

An H_∞ Controller for a Lane Keeping System

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Abstract. For a Lane Keeping System there are several strategies. But , as we will see , the presence of an exogenous input(the desired yaw rate $\dot{\psi}_{des}$), a simple static state feedback is not sufficient to maintain the vehicle in the lane with the correct orientation. In this article, we are going to derive for the process a series of Linear Matrix Inequalities Conditions (LMIs) , such that the gain K of the controller can be obtained as the solution of a constrained convex optimization problem. In particular , the gain has been selected to minimize a particular induced gain of the system the H_∞ norm of the transfer function between the yaw angle error and the disturbance.

I. Introduction

Lane departures are the number one cause of fatal accidents. Reports by the National Highway Transportation Safety Administration (NHTSA) state that a large percentage of annual accidents can be attributed to unintended lane departures.

With the aim to reduce the annually accidents due to lane departures , has been proposed and designed three different types of controller to maintain the vehicle in the current lane without departures : a (LDWS) Lane Departure Warning System , a (LKS) Lane Keeping System , a (YSCS) Yaw Stability Control System.

The first of the three systems stated before , is a system that monitors the vehicle's position with respect to the lane and provides a warning signal if the vehicle is about to leave the lane.

An example of LDW system is the AutoVue LDW system by Iteris. The AutoVue system consists of an integrated camera , an onboard computer and a proprietary software for lane recognition. The camera unit tracks the visible lane markings and feeds the information to the onboard computer. Using AI techniques like image recognition predict when the vehicle begins to drift towards an unintended lane departure.

While a LDW system limits its work to emit an acoustic warning signal , a LKS Lane Keeping System , controls the steering angle $\delta(t)$ to keep the vehicle in its lane and also follow the lane with the correct orientation as it curves around. Over the last ten years , several research groups at universities have developed and demonstrated different types of lane keeping systems. This kind of systems are also under development by several automotive manufacturers including Nissan. A lane keeping system called simply LKS , which has recently been introduced in Japan on Nissan , offers automatic steering in parallel with the driver steering input.

The last of this kind of control systems is the YSCS , the Yaw Stability Control System. This system prevents vehicles from spinning and drifting. Such stability control systems are also often referred to as yaw control systems or electronic stability control systems.

II. Mathematical Models

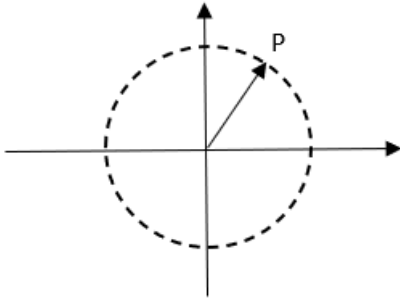
In this article we are going to present different mathematical models for vehicle that are travelling on a circular trajectory. In particular for the design of a static state feedback controller ,

we will focus on state space representations whose state space variables represent respectively the position and the yaw orientation of the vehicle, and the second state space model will describe the dynamics of the errors between the center of the lane and CG of the vehicle , and the current orientation of the vehicle respect to the desired orientation.

To understand better some concepts , let's review some notions about the circular motion of a point of mass.

A.1 Circular Trajectory Motion

For semplicity and without loss of generality , let's consider a point of mass $m = 1 \text{ Kg}$, that is traveling on a circular trajectory , as shown in the following figure



The trajectory done by the point of mass P can be represented as

$$r(t) = r e^{j\theta(t)} \quad (1.1)$$

with r , the radius of curvature and $\theta(t)$ the angle that the position vector forms with the Real axis.

By taking the time derivative of the precedent expression we obtain

$$\dot{r}(t) = r \dot{\theta}(t) e^{j(\theta(t) + \frac{\pi}{2})}$$

Let $v(t) = r \dot{\theta}(t)$, and remember that in Newton mechanics this quantity represents the tangential velocity of the body.

Taking the second derivative ,

$$\ddot{r}(t) = r \ddot{\theta}(t) e^{j(\theta(t) + \frac{\pi}{2})} + r \dot{\theta}(t)^2 e^{j(\theta(t) + \pi)}$$

and note that the acceleration is composed by two terms. The first , that represent the tangential acceleration and the second

$$\alpha_c = r \dot{\theta}(t)^2 = \frac{v(t)^2}{r}$$

which represent the centripetal component of the acceleration.

To better understand the pages that follow , look at this well-known reformulation of Newton Second Law , the Inertia Principle of d'Alembert.

