A bunch of small problems.

I. ELECTROSTATICS OF TWO CYLINDERS

Consider two charged cylindrical ideal conductors centered at A and B (with line charge density $\lambda_{a,b}$, and radius a, b, respectively) with infinite length and parallel axes (with distance c), here we focus solely on the analytically tractable case: $q_a = -q_b = q$.

This electrostatic problem can be solved by introducing two image (line) charges at C and D (between A and B, with C closed to A, D closed to B), I denote AC=x<a, DB=y<b, then $\frac{b^2}{c-x} = y$, $\frac{a^2}{c-y} = x$, we have $\frac{a^2}{c-\frac{b^2}{c-x}} = x$, or $x^2 - \frac{c^2+a^2-b^2}{c}x + a^2 = 0$, hence

$$x(a,b,c) = \frac{a^2 - b^2 + c^2 \pm \sqrt{\left(a^2 - b^2 + c^2\right)^2 - 4a^2c^2}}{2c} \tag{1}$$

but since $x_+x_- = a^2$ which means (since x<a) we can only take the minus solution, while $y = x(b,a,c)$. Now the system has been reduced to two such image charges seperated by

$$r = c - x - y = \frac{1}{2c}\left[\sqrt{\left(a^2 - b^2 + c^2\right)^2 - 4a^2c^2} + \sqrt{\left(b^2 - a^2 + c^2\right)^2 - 4b^2c^2}\right] \tag{2}$$

The force between A and B is $F = -\frac{q^2}{r}$ (attractive). The potential at the surface of A,B is

$$V_a = -q\ln\left(\frac{a-x}{c-y-a}\right), \quad V_b = q\ln\left(\frac{b-y}{c-x-b}\right) \tag{3}$$

The electrostatic energy of the system is (remembering that, the charge actually resides on the sufrace of the cylinders)

$$W = \frac{1}{2}q_iV_i = -\frac{1}{2}q\left[\ln\left(\frac{a-x}{c-y-a}\right) - \ln\left(\frac{b-y}{c-x-b}\right)\right] = -\frac{1}{2}q\ln\left[\frac{(a-x)(c-x-b)}{(b-y)(c-y-a)}\right] \tag{4}$$

We should be able to check that $F = -\partial_c W$.

In this way we solve only the case when $q_a = q = -q_b$, but in general

$$q_a = C_{ab}V_b + C_{aa}V_a, \quad q_b = C_{bb}V_b + C_{ba}V_a \tag{5}$$

where the capacitance constants C depends on a, b, c (note however $C_{ab} = C_{ba}$). the special solution thus gives the relation (likewise for C_{bb}):

$$1 = C_{ab}\ln\left(\frac{b-y}{c-x-b}\right) - C_{aa}\ln\left(\frac{a-x}{c-y-a}\right) \tag{6}$$

If we can find yet another special solution (for different set of q_a, q_b), this problem is compleltely solved, but such a solution would be way less trivial.

Surely the problem is simple if one of the cylinder (say, B) shrinks to zero-radius, in this case we need only one image charge $-q_b$ with $x = a^2/c$, an arbitrary charge $q_a + q_b$ can be placed in the center of A. So the potential at A's surface is

$$V_a = q_b \ln\left(\frac{a-x}{c-a}\right) - (q_a + q_b)\ln a \tag{7}$$

and the mutual force is $F = -\frac{q_b^2}{c-x} + \frac{(q_a+q_b)q_b}{c}$.

Notes Added: I found this problem has been treated long ago by John Lekner, see his monograph Electrostatics of Conducting Cylinders and Spheres (AIP Publishing, 2021). Pitifully I do not have access to this book.

This problem, being 2D, may also be solved via conformal mapping (for this tool in electrostatics, see, e.g., Conformal Mapping in Electrostatics and a Table of Conformal Maps), but still we can only treat the $q_a = -q_b$, even $V_{a,b}$ are both free parameters (what matters in this case is $V_a - V_b$), since this is not a Dirichlet boundary value problem, the boundary condition of **conductor** occupying space region M is $\mathbf{E} \perp \partial M$ and $\int_{\partial M} \mathbf{E}\cdot\hat{\mathbf{n}} = Q$ the total charge, where ∂M is the boundary of M. This conformal transformation is $w = \frac{z-a}{az-1}$ from $z = x+iy$ to $w = u+iv$, where $a = \frac{x_1x_2+1+\sqrt{(x_1^2-1)(x_2^2-1)}}{x_1+x_2}$, which maps the region outside both a unit circle centered at $(0,0)$ and a circle with radius $(x_1-x_2)/2$ centered at $(\frac{x_1+x_2}{2}, 0)$ into the region between two coaxial circles with radius R and 1 centered at $(0,0)$. Note the inverse mapping also has such property (see figure below). Poisson equation with the latter boundary condition can be easily solved, but pitifully naturally yeilds $q_a = -q_b$. This mapping can be visualized:

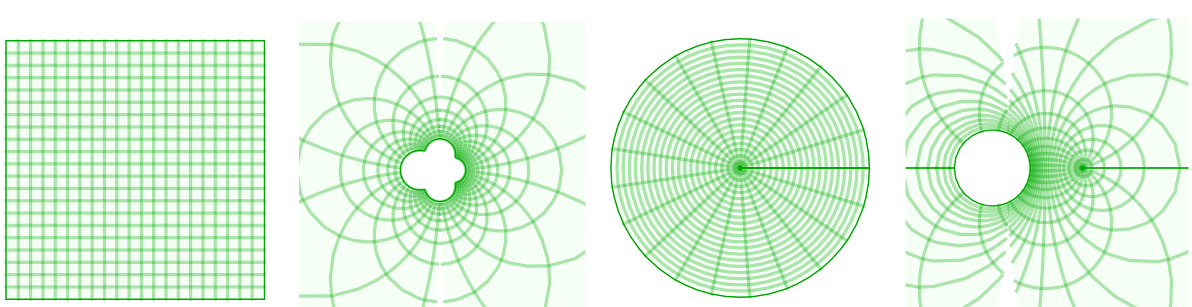


FIG. 1. Conformal mapping $w = u + iv = F(z = x + iy)$, left: (u, v), right: (x, y).

