Keyhole modeling for the temperature field. This is some of the first trials for the simple source terms that include different point and line moving sources. The basics are explained inside the book “Keyhole Welding: The Solid and Liquid Phases” by Alexander Kaplan.

Steady State 3d Solutions based on Moving Point Sources of Heat:

In[∘]:=

```
A = 0.4; (*absorptance*)
PL = 150; (*power*)
K = 20;  (* conductivity *)
κ =  10; (* thermal diffusivity *)
Ta = 298; (* ambient temperature *)
T[x_, y_, z_] :=
  Ta + 2 A PL / K Sqrt[x^2 + y^2 + z^2] × Exp[-v[x + Sqrt[x^2 + y^2 + z^2]] / (2 κ)]
```

Rosenthal solution for the line source model

The solution for the temperature field T(r,ϕ) in the polar coordinates is given by the analytical solution:

T(r,ϕ) = $T_a + P'/2\,\pi\lambda_{th}\,K_o(\mathrm{Pe} * r)\,\exp(-\mathrm{Pe} * r * \cos(\phi))$ where Pe is modified Peclet number:

Pe = v/2κ, T_a - ambient temperature, λ_{th} is the thermal diffusivity, $K_0()$ is the modified Bessel function

of second kind and zeroth order. The temperature has a singularity at the origin, where the line source

with power per unit depth P is located.

The previous equation can be calculated to explicitly give the strength P’ which is necessary to reach evaporation of the arbitrary point in polar coordinates.

P’(r,ϕ) = $(T_v - T_a)\,2\,\pi\lambda_{th}\,\frac{1}{K_o(\mathrm{Pe} * r)}\exp(\mathrm{Pe} * r\cos(\phi))$

The heat flow is determined by Fourier’s law of heat conduction that can be simplified by considering only of the radial component:

$q = -\lambda_{th}\,\nabla T = -\lambda_{th}\,\frac{\partial T}{\partial r}$

Spatial derivation of the temperature of the Rosenthal solution with respect to r leads to:

$\partial T/\partial r = P'(r,\ \phi)/(2\,\pi\lambda_{th})\,\mathrm{Pe}'[-K_0(\mathrm{Pe}'\,r)\cos(\phi) + (K')_0\,(\mathrm{Pe} * r)]\exp(-\mathrm{Pe}\,r\cos(\phi))$

$K'_o(x) = -K_1(x)$

q(r,ϕ) = $-\lambda_{th}\partial T/\partial r$ = P’(r,ϕ)/2π $\mathrm{Pe}'\,(K_0(\mathrm{Pe}'\,r)\cos(\phi) + K_1(\mathrm{Pe}'\,r))\exp(-\mathrm{Pe}'\,r\cos(\phi))$

If we consider the boundary condition that the evaporation temperature Tv shall be reached at keyhole wall.

$q_v(r,\ \phi) = (T_v - T_a)\,\lambda_{th}\,\mathrm{Pe}'\,(\cos(\phi) + K_1(\mathrm{Pe}'\,r)/K_0(\mathrm{Pe}'\,r))$

Keyhole profile in the x-z plane only the azimuthal angle ϕ=0,π are of interest, describing the heat

loses of any point xf at the front wall (ϕ=0)
$q_v(x_r)$=$(T_v - T_a)\lambda_{\text{th}}$ Pe'(-1 + K_1(Pe' x_r)/K_0(Pe' x_r))

In[ ]:=
```
Pe = v / (2 κ);
qvf[xf_] := (Tv - Ta) λ Pe (-1 + BesselK[1, Pe * xf]) / BesselK[0, Pe xf];
qvr[xr_] := (Tv - Ta) λ Pe (1 + BesselK[1, Pe * xr]) / BesselK[0, Pe xr];
```

The intensity of the laser beam with only Fresnel absorption:

In[ ]:=
```
I0 = 2 PL / (rf0^2 π);
Intensity[x_, z_] := I0 * (rf / rf0)^2 Exp[-2 r^2 / rf^2];
rf0 = 2 F * M^2 / π;
```

Beam radius is varying over the depth by:

In[ ]:=
```
zr = 2 rf0 * F;
rf[z_] := rf0 (1 + ((z - z0)/zr)^2)^(1/2)
```

In[ ]:=

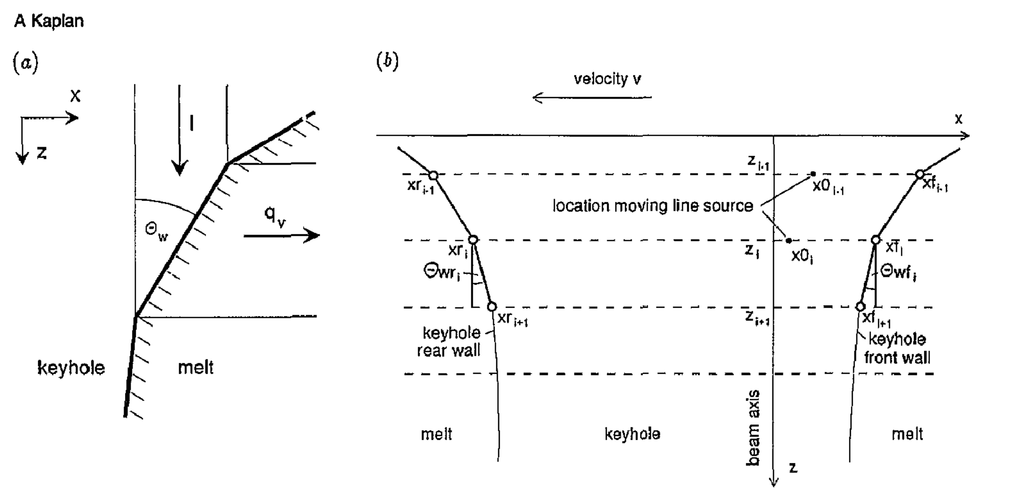

Figure 1. (*a*) Resulting keyhole wall angle θ_w due to the relation of the incident beam intensity I to the heat flow necessary for evaporation q_v; (*b*) pointwise wall calculation $x(z_i)$.

Out[ ]=

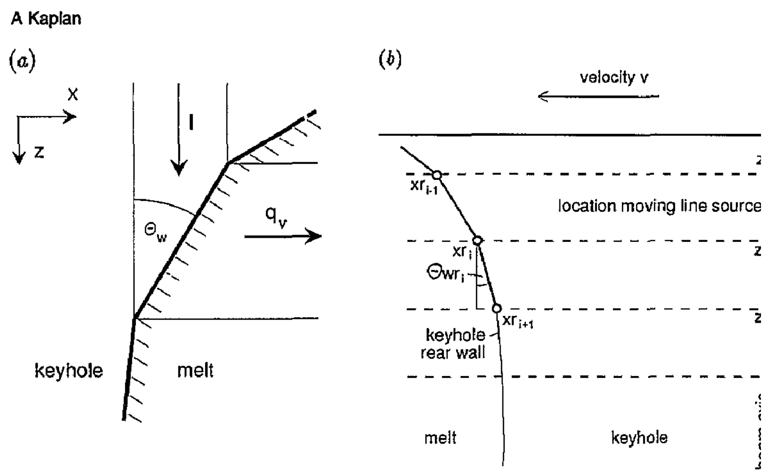

Figure 1. (*a*) Resulting keyhole wall angle θ_w due to the relation of the incident beam intensit
necessary for evaporation q_v; (*b*) pointwise wall calculation $x(z_i)$.

