Abstract

Here, we propose to use a pair of coplanar homogeneous equilateral triangular plates freely rotating about their fixed axes but which can be precisely balanced upon exerting some external torques, to contrast the statical consequences of different linear models of interaction potential.

CONTENTS

I. LINEAR MODIFICATIONS OF NEWTONIAN POTENTIAL

Besides a muthal force, there is a mutual ("tidal") gravitational torque between two extendend mass distribution, here we consider two rigid coplanar plates.

The plate is assumed to be a unifrom distribution of "mass" (or the charge of other force, if it completely overwhelms gravity). In Newton's theory of gravity, the mass distribution generates a potential accoring to the Poisson equation $\nabla^2\Phi = 4\pi G\rho$, the resulted interaction energy of two mass points with distance r is

$$V(r) = -\frac{Gm_1m_2}{r} \tag{1}$$

And the mutual force is $\mathbf{F} = -F(r)\hat{\mathbf{r}}$, with $F(r) = \frac{Gm_1m_2}{r^2}$. This formula has been tested at very high precession, still there may be some very small deviations which is currently undetectable. The first possible modification comes from post-Newtonian and the one-loop quantum correction, the result is [1]

$$V = -\frac{Gm_1m_2}{r}\left[1 + 3\frac{G(m_1+m_2)}{c^2r} + \frac{41}{10\pi}\frac{l_{pl}^2}{r^2}\right] \tag{2}$$

Note the Planck length is $l_{pl} = \sqrt{\hbar G/c^3}$. We may also consider the contribution from a cosmological constant, which leads to an extra expelling force on the body with mass m, relative to another body,

$$F_\Lambda = \frac{8\pi G}{3}\rho_{\Lambda 0} m r \tag{3}$$

where $\frac{8\pi G}{3}\rho_{\Lambda 0} \sim H_0^2$. Some other modifications that have been disccused seriously in the literature include [2]

$$V = -\frac{G m_1 m_2}{\sqrt{r^2+\epsilon^2}}\left(1 + \frac{k}{r^2+\epsilon^2}\right) \tag{4}$$

where k, ϵ are constants, and [3] (which actually focused on some more involved modifications)

$$V = -\frac{G m_1 m_2}{r}\left(1 + \alpha e^{-r/\lambda}\right) \tag{5}$$

where α, λ are constants, and in Modified Gravity (MOG) [4]

$$V = -\frac{G m_1 m_2}{r}\left[1 - \alpha\left(1 - (1+\mu r)e^{-\mu r}\right)\right] \tag{6}$$

where α, μ are constants, and [5]

$$V = -G m_1 m_2\left(\frac{1}{r} + \alpha \ln\frac{r}{r_0}\right) \tag{7}$$

where α, r_0 are constants. In some modified theories of gravity, the existence of additional massive scalar fields leads to fifth force, which translates into an additional interaction energy

$$V_\phi \sim m_1 m_2 \frac{q_1 q_2}{r} e^{-\mu r} \tag{8}$$

where q_i is the dimentionless charge carried by the i-th body and μ the mass of scalar field, conseqences of such coupling has been discussed thorougly in the context of astrophysical binary problem. Finally, we can also consider a coupling of the mass to a spatially homogeneous oscillating ultralight scalar dark matter background, leading to an oscillating mass [8]

$$m(t) = m_0\left(1 + \sum_n \beta_n \phi^n\right) \tag{9}$$

where β_n are coupling constants (there can also be momentum dispersion, of course).