This plot is shows the mapping from the regular (x, y) mesh to (u, v), the mma script is, first define the $F(z)$

```
x1 = 3; x2 = 2;
a = (x1*x2 + 1 + Sqrt[(x1^2 - 1) (x2^2 - 1)])/(x1 + x2)
F[z_] = (z - a)/(a*z - 1);
```

then

```
{fig1, fig2} =
With[{z = r*Cos[t] + I*r*Sin[t]},
ParametricPlot[ReIm[#], {r, 0, 1}, {t, 0, 2*Pi}, Mesh -> 20,
PlotStyle -> LightGreen,
MeshStyle -> Directive[Thick, Darker[Green]],
BoundaryStyle -> Directive[Thickness[.005], Darker[Green]],
Frame -> False, Axes -> False]] & /@ {z, F[z]};
GraphicsRow[{fig1, fig2}]
```

for circular mesh, or

```
{fig1, fig2} =
With[{z = x + I*y},
ParametricPlot[ReIm[#], {x, -2, 2}, {y, -2, 2}, Mesh -> 20,
PlotStyle -> LightGreen,
MeshStyle -> Directive[Thick, Darker[Green]],
BoundaryStyle -> Directive[Thickness[.005], Darker[Green]],
Frame -> False, Axes -> False]] & /@ {z, F[z]};
GraphicsRow[{fig1, fig2}]
```

for square mesh. For the inverse mapping, we can use simply

```
R = (x1*x2 - 1 - Sqrt[(x1^2 - 1) (x2^2 - 1)])/(x1 - x2);
reg = ImplicitRegion[
Abs[F[x + I y]] < 1 && Abs[F[x + I y]] > R, {x, y}];
RegionPlot[reg, PlotRange -> {{-5, 5}, {-5, 5}}, GridLines -> Automatic]
```

For unknown reasons, the script

```
RegionPlot[TransformedRegion[ImplicitRegion[R^2<x^2 + y^2 < 1, {x, y}],
ReIm[F[Indexed[#, 1] + I Indexed[#, 2]]] &]]
```

fails.

There is no ambiguities for using conformal mapping to solve a pure Dirichlet boudary value problem, but **what if the conformal mapping is not bijective, and there are point charges?** For a concrete example, we check the Joukowski transformation: $w = F(z) = \alpha z + \frac{\beta}{z}$, which maps a unit circle $x^2 + y^2 = 1$ into $\frac{u^2}{(\alpha+\beta)^2} + \frac{v^2}{(\alpha-\beta)^2} = 1$. The case $\alpha < \beta$ and $\alpha > \beta$ (assuming both positive) are different. To see this, the mma script:

```
alpha = 2;
beta = 1;
F[z_] := alpha z + beta/z;
With[{p = 1 - 0.5 I},
p1 = NSolve[F[z] == p, z][[1, 1, 2]];
p2 = NSolve[F[z] == p, z][[2, 1, 2]];
Print[{p1, p2}];
ParametricPlot[{ReIm[F[Cos[t] + I*Sin[t]]],
ReIm[Cos[t] + I*Sin[t]]}, {t, Pi/2, 2*Pi},
AspectRatio -> Automatic,
Epilog -> { {Red, Point[ReIm[p]]}, Point[ReIm[p1]],
Point[ReIm[p2]]}, PlotStyle -> {Red, Black, Black},
AxesLabel -> {"x", "y"}]]
```

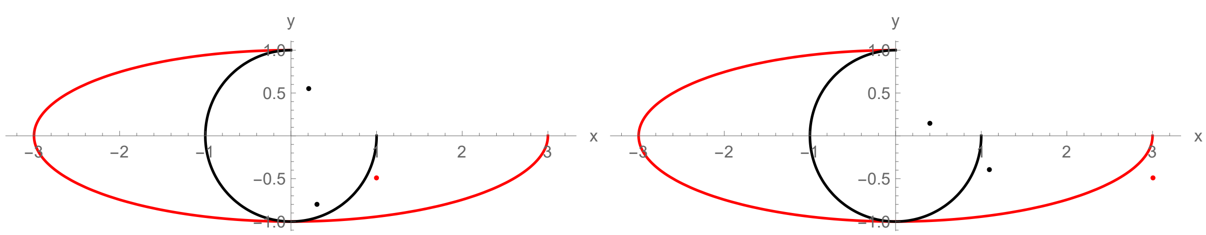


FIG. 2. $\alpha = 2, \beta = 1$, black: (x, y), red: (u, v). The black curve is the unit circle, the black points are the primary images of the red point. There is always one primary image inside the unit circle, and if (x, y) is inside the ellipse (mapped from the unit circle), then two primary images are both inside. In fact, the interior of the unit circle is mapped to the whole complex plane, but the outside of the unit circle is mapped to the ouside of the ellipse.

