Falling of a Pencil and Other Things in Scalar Ultralight Dark Matter Enviroment

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Abstract

This note investigate at first the motion of a rigid pencil on a horizontal table, assuming that all mass oscillates coherently from the coupling to a homogeneous ultralight scalar field background. The pencil is initially erect, then it undergoes a spontaneous symmetry breaking, i.e., it falls, the question is when would this pencil slide during the falling process. It is shown that the condition of sliding gets modified greatly, depending on the magnitude of coupling, its oscillation frequency and the phase of oscillation. In addition, the falling process and a similar problem of sliding ladder (both without friction), together whith some other processes (pendulum and oblique projectile motion) are simulated numerically.

CONTENTS

I. THE PROBLEM

A. Falling without Slide

The situation is depicted in the figure below. The pencil is modeled as a straight rigid rod with uniform mass ditribution, mass $m(t) = 1 + \beta \cos(\mu t + \phi)$, where β, μ, ϕ are constants, and length $l = 1$ (also we work in the unit $g = 1$). As the pencil falls, θ increases from 0 to $\pi/2$.

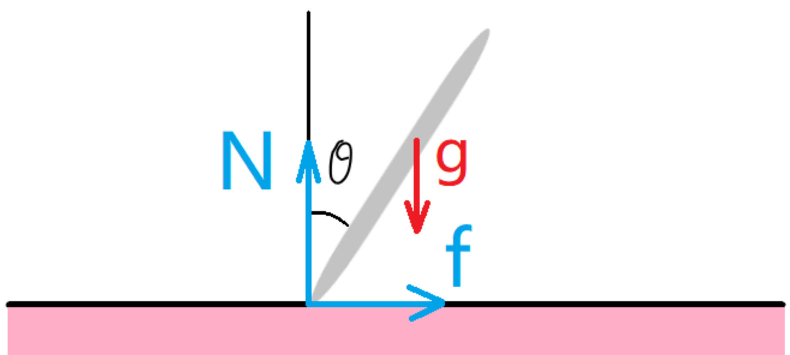

FIG. 1. The problem set-up.

To demonstrate the effect of $m(t)$, we do a lagrangian analysis for the frictionless case[1],

$$L = \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\theta}^2 + \frac{1}{2}mv^2 - \frac{1}{2}mgl\cos\theta \tag{1}$$

where $\mathbf{v}$ is the velocity of the mass center. Assuming the pencil does not slide, $v = \frac{l}{2}\dot{\theta}$, then

$$L = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 - \frac{1}{2}mgl\cos\theta \tag{2}$$

Now the rotation dynamics reads

$$\frac{d}{dt}\left(\frac{1}{3}ml^2\dot{\theta}\right) = \frac{1}{2}mgl\sin\theta \tag{3}$$

or

$$\ddot{\theta} + \frac{\dot{m}}{m}\dot{\theta} = \frac{3}{2}g\sin\theta \tag{4}$$

The vertical motion of the mass center is

$$\frac{d}{dt}\left(m\frac{d}{dt}(\frac{1}{2}l\cos\theta)\right) = N - mg \tag{5}$$

or

$$N/m = 1 - \frac{1}{2}\sin\theta\left(\ddot{\theta} + \frac{\dot{m}}{m}\dot{\theta} + \cot\theta\dot{\theta}^2\right) = g - \frac{1}{2}\left(\frac{3}{2}g\sin^2\theta + \cos\theta\dot{\theta}^2\right) \tag{6}$$

where N is the normal force exerted by the table. Now incorporating friction from the table, these equations do not change, but horizontal motion of the mass center does change. Assuming the contacting point doesn't slide, we have

$$\frac{d}{dt}\left(m\frac{d}{dt}(\frac{1}{2}l\sin\theta)\right) = f \tag{7}$$

or

$$f/m = \frac{1}{2}\cos\theta\left(\ddot{\theta} + \frac{\dot{m}}{m}\dot{\theta} - \tan\theta\dot{\theta}^2\right) = \frac{1}{2}\left(\frac{3}{2}g\sin\theta\cos\theta - \sin\theta\dot{\theta}^2\right) \tag{8}$$

where f is the friction force. In the case of static mass, if the pencil does not slide, the total energy of the system

$$H = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 + \frac{1}{2}mgl\cos\theta = \frac{1}{2}mgl \tag{9}$$

is conserved, then f, N are fixed as functions of the angle and time, the result is

$$\begin{aligned} \dot{\theta} &= \sqrt{3\,(1-\cos\theta)} \\ N/mg &= 1 - \frac{3}{4}\sin^2\theta + \frac{3}{2}\cos^2\theta - \frac{3}{2}\cos\theta \\ f/mg &= \frac{9}{4}\sin\theta\cos\theta - \frac{3}{2}\sin\theta \end{aligned} \tag{10}$$

The slide cannot happen if $r \equiv f/N$ is smaller then the critical friction factor of the table. Let us briefly remarks on this function, $f(\theta)$ is initially positive but changes sign near 47° while N remains postive until near 71°, at 40° $r(\theta)$ acquires a local maximum value of $r_{\mathrm{crit}} = 0.37$.

If mass varies with time, however, the Hamiltonian is no longer conserved, we have to solve the equation of motion (4) directly, which for the mass oscillation model above, is explicitly

$$\ddot{\theta} + \frac{-\beta\mu\sin(\mu t + \phi)}{1 + \beta\cos(\mu t + \phi)}\dot{\theta} - \frac{3}{2}g\sin\theta = 0 \tag{11}$$

and calculate f, N using (6) and (8).

