $y[n] = x[n] * h[n]$ Then $Y(e^{jw}) = X(e^{jw})H(e^{jw})$

  Example: Consider the following system



From the convolution property, we have $Y(e^{jw}) = X(e^{jw})H(e^{jw})$

$$= \frac{1}{(1 - ae^{-jw})(1 - be^{-jw})} = \frac{A}{1 - ae^{-jw}} + \frac{B}{1 - be^{-jw}}$$

using partial fraction $A = \frac{a}{a-b}$ and $B = -\frac{b}{a-b}$

Therefore, $y[n] = \frac{1}{a-b}\left[a^{n+1}u[n] - b^{n+1}u[n]\right] = \frac{a^{n+1} - b^{n+1}}{a-b}u[n]$

Ideal frequency-selective LTI-systems (or filters)

- Ideal frequency-selective filter have unity frequency response over a certain range of frequencies, and is zero at the remaining frequencies
  - Example: Ideal low-pass filter: passes only low frequencies and rejects high frequencies of an input signal x[n]

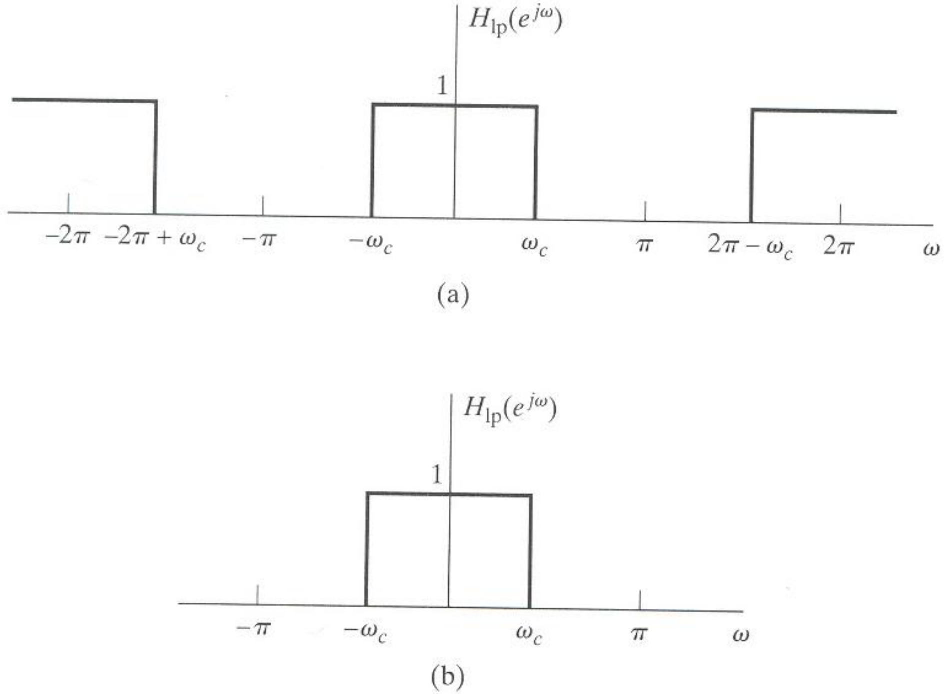


Figure 2.17 Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.

Example : ideal lowpass filter

- Frequency response

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

$$\leftrightarrow h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

- h[n] is not absolutely summable ➔ Filter noncausal

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DTFT: Summary

- DT Fourier Transform represents a discrete time aperiodic signal as a sum of infinitely many complex exponentials, with the frequency varying continuously in (-π, π)

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega t}\,d\omega, \quad X(e^{j\omega}) = \sum_{n} x[n]e^{-jn\omega}$$

  - DTFT is periodic

    ➔ only need to determine it for $\omega \in (-\pi, \pi)$

Summary: Signal & System representations

- Signal: A sum of scaled, delayed impulse $x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- Signal: A linear combination of weighted sinusoidal signals