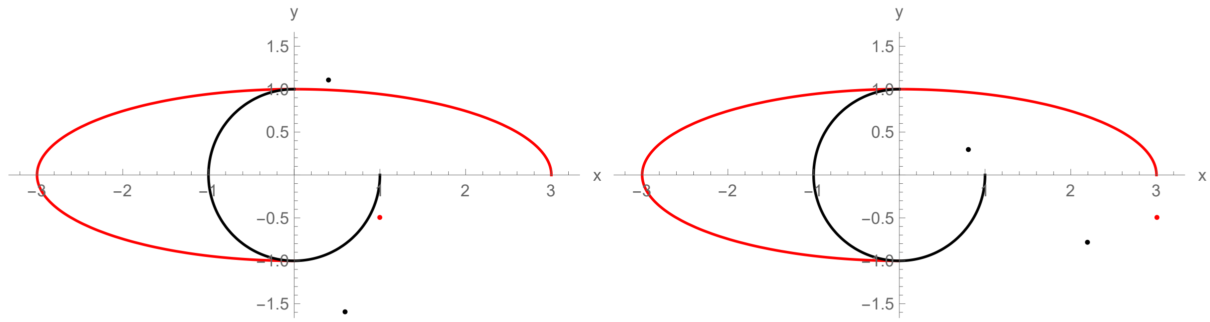


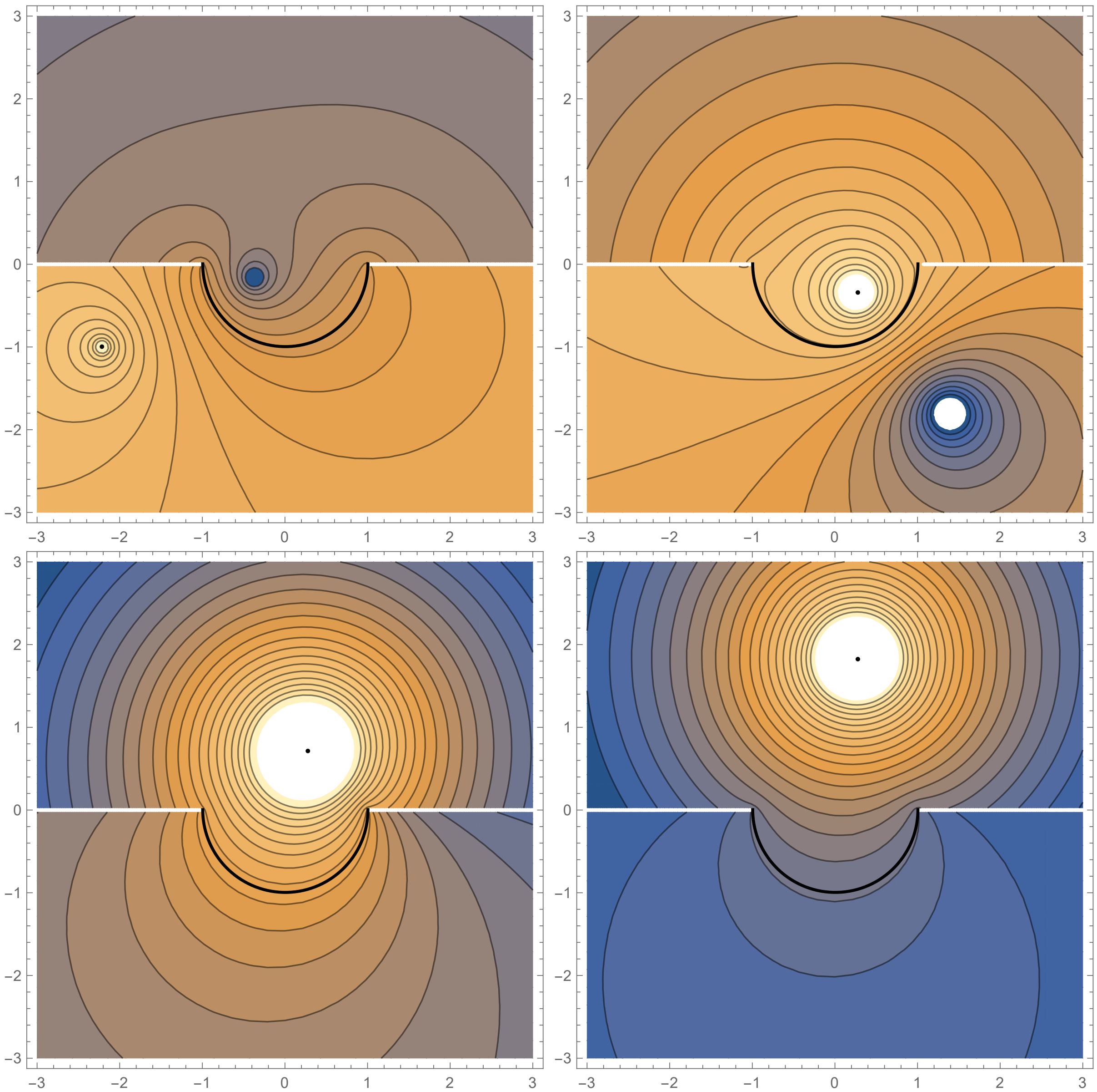
FIG. 3. $\alpha = 1, \beta = 2$, black: (x, y), red: (u, v). In fact, the interior (outside) of the unit circle is mapped to the region outside (inside) the ellipse.

The potential is more likely to be solved in (x, y), but this requires in the region we are interested the inverse mapping is a **single-valued** function (**hence it's much more desirable to conforamlly map to a region that is suitable for solving the poisson equ**). If the potential is solved in (x, y), to transform back to (u, v) we need $x = x(u.v), y = y(u, v)$, whose analytical form is however ugly, given by $\alpha z^2 - wz + \beta = 0$. Note the $\alpha > \beta$ version makes it possible to solve for a point charge outside an elliptical conductor (with arbitrary charge).

What if the boundary condition is incomplete? As a concrete example, consider the conformal mapping $w = F(z) = \frac{1}{2}(i + z + i\sqrt{1 - z^2})$. The authors of Is the electrostatic force between a point charge and a neutral metallic object always attractive? claimed this maps the region $x^2 + y^2 \neq 1 \cup y > 0$ (i.e., outside a semi-circle) into $u^2 + v^2 > 1/2$, hence use it to solve the problem of a neutral lower half circular conductor with a charge (and shows that the electrostatic force between a point charge and a neutral circular conductor can be repulsive, e.g., if the charge is at $z < 0$). But really? If so, the charge can be placed at any $Z = X + iY$, which is mapped to $W = U + iV$, with radius larger than $1/\sqrt{2}$, so the potential is

$$V(w) = -q \ln |w - W| + q \ln \left| w - \frac{1/2}{W^*} \right| - q \ln |w| \tag{8}$$

and $V(z) = V(F(z))$ with W replaced by $F(W)$ and W^* replaced with $(F(W))^*$. But it's simply not the case, e.g., $|F(-2I)| < 1/\sqrt{2}$. Or directly plotting the potential:



mma script:

```
F[z_] = (I + z + I*Sqrt[1 - z^2])/2;
\[Phi][z_, W_] =
Log[Abs[(F[z] - 1/2/Conjugate[h[W]])/((h[z] - h[W]) h[z])]];
Manipulate [
Show[ContourPlot[\[Phi][x + I y, X + I Y], {x, -3, 3}, {y, -3, 3},
Contours -> 30, PlotLegends -> Automatic],
Graphics[{Thick, Circle[{0, 0}, 1, {\[Pi], 2 \[Pi]}]}],
Graphics[{Point[{X, Y}]}]], {Y, -1, 3}, {X, -3, 3}]
```

In fact we find that actually only the region $x^2+y^2 < 1 \cup y > 0$ is mapped to $u^2+v^2 > 1/2$, in particular the boudary $x^2+y^2 = 1$ with $y < 0$ is mapped to a part of $u^2+v^2 = 1$, while the boundary $y = 0$ with $|x| > 1$ is mapped to part of $v = 1/2$. To see this,

```
RegionPlot[ImplicitRegion[Abs[F[x + I y]] > 1/Sqrt[2], {x, y}],
PlotRange -> {{-2, 2}, {-2, 2}}]
```

