# Classical Free Fields in a 1+1D Background Spacetime Flow

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# Abstract

The equaton of motion for some classical free fields in a 1+1D spacetime-dependent background flow are presented. Special attentions are paid to their non-relativistic limit.


**CONTENTS**



## I. THE BACKGROUND

We consider a metric of the form

$$ds^2 = -dt^2 + [dx - v(x,t)dt]^2 = -(1-v^2)dt^2 + dx^2 - 2vdtdx \tag{1}$$

which is actually a vacuum solution of the Einstein equation (for general $v(x,t)$, the metric does not have special symmetry). In 1+1D the Schwartzchild metric corresponds to $v(r) = -\sqrt{\frac{r_s}{r}}$ and the dS spacetime has $v(r) = Hr$. Under coordinate transformation $d\tau = dt + \frac{v}{1-v^2}dx$, the lime element turns into

$$ds^2 = -(1-v^2)d\tau^2 + \frac{1}{1-v^2}dx^2 \tag{2}$$

(however, the sign of $v$ is lost in this transforamtion) which can however be mapped conformally to an arbitrary diagonal 1+1D metric

$$ds^2 = -F(x,\tau)d\tau^2 + G(x,\tau)dx^2 \mapsto -\sqrt{\frac{F}{G}}dt^2 + \sqrt{\frac{G}{F}}dx^2 \tag{3}$$

provided that both $F$ and $G$ are positive-definite, with the correspondence

$$v = \sqrt{1 - \sqrt{\frac{F}{G}}} \tag{4}$$

if $F/G < 1$. As an example, the radial FLRW spacetime is

$$ds^2 = -d\tau^2 + \frac{a^2(\tau)}{1 - kx^2} dx^2 \tag{5}$$

then $v(\tau, x) = \sqrt{1 - \sqrt{\frac{1-kx^2}{a^2}}}$. It's possible to find the explicit form of $\tau(x,t)$. Hence, the background (1) is of general interest.

Note that a horizon is realized at $v(x) = 1$, the case for $v(t,x)$ is perhaps different, but the behaviour of solution near horizon is not our focus here.

We compute the geometrical quantities for this spacetime, the Ricci tensor is

$$R_{ab} = g_{ab}\left(v_{,x}^2 + vv_{,,x} + v_{,t,x}\right) \tag{6}$$

The Ricci scalar is

$$R = 2\left(v_{,x}^2 + vv_{,,x} + v_{,t,x}\right) \tag{7}$$

So $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 0$. Note for vacuum Einstein equation, $R - g^{ab} g_{ab} R/2 = 0$, but we get $R = 0$ only if the spacetime dimension $n \neq 2$, since otherwise $g^{ab} g_{ab} = 2$, which is the present case. This is as expected since $g_{ab} = \mathrm{diag}(-f(t,x), 1/f(t,x))$ is a vacuum solution of the Einstein equation. The Kretschmann scalar is

$$\mathcal{K} = R_{abcd} R^{abcd} = R^2 \tag{8}$$

E.g., for 1+1D Schwartzchild, $v(x) = -\sqrt{\frac{r_s}{x}}$, $\mathcal{K} = \frac{4r_s^2}{x^6}$ (in 1+3D, the value is $\frac{12r_s^2}{x^6}$). The Christopher symbols are

$$\begin{pmatrix} \{v^2 v_{,x}, v_{,x}(v^3 - v) - v_{,t}\} & \{-vv_{,x}, -v^2 v_{,x}\} \\ \{-vv_{,x}, -v^2 v_{,x}\} & \{v_{,x}, vv_{,x}\} \end{pmatrix} \tag{9}$$

($\Gamma^i_{jk}$ is the i-th element of the list in j-th row , k-th column) Only the gradient of the flow manifest itself in the covariant derivative $A^a_{;c} = A^a_{,c} + A^d \Gamma^a_{dc}$. However, the covariant divergence is same as flat spacetime, $\nabla^a A_a = A^1_{,1} + A^0_{,0}$. For $v = v(t)$, the only non-vanishing component is $\Gamma^1_{00} = -v_{,t}$, so $A^1_{;t} = A^1_{,t} - v_{,t} A^0$.

