Dipartimento di Ingegneria Informatica, Modellistica, Elettronica e Sistemistica

Master's Degree in Robotics & Automation Engineering

Vehicle Control

A Two Wheels Self-Balanced Robot

an LMIs approach

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Abstract

In this article we want to show the knowledge acquired during the Vehicle Control Course , Master's Degree in Robotics & Automation Engineering. The mathematical model taken in consideration is a "Two Wheels Self-Balanced Robot" , better known in the scientific community with the name of "Inverse Pendulum on a Cart". We'll start with a well-known reformulation of Newton Second Law , the Lagrange Reformulation; done it , we will focus on the identification of the body that compose the mechanical system and the generalized forces that act on them. Of course the aim in doing so , is the construction of a mathematical model for the physics phenomena , and as we will see , it consists in a set of non linear differential equations. Obtained the mathematical model , we are going to look for the equilibrium points of the system with the objective to construct the linearization in a neighborhood of one of them: the origin. We will procede with the stability analysis of this point of equilibrium by using the Lyapunov Theorem on the Linearized Model , in particular we will conclude the instability of this point. Done that , we are going to study the structured properties of the system : reachability and the osservability , with the aim to construct respectively a stabilizing action and an observer for the state of the system. After introducing a noise acting on the system , the objective is the definition of a gain K that stabilize the system and in particular minimize an induced gain of the process , the H_∞ norm of the transfer function that relates the noise with some state variable , for us the objective. Remember that some performance specifications can be translated into the dimension of a particular signals of interest. What kind of norm has to be used , of course , depends on the situation where we are. The part of interest of this presentation is to see the difference between the real process and the behaviour approximate by the mathematical model. Remember that the mathematical model is only an approximation of the real behaviour of the system ; the presence of unmodeled dynamics and the not perfect knowledge of some parameters drive us to work with a particular type of mathematical model : the uncertain systems.

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Mathematical Model

The construction of the mathematical model for a physical phenomena consists on the identification of different parts that compose the system , and trying to describe these parts by using the classical Mechanics Laws.

Let's start our discussion by presenting a well-known reformulation of Newton Second Law , the Lagrange Reformulation. In particular in this part of the presentation we want to derive it in a very simple way.

- ## Lagrange Reformulation of NSL

The Newton Second Law presented to the students in a classical mechanics course

$$\sum \vec{F} \quad = \quad M\,\vec{a}$$

can be reformulated as follow

$$\frac{\partial}{\partial t}\left(\frac{\partial K}{\partial \dot{q_k}}\right)-\frac{\partial K}{\partial q_k} \;=\; \sum_{i=1}^{n}\left(F_{ix}\,\frac{\partial f_i}{\partial q_k}+F_{iy}\,\frac{\partial g_i}{\partial q_k}+F_{iz}\frac{\partial h_i}{\partial q_k}\right)\;\; k=1{,}2\ldots r$$

with r , n respectively the number of generalized coordinates the number of parts that make up the system.

To better undestard the pages that follow , we need to proof this formula.

Proof.

Suppose that our system is composed by n parts each of mass m_i and cartesian coordinates (x_i, y_i, z_i) that describe the position of each part in the space.

These coordinates can be written as a function of another set of coordianates

$$x_i = f_i(q_1(t), q_2(t), \ldots \;\; , q_r(t), t)$$

$$y_i = g_i(q_1(t), q_2(t), \ldots \;\; , q_r(t), t)$$

$$z_i = h_i(q_1(t), q_2(t), \ldots \;\; , q_r(t), t)$$

this new set of coordinates is known with the name of $generalized\ coordinates$. Let $(F_{xi}\ , F_{yi}, F_{zi})$ the components of the total force acting on the i-th body; by appling the NSL we can write

$$F_{ix} = m_i\ a_x$$

$$F_{iy} = m_i\ a_y$$

$$F_{iz} = m_i\ a_z$$

Now , multiple both members of the precedent expressions respectively by

$$\frac{\partial f_i}{\partial q_k}, \frac{\partial g_i}{\partial q_k}, \frac{\partial h_i}{\partial q_k}$$

where q_k is the k-th generalized coordinate

$$F_{ix}\ \frac{\partial f_i}{\partial q_k} = m_i \ddot{x}_\iota\ \frac{\partial f_i}{\partial q_k}$$

$$F_{iy} \frac{\partial g_i}{\partial q_k} = m_i \ddot{y}_\iota\ \frac{\partial g_i}{\partial q_k}$$

$$F_{iz}\ \frac{\partial h_i}{\partial q_k} = m_i\ \ddot{z}_\iota\ \frac{\partial h_i}{\partial q_k}$$

the quantities on the first member are the well-known $generalized\ forces$.

By adding both members and by adding with respect to all parts that form the system , we have

$$\sum_{i=1}^{n}\left(m_i\ \ddot{x}_\iota \frac{\partial f_i}{\partial q_k} + m_i\ \ddot{y}_\iota\ \frac{\partial g_i}{\partial q_k} + m_i\ \ddot{z}_\iota\ \frac{\partial h_i}{\partial q_k}\right) = \quad \sum_{i=1}^{n}\left(F_{ix}\ \frac{\partial f_i}{\partial q_k}\ +\ F_{iy} \frac{\partial g_i}{\partial q_k} +\ F_{iz}\ \frac{\partial h_i}{\partial q_k}\right)$$

in particular , by some analytic considerations

$$\frac{\partial x_i}{\partial t} = \frac{\partial}{\partial t}\ f_i(q_1(t), q_2(t), \ldots\ \ , q_r(t), t) = \sum_{i=1}^{r} \frac{\partial f_i}{\partial q_i}\ \dot{q}_\iota\ + \frac{\partial f_i}{\partial t}$$

take now the derivative of the precedent expression with respet to the k-th coordinate

$$\frac{\partial \dot{x}_\iota}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left(\sum_{i=1}^{r} \frac{\partial f_i}{\partial q_i}\ \dot{q}_\iota\ + \frac{\partial f_i}{\partial t}\right) = \frac{\partial f_i}{\partial q_k}$$

and we obtain that

$$\ddot{x}_\iota \frac{\partial f_i}{q_k} = \ddot{x}_\iota \frac{\partial \dot{x}_\iota}{\partial \dot{q}_k} = \frac{\partial}{\partial t}\left(\dot{x}_\iota \frac{\partial \dot{x}_\iota}{\partial \dot{q}_k}\right) - \dot{x}_\iota \frac{\partial}{\partial t}\left(\frac{\partial \dot{x}_\iota}{\partial \dot{q}_k}\right)$$

but,

$$\dot{x}_\iota \frac{\partial}{\partial t}\left(\frac{\partial \dot{x}_\iota}{\partial \dot{q}_k}\right) = \dot{x}_\iota \frac{\partial}{\partial t}\left(\frac{\partial f_i}{\partial q_k}\right) = \dot{x}_\iota \left(\sum_{i=1}^{r} \frac{\partial^2 f_i}{\partial q_i \partial q_k}\ \dot{q}_\iota + \frac{\partial^2 f_i}{\partial t\, \partial q_k}\right)$$

by assuming that $f, g, h\ \in C^\infty$ (is sufficient C^2) the Schwartz theorem tell us that

$$\dot{x}_\iota \frac{\partial}{\partial t}\left(\frac{\partial \dot{x}_\iota}{\partial \dot{q}_k}\right) = \dot{x}_\iota \frac{\partial}{\partial t}\left(\frac{\partial f_i}{\partial q_k}\right) = \dot{x}_\iota \frac{\partial}{\partial q_k}\left(\frac{\partial f_i}{\partial t}\right) = \dot{x}_\iota\ \frac{\partial \dot{x}_\iota}{\partial q_k}$$

and so ,

$$\ddot{x}_\iota \frac{\partial f_i}{q_k} = \ddot{x}_\iota \frac{\partial \dot{x}_\iota}{\partial \dot{q}_k} = \frac{\partial}{\partial t}\left(\dot{x}_\iota \frac{\partial \dot{x}_\iota}{\partial \dot{q}_k}\right) - \dot{x}_\iota \frac{\partial}{\partial t}\left(\frac{\partial \dot{x}_\iota}{\partial \dot{q}_k}\right) = \frac{\partial}{\partial t}\left(\dot{x}_\iota \frac{\partial \dot{x}_\iota}{\partial \dot{q}_k}\right) - \dot{x}_\iota\ \frac{\partial \dot{x}_\iota}{\partial q_k}$$

in particular by the derivation of the composite function ,

$$\ddot{x}_\iota \frac{\partial \dot{x}_\iota}{\partial \dot{q}_k} = \frac{\partial}{\partial t}\left(\dot{x}_\iota \frac{\partial \dot{x}_\iota}{\partial \dot{q}_k}\right) - \dot{x}_\iota\ \frac{\partial \dot{x}_\iota}{\partial q_k} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{x}_\iota^{\ 2}\right)\right] - \frac{\partial}{q_k}\left(\frac{1}{2}\dot{x}_\iota^{\ 2}\right) = \ddot{x}_\iota \frac{\partial f_i}{q_k}$$

replacing all these quantities in the start equation ,

$$\sum_{i=1}^{n}\left(m_i\, \ddot{x}_\iota \frac{\partial f_i}{\partial q_k} + m_i\, \ddot{y}_\iota\ \frac{\partial g_i}{\partial q_k} + m_i\, \ddot{z}_\iota\ \frac{\partial h_i}{\partial q_k}\right) = \frac{\partial}{\partial t}\sum_{i=1}^{n}\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}m_i\dot{x}_\iota^{\ 2} + \frac{1}{2}m_i\dot{y}_\iota^{\ 2} + \frac{1}{2}m_i\dot{z}_\iota^{\ 2}\right) +$$

$$-\sum_{i=1}^{n}\frac{\partial}{\partial q_k}\left(\frac{1}{2}m_i\dot{x}_i^{\ 2} + \frac{1}{2}m_i\dot{y}_i^{\ 2} + \frac{1}{2}m_i\dot{z}_i^{\ 2}\right) = \sum_{i=1}^{n}\left(F_{ix}\ \frac{\partial f_i}{\partial q_k}\ +\ F_{iy}\frac{\partial g_i}{\partial q_k} +\ F_{iz}\ \frac{\partial h_i}{\partial q_k}\right)$$

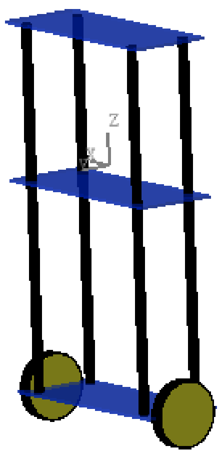
said $K = \sum_{i=1}^{n}(\ \frac{1}{2}m_i\dot{x}_i^{\ 2} + \frac{1}{2}m_i\dot{y}_i^{\ 2} + \frac{1}{2}m_i\dot{z}_i^{\ 2}\ )$ the kinetics energy of the system ,

$$\frac{\partial}{\partial t}\left(\frac{\partial K}{\partial \dot{q}_k}\right)-\frac{\partial K}{\partial q_k}=\sum_{i=1}^{n}\left(F_{ix}\,\frac{\partial f_i}{\partial q_k}+F_{iy}\,\frac{\partial g_i}{\partial q_k}+F_{iz}\,\frac{\partial h_i}{\partial q_k}\right)\quad k=1,2\ldots r$$

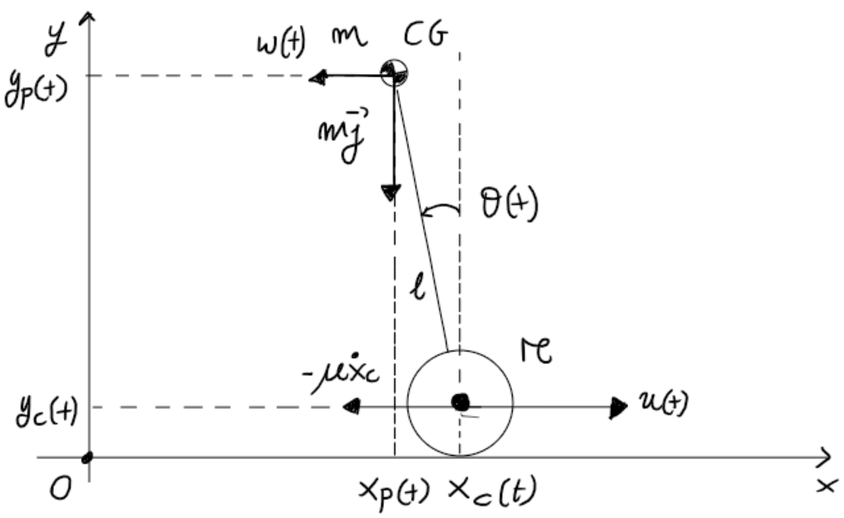
we obtain the lagrange reformulation of the Newton Second Law. Now we can apply this theory to the process in analysis.

- ## Mathematical Modelling Of The System

Let's start our discussion by presenting the physical structure of the Robot that we want to control; this kind of structure has been obtained by using SymMechanics Library on Matlab



Start the mathematical modelling of the system by using the following representation of the system



in the above picture we have respectively

- l is the distance between the CG of the pendulum and the cart.
- $w(t)$ is the noise acting on the system, force acting on the pendulum.
- $u(t)$ is the input of the system , is a force acting on the cart.
- μ is the road – tire friction coefficient.

The first step in the Lagrangian Formulation is the definition of the set of the generalized coordinates.

By a simple geometric analysis , we have

$$\begin{cases} x_c(t) = f_1\big(x_c(t), \theta(t)\big) = x_c(t) \\ y_c(t) = g_1\big(x_c(t), \theta(t)\big) = R \\ x_p(t) \quad = f_2\big(x_c(t), \theta(t)\big) = x_c(t) - lsin\big(\theta(t)\big) \\ y_p(t) = g_2\big(x_c(t), \theta(t)\big) = R + lcos\big(\theta(t)\big) \end{cases}$$

the generalized coordinates are of course { $x_c(t)$, $\theta(t)$ }.

Now , we are ready to evaluate the kinetic energy of the system , function of the generalized coordinates

$$K\big(x_c(t)\,,\theta(t)\big) = \frac{1}{2}M\dot{x_c}(t)^2 + \frac{1}{2}m\left(\dot{x_c}(t) - lcos\big(\theta(t)\big)\dot{\theta}(t)\right)^2 + \frac{1}{2}m\left(-lsin\big(\theta(t)\big)\dot{\theta}(t)\right)^2 + \frac{1}{2}I_p\dot{\theta}(t)^2$$

note how this quantity is composed by a translational part and e rotational energy.

Now , we are going to define the variation of the energy respect to the coordinates

$$\frac{\partial}{\partial t}\left(\frac{\partial K}{\partial \dot{\theta}}\right) - \frac{\partial K}{\theta} = \big(ml^2 + I_p\big)\ddot{\theta}(t) - mlcos\big(\theta(t)\big)\ddot{x_c}(t) = F_\theta$$

$$\frac{\partial}{\partial t}\left(\frac{\partial K}{\partial \dot{x_c}}\right) - \frac{\partial K}{x_c} = (M + m)\ddot{x_c} - mlcos\big(\theta(t)\big) + mlsin\big(\theta(t)\big)\dot{\theta}(t)^2 = F_{x_c}$$

at the second member we have the generalized forces. In particular,

$$F_\theta = \sum_{i=1}^{n=2}\left(F_{ix}\frac{\partial f_i}{\partial\theta} + F_{iy}\frac{\partial g_i}{\partial\theta}\right) = mglsin\big(\theta(t)\big) + w(t)lcos\big(\theta(t)\big)$$

while,

$$F_{x_c} = \sum_{i=1}^{n=2}\left(F_{ix}\frac{\partial f_i}{\partial x_c} + F_{iy}\frac{\partial g_i}{\partial x_c}\right) = u(t) - w(t) - \mu\dot{x}_c(t)$$

and we have the complete mathematical model of the system , as stated below

$$\begin{cases}(ml^2 + I_p)\ddot{\theta}(t) - mlcos\big(\theta(t)\big)\ddot{x_c}(t) - mglsin\big(\theta(t)\big) = w(t)lcos\big(\theta(t)\big) \\ (M+m)\ddot{x_c} - mlcos\big(\theta(t)\big) + mlsin\big(\theta(t)\big)\dot{\theta}(t)^2 + \mu\dot{x}_c(t) = u(t) - w(t)\end{cases}$$

Remember that it is only an approximation of the real behaviour of the system.

What we have to do now is to determine the equilibrium of the system and linearize the model in a neighborhood of one of them : the origin.

Linerarization And State Space Representation

As seen before , the system is a pendulum , so for $u(t) = 0$ we have two different points of equilibrium for the system , of course one is instable.

