A summary of part of my bachelor works (2021), motivated by [1].

I. SCATTERING PROBLEM

1+1D wave equations in the presence of a background flow $\mathbf{v} = v(x,t)\hat{\mathbf{x}}$:

Shrodinger equation in constant potential

$$i\left(\partial_t + v\partial_x\right)\psi = -\frac{1}{2}\partial_x^2\psi + \psi \tag{1}$$

Modified BdG equation

$$i\left(\partial_t + v\partial_x\right)\psi = -\frac{1}{2}\partial_x^2\psi + a\psi + b\psi^* \tag{2}$$

Extended Bogoliubov-de Gennes (BdG) equation

$$i\left(\partial_t + v\partial_x\right)\psi = -\frac{1}{2}v\partial_x\left(v^{-1}\partial_x\psi\right) + a\psi + b\psi^* \tag{3}$$

A first order wave equation

$$\partial_t\psi = -\frac{1}{2}f\partial_x\psi, \quad f(x,t) \in \mathbb{R} \tag{4}$$

For $f = f(x)$, this is actually a special solution of KG equation in background $g_{ab} = \mathrm{diag}(f, -1/f)$, which in the Edington-Finkelstein time coordinate $dv = dt + dx/f(x)$ reads $\Box = \frac{1}{\sqrt{-g}}\partial_a\sqrt{-g}g^{ab}\partial_b = -2\partial_v\partial_x - f'\partial_x - f\partial_x^2 = -\partial_x\left(2\partial_v + f\partial_x\right)$.

- Cherenkov excitation: $i\left(\partial_t + v\partial_x\right)\psi = -\frac{1}{2}v\partial_x^2\psi + \psi + S(x)$. For constant flow $v(x) = \mathrm{const}$, this is equivalent to $i\partial_t\psi = -\frac{1}{2}\partial_x^4\psi + \psi + S(x - vt)$. Also Cherenkov excitation of $\partial_t^2 f = -\frac{1}{4}\nabla^2\psi + a\partial_x^2 f + (a^2 - b^2)f + S(x)$, which is the decoupled equation for both the real and imaginary part of ψ in (2) if $a, b = \mathrm{const}$.

- For Eq. (1), plane wave scattering in steplike flow $v(x) = \Theta(-x)v_1 + \Theta(x)v_2$ and wave packet propagation in $v(x) = \frac{v_1+v_2}{2} + \left(\frac{v_2-v_1}{2}\right)\tanh x/\sigma$.

- For Eq. (2) with $a = 1$ (so the critical flow speed for Cherenkov excitation is $v_c = \sqrt{1+\sqrt{1-b^2}}$), plane wave scattering in three-layer configuration [2] $v(x) = \Theta(-x)v_1 + \Theta(L-x)\Theta(x)v_2 + \Theta(x-L)v_3$ with $-v_c < v_{1,3} < 0, v_2 < -v_c$. Position of (transmission and reflection) resonance peaks. "Laser" condtion for length L.

- For Eq. (2) with $v=-1$ and $a=b=c^2(x)$ [3], canonical quantization, plane wave mode-mixing scattering and radiation spectrum of in-vacuum state in steplike BH $c(x)=\Theta(-x)c_1+\Theta(x)c_2$ with $c_2>1>c_1$. Equal-time two-point correlation function $G^{(2)}(x,x')=\langle 0_{\rm in}|[\hat{\psi}(x'),\hat{\psi}^\dagger(x')][\hat{\psi}(x),\hat{\psi}^\dagger(x)]|0_{\rm in}\rangle + \text{c.c (i.e.,)}(x\leftrightarrow x')$. Measurement of the entanglement of Hawking pair.

- For Eq. (2) with $v=-1$, $b=\text{const}$ and steplike $a(x)$, similar things.

- Temporal steplike scattering?

- Double and multi-horizon scattering?

- Note Eq. (3) and (4) both has conserved current for an appropriately defined inner product of two solutions, $\partial_t\rho_{12}+\partial_x J_{12}=0$.