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

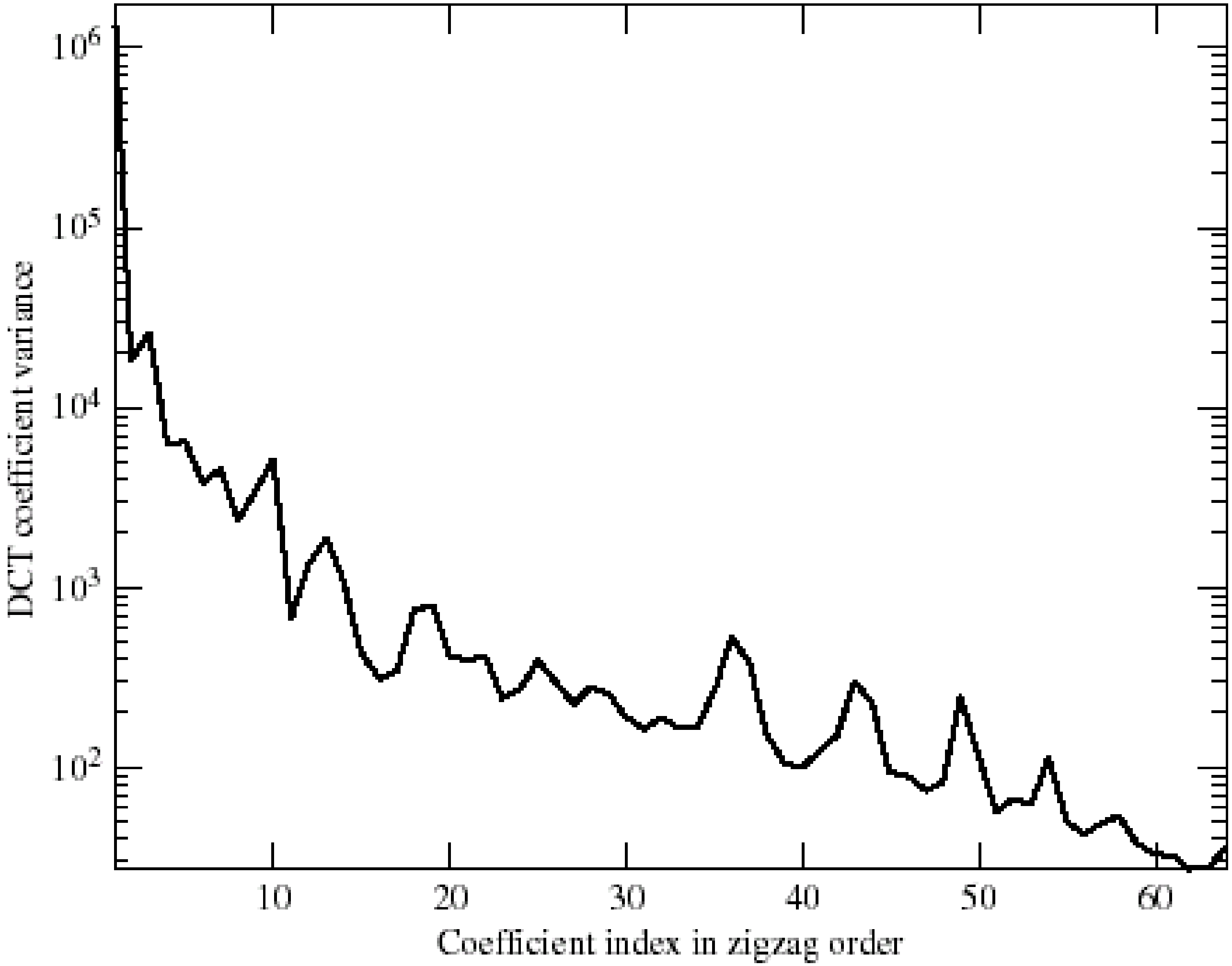
- LTI system: Convolution

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k] \leftrightarrow Y(e^{j\omega})=X(e^{j\omega})\cdot H(e^{j\omega})$$

- LTI system: Difference equation: $\sum_{k=0}^{N} a_k y[n-k]=\sum_{k=0}^{M} b_k x[n-k]$

$$y[n]-ay[n-1]=x[n]$$

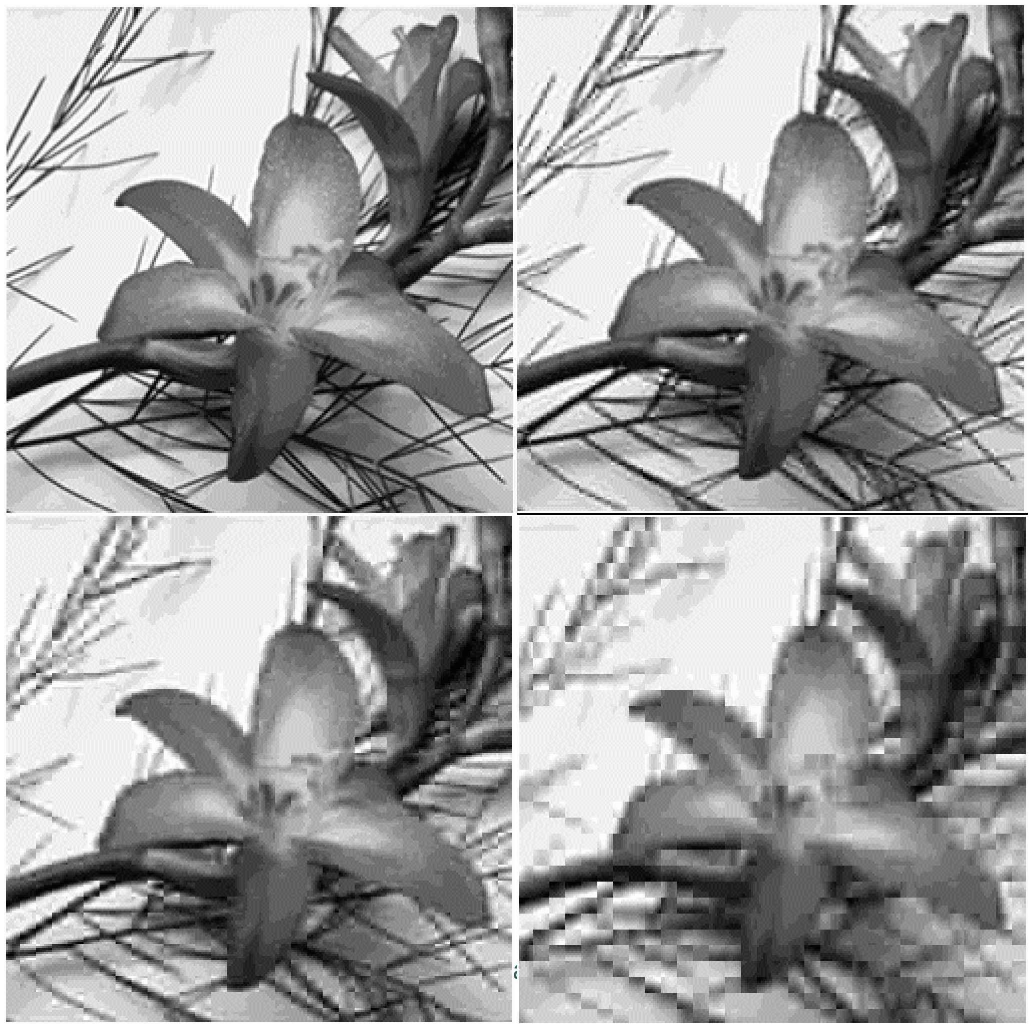
Real-world application: Image compression Energy Distribution of transform (DCT) Coefficients in Typical Images



