# Classical Free Fields in a 1+1D Background Spacetime Flow

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## Abstract

The equaton of motion for some classical free fields in a 1+1D spacetime-dependent background flow are presented. Special attentions are paid to their non-relativistic limit.


**CONTENTS**



## I. THE BACKGROUND

We consider a metric of the form

$$ds^2 = -dt^2 + [dx - v(x,t)dt]^2 = -(1-v^2)dt^2 + dx^2 - 2vdtdx \tag{1}$$

which is actually a vacuum solution of the Einstein equation, though in 1+1D the Schwartzchild metric corresponds to $v(r) = -\sqrt{\frac{r_s}{r}}$ and the dS spacetime has $v(r) = Hr$. Under coordinate transformation $d\tau = dt + \frac{v}{1-v^2}dx$, the lime element turns into

$$ds^2 = -(1-v^2)d\tau^2 + \frac{1}{1-v^2}dx^2 \tag{2}$$

which can however be mapped conformally to an arbitrary diagonal 1+1D metric

$$ds^2 = -F(x,\tau)d\tau^2 + G(x,\tau)dx^2 \mapsto -\sqrt{\frac{F}{G}}dt^2 + \sqrt{\frac{G}{F}}dx^2 \tag{3}$$

provided that both $F$ and $G$ are positive-definite, with the correspondence

$$v = \sqrt{1 - \sqrt{\frac{F}{G}}} \tag{4}$$

if $F/G < 1$. As an example, the radial FLRW spacetime is

$$ds^2 = -d\tau^2 + \frac{a^2(\tau)}{1 - kx^2} dx^2 \tag{5}$$

then $v(\tau, x) = \sqrt{1 - \sqrt{\frac{1-kx^2}{a^2}}}$. It's possible to find the explicit form of $\tau(x, t)$. Hence, the background (1) is of general interest.

Note that a horizon is realized at $v(x) = 1$, the case for $v(t, x)$ is perhaps different, but the behaviour of solution near horizon is not our focus here.

We compute the geometrical quantities for this spacetime, the Ricci tensor is

$$R_{ab} = g_{ab} \left( v_{,x}^2 + v v_{,,x} + v_{,t,x} \right) \tag{6}$$

The Ricci scalar is

$$R = 2 \left( v_{,x}^2 + v v_{,,x} + v_{,t,x} \right) \tag{7}$$

So $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 0$. Note that by taking the trace of vacuum Einstein equation, $R - g^{ab} g_{ab} R/2 = 0$, we get $R = 0$ only if the spacetime dimension $n \neq 2$, since otherwise $g^{ab} g_{ab} = 2$, which is the present case. This is as expected since $g_{ab} = \mathrm{diag}(-f(t, x), 1/f(t, x))$ is a vacuum solution of the Einstein equation.

## II. SCALAR WAVE EQUATION

### A. Relativistic Scalar

We consider at first a real scalar field, the action is given by

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \tag{8}$$

For metric (1), $g_{ab} = \left( \begin{smallmatrix} v^2-1 & -v \\ -v & 1 \end{smallmatrix} \right)$, $g = -1$, $g^{ab} = \left( \begin{smallmatrix} -1 & -v \\ -v & 1-v^2 \end{smallmatrix} \right)$, the effective lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \left[ (v^2 - 1)(\phi_{,x})^2 + 2v \dot{\phi} \phi_{,x} + \dot{\phi}^2 \right] - V \\ &= \frac{1}{2} \left[ ((\partial_t + v \partial_x) \phi)^2 - (\partial_x \phi)^2 \right] - V \end{aligned} \tag{9}$$

The Klein-Gordon equation reads

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}v\right)\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)\phi-\frac{\partial^2\phi}{\partial x^2}=-V_{,\phi} \tag{10}$$

The Hamiltonian is

$$\mathcal{H}=\frac{1}{2}\left[(\partial_t\phi)^2+\left(1-v^2\right)(\partial_x\phi)^2\right]+V \tag{11}$$

For a complex scalar field, the lagrangian is

$$\mathcal{L}=\left[|(\partial_t+v\partial_x)\phi|^2-|\partial_x\phi|^2\right]-V \tag{12}$$

the conjugate momentum $\pi=\frac{\partial\mathcal{L}}{\partial_t\phi^*}=(\partial_t+v\partial_x)\phi$, we can define a conserved U(1) KG norm

$$(\phi_1,\phi_2)\equiv i\int_{-\infty}^{\infty}dx\left[\phi_1^*\left(\partial_t+v\partial_x\right)\phi_2-\phi_2\left(\partial_t+v\partial_x\right)\phi_1^*\right] \tag{13}$$

