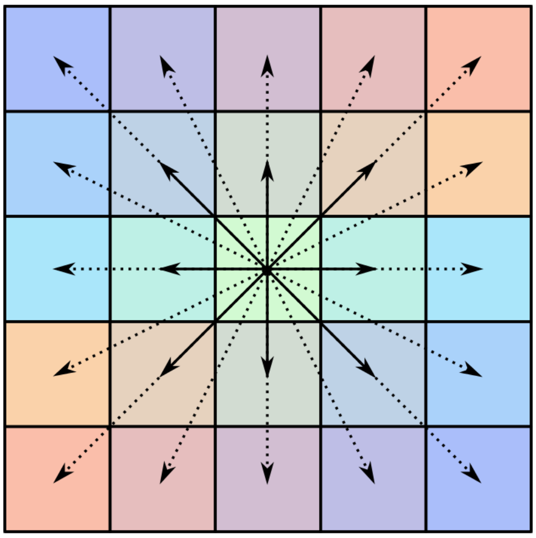
Gymnázium Evolution Jižní Město



Intro-ish To Linear Algebra

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Preface

This text covers selected topics from the curriculum of a typical undergraduate linear algebra course. Almost no pre-existing knowledge is strictly required save a superficial understanding of propositional logic and set theory. A reasonably good ability to manipulate algebraic expressions should prove advantageous, too.

Mathematics is an exact and rigorous language. Words and symbols have singular, precisely defined, meaning. Many students fail to grasp that intuition and imagination are paramount, but they serve as a *starting point*, with formal logical expression being the end. For example, an intuitive understanding of a *line* as an infinite flat 1D object is pretty much correct but not *formal*. It is indeed the formality of mathematics which puts many students off. Whereas high school mathematics is mostly algorithmic and non-argumentative, higher level maths tends to be the exact opposite – full of concepts and relations between those, which one is expected to be capable of grasping and formally describing. Owing to this, I wish this text would be a kind of synthesis of the formal and the conceptual. On one hand, rigorous definitions and proofs are given; on the other, illustrations, examples and applications serve as hopefully efficient conveyors of the former's geometric nature.

Linear algebra is a mathematical discipline which studies – as its name rightly suggests – the *linear*. Nevertheless, the word *linear* (as in 'line-like') is slightly misplaced. The correct term would perhaps be *flat* or, nigh equivalently, *not curved*. It isn't hard to imagine why curved objects (as in *geometric* objects, say) are more difficult to describe and manipulate than objects flat. For instance, the formula for the volume of a cube is just the product of the lengths of its sides. Contrast this with the volume of a still 'simple', yet curved, object – the ball. Its volume cannot even be *precisely* determined; its calculation involves approximating an irrational constant and the derivation of its formula is starkly unintuitive without basic knowledge of measure theory.

As such, linear algebra is a highly 'geometric' discipline and opportunities for visual interpretations abound. This is also a drawback in a certain sense. One should not dwell on visualisations alone as they tend to lead astray where imagination falls short. Symbolic representation of the geometry at hand is key.

The word *linear* however dons a broader sense in modern mathematics. It can be rephrased as reading, 'related by addition and multiplication by a scalar'. We trust kind readers have been acquainted with the notion of a *linear function*. A linear function is (rightly) called *linear* for it receives a number as input and outputs its *constant* multiple plus another *constant* number. Therefore, the output is in a *linear* relation to the input – it is multiplied by some fixed number and added to another. This understanding of the word is going to prove crucial already in the first chapter, where we study *linear systems*. Following are *vector spaces* and *linear maps*, concepts whose depth shall occupy the span of this text. Each chapter is further endowed with an *applications* section where I try to draw a simile between mathematics and common sense.

Contents

Chapter 1

Linear Systems

Linear systems are by definition sets of linear equations, that is, of equations which relate present variables in a *linear* way. It is important to understand what this means. Spelled out, an expression on either side of any of the equations is formed *solely* by

(1) multiplying the variables by a given number (**not another variable**),

(2) adding these multiples together.

Any such combination where variables are only allowed to be multiplied by a constant and added is called a *linear combination*. This term is extremely important and ubiquitous throughout the text; hence, it warrants an isolated definition.

Definition 1.0.1 (Linear combination)

Let $x_1, \ldots, x_n$ with $n \in \mathbb{N}$ be variables. Their *linear combination* is any expression of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n,$$

where $a_1, \ldots, a_n$ are numbers.

Remark 1.0.2

In the definition above, we have deliberately not specified what type of *numbers* we mean. In the future, we shall work extensively with real and complex numbers as well as elements of other fields, which dear readers might not have even recognised as 'numbers' thus far. The only important concept in this regard is the clear distinction between a *number* (later *scalar*) and a *variable* (later *vector*).

Example 1.0.3

Consider the variables x, y and z. The expression

$$3x + 2y - 0.5z$$

is their linear combination whereas

$$5x + 3y - yz + 7z^2$$

is not.

To reiterate, a *linear system* is any set of equations featuring only linear combinations of variables; these equations are consequently called *linear* as well. A *solution* of a linear system is the set of all possible substitutions of numbers (in place of variables) which make the equations true.

It is clear that every linear equation can be rearranged to

$$a_1x_1 + \cdots + a_nx_n = c$$

for some variables $x_1, \dots, x_n$ and numbers $a_1, \dots, a_n, c$ by simple subtraction. This is how we shall define it, for simplicity.

Definition 1.0.4 (Linear equation)

Any equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c, \tag{1.1}$$

where $x_1, \dots, x_n$ are variables and $a_1, \dots, a_n, c$ are numbers, is called *linear*. A *solution* of a linear equation is an n-tuple $(b_1, \dots, b_n)$ of numbers such that under the substitutions $x_i \coloneqq b_i$, for $i \in \{1, \dots, n\}$, the equation (1.1) is satisfied.

Example 1.0.5

The equation

$$3x_1 - 2x_2 + 4x_3 + x_4 = 5$$

is linear in variables x_1, x_2, x_3 and x_4. On the contrary,

$$3x_1x_2 - 4x_3^2 = 10$$

is **not** linear.

Definition 1.0.6 (Linear system)

Any set of linear equations in the given variables $x_1, \dots, x_n$ is called a *linear system*. A *solution* of a linear system is an n-tuple $(b_1, \dots, b_n)$ which solves every linear equation in the set.

Example 1.0.7

The set of equations

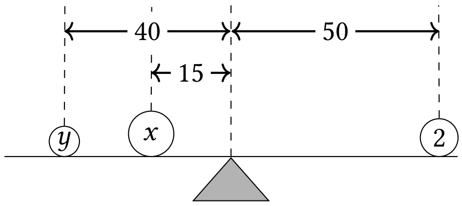
$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= 1 \\ x_1 \qquad - x_3 &= -1 \\ 2x_1 - 3x_2 + 3x_3 &= 0 \end{aligned}$$

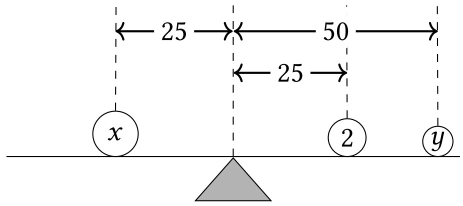
is a linear system whose solution is the triple $(0, 1, 1)$.

We proceed to discuss two trivial examples, which readers might have discussed in high school, naturally leading to linear systems. More sophisticated examples are presented in the applications section.

Example 1.0.8 (Static equations)

Suppose we have three objects – one with a mass of 2 and the other two with masses unknown. Experimentation produces these two balances.





For the weights to be in balance, the sums of *moments* on both sides of the scales must be identical one to another. A *moment* of an object is its distance from the centre of the scales times its mass. This condition yields a system of two linear equations

$$\begin{aligned} 15x + 40y &= 50 \cdot 2, \\ 25x &= 25 \cdot 2 + 50y. \end{aligned}$$

Or, after rearrangement (to stay true to our definition of linear equation),

$$\begin{aligned} 15x + 40y &= 50 \cdot 2, \\ 25x - 50y &= 25 \cdot 2. \end{aligned}$$

Example 1.0.9 (Chemical reactions)

Toluene, C_7H_8, mixes (under right conditions) with nitric acid, HNO_3, to produce trinitrotoluene (widely known as TNT), $C_7H_5O_6N_3$, along with dihydrogen monoxide, H_2O. If we want this chemical reaction to occur successfully, we must (among other things) ascertain we mix the constituents in the right proportion. In pseudo-chemical notation, the reaction to take place can be written as

$$x \cdot C_7H_8 + y \cdot HNO_3 \longrightarrow z \cdot C_7H_5O_6N_3 + w \cdot H_2O.$$

Comparing the number of atoms of each element before the reaction and afterwards (which must remain identical owing to the conservation of energy) yields the system

$$\begin{aligned} H:&\ 8x + 1y = 5z + 2w, \\ C:&\ 7x = 7z, \\ N:&\ 1y = 3z, \\ O:&\ 3y = 6z + 1w. \end{aligned}$$

In the next section, we devise an algorithm to solve any system of linear equations.

1.1 Gauss-Jordan Elimination

Probably the most well-known algorithm for solving a linear system is the *Gauss-Jordan elimination.* As its name partially implies, its heart lies in the successive *elimination* of variables until only a single linear equation in one variable stands unsolved. This is done by applying different *transformations* to the initial system that are guaranteed not to alter the solution. We're going to solve a linear system first and describe the general method second.

Problem 1.1.1

Solve the linear system

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2\,. \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

Solution. We aim to transform the system step by step to a form which allows us to (successively) eliminate all variables.

The first transformation entails a simple exchange of the first and third row.

`Swapped first and third row.`

$$\begin{array}{rcrcrcl} \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ & & & & 3x_3 & = & 9 \end{array}$$

Next, we scale the first row by a factor of 3.

`Scaled the first row by 3.`

$$\xrightarrow{\cdot 3} \quad \begin{array}{rcrcrcl} x_1 & + & 6x_2 & & & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ & & & & 3x_3 & = & 9 \end{array}$$

Finally, we subtract the first row from the second row. Said in a more foreshadowing manner, we add the (-1)-multiple of the first row to the second row.

`Subtracted the first row from the second.`

$$\cdot(-1) \quad \begin{array}{rcrcrcr} x_1 & + & 6x_2 & & & = & 9 \\ & - & x_2 & - & 2x_3 & = & -7 \\ & & & & 3x_3 & = & 9 \end{array}$$

These transformations have wrought the system into a state where it can be easily solved.

Indeed, we immediately see that the third equation implies $x_3 = 3$. Substituting into the second equation gives

$$-x_2 - 2 \cdot 3 = -7$$

whose solution is $x_2 = 1$. Finally, knowing the value of x_2, we can solve the first equation by another substitution. We get

$$x_1 + 6 \cdot 1 = 9,$$

thus $x_1 = 3$ and the triple $(3, 1, 3)$ is the *unique* solution of the system. ♠

Observant readers might have already identified the 'kinds' of transformations that were used in solving the linear system above. Nonetheless, we're about to spell them out.

The transformations that do not change the solution of a linear system include

(1) swapping two equations;

(2) scaling an equation by a non-zero constant;

(3) adding a multiple of an equation to *another* equation.

Note that transformations (2) and (3) come with sensible restrictions. Scaling an equation by 0 clearly changes the set of solutions of the system as it basically removes the equation entirely. Adding a multiple of an equation to *itself* suffers from the same problem; it might result in 'invalidating' the equation should the scaling factor be -1.

We know proceed to prove that transformations (1) - (3) truly do not alter the solutions of the initial system.

Theorem 1.1.2 (Gauss-Jordan)

The transformations (1) - (3) of a linear system outlined above do not change its solution set.

Proof. We will cover transformation (3) here. The proofs for transformations (1) and (2) are similar and thus left as an exercise.

Consider the linear system

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & c_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & c_2 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & c_m \end{array}$$

of m equations in variables $x_1, \ldots, x_n$ and let $(b_1, \ldots, b_n)$ be one of its solutions. Choose a constant k and add the k-multiple of the i-th equation to the j-th equation for some indices $i \neq j \in \{1, \ldots, m\}$. Hence, the j-th equation of the system gets replaced by

$$(a_{j,1} + k \cdot a_{i,1})x_1 + (a_{j,2} + k \cdot a_{i,2})x_2 + \cdots + (a_{j,n} + k \cdot a_{i,n})x_n = c_j + k \cdot c_i,$$

which can be rearranged to

$$a_{j,1}x_1 + a_{j,2}x_2 + \cdots + a_{j,n}x_n + k \cdot (a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n) = c_j + k \cdot c_i. \tag{1.2}$$

Since $(b_1, \ldots, b_n)$ is a solution of the original system, we know that

$$\begin{array}{ccccccccc} a_{i,1}b_1 & + & a_{i,2}b_2 & + & \cdots & + & a_{i,n}b_n & = & c_i \\ a_{j,1}b_1 & + & a_{j,2}b_2 & + & \cdots & + & a_{j,n}x_n & = & c_j \end{array}.$$

Substituting this into equation (1.2) gives

$$c_j + k \cdot c_i = c_j + k \cdot c_i,$$

hence $(b_1, \ldots, b_n)$ is also the solution of the transformed system, as required. ■

Exercise 1.1.3

Show that transformations (1) and (2) also don't change the set of solutions of the transformed linear system.

Definition 1.1.4 (Elementary operations)

The transformations (1) - (3) outlined above are called *elementary operations* or *row operations*.

As we've seen in problem 1.1.1, the application of transformations (1) - (3) has its purpose in preparing the system for a final back-substitution, where the values of all variables save the first in a row are known beforehand. A system which is 'ready' to be solved by back-substitution is said to be in *echelon form*.

Definition 1.1.5 (Echelon form)

In each row of a linear system, the first variable with a non-zero coefficient is called the row's *leading variable*.

A linear system is in *echelon form* (or *upper triangular form*) if the leading variable in each row is at least one column to the right of the leading variable in the row above and all rows filled with zeroes are at the bottom.

Example 1.1.6

The system

$$\begin{array}{rcrcrcr} x_1 & + & 6x_2 & & & = & 9 \\ & - & x_2 & - & 2x_3 & = & -7 \\ & & & & 3x_3 & = & 9 \end{array}$$

is in echelon form whereas

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & = & 9 \\ & & 3x_2 & - & 2x_3 & = & 2 \\ x_1 & & & - & x_3 & = & 0 \end{array}$$

is **not**.

For now, we shall employ intuition and a nibble of foresight to guide our transformation of a linear system into its echelon form. Later, we intend to present a precise algorithm (that computers also use) that achieves this.

Example 1.1.7

We're going to put the system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 0 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 3 \\ x_1 & - & 2x_2 & - & x_3 & = & 3 \end{array}$$

into echelon form and solve it using back-substitution. We'll label the rows of the system by Roman letters and denote transformations accordingly. For example, adding a 3-multiple of row one to row three would be written symbolically as $3 \cdot \mathtt{I} + \mathtt{III}$.

First, we need to get rid of the variable x_1 in rows $\mathtt{II}$ and $\mathtt{III}$. This can be done by subtracting adequate multiples of row $\mathtt{I}$.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 0 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 3 \\ x_1 & - & 2x_2 & - & x_3 & = & 3 \end{array} \xmapsto[-\mathtt{I}+\mathtt{III}]{-2\mathtt{I}+\mathtt{II}} \begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 0 \\ & - & 3x_2 & + & 3x_3 & = & 3 \\ & - & 3x_2 & - & x_3 & = & 3 \end{array}$$

We continue by subtracting row $\mathtt{II}$ from row $\mathtt{III}$.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 0 \\ & - & 3x_2 & + & 3x_3 & = & 3 \\ & - & 3x_2 & - & x_3 & = & 3 \end{array} \xmapsto{-\mathtt{II}+\mathtt{III}} \begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 0 \\ & - & 3x_2 & + & 3x_3 & = & 3 \\ & & & - & 4x_3 & = & 0 \end{array}$$

The system is now in echelon form. The equation in row $\mathtt{III}$ forces $x_3 = 0$. Substitution into row $\mathtt{II}$ immediately gives $x_2 = -1$ and one final substitution into row $\mathtt{I}$ yields $x_1 = 1$.

Hence, the solution of the system is the triple $(1, -1, 0)$.

Exercise 1.1.8

Using *Gauss-Jordan elimination* solve the systems from examples 1.0.8 and 1.0.9.

All the systems we've studied so far have had the same number of equations as variables. This of course need not be the case in general. Thankfully, Gauss-Jordan elimination can *always* be used to determine the solution set of a linear system. However, this set can also be empty or infinite in cases where the number of variables doesn't match the number of equations. The following examples illustrate this.

Example 1.1.9

This system has more equations than variables.

$$\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 1 \\ 2x_1 & + & x_2 & = & -3 \\ 2x_1 & + & 2x_2 & = & -2 \end{array} \tag{1.3}$$

Before we put it into echelon form and solve it, let us ponder what the solution set may look like. Intuitively, a linear equation is basically a 'restraint' or 'condition' on the range of possible values the present variables may attain. If there are three equations restraining only two variables, then this restraint may be too harsh and lead to the system having no solution at all. The only case where solution *does* exist involves one of the equations being *redundant* – providing no additional condition. Algebraically, this happens if said equation is a linear combination of the other two.

To draw a 'real-life' simile, imagine the price of an apple being \$5/kg and that of bananas, \$1.5/kg. Saying that 3 kg of apples and 4 kg of bananas cost, say, \$30 is simply false because we can calculate (by the information ere provided) that this amount actually costs \$21. The third condition on the price of apples and bananas contradicted the previous two; just as a third equation in a linear system in two variables can contradict the first two equations. We tend to call such systems *overdetermined* and will in time dedicate a section to finding a 'good' approximation of their solution.

To solve the system (1.3), we transform it into echelon form. First, we subtract twice the first row from the other two.

$$\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 1 \\ 2x_1 & + & x_2 & = & -3 \\ 2x_1 & + & 2x_2 & = & -2 \end{array} \xmapsto[-2\mathtt{I}+\mathtt{III}]{-2\mathtt{I}+\mathtt{II}} \begin{array}{rcrcr} x_1 & + & 3x_2 & = & 1 \\ & & -5x_2 & = & -5 \\ & & -4x_2 & = & -4 \end{array}$$

Finally, we add $(-4/5)$-times row **II** to row **III**.

$$\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 1 \\ & & -5x_2 & = & -5 \\ & & -4x_2 & = & -4 \end{array} \xmapsto{-(4/5)\mathtt{II}+\mathtt{III}} \begin{array}{rcrcr} x_1 & + & 3x_2 & = & 1 \\ & & -5x_2 & = & -5 \\ & & 0 & = & 0 \end{array}$$

Clearly, the third equation is *redundant* because it provides no condition binding the values of the variables. Back-substitution yields $x_2 = 1$ and $x_1 = -2$. As we've claimed (but not yet proven), row **III** is indeed a linear combination of rows **I** and **II**. In this particular case, it holds that $(2/5)\mathtt{I} + (4/5)\mathtt{II} = \mathtt{III}$.

Example 1.1.10

Contrast this system with the system (1.3) from example 1.1.9.

$$\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 1 \\ 2x_1 & + & x_2 & = & -3 \\ 2x_1 & + & 2x_2 & = & 0 \end{array}$$

In this case, the exact same row operations transform the system into

$$\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 1 \\ & & -5x_2 & = & -5\,, \\ & & 0 & = & 2 \end{array}$$

which clearly has no solution. This is a case of one equation of the system contradicting the other two.

Naturally, the ambitious, purposeful and *overdetermined* systems have their disinterested and vagrant sisters – the *underdetermined* systems. We style such the systems that are short on the number of variables as compared to the number of equations. As an example, consider the system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \end{array}, \tag{1.4}$$

which already **is** in echelon form. In spite of that, the typical back-substitution method is (without needed alterations) rendered unusable by the presence of two variables in the last row.

The system (1.4) is *underdetermined* in the sense that not enough equations are present to pinpoint a **unique** solution. Quite the opposite, this system has infinitely many solutions that all depend on as many parameters as many equations are missing to bind the values of the variables completely – in this case, *one*.

In cases like these, one typically proceeds the following way: let one of the variables (say, x_3) in the last equation be *a parameter*. For the sake of clarity, we shall rename x_3 to t to highlight its updated social status. The system thus looks like this.

$$\begin{aligned} x_1 + x_2 + t &= 0 \\ x_2 + t &= 0 \end{aligned}$$

Now, t is no longer a variable so the system is no longer underdetermined. We solve it briskly by setting $x_2 = -t$ and substituting this into the first equation to obtain $x_1 = 0$. Therefore, the solution to the system (1.4) is $(0, -t, t)$.

The dependence of the system's solution on a parameter naturally means that its solution set is infinite. Any choice for the value of t gives one particular solution – say $(0, -1, 1)$ or $(0, 0, 0)$.

We close the section off with a few exercises. The next section is dedicated to the geometric interpretation of linear systems.

Exercise 1.1.11

Use Gauss-Jordan elimination to solve the following system.

$$\begin{aligned} x_1 - x_3 &= 0 \\ 3x_1 + x_2 &= 1 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$$

Exercise 1.1.12

Each of the following systems is in echelon form. Determine their number of solutions (without calculation).

$$\begin{aligned} -3x_1 + 2x_2 &= 0 \\ - 2x_2 &= 0 \end{aligned} \qquad \begin{aligned} 2x_1 + 2x_2 &= 4 \\ x_2 &= 1 \\ 0 &= 4 \end{aligned} \qquad \begin{aligned} 2x_1 + x_2 &= 4 \end{aligned}$$

Exercise 1.1.13

Find the values of a, b and c that cause the graph of $f(x) = ax^2 + bx + c$ to pass through the points $(1, 2)$, $(-1, 6)$ and $(2, 3)$.

Exercise 1.1.14

Show that for all numbers a, b, c, d, j, k such that $ad - bc \neq 0$, the system

$$\begin{aligned} ax_1 + bx_2 &= j \\ cx_1 + dx_2 &= k \end{aligned}$$

has a *unique* solution.

1.2 Visualizing Linear Systems

In this, rather informal, section, we present a way to visualize linear systems in two and three variables and their solutions. Why two and three, you ask? The number of variables in a linear equation determines the *dimension* of the *geometric object* described by this equation. We shall soon provide the necessary definitions to make rigorous sense of the sentence previous. Intuitively, each variable represents a new 'direction' we're allowed to move in. Therefore, linear equations in two variables live in two-dimensional spaces and linear equations in three variables occupy three dimensions.

Nonetheless, the equations themselves (if non-trivial) never describe objects of the maximal possible dimension but of the dimension lower by one. This is because they establish a relationship between the variables – a relationship where one variable grows entirely dependent on the rest, essentially 'locking' a single direction of movement. Think of it like this: a linear equation in two variables is a sort of order, telling you that for every step forward you must also make (say) two steps to the right, thereby rendering you unable to ever walk in a direction different from the initial.

We proceed to show that the objects described by linear equations in two variables are *straight lines*. Said 'objects described' are formally the sets of points satisfying given equations. For instance, the object described by the equation $3x + 2y = 4$ is the set

$$L \coloneqq \{(x, y) \in \mathbb{R}^2 \mid 3x + 2y = 4\}.$$

Before we move on, we need establish an important fact. What is a *straight line* **exactly**? Wishing not to cheat and define straight line as the object described by a linear equation, we employ a more geometric approach to the definition. As we hope dear readers agree, a (one-dimensional) object is *straight* if moving along it requires 'keeping the initial direction', that is, always moving the same number of steps upward for a given number of steps rightward, or vice versa. In other words, the *ratio* between the number of steps upward and rightward must remain constant. We encourage kind readers to absorb that this particular property is what distinguishes *curved* objects from *straight* ones.

Figure 1.2 inspires the following definition.

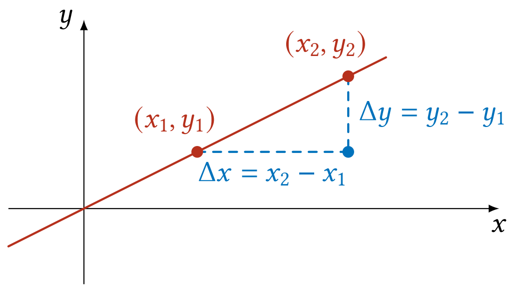


Figure 1.2: The 'definition' of straightness. The ratio $\Delta y/\Delta x$ must remain **constant**. It is habitually referred to as the *slope* of the line.

Definition 1.2.1 (Straight line)

An **infinite** subset $L \subseteq \mathbb{R}^2$ is called a *straight line* if for all triples of points (x_1, y_1), (x_2, y_2), $(x_3, y_3) \in L$ it holds true that either

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}, \tag{1.5}$$

or $x_1 = x_2 = x_3$ (a vertical line).

We proceed to show that the all the points in the plane satisfying a linear equation form a straight line. This is exceedingly easy. Suppose we have three solutions (x_1, y_1), (x_2, y_2) and (x_3, y_3) satisfying the equation $ax + by = c$, where $a, b, c \in \mathbb{R}$ and at least one of a, b is not zero. In other words, we have $ax_i + by_i = c$ for $i \in \{1, 2, 3\}$.

We've had to exclude the case $a = b = 0$ because the set of solutions of the linear equation $0 = c$ is never a straight line. If $c \neq 0$, it is empty, and if $c = 0$, it equals $\mathbb{R}^2$.

Assume first that $b = 0$. Then, $x_i = c/a$ and so $x_1 = x_2 = x_3$. Hence, in this case, the set of solutions is indeed a straight line.

In case $b \neq 0$, we may rearrange

$$y_i = \frac{c - ax_i}{b}.$$

Plugging this into (1.5) gives

$$\frac{(c - ax_2) - (c - ax_1)}{b(x_2 - x_1)} = \frac{(c - ax_3) - (c - ax_1)}{b(x_3 - x_1)}. \tag{1.6}$$

Simple calculation yields

$$\frac{(c - ax_2) - (c - ax_1)}{b(x_2 - x_1)} = \frac{a(x_1 - x_2)}{b(x_2 - x_1)} = -\frac{a}{b}$$

and similarly for $(y_3 - y_1)/(x_3 - x_1)$. Hence, both sides of (1.6) equal $-a/b$ and the proof is complete.

1.2.1 Two-dimensional Linear Systems

We dedicate a subsection to the visualization of linear systems in two variables and their solutions. As already established, a linear equation in two variables represents a straight line. A solution to a linear system in two variables is a pair of real numbers (equivalently, a point in the real plane) which lies on every straight line determined by the equations of the system. Simply put, the solution of a linear system in two variables is the *intersection* of all objects described by its equations.

An 'ideal' linear system in two variables contains two linear equations describing distinct lines. One such system is

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2 \end{aligned}$$

with solution $(1, 1)$ and whose visual depiction is provided in figure 1.3.

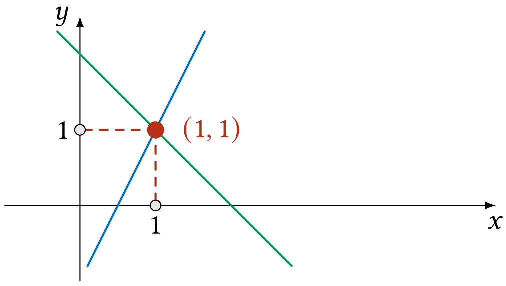


Figure 1.3: Well-determined linear system in two variables with solution $(1, 1)$.

An easily proven fact (which we shall eventually prove in greater generality) that follows immediately from the geometric view reads that a linear system in two variables with two *distinct* linear equations always has a solution – the intersection point of the corresponding lines.

A linear system in two variables can only be underdetermined should it feature just one non-trivial linear equation (or, equivalently, many identical linear equations). In this case, assuming the system consists of the single linear equation

$$ax + by = c,$$

its solution set is spanned by the points $(x, (c - ax)/b)$, for $x \in \mathbb{R}$, or $(c/a, y)$, for $y \in \mathbb{R}$, should $b = 0$. Geometrically, all points lying on the line determined by its sole equation solve the underdetermined linear system.

Overdetermined linear systems in two variables are considerably more interesting. There are four possible arrangements of three lines in the plane, they're depicted in figure 1.4.

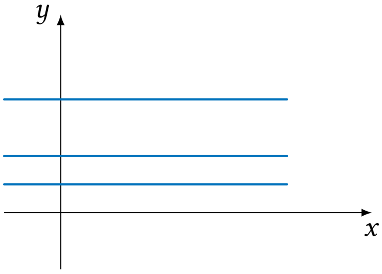
It is clear that in cases (a), (b) and (c) in figure 1.4, the linear system has no solution. In case (d), the system does have a solution but one of the lines is redundant – it can in fact (as we've claimed before) be written as a linear combination of the other two lines. By putting the linear system in question into echelon form, we can easily deduce which of the depicted cases emerged true.

Indeed, consider the system

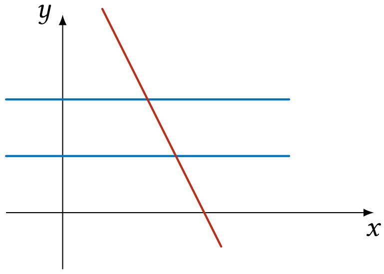
$$\begin{aligned} x + y &= 2 \\ 2x + 2y &= 3\,. \\ -x - y &= 1 \end{aligned}$$

By subtracting $2\mathtt{I}$ from $\mathtt{II}$ and adding $\mathtt{I}$ to $\mathtt{III}$, we put it into the following echelon form:

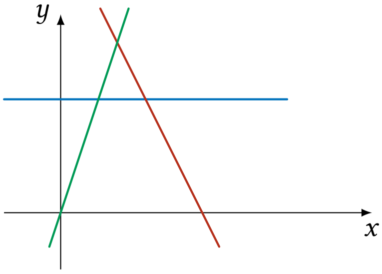
$$\begin{aligned} x + y &= 2 \\ 0 &= -1\,. \\ 0 &= 3 \end{aligned}$$



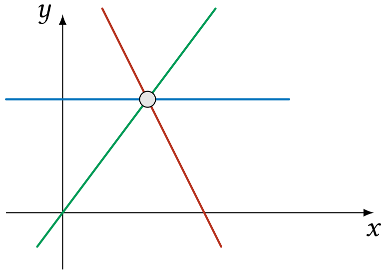
(a) All three lines parallel.



(b) Two of the three lines parallel.



(c) No lines parallel without a common intersection.



(d) No lines parallel with a common intersection.

Figure 1.4: All the possible arrangements of three lines in the real plane.

Since two of the three equations have no solutions, case (a) arises – the three lines are all parallel to one another.

