Falling of a Pencil in Axion Enviroment

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Abstract

This note investigate the motion of a rigid pencil on a horizontal table, assuming that all mass oscillates coherently from the coupling to a homogeneous axion background. The pencil is initially erect, then it undergoes a spontaneous symmetry breaking, i.e., it falls, the question is when would this pencil slide during the falling process. It is shown that the condition of sliding can be modified greatly, depends on the magnitude of coupling, its oscillation frequency and the phase of oscillation. In addition, the falling and a similar problem of sliding ladder, both without friction, are simulated numerically.

CONTENTS

I. THE PROBLEM

A. Falling without Slide

The situation is depicted in the figure below. The pencil is modeled as a straight rigid rod with uniform mass ditribution, mass $m(t) = 1 + \beta\cos(\mu t + \phi)$, where β, μ, ϕ are constants, and length $l = 1$ (also we work in the unit $g = 1$). As the pencil falls, θ increases from 0 to $\pi/2$.

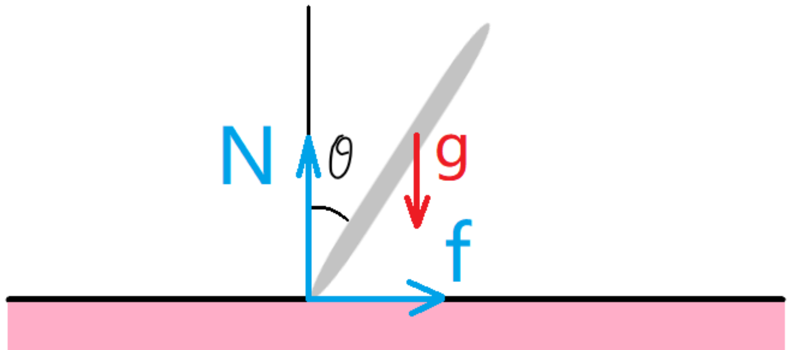

FIG. 1. The problem set-up.

To demonstrate the effect of $m(t)$, we do a lagrangian analysis for the frictionless case[1],

$$L = \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\theta}^2 + \frac{1}{2}mv^2 - \frac{1}{2}mgl\cos\theta \tag{1}$$

where $\mathbf{v}$ is the velocity of the mass center. Assuming the pencil does not slide, $v = \frac{l}{2}\dot{\theta}$, then

$$L = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 - \frac{1}{2}mgl\cos\theta \tag{2}$$

Now the rotation dynamics reads

$$\frac{d}{dt}\left(\frac{1}{3}ml^2\dot{\theta}\right) = \frac{1}{2}mgl\sin\theta \tag{3}$$

or

$$\ddot{\theta} + \frac{\dot{m}}{m}\dot{\theta} = \frac{3}{2}g\sin\theta \tag{4}$$

The vertical motion of the mass center is

$$\frac{d}{dt}\left(m\frac{d}{dt}(\frac{1}{2}l\cos\theta)\right) = N - mg \tag{5}$$

or

$$N/m = 1 - \frac{1}{2}\sin\theta\left(\ddot{\theta} + \frac{\dot{m}}{m}\dot{\theta} + \cot\theta\dot{\theta}^2\right) = g - \frac{1}{2}\left(\frac{3}{2}g\sin^2\theta + \cos\theta\dot{\theta}^2\right) \tag{6}$$

where N is the normal force exerted by the table. Now incorporating friction from the table, these equations do not change, but horizontal motion of the mass center does change. Assuming the contacting point doesn't slide, we have

$$\frac{d}{dt}\left(m\frac{d}{dt}(\frac{1}{2}l\sin\theta)\right) = f \tag{7}$$

or

$$f/m = \frac{1}{2}\cos\theta\left(\ddot{\theta} + \frac{\dot{m}}{m}\dot{\theta} - \tan\theta\dot{\theta}^2\right) = \frac{1}{2}\left(\frac{3}{2}g\sin\theta\cos\theta - \sin\theta\dot{\theta}^2\right) \tag{8}$$

where f is the friction force. In the case of static mass, if the pencil does not slide, the total energy of the system

$$H = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 + \frac{1}{2}mgl\cos\theta = \frac{1}{2}mgl \tag{9}$$

is conserved, then f, N are fixed as functions of the angle and time, the result is

$$\begin{aligned}
\dot{\theta} &= \sqrt{3\left(1-\cos\theta\right)} \\
N/mg &= 1 - \frac{3}{4}\sin^2\theta + \frac{3}{2}\cos^2\theta - \frac{3}{2}\cos\theta \\
f/mg &= \frac{9}{4}\sin\theta\cos\theta - \frac{3}{2}\sin\theta
\end{aligned} \tag{10}$$

The slide cannot happen if $r \equiv f/N$ is smaller then the critical friction factor of the table. Let us briefly remarks on this function, $f(\theta)$ is initially positive but changes sign near 47° while N remains postive until near 71°, at 40° $r(\theta)$ acquires a local maximum value of $r_{\text{crit}} = 0.37$.

