2D Orbit Dynamics of a Rigid Dumbbell around a Central Object with Oscillating Mass: An Example of Mechanics with Time-Varying Mass

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Abstract

Here, I investigate the (non-relativistic) motion of a dumbbell made of two rigidly connected mass-points in the newtonian potential of a central obejct, for simplicity the dumbbell is assumed to be coplanar with the orbit plane. Moreover the effects of mass oscillation are considered, which could result from the coupling of mass to a spatially homogeneous ultralight scalar field background. The general formulation of this problem is presented though the numerical study has been done only for the simple case of ideal dipole (basically same with two-body problem). A brief discussion of the implications of such mass oscillation in various mechanical systems is given in the appendix.

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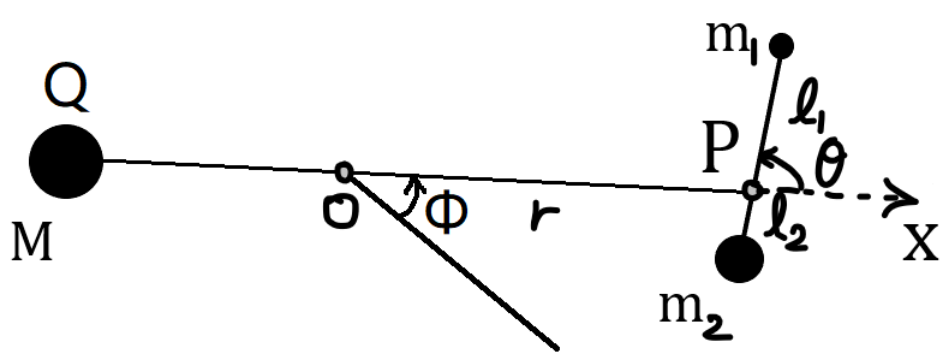


FIG. 1. Configuration of the system, the dumbbell is rigid, i.e., $l_1, l_2 = \text{const}$.

I. THE PROBLEM

This is a restricted three-body problem. The system is depicted in the figure below. The mass of the central object at $\mathbf{Q}$ is M, the dumbbell is made of two rigidly connected mass-points at $\mathbf{r}_{1,2}$ with mass $m_{1,2}$, we write $m = m_1 + m_2$. The line connceting $\mathbf{Q}$ and mass center of the dumbbell $\mathbf{P}$ defines the x axis, relative to which the rotation angle θ of the dumbbell is measured. The mass center $\mathbf{O}$ of the whole system is on this x axis and is choosed to be the coordinate center, and the coordinate of $\mathbf{P}$ is denoted by r. Rotation angle of the x axis in the background frame is denoted by ϕ, and the angular velocity $\omega = \dot{\phi}$.

In the background frame where $\mathbf{O}$ is static, the force experienced by m_i (i=1,2) is

$$\mathbf{F}_i = M m_i \frac{\mathbf{Q} - \mathbf{r}_i}{|\mathbf{Q} - \mathbf{r}_i|^3} \tag{1}$$

where we choose $G = 1$. For the analysis we'll use a lagrangian approach, but let's do a Newtonian analysis first assuming the mass is static. In the co-rotating frame of the x axis, all objects experience an inertial acceleration $\mathbf{a} = -\frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}) = -\dot{\boldsymbol{\omega}} \times \mathbf{r} + \omega^2 \mathbf{r}$. Newton's 2nd law reads in this co-rotating frame,

$$M\ddot{\mathbf{Q}} = -\sum_{i=1,2} \mathbf{F}_i + M(-\dot{\boldsymbol{\omega}} \times \mathbf{Q} + \omega^2 \mathbf{Q}) \tag{2}$$

(as should be checked the RHS is along the x direction) Now we consider a general time-dependence of the mass $m(t)$, with Chasles' theorem in mind, the system's lagraingian is

$$L = \frac{1}{2}M(\frac{d}{dt}(\frac{m}{M}r))^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}M(\frac{m}{M}r\dot{\phi})^2 + \frac{1}{2}(m)(r\dot{\phi})^2 + \frac{1}{2}I(\dot{\theta} + \dot{\phi})^2 + \frac{Mm_1}{|\mathbf{Q} - \mathbf{r}_1|} + \frac{Mm_2}{|\mathbf{Q} - \mathbf{r}_2|} \tag{3}$$

or using the distance between mass centers $R = |\mathbf{R}| = \frac{M+m}{M} r$, $\mathbf{R} \equiv \mathbf{Q} - \mathbf{P}$,

$$\begin{aligned} L &= \frac{1}{2} M (\frac{d}{dt} (\frac{\mu}{M} R))^2 + \frac{1}{2} m (\frac{d}{dt} (\frac{\mu}{m} R))^2 + \frac{1}{2} \mu R^2 \dot{\phi}^2 + \frac{1}{2} I (\dot{\theta} + \dot{\phi})^2 + \frac{M m_1}{|\mathbf{Q} - \mathbf{r}_1|} + \frac{M m_2}{|\mathbf{Q} - \mathbf{r}_2|} \\ &= \frac{\mu}{2} \left(\dot{R}^2 + R^2 \dot{\phi}^2 \right) + \frac{1}{2} I (\dot{\theta} + \dot{\phi})^2 - V, \quad V = - \frac{M m_1}{|\mathbf{Q} - \mathbf{r}_1|} - \frac{M m_2}{|\mathbf{Q} - \mathbf{r}_2|} \end{aligned} \tag{4}$$

where $I = m_1 l_1^2 + m_2 l_2^2$ and $\mu = mM/(m+M)$ the reduced mass.

For numerics we shall set $l = l_1 + l_2 = 1$, so $l_1 = \frac{m_2}{m}$, $l_2 = \frac{m_1}{m}$, $I = \frac{m_1 m_2}{m}$. Using the Law of Cosines,

$$|\mathbf{Q} - \mathbf{r}_1| = \sqrt{R^2 + l_1^2 + 2 R l_1 \cos\theta}, \quad |\mathbf{Q} - \mathbf{r}_2| = \sqrt{R^2 + l_2^2 - 2 R l_2 \cos\theta} \tag{5}$$