Real-world applications: Image compression Images Approximated by Different Number of transform (DCT) Coefficients

Original

With 16/64 Coefficients



With 8/64 Coefficients

With 4/64 Coefficients

DTFT: Summary

- Know how to calculate the DTFT of simple functions
  - Know the geometric sum:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad if \quad |a|<1$$

- Know Fourier transforms of special functions, e.g. δ[n], exponential
- Know how to calculate the inverse transform of rational functions using partial fraction expansion
- Properties of DT Fourier transform
  - Linearity, Time-shift, Frequency-shift, …

DT-FT Summary: a quiz

- A discrete-time LTI system has impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$
- Find the output y[n] due to input $x[n] = \left(\frac{1}{7}\right)^n u[n]$
- **Solution :** Use the convolution property:

$$y[n] = h[n] * x[n] \Rightarrow Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$m[n] = (a)^n u[n] \Rightarrow M(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, \qquad a < 1$$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \quad \text{and } X(e^{j\omega}) = \frac{1}{1 - \frac{1}{7} e^{-j\omega}}$$

DT-FT Summary: a quiz (cont.)

$$\Rightarrow \quad Y(e^{j\omega}) = \left(\frac{1}{1-\frac{1}{7}e^{-j\omega}}\right)\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right)$$

- Using <u>partial fraction expansion</u> method of finding inverse FT gives:

$$Y(e^{j\omega}) = \frac{-2/5}{1-\frac{1}{7}e^{-j\omega}} + \frac{7/5}{1-\frac{1}{2}e^{-j\omega}}$$

- Therefore,
  - since a FT is unique, (i.e. no two same signals in time give the same function in frequency) and since

$$m[n] = a^n u[n] \Rightarrow M(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$\Rightarrow$ It can be seen that a FT of the type $\frac{1}{1-ae^{-j\omega}}$ should correspond to a signal $a^n u[n]$.

- Therefore,
  - the inverse FT of $\frac{-2/5}{1-\frac{1}{7}e^{-j\omega}}$ is $-\frac{2}{5}\left(\frac{1}{7}\right)^n u[n]$
  - the inverse FT of $\frac{7/5}{1-\frac{1}{2}e^{-j\omega}}$ is $\frac{7}{5}\left(\frac{1}{2}\right)^n u[n]$

- Thus the complete output $y[n] = -\frac{2}{5}\left(\frac{1}{7}\right)^n u[n] + \frac{7}{5}\left(\frac{1}{2}\right)^n u[n]$

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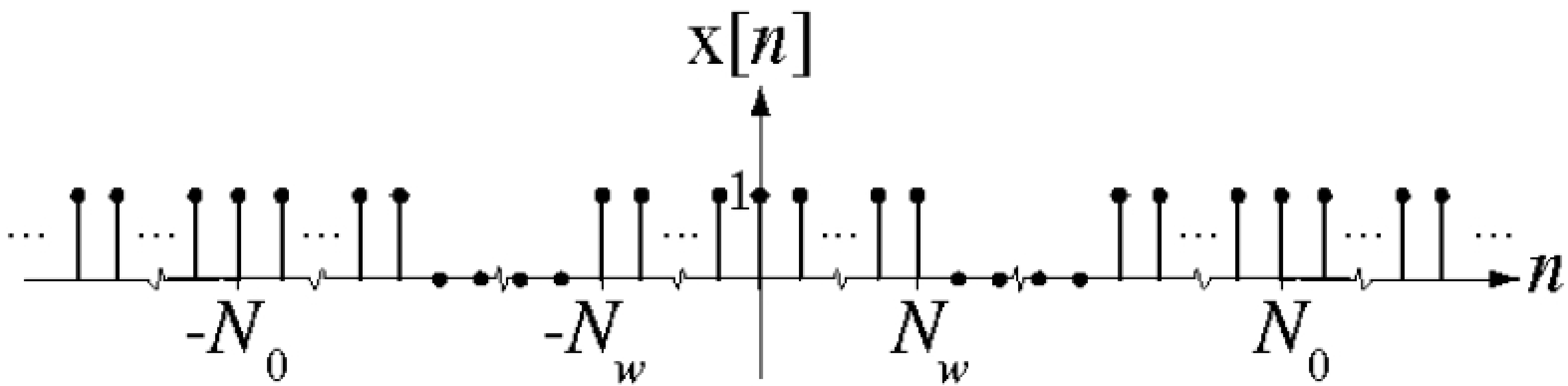
Transition: DT Fourier Series to DT Fourier Transform

