Number Sets

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Natural Numbers

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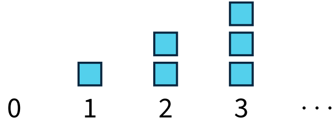
A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



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One of the popular ones is using a **successor** function, denoted s. If n is a natural number, then $s(n)$ basically means 'add another block on top of n'.
One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet**! In fact, you need the successor function to define addition in the first place.

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2. Every natural number has a successor which is also natural.
3. The number 0 is not the successor of any natural number.
4. If $s(n) = s(m)$, then $n = m$.
5. (Induction Axiom) If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

1

Unpacking The Axioms

Natural Numbers – Axiom 1

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Hopefully obvious.

Natural Numbers – Axiom 2

Every natural number has a successor which is also natural.

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Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

Natural Numbers – Axiom 3

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Basically means that the natural numbers are infinite only ‘in one direction’. There is a **first** natural number.

Natural Numbers – Axiom 4

If $s(n) = s(m)$, then $n = m$.

Natural Numbers – Axiom 4

$$\text{If } s(n) = s(m), \text{ then } n = m.$$

This means that the successor function is **injective** – each natural number has a different successor.

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This means that any feature of the natural numbers 'propagates' via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it true for 1, it is true for 2 as well, etc.

2

Operations On Natural Numbers

What Is An Operation?

By **operation**, we mean a function which takes **one or multiple** natural numbers and produces **one** natural number.

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In this sense, subtraction and division **are not operations**! They take two natural numbers but they **do not produce a natural number**.

Addition

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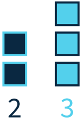
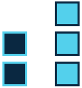
We can imagine addition as 'adding blocks *to the side*' and the successor function as 'adding one block *on top*'.
In this sense, $n + s(m) = s(n + m)$ only means that if you add one block atop m blocks and then n blocks to the side you have the same number of blocks as if you add n blocks next to m blocks and then another on top of that.

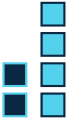
ADDITION



Addition

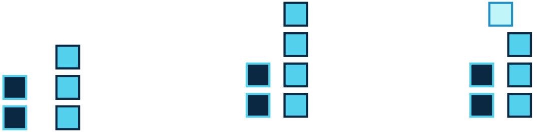


2 3



$2 + s(3)$

ADDITION

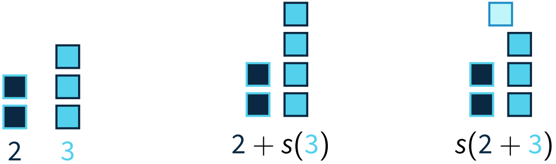


$2 \quad 3$

$2 + s(3)$

$s(2 + 3)$

ADDITION



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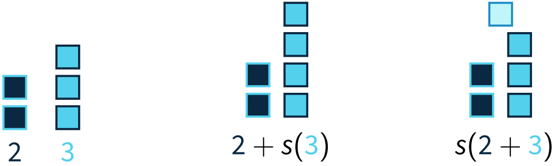

Using the formula $n + s(m) = s(n + m)$, one calculates $n + m$ by taking the successor of n, m times. Like this:

$$
\begin{aligned}
n + 0 &= n, \\
n + 1 &= n + s(0) = s(n + 0) = s(n), \\
n + 2 &= n + s(1) = s(n + 1) = s(n + s(0)) = s(s(n + 0)) = s(s(n)), \\
&\vdots
\end{aligned}
$$

Addition – Properties

Addition of natural numbers satisfies these two properties:

- **Commutativity**:

$$n + m = m + n.$$

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$$n + (m + k) = (n + m) + k.$$

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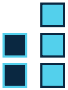

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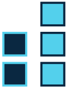
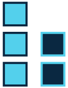


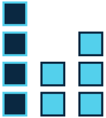
$2 + 3$



$3 + 2$

Addition – Properties

Using blocks, **associativity** just means that putting *m* blocks next to *k* blocks and then *n* more blocks next to those is the same as putting *m* blocks next to *n* blocks and then *k* more blocks next to those.



$4 + (2 + 3)$

Addition – Properties

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$4 + (2 + 3)$ $\qquad$ $(4 + 2) + 3$

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- $m \cdot s(n) = m \cdot n + m$.