The total angular momentum

$$J = \frac{\partial L}{\partial \dot{\phi}} = (\mu R^2 + I) \dot{\phi} + I \dot{\theta} \tag{6}$$

is conserved, which gives $\dot{\phi} = \frac{J - I\dot{\theta}}{\mu R^2 + I}$. The equation of motion of R and θ is

$$\frac{d}{dt} \left(\mu \dot{R} \right) = \mu R \dot{\phi}^2 - V_{,R} \tag{7}$$

$$\frac{d}{dt} \left[I (\dot{\theta} + \dot{\phi}) \right] = - V_{,\theta} \tag{8}$$

or, more explicitly, with $-V = \frac{M m_1}{(R^2 + l_1^2 + 2 R l_1 \cos\theta)^{1/2}} + \frac{M m_2}{(R^2 + l_2^2 - 2 R l_2 \cos\theta)^{1/2}}$,

$$\dot{\mu} R + \mu \ddot{R} = \mu R \left(\frac{J - I\dot{\theta}}{\mu R^2 + I} \right)^2 - \frac{M m_1 (R + l_1 \cos\theta)}{(R^2 + l_1^2 + 2 R l_1 \cos\theta)^{3/2}} - \frac{M m_2 (R - l_2 \cos\theta)}{(R^2 + l_2^2 - 2 R l_2 \cos\theta)^{3/2}} \tag{9}$$

$$\dot{I} \left(\frac{J + \mu R^2 \dot{\theta}}{\mu R^2 + I} \right) + I \frac{d}{dt} \left(\frac{J + \mu R^2 \dot{\theta}}{\mu R^2 + I} \right) = \frac{M m_1 l_1 \sin\theta}{(R^2 + l_1^2 + 2 R l_1 \cos\theta)^{3/2}} - \frac{M m_2 l_2 \sin\theta}{(R^2 + l_2^2 - 2 R l_2 \cos\theta)^{3/2}} \tag{10}$$

The initial condition is specified by $(R, \theta, \dot{R}, \dot{\theta})$ and J. The only free parameters are m_1, m_2, M.

For definiteness, we consider an ocsillating mass: [1]

$$m(t) = m_0 (1 + \beta \cos \mu t) \tag{11}$$

where β, μ are constants, the time-dependence is assumed to be universal for all bodies, and we set $M_0 = 1$.

II. SIMPLE CASES

A. Monopole

To begin with, we cosnider the simplest case where $l_1 = l_2 = 0$, i.e., a two-body Kepler problem. The lagrangian is

$$L = \frac{\mu}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2\right) + \frac{Mm}{R} \tag{12}$$

Angular momentum $J = \mu R^2\dot{\phi}$ is conserved. The equation of motion is

$$\mu\ddot{R} + \dot{\mu}\dot{R} = \frac{J^2}{\mu R^3} - \frac{Mm}{R^2} \tag{13}$$

Since $\partial_t = \frac{J}{\mu}u^2\partial_\phi$,

$$\dot{R} = -\frac{J}{\mu}u_{,\phi} \tag{14}$$

$$\ddot{R} = \frac{J}{\mu}u^2\partial_\phi(\frac{J}{\mu}u^2\partial_\phi\frac{1}{u}) = -\frac{J^2}{\mu^2}u^2u_{,\phi\phi} + \frac{J\dot{\mu}}{\mu}u_{,\phi} \tag{15}$$

The Binet equation is however unchanged

$$u_{,\phi\phi} = -u + \frac{Mm\mu}{J^2} \tag{16}$$

(note special relativity will modify the factor of u while GR in the leading order will add a cubic term of u) but the rightmost term is no longer a constant but representing a periodic driving force if the mass oscillates. In particular, the driven frequency will be three times of the frequency of mass oscillation.

B. Dipole

Then we cosnider a finite but small seperation of the two mass-points, $l_1, l_2 \ll r$, i.e., treating the dumbbell as an ideal mass dipole. The lagrangian is

$$L = \frac{\mu}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2\right) + \frac{1}{2}I(\dot{\theta} + \dot{\phi})^2 + \frac{Mm}{R} \tag{17}$$

In this case, besides $J = (\mu R^2 + I)\dot{\phi} + I\dot{\theta}$,

$$J_\theta = I(\dot{\theta} + \dot{\phi}) \tag{18}$$

is conserved as well, hence

$$\dot{\phi} = \frac{J - J_\theta}{\mu R^2}, \quad \dot{\theta} = \frac{J_\theta}{I} - \frac{J - J_\theta}{\mu R^2} \tag{19}$$

The radial motion is

$$\dot{\mu}\dot{R} + \mu\ddot{R} = \frac{(J - J_\theta)^2}{\mu R^3} - \frac{Mm}{R^2} \tag{20}$$

Of couse, the R-motion and θ-motion are still decoupled.

C. Quadrupole Interaction

There is another extreme situation in which the dubbell is extremely unbalanced, i.e., $m_2 \ll m_1$. A similar problem appears in [2] (however, our m_1 and m_2 are rigidly connected). The potential energy

$$V = -\frac{Mm_1}{|\mathbf{Q} - \mathbf{r}_1|} - \frac{Mm_2}{|\mathbf{Q} - \mathbf{r}_2|} \tag{21}$$

can be expanded, as

$$\begin{aligned} V(\mathbf{Q} - \mathbf{P}, \mathbf{r}_2 - \mathbf{P}) &= -\frac{Mm_1}{|\mathbf{Q} - \mathbf{P} + \frac{m_2}{m_1}(\mathbf{r}_2 - \mathbf{P})|} - \frac{Mm_2}{|\mathbf{Q} - \mathbf{P} - (\mathbf{r}_2 - \mathbf{P})|} \\ &= -M \sum_{\ell,m} \frac{4\pi}{2\ell + 1} \left[\frac{m_1(-m_2/m_1)^\ell + \mu}{R}\right] \left(\frac{\bar{r}}{R}\right)^\ell Y^*_{\ell m}(\Theta, \Phi) Y_{\ell m}(\bar{\theta}, \bar{\phi}) \end{aligned} \tag{22}$$

where $\bar{r} = |\mathbf{r}_2 - \mathbf{P}|$, (R, Θ, Φ) and $(\bar{r}, \bar{\theta}, \bar{\phi})$ are the spherical coordinates of $\mathbf{Q} - \mathbf{P}$ and $\mathbf{r}_2 - \mathbf{P}$ (relative to $\mathbf{P}$), respectively. Now in the $m_2/m_1 \ll 1$ limit, the potential turns out to be

$$V(R, l, \Theta, \Phi, \bar{\theta}, \bar{\phi}) = -\frac{Mm}{R} - \frac{Mm_2}{R} \sum_{\ell \geq 2, |m| \leq \ell} \frac{4\pi}{2\ell + 1} \left(\frac{l}{R}\right)^\ell Y^*_{\ell m}(\Theta, \Phi) Y_{\ell m}(\bar{\theta}, \bar{\phi}) \tag{23}$$

There is no contribution from $\ell = 1$ (dipole). In our 2D problem, we can choose $\Theta = \bar{\theta} = \pi/2$, $\Phi = \phi$ and $\bar{\phi} = \theta + \phi$ (the spherical harmonics $Y_{\ell m}(\bar{\theta}, \bar{\phi}) \propto e^{im\bar{\phi}}$, so $Y^*_{\ell m}(\Theta, \Phi) Y_{\ell m}(\bar{\theta}, \bar{\phi})$ depends solely on $\bar{\phi} - \Phi = \theta$, as it should be).

