Electron in Azimuthal Magnetic Field

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Abstract

In this note I investigate the motion of a single electron in an azimuthal external magnetic field. My first encounter to this problem (during high school) is on P186 of [1], while a formal discussion (to my knowledge) did not appear in the literature until very recently ("Motion of a charged particle in the static fields of an infinite straight wire") [2].

CONTENTS

I. AZIMUTHAL MAGNETIC FIELD

We use cylindric coordinate (R, ϕ, z), with $g_{\mu\nu} = \mathrm{diag}(-1, 1, R^2, 1)$. Consider an azimuthal magnetic field (e.g., in the Z-pinch),

$$\mathbf{B}(\mathbf{r}) = f(\mathbf{r})\mathbf{e}_\phi \tag{1}$$

$\nabla \times \mathbf{A} = \mathbf{B}$ reads

$$\nabla \times \mathbf{A} = \begin{pmatrix} \frac{1}{R}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \\ \frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \\ \frac{1}{R}\left(\frac{\partial R A_\phi}{\partial R} - \frac{\partial A_R}{\partial \phi}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix} \tag{2}$$

The Coulomb gauge condition is $\nabla \cdot \mathbf{A} = R^{-1}\partial_R(RA_R) + R^{-1}\partial_\phi A_\phi + \partial_z A_z = 0$, but this is not compulsory. Assuming $f = f(R)$, we can choose $A_R = A_\phi = 0$ and $dA_z(R)/dR = -f$, i.e., $\mathbf{A} = A\mathbf{e}_z$, with

$$A(R) = -\int^R f(R')dR', \quad A'(R) = -f(R) \tag{3}$$

For example, we take $f = k/R$, $k = \text{const}$, then $A = -k \ln R$. For the magnetic field generated by an infinite straight current I, $k = \mu_0 I/2\pi$.

II. CLASSICAL DYNAMICS

Some brief recaps. The Newtonian equation of motion reads

$$\gamma m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{4}$$

with $\mathbf{E} = -\nabla A^0 - \dot{\mathbf{A}}$ and $\mathbf{B} = \nabla \times \mathbf{B}$. If there is no electric field, since $\dot{\mathbf{v}} \cdot \mathbf{v} = 0$, v is constant. The lagrangian and hamiltonian are

$$L = -\frac{m}{\gamma} - q(A^0 - \mathbf{A} \cdot \mathbf{v}), \quad H = m\gamma \tag{5}$$

where we adopt the unit $c = 1$ and we shall set $A^0 = 0$.

Consider first a general time-dependent potential $A = A(R, t)$, then we have

$$P \equiv p_z = \gamma m\dot{z} + qA = \text{const}, \quad L \equiv p_\phi = \gamma m R^2\dot{\phi} = \text{const}, \quad p_R = \gamma m\dot{R} \tag{6}$$

with

$$\dot{p}_R = \frac{q}{m}A_{,R}\dot{z} + \gamma m R\dot{\phi}^2 \tag{7}$$

Note

$$v^2 = 1 - \frac{1}{\gamma^2} = \dot{z}^2 + \dot{R}^2 + R^2\dot{\phi}^2 = \left(\frac{P - qA}{m\gamma}\right)^2 + (\dot{R})^2 + R^2\left(\frac{L}{\gamma m R^2}\right)^2 \tag{8}$$

it follows that

$$\gamma^2 = \left(1 - \dot{R}^2\right)^{-1}\left[\left(\frac{P - qA}{m}\right)^2 + \left(\frac{L}{mR}\right)^2 + 1\right] \tag{9}$$

Now for a static potential $A = A(R)$, as we shall focus on below, the problem is greatly simplified, since also γ is constant, one finds that

$$E = \frac{1}{2}\gamma m \dot{R}^2 + V(R) = \text{const} \tag{10}$$

where the effective potential is

$$V(R) = \frac{1}{\gamma m}\left(\frac{L^2}{2}\frac{1}{R^2} + \frac{1}{2}q^2A^2 - qPA\right) \tag{11}$$

A. Periodicity

I think my study on this problem was (perhaps) earlier than [2], as a small EM course project at college, however I did not go into much depth as the aurthors of [2]. In that project I focused purely on the periodicity (see also next subsection), since in fact only R is dynamical, the motion is periodic in R, ϕ, z (with same period of R-motion). Let T be the period of radial motion, and $\mathbf{v}_0$ be the initial velocity, it is clear that T depends solely on $\mathbf{v}_0$ and the initial radial coordinate R_0. Moreover, the radial equation of motion reads

$$\gamma m \ddot{R} = -V_{,R} = \frac{1}{\gamma m}\left(\frac{L^2}{R^3} - q^2 A A_{,R} + qPA_{,R}\right) \tag{12}$$

or using variables $\rho = R/R_0$, $\tau = t/R_0$,

$$\gamma m \rho_{,,\tau} = \frac{1}{\gamma m}\left(\frac{\gamma^2 m^2 v_{\phi 0}^2}{\rho^3} - q^2 A A_{,\rho} + q[\gamma m v_{z0} + qA(\rho_0)]A_{,\rho}\right) \tag{13}$$

Now if $A = A(\rho)$, the radial motion is "scale-invariant", and the period T is simply proportional to R_0 for a given initial velocity. This is only possible for the magnetic field generated by an infinite straight current, since we can choose $A = -k \ln R/R_0$.

The axial and angular motion are given by

$$z_{,\tau} = R_0 \frac{\gamma m v_{z0} + qA(\rho_0) - qA}{\gamma m}, \quad \phi_{,\tau} = \frac{\rho_0 v_{\phi 0}}{\rho} \tag{14}$$

So for $A = A(\rho)$ and an given initial velocity, $\Delta\phi = \phi(t_0 + T) - \phi(t_0) = \int \phi_{,\tau} d\tau$ is same for all R_0, while the azimuthal precession $\Delta z = z(t_0 + T) - z(t_0)$ is proportional to R_0.