For a point pendulum with unit length, the potential energy is $-mgl\cos\theta$, the $-\frac{3}{2}g/l$ is replaced by g/l, where the meaning of θ is slightly different. For small angle, the equation of motion is $\ddot{\theta} = -(g/l)\theta$, with freuqency $\omega = \sqrt{g/l}$. This is the whole story for a rigid pendulum, while for a hanging pendulum one has to check the tension along the string, since when it vanishes the hanging mass undergoes a free falling.

B. Falling without Friction

Now we study the falling process assuming there is no friction at all. The situation is depicted in the figure below (in this section we neglects the wall).

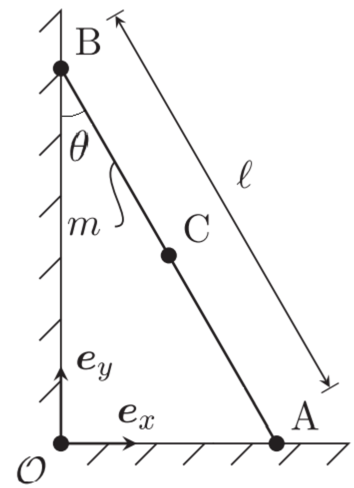

FIG. 2. The problem set-up.

With no friction, the mass center will not acquire a horizontal velocity and the problem is very simple to analyse, the lagrangian reads

$$L = \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\theta}^2 + \frac{1}{2}m\left(\frac{l}{2}\sin\theta\,\dot{\theta}\right)^2 - \frac{1}{2}mgl\cos\theta = \frac{1}{2}ml^2\left(\frac{1}{12}+\frac{1}{4}\sin^2\theta\right)\dot{\theta}^2 - \frac{1}{2}mgl\cos\theta \tag{12}$$

The equation of motion is

$$\frac{d}{dt}\left[ml^2\left(\frac{1}{12}+\frac{1}{4}\sin^2\theta\right)\dot{\theta}\right] = \frac{1}{2}mgl\sin\theta + \frac{1}{4}ml^2\sin\theta\cos\theta\,\dot{\theta}^2 \tag{13}$$

or $(g = l = 1)$

$$\ddot{\theta} + \frac{\dot{m}}{m}\dot{\theta} + \frac{\sin\theta\cos\theta}{4\left(\frac{1}{12}+\frac{1}{4}\sin^2\theta\right)}\dot{\theta}^2 - \frac{\sin\theta}{2\left(\frac{1}{12}+\frac{1}{4}\sin^2\theta\right)} = 0 \tag{14}$$

The normal force exerted by the table is given by 6. In the static mass case, the zero point of $N(\theta)$ would be different from that of last section, and it will actually be reached, when

$$g = \frac{l}{2}(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}) \tag{15}$$

then the pencil lose support from the table and rotates with uniform angular velocity, until it touches the table again, this process is somewhat intricate and may deserve careful investigation. If the mass varies with time, this process is again modified.

C. Sliding Ladder without Friction

Now we study a related problem, the situation is depicted in Fig. II A, but this time with the wall. For discussion of this problem (also including friction) in the static mass case, see, e.g., The sliding ladder problem revisited in phase space (2019, AJP). The postion of mass center is $(\frac{l}{2}\sin\theta, \frac{l}{2}\cos\theta)$, hence the lagrangian reads

$$L = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 - \frac{1}{2}mgl\cos\theta \tag{16}$$

which is same as a falling pencil without sliding (see above). In the present scenario, one might ask whether the ladder will lose contact with the vertical wall, mass oscillation will certainly have impacts on this, though I'll not check it here.

D. Oblique Projectile

The lagrangian of a free falling object is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \tag{17}$$

The equation of motion reads

$$\ddot{y} + \frac{\dot{m}}{m}\dot{y} = -g, \quad \ddot{x} + \frac{\dot{m}}{m}\dot{x} = 0 \tag{18}$$

Both are not quite straightforward to deal with analytically, but clearly there is no parametric resonance.

II. RESULTS AND DISCUSSION

A. Falling without Slide

For a very low mass oscillation frequency, we find numerically that $r(\theta)$ qualitatively does not change, even for a $\beta \sim 1$. The concrete effects though on r depends on the parameters, typically being small early on and the difference grows large after r turns negative.

In the high-frequency regime, some results are shown in Fig. II A, II A, II A, II A, II A. Basically, N and r oscillates around their values in the static-mass case with frequency and magnitude determined by μ, β, respectively. This has particular important consequences for the zero point of N, so if the critical friction factor is very large, the pencil will slide earlier than in the static mass case. Also, the quantity H/m is being amplified.