Actually in our case it's more convenient to use an expansion in terms of Legendre polynomials (which can be easily carried out without taking the extrem-mass-ratio limit):

$$V = -\frac{Mm_1}{|\mathbf{Q} - \mathbf{r}_1|} - \frac{Mm_2}{|\mathbf{Q} - \mathbf{r}_2|} = -M \sum_{n=1}^{\infty} \frac{m_1 l_1^n P_n(-\cos\theta) + m_2 l_2^n P_n(\cos\theta)}{R^{n+1}} \tag{24}$$

Clearly, the dipole term vanishes. Note $l_{1,2} = \frac{m_{2,1}}{m}$. The quadrupole term reads

$$V_{(2)}(R, \theta) = -M \frac{m_1 m_2 l^2}{m R^3} \left(\frac{3}{2}\cos^2\theta - 1\right) \tag{25}$$

The θ-dynamics resulted from this term is left for futural study.

III. NUMERICAL RESULTS

For simplicity, we study only the ideal dipole case. The equation of motion is

$$\ddot{R} - \frac{\beta\mu\sin\mu t}{1+\beta\cos\mu t}\dot{R} = \frac{(1+m_0)^2(J-J_\theta)^2}{m_0^2(1+\beta\cos\mu t)^2R^3} - \frac{(1+m_0)(1+\beta\cos\mu t)}{R^2} \tag{26}$$

The rotation is given by

$$\dot{\phi} = \frac{(1+m_0)(J-J_\theta)}{m_0(1+\beta\cos\mu t)R^2}, \quad \dot{\theta} = \frac{J_\theta}{I_0(1+\beta\cos\mu t)} - \frac{(1+m_0)(J-J_\theta)}{m_0(1+\beta\cos\mu t)R^2} \tag{27}$$

We focus on a particular problem, that is, how does the mass oscillation affects the escape velocity? If $\beta = 0$, $\ddot{R} = -U'(R)$, with the effective potential

$$U(R) = -\frac{1+m_0}{R} + \left(\frac{1+m_0}{m_0}\right)^2 \frac{(J-J_\theta)^2}{2R^2} \tag{28}$$

hence the circular velocity is $R\dot{\phi} = \sqrt{\frac{1+m_0}{R}}$, and the escape velocity is $R\dot{\phi} = \sqrt{\frac{2(1+m_0)}{R}}$. So we set the initial condition $\dot{R} = 0, R\dot{\phi} = \sqrt{\frac{k(1+m_0)(1+\beta\cos\mu t)}{R}}$ at $t = 0$, which means

$$J - J_\theta = m_0\sqrt{\frac{kR(1+\beta\cos\mu t)^3}{1+m_0}} \tag{29}$$

so $k = 2$ gives the escape condition in the static mass case. Still we are free to choose J_θ and I_0, m_0, β, μ (note that $I_0 \in [0, m_0/4]$). We simply would like to check the fate of our test body.

If $\beta = 0$, near the circular radius $R = R_0$, the effective potential is

$$U(R) = -\frac{1+m_0}{R} + \frac{(1+m_0)R_0}{2R^2} = \frac{1+m_0}{2R_0^3}(R-R_0)^2 + \text{const} \tag{30}$$

which gives the epicycle frequecy $\Omega = \sqrt{(1+m_0)/R_0^3}$ (but this also happends to be the orbit frequency) with period $T_R = 2\pi/\Omega$. In [3] it's pointed that resonance condition is $\mu \approx N\Omega$ with N an integer, and the resulted secular orbit energy change per unperturbed orbit period due to the mass oscillation as perturbation is estimated with

$$\delta E = \mu \int_0^T dt\, \dot{\mathbf{R}} \cdot \delta\ddot{\mathbf{R}} \tag{31}$$

where $\delta\ddot{\mathbf{R}} \approx -\dot{m}\dot{\mathbf{R}} - G(M_0 + m_0)\beta\cos\mu t\mathbf{R}/R^3$ is the extra acceleration due to mass ocsillation in the perturbation regime (note a spatially homogeneous gravitational perturbation sourced by the scalar field oscillation could contribute to the acceleration as $4\pi G\rho_{\rm DM}\cos 2\mu t\,\mathbf{R}/R$, independent with the binary seperation).

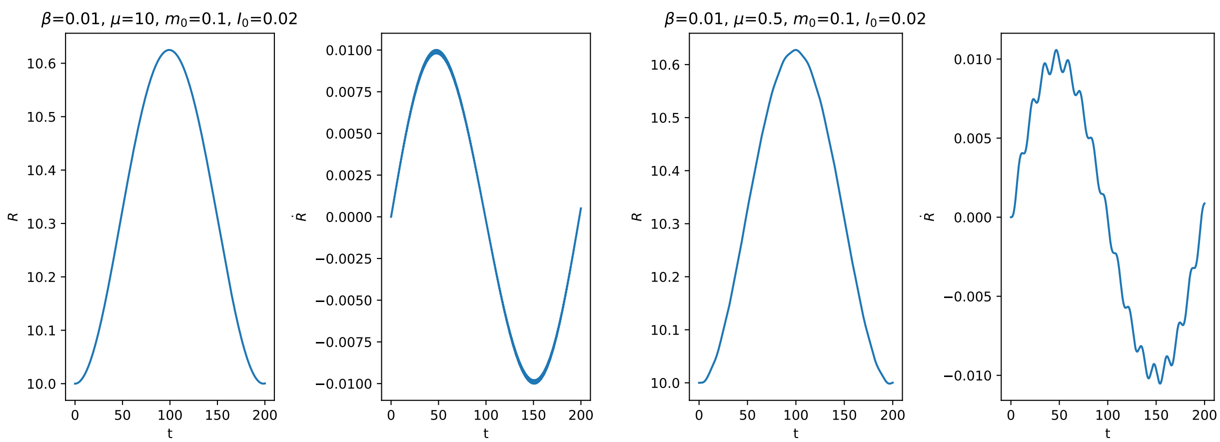


FIG. 2. $k = 1$. If $\beta = 0$, $R = \text{const}$, but now the radius oscillates, the period is comparable to the epicycle period $T_R = 189$. Also notice that this oscillation is not around R_0, but $R(t) \geq R_0$.

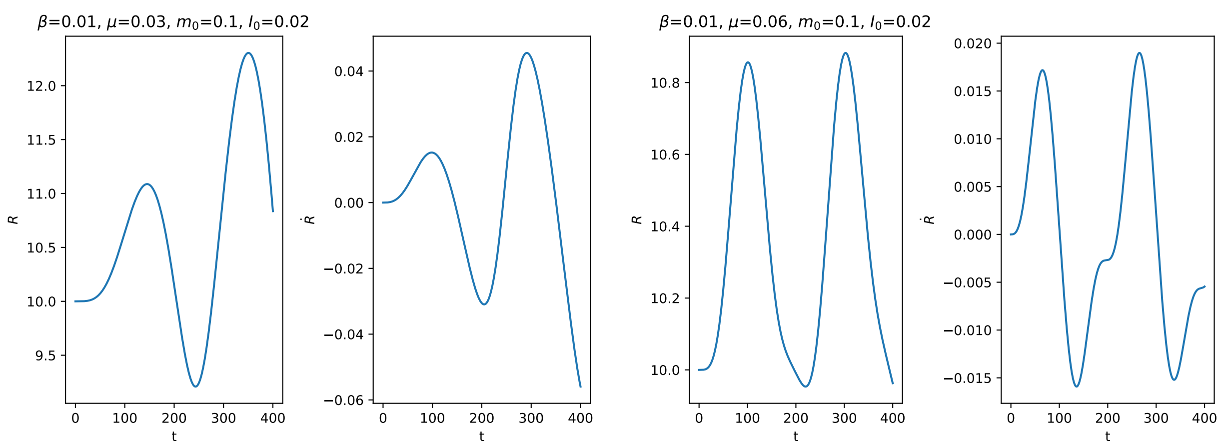


FIG. 3. $k = 1$. Left: $\mu = \Omega$, the orbit radius is being resonantly amplified. Right: $\mu = 2\Omega$, the resonance is much weaker.