If mass varies with time, however, the Hamiltonian is no longer conserved, we have to solve the equation of motion (4) directly, which for the mass oscillation model above, is explicitly

$$\ddot{\theta} + \frac{-\beta\mu\sin(\mu t+\phi)}{1+\beta\cos(\mu t+\phi)}\dot{\theta} - \frac{3}{2}g\sin\theta = 0 \tag{11}$$

and calculate f, N using (6) and (8).

Note for a point pendulum with unit length, the potential energy is $-mgl\cos\theta$, the $-\frac{3}{2}$ is replaced by 1.

B. Falling without Friction

Now we study the falling process assuming there is no friction at all. The situation is depicted in the figure below (in this section we neglects the wall).

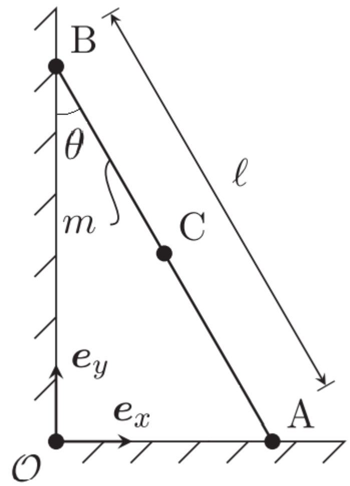

FIG. 2. The problem set-up.

With no friction, the mass center will not acquire a horizontal velocity and the problem is very simple to analyse, the lagrangian reads

$$L = \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\theta}^2 + \frac{1}{2}m\left(\frac{l}{2}\sin\theta\,\dot{\theta}\right)^2 - \frac{1}{2}mgl\cos\theta = \frac{1}{2}ml^2\left(\frac{1}{12}+\frac{1}{4}\sin^2\theta\right)\dot{\theta}^2 - \frac{1}{2}mgl\cos\theta \tag{12}$$

The equation of motion is

$$\frac{d}{dt}\left[ml^2\left(\frac{1}{12}+\frac{1}{4}\sin^2\theta\right)\dot{\theta}\right] = \frac{1}{2}mgl\sin\theta + \frac{1}{4}ml^2\sin\theta\cos\theta\,\dot{\theta}^2 \tag{13}$$

or $(g = l = 1)$

$$\ddot{\theta} + \frac{\dot{m}}{m}\dot{\theta} + \frac{\sin\theta\cos\theta}{4\left(\frac{1}{12}+\frac{1}{4}\sin^2\theta\right)}\dot{\theta}^2 - \frac{\sin\theta}{2\left(\frac{1}{12}+\frac{1}{4}\sin^2\theta\right)} = 0 \tag{14}$$

The normal force exerted by the table is given by 6. In the static mass case, the zero point of $N(\theta)$ would be different from that of last section, and it will actually be reached, when

$$g = \frac{l}{2}(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}) \tag{15}$$

then the pencil lose support from the table and rotates with uniform angular velocity, until it touches the table again, this process is somewhat intricate and may deserve careful investigation. If the mass varies with time, this process is again modified.

C. Slidding Ladder without Friction

Now we study a related problem, the situation is depicted in Fig. II A, but this time with the wall. For discussion of this problem (also including friction) in the static mass case, see, e.g., The sliding ladder problem revisited in phase space (2019, AJP). The postion of mass center is $(\frac{l}{2}\sin\theta, \frac{l}{2}\cos\theta)$, hence the lagrangian reads

$$L = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 - \frac{1}{2}mgl\cos\theta \tag{16}$$

which is same as a falling pencil without sliding (see above). In the present scenario, one might ask whether the ladder will lose contact with the vertical wall, mass oscillation will certainly have impacts on this, though I'll not check it here.

II. RESULTS AND DISCUSSION

A. Falling without Slide

For a very low mass oscillation frequency, we find numerically that $r(\theta)$ qualitatively does not change, even for a $\beta \sim 1$. The concrete effects though on r depends on the parameters, typically being small early on and the difference grows large after r turns negative.