the result is

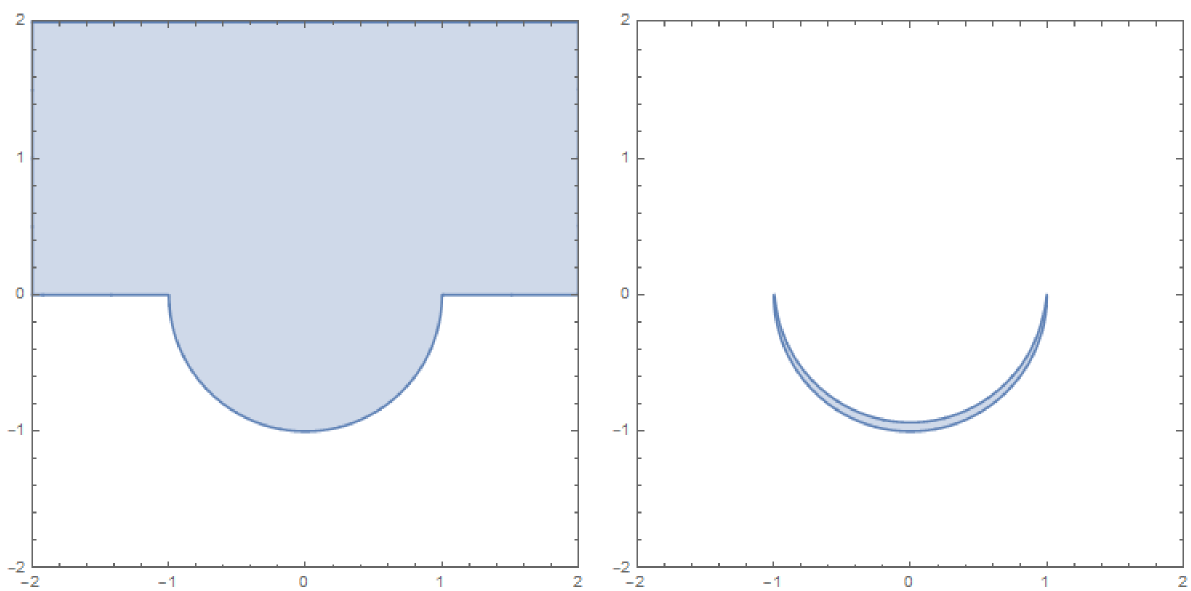
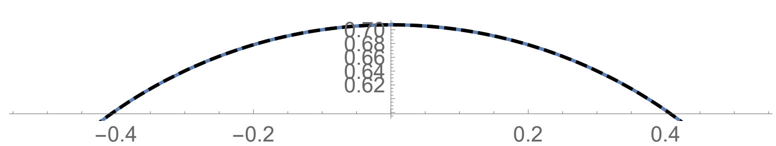


FIG. 4. Left: $|F(z)| > 1/\sqrt{2}$, right: $1/\sqrt{2} < |F(z)| < 1/\sqrt{2} + 0.01$.

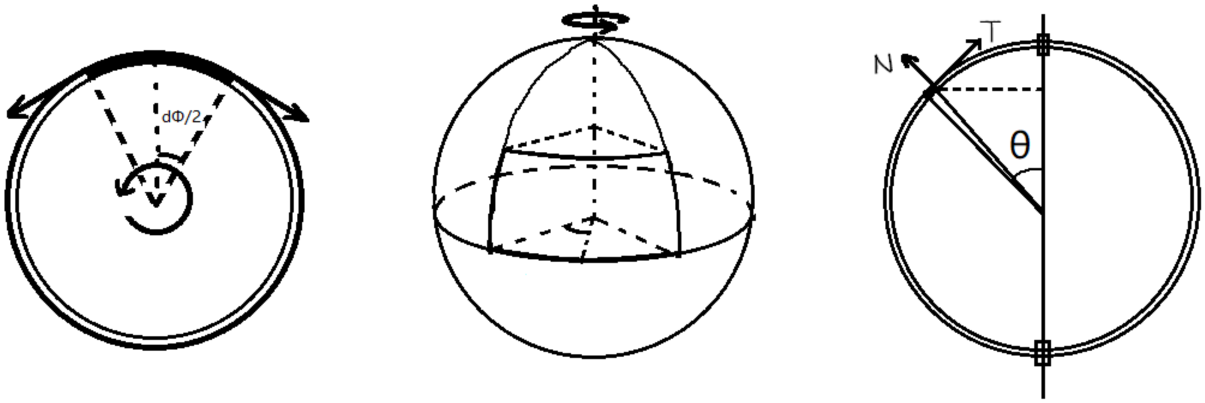
```
Show[ParametricPlot[ReIm[F[ Cos[t] + I Sin[t]]], {t, Pi, 2 Pi}],
ParametricPlot[{Cos[t], Sin[t]}/Sqrt[2], {t, 0, 2 Pi},
PlotStyle -> {Black, Dashed}]]
```

the result is



But in the problem on w plane, only the boundary condition at $u^2+v^2 = 1/2$ is specified. Hence we doubt that this solution is problematic. It is then curious that the analytical result of the aforementioned paper still matches numerical computation.

II. STRESS DISTRBIBUTION IN ROTATING 2D RIGID BODY



Consider a segment of a circular ring (with uniform line mass density ρ and radius R) with central angle 2θ, the distance between the mass center of this arc and center of the circle is $R\sin\theta/\theta$. This can be derived in a rather interesting way: consider a ring that is spinning about its axis (see the leftmost figure above) with constant angular velocity ω, there must be a uniform tangent stress T, so

$$2T\sin(d\theta) = 2Rd\theta\rho\omega^2 R \quad \Rightarrow \quad T = \rho\omega^2 R^2 \tag{9}$$

This is the stress distribution in a spinning ring. Now apply this result to a segment with finite θ, we have

$$2T\sin\theta = 2R\theta\rho\omega^2 R = 2\theta Tl/R \quad \Rightarrow \quad l = R\sin\theta/\theta \tag{10}$$

But we can also derive this result consistently from a circular ring spinning about its diameter (see the figure above). In this case, the stress distribution is a little more complicated since there should be both tangent and normal stress, we have

$$d(T\sin\theta + N\cos\theta) = 0, \quad d(-T\cos\theta + N\sin\theta) = \rho d\theta\omega^2\sin\theta \tag{11}$$

or

$$(T' - N)\sin\theta + (T + N')\cos\theta = 0, \quad (N - T')\cos\theta + (T + N')\sin\theta = \rho\omega^2\sin\theta \tag{12}$$

here, to be more general, we consider a density distribution $\rho(\theta)$. Then

$$T' - N = -\rho\omega^2\sin\theta\cos\theta, \quad T + N' = \rho\omega^2\sin^2\theta \tag{13}$$

The decoupled equaton for N is

$$N'' + N = \rho\omega^2 \sin\theta\cos\theta + \omega^2\partial_\theta(\rho\sin^2\theta) \tag{14}$$