By applying the NSL in the radial direction , we obtain that

$$F_{ext} = \frac{v(t)^2}{r}$$

in particular can be rewritten as ,

$$F_{ext} - F_{in} = 0$$

where $F_{in} = \frac{v(t)^2}{r}$.

As shown, the d'Alembert Principles states that in every time the body opposes to the variation of the direction of motion, it always want to proceed in a straight line.

A.2 Bicycle – Lateral Dynamics

In the precedent discussion we have seen that the acceleration of a body traveling on a circular trajectory is always composed by the sum of two quantities , the tangential component and the centripetal component.

By definition

$$\alpha_c = \frac{v(t)^2}{r}$$

, for a vehicle that travels with low speed this quantity can be neglected , so the tires don't need to develop lateral forces to counteract the inertia of the body. In this situation the wheels roll without slip angle $a_f = a_r = 0$.

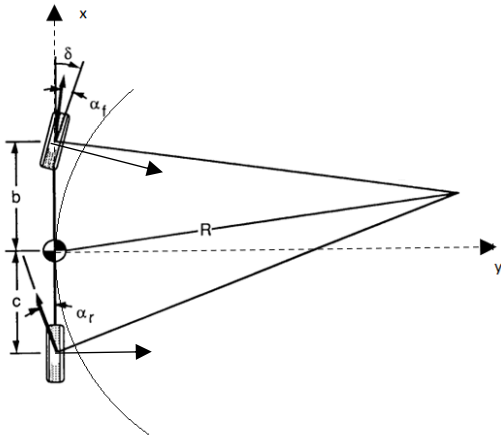
By assuming small angles , the Ackerman geometry gives us the correct orientation for

the right and left wheel

$$\delta_{out} = \frac{L}{R + \frac{t}{2}} \quad \delta_{in} = \frac{L}{R - \frac{t}{2}}$$

where L is the wheelbase distance, R radius of curvature and t is the rear axle length.

For a vehicle at low speed, the radius of curvature is greater than $R \gg L$, so, $\delta_{out} \approx \delta_{in}$ and the vehicle can be approximated with the following bicycle model



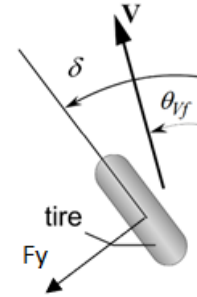
In particular note that the local reference frame as been selected such that the longitudinal V_x velocity is the tangential velocity of the body.

But when the velocity cannot be neglected, the tires must develop lateral forces, also known "cornering forces" to counteract the inertia of the vehicle. As consequence the direction of travel is always different from the direction of the driving wheel; the velocity vector forms an angle with the wheel's plane, the slip angle α .

For small slip angle the cornering forces are proportional to the angle itself (empirical result) and in particular equals to

$$F_y = C_\alpha \alpha$$

where C_α = "cornering stiffness". The slip angle, with reference to the following figure, is defined as



$$\alpha_f = \delta - \theta_{vf}.$$

Assuming small angle approximations, and by applying the Newton Second Law in the radial direction, we obtain

$$m \left(\ddot{y} + \frac{V_x^2}{R} \right) = F_{yf} + F_{yr} = C_{af} \alpha_f + C_{ar} \alpha_r$$

Of course the vehicle can move along the radial direction, so the acceleration is composed by the sum of two terms. In particular by assuming $V_x = c$ the acceleration is the sum of the centripetal acceleration which modify continuously the direction of the velocity vector and the acceleration along the radial direction. So the position of the CF of the vehicle can differ from the center of the lane.

Remember that $V_x = R \dot{\psi}$ and by replacing the relative expressions for the slip angles we have that

$$m (\ddot{y} + \dot{\psi} V_x) = C_{af} (\delta - \theta_{vf}) - C_{ar} \theta_{vr}$$

and by the Newton Second Law for rotating components we can write

$$I_z \ddot{\psi} = b C_{af} (\delta - \theta_{vf}) + c (C_{ar} \theta_{vr})$$

By remember that in a rigid body from any couple of point the distance is constant, we have

$$V_p = V_{CG} + \dot{\psi} \times b$$

In particular by the definition of vector product

$$V_p = V_{CG} + \dot{\psi} \times b =$$

$$= [V_x; V_y; 0]^T + \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \dot{\psi} \\ b & 0 & 0 \end{pmatrix}$$

and so , for the front wheel velocity , the following expressions is right

$$\begin{aligned} V_{Bx} &= V_x \\ V_{By} &= V_y + b\dot{\psi} \end{aligned}$$

where (V_x, V_y) are the velocity of the vehicle at the center of gravity. By simple geometric consideration , we obtaine the angle of the velocity vector that forms with respect to the longitudinal vehicle axis