As yet another example, we present in all its glory the system

$$\begin{array}{rcrcr} 2x & + & y & = & 5 \\ x & & & = & 2\,. \\ 3x & - & y & = & 0 \end{array}$$

By swapping $\mathtt{I}$ with $\mathtt{II}$, then subtracting $\mathtt{II} - 2\mathtt{I}$ and $\mathtt{III} - 3\mathtt{I}$, we get

$$\begin{array}{rcrcr} x & & & = & 2 \\ & & y & = & 1\,. \\ & - & y & = & -6 \end{array}$$

Each of the equations has a solution individually but the conditions bestowed on y by equations $\mathtt{II}$ and $\mathtt{III}$ are contradictory. Meaning, any pair of equations in the system can be satisfied simultaneously but all three equations can't. This is the situation depicted in figure 1.4, (c).

Exercise 1.2.2

Find examples of linear systems of three equations in two variables that correspond to parts (b) and (d) of figure 1.4.

1.2.2 Three-dimensional Linear Systems

Stepping up the game a little, we're taking a look at linear systems in three variables. Just as a linear equations in two variables are lines in the real plane, linear equations in three variables depict geometric objects called 'planes' in the three-dimensional real space, $\mathbb{R}^3$. They form the last class of linear systems that can be efficiently visualized; with linear systems in more variables being generally out of our perceptive reach.

Planes are the *straight* objects in three-dimensional kind of sense. They lock one direction of movement by making one variable wholly dependent on the other two. An illustration is provided in figure 1.5.

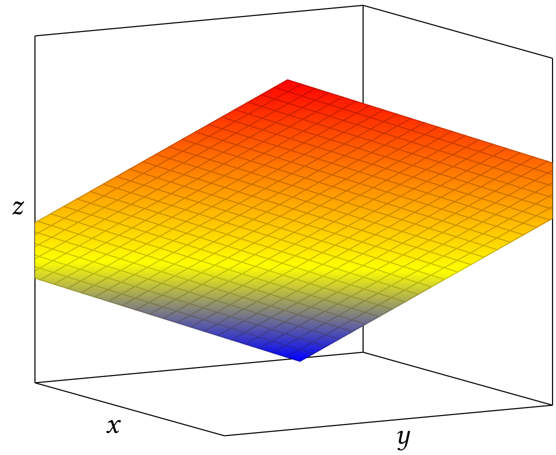


Figure 1.5: A plane defined by the equation $2x - y - 3z = -3$.

An *underdetermined* system in three variables can contain either one or two linear equations. In the former case, only one variable is dependent on the other two – we shall often call the independent variables by names such as *free variables* or *parameters*. Both these names signify that a substitution of any pair of real numbers in lieu of the two *free variables* yields a solution of the system.

For instance, the linear equation in figure 1.5 is effectively a linear system in three variables. We can choose any of the three variables to be dependent and leave the other two free, giving thus three different descriptions of *the same* solution set. The following equation (1.7) shows all of them with the chosen dependent variable written on the left in typewriter font.

$$\begin{aligned} \mathtt{x} &: \left\{\left(\frac{y+3z-3}{2}, y, z\right) \mid y, z \in \mathbb{R}\right\} \\ \mathtt{y} &: \{(x, 2x-3z+3, z) \mid x, z \in \mathbb{R}\} \\ \mathtt{z} &: \left\{\left(x, y, \frac{2x-y+3}{3}\right) \mid x, y \in \mathbb{R}\right\} \end{aligned} \tag{1.7}$$

Linear systems in three variables and two equations are also underdetermined. Geometrically, they correspond to arrangements of two planes in space. Those two planes can either be parallel – leading to the system having no solution – or not – intersecting in a straight line describable as a set of triples with exactly one free variable. In a case similar to two-dimensional linear systems, putting the system in question into echelon form *can* reveal (albeit not always) its geometric nature.

Consider the system

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 2 \\ 2x & - & 2y & + & 4z & = & 9 \end{array}. \tag{1.8}$$

Subtracting $\mathtt{II} - 2 \cdot \mathtt{I}$ produces

$$\begin{aligned} x - y + 2z &= 2 \\ 0 &= 5 \text{'} \end{aligned}$$

clearly a system with no solution. Subsequently, the two corresponding planes are parallel to each other. See them depicted in figure 1.6.

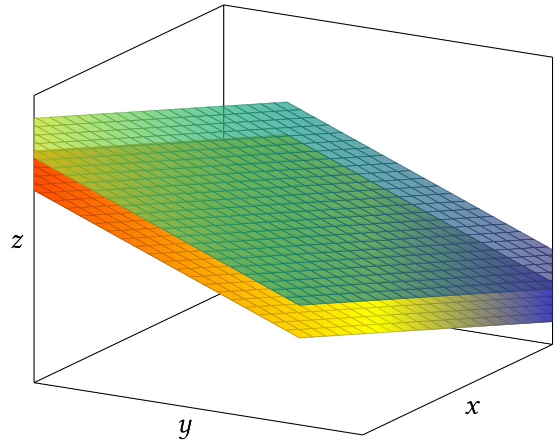


Figure 1.6: The two parallel planes from the system (1.8).

A system of two non-parallel planes is presented below.

$$\begin{aligned} x - y + 2z &= 2 \\ 2x + 3y - z &= -1 \end{aligned} \tag{1.9}$$

By subtracting, once again, $\mathtt{II} - 2 \cdot \mathtt{I}$, we put into the following echelon form.

$$\begin{aligned} x - y + 2z &= 2 \\ 5y - 5z &= -5 \end{aligned}$$

The algorithm of Gauss-Jordan elimination limits our choice of parameters to the ones left in the last row. We are hence to set either y or z loose while caging the latter. Custom dictates to label as parameters all variables but the first of the last row, making z the victor. The rest is just back-substitution. We calculate $y = z - 1$ and substitute into the first equation to receive

$$x - (z - 1) + 2z = 2, \quad \text{hence} \quad x = 1 - z.$$

It follows that *one possible* description of the solution set of the system (1.9) is

$$\{(1 - z, z - 1, z) \mid z \in \mathbb{R}\}.$$

See it depicted in figure 1.7.

Reaching the apex of 'ideal' linear systems in three variables and three equations, we stop to ponder the number of arrangements of three planes in three-dimensional space. There are two obvious ones:

(1) All three planes are parallel to each other.

(2) Only two planes are parallel to each other.

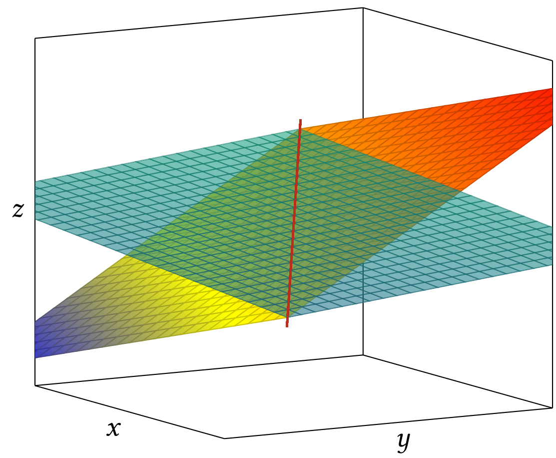


Figure 1.7: The two non-parallel planes from the system (1.9) and their intersection.

Corresponding to (1), resp. (2), is a linear system with two equations, resp. one equation, with no solution.

As an example, consider the system

$$\begin{aligned} -x + 2y - z &= 4 \\ x - 6y + z &= 1\,. \\ x - 2y + z &= 3 \end{aligned} \tag{1.10}$$

Its echelon form looks like this:

$$\begin{aligned} -x + \quad 2y - z &= 4 \\ -4y \qquad &= 5\,. \\ 0 &= 7 \end{aligned}$$

Clearly, the third equation has no solution while the second does. This fact alone, alas, carries not the full picture. To successfully determine that this system corresponds to case (2) above, one need additionally take note of the fact that the left side of row `III` of (1.10) is a (-1)-multiple of row `I`, meaning the two planes in question are parallel.

The system (1.10) is shown in figure 1.8.

Finally, there are three other possible arrangements of three planes in space:

(3) non-parallel planes that fail to have a common intersection (the so-often-called ‘tent' configuration);

(4) non-parallel planes that meet in a single point;

(5) non-parallel planes that meet in a single line.

The echelon form of a linear system is not enough to distinguish case (2) from case (3). For instance, the echelon form of the system

$$\begin{aligned} x + 2y - z &= -1 \\ 2x - 3y + 2z &= 4 \\ -x + 5y - 3z &= 0 \end{aligned} \tag{1.11}$$

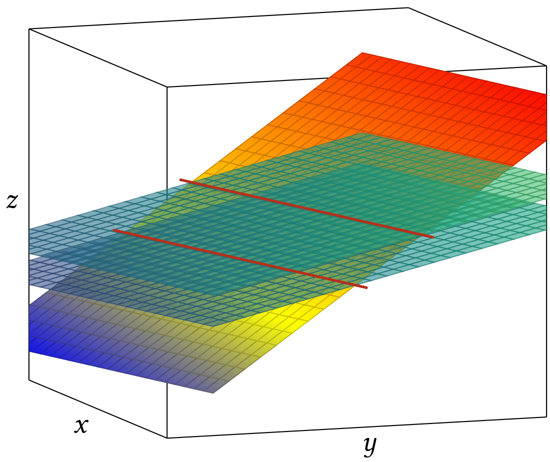


Figure 1.8: Depiction of the system (1.10).

can easily be computed to be

$$\begin{aligned} x + \ 2y - \ z &= -1 \\ -7y + 4z &= \ \ 6\,. \\ 0 &= \ \ 5 \end{aligned}$$

Notice the likeness to the echelon form of the previously studied system (1.10). One equation without solution, two solvable. Its visual representation is to be found in

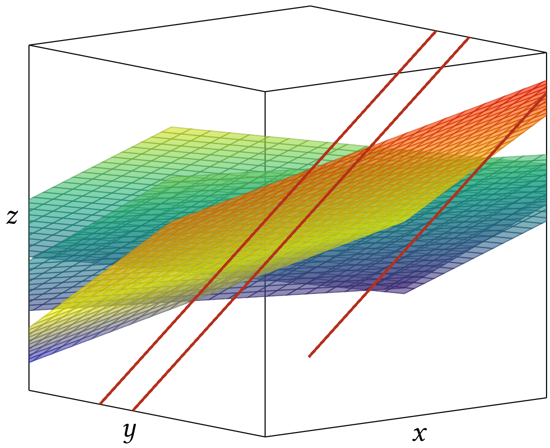


Figure 1.9: Example of the 'tent' arrangement of planes in (1.11).

Without delving into visualisations of linear systems in three variables and more than three equations – which do not actually bring anything new to the table – we conclude the section with a few exercises.

Exercise 1.2.3

Draw the following linear systems.

$$\begin{aligned} 2x + y &= 1 \\ 3x + 2y &= 3 \end{aligned} \qquad \begin{aligned} -x + y &= 2 \\ 2x - 2y &= 5 \end{aligned} \qquad \begin{aligned} -x - y &= 1 \\ 3x + 2y &= 0 \end{aligned}$$

Exercise 1.2.4

Without depicting them visually, determine the arrangement of planes corresponding to the linear system below.

$$\begin{aligned} 2x - y + z &= 3 \\ x - 3y + 4z &= 1 \\ x + 2y - 3z &= 2 \end{aligned}$$

Exercise 1.2.5

Find linear systems in three variables and three equations corresponding to cases (1), (4) and (5) in the text above.

1.3 Describing Solution Sets of Linear Systems

In section 1.2, we studied specific (simple) classes of linear systems and touched upon a few important concepts, including, but not limited to, *parameters*, *free variables*, *underdetermined* and *overdetermined* systems.

We continue down this road and bring a general description of solution sets of linear systems. Before we formulate the result we shall endeavour to prove in this section, we introduce a few pieces of notation which are going to allow us to manipulate linear systems more efficiently. Do note that behind these mere 'pieces of notation' there lies hidden a much deeper geometric meaning, to be uncovered in later chapters.

Definition 1.3.1 (Matrix)

An $m \times n$ *matrix* is an array of numbers with m rows and n columns. The numbers are then called *entries* of the matrix.

Matrices allow us to write linear systems in a much more succinct manner. For example, the system

$$\begin{aligned} -x + y &= 2 \\ 2x - 2y &= 5 \end{aligned}$$

can be written using a matrix like this:

$$\left(\begin{array}{rr|r} -1 & 1 & 2 \\ 2 & -2 & 5 \end{array}\right),$$

abusing the fact that the same variables are piled in a single column and each row is a single linear equation. The bar on the right side simply serves to divide left sides of the equations from right ones.

Matrices make (amongst other things) Gauss-Jordan elimination easier to perform and keep track of its progress. The matrix of the eliminated system looks like this

$$\left(\begin{array}{rr|r} -1 & 1 & 2 \\ 0 & 0 & 9 \end{array}\right)$$

and has been reached by the row operation $\mathtt{II} + 2\mathtt{I}$.

Certain matrices are special (for reasons soon to be revealed) and we call them *vectors*.

Definition 1.3.2 (Vector)

A *column vector* is an $n \times 1$ matrix (that is, matrix with a single column) and a *row vector* is a $1 \times n$ matrix (a matrix with a single row). As column vectors are the 'default', we call them simply *vectors*.

There exists an obvious bijection between tuples $(v_1, \ldots, v_n)$ and column vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. Consequently, we say that a vector $\mathbf{v}$ with entries $v_1, \ldots, v_n$ *solves* a linear equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$$

if the tuple $(v_1, \ldots, v_n)$ does.

The addition of vectors and their multiplication by a number are defined naturally.

Definition 1.3.3 (Adding vectors)

Given vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

their *sum* is defined as the vector

$$\mathbf{u} + \mathbf{v} \coloneqq \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}.$$

Definition 1.3.4 (Multiplying vector by a number)

Given a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

and a number c, the *scalar c-multiple of* $\mathbf{v}$ is the vector

$$c\mathbf{v} \coloneqq \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix}.$$

The multiplying number c is often referred to as a *scalar*.

We further need to discuss the concept of *free variables* and *parameters*.

In the previous section, we described the solution set of the system (1.7) using three different ways. In each case, two of the variables were independent and the third was their linear combination. We style the two independent variables, *parameters*. Vaguely said, a *parameter* is a variable on the value whereof other variables depend.

The question arises: 'Which variables to choose as *parameters*?' The answer descends: 'Why, of course, my child, choose the *free variables*!' After the process of Gauss-Jordan elimination, a preceding row always has more variables present than its neighbour downstairs. Occasionally, the number of additional variables is larger than one. It is clear that in such cases, back-substitution cannot determine the values of those additional variables exactly (as it leads to a linear equation in more than one variable). All save one of those variables are to be chosen as *parameters* and serve the noble purpose of describing the value of the last variable standing. Custom dictates that all but the leftmost variable in such a row are labelled *free* and the leftmost variable called a *pivot*. In light of this, the heavenly answer can be decrypted – the *free* variables shall serve as *parameters* and the value of the *pivot* be written as a linear combination of free variables.

To understand explicitly the preceding paragraph, consider the eliminated system

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Row `II` has two more variables than row `III` and row `I` also wins by two variables over row `II`. As the holy text states, the variable x_4 of the fourth column is *free*, whereas x_3 is a *pivot*. Therefore, x_4 now serves as a parameter and from row `II` we get the relation

$$x_3 = -x_4 + 4.$$

Row `I` brings in a new free variable – x_2 – and a new pivot – x_1. Using the fact that x_3, the pivot from row `II`, is already expressed as a linear combination of free variables, we substitute into row `I` to get

$$x_1 + 2x_2 - (-x_4 + 4) + 3x_4 = 1.$$

A tiny bit of cheap computation yields

$$x_1 = -2x_2 - 4x_4 + 5.$$

Thereby, all the pivots of the system are expressed as linear combinations of free variables. The set of solutions of this system can be described as the set of quadruples $(-2x_2 - 4x_4 + 5, x_2, -x_4 + 4, x_4)$.

Visualisation of the concepts of pivots and free variables is provided in figure 1.10.

Using vectors, the solution set of the currently studied system can be expressed quite elegantly. First, the quadruple $(-2x_2 - 4x_4 + 5, x_2, -x_4 + 4, x_4)$ corresponds to the column vector

$$\begin{pmatrix} -2x_2 - 4x_4 + 5 \\ x_2 \\ -x_4 + 4 \\ x_4 \end{pmatrix}.$$

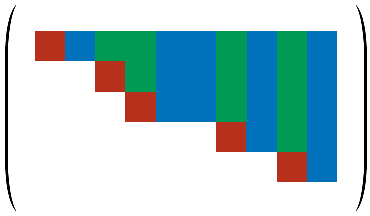


Figure 1.10: Visual depiction of an eliminated matrix. Red variables are pivots, blue ones are free and green ones are pivots from lower rows.

This vector can be further broken down into three vectors, two for the free variables and one for the constants. Explicitly,

$$\begin{pmatrix} -2x_2 - 4x_4 + 5 \\ x_2 \\ -x_4 + 4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 4 \\ 0 \end{pmatrix}.$$

Take note that the last vector is a *particular* solution of the system obtained by setting $x_2 = x_4 = 0$. Adding random multiples of the vectors

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -4 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

to this particular solution generates more solutions of the system.

Let's make another example, shall we? In this eliminated system of two equations in three variables,

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right),$$

the variables x_1 and x_3 are pivots and x_2 is free. Judging from the previous example, we should be able to express its solution as $\mathbf{u} + x_2\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are vectors and, furthermore, $\mathbf{u}$ is some particular solution of the system at hand.

Indeed, choosing x_2 to be a parameter, back-substitution yields $x_3 = 1$ and $x_1 = 2 - x_2 + x_3 = -x_2 + 3$. Hence, every vector of the shape

$$\begin{pmatrix} -x_2 + 3 \\ x_2 \\ 1 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

solves the system.

We're now equipped to formulate a result about the 'shape' of a linear system's solution set with a rather far-reaching importance.

Theorem 1.3.5 (Solution set of a linear system)

The solution set of every linear system can be written in the form

$$\{\mathbf{u} + t_1\mathbf{v_1} + t_2\mathbf{v_2} + \ldots + t_l\mathbf{v_l}\},$$

where $\mathbf{u}$ *is a particular solution,* $\mathbf{v_1}, \ldots, \mathbf{v_l}$ *are vectors and* $t_1, \ldots, t_l$ *are parameters corresponding to the free variables of the eliminated system.*

Before the proof, we formulate an immediate corollary.

Corollary 1.3.6 (Number of solutions of a linear system)

Every linear system has zero, one or infinitely many solutions.

Proof. Referring to the form of the solution set of a linear system from theorem 1.3.5, we distinguish three cases:

(1) The vector $\mathbf{u}$ doesn't exist, therefore the system has *no solution*.

(2) The vector $\mathbf{u}$ exists and there are no free variables (only pivots) in the eliminated system. In this case, the solution is *unique*.

(3) The vector $\mathbf{u}$ exists and there is at least one free variable to be found in the eliminated system. In this case, the substitution of any number in place of the free variables generates a solution. Hence, there are *infinitely many*. ■

On our way to the proof of theorem 1.3.5, we make a preparatory step. We call a linear system *homogeneous* if the right side of its every equation is 0. Concretely, a *homogeneous* linear system assumes the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & 0 \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & 0. \end{array}$$

Notice that this system always has at least one solution, namely the vector $\mathbf{0}$ – the vector whose every entry is 0. We shall first prove the following proposition.

Proposition 1.3.7 (Solution set of a homogeneous linear system)

The solution set of a homogeneous *linear system can be written in the form*

$$\{t_1\mathbf{v_1} + t_2\mathbf{v_2} + \ldots + t_l\mathbf{v_l}\},$$

where $\mathbf{v_1}, \ldots, \mathbf{v_l}$ *are vectors and* $t_1, \ldots, t_l$ *are parameters corresponding to the free variables of the eliminated system.*

Proof. We consider a homogeneous linear system as above:

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,n}x_n & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,n}x_n & = & 0 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \dots & + & a_{m,n}x_n & = & 0. \end{array} \tag{1.12}$$

Firstly, in the light of theorem 1.1.2, we may assume that the system has been reduced to echelon form. We shall prove that every pivot can be written as a linear combination of free variables by induction on the number k of rows (counting from the bottom) already substituted into. This approach basically mimics and formalises the traditional back-substitution process.

Without loss of generality, we may also assume that no rows full of zeroes are left at the bottom of the system, as those can be ignored. Hence, the last row of the eliminated linear system looks like this:

$$a_{m,j}x_j + a_{m,j+1}x_{j+1} + \dots + a_{m,n}x_n = 0$$

for adequate $1 \leq j \leq n$ and $a_{m,j} \neq 0$. Here, x_j is the pivot and $x_{j+1}, \dots, x_n$ are free. This gives the expression

$$x_j = -\frac{1}{a_{m,j}}(a_{m,j+1}x_{j+1} + \dots + a_{m,n}x_n)$$

of the pivot x_j as a linear combination of the free variables $x_{j+1}, \dots, x_n$. So, the result holds for $k = 0$.

Now, supposing all pivots in the last k rows of the system (1.12) have been expressed as linear combinations of free variables, we write the pivot of the $(m-k)$-th row (or $(k+1)$-st from the bottom) also as a linear combination of free variables. Again, there exists some smallest $1 \leq i \leq n$ such that $a_{m-k,i} \neq 0$. The $(m-k)$-th row is thus

$$a_{m-k,i}x_i + a_{m-k,i+1}x_{i+1} + \dots + a_{m-k,n}x_n = 0.$$

Performing an analogous computation gives

$$x_i = -\frac{1}{a_{m-k}}(a_{m-k,i+1}x_{i+1} + \dots + a_{m-k,n}x_n). \tag{1.13}$$

All the variables found on the right side of (1.13) are either free or pivots from lower rows. However, by the induction hypothesis, all pivots from lower rows have already been expressed as linear combinations of free variables. Simple substitution now yields an expression of x_i as a linear combination of free variables. With l denoting the number of free variables of the eliminated system and with the solution vector having been split into a sum of scalar multiples of free variables, the result is proven. ■

Remark 1.3.8

By proposition 1.3.7 above, a *homogeneous linear system* has either one or infinitely many solutions since the n-tuple $(0, 0, \dots, 0)$ always solves it.

Example 1.3.9

The echelon form of the homogeneous linear system

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ -2 & -3 & 1 & 0 \\ 3 & 7 & 1 & 0 \end{array}\right)$$

is equal to

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence, the variables x_1 and x_2 are *pivots* and x_3 is *free*. Back-substitution gives $x_2 = -x_3$ and $x_1 = 2x_3$. The solution set of this system is thus given by all the vectors

$$\begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

In the notation of proposition 1.3.7, we'd have

$$t_1 = x_3 \quad \text{and} \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

We've reached the climax of the section – the proof of theorem 1.3.5. Armed with proposition 1.3.7, it behoves us to merely work a link between the solution set of a linear system and its corresponding homogeneous system.

Proof (of theorem 1.3.5). Let

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \ldots & + & a_{1,n}x_n & = & c_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \ldots & + & a_{2,n}x_n & = & c_2 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \ldots & + & a_{m,n}x_n & = & c_m. \end{array} \tag{1.14}$$

be the linear system in question. We proceed to show that its every solution is of the form $\mathbf{u} + \mathbf{h}$, where $\mathbf{u}$ is a particular solution and $\mathbf{h}$ is a solution of the corresponding homogeneous linear system (as in (1.12)), and that, contrariwise, for every solution $\mathbf{h}$ of the homogeneous linear system, the vector $\mathbf{u} + \mathbf{h}$ solves the system (1.14), assuming $\mathbf{u}$ does.

Let's start with the former. Denote by $\mathbf{u}$ any fixed solution of system (1.14). We want to show that any other solution $\mathbf{v}$ of the same system can be written as $\mathbf{v} = \mathbf{u} + \mathbf{h}$ where $\mathbf{h}$ solves the corresponding homogeneous linear system (1.12). For this, it is clearly enough to show that $\mathbf{h} \coloneqq \mathbf{v} - \mathbf{u}$ solves the system (1.12). Substituting $x_i \coloneqq h_i = v_i - u_i$ into the left side of the j-th equation of (1.12) yields

$$a_{j,1}(v_1 - u_1) + a_{j,2}(v_2 - u_2) + \ldots + a_{j,n}(v_n - u_n),$$

which can be broken into

$$(a_{j,1}v_1 + \ldots + a_{j,n}v_n) - (a_{j,1}u_1 + \ldots + a_{j,n}u_n).$$

As both **u** and **v** solve (1.14), we know that

$$a_{j,1}v_1 + \ldots + a_{j,n}v_n = c_j = a_{j,1}u_1 + \ldots + a_{j,n}u_n,$$

and thus

$$(a_{j,1}v_1 + \ldots + a_{j,n}v_n) - (a_{j,1}u_1 + \ldots + a_{j,n}u_n) = c_j - c_j = 0.$$

This is true for all $1 \leq j \leq m$, hence the result.

As for the inverse inclusion, we must show that $\mathbf{u}+\mathbf{h}$ where $\mathbf{h}$ is an arbitrary solution of (1.12) also solves (1.14), assuming that **u** solves it. This time, we substitute $x_i \coloneqq u_i + h_i$ into the left side of the j-th equation of (1.14) and get

$$a_{j,1}(u_1 + h_1) + \ldots + a_{j,n}(u_n + h_n) = (a_{j,1}u_1 + \ldots + a_{j,n}u_n) + (a_{j,1}h_1 + \ldots + a_{j,n}h_n). \quad (1.15)$$

We know that

$$\begin{aligned} a_{j,1}u_1 + \ldots + a_{j,n}u_n &= c_j, \\ a_{j,1}h_1 + \ldots + a_{j,n}h_n &= 0. \end{aligned}$$

Thus, the expression (1.15) equals $c_j + 0 = c_j$ and $\mathbf{u} + \mathbf{h}$ solves the j-th equations of (1.14). Again, this being true for all $1 \leq j \leq m$ proves this inclusion and with it, the theorem. ■

Example 1.3.10

We change the right side of the system from example 1.3.9 to produce

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 5 \\ -2 & -3 & 1 & 0 \\ 3 & 7 & 1 & 5 \end{array}\right),$$

and after elimination:

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 5 \\ 0 & 5 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Typical back-substitution yields $x_2 = 2 - x_3$ and $x_1 = -3 + 2x_3$. The solution can thus be written as

$$\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$$

where $\mathbf{u} = \left(\begin{smallmatrix} -3 \\ 2 \\ 0 \end{smallmatrix}\right)$ is a particular solution of the system and $\mathbf{h} = x_3\left(\begin{smallmatrix} 2 \\ -1 \\ 1 \end{smallmatrix}\right)$ is for any x_3 a solution of the corresponding homogeneous linear system from example 1.3.9.

Exercise 1.3.11

Solve each of the systems below using matrix notation. Write the solution in the form of theorem 1.3.5.

$$\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned} \qquad \begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 - x_3 \phantom{{}+x_4} &= 3 \\ 2x_1 + x_2 \phantom{{}-x_3} + x_4 &= 4 \\ x_1 - x_2 + x_3 + x_4 &= 1 \end{aligned}$$

Exercise 1.3.12

Show that any five points in the plane $\mathbb{R}^2$ lie on a common *conic section*, that is, they all satisfy an equation of the form

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

for some $a, \ldots, f \in \mathbb{R}$.

Exercise 1.3.13

Prove that if $\mathbf{s}$ and $\mathbf{t}$ are solutions of a homogeneous linear system, then so are

(1) $\mathbf{s} + \mathbf{t}$,

(2) $3\mathbf{s}$,

(3) $k\mathbf{s} + m\mathbf{t}$ for any numbers k, m.

What is wrong with the following argument: 'These three show that if a homogeneous system has one solution, then it has many solutions – any multiple of a solution is another solution, and any sum of solutions is also a solution – so there are no homogeneous linear systems with exactly one solution.'?

1.4 Applications

In this final section of chapter 1, we focus on some 'real-world' applications of linear systems and, more generally, on methods of solving linear systems using computers.

The software we shall employ toward this end styles SageMath. It's a free open-source mathematics software capable of numeric and symbolic manipulation of objects from various fields of mathematics, linear algebra included. It can be installed on most operating systems following the official guide.

SageMath is essentially a terminal-based software and out of the box offers no graphical user interface. Upon launch, the user is greeted by a screen similar to this one:

```
┌────────────────────────────────────────────────────────────────────┐
│ SageMath version 10.4, Release Date: 2024-07-19                    │
│ Using Python 3.12.8. Type "help()" for help.                       │
└────────────────────────────────────────────────────────────────────┘
sage:
```

SageMath is mainly built upon C and Python and is *interpreted*, meaning every piece of code is immediately run without a need for compilation.

Before we focus on applications of linear systems in fields like *economics* and *physics*, we need to learn to solve them using SageMath. By far the simplest way to encode linear systems is using matrix notation. SageMath features the `Matrix` class which hosts a plethora of methods for matrix manipulation we are going to make great use of in time.

Example 1.0.8 contains the system

$$\left(\begin{array}{cc|c} 15 & 40 & 100 \\ 25 & -50 & 50 \end{array}\right).$$

Let us solve it using SageMath. The **Matrix** class expects a matrix to be defined as a list of rows which are themselves lists of elements. In addition, we may specify the number set wherein the elements lie. For example,

```
sage: A = Matrix(ZZ, [
....:     [15, 40],
....:     [25, -50],
....: ])
```

creates the matrix

$$\begin{pmatrix} 15 & 40 \\ 25 & -50 \end{pmatrix}$$

with entries in $\mathbb{Z}$, the integers. This has a caveat. When we tell SageMath our matrix contains entries *exclusively* in $\mathbb{Z}$, it will fulfil our wish with utmost conscientiousness. This means that `A` can never contain anything but integers. A problem might emerge should we wish to put it into echelon form for example. Gauss-Jordan elimination of the matrix `A` would clearly require subtracting $(25/15)$-multiple of row **I** from row **II**. Assuming the entries of `A` are solely integers, such an operation is not permitted. The **Matrix** class has an in-built method for Gauss-Jordan elimination. Let us try to use it.

```
sage: A.echelon_form()
[  5 130]
[  0 350]
```

The result is somewhat unexpected. Thankfully or unfortunately, SageMath is clever enough to know that simply following the algorithm of the Gauss-Jordan elimination does not yield an integer matrix. So, it instead follows the algorithm and then multiplies the matrix by the least common multiple of the denominators of all entries in order to yield an integer matrix. Beware however, that trying to solve linear systems whose solutions are rational with integer matrices might result in an error. To stay in the clear, we instead use the real numbers throughout the calculation. Not specifying the number set would lead to SageMath 'guessing' it based on the values of the entries – which are all integral.