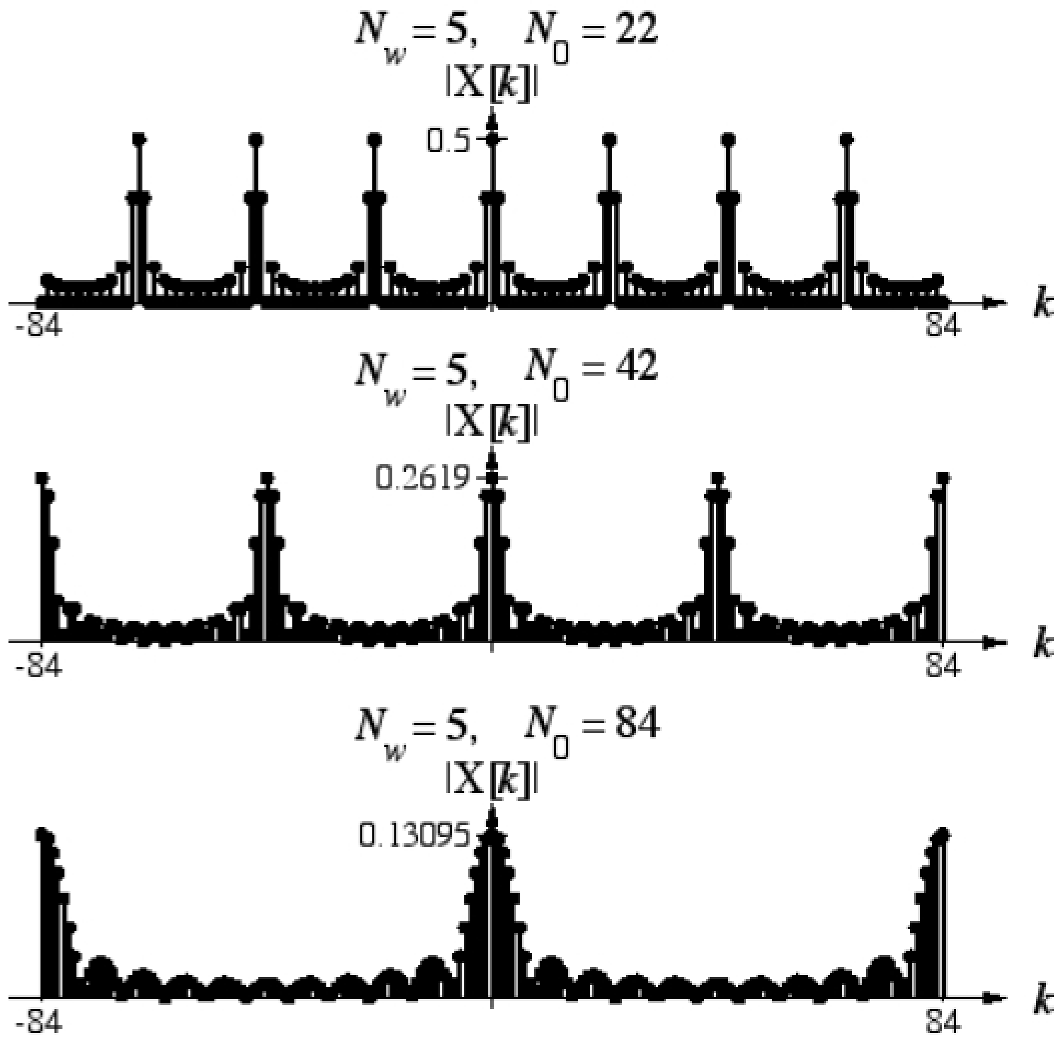
- DT Pulse Train Signal $x(n) = \text{rect}_{N_w}[n] * \text{comb}_{N_0}[n]$



- This DT periodic rectangular-wave signal is analogous to the CT periodic rectangular-wave signal used to illustrate the transition from the CT Fourier Series to the CT Fourier Transform

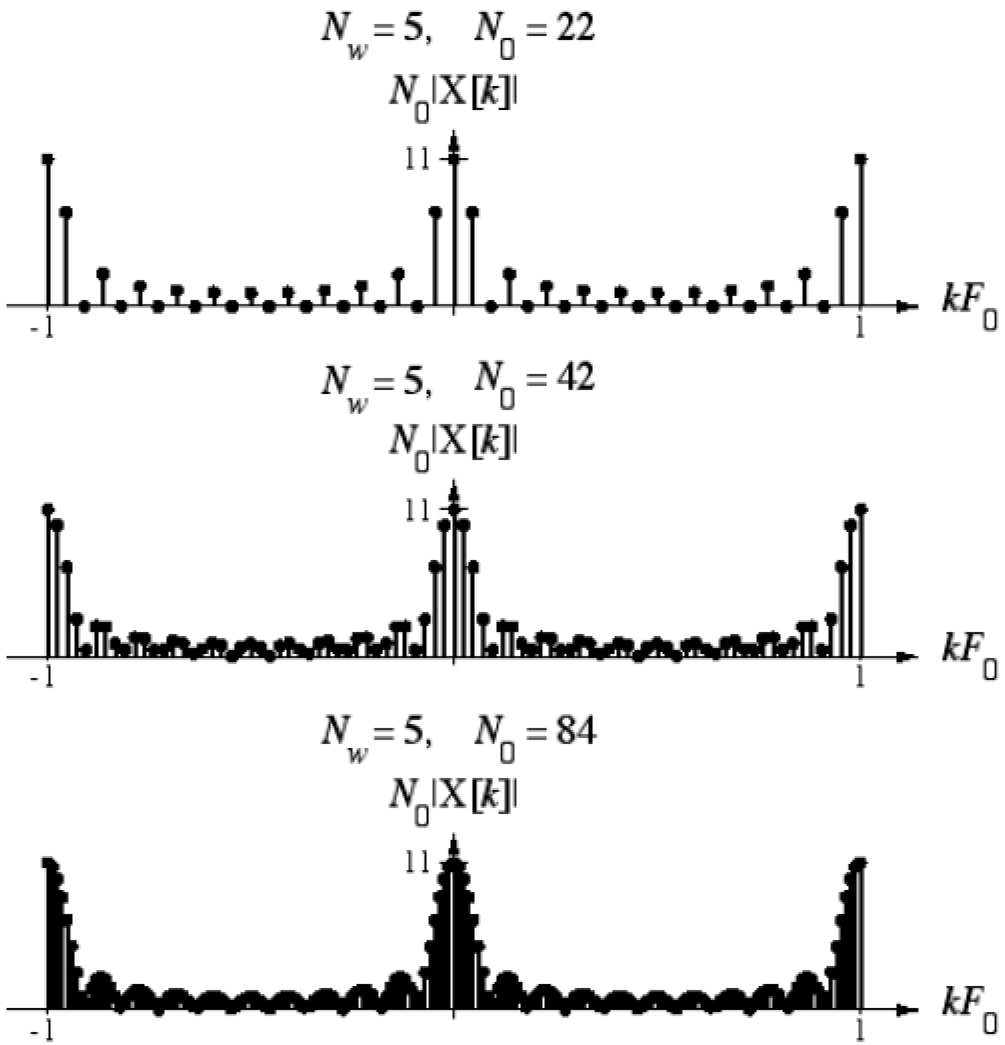
Transition: DT Fourier Series to DT Fourier Transform