$$\tan \theta_{vf} = \frac{(V_y + b\dot{\psi})}{V_x}$$

and assuming small angles ,

$$\theta_{vf} \approx \frac{(V_y + b\dot{\psi})}{V_x}$$

In the same manner for the rear wheel ,

$$\theta_{vr} \approx \frac{(V_y - c\dot{\psi})}{V_x}$$

By replacing $l_f = b$, $l_r = c$ and substituting in the preceding equations , we obtaine the following state space model for the lateral vehicle dynamics

$$\frac{d}{dt} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2C_{af} + 2C_{ar}}{mV_x} & 0 & -V_x - \frac{2C_{af}l_f - 2C_{ar}l_r}{mV_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2l_f C_{af} - 2l_r C_{ar}}{I_z V_x} & 0 & -\frac{2l_f^2 C_{af} + 2l_r^2 C_{ar}}{I_z V_x} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2C_{af}}{m} \\ 0 \\ \frac{2l_f C_{af}}{I_z} \end{bmatrix} \delta$$

With the aim to design a feedback controller , is convinient to define a mathematical model in terms of errors , such that the problem of define a state controller is translated into a stabilizing one.

A.3 Errors Model

When the objective is to develop a steering control system for automatic lane keeping , it is useful to utilize a dynamic model in which the state variables are in terms of position and orientation error with respect to the road center.

Suppose to consider an ideal situtation in which we have an ideal vehicle that turns around the curve with radius R and longitudinal velocity V_x .

As stated before , the longitudinal axis is fixed such that the velocity V_x is equals to the tangential velocity, so we can define the ideal yaw rate as follows

$$\dot{\psi}_{des} = \frac{V_x}{R}$$

In an ideal situation we assume that the CG of the vehicle is equals at each time instant to the center of the lane. Assuming that $V_x = c$, the acceleration of the vehicle is only the centripetal component

$$\ddot{\psi}_{des} = \dot{\psi}_{des} V_x$$

Now , define the following state variable

$e_1(t)$ is the distance between the CG of the real vehicle and the center of the lane.

and take the second time derivative ,

$$\ddot{e}_1(t) = (\ddot{y} + \dot{\psi} V_x) - \dot{\psi}_{des} V_x = \ddot{y} + V_x(\dot{\psi} - \dot{\psi}_{des})$$

as we can see , at the vehicle is given the possibility

to move along the radial direction , but as we as seen before , this direction coincide with the y vehicle axis.

Introduce the second error variable as

$$e_2(t) = \psi - \psi_{des}$$

the difference between the vehicle orientation and the desired orientation.

Integrating in the time the first error equation , we have ($V_x = c$)

$$\dot{e}_1(t) = \dot{\psi} + V_x(\psi - \psi_{des})$$

by replacing this equation into the NSL for translational and rotational object obtained in the previous section

$$m(\ddot{y} + \dot{\psi}V_x) = C_{af}(\delta - \theta_{vf}) - C_{ar}\theta_{vr}$$

and,

$$I_z\ddot{\psi} = l_f(\delta - \theta_{vf}) + l_r(C_{ar}\theta_{vr})$$

we obtaine the following state space model for the vehicle later dynamics in terms of errors form the CG position and orientaton with respect to the ideal behaviour.

$$\frac{d}{dt} \begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_2 \\ \dot{e}_2 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2C_{af}^\ell + 2C_{ar}^\ell}{mV_x} & \frac{2C_{af}^\ell + 2C_{ar}^\ell}{m} & -\frac{2C_{af}^\ell f + 2C_{ar}^\ell r}{mV_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2C_{af}^\ell f - 2C_{ar}^\ell r}{I_z V_x} & \frac{2C_{af}^\ell f - 2C_{ar}^\ell r}{I_z} & -\frac{2C_{af}^\ell f^2 + 2C_{ar}^\ell r^2}{I_z V_x} \end{bmatrix} \begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_2 \\ \dot{e}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2C_{af}^\ell}{m} \\ 0 \\ \frac{2C_{af}^\ell f}{I_z} \end{bmatrix} \delta + \begin{bmatrix} 0 & -\frac{2C_{af}^\ell f - 2C_{ar}^\ell r}{mV_x} \\ 0 & -\frac{2C_{af}^\ell f^2 + 2C_{ar}^\ell r^2}{I_z V_x} \end{bmatrix} \dot{\psi}_{des}$$

and as in the previous case , the input to this model is the steering angle $\delta(t)$.