We proceed to investigate the particle geodesics. For the particle lagrangian we use

$$L = \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b = \frac{1}{2}\left[(v^2-1)\dot{t}^2 + \dot{x}^2 - 2v\dot{x}\dot{t}\right] \tag{10}$$

where $\dot{x}^a = dx/ds$, for massive particle $L = 1/2$ and for massless one $L = 0$. For $v = v(t)$, $u \equiv \dot{x} - v\dot{t}$ is constant, with $\frac{d}{ds}\left[(v^2-1)\dot{t} - v\dot{x}\right] = vv_{,t}\dot{t}^2 - v_{,t}\dot{x}\dot{t}$, or

$$\frac{d}{ds}\left[(v^2-1)\frac{\dot{x}-u}{v} - v\dot{x}\right] = vv_{,t}\left(\frac{\dot{x}-u}{v}\right)^2 - v_{,t}\dot{x}\frac{\dot{x}-u}{v} \tag{11}$$

A detailed analysis is left for futural study. But for photon orbit, we have $(v^2-1)\dot{t}^2 + \dot{x}^2 - 2v\dot{x}\dot{t} = 0$, then

$$(v^2-1)\left(\frac{\dot{x}-u}{v}\right)^2 + \dot{x}^2 - 2v\dot{x}\frac{\dot{x}-u}{v} = \dot{x}^2 - 2u\dot{x} + (1-v^2)u^2 = 0 \tag{12}$$

which gives $\dot{x} = u \pm |uv|$, but to obtain $x(t)$ we still need $dt/ds$.

## II. SCALAR WAVE EQUATION

### A. Relativistic Scalar

We consider at first a real scalar field, the action is given by

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{13}$$

For metric (1), $g_{ab} = \left(\begin{smallmatrix} v^2-1 & -v \\ -v & 1\end{smallmatrix}\right)$, $g = -1$, $g^{ab} = \left(\begin{smallmatrix} -1 & -v \\ -v & 1-v^2\end{smallmatrix}\right)$, the effective lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left[(v^2-1)(\phi_{,x})^2 + 2v\dot{\phi}\phi_{,x} + \dot{\phi}^2\right] - V \\ &= \frac{1}{2}\left[\left((\partial_t + v\partial_x)\phi\right)^2 - (\partial_x\phi)^2\right] - V\end{aligned} \tag{14}$$

The Klein-Gordon equation reads

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}v\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\phi - \frac{\partial^2\phi}{\partial x^2} = -V_{,\phi} \tag{15}$$

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left[(\partial_t\phi)^2 + \left(1-v^2\right)(\partial_x\phi)^2\right] + V \tag{16}$$

For a complex scalar field, the lagrangian is

$$\mathcal{L} = \left[|(\partial_t + v\partial_x)\phi|^2 - |\partial_x\phi|^2\right] - V \tag{17}$$

the conjugate momentum $\pi = \frac{\partial\mathcal{L}}{\partial_t\phi^*} = (\partial_t + v\partial_x)\phi$, we can define a conserved U(1) KG norm

$$(\phi_1, \phi_2) \equiv i\int_{-\infty}^{\infty} dx\left[\phi_1^*\left(\partial_t + v\partial_x\right)\phi_2 - \phi_2\left(\partial_t + v\partial_x\right)\phi_1^*\right] \tag{18}$$

for two solutions $\phi_1, \phi_2$.