- DTFS of DT Pulse Train
- As the period of the rectangular wave increases, the period of the DT Fourier Series increases and the amplitude of the DT Fourier Series decreases



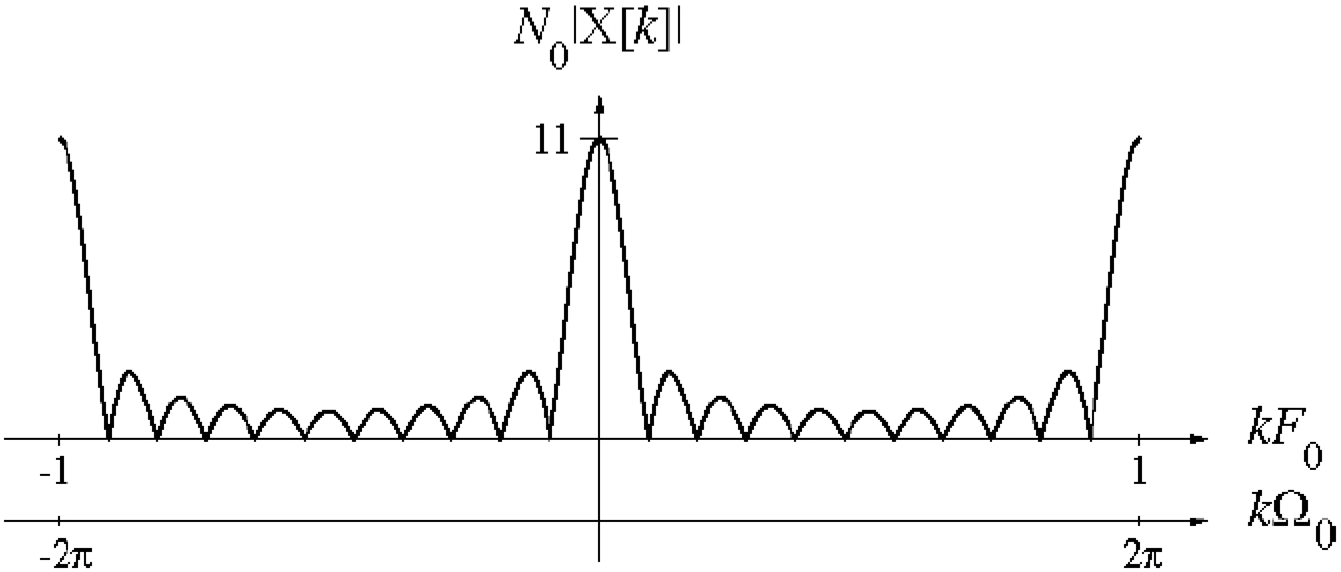
Transition: DT Fourier Series to DT Fourier Transform

- Normalized DT Fourier Series of DT Pulse Train
- By multiplying the DT Fourier Series by its period and plotting versus instead of *k*, the amplitude of the DT Fourier Series stays the same as the period increases and the period of the normalized DT Fourier Series stays at one



Transition: DT Fourier Series to DT Fourier Transform

- The normalized DT Fourier Series approaches this limit as the DT period approaches infinity



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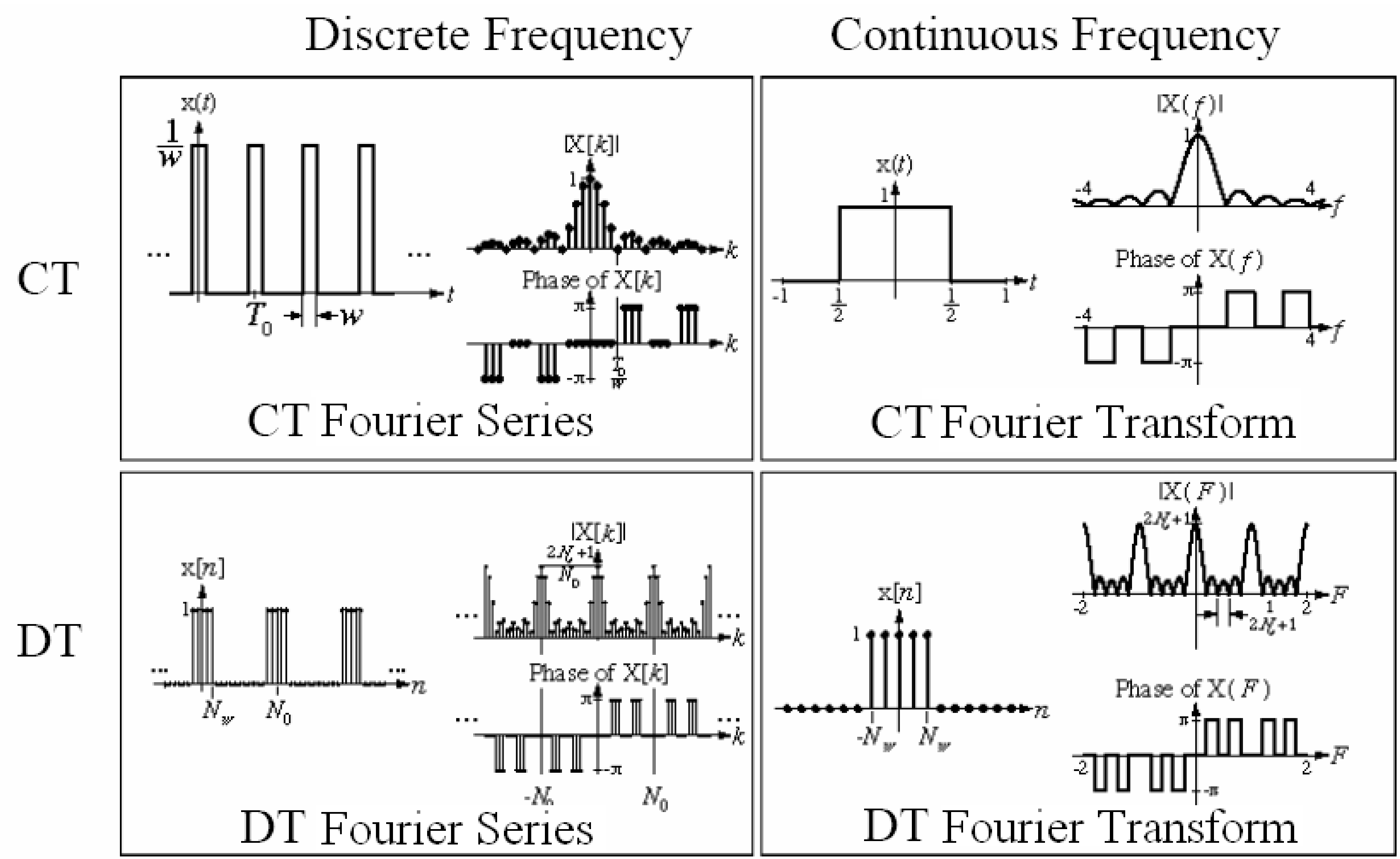
Relations Among Fourier Methods