According the figure local energy balance at the keyhole wall is given by:
$\tan(\theta_w) = q_v / I_a = f(x, z)$
which we will give angle θ_w. I_a is the intensity at the wall.

For further absorption we will have additional plasma absorption due to Bremsstralung and

Fresnel absorption during multiple reflections Iamr:

```
In[ ]:= tanθw = (qv[x, z] - IaiB[x, z] - Iamr[x, z]) / IaFr[θw];
```

FindRoot : Failed to converge to the requested accuracy or precision within 100 iterations.

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Algorithm for the calculation

For every z_i at keyhole wall calculate downwards for both walls:

Calculate new x (r) coordinate from the previous step.

For every slice calculate the location of the moving line source (which is different than beam axis)

Distance x_f and x_r decide about the position of the line source x_0 to satisfy the general equation for the qv flow.

Calculate the new θ_w from the absorbed power to the laser intensity Ia.

The algorithm is finished when back and front wall start cross.

This calculation should be done two times. The first time only with I_a we get the first description of the keyhole. The second run then estimate different absorption mechanisms and yields the final keyhole profile.

```
In[ ]:= Clear[x];

In[ ]:= rf0 = 2 * 1064 × 10^-6 F M^2 / π

Out[ ]= 0.0203209
```

In[•]:=
```
(* laser power *)
I0 = 2 PL / (rf0^2 π);
rf0 = 2 wave F M^2 / π;
(* absorption by the vapour plume / damping by plasma plume *)
αpl = 1 - Exp[-αiB * hpl];
(* the first run plasma absorption by Bremsstralung *)
αiB1 = 1 - Exp[-αiB d / 2];
(*partial absorption of the intensity*)
αiBmr = 1 - Exp[-αiB 3 d / 2];


Clear[Iamrl, IaiBl];
calculateBremsMRefl[x_List, z_, θ_List] :=
  Module[{αpl = αpl, αiB1 = αiB1, αFr = αFr, αiBmr = αiBmr},
   zr = 2 rf0 * F;
   rf[z] := rf0 (1 + ((z - z0)/zr)^2)^(1/2);

   Ia[x, z] := I0 * (rf[z] / rf0)^2 Exp[-2 x^2 / rf[z]^2];
   Iamrl[x_, z, α_] := (1 - αpl) (1 - αiB1) * (1 - αFr) α * Ia[x, z];
   IaiBl[x_, z, α_] :=
    (1 - αpl) (αiB1 + αiBmr * (1 - αiB1) (1 - αFr) (1 - α) ) * Ia[x, z];
   (* calculation of the reflection with average angle θ *)
   ρmr = ρFr ^ (π / (4 θ) - 1);
   αmr = 1 - ρmr;
   MapThread[(Iamrl[#1, z, #2] + IaiBl[#1, z, #2]) &, {x, αmr}]
  ];
```

Calculate the absorption of the laser:

In[•]:=
```
calculateAbsorption[x_List, z_, θ_List, power_] :=
  Module[{I0 = 2 power / (rf0^2 π), αpl = αpl , αiB1 = αiB1},
   zr = 2 rf0 * F;
   rf[z] := rf0 (1 + ((z - z0)/zr)^2)^(1/2);
   Ia[x_, z] := I0 * (rf[z] / rf0)^2 Exp[-2 x^2 / rf[z]^2];
   (*Ia[x,z]:= I0 * Exp[-2 x^2/rf[z]^2];*)
   IaFr[x_, z, α_] := (1 - αpl) (1 - αiB1) * αFr[π / 2 - α] * Ia[x, z];
   MapThread[IaFr[#1, z, #2] &, {x, θ}]
  ];
```

Find the power for particular position of the keyhole wall, it returns two results.

In[ ]:=
```
Clear[findPower];
findPower[v_, κ_, xk_List, Ta_, Tv_, 𝒜_, d_, λ_] :=
 Module[{xr = xk[[1]], xf = xk[[2]], x0},
  Pl[r_, θ_] := (Tv - Ta) 2 π λ d / 𝒜 1/BesselK[0, v r / (2 κ)] Exp[v r Cos[θ] / (2 κ)];
  fr = FindRoot[Pl[Abs[xr - x0], π] == Pl[xf - x0, 0], {x0, xr, xf}];
  {Re[Pl[xf - x0 /. fr, 0]], Re[x0 /. fr]}]
```

Calculate modified Peclet number for the problem:

In[ ]:=
```
Pe = v / (2 κ);
```

Calculate the front and back heat fluxes based on the Rosental's solution:

In[ ]:=
```
qvf[xf_] := (Tv - Ta) λ Pe (1 + BesselK[1, Pe * xf]) / BesselK[0, Pe xf];
qvr[xr_] := (Tv - Ta) λ Pe (-1 + BesselK[1, Pe * xr]) / BesselK[0, Pe xr];
```

Do simplified version of the above functions:

In[ ]:=
```
qvcr[rkh_] := (Tv - Ta) λ κ / 2 (v * rkh / (2 κ))^-0.7;
qvcf[rkh_] := (Tv - Ta) λ κ / 2 (2 + (v * rkh / (2 κ))^-0.7);
```

Tricky part find the rear and front position of the keyhole in x, z plane by finding were the temperature reaches evaporation temperature.Note this is the moving line source and d-parameter is representing depth/thickness of the plate.

In[ ]:=
```
findRearAndFront[v_, κ_, Ta_, Tv_, 𝒜_, power_, xk_List, off_, d_, λ_] :=
  Module[{xrstart = -2, xfstart = (xk[[1]] + xk[[2]]) / 2
    , xfmin = xk[[1]], xfmax = xk[[2]], r, θ},
   Tn[x_, y_] :=
    Ta + 𝒜 power/(2 π λ d) Exp[-v x / (2 κ)] × BesselK[0, v * Norm[{x, y}] / (2 κ)];
   (* find point where temperature is Tv *)
   xr = NSolve[Tn[x, 0] == Tv, x, Reals];
   {x /. xr[[1]], x /. xr[[2]]}
  ];
```

Test some of the previous functions :

In[ ]:=
```
x1 = {-0.5511059201951887, -0.004411116479318331`};
p1 = 2660.3611639948417;
off1 =-0.11137274624218212;
```

In[ ]:=
```
xl = findRearAndFront[v, κ, Ta, Tv, αFr[π / 2 - 0.1], 4000, {-3, 1}, off1, d, λ]
```

NSolve : NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

NSolve : Inverse functions are being used by NSolve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]=
```
{-0.0423758, 0.0311167}
```

In[ ]:=
```
calculateAbsorption[
 {-0.0015122920605260045`, 0.0014746971476816383`}, 0.02, {π / 2, π / 3}, 4000]
```
Out[ ]=
```
{6027.63, 6078.23}
```