For mapping to a real experiment, consider the time scale of falling without slide, in the static mass case it's shorter than (since the slide is unavoidable)

$$\int_0^{\pi/2} \frac{d\theta}{\dot{\theta}} = \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{3(1-\cos\theta)}} \approx 15.6\sqrt{\frac{l}{g}} \approx 5\sqrt{l/\mathrm{m}} \tag{19}$$

which translates into an angular frequency scale $\omega_{\text{pencil}} = 1.2/\sqrt{l/\mathrm{m}}$. In comparison, the harmonic freuqency of a pendulum with length l is $\sqrt{g/l} = 3.1/\sqrt{l/\mathrm{m}}$. For the falling pencil, there is no resonances, but for r to show oscillatory behaviour we need roughly $\mu > \omega_{\text{pencil}}$; for the pendulum, the resonance condition is $\mu = 2\sqrt{g/l}$. (as a sidenote, the Kepler frequency of binary orbit is $\omega = 2\pi\sqrt{GM/a^3}$, where a is the semi-major axis and M the sum of binary mass, the resonance condition is $\mu = \omega$) On the other hand,

$$\mu = \text{mass} \times \frac{c^2}{\hbar} = \frac{\text{mass} \times c^2}{\text{eV}} 1.5 \times 10^{15}\,\text{Hz} \tag{20}$$

A comparison is given in the table below, for pendulum I show the length $l = 4g/\mu^2$, and for the binary I show $a = (4\pi^2 GM_\odot/\mu^2)^{1/3}$ (note however QCD axion is pseudoscalar).

particle	mass (eV)	pendulum length (m)	binary orbit radius for $M = M_\odot$ (m)
fuzzy DM	10^{-22}	1.7×10^{15}	6.1×10^{11}
?	1.3×10^{-14}	0.1	2.4×10^{6}
QCD axion	10^{-11}	1.7×10^{-7}	2.8×10^{4}

In condlusion, we can always find a particle mass that could resonante with the pendulum or make the r of a falling pencil oscillates, the only problem being the coupling too weak and even more importantly, whether such particle exists (anyway, one can put constraints).

B. Falling without Friction

We solve Eq. 14, and plot the line connecting $\mathbf{A} = (\frac{l}{2}\sin\theta, 0)$ and $\mathbf{B} = (-\frac{l}{2}\sin\theta, l\cos\theta)$. Yet I have not take into account the possible detach of the pencil from table, this would eventually happen even if the mass is static. So what I actually simulated is a pencil freely rotating about a shaft which moves horizontally along a frictionless track, in this scenario $\theta \in (0, 2\pi)$. The movies can be found in https://www.pixiv.net/artworks/103764273.

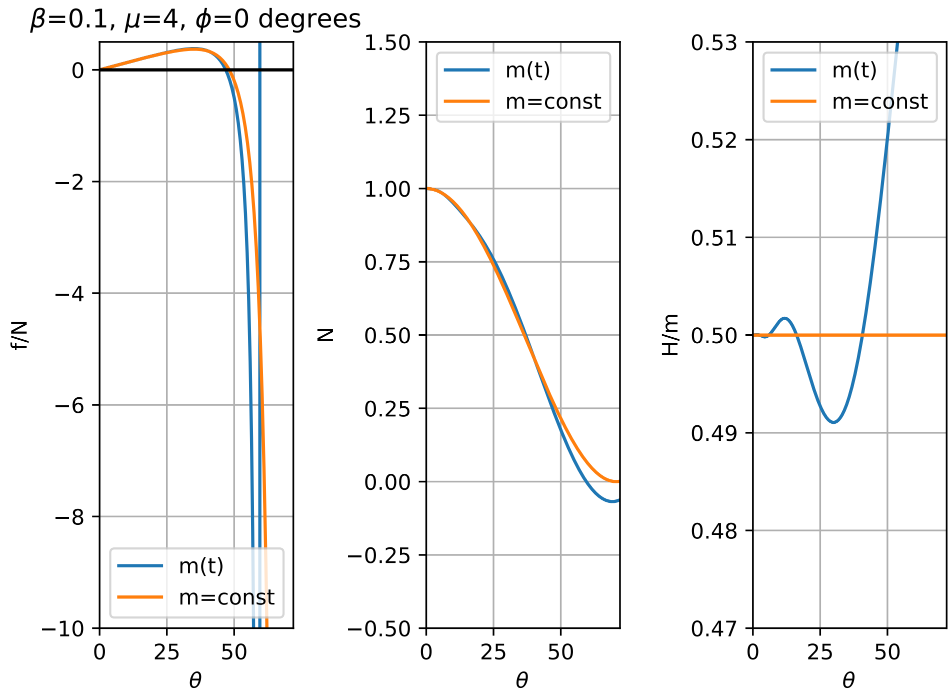

FIG. 3. Small coupling

C. Pendulum

At first we check the harmonic oscillator:

$$\ddot{\theta} + \frac{-\beta\mu\sin(\mu t+\phi)}{1+\beta\cos(\mu t+\phi)}\dot{\theta} + \theta = 0 \tag{21}$$

The resonance happends at $\mu = 2$, see Fig. II C.

For pendulum, replace θ with $\sin\theta$ (we consider only rigid pendulum, for which this equation of motion always holds), the large angle motion provides an opportunity to explore the interplay between parametric resonance (in the linear regime) and nonlinear dynamics, the reonant amplifiaction turns out to be limited and repeats itself periodically, see Fig. II C, II C.