### B. Nonrelativistic Scalar

For a real field, making the usual ansatz $\phi=\frac{1}{\sqrt{2m}}(e^{-imt}\psi+\text{c.c.})$, in the limit $\dot\psi\ll m\psi$, and assuming the potential is $V=\frac{1}{2}m^2\phi^2+m\phi^2U$, the lagrangian becomes

$$\mathcal{L}=\frac{1}{2}\left[i\dot\psi\psi^*-\frac{1-v^2}{m}|\psi_{,x}|^2-iv\psi^*_{,x}\psi+\frac{v}{m}\psi^*_{,x}\dot\psi+\text{c.c.}\right]-|\psi^2|U \tag{14}$$

The Schrodinger equation reads

$$i(\partial_t+v\partial_x)\psi=\frac{1}{2}\partial_x\left(-\frac{1-v^2}{m}\partial_x\psi-iv\psi+\frac{v}{m}\dot\psi\right)+\frac{1}{2}\partial_t\left(\frac{v}{m}\partial_x\psi\right)+U\psi \tag{15}$$

or

$$i[(1-\frac{v_{,x}}{2m})\partial_t+(\frac{3}{2}v-\frac{v_{,t}}{2m})\partial_x]\psi=-\frac{1-v^2}{2m}\partial_x^2\psi+\frac{v}{m}\partial_t\partial_x\psi+(U-\frac{iv_{,x}}{2})\psi \tag{16}$$

or with $\dot\psi\ll m\psi$,

$$i[\partial_t+(\frac{3}{2}v-\frac{v_{,t}}{2m})\partial_x]\psi\approx-\frac{1-v^2}{2m}\partial_x^2\psi+(U-\frac{iv_{,x}}{2})\psi \tag{17}$$

The exact U(1) current $j^a=\frac{\partial\mathcal{L}}{\partial(\partial_a\psi)}i\psi-\frac{\partial\mathcal{L}}{\partial(\partial_a\psi^*)}i\psi^*$ is

$$\rho=-j^0=|\psi|^2,\quad j=-j^x=v|\psi|^2 \tag{18}$$

which satisfies $-\partial_aj^a=\dot\rho+\partial_xj=0$.

### C. Slow Flow

We consider a low-speed approximation, $v \ll 1$, then

$$i[\partial_t + (\frac{3}{2}v - \frac{v_{,t}}{2m})\partial_x]\psi = -\frac{1}{2m}\partial_x^2\psi + (U - \frac{iv_{,x}}{2})\psi \tag{19}$$

which is of the form

$$i(\partial_t - u(t,x)\partial_x)\psi = -\frac{1}{2}\partial_x^2\psi + \mathcal{U}\psi \tag{20}$$

with $u \equiv \frac{3}{2}v - \frac{v_{,t}}{2m}$ and $\mathcal{U} \equiv U - \frac{iv_{,x}}{2}$, where for convenience, we have set $m = 1$. However, we note this approximated equation of motion cannot be derived from a lagrangian if the potential $\mathcal{U}$ is complex while the action is required to be real.

If $u = u(x)$, introducing $\psi = y \exp\left(-i\int^x u(x')dx'\right)$, the wave equation tunrs into

$$i\partial_t y = -\frac{1}{2}\partial_x^2 y + \left(-\frac{u^2}{2} + \mathcal{U} + \frac{iu_{,x}}{2}\right) y \tag{21}$$

hence the dissipation comes from both the x-dependence of $v$ and $u$.

If $v = v(t)$, in this case $u = u(t)$, $\mathcal{U} = U$, there is no dissipation. The drift term in the wave equation can be removed by a local Galilean transformation (i.e., in the comoving frame) $dx' = dx + udt, t' = t$, with $\partial_{t'} = -u\partial_x + \partial_t$, the potential however generally becomes time-dependent uncless it's constant.

The velocity-drift can have interesting consequences in the background frame, for which we consider a simple example here, an oscillating flow $u = \alpha + \beta \cos \mu t$, where $\alpha, \beta, \mu$ are constants (note it's required that $\alpha, \beta \ll 1$), with a harmonic oscillator potential $U = \frac{1}{2}\omega^2 x^2$. We can choose $x' = x + \alpha t + \frac{\beta}{\mu}\sin \mu t$, then in the co-moving frame, $i\partial_{t'}\psi = -\frac{1}{2}\partial_{x'}^2\psi + U\psi$, with

$$U(x', t') = \frac{1}{2}\omega^2(x' - \alpha t' - \frac{\beta}{\mu}\sin \mu t')^2 \tag{22}$$

Now at $t' = 0$, this is a harmonic potential, the potential minimum is at $x' = \alpha t' + \frac{\beta}{\mu}\sin \mu t'$. If the time-variation of the potential is slow enough, the state at $t' = 0$ may evolve adiabatically. For $\alpha = 0$ the perturabtion is periodic but a Berry phase of course does not exit since there is only one parameter.