In[ ]:=
```
calculateBremsMRefl[x1, 2, {0.3, 0.4}]
```
Out[ ]=

$$\left\{0.980199\,\mathrm{Ia}[-0.551106, 2]\left(0.240428 + 0.426701\left(1 - 0.5\left(\frac{59.2\,\mathrm{Cos}[\theta]}{459.29 + 29.6\,\mathrm{Cos}[\theta] + \mathrm{Cos}[\theta]^2} + \frac{59.2\,\mathrm{Cos}[\theta]}{1 + 29.6\,\mathrm{Cos}[\theta] + 459.29\,\mathrm{Cos}[\theta]^2}\right)\right)^{1.61799}(1 - \alpha_{\mathrm{Fr}})\right) + 0.744532\left(1 - \left(1 - 0.5\left(\frac{59.2\,\mathrm{Cos}[\theta]}{459.29 + 29.6\,\mathrm{Cos}[\theta] + \mathrm{Cos}[\theta]^2} + \frac{59.2\,\mathrm{Cos}[\theta]}{1 + 29.6\,\mathrm{Cos}[\theta] + 459.29\,\mathrm{Cos}[\theta]^2}\right)\right)^{1.61799}\right)\mathrm{Ia}[-0.551106, 2]\,(1 - \alpha_{\mathrm{Fr}}),\ 0.980199\,\mathrm{Ia}[-0.00441112, 2]\left(0.240428 + 0.426701\left(1 - 0.5\left(\frac{59.2\,\mathrm{Cos}[\theta]}{459.29 + 29.6\,\mathrm{Cos}[\theta] + \mathrm{Cos}[\theta]^2} + \frac{59.2\,\mathrm{Cos}[\theta]}{1 + 29.6\,\mathrm{Cos}[\theta] + 459.29\,\mathrm{Cos}[\theta]^2}\right)\right)^{0.963495}(1 - \alpha_{\mathrm{Fr}})\right) + 0.744532\left(1 - \left(1 - 0.5\left(\frac{59.2\,\mathrm{Cos}[\theta]}{459.29 + 29.6\,\mathrm{Cos}[\theta] + \mathrm{Cos}[\theta]^2} + \frac{59.2\,\mathrm{Cos}[\theta]}{1 + 29.6\,\mathrm{Cos}[\theta] + 459.29\,\mathrm{Cos}[\theta]^2}\right)\right)^{0.963495}\right)\mathrm{Ia}[-0.00441112, 2]\,(1 - \alpha_{\mathrm{Fr}})\right\}$$

In[ ]:=
```
findPower[v, κ, xl, Ta, Tv, αFr[π / 2 - 0.1], d, λ]
```
Out[ ]=

$$\{4000.,\ -1.15221 \times 10^{-17}\}$$

In[ ]:=
```
qv = {qvr[Abs[#1]], qvf[#2]} & @@ xl
```
Out[ ]=
```
{956.198, 1306.86}
```

In[ ]:=
```
Ia[#, 0] & /@ xl
```
Out[ ]=
```
{Ia[-0.153749, 0], Ia[-0.0802561, 0]}
```

In[ ]:=
```
IaFr[Abs[#], 0] & /@ xl
```
Out[ ]=
```
{{0.140051 Ia[0.153749, 0], 0.0868871 Ia[0.153749, 0]},
 {0.140051 Ia[0.0802561, 0], 0.0868871 Ia[0.0802561, 0]}}
```

Here is the main algorithm to calculate the shape of the keyhole :

```
δz = 0.005; 𝒳 = {}; 𝒵 = {}; 𝒪 = {}; 𝒫 = {}; ℛ = ℛc = {}; Θ = {};
(* initialize parameters *)
power = PL; offset = 0; eps = 10^-3; xkh = {-1, 0.5}; θ = {0.01, 0.01};
xkh = findRearAndFront[v, κ, Ta, Tv, αFr[π / 2 - 0.1], power, xkh, offset, d, λ];
Print["Walls:", xkh];
rkh = xkh⟦2⟧ - xkh⟦1⟧;
For[step = 0, step < 1, step = step + 1,
 Print["Step: ", step];
 For[z = 0, z < d, z = z + δz,
  (* xkh = findRearAndFront[v,κ,Ta,Tv, αFr,power, xkh, offset, d, λ]; *)
  (* xkh =
    findRearAndFront[v,κ,Ta,Tv, αFr[π/2-0.1],power, xkh, offset, d, λ]; *)
  If[Max[Abs[xkh]] > 10, Break[]];
  (* calculate the shape of keyhole, the heat losses *)
  (* qv = {qvcr[Abs[rkh]], qvcf[rkh]}; *)
  qv = {Re[qvr[Abs[#1]]], Re[qvf[#2]]} & @@ xkh;

  (* tanθw =( qv-IaiB[x,z]-Iamr[x,z] )/IaFr[x,z]; *)
  Print["Values:" , qv];
  Ifrens = calculateAbsorption[Abs[xkh], z, θ, power];
  (* Print[Ifrens,z,xkh]; *)
  If[step == 1,
   (*Ic = calculateBremsMRefl[xkh,z,θ]; Print[Ic, θ]*),
   Ic = {0.0, 0.0}];
  tanθw = qv / Ifrens;
  xkh = xkh - δz * Re[tanθw];
  (*  Print["New:",xkh,{offset,power}]; *)
  rkh = xkh⟦2⟧ - xkh⟦1⟧; xc = rkh / 2.0;

  avθ = Mean[θ];
  {power, offset} = findPower[v, κ, xkh, Ta, Tv, αFr[π / 2 - avθ], d, λ];
  θ = ArcTan[tanθw];
  AppendTo[Θ, θ];
  AppendTo[𝒳, xkh]; AppendTo[𝒵, z];
  AppendTo[𝒪, offset]; AppendTo[𝒫, power]; AppendTo[ℛ, rkh];
  AppendTo[ℛc, xc];
  (* curvature of keyhole closes *)
  If[ xkh⟦1⟧ > xkh⟦2⟧, Break[] ];] ]
```