Back to the harmonic oscillator, the interplay between parametric resoanance and forced oscillation may also be interesting. We consider

$$\ddot{\theta} + \frac{-\beta\mu\sin(\mu t+\phi)}{1+\beta\cos(\mu t+\phi)}\dot{\theta} + 2\gamma\dot{\theta} + \theta = f_0\cos\omega_0 t \tag{22}$$

where γ the friction coefficient, and RHS an external periodic driven force. If $\beta = 0$, due to the dissipation the phase trajectory approches a limit circle with frequency ω_0, as displayed

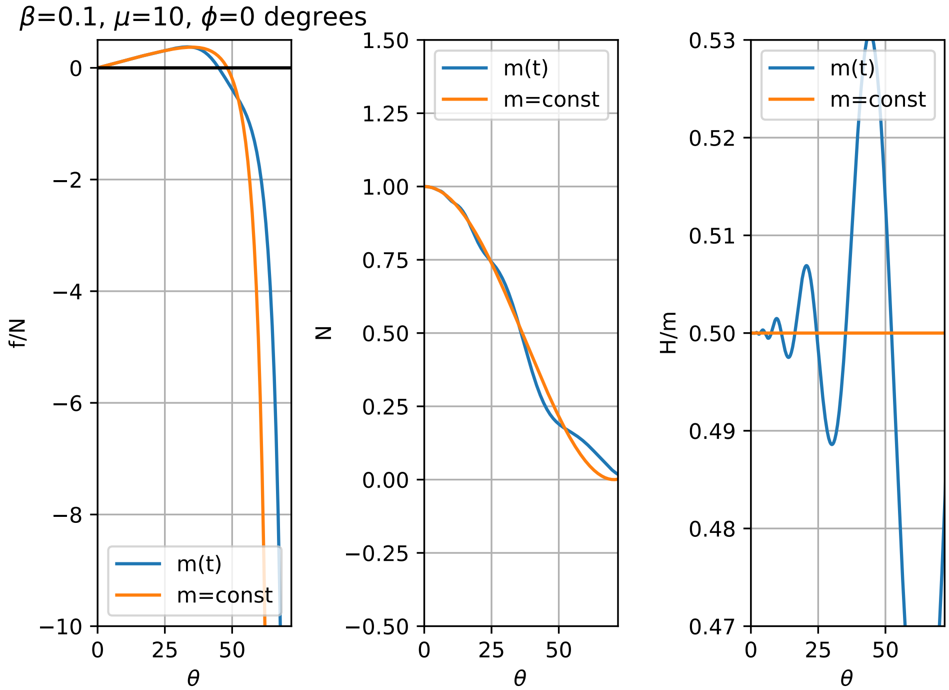

FIG. 4. Small coupling

in Fig. II C. For nonzero β, the parametric resonance seems to supressed at least for $\mu = \omega_0$, see Fig. II C.

D. Oblique Projectile

For a hanging pendulum, the object may undergo a free falling, which has not been incorporated into our pendulum simulation. As a first step in this direction we study a pure free falling process. To make this short investigation more concrete, we focus on how the mass ocillation would modify the maximum range of a cannon ball. The equation of motion is

$$\ddot{y} + \frac{\dot{m}}{m}\dot{y} = -1, \quad \ddot{x} + \frac{\dot{m}}{m}\dot{x} = 0 \tag{23}$$

For static mass, $y(t) = v_{y0}t - \frac{1}{2}t^2$, we choose $v = 1$, so that maximum horizontal range (at same height with the firing position) is realized for $v_{x0} = v_{y0}$ and is 1. Numerically we find that mass oscillation can both enhance or shrink the range, depending on the oscillation phase, the effect is greater for larger coupling and oscillation frequency (the frequency dependence becomes weak after it gets large enough). See Fig. II D.

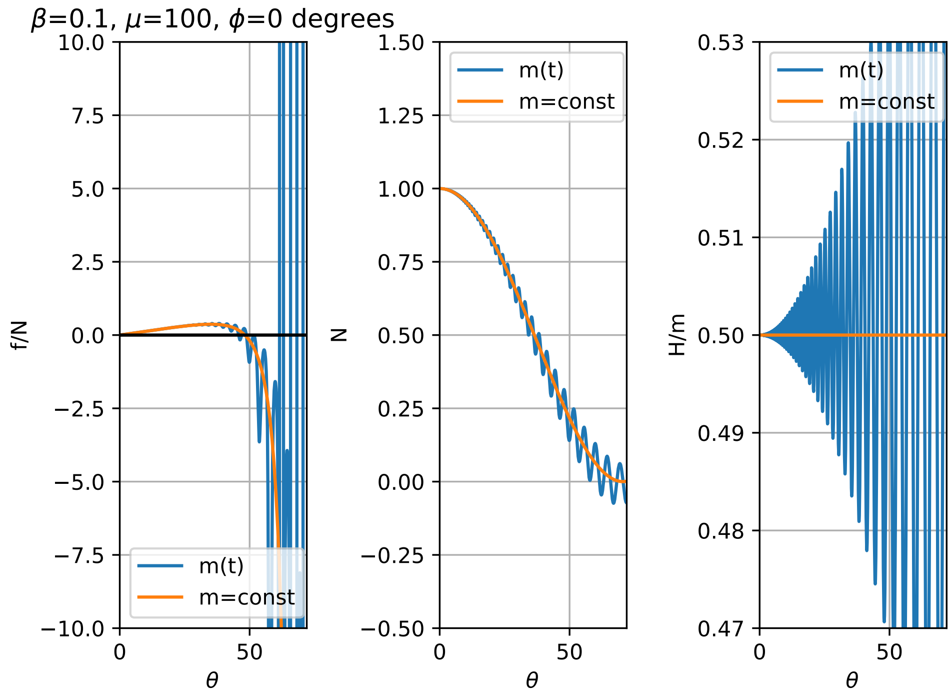

FIG. 5. Small coupling, high frequency.

