

---

# QBOUND: ENVIRONMENT-AWARE Q-VALUE BOUNDS FOR STABLE TEMPORAL DIFFERENCE LEARNING

---

A PREPRINT

Tesfay

tzemuy13@gmail.com

*Working Draft – Contributions Welcome*

<https://github.com/tzemuy13/QBound>

November 30, 2025

## ABSTRACT

We present QBound, a stabilization mechanism for value-based reinforcement learning that derives and enforces Q-value bounds from known reward structures. **Critical finding: QBound’s effectiveness fundamentally depends on reward sign.** Comprehensive evaluation across 8 environments with 5 seeds (40 runs) reveals that QBound improves performance only for positive dense rewards, while consistently degrading performance for negative rewards. This reward-sign dependence is the paper’s central empirical finding and limits QBound’s applicability.

QBound addresses overestimation bias [Thrun and Schwartz, 1993, Van Hasselt et al., 2016]—where bootstrapped Q-value estimates systematically exceed true values—by clipping next-state Q-values to environment-specific bounds  $[Q_{\min}, Q_{\max}]$ , which propagate naturally through temporal difference learning.

**Scope:** QBound targets *off-policy* value-based methods (DQN, DDQN, Dueling DQN) and off-policy actor-critic methods (DDPG, TD3) that use experience replay. On-policy methods (e.g., PPO, A2C, REINFORCE) are outside QBound’s scope because they do not suffer from the same overestimation dynamics.

**Results by reward sign:** For positive dense reward environments (e.g., CartPole:  $r = +1$  per timestep), QBound achieves mean improvements of 12% to 34% across DQN variants, but per-seed analysis reveals inconsistency: CartPole Dueling shows 100% win rate (5/5 seeds improve), while CartPole DQN shows only 80% win rate (4/5 seeds improve, with one seed degrading by 8.7%). Neural networks with linear output layers have no architectural constraint on positive values, making explicit upper bounds beneficial on average.

For negative reward environments (Pendulum:  $r \in [-16, 0]$ ), we tested architectural QBound (output activation:  $Q = -\text{softplus}(\text{logits})$ ) across off-policy algorithms. QBound degrades performance for most algorithms: architectural QBound degrades DQN by 3.3%, Double DQN by 7.1%, and DDPG by 8.0%. TD3 is an exception, showing 4.1% improvement with architectural QBound ( $-183.25 \rightarrow -175.66$ ) but with 72% higher variance ( $\pm 40.15$  vs  $\pm 23.36$ ), suggesting a unique interaction with TD3’s twin critic architecture.

The fundamental question of why QBound works for positive but not negative rewards remains open. Both  $Q_{\max}$  (positive rewards) and  $Q_{\max} = 0$  (negative rewards) are theoretical upper bounds derived from cumulative discounted rewards. We propose several hypotheses for future investigation: (1) network initialization bias interacts differently with positive vs negative targets, (2) gradient flow patterns differ across value ranges, (3) replay buffer composition effects, (4) exploration strategy interactions. Answering this question requires controlled ablation studies beyond the scope of this work.

This work provides: (1) systematic per-seed analysis across 40 runs showing QBound’s effectiveness varies by algorithm (100% win rate for Dueling DQN, 80% for standard DQN), (2) comprehensive negative results for negative rewards (40-60% win rates, no better than chance), (3) honest reporting

of seed-dependent failures even in successful cases, (4) clear recommendations based on win rates rather than mean improvements alone. Implementation imposes negligible overhead ( $<2\%$ ). All experiments use 5 seeds with full reproducibility protocols.

**Recommendations:** (1) Use hard clipping QBound for positive dense rewards with Dueling DQN (100% win rate, +22.5% mean improvement). (2) For standard DQN on positive rewards, QBound helps on average (+12%) but expect 20% of seeds to degrade. (3) Do not use QBound for negative rewards—win rates are 40-60% (no better than chance) with high variance. (4) QBound is designed for off-policy methods only; on-policy methods are outside scope. QBound is a specialized technique for positive dense rewards, not a universal improvement, and practitioners should run multiple seeds before deployment.

**Keywords** Reinforcement Learning · Temporal Difference Learning · Q-Learning · Overestimation Bias · Value Stability · Sample Efficiency

## 1 Introduction

### 1.1 Motivation: Instability in Value Learning Limits Sample Efficiency

Reinforcement learning has achieved remarkable successes in games [Mnih et al., 2015], robotics [Levine et al., 2016], and complex decision-making tasks [Vinyals et al., 2019]. However, a critical bottleneck remains: **sample efficiency**—the number of environment interactions required to learn effective policies. In many real-world applications, environment samples are the limiting resource:

- **Robotics:** Physical interactions cost time, energy, and risk hardware damage [Kalashnikov et al., 2018]
- **Clinical trials:** Patient interactions are limited by enrollment, ethics, and cost [Dulac-Arnold et al., 2019]
- **Financial trading:** Historical data is finite, live testing is risky
- **Industrial control:** Plant operations are expensive and safety-critical [Dulac-Arnold et al., 2019]
- **Autonomous vehicles:** Real-world testing is dangerous and expensive
- **Game design:** Human playtesting is time-consuming and costly

Current deep RL methods vary dramatically in sample efficiency [Duan et al., 2016, Achiam, 2018]. Pure policy gradient methods like REINFORCE [Williams, 1992] require 50M-100M+ environment steps due to high variance gradient estimates [Schulman et al., 2015]. Actor-critic methods like DDPG [Lillicrap et al., 2015], TD3 [Fujimoto et al., 2018], and SAC [Haarnoja et al., 2018a] achieve 2M-20M steps by combining policy gradients with value function learning [Haarnoja et al., 2018b]. Pure value-based methods like DQN [Mnih et al., 2015] and its variants achieve the highest sample efficiency at 1M-10M steps through bootstrap learning with experience replay [Lin, 1992, Mnih et al., 2015].

The sample efficiency hierarchy correlates directly with whether methods learn value functions. This suggests that improving value function learning improves sample efficiency across the entire spectrum of methods that use critics. However, value-based methods face a fundamental challenge: bootstrapped Q-value estimates suffer from instability and overestimation bias [Thrun and Schwartz, 1993, Van Hasselt et al., 2016], which directly undermines their sample efficiency advantage. Stabilizing value learning is therefore essential for achieving better sample efficiency.

### 1.2 The Bootstrapping Instability Problem

All methods that learn value functions face a fundamental challenge: bootstrapping with imperfect function approximators produces unbounded, inconsistent value estimates [Tsitsiklis and Van Roy, 1997]. During training, Q-values frequently:

1. Diverge to arbitrary magnitudes ( $Q(s, a) \rightarrow \pm\infty$ )
2. Violate theoretical constraints (e.g.,  $Q(s, a) > Q_{\max}$  when  $Q_{\max}$  is derivable from environment structure)
3. Exhibit high variance in bootstrap targets, leading to unstable learning
4. Create poorly scaled gradient signals that slow convergence

Prior stabilization work includes target networks [Mnih et al., 2015], clipped double-Q [Fujimoto et al., 2018], reward clipping [Mnih et al., 2013], and gradient clipping [Pascanu et al., 2013]. However, these approaches do not directly enforce theoretically-derived bounds based on environment structure.

### 1.3 Our Approach: QBound

We propose QBound, a stabilization mechanism that prevents overestimation bias by exploiting known environment structure. QBound addresses the root cause of instability in bootstrapped value learning—unbounded overestimation—by deriving and enforcing environment-aware Q-value bounds. Unlike generic pessimism (e.g., Double-Q [Van Hasselt et al., 2016]), QBound’s bounds are environment-specific, providing stabilization without excessive conservatism.

**Important scope limitation:** QBound is a *specialized technique*, not a universal improvement. Our comprehensive evaluation reveals that QBound provides consistent benefits *only* for positive dense reward environments (+12-34% on CartPole). For negative reward environments, QBound causes degradation (-3% to -8%). For sparse terminal rewards, QBound provides no meaningful benefit. Practitioners should carefully assess reward structure before applying QBound.

#### Core Mechanism:

- Derive tight bounds  $[Q_{\min}, Q_{\max}]$  from environment reward structure
- Clip next-state Q-values during bootstrapping:  $Q_{\text{next}} \leftarrow \text{clip}(Q_{\text{next}}, Q_{\min}, Q_{\max})$
- Compute bounded targets:  $Q_{\text{target}} = r + \gamma \cdot Q_{\text{next}}^{\text{clipped}}$
- Standard TD loss propagates bounds through the network naturally

**Key Insight:** Clipping next-state Q-values in the bootstrap target— $y = r + \gamma \cdot \text{clip}(Q(s', a'), Q_{\min}, Q_{\max})$ —naturally propagates bounds through temporal difference learning [Sutton and Barto, 2018]. Since today’s  $Q(s, a)$  is trained toward this clipped target, it becomes tomorrow’s  $Q(s', a')$  for other states, creating a self-reinforcing bounded value propagation through the Bellman backup chain.

#### Key Benefits:

- **Primary:** Prevents overestimation bias, stabilizing temporal difference learning [Sutton and Barto, 2018]
- **Outcome:** 12-34% improvement on positive dense reward environments (CartPole) with 5-seed validation; no benefit or degradation on negative/sparse reward environments
- Reduces variance in bootstrapped targets, particularly during early training
- Negligible computational overhead ( $< 2\%$ )
- Works with off-policy algorithms that learn Q-functions (DQN, DDQN, Dueling DQN, DDPG, TD3)

**Target Applications:** QBound is most effective for positive dense reward environments (e.g., CartPole with  $r = +1$  per timestep). For such environments, explicit upper bounds prevent unbounded Q-value growth. QBound is not recommended for negative reward or sparse terminal reward environments where it provides no benefit or causes degradation.

### 1.4 QBound Variants: A Taxonomy

We implement three QBound variants depending on action space and reward sign. Table 1 summarizes when to use each.

Table 1: QBound Implementation Variants

Variant	Mechanism	Use Case	Algorithms
<b>Hard QBound</b>	Direct clipping: $Q \leftarrow \text{clip}(Q, Q_{\min}, Q_{\max})$	Discrete actions, positive rewards	DQN, DDQN, Dueling
<b>Soft QBound</b>	Penalty loss for violations (preserves gradients)	Continuous actions (theoretical; not evaluated)	DDPG, TD3, SAC
<b>Architectural</b>	Output activation: $Q = -\text{softplus}(\cdot)$ enforces $Q \leq 0$	Negative rewards (empirically fails)	Any Q-learning

**Key distinction:** Hard and Soft QBound enforce bounds *algorithmically* (post-hoc on network outputs), while Architectural QBound enforces bounds *structurally* through network architecture. Our experiments reveal that Hard QBound succeeds for positive rewards (+12-34%), but Architectural QBound largely fails for negative rewards despite theoretical motivation (Sec. 3.4.3).

## 2 Related Work

### 2.1 Value-Based Reinforcement Learning

**Q-learning** [Watkins and Dayan, 1992] learns action-value functions through temporal difference bootstrapping, with convergence guarantees proven for tabular settings [Jaakkola et al., 1994, Melo, 2001]. The foundational analysis by Watkins [1989] established the theoretical framework that underlies modern value-based methods.

**Deep Q-Networks (DQN)** [Mnih et al., 2013, 2015] revolutionized RL by combining Q-learning with deep neural networks, experience replay [Lin, 1992], and target networks. **Double Q-Learning** [Van Hasselt et al., 2016] addresses overestimation bias but does not bound absolute value magnitudes. Recent advances include dueling architectures [Wang et al., 2016], distributional methods [Bellemare et al., 2017, Dabney et al., 2018], and Rainbow combinations [Hessel et al., 2018].

### 2.2 Actor-Critic Methods

**Actor-critic methods** [Konda and Tsitsiklis, 2000] combine policy gradients [Sutton et al., 2000] with value function learning. Classical methods include A2C/A3C [Mnih et al., 2016] for discrete control. For continuous control, **DDPG** [Lillicrap et al., 2015] pioneered deterministic policy gradients, while **TD3** [Fujimoto et al., 2018] added clipped double-Q estimation and delayed policy updates. **SAC** [Haarnoja et al., 2018a,b] maximizes entropy-augmented objectives for improved exploration. Trust region methods like TRPO [Schulman et al., 2015] and PPO [Schulman et al., 2017] provide stable policy updates.

### 2.3 Sample Efficiency and Experience Replay

Experience replay [Lin, 1992] dramatically improves sample efficiency by reusing transitions. **Prioritized experience replay** [Schaul et al., 2015] focuses on important transitions, while **hindsight experience replay** [Andrychowicz et al., 2017] creates synthetic successes for sparse reward environments. Recent work [Fedus et al., 2020] revisits replay fundamentals, showing that simple improvements can be highly effective.

### 2.4 Stabilization and Optimization

Deep RL stability has been improved through various techniques: target networks [Mnih et al., 2015], gradient clipping [Pascanu et al., 2013], batch normalization [Ioffe and Szegedy, 2015], and optimizers like Adam [Kingma and Ba, 2014]. Henderson et al. [2017] highlighted reproducibility issues and the importance of proper baselines, while theoretical work [Szepesvári, 2010] provides PAC-MDP analysis for tabular settings.

### 2.5 Activation Functions in Value Networks

The choice of output activation functions for value networks is rarely discussed in RL literature, with most work using linear (identity) output layers by default. However, recent work has begun exploring architectural constraints:

**Bounded activation functions:** Some early DQN implementations experimented with sigmoid or tanh output activations to bound Q-values [Mnih et al., 2013], but this was abandoned in favor of linear outputs in DQN [Mnih et al., 2015] due to representational limitations. The consensus has been that linear outputs provide the most flexibility.

**Transformed Bellman operators:** Pohlen et al. [2018] introduced a transformed Bellman operator that addresses varying Q-value scales across Atari games through reward transformation, enabling stable training with high discount factors ( $\gamma = 0.999$ ). Their approach bounds Q-values implicitly through invertible transformations rather than explicit architectural constraints. While effective for scale normalization, their method does not leverage environment-specific reward bounds or address reward-sign-dependent implementation strategies. Our work complements this by showing that explicit bounds derived from reward structure can provide additional benefits, particularly for known reward ranges.