As said before with this reformulation of the mathematical description , the problem of design a LKS controller is translated into a stabilizing one.

Note that the matrix in the state space decription are function of the longitudinal velocity , in our case assumed to be constant.

III. Control

By replacing the lateral vehicle dynamics with a state space model in terms of errors , and so by choosing the following state vector

$$x(t) = [e_1(t), \dot{e}_1(t), e_2(t), \dot{e}_2(t)]^T$$

we have discussed how the problem of design a LKS system has been translated into a stabilizing one.

But , as we can see from the previous model, the term

$$\begin{bmatrix} 0 \\ -\frac{2C_{af}^\ell f - 2C_{ar}^\ell r}{mV_x} - V_x \\ 0 \\ -\frac{2C_{af}^\ell f^2 + 2C_{ar}^\ell r^2}{I_z V_x} \end{bmatrix} \dot{\psi}_{des}$$

show how this kind of controller doesn't exist. Is no possible to use only a static state feedback such that the state of the model goes to zero when the time goes to infinity.

For the vehicle in analysis the following parameters has taken

$$m = 1573, I_z = 2873, l_f = 1.1$$

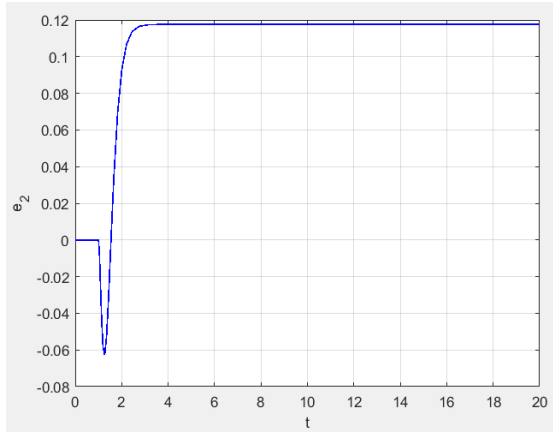
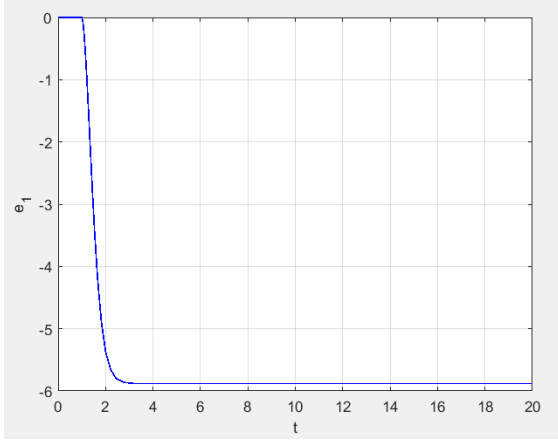
$$l_r = 1.58, C_{af} = 80000, C_{ar} = 80000$$

and suppose that the longitudinal velocity of the vehicle is constant and equals to $V_x = 30$ (m/sec).

Assuming that the radius of curvature of the road is $R = 1000$ (m) , we have for the desired yaw rate the following value

$$\dot{\psi}_{des} = \frac{V_x}{R} = \frac{30}{1000} = 0.03 \left(\frac{rad}{sec} \right) = 1.72 \left(\frac{deg}{sec} \right)$$

To confirm the assertion of the no existence of a stabilizing controller , take a look to these results



Then, a single static action of the form

$$\delta(t) = -K e(t)$$

is not sufficient to drive the state of the vehicle to the origin. But the system is linear , so we can add at the controller a feedforward component with the aim to project a controller such that the state value vanish.

Take now in consideration the following expression for the controller

$$\delta(t) = -K e(t) + \delta_{fw}$$

the closed loop system become

$$\dot{e}(t) = (A - B_1 K) e(t) + B_1 \delta_{fw} + B_2 \dot{\psi}_{des}$$

with $K = [k_1 \ k_2 \ k_3 \ k_4]$ the gain of the controller.