Appendix A: Gravitational Acceleration

We have tacitly assumed that g is constant, this will not be the case if the entire earth is coherently coupled to a spatially homogeneous scalar field background. In the latter situation,assuming the earth is still spinning with a uniform angular velocity Ω, in the co-rotating frame there is an inertial acceleration $\mathbf{a} = \Omega^2(\mathbf{r} - \mathbf{r}\cdot\hat{\mathbf{z}})$, and the resulted net vertical gravitational acceleration is

$$g = \frac{M}{r^2} - \Omega^2 \cos^2\varphi \tag{A1}$$

where φ is the latitude. The effects of oscillation of g can be easily examined.

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[1] The potential energy is taken to be mgh, since we assume that gravitational force is still mg even for $m(t)$. Indeed, g might not be a constant, either, see Appendix A.

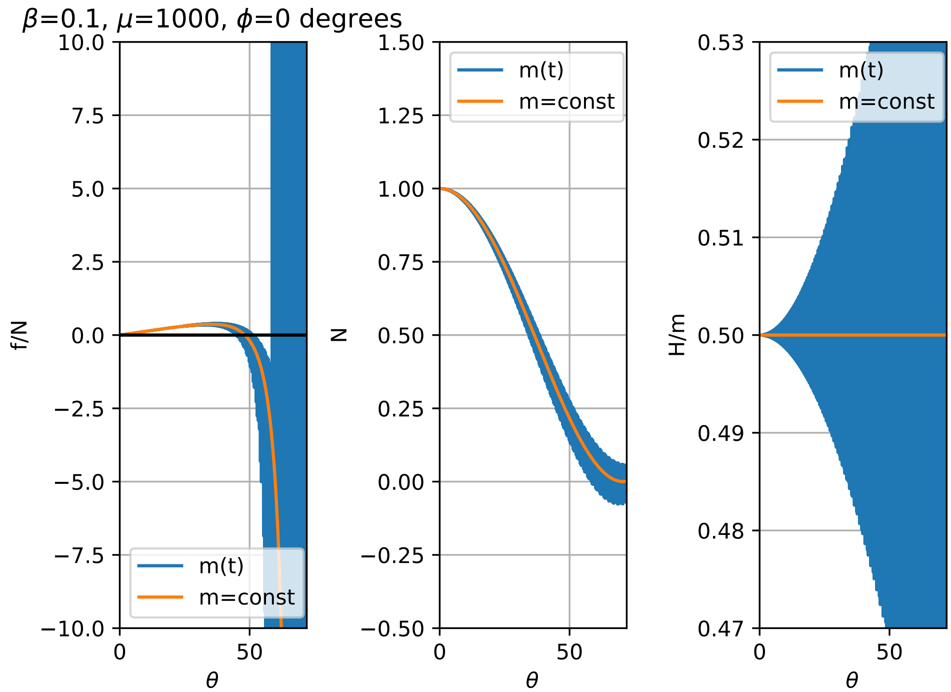

FIG. 6. Small coupling, even higher frequency.

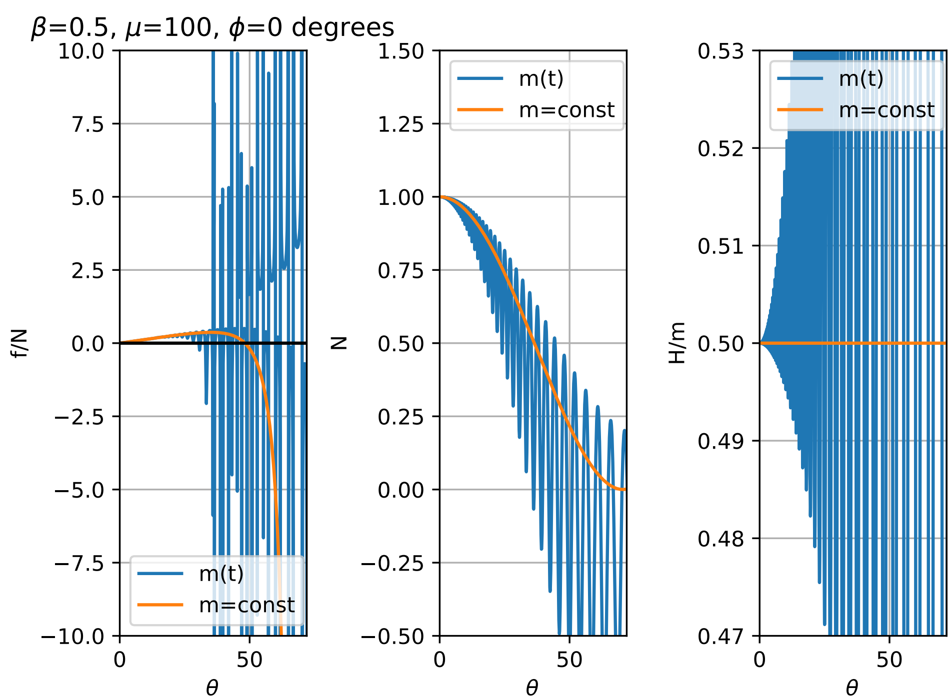

FIG. 7. Large coupling, high frequency.

