Orbit Dynamics of a Rigid Dumbbell around a Central Object with Oscillating Mass

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Abstract

Here, I investigate the (non-relativistic) motion of a dumbbell made of two rigidly connected mass-points in the newtonian potential of a central obejct, for simplivity the dumbbell is assumed to be coplanar with the orbit plane. Moreover I shall consider the effects of mass oscillation which could result from the coupling of mass to a spatially homogeneous axion background.

CONTENTS

I. THE PROBLEM

This is a restricted three-body problem. The system is depicted in the figure below. The mass of the central object at $\mathbf{Q}$ is M, the dumbbell is made of two rigidly connected mass-points at $\mathbf{r}_{1,2}$ with mass $m_{1,2}$, we write $m = m_1 + m_2$. The line connceting $\mathbf{Q}$ and mass center of the dumbbell $\mathbf{P}$ defines the x axis, relative to which the rotation angle θ of the dumbbell is measured. The mass center $\mathbf{O}$ of the whole system is on this x axis and is choosed to be the coordinate center, and the coordinate of $\mathbf{P}$ is denoted by r. Rotation angle of the x axis in the background frame is denoted by ϕ, and the angular velocity $\omega = \dot{\phi}$.

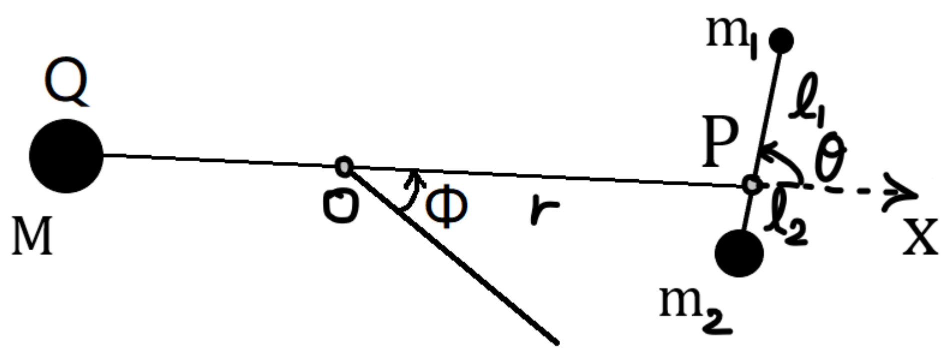

FIG. 1. Configuration of the system, the dumbbell is rigid, i.e., $l_1, l_2 = \text{const}$.

In the background frame where $\mathbf{O}$ is static, the force experienced by m_i (i=1,2) is

$$\mathbf{F}_i = Mm_i \frac{\mathbf{Q} - \mathbf{r}_i}{|\mathbf{R} - \mathbf{r}_i|^3} \tag{1}$$

where we choose $G = 1$. For the analysis we'll use a lagrangian approach, but let's do a Newtonian analysis first assuming the mass is static. In the co-rotating frame of the x axis, all objects experience an inertial acceleration $\mathbf{a} = -\frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}) = -\dot{\boldsymbol{\omega}} \times \mathbf{r} + \omega^2 \mathbf{r}$. Newton's 2nd law reads in this co-rotating frame,

$$M\ddot{\mathbf{Q}} = -\sum_{i=1,2} \mathbf{F}_i + M(-\dot{\boldsymbol{\omega}} \times \mathbf{Q} + \omega^2 \mathbf{Q}) \tag{2}$$

(as should be checked the RHS is along the x direction) Now we consider a general time-dependence of the mass $m(t)$, with Chasles' theorem in mind, the system's lagraingian is

$$L = \frac{1}{2}M(\frac{d}{dt}(\frac{m}{M}r))^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}M(\frac{m}{M}r\dot{\phi})^2 + \frac{1}{2}(m)(r\dot{\phi})^2 + \frac{1}{2}I(\dot{\theta}+\dot{\phi})^2 + \frac{Mm_1}{|\mathbf{Q}-\mathbf{r}_1|} + \frac{Mm_2}{|\mathbf{Q}-\mathbf{r}_2|} \tag{3}$$

or using the distance between mass centers $R = |\mathbf{Q} - \mathbf{P}| = \frac{M+m}{M} r$,