In this case the instable equilibrium point , corrisponds to the angle $\theta(t) = 0.$

Selecting for the system under analysis the state

$$[\, x_1(t) = \theta(t)\,; x_2(t) = \dot{\theta}(t)\,; x_3(t) = x_c(t)\,; x_4(t) = \dot{x_c}(t)\,]$$

and remember that linearizing the systen in a neighborhood of $\vec{x} = 0_x$ is equivalent to tale these approximation

$$\sin\theta(t) \approx \theta(t)\,, \qquad \cos\theta(t) \approx 1$$

We obtain the linearized model that we was looking for , and we can use it for the design of a stabilizing controller.

$$\begin{cases} \left(ml^2 + I_p\right) \ddot{\theta}(t) - ml\, \ddot{x}_c(t) - mgl\, \theta(t) = l\, w(t) \\ (M+m)\, \ddot{x}_c(t) - ml\, \ddot{\theta}(t) + \mu\, \dot{x}_c(t) = u(t) - w(t) \end{cases}$$

The state space model can be obtained in a very simple way by the precedent linear differential equations , let

$$C = \left[1 - \frac{(ml)^2}{\left(ml^2 + I_p\right)(M+m)} \right]$$

we have

$$\dot{x_1}(t) \;=\; x_2(t)$$

$$\dot{x_2}(t) \;=\; \frac{1}{C}\left[\frac{mgl}{(ml^2+I_p)}\, x_1(t) - \frac{ml\mu}{(ml^2+I_p)(M+m)} x_4(t) + \frac{ml}{(ml^2+I_p)(M+m)} u(t) + \left(\frac{l}{ml^2+I_p} - \frac{ml}{(ml^2+I_p)(M+m)} \right) w(t) \right]$$

$$\dot{x_3}(t) \;=\; x_4(t)$$

$$\dot{x_4}(t) \;=\; \frac{1}{C}\left[-\frac{\mu}{(M+m)} x_4(t) + \frac{(ml)^2 g}{(ml^2+I_p)(M+m)} x_1(t) + \frac{1}{(M+m)} u(t) + \left(\frac{ml^2}{(ml^2+I_p)(M+m)} - \frac{1}{(M+m)} \right) w(t) \right]$$

In particular with the following choose for the parameters of the model

$$\left\{ M = 3\,, m = 10\,, g = 9.81\,, l = 0.22\,, \mu = 0.90\,, W_p = 0.20, H_p = 0.03\,, I_p = \frac{1}{12} m\left(W_p^2 + H_p^2\right) \right\}$$

we obtain respectively

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 154.7737 & 0 & 0 & -1.0923 \\ 0 & 0 & 0 & 1 \\ 26.1925 & 0 & 0 & -0.2541 \end{bmatrix}$$

$$B1 = \begin{bmatrix} 0 \\ 1.2136 \\ 0 \\ 0.2823 \end{bmatrix}$$

$$B2 = \begin{bmatrix} 0 \\ 0.3641 \\ 0 \\ -0.0153 \end{bmatrix}$$

and we have finished the construction of the mathematical model for the system. We can now pass to the analysis of the stability of this point of equilibrium , by using the Lyapunov Theorem on the Linearized Model.

Dynamic Properties Of The Model

When we talk about dynamic properties of the model , we mean the study of the linearized model stability, the possibility to reach every state in the state space X (reachability analysis) , and the possibility to riconstruct the state of the system from input-output measurements (osservability analysis).

In fact as we will see , the reachability and the observability of the system are used to define respectively the existence of a stabilizing controller , in particular the possibility to allocate the maximum number of eigenvalues of the closed loop system , and the existance of an asymptotic observer.

- ## Lyapunov Theory

The stability analysis of the equilibrium point can be done in different ways , for example by applying the second Lyapunov Theorem which doesn't require the explicit solution of the differential equations.

But , we have the know the linearized model of the process , so we can use a different result , the Reduced Lyapunov Criteria.

Teorema (Reduced Lyapunov Criteria)

Suppose that the autonomous non linear system f is continuously differentiable and the origin an equilibrium point.

1. If the linearized system

$$\dot{x}(t) = F\, x(t)$$

   is Hurwitz stable (eigenvalues all with strict negative real part) then , the equilibrium point is asymptotically stable.

2. If there exists at least one eigenvalue with strictly positive real part then , the equilibrium point in unstable.

Remark
Note how the theorem doesn't permit to conclude anything about the stability in presence of eigenvalue with zero real part.

As we can image , but we proof formally , the origin of our system is not stable , by using the previous criteria.

Given the linearized model of the system , with the aim to study the stability of the origin , we have to find the eigenvalues of the Jacobian matrix A

$$eig(A) = \{ 0 , 12.3492 , -12.5341 , -0.0692 \}$$

So , the presence of the eigenvalue with positive real part , allow us to conclude the instability of the system.

Reachability Analysis

Without going to much in details , remember that the reachability of the system is equivalent to the possibility to reach every state of the system belonging to the state space X.

In particular , the complete reachability of the system is directly related to the possibility to allocate all the eigenvalue of the closed loop system.

Infact , if the system is not reachable , it can be turned in the standard reachability form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \qquad B_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with the couple (A_1, B_1) reachable.

The closed loop system becomes

$$A + B_1 K = \begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ 0 & A_{22} \end{bmatrix}$$

and as we can see , whatever is the choise of $K = [K1\ K2]$ is not possible to modify the eigenvalues of the matrix A_{22}.

With the aim to define a stabilizing controller , we are going to study the reachibility of the system

Remember that the reachability space at time t is given by

$$X_t^R = Im\, W_t$$

so , it is the image of the Gramian at the time t.

But , while in the discrete case the reachibility space depends of the length of the time interval , in the continuous case , the reachibility set not depends by the length of the time inteval $[0\,, t]$ during which the control action acts.

It can be proof that , the reachibility space is equals to

$$X^R = Im\,\{\, R = [B\; AB\; A^2B \;\dots\; A^{n-1}B]\,\}$$

so , the system is completely reachable if the rank of this matrix is full.

In our case $n = 4$, and the reachability matrix R is

$$R = \begin{bmatrix} 0 & 1.2136 & -0.3084 & 187.9157 \\ 1.2136 & -0.3084 & 187.9157 & -82.46545 \\ 0 & 0.2823 & -0.0717 & 31.8061 \\ 0.2823 & -0.0717 & 31.8061 & -16.1576 \end{bmatrix}$$

and the $rank(R) = 4 = n$, so the system is completely reachable. It of course, is equivalent to the possibility of modify all the eigenvalue of the closed loop sytem.

So we can conclude that the couple (A, B) can be stabilized by a static retroaction of the state. But remember that the complete reachability of the system is only a sufficient but not a necessary condition for the stabilization.It is necessary that the not reachability sub-space is asymptotically stable.

Observability Analysis

The observability of the system is a property related to the possibility of reconstructing the state of the system from input-output measurements. This concept , is of great important in a control problem , in particular when we need at each time istant the knowledge of the state of the system to implement the control action.

The state of the system is not always known. The osservability as we known , is directly related to the possibility to create an observer for the state.

More precisely , the osservability allow us to determine the state of the system at a generic time istant t_0 from input-output measurements acquired in time instants $t \geq t_0$.

Remember that , said

$$w_t : X \to C_{[0,t]}$$

the operator that at each state x associates the value $y_l(\sigma) = Ce^{A\sigma}x$, $\sigma \in [0, t]$ the free responce of the system , the non osservability subspace is defined as

$$X_t^{no} = Ker\ (w_t)$$

the kernel of the operator defined above. Remember that this subspace is composed by the state of the system whose free response is equals to the free response of the system generated by the origin.

In the continuous case , the dimension of the observability subspace not depends from the length of the time interval , so

$$X^{no} = Ker(O_n)$$

where $O_n = \left[\, C^T, A^T C^T\,,\ \ldots\ , A^{n-1^T} C^T \right]^T$ is the not observability matrix.

The system is reachable if $X^{no} = \{\, 0 \,\}$ or that is the same

$$rank(O_n) = n = \dim\,(X)$$

In our case we suppose that we can measure the following outputs of the system

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} w(t)$$

it is equals to assume $C = [1\ 0\ 0\ 0; 0\ 0\ 1\ 0]^T$.

The non observability matrix is given by

$$O_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 154.7737 & 0 & 0 & -1.0923 \\ 26.1925 & 0 & 0 & -0.2541 \\ -28.6091 & 154.7737 & 0 & 0.2775 \\ -6.6549 & 26.1925 & 0 & 0.0646 \end{bmatrix}$$

and the $rank(O_n) = 4 = n$, so , the system is observable. It is equivalent to the possibility to allocate all the eigenvalues of the observer for the state extimation.

Stabilization

As we seen in the previous section , the linearized model in unstable. But the complete reachibility of the system allow us to modify the position in the complex plane of all closed loop eigenvalues, e so , the possibility to make the linearized system asymptotically stable.

The control actions obtained in this section will be used to make comparison with the other controllers obtained for example my minimizing some induced gain of the system.

- **LMIs C**onditions

Let's start out discussion by presenting an important proposition , due to Lyapunov

Proposition

A linear time invariant system is asymptotically stable if and only if $\exists\ P = P^T , P > 0$:

$$A^T P + PA < 0$$

Note for example , how the necessary condition of this proposition is directly related by the following theorem , always due to Lyapunov

Theorem

A linear time invariant system is asymptotically stable if and only if $\forall\ Q = Q^T , Q > 0$, $\exists\ P = P^T, P > 0$:

$$A^T P + PA = -Q$$

in particular Lyapunov give us the expression of the solution $P = \int_0^{+\infty} e^{A^T t} Q \, e^{At} dt$. We will see after the application of this result to the calculus of the H_2 norm for linear time invariant system.

The aim of this section is the definition of a gain K that stabilize the system. By the complete reachibility of the system we are sure that this king of controller exists.

In particular what we want to do now , is the definition of a series of synthesis conditions for the gain K , and as we will see this kind of conditions can be translated into LMIs by using an appropriate change of variables.

Remarks

Suppose that the couple (A, B_1) is stabilizable , so there exist a gain K such that the closed loop system $(A + B_1 K, B_1)$ is asymptotically stable.

By the previous proposition then we can conclude that
$\exists \, P = P^T \, , P > 0 \, :$

$$(A + B_1 K)^T P + P \, (A + B_1 K) < 0$$

note that this matrix inequality is not an LMI , cause both K and P are unknown. So what we want to do now is to trasform this inequality into an LMI , and in particular translate the stabilization problem into a feasibility one.

Remember that two matrices related by a congruence tranformation have the same sign, so the problem of study the sign of the previous inequality is equals to study the sign of

$$P^{-1}(A + B_1 K)^T + (A + B_1 K) P^{-1} < 0$$

define now

$$X = P^{-1} \quad , \quad W = K \, X$$

and by remembering that the inverse of a positive matrix is itself positive , we have that is there exist K such that $(A + B_1 K, B_1)$ is asymptotically stable , then

$$\exists \, X \in S^n \, X > 0 \, , \exists \, W : \; X A^T + AX + B_1 W + W^T B_1^T < 0$$

so , at the end of the day , we have the following proposition

Proposizione
The couple (A, B_1) is stabilizable by a static state retroaction $u(t) = K\,x(t)$ if and only if

$$\exists\, X \in S^n\; X > 0 \,, \exists\, W : \;\; (AX + B_1 W) + (AX + B_1 W)^T < 0$$

and the stabilizing gain K is given by $K = W\,X^{-1}$.

- ## System Stabilization

In the previous section we have found the LMIs conditions for the synthesis of a stabilizing controller , now we want to define it for the robot taken in exam. As we have said before , all the controllers obtained in this section will be used to make comparison with the others obtained in the future sections: for example an H_∞ controller , obtained by minimizing the L_2 induced gain.

- **Pole Placement Stabilization**

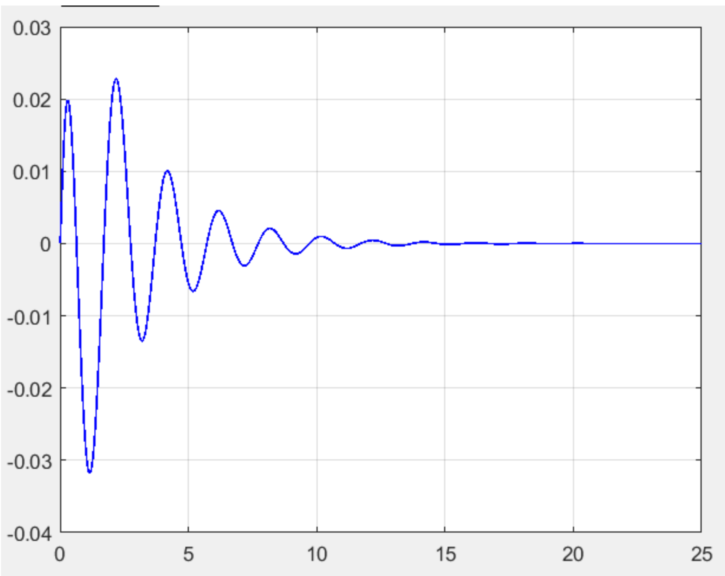
We want that the eigenvalues of the closed loop system are respectively

```
p      = [-2  -3  -4  -5];
```

of course , the complete reachability of the system guarantees the existence of such controller

K = [-191.1709 -15.2441   10.5370   14.4225];

the following picture graphs the behaviour of the angle $\theta(t)$ by assuming that on the system acts a disturbance with intensity $\bar{d} = 15$.



D-stability

In the last section we have presented LMI conditions for the well-known Hurwitz stability of the matrices. In this section , we will introduce the concept of D-stability , which is a generalization of Hurwitz stability and Schur stability. As in the sequential section , our main task is to establish the LMI conditions for the D-stability analysis with given D-regions.

- ## Introduction To D-stability

With the aim to present the concept of D-stability , let D a region of the complex plane C symmetric about the real axis.

Definition

A matrix A is said to be D-stable if all the eigenvalues of the matrix A are in the D-region, formally

$$\lambda_i(A) \in D \quad \forall i = 1 \dots n$$

Remark

Note how this definition is a generalization of the well-known Hurwitz and Schur stability. Infact , we can choose the condition to the Hurwitz and Schur stability.

With the aim to preset one of the most beautiful result of control , The generalized Lyapunov Theorem , start our discussion by presenting the concept of LMI-region.

Definition (LMI Region)
Let D a region of the complex plane C. If there exist matrices $\exists\, L \in S^n, \exists\, M \in R^{nxn}$ such that

$$D \coloneqq \{\, s \in C \mid L + sM + \bar{s}M^T < 0 \,\}$$

then D is called an LMI region. The complex function

$$F(s) = L + sM + \bar{s}M^T$$

is called the charactestic function of the LMI region D.

Note how this region not only is symmetric about the real axis , but is convex , due to its definition. Remember infact that an LMI defines a convex constrain in a constrained optimization problem.

The useful of the concept of LMI region stay in the Generalized Lyapunov Theorem. Infact as we will see , is sufficient to give in input to this theorem a genetic LMI region to obtain in output the LMI conditions whose solutions is equivalent to the D-stability of the system under analysis.

Theorem
Let D be an LMI region, whose characteristic function is

$$F(s) = L + sM + \bar{s}M^T$$

Then , a matrix A is D-stable if and only if there exist a symmetric positive definite matrix P such that

$$R_D(A,P) \coloneqq L \otimes P + M \otimes (AP) + M^T \otimes (AP)^T < 0$$

Given an LMI region of interest is sufficient to apply this theorem to obtain in output the LMI conditions relative to the analysis of the D-stability of the system. Of cource , again , the problem to check the D-stability as been translated into a feasibility one.

- ## LMI Region Of Interest

Suppose that for the our system we have the following specifications on the performance:

1) Maximum Overshoot $S \leq 0.10$
2) Settling Time at 5% , $T_{a5} \leq 2\ sec$
3) BandWidth of the system , $w_n \leq 50\ \left(\frac{rad}{sec}\right)$

Remember how these specifications are related to the parameters of the closed loop eigenvalues,

$$S = \exp\left(-\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)$$

with $\delta = \cos\theta(t)$ the damping of our poles , and

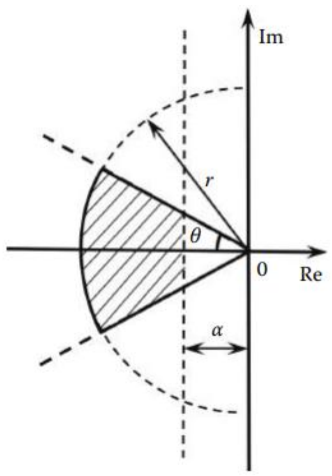
$$T_{a5} = -\frac{\ln(0.05)}{\delta w_n}$$

with $\frac{1}{\delta wn}$ the real part of the poles.

Applying the required specification and by solving with respect to θ , $Re(s)$ we have

$$\begin{cases} \theta \leq 53.84^{\circ} \\ |Re(s)| \geq 1.5 \end{cases}$$

So, in the complex plane C the LMI region of interest is $(r = 50)$



Now , we have to establish the single LMI region , and after we have to take the intersection.

- Hyperplane $\boldsymbol{H_\alpha}$

So let's start by first considering the requirement that the real part of the closed loop eigenvalue must be greater than 1.5.