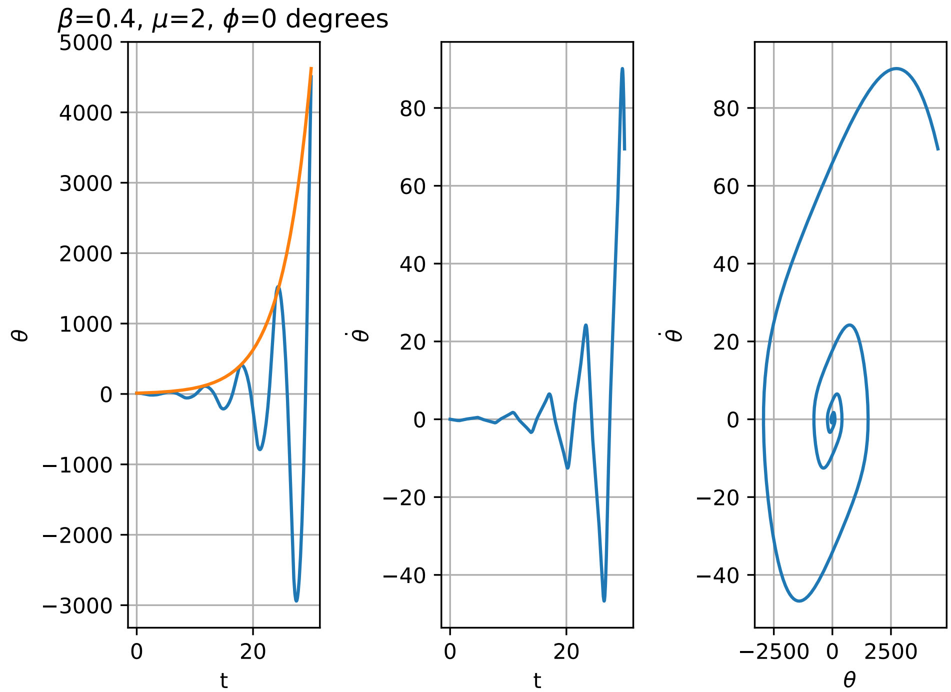

FIG. 8. $\mu = 2$, $\theta(0) = 0.2$ (harmonic oscillator). The trend of this parametric resonance is well-desceibed by the orange curve $\theta(t) = \theta(0)e^{\beta t/2}$, even for such a (relatively) large β.

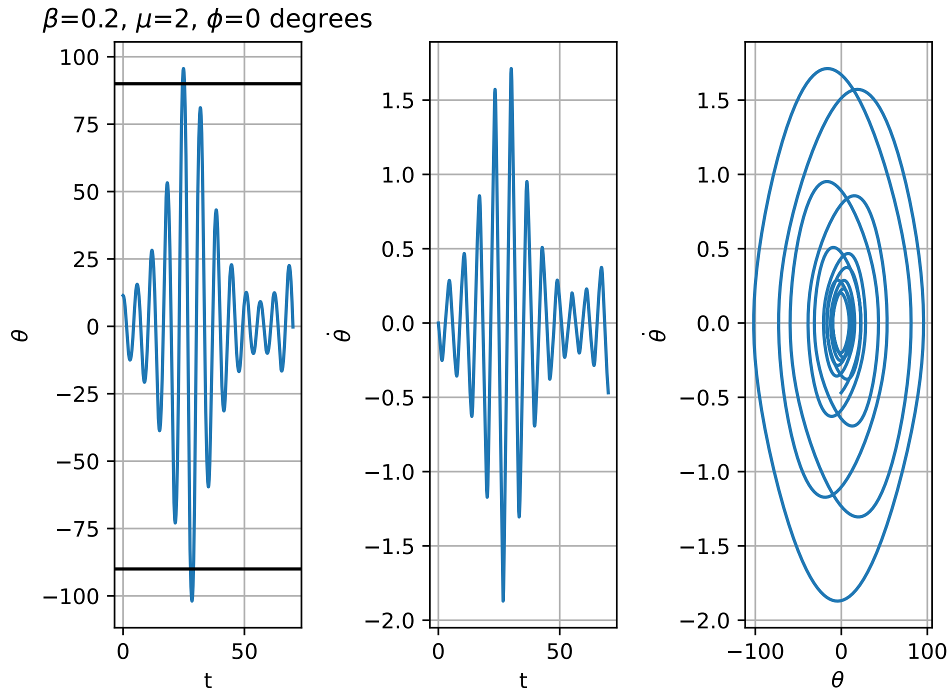

FIG. 9. $\mu = 2$, $\theta(0) = 0.2$, the parametric resonance ends (repeats periodically) due to the nonlinear dynamics.