NSolve : NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

NSolve : Inverse functions are being used by NSolve, so some solutions may not be found; use Reduce for complete solution information.

```
Walls:{-0.0901656, 0.0554101}

Step: 0

Values:{630.881, 1233.71}
```

```
Values:{628.922, 1238.9}
Values:{627.77, 1242.46}
Values:{626.539, 1245.96}
Values:{625.319, 1249.48}
Values:{624.101, 1253.06}
Values:{622.884, 1256.69}
Values:{621.67, 1260.38}
Values:{620.458, 1264.13}
Values:{619.249, 1267.94}
Values:{618.042, 1271.81}
Values:{616.837, 1275.75}
Values:{615.635, 1279.75}
Values:{614.434, 1283.82}
Values:{613.237, 1287.96}
Values:{612.041, 1292.17}
Values:{610.848, 1296.46}
Values:{609.656, 1300.82}
Values:{608.468, 1305.26}
Values:{607.281, 1309.79}
Values:{606.097, 1314.39}
Values:{604.914, 1319.08}
Values:{603.735, 1323.85}
Values:{602.557, 1328.72}
Values:{601.381, 1333.68}
Values:{600.208, 1338.74}
Values:{599.037, 1343.89}
Values:{597.868, 1349.15}
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Values:{592.056, 1377.1}
Values:{590.9, 1383.05}
Values:{589.746, 1389.13}
Values:{588.594, 1395.34}
Values:{587.445, 1401.69}
Values:{586.298, 1408.19}
Values:{585.152, 1414.84}
Values:{584.009, 1421.64}
```

```
Values:{582.869, 1428.61}
Values:{581.73, 1435.74}
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Values:{579.459, 1450.54}
Values:{578.327, 1458.23}
Values:{577.197, 1466.11}
Values:{576.069, 1474.2}
Values:{574.944, 1482.51}
Values:{573.82, 1491.04}
Values:{572.699, 1499.81}
Values:{571.58, 1508.84}
Values:{570.463, 1518.12}
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Values:{568.237, 1537.51}
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Values:{566.019, 1558.12}
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Values:{563.811, 1580.08}
Values:{562.711, 1591.6}
Values:{561.613, 1603.53}
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Values:{559.424, 1628.66}
Values:{558.333, 1641.92}
Values:{557.245, 1655.69}
Values:{556.159, 1669.99}
Values:{555.077, 1684.87}
Values:{553.996, 1700.36}
Values:{552.919, 1716.51}
Values:{551.844, 1733.37}
Values:{550.773, 1750.99}
Values:{549.704, 1769.43}
Values:{548.638, 1788.76}
Values:{547.576, 1809.06}
Values:{546.517, 1830.41}
Values:{545.461, 1852.9}
Values:{544.408, 1876.65}
Values:{543.36, 1901.79}
Values:{542.315, 1928.44}
Values:{541.274, 1956.79}
```

```
Values:{540.237, 1987.02}
Values:{539.205, 2019.35}
Values:{538.177, 2054.04}
Values:{537.154, 2091.41}
Values:{536.136, 2131.83}
Values:{535.124, 2175.74}
Values:{534.118, 2223.7}
Values:{533.119, 2276.38}
Values:{532.126, 2334.62}
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Values:{528.239, 2650.35}
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Values:{525.44, 3053.}
Values:{524.538, 3253.02}
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Values:{521.168, 5323.54}
Values:{520.417, 7465.94}
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Values:{519.13, -2172.27}
Values:{518.692, -2347.18}
Values:{525.979, -2966.24}
Values:{524.857, -3389.28}
Values:{530.692, -7798.57}
Values:{529.981, 47771.2}
Values:{531.137, -720.856}
Values:{531.214, -728.493}
Values:{514.97, -732.992}
Values:{513.2, -732.101}
Values:{511.882, -753.565}
Values:{517.797, -772.558}
Values:{515.955, -776.662}
Values:{514.695, -774.051}
Values:{513.487, -785.787}
Values:{514.041, -786.624}
```

```
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Values:{518., -854.42}
Values:{516.525, -862.948}
Values:{515.718, -861.627}
Values:{514.678, -886.506}
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Values:{518.446, -916.436}
Values:{517.21, -913.138}
Values:{516.014, -931.752}
Values:{517.497, -935.996}
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Values:{522.657, -1022.24}
Values:{521.103, -1034.73}
Values:{520.288, -1032.64}
Values:{519.235, -1069.15}
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Values:{547.164, -4202.43}
Values:{546.223, -5953.14}
Values:{549.009, 14295.7}
Values:{548.736, -4968.03}
```

```
Values:{547.896, -8920.42}
Values:{549.898, 6671.74}
Values:{549.737, 13134.4}
Values:{547.986, -5300.79}
Values:{547.485, -10391.2}
Values:{549.617, 5803.03}
Values:{549.449, 8701.53}
Values:{547.62, -29083.5}
Values:{547.046, 3485.83}
Values:{547.625, 3814.26}
Values:{547.951, 4402.66}
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Values:{549.534, -38815.6}
Values:{549.68, 2610.19}
Values:{548.785, 2709.3}
Values:{547.745, 2833.15}
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Values:{545.806, 3167.98}
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Values:{537.733, 14551.}
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Values:{538.952, 9627.72}
Values:{538.737, -18100.7}
Values:{537.648, 5110.43}
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Values:{538.462, 19635.2}
Values:{539.076, -2898.3}
```

```
Values:{539.203, -3365.71}
Values:{531.834, -5330.77}
Values:{545.089, 3260.63}
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Values:{728.389, -693.486}

Values:{726.779, -695.385}

Values:{726.256, -694.056}

Values:{725.588, -697.079}

Values:{725.696, -697.191}

Values:{725.236, -703.508}

Values:{727.239, -704.315}

Values:{726.685, -727.946}

Values:{731.494, -735.941}

Values:{730.274, -737.746}

Values:{729.844, -736.278}

Values:{729.148, -739.46}

Values:{729.197, -739.525}

Values:{728.719, -745.912}

Values:{730.339, -746.711}

Values:{729.92, -770.092}

Values:{736.141, -778.677}

Values:{734.81, -780.302}

Values:{734.329, -777.842}

Values:{733.558, -781.063}

Values:{733.501, -780.963}

Values:{732.986, -787.014}

Values:{734.089, -787.734}

Values:{733.743, -807.398}

Values:{741.87, -815.076}

Values:{740.408, -816.576}

```
Values:{739.885, -812.936}
Values:{739.05, -816.32}
Values:{738.94, -816.096}
Values:{738.404, -822.243}
Values:{739.339, -822.968}
Values:{738.991, -841.643}
Values:{747.049, -848.678}
Values:{745.539, -850.62}
Values:{745.026, -847.516}
Values:{744.241, -851.386}
Values:{744.211, -851.325}
Values:{743.695, -858.646}
Values:{744.951, -859.591}
Values:{744.593, -883.806}
Values:{752.62, -893.922}
Values:{751.201, -895.745}
Values:{750.698, -891.642}
Values:{749.849, -895.802}
Values:{749.771, -895.617}
Values:{749.235, -903.252}
Values:{750.34, -904.242}
Values:{749.987, -928.173}
Values:{758.298, -938.261}
Values:{756.842, -940.349}
Values:{756.339, -936.032}
Values:{755.492, -940.712}
Values:{755.443, -940.589}
Values:{754.911, -949.291}
Values:{756.165, -950.487}
Values:{755.803, -978.499}
Values:{764.033, -991.005}
Values:{762.62, -993.326}
Values:{762.132, -988.644}
Values:{761.271, -993.948}
Values:{761.239, -993.86}
Values:{760.706, -1003.78}
Values:{762.078, -1005.23}
Values:{761.704, -1037.53}
Values:{769.763, -1052.66}
```