$$\begin{aligned} L &= \frac{1}{2}M(\frac{d}{dt}(\frac{\mu}{M}R))^2 + \frac{1}{2}m(\frac{d}{dt}(\frac{\mu}{m}R))^2 + \frac{1}{2}\mu R^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\theta}+\dot{\phi})^2 + \frac{Mm_1}{|\mathbf{Q}-\mathbf{r}_1|} + \frac{Mm_2}{|\mathbf{Q}-\mathbf{r}_2|} \\ &= \frac{\mu}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2\right) + \frac{1}{2}I(\dot{\theta}+\dot{\phi})^2 - V, \quad V = -\frac{Mm_1}{|\mathbf{Q}-\mathbf{r}_1|} - \frac{Mm_2}{|\mathbf{Q}-\mathbf{r}_2|} \end{aligned} \tag{4}$$

where $I = m_1 l_1^2 + m_2 l_2^2$ and $\mu = mM/(m+M)$ the reduced mass.

For numerics we shall set $l = l_1 + l_2 = 1$, so $l_1 = \frac{m_2}{m}, l_2 = \frac{m_1}{m}$, $I = \frac{m_1 m_2}{m}$. Using the Law of Cosines,

$$|\mathbf{Q} - \mathbf{r}_1| = \sqrt{R^2 + l_1^2 + 2Rl_1\cos\theta}, \quad |\mathbf{Q} - \mathbf{r}_2| = \sqrt{R^2 + l_2^2 - 2Rl_2\cos\theta} \tag{5}$$

The total angular momentum

$$J = \frac{\partial L}{\partial \dot{\phi}} = (\mu R^2 + I)\dot{\phi} + I\dot{\theta} \tag{6}$$

is conserved, which gives $\dot{\phi} = \frac{J - I\dot{\theta}}{\mu R^2 + I}$. The equation of motion of R and θ is

$$\frac{d}{dt}\left(\mu \dot{R}\right) = \mu R \dot{\phi}^2 - V_{,R} \tag{7}$$

$$\frac{d}{dt}\left[I(\dot{\theta} + \dot{\phi})\right] = -V_{,\theta} \tag{8}$$

or, more explicitly, with $-V = \frac{Mm_1}{(R^2+l_1^2+2Rl_1\cos\theta)^{1/2}} + \frac{Mm_2}{(R^2+l_2^2-2Rl_2\cos\theta)^{1/2}}$,

$$\dot{\mu} R + \mu \ddot{R} = \mu R \left(\frac{J - I\dot{\theta}}{\mu R^2 + I}\right)^2 - \frac{Mm_1(R + l_1\cos\theta)}{(R^2 + l_1^2 + 2Rl_1\cos\theta)^{3/2}} - \frac{Mm_2(R - l_2\cos\theta)}{(R^2 + l_2^2 - 2Rl_2\cos\theta)^{3/2}} \tag{9}$$

$$\dot{I}\left(\frac{J + \mu R^2\dot{\theta}}{\mu R^2 + I}\right) + I\frac{d}{dt}\left(\frac{J + \mu R^2\dot{\theta}}{\mu R^2 + I}\right) = \frac{Mm_1 l_1 \sin\theta}{(R^2 + l_1^2 + 2Rl_1\cos\theta)^{3/2}} - \frac{Mm_2 l_2 \sin\theta}{(R^2 + l_2^2 - 2Rl_2\cos\theta)^{3/2}} \tag{10}$$

The initial condition is specified by $(R, \theta, \dot{R}, \dot{\theta})$ and J. The only free parameters are m_1, m_2, M.

For definiteness, we consider an ocsillating mass: [1]

$$m(t) = m_0(1 + \beta \cos \mu t) \tag{11}$$

where β, μ are constants, the time-dependence is assumed to be universal for all bodies, and we set $M_0 = 1$.