From the final value Theorem the steady state value of the state is given by

$$x_{\infty} = \lim_{s \rightarrow 0} s X(s) = -(A - B_1 K)^{-1} (B_1 \delta_{fw} + B_2 \dot{\psi}_{des})$$

and in particular

$$x_{ss} = \begin{Bmatrix} \frac{\delta_{ff}}{k_1} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -\frac{1}{k_1} \frac{mV_x^2}{R(\ell_f + \ell_r)} \left[\frac{\ell_r}{2C_{\alpha f}} - \frac{\ell_f}{2C_{\alpha r}} + \frac{\ell_f}{2C_{\alpha r}} k_3 \right] - \frac{1}{k_1 R} [\ell_f + \ell_r - \ell_r k_3] \\ 0 \\ \frac{1}{2RC_{\alpha r}(\ell_f + \ell_r)} [-2C_{\alpha r}\ell_f\ell_r - 2C_{\alpha r}\ell_r^2 + \ell_f mV_x^2] \\ 0 \end{Bmatrix}$$

we can see that there is no way to influence the value of the steady state value for the third component of the state , but by choosing

$$\delta_{ff} = \frac{mV_x^2}{RL} \left[\frac{\ell_r}{2C_{\alpha f}} - \frac{\ell_f}{2C_{\alpha r}} + \frac{\ell_f}{2C_{\alpha r}} k_3 \right] + \frac{L}{R} - \frac{\ell_r}{R} k_3$$

the first component of the state vanish. So with this kind of controller , in particular with this kind of feedforward component , we can ensure that the center of gravity CG of the vehicle and the center of the lane are in each time instant equals.

The steady state error for the orientation is given by

$$e_{2\infty} = -\frac{\ell_r}{R} + \frac{\ell_f}{2C_{\alpha r}(\ell_f + \ell_r)} \frac{mV_x^2}{R}$$

With the enthusiasm of introducing the concept of understeering gradient, rewrite the previous expression for the feedforward component as follow

$$\delta_{ff} = \frac{mV_x^2}{RL} \left[\frac{\ell_r}{2C_{\alpha f}} - \frac{\ell_f}{2C_{\alpha r}} + \frac{\ell_f}{2C_{\alpha r}} k_3 \right] + \frac{L}{R} - \frac{\ell_r}{R} k_3$$

and,

$$\delta_{ff} = \frac{L}{R} + K_V a_y - k_3 \left[\frac{\ell_r}{R} - \frac{\ell_f}{2C_{\alpha r}} \frac{mV_x^2}{R\ell} \right]$$

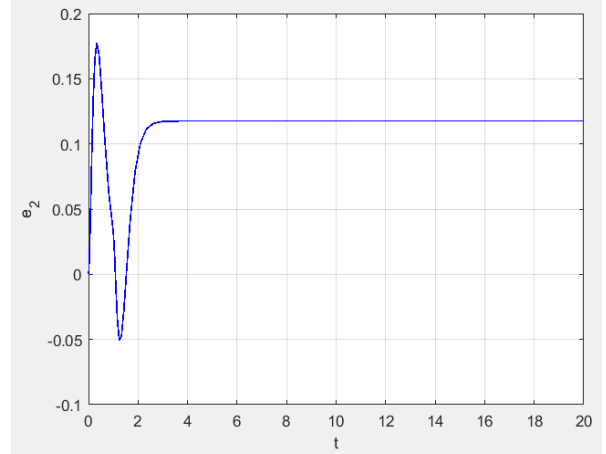
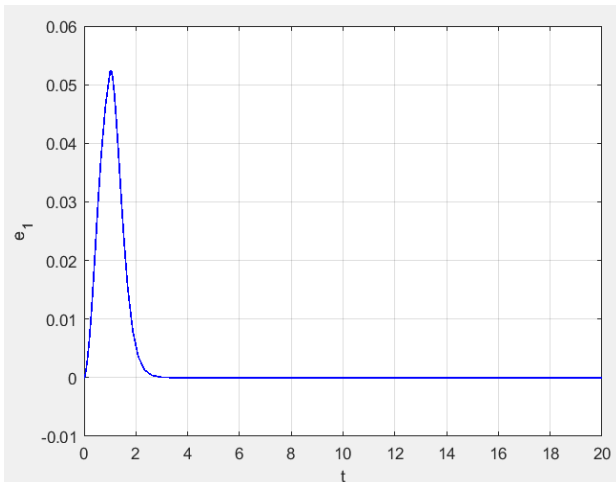
with

$$K_V = \frac{\ell_r m}{2C_{\alpha f}(\ell_f + \ell_r)} - \frac{\ell_f m}{2C_{\alpha r}(\ell_f + \ell_r)}$$

known in literature with the name of understeering gradient. In particular note how in the second member of the feedforward component appears the mean value of the Ackerman geometry and the steady state value of $e_2(t)$

$$\delta_{fw} = \frac{L}{R} + K_V a_y + k_3 e_{2\infty}$$

We can pass now to the simulation, and compare the result with the previous and simple stabilizing controller.



IV. H_∞ Controller

As stated before there is no way to influence the steady state value of the third component of the state.