Curiously, this has the form os "forced oscillation", for periodic time θ. If density is uniform, we have

$$N'' + N = \frac{3}{2}\rho\omega^2 \sin 2\theta \tag{15}$$

The solution for $y'' + py' + qy = e^{\lambda x}\sin\omega x$ with constant coeffients is well known, if $\lambda + i\omega$ is a solution of $r^2 + pr + q = 0$, the solution is $f(x) = xe^{\lambda x}(a\sin\omega x + b\cos\omega x)$, if not, the solution is $f(x) = e^{\lambda x}(a\sin\omega x + b\cos\omega x)$. In the present case, the general solution is

$$N(\theta) = a\sin 2\theta + b\cos 2\theta \tag{16}$$

Imposing the physical initial condition $N(0) = 0$, we obtain

$$N(\theta) = -\frac{1}{2}\rho\omega^2\sin 2\theta \quad \Rightarrow \quad T(\theta) = \rho\omega^2\cos^2\theta \tag{17}$$

Now, e.g., apply this result to a half circle $\theta \in (0, \pi)$,

$$2T(0) = \rho\pi\omega^2 l \quad \Rightarrow \quad l = 2R/\pi \tag{18}$$

which is consistent. For density $\rho(\theta) \propto \sin\theta$, we should be able to arrive at $l = \pi R/4$. Thus, splitting the half spherical shell into many semi-circles, the distance between the half-sphere's mass center to the spherical center is $\frac{2}{\pi}R \cdot \frac{\pi}{4} = R/2$.

However, we note that the stress distribution in a spinning spherical shell is ambigious, since there is no restrictions for azimuthal stress, e.g., the centrifugal force can be balanced solely by the azimuthal stress or completely without it.

The analysis is similar for other shapes, e.g, a triangle frame.

III. ZLZ'S RING AND NJX'S CHAIR

In ZLZ's ring, a ball is sealed inside a ring. The ball however has finite kinetic energy thus jumps and collide with the ring repeatedly. Seen from the outside (assuming the observer cannot see the inner ball), the ring appears to roll or even jumps "automatically" (this can be simulated conveniently with Algodoo). In contrast to a 1D system of this type, the horizontal motion of the ring's mass center can be made unidirectional, thus provides a special way to drive a wheel provided the collision process is efficient (perhaps?).

In a NJX's chair, one part of the whole system (called A) acquires a velocity while the other parts of system (called B) remains static. But A and B then collides, which makes B move too. Such a system may be realized in many ways.

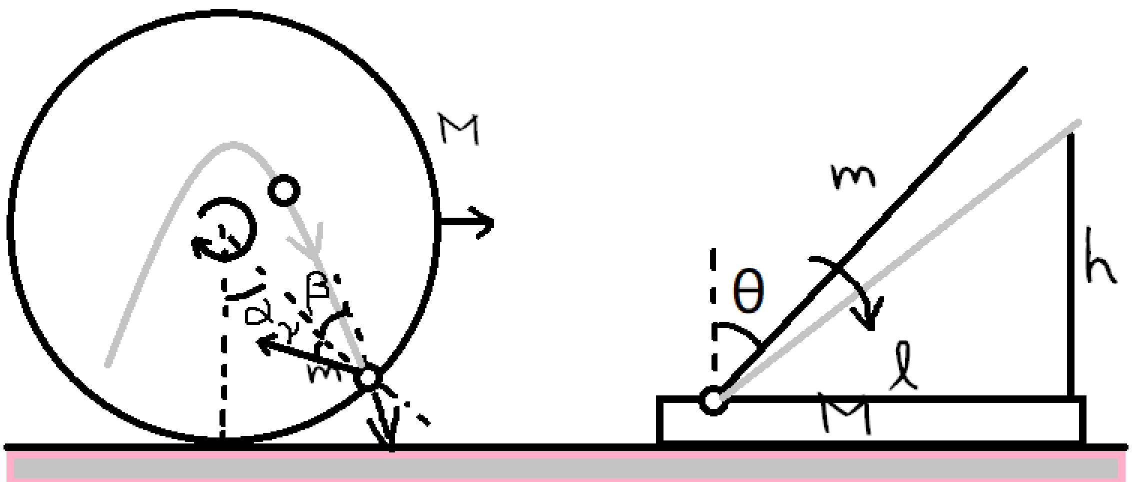


FIG. 5. Left: ZLZ's ring. Right: a NJX's chair.

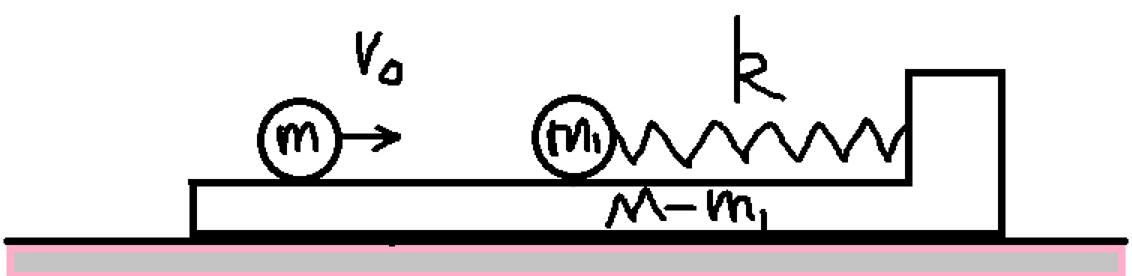


FIG. 6. Another NJX's chair, the ball with mass m (and no friction) is accelerated to velocity v_0 by an invisible hand attached to the base (the hand would certainly consumes energy, but the point is it does not affect other things, only the ball gets moving).

We study here specifically the system depicted in the figure above. Assuming a sudden collision, the external forces such as friction will has no effects, but we are interested in the critical condition for the base to move against friction, hence we model the finite-time response with a spring. The collision of m and m_1 is still assumed to be sudden and elastic, thus after the collison, the velofity of m_1 is

$$v_1 = \frac{2}{m_1/m+1} v_0 \tag{19}$$

Assuming the base is static, the maximum force exerted by the spring is

$$kx = k\sqrt{m_1 v_1^2/k} = \sqrt{km_1}\frac{2}{m_1/m+1} v_0 \tag{20}$$

As expected, the more rigid the spring, the larger the force.