II. SIMPLE CASES

A. Monopole

To begin with, we cosnider the simplest case where $l_1 = l_2 = 0$, i.e., a two-body Kepler problem. The lagrangian is

$$L = \frac{\mu}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2\right) + \frac{Mm}{R} \tag{12}$$

Angular momentum $J = \mu R^2 \dot{\phi}$ is conserved. The equation of motion is

$$\mu \ddot{R} + \dot{\mu}\dot{R} = \frac{J^2}{\mu R^3} - \frac{Mm}{R^2} \tag{13}$$

Since $\partial_t = \frac{J}{\mu} u^2 \partial_\phi$,

$$\dot{R} = -\frac{J}{\mu} u_{,\phi} \tag{14}$$

$$\ddot{R} = \frac{J}{\mu} u^2 \partial_\phi \left(\frac{J}{\mu} u^2 \partial_\phi \frac{1}{u}\right) = -\frac{J^2}{\mu^2} u^2 u_{,\phi\phi} + \frac{J\dot{\mu}}{\mu} u_{,\phi} \tag{15}$$

The Binet equation is however unchanged

$$u_{,\phi\phi} = -u + \frac{Mm\mu}{J^2} \tag{16}$$

(note special relativity will modify the factor of u while GR in the leading order will add a cubic term of u) but the rightmost term is no longer a constant but representing a periodic driving force if the mass oscillates. In particular, the driven frequency will be three times of the frequency of mass oscillation.

B. Dipole

Then we cosnider a finite but small seperation of the two mass-points, $l_1, l_2 \ll r$, i.e., treating the dumbbell as an ideal mass dipole. The lagrangian is

$$L = \frac{\mu}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2\right) + \frac{1}{2} I (\dot{\theta} + \dot{\phi})^2 + \frac{Mm}{R} \tag{17}$$

In this case, besides $J = (\mu R^2 + I)\dot{\phi} + I\dot{\theta}$,

$$J_\theta = I(\dot{\theta} + \dot{\phi}) \tag{18}$$

is conserved as well, hence

$$\dot{\phi} = \frac{J - J_\theta}{\mu R^2}, \quad \dot{\theta} = \frac{J_\theta}{I} - \frac{J - J_\theta}{\mu R^2} \tag{19}$$

The radial motion is

$$\dot{\mu}\dot{R} + \mu\ddot{R} = \frac{(J - J_\theta)^2}{\mu R^3} - \frac{Mm}{R^2} \tag{20}$$

Of couse, the R-motion and θ-motion are still decoupled.

C. Extreme-Mass-Ratio Dumbbell

There is another interesting situation, similar to that analysed in [2] (however, there m_1 and m_2 are not rigidly connected, so our problem is much simpler). The potential energy

$$V = -\frac{Mm_1}{|\mathbf{Q} - \mathbf{r}_1|} - \frac{Mm_2}{|\mathbf{Q} - \mathbf{r}_2|} \tag{21}$$

can be expanded, as

$$
\begin{aligned}
V(\mathbf{Q}-\mathbf{P},\mathbf{r}_2-\mathbf{P}) &= -\frac{Mm_1}{|\mathbf{Q}-\mathbf{P}+\frac{m_2}{m_1}(\mathbf{r}_2-\mathbf{P})|}-\frac{Mm_2}{|\mathbf{Q}-\mathbf{P}-(\mathbf{r}_2-\mathbf{P})|} \\
&= -M\sum_{\ell,m}\frac{4\pi}{2\ell+1}\left[\frac{m_1(-m_2/m_1)^l+\mu}{R}\right]\left(\frac{\bar{r}}{R}\right)^{\ell} Y^*_{\ell m}(\Theta,\Phi)Y_{\ell m}(\bar{\theta},\bar{\phi})
\end{aligned} \tag{22}
$$

where $\bar{r}=|\mathbf{r}_2-\mathbf{P}|$, (R,Θ,Φ) and $(\bar{r},\bar{\theta},\bar{\phi})$ are the spherical coordinates of $\mathbf{Q}-\mathbf{P}$ and $\mathbf{r}_2-\mathbf{P}$ (relative to $\mathbf{P}$), respectively. Note this expansion is exact. Now in the $m_2/m_1 \ll 1$ limit, the potential turns out to be

$$
V(R,l,\Theta,\Phi,\bar{\theta},\bar{\phi}) = -\frac{Mm}{R}-\frac{Mm_2}{R}\sum_{\ell\geq 2,|m|\leq\ell}\frac{4\pi}{2\ell+1}\left(\frac{l}{R}\right)^{\ell} Y^*_{\ell m}(\Theta,\Phi)Y_{\ell m}(\bar{\theta},\bar{\phi}) \tag{23}
$$

There is no contribution from $\ell=1$ (dipole). Here, we have $\Theta=\bar{\theta}=\pi/2$, $\Phi=-\phi$ and $\bar{\phi}=\theta+\phi-\pi$.