Note however there is a large set of theories of modified gravity that give non-linear Poisson equations, and hence can not be treated in the simple fasion of Sect. II. For instance, in the famous MoND theory,

$$\nabla\left[\mu\left(\frac{|\nabla\Phi|}{a_0}\right)\nabla\Phi\right] = 4\pi G\rho \tag{10}$$

where a_0 is a constant and $\mu(x)$ some given interpolating function. The gravitational force is $m\mathbf{g} = -m\nabla\Phi$. In some theories there will be a Chemeleon field generated by the mass distribution, e.g., according to [6]

$$\nabla^2\phi = V'_{\rm eff}(\phi), \quad V_{\rm eff} = \Lambda^4\left(1+\frac{\Lambda^n}{\phi^n}\right) + \rho\left(1+\frac{\phi}{M_\beta}\right) \tag{11}$$

where n, M_β, λ are constants, the field exerts an additional force $F_\phi = -\nabla\phi/M_\beta$. A nonlinear Poisson also arises in non-local gravity [7]. These can only be handled with a powerful PDE solver together with an appropriate set of boundary conditions. In the theory of Emergent Gravity (EG), "baryonic matter displaces dark energy to give an additional elastic force", which cannot even be described by a formula (for extended mass distribution).

We shall demonstarte the procedure by considering a general single power-law force between two mass points:

$$F(r) = m_1 m_2 r^k \tag{12}$$

II. SET-UP AND RESULTS

We begin with the most general configuration of two ETs (named as A and B), as depicted in the figure below. The centers of A and B are fixed in the y axis, at distance r, with the rotation angles denoted by α, β respectively. Due to rotational symmetry, we need to consider only $\alpha, \beta \in (0, \frac{2}{3}\pi)$. The side length of both ET will be set as 1 (i.e., as the length unit), then $r > d = \frac{2}{3}\sqrt{3}$. The only nontrivial parameters are r, α, β.

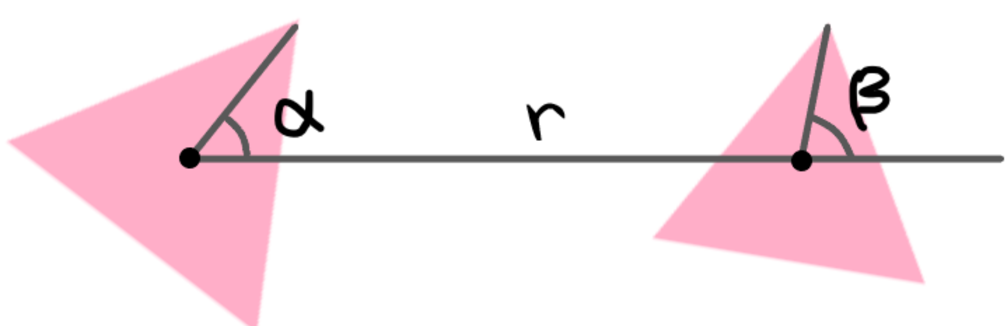

FIG. 1. Configuration of the coplanar ET plates.

We aim to calculate the torque exerted on B by A assuming the model (12). To this end, we have discretized each ET into N^2 identicle mass points with mass $1/N^2$. Hence, the torque (relative to the center of A, which we take to be $\mathbf{0}$) exerted on i-th point of A by j-th point of B is

$$\mathbf{r}_i \times \mathbf{F} = x_i F_y - y_i F_x \tag{13}$$

with $\mathbf{F} = \frac{1}{N^4}|\mathbf{R}_j - \mathbf{r}_i|^{k-1}(\mathbf{R}_j - \mathbf{r}_i)$. On the other hand, the torque (relative to the center of B) exerted on j-th point of B by i-th point of A is

$$-X_j F_y + (Y_i - r)F_x \tag{14}$$

For simplicity, we consider only the case $\alpha = 0$. Some results are presented in the figures below.