The LMI region of interest will be

$$H_a := \{ s \in C \mid Re(s) < -a < 0 \} , \qquad a = 1.5$$

In particular note that

$$A \text{ is } H_a - stable \quad \leftrightarrow \quad (A + aI) \text{ is Hurwitz stable}$$

But we know that the matrix $(A + \alpha I)$ is Hurwitz stable if and only if $\exists\ P = P^T\ P > 0\ :$

$$(A + aI)^T P + P\,(A + aI) = A^T P + PA + 2aP < 0$$

and this is an LMI in the unknown matrix P.

So we have obtain the following result

$$A\ is\ H_a - stable\ \leftrightarrow\ \exists\ P = P^T\ , P > 0 : A^T P + PA + 2aP < 0$$

- Circumference $\boldsymbol{C_r}$

Now , consider the third requirement , the closed loop eigenvalue must lie in the circumference with center in (0,0) and radius $r = 50$.

The LMI region of interest is

$$\boldsymbol{C_r} := \{ \mathrm{s} \in \mathrm{C} \mid Re(s)^2 + Im(s)^2 < r^2 \} \quad , r = 50$$

and in particular we have that

$$A\ is\ \ C_r - stable\ \ \leftrightarrow \frac{1}{r}\ A\ \ is\ \ Schur - Stable$$

We known that $\frac{1}{r}$ A is Schur stable if and only if $\exists\ P = P^T$:

$$\begin{cases} -P & < 0 \\ A^T PA - r^2 P < 0 \end{cases}$$

but , the previous conditions can be seen as the Schur factor of the matrix

$$\begin{bmatrix} -rP & PA \\ A^T P & -rP \end{bmatrix} < 0$$

that is an LMI in the unknown matrix P.

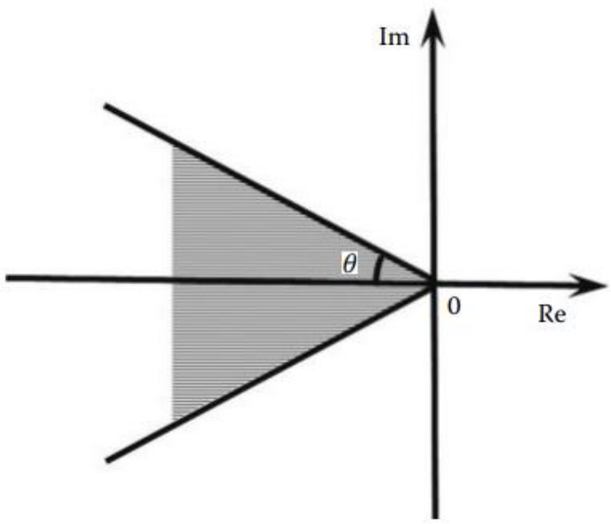
At the end , we have obtained that

$$A\ is\ C_r - stable\ \leftrightarrow\ \exists\ P = P^T : \begin{bmatrix} -rP & PA \\ A^T P & -rP \end{bmatrix} < 0$$

Now we can derive the last condition for our closed loop poles by applying the Generalized Lyapunov Theorem.

- **Cone D_θ**

Now , we want to obtain the LMI conditions to solve the requirement related to the maximum overshoot. As we have seen before , we want that our pole lies in the following LMI region ($\theta = 53.84$)



The sector showed above , is

$$D_\theta \coloneqq \{\ s\ \in C\ |\ |Im(s)| <\ -Re(s)\tan(\theta)\}$$

that can be rewritten as

$$D_\theta = \{ s \in C \mid \begin{bmatrix} (s+\bar{s})sin\theta & (s-\bar{s})cos\theta \\ (\bar{s}-s)cos\theta & (s+\bar{s})sin\theta \end{bmatrix} < 0 \}$$

note that is an LMI region. So it can be given in input to the Lyapunov Theorem , which give us in output the LMI Condition for the D_θ stability.

$$\exists\, P = P^T, P > 0 : \begin{bmatrix} (A^TP + PA)\sin\,\theta & (PA - A^TP)cos\theta \\ (A^TP - PA)cos\theta & (A^TP + PA)sin\theta \end{bmatrix} < 0$$

So now we have to take the intersection of the previous LMI regions and we have obtained the analysis conditions for the D-stability with D LMI region of interest.

LMI Conditions & D-stabilization

In the last section we have obtained the LMI conditions for the single LMI region. What we have to do now is to take the intersection.

In particular we have the following proposition

Proposition

Given two LMI region D_1 , D_2 , a matrix A is $D = D_1 \cap D_2$ stable if and only if

$\exists\, P = P^T, P > 0$:

$$R_{D1}(A,P) < 0 \quad , \qquad R_{D2}(A,P) < 0$$

this result can be extended to a generic number of LMI region.

So at the end of the day we have that

Proposition

Let $D = H_\alpha \cap C_r \cap D_\theta$ the LMI region of interest. The matrix A is D-stable if and ony if $\exists\ P = P^T$:

$$\begin{cases} A^T P + PA + 2aP & < 0 \\ \begin{bmatrix} -rP & PA \\ A^T P & -rP \end{bmatrix} & < 0 \\ \begin{bmatrix} (A^T P + PA)\sin\ \theta & (PA - A^T P)cos\theta \\ (A^T P - PA)cos\theta & (A^T P + PA)sin\theta \end{bmatrix} & < 0 \end{cases}$$

with $a = 1.5$, $r = 200$, $\theta = 53.84^\circ$.

Given the analysis conditions we can derive the synthesis conditions as we have done in the last sections. By applying a congruence tranformation and by a well-known change of variables the synthesis conditions can be translated into LMI conditions and the same time is given the expression of the gain K.

$$Matrix\ Inequalities \rightarrow Congruence\ Transormation \rightarrow Change\ of\ Variables$$

Proposizione

The couple (A, B_1) is D-stabilizable if and only if $\exists\ X = X^T$, $\exists\ W$:

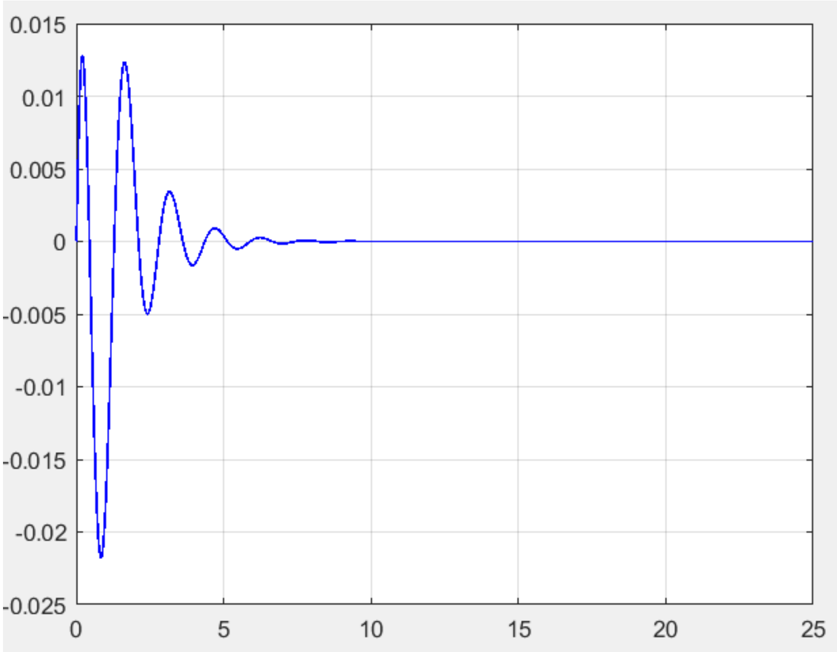
$$\begin{cases} (AX + B_1 W) + (AX + B_1 W)^T + 2aX & < 0 \\ \begin{bmatrix} -rX & (AX + B_1 W) \\ (AX + B_1 W)^T & -rX \end{bmatrix} & < 0 \\ \begin{bmatrix} [(AX + B_1 W) + (AX + B_1 W)^T]\sin\ \theta & [(AX + B_1 W) - (AX + B_1 W)^T]cos\theta \\ [(AX + B_1 W)^T - (AX + B_1 W)]cos\theta & [(AX + B_1 W) + (AX + B_1 W)^T]\sin\ \theta \end{bmatrix} & < 0 \end{cases}$$

the gain of the controller is given by $K = W\ X^{-1}$.

The gain obtained by solving these LMI conditions is

K = [-370.6183 -47.5533   65.9474   65.1588];

and the responce of the system by assuming that on it is acting a disturbace with intensity $\bar{d} = 15$ is



Design H_∞ , H_2 , L_1 Controllers

In this section the aim is the definition of static controller K that stabilize the plant and in particular minimize the value of some induced gains of the system. In particular , after a brief introduction to the definition of the induced norm for a linear operator , the attention will be focused on the design of static state retroaction such that the effect of the disturbance $d(t)$ that acts on the system is minimized.

- H_∞ Controller

Let's start our discussion by defining the objectives that we want to reach. From this point the interest is the definition of a gain K such that the effect of the disturbance is minimum.

We choose the following outputs of interest

$z_\infty(t) = C_1 x(t) + D_{11} d(t) \ = \ x_1(t)$

so , the aim is to reduce the effect of $d(t)$ to $\theta(t)$.

Before to present the LMI conditions that limit the value of the H_∞ norm of the transfer function between $d(t)$ and $\theta(t)$, why we are interest to minimize this quantity?

Remember that the H_∞ plays the role of induced gain when the input-output function spaces we select the L_2 norm

$$||y||_2 \leq ||G||_\infty ||d||_2$$

So , manteining this quantity below a generic positive scalar γ , is equals to guarantee an attenuation (in terms of energy) of the disturbance $d(t)$ on the output of interest , in this case $\theta(t)$.

Given a BIBO stable transfer function $G(s)$, its H_∞ norm is defined as

$$||G||_\infty := \underset{\omega > 0}{sup} \left\{ \underset{\sigma > 0}{\sup} |G(\sigma + j\,\omega)| \right\} = \underset{\omega > 0}{\sup} |G(j\,\omega)|$$

in particular remember how the second equality is given by the application of the Fatou Lemma and Maximum Modulus Principle.

With reference to the following state space representation

$$\begin{cases} \dot{x}(t) &= Ax + B\,w \\ z_\infty(t) &= Cx + D\,w \end{cases}$$

and said $G(s) = C\,(sI - A)^{-1}B + D$ the transfer function between the disturbance and the objective, then

The system is asymptotically stable and

$$||G(s)||_\infty < \gamma$$

with γ fixed positive real scalar if and only if $\exists\ P = P^T , P > 0$ such that

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

As seen in the last sections , known the analysis condition , now we define the synthesis condition and by applying always the same change of variable , we can translate this synthesis matrix inequalities into LMIs.

Problem

Given the following state space model

$$\begin{cases} \dot{x}(t) & = Ax(t) + B_1u(t) + B_2w(t) \\ z_\infty(t) = Cx(t) + D_1u(t) + D_2w(t) \end{cases}$$

and said

$$G(s) = (C + D_1K)\big(sI - (A + B_1K)\big)^{-1}B_2 + D_2$$

the transfer function between the disturbace and the outputs of insterest , then define a static state controller

$$u(t) = K\ x(t)$$

such that the closed loop system is asymptotically stable and

$$\left\|G(s)\right\|_{H\infty} < \gamma$$

With the control action $u(t) = K\ x(t)$ the closed loop system becomes

$$\begin{cases} \dot{x}(t) & = (A + B_1\ K)x(t) + B_2w(t) \\ z_\infty(t) = (C + D_1K)x(t) + D_2w(t) \end{cases}$$

then the transfer function $G(s)$ is

$$G(s) = (C + D_1K)\big(sI - (A + B_1K)\big)^{-1}B_2 + D_2$$

from the aforementioned condition , we have that , the system is asymptotically stable and

$$\left\|G(s)\right\|_{\infty} \leq \gamma$$

if and only if $P = P^T, P > 0$:

$$\begin{bmatrix} (A + B_1K)^TP + P(A + B_1K) & PB_2 & (C + D_1K)^T \\ B_2^TP & -\gamma I & D_2^T \\ (C + D1K) & D_2 & -\gamma I \end{bmatrix} < 0$$

of course if the gain K is not fixed , the above matrix inequality is not an LMI , but pre and post multipling the precedend inequality by P^{-1} and by using the following change of variables

$$X = P^{-1}, W = KX$$

the last problem can be reformulated as

$\exists\, X = X^T, X > 0 , \exists\, W :$

$$\begin{bmatrix} (AX + B_1W) + (AX + B_1W)^T & B_2 & (CX + D_1W)^T \\ B_2^T & -\gamma I & D_2^T \\ (CX + D_1W) & D_2 & -\gamma I \end{bmatrix} < 0$$

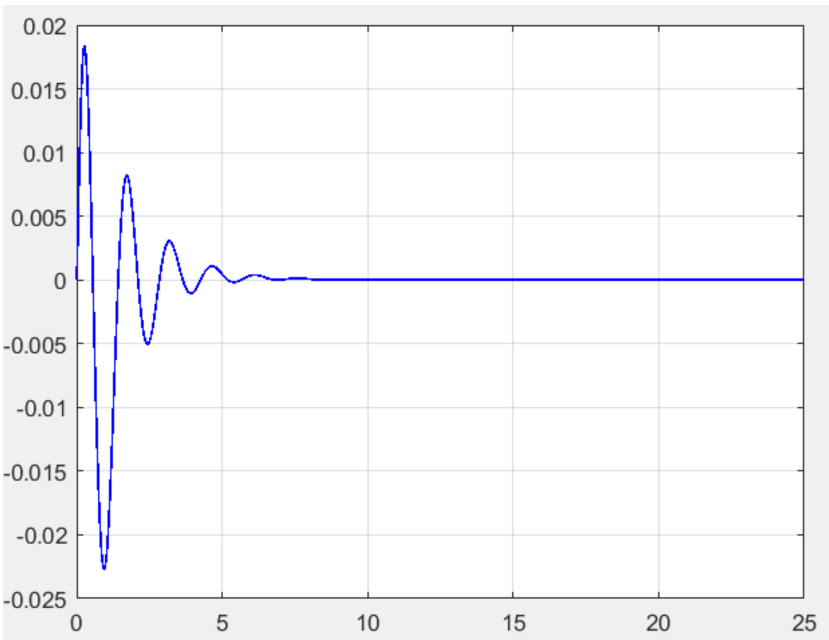
now the precedend inequality is a LMI , and in particular the gain K is given by $K = W\,X^{-1}$.

Now what we want to do is to apply this theory to our system. In particular we assume that on the system acts a disturbance with intensity $\bar{d} = 15$.

The controller K is

K = [-211.0521 -10.9077 5.2228 15.0298];

and below we have the responce of the system to these inputs



- H_2 Controller

In the last section the aim was to minimize the L_2 induced gain. Now our attenction will be focused on the minimization of the H_2 norm. Of course is always of interest minimize this quantity because remember that the H_2 plays the role of induced gain for a particular choise of the norms for the input-output function spaces, in particular we have

$$||y||_\infty \leq ||G(s)||_{H_2} ||w||_2$$

Remember that , while is not possibly to evaluate exactly the value of the H_∞ norm , in the case of H_2 norm we have two different ways to evaluate it: the reachability gramian and the observability gramian.

But , let's start with the definition of the H_2 norm for a BIBO transfer function $G(s)$

$$||G(s)||_{H2} := \left\{ \sup_{\sigma > 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} tr[\, \bar{G}(\sigma + j\omega)^T G(\sigma + j\omega)] \; d\omega \right\}^{\frac{1}{2}} = \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} tr[\, \bar{G}(j\omega)^T G(j\omega)] \, d\omega \right]^{\frac{1}{2}}$$

Like in the previous case the last inequality is right due to Fatou Lemma and the Maximum Modulus Principle.

In particular note from the above definition that the transfer function $G(s)$ has to be strictly proper; it is necessary to the convergence of that quantity.

Let's consider a minimal realization of $G(s)$

$$\begin{cases} \dot{x}(t) \; = Ax(t) + B\,u(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

such that $G(s) = C(sI - A)^{-1}B + D$.

Due to the Parseval Theorem we have

$$||G(s)||^2_{H2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} tr[\, \bar{G}(j\omega)^T G(j\omega)] \, d\omega \;\; = \int_0^{+\infty} tr[(Ce^{At}B + D)^T (Ce^{At} + D)] dt$$

Note that if $D \neq 0$, the precedent expression is not finite. So in this case the disturbace doesn't act directly on the output of interest ($D = 0$).
Ok , now we want to evaluate exactly this quantity

$$\left|\left|G(s)\right|\right|^2_{H2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} tr[\,\bar{G}(j\omega)^T G(j\omega)]\,d\omega \;=\; \int_0^{+\infty} tr[(Ce^{At}B)^T(Ce^{At})]dt \;=$$

$$= \int_0^{+\infty} tr\left[B^T e^{A^T t} C^T C\, e^{At} B\right] dt = tr\left\{B^T \int_0^{+\infty} e^{A^T t} C^T C\, e^{At} dt \;\; B\right\}$$

defining with $X = \int_0^{+\infty} e^{A^T t} C^T C\, e^{At} dt$ the observability gramian , we have

$$\left|\left|G(s)\right|\right|^2_{H2} = tr\,\{B^T X\, B\}$$

Remember that in the case of two matrices A , B that commute , we have that the $tr(AB) = tr(BA)$, we obtain for the H_2 norm a ìn equivalent expression , in terms of the reachability gramian

$$\left|\left|G(s)\right|\right|^2_{H2} = tr\,\{CXC^T\}$$

with $X = \int_0^{+\infty} e^{At} B B^T e^{A^T t} dt$.