B. A Kind of Recurrence

The above properties leads to the following observation (not a physics problem, indeed). If two particles (assuming non interacting, of course) are released with same initial velocity and at same ϕ_0 and z_0 (at radius R_1 and R_2, respectively), can they meet again at same ϕ and z? My conjecture is yes, they can meet at arbitrary precession.

I can only give a proof for the case $R_1/R_2 = n_1/n_2$, where $n_1, n_2 \in \mathbb{N}$, hence $T_1/n_1 = T_2/n_2 \equiv \eta = \text{const}$. Then after time $t = N\eta\times$ least common multiple of n_1 and n_2, where $N \in \mathbb{N}$, the two particles reach a same z-value while their relative azimuthal angle is

$$\alpha(t) = \left(\frac{1}{n_1} - \frac{1}{n_2}\right)\frac{\Delta\phi}{\eta}t \equiv 2M\pi + \epsilon, \quad \epsilon \in (0, 2\pi), \quad M \in \mathbb{N} \tag{15}$$

But as can be proved, $\forall\epsilon, \exists K \in \mathbb{N}$, so that $\mod(K\epsilon, 2\pi) < \epsilon$, where $\mod(a,b) \equiv |a - [a/b]b|$, $[x]$ denotes the nearst integer around x.

C. 2D Motion

If $L = 0$, the motion is confined to a plane. Numerically I found a trajectory that is somewhat special, if $\mathbf{v}_0 = k\mathbf{e}_z$ (we set $q/m = 1$) and $R_0 = ae$ (maximum radius), then the minimum radius is a/e, the radius at which $v_z = 0$ is a, and the raidus of kink is a, here a is a constant. This awaits for an analytical proof.

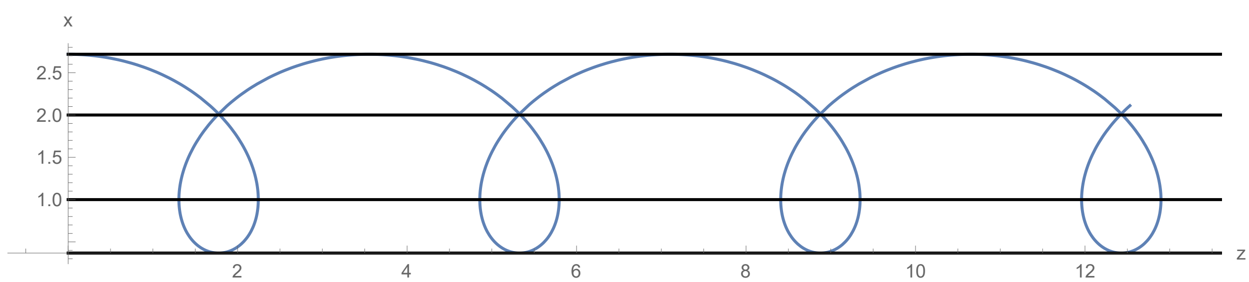

FIG. 1. $L = 0$, $k = a = v = 1$. The black lines correspond to $R = ae, 2a, a, a/e$. The numerical procedure to obtain this trajectory is described in Appendix A. As can be easily seen from the effective potential with $A = -k\ln R$, the extreme radius for $L = 0$ is $R_0 e^{(\pm v - v_{z0})/k}$.

If the gyrofrequency $\Omega = \frac{|\mathbf{B}|q}{m}$ is very large and the distance the particle moves within a gyroperiod is very small compared to the scale of magnetic field gradient, adiabatically the particle motion can be decomposed into a smooth guiding center trajectory $\mathbf{X}(t)$ (referred as drift) with gyromotion around the guiding center (see f.e., W. Dittrich, M. Reuter, Classical and Quantum Dynamics (Springer, 2017), p.404). The gyrophase $\theta(t) = \int_0^t dt'\, \Omega(\mathbf{X}(t'))$ receives a geometrical contribution

$$\mathbf{R} \cdot \dot{\mathbf{X}}(t) \subset \dot{\theta}(t) \tag{16}$$

where

$$\mathbf{R} \equiv (\nabla \mathbf{e}_1) \cdot \mathbf{e}_2 \equiv (\partial_i e_{1j}) e_{2j}, \quad \mathbf{e}_1 = \frac{\mathbf{b} \cdot \nabla \mathbf{b}}{|\mathbf{b} \cdot \nabla \mathbf{b}|}, \quad \mathbf{e}_2 = \mathbf{b} \times \mathbf{e}_1, \quad \mathbf{b} = \frac{\mathbf{B}}{|\mathbf{B}|} \tag{17}$$

In the present case, $\mathbf{b} = \mathbf{e}_\phi$, $\mathbf{b} \cdot \nabla \mathbf{b} = R^{-1} \partial_\phi \mathbf{e}_\phi = -R^{-1} \mathbf{e}_R$, hence

$$\mathbf{e}_1 = -\mathbf{e}_R, \quad \mathbf{e}_2 = \mathbf{e}_z, \quad \mathbf{R} = \mathbf{0} \tag{18}$$

hence there is no geometrical phase. The guiding center motion is along $\mathbf{e}_2$, and from the discussion above, the drift velocity is independent of R_0, this is consistene with the general result of gradient drift in slowly varing magnetic field (a canonical example being $\mathbf{F} = q\mathbf{E}$ in a uniform EM field),

$$\mathbf{v}_{\mathrm{grad}} = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \tag{19}$$

with the gradient force $\langle \mathbf{F} \rangle = \frac{1}{T} \int_0^T \mathbf{F} dt = -\frac{1}{2} m v_\perp^2 \frac{\nabla B}{B}$, where $v_\perp$ is the magneitude of velocity perpendicular to the magnetic field,

$$\mathbf{v}_{\mathrm{grad}} = -\frac{\frac{1}{2} m v_\perp^2 \nabla B \times \mathbf{B}}{qB^3} = \frac{mv^2}{2kq} \mathbf{e}_z \tag{20}$$

There is also curvature drift resulted from the curvature of magnetic field lines, which should be relevant for the 3D motion.