We find numerically that if μ matches Ω, the orbit resonance is actually very strong hence in the non-perturbative regime. But this is not a simple Mathieu-type parametric resonance and an analytical understanding via Eq. (26) is left for futural study.

To plot the trajectory, one should solve instead the Binet equation. The effects of mass oscillation on elliptical orbit togther with its interplay with standard GW dissipation is left for further investigation.

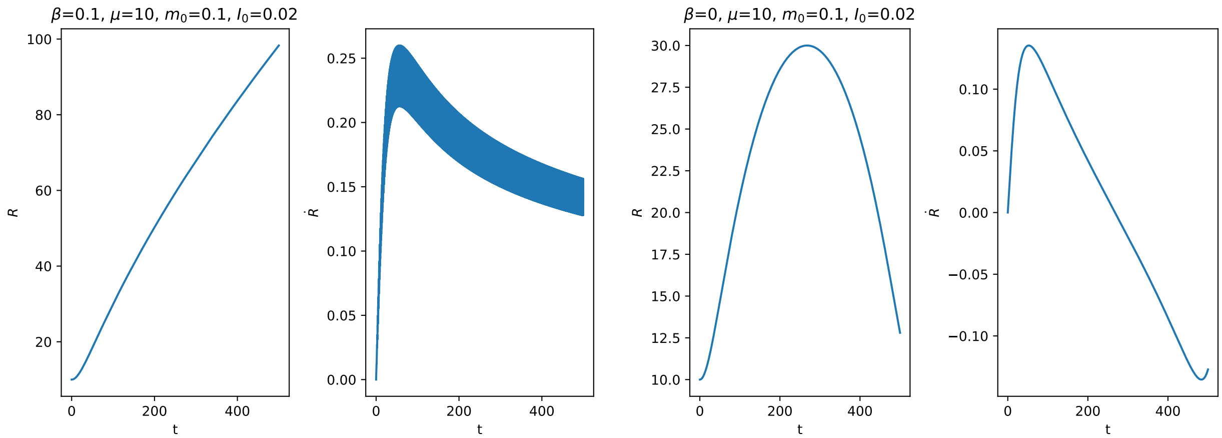


FIG. 4. $k = 1.5$. If $\beta = 0$, this test body cannot escape (right figure), but it does if the oscillation is quick enough (left figure).

IV. GRAVITATIONAL WAVE SIGNAL

The waveforms and luminosities of gravitational waves emitetd by Newtonian three-body system in periodic orbits was investigated in [4], here the situation is similar. The quadrupole moment is

$$I_{ab} = \sum_i m_i x_a^{(i)} x_b^{(i)} \tag{32}$$

where $\mathbf{x}^{(i)}$ is the position of i-th mass-point. The strain signal (in the LIGO frame) is

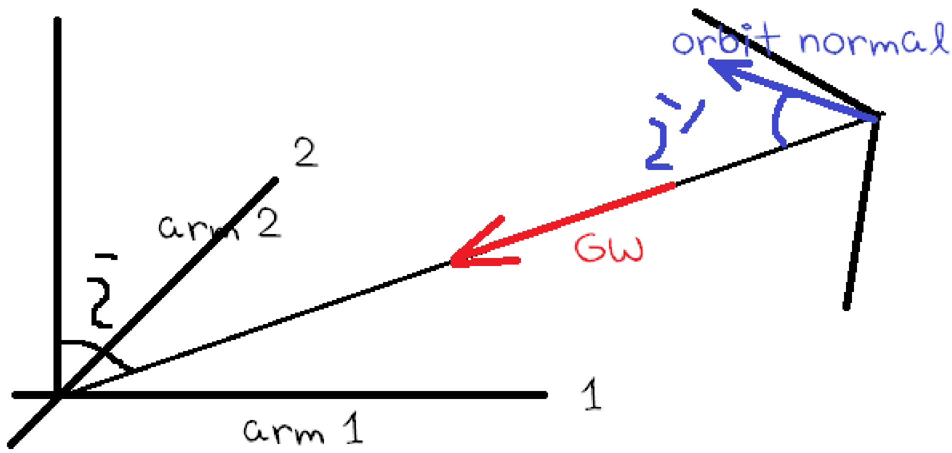
$$s(t) = h_{11} - h_{22} = \cos 2\varphi \left(1 + \cos^2 i\right) \tilde{h}_+ - 2 \sin 2\varphi \cos i \tilde{h}_\times \tag{33}$$

where (i, φ) are the spherical coordinates of the source, and $\tilde{h}$ is measured in a frame where GW travels in z direction. In quadrupole approximation,

$$h_+ = \frac{G}{d}\left[\ddot{I}_{11}(t-d) - \ddot{I}_{22}(t-d)\right], \quad h_\times = \frac{2G}{d}\ddot{I}_{12}(t-d) \tag{34}$$

where d is the source distance. So the inputs of waveform calculation are $\mathbf{x}^{(i)}(t)$ on the orbit plane, in the present case,

$$\begin{aligned} \mathbf{Q} &= -\frac{m}{m+M} R(\cos\phi, \sin\phi) \\ \mathbf{r}_1 &= \frac{M}{m+M} R(\cos\phi, \sin\phi) + l_1[\cos(\phi+\theta), \sin(\phi+\theta)] \\ \mathbf{r}_2 &= \frac{M}{m+M} R(\cos\phi, \sin\phi) - l_2[\cos(\phi+\theta), \sin(\phi+\theta)] \end{aligned} \tag{35}$$



In the two-body case,

$$\mathbf{Q} = -\frac{m}{M}\mathbf{P} = \frac{m}{m+M}\mathbf{R}, \quad I_{ij} = \mu(t)R_iR_j, \quad \mathbf{R} = \begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix} \tag{36}$$

$s(t)$ should be a function of $(i, \varphi, i', \alpha, t^* = t - d)$, where (i', α) is the spherical direction normal to the binary in a frame where GW travels in 3-direction (see figure below), but the general expression is a mess, hence for demonstration we choose $i = \alpha = i' = \alpha = 0$, then

$$s(t) = \frac{4G\mu}{d}\left[\cos 2\phi\left(R\ddot{R} - 2R^2\dot{\phi}^2 + \dot{R}^2 + 2R\dot{R}\frac{\dot{\mu}}{\mu} + \frac{R^2\ddot{\mu}}{2\mu}\right) - \sin 2\phi\left(R^2\ddot{\phi} + 4R\dot{\phi}\dot{R} + 2R^2\frac{\dot{\mu}}{\mu}\dot{\phi}\right)\right] \tag{37}$$

where $\ddot{R}$ is given by (26) and $\dot{\phi}$ given by (27). It will be particular interesting to check the GW signal for a resonantely mateched ($\Omega = \mu$) bianry orbit, which we left for futural works.