In particular the precedent two equations satisfy particular Lyapunov equations. In the first case we have that

$$A^T X + XA = \; -C^T C$$

Infact ,

$$A^T\left(\int_0^{+\infty} e^{A^T t} C^T C\, e^{At} dt\;\right) + \left(\int_0^{+\infty} e^{A^T t} C^T C\, e^{At} dt\;\right) A = \int_0^{+\infty} \left(A^T e^{A^T t} C^T C\, e^{At} + \; e^{A^T t} C^T C\, e^{At} A\right) dt =$$

$$= \int_0^{+\infty} \frac{\partial}{\partial t}\left\{ \,\left(e^{A^T t} C^T C\, e^{At}\right)\right\} dt = \; -C^T C$$

In the second case instead we have

$$AX + XA^T \; = -BB^T$$

the proof is the same as in the previous case.

So now we are ready to present the following proposition , the LMI conditions that guarantee an attenuation level on the norm $H_2 < \gamma$.

Proposition

The system is asymptotically stable and $||G(s)||_{H2} < \gamma$ if and only if one of the two equivalent conditions is satisfied

1) $\exists\, X = X^T\; X > 0\; : AX + XA^T + BB^T < 0\;\; , tr(CXC^T) < \gamma^2$
2) $\exists\, X = X^T\; X > 0\; : A^TX + XA + C^TC < 0\;\; , tr(B^TXB) < \gamma^2$

Given the analysis conditions , now we are ready to obtain the LMIs condition for the synthesis of the controller $u(t) = K\,x(t)$. But before doing so , we have the following proposition

Proposition

Let $A(x) \in R^{nxn}$, $A(x) = A(x)^T$, the following statements are equivalent

1) $\exists\, x\; \in R^n : tr\big(A(x)\big) < \gamma$
2) $\exists\, x\; \in R^n, \exists\, Z = Z^T :\;\; A(x) - Z < 0\, , tr(Z) < \gamma$

So now we can establish the LMI synthesis conditions for the gain K such that the closed loop system $(A + B_1K, B_1)$ is asymptotically stable and $||G(s)||_{H2} < \gamma$.

With reference to the following state space model

$$\begin{cases} \dot{x}(t) & = Ax(t) + B_1u(t) + B_2w(t) \\ & z_2 = C_2x(t) + D_{22}u(t) \end{cases}$$

then,

Proposition

The closed loop system $(A + B_1K, B_1)$ is asymptotically stable and $||G(s)||_{H2} < \gamma$ if and only if

$\exists\, X = X^T\, , \exists\, Z = Z^T\, , \exists\, W\, :$

$$\begin{bmatrix} (AX + B_1W) + (AX + B_1W)^T & B_2 \\ B_2^T & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} Z & (C_2X + D_{22}W) \\ (C_2X + D_{22}W)^T & X \end{bmatrix} > \; 0$$

$$tr(Z) < \gamma^2$$

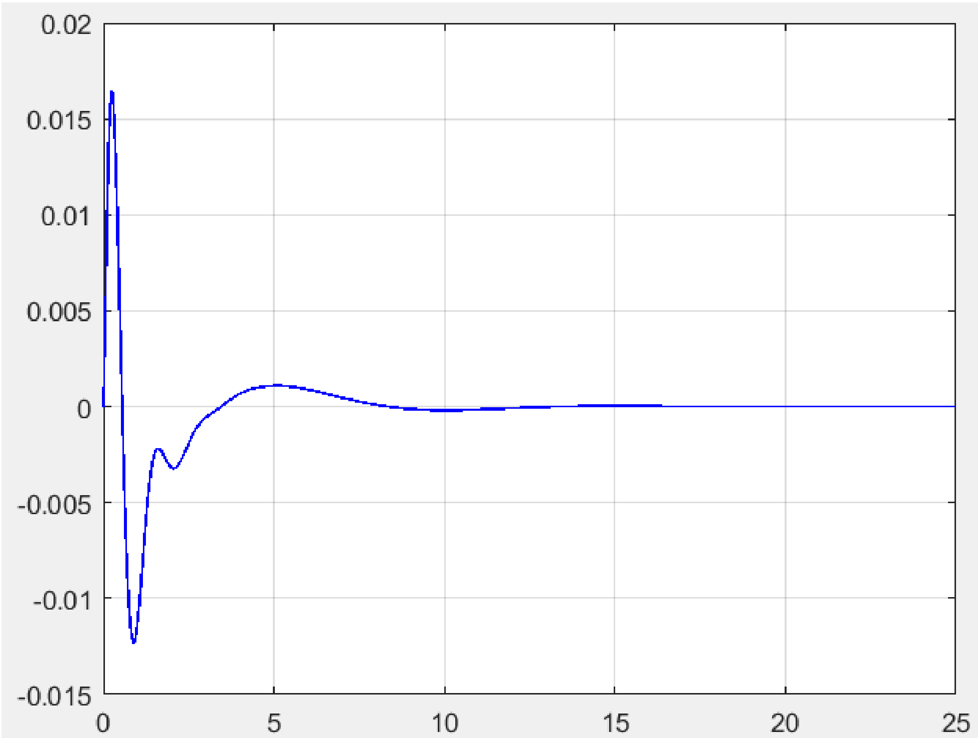
with $K = W\, X^{-1}$.

What we want to do now is to obtain this type of controller for our system under analysis , in particular suppose that on the system acts a disturbance with intensity $\bar{d} = 15$.

The gain K is,

K = [-257.5629 -10.2175   6.6725   10.1981];

and we have the following result



- L_1 Controller

In the last section we have defined K such that the induced gain H_2 is minimized. Now the objective is the definition of a controller that minimize the L_1 norm (L_∞ induced norm) of the linear operator $H(t)$ that define the behaviour of the forced responce of the system.

In particular , remember that the linear operator defined above is given by the convolution integral

$$y(t) = \int_0^t G(t-\tau)u(\tau)\, d\tau$$

As done in the case of reachability analysis , define the following operator

$$H : U \rightarrow Y \quad , \qquad u \in U \rightarrow \int_0^t G(t-\tau)\, u(\tau)\, d\tau$$

that relates at a given input $u(t)$ belonging to the input space function the function $y(t)$ in the output functional space , with $(t) = L^{-1}\big(G(s)\big) = (C\, e^{At}B + D)$.

The aim is the definition of a controller K such that the closed loop system is asymptotically stable and

$$||H||_1$$

is minimum.

Of cource this problem have a pratical meaning , because remember that the L_1 norm plays the role of induced gain when on the input-output functional spaces we choose the infinity norm.

$$||y||_\infty \leq ||H||_1 ||u||_\infty$$

Infact , remember , to this operator is associated the name of peak-to-peak gain.

Now , consider the following state space representation of the system

$$\begin{cases} \dot{x}(t) = A\, x(t) + B_1 u(t) + B_2 w(t) \\ z_1 = C_3 x(t) + D_{31} w(t) + D_{32} u(t) \end{cases}$$

and by choosing the input $u(t) = K\, x(t)$, the closed loop system becomes

$$\begin{cases} \dot{x}(t) = (A + B_1 K)x(t) + B_2 w(t) \\ z_1 = (C_3 + D_{32}(t)K)x(t) + D_{31} w(t) \end{cases}$$

We have proofed that a gain K such that the closed loop system is asymptotically stable and the value of L_1 norm is minimum is obtained by solving the following constrained optimization problem

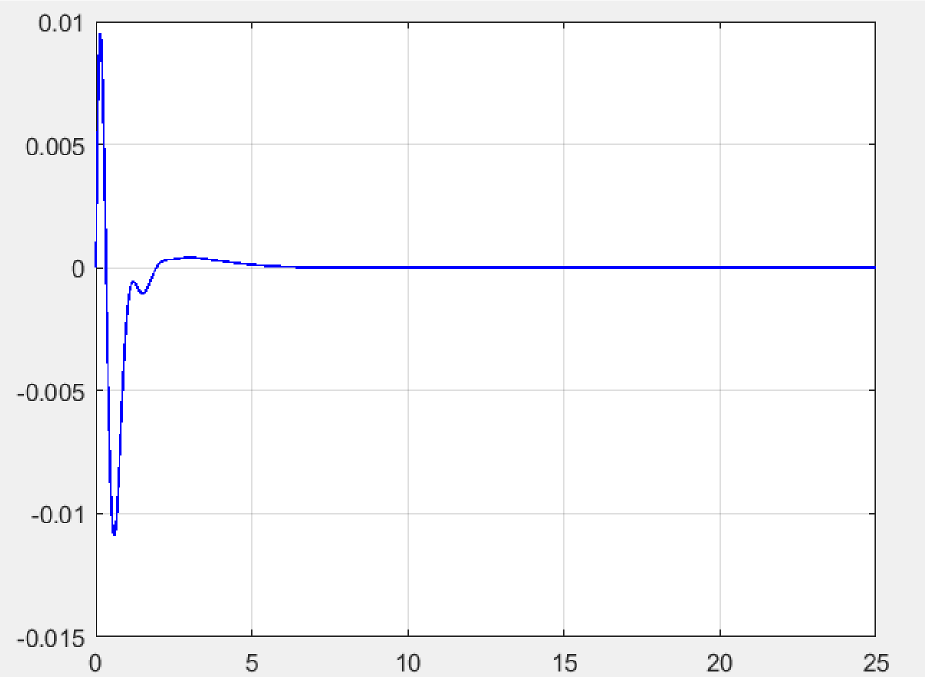
$$
\begin{cases}
\min_{X,Y} \gamma_1 \\
s.v. \\
\begin{bmatrix} (AX+B_1Y)+(AX+B_1Y)^T+\lambda X & B_2 \\ B_2{}^T & -\mu I \end{bmatrix} < 0 \\
\begin{bmatrix} \lambda X & 0 & (C_3X+D_{32}Y)^T \\ 0 & (\gamma_1-\mu)I & D_{31}{}^T \\ (C_3X+D_{32}Y) & D_{31} & \gamma_1 I \end{bmatrix} > 0 \\
X > 0
\end{cases}
$$

with γ fixed , the conditions are of course a series of LMIs conditions.

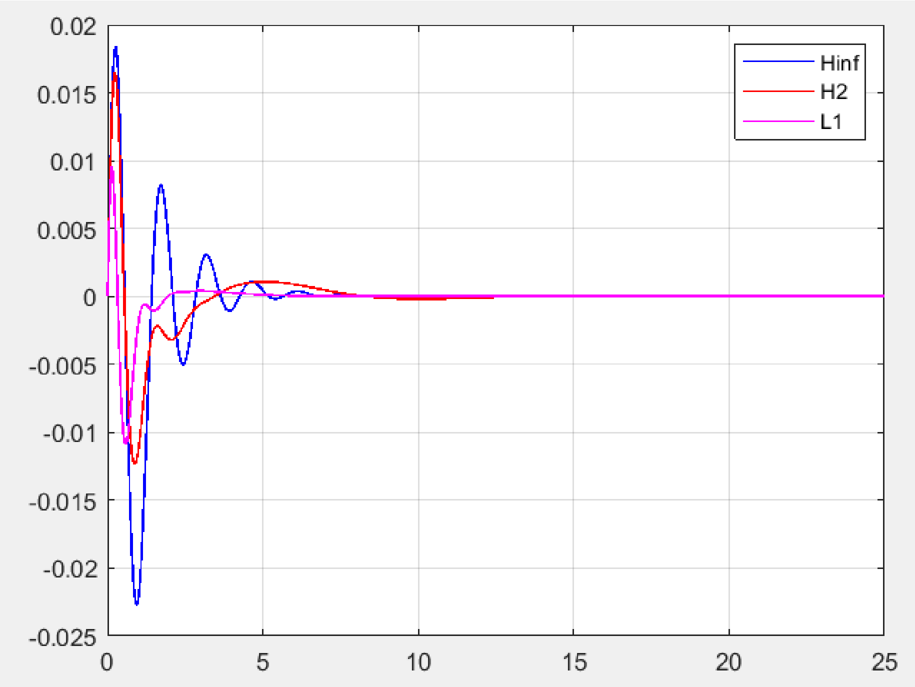
Solved the optimization problem , the controller is given by

$$K = Y\,X^{-1}.$$

Obtained the controller , we can simulate the behaviour of the system by supposing that on the system acts a disturbace of intensity $\bar{d} = 15.$



Now we are ready to make comparisons between the controller obtained in the last two sections.



And of course the results show how the L_1 controller in this case is the best controller ever.

Multiobjective Control

A classical optimization problem can be stated as follow

$$\min f(x)$$
$$x \in S$$

where $S \subseteq R^n$ is the ammisible region and , $f(x)$ is a single scalar function to be minimized.

In a multiobjective optimization problem , the objective function is not one , but we want to minimize a list of objectives, such stated below

$$\min[f_1(x), f_2(x), .. f_N(x)]$$
$$x \in S$$

where $N > 1$. In particular note that the ammisible region S is the same in both optimization problem.

The most important thing is that the definition of optimal in the case of mono-objective optimization problem cannot be applied in the multiobjective case. So the first step in the solution is the definition of optimal for the multiobjective case.

Definition (Weakly Pareto's Optimum)
A point $x^*\ \in S$ is said to be a weakly Pareto's optimum for the multiobjective problem if and only if there not exist a point $x\ \in S$ such that

$$f_i(x) < f_i(x^*)\,, i = 1 \dots N$$

Definition (Strong Pareto's Optimum)
A point $x^*\ \in S$ is said to be a strong Pareto's optimum for the multiobjective problem if and only if there not exist a point $x\ \in S$ such that

$$f_i(x) \leq f_i(x^*)\,, i = 1 \dots N$$

but at least on has to be verified strictly.

The locus of all strong Pareto's solutions is known as **Pareto Front.**

- Scalarization Method

The problem can be solved by converting the multiobjective problem into a monobjective one , by the definition of the following cost function

$$J_\gamma(x) = \sum_{i=1}^{N} \gamma_i\, f_i(x) \quad , \gamma_i \geq\ 0 \,, \sum \gamma_i = 1$$

and by solving

$$\begin{gathered}\min J_\gamma(x)\\ x\ \in S\end{gathered}$$

In particular we have , if

$$\gamma = \begin{bmatrix}\gamma_1\\ \gamma_2\\ \cdot\\ \gamma_N\end{bmatrix} >\ 0$$, then the solution x^* is a strong Pareto's optimum.

instead , if

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdot \\ \gamma_N \end{bmatrix} \geq 0 \,,\ \text{then the solution } x^* \text{ is a weakly Pareto's optimum}$$

Of course , for any choise of the weight vector γ we have different solutions to the problem. In particular all the consideration done before , has been rigorously formulated by the **Geoffrion's Theorem.**

- ## $H_\infty - H_2$ Multiobjective Control

Consider now the following state space representation of the model

$$\begin{cases} \dot{x}(t) = & Ax(t) + B_1u(t) + B_2w(t) \\ z_\infty(t) = & C_1x(t) + D_{11}w(t) + D_{12}u(t) \\ z_2(t) = & C_2x(t) + D_{22}u(t) \end{cases}$$

with $C_1 = [1\ 0\ 0\ 0]$, $D_{11} = D_{12} = 0$ and $C_2 = [0\ 0\ 1\ 0]$, $D_{22} = 0$. Said

$$G_\infty(s) = C_1(sI - A)^{-1}B_2 + D_{11}$$

$$G_2(s) = C_2(sI - A)^{-1}B_2$$

the transfer function that relates our objectives to the noise , the problem that we want to solve is

find a controller $u(t) = K\,x(t)$ such that the closed loop system is asymptotically stable and minimize

$$||G_\infty||_\infty < \gamma_\infty \ , ||G_2||_2 < \gamma_2^2$$

Of course it is a multiobjective optimization method , so we can use for example the scalarization method by defining

$$J = \alpha\gamma_\infty + \beta\gamma_2^2$$

where $\alpha > 0$, $\beta > 0$ are given.