We thus rewrite our matrix `A` like this:

```
sage: A = Matrix(RR, [
....:     [15, 40],
....:     [25, -50],
....: ])
```

We will also create a **vector** (a **Matrix** with a single row basically) of the right hand side of the studied system.

```
sage: b = vector(RR, [100, 50])
```

The **Matrix** method for solving a system with a given **vector** of right hand side is called **solve_right**. Using it gives

```
sage: A.solve_right(b)
(4.00000000000000, 1.00000000000000)
```

Since we explicitly required SageMath solve the system over the real numbers, it returned the solution as a pair of decimals rounded based on a default precision parameter. We would instead prefer to write the solution as $(4, 1)$. Should we wish to record the solution as a pair of fractions or integers instead, we would need to define `A` and `b` over $\mathbb{Q}$.

```
sage: A = Matrix(QQ, [
....:     [15, 40],
....:     [25, -50],
....: ])
sage: b = vector(QQ, [100, 50])
sage: A.solve_right(b)
(4, 1)
```

1.4.1 Numerical Stability

Numerical stability (of a linear system) refers to one of its computational qualities – the quality described often as 'small change in input causes a small change in output'. As real numbers are represented in computer memory with a given precision (more or less the number of decimal places), deviations in input data small enough to go unnoticed may cause issues. We shall highlight two of said 'issues' (and possible countermeasures) in this subsection.

Consider the system

$$\begin{aligned} 2x + y &= 3 \\ 2x + y &= 3 \end{aligned} \tag{1.16}$$

with infinitely many solutions of the form $((3 - y)/2, y)$. Now, altering the system slightly

$$\begin{aligned} 2x + \quad\quad\quad\; y &= \quad\quad\quad 3 \\ 2.000000002x + 1.000000001y &= 3.000000003 \end{aligned}$$

yields a system with exactly one solution – $(1, 1)$. We see that immediately but a computer with limited precision might regard this altered system exactly the same way as the previous one. Should we draw the system, we would basically see just one line given that the size of the angle between the lines corresponding to the two equations is negligible.

Systems where two or more equations are indistinguishable with low enough precision are typically called *ill-conditioned.* In this case, there is not much that can be done to alleviate the problem. See for yourself.

```
sage: A = Matrix(RR, [
....:     [2, 1],
....:     [2 + 2*10**-18, 1 + 10**-18],
....: ])
sage: b = vector(RR, [3, 3 + 3*10**-18])
sage: A.solve_right(b)
(1.50000000000000, 0.000000000000000)
```

The solution given by SageMath is clearly wrong because of the tiny deviation in input data. It instead computed the solution to the system (1.16) and substituted $y = 0$, which is default behaviour.

Next, we take a look at the system

$$\begin{aligned} \tfrac{1}{1000}x + y &= 1 \\ x - y &= 0 \end{aligned}$$

with unique solution $(1000/1001, 1000/1001)$. Here, depending on the order of the equations, computers can arrive at a wrong solution. In the first step of Gauss-Jordan elimination, we subtract a 1000-multiple of row `I` from row `II`, obtaining

$$\begin{aligned} \tfrac{1}{1000}x + \quad y &= \quad 1 \\ -1001y &= -1000. \end{aligned} \tag{1.17}$$

Even if we are working with enough precision to represent thousandths of integers, the result of the computation

$$y = \frac{-1000}{-1001}$$

may easily be rounded to 1 due to the way computers perform division. As three decimal places are hardly enough to push modern computers to their limits, see the following example instead.

```
sage: a = -1 * 10**18
sage: b = -1 * 10**18 - 1
sage: numerical_approx(a / b)
1.00000000000000
```

The `numerical_approx` function tells SageMath to represent a/b as a real number, otherwise it would have stored it as a fraction.

Should we now begin the process of back-substitution in the system (1.17), we would inevitably get a wrong solution. If the second equation yields (with low precision) that $y = 1$, then from the first equation, we get $x = 0$. This is a *completely* different solution from the exact one. The difference between $(0, 1)$ and $(1000/1001, 1000/1001)$ might not seem too high but imagine x and y represented *percentages* for example. Then, instead of both x and y being nearly 100%, x gets smashed down all the way to 0%.

Perhaps a little surprisingly, this problem can be *thoroughly* solved by simply changing the order of the equations. If we had instead used Gauss-Jordan elimination to solve the system

$$\begin{aligned} x - y &= 0 \\ \tfrac{1}{1000}x + y &= 1, \end{aligned}$$

we wouldn't have run into any issues. Indeed, the first step here entails subtracting $(1/1000)$-multiple of row `I` from row `II`. This yields

$$\begin{aligned} x - \quad y &= 0 \\ \tfrac{1001}{1000}y &= 1. \end{aligned}$$

This time, even if $1001/1000$ *does* get rounded to one, the exact solution will still be reached with sufficient degree of accuracy. Supposing the second equation is evaluated to be true if $y = 1$, the first equation then gives $x = 1$. Clearly, the number $1000/1001$ is much closer to 1 than it is to 0.

All in all, there exist cases where additional steps performed during Gauss-Jordan elimination greatly increase the accuracy of the approximation of potential solutions of a linear system. One very simple and statistically effective method is to always swap the row which is to be used for elimination of other rows with the row with highest (in absolute value) pivot coefficient. The reason this works is that computers are, vaguely speaking, prone to rounding numbers that *are not* close to 0. This method is exactly what we employed here, by the way. Instead of solving

$$\begin{aligned} \tfrac{1}{1000}x + y &= 1 \\ x - y &= 0 \end{aligned}$$

we swapped row **I** with row **II** as row **II** has a 1000-times larger coefficient of the variable x than row **II**. In the next section, we intend to show how Gauss-Jordan elimination can be coded in SageMath while also including the aforementioned 'accuracy-improving' step.

Exercise 1.4.1

Devise a linear system the accuracy of the solution whereof suffers from insufficient precision but falls into neither of the two categories described.

1.4.2 Gauss-Jordan Elimination Revisited

Here, we provide a fully algorithmic description of the Gauss-Jordan elimination algorithm discussed in section 1.1 and also one possible way of encoding it in SageMath. Here goes nothing.

Algorithm 1: Gauss-Jordan Elimination.

input : An $n \times m$ matrix $A = (a_{i,j})_{i=1,j=1}^{n,m}$ with real entries.
output: The matrix A in echelon form.

```
/* Row to be used for elimination of other rows.                    */
r ← 1;
/* Traverse the columns.                                            */
for c ∈ {1,...,m} do
    /* Find the row (below r) with maximal value in column c. Denote by
       b the row with the maximal currently known value.            */
    b ← r;
    /* Traverse the rows below r.                                   */
    for i ∈ {r+1,...,n} do
        if |a_{i,c}| > |a_{b,c}| then
            /* Found a row with higher value in column c. Replace b with
               i.                                                   */
            b ← i;
    /* If a_{b,c} = 0, then move to next column since this column is full
       of zeroes.                                                   */
    if a_{b,c} = 0 then
        continue;
    swap rows with indices r and b;
    /* Eliminate variables in column c in all rows below r.         */
    for i ∈ {r+1,...,n} do
        for j ∈ {c,...,m} do
            a_{i,j} ← a_{i,j} − (a_{i,c}/a_{r,c}) a_{r,j};
    /* Row r now contains the pivot in column c so it will remain the
       same for the rest of the algorithm. Move to the next row.    */
    r ← r + 1;
return A;
```

Example 1.4.2

Let's put the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 4 & 2 & 2 \end{pmatrix}$$

into echelon form using algorithm 1. At first, we have $r = 1$ and $c = 1$. Going through rows 2 and 3, we see that the number with the highest value in column 1 lies in row 3. Hence, we first swap row 1 with row 3.

$$A = \begin{pmatrix} 4 & 2 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

Now begins the process of elimination. Since $r = 1$, the index i exhausts the set $\{2, 3\}$. For $i = 2$, we calculate $a_{i,c}/a_{r,c} = a_{2,1}/a_{1,1} = -1/4$. We thus subtract $(-1/4)$-multiple of row 1 from row 2.

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 2 & 0 & 1 \end{pmatrix}$$

Next, we set $i \coloneqq 3$ and calculate $a_{i,c}/a_{r,c} = 1/2$; we then subtract $(1/2)$-multiple of row 1 from row 3.

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -1 & 0 \end{pmatrix}$$

Since all rows in column c below r have been eliminated, we move to the next row by setting $r \coloneqq 2$. We also move to the next column via $c \coloneqq 2$.

Now, the number with the largest absolute value in column c and all rows below (and including) r already lies in row r, so no swap is needed. We perform the elimination of row 3 by calculating $a_{3,c}/a_{r,c} = a_{3,2}/a_{2,2} = -2/3$ and subtracting the $(-2/3)$-multiple of row 2 from row 3.

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

We move to the next row via $r \coloneqq 3$ and the next column via $c \coloneqq 3$. No further elimination takes place because there are no rows below row 3. The matrix A has been put into echelon form.

Before finally marching on, we present a way of implementing algorithm 1 in SageMath. This implementation almost aligns with the implementation of the algorithm in Python. Let us be responsible adults and break it into functions.

The function for finding the row with the highest pivot coefficient in the current row might look like this:

```
sage: def find_best_row(A: Matrix, cur_row: int, cur_col: int):
....:     best_row = cur_row
....:
....:     for row_below in range(cur_row + 1, len(A.rows())):
....:         if abs(A[row_below][cur_col]) > abs(A[best_row][cur_col]):
```

```
....:             best_row = row_below
....:
....:     return best_row
```

We also implement the function to eliminate the first non-zero element in all rows below a given row.

```
sage: def eliminate_rows(A: Matrix, cur_row: int, cur_col: int):
....:     for row_below in range(cur_row + 1, len(A.rows())):
....:         scalar = A[row_below][cur_col] / A[cur_row][cur_col]
....:         A[row_below] = A[row_below] - scalar * A[cur_row]
```

These are all the functions we need to cleanly implement Gauss-Jordan elimination in SageMath.

```
sage: def eliminate(A: Matrix):
....:     cur_row = 0
....:
....:     for cur_col in range(len(A.columns())):
....:         best_row = find_best_row(A, cur_row, cur_col)
....:         A[cur_row], A[best_row] = A[best_row], A[cur_row]
....:
....:         if A[cur_row][cur_col] == 0:
....:             continue
....:
....:         eliminate_rows(A, cur_row, cur_col)
....:         cur_row += 1
```

And, to wrap things up, a short application on the matrix from example 1.4.2.

```
sage: A = Matrix(QQ, [
....:     [2, 0, 1],
....:     [-1, 1, 1],
....:     [4, 2, 2],
....: ])
sage: eliminate(A)
sage: A
[  4   2   2]
[  0 3/2 3/2]
[  0   0   1]
```

1.4.3 Input-Output Analysis

A place where linear systems naturally flourish is *economics*. Put briefly, economy is a network of mutually influenced industries. An important observation is that this ‘influence’ is mostly of *linear* nature. We take as an example the *steel* and *automobile* industries. Both of these industries use its own output and the other industry’s output to optimize production. The steel industry might use steel to produce factories, and use cars for the transport of goods between them. Similarly, the automobile industry uses its own cars to transports its other cars and uses steel to produce them in the first place. In economics, we’re typically interested in predicting the future value of an industry. However, in cases like these, it isn’t intuitively evident how the total value of steel used by external actors (meaning not the steel or automobile industries) would influence the system, for example.

Suppose we accumulated the following data:

	used by steel	**used by auto**	**used by others**	**total**
value of steel (in billions of $)	6.90	1.28	10.51	18.69
value of auto (in billions of $)	2.24	4.40	7.63	14.27

Table 1.1: The annual summary of the value of steel and automobile industries.

Based on this data, how ought we to attempt to predict the total values of steel and automobile industries based on shifting external demand? First and foremost, why do we care primarily about external demand? The answer is simple. As long as external demand stays stable, it is improbable that the automobile industry would suddenly produce more cars or that the steel industry more steel. It is indeed mostly individual customers and other affiliated industries which cause a change in production.

Suppose that the value of steel and automobile industries used externally in the next year shifts by d_s and d_a, respectively. How does this affect their total value? To answer this, we need observe that the steel and automobile industries form a linear system. Under the premise that the steel industry uses the same *fraction* of its own output and the automobile industry also uses the same fraction of the steel industry output as this year, we can predict its value next year (which we denote s) to equal

$$s = (6.90/18.69)s + (1.28/14.27)a + (10.51 + d_s).$$

This formula essentially says the obvious:

$$\begin{aligned} \text{next year's value of steel} &= \text{next year's value of steel used by steel} \\ &\quad + \text{next year's value of steel used by auto} \\ &\quad + \text{next year's value of steel used by others.} \end{aligned}$$

We are just predicting the next year values based on this year's ones while keeping the ratios of output distribution stable.

Similarly, the equation for the predicted next year's total automobile industry value (denoted a) is

$$a = (2.24/18.69)s + (4.40/14.27)a + (7.63 + d_a).$$

Both of these linear equations put together form the linear system

$$\begin{aligned} s &= (6.90/18.69)s + (1.28/14.27)a + (10.51 + d_s) \\ a &= (2.24/18.69)s + (4.40/14.27)a + (7.63 + d_a) \end{aligned}$$

An easy computation and rearrangement gives

$$\begin{aligned} (11.79/18.69)s - (1.28/14.27)a &= (10.51 + d_s) \\ -(2.24/18.69)s + (9.87/14.27)a &= (7.63 + d_a) \end{aligned}$$

As we did many a time already, we collect the equations into a matrix A and a vector b like so:

$$A = \begin{pmatrix} 11.79/18.69 & -1.28/14.27 \\ -2.24/18.69 & 9.87/14.27 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 10.51 + d_s \\ 7.63 + d_a \end{pmatrix}.$$

Fortunately, SageMath has in-built support for variables. We can thus let it solve the system for us and represent the solution in terms of variables d_s and d_a.

```
sage: var('ds da')
(ds, da)
sage: A = Matrix([
....:     [11.79/18.69, -1.28/14.27],
....:     [-2.24/18.69, 9.87/14.27],
....: ])
sage: b = vector([10.51 + ds, 7.63 + da])
sage: sol = A.solve_right(b)
(0.210776906804487*da + 1.62528755481273*ds + 18.6900000000000,
1.48231851778104*da + 0.281627945702251*ds + 14.2700000000000)
```

We can now easily get a solution for *concrete* values of d_s and d_a by using SageMath's symbolic substitution capabilities. For example, if we expect the external output value of automobile industry will rise by $d_a = 0.05$ and the external output value of steel will fall by 0.10, that is $d_s = -0.10$, we can calculate the predicted future total values of the industries by setting

```
sage: sol(da=0.05,ds=-0.10)
(18.5380100898590, 14.3159531313188)
```

In this case, we predict the total value of the steel industry to fall by about \$0.15 billion and the value of the automobile industry to rise by roughly \$0.045 billion.

Exercise 1.4.3

Predict next year's total productions of each of the three sectors of the hypothetical economy shown in table 1.2.

value of / **used by**	**farm**	**rail**	**shipping**	**others**	**total**
farm	25	50	100		800
rail	25	50	50		300
shipping	15	10	0		500

Table 1.2: The output data of a hypothetical economy.

if next year's external demands are as stated.

(a) 625 for farm, 200 for rail, 475 for shipping,

(b) 650 for farm, 150 for rail, 450 for shipping.

Can you solve the system with data presented in (a) and (b) simultaneously by making the given external demands into parameters?

1.4.4 Electric Networks

The final presented application comes from engineering. In figure 1.11, you can see a simplified version of a car's electric network.

A designer of this network must be able to answer questions similar to: 'How much electricity flows when both the hi-beam headlights and the brake lights are on?' Even very sophisticated electric networks can be analysed using Kirchhoff's laws and the theory of linear systems.

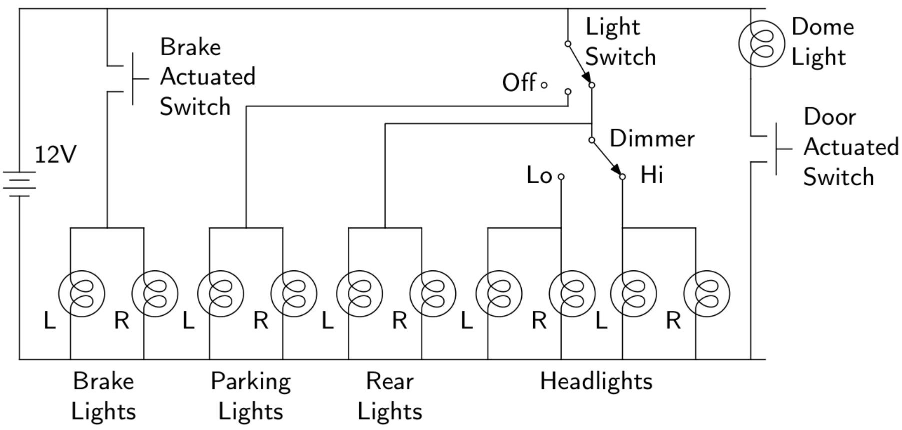


Figure 1.11: An excerpt from a car's electric network.

The intuitive explanation of electric circuits (which suffices for our purposes) tells that there are three interconnected forces at play – voltage (U), resistance (R) and current (I). At any point of the circuit, these are tied by the formula $U = RI$. The battery serves as a capacitor; it provides *voltage* – or electric potential – to the circuit, making electricity flow as long as there is a path. The moment a path is formed (we say the circuit is closed), the battery creates a force through the circuit – the *current*. Finally, some components of the circuit act as *resistors*, effectively limiting the amount of voltage that is 'available' to the subsequent components of the circuit. This limiting factor is the *resistance* of the component and is often proportional to the force provided by the battery. We can think of the resistors causing *voltage drops* throughout the current whilst the battery provides a *voltage rise*.

To interpret electric networks (basically meshes of electric circuits) as linear systems, two physical laws are needed – *Kirchhoff's Current Law* and *Kirchhoff's Voltage Law*. The former states that at any point in the network, the flow in equals the flow out. The latter then that around any circuit in the network, the total voltage rise equals the total voltage drop.

Let us start with a simple network consisting of a single circuit.

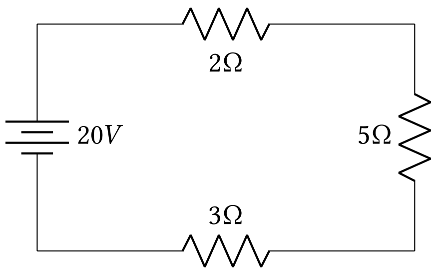


Figure 1.12: An electric circuit with a battery and three resistors.

The component represented by  is the battery and  depicts a resistor. We measure voltage provided by the battery in *volts* (V) and the resistance of the other components in *ohms* (Ω).

Since this network features only a single closed circuit, the current – measured in *amperes* (A) – is consistent throughout by Kirchhoff's Current Law. By Kirchhoff's Voltage Law, the total voltage rise (which is $20V$) equals the total voltage drop. In this circuit, there are three voltage drops, each equal to the resistance of the component times the current flowing through it. This gives us a linear system consisting of the single equation

$$20 = 2I + 5I + 3I$$

wherefrom we infer that $I = 2A$; the current around the circuit equals 2 amperes.

An example of a network leading to a more elaborate linear system requires connecting the resistors *in parallel* which automatically creates more circuits in the network.

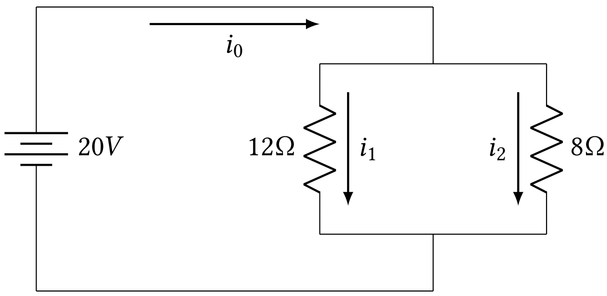
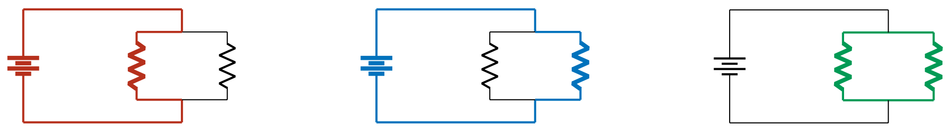


Figure 1.13: An electric network with resistors connected in parallel.

It might not look like it but the network in figure 1.13 actually hosts three circuits depicted in figure 1.14. Each of those circuits obeys Kirchhoff's Voltage Law. Spelt out for the first circuit, it



(a) First circuit. (b) Second circuit. (c) Third circuit.

Figure 1.14: The three circuits of an electric network.

says the total voltage rise of $20V$ must equal the total voltage drop of 12Ω times the current flowing through this circuit, which we labelled i_1. Similarly, the voltage rise in the second circuit is $20V$ and equals $8i_2$. Finally, the voltage rise in circuit three is $0V$ and equals the voltage drop through the first resistor plus the voltage drop through the second resistor. The only caveat here is the choice of orientation of the current. The current flowing through the first resistor must do so in direction opposite to the second resistor as the circuit forms a closed oriented loop. This means that one of the currents (for instance i_2) must be given a negative sign, signifying a direction of flow opposite to the one in circuit two. This gives a total voltage drop in the third circuit as $12i_1 - 8i_2$.

Finally, there are two points in the network where the flow splits. Applying Kirchhoff's Current Law thus awards two more equations: $i_0 = i_1 + i_2$ and $i_1 + i_2 = i_0$. All in all, we ended up with a linear system of five equations.

$$\begin{array}{rcrcrcr} & & 12i_1 & & & = & 20 \\ & & & & 8i_2 & = & 20 \\ & & 12i_1 & - & 8i_2 & = & 0 \\ i_0 & - & i_1 & - & i_2 & = & 0 \\ -i_0 & + & i_1 & + & i_2 & = & 0 \end{array} \tag{1.18}$$

Clearly, there are redundant equations in the system (1.18). This just goes to show that redundancy arises in practice and the problem of determining which equations are redundant is generally not entirely trivial; we shall discuss it later in the book.

In this case, of course, the first two equations already give us equalities $i_1 = \frac{5}{3}A$ and $i_2 = \frac{5}{2}A$. Finally, the fourth equation (or the fifth for that matter) ensures that $i_0 = \frac{25}{6}A$. Hence, the total current through the entire network is $\frac{25}{6}A$.

The final example to discuss is the so-called Wheatstone Bridge. There is *a lot* of circuits in this

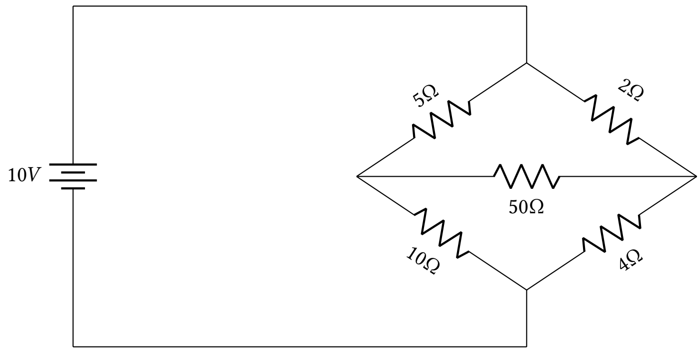


Figure 1.15: The Wheatstone Bridge network.

network. We first choose an arbitrary orientation of the currents through each of the branches as in figure 1.16.

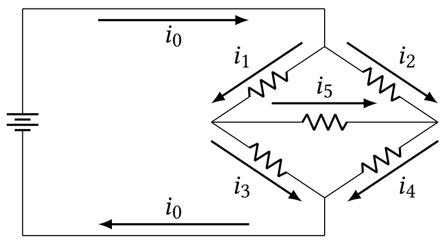


Figure 1.16: A choice of current orientation in the Wheatstone Bridge network.

We can't yet be sure how many or which equations we will need to calculate i_0 – the total current. We definitely need at least 6 given the number of variables. Kirchhoff's Current Law yields many equations but we (based mostly on intuition) pick these three:

$$\begin{aligned} i_0 &= i_1 + i_2 \\ i_3 + i_4 &= i_0 \\ i_2 + i_5 &= i_4 \end{aligned}$$

We've chosen these particular equations in a way that makes every variable appear at least once. Using Kirchhoff's Voltage Law on the inner, outer and the upper 'triangle-shaped' circuit gives respectively:

$$\begin{aligned} 10 &= 5i_1 + 10i_3 \\ 10 &= 2i_2 + 4i_4 \\ 0 &= 5i_1 + 50i_5 - 2i_2 \end{aligned}$$

Again, we have chosen these equations in order to make the resistance of every component appear at least once. Having collected the six equations into a linear system, we pray that we get a unique solution.

$$\begin{array}{rcrcrcrcrcrcr}
i_0 & - & i_1 & - & i_2 & & & & & & & = & 0 \\
-i_0 & & & & & + & i_3 & + & i_4 & & & = & 0 \\
& & & & i_2 & & & - & i_4 & + & i_5 & = & 0 \\
& & 5i_1 & & & + & 10i_3 & & & & & = & 10 \\
& & & & 2i_2 & & & + & 4i_4 & & & = & 10 \\
& & 5i_1 & - & 2i_2 & & & & & + & 50i_5 & = & 0
\end{array}$$

And… yes! We do. As SageMath confirms.

```
sage: A = Matrix(QQ, [
....:     [1, -1, -1, 0, 0, 0],
....:     [-1, 0, 0, 1, 1, 0],
....:     [0, 0, 1, 0, -1, 1],
....:     [0, 5, 0, 10, 0, 0],
....:     [0, 0, 2, 0, 4, 0],
....:     [0, 5, -2, 0, 0, 50],
....: ])
sage: b = vector(QQ, [0, 0, 0, 10, 10, 0])
sage: A.solve_right(b)
(7/3, 2/3, 5/3, 2/3, 5/3, 0)
```

A somewhat surprising fact about this solution is the equality $i_5 = 0$, meaning no electricity flows through the corresponding component.

Exercise 1.4.4

Figure 1.17 depicts an electric network.

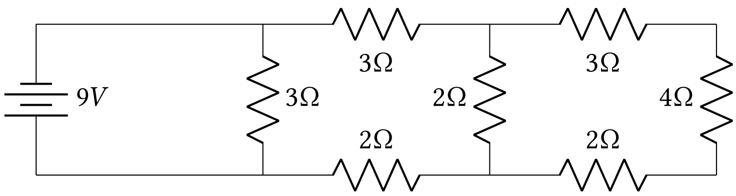


Figure 1.17: An electric network with 7 resistors.

Calculate the current in each branch of the network.

Chapter 2

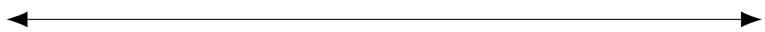
Linear Geometry

This chapter is dedicated to fostering a geometric intuition about vectors as introduced in section 1.3. There, we used them as a convenient way of grouping data. They, however, are also apt representations of a geometric concept – the concept of a 'shift in space'.

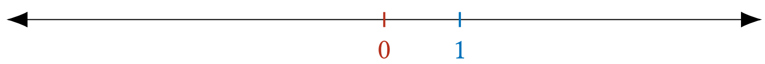
Before we move on, though, we need elucidate what we mean by 'geometry' and 'space'. The former word has a widespread connotation of 'drawing stuff using a ruler and a compass on a sheet of paper'. This is hardly the *geometry* we have in mind in this text. Although circles, lines and triangles are geometric objects to us as well, the fundamental notion of geometry, which is scarcely properly discussed in high-school setting, is *space*.

Space is the 'place where we do geometry', in essence. We begin by arguing the most intuitive way of describing space is that of a set of dimensions (or directions of movement) where each dimension is a *continuum*, meaning, in whichever direction I move, there are no holes along the way. We trust the first *continuum* (a 'set without holes') dear readers encountered, are the real numbers, $\mathbb{R}$. They seem a good candidate for the definition of space.

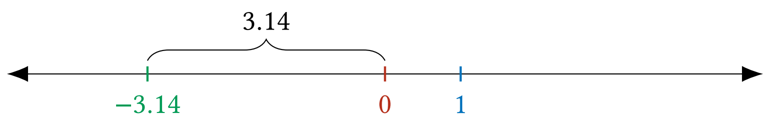
Firstly, we define the *one-dimensional* space to be exactly $\mathbb{R}$. You might wonder how this makes any sense. Well, a space with just one dimension is an infinite line like below.



How is this related to $\mathbb{R}$? Quite trivially. Pick any point on the line and label it 0. Then pick yet another different point and label it 1. Like so.

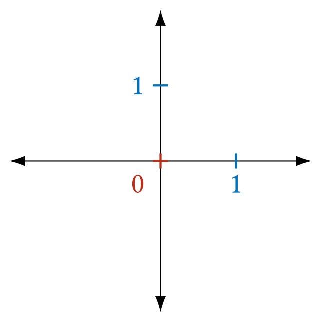


Just like that, we have forged a *correspondence* between an infinite line and the real numbers. Any real number $r \in \mathbb{R}$ corresponds to the point on the line distant exactly $|r|$ from 0 – to the right if it's positive, and to the left if it's negative. For example, the number -3.14 is represented as the following point.

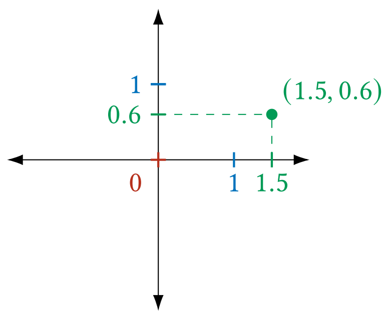


Why stop at one dimension? We may add as many dimensions as we like by simply inventing new directions of movement. Knowing that one direction of movement amounts to the set of real numbers, we get as many directions as we choose to include copies of the set of real numbers.

To make the preceding paragraph more tangible, let's first add a second dimension, which we typically depict as another line perpendicular to the first one and meeting it at 0. Like this.



Now, we have the entire *plane* to move about. We are free to tread as far right (or left) as we wish and as far up (or down) as we wish. Therefore, each point here is given two numbers – one on the axis of horizontal movement and one on the axis of vertical movement.



We call the set of all ordered pairs of real numbers the *cartesian product* of real numbers with themselves and denote it $\mathbb{R} \times \mathbb{R}$ (or $\mathbb{R}^2$ for short). Hence, the two-dimensional space (also called *plane*) is just equivalent to $\mathbb{R}^2$.

We needn't stop at two dimensions. As stated before, we may add as many directions of movement as desired. The preceding small examples justify the following definition.

Definition 2.0.1 (Space)

We call the set $\mathbb{R}^n$ of all (ordered) n-tuples of real numbers the *n-dimensional (real) space.*

We claimed that a *vector* represents a shift in space. We now make that idea precise. Consider the points (a_1, a_2) and (b_1, b_2) in $\mathbb{R}^2$. There exists a straight path from one to the other. The trajectory of such path can be represented as a pair of numbers where the first one signifies the distance we are to travel horizontally to reach b_1 from a_1 and the second number the vertical distance from a_2 to b_2. Clearly, the first distance equals $b_1 - a_1$ and the second $b_2 - a_2$. We can collect these numbers into the vector

$$\mathbf{v} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

which now serves as a representation of the *movement* or *shift* from the point (a_1, a_2) to the point (b_1, b_2). In this scenario, we call (a_1, a_2) the *root* or *source* of the vector $\mathbf{v}$ and the point (b_1, b_2) its *end* or *target*.