D. Radiation Damping

This appears to be have not been discussed before, for this particular motion. The radiation power (at far field) is

$$P = \frac{dE}{dt} = \frac{q^2 \gamma^6}{6\pi} \left(a^2 - |\mathbf{v} \times \mathbf{a}|^2 \right) \tag{21}$$

under the Lorentz-Heaviside convention $c = \epsilon_0 = \mu_0 = 1$. Since the motion is not periodic, the analysis will be more complicated than the circular motion in a constant magnetic field (in which case we have cyclotron or synchrotron radiation). The form of radiation damping force in classical electrodynamics is notoriously ambigious, hence we'd better work in an adiabatic approxiamtion, assuming the orbit qualitatively does nos change much in the time scale of orbit period. The acceleration is

$$\mathbf{a} = \ddot{\mathbf{r}} = \begin{pmatrix} \ddot{R} - R\dot{\phi}^2 \\ 2\dot{R}\dot{\phi} + R\ddot{\phi} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \ddot{R} - \frac{L^2}{M^2R^3} \\ 2\dot{R}\frac{L}{MR^2} + R\frac{d}{dt}\frac{L}{MR^2} \\ -\frac{q}{M}\dot{A} \end{pmatrix} = \begin{pmatrix} -\frac{V'(R)}{M} - \frac{L^2}{M^2R^3} \\ 0 \\ \frac{q}{M}f(R)\dot{R} \end{pmatrix} \tag{22}$$

Hence to find the averaged dissipation power

$$\bar{P} = \frac{1}{T}\int_0^T P(t)dt \tag{23}$$

we need $R(t)$ and $\dot{R}(t) = \pm\sqrt{2(E-V)/M}$ within a period, this can only be computed numerically.

III. NONRELATIVISTIC QUANTUM DYNAMICS

To formulate this problem in the nonrelativitic quantum mechanics, and specifically for a charged spin-1/2 particle, we write the Pauli equation

$$i\hbar\dot{\Psi} = \mathcal{H}_{\text{Pauli}}\Psi = \left[\frac{1}{2m}(-i\hbar\nabla - q\mathbf{A})^2 + \kappa\boldsymbol{\sigma}\cdot\mathbf{B}\right]\Psi \tag{24}$$

with $\Psi = (\psi_1, \psi_2)^T$, and where $\kappa = g\mu_B/2$, $\mu_B = \hbar q/2m$, with the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The off-diagonal coupling is

$$\boldsymbol{\sigma}\cdot\mathbf{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} = if\begin{pmatrix} 0 & -e^{i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \tag{25}$$

Let $\Phi_\pm = \psi_\pm e^{\pm i\phi/2}$, $\Phi = (\Phi_+, \Phi_-)^T$, since $(-i\hbar\nabla - q\mathbf{A})^2 = -\hbar^2\nabla^2 + q^2A^2 + i2q\hbar\mathbf{A}\cdot\nabla$, Eq. (24) now reads

$$i\hbar\dot{\Phi} = \left\{\frac{1}{2m}\left[-\hbar^2(R^{-1}\partial_R R\partial_R + R^{-2}\partial_\phi^2 + \partial_z^2) + q^2A^2 + i2q\hbar A\partial_z\right] + \kappa f\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}\Phi \tag{26}$$

Here A, f are functions of R only, this means $[\mathcal{H}, \partial_z] = [\mathcal{H}, \partial_\phi] = 0$, the equation is seperable. Since $x^{-1}\partial_x[x\partial_x(y/\sqrt{x})] = (\partial_x^2 y + y/4x^2)/\sqrt{x}$, we make the ansatz $\Phi(\mathbf{r}, t) = \Phi_z(z)\Phi_\phi(\phi)\Phi(R, t)/\sqrt{R}$ with $-i\hbar\partial_z\Phi_z = P\Phi_z$, $-i\hbar\partial_\phi\Phi_\phi = L\Phi_\phi$. Then we get

$$i\hbar\dot{\Phi} = \left[-\frac{\hbar^2}{2m}\partial_R^2 + \left(\frac{q^2A^2}{2m} - \frac{qPA}{m} + \frac{1}{2m}\frac{L^2 - \hbar^2/4}{R^2}\right) + \kappa f \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]\Phi \tag{27}$$

Indeed, for $\hbar \to 0$, the potential coincides with the classical result, if the spin is zero [3], this will be the case for a charged complex scalar particle.

A. 2D Dynamics with Spin

If $L = 0$, in the classical problem the motion is confined to a plane containing z-axis, say, the y-z plane, now we consider the counterpart in quantum mechanics assuming the system is also confined to a 2D plane (the 3D problem, of course, cannot get confined quantum-mechanially, though what is the exact quantum state corresponding to the classically-confined motion is an interesting question). For this we can simply take $L = 0$, but surely it's more convenient if we switch to the y-z plane and choose $\sigma_x = \mathrm{diag}(1, -1)$, so that Pauli equation reads, in the (z, y) coordinates, after separating $\Phi(z, y) = \Phi_z(z)\Phi(y)$ with $-i\hbar\partial_z\Phi_z = P\Phi_z$ (the constant term in the potential has been omitted), we are left with a 1+1D equation

$$i\hbar\dot{\Phi} = \left[-\frac{\hbar^2}{2m}\partial_y^2 + \left(\frac{q^2A^2}{2m} - \frac{qPA}{m}\right) + \kappa f\sigma_x\right]\Phi \tag{28}$$

where $A(y) = A(R = y)$. For the numerics we shall work in the unit $\hbar = m = q = k = 1$ (so that $\kappa = g/4 = 1/2$ for electron), then the time-independent Schrodinger equation with $\Phi(t, y) = \Phi(y)e^{-iEt}$ reads

$$E\Phi = \left(-\frac{1}{2}\partial_y^2 + V_\pm\right)\Phi, \quad V_\pm(y) = \frac{1}{2}A^2 - PA \pm \frac{1}{2}f \tag{29}$$