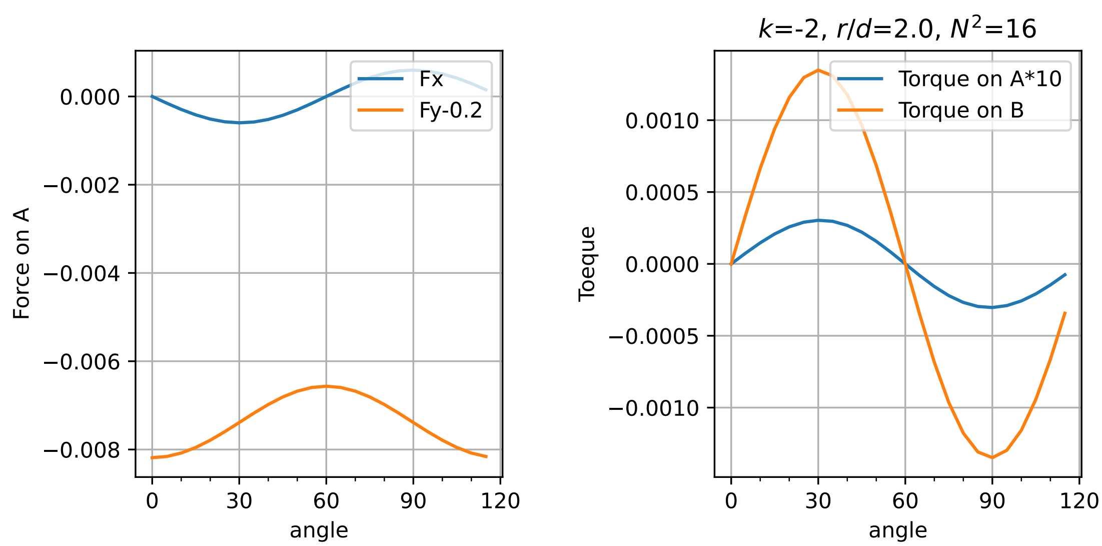

FIG. 2. The case for $\alpha = 0$ and Newtonian inverse-square force. $\beta = \frac{\pi}{3}$ is the fixed point.

Remarkably, for $k < 1$ and $k > 1$, the fixed point of β changes (in both cases we obtain a smooth function of angle, indicating the choice $N = 4$ for $r = 2d$ is ample), and in the critical case $k = 1$, i.e., when the force is dominated by a (negative) cosmological constant, the toques experienced by the whole body are exremely small (this is explained in next section), and the numerical fluctuations are very large, so a larger number of N is required in order to get more precise result.

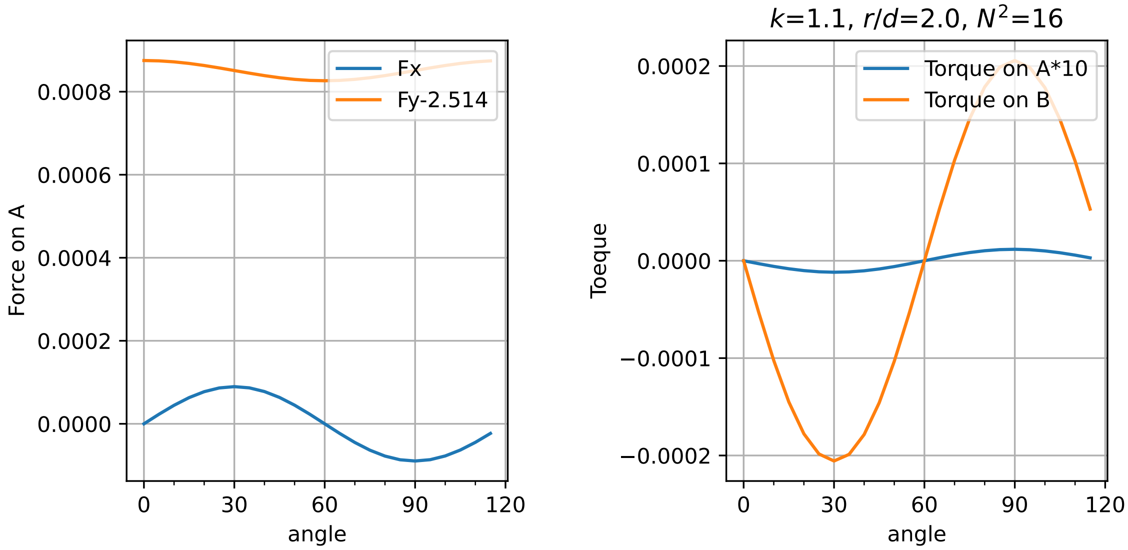

FIG. 3. The case for $\alpha = 0$ and $k = 1.1$. $\beta = 0$ is now the fixed point.