```
Values:{768.387, -1055.43}
Values:{767.92, -1050.69}
Values:{767.065, -1056.88}
Values:{767.078, -1056.91}
Values:{766.551, -1068.74}
Values:{768.173, -1070.61}
Values:{767.764, -1109.97}
Values:{775.14, -1129.3}
Values:{773.844, -1133.03}
Values:{773.427, -1129.04}
Values:{772.621, -1136.75}
Values:{772.754, -1137.15}
Values:{772.253, -1152.75}
Values:{774.564, -1155.68}
Values:{773.961, -1206.02}
Values:{779.437, -1232.24}
Values:{778.352, -1238.89}
Values:{778.129, -1237.43}
Values:{777.474, -1249.71}
Values:{778.2, -1251.45}
Values:{777.799, -1283.29}
Values:{784.541, -1296.45}
Values:{783.031, -1305.29}
Values:{782.709, -1303.28}
Values:{782.074, -1318.18}
Values:{783.05, -1320.54}
Values:{782.662, -1361.87}
Values:{790.687, -1383.31}
Values:{789.256, -1389.48}
Values:{788.836, -1383.71}
Values:{788.013, -1396.68}
Values:{788.308, -1397.93}
Values:{787.819, -1425.92}
Values:{791.409, -1433.7}
Values:{790.295, -1486.67}
Values:{794.455, -1515.34}
Values:{793.409, -1529.33}
Values:{793.428, -1529.46}
Values:{792.859, -1556.6}
```

```
Values:{795.016, -1562.88}
Values:{794.469, -1643.18}
Values:{800.934, -1699.41}
Values:{799.867, -1715.38}
Values:{799.824, -1714.94}
Values:{799.197, -1746.99}
Values:{800.982, -1754.55}
Values:{800.513, -1847.63}
Values:{808.029, -1920.48}
Values:{806.905, -1940.97}
Values:{806.847, -1940.14}
Values:{806.184, -1982.62}
Values:{808.056, -1993.98}
Values:{807.554, -2111.35}
Values:{814.928, -2216.59}
Values:{813.862, -2251.14}
Values:{814.068, -2254.3}
Values:{813.467, -2330.91}
Values:{817.152, -2366.45}
Values:{816.071, -2464.68}
Values:{818.736, -2511.34}
Values:{817.928, -2607.85}
Values:{820.759, -2652.88}
Values:{819.924, -2788.2}
Values:{823.961, -2886.97}
Values:{822.951, -2981.62}
Values:{824.119, -3008.99}
Values:{823.579, -3228.98}
Values:{830.778, -3510.06}
Values:{829.782, -3665.19}
Values:{831.32, -3730.35}
Values:{830.698, -4091.7}
Values:{837.502, -4740.91}
Values:{836.694, -5272.68}
Values:{842.019, -6401.31}
Values:{841.335, -8179.15}
Values:{847.502, -56415.4}
Values:{847.169, 4346.18}
Values:{848.114, 4577.79}
```

```
Values:{848.484, 4816.87}
Values:{849.742, 5094.54}
Values:{849.989, 5427.45}
Values:{852.04, 5867.51}
Values:{852.191, 6459.57}
Values:{855.763, 7371.42}
Values:{855.941, 8996.19}
Values:{858.786, 13050.5}
Values:{858.933, 1.29568×10^6}
```

FindRoot : Failed to converge to the requested accuracy or precision within 100 iterations.

In[ ]:=
```
rear = ListLinePlot[Partition[Riffle[𝒳[[All, 1]], 𝒵], 2], AspectRatio → True]
```

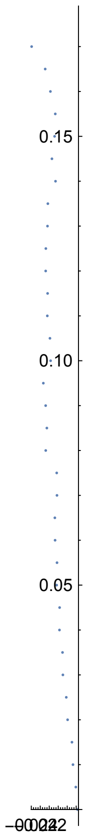
Out[ ]=

In[∘]:=
```
front = ListLinePlot[Partition[Riffle[𝒳⟦All, 2⟧, 𝒵], 2] , AspectRatio → True]
```

Out[∘]=

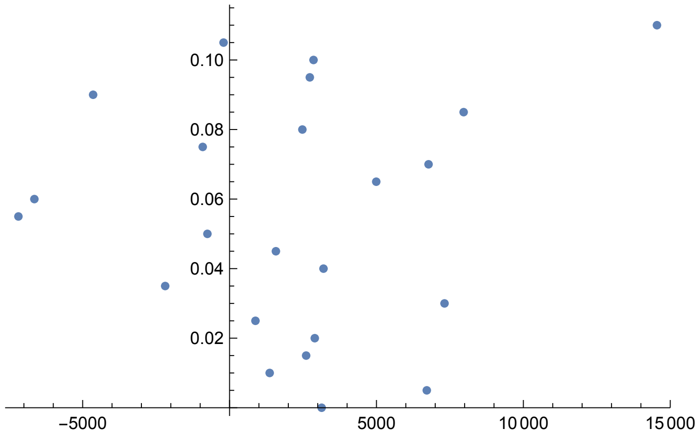
In[∘]:=
```
ListPlot[Partition[Riffle[𝒪, 𝒵], 2] , AspectRatio → True]
```

Out[∘]=



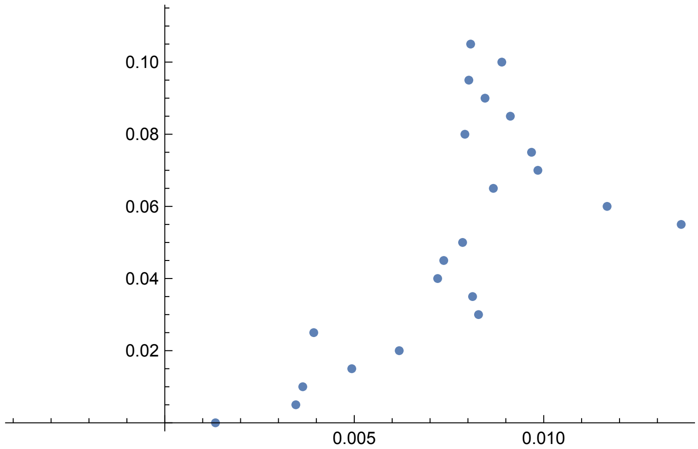
In[ ]:= ListPlot[Partition[Riffle[𝒫, 𝒵], 2]]

Out[ ]=



In[ ]:= ListPlot[Partition[Riffle[ℛc, 𝒵], 2]]

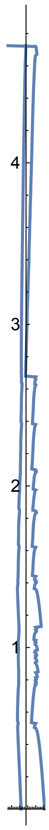
Out[ ]=



In[ ]:=

```
Show[{rear, front}, AspectRatio → True, PlotRange → All]
```

Out[ ]=



In[ ]:=

```
ArcTan[Θ] // Mean
```

Out[ ]=

```
{0.0620616, 0.0393942}
```

In[ ]:=

```
Last[𝒳]
```

Last : {} has zero length and no last element.

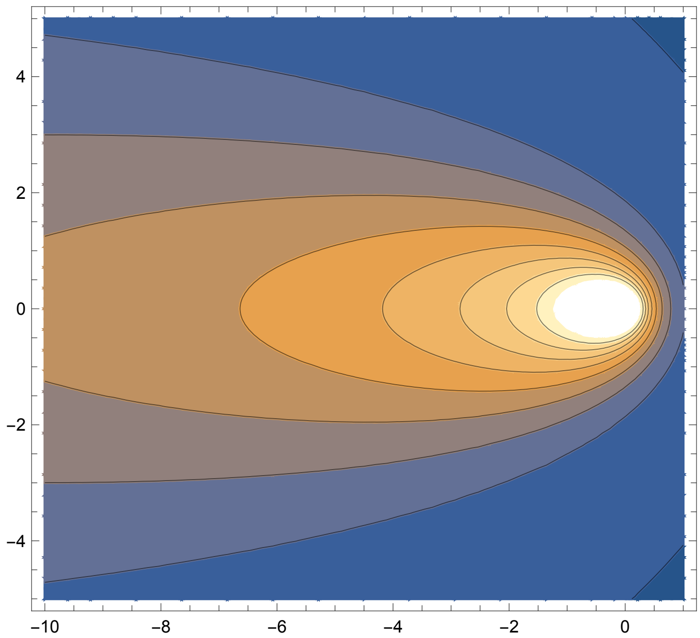
Out[ ]=