Note $f(y) = -A'(y)$. Now, $f(y) = 1/|y|$, $A(y) = -\ln|y|$, the potential is

$$V_\pm(y) = \frac{1}{2}(\ln|y|)^2 + P\ln|y| \pm \frac{1}{2|y|} \tag{30}$$

The potential depends on P. For the "+" spin state, we have a confining potential; for the "-" spin state, on the other hand, the potential at $y = 0$ tends to $-\infty$. Near $y = 0$, the potential is dominated by $1/|y|$ and the problem looks same as the tunneling of α particle. Yet there is no scattering problem since the potential admitts only bound states. In contrast, a uniform magnetic field corresponds to $A(y) \propto y$, leading to a harmonic potential with Landau levels (though this is not the preferred gauge), in this case f = const hence the spin merely introduces a constant energy split.

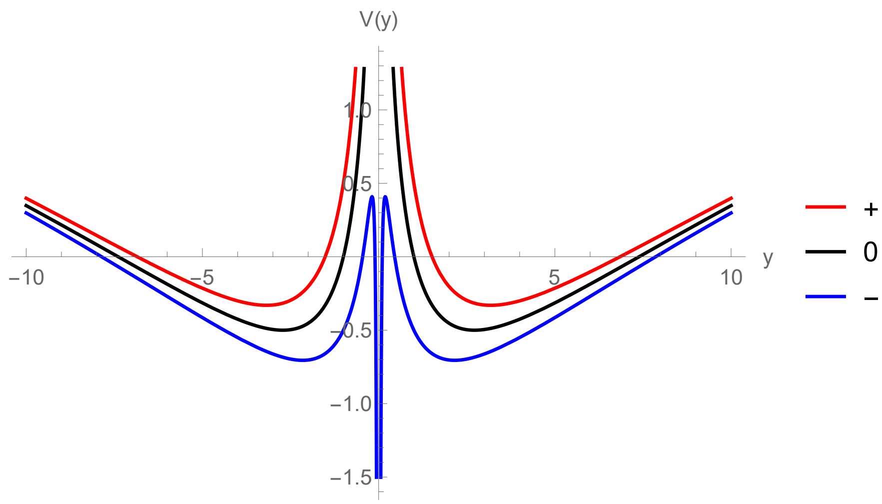

FIG. 2. Effective potential, $P = -1$. The red, black and blue curves correspond to plus, spinless and minus state, resectively.

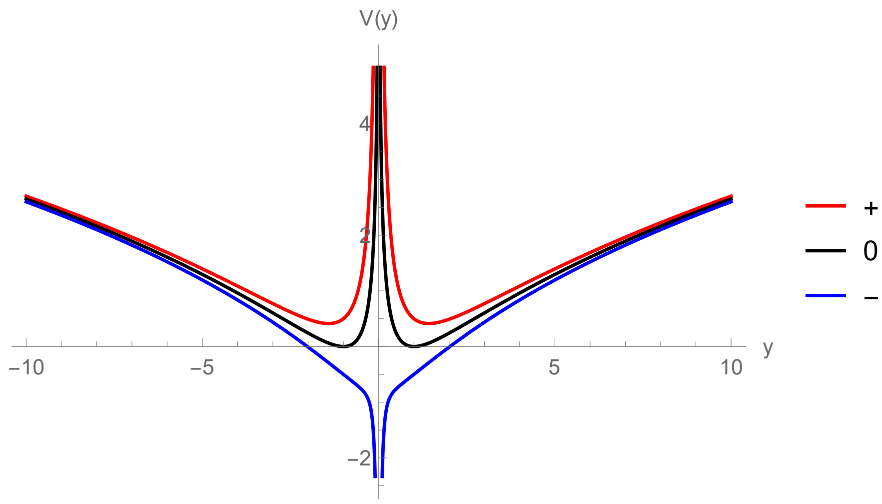

FIG. 3. Effective potential, $P = 0$.

We computed numerically (with limited precession) some eigenvalues, states and their Wigner distribution for such potential, see figures below ($y = 0$ is at the box center).