**Architectural inductive biases:** Recent work in supervised learning has emphasized the importance of architectural inductive biases matching problem structure [Battaglia et al., 2018]. Goyal and Bengio [2022] showed that architectural choices encoding prior knowledge outperform learned representations in many domains. However, this principle has seen limited application to value network design in RL.

**Initialization bias and exploration:** Kumar et al. [2020a] showed that network initialization creates implicit regularization in RL, affecting the learned Q-function. Fortunato et al. [2017] demonstrated that proper initialization

and exploration mechanisms significantly impact learning dynamics. Our work extends this by showing that *output activation functions* can serve as architectural inductive biases that guide exploration space from initialization.

**Value clipping vs. architectural constraints:** Most RL algorithms apply value clipping *algorithmically* (e.g., PPO’s value function clipping [Schulman et al., 2017], SAC’s target entropy [Haarnoja et al., 2018a]). However, these operate post-hoc on network outputs. Our work demonstrates that *architectural enforcement* through output activations can be more effective than algorithmic clipping for certain reward structures.

## 2.6 Recent Work on Value Bounding and Constraints

Recent work has explored constraining Q-values in various contexts. Liu et al. [2024] proposed bounding techniques for soft Q-learning in offline settings, demonstrating improved stability through bounded value function approximation. Adamczyk et al. [2023] derived theoretical bounds for compositional RL, providing double-sided inequalities relating optimal composite value functions to primitive task values.

**Adaptive pessimism:** Wang et al. [2024] introduced APTQ (Adaptive Pessimism via Target Q-value), which balances pessimism constraints with RL objectives by adaptively weighting conservatism based on task-specific Q-value distributions. Their key insight—that optimal constraint strength varies across tasks—parallels our finding that optimal *implementation* (hard clipping vs. architectural constraints) varies across reward structures. However, their approach targets offline RL with out-of-distribution action mitigation, while QBound addresses online learning with environment-derived bounds. Both works highlight that Q-value constraints should be task-aware rather than fixed.

Work on overestimation bias has also progressed. Kumar et al. [2023] proposed Elastic Step DQN, which alleviates overestimation by dynamically varying multi-step horizons based on state similarity. Feng et al. [2024] addressed maximization bias through two-sample testing, framing the problem statistically and proposing the T-Estimator that flexibly interpolates between over- and underestimation. Zhang et al. [2025] introduced Imagination-Limited Q-Learning, which envisions reasonable values and appropriately limits potential over-estimations using maximum behavior values.

For sparse reward environments, Wan et al. [2024] achieved  $2\times$  better sample efficiency through high replay ratio methods with regularization, while Huang et al. [2024] demonstrated significant improvements by extracting heuristics from large language models for reward shaping.

## 2.7 Related Bias Mitigation Methods

Several recent methods address Q-value overestimation through alternative mechanisms. We discuss their relationship to QBound and acknowledge that direct empirical comparisons remain future work.

**Q-Clip in Discrete SAC (SD-SAC):** Zhou et al. [2022] introduced Q-clip for discrete Soft Actor-Critic, explicitly bounding critic updates to stabilize training. Unlike QBound’s environment-derived bounds, Q-clip uses fixed or learned bounds without leveraging reward structure. A head-to-head comparison on shared discrete benchmarks would clarify whether environment-aware bounds provide additional benefits over generic Q-clip.

**Doubly Bounded Estimators (DB-ADP):** Ren et al. [2022] employ both upper and lower bounds on Q-targets via a conservative dynamic programming estimator, demonstrating robust gains on Atari. DB-ADP provides complementary lower-bound enforcement that QBound does not address. Combining QBound’s environment-aware upper bounds with DB-ADP’s lower bounds is a promising direction.

**Distributional Ensemble Methods (CTD4):** Wu et al. [2023] use distributional ensembles with variance-aware Kalman-style fusion to reduce overestimation. This approach is orthogonal to QBound: distributional methods model return uncertainty, while QBound enforces hard constraints from environment structure. Combining both could provide complementary benefits.

**Adaptive Bias Control (BE-TD3):** Lee et al. [2022] adaptively select between min and max operators based on estimated bias direction. This online adaptation contrasts with QBound’s static, environment-derived bounds. Whether QBound plus adaptive bias control achieves additive benefits is an open question.

**Relationship to QBound:** These methods differ from QBound in key aspects: (1) they typically use learned or fixed bounds rather than environment-derived bounds; (2) they focus on the bias-variance tradeoff rather than theoretical constraint satisfaction; (3) they do not explicitly leverage reward structure information. QBound’s contribution is showing that environment-aware bounds can provide benefits for specific reward structures (positive dense rewards), though our empirical evaluation is limited to simpler benchmarks. Direct comparisons on standard benchmarks (Atari, DMControl) remain important future work.

## 2.8 Positioning of QBound

QBound differs from prior work in several key aspects:

1. **Environment-aware bounds:** Unlike generic stabilization techniques or learned bounds, QBound derives bounds directly from environment reward structure, ensuring theoretical correctness
2. **Reward-sign analysis:** We investigate how QBound effectiveness depends on reward sign. Hard clipping succeeds for positive rewards (+12-34%), but architectural constraints for negative rewards largely fail despite theoretical motivation. While prior work has explored Q-value constraints [Pohlen et al., 2018, Wang et al., 2024], we provide systematic analysis revealing this fundamental limitation
3. **Architectural inductive bias investigation:** We investigate whether output activation functions ( $Q = -\text{softplus}(\text{logits})$ ) can enforce value bounds for negative rewards. Despite theoretical motivation, our experiments show this approach largely fails, providing important negative results for the community
4. **Bootstrapping-based enforcement:** QBound leverages the natural propagation of bootstrapped targets through temporal difference learning [Sutton and Barto, 2018], requiring only simple operations (clipping or activation)
5. **Online learning focus:** Unlike recent work on offline RL [Wang et al., 2024, Zhang et al., 2025], QBound targets online learning in both sparse and dense reward settings
6. **Comprehensive evaluation:** We provide extensive analysis across 8 environments with 5 seeds each (40 runs), including honest reporting of failure modes and implementation-dependent effectiveness

**Findings on activation functions:** Our experiments reveal that architectural constraints ( $Q = -\text{softplus}(\cdot)$ ) for negative rewards largely fail despite theoretical motivation, while hard clipping succeeds for positive rewards. This asymmetry provides practical guidance: QBound is effective for positive dense rewards but should not be applied to negative reward environments.

## 3 Theoretical Foundations

### 3.1 Preliminaries and Notation

**Definition 1** (Markov Decision Process). A Markov Decision Process is a tuple  $\mathcal{M} = (\mathcal{S}, \mathcal{A}, P, r, \gamma)$  where:

- $\mathcal{S}$ : State space (finite or continuous)
- $\mathcal{A}$ : Action space (discrete:  $\mathcal{A} = \{a_1, \dots, a_{|\mathcal{A}|}\}$  or continuous:  $\mathcal{A} \subseteq \mathbb{R}^d$ )
- $P(s'|s, a)$ : Transition dynamics
- $r(s, a, s') \in \mathbb{R}$ : Reward function
- $\gamma \in [0, 1)$ : Discount factor

**Definition 2** (Value Functions). For policy  $\pi : \mathcal{S} \rightarrow \Delta(\mathcal{A})$  (stochastic) or  $\mu : \mathcal{S} \rightarrow \mathcal{A}$  (deterministic):

$$V^\pi(s) = \mathbb{E}_\pi \left[ \sum_{t=0}^{\infty} \gamma^t r_t \mid s_0 = s \right] \quad (1)$$

$$Q^\pi(s, a) = \mathbb{E}_\pi \left[ \sum_{t=0}^{\infty} \gamma^t r_t \mid s_0 = s, a_0 = a \right] \quad (2)$$

**Definition 3** (Optimal Value Functions).

$$Q^*(s, a) = \max_{\pi} Q^\pi(s, a) \quad (3)$$

$$V^*(s) = \max_a Q^*(s, a) \quad (4)$$

The Bellman optimality equation [Bellman, 1957, Sutton and Barto, 2018] provides the foundation for Q-learning:

$$Q^*(s, a) = \mathbb{E}_{s' \sim P(\cdot|s, a)} \left[ r(s, a, s') + \gamma \max_{a'} Q^*(s', a') \right]$$

**Assumption 4** (Bounded Rewards). We assume that worst-case and best-case cumulative returns over all possible trajectories are finite and can be computed or bounded. This is satisfied by most practical environments.

### 3.2 Environment-Specific Q-Value Bounds

The key theoretical contribution is deriving tight bounds  $[Q_{\min}, Q_{\max}]$  such that all possible Q-values lie within this range.

**Definition 5** (Trajectory). A trajectory  $\tau = (s_0, a_0, r_0, s_1, a_1, r_1, \dots)$  is a sequence of states, actions, and rewards following dynamics  $P$  and policy  $\pi$ .

**Definition 6** (Trajectory Return). For finite horizon  $H$  or until termination:

$$G(\tau) = \sum_{t=0}^{H-1} \gamma^t r_t$$

**Definition 7** (Environment-Specific Bounds).

$$Q_{\min} = \inf_{\pi \in \Pi, s \in \mathcal{S}, a \in \mathcal{A}} Q^\pi(s, a) = \inf_{\tau \in \mathcal{T}(s, a)} G(\tau) \quad (5)$$

$$Q_{\max} = \sup_{\pi \in \Pi, s \in \mathcal{S}, a \in \mathcal{A}} Q^\pi(s, a) = \sup_{\tau \in \mathcal{T}(s, a)} G(\tau) \quad (6)$$

where  $\mathcal{T}(s, a)$  is the set of all trajectories starting with  $(s, a)$ .

**Theorem 8** (Bound Correctness). If  $Q_{\min}$  and  $Q_{\max}$  are computed according to the above definition, then:

$$Q^*(s, a) \in [Q_{\min}, Q_{\max}] \quad \forall s, a$$

*Proof.* Follows directly from definition:  $Q^*(s, a) = \max_{\pi} Q^\pi(s, a) \leq \sup_{\pi} Q^\pi(s, a) = Q_{\max}$ , and similarly  $Q^* \geq Q_{\min}$ .  $\square$

**Corollary 9.** Clipping Q-values to  $[Q_{\min}, Q_{\max}]$  cannot remove the optimal value  $Q^*$ .

### 3.3 Analysis of the Clipped Bellman Operator

A key theoretical question is whether the clipped target operator preserves the contraction properties essential for convergence. We analyze this formally.

**Definition 10** (Clipped Bellman Operator). Define the clipped Bellman optimality operator  $\mathcal{T}_{\text{clip}} : \mathbb{R}^{|\mathcal{S}||\mathcal{A}|} \rightarrow \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$  as:

$$(\mathcal{T}_{\text{clip}} Q)(s, a) = \mathbb{E}_{s' \sim P(\cdot|s, a)} \left[ r(s, a, s') + \gamma \cdot \text{clip} \left( \max_{a'} Q(s', a'), Q_{\min}, Q_{\max} \right) \right]$$

**Theorem 11** (Contraction Property of Clipped Operator). The clipped Bellman operator  $\mathcal{T}_{\text{clip}}$  is a  $\gamma$ -contraction in the  $\ell_\infty$  norm:

$$\|\mathcal{T}_{\text{clip}} Q_1 - \mathcal{T}_{\text{clip}} Q_2\|_\infty \leq \gamma \|Q_1 - Q_2\|_\infty$$

*Proof.* For any  $s, a$ :

$$|(\mathcal{T}_{\text{clip}} Q_1)(s, a) - (\mathcal{T}_{\text{clip}} Q_2)(s, a)| = \gamma \left| \mathbb{E}_{s'} \left[ \text{clip}(\max_{a'} Q_1(s', a')) - \text{clip}(\max_{a'} Q_2(s', a')) \right] \right|$$

Since  $\text{clip}(\cdot, Q_{\min}, Q_{\max})$  is a non-expansive mapping (i.e.,  $|\text{clip}(x) - \text{clip}(y)| \leq |x - y|$ ), and max over a finite set is also non-expansive:

$$|\text{clip}(\max_{a'} Q_1(s', a')) - \text{clip}(\max_{a'} Q_2(s', a'))| \leq |\max_{a'} Q_1(s', a') - \max_{a'} Q_2(s', a')| \leq \|Q_1 - Q_2\|_\infty$$

Therefore:

$$|(\mathcal{T}_{\text{clip}} Q_1)(s, a) - (\mathcal{T}_{\text{clip}} Q_2)(s, a)| \leq \gamma \|Q_1 - Q_2\|_\infty$$

Taking the supremum over all  $(s, a)$  yields the result.  $\square$

**Theorem 12** (Fixed Point of Clipped Operator). Let  $Q^*$  be the optimal Q-function and  $\tilde{Q}^*$  be the fixed point of  $\mathcal{T}_{\text{clip}}$ . If bounds are correctly specified (i.e.,  $Q^*(s, a) \in [Q_{\min}, Q_{\max}]$  for all  $s, a$ ), then  $\tilde{Q}^* = Q^*$ .

*Proof.* When  $Q^*(s, a) \in [Q_{\min}, Q_{\max}]$  for all  $(s, a)$ , we have  $\max_{a'} Q^*(s', a') \in [Q_{\min}, Q_{\max}]$  for all  $s'$ . Therefore,  $\text{clip}(\max_{a'} Q^*(s', a'), Q_{\min}, Q_{\max}) = \max_{a'} Q^*(s', a')$ , which implies  $\mathcal{T}_{\text{clip}} Q^* = \mathcal{T} Q^* = Q^*$ . Since  $Q^*$  satisfies the clipped Bellman equation, it is the unique fixed point by the Banach fixed point theorem.  $\square$

*Remark 13 (Bias Under Mis-specified Bounds).* If bounds are mis-specified such that  $Q^*(s, a) > Q_{\max}$  for some  $(s, a)$ , then the fixed point  $\tilde{Q}^*$  satisfies  $\tilde{Q}^*(s, a) \leq Q^*(s, a)$ . Specifically, for states where clipping is active, the learned Q-function underestimates the true optimal value. This bias is bounded by:

$$Q^*(s, a) - \tilde{Q}^*(s, a) \leq \frac{\gamma}{1 - \gamma} (Q_{\max}^* - Q_{\max})$$

where  $Q_{\max}^* = \max_{s,a} Q^*(s, a)$  is the true maximum Q-value. Overly conservative bounds thus introduce pessimistic bias, while correctly specified bounds preserve optimality.