In the high-frequency regime, the friction term actually acts as a driving force, and the effects of β and ϕ are way more significant. Some results are shown in Fig. II A, II A, II A, II A, II A. Basically, N and r oscillates around their values in the static-mass case with frequency and magnitude determined by μ, β, respectively. This has particular important consequences for the zero point of N, so if the critical friction factor is very large, the pencil will slide earlier than in the static mass case. Also, the quantity H/m is being amplified.

For mapping to a real experiment, consider the time scale of falling without slide, in the static mass case it's shorter than (since the slide is unavoidable)

$$\int_0^{\pi/2} \frac{d\theta}{\dot{\theta}} = \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{3(1-\cos\theta)}} \approx 15.6\sqrt{\frac{l}{g}} \approx 5\sqrt{l/\mathrm{m}} \tag{17}$$

which translates into a frequency scale $f = 0.2/\sqrt{l/\mathrm{m}}$.

The oscillation frequency is given by the particle mass

$$\mu = \text{mass} \times \frac{c^2}{\hbar} = \frac{\text{mass} \times c^2}{\text{eV}} 1.5 \times 10^{15}\,\text{Hz} \tag{18}$$

Thus, for an ultralight particle with mass $\sim 10^{-20}\,$eV, the freuency is matched for a pencil of length 10^8 meters (at least). But for larger mass, e.g., 10^{-10} eV, the oscaillation is much faster then the falling even for a normal-sized pencil, in this case the only problem is the coupling being too weak.

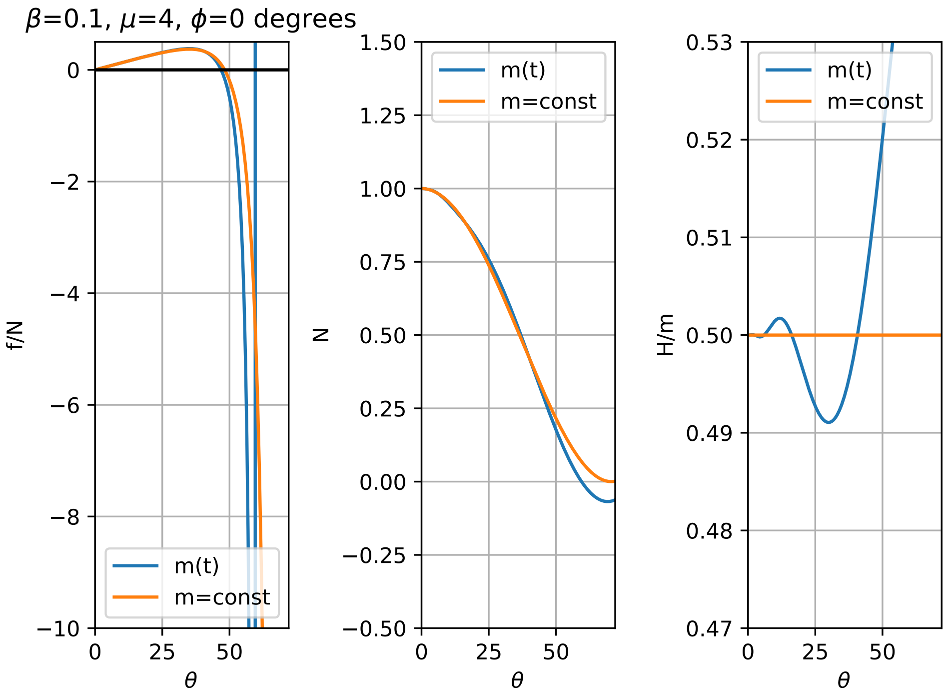

FIG. 3. Small coupling

B. Falling without Friction

We solve Eq. 14, and plot the line connecting $\mathbf{A} = (\frac{l}{2}\sin\theta, 0)$ and $\mathbf{B} = (-\frac{l}{2}\sin\theta, l\cos\theta)$. Yet I have not take into account the possible detach of the pencil from table, this would eventually happen even if the mass is static. So what I actually simulated is a pencil freely rotating about a shaft which moves horizontally along a frictionless track, in this scenario $\theta \in (0, 2\pi)$.