  For (3), the field can be seperated into a positive and a negative frequency part:

$$\psi=\int_0^\infty d\omega\left(e^{-i\omega t}\phi_\omega+e^{i\omega t}\varphi^*_\omega\right)=\underbrace{\int_0^\infty d\omega e^{-i\omega t}\phi_\omega}_{\equiv\phi}+\underbrace{\int_0^\infty d\omega e^{i\omega t}\varphi^*_\omega}_{\equiv\varphi^*} \tag{5}$$

  then

$$\rho_{12}(x,t)=\frac{i}{v}(\varphi_1\varphi_2^*-\phi_1\phi_2^*) \tag{6}$$

$$J_{12}(x,t)=\frac{1}{2v}\left(\varphi_1\partial_x\varphi_2^*-\varphi_2^*\partial_x\varphi_1+\phi_1\partial_x\phi_2^*-\phi_2^*\partial_x\phi_1\right)+i\left(\varphi_2^*\varphi_1-\phi_1\phi_2^*\right) \tag{7}$$

  here $v=v(x,t)$, this result is independent from the functional form of $a(x,t)$ and $b(x,t)$.

  For (4),

$$\rho_{12}(x,t)=\frac{2i}{f}(\psi_1^*\partial_t\psi_2-\psi_2\partial_t\psi_1^*),\quad J_{12}(x,t)=\frac{f}{2i}(\psi_1^*\partial_x\psi_2-\psi_2\partial_x\psi_1^*) \tag{8}$$

  Hence they can both be canonically quantized (with respect to these norms), provided the mode solutions to wave equation can be found.

II. TUNNELING IN BOSE-HUBBARD MODEL

The Hamiltonian is

$$H=-\sum_n\left[\kappa_n\left(\hat{a}_n^\dagger\hat{a}_{n-1}+\hat{a}_{n-1}^\dagger\hat{a}_n\right)+\mu\hat{a}_n^\dagger\hat{a}_n\right] \tag{9}$$

(wlog we set $\mu = 0$) where the lattice points are $x_n = nd$, $n = -L, \cdots, 0, 1, \cdots, L$, and the hopping parameters $\kappa_n = \frac{1}{4d} f[(n-1/2)d]$. In Fock basis $\left\langle e_i \left| \hat{a}_n^\dagger \hat{a}_n \right| e_j \right\rangle = \delta_{ni}\delta_{nj}$, with

$$|e_{-L}\rangle = (1, 0, \ldots, 0)^T, \quad \ldots, \quad |e_L\rangle = (0, \ldots, 0, 1)^T \tag{10}$$

so $H_{ij} = -(\kappa_i \delta_{i,j+1} + \kappa_{i+1}\delta_{i,j-1} + \mu\delta_{ij})$, explicitly

$$H = \begin{pmatrix} H_{-L,-L} & H_{-L,-L+1} & & & & & & \\ H_{-L+1,-L} & \ddots & \ddots & & & & & \\ & \ddots & & & & & & \\ & & \ddots & H_{-1,-1} & H_{-1,0} & & & \\ & & & H_{0,-1} & H_{0,0} & H_{0,1} & & \\ & & & & H_{1,0} & H_{1,1} & \ddots & \\ & & & & & \ddots & \ddots & H_{L-1,L} \\ & & & & & & H_{L,L-1} & H_{L,L} \end{pmatrix} = \begin{pmatrix} \mu & \kappa_{-L+1} & & & & & & \\ \kappa_{-L+1} & \ddots & \ddots & & & & & \\ & \ddots & & & & & & \\ & & \ddots & \mu & \kappa_0 & & & \\ & & & \kappa_0 & \mu & \kappa_1 & & \\ & & & & \kappa_1 & \mu & \ddots & \\ & & & & & \ddots & \ddots & \kappa_L \\ & & & & & & \kappa_L & \mu \end{pmatrix} \tag{11}$$

The time evolution of an initial state reads $|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle$.

- For $f(x) = \alpha \tanh x/\sigma$ [4], eigenvalue and eigenstates, wave packet propagaion, tunneling of an initial localized delta state (frequency spectrum of tunneling probability and its high (WKB, failed) /low (linear, Hawking) frequency analytical approximation, time evolution of entanglement entropy for left and right half system: $S = -\mathrm{Tr}_{\mathrm{right}}(\rho \ln \rho)$, where the reduced density matrix $\rho = \mathrm{Tr}_{\mathrm{left}}(|\psi\rangle\langle\psi|)$, $\mathrm{Tr}_{\mathrm{left}} X = \sum_i \langle v_i^{\mathrm{left}}|X|v_i^{\mathrm{left}}\rangle$, $|v_i^{\mathrm{left}}\rangle$ the eigenstates of left half system).

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[3] C. Mayoral, A. Fabbri, and M. Rinaldi, Phys. Rev. D **83**, 124047 (2011).

[4] R.-Q. Yang, H. Liu, S. Zhu, L. Luo, and R.-G. Cai, Phys. Rev. Res. **2**, 023107 (2020).