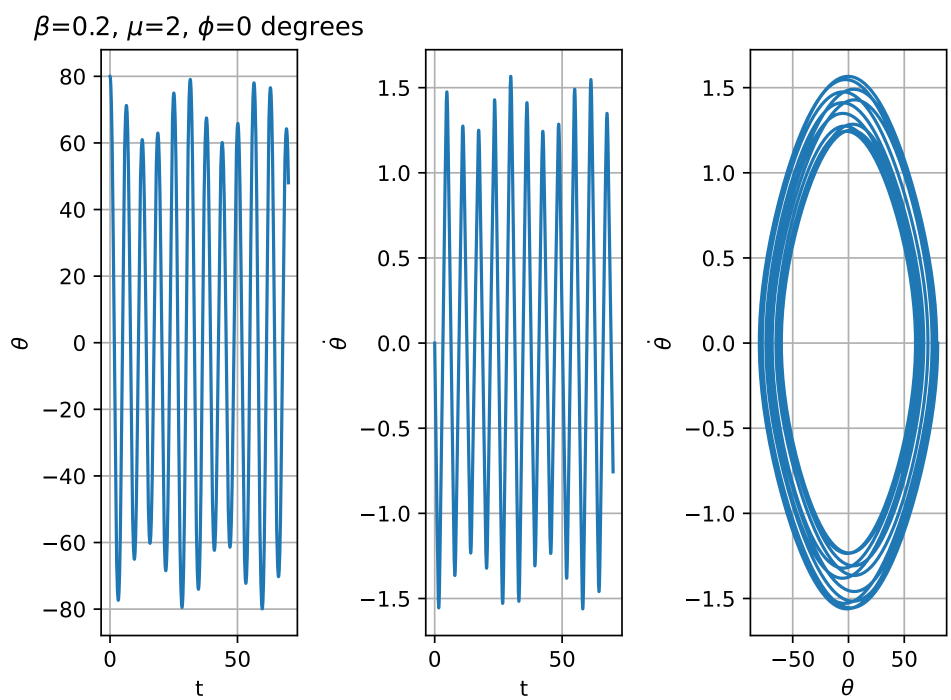

FIG. 10. $\mu = 2$, $\theta(0) = 80°$. For large initial angle, the parametric resonance is less pronounced.

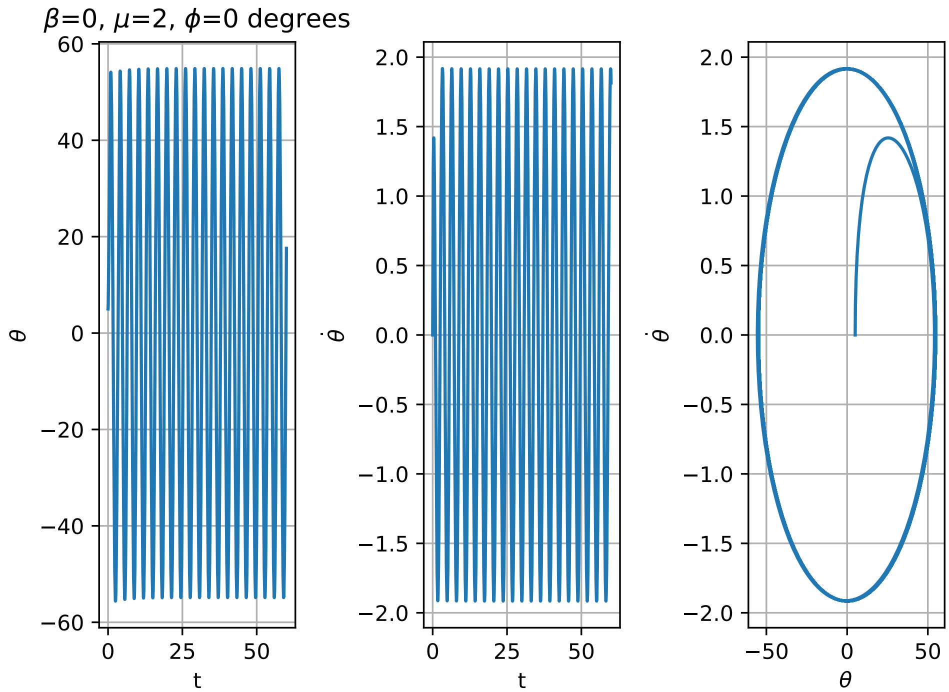

FIG. 11. $\omega_0 = 2$, $\theta(0) = 5^\circ$ (harmonic oscillator), $\gamma = 5$, $f_0 = 10$.