Naturally, this idea is easily scaled to higher dimensions. Given points

$$\begin{aligned} a &= (a_1, a_2, \ldots, a_n), \\ b &= (b_1, b_2, \ldots, b_n) \end{aligned}$$

in $\mathbb{R}^n$, the vector with *source* a and *target* b is exactly

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix}.$$

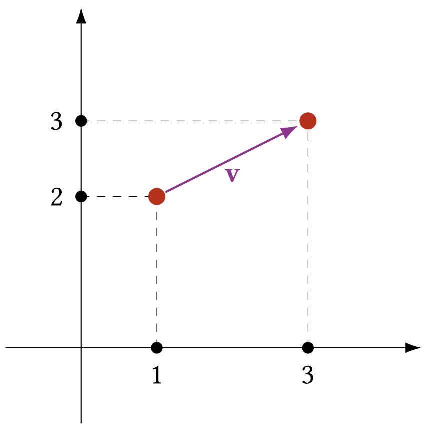


Figure 2.1: Depiction of a vector $\mathbf{v}$ with source $(1, 2)$ and target $(3, 3)$.

Observe how this idea of vector being a *shift* sings in unison with the definitions of vector addition and scalar multiplication. The sum of the vector of source a and target b with the vector of source b and target c is the vector of source a and target c (cf. figure 2.2). This is testified by the simple calculation

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix} + \begin{pmatrix} c_1 - b_1 \\ c_2 - b_2 \\ \vdots \\ c_n - b_n \end{pmatrix} = \begin{pmatrix} b_1 - a_1 + c_1 - b_1 \\ b_2 - a_2 + c_2 - b_2 \\ \vdots \\ b_n - a_n + c_n - b_n \end{pmatrix} = \begin{pmatrix} c_1 - a_1 \\ c_2 - a_2 \\ \vdots \\ c_n - a_n \end{pmatrix}$$

for points $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ and $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$.

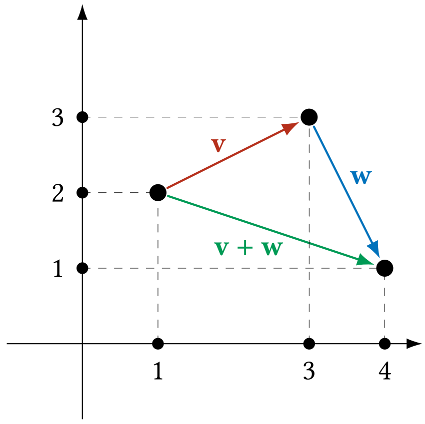


Figure 2.2: The sum $\mathbf{v} + \mathbf{w}$ of vectors $\mathbf{v}$ and $\mathbf{w}$.

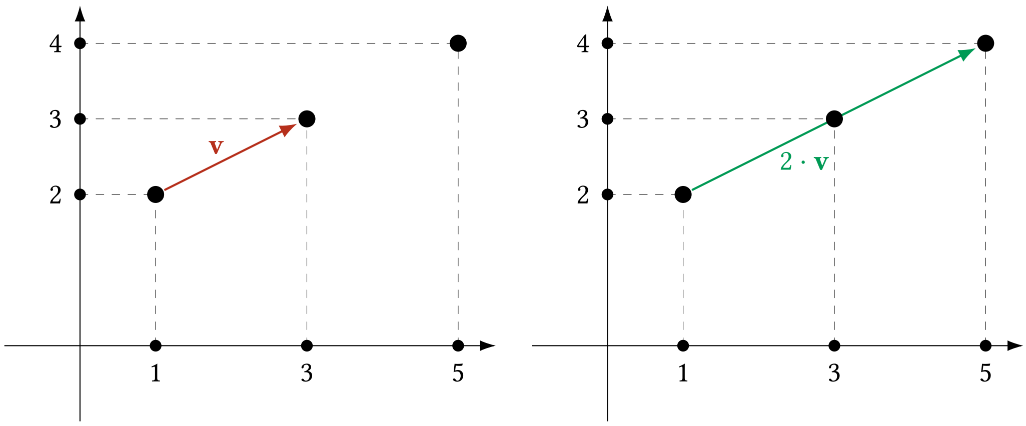


Figure 2.3: The multiple $2 \cdot \mathbf{v}$ of the vector $\mathbf{v}$.

Similarly, the concept of scalar multiplication is transmitted as enlarging or shortening (possibly reversing if the scalar is negative) of the vector in question (cf. figure 2.3).

There is a point to be made here. Figure 2.2 only shows addition of vectors such that the first one ends where the second begins. You may object that this aligns not with the general definition you gave earlier. There was no talk about sources or targets. Fret not, as a simple observation alleviates the issue – a vector itself in fact carries no information whatsoever about either its source or its target. For example, the vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

can have as its source the point $(0, 1, 2)$ and as its target $(1, 3, 7)$; it can equally well be rooted at $(8, -2, 3)$ and end in $(9, 0, 7)$ or start at just about any point (x, y, z) as long as it ends in $(x + 1, y + 2, z + 3)$.

It follows that we have rightly interpreted the sum of $\mathbf{v}$ and $\mathbf{w}$ as 'moving along $\mathbf{v}$ and then continuing along $\mathbf{w}$' since we can always move the start of $\mathbf{w}$ to the end of $\mathbf{v}$. This interpretation also hopefully drives home the idea that a vector is simply a description of a *shift* or *displacement* in space, not exactly a *segment* or a *path*. We tend to regard the latter two as having a fixed start and

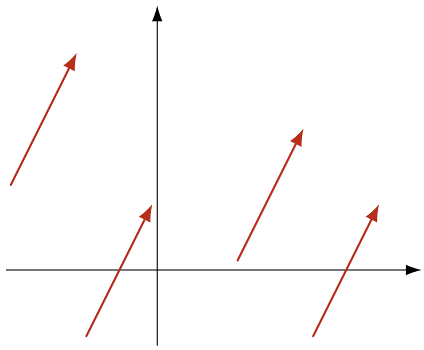


Figure 2.4: The vector $\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ drawn with four different sources and targets.

end which, as we just saw, vectors don't do.

There is one formal consequence of this idea. Since vectors are free to be rooted anywhere, why don't we simply make our lives easy and root them at the origin – the point $(0, \ldots, 0)$? Doing so would make vectors formally indistinguishable from points in space. Indeed, we may now trivially draw a relationship between a point $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and the vector

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

by consensus rooted at $(0, \ldots, 0)$. Intuitively, the former is a specific place in n-dimensional space and the latter is a shift from the origin to that place. Formally, however, there is no difference at all to be found. That said, we shall henceforth regard $\mathbb{R}^n$ as the set of all points with n real coordinates as well as the set of all vectors with n real entries, whichever one is more convenient in a given context. Expressed symbolically, we may write

$$\mathbb{R}^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{R}\} = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \mid a_1, \ldots, a_n \in \mathbb{R} \right\}.$$

The concept of a *shift* typically brings with itself two important characteristics – *length* and *direction*. In natural language, we habitually express shifts by saying 'Make three steps forward.' or 'Turn slightly right and then keep driving for 4 kilometres.' We should ask: 'What is the *length* and *direction* of a vector?' The rest of this chapter lays down an answer to this question.

2.1 Length Of A Vector

In this section we generalise the 'intuitive' understanding of a vector's length readers might have acquired in dimensions one and two.

Starting low, in dimension one, a vector with one entry is basically just a real number. It represents a shift on an infinite line to the right or left starting at 0. In this case, its length is clearly just the *absolute value* of its single entry. Nonetheless, it's appropriate to remind ourselves how absolute value is actually defined. For a number $x \in \mathbb{R}$, we define its *absolute value* to be

$$|x| \coloneqq \sqrt{x^2}.$$

This means that the length of a vector $\mathbf{v} = (v_1)$ with a single entry $v_1 \in \mathbb{R}$ comes out to be exactly $\sqrt{v_1^2}$. We shall denote the length of a vector $\mathbf{v}$ as $\|\mathbf{v}\|$, also called its *norm*.

In dimension two, things complicate a tad. A vector now comprises two real entries, the horizontal shift and the vertical one. Thankfully, the well-known and loved *Pythagorean Theorem* comes to the rescue. The important idea here is to literally split a vector into its horizontal and vertical part. We mean it like this: given a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

we construct vectors $\mathbf{v}_x$ and $\mathbf{v}_y$ like this:

$$\mathbf{v}_x \coloneqq \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_y \coloneqq \begin{pmatrix} 0 \\ v_2 \end{pmatrix}.$$

Now, we have $\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y$ and we know that $\|\mathbf{v}_x\| = |v_1|$ and $\|\mathbf{v}_y\| = |v_2|$. Since $\mathbf{v}_x$ and $\mathbf{v}_y$ are the legs of a right triangle with hypotenuse $\mathbf{v}$ (see figure 2.5), we arrive at the equation

$$\|\mathbf{v}\|^2 = \|\mathbf{v}_x\|^2 + \|\mathbf{v}_y\|^2,$$

from which it follows that

$$\|\mathbf{v}\| = \sqrt{\|\mathbf{v}_x\|^2 + \|\mathbf{v}_y\|^2} = \sqrt{|v_1|^2 + |v_2|^2} = \sqrt{v_1^2 + v_2^2},$$

where the last equality holds because v_1^2 and v_2^2 are positive regardless of whether v_1 and v_2 are.

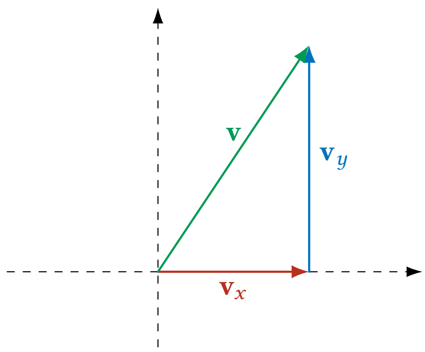


Figure 2.5: Computing the length of $\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y$ using the Pythagorean Theorem.

An analogous approach will also work in dimension three. Here we instead break a vector into three components. That is, given a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

we break it up into

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} = \mathbf{v}_x + \mathbf{v}_y + \mathbf{v}_z.$$

Just as before, we know that $\|\mathbf{v}_x\| = |v_1|$, $\|\mathbf{v}_y\| = |v_2|$ and $\|\mathbf{v}_z\| = |v_3|$. The vector

$$\mathbf{v}_{xy} = \mathbf{v}_x + \mathbf{v}_y$$

is the hypotenuse of the right triangle with legs $\mathbf{v}_x$ and $\mathbf{v}_y$. This means that

$$\|\mathbf{v}_{xy}\| = \sqrt{\|\mathbf{v}_x\|^2 + \|\mathbf{v}_y\|^2} = \sqrt{v_1^2 + v_2^2}.$$

Similarly, the vector $\mathbf{v}$ itself is a hypotenuse of the right triangle formed by vectors $\mathbf{v}_{xy}$ and $\mathbf{v}_z$. It follows that

$$\|\mathbf{v}\| = \sqrt{\|\mathbf{v}_{xy}\|^2 + \|\mathbf{v}_z\|^2} = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

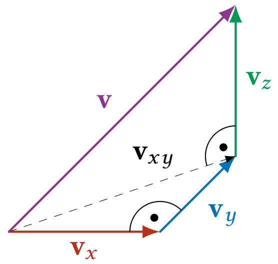


Figure 2.6: Computing the length of the vector $\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y + \mathbf{v}_z$ using the Pythagorean Theorem twice.

Hopefully kind readers have begun to see the pattern. We can split a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

into 'basically one-dimensional' vectors

$$\mathbf{v}_{x_1} = \begin{pmatrix} v_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{v}_{x_2} = \begin{pmatrix} 0 \\ v_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{v}_{x_n} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix}.$$

and calculate its length as the length of the body diagonal of the n-dimensional cuboid with side lengths $\|\mathbf{v}_{x_1}\|, \|\mathbf{v}_{x_2}\|, \ldots, \|\mathbf{v}_{x_n}\|$. Let us first prove that said body diagonal indeed has the length we expect.

Lemma 2.1.1 (Body diagonal of a cuboid)

The length of the body diagonal of an n-dimensional cuboid with side lengths $a_1, a_2, \ldots, a_n$ is exactly $\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$.

Proof. We prove the lemma by induction on the dimension of the cuboid.

A cuboid of dimension one has only one side of length a_1 and thus its body diagonal consists of just this single side and is therefore long exactly $a_1 = |a_1| = \sqrt{a_1^2}$. Thus, the base case is handled.

Now, consider an $(n-1)$-dimensional cuboid C_{n-1} with side lengths $a_1, \ldots, a_{n-1}$ and assume its body diagonal has length $\sqrt{a_1^2 + a_2^2 + \ldots + a_{n-1}^2}$. Adding a dimension to C_{n-1} means regarding C_{n-1} as the base of the n-dimensional cuboid C_n with side lengths $a_1, a_2, \ldots, a_n$ (imagine the square being a base for the cube). By definition, this added side is perpendicular to the entirety of C_{n-1}. In particular, the body diagonal of C_{n-1} is perpendicular to the newly added side of length a_n. This means that the body diagonal of C_n is the hypotenuse in the right triangle formed by the body diagonal of C_{n-1} and the side of length a_n of C_n. The Pythagorean theorem now reads that the length of the body diagonal of C_n is exactly

$$\sqrt{\left(\sqrt{a_1^2 + a_2^2 + \ldots + a_{n-1}^2}\right)^2 + a_n^2} = \sqrt{a_1^2 + a_2^2 + \ldots + a_{n-1}^2 + a_n^2}$$

and the lemma is hence proven. ■

The previous lemma justifies the definition of the length of a vector which we promptly proceed to utter.

Definition 2.1.2 (Length of a vector)

The length of a vector $\mathbf{v} \in \mathbb{R}^n$ with entries $v_1, \ldots, v_n$ is defined as

$$\|\mathbf{v}\| := \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

2.2 Angle Between Vectors

Having defined the length of a vector, we now turn to its *direction*. Do note that direction, as well as length, can only be defined relative to an initial value. In case of length, we use the real numbers 0 and 1 for reference. In light of this, it seems apt to dedicate some time to the study of *angle* between two vectors. This way, we can say what direction a given vector has relative to any other vectors we choose as our initial setup. This 'initial setup', we shall call a *basis* in the next chapter.

As a matter of fact, this is how we naturally specify directions when navigating, for example. The sentence 'Turn slightly right,' consists of two messages – one implicit and one explicit:

(1) Regard the direction you're facing as *initial* – forming an angle of 0° with your line of sight.

(2) Turn clockwise by an angle we might consider 'slight', say, by 30°.

Most of us clearly see the value in the second message and treat the first one as obvious (ehm... because it is). But, had we instead decided that the straight line to any other surrounding point is the initial direction, step (2) would have possibly sent us marching where no one has gone before. This benignly intrusive introduction only served the purpose of elucidating that an initial *point of* *reference* is equally as important as the later specified length or direction, despite ours taking the former for granted.

That said, this section treats all vectors as possible points of reference and only discusses the issue of an angle of a vector relative to some other specified vector, or said naturally, the angle between two vectors.

Dimension one being trivial – two vectors are always collinear and can thus only form an angle of 0° or 180° – we start in dimension two. Just as in the previous section, the geometry of triangles is playing an important role here. The triangle we focus on now is formed by three vectors: $\mathbf{v}, \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ (cf. figure 2.7).

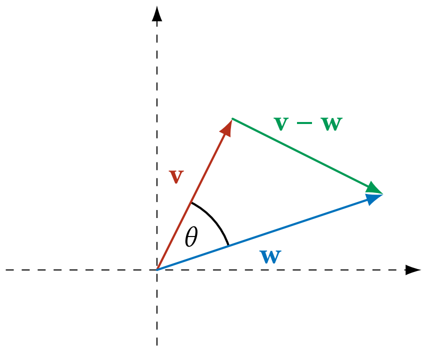


Figure 2.7: The triangle defined by the vectors $\mathbf{v}$ and $\mathbf{w}$.

The properties of this triangle will allow us to calculate the angle between $\mathbf{v}$ and $\mathbf{w}$, which we label θ. The paramount ingredient here is the *Law Of Cosines*. We shall remind dear readers what it says.

Theorem 2.2.1 (Law Of Cosines)

In a triangle with side lengths a, b, c and angles α, β, γ (as in figure 2.8), the following equality holds

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

Proof. Trivial. See, for instance, one of the many proofs on Wikipedia. ■

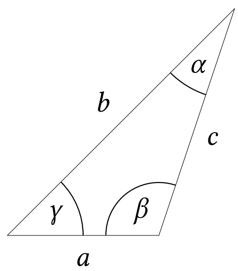


Figure 2.8: Auxiliary illustration to the Law Of Cosines.

Using the Law Of Cosines, we shall now proceed to calculate the angle θ based on the entries of $\mathbf{v}$ and $\mathbf{w}$. Substituting $a = \|\mathbf{v}\|$, $b = \|\mathbf{w}\|$ and $c = \|\mathbf{v} - \mathbf{w}\|$ in the theorem, we get

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos \theta.$$

Expanding gives

$$(v_1 - w_1)^2 + (v_2 - w_2)^2 = v_1^2 + v_2^2 + w_1^2 + w_2^2 - 2\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}\cos\theta.$$

Now, the left side equals

$$(v_1^2 - w_1^2) + (v_2^2 - w_2^2) = v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2.$$

The squares cancel out with those on the right hand side and we reach

$$-2v_1w_1 - 2v_2w_2 = -2\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}\cos\theta.$$

A final rearrangement leads to the formula for θ:

$$\theta = \arccos\left(\frac{v_1w_1 + v_2w_2}{\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}}\right).$$

However, this formula works only for vectors $\mathbf{v}$ and $\mathbf{w}$ of dimension two. To proceed further, we need make an observation. The fact of the matter is that the calculation above is almost entirely independent of the dimensions of $\mathbf{v}$ and $\mathbf{w}$. Why? Well, the vectors $\mathbf{v}$ and $\mathbf{w}$ – let them be n-dimensional – define a two-dimensional plane in $\mathbb{R}^n$ because each contributes one direction of movement. A good example to make is that the vectors $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$ define the 'floor' in a three-dimensional (infinite) 'room' as the first one allows horizontal movement and the second one permits moving forward and backward. The vector $\mathbf{v} - \mathbf{w}$ lies on the same plane simply because it's a vector rooted at the tip of $\mathbf{v}$ and ending in the tip of $\mathbf{w}$. This means that no matter what dimension $\mathbf{v}$ and $\mathbf{w}$ lie in, they still form the same triangle as in figure 2.7, which now lies on a plane in $\mathbb{R}^n$. As a consequence, the calculation we just did is still almost valid; it only need be upgraded to vectors with n real entries.

We've just observed that for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ the equality

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\| + \|\mathbf{w}\| - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

still stands. Applying the same transformations as before, we reach the expression for θ:

$$\theta = \arccos\left(\frac{v_1w_1 + v_2w_2 + \ldots + v_nw_n}{\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}\sqrt{w_1^2 + w_2^2 + \ldots + w_n^2}}\right).$$

The denominator of the fraction is of course simply $\|\mathbf{v}\|\|\mathbf{w}\|$ and the nominator is the output of a function called the *dot* (or *scalar*) product. It has beautiful geometric properties and is paramount to a deeper study of linear systems but for now it only serves the purpose of convenient notation.

Definition 2.2.2 (Dot product)

The *dot product* of vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{R}^n$$

is defined as

$$\mathbf{v} \cdot \mathbf{w} \coloneqq v_1 w_1 + v_2 w_2 + \ldots + v_n w_n.$$

Note that the dot product of two vectors is a **number**, not a vector, and it is defined only for vectors with the same number of entries. More interestingly, it is tied to the length of a vector by the formula

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

We intend to make use of this formula down the line.

There is one last problem to be solved before we can properly define the angle between two vectors. As educated readers well know, the function arccos is only takes input from the closed interval $[-1, 1]$. Since we intend to define θ as $\arccos(\mathbf{v} \cdot \mathbf{w} / \|\mathbf{v}\| \|\mathbf{w}\|)$, we must make sure that the argument is always in said interval.

We actually aim to present a little stronger result the proof whereof will contain the desired inequality. This result styles the *triangle inequality* and is the cornerstone of Euclidean geometry – 'The shortest distance between two points is a straight line.'

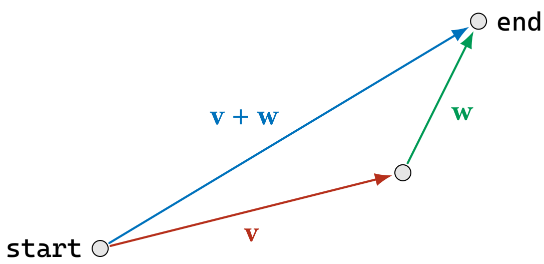


Figure 2.9: The triangle inequality.

Theorem 2.2.3 (Triangle inequality)

For any vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *the inequality*

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\| \tag{2.1}$$

holds. Furthermore, the two sides are equal if and only if $\mathbf{v}$ *is a scalar multiple of* $\mathbf{w}$.

Proof. We shall use a few properties of the dot product we haven't proven. Namely,

- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$;
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;

these are left as an exercise.

We now proceed to make a few algebraic manipulations to the inequality (2.1). First, both sides are positive numbers, hence the inequality is equivalent to

$$\|\mathbf{v}+\mathbf{w}\|^2 \leq (\|\mathbf{v}\| + \|\mathbf{w}\|)^2.$$

Rewriting slightly (and using $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$)

$$\begin{aligned}(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) &\leq \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| \\ \mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}+\mathbf{w}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w} &\leq \mathbf{v}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w}+2\|\mathbf{v}\|\|\mathbf{w}\| \\ 2(\mathbf{v}\cdot\mathbf{w}) &\leq 2\|\mathbf{v}\|\|\mathbf{w}\|.\end{aligned}$$

Multiplying both sides by $\|\mathbf{v}\|\|\mathbf{w}\|$ gives

$$\begin{aligned}2\|\mathbf{v}\|\|\mathbf{w}\|(\mathbf{v}\cdot\mathbf{w}) &\leq 2\|\mathbf{v}\|^2\|\mathbf{w}\|^2 \\ 2(\|\mathbf{w}\|\mathbf{v}\cdot\|\mathbf{v}\|\mathbf{w}) &\leq 2\|\mathbf{v}\|^2\|\mathbf{w}\|^2,\end{aligned}$$

and further manipulation then

$$0 \leq \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - 2(\|\mathbf{w}\|\mathbf{v}\cdot\|\mathbf{v}\|\mathbf{w}) + \|\mathbf{v}\|^2\|\mathbf{w}\|^2.$$

Finally, as

$$\|\mathbf{v}\|\mathbf{w}\cdot\|\mathbf{v}\|\mathbf{w} = \|\mathbf{v}\|^2(\mathbf{w}\cdot\mathbf{w}) = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \quad \text{and} \quad \|\mathbf{w}\|\mathbf{v}\cdot\|\mathbf{w}\|\mathbf{v} = \|\mathbf{w}\|^2(\mathbf{v}\cdot\mathbf{v}) = \|\mathbf{w}\|^2\|\mathbf{v}\|^2,$$

we can complete the square and get

$$0 \leq (\|\mathbf{w}\|\mathbf{v} - \|\mathbf{v}\|\mathbf{w})\cdot(\|\mathbf{w}\|\mathbf{v} - \|\mathbf{v}\|\mathbf{w}) = \big\|\|\mathbf{w}\|\mathbf{v} - \|\mathbf{v}\|\mathbf{w}\big\|^2. \tag{2.2}$$

The right hand side is the length of the vector $\|\mathbf{w}\|\mathbf{v} - \|\mathbf{v}\|\mathbf{w}$ squared, that is, definitely a non-negative number. This proves the inequality.

As for the conditional equality statement, the inequality (2.2) suggests that the two sides of the original inequality (2.1) are equal if and only if

$$\|\mathbf{w}\|\mathbf{v} - \|\mathbf{v}\|\mathbf{w} = 0$$

but this clearly happens if and only if

$$\begin{aligned}\|\mathbf{w}\|\mathbf{v} &= \|\mathbf{v}\|\mathbf{w} \\ \mathbf{v} &= \frac{\|\mathbf{v}\|}{\|\mathbf{w}\|}\mathbf{w},\end{aligned}$$

that is, if and only if $\mathbf{v}$ is the $(\|\mathbf{v}\|/\|\mathbf{w}\|)$-multiple of $\mathbf{w}$.

This concludes the proof. ■

Exercise 2.2.4 (Some properties of dot product)

Prove that for any three vectors $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$, the following equalities hold:

- $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$,

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

As a corollary, we get the inequality which is widely remembered as the *Cauchy-Schwarz inequality* and plays an indispensable role in linear algebra as well as other branches of mathematics, such as the theory of metric spaces and, by extension, the theory of Lebesgue integration.

Corollary 2.2.5 (Cauchy-Schwarz inequality)

For any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, *the inequality*

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$$

holds and the two sides are equal if and only if $\mathbf{v}$ *is a scalar multiple of* $\mathbf{w}$.

Proof. The proof of theorem 2.2.3 suggests that

$$\mathbf{v} \cdot \mathbf{w} \leq \|\mathbf{v}\|\|\mathbf{w}\|$$

so if $\mathbf{v} \cdot \mathbf{w}$ is non-negative, we don't have anything to prove. On the other hand, if $\mathbf{v} \cdot \mathbf{w}$ is negative, we compute

$$|\mathbf{v} \cdot \mathbf{w}| = -(\mathbf{v} \cdot \mathbf{w}) = (-\mathbf{v}) \cdot \mathbf{w} \leq \|-\mathbf{v}\|\|\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|$$

and we're done. ∎

We've reached the end of the section where we finally properly define the angle between two vectors. This definition is justified by the last corollary 2.2.5 since it assures that

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \in [-1, 1]$$

and so the real number

$$\arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

exists for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

Definition 2.2.6 (Angle between vectors)

For any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we define the angle θ between $\mathbf{v}$ and $\mathbf{w}$ as the number

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right).$$

2.3 Visualisation of Linear Systems Revisited

In section 1.2, we discussed geometric properties of the sets of solutions of linear systems. In section 1.3, we described them as sets of vectors. Finally, now that we have revealed the geometric side of vectors as well, the two different ways of looking at sets of solutions of linear systems should align. The conception of this alignment is the content of this, rather brief, section.

The solution of the linear equation

$$x + 3y = 4$$

is the set $\{(4-3y, y) \mid y \in \mathbb{R}\}$ and also the set

$$\left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix} + y \begin{pmatrix} -3 \\ 1 \end{pmatrix} \mid y \in \mathbb{R} \right\}.$$

We've already proven that the first set describes a line. Under our current geometric interpretation of vectors, does the second set describe the same line? As you may expect, the answer is *yes*, but only if we identify (as we already have multiple times) the targets of vectors with the vectors themselves. You see, the second set is a set of *vectors* while the first one is a set of *points*. The idea here is that these two sets are the same as long as we consider the second set as a line formed by the ends or targets of the vectors within.

Now, any vector of the form $\left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right) + y\left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)$ is a vector which is rooted at the end of $\left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right)$ and then extended by an arbitrary length in the direction of $\left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)$ (see figure 2.10). This means that in order to reach any point on the line, we must travel 4 steps to the right (in the direction of $\left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right)$) and then some distance in the direction of $\left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)$. Clearly, if we separate the directions, we see that we move by $4 + y \cdot (-3)$ in the horizontal direction and by $y \cdot 1$ in the vertical direction. The point we reach this way has coordinates $(4-3y, y)$ for some choice of $y \in \mathbb{R}$. This shows that we are indeed moving along the same line; as well we should.

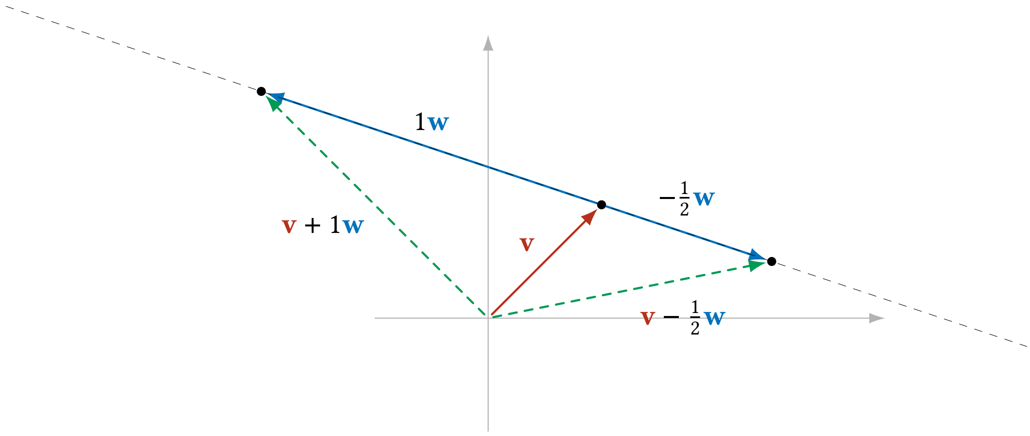


Figure 2.10: Line as a set of vectors. Every vector whose end lies on the line is of the form $\mathbf{v} + y\mathbf{w}$ for $y \in \mathbb{R}$.

Since we already proved in section 1.3 that the two descriptions (using points vs. using vectors) of the sets of solutions of linear systems are equivalent, we shan't dwell on this matter much longer. Let us close this section with two examples from $\mathbb{R}^3$, the kind we studied and visualised in subsection 1.2.2.

The linear equation

$$x - y + z = 4$$

defines a plane in $\mathbb{R}^3$. Its solution set can be represented as the set of points $\{(4+y-z, y, z) \mid y, z \in \mathbb{R}\}$ or the set of (ends of) vectors

$$\left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

This second representation reveals that we're dealing with a plane created by moving freely in the directions of $\left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right)$, shifted 4 steps to the right from the origin (in the direction of $\left(\begin{smallmatrix} 4 \\ 0 \\ 0 \end{smallmatrix}\right)$).

As a matter of fact, the triangle inequality assures that the geometric object defined as the set of all vectors of the form $\mathbf{u} + y\mathbf{v} + z\mathbf{w}$, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $y, z \in \mathbb{R}$, is always a plane (that is a 'two-dimensional flat object') because the shortest distance between two points on such an object is always the straight segment connecting them. Kind readers would do well to realize this is the very definition of 'flatness'.