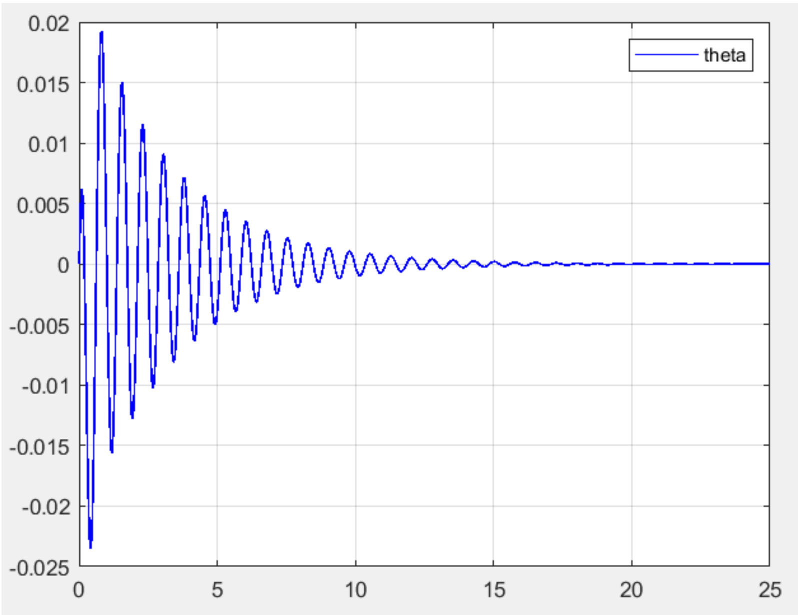
Of course the constraint are the LMI conditions for the minimization of the H_∞ and H_2 norms.

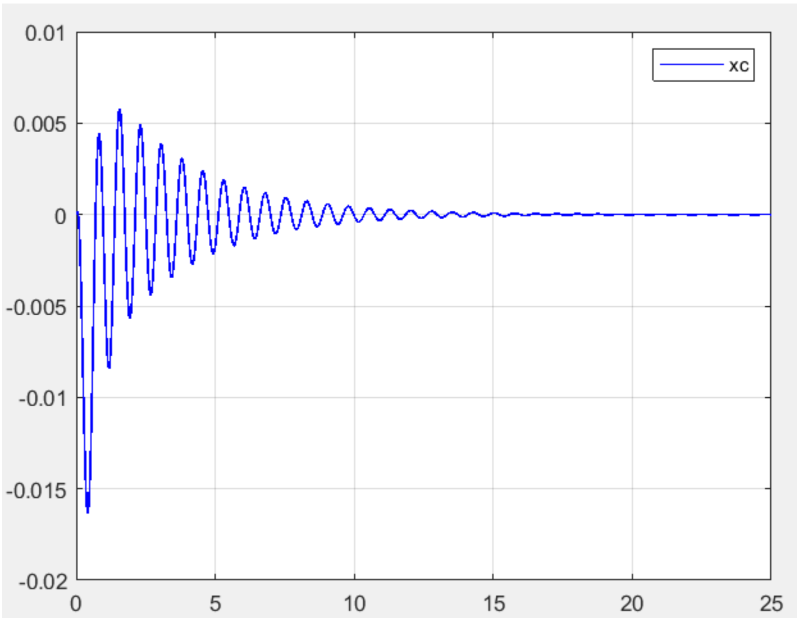
In our case the coeffiecient has been selected respectively $\alpha = 50$, $\beta = 25$. We have supposed that on the system acts a disturbace with intensity $\bar{d} = 15$.

The controller

K = 1.0e+03 *[-1.3692   -0.2018   1.2092   0.6475];

and the behaviour of the system



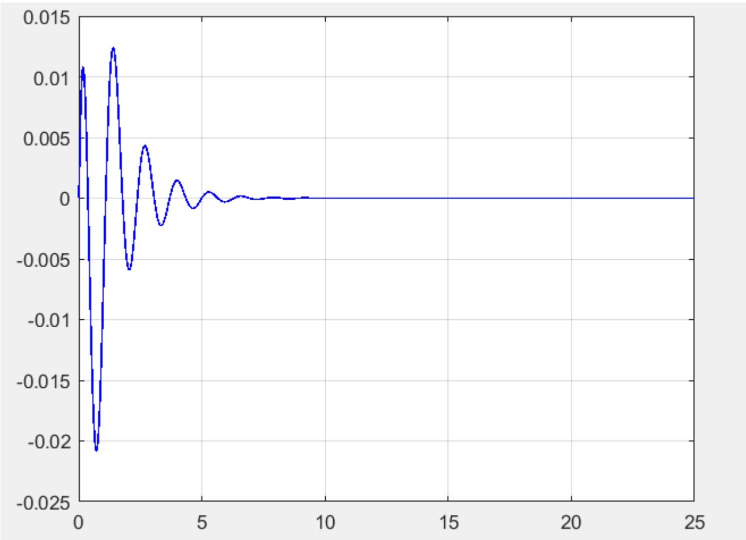


- Minimization and D-Stability Controller

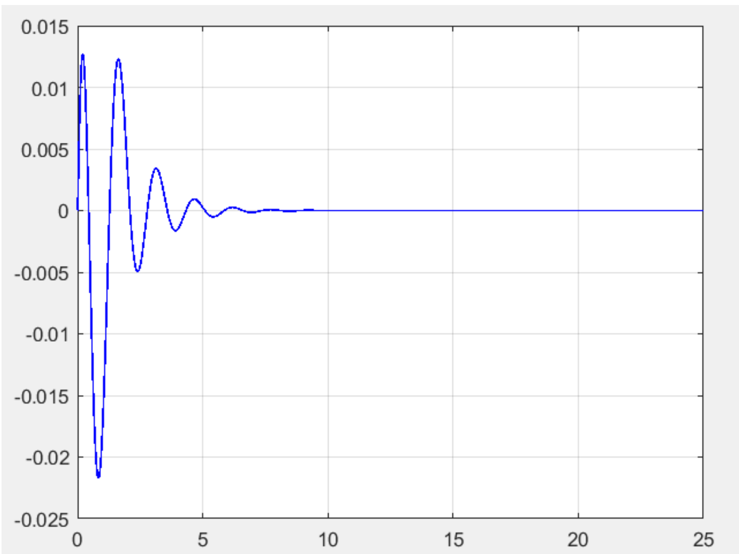
What we want to do now is to define a static controller $u(t) = K\,x(t)$ such that minimize the H_∞ , H_2 norm of the transfer function $G(s)$ between the disturbance and the outputs of interest but , at the same time , guarantees some specifications on the performances.

So , in this section we combine the LMI consitions for an H_∞ controller and D-stabilization.

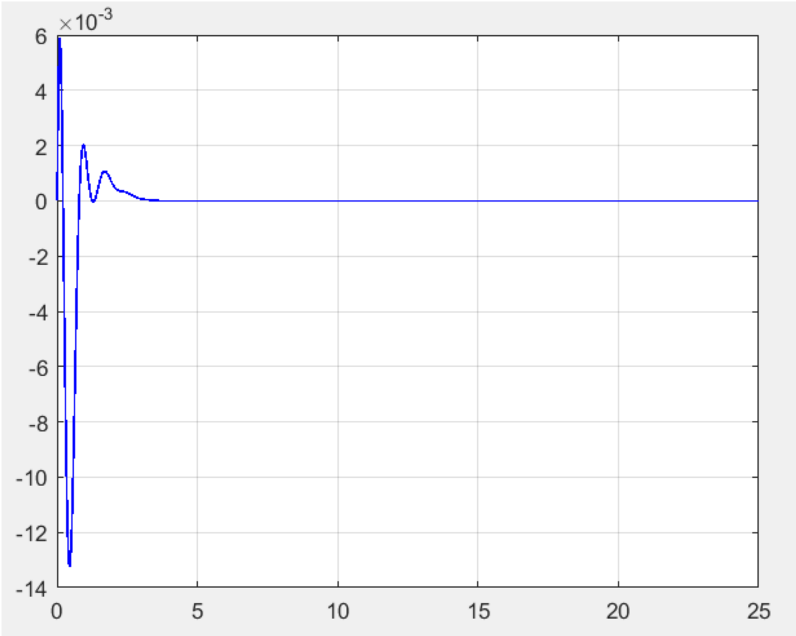
- H_∞ and D-stability



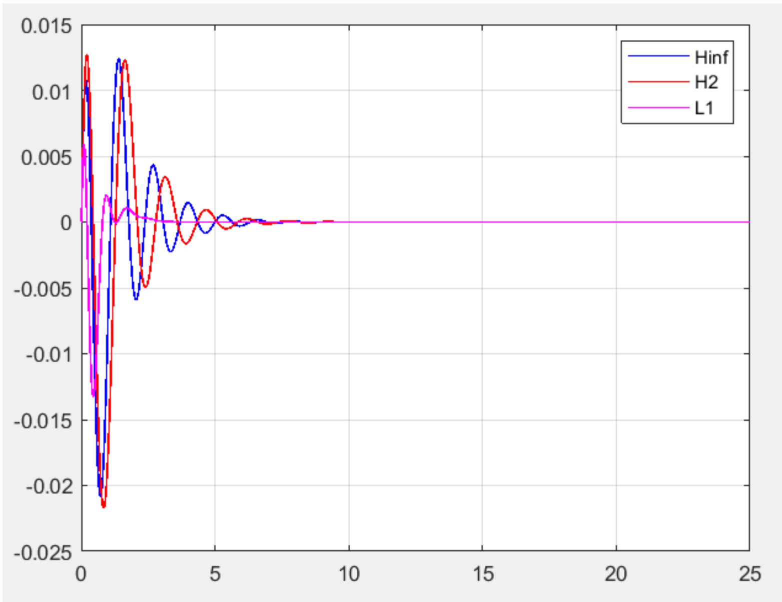
- H_2 and D-stability



- L_1 and D-stability



Now we can conclude this section by doing some comparisons between the last three controllers.



Dynamic Controllers

In this section we are not interested in controller of the form $u(t) = K\,x(t)$ but we want to design an output feedback controller. But , from System Theory , we known that with a controller of the form

$$u(t) = K\,y(t) = K\,C\,x(t)$$

not always is possible to make the closed loop system asymptotically stable , 'cause remember , with this type of controller the eigenvalue must lie on the Evans Locus. The degree of freedom with this type of controller are reduced. Then in this case , we use a dynamical output feedback controller , of the form

$$\begin{cases} \dot{\varphi} = A_k\varphi + B_k y \\ u = C_k\varphi + D_k y \end{cases}$$

and in particular as we will see the matrices of the controller (A_k, B_k, C_k, D_k) will be derived by the solution of appropriate constrained optimization problem.

- ## H_∞ Dynamic Controller

In this case the matrices of the controller will be derived such that the closed loop system will be asymptotically stable and in particular , the H_∞ norm of the transfer function $G(s)$ that relates the noise to the outputs of interest will be minimized.

As in the last section the objective will be the attenuation of the disturbance $d(t)$ on the angle $\theta(t)$.

Let's start by assuming that the system is modelled in the state space as follow

$$\begin{cases} \dot{x}(t) = & A\,x(t) + B_1 u(t) + B_2 w(t) \\ y(t) = & C\,x(t) + D\,w(t) \\ z_\infty(t) = & C_1 x(t) + D_{11} w(t) + D_{12} u(t) \end{cases}$$

and consider the following realization for the controller $C(s)$

$$\begin{cases} \dot{\xi}(t) = A_k \xi(t) + B_k y(t) \\ u(t) = C_k \xi(t) + D_k y(t) \end{cases}$$

the state of the overhall system is given by the composition of the single state of the single system, so

$$x_{cl} = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$$

whose evolution is described by the following dynamical model

$$\dot{x}_{cl} = \begin{bmatrix} \dot{x(t)} \\ \xi(t) \end{bmatrix} = \begin{bmatrix} A + B_1 D_k C & B_1 C_k \\ B_k C & A_k \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} B_1 D_k D + B_2 \\ B_k D \end{bmatrix} w(t)$$

$$z_\infty(t) = \qquad [C_1 + D_{12} D_k C \quad D_{12} C_k] \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + [D_{11} + D_{12} D_k D] w(t)$$

or , in a more compact way

$$\begin{cases} \dot{x}_{cl}(t) = A_{cl} x_{cl}(t) + B_{cl} w(t) \\ z_\infty(t) = \overline{C_1} x_{cl}(t) + \overline{D_1} w(t) \end{cases}$$

The transfer function that relates the disturbance to the output of interest is

$$G(s) = \overline{C_1}\,(sI - A_{cl})^{-1} B_{cl} + \overline{D_1}$$

from which derives immediately the LMIs conditions whose solution will be equals to the existence of a dynamic controller that stabilize the closed loop system and in particular minimize the H_∞ norm of the function $G(s)$.

Proposition

The closed loop system is asymptotically stable and $\left\| G(s) \right\|_\infty < \gamma$ if and only if $\exists\, P = P^T, P > 0$:

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} & \overline{C_1}^T \\ B_{cl}^T P & -\gamma I & \overline{D_1^T} \\ \overline{C_1} & \overline{D_1} & -\gamma I \end{bmatrix} < 0$$

Of course if the controller in unknown the precedent matrix inequality is not an LMI. So we have to transform the precedent expression into an LMI , and obtain at the same time the matrices of the dynamic controller.

With the aim to translate the last inequality into an LMI , consider the following partition of the matrix $P \in R^{2nx2n}$ and of its inverse

$$P = \begin{bmatrix} X & N \\ N^T & * \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} Y & M \\ M^T & * \end{bmatrix}$$

with $X = X^T > 0\,, Y = Y^T > 0$.

By definition $P\,P^{-1} = I_{2n}$, so

$$P\,\Pi_y = P\begin{bmatrix} Y & I \\ M^T & 0 \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & N^T \end{bmatrix} = \Pi_x$$

Remember that a congruence transformation mantains the sign of an inequality , so we have

$$\begin{bmatrix} \Pi_y^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} & \overline{C_1}^T \\ B_{cl}^T P & -\gamma I & \overline{D_1^T} \\ \overline{C_1} & \overline{D_1} & -\gamma I \end{bmatrix} \begin{bmatrix} \Pi_y & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0$$

by doing the product and by definig the following quantities

$$\overline{A_k} = N A_k M^T + N B_k C Y + X B_1 C_k M^T + X(A + B_1 D_k C) Y$$

$$\overline{B_k} = N B_k + X B_1 D_k$$

$$\overline{C_k} = C_k M^T + D_k C Y$$

$$\overline{D_k} = D_k$$

we obtain the following proposition

Proposition

The closed loop system is asymptotically stable and $\left\|G(s)\right\|_\infty < \gamma$ if and only if $\exists\, X = X^T, \exists\, Y = Y^T, \exists\, \overline{A_k}, \overline{B_k}, \overline{C_k}, \overline{D_k}$:

$$\begin{bmatrix} (AY + B_1\overline{C_k}) + (AY + B_1\overline{C_k})^T & \overline{A_k}^T + (A + B_1\overline{D_k}C) & (B_2 + B_1\overline{D_k}D) & (C_1 Y + D_{12}\overline{C_k})^T \\ \overline{A_k} + (A + B_1\overline{D_k}C)^T & (XA + \overline{B_k}C) + (XA + \overline{B_k}C)^T & (XB_2 + \overline{B_k}D) & (C_1 + D_{12}\overline{D_k}C)^T \\ (B_2 + B_1\overline{D_k}D)^T & (XB_2 + \overline{B_k}D)^T & -\gamma I & (D_{11} + D_{12}\overline{D_k}D)^T \\ (C_1 Y + D_{12}\overline{C_k}) & (C_1 + D_{12}\overline{D_k}C) & (D_{11} + D_{12}\overline{D_k}D) & -\gamma I \end{bmatrix} < 0$$

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0$$

In particular , the last LMI , not only guarantees that the matrix $X\,, Y$ are positive defined , but guarantees that the matrix N, M are not singular. Infact, by remembering that

$$P\,\Pi_y = \begin{bmatrix} X & N \\ N^T & * \end{bmatrix}\begin{bmatrix} Y & I \\ M^T & 0 \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & N^T \end{bmatrix} = \Pi_x$$

from which follow immediately that

$$XY + N\,M^T = I$$

and,

$$NM^T = I - XY$$

In particular ,

$$\det(NM^T) = \det(N)\det(M^T) = \det(I - XY)$$

so , the invertibility of the matrix $I - XY$ is equals to the non singularity of the matrices $M\,,N$.

Why the last LMI guarantees the non singularity of the matrix $I - XY$?

Start with the LMI

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0$$

from the Schur Lemma follow that

$$X > 0\,, Y - X^{-1} > 0$$

from which,

$$X > 0\,, Y - X^{-1} > 0 \;\rightarrow X > 0\,, X^{-1}(XY - I) > 0 \;\rightarrow (XY - I) > 0$$

and so the non singularity of the matrix

$$I - XY$$

The matrices M, N can be obtained from the matrix $I - XY$ by applying some matrix factorization , for example the LU factorization.

Now we are ready to apply this theory to the system that we are analyzing.

As said before , our objective is to reject as much as possible the effect of the disturbance $d(t)$ on the variable state $\theta(t)$, so we choose

$$C_1 = [1\ 0\ 0\ 0]\ , \qquad D_{12} = D_{11} = 0$$

For the construction of the outputs vector $y(t)$, we assume that we can measure the angle $\theta(t)$ and the cart position $x_c(t)$, so

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\ , \qquad D = [0\ 0]^T$$

We present now the matrices of the controller obtained by solving the precedent optimization problem , and by assuming that on the system acts a disturbance of intensity $\bar{d} = 15$.