III. NUMERICAL RESULTS

For simplicity, we study only the ideal dipole case. The equation of motion is

$$
\ddot{R}-\frac{\beta\mu\sin\mu t}{1+\beta\cos\mu t}\dot{R}=\frac{(1+m_0)^2(J-J_\theta)^2}{m_0^2(1+\beta\cos\mu t)^2R^3}-\frac{(1+m_0)(1+\beta\cos\mu t)}{R^2} \tag{24}
$$

The rotation is given by

$$
\dot{\phi}=\frac{(1+m_0)(J-J_\theta)}{m_0(1+\beta\cos\mu t)R^2},\quad \dot{\theta}=\frac{J_\theta}{I_0(1+\beta\cos\mu t)}-\frac{(1+m_0)(J-J_\theta)}{m_0(1+\beta\cos\mu t)R^2} \tag{25}
$$

We focus on a particular problem, that is, how does the mass oscillation affects the escape velocity? If $\beta=J_\theta=0$, $\ddot{R}=-U'(R)$, with the effective potential

$$
U(R)=-\frac{1+m_0}{R}+\left(\frac{1+m_0}{m_0}\right)^2\frac{(J-J_\theta)^2}{2R^2} \tag{26}
$$

hence the circular velocity is $R\dot{\phi}=\sqrt{\frac{(1+m_0)}{R}}$, and the escape velocity is $R\dot{\phi}=\sqrt{\frac{2(1+m_0)}{R}}$. So we set the initial condition $\dot{R}=0, R\dot{\phi}=\sqrt{\frac{k(1+m_0)(1+\beta\cos\mu t)}{R}}$ at $t=0$, which means

$$
J-J_\theta=m_0\sqrt{\frac{kR(1+\beta\cos\mu t)^3}{1+m_0}} \tag{27}
$$

so $k=2$ gives the escape condition in the static mass case. Still we are free to choose J_θ and I_0, m_0, β, μ (note that $I_0\in[0,m_0/4]$). We simply would like to check the fate of our test body.

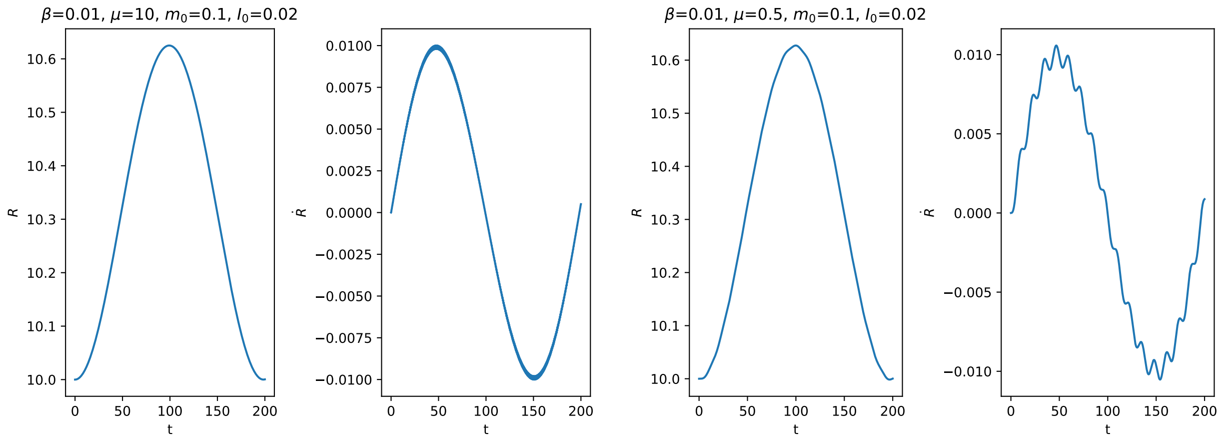

FIG. 2. $k = 1$. If $\beta = 0$, $R = \text{const}$, but now the radius oscillates, the period is comparable to the epicycle period $T_R = 189$. Also notice that this oscillation is not around R_0, but $R(t) \geq R_0$.