### B. Nonrelativistic Scalar

For a real field, making the usual ansatz $\phi = \frac{1}{\sqrt{2m}}(e^{-imt}\psi + \text{c.c.})$, in the limit $\dot{\psi} \ll m\psi$, and assuming the potential is $V = \frac{1}{2}m^2\phi^2 + m\phi^2 U$, the lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left[i\dot{\psi}\psi^* - \frac{1-v^2}{m}|\psi_{,x}|^2 - iv\psi^*_{,x}\psi + \frac{v}{m}\psi^*_{,x}\dot{\psi} + \text{c.c.}\right] - |\psi^2|U \tag{19}$$

The Schrodinger equation reads

$$i(\partial_t + v\partial_x)\psi = \frac{1}{2}\partial_x\left(-\frac{1-v^2}{m}\partial_x\psi - iv\psi + \frac{v}{m}\dot{\psi}\right) + \frac{1}{2}\partial_t\left(\frac{v}{m}\partial_x\psi\right) + U\psi \tag{20}$$

or

$$i[(1 - \frac{v_{,x}}{2m})\partial_t + (\frac{3}{2}v - \frac{v_{,t}}{2m})\partial_x]\psi = -\frac{1-v^2}{2m}\partial_x^2\psi + \frac{v}{m}\partial_t\partial_x\psi + (U - \frac{iv_{,x}}{2})\psi \tag{21}$$

or with $\dot{\psi} \ll m\psi$,

$$i[\partial_t + (\frac{3}{2}v - \frac{v_{,t}}{2m})\partial_x]\psi \approx -\frac{1-v^2}{2m}\partial_x^2\psi + (U - \frac{iv_{,x}}{2})\psi \tag{22}$$

The exact U(1) current $j^a = \frac{\partial\mathcal{L}}{\partial(\partial_a\psi)}i\psi - \frac{\partial\mathcal{L}}{\partial(\partial_a\psi^*)}i\psi^*$ is

$$\rho = -j^0 = |\psi|^2, \quad j = -j^x = v|\psi|^2 \tag{23}$$

which satisfies $-\partial_a j^a = \dot{\rho} + \partial_x j = 0$.

### C. Slow Flow

We consider a low-speed approximation, $v \ll 1$, then

$$i[\partial_t + (\frac{3}{2}v - \frac{v_{,t}}{2m})\partial_x]\psi = -\frac{1}{2m}\partial_x^2\psi + (U - \frac{iv_{,x}}{2})\psi \tag{24}$$

which is of the form

$$i(\partial_t - u(t,x)\partial_x)\psi = -\frac{1}{2}\partial_x^2\psi + \mathcal{U}\psi \tag{25}$$

with $u \equiv \frac{3}{2}v - \frac{v_{,t}}{2m}$ and $\mathcal{U} \equiv U - \frac{iv_{,x}}{2}$, where for convenience, we have set $m = 1$. However, we note this approximated equation of motion cannot be derived from a lagrangian if the potential $\mathcal{U}$ is complex while the action is required to be real.

If $u = u(x)$, introducing $\psi = y\exp\left(-i\int^x u(x')dx'\right)$, the wave equation tunrs into

$$i\partial_t y = -\frac{1}{2}\partial_x^2 y + \left(-\frac{u^2}{2} + \mathcal{U} + \frac{iu_{,x}}{2}\right)y \tag{26}$$

hence the dissipation comes from both the x-dependence of $v$ and $u$.

If $v = v(t)$, in this case $u = u(t)$, $\mathcal{U} = U$, there is no dissipation. The drift term in the wave equation can be removed by a local Galilean transformation (i.e., in the comoving frame) $dx' = dx + udt, t' = t$, with $\partial_{t'} = -u\partial_x + \partial_t$, the potential however generally becomes time-dependent uncless it's constant.

The velocity-drift can have interesting consequences in the background frame, for which we consider a simple example here, an oscillating flow $u = \alpha + \beta \cos \mu t$, where $\alpha, \beta, \mu$ are constants (note it's required that $\alpha, \beta \ll 1$), with a harmonic oscillator potential $U = \frac{1}{2}\omega^2 x^2$. We can choose $x' = x + \alpha t + \frac{\beta}{\mu} \sin \mu t$, then in the co-moving frame, $i\partial_{t'}\psi = -\frac{1}{2}\partial_{x'}^2\psi + U\psi$, with

$$U(x', t') = \frac{1}{2}\omega^2(x' - \alpha t' - \frac{\beta}{\mu} \sin \mu t')^2 \tag{27}$$

Now at $t' = 0$, this is a harmonic potential, the potential minimum is at $x' = \alpha t' + \frac{\beta}{\mu} \sin \mu t'$. If the time-variation of the potential is slow enough, the state at $t' = 0$ may evolve adiabatically. For $\alpha = 0$ the perturabtion is periodic but a Berry phase of course does not exit since there is only one parameter.