IV. CLASSICAL LIMIT OF A NON-HERMITIAN SCHRODINGER EQUATION

The complex conjugation is a non-hermitian operator. Recall the defining condition of a hermitian operator A is $\langle a|A|b\rangle = \langle Aa|b\rangle$, but we have $\langle a| * |b\rangle = \langle a|b^*\rangle = (\langle a^*|b\rangle)^*$. Thus,

$$i\hbar\partial_t\phi = -\frac{1}{2}\nabla^2\phi + a\phi + b\phi^*, \quad a, b \in \mathbb{R} \tag{21}$$

is a non-hermitian Schrodinger equation. Now we consider its classical limit $\hbar \to 0$. First we introduce $\phi = \psi + i\varphi$ with $\psi, \varphi \in \mathbb{R}$, then the wave equation becomes

$$\hbar\partial_t\psi = \left[-\frac{1}{2}\nabla^2 + (a-b)\right]\varphi, \quad -\hbar\partial_t\varphi = \left[-\frac{1}{2}\nabla^2 + (a+b)\right]\psi \tag{22}$$

We consider here only the case that a, b are time-independent, then the decoupled equation for ψ is

$$-\hbar^2\partial_t^2\psi = \left[-\frac{1}{2}\nabla^2 + (a-b)\right]\left[-\frac{1}{2}\nabla^2 + (a+b)\right]\psi = \frac{1}{4}\nabla^4\psi - a\nabla^2\psi + \left[a^2 - b^2 - \frac{1}{2}\nabla^2(a+b)\right]\psi \tag{23}$$

Introduce $\psi = e^{iS/\hbar}$ with $S = \sum_{n=0}^{\infty}(i\hbar)^n S_n$, then we have $\partial_t\psi = (i/\hbar)S_{,t}\psi$, $\partial_t^2\psi = (i/\hbar)S_{,,t}\psi + (i/\hbar)^2 S_{,t}^2\psi$, and to zero-th order in $\hbar$, we find (use the mma script:

```
CoefficientList[
Simplify[(D[Exp[I*(S0[t] + I*h*S1[t] - h^2*S2[t] - I*h^3 S3[t] + h^4 S4[t])/
h], {t, 4}]/Exp[I*(S0[t] + I*h*S1[t] - h^2*S2[t] - I*h^3 S3[t] + h^4 S4[t])/h])*h^4],
{h}]
```

) $\nabla S_0 = 0$ (in h^0 order), and the dynamics starts at h^2 order, which reads

$$(\partial_t S_0)^2 = -\frac{3}{4}(\nabla^2 S_0)^2 + a|\nabla S_0|^2 \tag{24}$$

b shows up at h^4 order, hence we may conclude that in the classical limit b has no effects at all.

V. THREE-BODY PENROSE (DECAY) PROCESS IN KERR

In an axially symmetric spacetime:

$$ds^2 = g_{ab}dx^a dx^b = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi \tag{25}$$

The particle lagrangian is

$$\mathcal{L} = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b = \frac{1}{2}\left(g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{\phi\phi}\dot{\phi}^2 + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2\right) \tag{26}$$

where $\dot{x} \equiv dx/ds$, the conserved energy and angular momentum are then

$$E = -\frac{\partial\mathcal{L}}{\partial\dot{t}} = -g_{tt}\dot{t} - g_{t\phi}\dot{\phi}, \quad L = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = g_{\phi\phi}\dot{\phi} + g_{t\phi}\dot{t} \tag{27}$$

E.g., in the flat spacetime, $ds = i\sqrt{1-v^2}dt$, use $d\tau = -ids = \sqrt{1-v^2}dt$ instead of ds as affine parameter, $g_{tt} = -1$, we have

$$mE = \frac{m}{\sqrt{1-v^2}}, \quad p^\mu = \frac{mx^\mu_{,t}}{\sqrt{1-v^2}}, \quad (p^\mu)^2 = -m^2 \tag{28}$$

and it's more convenient to use Cartesian coordinates. In the Kerr spacetime, however, we'd better to stick to spherical (actually BL) coordinates,

$$g_{tt} = -1 + \frac{2Mr}{r^2 + a^2\cos^2\theta}, \quad g_{\phi\phi} = \sin^2\theta\left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right), \quad g_{t\phi} = -\frac{2Mra\sin^2\theta}{r^2 + a^2\cos^2\theta} \tag{29}$$

on equilateral plane $\theta = \pi/2$,

$$g_{tt} = -1 + \frac{2M}{r}, \quad g_{\phi\phi} = r^2 + a^2 + \frac{2Ma^2}{r}, \quad g_{t\phi} = -\frac{2Ma}{r} \tag{30}$$

In flat spacetime, the analysis of three-body decay $m_0 \to m_1 + m_2 + m_3$ is already a headache. Assuming the decaying body is at rest (i.e., in the COM frame, here M is momentum), the (relativistically) invariant mass $m_{ij} = (p_i^\mu + p_j^\mu)^2 = (p_0^\mu - p_k^\mu)^2 = -m_0^2 - m_k^2 - 2p_0^\mu \cdot p_k^\mu$ satisfying $m_{12}^2 + m_{23}^2 + m_{13}^2 = -m_0^2 - \sum_i m_i^2$, the momentum phase space plot for the possible final states in (m_{12}, m_{23}) plane is known as Dalitz diagram, the region of possible (m_{12}, m_{23}) depends on m_0, m_i.

On the other hand, in the two-body elastic decay penrose process, the two decay products are assumed to be of identicle mass (what if not?) and the focus is on the energy partition of decay products. In Kerr spacetime, we have

$$r^2\dot{r}^2 = r^2E^2 + \frac{2M}{r}(aE - L)^2 + \left(a^2E^2 - L^2\right) - \delta\Delta, \quad \Delta = r^2 + a^2 - 2Mr \tag{31}$$

where $\delta = 1, 0$ for time-like (i.e., massive) and null (i.e., massless) geodesics, respectively, M the BH mass, a the BH spin ($L>0$ is called corotating, $E < 0$ will fall into hoizon). Hence the initially rest particle can only exist in the turning point $\dot{r} = 0$, and the usual analysis focus on equitorial plane and even the decay products have $\dot{r} = 0$ (why?). Instead of the two-body decay, one can also conisder the collisional penrose process.

Now, even in the Kerr spacetime, the three-body decay should repect energy-momentum conservation law. Assuming the decay is elastic, and in the turning point r_0 on equitorial plane (so only two of the 4-momentum are relevant), we have

$$m_0 E_0 = \sum_i m_i E_i, \quad m_0 L_0 = \sum_i m_i L_i \tag{32}$$

Here we just count the DOFs (assuming all m_i are given). For three-body decay in flat spacetime, there are $3 \times 3 = 9$ DOF for the final states with 3 DOF redundant (due to the rotational symmetry of spacetime), and $1 + 3$ constraints, hence only $9 - 7 = 2$ DOF, as plotted in Dalitz diagram. For the aforementioned resticted case of two-body decay in Kerr spacetiem, $2 \times 2 = 4$ DOF for the final states, 2 constraint from conservation, there are 2DOFs; but if imposing the turning point condition the E and L are then not independent, and there remains no DOF. Now for the resticted case of there-body decay, $3 \times 2 = 6$ DOF for the final states, 2 constraints from conservation, with 3 extra turning point constraints, we would have one DOF for the final states.