```
Last[{}]
```

This represents the solution of the moving line source with thickness of the plate thick, this is 3d solution in cylindrical coordinates:

In[∘]:=
```
Tn[x_, y_, P_, thick_] :=
  Ta + P/(2 π λ thick) Exp[-v x / (2 κ)] × BesselK[0, v * Norm[{x, y}] / (2 κ)];
ContourPlot[Tn[x, y, 245, 1.], {x, -10, 1}, {y, -5, 5}, AspectRatio → True]
```

Out[∘]=



In[∘]:=
```
r1 = FindRoot[Tn[x, 0, 2450, 1.] == Tv, {x, 4}]
```

Out[∘]=

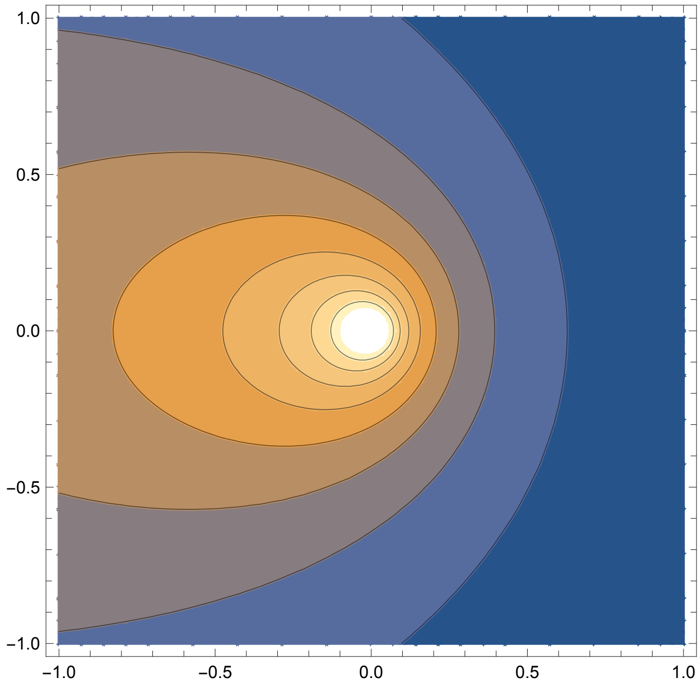
```
{x → -13.1836}
```

Solution of the moving point in 2d case :

In[∘]:=
```
Clear[T2D]
```

In[∘]:=
```
T2D[x_, y_, P_] := Ta + P/(2 π λ) Exp[-v x / (2 κ)] × BesselK[0, v * Norm[{x, y}] / (2 κ)];
```

In[ ]:=
```
ContourPlot[T2D[x, y, 245], {x, -1., 1}, {y, -1, 1}, AspectRatio → True]
```
Out[ ]=

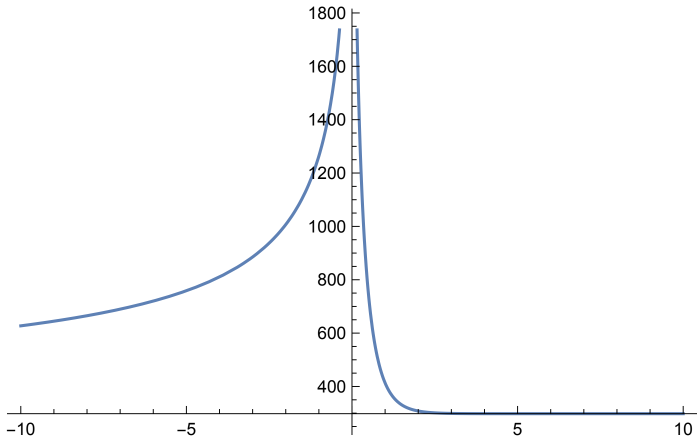


In[ ]:=
```
r1 = NSolve[T2D[x, 0, PL] == Tv, x, Reals]
```
NSolve : NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

NSolve : Inverse functions are being used by NSolve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]=
```
{{x → -28.4159}, {x → 0.668721}}
```

In[ ]:=
```
Plot[T2D[x, 0, 245], {x, -10, 10}]
```
Out[ ]=



In[ ]:=
```
T2D[-1, 0, 2450]
```
Out[ ]=
```
10519.4
```

Different Rosenthal solution:

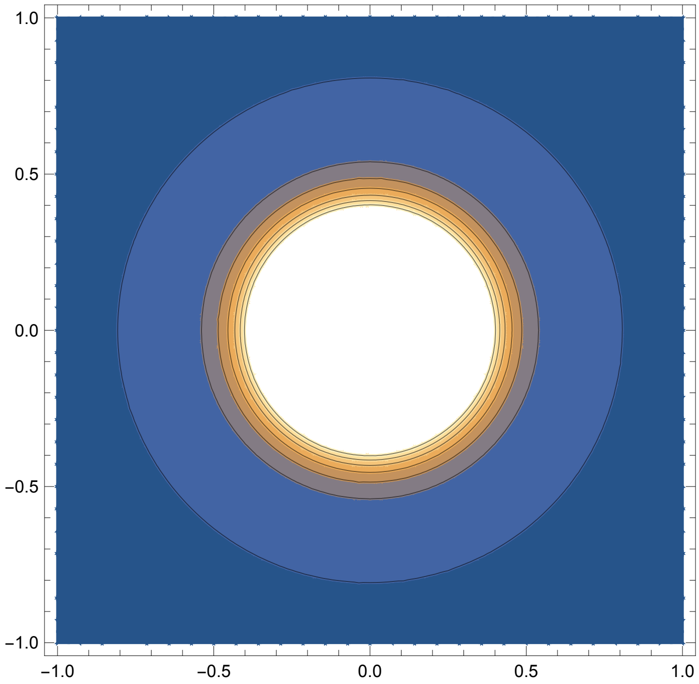
In[∘]:=
```
v = 40;
```

In[∘]:=
```
Trosen[x_, y_, P_, dis_] :=
```

$$Ta + \frac{P}{2\pi\lambda\,\text{Norm}[\{x, y\}]}\,\text{Exp}[-v * (\text{Norm}[\{x, y\}] + \text{dis}) / (2\kappa)]$$

In[∘]:=
```
ContourPlot[Trosen[x, y, 2450, 0], {x, -1, 1}, {y, -1, 1}, AspectRatio → True]
```

Out[∘]=



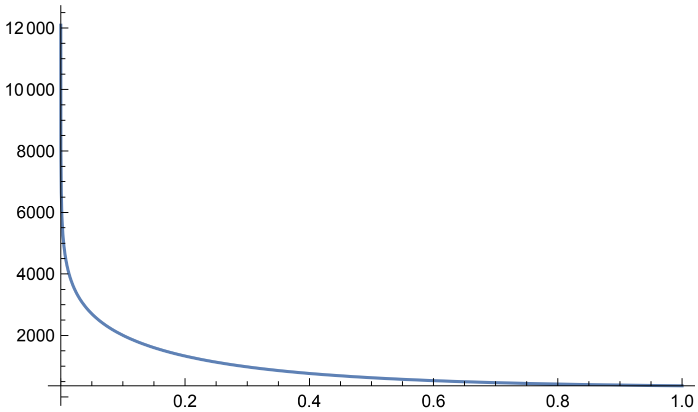
In[∘]:=
```
Tn[-1, 0.000001]
```

Out[∘]=

1215.08 − 16 090.8 i

In[∘]:=
```
Plot[Tn[x, 0], {x, -1, 1}, PlotRange → All]
```

Out[∘]=