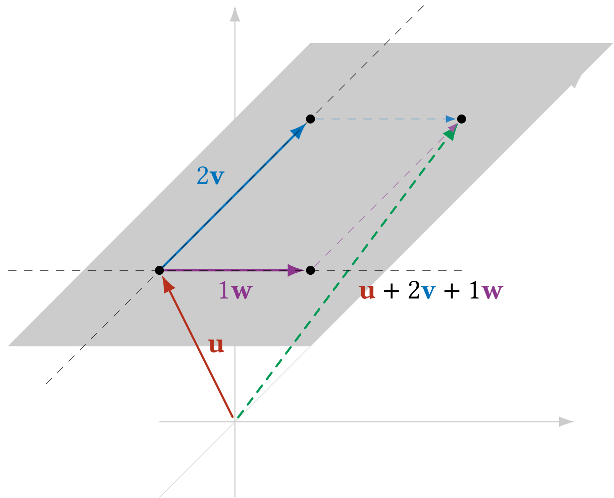


Figure 2.11: Plane as a set of vectors. Every vector lying on the plane is of the form $\mathbf{u} + y\mathbf{v} + z\mathbf{w}$ for some $y, z \in \mathbb{R}$.

Adding another linear equation creates an intersection of two planes – a line in $\mathbb{R}^3$. This is actually best seen from its vector representation. The system

$$\begin{aligned} x - y + z &= 4 \\ -x + 3y - 3z &= 0 \end{aligned}$$

has the solution set $\{(6, z+2, z) \mid z \in \mathbb{R}\}$ or

$$\left\{ \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

The latter description immediately suggests that the geometric object in question is indeed a line – we reach every solution by first moving along $\begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$ and then any distance whatsoever in the direction of $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

We leave the geometric review of vectors on this note. We advise readers to keep this idea in mind, however, as we enter a more abstract space, quite literally, in the next chapter. A few exercises to keep you entertained.

Exercise 2.3.1

Describe the plane passing through points $(1, 1, 5, -1)$, $(2, 2, 2, 0)$ and $(3, 1, 0, 4)$ as

(a) a set of points,

(b) a set of vectors.

Does the origin $(0, 0, 0, 0)$ lie in the plane?

Exercise 2.3.2

Describe the plane (as a set of points or vectors, as you wish) that contains

$$\text{the point } \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \text{ and the line } \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Exercise 2.3.3

A person travelling eastward at a rate of 3 miles per hour finds that the wind appears to blow directly from the north. On doubling his speed it appears to come from the north east. What was the wind's velocity?

Exercise 2.3.4

Find the length of each of the vectors

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

Exercise 2.3.5

Find the angle between each two of these vectors, if it is defined.

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

Exercise 2.3.6

Suppose that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ for some $\mathbf{u} \neq \mathbf{0}$. Is it necessarily true that $\mathbf{v} = \mathbf{w}$? Prove or provide a counterexample.

Exercise 2.3.7

Find the midpoint of the line segment connecting (x_1, y_1) to (x_2, y_2). Generalize to $\mathbb{R}^n$.

Exercise 2.3.8

Generalize the Pythagorean Theorem: if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are perpendicular, then

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

Exercise 2.3.9

Show that the dot product is *linear*, that is, given $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $k, m \in \mathbb{R}$, the equality

$$\mathbf{u} \cdot (k\mathbf{v} + m\mathbf{w}) = k(\mathbf{u} \cdot \mathbf{v}) + m(\mathbf{u} \cdot \mathbf{w})$$

holds. You may use the properties of dot product from exercise 2.2.4.

Chapter 3

Abstract Vector Spaces

The geometric interpretation of vectors developed in the previous chapter leaves a lot of tones unsung. We defined the space the vectors occupy as $\mathbb{R}^n$. This definition works well for building intuition but can't get us very far in the theory of linear systems. You see, the solution sets to linear systems are rarely equal to $\mathbb{R}^n$ for any n. But they *are*, in a very literal sense, 'spaces of vectors'.

To illustrate what we mean, consider again the linear equation

$$x - y + z = 4$$

from section 2.3. Its solution set is the set of vectors

$$\left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R} \right\}.$$

The vector $\left(\begin{smallmatrix} 4 \\ 0 \\ 0 \end{smallmatrix}\right)$ is just a single shift but the set

$$S \coloneqq \left\{ y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R} \right\}$$

indeed is an entire 'space' of vectors, in the sense that it behaves essentially the same as $\mathbb{R}^2$. To highlight certain qualities:

- It's a plane in $\mathbb{R}^3$; a two-dimensional object, just like $\mathbb{R}^2$ is a two-dimensional space.
- It contains the origin – the vector $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$.
- Adding vectors in S gives a vector in S. We cannot ever leave the plane just by moving along the vectors in the same plane.
- Multiplying vectors from S by real numbers also gives a vector in S. Enlarging or shortening a vector also doesn't allow us to leave the plane.

In this chapter, we shall endeavour to formalize the just outlined concept of a 'set of vectors which behaves just like a space does'. We're going to call these spaces *abstract vector spaces* or just *vector spaces* for short.

The most important idea behind the definition of a vector space (or a space in general) is 'closedness'. A space ought to be a universe in itself, interactions between elements cannot ever lead to the creation of an element which is not present. Mathematicians tend to call these interactions, *operations*, and label sets which meet this criterion as *closed*. There are a few other formal requirements we must enforce (e.g. commutativity and associativity of operations which we take for granted in the real numbers) but the primary aim remains to define a *closed* set, a space moving along whose vectors allows one not to escape it.

We thus proceed to define an abstract vector space as a set of vectors which can be added together and scaled by real numbers by listing axioms (basically enforced rules of behaviour) the set must satisfy.

Definition 3.0.1 (Abstract vector space)

An *(abstract) vector space* over $\mathbb{R}$ is a set V (whose elements style *vectors*) together with operations $\oplus : V \times V \longrightarrow V$ (called *vector addition*) and $\odot : \mathbb{R} \times V \longrightarrow V$ (called *scalar multiplication*) satisfying the following axioms.

(1) The operation $\oplus$ is *commutative*, i.e. $\mathbf{v} \oplus \mathbf{w} = \mathbf{w} \oplus \mathbf{v}$ for every $\mathbf{v}, \mathbf{w} \in V$.

(2) The operation $\oplus$ is *associative*, i.e. $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

(3) The set V is *closed* under the operation $\oplus$, i.e. $\mathbf{v} \oplus \mathbf{w} \in V$ whenever $\mathbf{v}, \mathbf{w} \in V$.

(4) There exists a *zero vector*, i.e. a vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for every $\mathbf{v} \in V$.

(5) Each vector $\mathbf{v} \in V$ has an *additive inverse*, i.e. a vector $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

(6) The operation $\odot$ distributes over $+$, that is, $(r + s) \odot \mathbf{v} = r \odot \mathbf{v} \oplus s \odot \mathbf{v}$ for all $r, s \in \mathbb{R}$ and $\mathbf{v} \in V$.

(7) The operation $\odot$ distributes over $\oplus$, i.e. $r \odot (\mathbf{v} \oplus \mathbf{w}) = r \odot \mathbf{v} \oplus r \odot \mathbf{w}$ for every $r \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$.

(8) Ordinary multiplication (of real numbers) associates with $\odot$, i.e. $(rs) \odot \mathbf{v} = r \odot (s \odot \mathbf{v})$ for every $r, s \in \mathbb{R}$ and $\mathbf{v} \in V$.

(9) The set V is *closed* under $\odot$, that is, $r \odot \mathbf{v} \in V$ whenever $r \in \mathbb{R}$ and $\mathbf{v} \in V$.

(10) Scalar multiplication by 1 acts as the *identity operation*, that is, $1 \odot \mathbf{v} = \mathbf{v}$ for every $\mathbf{v} \in V$.

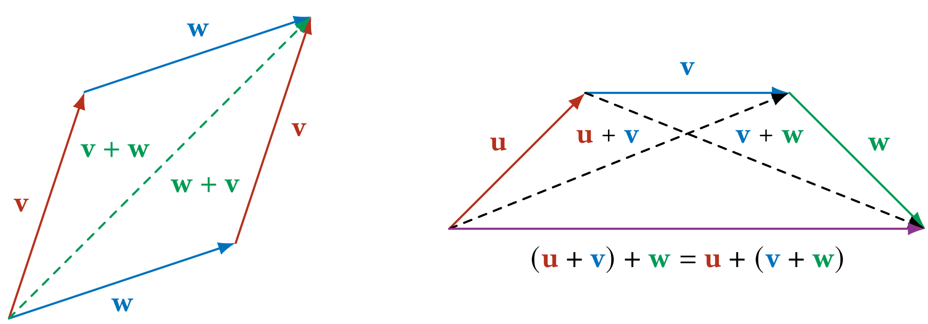


Figure 3.1: The visualization of commutativity and associativity of vector addition.

Remark 3.0.2

The definition of vector space hosts a plethora of axioms; some for different 'mathematical' reasons than others.

The axioms (3) and (9) are the 'most important' in a sense. They ascertain that the resulting set is indeed a space, in the sense discussed in the introduction to this chapter.

The axioms (1), (2), (6), (7), (8) and (10) are technical. Their role is to make vectors and numbers interact in a way we find intuitive hailing from n-dimensional real spaces. There, we take for granted the sum of two vectors is invariant under change of the order of summation but, on the other hand, one can easily design sets of elements with an addition operation not commutative. The gist of it is that we still wish to think of elements of vector spaces as ... well ... vectors, and once a vector, you should behave like a vector.

Finally, axioms (4) and (5) are there so that we can 'reverse arrows', in a sense. Again, we tend to treat vectors as measures of length and direction so it makes sense to be able to travel the same distance in a direction opposite. The condition of being able to reverse brings with it the necessity to have an 'original point' since otherwise the sum of a vector with its additive inverse would send us flying out of the space we're in. That ought not to happen.

Remark 3.0.3

Diligent readers have surely noticed that we denoted the operations $\oplus$ and $\odot$ on an abstract vector space differently than we would before. For predominantly didactic reasons. When we first defined vector addition, we didn't feel the need to distinguish adding two real numbers from adding two vectors because vector addition is just component-wise addition of numbers anyway. However, elements of an abstract vector space can (as we shall soon see) actually be somewhat distant from the intuitive image of vectors we harbour. It seemed apt to fully convey the perception of difference between vector addition and 'normal' addition. Scalar multiplication falls under the same argument.

Nonetheless, it is common in literature to write the vector addition operation $\oplus$ the same way as the addition of real numbers and we purport to adhere to the norm. However, we shall at least make the distinction between scalar multiplication and real multiplication by using the symbol $\cdot$ for the former and lack of a symbol for the latter. The operation $\cdot$ of scalar multiplication ought not to be confused with dot product which we have not defined for abstract vectors.

We fare ahead and list quite a few examples of abstract vector spaces. Some should come as no surprise, some as quite it.

Example 3.0.4 (n-dimensional real space)

An obvious example of a vector space is the n-dimensional real space $\mathbb{R}^n$. Veracity of the ten axioms is trivially checked given that their conception is based on the $\mathbb{R}^n$ archetype.

Example 3.0.5 (Solution sets of homogeneous linear systems)

The motivating example behind vector spaces have exactly been the sets of vectors like

$$\{r_1 \cdot \mathbf{v}_1 + r_2 \cdot \mathbf{v}_2 + \ldots + r_k \cdot \mathbf{v}_k\},$$

which should be 'correctly' written as

$$\{r_1 \odot \mathbf{v}_1 \oplus r_2 \odot \mathbf{v}_2 \oplus \ldots \oplus r_k \odot \mathbf{v}_k\},$$

for some $r_1, \ldots, r_n \in \mathbb{R}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. These are not *exactly* equal to $\mathbb{R}^k$ although they do describe a space with k different directions of movement. Since we defined vector spaces with the primary aim of accommodating such examples, it would be quite the sorry situation should they fail to be them. Luckily, all such sets indeed are vector spaces. Naturally, by proposition 1.3.7, these sets arise as sets of solutions of homogeneous linear systems.

Since they are sets of vectors in $\mathbb{R}^n$, the technical axioms (1), (2), (6), (7), (8) and (10) are trivially satisfied. We are going to check axioms (3) and (9) first. They say that the sum of solutions of a homogeneous linear system is also a solution of the same system and so are multiples of solutions. Suppose thus that the vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are solutions of the homogeneous linear system

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \ldots & + & a_{1,n}x_n & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \ldots & + & a_{2,n}x_n & = & 0 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \ldots & + & a_{m,n}x_n & = & 0 \end{array}$$

Then, for every $i \leq n$, the equality

$$\begin{aligned} a_{i,1}(v_1 + w_1) + a_{i,2}(v_2 + w_2) + \ldots + a_{i,n}(v_n + w_n) &= a_{i,1}v_1 + a_{i,2}v_2 + \ldots + a_{i,n}v_n \\ &\quad + a_{i,1}w_1 + a_{i,2}w_2 + \ldots + a_{i,n}w_n \\ &= 0 + 0 = 0 \end{aligned}$$

is satisfied, which proves (3). Similarly, if $\mathbf{v}$ is a solution, then for any $r \in \mathbb{R}$ and all $i \leq n$ we have

$$\begin{aligned} a_{i,1}(rv_1) + a_{i,2}(rv_2) + \ldots + a_{i,n}(rv_n) &= ra_{i,1}v_1 + ra_{i,2}v_2 + \ldots + ra_{i,n}v_n \\ &= r(a_{i,1}v_1 + a_{i,2}v_2 + \ldots + a_{i,n}v_n) = 0, \end{aligned}$$

which means that $r \cdot \mathbf{v}$ is also a solution, and thus (9) holds.

Finally, as far as axioms (4) and (5) are concerned, the vector $\mathbf{0}$ is always a solution of a homogeneous linear system and the inverse to a solution $\mathbf{v}$ is, of course, the solution $-1 \cdot \mathbf{v}$ (which is indeed a solution by the preceding paragraph).

Remark 3.0.6

Do note that by example 3.0.5, only the solutions of **homogeneous** linear systems are vector spaces. Solutions to non-homogeneous linear systems **always** fail to be vector spaces; they do not contain the vector $\mathbf{0}$ for example. They are so-called *affine* spaces which we shan't study in this text.

Example 3.0.7 (Sets of polynomials)

Sets of polynomials in one variable of a given degree make an interesting example of an abstract vector space. To recall, a real polynomial p in one variable of degree n is the expression

$$p(x) = r_0 + r_1 x + r_2 x^2 + \ldots + r_n x^n$$

for some $r_0, \ldots, r_n \in \mathbb{R}$. We claim that the set

$$\{p \mid p \text{ is a real polynomial of degree } n\},$$

with the addition operation being the typical addition of polynomials and scalar multiplication being simple multiplication of polynomials by real numbers, is a vector space.

Similarly to the previous example, axioms (1), (2), (6), (7), (8) and (10) are easily verified to hold. Clearly, the sum of two polynomials of degree n is a polynomial of degree n and the multiple of a polynomial of degree n is again a polynomial degree n. The zero vector is the polynomial $0 + 0x + 0x^2 + \ldots + 0x^n$ and the inverse to the polynomial p is of course $-p$.

Do note however that we **forbid polynomial multiplication** since the product of two polynomials of degree n is no longer a polynomial of degree n. We may only add polynomials and multiply them by a real number.

As a matter of fact, polynomials of degree n are in a sense 'the same' as vectors over $\mathbb{R}^{n+1}$. The correspondence is easy to forge – we simply encode the coefficients of the polynomial into a vector like so:

$$r_0 + r_1 x + r_2 x^2 + \ldots + r_n x^n \mapsto \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \in \mathbb{R}^{n+1}.$$

We can now rewrite polynomial addition and scalar multiplication as the same operations on vectors in $\mathbb{R}^{n+1}$. For example, on the set of polynomials of degree 3, the addition

$$(2x + x^2 + 3x^3) + (-1 + 3x - 5x^3) = -1 + 5x + x^2 - 2x^3$$

can be written in the following vector form.

$$\begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 1 \\ -2 \end{pmatrix}$$

Example 3.0.8 (Matrices)

The set

$$\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$$

of 2×2 matrices with real entries and entry-wise addition and scalar multiplication is a vector space over $\mathbb{R}$. This space is often denoted as $\mathbb{R}^{2\times 2}$. To give an example, the addition of matrices behaves like this:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1+2 & 2+1 \\ 3+4 & 4+3 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix};$$

and scalar multiplication like this:

$$2 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 & 2\cdot 2 \\ 2\cdot 3 & 2\cdot 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}.$$

Upon closer inspection, 2×2 matrices behave exactly like 4-component real vectors. Indeed, we can find a correspondence

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longleftrightarrow \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}$$

and treat the space $\mathbb{R}^{2\times 2}$ exactly the same as $\mathbb{R}^4$ with vector addition and multiplication. This correspondence also shows that all the defining axioms are satisfied.

This example can naturally be scaled to spaces $\mathbb{R}^{m\times n}$ of real matrices with m rows and n columns. One always gets a correspondence between $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{mn}$.

Example 3.0.9 (The trivial space)

The set $\{\mathbf{0}\}$ containing only the zero vector is a vector space. It is actually the smallest (by element count) vector space that can exist. The empty set is not a vector space precisely due to its lack of the zero vector.

Example 3.0.10 (Natural functions)

The set

$$\{f \mid f \text{ is a function } \mathbb{N} \to \mathbb{R}\},$$

with addition defined by $(f+g)(n) = f(n) + g(n)$ and scalar multiplication by $(r \cdot f)(n) = rf(n)$, is a vector space. It differs from previous examples by its *dimension*. Without proper means to define the dimension of a vector space just yet, we just vaguely state that this vector space has *infinite* dimension.

Each function $f : \mathbb{N} \to \mathbb{R}$ *can* actually be represented as a vector with real entries, but with an infinite number of them. For example, the function $f(n) = n^2 + 1$ can be represented as a vector

$$\begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \end{pmatrix} = \begin{pmatrix} 0^2+1 \\ 1^2+1 \\ 2^2+1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ \vdots \end{pmatrix}$$

whose i-th entry is $f(i)$. Since there are infinitely many natural numbers, these vectors themselves must have infinite entries. We leave the verification of the vector space axioms to our hard-working readers.

Example 3.0.11 (Real functions)

Our final example features the set of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with addition and scalar multiplication defined as in example 3.0.10. This is again a vector space (exercise) which has not just infinite, but *uncountable* dimension. The consequence of this fact is that such functions can no longer be reasonably represented as vectors. As real numbers cannot be *enumerated*, we have no clear idea what the i-th entry of such a vector should be.

Exercise 3.0.12

Prove (by checking the axioms) that the sets of functions mentioned in examples 3.0.10 and 3.0.11 are indeed vector spaces by definition.

The string of examples hopefully shed some light on the power of *abstraction* or *generalisation* in mathematics. It is not that (most) mathematicians enjoy working with abstract and unintuitive concepts (most of them refer to them as 'abstract nonsense' actually) but that this approach yields results about many structures at once. Everything we henceforth prove about vector spaces is going to be valid for all the sets mentioned here as well as absolutely any set that fits the definition of a vector space. To give a few examples:

- the theory of differential equations from calculus relies heavily on the description of solutions of linear systems;
- multivariable real calculus (differentiation and integration of function $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$) uses theory of matrices and their determinants (to be introduced way later) as derivatives of multivarible functions are matrices;
- vectors and matrices whose entries are module homomorphisms are a regular occurrence in my own branch of representation theory of algebras.

Had we kept working only in the spaces $\mathbb{R}^n$, we would have had to be constantly doubting whether any one result wasn't particular to this scenario.

We close this introductory passage with some 'intuitively obvious' statements about qualities of vectors that are nevertheless not mentioned as axioms and, as such, must be proven.

Lemma 3.0.13 (Abstract nonsense)

Let V be a vector space over $\mathbb{R}$. Then, for any $r \in \mathbb{R}$ and $\mathbf{v} \in V$, we have

(a) $0 \cdot \mathbf{v} = \mathbf{0}$;

(b) $(-1 \cdot \mathbf{v}) + \mathbf{v} = \mathbf{0}$;

(c) $r \cdot \mathbf{0} = \mathbf{0}$.

Proof. As for (a), observe that

$$\mathbf{v} \overset{(10)}{=} 1 \cdot \mathbf{v} = (1+0) \cdot \mathbf{v} \overset{(6)}{=} 1 \cdot \mathbf{v} + 0 \cdot \mathbf{v} \overset{(10)}{=} \mathbf{v} + 0 \cdot \mathbf{v}, \tag{3.1}$$

where the numbers above equal signs refer to axioms in the definition of a vector space. Let now $\mathbf{w}$ be the *additive inverse* of $\mathbf{v}$, i.e. $\mathbf{v} + \mathbf{w} = \mathbf{0}$; such exists by axiom (5). Adding $\mathbf{w}$ to both sides of the equation (3.1) gives

$$\begin{aligned}\mathbf{w} + \mathbf{v} &= \mathbf{w} + \mathbf{v} + 0 \cdot \mathbf{v} \\ \mathbf{0} &= \mathbf{0} + 0 \cdot \mathbf{v}\end{aligned}$$

By axiom (4), the right side equals $0 \cdot \mathbf{v}$ and thus we have proven that $\mathbf{0} = 0 \cdot \mathbf{v}$.

The calculation

$$(-1 \cdot \mathbf{v}) + \mathbf{v} \overset{(6)}{=} (-1 + 1) \cdot \mathbf{v} = 0 \cdot \mathbf{v} \overset{(a)}{=} \mathbf{0}$$

proves point (b).

As for (c), we have

$$r \cdot \mathbf{0} \overset{(a)}{=} r \cdot (0 \cdot \mathbf{0}) \overset{(8)}{=} (r0) \cdot \mathbf{0} = 0 \cdot \mathbf{0} \overset{(a)}{=} \mathbf{0},$$

which proves the statement. ■

3.1 Subspaces And Spans

In this section, we set out to formalise two concepts. First, in section 2.3, we interpreted the set

$$\left\{ y \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R} \right\}$$

as a plane in $\mathbb{R}^3$. In the introduction to this chapter, we explained the 'plane' part. However, we have not yet made it clear what we mean by 'in $\mathbb{R}^3$'. What does it *mean* for a vector space to *lie inside* another vector space? Second, we've often used the phrase 'vectors define dimensions or directions of movement'. We shall return to this idea very soon as well.

Now, a vector space which is wholly contained in another vector space is called a subspace and its definition is quite simple and natural.

Definition 3.1.1 (Subspace)

Let V be a vector space. A set S is a *subspace* of V if $S \subseteq V$ and, additionally, S is a vector space in its own right. We write $S \leq V$ to indicate that S is a subspace of V.

Simply put, subspaces of V are its subsets that are also closed under vector addition and scalar multiplication. Before we proceed to illustrate the concept on a few examples, we prove a lemma which allows us to check whether a given subset is also a subspace without having to go through all the axioms in the definition of vector space.

Lemma 3.1.2 (Characterisation of subspaces)

Let V be a vector space and S a subset of V with inherited operations of vector addition and scalar multiplication. Then, the following statements are equivalent.

(a) S is a subspace of V.

(b) *S is closed under linear combinations of pairs of vectors: for any $\mathbf{s}_1, \mathbf{s}_2 \in S$ and $r_1, r_2 \in \mathbb{R}$, the vector $r_1 \cdot \mathbf{s}_1 + r_2 \cdot \mathbf{s}_2$ lies in S.*

(c) *S is closed under linear combinations of any number of vectors: for any $n \in \mathbb{N}$, vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n \in S$ and $r_1, r_2, \ldots, r_n \in \mathbb{R}$, the vector $r_1 \cdot \mathbf{s}_1 + r_2 \cdot \mathbf{s}_2 + \ldots + r_n \cdot \mathbf{s}_n$ lies in S.*

Proof. Instead of proving $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$ and $(b) \Leftrightarrow (c)$, it is simpler to establish the claim by proving

$$(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a).$$

As for $(a) \Rightarrow (c)$, since S is a vector space, we need only invoke the axioms. Given $\mathbf{s}_1 \in S$ and $r_1 \in \mathbb{R}$, we use the axiom (9) to ascertain that $r_1 \cdot \mathbf{s}_1 \in S$. Similarly for the other vectors $\mathbf{s}_2, \ldots, \mathbf{s}_n \in S$ and numbers $r_2, \ldots, r_n \in \mathbb{R}$. Then, by repeatedly using axiom (3), we prove that the linear combination $r_1 \cdot \mathbf{s}_1 + r_2 \cdot \mathbf{s}_2 + \ldots + r_n \cdot \mathbf{s}_n$ lies in S.

The implication $(c) \Rightarrow (b)$ is obvious, simply substitute $n = 2$.

The proof of $(b) \Rightarrow (a)$ takes the most work. We must check that S satisfies all the axioms of a vector space. The technical axioms (1), (2), (6), (7), (8) and (10) hold in S because they hold in V. We shall illustrate this on axiom (1), the rest is proven in a very similar manner. Given $\mathbf{s}_1, \mathbf{s}_2 \in S$, we know by (b) that $\mathbf{s}_1 + \mathbf{s}_2 \in S$. However, since $+$ is commutative in V and $\mathbf{s}_1 + \mathbf{s}_2 \in V$, we have the equality $\mathbf{s}_1 + \mathbf{s}_2 = \mathbf{s}_2 + \mathbf{s}_1$ in the vector space V. Because (again by (b)) $\mathbf{s}_2 + \mathbf{s}_1 \in S$, the same equality must also hold in S as it is a subset of V. This proves that vector addition is commutative in S. Axiom (4) is proven by setting $r_1 = r_2 = 0$ and using (b) together with lemma 3.0.13 (a). Similarly, axiom (5) follows (for a given $\mathbf{s} \in S$) by setting $r_1 = 0, \mathbf{s}_1 = \mathbf{0}$ and $r_2 = -1, \mathbf{s}_2 = \mathbf{s}$ while using lemma 3.0.13 (b). Finally, given $\mathbf{s}_1, \mathbf{s}_2 \in S$, axiom (3) follows by $r_1 = r_2 = 1$ and given $r \in \mathbb{R}, \mathbf{s} \in S$, axiom (9) is verified by $r_1 = r, \mathbf{s}_1 = \mathbf{s}, r_2 = 0, \mathbf{s}_2 = \mathbf{0}$. S is thus a vector space and a subset of V so (a) holds. ■

Warning 3.1.3

By definition, $\mathbb{R}^m$ is **never** a subspace of $\mathbb{R}^n$ as long as $m \neq n$. This is because it is not a subset. The space $\mathbb{R}^n$ contains vectors with *exactly* n entries and no vector with m entries.

This issue is alleviated by 'filling' the vectors in $\mathbb{R}^m$ with zeroes so that they have n entries if $m \leq n$. Formally, the set

$$\left\{ \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n \mid v_1, \ldots, v_m \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^n$ and is all but formally equivalent to $\mathbb{R}^m$.

Example 3.1.4

By example 3.0.5, the solution set of a homogeneous linear system with n variables is a subspace of $\mathbb{R}^n$. To give a concrete example, the solution set of the system

$$\begin{aligned} x + 3y - z &= 0 \\ 2x + y - 2z &= 0 \end{aligned}$$

is the set

$$S := \left\{ z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mid z \in \mathbb{R} \right\}$$

and it is a subspace of $\mathbb{R}^3$. We can prove this using lemma 3.1.2 (b). Given $\mathbf{s}_1, \mathbf{s}_2 \in S$, there exist $z_1, z_2 \in \mathbb{R}$ such that

$$\mathbf{s}_1 = z_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{s}_2 = z_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then, for any $r_1, r_2 \in \mathbb{R}$, we have

$$r_1 \cdot \mathbf{s}_1 + r_2 \cdot \mathbf{s}_2 = (r_1 z_1) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (r_2 z_2) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (r_1 z_1 + r_2 z_2) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and so $r_1 \cdot \mathbf{s}_1 + r_2 \cdot \mathbf{s}_2 \in S$.

Example 3.1.5

The set of polynomials of degree at most 2 is a subspace of the set of polynomials of degree at most 4. Indeed, given two polynomials $a_0 + a_1 x + a_2 x^2$ and $b_0 + b_1 x + b_2 x^2$ and numbers $r_1, r_2 \in \mathbb{R}$, we have

$$r_1 \cdot (a_0 + a_1 x + a_2 x^2) + r_2 \cdot (b_0 + b_1 x + b_2 x^2) = (r_1 a_0 + r_2 b_0) + (r_1 a_1 + r_2 b_1) x + (r_1 a_2 + r_2 b_2) x^2$$

which is a polynomial of degree at most 2. Clearly, every polynomial of degree at most 2 is a polynomial of degree at most 4.

Example 3.1.6

The set of 2×2 real matrices

$$L := \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a + b + c = 0 \right\}$$

is a subspace of $\mathbb{R}^{2\times 2}$. Indeed, this is because we can substitute $a = -b - c$ and write

$$\begin{pmatrix} -b-c & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} -b & 0 \\ b & 0 \end{pmatrix} + \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} = b \cdot \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and so every matrix in L is a linear combination of the matrices

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which proves that L is a subspace by lemma 3.1.2 (b).

Example 3.1.7

The set of real functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which have a finite derivative at 0 is a subspace of the set of all real functions. Indeed, if $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are real functions and $f'(0), g'(0) \in \mathbb{R}$, then for any $r_1, r_2 \in \mathbb{R}$ we can compute

$$(r_1 \cdot f + r_2 \cdot g)'(0) = r_1 f'(0) + r_2 g'(0) \in \mathbb{R}$$

so the derivative of $r_1 \cdot f + r_2 \cdot g$ at 0 is finite.

We now return to the idea of vectors as 'directions of movement'. The overused poor little set

$$S := \left\{ y \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$ and its description tells us that every vector in that space is a linear combination of

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

If such is the case, we say that S is *generated* by these vectors and write

$$S = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right).$$

More generally, if a vector space V is the *span* of a set of vectors S, it means that every vector in V is a linear combination of vectors from S. Intuitively, this formalises the idea that every point in the space V is reachable just by moving along vectors from S.

Definition 3.1.8 (Span)

Let V be a vector space and S a subset of V (**not** necessarily a subspace). The *span of* S is the set of all linear combinations of vectors from S. Symbolically,

$$\operatorname{span} S := \left\{ \sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s} \mid r_{\mathbf{s}} \in \mathbb{R} \ \forall \mathbf{s} \in S \right\}$$

To avoid technical details, we only consider **finite** sums of the form above to form the set $\operatorname{span} S$. If $S = \emptyset$, we define $\operatorname{span} S := \{\mathbf{0}\}$.

Remark 3.1.9

Proposition 1.3.7 states that the set of solutions of a homogeneous linear system is of the form

$$\{r_1 \cdot \mathbf{v}_1 + r_2 \cdot \mathbf{v}_2 + \ldots + r_k \cdot \mathbf{v}_k\}.$$

We can now write the same set more succinctly as

$$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

In light of lemma 3.1.2, it should come as no surprise that spans of sets are always vector spaces. The verification of this fact essentially boils down to 'a linear combination of linear combinations is a linear combination'.

Lemma 3.1.10

Let V be a vector space. Then, for every subset $S \subseteq V$, the set $\operatorname{span} S$ is a subspace of V.