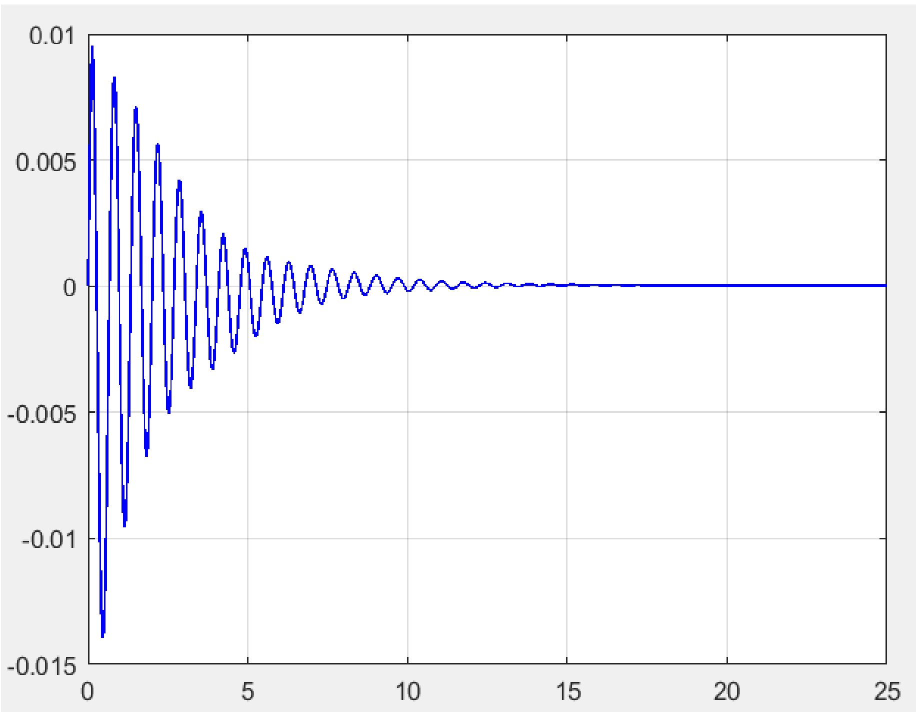
$$A_k = 1.0e03 \begin{bmatrix} 0.0075 & 0.0067 & 0.0018 & 0.4734 \\ -0.1140 & -0.0208 & -0.0064 & -1.7771 \\ -0.0001 & 0.0019 & -0.0002 & 0.0674 \\ -0.0013 & -0.0001 & 0.0001 & -0.0035 \end{bmatrix}$$

$$B_k = \begin{bmatrix} 93.5123 & -5.0572 \\ -589.5364 & -31.5477 \\ 8.2500 & -41.7833 \\ -107.2670 & 5.2665 \end{bmatrix}$$

$$C_k = [4.9330 \quad 1.0187 \quad 0.2669 \quad 85.8861]$$

$$D_k = [-71.4366 \quad -1.6872]$$

and of course the simulation,



- H_2 Dynamic Controller

In this case the matrices (A_k, B_k, C_k, D_k) of our dynamic controller will be derived by the resolution of a constrained optimization problem , whole constrains are the LMI conditions used for the minimization of the H_2 norm of the transfer function $G(s)$ that relates the disturbance to the output of interest.

As in the previous case the objective will be the angle $\theta(t)$.

The step that transform the matrix inequality condition into LMIs are the same as in the previous case , so will be omitted.

Now , consider the following state space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1u(t) + B_2w(t) \\ y(t) = \qquad C\,x(t) + D\,w(t) \\ z_2(t) = \qquad C_2x(t) + D_{21}w(t) \end{cases}$$

with $C_2 = [1 \quad 0 \quad 0 \quad 0]$, $D_{21} = 0$.

The problem is the definition of a dynamic controller

$$\begin{cases} \dot{\xi}(t) = A_k\xi(t) + B_ky(t) \\ u(t) = C_k\xi(t) + D_ky(t) \end{cases}$$

such that the closed loop system is asymptotically stable and the H_2 norm of the transfer function of the augmented system is minimized.

By defining , as in the previous case , augmented system and by doing the same step (congruence transformation , change of variables) we obtain the following proposition for the definition of the dynamic controller

Proposition

The closed loop system is asymptotically stable and $||G(s)||_{H_2} < \gamma$ if and only if $\exists\, X = X^T, Y = Y^T, Q = Q^T, \overline{A_k}, \overline{B_k}, \overline{C_k}, \overline{D_k}$:

$$\begin{bmatrix} (AY + B_1\overline{C_k}) + (AY + B_1\overline{C_k})^T & \overline{A_k}^T + (A + B_1\overline{D_k}C) & (B_2 + B_1\overline{D_k}D) \\ \overline{A_k} + (A + B_1\overline{D_k}C)^T & (XA + \overline{B_k}C) + (XA + \overline{B_k}C)^T & (XB_2 + \overline{B_k}D) \\ (B_2 + B_1\overline{D_k}D)^T & (XB_2 + \overline{B_k}D)^T & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} Q & (C_2Y + D_{22}\overline{C_k}) & (C_2 + D_{22}\overline{D_k}C) \\ (C_2Y + D_{22}\overline{C_k})^T & Y & I \\ (C_2 + D_{22}\overline{D_k}C)^T & I & X \end{bmatrix} > 0$$

$$trace(Q) < \gamma^2$$

We present now the matrices of the controller obtained by solving the precedent optimization problem , and by assuming that on the system acts a disturbance of intensity $\bar{d} = 15$.

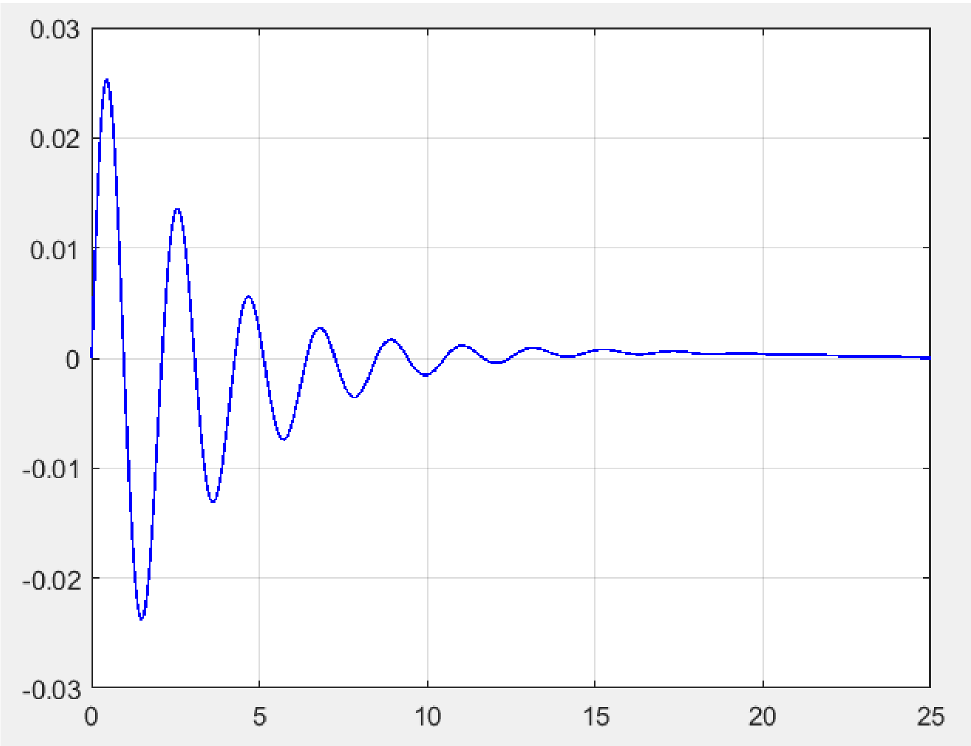
$$A_k = 1.0e03 \begin{bmatrix} 0.0048 & 0.0010 & 0.0103 & 0.3849 \\ -0.0770 & -0.0056 & -0.1501 & -5.8098 \\ -0.0066 & 0.0080 & -0.0168 & -0.1928 \\ -0.0040 & 0.0054 & -0.0114 & -0.1482 \end{bmatrix}$$

$$B_k = \begin{bmatrix} -70.2496 & -29.9944 \\ 553.3059 & -56.7589 \\ 391.1128 & -27.6728 \\ -232.6209 & 13.8696 \end{bmatrix}$$

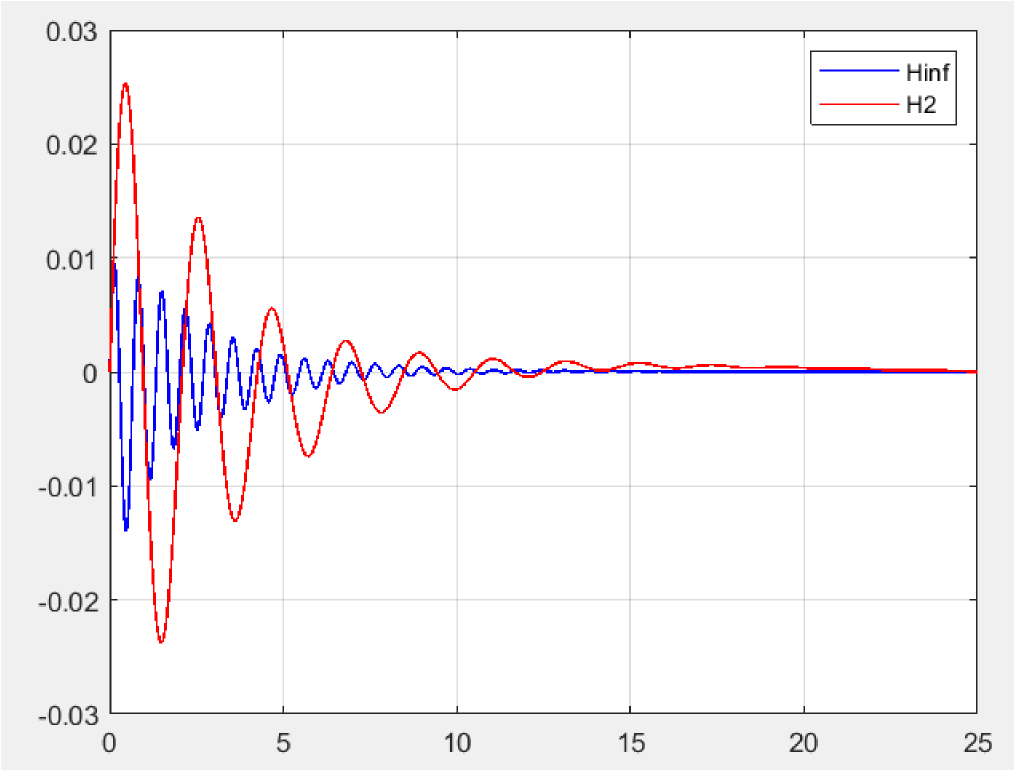
$$C_k = [3.4636 \quad 0.2498 \quad 6.8408 \quad 265.9498]$$

$$D_k = [-123.9455 \quad -0.2754]$$

and of course the simulation,



We conclude this section by making some comparisons between the behaviour of the system by applying to it the precedent controllers



While in the case of static controllers , the H_2 controller was better than H_∞ controller , in this case the H_∞ give us better performance than the H_2 controller.

Robust Control

A mathematical model represents only an approximation of the real behaviour of the physical system. The difference between the real and the behaviour given by the mathematical model is due to:

1) Unmodelled dynamics (Non Parametric Uncertainty)

2) Some parameters may be partially known , but we known the range of variability (Parametric Uncertainty)

In the design of the controller if we don't take in account the presence of such differences , the control law obtained not only couldn't guarantee some performance specification but the closed loop system may be unstable.

In this section we want to project robust controller , so , we want to take in account the differences between the real system and the mathematical model during the phase of design. Infact , as we have seen , the objective of a robust controller is to guarantee performance specification and closed loop stability for each value of the uncertainty.

In this document we'll consider only the Parametric Uncertainty , in particular we suppose that the following parameters are not perfectly known:

- l distance between the CG of the pendulum and the cart
- I_p the moment of inertia of the pendulum respect to the yaw direction

but, of this parameters we known the range of variability,

$$l \in [\underline{l}\,;\,\bar{l}\,]$$

$$I_p \in \left[\,\underline{I_p}\,;\,\overline{I_p}\,\right]$$

So , the objective is the definition of a static controller $u(t) = K\,x(t)$ such that the closed loop system is asymptotically stable for each value of the uncertanty in the range defined above. Of course the controller as we will see , will be designed such that some induced gain are minimized.

Parametric Uncertainty

As stated before , the parametric uncertainty regards the difference between the mathematical model and the physical phenomenum due to the not perfect knowledge of some parameters of the model. Of course , better is the characterization of the uncertainty best are the performance of the controller.

As said in the last section , if we don't take in account the uncertainty in the design phase , the controller obtained couldn't work. So the first step in the design of a robust controller is to take in account the uncertainties of the model , by introducing the so called **Uncertainty State Space Representation.**

- ## LDI - Linear Differential Inclusion

Suppose we have some uncertain parameters , collected in the uncertain parameters vector p. Consider now the following state space representation for a dynamical system

$$x(t+1) = A(p)x(t) \quad , x(0) = x_o$$

of course , the matrices of the system are not constant , but they are function of the uncertain parameters vector p. When p varies in the ammisible set P we have different matrices and so different autonomous dynamical systems.

In particular all the matrices , belong to the set

$$\Omega := \{ A(p) \mid p \in P \}$$

Ω represents a family of dynamical systems. Of course the value that the state assume at the time instant $k+1$ depends on the particular value assumed by the uncertain parameters. As conseguence at the time instant $k+1$ for the system state $x(k+1)$ we have a set of possible values each of which is associated to a particular value of the uncertain parameters vector p.

$$x(t+1) \in \Omega \, x(t) \quad , x(0) = x_o$$

Ω is said to represent a Linear Differential Inclusion , the mathematical definition of a family of dynamical systems.

Of course if we proof that every trajectory of the Linear Differential Inclusion converges to zero , because the real trajectory of real system is a trajectory of the LDI , we've proofed the convergence to zero of the trajectory of the real system under analysis.

So , the idea is to consider a family of dynamical systems , each of which is associated to a particular value of the uncertainty.