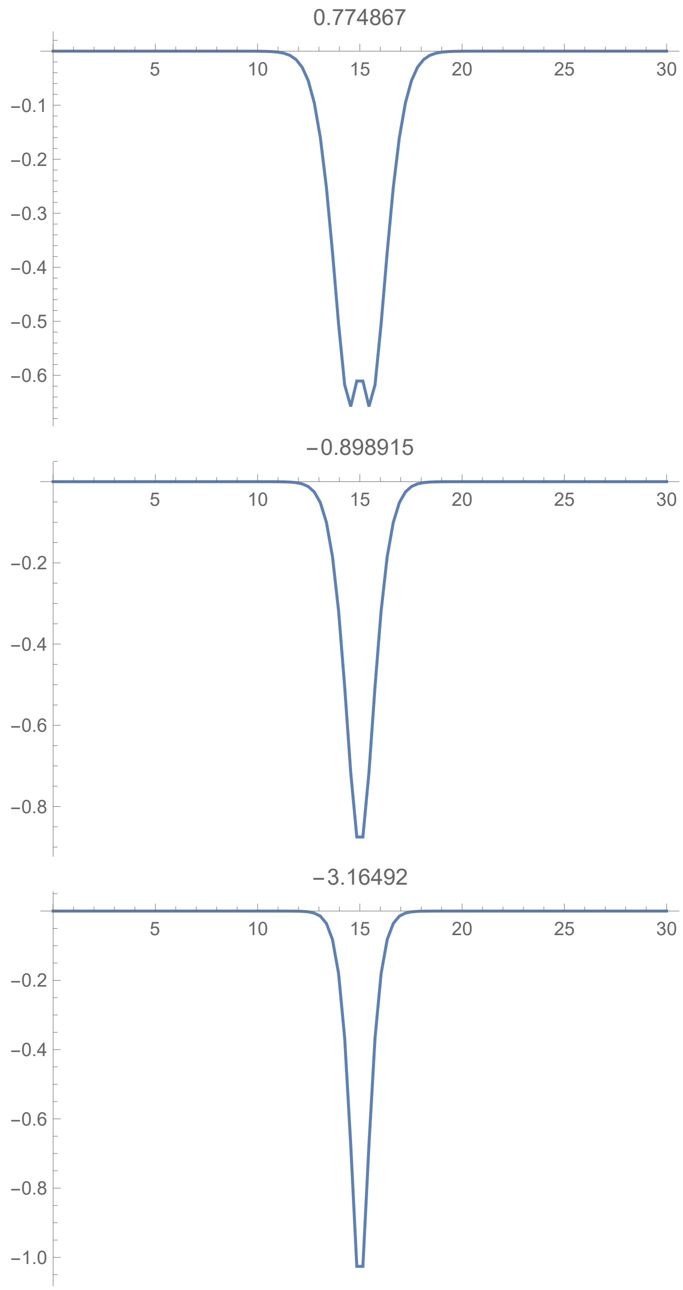

FIG. 4. Ground state and energy for plus (upper), spinless (middle) and minus (lower) spin state. $P = 2$. (since ptential at the center diverges, these solutions seem to be problematic, also the energy level ordering is strange, more precise calculation is obviously needed)

We also simulated the propagation of an initial Gaussian wave packet in this potential, see Fig. III A. In this 2D problem the plus and minus spin states gets decoupled, but in the 3D case we have to solve the full equation Eq.(27), whose solution might exhibit interesting features, such as oscillation between spin states.

IV. RELATIVISTIC QUANTUM DYNAMICS

To formulate this problem in the relativitic quantum mechanics, and specifically for a charged spin-1/2 particle, we write the Dirac equation (in mostly-minus convention)

$$\gamma^\mu(i\partial_\mu + qA_\mu)\Psi - m\Psi = 0 \tag{31}$$

Note that $[\mathbf{A}]_i = A^i = -A_i$. The goal of this section is small: just try to explicitly write down the Dirac equation.

For the gamma matrices, we try first the Dirac representation:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{32}$$

The Dirac equation can also be written (in a Shrodinger form) as

$$i\hbar\dot{\Psi} = \mathcal{H}_{\text{Dirac}}\Psi = \left[c\boldsymbol{\alpha}\cdot\left(-i\hbar\nabla - \frac{q}{c}\mathbf{A}\right) + qA^0 + mc^2\beta\right]\Psi \tag{33}$$

with $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, where

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{34}$$

Then for zero electric potential and $\mathbf{A} = A\mathbf{e}_z$, explicitly (set $c = \hbar = 1$),

$$\mathcal{H}_{\text{Dirac}} = \begin{pmatrix} m & 0 & p_z - qA & p_x - ip_y \\ 0 & m & p_x + ip_y & -(p_z - qA) \\ p_z - qA & p_x - ip_y & -m & 0 \\ p_x + ip_y & -(p_z - qA) & 0 & -m \end{pmatrix} \tag{35}$$

where $p_i = -i\partial_i$. This equation is also seperable, by introducing $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T = (e^{\phi/2}\psi_1, e^{-\phi/2}\psi_2, e^{\phi/2}\psi_3, e^{-\phi/2}\psi_4)^T$. Then, in cylindrical coordinate, Eq. (33) reads

$$i\dot{\Phi} = \begin{pmatrix} m\Phi_1 - (i\partial_z + qA)\Phi_3 - [i\partial_R + \frac{1}{R}(\frac{1}{2} + \partial_\phi)]\Phi_4 \\ m\Phi_2 + (i\partial_z + qA)\Phi_4 - [i\partial_R + \frac{1}{R}(\frac{1}{2} - \partial_\phi)]\Phi_3 \\ -m\Phi_3 - (i\partial_z + qA)\Phi_1 - [i\partial_R + \frac{1}{R}(\frac{1}{2} + \partial_\phi)]\Phi_2 \\ -m\Phi_4 + (i\partial_z + qA)\Phi_2 - [i\partial_R + \frac{1}{R}(\frac{1}{2} - \partial_\phi)]\Phi_1 \end{pmatrix} \tag{36}$$

Again by choosing $-i\partial_z\Phi_z = P\Phi_z$, $-i\partial_\phi\Phi = L\Phi$ and $\Phi(\mathbf{r},t) = e^{iEt}\Phi(R)\Phi_\phi\Phi_z$, we are left with

$$\begin{pmatrix} (E+m)\Phi_1 - (-P+qA)\Phi_3 - [i\partial_R + \frac{1}{R}(\frac{1}{2}+iL)]\Phi_4 \\ (E+m)\Phi_2 + (-P+qA)\Phi_4 - [i\partial_R + \frac{1}{R}(\frac{1}{2}-iL)]\Phi_3 \\ (E-m)\Phi_3 - (-P+qA)\Phi_1 - [i\partial_R + \frac{1}{R}(\frac{1}{2}+iL)]\Phi_2 \\ (E-m)\Phi_4 + (-P+qA)\Phi_2 - [i\partial_R + \frac{1}{R}(\frac{1}{2}-iL)]\Phi_1 \end{pmatrix} = 0 \tag{37}$$

or

$$\begin{pmatrix} E+m & 0 & P-qA & -[i\partial_R + \frac{1}{R}(\frac{1}{2}+iL)] \\ 0 & E+m & -[i\partial_R + \frac{1}{R}(\frac{1}{2}-iL)] & -P+qA \\ P-qA & -[i\partial_R + \frac{1}{R}(\frac{1}{2}+iL)] & E-m & 0 \\ -[i\partial_R + \frac{1}{R}(\frac{1}{2}-iL)] & -P+qA & 0 & E-m \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix} = 0 \tag{38}$$

But I cannot see how this can be further simplified, even without A.