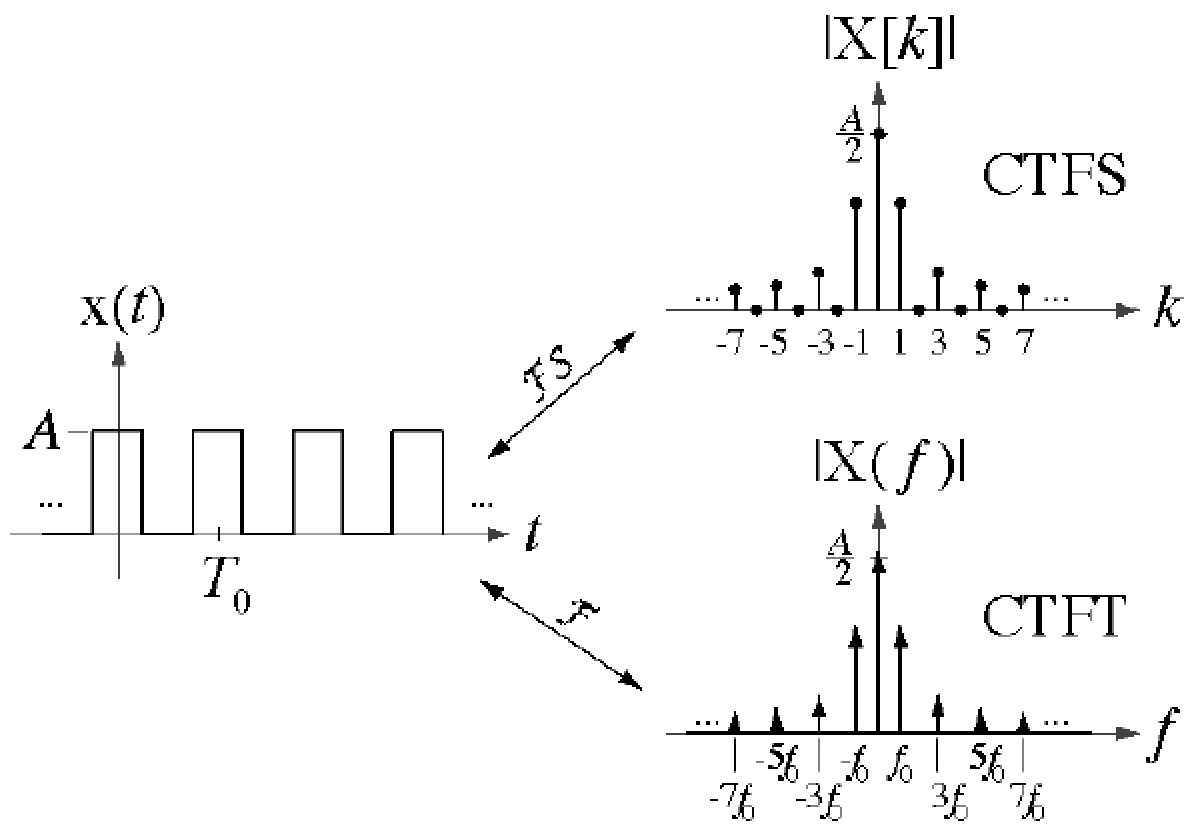
	Periodic in Time Discrete in Frequency	Aperiodic in Time Continuous in Frequency
Continuous in Time Aperiodic in Frequency	$\otimes$ CT Fourier Series : CT - $P_T \Rightarrow$ DT $a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$ $\otimes$ CT Inverse Fourier Series : DT $\Rightarrow$ CT - P_T $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\otimes$ CT Fourier Transform: CT $\Rightarrow$ CT $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $\otimes$ Inverse CT Fourier Transform: CT $\Rightarrow$ CT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
Discrete in Time Periodic in Frequency	$\otimes$ DT Fourier Series DT - $P_N \Rightarrow$ DT - P_N $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_0 kn}$ $\otimes$ Inverse DT Fourier Series DT - $P_N \Rightarrow$ DT - P_N $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\omega_0 kn}$	$\otimes$ DT Fourier Transform: DT $\Rightarrow$ CT + $P_{2\pi}$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $\otimes$ Inverse DT Fourier Transform: CT + $P_{2\pi} \Rightarrow$ DT $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