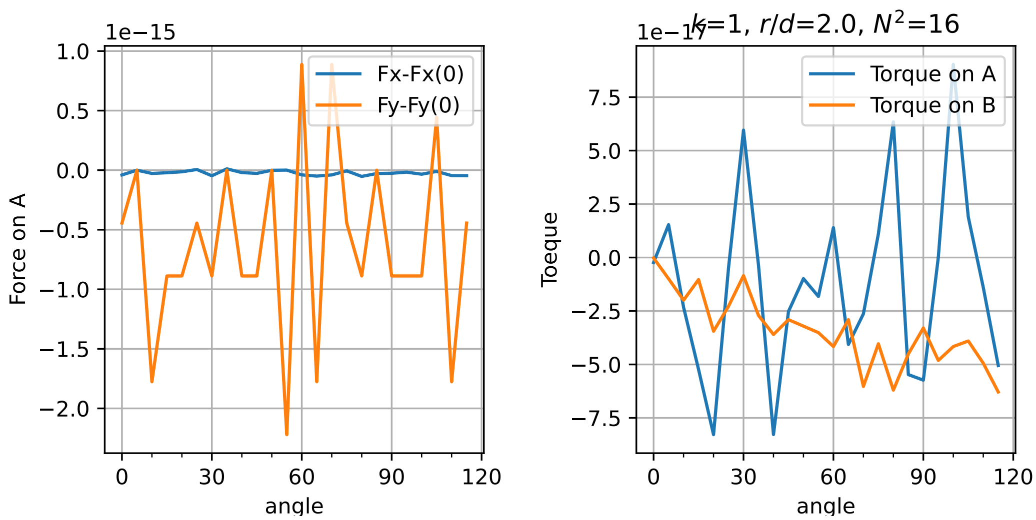

FIG. 4. The case for $\alpha = 0$ and $k = 1$.

III. DISCUSSIONS AND OUTLOOKS

Several extensions can be made. First, the case of varing α and d can be explored. One can also compare the result of different $F(r)$ models more carefully. We have only calculated the static torques, while the dynamical simulation is left for futural studies. In principle, the study can also be extended to other shapes, e.g., straight rods and square plates.

As a first step to understand the numerical results presented above, we consider the potential generated by three equally-sapced and equal-mass points. For newtonian force, we have

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\sum_{n=1}^{\infty}\frac{r_0^n}{r^{n+1}}P_n(\cos\theta_0),\quad \theta_0=\angle(\mathbf{r},\mathbf{r}_0) \tag{15}$$

for $r_0 < r$ (since we are confined to 2D planes, we don't need the expansion in terms of spherical harmonics). The first a few Legendre polynomials are $P_0(x) = 1$ (monopole), $P_1(x) = x$ (dipole), $P_2(x) = (3x^2 - 1)/2$ (quadrupole), $P_3(x) = (5x^3 - 2x)/2$ (octupole), $P_4(x) = \left(35x^4 - 30x^2 + 3\right)/8$ (hexadecapole). For three mass points (arranged in an equilateral triangle with side lenth 1), relative to the geometrical center, we have $r_0 = \frac{1}{3}\sqrt{3}$, and $\theta_2 = \theta_1 + \frac{2\pi}{3}$, $\theta_3 = \theta_1 - \frac{2\pi}{3}$, since $\cos(\theta + \frac{2}{3}\pi) = -\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$, $\sum_i \cos\theta_i = 0$ (as it should be). The leading order correction to the monopole potential $3/r$ then comes from the quadrupole term, which turns out to be isotropic:

$$\frac{3r_0^2}{2r^3}\left(\sum_i \cos^2\theta_i - 1\right) = \frac{1}{4r^3} \tag{16}$$

So the angular dependence stars at octupole level:

$$\frac{5r_0^3}{2r^4}\sum_i \cos^3\theta_i = \frac{5\sqrt{3}}{24r^4}\cos(3\theta_1) \tag{17}$$