Another way to write this is through a coordinate transformation, from the Cartesian coordinate to the cylindrical coordinate (i.e., spatial rotation, from $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ to $ds^2 = -dt^2 + dR^2 + R^2 d\phi^2 + dz^2$), with Dirac operator being invariant:

$$\gamma^\mu \nabla_\mu = \gamma^0\partial_t + \gamma^x\partial_x + \gamma^y\partial_y + \gamma^z\partial_z = \gamma^0\partial_t + \gamma^R\partial_R + \frac{1}{R}\gamma^\phi\partial_\phi + \gamma^z\partial_z \tag{39}$$

Since $\partial_x = \cos\phi\,\partial_R - \frac{\sin\phi}{R}\partial_\phi$, $\partial_y = \sin\phi\,\partial_R + \frac{\cos\phi}{R}\partial_\phi$, we get

$$\gamma^R = \cos\phi\,\gamma^x + \sin\phi\,\gamma^y = \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & -e^{-i\phi} & 0 & 0 \\ -e^{i\phi} & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^\phi = -\sin\phi\,\gamma^x + \cos\phi\,\gamma^y = i\begin{pmatrix} 0 & 0 & 0 & -e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ -e^{i\phi} & 0 & 0 & 0 \end{pmatrix} \tag{40}$$

(in fact this is just a spatial rotation of $\boldsymbol{\gamma}$) So one may write $\sigma_R = \left(\begin{smallmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{smallmatrix}\right)$ and $\sigma_R = \left(\begin{smallmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{smallmatrix}\right)$. Note however ψ itself is not a Lorentz vector!

So we try another conventional choice for the gamma matrices, the chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{41}$$

This has been used to solve Dirac equation in cylindrical problem [4]. It was found that a unitary transformation

$$\psi' = S\psi, \quad S = \frac{1}{\sqrt{R}} \mathrm{e}^{\frac{1}{2}\phi\gamma^1\gamma^2} \tag{42}$$

brings the Dirac equation into

$$\left(\gamma^0\partial_t + \gamma^1\partial_R + \frac{\gamma^2}{R}\partial_\phi + \gamma^3(\partial_z - iqA) + im + iq\gamma^0 A^0\right)\psi' = 0 \tag{43}$$

[4] focuses on a delta-like $A^0(R)$, while our problem is about A^3. If $A = 0$, we can introduce the ansatz: $\psi' = \gamma^1\gamma^2\psi''$, then

$$(H_1 + H_2)\psi'' = 0, \quad H_1 = \left(\gamma^0\partial_t + \gamma^3(\partial_z - iqA)\right)\gamma^1\gamma^2, \quad H_2 = \left(\gamma^1\partial_R + \frac{1}{R}\gamma^2\partial_\phi + im\right)\gamma^1\gamma^2 \tag{44}$$

$[H_1, H_2] = 0$, since (we wtite for short $\gamma^i\gamma^j = ij$) $[012, 2] = [012, 1] = [312, 2] = [312, 1] = [012, 12] = [312, 12] = 0$, thus one can set $H_1\psi'' = \lambda\psi'' = -H_2\psi''$, which greatly simplifies the problem. But once $A = A(R)$, $[H_1, H_2] \neq 0$, this technique fails. Anyway, we make the ansatz $\psi'' = e^{iEt}e^{iPz}e^{iL\phi}\left(\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right)$, then

$$\left(E\gamma^0 - i\gamma^1\partial_R + \frac{L}{R}\gamma^2 + (P - qA)\gamma^3 + m\right)\begin{pmatrix}\epsilon\\ \eta\end{pmatrix} = 0 \tag{45}$$

Now ϵ, η are coupled, unless we consider (i) $m = 0$, (ii) Majorana field. Even so, (45) is much better than (38), since for the latter even $m = 0$ the 4 components are still coupled.

A. Majorana Equation and Neutrino

It will be interesting to study the neutrino oscillation in this magnetic field. To this end, we have to solve the wave equation for Majorana field ψ_a (as mass eigenstates basis)

$$(i\gamma^\mu\partial_\mu - m_a)\psi^a - \frac{1}{2}\mu_{ab}\sigma_{\mu\nu}F^{\mu\nu}\psi_b = 0 \tag{46}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and μ_{ab} the anomalous magnetic moments (neutrinos are neutral), which can then be used for the canonical quantization, but for a spatially non-homogeneous magnetic field the problem should be considerably more challenging [5]. For Majorana fermion, one can write $\psi_a = \left(\begin{smallmatrix}\mathrm{i}\sigma_2\eta_a^*\\ \eta_a\end{smallmatrix}\right)$, the wave equation then reads

$$\dot{\eta}_a - \sigma_i\partial_i\eta_a + m_a\sigma_2\eta_a^* - \mu_{ab}\boldsymbol{\sigma}\cdot(\mathbf{B} - \mathrm{i}\mathbf{E})\sigma_2\eta_b^* = 0 \tag{47}$$