Relations Among Fourier Methods



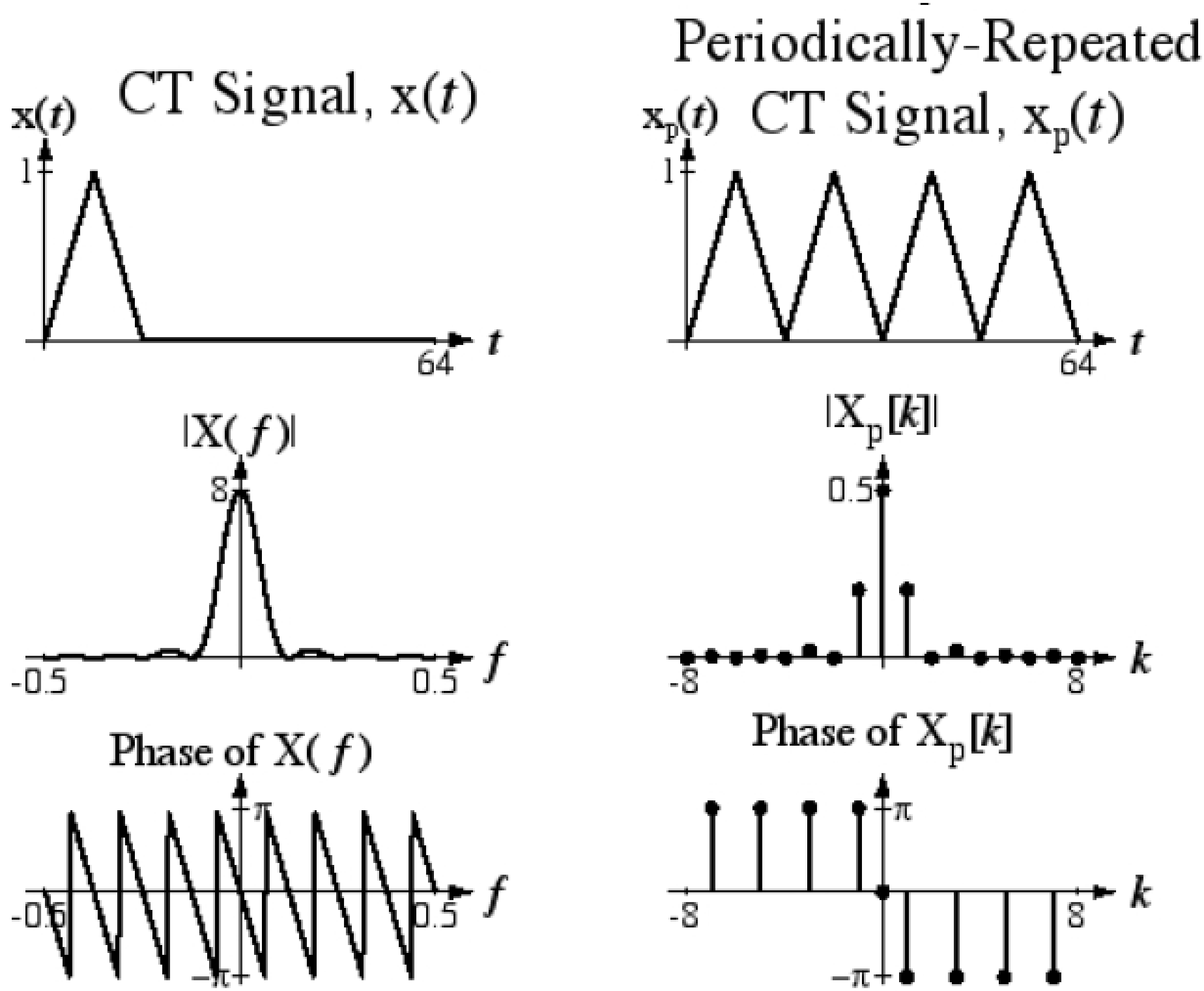
CT Fourier Transform - CT Fourier Series

$$\mathrm{X}(f) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(f - kf_0)$$



CT Fourier Transform - CT Fourier Series

$$X_p[k] = f_p\, X(kf_p)$$



CT Fourier Transform - DT Fourier Transform

Let $x_\delta(t) = x(t)\dfrac{1}{T_s}\operatorname{comb}\left(\dfrac{t}{T_s}\right) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$