Proof. By lemma 3.1.2 (b), we need only prove that $a \cdot \mathbf{s}_1 + b \cdot \mathbf{s}_2 \in \operatorname{span} S$ for $a, b \in \mathbb{R}$ and $\mathbf{s}_1, \mathbf{s}_2 \in \operatorname{span} S$. By definition, we can write

$$\mathbf{s}_1 = \sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s}, \quad \mathbf{s}_2 = \sum_{\mathbf{s} \in S} t_{\mathbf{s}} \cdot \mathbf{s}$$

for adequate numbers $r_{\mathbf{s}}, t_{\mathbf{s}} \in \mathbb{R}$. Then,

$$\begin{aligned} a \cdot \mathbf{s}_1 + b \cdot \mathbf{s}_2 &= a \cdot \left(\sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s}\right) + b \cdot \left(\sum_{\mathbf{s} \in S} t_{\mathbf{s}} \cdot \mathbf{s}\right) = \sum_{\mathbf{s} \in S} a \cdot (r_{\mathbf{s}} \cdot \mathbf{s}) + \sum_{\mathbf{s} \in S} b \cdot (t_{\mathbf{s}} \cdot \mathbf{s}) \\ &= \sum_{\mathbf{s} \in S} (a r_{\mathbf{s}}) \cdot \mathbf{s} + \sum_{\mathbf{s} \in S} (b t_{\mathbf{s}}) \cdot \mathbf{s} = \sum_{\mathbf{s} \in S} (a r_{\mathbf{s}} + b t_{\mathbf{s}}) \cdot \mathbf{s}. \end{aligned}$$

Relabelling $p_{\mathbf{s}} := a r_{\mathbf{s}} + b t_{\mathbf{s}}$, we get $p_{\mathbf{s}} \in \mathbb{R}$ and

$$a \cdot \mathbf{s}_1 + b \cdot \mathbf{s}_2 = \sum_{\mathbf{s} \in S} p_{\mathbf{s}} \cdot \mathbf{s},$$

hence $a \cdot \mathbf{s}_1 + b \cdot \mathbf{s}_2 \in \operatorname{span} S$ which proves that $\operatorname{span} S$ is a vector space.

By lemma 3.1.2 (c), the space V is closed under linear combinations of finite numbers of vectors. Since $\operatorname{span} S$ is defined as the set of finite linear combinations of vectors of S (which also lie in V), it is clear that $\operatorname{span} S \leq V$. ■

Quite trivially, the converse of the previous lemma also holds: any subspace of V is a span of some set of vectors from V. This is because any subspace of V is obviously its own span.

Example 3.1.11 (Line)

In any vector space V, the set $\operatorname{span}(\mathbf{v})$ is a subspace for any $\mathbf{v} \in V$. It is a line passing through the origin in the direction of $\mathbf{v}$.

Example 3.1.12

The space
$$\operatorname{span}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right)$$
is the entirety of $\mathbb{R}^2$. How do we prove this? By definition of a span, every vector $\mathbf{v} \in \mathbb{R}^2$ should be a linear combination of the vectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$. That is, there should exist numbers $a, b \in \mathbb{R}$ such that
$$\mathbf{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix} = a \cdot \begin{pmatrix}1\\1\end{pmatrix} + b \cdot \begin{pmatrix}1\\-1\end{pmatrix}.$$
Rewriting the equation in a more instructive manner:
$$\begin{aligned} a + b &= v_1 \\ a - b &= v_2 \end{aligned}$$
This bespeaks the fact that $\mathbf{v} \in \operatorname{span}(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix})$ if and only if the system above has a solution in variables a and b. Gauss-Jordan elimination gives
$$\begin{aligned} a + b &= v_1 \\ -2b &= v_2 - v_1 \end{aligned}$$
and so $b = (v_1 - v_2)/2$ and $a = (v_1 + v_2)/2$. This proves that $\operatorname{span}(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}) = \mathbb{R}^2$ since we can express any vector in $\mathbb{R}^2$ as a linear combination of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$.

Example 3.1.13

Consider the set
$$S := \{3x - x^2, 2x\}$$
as a subset of the set of polynomials of degree at most 2. We wish to describe polynomials lying in $\operatorname{span} S$. This entails looking at all polynomials of the form
$$a \cdot (3x - x^2) + b \cdot (2x)$$
for $a, b \in \mathbb{R}$. A polynomial of degree at most 2 has the general form $r_0 + r_1x + r_2x^2$. We can compare coefficients on either side of the equation
$$\begin{aligned} r_0 + r_1x + r_2x^2 &= a \cdot (3x - x^2) + b \cdot (2x) \\ r_0 + r_1x + r_2x^2 &= 0 + (3a + 2b)x + (-a)x^2 \end{aligned}$$
to reach the linear system
$$\begin{aligned} -a &= r_2 \\ 3a + 2b &= r_1 \\ 0 &= r_0 \end{aligned}$$
whose solution is $(a, b) = (-r_2, (3/2)r_2 + (1/2)r_1)$ but only in the case that $r_0 = 0$, otherwise it has none. It follows that $\operatorname{span}(3x - x^2, 2x)$ is the subspace of all polynomials $r_1x + r_2x^2$ for $r_1, r_2 \in \mathbb{R}$.

The preceding two examples hint at a general way to check whether a given vector $\mathbf{v} \in V$ lies in $\operatorname{span} S$ for some $S \subseteq V$. If S is finite, say, $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_k\}$, then $\mathbf{v}$ lies in $\operatorname{span} S$ if and only if there exist numbers $r_1, \ldots, r_k \in \mathbb{R}$ such that

$$\mathbf{v} = r_1 \cdot \mathbf{s}_1 + r_2 \cdot \mathbf{s}_2 + \ldots + r_k \cdot \mathbf{s}_k.$$

If V is further a subspace of $\mathbb{R}^n$, that is, if we can write the vector $\mathbf{v}$ as

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

and the vectors $\mathbf{s}_1, \ldots, \mathbf{s}_k$ as

$$\mathbf{s}_1 = \begin{pmatrix} s_{11} \\ s_{12} \\ \vdots \\ s_{1n} \end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix} s_{21} \\ s_{22} \\ \vdots \\ s_{2n} \end{pmatrix}, \ldots, \mathbf{s}_k = \begin{pmatrix} s_{k1} \\ s_{k2} \\ \vdots \\ s_{kn} \end{pmatrix},$$

then $\mathbf{v} \in \operatorname{span} S$ if and only if the linear system

$$\left(\begin{array}{cccc|c} s_{11} & s_{21} & \cdots & s_{k1} & v_1 \\ s_{12} & s_{22} & \cdots & s_{k2} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & s_{2n} & \cdots & s_{kn} & v_n \end{array}\right)$$

has **at least one** solution. Any one solution of this system is then naturally the n-tuple $(r_1, \ldots, r_n)$ of the coefficients of the linear combination which gives rise to the vector $\mathbf{v}$. We can also write the matrix above succinctly as

$$\left(\begin{array}{cccc|c} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_k & \mathbf{v} \end{array}\right).$$

Put into words, a vector $\mathbf{v}$ lies in $\operatorname{span}(\mathbf{s}_1, \ldots, \mathbf{s}_k)$ if and only if the system with columns the vectors $\mathbf{s}_1, \ldots, \mathbf{s}_k$ and right side the vector $\mathbf{v}$ has at least one solution.

Problem 3.1.14

Determine whether the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

lies in

$$\operatorname{span}\left(\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right).$$

If so, find $x_1, x_2, x_3, x_4 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Solution. We simply attempt to solve the system

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 4 & -2 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & -2 \end{array}\right).$$

Gauss-Jordan elimination gives

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 0 & -6 & 1 & -7 & -1 \\ 0 & 0 & 13 & -1 & -13 \end{array}\right).$$

For we're only interested in a single solution and x_4 is a free variable, we substitute $x_4 = 0$. Then, from $13x_3 - 1 \cdot 0 = -13$ follows $x_3 = -1$. Further computation gives $x_2 = 0$ and $x_1 = 1$, which means that

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

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Problem 3.1.14 shows that there may often be multiple ways to express a vector as a linear combination of the spanning vectors. This naturally leads to a question computational in nature: 'Given a set of vectors S, can we find another set T such that $\operatorname{span} S = \operatorname{span} T$ but T has as few vectors as possible?' The reason we'd wish to do something like this is simple; in checking whether a vector lies in a given span, free variables in the resulting system are *redundancies* that increase both the computational time and required memory. The next section is dedicated to answering the question at hand.

3.2 Linear Independence, Basis And Dimension

Given a set $S \subseteq V$ of vectors, we first answer the question of *which vectors can be removed from S while not altering its span.* That is, given a vector $\mathbf{s} \in S$, how do we find out whether $\operatorname{span}(S \setminus \{\mathbf{s}\}) = \operatorname{span} S$? Vaguely speaking, provided that $\operatorname{span} S$ is a set of linear combinations of vectors from S, should some vector in S already *be a linear combination* of the other vectors in S, it wouldn't be needed. Turns out this statement is not so vague after all, as we proceed to demonstrate.

Lemma 3.2.1

Let V be a vector space, $S \subseteq V$ and $\mathbf{v} \in V$. Then, $\operatorname{span}(S \cup \{\mathbf{v}\}) = \operatorname{span} S$ *if and only if* $\mathbf{v} \in \operatorname{span} S$.

Proof. We must prove two implications.

As for the implication ($\Rightarrow$), it is simpler to prove it in contrapositive form. If $\mathbf{v} \notin \operatorname{span} S$, then clearly $\operatorname{span} S \neq \operatorname{span}(S \cup \{\mathbf{v}\})$ simply because the latter contains the vector $\mathbf{v}$ while the former does not.

In proving ($\Leftarrow$), assume that $\mathbf{v} \in \operatorname{span} S$. Clearly, $\operatorname{span} S \subseteq \operatorname{span}(S \cup \{\mathbf{v}\})$ as the latter set contains every linear combination of the vectors in S. We must show that also $\operatorname{span}(S \cup \{\mathbf{v}\}) \subseteq$

span S. To this end, choose a vector $\mathbf{w} \in \operatorname{span}(S \cup \{\mathbf{v}\})$. This vector $\mathbf{w}$ is a linear combination of vectors from $S \cup \{\mathbf{v}\}$, i.e.

$$\mathbf{w} = \sum_{\mathbf{s} \in S \cup \{\mathbf{v}\}} r_{\mathbf{s}} \cdot \mathbf{s}$$

for some $r_{\mathbf{s}} \in \mathbb{R}$. We can break this linear combination into two parts like so:

$$\mathbf{w} = \sum_{\mathbf{s} \in S \cup \{\mathbf{v}\}} r_{\mathbf{s}} \cdot \mathbf{s} = \sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s} + r_{\mathbf{v}} \cdot \mathbf{v}. \tag{3.2}$$

Now, $\mathbf{v} \in \operatorname{span} S$ by assumption so there also exist numbers $t_{\mathbf{s}} \in \mathbb{R}$ such that

$$\mathbf{v} = \sum_{\mathbf{s} \in S} t_{\mathbf{s}} \cdot \mathbf{s}.$$

Substituting this into the equation (3.2) gives

$$\mathbf{w} = \sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s} + r_{\mathbf{v}} \cdot \left(\sum_{\mathbf{s} \in S} t_{\mathbf{s}} \cdot \mathbf{s} \right) = \sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s} + \sum_{\mathbf{s} \in S} (r_{\mathbf{v}} t_{\mathbf{s}}) \cdot \mathbf{s} = \sum_{\mathbf{s} \in S} (r_{\mathbf{s}} + r_{\mathbf{v}} t_{\mathbf{s}}) \cdot \mathbf{s}.$$

The last sum is a linear combination of vectors from S and thus $\mathbf{w} \in \operatorname{span} S$, as desired. ■

As a corollary, we get a formalisation of the idea from the introductory paragraph.

Corollary 3.2.2

Given $\mathbf{s} \in S$, *it holds that* $\operatorname{span} S = \operatorname{span}(S \setminus \{\mathbf{s}\})$ *if and only if* $\mathbf{s} \in \operatorname{span}(S \setminus \{\mathbf{s}\})$.

Proof. Follows immediately from lemma 3.2.1. Simply substitute $\mathbf{v} \coloneqq \mathbf{s}$ and $S \coloneqq S \setminus \{\mathbf{s}\}$. ■

The just uttered corollary has algorithmic vibes. Can't we just keep removing vectors from S which are linearly dependent on other vectors until there are no longer any? Indeed, we can. First however, we should devise a computationally sound way to determine which vectors we may omit. As we now stand, the best we can do is guess at random. Pick a vector $\mathbf{s} \in S$ and check if it lies in $\operatorname{span}(S \setminus \{\mathbf{s}\})$ (as we learnt to do in the the previous section). If it doesn't, tough luck, try again. We might potentially have to go through *every* vector in S before we find one that can be left out, if there even were one to begin with. This is about as algorithmic as cooking a soup by mixing random ingredients until we stumble upon a combination which is reasonably non-lethal.

Fortunately, there is an algorithmic approach to the problem and we are unveiling it promptly. Before that however, we should label sets with no 'unnecessary' vectors somehow.

Definition 3.2.3 (Linear independence)

Let V be a vector space and $S \subseteq V$. If no vector $\mathbf{s} \in S$ can be written as a linear combination of vectors from $S \setminus \{\mathbf{s}\}$, we call S *linearly independent.* If such is not the case, it is called *linearly dependent.*

There lies just a simple observation between us and a feasible algorithm for determining linear independence of a given set of vectors. Suppose $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ and may the vector $\mathbf{s}_i$ be a linear combination of the other vectors, that is to say, there are numbers $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n \in \mathbb{R}$

satisfying the equation

$$\mathbf{s}_i = r_1 \cdot \mathbf{s}_1 + r_2 \cdot \mathbf{s}_2 + \ldots + r_{i-1} \cdot \mathbf{s}_{i-1} + r_{i+1} \cdot \mathbf{s}_{i-1} + \ldots + r_n \cdot \mathbf{s}_n.$$

We can naturally put $\mathbf{s}_i$ to the right hand side and set $r_i \coloneqq -1$ to arrive at the equality

$$\mathbf{0} = r_1 \cdot \mathbf{s}_1 + \ldots + r_{i-1} \cdot \mathbf{s}_{i-1} + r_i \cdot \mathbf{s}_i + r_{i+1} \cdot \mathbf{s}_{i+1} + \ldots + r_n \mathbf{s}_n.$$

To express this equality in words: we have found a linear combination (with non-zero coefficients) of vectors from S that gives the zero vector. Could this happen were S linearly independent? Of course it couldn't! If it did, then we could just rearrange the last equality to the first one and get $\mathbf{s}_i$ as a linear combination of the other vectors from S, proving thus that S, in fact, had *not* been linearly independent. Let us dock this train of thought in the following, computationally indispensable, proposition.

Proposition 3.2.4

Let V be a vector space and $S \subseteq V$. Then S is linearly independent if and only if the equality

$$\sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot s = \mathbf{0}$$

enforces $r_{\mathbf{s}} = 0$ for every $\mathbf{s} \in S$. In other words, the only linear combination that gives the zero vector has all coefficients equal to 0.

Proof. The paragraph preceding this proposition already illustrates the idea of the proof.

To prove the implication $(\Leftarrow)$, suppose that S is linearly dependent. That is, there exists $\mathbf{v} \in S$ such that $\mathbf{v} \neq \mathbf{0}$ and

$$\mathbf{v} = \sum_{\mathbf{s} \in S \setminus \{\mathbf{v}\}} r_{\mathbf{s}} \cdot \mathbf{s}.$$

If we put $\mathbf{v}$ to the right hand side and set $r_{\mathbf{v}} \coloneqq -1$, we get

$$\mathbf{0} = \sum_{\mathbf{s} \in S \setminus \{\mathbf{v}\}} r_{\mathbf{s}} \cdot \mathbf{s} + (-1) \cdot \mathbf{v} = \sum_{\mathbf{s} \in S \setminus \{\mathbf{v}\}} r_{\mathbf{s}} \cdot \mathbf{s} + r_{\mathbf{v}} \cdot \mathbf{v} = \sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s}.$$

We've thus wrought a linear combination of vectors from S which has non-zero coefficients but equals the zero vector.

As for $(\Rightarrow)$, assume that there exists a linear combination

$$\sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s} = \mathbf{0}$$

with at least one $r_{\mathbf{v}} \neq 0$. This means that we can rearrange

$$\begin{aligned}
\sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s} &= \mathbf{0} \\
\sum_{\mathbf{s} \in S \setminus \{\mathbf{v}\}} r_{\mathbf{s}} \cdot \mathbf{s} + r_{\mathbf{v}} \cdot \mathbf{v} &= \mathbf{0} \\
\sum_{\mathbf{s} \in S \setminus \{\mathbf{v}\}} r_{\mathbf{s}} \cdot \mathbf{s} &= -r_{\mathbf{v}} \cdot \mathbf{v} \\
-\frac{1}{r_{\mathbf{v}}} \cdot \left(\sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s} \right) = \sum_{\mathbf{s} \in S \setminus \{\mathbf{v}\}} -\frac{r_{\mathbf{s}}}{r_{\mathbf{v}}} \cdot \mathbf{s} &= \mathbf{v}
\end{aligned}$$

and thus $\mathbf{v} \in \operatorname{span}(S \setminus \{\mathbf{v}\})$ which shows that S is linearly dependent. ■

Corollary 3.2.5 (Computing linear independence)

Let $V \leq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_k\} \subseteq V$. Then, S is linearly independent if and only if the linear system

$$\left(\mathbf{s}_1 \;\; \mathbf{s}_2 \;\; \cdots \;\; \mathbf{s}_k \;\middle|\; \mathbf{0}\right)$$

has the unique solution $\mathbf{0}$.

Proof. The proof just amounts to rewriting the equality

$$\sum_{\mathbf{s} \in S} r_{\mathbf{s}} \cdot \mathbf{s} = \sum_{i=1}^{k} r_{\mathbf{s}_i} \cdot \mathbf{s}_i = \mathbf{0}$$

into a linear system and applying proposition 3.2.4. ■

Example 3.2.6

The set

$$S \coloneqq \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$$

is linearly dependent in $\mathbb{R}^2$. Indeed, the system

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right)$$

is solved by $(z, -z, -z)$ for any $z \in \mathbb{R}$. Corollary 3.2.5 states that S is linearly dependent.

Example 3.2.7

The set $S \coloneqq \{1 - x, 1 + x\}$ is linearly independent in the vector space of quadratic polynomials. To see this, consider a linear combination

$$\begin{aligned} r_1 \cdot (1 - x) + r_2 \cdot (1 + x) &= 0 + 0x + 0x^2 \\ (r_1 + r_2) + (-r_1 + r_2)x + 0x^2 &= 0 + 0x + 0x^2. \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} r_1 + r_2 &= 0 \\ -r_1 + r_2 &= 0 \\ 0 &= 0. \end{aligned}$$

This system has the unique solution $(0, 0)$, hence the only way to linearly combine the polynomials $1 - x$ and $1 + x$ into the zero polynomial requires multiplying them both by 0. Proposition 3.2.4 takes the reins.

Now that we have an algorithmic way of determining whether a given set is linearly independent or not, we should tackle the problem of which vectors can be removed from the set without shrinking its span. Before we do that, let us first ascertain that indeed every (at least **finite**) linearly dependent set can be made linearly independent by successive removal of redundant vectors.

Lemma 3.2.8 (Linearly independent subset)

Given a vector space V and a ***finite subset*** $S \subseteq V$*, there exists a set* $T \subseteq S$ *that is linearly independent and* $\operatorname{span} S = \operatorname{span} T$.

Proof. Label $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_k\}$. If S is linearly independent, we're done. Otherwise, set $S_0 \coloneqq S$ and find $i \in \{1, \ldots, k\}$ such that

$$\mathbf{s}_i = \sum_{j \neq i} r_j \cdot \mathbf{s}_j$$

for some $r_j \in \mathbb{R}$. Set $S_1 \coloneqq S_0 \setminus \{\mathbf{s}_i\}$. By corollary 3.2.2, $\operatorname{span} S_1 = \operatorname{span} S_0$.

Repeat this process until S_m is linearly independent for some $m \in \mathbb{N}$. Such m necessarily exists because S has a finite number of elements and a one-vector set is always linearly independent. Again, by corollary 3.2.2 (applied m times), we have $\operatorname{span} S_m = \operatorname{span} S$ and thus we have found a linearly independent subset of S with the same span as S. ■

Recall from section 1.3 that some variables of linear systems are pivots and some are free. Pivots have their value expressed as a linear combination of free variables. If we put vectors from a given finite set $S \subseteq V$ into columns of a matrix (as in corollary 3.2.2), we claim that columns hosting free variables mark vectors that can be removed without shrinking the span of S. Why is it so? The answer is actually quite easy. We show it on an example.

Consider the set

$$S = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$

Upon organizing the vectors of S into the matrix

$$\begin{pmatrix} 1 & 1 & 2 & -1 & 1 \\ 3 & -1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 & 3 \end{pmatrix}$$

and performing Gauss-Jordan elimination, we get

$$\begin{pmatrix} 1 & 1 & 2 & -1 & 1 \\ 0 & -4 & -6 & 4 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

It follows that the variables x_3 and x_5 are free. Why does it mean that the third and fifth vector of S are redundant? Well, the solution of the homogeneous linear system with this matrix is

$$\left\{ x_3 \cdot \begin{pmatrix} 1 \\ 0 \\ -5 \\ -6 \\ 3 \end{pmatrix} + x_5 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ -1 \end{pmatrix} \;\middle|\; x_3, x_5 \in \mathbb{R} \right\}. \tag{3.3}$$

By corollary 3.2.2, the set S is linearly independent if and only if the set above contains only the vector $\mathbf{0}$. However, that happens if and only if we force $x_3 = x_5 = 0$. Next, every linear combination of vectors from S is of the form

$$x_1 \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + x_5 \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

If, to ensure linear independence, we must require that x_3 and x_5 both be always equal to 0, it is completely pointless that we include the vectors $\left(\begin{smallmatrix}2\\0\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1\\1\\3\end{smallmatrix}\right)$ in the linear combination in the first place.

Furthermore, observe that the vectors in the set (3.3) also hint at how we can express the third and fifth vectors as linear combinations of the other three. Indeed, the vector

$$\begin{pmatrix}1\\0\\-5\\-6\\3\end{pmatrix}$$

in fact contains the coefficients of the linear combination of vectors in S that gives the zero vector (as it is the solution of the corresponding homogeneous system). This means that

$$1\cdot\begin{pmatrix}1\\3\\2\end{pmatrix}+0\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}+(-5)\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}+(-6)\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}+3\cdot\begin{pmatrix}1\\1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Rearranging (and dividing by -3) gives

$$-\frac{1}{3}\cdot\begin{pmatrix}1\\3\\2\end{pmatrix}+\frac{5}{3}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}+2\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\3\end{pmatrix}$$

and thus we have expressed the vector $\left(\begin{smallmatrix}1\\1\\3\end{smallmatrix}\right)$ as a linear combination of the other four vectors. We could do the same for the vector $\left(\begin{smallmatrix}2\\0\\1\end{smallmatrix}\right)$ which we also know to be redundant. Note, however, that we would have to in addition substitute the vector $\left(\begin{smallmatrix}1\\1\\3\end{smallmatrix}\right)$ in the resulting linear combination as it should have been already removed from S.

To breathe some clarity into the concluded discussion, we shall show the described procedure in a more algorithmic way.

Problem 3.2.9

Prove that the set

$$S := \left\{\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}-2\\1\end{pmatrix},\begin{pmatrix}0\\-3\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right\}\subseteq\mathbb{R}^2$$

is linearly dependent and reduce it to a linearly independent set $S' \subseteq S$ with $\operatorname{span} S' = \operatorname{span} S$. In addition, express the removed vectors as linear combinations of the remaining ones.

Solution. We compute the solution of the homogeneous linear system

$$\left(\begin{array}{cccc|c}1&-2&0&3&0\\3&1&-3&0&0\end{array}\right).$$

After Gauss-Jordan elimination, we're left with

$$\left(\begin{array}{cccc|c}1&-2&0&3&0\\0&7&-3&-9&0\end{array}\right).$$

This means that columns 1 and 2 host pivots and columns 3 and 4 the free variables. We shall thus remove the third and the fourth vector from S. To finish the calculation, back-substitute and arrive at the set

$$\left\{ x_3 \cdot \begin{pmatrix} 3 \\ 0 \\ 3 \\ -1 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \\ 2 \end{pmatrix} \;\middle|\; x_3, x_4 \in \mathbb{R} \right\}.$$

We get rid of the fourth vector first. From the shape of the just computed solution, we infer that

$$0 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 3 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ -3 \end{pmatrix} + 2 \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = -\frac{3}{2} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \frac{1}{2} \cdot \begin{pmatrix} 0 \\ -3 \end{pmatrix}, \tag{3.4}$$

which by corollary 3.2.2 proves that

$$\operatorname{span}\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right) = \operatorname{span} S.$$

We now proceed to further remove $\left(\begin{smallmatrix} 0 \\ -3 \end{smallmatrix} \right)$ and express it as a linear combination of the remaining two vectors. The second vector in the computed solution gives the equality

$$3 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ -3 \end{pmatrix} - 1 \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

hence

$$\begin{pmatrix} 0 \\ -3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{3} \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Substituting for $\left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix} \right)$ the linear combination in (3.4) and merging the coefficients yields

$$\begin{pmatrix} 0 \\ -3 \end{pmatrix} = -\frac{6}{7} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{3}{7} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Finally, the desired linearly independent set S' with $\operatorname{span} S' = \operatorname{span} S$ is

$$S' = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}.$$

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Remark 3.2.10

Observe that the solution of problem 3.2.9 basically mimics the proof of lemma 3.2.8 with an algorithmic approach to the selection of redundant vectors.

Warning 3.2.11

The indices of columns with pivots vs. free variables only point at the vectors of the original set which **are sure not to** shrink the span but they maken't the choice of vectors unique in any way. As a matter of fact, in many cases any of the present vectors can be removed without altering the span of the original set.

To give one trivial example, consider the set

$$S := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}.$$

Any one of the vectors in S is linearly dependent on the other two (on either of them actually); all the vectors in S are redundant.

On the other hand, in the set

$$S := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right\},$$

the first vector **is not** redundant. Only either of the second and third vectors may be mercilessly cut down without shrinking the span of S. It is in cases like these that the procedure outlined in problem 3.2.9 has its merit.

We conclude the introduction to the concept of linear independence with one last brief little inconsequential unimportant and just barely appealing discussion. We've proven that removing a vector from a (finite) linearly dependent set can make it independent. Adding a vector to a linearly dependent set on the other hand cannot fix linear dependence. We shall summarise the link between subsets and linear independence of the original set in table 3.1.

	$\hat{S} \subseteq S$	$\tilde{S} \supseteq S$
S linearly independent	$\hat{S}$ also linearly independent	$\tilde{S}$ can be either
S linearly dependent	$\hat{S}$ can be either	$\tilde{S}$ also linearly dependent

Table 3.1: Linear dependence/independence of subsets.

3.2.1 Basis Of A Vector Space

The study of linearly independent sets in section 3.2 carries on its back yet another question: 'Can *every* vector space be expressed as the span of a linearly independent set?' The answer this time is *almost*. As we've made customary, before we proceed to elucidate the given answer, we establish some nomenclature to achieve a manageable level of brevity.

Definition 3.2.12 (Basis)

Let V be a vector space. An ordered n-tuple $(\mathbf{v}_1, \dots, \mathbf{v}_n)$, where $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$, which is both linearly independent and spans V is called the *basis* of V.

Warning 3.2.13

We've defined the basis of a vector space specifically to be an **ordered tuple** and **not just a set**. The reason for this will be given later in the chapter when we discuss representation of vectors with respect to distinct bases. Practically, this means that a basis, for example,

$$\left(\begin{pmatrix}69\\0\end{pmatrix},\begin{pmatrix}0\\420\end{pmatrix}\right)$$

is **different** from

$$\left(\begin{pmatrix}0\\420\end{pmatrix},\begin{pmatrix}69\\0\end{pmatrix}\right).$$

Example 3.2.14

The pair

$$B := \left(\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)$$

is a basis of $\mathbb{R}^2$. Verification of this statement entails making sure that B is linearly independent and that the linear system

$$\left(\begin{array}{cc|c}2 & 1 & v_1\\4 & 1 & v_2\end{array}\right).$$

has a solution for every $v_1, v_2 \in \mathbb{R}$.

Every n-dimensional space has a basis – many of them in fact. One particular basis is considered the 'most natural', for chiefly geometric reasons. It is the basis whose vectors have directions of the coordinate axes; it bears many names, e.g. *standard*, *canonical* or *natural*.

Definition 3.2.15 (Standard basis)

The n-tuple

$$\mathcal{E}_n := \left(\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix},\dots,\begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}\right)$$

is a basis of $\mathbb{R}^n$ and is called the *standard* (or *canonical* or *natural*) basis. We denote the vectors of $\mathcal{E}_n$ (in order) $\mathbf{e}_1$ up to $\mathbf{e}_n$.

Example 3.2.16

The natural basis of the vector space of cubic polynomials is $(1, x, x^2, x^3)$. Other bases of the same space include $(x^3, 3x^2, 6x, 6)$ or $(1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3)$.

Example 3.2.17

The trivial space $\{\mathbf{0}\}$ has only one basis – the empty set $\emptyset$.

Example 3.2.18

The vector spaces of functions $f : \mathbb{N} \to \mathbb{R}$ and of functions $f : \mathbb{R} \to \mathbb{R}$ **do not** have bases because there is no reasonable way to enumerate functions with outputs in the real numbers.

Example 3.2.19

We have met bases before when studying sets of solutions of homogeneous systems; we only wouldn't call them such. The solution set of the linear system

$$\begin{aligned} x_1 + x_2 - x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

is

$$\left\{ x_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}.$$

Notice that the set is written as a span of two linearly independent vectors. In other words, its basis is the pair

$$\left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right).$$

Before we return to the original question of *existence* of a basis, we merge our current knowledge into a very important theorem which has both theoretical and computational consequences.

Theorem 3.2.20 (Characterisation of a basis)

Given a vector space V, an n-tuple $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is a basis of V if and only if every vector in V can be written as a linear combination of vectors in B in a ***unique*** *way.*

Proof. By definition of a basis, $\operatorname{span} B = V$ so indeed every vector in V must be expressible as a linear combination of vectors in B.