B. Axion in Azimuthal Magnetic Field

The axion is a hypothetical pseudo-scalar particle that couples to $\mathbf{E}\cdot\mathbf{B}$. In a fixed EM field, the axion wave equation would be $(-\partial_t^2 + \nabla^2)a = m^2 a - g\mathbf{E}\cdot\mathbf{B}$, where g is the coupling constant. For a pure magnetic field this is uninteresting, yet in a next-order approximation, seperating the total EM field as a sum of background and radiation field, the latter satisfies

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{j}_{\text{eff}}, \quad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{E} = \rho_{\text{eff}}, \quad \nabla \cdot \mathbf{B} = 0 \tag{48}$$

with

$$\mathbf{j}_{\text{eff}} = g(\dot{a}\mathbf{B}_0 + \nabla a \times \mathbf{E}_0), \quad \rho_{\text{eff}} = -g \nabla a \cdot \mathbf{B}_0 \tag{49}$$

then the axion-to-photon conversion can be analyzed, e.g., given a fixed external axion field profile. As commonly set, we choose $a = a(t) = a_0 \cos \omega t$ (with a_0, ω being constant), then an azimuthal magnetic field would induce an azimuthal current distribution, and for $\mathbf{B}_0$ generated by an infinite stright wire,

$$\mathbf{j}_{\text{eff}}(t, R) = -k g a_0 \omega \frac{\sin \omega t}{R} \mathbf{e}_\phi \tag{50}$$

The resulted radiation field is an interesting question, due to the symmetry, it should depend solely on R and t.

V. SOME OTHER MAGNETIC FIELDS

There are many other interesting configurations of magnetic fields, for which the present study may be exttended to. For example,

$$\nabla \times \mathbf{A} = \begin{pmatrix} \frac{1}{R}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \\ \frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \\ \frac{1}{R}\left(\frac{\partial R A_\phi}{\partial R} - \frac{\partial A_R}{\partial \phi}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ a_1/r \\ a_2/r^2 \end{pmatrix} \tag{51}$$

was used to model the jet enviroment around a supermassive black hole in the galaxy center, where $r = \sqrt{R^2 + z^2}$ is the radial distance to the BH [6]. One choice is

$$\mathbf{A} = \begin{pmatrix} A_R \\ A_\phi \\ A_z \end{pmatrix} = a_1 \begin{pmatrix} 0 \\ 0 \\ -\operatorname{arcsinh}(R/z) \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ \ln(R^2 + z^2)/2R \\ z\phi/(R^2 + z^2) \end{pmatrix} \tag{52}$$

Motion of an electric charge in the magnetic field generated by a magnetic monopole is a classical problem, but the quantum-mechanical version may also be considered (perhaps have been done).

A slight but nevertheless interesting extention of the static azimuthal magnetic field is to consider a time-dependent one. E.g., for a monochromatic current source

$$\mathbf{j} = I_0 \sin \omega t \, \delta(x)\delta(y)\mathbf{e}_z, \quad \rho = 0 \tag{53}$$

(the more physical scenario would be the region near a wire loop, see, e.g., Griffith's electrodynamics, p. 473) The vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{j}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} = -\frac{\mu_0}{4} I_0 \left[J_0(\omega R) \cos \omega t + Y_0(\omega R) \sin \omega t \right] \mathbf{e}_z \tag{54}$$

and the EM fields are

$$\mathbf{E} = -\dot{\mathbf{A}} = (\cdots)\mathbf{e}_z, \quad \mathbf{B} = \nabla \times \mathbf{A} = -\frac{\omega\mu_0}{4} I_0 \left[J_1(\omega R) \cos \omega t + Y_1(\omega R) \sin \omega t \right] \mathbf{e}_\phi \tag{55}$$

where $J_n(x)$ and $Y_n(x)$ are n-th Bessel functions. This is different to the EM fields in a coaxial transmission line (see, e.g., Griffith's electrodynamics, p. 431), which read

$$\mathbf{E}(\mathbf{r}, t) = \frac{A}{R} \cos(kz - \omega t)\mathbf{e}_R, \quad \mathbf{B}(\mathbf{r}, t) = \frac{A/c}{R} \cos(kz - \omega t)\mathbf{e}_\phi \tag{56}$$

where A, ω, k are constants, and are sourced by

$$j \text{ and } \rho \propto \cos(kz - \omega t)\delta(x)\delta(y) \tag{57}$$

Still another modification is to consider a stright wire carrying a stationary current wih finite length l (with center at $z = 0$). Though being a bit less physical, the generated magnetic field is (Biot-Savart Law)

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{z} \times (\mathbf{r} - \mathbf{z})}{|\mathbf{r} - \mathbf{z}|^3} = \frac{\mu_0 I}{4\pi R} \mathbf{e}_\phi \int_{-\arctan(z+l/2)/R}^{\arctan(l/2-z)/R} d\theta \cos\theta \tag{58}$$

Finally, if the wire is extremely massive, so that we are in the strong-field regime of gravity, we should consider a cylindrically-symmetrical spacetime (so called cosmic string) that accounts for the wire's mass distirbution. The wire may still carry a current, and one needs to solve the Einstein-Maxwell system for the resulted EM field profiles. The classical motion of an electrically charged mass-point in this spacetime is given by the geodesic eqaution under Lorentz force.