Appendix A: Gravitational Acceleration

We have tacitly assumed that g is constant, this will not be the case if the entire earth is coherently coupled to a spatially homogeneous axion background. In the latter situation,assuming the earth is still spinning with a uniform angular velocity Ω, in the co-rotating frame there is an inertial acceleration $\mathbf{a} = \Omega^2(\mathbf{r} - \mathbf{r}\cdot\hat{\mathbf{z}})$, and the resulted net vertical gravitational acceleration is

$$g = \frac{M}{r^2} - \Omega^2 \cos^2\varphi \tag{A1}$$

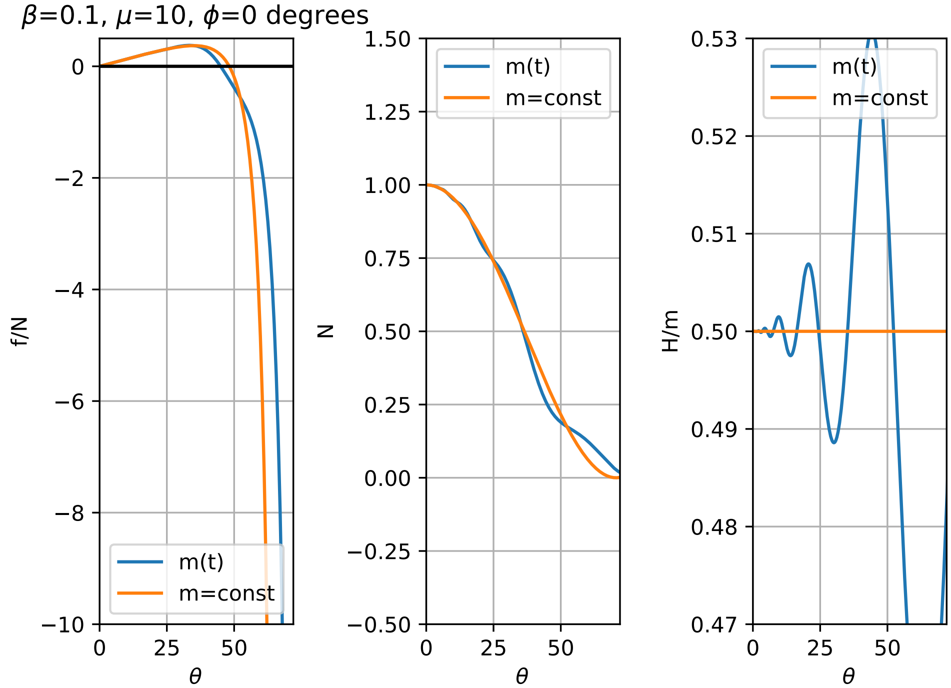

FIG. 4. Small coupling

where φ is the latitude. We have explicitly kept g in (4), (6), (8), the effect of its oscillation can be readily examined.

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[1] The potential energy is taken to be mgh, since we assume that gravitational force is still mg even for $m(t)$. Indeed, g might not be a constant, either, see Appendix A.

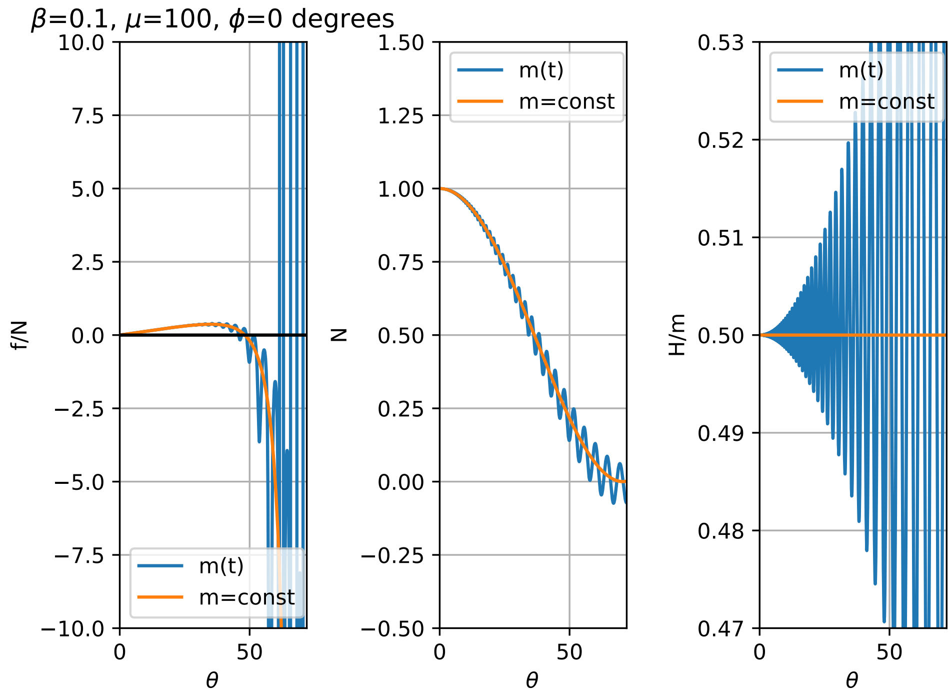

FIG. 5. Small coupling, high frequency.