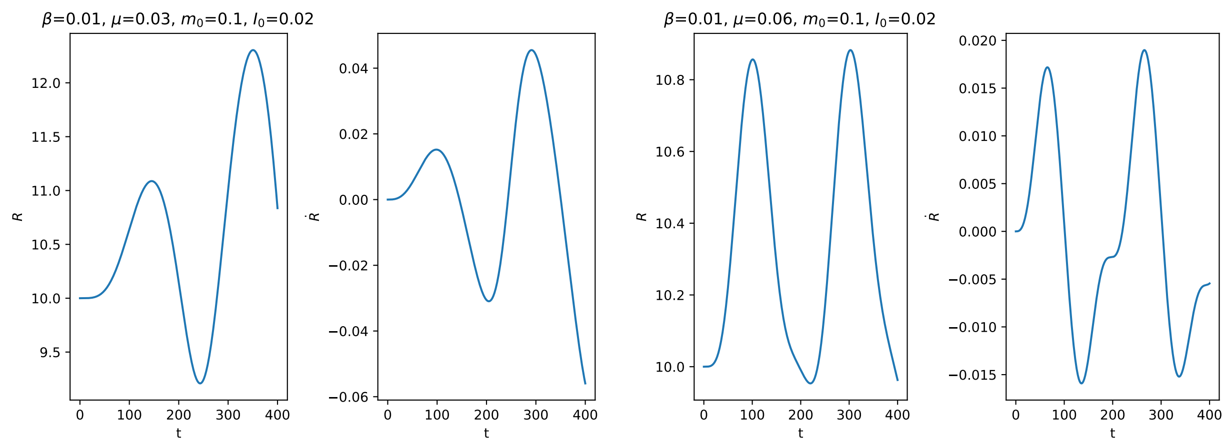

FIG. 3. $k = 1$. Left: $\mu = \Omega$, the orbit radius is being resonantly amplified. Right: $\mu = 2\Omega$, no resonance.

If $\beta = 0$, near the circular radius $R = R_0$, the effective potential is

$$U(R) = -\frac{1 + m_0}{R} + \frac{(1 + m_0)R_0}{2R^2} = \frac{1 + m_0}{2R_0^3}(R - R_0)^2 + \text{const} \tag{28}$$

which gives the epicycle frequecy $\Omega = \sqrt{(1 + m_0)/R_0^3}$ (but this also happends to be the orbit frequency) with period $T_R = 2\pi/\Omega$. If μ matches Ω, there appears to be orbit resonance.

As the numerical results show, the test body tends to be ionized. In general, however, it's challenging the determine the escape condition. To plot the trajectory, one should solve instead the Binet equation.

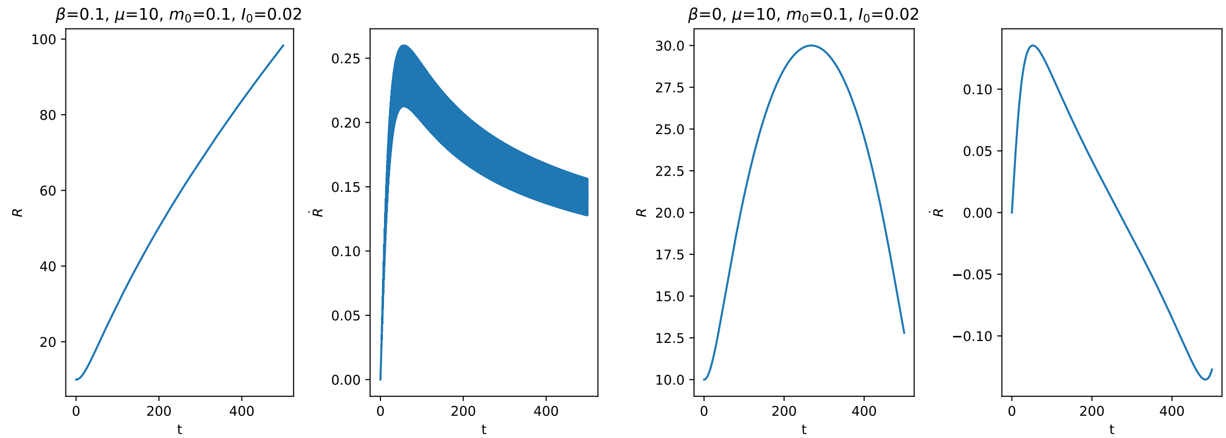

FIG. 4. $k = 1.5$. If $\beta = 0$, this test body cannot escape (right figure), but it does if the oscillation is quick enough (left figure).