```
T[r, π] /. r → r1
```

Out[∘]=

$$\{298 + 907.258\, e^{1.32979\,(x \to 0.0311167 + 8.72693 \times 10^{-17}\, i)}\ \text{BesselK}[0, 1.32979\,(x \to 0.0311167 + 8.72693 \times 10^{-17}\, i)]\}$$

In[ ]:= $\frac{2\,PL}{\pi\, rf0^2}$

Out[ ]= $\frac{75\,\pi}{F^2\, M^4\, wave^2}$

In[ ]:= $\frac{\pi\, PL}{2\, F^2\, M^4}$

Out[ ]= $\frac{75\,\pi}{F^2\, M^4}$

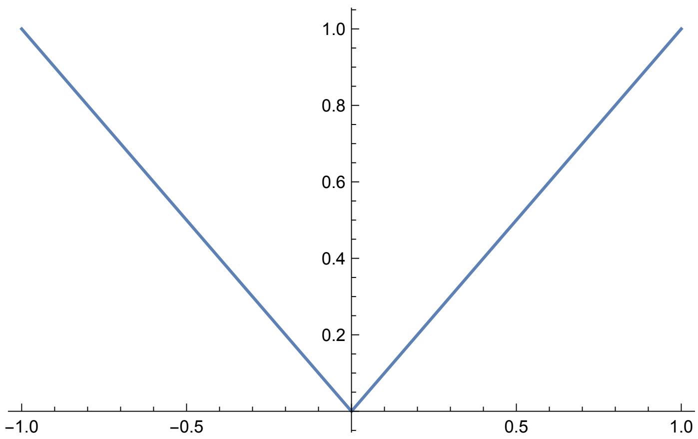
In[ ]:= `qvr[0.8]`

Out[ ]= $\frac{(-298 + Tv)\, v\, \lambda\, (-1 + \mathrm{BesselK}[1,\, 0.04\, v])}{20\, \mathrm{BesselK}[0,\, 0.04\, v]}$

In[ ]:= `f[x_] := Abs[x]`

In[ ]:= `Plot[f[x], {x, -1, 1}]`

Out[ ]=



Absorption

The plasma absorption can be described by Beer-Lambert law:

$I_a(z)/I_i(0) = 1 - I_t(z)/I_i(0)$=1-exp(-$\alpha_{iB}\, z$)

where I_i – is the incident intensity, I_t – the transmitted intensity and

I_a is the intensity absorbed when passing path z. For the plasma absorption coefficient due to inverse Bremsstrahlung α_{iB} that is temperature dependent. Mean value of 100 m^{-1} is used.

Absorbed fraction of the metal vapour plume over the workpiece is known by estimating height h_{pl} of the plume :

$\alpha_{pl} = 1 - \exp(-\alpha_{iB}\, h_{pl})$

The remaining radiation is transmitted and passes the plume. The power hitting the workpiece outside

the keyhole is strongly reflected and only small part is absorbed. The first plasma absorption before

hitting the keyhole wall:
$\alpha_{iB,1} = 1 - \exp(-\alpha_{iB}\, d/2)$

The first Fresnel absorption at each point is included in the energy balance by multiplying the local beam intensity by the Fresnel absorption coefficient.

After n_{mr} reflections, assuming geometrical optics, the part of the remaining intensity I_{rn} :
$I_{rn}/I_i = (\rho_{Fr})^{nmr}$
where ρ_{Fr} is the Fresnel reflection coefficient, which is in general angle-dependent.
The approximation of the keyhole profile with mean wall angle θ_w the angle of reflection
$\theta_r = 2\, n_{mr}\, \theta_w$

The number of the reflections is defined by counting them up to the limiting angle of the reflected beam:
$\theta_r \geq \pi/2$..
$n_{mr} = \frac{\pi/2}{2\,\theta_w} = \frac{\pi}{4\,\theta_w}$
If the first reflection, which is separately considered in the energy balance.
Using previous equation for the number of reflections:
$\rho_{mr} = [\rho_{Fr}\,(\theta = \pi/2)]^{nmr-1}$
Consequently the absorption coefficient α_{mr} :
$\alpha_{mr} = 1 - \rho_{mr}$

During the reflections the rays cross plasma and are partially absorbed by it.
$\alpha_{iB,mr} = 1 - \exp(-\alpha_{iB}\, 3\, d/2)$

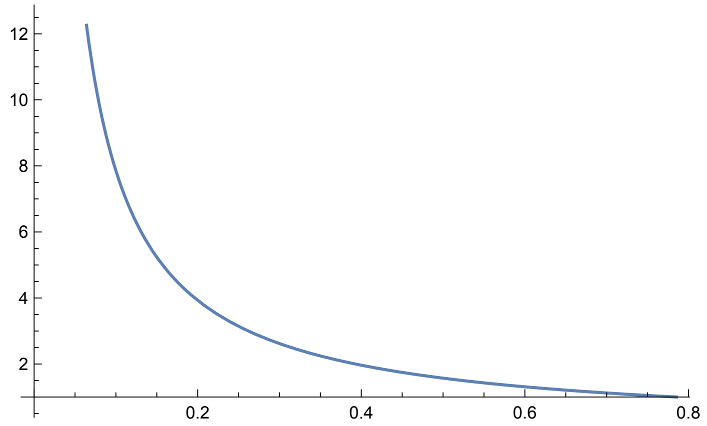
The of the incident intensities damped by the plume minus the remaining intensity I_r leaving the keyhole:
$I_{a,Fr} + I_{a,mr} + I_{a,iB} = \left(1 - \alpha_{pl}\right) I(x, z) - I_r = f(\theta_w)$
where:
$I_r = \left(1 - \alpha_{pl}\right)\left(1 - \alpha_{iB,1}\right)(1 - \alpha_{Fr})\, x(1 - \alpha_{mr})\left(1 - \alpha_{iB,mr}\right) I(x, z)$

$I_{a,Fr} = \left(1 - \alpha_{pl}\right) * \left(1 - \alpha_{iB,1}\right) \alpha_{Fr}\, I(x, z)$
$I_{a,mr} = \left(1 - \alpha_{pl}\right)^{*}\left(1\text{-}\alpha_{iB,1}\right)(1 - \alpha_{Fr})\alpha_{mr} I(x,z)$
$I_{a,iB} = \left(1 - \alpha_{pl}\right)(\alpha_{iB,1} + \alpha_{iB,mr}\left(1 - \alpha_{iB,1}\right) x(1\text{-}\alpha_{Fr})(1\text{-}\alpha_{mr}))\, I(x,z)$