V. CONCLUSION AND OUTLOOKS

Here I assumed the spin of the dumbbell is normal to the orbit plane, if it's not so, the general sitation would likely be rather messy, but the motion may also be interesting.

The dumbbell can serve as a basic model for restricted three-body system such as the earth-moon-sun, as treated nicely in Greiner's Classical Mechanics II (Springer, 2010), p.230, note also contained in this book (p.544) is an interesting discussion of the chaotic rotation of Hyperion (a moon of Saturn) formulated as a 2D problem, with the Hyperion modeled as a dumbbell.

It will also be interesting to consider an electric counterpart of this problem, with $m_{1,2}$ replaced by $q_{1,2}$ and $q_1 q_2 < 0$, in particular for $q_1 = -q_2$, the monopole term of potential vanishes and the leading order contribution comes from dipole, also the effect of mass oscillation may be different (since the mass then appears only in the kinetic energy). This is left for futural study.

Appendix A: Mechanics with Time-Varying Mass

a. Newtonian Mechanics We consider the Newtonian mechanics of many particles with mass $m_i(t) = g(t)m_i(0)$ [5]. Many of the results do not change as compared to $g = 1$, but

$$\mathbf{F} = \frac{dm\mathbf{v}}{dt} = m\mathbf{a} + \dot{m}\mathbf{v} \tag{A1}$$

or $\mathbf{F} - \dot{m}\mathbf{v} = m\mathbf{a}$, introducing an effective extra force $-\dot{m}\mathbf{v}$. However, this desciption may not be the most convenient one, since if the system is conservative when $g = 1$, the lagrangian for general $g(t)$ still takes the same form, just with $m = m(t)$. In particular, the conserved quantities associated with cyclic coordiantes will still be conserved. The Euler-Lagrange equation still holds, even if the mass varies with time (more generally it can even be space-time dependent. A time-dependent mass also emerge naturally in cosmology). However, the Hamiltonian (or in a field theory, the energy-momentum) is generally no longer conserved.

The term $\frac{\dot{m}}{m}\dot{x}$ is a main consequence of time variation and will frequently appears below. Consider the equation of motion:

$$\ddot{x} + \frac{\dot{m}}{m}\dot{x} + \omega^2 x = f(t) \tag{A2}$$

Introducing $X = x/\sqrt{m}$, it reads

$$\ddot{X} + \left(\omega^2 + C\right) X = \frac{f}{\sqrt{m}}, \quad C(t) = \frac{\dot{m}^2}{4m^2} - \frac{\ddot{m}}{2m} \tag{A3}$$

If $m(t) = m_0(1 + \beta \cos \mu t)$, $f = 0$, it's

$$\ddot{X} + \left[\omega^2 + \frac{\beta^2 \mu^2 (1 - \cos 2\mu t)}{8(1 + \beta \cos \mu t)^2} + \frac{\beta \mu^2 \cos \mu t}{2(1 + \beta \cos \mu t)}\right] X = 0 \tag{A4}$$

If ω is constant, and $\beta \ll 1$ so the denominators are constant, this is a Whittaker-Hill equation, or more approximately a Mathieu equation:

$$\ddot{X} + \left(\frac{4\omega^2}{\mu^2} + 2\beta \cos 2\tau\right) X = 0 \tag{A5}$$

where we have introduced a non-dimensional time coordinate $\tau = \mu t/2$. The first (most pronounced) resonance band is

$$\frac{4\omega^2}{\mu^2} = 1 \pm \beta \tag{A6}$$

This resonance peak is at $\omega = \mu/2$, the Floquet exponent is approximately given by $\beta/2$, i.e., the (initial) exponential growth is $e^{\beta\tau/2}$. The n-th resonance peak is at $\frac{4\omega^2}{\mu^2} = n$.

b. Examples As a first example we consider a "free" particle, the lagrangian reads

$$L = \frac{1}{2} m v^2, \quad v = \dot{x} \tag{A7}$$

The equation of motion is $d(mv)/dt = \dot{m}v + m\dot{v} = 0$, mv is constant, hence for an oscillating mass the velocity oscillates as well so long as it does vanish (also we see that "energy" is not conserved). Next we consider a harmonic oscillator:

$$L = \frac{1}{2} m v^2 - \frac{1}{2} k x^2 \tag{A8}$$

The equation of motion is $\ddot{x} + \frac{\dot{m}}{m}\dot{x} + \frac{k}{m}x = 0$ (also one can study the quantum version, for the canonical momentum we can use $m(t)v$ or $m_0 v$). For a relativistic oscillator, the lagrangian is (with $c = 1$)

$$L = -m\sqrt{1 - v^2} - \frac{1}{2} k x^2 \tag{A9}$$

The equation of motion is $\frac{d}{dt}\left(\frac{mv}{\sqrt{1-v^2}}\right) + kx = 0$. While in a spatially flat FLRW background (using conformal time), it is

$$L = a(\eta)\left(-m\sqrt{1 - v^2} - \frac{1}{2} k x^2\right), \quad v = x' \tag{A10}$$

and the action $S = \int d\eta\, L$.

It will also be interesting to study the unrestricted three-body problem:

$$L = \sum_{i=1,2,3} \frac{1}{2} m_i v_i^2 + \frac{G m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} + \frac{G m_1 m_3}{|\mathbf{r}_1 - \mathbf{r}_3|} + \frac{G m_2 m_3}{|\mathbf{r}_2 - \mathbf{r}_3|} \tag{A11}$$

and the rigid body mechanics (since the Hamiltonian is no longer conserved, things are generally more chaotic). I'll study the dynamics of a falling pencil on a horizontal fricative table along with some other problems elsewhere. Here as an example, I consider the rotation of a "free" top relative to its mass center (at the coordinate center), for which we can use an effective Newtonian equation $m\mathbf{a} = -\dot{m}\mathbf{v}$. The torque is

$$\mathbf{D} = -\frac{\dot{m}}{m} \int d^3r'\, \rho(\mathbf{r}')\mathbf{r}' \times \mathbf{v}(\mathbf{r}') = -\frac{\dot{m}}{m}\mathbf{L} = m\frac{d}{dt}\left(\frac{\mathbf{L}}{m}\right) \tag{A12}$$

Hence the Euler equation reads

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = -\frac{\dot{m}}{m}\begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} = \begin{pmatrix} I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\omega_1 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 \end{pmatrix} \tag{A13}$$