### 3.4 Fundamental Q-Value Bounds for Common Reward Structures

#### 3.4.1 Case 1: Sparse Binary Rewards (Primary Use Case)

**Environment Structure:** Single reward at episode end, zero otherwise:

$$r(s, a, s') = \begin{cases} 1 & \text{if } s' \text{ is goal state} \\ 0 & \text{otherwise} \end{cases}$$

This is the most common sparse reward structure in robotics, games, and goal-reaching tasks.

**Theorem 14 (Sparse Binary Reward Bounds).** *For sparse binary reward environments with discount factor  $\gamma$ , the bounds depend on episode termination:*

**Case 1a (Reach-Once):** Episode terminates upon reaching goal:

$$Q_{\min} = 0, \quad Q_{\max} = 1$$

**Case 1b (Stay-at-Goal):** Agent can remain at goal and continue receiving rewards until episode end or indefinitely:

$$Q_{\min} = 0, \quad Q_{\max} = \frac{1}{1 - \gamma}$$

*Proof. Lower bound (both cases):* Since all immediate rewards are non-negative, any trajectory return  $G(\tau) = \sum_{t=0}^{\infty} \gamma^t r_t \geq 0$ , hence  $Q_{\min} = 0$ .

**Upper bound (Case 1a - Reach-Once):** The agent receives reward  $r = 1$  once when reaching the goal, then the episode terminates:

$$Q_{\max} = 1 \cdot \gamma^0 = 1$$

**Upper bound (Case 1b - Stay-at-Goal):** The agent receives reward  $r = 1$  at every timestep after reaching the goal:

$$Q_{\max} = \sum_{t=0}^{\infty} \gamma^t \cdot 1 = \frac{1}{1 - \gamma}$$

This bound is achieved when the agent reaches the goal immediately and remains there. □

**Example 15 (Robot Navigation - Reach-Once).** A mobile robot navigating to a goal location where the episode ends upon arrival. With  $\gamma = 0.99$ :

$$Q_{\min} = 0, \quad Q_{\max} = 1$$

Any Q-value outside  $[0, 1]$  is impossible given the reward structure and can be safely clipped. This provides extremely tight bounds.

**Example 16 (Robot Navigation - Stay-at-Goal).** A mobile robot that must reach and *maintain* position at the goal, receiving  $r = 1$  per timestep while at goal. With  $\gamma = 0.99$ :

$$Q_{\min} = 0, \quad Q_{\max} = \frac{1}{1 - 0.99} = 100$$

Any Q-value outside  $[0, 100]$  can be safely clipped.

**Example 17 (Game Playing - Reach-Once).** A chess engine with binary win/loss outcomes (+1 for win, 0 for loss/draw) where each game is a single episode. With  $\gamma = 0.995$ :

$$Q_{\min} = 0, \quad Q_{\max} = 1$$

*Note:* The discount factor here primarily affects temporal credit assignment during the game, but the final outcome is binary, so  $Q_{\max} = 1$ .



### 3.4.2 Case 2: Dense Per-Step Costs with Terminal Reward

**Environment Structure:** Negative cost per step, positive reward at goal:

$$r(s, a, s') = \begin{cases} R_{\text{goal}} & \text{if } s' \text{ is goal state} \\ -c & \text{otherwise} \end{cases}$$

**Theorem 18** (Cost-Plus-Reward Bounds). *For maximum episode length  $H$ :*

$$Q_{\min} = -cH + \gamma^H R_{\text{goal}} \approx -cH \text{ if } c \gg R_{\text{goal}} \quad (7)$$

$$Q_{\max} = R_{\text{goal}} \quad (8)$$

**Example 19** (MountainCar). With  $r = -1$  per step,  $r = 0$  at goal,  $H = 200$ :

$$Q_{\min} = -200, \quad Q_{\max} = 0$$

### 3.4.3 Case 3: Dense Positive Rewards (Survival Tasks)

**Environment Structure:** Positive reward per step until failure:

$$r(s, a, s') = r_{\text{step}} > 0$$

**Theorem 20** (Survival Task Bounds). *For finite horizon  $H$ :*

$$Q_{\min} = 0 \text{ (immediate failure)} \quad (9)$$

$$Q_{\max} = r_{\text{step}} \sum_{k=0}^{H-1} \gamma^k = r_{\text{step}} \frac{1 - \gamma^H}{1 - \gamma} \quad (10)$$

*For infinite horizon (no termination):*

$$Q_{\max} = \frac{r_{\text{step}}}{1 - \gamma}$$

**Example 21** (CartPole). With  $r = +1$  per step,  $\gamma = 0.99$ , maximum episode length  $H = 500$ :

$$Q_{\min} = 0, \quad Q_{\max} = \frac{1 - 0.99^{500}}{1 - 0.99} \approx 100$$

**Dynamic Bounds:** For survival tasks with fixed start states (e.g., CartPole), we can use step-aware dynamic bounds:  $Q_{\max}(t) = \frac{1 - \gamma^{(H-t)}}{1 - \gamma}$  at timestep  $t$ , which provides tighter constraints than the static bound. This accounts for the discounted sum of remaining rewards. This is possible because the remaining episode potential is determined by the timestep, not by state proximity to a goal. For sparse reward tasks (e.g., GridWorld), remaining potential depends on unknown state-to-goal distance, making dynamic bounds infeasible.

## 3.5 Critical Insight: Reward Sign Determines QBound Effectiveness

Our comprehensive empirical evaluation (Section 6) reveals that QBound’s effectiveness fundamentally depends on the *sign* of the reward signal. This section provides theoretical justification.

### 3.5.1 The Upper Bound is What Matters for Maximization

**Proposition 22** (Upper Bound Primacy). *In RL, the agent maximizes  $\mathbb{E}[\sum_{t=0}^{\infty} \gamma^t r_t]$ . The upper bound  $Q_{\max}$  directly constrains this objective. The lower bound  $Q_{\min}$  is largely irrelevant because: (1) the agent seeks to maximize, not avoid low values, and (2) overestimation (predicting above  $Q_{\max}$ ) causes suboptimal policies, while underestimation merely slows convergence.*

### 3.5.2 Positive Rewards: QBound Provides Essential Upper Bound

For environments with positive dense rewards (e.g., CartPole:  $r = +1$  per timestep), neural networks with linear output layers have no architectural constraint on the upper bound. Q-values can grow unbounded during training.

**Theorem 23** (Overestimation Vulnerability with Positive Rewards). *For  $r_t > 0$ , the Bellman equation  $Q(s, a) = \mathbb{E}[r + \gamma \max_{a'} Q(s', a')]$  allows unbounded growth. Function approximation errors  $\epsilon$  compound through bootstrapping. QBound prevents this by enforcing  $Q_{\text{target}} = r + \gamma \cdot \text{clip}(\max_{a'} Q(s', a'), Q_{\min}, Q_{\max})$ .*

**Example:** CartPole achieves +12% to +34% improvement across DQN variants (Section 6.3.2).

### 3.5.3 Negative Rewards: Implementation Matters

For negative rewards (e.g., Pendulum:  $r \in [-16, 0]$ ), there exists a theoretical upper bound on Q-values.

**Theorem 24** (Theoretical Upper Bound for Negative Rewards). *If  $r(s, a, s') \leq 0$  for all transitions, then  $Q^\pi(s, a) \leq 0$  for any policy  $\pi$ .*

**Proof:** By induction:  $Q^\pi(s, a) = \mathbb{E}[r + \gamma Q^\pi(s', a')] \leq \mathbb{E}[0 + \gamma Q^\pi(s', a')] = \gamma \mathbb{E}[Q^\pi(s', a')]$ . If  $Q^\pi(s', a') \leq 0$ , then  $Q^\pi(s, a) \leq 0$ .

**Open question:** While this theoretical bound exists, whether neural network function approximators learn to satisfy it without explicit constraints remains unclear. Our experiments show QBound provides no benefit for negative rewards, but *why* this occurs is left for future investigation (see Section 1 for hypotheses).

**Hard Clipping QBound Fails.** Algorithmic clipping ( $Q \leftarrow \text{clip}(Q, -\infty, 0)$ ) interferes with learning due to *exploration-correction conflicts*:

- **Positive initialization bias:** Neural networks with He/Xavier initialization [He et al., 2015, Glorot and Bengio, 2010] typically produce positive outputs initially due to random weight distributions
- **Exploration in wrong space:** Network explores unbounded space, frequently violating  $Q > 0$  (56.79% violation rate observed)
- **Post-hoc correction:** Clipping corrects violations after they occur, but network never learns to naturally output  $Q \leq 0$
- **Persistent conflict:** Network continues exploring positive regions, clipping continues correcting  $\rightarrow$  instability
- **Empirical result:** Pendulum DQN shows **-3.3% degradation** with architectural QBound (5 seeds)

**Architectural QBound: Theoretical Motivation vs. Empirical Reality.** Output activation function ( $Q = -\text{softplus}(\text{logits})$ , enforcing  $Q \in (-\infty, 0]$ ) constrains *exploration space*. In principle, this should help:

- **Constrained exploration:** Network explores WITHIN correct range  $(-\infty, 0]$  from first forward pass
- **Learning correct magnitude:** Network learns “how negative” Q should be, not “whether it should be negative”
- **Zero violations by construction:** Activation function mathematically enforces  $Q \leq 0$ , no conflict possible
- **Smooth gradients:**  $\frac{\partial Q}{\partial \text{logits}} = -\text{sigmoid}(\text{logits}) \in (-1, 0)$  is never zero

**However, empirical results contradict this reasoning.** On Pendulum (5 seeds), architectural QBound *degrades* performance for most algorithms: DQN by -3.3%, Double DQN by -7.1%, and DDPG by -8.0%. Only TD3 shows improvement (+4.1%), but with 72% higher variance. The gap between theoretical motivation and empirical outcomes suggests that additional factors—possibly related to optimization dynamics, initialization interactions, or algorithm-specific properties—dominate the theoretical benefits of constrained exploration.

**Key insight:** QBound shows asymmetric results between positive and negative rewards. For positive rewards (Cart-Pole), hard clipping provides 12-34% improvement. For negative rewards (Pendulum), QBound generally degrades performance (-3% to -8%). TD3 is an exception showing modest improvement (+4.1%), but with 72% higher variance. The fundamental question of *why* this asymmetry exists remains open for future investigation.

### 3.5.4 Summary: When QBound Works

Table 2: QBound Effectiveness by Reward Type and Implementation (5 seeds)

Reward Type	Theoretical Bound	Hard Clipping	Architectural	Best Approach
<b>Positive Dense</b> (CartPole)	Unbounded	+12% to +34%	N/A	Hard Clipping
<b>Negative Dense</b> (Pendulum DQN)	$Q \leq 0$	N/A	-3.3%	Neither
<b>Negative Dense</b> (Pendulum DDPG)	$Q \leq 0$	N/A	-8.0%	Neither
<b>Negative Dense</b> (Pendulum TD3)	$Q \leq 0$	N/A	+4.1%*	Architectural*
<b>Sparse Terminal</b> (GridWorld)	$Q \leq 1$	$\approx 0\%$	N/A	Neither

*\*TD3 shows improvement but with 72% higher variance; use with caution.*

**Recommendation:** Use QBound (hard clipping) for positive dense rewards like CartPole where it provides consistent 12-34% improvements. Do not use QBound for negative rewards (Pendulum-style environments) as it generally degrades performance. TD3 is an exception that may benefit but with high variance.

## 4 QBound Bound Selection Strategy

This section explains how to derive appropriate Q-value bounds for different environment types, focusing on the theoretical foundations demonstrated in our experimental evaluation.

### 4.1 Sparse Binary Reward Environments

Sparse binary reward environments (e.g., GridWorld, FrozenLake) provide extremely tight bounds since the agent receives reward only at terminal states.

#### 4.1.1 Example: Navigation Tasks (GridWorld, FrozenLake)

In our experimental evaluation, we tested GridWorld (deterministic 10×10 navigation) and FrozenLake (stochastic 4×4 navigation with slippery ice).

**Reward Structure:**

- $r = 1$  when agent reaches goal (success)
- $r = 0$  for all other states/actions
- Episode terminates upon reaching goal (reach-once semantics)

**QBound Bounds:**

$$Q_{\min} = 0, \quad Q_{\max} = 1$$

Since the episode terminates immediately upon success, the maximum return is exactly 1 regardless of discount factor. These extremely tight bounds prevent Q-value explosions common in sparse reward exploration.

**Results:** GridWorld achieved 20.2% faster convergence; FrozenLake achieved 5.0% improvement and 76% better final performance than baseline (see Section 5 for details).

### 4.2 Dense Reward Environments: Survival Tasks

For environments with per-timestep rewards (e.g., CartPole), QBound uses step-aware dynamic bounds.

#### 4.2.1 Example: CartPole Balance Task

**Reward Structure:**

- $r = +1$  per timestep (dense rewards)
- Episode terminates on failure or after  $H = 500$  steps
- Discount factor  $\gamma = 0.99$

**QBound Bounds (Step-Aware Dynamic):**

$$Q_{\min} = 0, \quad Q_{\max}(t) = \frac{1 - \gamma^{(H-t)}}{1 - \gamma}$$

At episode start ( $t = 0$ ):  $Q_{\max}(0) = 99.34$ . At the final timestep ( $t = 499$ ):  $Q_{\max}(499) = 1.0$ .

The bounds adapt to remaining episode potential, allowing high Q-values early while constraining them appropriately as the episode progresses.

**Results:** CartPole achieved 31.5% higher cumulative reward than baseline (172,904 vs 131,438 total reward over 500 episodes).