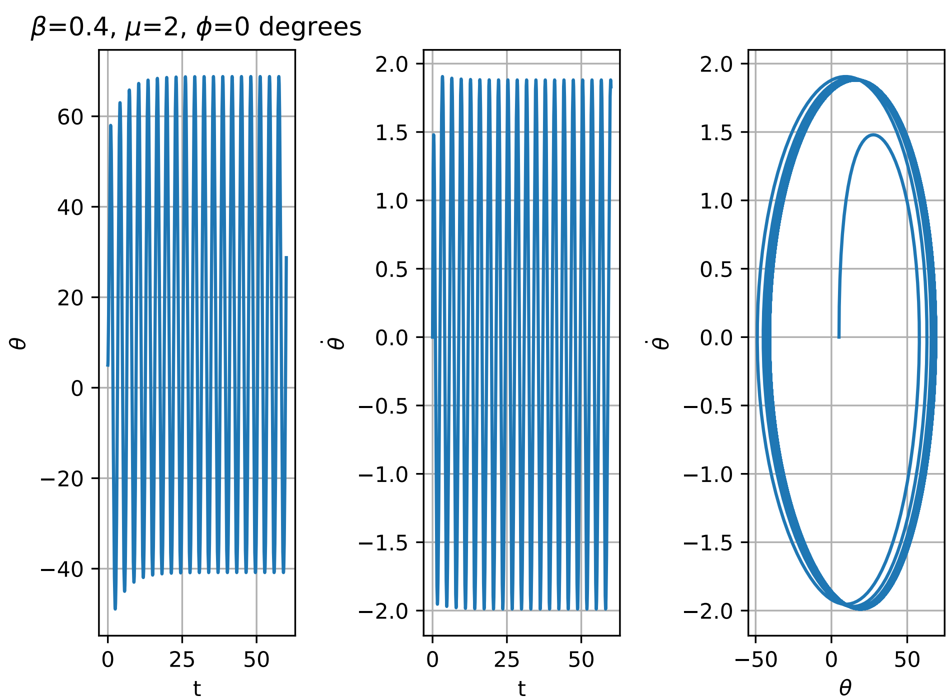

FIG. 12. $\omega_0 = \mu = 2$, $\theta(0) = 5^\circ$ (harmonic oscillator), $\gamma = 5$, $f_0 = 10$.

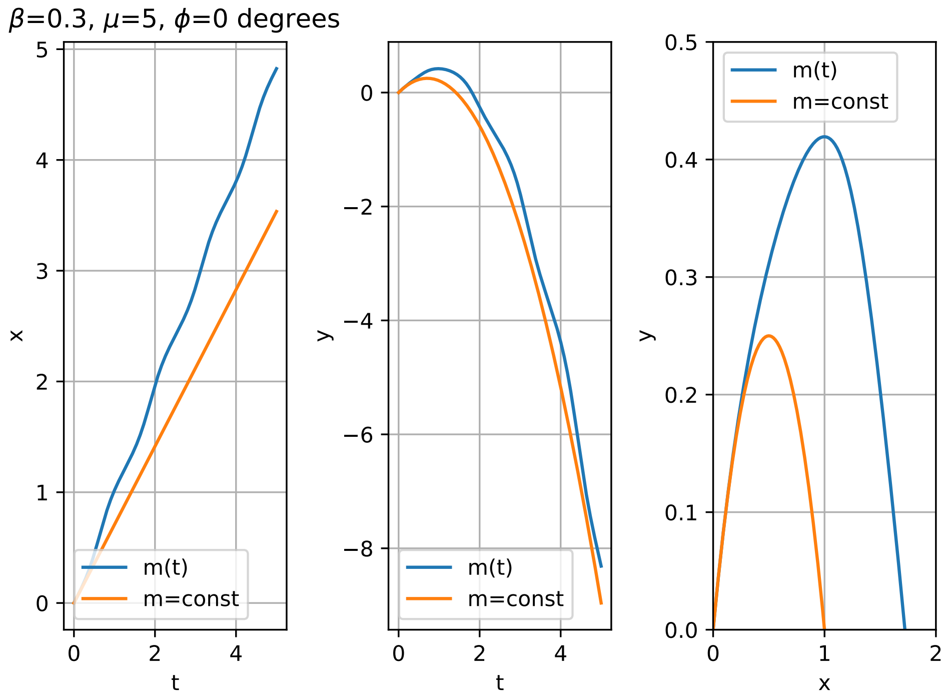

FIG. 13. oblique projectile trajectory, launch angle: 45°.

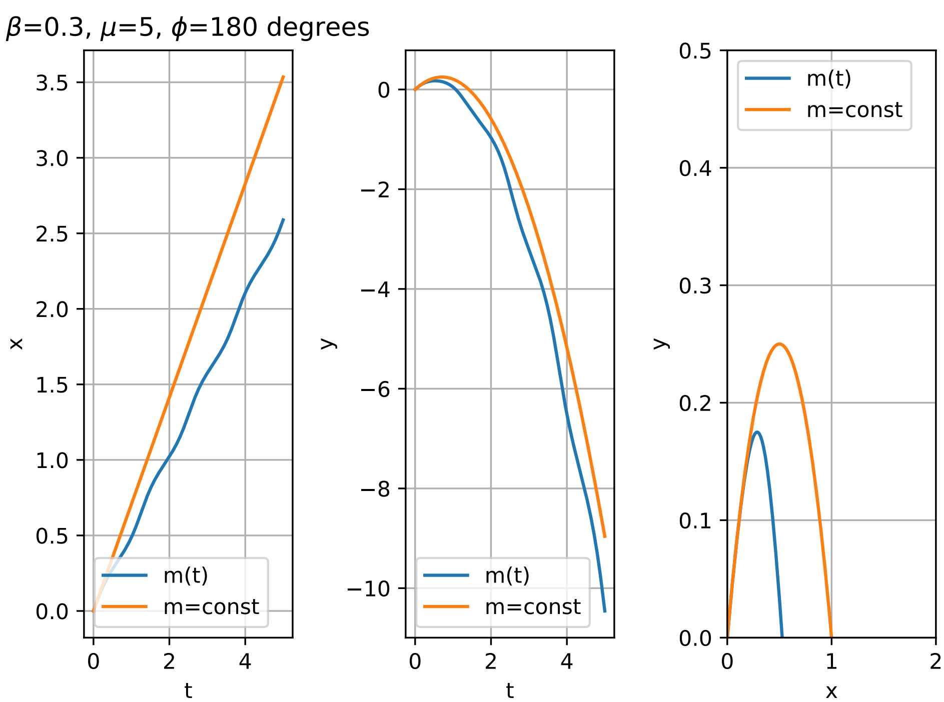

FIG. 14. oblique projectile trajectory, launch angle: 45°.