For simplicity I choose $\alpha = 0$, and an inital eigenstate

$$\psi_n(x) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{n!2^n}} H_n(x) e^{-i\omega(1/2+n)t} \tag{23}$$

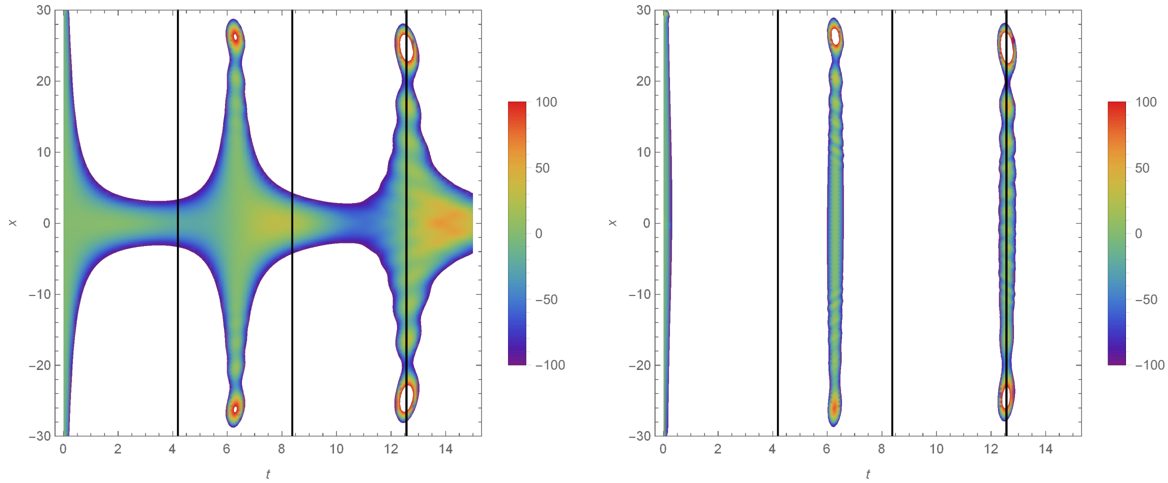


FIG. 1. $\omega = 3$, $\mu = 100$. Left: $\beta = 0$, right: $\beta = 30$. This plot only shows value of $\mathrm{Re}(\psi)$ in the range $(-100, 100)$. Interval between the black lines is $T = 2\pi/(\omega/2)$, which should be the oscillation period for $\beta = 0$, the numerical error, however, is clearly too large (I used periodic boundary condition with a finite box length). After $\beta$ is switched on, the wavefront is more flat, but with original oscillation frequency (whatever it is).

where $H_n(x)$ is the Hermite polynomial, is evolved. For simplicity I choose $n = 0$, the ground state:

$$\psi_0(x) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2}} e^{-i\omega t/2} \tag{24}$$

The figures above are computed in mma. More precise simulation is needed.

## III. DIRAC EQUATION

For a diagonal metric, the concrete form of Dirac equation is easily obtained, see, e.g., [ref]. The key is to find a proper set of dyad $e_\mu^{(a)}$ satisfying

$$g_{\mu\nu} = e_\mu^{(a)} e_\nu^{(b)} \eta_{ab}, \quad \gamma^\mu = e^\mu_{(a)} \gamma^a, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{25}$$

The first one is the defining condition. For 1+1D Minkovski spacetime, we can choose the $\gamma^a$ to be $\gamma^0 = -i\sigma_z = -i\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $\gamma^1 = \sigma_y = -i\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Let $e_\mu^{(0)} = (a, b), e_\mu^{(1)} = (c, d)$, then

$$\begin{aligned} g_{00} &= -e_0^{(0)} e_0^{(0)} + e_0^{(1)} e_0^{(1)} = b^2 - a^2 = v^2 - 1 \\ g_{11} &= -e_1^{(0)} e_1^{(0)} + e_1^{(1)} e_1^{(1)} = d^2 - c^2 = 1 \\ g_{01} &= -e_0^{(0)} e_1^{(0)} + e_0^{(1)} e_1^{(1)} = cd - ab = -v \end{aligned} \tag{26}$$

We choose $d = 1$, so $c = 0, a^2 = 1, b^2 = v^2$, we choose $a = 1, b = v$. Now $e^{(0)}_\mu = (1, v), e^{(1)}_\mu = (0, 1)$. Note $e^\mu_{(a)} \neq g^{\mu\nu} e^{(a)}_\nu$, but its inverse: $e^{(a)}_\mu e^\nu_{(a)} = \delta^\nu_\mu$. Hence $e^\mu_{(0)} = (1, v), e^\mu_{(1)} = (0, 1)$.