We now prove that this expression need be unique. For contradiction, assume that there exists a vector $\mathbf{v} \in V$ such that

$$\mathbf{v} = \sum_{i=1}^{n} r_i \cdot \mathbf{b}_i \quad \text{and also} \quad \mathbf{v} = \sum_{i=1}^{n} t_i \cdot \mathbf{b}_i$$

with $r_j \neq t_j$ for at least one index $j \leq n$. We may rearrange

$$\begin{aligned} \sum_{i=1}^{n} r_i \cdot \mathbf{b}_i &= \sum_{i=1}^{n} t_i \cdot \mathbf{b}_i \\ \sum_{i=1}^{n} r_i \cdot \mathbf{b}_i - \sum_{i=1}^{n} t_i \cdot \mathbf{b}_i &= \mathbf{0} \\ \sum_{i=1}^{n} (r_i - t_i) \cdot \mathbf{b}_i &= \mathbf{0}. \end{aligned}$$

Since $r_j \neq t_j$ and thus $r_j - t_j \neq 0$, the linear combination on the left hand side has non-zero coefficients. By proposition 3.2.4, this means that B is linearly dependent. That's a contradiction with the assumption that it is a basis, hence such a vector $\mathbf{v}$ can't exist and the theorem is proven. ■

Unfortunately, we lack the theoretical background to fully answer the question of which vector spaces have bases and which don't. We can only define a class of vector spaces that **always** do have bases. Nevertheless, we can't prove that vector spaces outside of this class do not have bases – perhaps because it is not true ...

What we can say is that vector spaces which can be written as spans of vectors have a basis. This is actually a trivial consequence of lemma 3.2.8. Suppose a vector space V is spanned by a finite set of vectors $S \subseteq V$. We can keep removing vectors from S until we reach a set $S' \subseteq S$ which is linearly independent and $\operatorname{span} S' = \operatorname{span} S$. Any ordering of the set S' is now a basis of V. Indeed, it spans V and every vector in V can be written as a linear combination of vectors from S' in a unique way. The former statement is clear (by corollary 3.2.2) and the latter follows from the proof of theorem 3.2.20. Should a vector $\mathbf{v} \in V$ have two different expressions in terms of vectors of S', we could subtract one from the other and get a non-trivial linear combination giving the zero vector – a contradiction with the linear independence of S' by proposition 3.2.4.

There is a point relevant to bases we should address. Intuitively, a basis of a space is the set of all possible *unique* directions of movement in that space. Wouldn't it be weird were we able to move in n possible ways in $\mathbb{R}^n$ when using the standard basis and, say, $n + 2$ ways when using some different basis? We tend to think of the dimension (or the total number of distinct directions of travel) of a space as something *fixed*, something inherent to the space itself, unrelated to any specific choice of vectors representing the directions.

As is thankfully often the case in linear algebra, our geometric intuition is correct. The formal way to express it is to say that all bases of a space should have the same number of elements. This is indeed the case and the number of elements in a basis is then called the *dimension* of said vector space.

First, we classify the vector spaces whereof we know they have a basis.

Definition 3.2.21 (Finitely generated vector space)

A vector space V is called *finitely generated* if it has a basis with finite number of vectors.

Remark 3.2.22

In the definition above, we specifically said 'has **a** basis' because we have not yet proven that all bases of a vector space have the same number of elements. Once we do so, finitely generated vector spaces can be seen as vector spaces of finite dimension.

Example 3.2.23

Every n-dimensional real space is finitely generated (take its standard basis for example) while the space of all functions $f : \mathbb{R} \to \mathbb{R}$ is not.

We prove the statement of equal number of elements across all bases in a somewhat roundabout way. This has the advantage of introducing a method of – somewhat algorithmically – transforming one basis of a space into another vector by vector.

Lemma 3.2.24 (Exchange lemma)

Assume V is a finitely generated vector space with basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ and pick a vector $\mathbf{v} \in V$ given by the linear combination

$$\mathbf{v} = r_1 \cdot \mathbf{b}_1 + r_2 \cdot \mathbf{b}_2 + \dots + r_n \cdot \mathbf{b}_n$$

with $r_i \neq 0$ for some $i \leq n$. Then, $\hat{B} := \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{i-1}, \mathbf{v}, \mathbf{b}_{i+1}, \dots, \mathbf{b}_n\}$ is also a basis of V.

Proof. We need to show that

(a) $\hat{B}$ is linearly independent.

(b) $\hat{B}$ spans V.

As for (a), assume we have a linear combination

$$t_1 \cdot \mathbf{b}_1 + \dots + t_{i-1} \cdot \mathbf{b}_{i-1} + t_i \cdot \mathbf{v} + t_{i+1} \cdot \mathbf{b}_{i+1} + \dots + t_n \cdot \mathbf{b}_n = \mathbf{0} \tag{3.5}$$

for some $t_1, \dots, t_n \in \mathbb{R}$. Substituting for $\mathbf{v}$ gives

$$t_1 \cdot \mathbf{b}_1 + \dots + t_{i-1} \cdot \mathbf{b}_{i-1} + t_i \cdot (r_1\mathbf{b}_1 + \dots + r_n\mathbf{b}_n) + t_{i+1} \cdot \mathbf{b}_{i+1} + \dots + t_n \cdot \mathbf{b}_n = \mathbf{0}.$$

Rearranging then

$$\begin{aligned}(t_1 + t_ir_1) \cdot \mathbf{b}_1 + \dots + (t_{i-1} + t_ir_{i-1}) \cdot \mathbf{b}_{i-1} + t_ir_i \cdot \mathbf{b}_i \\ + (t_{i+1} + t_ir_{i+1}) \cdot \mathbf{b}_{i+1} + \dots + (t_n + t_ir_n) \cdot \mathbf{b}_n = \mathbf{0}.\end{aligned} \tag{3.6}$$

This is a linear combination of vectors from the linearly independent basis B and thus by proposition 3.2.4, every coefficient of this combination is equal to 0. In particular, this means that $t_ir_i = 0$ and, since we've assumed $r_i \neq 0$, necessarily $t_i = 0$. However, in the wake of this, the combination (3.5) becomes

$$t_1 \cdot \mathbf{b}_1 + \dots + t_{i-1} \cdot \mathbf{b}_{i-1} + t_{i+1} \cdot \mathbf{b}_{i+1} + \dots + t_n\mathbf{b}_n = \mathbf{0},$$

i.e. a linear combination of vectors from B. Using proposition 3.2.4 again gives $t_j = 0$ for all $j \leq n$ since we already knew that $t_i = 0$. It follows that also $t_j + t_ir_j = 0$ for every $j \leq n$ and the linear combination in (3.6) has all coefficients equal to 0. This proves that $\hat{B}$ is linearly independent.

To prove (b), we check that $\operatorname{span}\hat{B} \subseteq \operatorname{span} B$ and $\operatorname{span} B \subseteq \operatorname{span}\hat{B}$. The inclusion $\operatorname{span}\hat{B} \subseteq \operatorname{span} B$ is obvious as $\mathbf{v}$ lies in $\operatorname{span} B$ (and so do all the vectors $\mathbf{b}_i$ of course). For the reverse inclusion to hold, it is enough to represent the exchanged vector $\mathbf{b}_i$ as linear combination of vectors from $\hat{B}$ because B and $\hat{B}$ share all the other vectors besides $\mathbf{b}_i$. In the linear combination

$$\mathbf{v} = r_1 \cdot \mathbf{b}_1 + \dots + r_i \cdot \mathbf{b}_i + \dots + r_n \cdot \mathbf{b}_n,$$

we assumed that $r_i \neq 0$. We can thus rearrange

$$\begin{aligned}\mathbf{v} &= r_1 \cdot \mathbf{b}_1 + \dots + r_i \cdot \mathbf{b}_i + \dots + r_n \cdot \mathbf{b}_n \\ -r_i \cdot \mathbf{b}_i &= r_1 \cdot \mathbf{b}_1 + \dots + r_{i-1} \cdot \mathbf{b}_{i-1} + (-1) \cdot \mathbf{v} + r_{i+1} \cdot \mathbf{b}_{i+1} + \dots + r_n \cdot \mathbf{b}_n \\ \mathbf{b}_i &= -\frac{r_1}{r_i} \cdot \mathbf{b}_1 + \dots + \left(-\frac{r_{i-1}}{r_i}\right) \cdot \mathbf{b}_{i-1} + \left(-\frac{1}{r_i}\right) \cdot \mathbf{v} + \left(-\frac{r_{i+1}}{r_i}\right) \cdot \mathbf{b}_{i+1} + \dots + \left(-\frac{r_n}{r_i}\right) \cdot \mathbf{b}_n\end{aligned}$$

which proves that $\mathbf{b}_i \in \operatorname{span}\hat{B}$ and with it, the lemma. ∎

We intend to use the exchange lemma to prove that all bases of a finitely generated vector space have the same number of vectors by inductively exchanging the vectors of one basis for the vectors of another.

Theorem 3.2.25 (The dimension theorem)

All bases of a finitely generated vector space have the same number of elements.

Proof. Fix a vector space V and its basis $B \coloneqq (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ with minimal number of elements. Given another basis $D = (\mathbf{d}_1, \ldots, \mathbf{d}_m)$, necessarily $n \leq m$ because the number of elements of B is assumed to be minimal. We shall prove that $m \leq n$.

The idea of the proof is to exchange all vectors in B for vectors in D until we get a basis of V consisting of only n vectors of D.

We proceed by induction on the number of exchanged vectors. Set $B_0 \coloneqq B$. So far no vectors have been exchanged. Since B spans V and $\mathbf{d}_1 \in V$, there exists a linear combination

$$\mathbf{d}_1 = d_{1,1} \cdot \mathbf{b}_1 + d_{1,2} \cdot \mathbf{b}_2 + \ldots + d_{1,n} \cdot \mathbf{b}_n$$

with at least one $d_{1,i}$ non-zero (as the zero vector is always linearly dependent on others, thus $\mathbf{d}_1 \neq \mathbf{0}$). By the exchange lemma, we may exchange $\mathbf{d}_1$ for $\mathbf{b}_i$ and get the basis

$$B_1 \coloneqq (\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}, \mathbf{d}_1, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_n)$$

of V.

For the induction step, suppose the basis B_k has been formed by exchanging the vectors $\mathbf{d}_1, \ldots, \mathbf{d}_k \in D$ for exactly k vectors from B. Let us denote the set of indices of the remaining original vectors as $I \subseteq \{1, \ldots, n\}$. That is, $\mathbf{b}_i \in B_k$ if and only if $i \in I$. Pick $\mathbf{d}_{k+1} \in D$ and write

$$\mathbf{d}_{k+1} = \sum_{i=1}^{k} d_{k+1,i} \cdot \mathbf{d}_i + \sum_{i \in I} d_{k+1,i} \cdot \mathbf{b}_i$$

as a linear combination of vectors from B_k. The important observation to make is that at least one of the coefficients $d_{k+1,i}$, $i \in I$, must be non-zero. To see why, assume we have $d_{k+1,i} = 0$ for all $i \in I$. Then, the linear combination above assumes the form

$$\mathbf{d}_{k+1} = \sum_{i=1}^{k} d_{k+1,i} \cdot \mathbf{d}_i.$$

But, this means that $\mathbf{d}_{k+1}$ is a linear combination of other vectors from D. This contradicts the assumption that D is linearly independent and so this situation cannot arise.

Now that we know that there exists an index $i \in I$ such that $d_{k+1,i} \neq 0$, we may (again by the exchange lemma) exchange the vector $\mathbf{b}_i$ for $\mathbf{d}_{k+1}$ and form the basis B_{k+1}.

Upon having exchanged the last remaining vector $\mathbf{b}_i$ for $\mathbf{d}_n$, we have constructed the basis

$$B_n = (\mathbf{d}_1, \ldots, \mathbf{d}_n)$$

of the space V. Since B_n is linearly independent and spans V, it follows that $\mathbf{d}_{n+1}, \ldots, \mathbf{d}_m \in \operatorname{span} B_n$ which is a contradiction because B_n is a subset of D and D is assumed to be linearly independent. Thus, there must be no more vectors in D after $\mathbf{d}_n$ which proves that $m \leq n$ and with that also that $m = n$, as desired. ■

The dimension theorem has a few immediate consequences. For instance, we can finally define the dimension of any finitely generated vector space.

Definition 3.2.26 (Dimension)

Given a finitely generated vector space V, its *dimension* is the number of elements of any of its bases. We label it $\dim V$.

Example 3.2.27

The n-dimensional real space has dimension n. The testifying basis is $\mathcal{E}_n$, for example.

Example 3.2.28

The space of polynomials of degree at most n has dimension $n+1$. As we've partially observed, its standard basis is $(1, x, x^2, \ldots, x^n)$ which has $n + 1$ elements.

Corollary 3.2.29

In a finitely generated vector space V, no linearly independent set $S \subseteq V$ can have more elements than the dimension of V.

Proof. Follows from the proof of the dimension theorem. Observe that in the proof we have never used the assumption that D spans V, only that it is linearly independent. ■

Corollary 3.2.30

Any linearly independent set $S \subseteq V$ in a finitely generated vector space V can be expanded to a basis of V.

Proof. If $\operatorname{span} S \neq V$, then there exists a vector $\mathbf{v} \in V$ such that $\mathbf{v} \notin \operatorname{span} S$. By lemma 3.2.1, $S \subsetneq S \cup \{\mathbf{v}\}$ and $S \cup \{\mathbf{v}\}$ is linearly independent because $\mathbf{v} \notin \operatorname{span} S$. Hence, we simply keep adding vectors to S until $\operatorname{span} S = V$ and $\#S = \dim V$. ■

Corollary 3.2.31

Any set $S \subseteq V$ with $\operatorname{span} S = V$ can be shrunk to a basis of the finitely generated vector space V.

Proof. If S is empty, then it spans the space $\{\mathbf{0}\}$ and is already a basis of it. If $S = \{\mathbf{0}\}$, then it also spans just the space $\{\mathbf{0}\}$ and we can remove the vector $\mathbf{0}$ from it, keeping its span.

Otherwise, S contains a vector $\mathbf{s}_1 \neq \mathbf{0}$. We form a basis $B_1 := (\mathbf{s}_1)$. If $\operatorname{span} B_1 = \operatorname{span} S$, we're done. Otherwise, there exists a vector $\mathbf{s}_2 \in S$ such that $\mathbf{s}_2 \notin \operatorname{span} B_1$. Form $B_2 := (\mathbf{s}_1, \mathbf{s}_2)$. This pair is linearly independent by the same argument as in the proof of corollary 3.2.30. We repeat this process until $\operatorname{span} B_n = \operatorname{span} S$ which takes exactly $\dim V$ steps. ■

Corollary 3.2.32

In a vector space V with $\dim V = n$, an n-element set is linearly independent if and only if it spans V.

Proof. As for (⇒), any linearly independent set S can be expanded to a basis of V by corollary 3.2.30. Since a basis of V has n elements and so does S, there is no expansion to be done and any ordering of S is already a basis of V; in particular $\operatorname{span} S = V$.

The implication (⇐) is also immediate. If $\operatorname{span} S = V$, then by corollary 3.2.31, it can be shrunk to a basis of V, which has n elements. Since S also has n elements, no shrinking takes place and any ordering of S is again a basis of V and is thus linearly independent. ■

3.2.2 Representation With Respect To A Basis

Theorem 3.2.20 leads to a corollary of mainly computational importance: **every** vector in a vector space V with basis B corresponds to **exactly one** sequence of real coefficients of the linear combination of vectors from B that equals this vector.

To put this symbolically, denote $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ and consider a vector $\mathbf{v} \in V$. By the mentioned theorem 3.2.20, there exists exactly one n-tuple $(r_1, \ldots, r_n) \in \mathbb{R}^n$ such that

$$\mathbf{v} = r_1 \cdot \mathbf{b_1} + r_2 \cdot \mathbf{b}_2 + \ldots + r_n \cdot \mathbf{b}_n.$$

However, in chapter 1, we observed that elements of $\mathbb{R}^n$ are really just n-dimensional vectors with entries in $\mathbb{R}$. These two facts brought together beget an important idea we shall formalise in due time – *vector spaces of dimension n are 'equivalent' to* $\mathbb{R}^n$. The last sentence should be read as such: in every vector space V, we can choose a basis B and write every vector in V as a linear combination of vectors from B. The coefficients of this linear combination (that are unique for every vector) can be assembled into a vector in $\mathbb{R}^n$. This forges a two-way relationship (a correspondence, if you will) between vectors in V and vectors in $\mathbb{R}^n$. We call this relationship a *representation* of the vector $v \in V$ for the reason that it gives a concrete form to an abstract vector.

Definition 3.2.33 (Representation of a vector)

Let V be a vector space with basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $\mathbf{v} \in V$. We call the vector

$$[\mathbf{v}]_B \coloneqq \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \in \mathbb{R}^n$$

a *representation of* $\mathbf{v}$ *with respect to* B if $\mathbf{v} = r_1 \cdot \mathbf{b}_1 + r_2 \cdot \mathbf{b}_2 + \ldots + r_n \cdot \mathbf{b}_n$.

Remark 3.2.34

The preceding definition underlines the necessity of defining a basis as a **sequence**, not just a set. A permutation of the elements of a basis changes the representation of many vectors with respect to it.

The notion of *representation* formalises the approach we've taken many times ere of 'writing' polynomials or matrices as vectors of coefficients. Confront the following example.

Example 3.2.35

In the space of cubic polynomials, the representation of the polynomial $x + x^2$ with respect to the basis $B = (1, 2x, 2x^2, 2x^3)$ is given by

$$[x + x^2]_B = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix}.$$

With respect to a different basis $C = (1 + x, 1 - x, x + x^2, x + x^3)$, it instead looks like this:

$$[x + x^2]_C = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 3.2.36

Find the representation of the vector

$$\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

with respect to

$$B = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right).$$

Solution. We need to find real scalars $r_1, r_2 \in \mathbb{R}$ such that

$$r_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

This is tantamount to solving the linear system

$$\begin{array}{rcrcl} r_1 & & & = & 3 \\ r_1 & + & 2r_2 & = & 2 \end{array}$$

with obvious solution $r_1 = 3$ and $r_2 = -1/2$. With this, we've affirmed the equality

$$\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} \right]_B = \begin{pmatrix} 3 \\ -1/2 \end{pmatrix}.$$

♠

Example 3.2.37 (Representation with respect to canonical basis)

Since every vector $\mathbf{v} \in \mathbb{R}^n$ can be trivially broken into a linear combination of canonical basis vectors, its representation with respect to this basis are exactly its coordinates.

Expressed symbolically,

$$[\mathbf{v}]_{\mathcal{E}_n} = \left[\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}\right]_{\mathcal{E}_n} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

for every $\mathbf{v} \in \mathbb{R}^n$ because

$$\mathbf{v} = v_1 \cdot \mathbf{e}_1 + v_2 \cdot \mathbf{e}_2 + \ldots + v_n \cdot \mathbf{e}_n.$$

We intend not to dwell on the idea of representation any longer for now. It shall emerge again when we discuss linear transformations known as *changes of basis*. We close with a result concerning a link between linear independence and vector representation. In fact, linear independence of vectors is equivalent to the linear independence of their representations with respect to any basis.

Lemma 3.2.38

Let V be a vector space of dimension $n \in \mathbb{N}$ with basis B, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ and $r_1, \ldots, r_k \in \mathbb{R}$. Then,

$$r_1 \cdot \mathbf{v}_1 + r_2 \cdot \mathbf{v}_2 + \ldots + r_k \cdot \mathbf{v}_k = \mathbf{0}_V$$

if and only if

$$r_1 \cdot [\mathbf{v}_1]_B + r_2 \cdot [\mathbf{v}_2]_B + \ldots + r_k \cdot [\mathbf{v}]_k = \mathbf{0}_{\mathbb{R}^n}$$

where $\mathbf{0}_V$ is the zero vector of the space V and $\mathbf{0}_{\mathbb{R}^n}$ that of $\mathbb{R}^n$.

Proof. Write $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ and also denote

$$[\mathbf{v}_1]_B = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{pmatrix}, [\mathbf{v}_2]_B = \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \end{pmatrix}, \ldots, [\mathbf{v}_k]_B = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{n,k} \end{pmatrix}.$$

Then, the condition

$$r_1 \cdot \mathbf{v}_1 + r_2 \cdot \mathbf{v}_2 + \ldots + r_k \cdot \mathbf{v}_k = \mathbf{0}_V$$

is equivalent to

$$\begin{aligned} &r_1 \cdot (a_{1,1} \cdot \mathbf{b}_1 + \ldots + a_{n,1} \cdot \mathbf{b}_n) + \\ &r_2 \cdot (a_{1,2} \cdot \mathbf{b}_1 + \ldots + a_{n,2} \cdot \mathbf{b}_n) + \\ &\ldots + \\ &r_k \cdot (a_{1,k} \cdot \mathbf{b}_1 + \ldots + a_{n,k} \cdot \mathbf{b}_n) = \mathbf{0}_V. \end{aligned}$$

Grouping together coefficients of the basis vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ gives

$$\begin{aligned} &(r_1 \cdot a_{1,1} + r_2 \cdot a_{1,2} + \ldots + r_k \cdot a_{1,k}) \cdot \mathbf{b}_1 + \\ &(r_1 \cdot a_{2,1} + r_2 \cdot a_{2,2} + \ldots + r_k \cdot a_{2,k}) \cdot \mathbf{b}_2 + \\ &\ldots + \\ &(r_1 \cdot a_{n,1} + r_2 \cdot a_{n,2} + \ldots + r_k \cdot a_{n,k}) \cdot \mathbf{b}_n = \mathbf{0}_V. \end{aligned}$$

By proposition 3.2.4, this equality is satisfied if and only if each of the coefficients is equal to 0. On the horizon there glitters the homogeneous linear system

$$\begin{array}{ccccccccc} r_1 \cdot a_{1,1} & + & r_2 \cdot a_{1,2} & + & \ldots & + & r_k \cdot a_{1,k} & = & 0 \\ r_1 \cdot a_{2,1} & + & r_2 \cdot a_{2,2} & + & \ldots & + & r_k \cdot a_{2,k} & = & 0 \\ & & & & & & & \vdots & \\ r_1 \cdot a_{n,1} & + & r_2 \cdot a_{n,2} & + & \ldots & + & r_k \cdot a_{n,k} & = & 0, \end{array}$$

which can be rewritten (as we've done many times before) into vector form as

$$r_1 \cdot \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \end{pmatrix} + \ldots + r_k \cdot \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{n,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Shown vectors are of course just representations of the vectors $\mathbf{v}_1$ up to $\mathbf{v}_n$ with respect to B and so the result is proven. ■

Exercise 3.2.39

Decide which of the following sets are linearly independent.

(a) $\left\{ \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 14 \end{pmatrix} \right\}$;

(b) $\left\{ \begin{pmatrix} 1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix} \right\}$;

(c) $\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}$;

(d) $\left\{ \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \\ -1 \end{pmatrix} \right\}$.

Exercise 3.2.40

Determine which of the sets are linearly independent in the space of quadratic polynomials.

(a) $\{3 - x + 9x^2, 5 - 6x + 3x^2, 1 + 1x - 5x^2\}$;

(b) $\{-x^2, 1 + 4x^2\}$;

(c) $\{2 + x + 7x^2, 3 - x + 2x^2, 4 - 3x^2\}$.

Exercise 3.2.41

Prove that each of the following sets is linearly independent in the vector space of all functions $f : (0, \infty) \to \mathbb{R}$.

(a) $\{x \mapsto x, x \mapsto \frac{1}{x}\}$;

(b) $\{x \mapsto \cos x, x \mapsto \sin x\}$;

(c) $\{x \mapsto \exp x, x \mapsto \log x\}$.

Exercise 3.2.42

Prove that the rows of a real-valued matrix in echelon form are a linearly independent set.

Exercise 3.2.43

Prove that if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly independent set, then so are all its proper subsets: $\{\mathbf{x}, \mathbf{y}\}$, $\{\mathbf{x}, \mathbf{z}\}$, $\{\mathbf{y}, \mathbf{z}\}$, $\{\mathbf{x}\}$, $\{\mathbf{y}\}$, $\{\mathbf{z}\}$ and $\emptyset$. Is the converse also true?

Exercise 3.2.44

Is there a set of four vectors in $\mathbb{R}^3$ such that any three of them form a linearly independent set?

Exercise 3.2.45

Prove that a set of two perpendicular non-zero vectors in $\mathbb{R}^n$ is always linearly independent as long as $n > 1$. Generalise the result to more than two vectors.

Exercise 3.2.46

Decide whether $\{x^2 - x + 1, 2x + 1, 2x - 1\}$ and $\{x + x^2, x - x^2\}$ are bases of the space of quadratic polynomials.

Exercise 3.2.47

Find a basis for the solution set of the linear system

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0. \end{aligned}$$

Exercise 3.2.48

Find a basis for $\mathbb{R}^{2\times 2}$, the space of 2×2 real matrices.

Exercise 3.2.49

Let $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ be a basis.

(a) Show that $(r_1 \cdot \mathbf{b}_1, r_2 \cdot \mathbf{b}_2, r_3 \cdot \mathbf{b}_3)$ is also a basis as long $r_1, r_2, r_3 \neq 0$. What happens if at least one of r_i **is** zero?

(b) Prove that $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is also a basis where $\mathbf{a}_i = \mathbf{b}_1 + \mathbf{b}_i$, $i \in \{1, 2, 3\}$.

Exercise 3.2.50

Theorem 3.2.20 shows that, with respect to a basis, every linear combination is unique. If a subset is not a basis, can linear combinations be not unique? If so, must they be?

Exercise 3.2.51

Represent the polynomials

a) $2 + 4x^2$ b) $1 + 3x^2$ c) $1 + 5x^2$

with respect to the basis $B = (1 - x, 1 + x, x^2)$ of the space of quadratic polynomials. Use these representations to show that the three featured polynomials are linearly dependent.

Exercise 3.2.52

Represent the vector

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

with respect to the basis

$$B = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

of $\mathbb{R}^2$.

3.3 Linear Systems As Vector Spaces

We've defined an abstract vector space with the primary goal of accommodating the structure of sets of solutions of homogeneous linear systems. It might come as a surprise that linear systems *themselves* – not just their solutions – exhibit vector space structure. Understanding of said structure brings to light many properties of their solution sets, even.

Looking at the left hand side of linear systems (assembled into matrices), one immediately sees two sets of vectors – the **rows** of the matrix, and the **columns**. Enlightened as we have been by the late section 3.1 dealing with spans, we begin to study the vector spaces given by spans of these two sets. Believe it or not, they're actually closely related. Choosing a 'random' table of numbers, you cannot prevent its rows bearing a similar structure as the columns. Isn't that weird?

Definition 3.3.1 (Row space)

The *row space* of a matrix is the span of its rows.

Example 3.3.2

The row space of the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

is the vector space

$$\operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right).$$

The row space of a matrix is tied to the solution of the corresponding linear system in a tight manner. Observe that the three 'row transformations' defined just prior to theorem 1.1.2 really just replace rows with linear combinations of other rows. That said, they do not alter the row space of a matrix. We shall formulate this observation as a lemma.

Lemma 3.3.3

Row transformations (1) - (3) defined above theorem 1.1.2 do not change the row space of a matrix. I.e. if matrix B is a matrix derived from A by a series of row operations, then the row space of A equals the row space of B.

Proof. We go through the row transformations one by one and check that they indeed do not shrink or enlarge the row space.

The operation of swapping two rows obviously doesn't change the row space as the span of a set of vectors is independent of their order.

Multiplying a vector of a set by a non-zero constant also clearly doesn't affect the span of the set.

Finally, assume $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the rows of A. The third row transformation amounts to replacing the row $\mathbf{a}_i$ with the row $\mathbf{a}_i + c \cdot \mathbf{a}_j$ for some $j \neq i$ and a constant $c \in \mathbb{R}$. Clearly, $\mathbf{a}_i + c \cdot \mathbf{a}_j$ lies in the row space of A. On the other hand, if B is the resulting matrix with rows $\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_i + c \cdot \mathbf{a}_j, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n$, then $\mathbf{a}_i$ lies in the row space of B as

$$\mathbf{a}_i = 1 \cdot (\mathbf{a}_i + c \cdot \mathbf{a}_j) - c \cdot \mathbf{a}_j$$

and $\mathbf{a}_i$ is thus a linear combination of rows of B. This concludes the proof. ■

In light of lemma 3.3.3, we may wish to formalise the intuition that Gauss-Jordan elimination in fact finds a **basis** of the row space of a matrix as it procedurally nullifies rows that can be expressed as linear combinations of preceding rows. The following lemma is an ingredient to that dish.

Lemma 3.3.4

The non-zero rows of a matrix in echelon form are linearly independent.

Proof. Each row of a matrix in echelon form has at least one more leading zero than the preceding row. That is, labelling the non-zero rows of the eliminated matrix A with n rows as

$\mathbf{a}_1, \ldots, \mathbf{a}_k$, consider the linear combination

$$r_1 \cdot \mathbf{a}_1 + r_2 \cdot \mathbf{a}_2 + \ldots + r_k \cdot \mathbf{a}_k = 0.$$

Rewriting this system in the form of a matrix gives

$$\left(\begin{array}{ccccc|c} a_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k} & 0 \end{array}\right).$$

We're certain that $a_{i,j} = 0$ if $j > i$ because the i-th row $\mathbf{a}_i$ (which is now the i-th column in the matrix) must have at least $i - 1$ leading zeroes.

Simple back substitution (starting on the first row) immediately yields $r_1 = r_2 = \ldots = r_k = 0$ and thus the row vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are linearly independent by proposition 3.2.4. ■

Let us now take a look at the vector space spanned by the columns of a matrix. We shall uncover interesting links to the row space.

Definition 3.3.5 (Column space)

The column space of a matrix is the span of its columns.

Example 3.3.6

The column space of the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

is the vector space

$$\operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right).$$

There is an obvious reason we care about studying the column space. Picking a matrix $A = (\tilde{\mathbf{a}}_1 | \cdots | \tilde{\mathbf{a}}_n)$ (i.e. we label its columns as $\tilde{\mathbf{a}}_1$ up to $\tilde{\mathbf{a}}_n$), every solution of the corresponding homogeneous linear system assumes the form

$$r_1 \cdot \tilde{\mathbf{a}}_1 + r_2 \cdot \tilde{\mathbf{a}}_2 + \ldots + r_n \cdot \tilde{\mathbf{a}}_n.$$

Consequently, the column space is **exactly** the vector space of solutions of the homogeneous linear system with matrix A.

In order to find a *basis* of a vector space given as a span of some set, we would assemble the spanning vectors into rows of a matrix and then put that into echelon form. Should we thus wish to find the basis for the column space, we would 'assemble the columns of a matrix into rows'. This matrix operation is called the *transpose*.

Definition 3.3.7 (Transpose of a matrix)

Given matrix $A = (\tilde{\mathbf{a}}_1 | \tilde{\mathbf{a}}_2 | \cdots | \tilde{\mathbf{a}}_n)$, its *transpose* is the matrix A^T with rows $\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_n$.

Example 3.3.8

The transpose of the matrix
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$
is the matrix
$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

The circle has gone closed with the following lemma which establishes that row operations do not alter the column space. Restated, row operations do not change the solution set of a linear system. That is, in fact, the first theorem of the book – theorem 1.1.2.

Lemma 3.3.9

Row transformations (1) - (3) defined above theorem 1.1.2 do not change the column space of a matrix.

Proof. See the mentioned theorem 1.1.2. ■

We are now ready to present an important observation, one that ties together the dimension of row space to that of the column space. The crux of the matter is that Gauss-Jordan elimination actually doesn't find only the basis of the row space, it also finds the basis of the **column** space. We first illustrate why this is the case on an example.