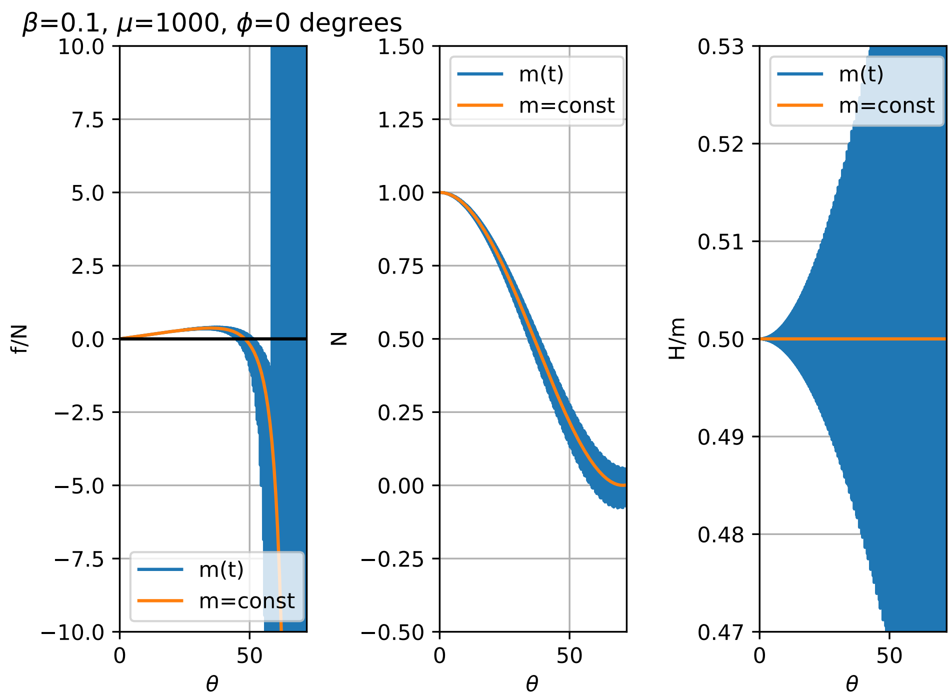

FIG. 6. Small coupling, even higher frequency.

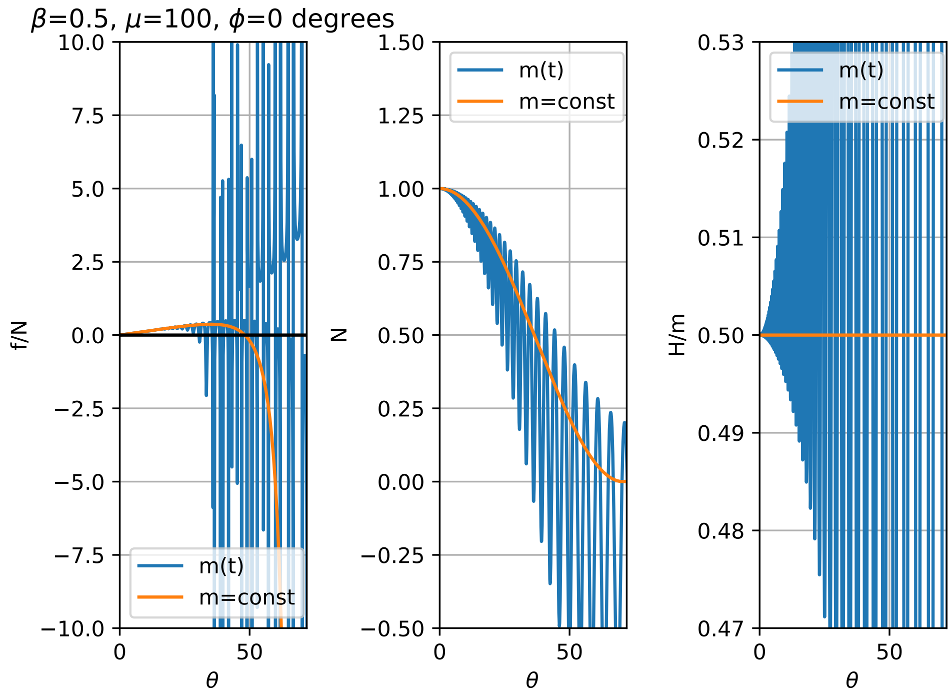

FIG. 7. Large coupling, high frequency.