IV. GRAVITATIONAL WAVE SIGNAL

The waveforms and luminosities of gravitational waves emitetd by Newtonian three-body system in periodic orbits was investigated in [3], here the situation is similar. The quadrupole moment is

$$I_{ab} = \sum_i m_i x_a^{(i)} x_b^{(i)} \tag{29}$$

where $\mathbf{x}^{(i)}$ is the position of i-th mass-point. The strain signal (in the LIGO frame) is

$$s(t) = h_{11} - h_{22} = \cos 2\phi \left(1 + \cos^2 i\right) \tilde{h}_+ - 2 \sin 2\phi \cos i \tilde{h}_\times \tag{30}$$

where (i, ϕ) are the spherical coordinates of the source, and $\tilde{h}$ is measured in a frame where GW travels in z direction. In quadrupole approximation,

$$h_+ = \frac{G}{d} \left[\ddot{I}_{11}(t-d) - \ddot{I}_{22}(t-d)\right], \quad h_\times = \frac{2G}{d} \ddot{I}_{12}(t-d) \tag{31}$$

where d is the source distance. So we only need to input $\mathbf{x}^{(i)}(t)$ on the orbit plane, in the present case,

$$\begin{aligned} \mathbf{Q} &= -\frac{m}{m+M} R(\cos\phi, \sin\phi) \\ \mathbf{r}_1 &= \frac{M}{m+M} R(\cos\phi, \sin\phi) + l_1[\cos(\phi+\theta), \sin(\phi+\theta)] \\ \mathbf{r}_2 &= \frac{M}{m+M} R(\cos\phi, \sin\phi) - l_2[\cos(\phi+\theta), \sin(\phi+\theta)] \end{aligned} \tag{32}$$

In the ideal dipole case,

$$\mathbf{Q} = -\frac{m}{M}\mathbf{P} = \frac{m}{m+M}\mathbf{Q} - \mathbf{P}, \quad I_{ij} = \mu(Q_i - P_i)(Q_j - P_j) \tag{33}$$

The actual computation is left for futural work.

V. CONCLUSION AND OUTLOOKS

Here I assumed the spin of the dumbbell is normal to the orbit plane, if it's not so, the general sitation would likely be rather messy, but the motion may also be interesting.

The dumbbell can serve as a basic model for restricted three-body system such as the earth-moon-sun, as treated nicely in Greiner's Classical Mechanics II (Springer, 2010), p.230, note also contained in this book (p.544) is an interesting discussion of the chaotic rotation of Hyperion (a moon of Saturn) formulated as a 2D problem, with the Hyperion modeled as a dumbbell.

It will also be interesting to consider an electric counterpart of this problem, with $m_{1,2}$ replaced by $q_{1,2}$ and $q_1 q_2 < 0$, in particular for $q_1 = -q_2$, the monopole term of potential vanishes and the leading order contribution comes from dipole, also the effect of mass oscillation may be different (since the mass then appears only in the kinetic energy). This is left for futural study.

Appendix A: Mechanics with Time-Varying Mass

We consider the Newtonian mechanics of many particles with mass $m_i(t) = g(t)m_i(0)$ [4]. Many of the results do not change as compared to $g = 1$, but

$$\mathbf{F} = \frac{dm\mathbf{v}}{dt} = m\mathbf{a} + \dot{m}\mathbf{v} \tag{A1}$$

or $\mathbf{F} - \dot{m}\mathbf{v} = m\mathbf{a}$, introducing an effective extra force $-\dot{m}\mathbf{v}$. However, this desciption may not be the most convenient one, since if the system is conservative when $g = 1$, the lagrangian for general $g(t)$ still takes the same form, just with $m = m(t)$. In particular, the conserved quantities associated with cyclic coordiantes will still be conserved. The Euler-Lagrange equation still holds, even if the mass varies with time (more generally it can even be space-time dependent. A time-dependent mass also emerge naturally in Cosmology). However, the