### 4.3 Implementation Guidelines

#### 4.3.1 DQN and Value-Based Methods

For discrete action spaces, QBound requires minimal code changes:

```
# DQN with QBound integration
def qbound_dqn_update(states, actions, rewards, next_states, dones):
    # Standard DQN target computation
    next_q_values = target_net(next_states).max(1)[0]

    # QBound: clip next-state Q-values
    next_q_values = torch.clamp(next_q_values, Q_min, Q_max)

    # Compute bounded targets
    targets = rewards + gamma * next_q_values * (1 - dones)

    # QBound: clip targets for safety
    targets = torch.clamp(targets, Q_min, Q_max)

    # Current Q-values (unclipped)
    current_q_values = q_net(states).gather(1, actions)

    # Standard TD loss
    loss = F.mse_loss(current_q_values, targets)
    return loss
```

**Key Point:** Bootstrapping naturally propagates bounds through the network via temporal difference learning [Sutton and Barto, 2018]. Since agents select actions using current Q-values (not next-state Q-values), clipping the bootstrapped targets ensures bound compliance during training.

## 5 Algorithm and Implementation Details

### 5.1 Complete QBound Algorithm

---

#### Algorithm 1 QBound: Bounded Q-Value Learning

---

**Require:** MDP  $\mathcal{M}$ , Q-network  $Q_\theta$ , target network  $Q_{\theta'}$ , replay buffer  $\mathcal{D}$

**Require:** Bounds  $[Q_{\min}, Q_{\max}]$ , batch size  $B$ , learning rate  $\alpha$

```
1: function QBOUNDUPDATE( $\mathcal{D}, Q_\theta, Q_{\theta'}$ )
2:   Sample batch  $\{(s_i, a_i, r_i, s'_i, d_i)\}_{i=1}^B \sim \mathcal{D}$ 
3:   Initialize loss  $L \leftarrow 0$ 
4:   for each transition  $(s_i, a_i, r_i, s'_i, d_i)$  do
5:     // Compute bounded Bellman target
6:      $Q_{\text{next}} \leftarrow \max_{a'} Q_{\theta'}(s'_i, a')$  ▷ From target network
7:      $Q_{\text{next}}^{\text{clipped}} \leftarrow \text{clip}(Q_{\text{next}}, Q_{\min}, Q_{\max})$  ▷ Enforce bounds
8:      $Q_{\text{target}} \leftarrow r_i + (1 - d_i) \cdot \gamma \cdot Q_{\text{next}}^{\text{clipped}}$ 
9:      $Q_{\text{target}}^{\text{final}} \leftarrow \text{clip}(Q_{\text{target}}, Q_{\min}, Q_{\max})$  ▷ Safety clip
10:    // Standard TD loss
11:     $Q_{\text{current}} \leftarrow Q_\theta(s_i, a_i)$  ▷ Current Q-value (unclipped)
12:     $L \leftarrow L + (Q_{\text{current}} - Q_{\text{target}}^{\text{final}})^2$ 
13:  end for
14:  // Update network
15:   $\theta \leftarrow \theta - \alpha \cdot \nabla_\theta L$ 
16:  // Periodically update target network
17:  if update step then
18:     $\theta' \leftarrow \theta$ 
19:  end if
20: end function
```

---

**Key Insight:** Action selection uses current Q-values  $Q_\theta(s, \cdot)$ , but learning uses clipped next-state Q-values in targets. This means bounded targets naturally propagate through the network via temporal difference bootstrapping [Sutton and Barto, 2018], ensuring the Q-function converges to bounded estimates.

## 5.2 Key Implementation Considerations

### 5.2.1 Bound Computation Strategies

**1. Exact Bounds (Preferred):** For environments with known reward ranges  $[r_{\min}, r_{\max}]$ :

$$Q_{\min} = \frac{r_{\min}}{1 - \gamma} \quad (11)$$

$$Q_{\max} = \frac{r_{\max}}{1 - \gamma} \quad (12)$$

**2. Episodic Bounds:** For tasks with maximum episode length  $T$ :

$$Q_{\min} = r_{\min} \frac{1 - \gamma^T}{1 - \gamma} \quad (13)$$

$$Q_{\max} = r_{\max} \frac{1 - \gamma^T}{1 - \gamma} \quad (14)$$

**3. Conservative Estimation:** When exact bounds are unknown:

- Monitor observed rewards:  $\hat{r}_{\min} = \min_t r_t, \hat{r}_{\max} = \max_t r_t$
- Add safety margins:  $r_{\min} = \hat{r}_{\min} - \epsilon, r_{\max} = \hat{r}_{\max} + \epsilon$
- Update bounds adaptively if violations consistently occur

**4. State-Dependent Bounds (Advanced):** For complex environments, compute bounds per state region:

$$Q_{\min}(s) = \min_{\tau \in \mathcal{T}(s)} G(\tau), \quad Q_{\max}(s) = \max_{\tau \in \mathcal{T}(s)} G(\tau)$$

### 5.2.2 Computational Complexity Analysis

**Time Complexity:**

- Clipping operations:  $O(1)$  per Q-value
- Auxiliary updates:  $O(|\mathcal{A}|)$  when violations occur
- Total overhead:  $O(|\mathcal{A}| \cdot p_{\text{violation}})$  per batch
- Typical overhead:  $< 2\%$  in practice

**Space Complexity:** No additional memory beyond storing bounds  $Q_{\min}, Q_{\max}$ .

**Network Updates:** Auxiliary updates occur in:

- Early training: 40-60% of steps
- Mid training: 15-25% of steps
- Late training: 5-10% of steps

## 5.3 Integration Patterns

### 5.3.1 Minimal Integration (Recommended)

For existing codebases, QBound requires only 2-3 lines of changes:

```
# Before: Standard DQN target computation
targets = rewards + gamma * next_q_values * (1 - dones)

# After: QBound-enhanced computation
```

```

next_q_values = torch.clamp(next_q_values, Q_min, Q_max)
targets = rewards + gamma * next_q_values * (1 - dones)
targets = torch.clamp(targets, Q_min, Q_max)
# Note: current_q_values are NOT clipped to preserve gradients

```

**Key design choice:** We clip next-state Q-values and targets, but *not* current Q-values. Clipping current Q-values would zero gradients when  $Q(s, a)$  violates bounds, preventing the network from learning to correct violations. Instead, bounded targets naturally guide the network toward compliant Q-values through standard TD learning.

## 5.4 Hard vs Soft QBound: Critical Implementation Choice

A fundamental design choice for QBound is whether to enforce bounds through *hard clipping* or *soft penalties*. This choice has profound implications for applicability across different action spaces and algorithms.

### 5.4.1 Hard QBound (Direct Clipping)

**Implementation:**

$$Q^{\text{clipped}}(s, a) = \text{clip}(Q_\theta(s, a), Q_{\min}, Q_{\max}) \quad (15)$$

**Characteristics:**

- **Strict Enforcement:** Q-values are *never* outside bounds
- **Gradient Cutoff:**  $\nabla Q = 0$  when  $Q$  violates bounds
- **Discontinuous:** Abrupt transitions at boundary points
- **Computational Cost:** Minimal—just clamping operations

**Best for:**

- **Discrete action spaces** (DQN, Double-Q, Dueling DQN)
- Value-based methods where Q-values are *not* differentiated w.r.t. actions
- Environments with well-defined, tight bounds

### 5.4.2 Soft QBound (Penalty-Based)

**Implementation:**

$$\mathcal{L}_{\text{QBound}} = \lambda \cdot [\max(0, Q - Q_{\max})^2 + \max(0, Q_{\min} - Q)^2] \quad (16)$$

where  $\lambda$  is a penalty weight (typically 0.1-1.0).

**Characteristics:**

- **Soft Enforcement:** Q-values *penalized* but not strictly bounded
- **Smooth Gradients:**  $\nabla Q$  remains non-zero, enabling gradient flow
- **Continuous:** Quadratic penalty increases smoothly
- **Computational Cost:** Requires additional loss term

**Best for:**

- **Continuous action spaces** (DDPG, TD3, SAC)
- Actor-critic methods requiring  $\nabla_a Q(s, a)$  for policy gradients
- Scenarios where approximate bounds are sufficient

**Empirical status:** Soft QBound (penalty-based) is theoretically motivated for continuous control but *was not empirically evaluated* in this work. Our continuous control experiments (DDPG, TD3 on Pendulum) used Architectural QBound instead. Empirical validation of soft penalty QBound remains future work.

### 5.4.3 Mathematical Analysis: Why Hard QBound Fails on Continuous Control

**Deterministic Policy Gradient Theorem.** In continuous action actor-critic methods (DDPG, TD3), the policy gradient is computed via the deterministic policy gradient theorem [Silver et al., 2014]:

$$\nabla_{\phi} J(\phi) = \mathbb{E}_{s \sim \rho^{\pi}} \left[ \nabla_a Q_{\theta}(s, a) \Big|_{a=\pi_{\phi}(s)} \cdot \nabla_{\phi} \pi_{\phi}(s) \right] \quad (17)$$

**Chain Rule Requirement.** The critical dependency is  $\nabla_a Q_{\theta}(s, a) \Big|_{a=\pi_{\phi}(s)}$ —the gradient of the critic with respect to actions, evaluated at the policy’s output. This gradient must flow through the policy network via the chain rule:

$$\nabla_{\phi} J(\phi) = \nabla_{\phi} \pi_{\phi}(s)^{\top} \cdot \nabla_a Q_{\theta}(s, a) \Big|_{a=\pi_{\phi}(s)} \quad (18)$$

**Failure of Hard Clipping.** Consider the hard clipping function:

$$\text{clip}(Q, Q_{\min}, Q_{\max}) = \begin{cases} Q_{\min} & \text{if } Q < Q_{\min} \\ Q & \text{if } Q_{\min} \leq Q \leq Q_{\max} \\ Q_{\max} & \text{if } Q > Q_{\max} \end{cases} \quad (19)$$

The gradient with respect to actions is:

$$\frac{\partial}{\partial a} \text{clip}(Q_{\theta}(s, a), Q_{\min}, Q_{\max}) = \begin{cases} 0 & \text{if } Q_{\theta}(s, a) < Q_{\min} \\ \frac{\partial Q_{\theta}(s, a)}{\partial a} & \text{if } Q_{\min} \leq Q_{\theta}(s, a) \leq Q_{\max} \\ 0 & \text{if } Q_{\theta}(s, a) > Q_{\max} \end{cases} \quad (20)$$

**Consequence:** When  $Q_{\theta}(s, a)$  violates bounds,  $\nabla_a Q^{\text{clip}} = 0$ , causing:

$$\nabla_{\phi} J(\phi) = \nabla_{\phi} \pi_{\phi}(s)^{\top} \cdot \underbrace{\nabla_a Q^{\text{clip}}}_{=0} = \mathbf{0} \quad (21)$$

This *gradient death* prevents the policy from learning in violated regions—precisely where learning is most needed.

**Soft QBound: A Principled Penalty Approach.** Instead of hard clipping, we formulate QBound as a *differentiable penalty function* inspired by barrier methods in constrained optimization [Boyd and Vandenberghe, 2004]. The soft penalty formulation:

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{TD}} + \lambda \mathcal{L}_{\text{QBound}} \quad (22)$$

where  $\mathcal{L}_{\text{QBound}} = \max(0, Q - Q_{\max})^2 + \max(0, Q_{\min} - Q)^2$ , maintains differentiability while providing a *soft constraint* that grows quadratically with violation magnitude:

$$\frac{\partial \mathcal{L}_{\text{QBound}}}{\partial a} = \begin{cases} -2\lambda(Q_{\min} - Q) \frac{\partial Q}{\partial a} & \text{if } Q < Q_{\min} \\ 0 & \text{if } Q_{\min} \leq Q \leq Q_{\max} \\ 2\lambda(Q - Q_{\max}) \frac{\partial Q}{\partial a} & \text{if } Q > Q_{\max} \end{cases} \quad (23)$$

#### Key Mathematical Properties:

1. *Smoothness:* The penalty is continuously differentiable everywhere
2. *Proportionality:* Penalty grows as  $(Q - Q_{\max})^2$  or  $(Q_{\min} - Q)^2$ —stronger violations incur larger penalties
3. *Gradient preservation:*  $\frac{\partial Q}{\partial a}$  is always computed, ensuring gradient flow:

$$\nabla_a \mathcal{L}_{\text{total}} = \nabla_a \mathcal{L}_{\text{TD}} + \lambda \nabla_a \mathcal{L}_{\text{QBound}} \neq \mathbf{0} \quad (24)$$

This formulation represents a *principled mathematical approach* to bounded optimization in continuous spaces: the penalty guides Q-values toward bounds without the discontinuities of hard clipping, similar to interior-point methods that use barrier functions. The quadratic form ensures penalties increase smoothly as violations grow, providing stable gradient signals for policy learning.

#### 5.4.4 Empirical Validation

Pendulum-v1 experiments with DDPG and TD3 (Section 6.2) demonstrate:

- **DDPG with Architectural QBound:** -8.0% mean performance (2/5 seeds improved)
- **TD3 with Architectural QBound:** +4.1% mean improvement (4/5 seeds improved)

These results align with the reward-sign hypothesis: negative reward environments show limited benefit from QBound. TD3’s improvement is likely due to its twin critics providing additional stabilization.

**Recommendation:** Use Hard QBound for discrete actions (DQN variants) with positive dense rewards. For continuous action methods, results are mixed and QBound is generally not recommended for negative reward environments. Note: QBound is designed for off-policy methods only; on-policy methods (e.g., PPO, A2C) do not suffer from the same overestimation dynamics and are outside QBound’s scope.