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If we write $s(n) = n + 1$, then the second formula just means that

$$m \cdot s(n) = m \cdot (n + 1) = m \cdot n + m.$$

Multiplication

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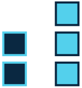
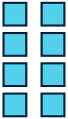
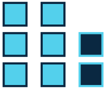


$2 \quad 3$

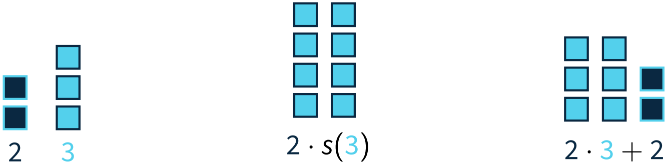


$2 \cdot s(3)$



$2 \cdot 3 + 2$

MULTIPLICATION



The formula $m \cdot s(n) = m \cdot n + m$ allows us to compute $m \cdot n$ by applying it n times.

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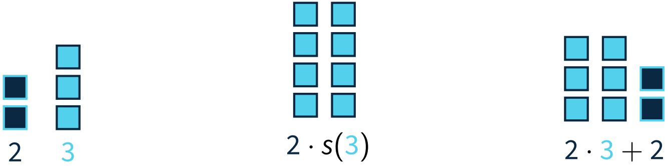

The formula $m \cdot s(n) = m \cdot n + m$ allows us to compute $m \cdot n$ by applying it n times. More precisely,

$$
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m \cdot 1 &= m \\
m \cdot 2 &= m \cdot s(1) = m \cdot 1 + m = m + m \\
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- **Associativity:**

$$m \cdot (n \cdot k) = (m \cdot n) \cdot k.$$

- **Distributivity:**

$$m \cdot (n + k) = m \cdot n + m \cdot k.$$

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Beware! 'Taking inverse' is a **unary** operation (meaning it acts on **one** element), not a binary one.

1

Digression

↓

Mathematical Structures
Destructive & Symmetric Transformations

Mathematical Structures

Mathematical Structure (also 'Universe')

A **mathematical** structure is a **set** with **operations**. If X is the set and $op_1, \ldots, op_n$ the operations on X, we write the resulting structure as

$$(X, op_1, \ldots, op_n).$$

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$$(X, op_1, \ldots, op_n).$$

The reason mathematical structures are called by some enthusiasts as mathematical 'universes' is that they really *are* universes in the broadest sense possible – a bunch of elements with prescribed rules of interaction.

Operation On A Set

Operation

An **operation** on a set X is really just a **rule of interaction** between its elements. In symbols, it is a **function**

$$op : X^n \to X$$

where $X^n \to X$ just means 'Take n elements of X and give me back one.'

Operation On A Set

Examples:

- $(\mathbb{N}, +, \cdot)$ is a structure where $+$ and $\cdot$ are **binary** operations (meaning they take two elements and return one). They can be seen as functions $\mathbb{N}^2 \to \mathbb{N}$.

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- $(\{\text{orderings of vertices of a regular } n\text{-gon}\}, r, r^2, \ldots, r^n, s_1, \ldots, s_n)$ is a structure where r is the rotation by $360^\circ/n$ and $s_1, \ldots, s_n$ are all the reflections. They are all **unary** operations (they take one element and return one).

Operation As A Transformation

Operations basically describe interactions between set elements – two or more elements combine or **transform** into another one.

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- The operation r on the vertices of a regular polygon is **symmetric** – it can be reversed or **inverted**.If, for example, vertex A is sent to G by this rotation, then rotation in the opposite direction sends G back to A.

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  - Elements x^* and y^* (the **inverse** elements to x and y) such that

$$op(x, x^*) = op(y, y^*) = e.$$

- It's possible X doesn't have those things – which means the operation *op* cannot be symmetric **on** X, but it can be symmetric on a larger set which does contain those elements.

Inverse As A Function

Inverse can be thought of as a **unary** operation – it takes an element x and gives back its inverse x^*.

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Clearly, the inverse to x^* is x, again.

Operation As A Transformation

Motivating examples:

- The operation $+$ on $\mathbb{N}$ **cannot be made symmetric.** Natural numbers do not contain inverses to $+$.

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- The operation $+$ on $\mathbb{N}$ **cannot be made symmetric**. Natural numbers do not contain inverses to $+$.We fix this by introducing **integers**, basically forcefully adding to each number n its inverse with respect to $+$, that is, the negative number $-n$. The new set is called the **integers**, denoted . The identity element is 0 in this case.

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- The operation $+$ on $\mathbb{N}$ **cannot be made symmetric**. Natural numbers do not contain inverses to $+$.We fix this by introducing **integers**, basically forcefully adding to each number n its inverse with respect to $+$, that is, the negative number $-n$. The new set is called the **integers**, denoted . The identity element is 0 in this case.
- The operation $\cdot$ on cannot be made symmetric. We fix this by introducing **fractions/reciprocals** to elements. The inverse with respect to $\cdot$ of an element z is $\frac{1}{z}$. The identity element is 1.