But we have proved as with the introduction of a feedforward component the steady state of the first component vanish

$$e_{1\infty} = 0$$

Of course, and as stated before, a stabilizing controller is not sufficient to drive the state to the origin, but we can reduce the effect of the presence $\dot{\psi}_{des}$ by define K such that the H_∞ of the transfer function between the disturbance and the objectives is minima.

$$z_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} e(t)$$

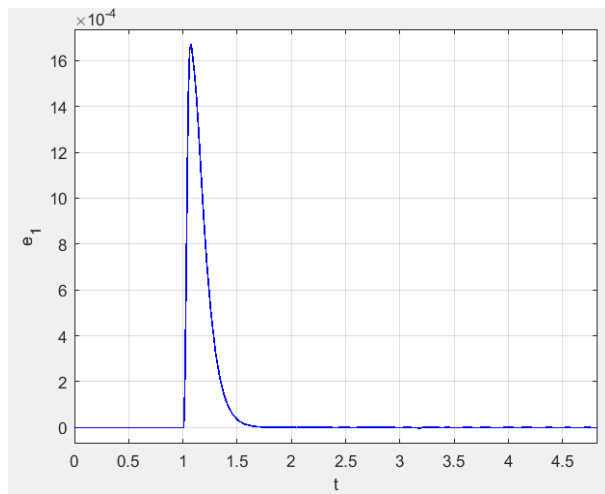
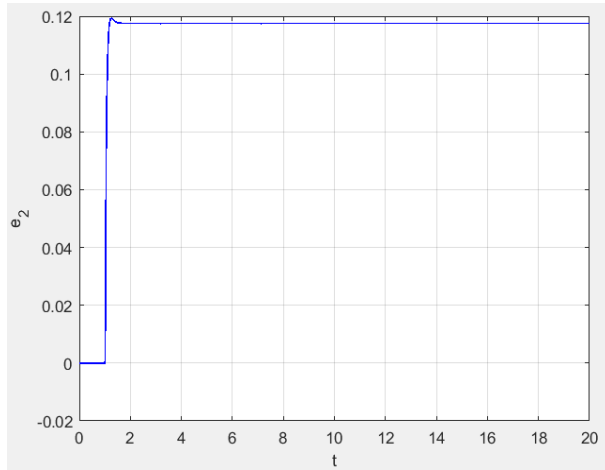
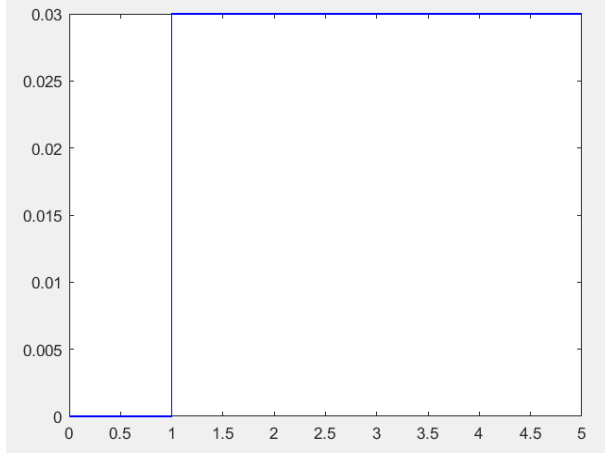
Infact, remember that

$$\|z_\infty\|_2 \leq \|G(s)\|_\infty \|\dot{\psi}_{des}\|_2$$

by reducing this quantity we are reducing the effect of this disturbance to the output of interest.

The gain K of the controller in this case is obtained by solving the following problem

$$\begin{cases} \min \gamma \\ \begin{bmatrix} (AX + B_1W) + (AX + B_1W)^T & B_2 & (CX + D_1W)^T \\ B_2^T & -\gamma I & D_2^T \\ (CX + D_1W) & D_2 & -\gamma I \end{bmatrix} < 0 \\ X > 0 \\ \gamma > 0 \end{cases}$$



V. Matlab Driving Toolbox

In this section we want to verify how our controller works in conditions where the desired yaw rate is not fixed but varying at each time instant.