We can define a GR version of the invariant mass: $m_{ij} = (p_i^\mu + p_j^\mu)^2$, where $p_a \equiv m \frac{\partial \mathcal{L}}{\partial x^a} = m g_{ab} \dot{x}^b$, then

$$p^a p_a = m^2 (g_{ab} \dot{x}^b)(g^{ac} \dot{x}_c) = m^2 \dot{x}^b \dot{x}_c = m^2 \tag{33}$$

(or if choosing the proper time as the affine parameter, $p^2 = -m^2$) and all is same as in flat spacetime.

VI. TWO-BODY DECAY PENROSE PROCESS IN PC-KERR

For pc-Kerr metric, on equilateral plane,

$$g_{tt} = -1 + \frac{2M}{r} - \frac{f(r)}{r^2}, \quad g_{\phi\phi} = r^2 + a^2 + \frac{2Ma^2}{r} - \frac{a^2 f(r)}{r^2}, \quad g_{t\phi} = -\frac{2Ma}{r} + \frac{af(r)}{r^2} \tag{34}$$

and

$$g_{rr} = \frac{r^2}{r^2 - 2Mr + a^2 + f(r)} \tag{35}$$

where $f(r) = \frac{B_n}{(n-1)(n-2)r^{n-2}}$. To repeat the Kerr analysis, we need the turning point condition. Note

$$\dot{\phi} = \frac{g_{tt}L + g_{t\phi}E}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}, \quad \dot{t} = \frac{E + g_{t\phi}\dot{\phi}}{-g_{tt}} = -\frac{g_{t\phi}L + g_{\phi\phi}E}{g_{tt}g_{\phi\phi} - g_{t\phi}^2} \tag{36}$$

The particle lagrangian on equilateral plane reads

$$\mathcal{L} = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b = \frac{1}{2}\left(g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{\phi\phi}\dot{\phi}^2 + g_{rr}\dot{r}^2\right) = AE^2 + BL^2 + CLE + \frac{1}{2}g_{rr}\dot{r}^2 \tag{37}$$

where

$$A = \frac{g_{\phi\phi}}{2(g_{tt}g_{\phi\phi} - g_{t\phi}^2)}, \quad B = \frac{g_{tt}}{2(g_{tt}g_{\phi\phi} - g_{t\phi}^2)}, \quad C = g_{t\phi} \tag{38}$$

Now we determine the turning-point condition $\dot{r}^2 = 0$ at $r = r_0$, i.e.,

$$\mathcal{L} = \frac{1}{2} = AE^2 + BL^2 + CLE \tag{39}$$

which relates E and L,

$$L = \frac{-CE \pm \sqrt{(C^2 - 4AB)E^2 + 2B}}{2B} \tag{40}$$

Now, imposing energy-momentum conservation:

$$m_0E_0 = m(E_1 + E_2), \quad m_0L = m(L_1 + L_2) \tag{41}$$

These apply to general axial-symmetric specetime. Wlog we can take $E_0 = 1$, the sign of L brings uncertainty. Yet, observe that adding f(r) is simply equivalent to change $M \to M' = M - \frac{f}{2r_0}$, hence we use directly the result for Kerr:

$$E_1 = \frac{1}{2}\left(1 \pm \sqrt{\frac{2M'\left(1 - 4m^2/m_0^2\right)}{r_0}}\right), \quad E_2 = \frac{1}{2}\left(1 \mp \sqrt{\frac{2M'\left(1 - 4m^2/m_0^2\right)}{r_0}}\right) \tag{42}$$

There will be a gain in energy at infinity provided that the turning point satisfies

$$r_0 < 2M'\left(1 - 4m^2/m_0^2\right) < 2M' \tag{43}$$

note $2M'$ is the radius of ergosurface of Kerr BH with mass M', so for Kerr case this means the decay happends inside ergoregion. and the maximum gain of energy $\eta = E_{1+}/E_0$ is obtained at horizon. Yet, in the present case the horizon is given by $r^2 - 2Mr + a^2 + f = 0$ and the ergosurface by $-1 + 2M/r - f/r^2 = 0$. Explicitly,

$$\eta = E_{1+} = \frac{1}{2}\left(1 + \sqrt{2(1 - 4m^2/m_0^2)\left(\frac{M}{r_0} - \frac{f(r_0)}{2r_0^2}\right)}\right) \tag{44}$$

For pc-Kerr, $f(r) = \frac{b_n M^n}{(n-1)(n-2)r^{n-2}}$, it was found that if $\chi = a/M = 0$, for n=3, the horizon only exists for $b_n \in (0, 64/27)$ and for n=4, for $b_n \in (0, 81/8)$ (for finite BH spin, the parameter space for which horizon exits is smaller). The comparison with Kerr for $m = 0$ (decaying into photons) is showed in the figure below, with $x \equiv r_0/M$.

```
n=3;
b=1.5;
chi=0.5;
xp=Max[x/.NSolve[(x^2+chi^2-2x)+b/((n-1)(n-2)x^(n-2))==0&&x>0,x]]
xe=Max[x/.NSolve[-1+2/x-b/((n-1)(n-2)x^n)==0&&x>0,x]]

Show[Plot[{1/2 (1+Sqrt[2 1/x]),1/2 (1+Sqrt[2(1/x-b/(2(n-1)(n-2)x^n))])},
{x,0,3}, AxesLabel->{"x","eta"},
PlotLegends->{"Kerr","pc-Kerr"},PlotStyle->{Black,Red}],
ParametricPlot[{{1+Sqrt[1-chi^2],y},{2,y}},{y,0,2},PlotStyle->Black],
ParametricPlot[{{xp,y},{xe,y}},{y,0,2},PlotStyle->Red]]
```