- ## Specific LDIs

Let's see now the different ways to take in account the uncertainty of the model, the different structures of Ω. In particular we suppose to choose for the uncertainty the **polytopical description.**

Our differential inclusion will be composed by the following dynamical systems

$$\begin{cases} \dot{x}(t) = A(p)x(t) + B(p)u(t) \\ y(t) = C(p)x(t) + D(p)u(t) \end{cases}$$

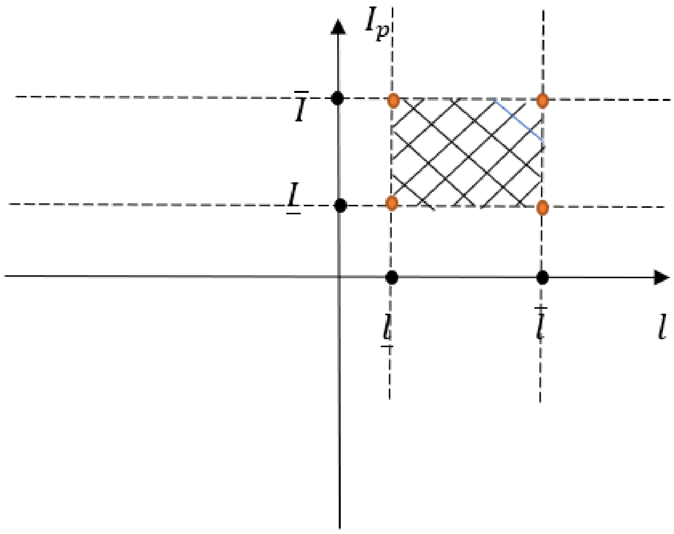
where,

$$\begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} \in \Omega \coloneqq \left\{ \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} \mid \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} = \sum_{i=1}^{l} p_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, p_i \geq 0, \sum_{i=1}^{l} p_i = 1 \right\}$$

The first step is to understand how to construct that representation for our system. As stated before , we suppose that our uncertain parameters are l the distance between the CG of the pendulum respect to the cart , and the moment of inertia I_p. Suppose for these parameters to consider the following ranges of variability

$$l \in [\underline{l}\,;\bar{l}\,], \; I_p \in [\underline{I}\,;\bar{I}\,]$$

Report now the extreme point of thete sets in a cartesian plane



the colored region in the graph represents the ammisible region for the uncertainty, and the orange circles are the vertices. Given the the vertices we can take the convex hull of these points , so that the uncertain parameters vector will belong to the following polytope

$$\Delta_p \coloneqq \left\{ \begin{bmatrix} l \\ I_p \end{bmatrix} \mid \begin{bmatrix} l \\ I_p \end{bmatrix} = p_1 \begin{bmatrix} \underline{l} \\ \underline{I} \end{bmatrix} + p_2 \begin{bmatrix} \underline{l} \\ \bar{I} \end{bmatrix} + p_3 \begin{bmatrix} \bar{l} \\ \underline{I} \end{bmatrix} + p_4 \begin{bmatrix} \bar{l} \\ \bar{I} \end{bmatrix} \;, p_i \geq 0, \sum_{i=1}^{4} p_i = 1 \right\}$$

from this , and from the linearity , follow

$$A\,(p) = A\left(\begin{bmatrix} l \\ I_p \end{bmatrix}\right) = p_1A_1 + p_2A_2 + p_3A_3 + p_4A_4$$

where,

$$A_1 = A\left(\begin{bmatrix} \underline{l} \\ \underline{I} \end{bmatrix}\right), A_2 = A\left(\begin{bmatrix} \underline{l} \\ \bar{I} \end{bmatrix}\right), A_3 = A\left(\begin{bmatrix} \bar{l} \\ \underline{I} \end{bmatrix}\right), A_4 = A\left(\begin{bmatrix} \bar{l} \\ \bar{I} \end{bmatrix}\right)$$

with $p_i \geq 0\,, \sum p_i = 1.$

For the process in analysis , for the uncertain parameters we'll consider the following range of ammisibility

$$l \in [0.005\,, 0.35]\ , \qquad I_p \in [0.005\,, 0.35]$$

and with reference to the couple (A, B_1),to the vertices of our polytope correspond the matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 96.9810 & 0 & 0 & -0.6844 \\ 0 & 0 & 0 & 1 \\ 0.3730 & 0 & 0 & -0.0719 \end{bmatrix}, \ B_{11} = \begin{bmatrix} 0 \\ 0.7605 \\ 0 \\ 0.0798 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 119.3463 & 0 & 0 & -0.8422 \\ 0 & 0 & 0 & 1 \\ 32.1317 & 0 & 0 & -0.2960 \end{bmatrix}, B_{12} = \begin{bmatrix} 0 \\ 0.9358 \\ 0 \\ 0.3289 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1.4012 & 0 & 0 & -0.0099 \\ 0 & 0 & 0 & 1 \\ 0.0054 & 0 & 0 & -0.0693 \end{bmatrix}, \ B_{13} = \begin{bmatrix} 0 \\ 0.0110 \\ 0 \\ 0.0770 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 54.2681 & 0 & 0 & -0.3830 \\ 0 & 0 & 0 & 1 \\ 14.6106 & 0 & 0 & -0.1723 \end{bmatrix}, B_{14} = \begin{bmatrix} 0 \\ 0.4255 \\ 0 \\ 0.1915 \end{bmatrix}$$

Now , we are ready to design a static state feedback $u(t) = K\,x(t)$ that stabilize the closed loop system for each value of the uncertainty.

Robust Control

In this section we want to derive sufficient conditions to the existence of a stabilizing controller for the uncertain model. Like in the previous case , also here the Lyapunov Theory gives us the solution.

We have stated before as the trajectory of the uncertain model can be seen as a trajectory of a family of dyamical systems (LDI) , in particular the trajecotry associated to the real value assumed by the uncertain parameters vector $\boldsymbol{p}$.

So , if we proof that all the trajectories of the LDI converge to zero $\forall\, x_o$ then we can conclude the asymptotic stability for the uncertain system.

Let's start our discussion by presenting a sufficient condition for the asymptotic stability , the **quadratic stability.**

An uncertain system is quadratically stable if

$$\exists\, V\big(x(t)\big) = x(t)^T P\, x(t)\,, P = P^T\,, P > 0 : \frac{\partial\, V\big(x(t)\big)}{\partial t} < 0 \quad , \qquad \forall\, p\ \in P$$

so , the trajectory of our LDI converge to zero at least quadratically.

Of course by the second method of Lyapunov , a system that is quadratically stable is of course asymptotically stable. The viceversa is not true in general.

A system should be Asymptotically Stable when it is not Quadratically Stable.

To undertand better the following pages , we have to define a beautiful result related to the convex optimization.

Proposition

Let $\Omega\ \subseteq R^n$ be a convex and compact set. Let $f : \Omega \rightarrow R$ be a convex function, then we have

$$f(x) < 0\ \ \forall x\ \in \Omega\ \ se\ e\ solo\ se\ \ f(x) < 0\ \ \forall\, x\ \in \Omega_e$$

So, as stated below , verify the negative of a function on a compact and convex set is equals to verify the negativity on the exteme points.

Robust Stabilization

Suppose that we have choosen for the uncertainty representaion the **polytopical description ,** so our dynamical system in free evolution will be represented by the following state space representation

$$\begin{cases} \dot{x}(t) = A(p)x(t) \\ x(0) = \qquad x_o \end{cases}$$

with,

$$A(p) \in \Omega := \left\{ A(p) | \, A(p) = \sum p_i A_i \, , p_i \geq 0 \, , \sum p_i = 1 \right\}$$

So , by definition given above, the system is quadratically stable if

$\exists \, V(x) = x^T P \, x \, , \; P = P^T > 0 :$

$$\dot{V}\big(x(t)\big) = x(t)^T \big(A(p)^T P + PA(p)\big) x(t) < 0 \;\; , \forall \, p \in P$$

that is equals to require

$$A(p)^T P + PA(p) < 0 \;\; , \qquad \forall \, p \; \in P$$

But , $\dot{V}\big(x(t)\big)$ is linear in p , so it is convex respect to p**.** And by the fact that a polytope is a convex and a compact set , from the last proposition we need only to verify the negativity of the function on the vertices of the polytope. So we have the following proposition

Proposition

The system is quadratically stable if $\exists \, P = P^T > 0$:

$$A(p)^T P + PA(p) = \sum_{i=1}^{4} p_i (A_i^T P + PA_i) < 0 \;\; , \, \forall p \;\in P$$

but it is equals to verify that

$$\exists \, P = P^T > 0 \;\; : \;\; A_i^T P + PA_i < 0 \;\;\; , \qquad i = 1 \, ,2 \; 3 \, ,4$$

is negative on the polytope’s vertices.

Note that are a set of LMI conditions on the unknown matrix P.

Defined the analysis conditions , in the hypothesis of existence , we can obtain a controller $u(t) = K\, x(t)$ such that the closed loop system is quadratically stable.

With reference to the following uncertain model

$$\begin{cases} \dot{x}(t) = A(p)x(t) + B_1(p)u(t) \\ x(0) = \qquad\qquad\qquad x_0 \end{cases}$$

if $\exists\, K$:

$$\begin{cases} \dot{x}(t) = (A(p) + B_1(p)\, K)x(t) \\ x(0) = \qquad\qquad\qquad x_0 \end{cases}$$

is quadratically stable , then $\exists\, P = P^T > 0$:

$$(A_i + B_{1i}K)^T P + P(A_i + B_{1i}K) < 0 \quad i = 1\,,2\ 3\,,4$$

pre-post multiplying by P^{-1} ,

$$P^{-1}(A_i + B_{1i}K)^T + (A_i + B_{1i}K)\, P^{-1} < 0 \quad i = 1,2,3,4$$

and by defining with $X = P^{-1}, W = K\, X$, we obtain

$$XA_i^T + W^T B_{1i}^T + A_i X + B_{1i} W < 0 \quad i = 1,2,3,4$$

and the controller is given by $K = W\, X^{-1}$.

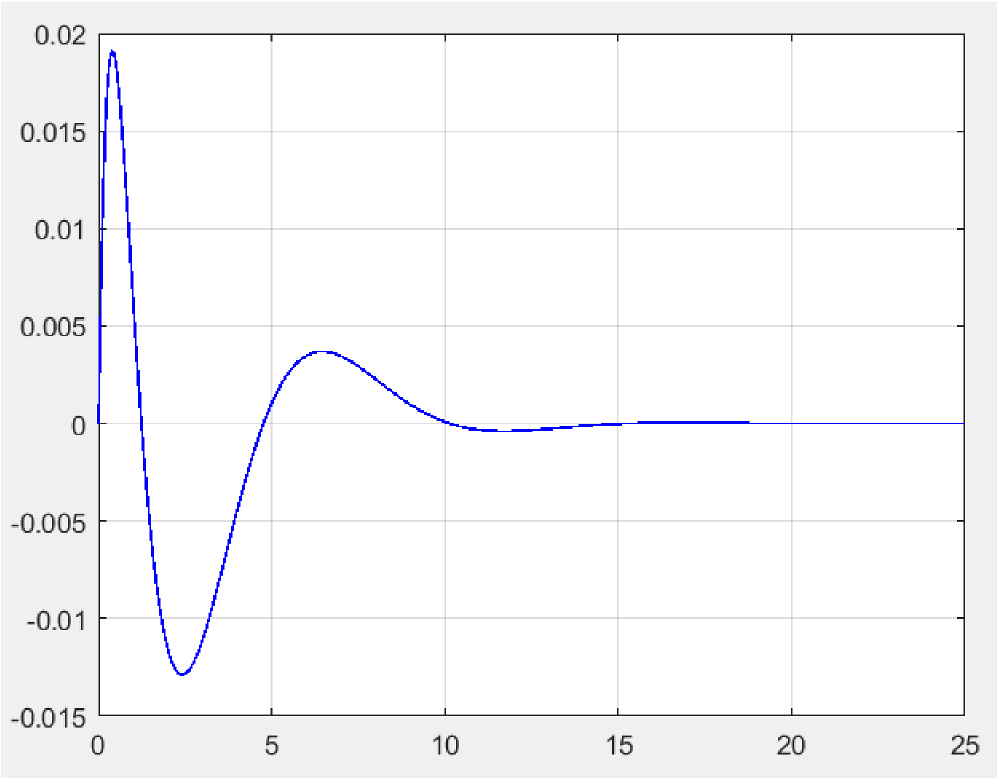
Now , we are ready to design a robust stabilizing controller for the process in analysis. In particular we have for the uncertain parameters the following values

$$l = 0.22 \;, \qquad I_p = \; 0.0341.$$

The controller obtained by solving a constrained optimization problem with 4 LMIs as constraints is

K = 1.0e+03 * [-1.9077 -1.2209 0.0441 0.1280];

and by supposing that on the system acts a disturbance of intensity $\bar{d} = 15$, the behaviour of the system is



- H_∞ Robust Control

In this section the objective is the definition of a controller $u(t) = K\,x(t)$ such that the closed loop system is asymptotically stable and the H_∞ norm of the transfer function that relates the disturbance with the outputs of interest is minimized.

With reference to the following state space representation

$$\begin{cases} \dot{x}(t) = & A(p)x(t) + B_1(p)u(t) + B_2(p)w(t) \\ z_\infty = & C_1(p)x(t) + D_{11}(p)w(t) + D_{12}(p)u(t) \end{cases}$$

with,

$$\begin{bmatrix} A(p) & B_1(p) & B_2(p) \\ C_1(p) & D_{11}(p) & D_{12}(p) \end{bmatrix} \in \Omega := \left\{ \begin{bmatrix} A(p) & B_1(p) & B_2(p) \\ C_1(p) & D_{11}(p) & D_{12}(p) \end{bmatrix} \mid \begin{bmatrix} A(p) & B_1(p) & B_2(p) \\ C_1(p) & D_{11}(p) & D_{12}(p) \end{bmatrix} = \sum_{i=1}^{4} p_i \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & D_{11i} & D_{12i} \end{bmatrix}, p_i \geq 0, \sum_{i=1}^{4} p_i = 1 \right\}$$

let $u(t) = K\,x(t)$, the closed loop system will be

$$\begin{cases} \dot{x}(t) = & (A(p) + B_1(p)K)x(t) + B_2(p)w(t) \\ z_\infty = & (C_1(p) + D_{12}(p)K)x(t) + D_{11}(p)w(t) \end{cases}$$

and the "transfer functions" between the disturbance and the objective are

$$T_\infty(s) = (C_1(p) + D_{12}(p)K)\big(sI - (A(p) + B_1(p)K)\big)^{-1} B_2(p) + D_{11}(p)$$

In the uncertain case , the induced gains to use as objective functions should be robust respect to the variations of the uncertainty

$$||T_\infty||_\infty^{wc} := \frac{\sup||T_\infty w||_2}{||w||_2, p \in P}$$

So , minimizing an induced gain in the worst case is equivalent to minimize the maximum induced gain given by the worst value of the uncertain parameters vector p in the ammisible range.

Based on the results obtained in the last section , we have the following proposition

Proposition

$\exists\, K$ such that the closed loop system is asymptotically stable and $||T_\infty||_\infty^{wc} < \gamma$ if

$\exists\, X = X^T > 0 , \exists\, W :$

$$\begin{bmatrix} (A_iX + B_{1i}W) + (A_iX + B_{1i}W)^T & B_{2i} & (C_{1i}X + D_{12i}W)^T \\ B_{2i}^T & -\gamma I & D_{11i}^T \\ (C_{1i}X + D_{12i}W) & D_{11i} & -\gamma I \end{bmatrix} < 0 \quad i = 1,2,3,4$$

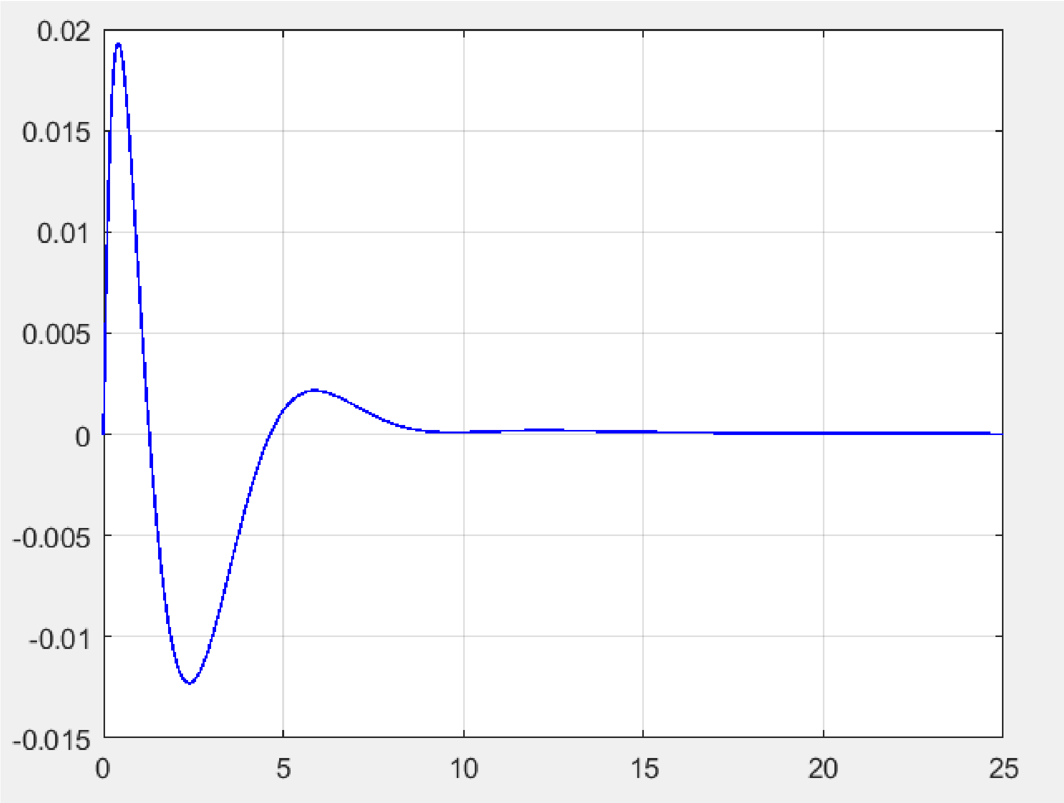
the robust controller will be given by $K = W\,X^{-1}$.

Of course in the practice we are interested to determine the smallest γ , so we have to solve a constrainted optimization problem whose constrains are the LMIs defined above.

Supposing that on the system acts a disturbance with intensity $\bar{d} = 15$, the controller is given by

K = 1.0e+03 *[-8.7267   -6.6318   0.1086   0.7481];

and the system behaviour will be



By doing the same considerations, our attention now is translated into the definition of a controller $u(t) = K\, x(t)$ that stabilize the closed loop system and minimize the H_2 norm of the function $G(s)$.