The hexadecapole term is also isotropic with value $\frac{3}{64}r^{-5}$. The $n = 5$ term is $\frac{105}{128}r_0^5 r^{-6}\cos(3\theta_1)$. Hence the full potential can be written as

$$\Phi(\mathbf{r}) \equiv \sum_i \frac{1}{|\mathbf{r}-\mathbf{r}_i|} = \underbrace{\left(\frac{3}{r}+\frac{1}{4r^3}+\cdots\right)}_{\text{isotropic}} + \underbrace{\left(\frac{5\sqrt{3}}{24r^4}+\cdots\right)}_{\equiv f(r)}\cos(3\theta_1) \tag{18}$$

Now for two ETs, in the lowest order approximation one of the ET may be taken as a point, so the interaction energy is $-\Phi(\mathbf{r})$ times mass, and the torque is given by $\partial_{\theta_1}\Phi = -3f(r)\sin(3\theta_1)$. So the torque takes extreme value at $\theta_1 = \pi/6 \pm \pi/3$, compatible with the figure above.

The potential of central force $\propto r^k$ can be also expanded. We focus here on $k = 1$, for which the expansion is simple: $|\mathbf{r} - \mathbf{r_0}|^2 = r^2 - 2r_0 r\cos\theta + r_0^2$, hence for three mass points,

$$\Phi(\mathbf{r}) = 3r^2 + 3r_0^2 \tag{19}$$

which is purely isotropic, and explains why the mutual torque between two ETS is very small.

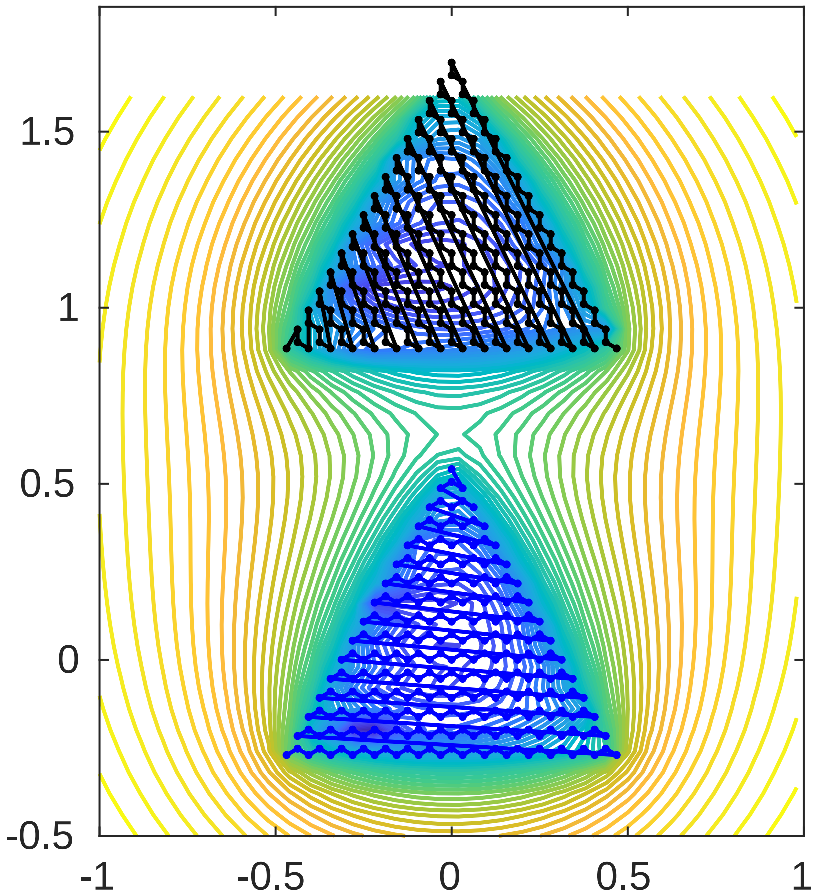

FIG. 5. Newtonian gravitational potential generated by double ETs.

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