Whether this new dynamics admitts a geometrical interpretation is an interesting question. Note here the principal moments of inertial I_i is calculated using $m(t)$, so $\dot{I}_i = \frac{\dot{m}}{m} I_i$ since the time dependence of mass is assumed to be same for all bodies. So it also reads

$$\begin{pmatrix} \frac{d}{dt}(I_1\omega_1) + (I_3 - I_2)\,\omega_2\omega_3 \\ \frac{d}{dt}(I_2\omega_2) + (I_1 - I_3)\,\omega_3\omega_1 \\ \frac{d}{dt}(I_3\omega_3) + (I_2 - I_1)\,\omega_1\omega_2 \end{pmatrix} = 0, \quad \mathbf{L} = \sum_i I_i \boldsymbol{\omega}_i = \mathbf{const} \tag{A14}$$

Note the rotation kinetic energy $\frac{1}{2}\mathbf{L}\cdot\boldsymbol{\omega}$ is not conserved. Then, the angular velocity of a spherically symmetric body has fixed direction but oscillating magnitude. For $I_1 = I_2 = I \neq I_3$ (so z is the figure axis), we have

$$(I_3\omega_3)^{-1}\frac{d}{dt}(I_1\omega_1) + \left(\frac{1}{I_2} - \frac{1}{I_3}\right) I_2\omega_2 = (I_3\omega_3)^{-1}\frac{d}{dt}(I_2\omega_2) + \left(\frac{1}{I_3} - \frac{1}{I_1}\right) I_1\omega_1 = 0 \tag{A15}$$

The decoupled equation is

$$(I_3\omega_3)^{-2}\frac{d}{dt}\left[\frac{\frac{d}{dt}(I\omega_1)}{\frac{1}{I_3} - \frac{1}{I}}\right] + \left(\frac{1}{I_3} - \frac{1}{I}\right) I\omega_1 = 0 \tag{A16}$$

more explicitly

$$\frac{d^2}{dt^2}(I\omega_1) + \frac{\dot{m}}{m}\frac{d}{dt}(I\omega_1) + \left[(I_3\omega_3)^2\left(\frac{1}{I_3} - \frac{1}{I}\right)^2\right] I\omega_1 = 0 \tag{A17}$$

The situation is similar to the harmonic oscillator A8. As another example we consider a charged particle in an external EM field, the lagrangian is

$$L = -\frac{m}{\gamma} - qA_\mu v^\mu = -m\sqrt{1 - v^2} - q\left(A^0 - \mathbf{A}\cdot\mathbf{v}\right) \tag{A18}$$

where q is the electric charge[6], the EM fields is given by $\mathbf{E} = -\nabla A^0 - \dot{\mathbf{A}}$, $\mathbf{B} = \nabla\times\mathbf{A}$. Specifically for a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$, we can choose $\mathbf{A} = \frac{1}{2}B(-y\mathbf{e}_x + x\mathbf{e}_y)$, the equation of motion reads

$$\frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1 - v^2}}\right) = qB\dot{y}, \quad \frac{d}{dt}\left(\frac{m\dot{y}}{\sqrt{1 - v^2}}\right) = -qB\dot{x} \tag{A19}$$

Note if the mass is static, v = const, hence there is no qualitative difference between the non-relativistic and relativistic cases, however here we have to use relativitic formula if the velocity is large. It is expected that the radius of gyromotion gets resonantly enhanced by the mass oscillation if their frequency matches, a detailed investigation (and the quantum version) is left for futural study. Finally we note that the mass appears in Bohr magneto $\mu_B = \frac{q\hbar}{2m}$ (more precisely, we have to check the non-relativistic limit of Dirac equation, there should be additional contributon from $m(t)$), thus will affect, e.g., the Larmor precession of a spin ngular momentum in magetic field and the spin flip in electron spin resonance. For the former, classically,

$$\frac{d\mathbf{s}}{dt} = \left(\frac{g_s\mu_B}{\hbar}\mathbf{B}\right) \times \mathbf{s} \tag{A20}$$

where $\frac{g_s\mu_B}{\hbar}\mathbf{s}$ is the magnetic dipole moment, and $\frac{g_s\mu_B}{\hbar}\mathbf{B}$ is the angular velocity of precession if mass is static. For quantum treatment, the relevant Hamiltonian is $H = \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$ (note however the coupling of scalar particle to photon can modify the effective value of electron-photon coupling, i.e., the fine structure constant, then this hamiltonian acquires another oscillating factor. But agian the full effects may be more complicated, here we focus solely on the scalar coupling to fermion mass), but we can define an effective field $\mathbf{B}_{\text{eff}} = \mathbf{B}/m$ (the Berry analysis is then the same), this situation normally does not exist since a time-varing magnetic field would induce also an electric field.

Last but not the least, the mass oscillation may have consequences in spontaneous periodic systems such as time crystals.

c. Thermodynamics Moving from newtonian mechanics to statistical mechnanics, the obvious problem is that the Hamiltonian depends explicitly on time, hence the usual formulation of classical statistical mechanics (e.g., based on splits of energy hypersurfaces) cannot be applied. The thermodynamics though may still be discussible if the mass oscillation is treated as an external perturbation, but I will not try it here.

Note however the hamiltonian formulation generally does not require that the Hamiltonian depends only implicitly on time, hence things like Poisson backets, canonical transforamtions, Liouville theorem (conservation of phase space volume) generally hold.

d. Brownian Motion Consider the Langevin dynamics

$$\ddot{x} + \frac{\dot{m}}{m}\dot{x} = -\gamma\dot{x} + \xi(t) \tag{A21}$$

where $\xi(t)$ is a stochastic term. If follows that, with $y \equiv x^2$,

$$\frac{1}{2}m_0\ddot{y} - m_0(\dot{x})^2 = -\frac{m_0}{2m}\left(\gamma - \frac{\dot{m}}{m}\right)\dot{y} + \frac{m_0}{m}x\xi(t) \tag{A22}$$

As usual we take a time average for which $\langle x\xi\rangle = 0$ and assume that the system is in thermal equilibrium $\frac{1}{2}m_0\langle(\dot{x})^2\rangle = \frac{1}{2}k_BT$, then

$$\ddot{y} + \frac{1}{m}\left(\gamma - \frac{\dot{m}}{m}\right)\dot{y} = \frac{2k_BT}{m} \tag{A23}$$

same as a free falling with a friction proportional to velocity if the oscillation is small. For static mass, y approches to $2k_BTt/\gamma$, while if mass oscillates, we find numerically that y oscillates around this critical value.

e. Lattice Oscillation Ths crystal lattice provdies a large discrete set of harmonic oscillators with a wide range of frequencies, the natural frequency of a k-mode depends on the specific dispersion relation (e.g., for a lattice composed of idential mass points with individual mass m connected via atomic spring $F = -\kappa x$, typically $\omega \sim \sqrt{\kappa/m}$ and the wavevector resolution $\Delta k = 2\pi/Na$, where a the lattice spacing and N the number of unit cells). So we might expect that phonons get resonantly excited, but nonlinear effects (phonon self-interactions) may supress it.