The Dirac equation is

$$\left(-\gamma^a e^\mu_{(a)} \partial_\mu - \frac{1}{2}\gamma^a \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} e^\mu_{(a)}\right) - m\right)\psi = 0 \tag{27}$$

Now we have

$$\gamma^a \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} e^\mu_{(a)}\right) = \gamma^0 \partial_\mu \left(e^\mu_{(0)}\right) + \gamma^1 \partial_\mu \left(e^\mu_{(1)}\right) = \gamma^0 v_{,x} \tag{28}$$

$$\gamma^a e^\mu_{(a)} \partial_\mu = \gamma^0 e^\mu_{(0)} \partial_\mu + \gamma^1 e^\mu_{(1)} \partial_\mu = \gamma^0(\partial_t + v\partial_x) + \gamma^1 \partial_x \tag{29}$$

the Dirac equation then reads

$$\left(\gamma^0(\partial_t + v\partial_x) + \gamma^1 \partial_x + \frac{1}{2}\gamma^0 v_{,x} + m\right)\psi = 0 \tag{30}$$

or

$$\begin{pmatrix} im + \frac{1}{2}v_{,x} + \partial_t + v\partial_x & \partial_x \\ -\partial_x & im - (\frac{1}{2}v_{,x} + \partial_t + v\partial_x) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \tag{31}$$

or

$$i\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -i\partial_x \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} - i\left(\frac{1}{2}v_{,x} + v\partial_x\right) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + m \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix} \tag{32}$$

The Dirac representation of gamma matrices we have adopted is the standard choice to perform the NR limit. Making the ansatz $\psi = \Psi e^{-imt}$, the Dirac equation is now

$$i\partial_t \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = -i\partial_x \begin{pmatrix} \Psi_2 \\ \Psi_1 \end{pmatrix} - i\left(\frac{1}{2}v_{,x} + v\partial_x\right) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} - 2m \begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix} \tag{33}$$

For $v = 0$, the usual Pauli equation originates from the approximation $-i\partial_x \Psi_1 - 2m\Psi_2 = 0$, with $\Psi_2$ being the "small" component. Now if we assume the $\partial_t \Psi_2 \ll m\Psi_2$, then

$$\left[2m + i\left(\frac{1}{2}v_{,x} + v\partial_x\right)\right]\Psi_2 = -i\partial_x \Psi_1 \tag{34}$$

But it's not strightforward to obtain a single decoupled equation for the "large" component $\Psi_1$, unless $v = \text{const}$, for which case (but there is no need since we could adopt a new time coordinate in which $v = 0$)

$$i(\partial_t + v\partial_x)\Psi_1 = -\frac{\partial_x^2 \Psi_1}{2m + iv\partial_x} \tag{35}$$

## IV. PROCA EQUATION

The Proca Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F_{ab})^2 - \frac{1}{2}m^2 A_a^2 = -\frac{1}{4}(\partial_a A_b - \partial_b A_a)\left[g^{ac}\partial_c\left(g^{bd}A_d\right) - g^{bd}\partial_d\left(g^{ac}A_c\right)\right] - \frac{1}{2}m^2 A_a^2 \quad (36)$$

The Proca equation reads

$$\nabla^d \nabla_d A^a - m^2 A^a - g^{ca} R_{dc} A^d = 0 \quad (37)$$

Explicitly, we find

$$\begin{pmatrix} \partial_x^2 + m^2 - \frac{1}{2}R & m^2 v - \partial_t \partial_x \\ m^2 v - \partial_t \partial_x & \partial_t^2 - m^2(v^2-1) + \frac{1}{2}R \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \end{pmatrix} = 0 \quad (38)$$

But the Lorenz condition follows:

$$\nabla^a A_a = A^1_{,1} + A^0_{,0} = 0 \quad (39)$$

Then

$$\begin{pmatrix} \partial_x^2 + \partial_t^2 + m^2 - \frac{1}{2}R & m^2 v \\ m^2 v & \partial_t^2 + \partial_x^2 - m^2(v^2-1) + \frac{1}{2}R \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \end{pmatrix} = 0 \quad (40)$$

But then $A^0$ and $A^1$ are not decoupled (in contrast to the case when $v = 0$), so massive EM waves do exist!

## V. OUTLOOKS

Explicit solutions may be found for some interesting choices of $v(x,t)$. The metric perturbation of this background can also be studied.