For simplicity I choose $\alpha = 0$, and an inital eigenstate

$$\psi_n(x) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{n!2^n}} H_n(x) e^{-i\omega(1/2+n)t} \tag{28}$$

where $H_n(x)$ is the Hermite polynomial, is evolved. For simplicity I choose $n = 0$, the ground state:

$$\psi_0(x) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2}} e^{-i\omega t/2} \tag{29}$$

The figures above are computed in mma. More precise simulation is needed. For that purposes, the Hamiltonian may be written as

$$H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\omega^2[x - X(t)]^2 = \frac{1}{2}(1 + \omega^2 X^2) + a^\dagger a - \frac{\omega^2 X}{\sqrt{2\omega}}(a + a^\dagger) \tag{30}$$

where the system has been secondly quantized (the problem is however the same), with $[a, a^\dagger] = 1$. This can now be simluated with, e.g., QuTiP.

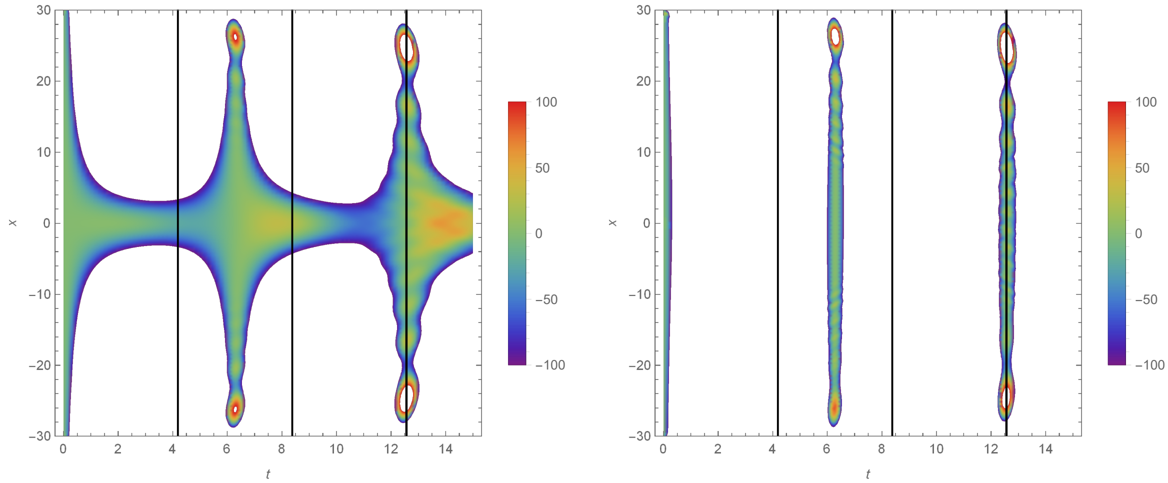


FIG. 1. Direct simulation in position space. $\omega = 3$, $\mu = 100$. Left: $\beta = 0$, right: $\beta = 30$. This plot only shows value of $\mathrm{Re}(\psi)$ in the range $(-100, 100)$. Interval between the black lines is $T = 2\pi/(\omega/2)$, which should be the oscillation period for $\beta = 0$, the numerical error, however, is clearly too large (I used periodic boundary condition with a finite box length), since exatcly the same oscillating pattern persists for different sets of model parameteres. Moreover, the errors are even larger for smaller $\mu$, but which is the more interesting regime. After $\beta$ is switched on, the wavefront is just flatter, but with original oscillation frequency (if this simulation really makes sense).