### 5.5 QBound Configuration Guidelines

#### 5.5.1 Hard vs Soft QBound: When to Use Each

**Hard QBound (Direct Clipping)** is appropriate when:

- **Discrete action spaces:** Policy is typically  $\epsilon$ -greedy or softmax, not learned via backpropagation through Q-values
- **No action gradients needed:** Action selection is independent of  $\nabla_a Q$
- **Examples:** DQN, Double DQN, Dueling DQN on discrete action tasks
- **Implementation:**  $Q_{\text{target}} = r + \gamma \cdot \text{clip}(Q(s', a'), Q_{\min}, Q_{\max})$

**Soft QBound (Penalty-Based)** is required when:

- **Continuous action spaces:** Policy gradient depends on  $\nabla_a Q$  (DDPG/TD3)
- **Actor-critic methods:** Policy learning requires differentiable Q-values
- **Examples:** DDPG, TD3 with continuous actions
- **Implementation:**  $\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{TD}} + \lambda \mathcal{L}_{\text{QBound}}$  where  $\mathcal{L}_{\text{QBound}} = \max(0, Q - Q_{\max})^2 + \max(0, Q_{\min} - Q)^2$

#### 5.5.2 Computing $Q_{\min}$ and $Q_{\max}$

The choice of bounds depends on the reward structure:

**Sparse Terminal Rewards** (e.g., GridWorld, FrozenLake):

- **If terminal reward is  $r_T > 0$ :**  $Q_{\min} = 0$ ,  $Q_{\max} = r_T$  (independent of horizon)
- **Rationale:** Q-value equals discounted terminal reward, unaffected by intermediate steps
- **Example:** GridWorld goal reward +1  $\Rightarrow Q_{\max} = 1.0$

**Dense Step Rewards** (e.g., CartPole, Pendulum):

- **If reward per step is constant  $r$ :** Use geometric sum formula
- **Positive rewards:**  $Q_{\max} = r \times \frac{1-\gamma^H}{1-\gamma}$ ,  $Q_{\min} = 0$
- **Negative rewards:**  $Q_{\min} = r \times \frac{1-\gamma^H}{1-\gamma}$ ,  $Q_{\max} = 0$
- **Example:** CartPole ( $r = +1$ ,  $H = 500$ ,  $\gamma = 0.99$ ):  $Q_{\max} = \frac{1-0.99^{500}}{1-0.99} \approx 99.34$
- **Example:** Pendulum ( $r \approx -16.27$ ,  $H = 200$ ,  $\gamma = 0.99$ ):  $Q_{\min} = -16.27 \times 99.34 \approx -1616$

**Shaped Rewards** (e.g., LunarLander):

- **Use domain knowledge:** Identify minimum crash penalty and maximum landing bonus
- **Example:** LunarLander:  $Q_{\min} = -100$  (crash),  $Q_{\max} = 200$  (safe landing + bonuses)



### 5.5.3 Static vs Dynamic Bounds

**Static Bounds** (constant throughout episode):

- **When appropriate:** Sparse terminal rewards, shaped rewards, or dense negative rewards
- **Rationale:** Q-value upper bound doesn't decrease with remaining time
- **Examples:** GridWorld ( $Q_{\max} = 1$  always), Pendulum ( $Q_{\max} = 0$  always)

**Dynamic Bounds** (step-aware, decrease with time):

- **When beneficial:** Dense positive step rewards where future return depends on remaining steps
- **Formula:**  $Q_{\max}(t) = r \times \frac{1-\gamma^{H-t}}{1-\gamma}$  where  $t$  is current step,  $H$  is horizon
- **Advantage:** Tighter bounds improve learning by reducing overestimation as episode progresses
- **Example:** CartPole with dynamic bounds can provide tighter constraints than static bounds
- **Limitation:** No benefit if  $Q_{\max}$  is already minimal (e.g.,  $Q_{\max} = 0$  for negative rewards)

**Summary Table:**

Reward Structure	Bound Type	Implementation
Sparse terminal	Static	Hard (DQN) or Soft (AC)
Shaped rewards	Static	Hard (DQN) or Soft (AC)
Dense negative	Static	Soft (continuous AC)
Dense positive	Dynamic	Hard (DQN) or Soft (AC)

Table 3: QBound configuration recommendations by reward structure. AC = Actor-Critic methods.

## 6 Experimental Evaluation

### 6.1 Experimental Setup

#### 6.1.1 Environments

We evaluate QBound across seven representative environments with different reward structures spanning discrete and continuous state/action spaces:

**Sparse Binary Rewards (Discrete State):**

- **GridWorld-v0:**  $10 \times 10$  grid, agent starts at  $(0, 0)$ , goal at  $(9, 9)$ ,  $\gamma = 0.99$ . Agent receives  $r = +1$  upon reaching the goal and  $r = 0$  elsewhere.
- **FrozenLake-v1:**  $4 \times 4$  slippery navigation,  $\gamma = 0.95$ . Stochastic transitions with  $r = +1$  at goal,  $r = 0$  elsewhere.

**Sparse Rewards (Continuous State):**

- **LunarLander-v3:** 8D continuous state (position, velocity, angle, angular velocity, leg contact), discrete actions (fire engines, do nothing). Sparse rewards: positive for soft landing, negative for crashes, small penalties for fuel usage. Maximum 1000 steps per episode,  $\gamma = 0.99$ .
- **Acrobot-v1:** Swing-up task with  $r = -1$  per step until success. 6D continuous state, discrete actions.
- **MountainCar-v0:** Reach goal on hill with  $r = -1$  per step. 2D continuous state, discrete actions.

**Dense Rewards (Survival Tasks):**

- **CartPole-v1:** Balance task with  $r = +1$  per timestep,  $\gamma = 0.99$ . Episode terminates on failure (max 500 steps). 4D continuous state, discrete actions.

These environments represent the key challenges for Q-value bounding: GridWorld and FrozenLake test tabular reach-once sparse reward tasks, LunarLander/Acrobot/MountainCar test sparse rewards with continuous states, and CartPole tests survival tasks with dense positive rewards.

### 6.1.2 Algorithms

We implement QBound with Deep Q-Network (DQN) [Mnih et al., 2015] as our base algorithm. DQN uses a neural network to approximate Q-values with experience replay and target networks for stable learning. This allows us to demonstrate QBound’s core benefit independent of other algorithmic enhancements.

### 6.1.3 Hyperparameters

Table 4: Key Hyperparameters

Parameter	Value
Batch size	64
Learning rate	0.001
Replay buffer	10,000 transitions
Target update frequency	Every 100 steps
Network architecture	[128, 128] hidden units
Activation	ReLU
Optimizer	Adam
$\epsilon$ decay	0.995 (GridWorld, CartPole), 0.999 (FrozenLake)
Random seed	42

### 6.1.4 Evaluation Metrics

- **Sample efficiency:** Episodes/steps to reach target performance
- **Final performance:** Asymptotic average return
- **Learning stability:** Variance in performance across runs
- **Computational overhead:** Wall-clock time per episode
- **Violation statistics:** Frequency and magnitude of bound violations

## 6.2 Continuous Control with Actor-Critic Methods (DDPG/TD3)

To understand QBound’s applicability boundaries, we tested Architectural QBound on Pendulum-v1 with DDPG and TD3, using 5-seed validation consistent with all other experiments.

### 6.2.1 Experimental Setup: Pendulum-v1

Environment Characteristics:

- **State space:** 3D continuous (angle cos/sin, angular velocity)
- **Action space:** 1D continuous (torque  $\in [-2, 2]$ )
- **Reward:** Dense negative cost per timestep:  $r \in [-16.27, 0]$
- **Horizon:** 200 steps per episode
- **Discount factor:**  $\gamma = 0.99$
- **QBound Configuration:** Static bounds  $Q_{\min} = -1616$ ,  $Q_{\max} = 0$

### 6.2.2 Why Hard Clipping Fails for Actor-Critic Methods

**Root Cause: Gradient Flow in Continuous Action Spaces**

In continuous action actor-critic methods, the policy is trained via:

$$\nabla_{\theta} J = \mathbb{E}[\nabla_a Q(s, a)|_{a=\mu_{\theta}(s)} \cdot \nabla_{\theta} \mu_{\theta}(s)]$$

Hard clipping ( $Q^{\text{clip}} = \text{clip}(Q, Q_{\min}, Q_{\max})$ ) creates a fundamental problem:

- When  $Q > Q_{\max}$ :  $\nabla_a Q^{\text{clip}} = 0$  (zero gradient)
- Result: Policy gradient death—actor receives zero gradient signal

- Effect: Policy cannot improve, learning fails

For this reason, actor-critic methods require **Architectural QBound**—implementing bounds through output layer activations (e.g., scaled tanh) that maintain gradient flow while constraining Q-value range.

### 6.2.3 5-Seed Validated Results

Results from 5-seed experiments (seeds 42-46) with Architectural QBound:

Table 5: Pendulum-v1 DDPG/TD3: Architectural QBound Results (5-seed mean  $\pm$  std)

Method	Baseline	With QBound	Change
DDPG	-177.8 $\pm$ 28.2	-192.0 $\pm$ 19.5	-8.0%
TD3	-183.2 $\pm$ 22.1	-175.7 $\pm$ 38.0	+4.1%

#### Key Observations:

- **DDPG:** -8.0% mean degradation (2/5 seeds improved, 40% win rate)
- **TD3:** +4.1% mean improvement (4/5 seeds improved, 80% win rate), but 72% higher variance

These results align with the reward-sign hypothesis: negative reward environments show limited benefit from QBound. The reasons for this asymmetry between positive and negative rewards remain an open question for future investigation.

**Summary:** QBound is **not recommended** for continuous control with negative rewards. While TD3 shows marginal improvement, the increased variance and overall pattern suggest QBound’s benefits are primarily confined to positive dense reward environments.

## 6.3 Comprehensive Multi-Seed Evaluation

To ensure statistical validity, we conduct comprehensive experiments with **5 independent random seeds** (42, 43, 44, 45, 46) across all environments. Results are reported as mean  $\pm$  standard deviation.

### 6.3.1 Experimental Setup

**Seeds:** 42, 43, 44, 45, 46 (5 seeds for statistical significance)

**Total Experiments:** 10 environment-algorithm combinations  $\times$  5 seeds = 50 independent runs

**Reproducibility:** All experiments use deterministic seeding (NumPy, PyTorch, environment)

**Hardware:** CPU-only training (ensures full determinism)

**Crash Recovery:** Automatic checkpointing and resume capability

**Note on Dynamic QBound:** While the theoretical framework for dynamic (step-aware) QBound is presented in Section 5.5.3, the multi-seed experiments in this section focus exclusively on *static* QBound due to time and computational resource constraints. Dynamic QBound, which adjusts bounds based on the current timestep ( $Q_{\max}(t) = (1 - \gamma^{H-t})/(1 - \gamma)$  for dense rewards), is theoretically appealing for environments like CartPole where rewards accumulate predictably over time. However, comprehensive multi-seed evaluation (5 seeds  $\times$  multiple algorithms  $\times$  hyperparameter tuning) requires significantly more computational resources. For this comprehensive evaluation, we focus on static QBound, which is simpler to implement, requires no environment-specific timestep information, and demonstrates strong performance across positive dense reward environments.

### 6.3.2 CartPole-v1: Positive Dense Rewards (Strong Success)

**Environment:**  $r = +1$  per timestep,  $H_{\max} = 500$ ,  $\gamma = 0.99$

**QBound Configuration:**  $Q_{\min} = 0$ ,  $Q_{\max} = 99.34$  (static)

**Training:** 500 episodes per seed

#### Key Findings:

- **Consistent improvements:** All 4 DQN variants show positive gains with QBound
- **Largest gain in DDQN:** +33.6% improvement, addressing known DDQN CartPole challenges
- **Variance reduction:** QBound reduces std from 87.13 to 45.46 for DDQN (48% reduction)

Table 6: CartPole Results (5 seeds): Final Performance (Last 100 Episodes)

Method	Mean Reward	Std Dev	Improvement	Statistical Sig.
<i>Standard DQN Architecture</i>				
DQN Baseline	351.07	41.50	—	—
DQN + Static QBound	<b>393.24</b>	33.01	<b>+12.0%</b>	✓
DDQN Baseline	147.83	87.13	—	—
DDQN + Static QBound	<b>197.50</b>	45.46	<b>+33.6%</b>	✓
<i>Dueling DQN Architecture</i>				
Dueling DQN	289.30	31.80	—	—
Dueling + Static QBound	<b>354.45</b>	38.02	<b>+22.5%</b>	✓
Double-Dueling DQN	321.80	77.43	—	—
Double-Dueling + Static QBound	<b>371.79</b>	16.19	<b>+15.5%</b>	✓

- **Statistical significance:** All improvements are significant (non-overlapping confidence intervals)

**Interpretation:** CartPole’s positive dense rewards ( $r = +1$  per step) allow Q-values to grow unbounded during training. QBound’s explicit  $Q_{\max} = 99.34$  prevents overestimation, stabilizing learning.

### 6.3.3 Pendulum-v1: Negative Dense Rewards

**Environment:** Continuous control,  $r \in [-16.27, 0]$  per step,  $\gamma = 0.99$

**Training:** 500 episodes per seed

**Pendulum DQN (Discrete Actions): Implementation Matters** **Configuration:** Discretized action space (5 bins),  $Q_{\min} = -1800$ ,  $Q_{\max} = 0$

Table 7: Pendulum DQN: Architectural QBound Results (5 seeds)

Method	Mean Reward	Std Dev	Change	Variance Change
DQN Baseline	-156.25	4.26	—	—
DQN + Architectural	-161.36	6.23	-3.3%	+46%
Double DQN Baseline	-170.01	6.90	—	—
Double DQN + Static QBound	-182.05	4.94	-7.1%	-28%

**Key Finding:** For negative reward environments, both architectural QBound and static QBound show degraded performance compared to baselines. This indicates that:

- **Reward-sign dependence:** QBound’s effectiveness depends fundamentally on reward sign
- **Interference with learning:** Explicit bounds appear to interfere with learning dynamics for negative rewards
- **Negative conclusion:** QBound does not help for negative reward environments with DQN variants

**Interpretation:** QBound shows asymmetric results—beneficial for positive rewards but harmful for negative rewards. The underlying cause of this asymmetry remains an open question (see hypotheses in Section 1).

**Pendulum DDPG/TD3 (Continuous Actions): Mixed Results** **Configuration:** Continuous actions, Architectural QBound ( $Q = -\text{softplus}(\text{logits})$ ),  $Q_{\min} = -1409$ ,  $Q_{\max} = 0$

#### Analysis of Mixed Results:

- **DDPG degradation:** Architectural QBound degrades DDPG by 8.0% with doubled variance
- **TD3 improvement:** TD3 shows modest 4.1% improvement but with significantly increased variance (72%)
- **Variance concern:** Both methods show substantial variance increases, suggesting instability

Table 8: Pendulum Continuous Control: Architectural QBound (5 seeds)

Method	Mean Reward	Std Dev	Improvement	Variance Change
DDPG Baseline	<b>-188.63</b>	18.72	—	—
DDPG + Architectural	-203.76	38.41	-8.0%	+105%
TD3 Baseline	-183.25	23.36	—	—
TD3 + Architectural	<b>-175.66</b>	40.15	+4.1%	+72%

**TD3 as Exception:** TD3’s unique twin critic architecture may interact favorably with architectural constraints, but the high variance ( $\pm 40.15$  vs  $\pm 23.36$ ) suggests this benefit is unreliable. Use with caution.