```
In[]:= αpl = 1 - Exp[-αiB * hpl];
nmr = π / (4 θw);
```

In[ ]:=
```
Plot[π / (4 θ), {θ, 0, π / 4}]
```

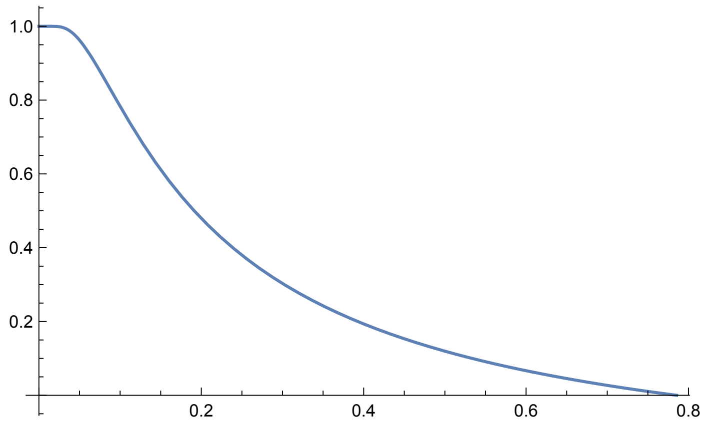
Out[ ]=



In[ ]:=
```
With[{ρ = 0.8}, Plot[1 - ρ^(π / (4 θ) - 1), {θ, 0, π / 4}]]
```

General : $0.8^{48950.}$ is too small to represent as a normalized machine number; precision may be lost.

Out[ ]=



Reflections from the keyhole, the first reflection is calculated separately in the energy balance. The rest reflections
can be approximated from the multiple reflections:

In[ ]:=
```
ρmr = ρFr^(nmr-1);
αmr = 1 - ρmr;
```

This effect of the partial absorption of the intensity can be modeled:

In[ ]:=
```
αiBmr = 1 - Exp[-αiB 3 d / 2];
```

After the first iteration of the algorithm we calculate the absorptions using the next sequence:

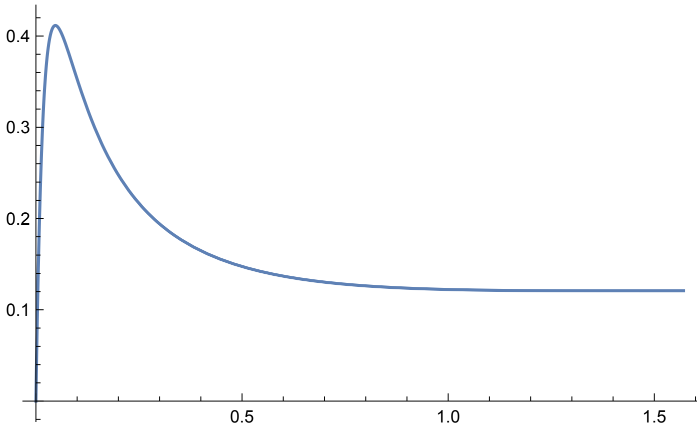
In[ ]:=
```
IaFr[x_, z_] := (1 - αpl) (1 - αiB1) * αmr * Ia[x, z];
Iamr[x_, z_] := (1 - αpl) (1 - αiB1) * (1 - αFr) αmr * Ia[x, z];
IaiB[x_, z_] := (1 - αpl) (αiB1 + αiBmr * (1 - αiB1) (1 - αFr) (1 - αmr) ) * Ia[x, z];
Ir[x_, z_] := (1 - αpl) (1 - αiB1) * (1 - αFr) (1 - αmr) (1 - αiBmr) * Ia[x, z];
```

Parameters used in the calculation:
Laser power (cw) Pl 4kW
Wavelength λ 10.6 μm
Polarisation - unpolarized -

Beam quality M^2 5.0
Beam mode TEM TEM_{00}
Focal length f 200 mm
Beam diameter on optics D_b 34
Focusing number $F = f / D_b$ 6.0
Focal radius r_{f0} 203 μm
Rayleigh length z_r 2.5 mm
Absorption coefficient Bremsstrahlung α_{iB} 100 m^{-1}
Focal plane z_0 optimized 0.7
welding speed v 50 mm/s
Initial depth of keyhole d 3.5 mm
Material mild steel
λ 45 W/mK
κ 18.8 mm^2/s
Weld blind weld

In[ ]:=
```
PL = 4 × 10^3;
M = Sqrt[5];
F = 6.0;
wave = 10.6 × 10^-3;
rf0 = 0.203;
zr = 2.5;
z0 = 0.7;
v = 50.0;
αiB = 100 × 10^-3;
hpl = 0.2;
d = 5.5;
(* this is the function of the incident angle *)
αFr[θ_] :=
  Module[{n = 14.8, k = 15.5}, Ap[θ] := 4 n Cos[θ] / ((n^2 + k^2) Cos[θ]^2 + 2 n Cos[θ] + 1);
   As[θ] := 4 n Cos[θ] / ((n^2 + k^2) + Cos[θ]^2 + 2 n Cos[θ]); (As[θ] + Ap[θ]) * 0.5];
ρFr = 1 - αFr[θ];
Ta = 298;
Tv = 2900 + 273;
κ = 18.8; λ = 45 * 10^-3;
```

In[∘]:=
```
Plot[αFr[π / 2 - θ], {θ, 0, π / 2}, PlotRange → All]
```
Out[∘]=



Gaseous phase

The state of the vapour inside the keyhole is calculated for each element after having calculate the keyhole profile.

The degree of ionization as a function of temperature is Saha's equation:

$n_e n_i/n_n = 2\,(g_o)_i/g_n \,\frac{2\,\pi \mathrm{mkT}}{h^3}\,\exp(-W_i/\mathrm{kT})$

where n_e,n_i and n_n are the particle densities of electrons, ions and ground state atoms respectively and g_0 is the statistical weight. The electron mass m_e, Planck's constant $\hbar = \mathrm{h}/2\pi$ and Boltzmann constant k.

W_i denotes the work for ionization.

The dominant absorption mechanism in a plasma for CO2 laser radiation is the mechanism of inverse Bremsstrahlung. Linear plasma absorption coefficient α_{iB} :

$$\alpha_{\mathrm{iB}} = \frac{Z^2\, e^6\, n_e}{6\,\sqrt{3}\,\hbar\omega^3\, m_e^2\, c_0\, \epsilon_0^3\left(1-(\omega_{\mathrm{pe}}/\omega)^2\right)}\; x\left(\frac{m}{2\,\pi \mathrm{kT}}\right)\left(1-\exp\left(\frac{-\hbar\omega}{\mathrm{kT}}\right)\right)\overline{g}$$

In[∘]:=
```
l1 = {1, 2, 3, 4}; l2 = {5, 6, 7, 8};
Mean[{l1, l2}]
```
Out[∘]=
```
{3, 4, 5, 6}
```