Hamiltonian (or in a field theory, the energy-momentum) is generally no longer conserved. As a first example we consider a “free” particle, the lagrangian reads

$$L = \frac{1}{2}mv^2, \quad v = \dot{x} \tag{A2}$$

The equation of motion is $d(mv)/dt = \dot{m}v + m\dot{v} = 0$, mv is constant, hence for an oscillating mass the velocity oscillates as well so long as it does vanish (also we see that “energy” is not conserved). Next we consider a harmonic oscillator:

$$L = \frac{1}{2}mv^2 - \frac{1}{2}kx^2 \tag{A3}$$

The equation of motion is $\ddot{x} + \frac{\dot{m}}{m}\dot{x} + \frac{k}{m}x = 0$. Oscillation of $\omega^2 \equiv k/m$ alone can lead to parametric resonance, but a friction term presents as well (also one can study the quantum version). For a relativistic oscillator, the lagrangian is (with $c = 1$)

$$L = -m\sqrt{1-v^2} - \frac{1}{2}kx^2 \tag{A4}$$

The equation of motion is $\frac{d}{dt}\left(\frac{mv}{\sqrt{1-v^2}}\right) + \frac{k}{m}x = 0$. While in a spatially flat FLRW background (using conformal time), it is

$$L = a(\eta)\left(-m\sqrt{1-v^2} - \frac{1}{2}kx^2\right), \quad v = x' \tag{A5}$$

and the action $S = \int d\eta\, L$.

It will also be interesting to study the unrestricted three-body problem:

$$L = \sum_{i=1,2,3} \frac{1}{2}m_i v_i^2 + \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} + \frac{Gm_1m_3}{|\mathbf{r}_1 - \mathbf{r}_3|} + \frac{Gm_2m_3}{|\mathbf{r}_2 - \mathbf{r}_3|} \tag{A6}$$

and the rigid body mechanics (since the Hamiltonian is no longer conserved, things are generally more chaotic). I’ll treat the problem of a falling pencil on a horizontal fricative table and a sliding ladder elsewhere. Here as an example, I consider the rotation of a “free” top relative to its mass center (at the coordinate center), for which we can use an effective Newtonian equation $m\mathbf{a} = -\dot{m}\mathbf{v}$. The torque is

$$\mathbf{D} = -\frac{\dot{m}}{m}\int d^3r'\, \rho(\mathbf{r}')\mathbf{r}' \times \mathbf{v}(\mathbf{r}') = -\frac{\dot{m}}{m}\mathbf{L} = m\frac{d}{dt}\left(\frac{\mathbf{L}}{m}\right) \tag{A7}$$

Hence the Euler equation reads

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = -\frac{\dot{m}}{m}\begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} = \begin{pmatrix} I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\omega_1 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 \end{pmatrix} \tag{A8}$$

Whether this new dynamics admitts a geometrical interpretation is an interesting question. Note here the principal moments of inertial I_i is calculated using $m(t)$, so $\dot{I}_i = \frac{\dot{m}}{m} I_i$ since the time dependence of mass is assumed to be same for all bodies. So it also reads

$$\begin{pmatrix} \frac{d}{dt}(I_1\omega_1) + (I_3 - I_2)\,\omega_2\omega_3 \\ \frac{d}{dt}(I_2\omega_2) + (I_1 - I_3)\,\omega_3\omega_1 \\ \frac{d}{dt}(I_3\omega_3) + (I_2 - I_1)\,\omega_1\omega_2 \end{pmatrix} = 0, \quad \mathbf{L} = \sum_i I_i \boldsymbol{\omega}_i = \mathbf{const} \tag{A9}$$

Note the rotation kinetic energy $\frac{1}{2}\mathbf{L}\cdot\boldsymbol{\omega}$ is not conserved. Then, the angular velocity of a spherically symmetric body has fixed direction but oscillating magnitude. For $I_1 = I_2 = I \neq I_3$ (so z is the figure axis), we have

$$\begin{pmatrix} (I_3\omega_3)^{-1}\frac{d}{dt}(I_1\omega_1) + \left(\frac{1}{I_2} - \frac{1}{I_3}\right) I_2\omega_2 \\ (I_3\omega_3)^{-1}\frac{d}{dt}(I_2\omega_2) + \left(\frac{1}{I_3} - \frac{1}{I_1}\right) I_1\omega_1 \end{pmatrix} = 0 \tag{A10}$$

The decoupled equatio is

$$(I_3\omega_3)^{-2}\frac{d}{dt}\left[\frac{\frac{d}{dt}(I\omega_1)}{\frac{1}{I_3} - \frac{1}{I}}\right] + \left(\frac{1}{I_3} - \frac{1}{I}\right) I\omega_1 = 0 \tag{A11}$$

more explicitly

$$\frac{d^2}{dt^2}(I\omega_1) + \frac{\dot{m}}{m}\frac{d}{dt}(I\omega_1) + \left[(I_3\omega_3)^2\left(\frac{1}{I_3} - \frac{1}{I}\right)^2\right] I\omega_1 = 0 \tag{A12}$$

The situation is similar to the harmonic oscillator A3.