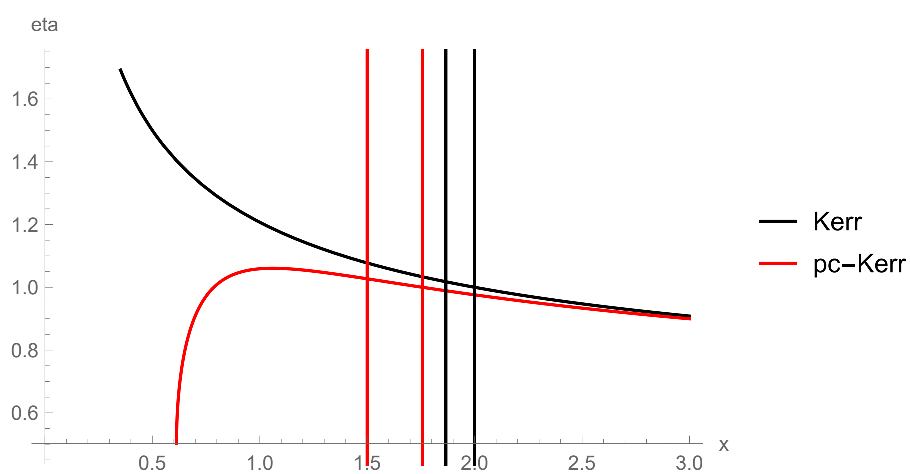
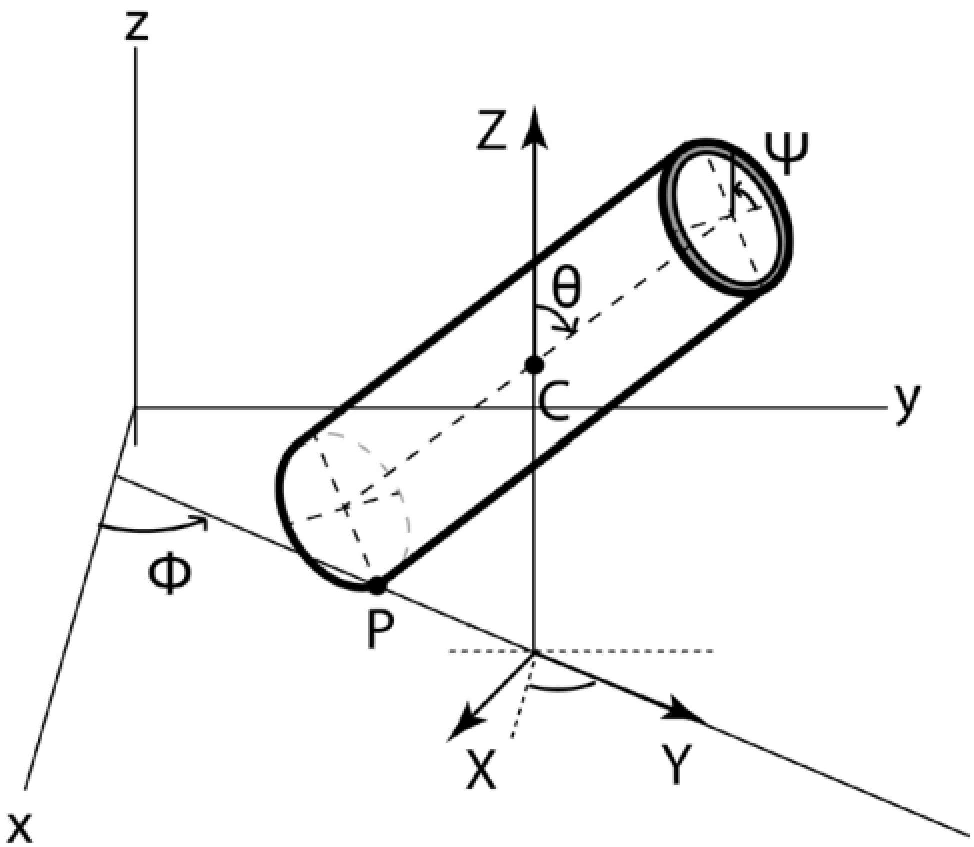


FIG. 7. $n = 3, b = 1.5, \chi = 0.5$. The vertical lines mark the horizon and ergosurface.

It is found that in pc-Kerr background, the energy gain is supressed.

One can also study the collisional Penrose process.

VII. RIGID CYLINDER ON TABLE



Concerning the motion of a rigid homogeneous cylinder without slip on a horizontal table, two questions remain unclear:

- How does the mass center move? The particularly interesting case is a coin initially with $\theta \sim \pi/2$ and pure precession.
- Would the cyliner unavoidably slide? When would this happen? (the critical condition is when the required friction grows to $\mu_{\rm crit} N$)

VIII. LANDAU LEVEL IN STEP-LIKE UNIFORM MAGNETIC FIELD

So we consider such a magnetic field: $\mathbf{B}(\mathbf{r}) = \mathbf{B}(x) = B(x)\mathbf{e}_z$ with $B(x) = B_1\Theta(-x) + B_2\Theta(x)$, since $B_z = \partial_x A_y - \partial_y A_x$, this can be generated from, e.g.,

$$A_x = A_z = 0, \quad A_y = [B_1\Theta(-x) + B_2\Theta(x)]x = \frac{B_1 + B_2}{2}x + \frac{B_2 - B_1}{2}|x| \tag{45}$$

Note as a more smooth version, we can use

$$B(x) = \frac{B_1 + B_2}{2} + \frac{B_2 - B_1}{2}\tanh\frac{x}{\sigma}, \quad A_y = \frac{B_1 + B_2}{2}x + \frac{B_2 - B_1}{2}\sigma\ln\left(\cosh\frac{x}{\sigma}\right) \tag{46}$$

Anyway, the Pauli equation without spin (in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$) reads

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + q^2A^2 + i2q\hbar\mathbf{A}\cdot\nabla\right)\psi = i\hbar\partial_t\psi = E\psi \tag{47}$$

If $B_1 = B_2$, i.e., a uniform magnetic field, we have (with $-i\hbar\partial_y = p_y$)

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{q^2B^2}{2m}x^2 - \frac{p_yqB}{m}x\right)\psi = i\hbar\partial_t\psi = E\psi \tag{48}$$

The quantized level of x-motion is known as Landau level. But for a step-like configuration, the potential is

$$V = \frac{q^2}{2m}\left(\frac{B_1 + B_2}{2}x + \frac{B_2 - B_1}{2}|x|\right)^2 - \frac{p_yq}{m}\left(\frac{B_1 + B_2}{2}x + \frac{B_2 - B_1}{2}|x|\right) \tag{49}$$

which contains, besides harmonic terms $ax^2 + bx$, the $|x|$, $x|x|$ terms. As an extreme case consider $B_2 = B = -B_1$, then

$$V = \frac{q^2B^2}{2m}x^2 - \frac{p_yqB}{m}|x| \tag{50}$$

Further we consider the limit $|x| \to 0$, then

$$V(x) = -F|x|, \quad F = \frac{p_yqB}{m} \tag{51}$$

What are the bound states?