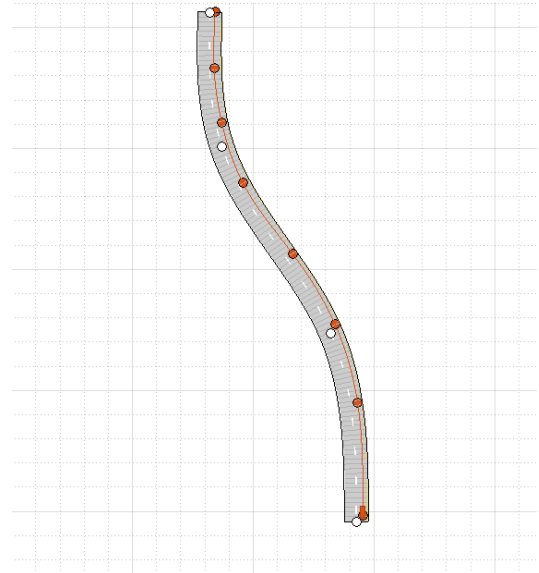
In particular we start by assuming that the desired

$$\dot{\psi}_{des}$$

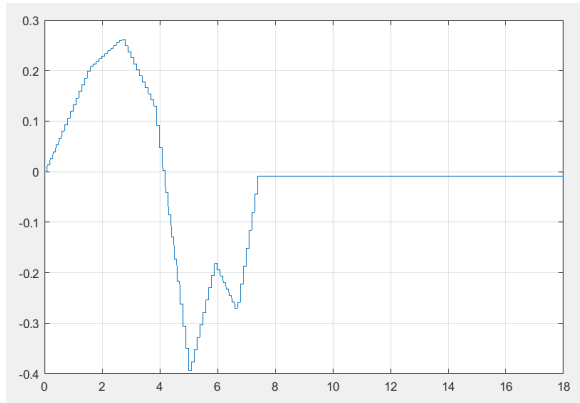
is obtained in an offline phase , and using this behaviour to compare the result after , when the desired yaw rate is calculated online by using measurements from camera sensor situated on the ego vehicle.

A. $\dot{\psi}_{des}$ Evaluated Offline

The path used for the simulations is the following one



As we can see in this first situation the path has been setted , with the aim to obtain the desired yaw rate for the ego vehicle. In particular the desired behaviour of $\dot{\psi}_{des}$ is the following

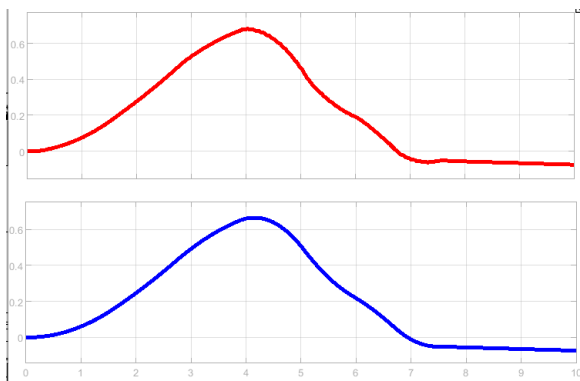


B. $\dot{\psi}_{des}$ Online Phase

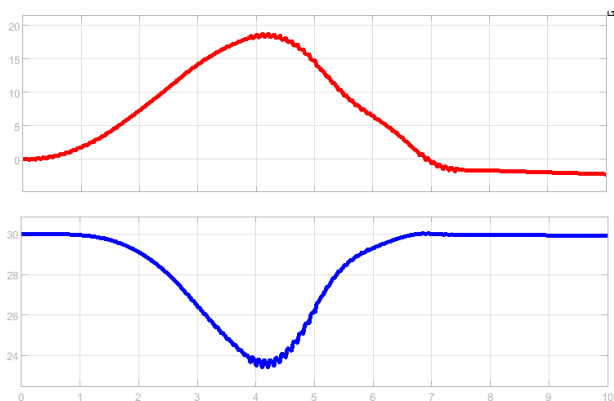
The part of interest of this article is , of course, the definition of an algorithm able to generate the yaw rate reference for the ego vehicle from measurements obtained from a camera.

So , in the “proseguio” we assume that the ego vehicle is equipped with a camera and a software for lane detections.

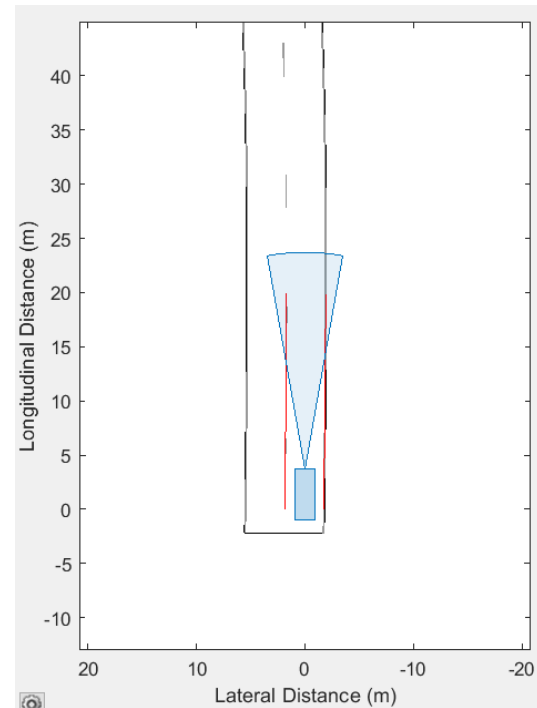
Now we can simulate the ego car and in particular the controller obtained in precedent sections