With reference to the following state space model

$$\begin{cases} \dot{x}(t) = A(p)x(t) + B_1(p)u(t) + B_2(p)w(t) \\ z_2(t) = \qquad\qquad C_2(p)x(t) + D_{22}(p)u(t) \end{cases}$$

with,

$$\begin{bmatrix} A(p) & B_1(p) & B_2(p) \\ C_2(p) & & D_{22}(p) \end{bmatrix} \in \Omega := \left\{ \begin{bmatrix} A(p) & B_1(p) & B_2(p) \\ C_2(p) & & D_{22}(p) \end{bmatrix} \mid \begin{bmatrix} A(p) & B_1(p) & B_2(p) \\ C_2(p) & & D_{22}(p) \end{bmatrix} = \sum_{i=1}^{4} p_i \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_{2i} & & D_{22i} \end{bmatrix}, p_i \geq 0\,, \sum_{i=1}^{4} p_i = 1 \right\}$$

said $u(t) = K\, x(t)$, the closed loop system becomes

$$\begin{cases} \dot{x}(t) = (A(p) + B_1(p)K)x(t) + B_2(p)w(t) \\ z_2(t) = \qquad\qquad C_2(p)x(t) + D_{22}(p)u(t) \end{cases}$$

and by defining the worst case induced gain , we have the following proposition

Proposition

$\exists\, K$ such that the closed loop system is asymptotically stable and $\left|\left|T_2\right|\right|_{H_2}^{wc} < \gamma$ if

$\exists\, X = X^T > 0 , \exists\, W , \exists\, Q = Q^T$:

$$\begin{bmatrix} (A_i X + B_{1i} W) + (A_i X + B_{1i} W)^T & B_{2i} \\ B_{2i}^T & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} Q & (C_{2i} X + D_{22i} W) \\ (C_{2i} X + D_{22i} W)^T & X \end{bmatrix} > 0$$

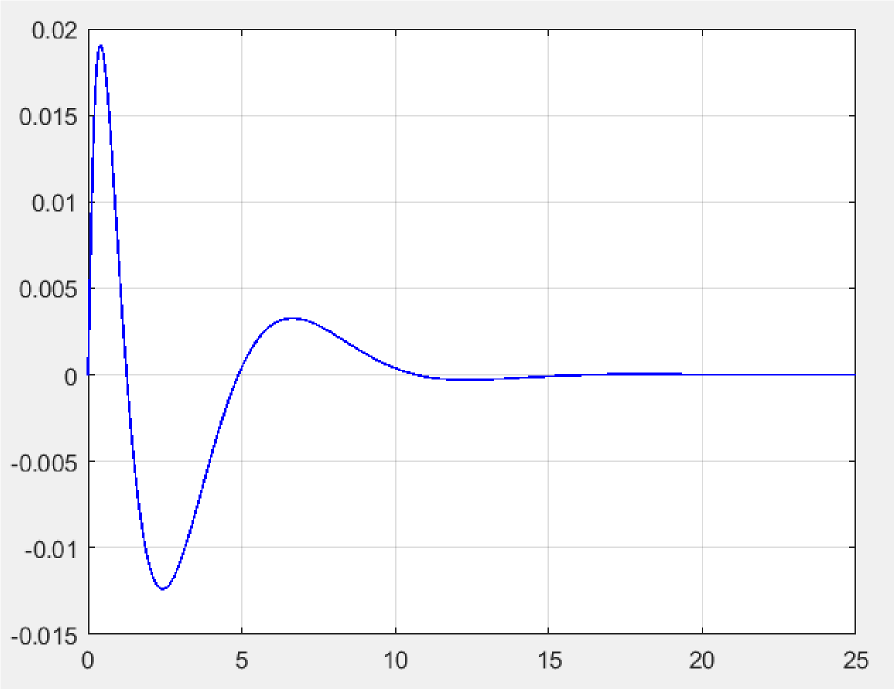
$$trace(Q) < \gamma^2$$

and the controller will be $K = W\, X^{-1}$.

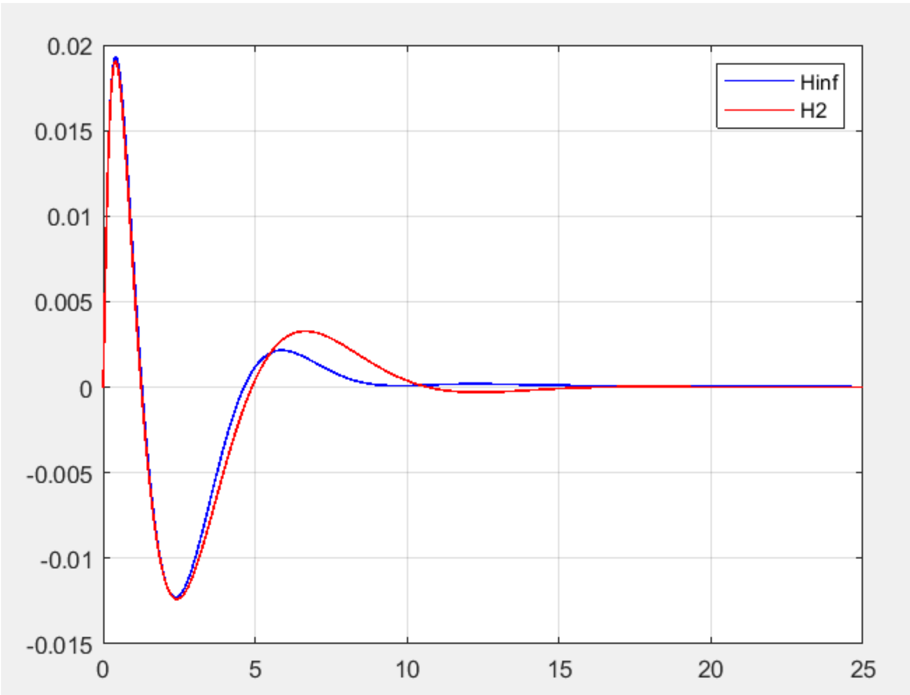
The controller K is

K = 1.0e+03 * [-1.8818   -1.1877   0.0401   0.1210];

and by assuming that on the system acts a disturbance with intensity $\bar{d} = 15$, the behaviour



We conclude this section by making some comparisons between the controller obtained in the last two section.



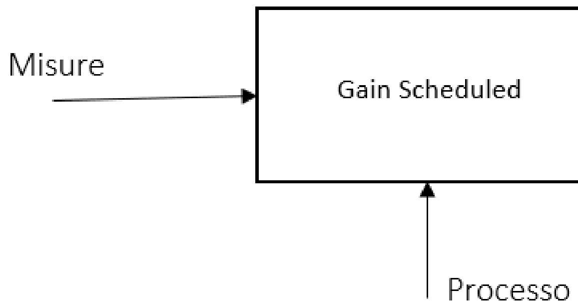
As in the previous case, the H_∞ controller give us better performance than the H_2 controller.

LPV Systems and Gain – Scheduled Control

In the last section , with the Robust Control , we have obtained controllers $u(t) = Kx(t)$ that guarantee performance specification and closed loop stability for each value of the uncertainty in the ammisible range.

But as we have seen , the controller gain K is constant for each value of p in P. So , from this point of view , the robust controllers doesn't take in account the non stationary of the process.

The Gain-Scheduled Control strategies have been introduced to increase / improve the usually weakly performance of the robust control. The main idea is to modify the controller gain K on the base of measurements acquired in real-time.



In particular the controller has some informations about the process. In this way we obtain a controller that reacts to the non stationary of the model , and so give us better performance than a robust controller.

Gain Scheduled controllers are natural extenions of previous theory when we consider the class of LPV or Quasi-LPV dynamical systems.

Linear Parameters Varying (LPV) Systems

This class of dynamical systems are represented by state space representation , whose matrices (A, B, C, D) are not constant but function of a system parameters vector $p(t)$ whose components are measurable in real time

$$\Sigma_{LPV} \begin{cases} \dot{x}(t) = A\big(p(t)\big)x(t) + B\big(p(t)\big)u(t) \\ y(t) = C\big(p(t)\big)x(t) + D\big(p(t)\big)u(t) \end{cases}$$

where , as said before , the components $p_i(t)$ of $p(t)$ represent parameters of the system , that varying in the time , but measurable. In particular these parameters are usually bounded

$$\underline{p_i} \leq p_i \leq \overline{p_i}$$

But , what is the difference between an LPV , LTI and LTV system?

The difference between an LPV and LTI system is very easy. While an LTI system is characterized by constant matrices , in the LPV case the matrices are not constant but function of the vector $p(t)$.

In particular an LPV system , becomes a LTI system when

$$p(t) = p^* \ \forall t$$

The difference between an LPV and LTV system is little less visible. For each trajectory of the vector $p(t)$ we have an LTV system

$$p(t) = p^*(t) \ \forall t$$

So we can conclude that an LPV system represents a family of LPV systems each of which is associated to a particular trajectory of the vector $p(t)$. So , verify the stability of an LPV system is equivalent to verify the stability of a family of linear time varying system.

- ## LPV Stabilization

Remember that the stability is a property related to the free state response of the system , so now we can consider the following autonomous LPV system

$$\begin{cases} \dot{x}(t) = A\big(p(t)\big)x(t) \\ x(0) = \qquad\qquad x_o \end{cases}$$

with the vector $p(t)$ measurable $p(t) \in P \ \forall \ t \geq 0$, and

$$A\big(p(t)\big) = \sum p_i(t)A_i \, , p_i(t) \geq 0 \, , \sum p_i(t) = 1$$

so we have choosen for the system the **polytopical description.**

Definition

The LPV system is asymptotically stable if $\forall\, x_o \in R^n$, $\forall\, p(.) : p(t)\ \in P\ \forall\, t \geq 0$ we have

$$\lim_{x \to x_o} x(t) = 0_x$$

Remember that the Quadratic Stability implies the Asymptotical Stability of the system , and in particular is musch easier to verify. So we can procede our discussion by deriving some conditions for the quadratic stability for the LPV system.

In particular the objective is the definition of a control law

$$u(t) = K\big(p(t)\big)x(t)$$

where

$$K\big(p(t)\big) = \sum p_i(t)K_i$$

such that the closed loop system is asymptotically stable for each value of the parameters vector $p(t)$, measurable in real time.

Let's start by remember the definition of quadratic stability

Definition

The LPV system is quadratically stable if $\exists\, P = P^T\,, P > 0\ \big(\, V\big(x(t)\big) = x(t)^T P x(t)\,\big)$:

$$\frac{\partial V\big(x(t)\big)}{\partial t} < 0\,, \forall\, p(.) : p(t) \in P\ \forall\, t \geq 0$$

Suppose now there exist a controller $u(t) = K\big(p(t)\big)x(t)$ such that the closed loop system is Q. S. , so

$\exists\, P = P^T, P > 0$:

$$\Big(A\big(p(t)\big) + B\big(p(t)\big)K\big(p(t)\big)\Big)^T P + P\,\Big(A\big(p(t)\big) + B\big(p(t)\big)K\big(p(t)\big)\Big) < 0\,, \forall p(t) \in P$$

pre and post multiplying by P^{-1},

$$P^{-1}\Big(A\big(p(t)\big) + B\big(p(t)\big)K\big(p(t)\big)\Big)^T + \Big(A\big(p(t)\big) + B\big(p(t)\big)K\big(p(t)\big)\Big)P^{-1} < 0$$

and by defining $X = P^{-1}$

$$XA(p(t))^T + XK^T(p(t))B^T(p(t)) + A(p(t))X + B(p(t))K(p(t))X < 0$$

substituting the relative expressions for A , B we obtain

$$X \left(\sum p_i(t)A_i\right)^T + X\left(\sum p_iK_i\right)^T \left(\sum p_jB_j\right)^T + \left(\sum p_iA_i\right)X + \left(\sum p_iB_i\right)\left(\sum p_jK_j\right)X < 0$$

and , said $K_i = W_iX^{-1}$

$$X \left(\sum p_i(t)A_i\right)^T + X\left(\sum p_iW_iX^{-1}\right)^T \left(\sum p_jB_j\right)^T + \left(\sum p_iA_i\right)X + \left(\sum p_iB_i\right)\left(\sum p_jW_jX^{-1}\right)X < 0$$

remembering that

$$\sum p_j(t) = 1 \ \ \forall t \geq 0$$

the matrix inequality can be rewritten as

$$\sum_i \sum_j p_i(t)p_j(t)[XA_i^T + W_i^TB_j^T + A_iX + B_iW_j] < 0 \, , \forall p_i(t) \, , p_j(t)$$

Now, fixed $p_i(t)$ the last expression , is linear in $p_j(t)$ and so convex in $p_j(t)$, so is sufficient to verify the last inequality only on the vertices of the polytope

$$(XA_i + B_iW_j) + (XA_i + B_iW_j)^T < 0 \, , i = 1 \, ,2 \, .. \, l \, , j = 1 \, , 2 \, ,.. \, l$$

and , at the end we arrive to the following proposition

Proposition

The LPV polytopical system is quadratically stabilizable by a state retroaction $u(t) = K(p(t))x(t)$ if

$\exists \ X = X^T > \ 0 \, , \exists \ W_1, W_2 \, , \dots \, , W_l$:

$$(XA_i + B_iW_j) + (XA_i + B_iW_j)^T < 0 \, , i = 1 \, ,2 \, .. \, l \, , j = 1 \, , 2 \, ,.. \, l$$

and of course the controllers are given by $K_i = W_iX^{-1}$.

Now we are ready to apply all this theory to our inverted pendulum. As in the last section , we suppose that the uncertaint parameters (but in this case measurable) are the distance l between the CG of the pendulum and the cart a, and yaw moment of inertia I_p.

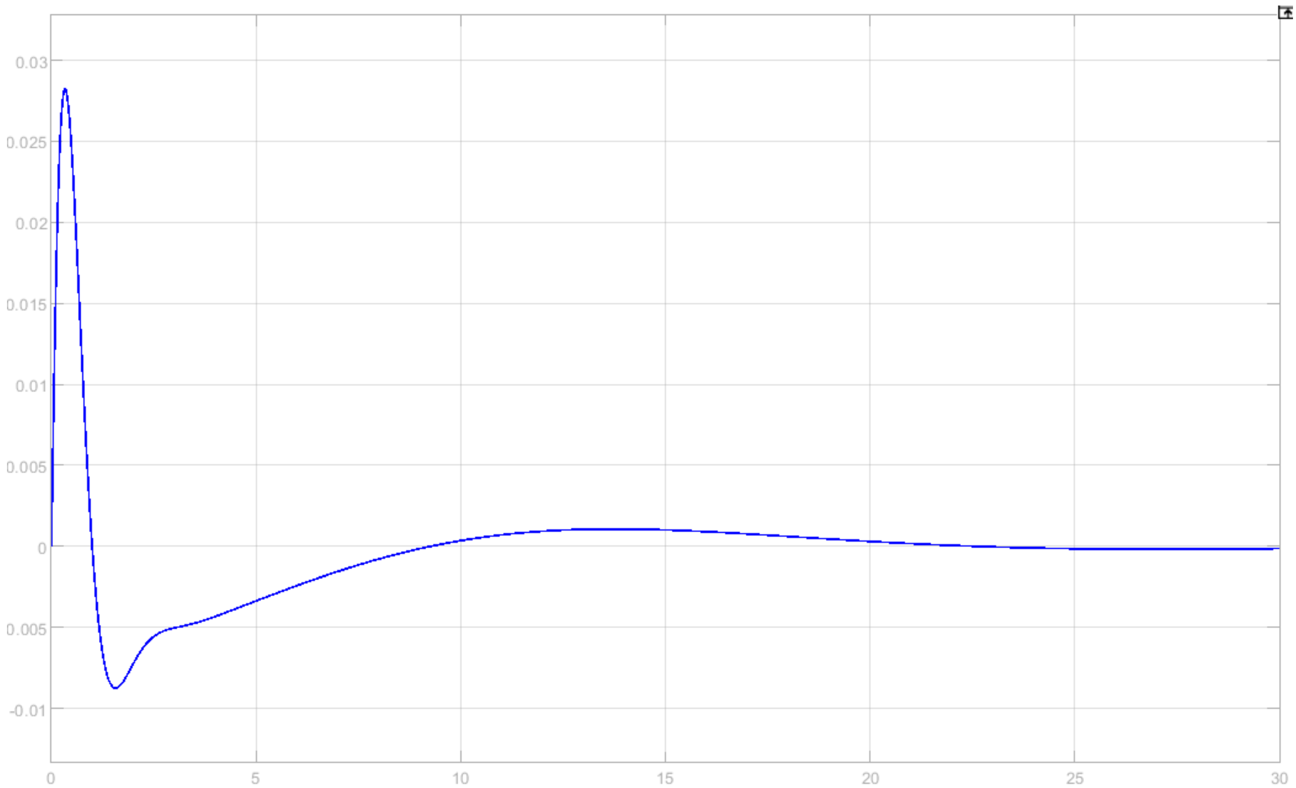
We suppose that these quantities belong to

$$l \in [\underline{l} \,;\, \bar{l}] \quad , I_p \in \left[\underline{I_p} \,;\, \overline{I_p}\right]$$

Defined the LMI conditions and the expressions of the controllers K_i , we have for the process the following controller

$$K = \sum p_i K_i = 1.0e03\,[-1.5663 \quad -1.0407 \quad 0.0216 \quad 0.0860];$$

and supposing that on the system acts a disturbance of intensity $\bar{d} = 15$, the time behaviour of the angle $\theta(t)$ is



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