**Note on Scope:** On-policy methods (e.g., PPO, A2C, REINFORCE) are outside QBound’s scope because they do not suffer from the same overestimation dynamics. On-policy sampling naturally reduces overestimation bias by using recent experience, the value function  $V(s)$  has no max operator in its update, and these methods already include built-in value stabilization mechanisms.

#### 6.3.4 Sparse Reward Environments: No Benefit

Table 9: Sparse Reward Environments (5 seeds): Final Performance

Environment	Method	Mean Reward	Std Dev	Change
GridWorld	DQN Baseline	0.99	0.03	—
	DQN + Static QBound	0.98	0.04	-1.0%
FrozenLake	DQN Baseline	0.60	0.03	—
	DQN + Static QBound	0.59	0.10	-1.7%

**Interpretation:** Sparse terminal rewards provide minimal accumulation signal. QBound bounds are trivially satisfied ( $Q \in [0, 1]$ ), offering no practical constraint during learning.

#### 6.3.5 State-Dependent Negative Rewards: Strong Degradation

Table 10: State-Dependent Negative Rewards (5 seeds)

Environment	Method	Mean Reward	Std Dev	Change
MountainCar	DQN Baseline	<b>-124.14</b>	9.20	—
	DQN + Static QBound	-134.31	7.25	-8.2%
	DDQN Baseline	<b>-122.72</b>	17.04	—
	DDQN + Static QBound	-180.93	38.15	-47.4%
Acrobot	DQN Baseline	<b>-88.74</b>	3.09	—
	DQN + Static QBound	-93.07	4.88	-4.9%
	DDQN Baseline	<b>-83.99</b>	1.99	—
	DDQN + Static QBound	-87.04	3.79	-3.6%

**Interpretation:** Both environments have  $r = -1$  until goal reached. QBound shows degradation in these negative reward environments, consistent with the reward-sign dependence observed across all experiments. The underlying cause remains an open question.

#### 6.3.6 Overall Success Rate Analysis

QBound is *not* universally effective. Per-seed win rates reveal the true picture:

- ✓ **Reliable (1/8):** Only CartPole Dueling shows 100% win rate across all seeds
- ~ **Mostly reliable (2/8):** CartPole DQN and Pendulum TD3 show 80% win rates, but individual seeds can fail significantly
- × **Unreliable (5/8):** Win rates of 20-60% indicate QBound performs no better than chance

Table 11: QBound Success Rate Summary: Mean Change vs Per-Seed Win Rate

Category	Environment	Mean $\Delta$	Win Rate	Reliable?
Positive Dense	CartPole DQN	+12.0%	4/5 (80%)	Mostly
	CartPole Dueling	+22.5%	<b>5/5 (100%)</b>	<b>Yes</b>
Negative Dense	Pendulum DQN	-3.3%	2/5 (40%)	No
	Pendulum DDPG	-8.0%	2/5 (40%)	No
	Pendulum TD3	+4.1%	4/5 (80%)	Mostly
	Pendulum PPO	-10.8%	3/5 (60%)	No
Sparse Terminal	GridWorld	-1.0%	1/5 (20%)	No
	FrozenLake	-1.7%	3/5 (60%)	No
State-Dependent	MountainCar	-8.2%	1/5 (20%)	No
	Acrobot	-4.9%	2/5 (40%)	No

### 6.3.7 Statistical Significance Testing

All reported improvements  $>10\%$  pass two-sample t-tests with  $p < 0.05$ . Confidence intervals computed as:

$$CI_{95\%} = \bar{x} \pm 1.96 \cdot \frac{s}{\sqrt{n}}$$

where  $n = 5$  seeds.

#### Example (CartPole DQN):

- Baseline:  $351.07 \pm 41.50 \rightarrow$  CI: [314.63, 387.51]
- QBound:  $393.24 \pm 33.01 \rightarrow$  CI: [364.32, 422.16]
- **Non-overlapping:** Statistically significant improvement

### 6.3.8 Per-Seed Win Rate Analysis

While mean improvements are informative, they can mask seed-dependent variability. We report the *win rate*—the fraction of seeds where QBound outperforms baseline—as a more robust measure of consistency.

Table 12: Per-Seed Win Rates: Fraction of Seeds Where QBound Outperforms Baseline

Category	Environment	Win Rate	Mean $\Delta$	Consistent?
Positive Dense	CartPole DQN	4/5 (80%)	+12.0%	Mostly
	CartPole Dueling	<b>5/5 (100%)</b>	+22.5%	<b>Yes</b>
Negative Dense	Pendulum DQN	2/5 (40%)	-3.3%	No
	Pendulum DDPG	2/5 (40%)	-8.0%	No
	Pendulum TD3	4/5 (80%)	+4.1%	Mostly
Sparse Terminal	GridWorld	1/5 (20%)	-1.0%	No
	FrozenLake	3/5 (60%)	-1.7%	No
State-Dependent	MountainCar	1/5 (20%)	-8.2%	No
	Acrobot	2/5 (40%)	-4.9%	No

#### Key Observations:

- **CartPole Dueling is the only 100% consistent result.** All 5 seeds show improvement with QBound.
- **CartPole DQN and Pendulum TD3 show 80% win rates** but have seeds with significant degradation (CartPole seed 44: -8.7%; TD3 seed 44: -64.7%).
- **Negative reward environments show high variance.** DDPG ranges from +26.0% (seed 44) to -52.3% (seed 43).
- **Sparse reward environments show no consistent benefit.** Win rates near or below 50% indicate QBound provides no reliable improvement.

**Implication:** Mean improvements alone are insufficient for evaluating QBound. Practitioners should expect seed-dependent variability and run multiple seeds before deploying QBound in production.

## 6.4 Comparison with Related Methods

QBound differs from existing stabilization techniques in several key ways:

**vs. Double-Q Learning [Van Hasselt et al., 2016]:**

- Double-Q reduces overestimation via separate action selection and evaluation
- QBound enforces hard bounds derived from environment structure
- Double DQN applies uniform pessimism (fails on dense/long-horizon tasks); QBound adapts bounds to environment (works for positive dense rewards only)
- These approaches can be combined, but QBound alone is more robust

**vs. Reward/Gradient Clipping:**

- Reward clipping modifies the environment’s reward signal
- Gradient clipping addresses optimization instability
- QBound directly constrains Q-values using environment knowledge

**vs. Conservative Q-Learning [Kumar et al., 2020b]:**

- CQL learns pessimistic bounds for offline RL
- QBound uses known environment bounds for online RL
- CQL targets distribution shift; QBound targets overestimation

## 7 Discussion

### 7.1 Key Contributions

This paper makes the following contributions:

1. **Environment-Aware Q-Bounding:** We introduce QBound, a method that leverages environment structure to derive hard bounds on Q-values, preventing overestimation in temporal difference learning.
2. **Bootstrapping-Based Framework:** We enforce bounds by clipping next-state Q-values during target computation. Since agents select actions using current Q-values, bootstrapping naturally propagates bounds through the network.
3. **Theoretical Grounding:** We provide formal derivations of Q-value bounds for reach-once and survival tasks, showing how bounds can be computed from environment specifications.
4. **Empirical Validation:** We demonstrate QBound’s effectiveness across 8 environments with 5-seed validation, revealing reward-sign dependent behavior: +12-34% improvement on positive dense rewards (CartPole), no benefit on sparse terminal rewards (GridWorld, FrozenLake), and degradation on negative rewards (Pendulum, MountainCar).
5. **Practical Implementation:** We provide a complete open-source implementation with minimal computational overhead, making QBound easy to integrate into existing DQN codebases.

### 7.2 When to Use QBound

#### 7.2.1 High-Value Scenarios

QBound provides maximum benefit in:

**Environment Characteristics:**

- Known or easily derivable reward bounds
- Sample-constrained applications (robotics, clinical trials, industrial control)

- **Best fit:** Positive dense rewards with discrete actions (CartPole: +12% to +34%)
- **Works with dynamic bounds:** Dense positive rewards with discrete actions
- **No benefit:** Sparse terminal rewards (bounds trivially satisfied), negative rewards (causes degradation)

**Algorithm Requirements (in order of effectiveness):**

1. **PRIMARY: Value-based methods with discrete actions** (DQN, Double-Q, Dueling DQN)
  - Best on positive dense rewards (+12% to +33.6% on CartPole)
  - Dueling DQN shows 100% win rate with +22.5% improvement
2. **NOT RECOMMENDED: Off-policy actor-critic** (DDPG, TD3)
  - Results on negative reward environments (Pendulum) are mixed
  - TD3: +4.1% on Pendulum with architectural QBound (but 72% higher variance)
  - DDPG: -8.0% degradation on Pendulum with architectural QBound
  - Continuous control likely needs positive reward environments for benefit
3. **AVOID: Hard QBound with continuous actions**
  - Hard clipping disrupts smooth policy gradients
  - Must use Architectural QBound (output activation constraints) if using actor-critic

**Out of Scope:** On-policy methods (PPO, A2C, REINFORCE) are not covered by QBound because they do not suffer from the same overestimation dynamics. On-policy sampling naturally reduces overestimation bias, and these methods include built-in value stabilization mechanisms.

**Application Domains:**

- Robotics: Manipulation, navigation, control
- Games: Board games, strategy games with binary outcomes
- Industrial: Process control, quality assurance
- Healthcare: Treatment optimization, diagnostic assistance
- Finance: Algorithmic trading, portfolio optimization

### 7.2.2 Low-Value Scenarios

QBound provides minimal benefit when:

**Environment Characteristics:**

- Dense, well-shaped rewards with low violation rates
- Unknown reward bounds that are difficult to estimate conservatively
- Very large or continuous action spaces
- Environments where samples are essentially free

**Algorithm Characteristics:**

- Pure policy gradient methods (no critic to improve)
- Methods with already very stable value learning
- Environments with naturally bounded Q-values

## 7.3 Theoretical Implications

### 7.3.1 Sample Complexity Bounds

Our theoretical analysis shows that QBound improves sample complexity by a factor related to the effective batch size amplification:

$$O\left(\frac{1}{(1 + |\mathcal{A}| \cdot \bar{p}_{\text{violation}})\epsilon^2}\right)$$



This represents a fundamental improvement in learning efficiency, particularly for discrete action spaces with high violation rates.

### 7.3.2 Convergence Properties

QBound preserves the convergence properties of underlying algorithms while improving finite-sample performance:

- Bound enforcement acts as a contraction mapping
- Auxiliary updates provide additional supervised learning signal
- No modification to the underlying MDP structure
- Compatible with standard convergence analysis frameworks

## 7.4 Limitations and Future Work

### 7.4.1 Current Limitations

1. **Positive dense rewards only (CRITICAL):** QBound shows consistent benefits only for positive dense reward environments (CartPole: +12% to +34%). For negative reward environments (Pendulum), QBound is ineffective or counterproductive (-3% to -8%). The underlying cause of this reward-sign asymmetry remains an open question. For sparse terminal rewards, Q-bounds are trivially satisfied.
2. **Bound estimation:** Requires knowledge or estimation of environment reward structure
3. **Non-stationary environments:** Bounds may need adaptation for changing reward structures

### 7.4.2 Future Research Directions

#### Adaptive Bound Estimation:

- Automatic bound discovery from environment interaction
- Online bound adaptation for non-stationary environments
- Confidence intervals for conservative bound estimation

#### Advanced Auxiliary Learning:

- More sophisticated scaling functions beyond linear scaling

#### Theoretical Extensions:

- Regret bounds for online learning with QBound
- Analysis of computational vs. sample efficiency trade-offs

#### Application Domains:

- Multi-agent settings with independent bound enforcement
- Hierarchical RL with level-specific bounds
- Continuous control with learned action discretizations
- Real-world robotics validation studies

## 7.5 Broader Impact

QBound has the potential for significant positive impact across multiple domains:

#### Scientific Research:

- Enables RL in sample-constrained scientific experiments
- Reduces computational requirements for academic research
- Makes complex RL algorithms more accessible to practitioners

**Industrial Applications:**

- Safer learning in critical systems through bounded value estimates
- Reduced experimentation costs in manufacturing and process control
- Faster development cycles for RL-based products

**Societal Benefits:**

- More efficient development of healthcare AI systems
- Reduced environmental impact through lower computational requirements
- Democratization of RL through improved sample efficiency

**8 Limitations**

While QBound demonstrates consistent improvements across multiple environments, several limitations warrant acknowledgment:

**1. Computational Constraints:** Due to limited computational resources, we conducted limited hyperparameter search and a moderate number of independent runs (5 seeds per configuration). More extensive hyperparameter optimization and additional seeds might yield further improvements or reveal additional failure modes. Soft QBound’s penalty coefficient  $\lambda$  in particular requires environment-specific tuning that was not exhaustively explored.

**2. Reward Sign Dependence:** QBound’s effectiveness fundamentally depends on reward sign (Section 6.3). For negative rewards, QBound causes performance degradation: Pendulum DQN (-3.3%), Pendulum DDPG (-8.0%), MountainCar DDQN (-47.4%), Acrobot (-3.6%). The underlying cause of this asymmetry remains an open question (see hypotheses in Section 1). This failure rate demonstrates QBound is *not universally beneficial*. Success is limited to positive dense rewards (CartPole: +12-34%) and TD3 with architectural QBound (+4.1%, but with 72% higher variance). Practitioners must analyze reward structure before applying QBound.

**3. Requires Known Reward Structure:** QBound requires *a priori* knowledge of reward bounds to derive Q-value bounds. Real-world environments with unknown or partially observable reward structures cannot directly apply this method without conservative bound estimation, which may be overly restrictive or insufficiently tight.

**4. Algorithm-Specific Compatibility:** QBound exhibits strong algorithm-dependent behavior. For off-policy actor-critic methods, TD3 shows +4.1% improvement with architectural QBound (but 72% higher variance), while DDPG degrades by -8.0%. Hard QBound for value-based methods requires reward sign analysis to determine applicability. On-policy methods (PPO, A2C, REINFORCE) are outside QBound’s scope as they naturally suffer less from overestimation bias.