Appendix A: Numerical Guide to Simulate the Classical Particle Trajectories

In cartesian coordinates, the Newtonian equation of motion under Lorentz force reads

$$\begin{cases} \ddot{x} = \frac{q}{m}\left(\dot{y}B_z - \dot{z}B_y\right) \\ \ddot{y} = \frac{q}{m}\left(\dot{z}B_x - \dot{x}B_z\right) \\ \ddot{z} = \frac{q}{m}\left(\dot{x}B_y - \dot{y}B_x\right) \end{cases} \tag{A1}$$

The basis vectors of cylindrical coordinate (R, ϕ, z) read

$$\mathbf{e}_z = \mathbf{z}, \quad \mathbf{e}_R = \frac{\mathbf{x}+\mathbf{y}}{\sqrt{x^2+y^2}}, \quad \mathbf{e}_\phi = \frac{-y\mathbf{e}_x + x\mathbf{e}_y}{\sqrt{x^2+y^2}} \tag{A2}$$

The basis vectors of spherical coordinate (r, θ, ϕ) (with z axis as polar axis) read

$$\mathbf{e}_r = \frac{\mathbf{x}+\mathbf{y}+\mathbf{z}}{\sqrt{x^2+y^2+z^2}}, \quad \mathbf{e}_\theta = \frac{z\frac{\mathbf{x}+\mathbf{y}}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}\mathbf{e}_z}{\sqrt{x^2+y^2+z^2}}, \quad \mathbf{e}_\phi = \frac{-y\mathbf{e}_x + x\mathbf{e}_y}{\sqrt{x^2+y^2}} \tag{A3}$$

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[1] P. Gndig, G. Honyek, and K. Riley, *200 puzzling physics problems. With hints and solutions* (Cambridge University Press, 2001).

[2] J. Franklin, D. J. Griffiths, and N. Mann, American Journal of Physics **90**, 513 (2022), https://doi.org/10.1119/5.0077042.

[3] It's possible to embed a "spin" also in the classical dynamics (Bargmann-Michel-Telegdi theory), but here we don't bother to consider it.

[4] M. Loewe, F. Marquez, and R. Zamora, Journal of Physics A: Mathematical and Theoretical **45**, 465303 (2012).

[5] M. Dvornikov 10.48550/ARXIV.1110.5859 (2011).

[6] M. Meyer, D. Montanino, and J. Conrad, Journal of Cosmology and Astroparticle Physics **2014** (09), 003.

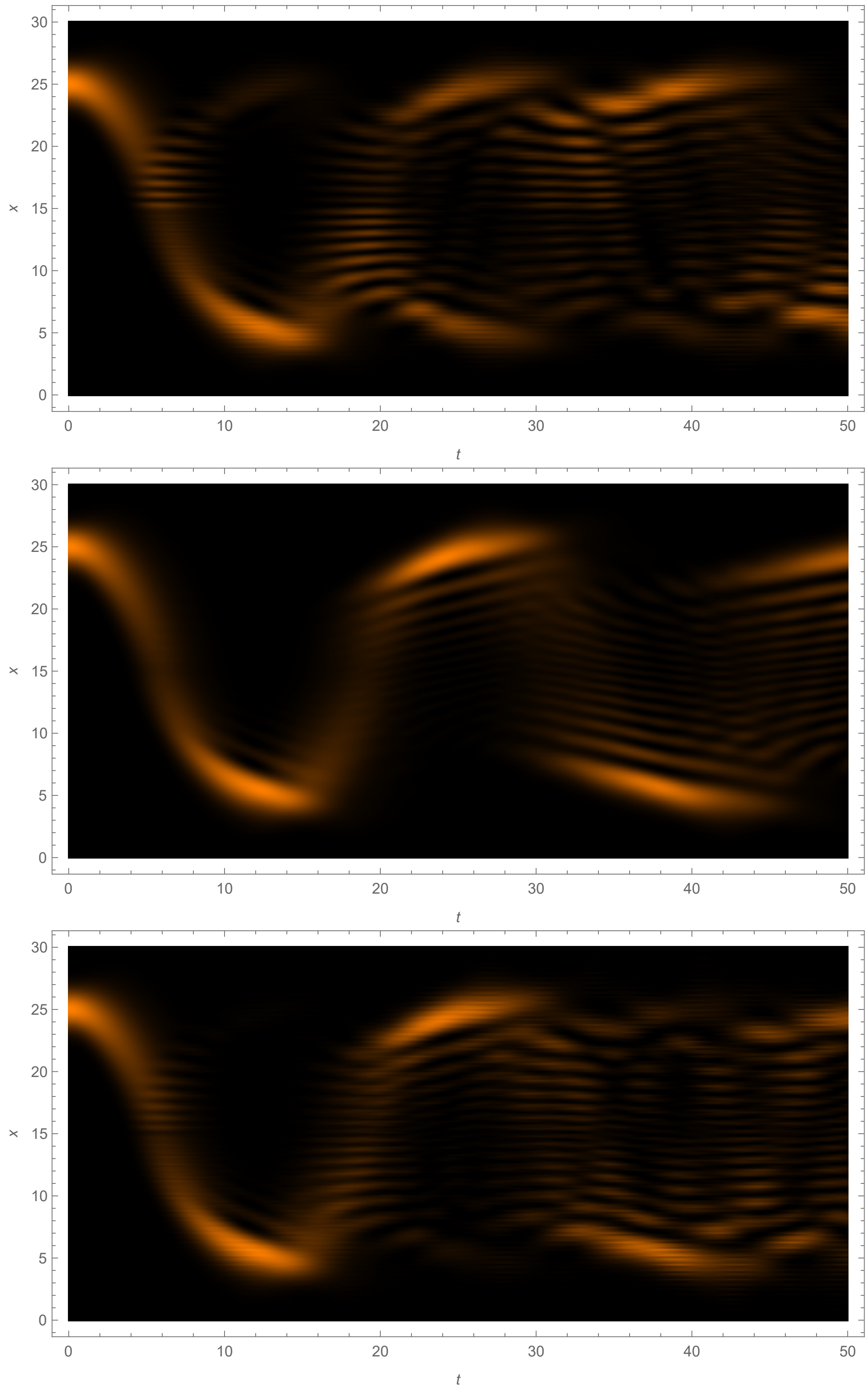

FIG. 5. Propagation of a wave packet of plus (upper), spinless (middle) and minus (lower) spin state. $P = 2$. (the spatial spreading of wavepacket appears to be too large, so a considerable part gets transmitted even for the plus and spinless cases)