f. Fluid Mechanics There are at least two approaches to fluid mechanics, one starts from the Boltzmann equation, the other from Schrodinger equation (or for relativistic BEC from KG equation). The latter via the density-phase representation of wave fucntion is quite strightforward, so we consider briefly the former. From the Liouville theorem,

$$\mathbf{F}\cdot\nabla_{\mathbf{p}}f + \mathbf{v}\cdot\nabla f + \partial_t f = 0 \tag{A24}$$

where $f(\mathbf{r}, \mathbf{p} = m(t)\mathbf{v}, t)$ is the distribution function, if there are no collisions. Define $n(\mathbf{r}, t) = \int d^3p f(\mathbf{r}, \mathbf{p}, t)$, we have

$$\int d^3p\,(\mathbf{F}\cdot\nabla_{\mathbf{p}}f + \mathbf{v}\cdot\nabla f + \partial_t f) = \int d^3p\,p_i(\mathbf{F}\cdot\nabla_{\mathbf{p}}f + \mathbf{v}\cdot\nabla f + \partial_t f) = 0 \tag{A25}$$

The continuity equation arises from the first:

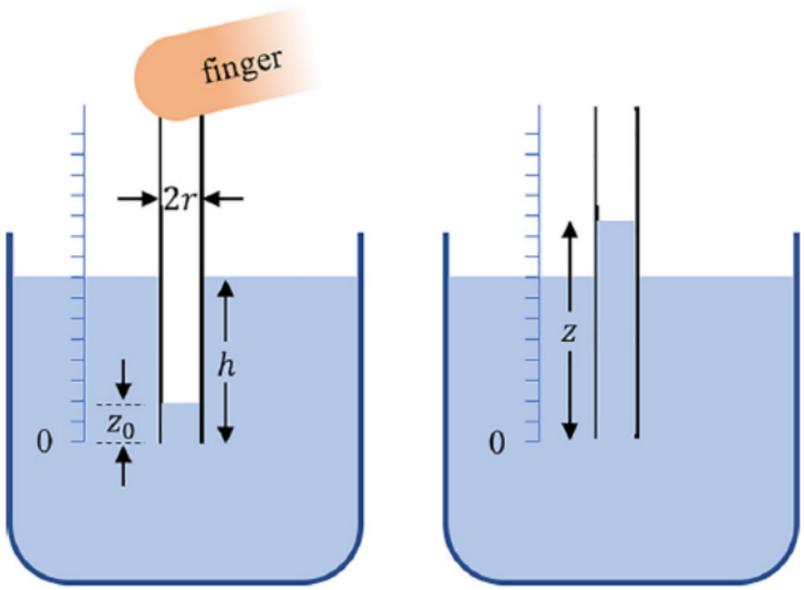
$$\partial_i \int d^3p\,(v_i f) + \partial_t n = \nabla(n\bar{\mathbf{v}}) + \partial_t n = 0, \quad \bar{\mathbf{v}} \equiv \frac{\int d^3p\,\mathbf{v}f}{n} \tag{A26}$$

While the Euler equation follows from the second:

$$\partial_t(nm\bar{\mathbf{v}}) + \partial_j(nm\overline{v_i v_j}) = n\mathbf{F} \tag{A27}$$

the effects of time varying mass is (not suprisingly) agian the familiar form.

Then we can consider the oscillation of fluid. E.g., a liquid column in a vertical tube (see figure below, source: [7]).



In a simplest Newtonian model, the dynamics of the liquid column is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho z\dot{z}) = \rho g(h-z) - b\dot{z} \quad \Rightarrow \quad \ddot{z} + \frac{\dot{m}}{m}\dot{z} = -\frac{b}{\rho}\frac{\dot{z}}{z} - g\left(1-\frac{h}{z}\right) - (\dot{z})^2 \tag{A28}$$

where $\rho(t)$ the mass density and b a phenomelogical friction coefficient. For $z \sim h$ the RHS is harmonic, with natural freuqency $\omega = \sqrt{g/h}$ (which can be easily adjusted). The parametric resonance should stop in the nonlinear regime.

g. Nonrelativistic Quantum Mechanics Next we discuss breifly the nonrelativistic quantum mechanics with a time-varying mass $\mathcal{M}$. We begin with a free relativistic real scalar field in the flat FLRW spacetime

$$\mathcal{L} = a^2\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - mV\phi^2\right] = \frac{a^2}{2}\left[(\phi')^2 - (\nabla\phi)^2 - (m^2+2mV)\phi^2\right] \tag{A29}$$

with $S = \int d\eta d^3x\,\mathcal{L}$, where $m = m(\eta) \equiv \mathcal{M}a$. Making the usual ansatz $\phi = \frac{1}{\sqrt{2m}}(\psi e^{-im\eta} +$ c.c), in the non-relativistic limit $\psi' \ll m\psi$,

$$\phi^2 = \frac{|\psi|^2}{m}, \quad (\nabla\phi)^2 = \frac{1}{m}\nabla\psi\cdot\nabla\psi^* \tag{A30}$$

but

$$\phi' = \left[\psi' - \left(\frac{m'}{2m} + i(m'\eta + m)\right)\psi\right]\frac{e^{-im\eta}}{\sqrt{2m}} + \mathrm{c.c} \tag{A31}$$

and

$$(\phi')^2 = \left[\left(\frac{m'}{2m}\right)^2 + (m'\eta + m)^2\right] \frac{|\psi|^2}{m} - \frac{1}{m}\left[\left(\frac{m'}{2m} - i(m'\eta + m)\right)\psi'\psi^* + \left(\frac{m'}{2m} + i(m'\eta + m)\right)(\psi')^*\psi\right] \tag{A32}$$

If $m' = 0$, we have $\mathcal{L} = a^2\left[i\psi^*\psi' + \text{c.c.} - \frac{1}{2m}\nabla\psi \cdot \nabla\psi^* - V|\psi|^2\right]$, which leads to the familiar Schrodinger equation

$$-\frac{1}{2m}\nabla^2\psi + V\psi = i\frac{(a^2\psi)'}{a^2} = i(2\mathcal{H}\psi + \psi') \tag{A33}$$

But if $m = m(0)(1 + \beta\cos\mu\eta) \approx m$, $m'/m \approx -\beta\mu\sin\mu\eta$, the situaton depends on the magnitude of μ and η. Only when the mass variation is negligible in $(\phi')^2$, we have the usual form of Schrodinger equation.