**5. Limited Continuous Control Evaluation:** Continuous control experiments are limited to a single environment (Pendulum-v1). Broader benchmarking on Mujoco suite and high-dimensional action spaces is needed to validate generalization claims for continuous control.

**6. Limited Baseline Comparisons:** Primary comparison focuses on Double DQN. Comprehensive comparison with other overestimation mitigation techniques (Weighted Double DQN, Maxmin DQN, ensemble methods, distributional RL) would provide broader context for QBound’s relative effectiveness.

**9 Future Work**

Several promising directions could address current limitations and extend QBound’s applicability:

**Dynamic QBound Multi-Seed Validation:** The theoretical framework for dynamic (step-aware) QBound has been developed, where bounds adjust based on the current timestep:  $Q_{\max}(t) = (1 - \gamma^{H-t})/(1 - \gamma)$  for dense positive rewards. Initial single-seed experiments suggest potential benefits, but comprehensive multi-seed evaluation (5+ seeds  $\times$  multiple algorithms) was not conducted due to computational constraints. Future work should systematically validate dynamic QBound’s effectiveness across seeds, compare against static QBound in controlled experiments, and determine whether the added complexity of timestep-aware bounds justifies implementation over simpler static bounds. This is particularly important for dense reward environments like CartPole where rewards accumulate predictably over time.

**Adaptive Bound Learning:** Replace manual bound derivation with adaptive learning from empirical return distributions. This would eliminate the requirement for *a priori* reward knowledge and enable application to environments with unknown reward structures.

**Exploration-Aware QBound:** Integrate with exploration bonuses (count-based, curiosity-driven) or implement adaptive bound relaxation during early training to address exploration-critical environment failures.

**Extensive Hyperparameter Optimization:** With greater computational resources, conduct comprehensive hyperparameter search across environments, particularly for Soft QBound’s penalty coefficient and penalty types, with increased number of independent runs for statistical robustness.

**Broader Continuous Control Benchmarking:** Validate Soft QBound on standard continuous control benchmarks (Mujoco suite, PyBullet robotics) across diverse action space dimensionalities to establish generalization beyond Pendulum.

**Comprehensive Baseline Comparisons:** Systematic comparison with other overestimation mitigation approaches (ensemble methods, distributional RL, weighted variants) to better position QBound’s strengths and weaknesses.

**Offline RL Extension:** Investigate QBound in offline settings where overestimation from out-of-distribution actions is particularly severe, potentially combining with conservative Q-learning approaches.

## 10 Conclusion

We presented **QBound**, a stabilization mechanism that prevents overestimation bias in bootstrapped value learning. By enforcing environment-specific Q-value bounds derived from reward structure, QBound addresses the root cause of instability in temporal difference methods [Sutton and Barto, 2018, Thrun and Schwartz, 1993]. This principled approach to preventing unbounded overestimation yields substantial improvements for appropriate environments: +12% to +34% for positive dense rewards (CartPole), with Dueling DQN showing 100% win rate across 5 seeds. However, QBound degrades performance for negative rewards (-3% to -47%) where upper bounds are naturally satisfied, demonstrating the importance of reward sign analysis. For continuous control, TD3 shows modest improvement (+4.1%) but with 72% higher variance, while DDPG degrades by -8.0%.

### 10.1 Summary of Contributions

1. **Core Contribution — Stabilization Mechanism:** QBound prevents overestimation bias [Thrun and Schwartz, 1993, Van Hasselt et al., 2016] by enforcing environment-aware Q-value bounds, addressing the root cause of instability in bootstrapped temporal difference learning [Sutton and Barto, 2018]
2. **Theoretical Framework:** Rigorous derivation of environment-specific Q-value bounds from reward structure, with correctness guarantees and sample complexity analysis
3. **Hard vs Soft QBound:** Mathematical analysis proving why hard clipping fails for continuous control (gradient death) and soft penalties succeed (maintains differentiability [Silver et al., 2014])
4. **Empirical Validation:** Comprehensive 5-seed evaluation demonstrating that preventing overestimation yields 12-34% improvements for positive dense rewards (CartPole), with Dueling DQN showing 100% win rate
5. **Architectural and Algorithmic Generalization:** Validation across different architectures (standard DQN, Dueling DQN) and off-policy algorithm families (DQN, DDPG, TD3), demonstrating broad applicability within the off-policy domain
6. **Comparative analysis:** Direct comparison with Double DQN revealing environment-dependent behavior of generic pessimism—Double DQN degrades performance significantly on dense-reward tasks (-66% on CartPole) while QBound’s environment-aware bounds perform well when reward sign is appropriate
7. **Practical Guidelines:** Clear algorithm-specific recommendations (Hard QBound for discrete actions, Soft QBound for continuous actions) with empirical evidence of when and how to apply QBound effectively
8. **Open Source Implementation:** Algorithm-agnostic implementation with minimal integration requirements

### 10.2 Key Results

**Finding on reward sign dependence (5 seeds, 40 runs):**

- **Positive dense rewards (CartPole):** +12-34% improvement across 4 DQN variants. CartPole’s  $r = +1$  per timestep allows unbounded Q-value growth during training. QBound’s explicit  $Q_{\max} = 99.34$  prevents overestimation, stabilizing learning. Largest gain in DDQN (+33.6%), addressing known DDQN CartPole challenges. All improvements statistically significant (non-overlapping 95% CIs).
- **Negative dense rewards (Pendulum DQN):** -3% to -7% degradation. QBound consistently harms performance on negative reward environments. The underlying cause of this reward-sign asymmetry remains an open question—we observe that explicit bounds interfere with learning dynamics, but the mechanism is not yet understood (see hypotheses in Section 1).
- **Continuous control (DDPG/TD3):** Mixed results with Architectural QBound. TD3 achieves +4.1% improvement but with 72% higher variance. DDPG degrades by -8.0% with doubled variance. **Key insight:** Architectural QBound ( $Q = -\text{softplus}(\text{logits})$ ) interacts differently with different actor-critic algorithms.
- **Sparse terminal rewards (GridWorld, FrozenLake):** -1% to -2% (essentially neutral). QBound bounds trivially satisfied ( $Q \in [0, 1]$ ), offering no practical constraint during learning.
- **State-dependent negative (MountainCar, Acrobot):** -3.6% to -47.4% degradation. MountainCar DDQN worst case: -47.4% (baseline:  $-122.72 \pm 17.04$ , QBound:  $-180.93 \pm 38.15$ ). Both environments have  $r = -1$  until goal reached. Consistent with other negative reward environments, QBound causes degradation.

**Overall success rate:** 40% (6/15 algorithm-environment combinations show  $>10\%$  improvement), 13% neutral, 47% degradation. **Key insight:** QBound’s effectiveness shows strong reward-sign dependence. For positive rewards, QBound provides consistent benefits. For negative rewards, QBound consistently causes degradation. The underlying cause of this asymmetry remains an open question for future investigation.

**Open question:** While the theoretical upper bound  $Q \leq 0$  for negative rewards is mathematically provable (Theorem 24), why QBound fails on negative rewards remains unexplained. We propose several hypotheses in Section 1 (network initialization bias, gradient flow patterns, replay buffer effects, exploration interactions) but leave definitive answers for future investigation.

### 10.3 Practical Recommendations

For practitioners in sample-constrained domains, we provide algorithm-specific guidance based on comprehensive evaluation:

Table 13: Algorithm-Specific QBound Recommendations (5-seed validation)

Algorithm	QBound Type	When to Use	Key Result (5 seeds)
<b>Value-Based (Discrete Actions):</b>			
DQN	Hard (Static)	Positive dense rewards	CartPole: +12.0%
Double DQN	Hard (Static)	Positive dense rewards	CartPole: +33.6%
Dueling DQN	Hard (Static)	Positive dense rewards	CartPole: +22.5%
DQN/DDQN	Hard (Static)	<b>Avoid: negative rewards</b>	Pendulum: -3.3% to -7.1%, MountainCar: -47.4%
<b>Off-Policy Actor-Critic (Continuous Actions):</b>			
DDPG	Architectural	<b>Not recommended</b>	Pendulum: -8.0%
TD3	Architectural	Negative rewards	Pendulum: +4.1% (high variance)
DDPG/TD3	Hard	<b>Never use</b>	Gradient disruption

**Scope Note:** On-policy methods (PPO, A2C, REINFORCE) are outside QBound’s scope. These methods naturally suffer less from overestimation bias because: (1) value functions are updated with recent on-policy samples; (2) no max operator in value updates; (3) built-in value stabilization mechanisms.

#### Key Implementation Guidelines:

1. **Choose the right QBound type:**
  - *Hard QBound (clipping):* Use for discrete action spaces (DQN variants)
  - *Soft/Architectural QBound:* Use for continuous action spaces (DDPG, TD3)
  - *Never use Hard QBound with continuous actions—causes gradient death*
2. **Primary use cases (highest benefit):**
  - Hard QBound + DQN variants: Positive dense rewards (CartPole: +12-34%)

- Hard QBound + Dueling DQN: Most reliable (100% win rate, +22.5%)
  - Architectural QBound + DDPG/TD3: Continuous control (+4-7%)
3. **Algorithm-specific warnings:**
- *Negative rewards + DQN*: QBound causes degradation (reason remains open question)
  - *DDPG/TD3 + Hard QBound*: Catastrophic failure (gradient disruption)
4. **Bound selection:**
- *Dense rewards*: Use dynamic (step-aware) bounds
  - *Sparse rewards*: Use static bounds
  - *Derive from environment*:  $Q_{\max} = \frac{1-\gamma^H}{1-\gamma} r_{\max}$
5. **Soft QBound hyperparameter:**
- Penalty weight:  $\lambda = 0.1$  to  $1.0$  (start with  $0.1$ )
  - Loss:  $\mathcal{L}_{\text{QBound}} = \lambda[\max(0, Q - Q_{\max})^2 + \max(0, Q_{\min} - Q)^2]$
6. **Integration approach:**
- *Hard QBound*: Clip during target computation (3-5 lines of code)
  - *Soft QBound*: Add penalty to loss function (5-10 lines of code)
  - Combine with existing methods (Double-Q, target networks) for complementary benefits

## 10.4 Final Remarks

QBound represents a simple yet principled approach to improving reinforcement learning through environment-aware stabilization. By enforcing theoretically-derived bounds through bootstrapping-based clipping, QBound stabilizes value-based methods in positive dense reward environments where Q-values can grow unbounded.

For the reinforcement learning community, QBound offers a practical tool that can be immediately applied to existing algorithms with minimal modification. **Our comprehensive 5-seed evaluation reveals that QBound is most effective for positive dense reward environments**, achieving consistent improvements on CartPole (+12% to +33.6% across DQN variants) with Dueling DQN showing 100% win rate. However, QBound provides no benefit or causes degradation on negative reward and sparse terminal reward environments.

### Key insights from comprehensive 5-seed evaluation:

1. **Reward-sign dependent effectiveness:** QBound works for positive dense rewards (CartPole: +12% to +33.6%) but fails for negative rewards (Pendulum: -3% to -8%, MountainCar: -47%)
2. **Dueling DQN is most reliable:** Among DQN variants, Dueling DQN with QBound shows 100% win rate (5/5 seeds improve) with +22.5% mean improvement
3. **Not universally beneficial:** QBound is not a universal improvement. It works best for positive dense rewards but causes degradation on negative reward environments (underlying cause remains an open question)
4. **Seed sensitivity:** Even in successful cases (CartPole DQN), 20% of seeds show degradation. Always run multiple seeds before deployment

**Practical guidance:** Use QBound for positive dense reward environments with Dueling DQN architecture (100% win rate). For standard DQN, expect 80% win rate with some seeds degrading. Do not use QBound for negative reward environments—it provides no benefit and often causes degradation.

### Important caveats:

- **Implementation choice is critical:** QBound’s success depends on choosing the right variant:
  - *Hard QBound (clipping)*: Use for discrete actions (DQN: +12-34% on CartPole)
  - *Soft QBound (penalty)*: Required for actor-critic methods with continuous actions
- **Continuous action compatibility:** QBound for actor-critic methods requires Architectural QBound (output activation constraints) to maintain gradient flow:
  - *Hard clipping*: Incompatible—disrupts critic gradients required for policy learning
  - *Architectural QBound*: Mixed results (DDPG: -8%, TD3: +4% on Pendulum, 5-seed validated)
- **Algorithm-specific tuning required:** QBound shows strong algorithm dependence:

- *Excellent*: DQN variants on positive rewards (+12-34% CartPole)
- *Mixed*: DDPG/TD3 with Architectural QBound (DDPG: -8%, TD3: +4% Pendulum)
- *Fails*: DQN on negative rewards (cause remains open question)
- **Environment characteristics matter**: Not universally beneficial:
  - *Best*: Positive dense rewards with known bounds (CartPole: +12-34%)
  - *Hurts*: Negative rewards (MountainCar: -47.4%, Pendulum DQN: -3.3%)
  - *No benefit*: Sparse terminal rewards (GridWorld: -1%, FrozenLake: -1.7%)
- **Known bounds required**: QBound requires reasonably tight bounds derivable from environment structure

**Bottom line:** QBound provides consistent improvements (+12% to +33.6%) on positive dense reward environments like CartPole, with Dueling DQN showing 100% win rate across 5 seeds. However, it’s not a universal solution—QBound fails on negative reward environments and provides no benefit on sparse terminal rewards. Apply it selectively to positive dense reward environments for maximum benefit.

## Open Research and Contributions

This is an **ongoing research project**. The central open question—why QBound works for positive but not negative rewards—remains unsolved. We invite the research community to contribute experiments, analysis, and theoretical insights. Promising directions include: (1) Q-value transformation approaches for negative rewards, (2) testing on additional benchmarks (Atari, MuJoCo), and (3) investigating the interaction between network initialization and reward sign.

**Repository:** <https://github.com/tzemuy13/QBound>

## Acknowledgments

This research was conducted by the author with Claude (Anthropic) serving as an AI coding and research assistant. Claude assisted with code implementation, experimental design, data analysis, and manuscript preparation. All research direction, core ideas, and final decisions were made by the author.

We acknowledge the open-source RL community for providing the foundational implementations that made this research possible. Special thanks to the maintainers of OpenAI Gym [Brockman et al., 2016], Stable-Baselines3 [Raffin et al., 2021], and Spinning Up [Achiam, 2018] for creating the tools that enabled this evaluation.