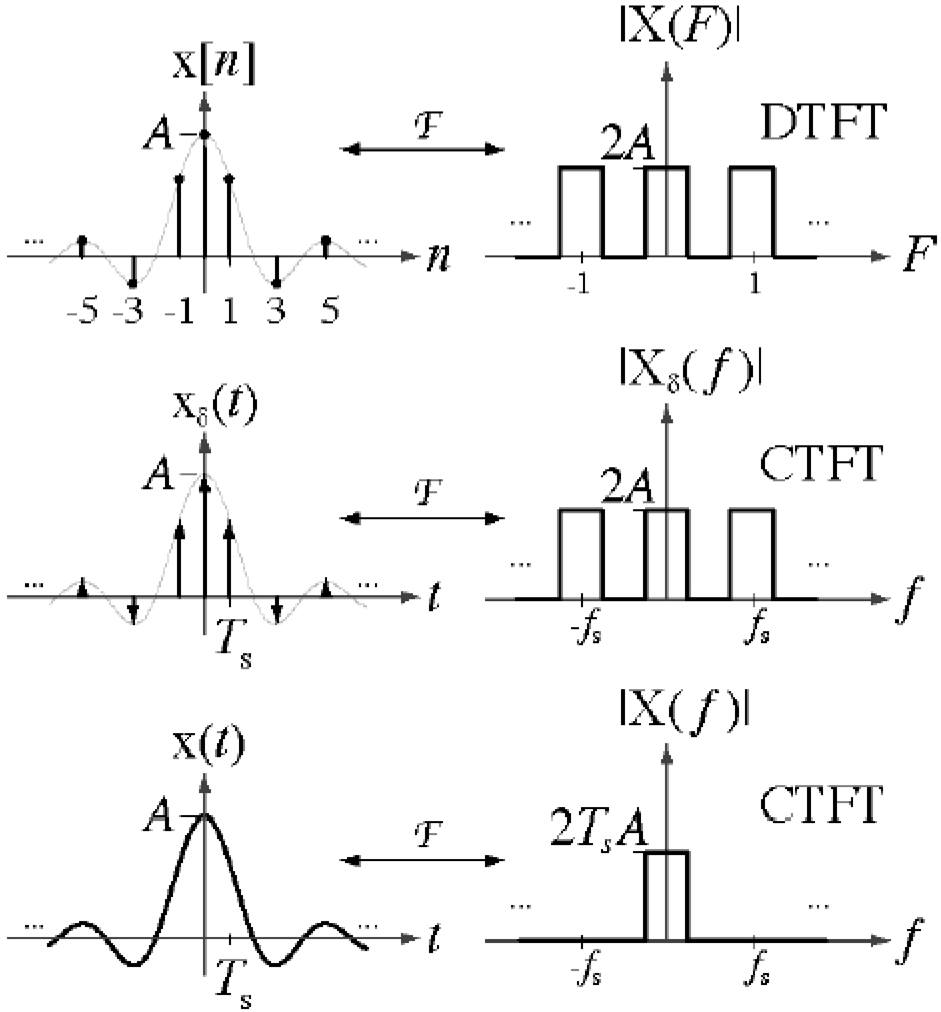
and let $x[n] = x(nT_s)$

There is an "information equivalence" between $x_\delta(t)$ and $x[n]$. They are both completely described by the same set of numbers.

$$X_{DTFT}(F) = X_\delta(f_s F) \qquad X_\delta(f) = X_{DTFT}\left(\frac{f}{f_s}\right)$$

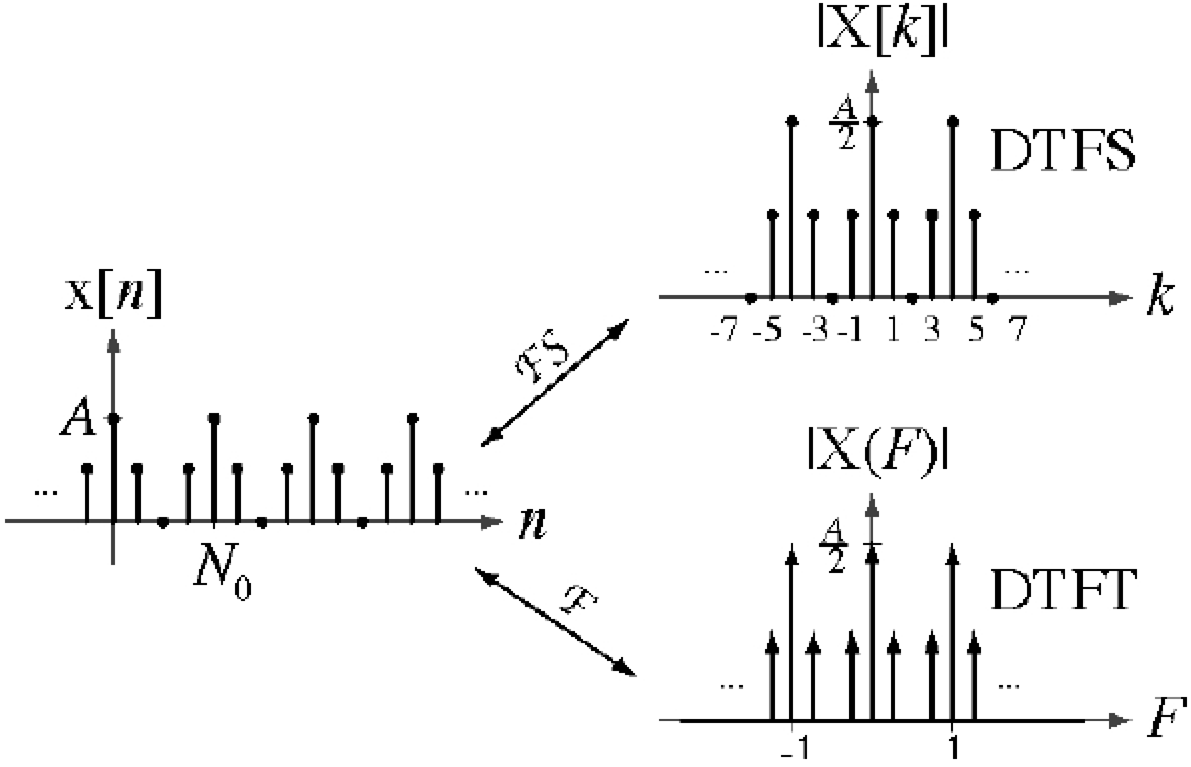
$$X_{DTFT}(F) = f_s \sum_{k=-\infty}^{\infty} X_{CTFT}(f_s(F - k))$$

CT Fourier Transform - DT Fourier Transform



DT Fourier Series - DT Fourier Transform

$$\mathrm{X}(F) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\,\delta(F - kF_0)$$



DT Fourier Series - DT Fourier Transform

$$X_p[k] = \frac{1}{N_p} X(kF_p)$$