If we use cosmic time t instead,

$$\mathcal{L} = \frac{a}{2}\left[a^2(\dot{\phi})^2 - (\nabla\phi)^2 - (m^2 + 2mV)\phi^2\right] \tag{A34}$$

with $S = \int dt d^3x\, \mathcal{L}$, and making the ansatz $\phi = \frac{1}{\sqrt{2m}}(\psi e^{-imt} + \text{c.c})$, in the non-relativistic limit $\dot{\psi} \ll m\psi$, things are basically similar. If $m' = 0$, the Schrodinger equation is

$$-\frac{1}{2m}\nabla^2\psi + V\psi = i\frac{\frac{d}{dt}(a^3\psi)}{a} = i(3Ha^2\psi + a^2\dot{\psi}) \tag{A35}$$

h. Backreaction on Scalar Field So far we have treated the scalar field as a completely fixed background, however the mass may also backreacts on the field. We consider a simple example (in flat spacetime):

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \left[m_0(1 + \alpha\phi)\sqrt{1 - (\dot{X})^2} + \frac{1}{2}kX^2\right]\delta(\mathbf{r} - \mathbf{R}) \tag{A36}$$

where $\mathbf{R}(t) = (X, Y, Z)$ is the position of mass point. The scalar wave equation is

$$(-\partial_t^2 + \nabla^2 - m^2)\phi = \alpha m_0\sqrt{1 - (\dot{X})^2}\delta(\mathbf{r} - \mathbf{R}) = -j(\mathbf{r}, t) \tag{A37}$$

while the mass point dynamics is

$$\frac{d}{dt}\left(\frac{m_0(1 + \alpha\phi)\dot{X}}{\sqrt{1 - (\dot{X})^2}}\right) + kX = 0 \tag{A38}$$

In the opposite regime, we may treat the particle motion as fixed (i.e., neglecting the scalar coupling in the particle dynamics), and calcluate the resulted massive scalar wave excitation, this is given by

$$\phi = \int d^3r'\, \frac{j(\mathbf{r}', t_{\text{ret}})}{4\pi|\mathbf{r} - \mathbf{r}'|}e^{-m|\mathbf{r} - \mathbf{r}'|}, \quad t_{\text{ret}} = t - |\mathbf{r} - \mathbf{r}'| \tag{A39}$$

In the massless (or ultralight) limit, similar to Liénard-Wiechert potential we have

$$\phi(\mathbf{r},t) = -\frac{\alpha m_0\sqrt{1-|\mathbf{v}(t_{\mathrm{ret}})|^2}}{|\mathbf{D}|\kappa} \tag{A40}$$

where $\mathbf{D}(\mathbf{r},t) = \mathbf{r} - \mathbf{R}(t_{\mathrm{ret}})$, $\kappa(\mathbf{r},t) = 1 - \mathbf{n}\cdot\mathbf{v}(t_{\mathrm{ret}})$, $\mathbf{v} = \dot{\mathbf{R}}$ and $\mathbf{n} = \mathbf{D}/|\mathbf{D}|$.

i. Quantum Noise Above the ultralight scalar background has been treated as purely classical field, this would be the case if the field is in its coherent state. Let's still focus on the coupling between the scalar field with a mass. For one k-mode $\phi = qe^{i\mathbf{k}\cdot\mathbf{r}}$ (e.g., the zero mode relevant for a homogeneous background) and non-relativistic particle, the action is

$$S = \int dt\,\left[\frac{1}{2}(\dot{q}^2 - \omega^2 q^2) - m_0(1+\alpha q)\left(1 - \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j\right)\right] \tag{A41}$$

Introducing $\xi^i = x^i/(1+\alpha\phi/2)$, it reads

$$L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) - m_0\alpha q + \frac{1}{2}m_0(|\dot{\boldsymbol{\xi}}|^2 - \alpha\dot{q}\xi^i\dot{\xi}^i) \tag{A42}$$

This system can now be quantized using path integral formalism as in [8], and the quantum noise signal in non-classical states may be derived.

[1] This oscillation could result from a direct coupling of the ultralight scalar dark matter field to the mass of ordinary matter [3] (e.g., the fermion mass $m_\psi \mapsto m_\psi(1+g_\psi\phi)$ resulted from the linear-in-ϕ scalar-fermion coupling $-g_\psi\phi\bar{\psi}\psi \subset \mathcal{L}$; also, scalar-photon coupling $g_\gamma\phi\frac{1}{4}(F_{\mu\nu})^2 \subset \mathcal{L}$ can modify the effective electromagnetic fine structure constant: $\alpha \mapsto \frac{1}{1-g_\gamma\phi}\alpha$, since $-\frac{1}{4}(F_{\mu\nu})^2 \mapsto -\frac{1}{4}(1-g_\gamma\phi)(F_{\mu\nu})^2$, the photon-electron coupling is effectively $\frac{1}{1-g_\gamma\phi}$ of the original value), the authors here also considered a quadratic coupling of the field to mass. The case for higher-spin ultralight dark matter shoud be different. The gravitational perturabtion of the DM background $\phi \propto \cos\mu t$ can also have effects on the orbit dynamics, the effective lagrangian is (assuming spatially homogenity) $4\pi G\rho_{\mathrm{DM}}\cos(2\mu t)\,R$ with oscillating freuquency 2μ, but here we neglect it completely, i.e., we assume that ρ_{DM} is negligible. One thing to be stressed is that we are assuming a scalar field, not pseudo-scalar one like QCD axion, the latter may couple to fermion in a different manner (in the nonrelativistic limit, couples directly to spin), and has direct coupling to photon.

[2] D. Baumann, H. S. Chia, and R. A. Porto, Phys. Rev. D **99**, 044001 (2019), arXiv:1804.03208 [gr-qc].

[3] D. Blas, D. L. Nacir, and S. Sibiryakov, Phys. Rev. Lett. **118**, 261102 (2017).

[4] V. Dmitrašinović, M. Šuvakov, and A. Hudomal, Phys. Rev. Lett. **113**, 101102 (2014).

[5] Note this is not the so called variable-mass system such as rokects, in which the mass is just being transfered from one place to another.

[6] If we also include a magnetic charge, the Lorentz force reads: $\mathbf{F} = q_e(\mathbf{E}+\mathbf{v}\times\mathbf{B})+q_m(\mathbf{B}-\mathbf{v}\times\mathbf{E})$ or covariantly: $F^a = (q_e F^{ab} + q_m G^{ab})v_b$ where G^{ab} the dual of F^{ab}, pitifully (to my knowledge) this cannot be derived from a Lagrangian of EM potential. As in this paper, one could introduce two potentials, then EM fields are given by $\mathbf{E} = -\nabla A^0 - \dot{\mathbf{A}} - \nabla\times\mathbf{C}$ and $\mathbf{B} = -\nabla C^0 - \dot{\mathbf{C}} + \nabla\times\mathbf{A}$, the Lorentz force law can be derived from full extended Maxwell lagrangian via continuity of energy-momnetum tensor, but not from a local lagrangian contains only $\mathbf{A}$ and $\mathbf{C}$.

[7] R. Smith and E. Matlis, American Journal of Physics **87**, 433 (2019).

[8] M. Parikh, F. Wilczek, and G. Zahariade, Phys. Rev. D **104**, 046021 (2021), arXiv:2010.08208 [hep-th].