## III. DIRAC EQUATION

For a diagonal metric, the concrete form of Dirac equation is easily obtained, see, e.g., [ref]. The key is to find a proper set of dyad $e^{(a)}_\mu$ satisfying

$$g_{\mu\nu} = e^{(a)}_\mu e^{(b)}_\nu \eta_{ab}, \quad \gamma^\mu = e^\mu_{(a)}\gamma^a, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{31}$$

The first one is the defining condition. For 1+1D Minkovski spacetime, we can choose the $\gamma^a$ to be $\gamma^0 = -i\sigma_z = -i\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $\gamma^1 = \sigma_y = -i\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Let $e^{(0)}_\mu = (a, b), e^{(1)}_\mu = (c, d)$, then

$$\begin{aligned} g_{00} &= -e^{(0)}_0 e^{(0)}_0 + e^{(1)}_0 e^{(1)}_0 = b^2 - a^2 = v^2 - 1 \\ g_{11} &= -e^{(0)}_1 e^{(0)}_1 + e^{(1)}_1 e^{(1)}_1 = d^2 - c^2 = 1 \\ g_{01} &= -e^{(0)}_0 e^{(0)}_1 + e^{(1)}_0 e^{(1)}_1 = cd - ab = -v \end{aligned} \tag{32}$$

We choose $d = 1$, so $c = 0, a^2 = 1, b^2 = v^2$, we choose $a = 1, b = v$. Now $e^{(0)}_\mu = (1, v), e^{(1)}_\mu = (0, 1)$. Note $e^\mu_{(a)} \neq g^{\mu\nu} e^{(a)}_\nu$, but its inverse, defined by $e^{(a)}_\mu e^\nu_{(a)} = \delta^\nu_\mu$ and $e^{(a)}_\mu e^\mu_{(b)} = \delta^a_b$. We find that $e^{\mu}_{(0)} = (1,0), e^{\mu}_{(1)} = (-v, 1)$.

The Dirac equation is

$$\left(\gamma^a e^{\mu}_{(a)}\partial_\mu + \frac{1}{2}\gamma^a \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}e^{\mu}_{(a)}\right) + m\right)\psi = 0 \tag{33}$$

Now we have

$$\gamma^a \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}e^{\mu}_{(a)}\right) = \gamma^0\partial_\mu\left(e^{\mu}_{(0)}\right) + \gamma^1\partial_\mu\left(e^{\mu}_{(1)}\right) = -\gamma^1 v_{,t} \tag{34}$$

$$\gamma^a e^{\mu}_{(a)}\partial_\mu = \gamma^0 e^{\mu}_{(0)}\partial_\mu + \gamma^1 e^{\mu}_{(1)}\partial_\mu = \gamma^0\partial_t + \gamma^1(-v\partial_t + \partial_x) \tag{35}$$

the Dirac equation then reads

$$\left(\gamma^0\partial_t + \gamma^1(-v\partial_t + \partial_x) - \frac{1}{2}\gamma^1 v_{,t} + m\right)\psi = 0 \tag{36}$$

or

$$\begin{pmatrix} im + \partial_t & \partial_x - v\partial_t - \frac{1}{2}v_{,t} \\ -\left(\partial_x - v\partial_t - \frac{1}{2}v_{,t}\right) & im - \partial_t \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = 0 \tag{37}$$

or

$$i\partial_t\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = -i\left(\partial_x - v\partial_t - \frac{1}{2}v_{,t}\right)\begin{pmatrix}\psi_2 \\ \psi_1\end{pmatrix} + m\begin{pmatrix}\psi_1 \\ -\psi_2\end{pmatrix} \tag{38}$$

The Dirac representation of gamma matrices we have adopted is the standard choice to perform the NR limit. Making the ansatz $\psi = \Psi e^{-imt}$, the Dirac equation is now

$$i\partial_t\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = -i\left(\partial_x + imv - \frac{1}{2}v_{,t}\right)\begin{pmatrix}\Psi_2 \\ \Psi_1\end{pmatrix} - 2m\begin{pmatrix}0 \\ \Psi_2\end{pmatrix} \tag{39}$$