In this picture we can see the difference between the orientation of the ego vehicle (blue) and the desired value (red).



In this second picture has been reported the behaviour of the cartesian coordinate of the ego vehicle (global coordinates).



Now we are going to present the algorithm that generates the reference for the ego vehicle

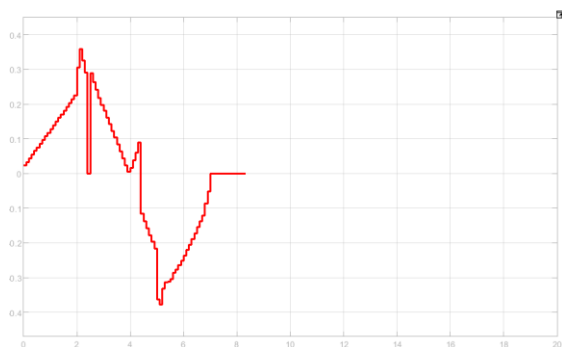
```
function yawRate = getReference(lanes)
% This Function Generates yawRate reference From Lanes
Detections
yawRate = 0;
% Get Lanes Detected
left = lanes.LaneBoundaries(1);
right = lanes.LaneBoundaries(2);
% Controlling Which Lanes are Detected
vLeft = left.Strength;
vRight = right.Strength;
% Control Visibility of Lanes
if (vLeft > 0 && vRight > 0)
% Both Lanes Are Detected
laneDepartureFromLeft = abs(left.LateralOffset);
laneDepartureFromRight = abs(right.LateralOffset);
% Evaluta Distance Ego Vehicle
if (laneDepartureFromLeft > laneDepartureFromRight)
% Much closed to Right Lane
curvature = right.Curvature * right.CurveLength;
% Control The Orientation
if (curvature >= 0)
yawRate = 0.15*(curvature/0.1);
else
yawRate = -0.15*(curvature/0.1);
end
end
```

```

elseif(laneDepartureFromLeft < laneDepartureFromRight)
    % Much closed to Left Lane
    curvature = left.Curvature * left.CurveLength;
    % Control The Orientation
    if(curvature >= 0)
        yawRate = -0.15*(curvature/0.1);
    else
        yawRate = 0.15*(curvature/0.1);
    end
end
elseif (vLeft > 0)
    % Only Left Lane
    leftCurvature = left.Curvature * left.CurveLength; %
meter
    % Evaluate An Approximation to yaw Angle
    curvature = (-1)*leftCurvature;
    % Control Of Curvature
    if(curvature >= 0)
        yawRate = (curvature / 0.1)*-0.15;
    else
        yawRate = (curvature / 0.1)*0.15;
    end
end
elseif (vRight > 0)
    % Only Right Lane
    rightCurvature= right.Curvature * right.CurveLength; %
meter
    % Evaluate An Approximation to yaw Angle
    curvature = rightCurvature;
    % Control Of Curvature
    if(curvature >= 0)
        yawRate = (curvature / 0.1)*0.15;
    else
        yawRate = (curvature / 0.1)*-0.15;
    end
end
end
end

```

In the below figure we can see the results obtained by the application of the precedent algorithm.



Of course in the previous section we have used the reference generated by an ideal situation to determine a criteria to compare the result in this other case. As we can see, the desired yaw rate given by the algorithm is closer to the yaw rate obtained in the offline phase. So the algorithm written is very good for this kind of problem.

Riferimenti

1. Rajesh Rajamani , Vehicle Dynamics & Control, Second Edition , Springer.
2. Thomas D. Gillespie , Fundamentals of Vehicle Dynamics , Revised Edition , SAE International.
3. Guag-Ren Duan Hai-Hua Yu , LMIs In Control Systems , Analysis Design and Applications , CRC Press.
4. E. Fornasini , Appunti di Teoria dei Sistemi , Edizioni Libreria Progetto Padova.
5. L. Magni R. Scattolini , Advanced & Multivariable Control , Pitagora Editrice Bologna.