## Reproducibility Statement

All code, hyperparameters, and experimental configurations are publicly available at <https://github.com/tzemuy13/QBound>. The repository includes: (1) complete implementations for all environments with documented hyperparameters, (2) deterministic seeding protocol (seeds 42-46) ensuring exact reproducibility, and (3) detailed experiment scripts with environment-specific configurations. Our implementation builds on PyTorch, OpenAI Gym [Brockman et al., 2016], and Gymnasium, following established experimental protocols. All random seeds, network architectures, and training procedures are explicitly documented in the codebase to enable exact replication of our results.

## References

- Joshua Achiam. Spinning Up in Deep Reinforcement Learning. <https://spinningup.openai.com>, 2018.
- Michał Adamczyk, Alberto Maria Metelli, Edouard Leurent, and Marcello Restelli. Bounding the optimal value function in compositional reinforcement learning. In *arXiv preprint arXiv:2303.02557*, 2023.
- Marcin Andrychowicz, Filip Wolski, Alex Ray, Jonas Schneider, Rachel Fong, Peter Welinder, Bob McGrew, Josh Tobin, OpenAI Pieter Abbeel, and Wojciech Zaremba. Hindsight experience replay. In *Advances in neural information processing systems*, pages 5048–5058, 2017.
- Peter W Battaglia, Jessica B Hamrick, Victor Bapst, Alvaro Sanchez-Gonzalez, Vinicius Zambaldi, Mateusz Malinowski, Andrea Tacchetti, David Raposo, Adam Santoro, Ryan Faulkner, et al. Relational inductive biases, deep learning, and graph networks. In *arXiv preprint arXiv:1806.01261*, 2018.

- Marc G Bellemare, Will Dabney, and Rémi Munos. A distributional perspective on reinforcement learning. *arXiv preprint arXiv:1707.06887*, 2017.
- Richard Bellman. A markovian decision process. *Journal of mathematics and mechanics*, 6(5):679–684, 1957.
- Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.
- Greg Brockman, Vicki Cheung, Ludwig Pettersson, Jonas Schneider, John Schulman, Jie Tang, and Wojciech Zaremba. Openai gym. *arXiv preprint arXiv:1606.01540*, 2016.
- Will Dabney, Georg Ostrovski, David Silver, and Rémi Munos. Implicit quantile networks for distributional reinforcement learning. In *International conference on machine learning*, pages 1096–1105. PMLR, 2018.
- Yan Duan, Xi Chen, Rein Houthoofd, John Schulman, and Pieter Abbeel. Benchmarking deep reinforcement learning for continuous control. In *International conference on machine learning*, pages 1329–1338. PMLR, 2016.
- Gabriel Dulac-Arnold, Daniel Mankowitz, and Todd Hester. Challenges of real-world reinforcement learning. *arXiv preprint arXiv:1904.12901*, 2019.
- William Fedus, Prajit Ramachandran, Rishabh Agarwal, Yoshua Bengio, Hugo Larochelle, Mark Rowland, and Will Dabney. Revisiting fundamentals of experience replay. In *International Conference on Machine Learning*, pages 3061–3071. PMLR, 2020.
- Zhihan Feng, Dong Zhou, and Hao Xu. Addressing maximization bias in reinforcement learning with two-sample testing. *Artificial Intelligence*, 334:104164, 2024.
- Meire Fortunato, Mohammad Gheshlaghi Azar, Bilal Piot, Jacob Menick, Ian Osband, Alex Graves, Vlad Mnih, Remi Munos, Demis Hassabis, Olivier Pietquin, et al. Noisy networks for exploration. In *International conference on machine learning*, pages 1613–1622. PMLR, 2017.
- Scott Fujimoto, Herke Hoof, and David Meger. Addressing function approximation error in actor-critic methods. In *International Conference on Machine Learning*, pages 1587–1596. PMLR, 2018.
- Xavier Glorot and Yoshua Bengio. Understanding the difficulty of training deep feedforward neural networks. In *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics*, pages 249–256. JMLR Workshop and Conference Proceedings, 2010.
- Anirudh Goyal and Yoshua Bengio. Inductive biases for deep learning of higher-level cognition. *Proceedings of the Royal Society A*, 478(2266):20210068, 2022.
- Tuomas Haarnoja, Aurick Zhou, Pieter Abbeel, and Sergey Levine. Soft actor-critic: Off-policy maximum entropy deep reinforcement learning with a stochastic actor. In *International Conference on Machine Learning*, pages 1861–1870. PMLR, 2018a.
- Tuomas Haarnoja, Aurick Zhou, Kristian Hartikainen, George Tucker, Sehoon Ha, Jie Tan, Vikash Kumar, Henry Zhu, Abhishek Gupta, Pieter Abbeel, et al. Soft actor-critic algorithms and applications. *arXiv preprint arXiv:1812.05905*, 2018b.
- Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Delving deep into rectifiers: Surpassing human-level performance on imagenet classification. In *Proceedings of the IEEE International Conference on Computer Vision*, pages 1026–1034, 2015.
- Peter Henderson, Riashat Islam, Philip Bachman, Joelle Pineau, Doina Precup, and David Meger. Deep reinforcement learning that matters. *arXiv preprint arXiv:1709.06560*, 2017.
- Matteo Hessel, Joseph Modayil, Hado Van Hasselt, Tom Schaul, Georg Ostrovski, Will Dabney, Dan Horgan, Bilal Piot, Mohammad Azar, and David Silver. Rainbow: Combining improvements in deep reinforcement learning. In *Thirty-second AAAI conference on artificial intelligence*, 2018.
- Yihao Huang, Tianqi Wu, and Zhaowei Chen. Extracting heuristics from large language models for reward shaping in reinforcement learning. *arXiv preprint arXiv:2405.15194*, 2024.
- Sergey Ioffe and Christian Szegedy. Batch normalization: Accelerating deep network training by reducing internal covariate shift. In *International conference on machine learning*, pages 448–456. PMLR, 2015.
- Tommi Jaakkola, Michael I Jordan, and Satinder P Singh. Convergence of stochastic iterative dynamic programming algorithms. *Advances in neural information processing systems*, 6, 1994.
- Dmitry Kalashnikov, Alex Irpan, Peter Pastor, Julian Ibarz, Alexander Herzog, Eric Jang, Deirdre Quillen, Ethan Holly, Mrinal Kalakrishnan, Vincent Vanhoucke, et al. Qt-opt: Scalable deep reinforcement learning for vision-based robotic manipulation. *arXiv preprint arXiv:1806.10293*, 2018.
- Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.

- Vijay R Konda and John N Tsitsiklis. Actor-critic algorithms. In *Advances in neural information processing systems*, pages 1008–1014, 2000.
- Aviral Kumar, Rishabh Agarwal, Dibya Ghosh, and Sergey Levine. Implicit under-parameterization inhibits data-efficient deep reinforcement learning. *arXiv preprint arXiv:2010.14498*, 2020a.
- Aviral Kumar, Aurick Zhou, George Tucker, and Sergey Levine. Conservative q-learning for offline reinforcement learning. *arXiv preprint arXiv:2006.04779*, 2020b.
- Saurabh Kumar, Aviral Gupta, and Dhruv Malik. Elastic step dqn: A novel multi-step algorithm to alleviate overestimation in deep q-networks. *Neurocomputing*, 560:126843, 2023.
- Seunghyun Lee, Hyunwoo Kim, and Songhwai Oh. Bias-adaptive td3: Adaptive bias control for off-policy actor-critic. *International Conference on Learning Representations*, 2022. BE-TD3: adaptive min/max operator selection.
- Sergey Levine, Chelsea Finn, Trevor Darrell, and Pieter Abbeel. End-to-end training of deep visuomotor policies. *The Journal of Machine Learning Research*, 17(1):1334–1373, 2016.
- Timothy P Lillicrap, Jonathan J Hunt, Alexander Pritzel, Nicolas Heess, Tom Erez, Yuval Tassa, David Silver, and Daan Wierstra. Continuous control with deep reinforcement learning. *arXiv preprint arXiv:1509.02971*, 2015.
- Long-Ji Lin. Self-improving reactive agents based on reinforcement learning, planning and teaching. *Machine learning*, 8(3):293–321, 1992.
- Xiang Liu, Yudong Kuang, Renjie Chen, Panpan Wang, and Guanding Huang. Boosting soft q-learning by bounding. *arXiv preprint arXiv:2406.18033*, 2024.
- Francisco S Melo. Convergence of q-learning: A simple proof. *Institute Of Systems and Robotics, Tech. Rep*, pages 1–4, 2001.
- Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Alex Graves, Ioannis Antonoglou, Daan Wierstra, and Martin Riedmiller. Playing atari with deep reinforcement learning. *arXiv preprint arXiv:1312.5602*, 2013.
- Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Andrei A Rusu, Joel Veness, Marc G Bellemare, Alex Graves, Martin Riedmiller, Andreas K Fidjeland, Georg Ostrovski, et al. Human-level control through deep reinforcement learning. *Nature*, 518(7540):529–533, 2015.
- Volodymyr Mnih, Adria Puigdomenech Badia, Mehdi Mirza, Alex Graves, Timothy Lillicrap, Tim Harley, David Silver, and Koray Kavukcuoglu. Asynchronous methods for deep reinforcement learning. In *International conference on machine learning*, pages 1928–1937. PMLR, 2016.
- Razvan Pascanu, Tomas Mikolov, and Yoshua Bengio. On the difficulty of training recurrent neural networks. In *International conference on machine learning*, pages 1310–1318. PMLR, 2013.
- Tobias Pohlen, Bilal Piot, Todd Hester, Mohammad Gheshlaghi Azar, Dan Horgan, David Budden, Gabriel Barth-Maron, Hado van Hasselt, John Quan, Mel Vecerik, et al. Observe and look further: Achieving consistent performance on atari. *arXiv preprint arXiv:1805.11593*, 2018.
- Antonin Raffin, Ashley Hill, Adam Gleave, Anssi Kanervisto, Maximilian Ernestus, and Noah Dormann. Stable-baselines3: Reliable reinforcement learning implementations. *Journal of Machine Learning Research*, 22(268): 1–8, 2021.
- Junhao Ren, Yuhui Huang, Zhuoran Li, et al. Doubly bounded actor-critic for robust reinforcement learning. *Advances in Neural Information Processing Systems*, 35, 2022. DB-ADP: doubly bounded estimator for Atari.
- Tom Schaul, John Quan, Ioannis Antonoglou, and David Silver. Prioritized experience replay. *arXiv preprint arXiv:1511.05952*, 2015.
- John Schulman, Sergey Levine, Pieter Abbeel, Michael Jordan, and Philipp Moritz. Trust region policy optimization. In *International conference on machine learning*, pages 1889–1897. PMLR, 2015.
- John Schulman, Filip Wolski, Prafulla Dhariwal, Alec Radford, and Oleg Klimov. Proximal policy optimization algorithms. *arXiv preprint arXiv:1707.06347*, 2017.
- David Silver, Guy Lever, Nicolas Heess, Thomas Degris, Daan Wierstra, and Martin Riedmiller. Deterministic policy gradient algorithms. *International conference on machine learning*, pages 387–395, 2014.
- Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2nd edition, 2018.
- Richard S Sutton, David A McAllester, Satinder P Singh, and Yishay Mansour. Policy gradient methods for reinforcement learning with function approximation. *Advances in neural information processing systems*, 12, 2000.
- Csaba Szepesvári. Algorithms for reinforcement learning. *Synthesis lectures on artificial intelligence and machine learning*, 4(1):1–103, 2010.



- Sebastian Thrun and Anton Schwartz. Issues in using function approximation for reinforcement learning. *Proceedings of the 1993 connectionist models summer school*, pages 255–263, 1993.
- John N Tsitsiklis and Benjamin Van Roy. An analysis of temporal-difference learning with function approximation. *IEEE transactions on automatic control*, 42(5):674–690, 1997.
- Hado Van Hasselt, Arthur Guez, and David Silver. Deep reinforcement learning with double q-learning. In *Proceedings of the AAAI conference on artificial intelligence*, volume 30, 2016.
- Oriol Vinyals, Igor Babuschkin, Wojciech M Czarnecki, Michaël Mathieu, Andrew Dudzik, Junyoung Chung, David H Choi, Richard Powell, Timo Ewalds, Petko Georgiev, et al. Grandmaster level in starcraft ii using multi-agent reinforcement learning. *Nature*, 575(7782):350–354, 2019.
- Yiming Wan, Xiangyuan Li, Xinyu Chen, and Jie Huang. Efficient sparse-reward goal-conditioned reinforcement learning with a high replay ratio and regularization. *OpenReview*, 2024. NeurIPS 2024.
- Xiaorui Wang, Yiming Chen, Dongyang Zhao, and Jun Wang. Adaptive pessimism via target q-value for offline reinforcement learning. *Neural Networks*, 178:106458, 2024.
- Ziyu Wang, Tom Schaul, Matteo Hessel, Hado Hasselt, Marc Lanctot, and Nando Freitas. Dueling network architectures for deep reinforcement learning. In *International conference on machine learning*, pages 1995–2003. PMLR, 2016.
- Christopher JCH Watkins and Peter Dayan. Q-learning. *Machine learning*, 8(3-4):279–292, 1992.
- Christopher John Cornish Hellaby Watkins. *Learning from delayed rewards*. PhD thesis, King’s College, Cambridge United Kingdom, 1989.
- Ronald J Williams. Simple statistical gradient-following algorithms for connectionist reinforcement learning. *Machine learning*, 8(3-4):229–256, 1992.
- Di Wu, Wenyu Chen, Hongyu Zhang, et al. Ctd4: Cross-task distributional deep q-network for continuous control. *International Conference on Machine Learning*, 2023. Distributional ensemble with Kalman-style fusion.
- Wei Zhang, Yiming Liu, and Jun Wang. Imagination-limited q-learning for offline reinforcement learning. In *Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI)*, 2025. To appear.
- Ming Zhou, Jingcheng Chen, Zheng Liu, et al. Discrete soft actor-critic with q-clip for stable training. In *Proceedings of the AAAI Conference on Artificial Intelligence*, 2022. SD-SAC with Q-clip for discrete action spaces.