For $v = 0$, the usual Pauli equation originates from the approximation $-i\partial_x\Psi_1 - 2m\Psi_2 = 0$, with $\Psi_2$ being the "small" component. Now if we assume the $\partial_t\Psi_2 \ll m\Psi_2$, then

$$-i\left(\partial_x + imv - \frac{1}{2}v_{,t}\right)\Psi_1 = 2m\Psi_2 \tag{40}$$

Now we have a single decoupled equation for the "large" component $\Psi_1$:

$$i\partial_t\Psi_1 = -\frac{1}{2m}\left(\partial_x + imv - \frac{1}{2}v_{,t}\right)^2\Psi_1 \tag{41}$$

## IV. PROCA EQUATION

The Proca Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F_{ab})^2 - \frac{1}{2}m^2 A_a^2 = -\frac{1}{4}(\partial_a A_b - \partial_b A_a)\left[g^{ac}\partial_c\left(g^{bd}A_d\right) - g^{bd}\partial_d\left(g^{ac}A_c\right)\right] - \frac{1}{2}m^2 A_a^2 \quad (42)$$

The Proca equation reads

$$\nabla_b F^{ba} = \nabla_b(\nabla^b A^a - \nabla^a A^b) = \nabla^d \nabla_d A^a - g^{ca} R_{dc} A^d = m^2 A^a \quad (43)$$

Explicitly, we find

$$\begin{pmatrix} (1-v^2)\partial_x^2 - \partial_t^2 - v_{,x}\partial_t - (v_{,t} + 4vv_{,x})\partial_x - (m^2 + R - v_{,t,x}) - 2v\partial_t\partial_x & v_{,,x} + 2v_{,x}\partial_x \\ v_{,,t} + 2v_{,x}v_{,t} + 2vv_{,t,x} + 2(vv_{,x} + v_{,t})\partial_t + 2vv_{,t}\partial_x & (1-v^2)\partial_x^2 - \partial_t^2 - v_{,x}\partial_t - v_{,t}\partial_x - (m^2 + v_{,x,t}) - 2v\partial_t\partial_x \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \end{pmatrix} = 0 \quad (44)$$

where we have used the Lorenz condition:

$$\nabla^a A_a = A^1_{,1} + A^0_{,0} = 0 \quad (45)$$

But then $A^0$ and $A^1$ are not decoupled, unless $v = \text{const}$. Recall the definition of electric field is $E = -F^{01} = -(\partial^0 A^1 - \partial^1 A^0)$ (in 1+1D there is only one component, and no magnetic field), if $v = 0$ we have $E = -F^{01} = A^1_{,0} + A^0_{,1}$,

$$\partial_x E = \partial_1 \partial_0 A^1 + \partial_x^2 A^0 = m^2 A^0, \quad \partial_t E = \partial_0^2 A^1 + \partial_0 \partial_x A^0 = -m^2 A^1$$

which for $m = 0$ means $E$ is purely constant and for massive field $E = 0$, which then implies that $A^\mu = 0$ (i.e., no EM or Proca wave in 1+1D). In the massless case we don't have $A^\mu = 0$ but what remains is a purely gauge DOF. If $v \neq 0$, the Proca equation turns out to be the same,

$$\nabla_\mu \left(\begin{smallmatrix} 0 & -E \\ E & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \partial_x E \\ -\partial_t E \end{smallmatrix}\right) = m^2 \left(\begin{smallmatrix} A^0 \\ A^1 \end{smallmatrix}\right) \quad (46)$$

Hence, there is literally nothing for Proca field.

## V. OUTLOOKS

Explicit solutions may be found for interesting choices of $v(x,t)$, in particular, if a horizon (v=1) is presented, the field near horizon and the Hawking temperature, the latter is given by

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\sqrt{-\frac{1}{2}n^{\mu;\nu}n_{\mu;\nu}} \quad (47)$$

evaluated at horizon, where $\kappa$ the surface gravity and $n^\mu$ a Killing vector being null at horizon. A flow with multi-horizon is easily realized, whose causal structure can be rich, but the stability of such flow needs to be checked. To this end, energy-momentum tensor of various fields may be derived explicitly, and the metric perturbation of this background may be studied.