Next we discuss breifly the nonrelativistic quantum mechanics with a time-varying mass $\mathcal{M}$. We begin with a free relativistic real scalar field in the flat FLRW spacetime

$$\mathcal{L} = a^2\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - mV\phi^2\right] = \frac{a^2}{2}\left[(\phi')^2 - (\nabla\phi)^2 - (m^2 + 2mV)\phi^2\right] \tag{A13}$$

with $S = \int d\eta d^3x\,\mathcal{L}$, where $m = m(\eta) = \mathcal{M}a$. Making the usual ansatz $\phi = \frac{1}{\sqrt{2m}}(\psi e^{-im\eta} +$ c.c), in the non-relativistic limit $\psi' \ll m\psi$,

$$\phi^2 = \frac{|\psi|^2}{m}, \quad (\nabla\phi)^2 = \frac{1}{m}\nabla\psi\cdot\nabla\psi^* \tag{A14}$$

but

$$\phi' = \left[\psi' - \left(\frac{m'}{2m} + i(m'\eta + m)\right)\psi\right]\frac{e^{-im\eta}}{\sqrt{2m}} + \text{c.c} \tag{A15}$$

and

$$(\phi')^2 = \left[\left(\frac{m'}{2m}\right)^2 + (m'\eta+m)^2\right]\frac{|\psi|^2}{m} - \frac{1}{m}\left[\left(\frac{m'}{2m} - i(m'\eta+m)\right)\psi'\psi^* + \left(\frac{m'}{2m} + i(m'\eta+m)\right)(\psi')^*\psi\right] \tag{A16}$$

If $m' = 0$, we have $\mathcal{L} = a^2\left[i\psi^*\psi' + \text{c.c.} - \frac{1}{2m}\nabla\psi\cdot\nabla\psi^* - V|\psi|^2\right]$, which leads to the familiar Schrodinger equation

$$-\frac{1}{2m}\nabla^2\psi + V\psi = i\frac{(a^2\psi)'}{a^2} = i(2\mathcal{H}\psi + \psi') \tag{A17}$$

But if $m = m(0)(1+\beta\cos\mu\eta) \approx m$, $m'/m \approx -\beta\mu\sin\mu\eta$, the situaton depends on the magnitude of μ and η. Only when the mass variation is negligible in $(\phi')^2$, we have the usual form of Schrodinger equation.

If we use cosmic time t instead,

$$\mathcal{L} = \frac{a}{2}\left[a^2(\dot{\phi})^2 - (\nabla\phi)^2 - (m^2+2mV)\phi^2\right] \tag{A18}$$

with $S = \int dtd^3x\,\mathcal{L}$, and making the ansatz $\phi = \frac{1}{\sqrt{2m}}(\psi e^{-imt} + \text{c.c})$, in the non-relativistic limit $\dot{\psi} \ll m\psi$, things are basically similar. If $m' = 0$, the Schrodinger equation is

$$-\frac{1}{2m}\nabla^2\psi + V\psi = i\frac{\frac{d}{dt}(a^3\psi)}{a} = i(3Ha^2\psi + a^2\dot{\psi}) \tag{A19}$$

[1] This oscillation could result from a direct coupling of the ultralight dark matter field to the mass of ordinary matter [5], the authors here also considered a quadratic coupling replacing the cosine with its square. The case for higher-spin ultralight dark matter shoud be different. The gravitational perturabtion of the DM background can also have effects on the orbit dynamics, here we neglect it completely.

[2] D. Baumann, H. S. Chia, and R. A. Porto, Phys. Rev. D **99**, 044001 (2019), arXiv:1804.03208 [gr-qc].

[3] V. Dmitrašinović, M. Šuvakov, and A. Hudomal, Phys. Rev. Lett. **113**, 101102 (2014).

[4] Note this is not the so called variable-mass system such as rokects, in which the mass is just being transfered from one place to another.

[5] D. Blas, D. L. Nacir, and S. Sibiryakov, Phys. Rev. Lett. **118**, 261102 (2017).