Example 3.3.10

Let us put the matrix
$$A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix}$$
into echelon form. Following the algorithm of Gauss-Jordan elimination gives
$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By lemma 3.3.4, the first and second rows of the eliminated matrix form the basis of the row space of the original matrix. That is,
$$\text{the row space of } A = \operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ -12 \end{pmatrix}\right).$$

However, by the preceding lemma 3.3.9, the performed operations also left the column space intact. Can we see the basis for the column space in the eliminated matrix? Why, of course we can! The columns that correspond to **free variables** (in this case the third and fourth columns) are necessarily linearly dependent on previous columns. The reason for that is simple – their last non-zero entry is a pivot in the same row and some previous column, their penultimate non-zero entry is again a pivot in the same row as that entry and some previous column, etc. Therefore, for every non-zero entry of a free variable column, there exists some previous **pivot** column which has a non-zero entry at the same coordinate.

It follows that

$$\text{the column space of } A = \operatorname{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\end{pmatrix}\right).$$

Observe that the dimension of the column space is equal to that of the row space. This is not a coincidence – the echelon form of A has as many non-zero rows as there are pivots. But that is also exactly the number of columns that are linearly independent on other columns. In other words, the dimension of **both** the row space **and** the column space is the number of pivots.

Let us summarise our findings in the following proposition.

Proposition 3.3.11

For every matrix A, the dimension of the row space of A is equal to the dimension of the column space of A.

Proof. The idea of the proof is pretty much contained in example 3.3.10. The number of non-zero rows in the echelon form of A equals the number of pivots (by definition of a *pivot*), and, by lemma 3.3.4, it also equals the dimension of its row space.

As we've observed in example 3.3.10, the number of pivots also equals the number of linearly independent columns and thus the dimension of the column space of A. ■

Definition 3.3.12 (Rank)

The *rank* of a matrix A equals the dimension of its row space or its column space and is denoted $\operatorname{rank} A$.

We finish the section strong by explicitly stating the relation between the rank of a matrix and the solution set of its associated homogeneous linear system.

Theorem 3.3.13

Let A be an $m \times n$ matrix. Then, the following claims are equivalent.

(1) $\operatorname{rank} A = r$.

(2) The vector space of solutions of the associated homogeneous linear system has dimension $n - r$.

Proof. By proposition 3.3.11 and the preceding example, $\operatorname{rank} A = r$ if and only if Gauss-Jordan elimination process of the matrix A ends with r non-zero rows. This in turn happens if and only if the number of pivots is exactly r. Finally, the number of pivots is r if and only if the number of free variables is $n - r$. The number of free variables is of course precisely the dimension of the set of solutions of the homogeneous linear system with matrix A. ■

Definition 3.3.14 (Regular matrix)

An $m \times n$ matrix is called *regular* if $\operatorname{rank} A = \min(m, n)$ (that is, the maximum possible). If A is not regular, it is called *singular*.

Remark 3.3.15

By theorem 3.3.13, a matrix A is *singular* if and only if the associated homogeneous linear system has infinitely many solutions.

Corollary 3.3.16

*For a **square** matrix A with n rows and n columns, the following claims are equivalent.*

(1) $\operatorname{rank} A = n$ *(A is regular).*

(2) The rows of A are linearly independent.

(3) The columns of A are linearly independent.

*(4) **Any** linear system (that is, not just homogeneous) with left side A has exactly one solution.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3) follow from the fact that A is regular if and only if the row and column spaces of A both have dimension n. Since A has n rows and n columns, this means that both its rows and its columns must be linearly independent.

It remains to prove (3) $\Leftrightarrow$ (4). The columns of A (labelled $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_n$) are linearly independent if and only if they form a basis of $\mathbb{R}^n$. Moreover, for any $\mathbf{b} \in \mathbb{R}^n$, the system

$$(\tilde{\mathbf{a}}_1 | \cdots | \tilde{\mathbf{a}}_n | \mathbf{b})$$

has a **unique** solution if and only if $\mathbf{b}$ can be represented as a linear combination of $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_n$ in a **unique** way, that is, if and only if the columns of A form a basis of $\mathbb{R}^n$. ■

Exercise 3.3.17

Decide if the vector

(a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ lies in the row space of the matrix $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ lies in the row space of the matrix $\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}$.

Exercise 3.3.18

Decide if the vector

(a) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ lies in the column space of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ lies in the column space of the matrix $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ 1 & -3 & -3 \end{pmatrix}$.

Exercise 3.3.19

Find the basis of both the row space and column space of the matrix

$$\begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix}.$$

Exercise 3.3.20

Given $a, b, c \in \mathbb{R}$ what choice of $d \in \mathbb{R}$ will cause the following matrix to have rank one?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Exercise 3.3.21

Find the column rank of the matrix

$$\begin{pmatrix} 1 & 3 & -1 & 5 & 0 & 4 \\ 2 & 0 & 1 & 0 & 4 & 1 \end{pmatrix}$$

Exercise 3.3.22

An $m \times n$ has *full row rank* if its row rank is m and has *full column rank* if its column rank is n.

(a) Show that a matrix can have both full row rank and full column rank only if it is square (that is, $m = n$).

(b) Prove that a linear system with matrix of coefficients A has a solution for **any** right side if and only if A has full row rank.

(c) Prove that a homogeneous system has a unique solution if and only if its matrix of coefficients A has full column rank.

(d) Prove that the statement '*if a system with matrix of coefficients A has any solution, then it has a unique solution*' holds if and only if A has full column rank.

Exercise 3.3.23

What is the relationship (if any) between

(a) $\operatorname{rank} A$ and $\operatorname{rank}(-A)$?

(b) $\operatorname{rank} A$ and $\operatorname{rank}(kA)$ for $k \neq 0$?

(c) $\operatorname{rank} A + \operatorname{rank} B$ and $\operatorname{rank}(A + B)$?

Chapter 4

Homomorphisms (chybějí obrázky)

In this chapter, our aim is to study and understand maps between vector spaces. Not just any kind of maps, however, but maps that *preserve structure*.

Most of modern mathematics is dedicated to the study of *structures* – basically prescribed rules of interaction between elements of a set. We call these rules, *operations*, and when moving from a set with structure to a set with structure by a map, we tend to require that said map somehow respects the structures of both sets. Such maps are often called *homomorphisms*, from Greek ὁμός („same“) and μορφή („form, shape“).

The only structure we consider in this book is that of a vector space given by two operations: scalar multiplication and vector addition. A *homomorphism between vector spaces* V and W (also called a *linear map*) is thus a map which respects both operations; in practice, this means that the image of a scalar multiple should be the same scalar multiple of the image and that the image of a sum of vectors should be the sum of the images.

One last note: we ought to be careful when comparing two structures. We labelled the operations on a vector space by symbols $\cdot$ and $+$ but these two symbols **mean different things in different vector spaces**! To keep the text tidy, we shan't resort to using yet another distinct pair of symbols. However, we *are* going to distinguish the structure in a small number of ensuing lemmata and definitions, to drive the point home.

Definition 4.0.1 (Homomorphism)

Let V and W be vector spaces over the field $\mathbb{F}$. We denote the operations of scalar multiplication and vector addition on V by $\cdot_V$ and $+_V$ and those on W by $\cdot_W$ and $+_W$. A map $f : V \to W$ is a *homomorphism* (or a *linear map*) if

(1) $f(\mathbf{v}_1 +_V \mathbf{v}_2) = f(\mathbf{v}_1) +_W f(\mathbf{v}_2)$ for every two vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$.

(2) $f(t \cdot_V \mathbf{v}) = t \cdot_W f(\mathbf{v})$ for every $t \in \mathbb{F}$ and $\mathbf{v} \in V$.

Example 4.0.2

The following maps are homomorphisms:

(a) the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(\mathbf{v}) = 2 \cdot \mathbf{v}$;

(b) the map $f : \mathcal{P}_3(\mathbb{F}) \to \mathbb{F}^4$ given by

$$f(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

where $\mathcal{P}_3(\mathbb{F})$ denotes the space of polynomials of degree 3 with coefficients in the field $\mathbb{F}$;

(c) the map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\pi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix};$$

maps that 'forget coordinates' are often called *projections*.

The following maps are **not** homomorphisms:

(a) the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix};$$

(b) the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x^2 + y^3 - 6.$$

(c) the map $f : \mathbb{R}^{2\times 2} \to \mathbb{R}^2$ given by

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a \cdot b + c \cdot d \\ a \cdot d - b \cdot c \end{pmatrix}.$$

In the previous example, we claimed that certain maps were homomorphisms without giving a proof. We did so because we first want to provide a characterisation of homomorphisms which makes checking whether a given map is a homomorphism somewhat easier. Hence, we now collect two qualities only homomorphisms possess.

Lemma 4.0.3 (Zero to zero)

Let $f : V \to W$ be a homomorphism and label the zero vector of V by $\mathbf{0}_V$ and the zero vector of W by $\mathbf{0}_W$. Then,

$$f(\mathbf{0}_V) = \mathbf{0}_W.$$

Proof. Exploiting axiom (2) in the definition of homomorphism, we get

$$f(\mathbf{0}_V) = f(0 \cdot_V \mathbf{0}_V) \overset{(2)}{=} 0 \cdot_W f(\mathbf{0}_V) = \mathbf{0}_W,$$

as required. ■

Lemma 4.0.4

For two vector spaces V, W over $\mathbb{F}$ and a map $f : V \to W$, the following statements are equivalent.

(a) The map f is a homomorphism.

(b) For any two vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ and two numbers $t_1, t_2 \in \mathbb{F}$, we have

$$f(t_1 \cdot_V \mathbf{v}_1 +_V t_2 \cdot_V \mathbf{v}_2) = t_1 \cdot_W f(\mathbf{v}_1) +_W t_2 \cdot_W f(\mathbf{v}_2).$$

(c) For any vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and numbers $t_1, \dots, t_n \in \mathbb{F}$, we have

$$f(t_1 \cdot_V \mathbf{v_1} +_V \dots +_V t_n \cdot_V \mathbf{v}_n) = t_1 \cdot_W f(\mathbf{v}_1) +_W \dots +_W t_n \cdot_W f(\mathbf{v}_n).$$

Proof. In the proof (as well as the following text), we stop distinguishing between $\cdot_V$, $\cdot_W$ and $+_V$, $+_W$ for the sake of clarity. The readers should do well to keep in mind that V and W host different structures, though.

We prove $(a) \Leftrightarrow (b)$ and $(b) \Leftrightarrow (c)$.

As for $(a) \Rightarrow (b)$, we shall, naturally, invoke the axioms (1) and (2) of the definition of homomorphism. We compute

$$f(t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2) \overset{(1)}{=} f(t_1 \cdot \mathbf{v}_1) + f(t_2 \cdot \mathbf{v}_2) \overset{(2)}{=} t_1 \cdot f(\mathbf{v}_1) + t_2 \cdot f(\mathbf{v}_2).$$

The implication $(b) \Rightarrow (a)$ is proven by simply substituting $t_1 = t_2 = 1$ in axiom (1) and $\mathbf{v}_2 = \mathbf{0}$ in axiom (2).

Similarly, the implication $(c) \Rightarrow (b)$ follows trivially by setting $n = 2$. We prove the last implication $(b) \Rightarrow (c)$ by induction on n. The base case $n = 2$ is covered completely by statement (b). For the induction step, label $\mathbf{w} = t_1 \cdot \mathbf{v}_1 + \dots + t_n \cdot \mathbf{v}_n$. Then,

$$f(t_1 \cdot \mathbf{v}_1 + \dots + t_n \cdot \mathbf{v}_n + t_{n+1} \cdot \mathbf{v}_{n+1}) = f(\mathbf{w} + t_{n+1} \cdot \mathbf{v}_{n+1}).$$

Using statement (b) again, we get

$$f(\mathbf{w} + t_{n+1} \cdot \mathbf{v}_{n+1}) = f(\mathbf{w}) + t_{n+1} \cdot f(\mathbf{v}_{n+1}).$$

By the induction hypothesis,

$$f(\mathbf{w}) = f(t_1 \cdot \mathbf{v}_1 + \dots + t_n \cdot \mathbf{v}_n) = t_1 \cdot f(\mathbf{v}_1) + \dots + t_n \cdot f(\mathbf{v}_n).$$

And thus,

$$\begin{aligned} f(\mathbf{w}) + t_{n+1} \cdot f(\mathbf{v}_{n+1}) &= f(t_1 \cdot \mathbf{v}_1 + \dots + t_n \cdot \mathbf{v}_n) + t_{n+1} \cdot \mathbf{v}_{n+1} \\ &= t_1 \cdot f(\mathbf{v}_1) + \dots + t_n \cdot f(\mathbf{v}_n) + t_{n+1} \cdot f(\mathbf{v}_{n+1}) \end{aligned}$$

and we're done. ■

Remark 4.0.5

The statement (b) in lemma 4.0.4 can be geometrically interpreted as saying that a homomorphism 'transforms parallelepipeds into parallelepipeds'. Let's see this on an example.

Any two linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ define a parallelogram as the set of all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ with coefficients between 0 and 1, i.e.

$$\mathbf{P}(\mathbf{u}, \mathbf{v}) \coloneqq \{a \cdot \mathbf{u} + b \cdot \mathbf{v} \mid a, b \in [0, 1]\}.$$

TODO obrazek

Then, the mentioned statement (b) says the following about a homomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$.

$$f(a \cdot \mathbf{u} + b \cdot \mathbf{v}) = a \cdot f(\mathbf{u}) + b \cdot f(\mathbf{v}).$$

However, this can be read to say that the image of every point in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is a point in the parallelogram determined by $f(\mathbf{u})$ and $f(\mathbf{v})$. Symbolically,

$$f(\mathbf{P}(\mathbf{u}, \mathbf{v})) = \mathbf{P}(f(\mathbf{u}), f(\mathbf{v})).$$

Let us return to example 4.0.2. There, we claimed that certain maps were homomorphisms without proof. For some of them, we're providing the proof now.

It is easily checked that the projection

$$\pi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$$

is a homomorphism. Indeed, we can calculate

$$\begin{aligned}
\pi\left(a \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + b \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) &= \pi\left(\begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ az_1 + bz_2 \end{pmatrix}\right) = \begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{pmatrix} \\
&= a \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + b \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = a \cdot \pi\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + b \cdot \pi\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right).
\end{aligned}$$

The rest is up to lemma 4.0.4.

In a very similar vein, the map $f(\mathbf{v}) = 2 \cdot \mathbf{v}$ is a homomorphism for any vector space V over $\mathbb{R}$. Indeed, we may compute

$$f(a \cdot \mathbf{u} + b \cdot \mathbf{v}) = 2 \cdot (a \cdot \mathbf{u} + b \cdot \mathbf{v}) = a \cdot (2 \cdot \mathbf{u}) + b \cdot (2 \cdot \mathbf{v}) = a \cdot f(\mathbf{u}) + b \cdot f(\mathbf{v}).$$

This last homomorphism is an example of an *automorphism* – a bijective homomorphism from a space to itself. Automorphisms are just one interesting class of homomorphisms we shall present shortly.

The map

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

is not a homomorphism because it doesn't send $\mathbf{0}$ to $\mathbf{0}$ (and thus contradicts lemma 4.0.3). It serves as a good example of a more general notion – homomorphisms *cannot* translate vectors. This is quite a stark restriction, yet it is a necessary condition for the images of homomorphisms to be vector spaces.

Also, the map

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x^2 + y^2 - 6$$

is equally **not** a homomorphism as it 'curves' the space. We may easily check that it breaks both axioms (1) and (2) in the definition. As we've observed by virtue of remark 4.0.5, homomorphisms send 'flat objects' to 'flat objects', not to misshapen ellipses.

The quality of the maps from example 4.0.2 we haven't fully commented on is left for kind readers to determine.

Exercise 4.0.6

Prove that the map $f : \mathcal{P}_3(\mathbb{F}) \longrightarrow \mathbb{F}^4$ from example 4.0.2 is a homomorphism.

Exercise 4.0.7

Prove that the map $f : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^2$ from example 4.0.2 is **not** a homomorphism.

Now, onto learning some new words, kids, shall we?

Definition 4.0.8 (Classes of homomorphisms)

A homomorphism $f : V \longrightarrow W$ is called

(1) an *isomorphism*, if it is bijective,

(2) an *endomorphism*, if $W = V$,

(3) an *automorphism*, if it is both an *isomorphism* and an *endomorphism*.

If there exists an *isomorphism* between two vector spaces V and W, we call these spaces *isomorphic*. Intuitively, this means that the two spaces behave exactly the same, we have only chosen to represent the vectors of one a little differently than the other.

One immediate example is the correspondence between polynomials of degree n and vectors with $n + 1$ entries we have mentioned many times throughout the text.

The map

$$f : \mathcal{P}_3(\mathbb{F}) \to \mathbb{F}^4$$
$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

from example 4.0.2 is an isomorphism between $\mathcal{P}_3(\mathbb{F})$ and $\mathbb{F}^4$. The fact that it's both *injective* and *surjective* is almost obvious since it just sends a polynomial to the vector of its coefficients.

We shan't spend more time discussing different classes of homomorphisms now but we certainly shall later. The readers are encouraged to make up examples themselves.

Exercise 4.0.9

Find an example of a homomorphism $f : V \to W$ which is

(a) an *isomorphism* but **not** an automorphism,

(b) an *endomorphism* but **not** an automorphism,

(c) an *automorphism*.

Now, there are quite a few special vector spaces tied to a homomorphism $f : V \to W$. As we've mentioned earlier, its image is a subspace of W, the preimage via f of a subspace of W is a subspace of V, and, finally, the set of homomorphisms itself is a subspace of the vector space of all maps from $V \to W$. We are proving all these assertions now.

Before that however, we just briefly recall the important definitions. Given a map $f : V \to W$, its *image* is the set

$$\operatorname{im} f = f(V) = \{f(\mathbf{v}) \mid \mathbf{v} \in V\} \subseteq W.$$

The *preimage* of a subset $S \subseteq W$ via f is the set

$$f^{-1}(S) \coloneqq \{\mathbf{v} \in V \mid f(\mathbf{v}) \in S\} \subseteq V.$$

Finally, the set of all homomorphisms from V to W is denoted $\operatorname{Hom}(V, W)$.

Lemma 4.0.10

Let $f : V \to W$ be a homomorphism. Then,

(a) $f(V)$ is a subspace of W.

(b) $f^{-1}(U)$ is a subspace of V whenever U is a subspace of W.

(c) $\operatorname{Hom}(V, W)$, the set of all homomorphisms $V \to W$, is a subspace of the vector space of all maps $V \to W$.

Proof. To prove (a), pick two vectors $\mathbf{w}_1, \mathbf{w}_2 \in f(V)$. We shall prove that $t_1 \cdot \mathbf{w}_1 + t_2 \cdot \mathbf{w}_2 \in f(V)$. This will mean that $f(V)$ is a subspace by lemma 3.1.2. Since $\mathbf{w}_1, \mathbf{w}_2 \in f(V)$, there exist by definition vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $f(\mathbf{v}_1) = \mathbf{w}_1$ and $f(\mathbf{v}_2) = \mathbf{w}_2$. We thus rewrite

$$t_1 \cdot \mathbf{w}_1 + t_2 \cdot \mathbf{w}_2 = t_1 \cdot f(\mathbf{v}_1) + t_2 \cdot f(\mathbf{v}_2).$$

Since f is a homomorphism by assumption, the right side of the above equality can by virtue of lemma 4.0.4 be reshaped as such:

$$t_1 \cdot f(\mathbf{v}_1) + t_2 \cdot f(\mathbf{v}_2) = f(t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2)$$

and thus f sends the vector $t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2$ to $t_1 \cdot \mathbf{w}_1 + t_2 \cdot \mathbf{w}_2$. This, in particular, ascertains that $t_1 \cdot \mathbf{w}_1 + t_2 \cdot \mathbf{w}_2 \in f(V)$, as desired.

We continue with statement (b). Pick $\mathbf{v}_1, \mathbf{v}_2 \in f^{-1}(U)$. By definition of $f^{-1}(U)$, there exist $\mathbf{u}_1, \mathbf{u}_2 \in U$ such that $f(\mathbf{v}_1) = \mathbf{u}_1$ and $f(\mathbf{v}_2) = \mathbf{u}_2$. We thus compute

$$f(t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2) = t_1 \cdot f(\mathbf{v}_1) + t_2 \cdot f(\mathbf{v}_2) = t_1 \cdot \mathbf{u}_1 + t_2 \cdot \mathbf{u}_2.$$

The last linear combination lies in U as it is a subspace of W by assumption. It follows that $t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2 \in f^{-1}(U)$ since $f(t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2) \in U$.

As for (c), we are given two homomorphisms $f, g \in \mathrm{Hom}(V, W)$. We must prove that $a \cdot f + b \cdot g$ is also a homomorphism for any $a, b \in \mathbb{F}$. For a change, we prove the axioms (1) and (2) in the definition of homomorphism. Taking $\mathbf{v}_1, \mathbf{v}_2 \in V$, we compute

$$\begin{aligned}(a \cdot f + b \cdot g)(\mathbf{v}_1 + \mathbf{v}_2) &= (a \cdot f)(\mathbf{v}_1 + \mathbf{v}_2) + (b \cdot g)(\mathbf{v}_1 + \mathbf{v}_2) \\ &= f(a \cdot \mathbf{v}_1 + a \cdot \mathbf{v}_2) + g(b \cdot \mathbf{v}_1 + b \cdot \mathbf{v}_2) \\ &= a \cdot f(\mathbf{v}_1) + a \cdot f(\mathbf{v}_2) + b \cdot g(\mathbf{v}_1) + b \cdot g(\mathbf{v}_2) \\ &= (a \cdot f(\mathbf{v}_1) + b \cdot g(\mathbf{v}_1)) + (a \cdot f(\mathbf{v}_2) + b \cdot g(\mathbf{v}_2)) \\ &= (a \cdot f + b \cdot g)(\mathbf{v}_1) + (a \cdot f + b \cdot g)(\mathbf{v}_2),\end{aligned}$$

hence (1) holds. The proof of (2) is left as an exercise. The validity of both axioms ascertains that $\mathrm{Hom}(V, W)$ is really a vector space and thus a subspace of the space of all maps $V \longrightarrow W$. ■

Exercise 4.0.11

Prove that for two homomorphisms $f, g \in \mathrm{Hom}(V, W)$, $a, b \in \mathbb{F}$, $\mathbf{v} \in V$ and $t \in \mathbb{F}$, we have

$$(a \cdot f + b \cdot g)(t \cdot \mathbf{v}) = t \cdot (a \cdot f + b \cdot g)(\mathbf{v}).$$

The fact that images and preimages of homomorphisms are vector spaces have geometric consequences. We now illustrate those on a few examples.

Example 4.0.12

Consider the projection $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ prescribed as

$$\pi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The image of $\pi(\mathbb{R}^3)$ is of course the entirety of $\mathbb{R}^2$. One may think of this projection as the 'squishing' of an entire room onto its floor. If we're allowed to move only along one of the walls in the room, we're then confined to just one edge of the floor of the 'squished' image. This is formalised by the fact that the image of the set of vectors with $y = 0$ is the subspace of $\mathbb{R}^2$ of vectors with $y = 0$, and similarly for x.

TODO obrazky

The preimage of a given vector $\mathbf{w} \in \mathbb{R}^2$ via π is the set of all vectors whose tips sit on the vertical line rooted at the tip of $\mathbf{w}$. This is because any vector

$$\begin{pmatrix} w_1 \\ w_2 \\ z \end{pmatrix}$$

gets mapped onto

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Of course, a set consisting of a single non-zero vector is not a subspace, just as the vertical line rooted at the tip of such vector is not.

On a similar note, the preimage of a line $\{t \cdot \mathbf{v} \mid t \in \mathbb{R}\}$ for a given $\mathbf{v} \in \mathbb{R}^2$ is the entire vertical plane containing said line in $\mathbb{R}^3$.

Example 4.0.13

The image of the homomorphism

$$\begin{aligned} h : \mathbb{R}^2 &\to \mathbb{R} \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto x + y \end{aligned}$$

is the entirety of $\mathbb{R}$. Fixing a number $r \in \mathbb{R}$, its preimage via h is the set of vectors whose coordinates add up to r. Their tips form a line in $\mathbb{R}^2$ given by the equation $x + y = r$.

The fact that h is a homomorphism can be expressed in a neat geometric manner – fix now two numbers $r_1, r_2 \in \mathbb{R}$. Think of the vectors whose tips lie on the line $x+y = r_1$ as 'r_1' vectors. Analogously, vectors whose tips form the line $x + y = r_2$ are regarded as 'r_2' vectors. Axiom (1) in the definition of homomorphism can be, in this particular case, restated as 'r_1' vectors plus 'r_2' vectors equal '$r_1 + r_2$' vectors. That is, $\mathbf{v}_1$ is an 'r_1' vector if $h(\mathbf{v}_1) = r_1$ and $\mathbf{v}_2$ is an 'r_2' vector if $h(\mathbf{v}_2) = r_2$; the previous sentence thus signifies exactly that $h(\mathbf{v}_1+\mathbf{v}_2) = h(\mathbf{v}_1)+h(\mathbf{v}_2)$ since $\mathbf{v}_1 + \mathbf{v}_2$ is clearly a '$r_1 + r_2$' vector.

TODO obrazek

Example 4.0.14

Define the 'derivative' homomorphism

$$\frac{\partial}{\partial x} : \mathcal{P}_3 \to \mathcal{P}_3$$
$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mapsto a_1 + 2a_2 x + 3a_3 x^2.$$

Its image is the subspace $\mathcal{P}_2$ (polynomials of degree 2 over $\mathbb{R}$) of $\mathcal{P}_3$. The inverse image of the set of polynomials of degree 2 is the image of the 'integral' homomorphism (with constant 0). Said formally, there exists a homomorphism

$$\int : \mathcal{P}_2 \to \mathcal{P}_3$$
$$a_0 + a_1 x + a_2 x^2 \mapsto a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3$$

such that $\frac{\partial}{\partial x} \circ \int$ is the identity map on $\mathcal{P}_2$. Notice however that it is **not** the case that $\int \circ \frac{\partial}{\partial x}$ is the identity map on $\mathcal{P}_3$. The derivative $\frac{\partial}{\partial x}$ is **not** an injective map and as such it doesn't have a 'two-sided' inverse, it only has a 'right' inverse in the form of $\int$.

The last example provokes a question: 'When does a homomorphism have an inverse which is also a homomorphism?' We remind dear readers that an inverse to a map $f : V \to W$ is a map $f^{-1} : W \to V$ such that $(f \circ f^{-1})(\mathbf{w}) = \mathbf{w}$ for every $\mathbf{w} \in W$ and $(f^{-1} \circ f)(\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v} \in V$. As seen in the mentioned example 4.0.14, **just one** of these equalities **is not enough**. A homomorphism may be invertible from just one side.

The rest of the introductory section to homomorphisms is dedicated to answering the raised question. First, we must establish a strong connection between homomorphisms and bases of their domains. We illustrate this connection first before conjuring a proof.

Consider a homomorphism f with domain $\mathbb{R}^3$. By lemma 4.0.4, the image of a vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

via this homomorphism can be broken into the linear combination

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = f\left(\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}\right) + f\left(\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}\right) + f\left(\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}\right) = x \cdot f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + y \cdot f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + z \cdot f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

This immediately suggests that the image of any vector via a homomorphism is determined purely by the images of standard basis vectors. Indeed, this statement is true in general, for any homomorphism and any basis.

Theorem 4.0.15

*Let V, W be vector spaces over $\mathbb{F}$, $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ be a basis of V. Given (not necessarily distinct) vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \in W$, there exists **just a single** homomorphism $f \in \mathrm{Hom}(V, W)$ such that $f(\mathbf{b}_1) = \mathbf{w}_1, f(\mathbf{b}_2) = \mathbf{w}_2, \dots, f(\mathbf{b}_n) = \mathbf{w}_n$.*

Proof. First, given $\mathbf{v} \in V$, we can write it uniquely as a linear combination of vectors from B, that is, there exist unique coefficients $t_1, \dots, t_n \in \mathbb{F}$ such that

$$\mathbf{v} = t_1 \cdot \mathbf{b}_1 + \ldots + t_n \cdot \mathbf{b}_n.$$

We define $f(\mathbf{v})$ by

$$f(\mathbf{v}) = t_1 \cdot \mathbf{w}_1 + \ldots + t_n \cdot \mathbf{w}_n.$$

We must prove three statements:

(1) f is well-defined (i.e. it's *actually* a map $V \to W$);

(2) f is a homomorphism;

(3) f is unique.

The statement (1) follows from the uniqueness of representation of a vector with respect to a basis – content of theorem 3.2.20. Consequently, as the vector $\mathbf{v}$ can only ever be represented by the same n-tuple of coefficients, the image of $\mathbf{v}$ under f is always the same.

Ad (2), choose $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $t_1, t_2 \in \mathbb{F}$. Let

$$\begin{aligned} \mathbf{v}_1 &= a_1 \cdot \mathbf{b}_1 + \ldots + a_n \cdot \mathbf{b}_n, \\ \mathbf{v}_2 &= c_1 \cdot \mathbf{b}_1 + \ldots + c_n \cdot \mathbf{b}_n \end{aligned}$$

for adequate coefficients $a_1, \dots, a_n, c_1, \dots, c_n \in \mathbb{F}$. Then,

$$\begin{aligned} t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2 &= t_1 \cdot (a_1 \cdot \mathbf{b}_1 + \ldots + a_n \cdot \mathbf{b}_n) + t_2 \cdot (c_1 \cdot \mathbf{b}_1 + \ldots + c_n \cdot \mathbf{b}_n) \\ &= (t_1 a_1 + t_2 c_1) \cdot \mathbf{b}_1 + (t_1 a_2 + t_2 c_2) \cdot \mathbf{b}_2 + \ldots + (t_1 a_n + t_2 c_n) \cdot \mathbf{b}_n. \end{aligned}$$

Hence, by the definition of f,

$$f(t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2) = (t_1 a_1 + t_2 c_1) \cdot \mathbf{w}_1 + \ldots + (t_1 a_n + t_2 c_n) \cdot \mathbf{w}_n.$$

The last expression can be rewritten back to

$$t_1 \cdot (a_1 \cdot \mathbf{w}_1 + \ldots + a_n \cdot \mathbf{w}_n) + t_2 \cdot (c_1 \cdot \mathbf{w}_1 + \ldots + c_n \cdot \mathbf{w}_n) = t_1 \cdot f(\mathbf{v}_1) + t_2 \cdot f(\mathbf{v}_2),$$

where the last equality holds because

$$\begin{aligned} f(\mathbf{v}_1) &= a_1 \cdot \mathbf{w}_1 + \ldots + a_n \cdot \mathbf{w}_n, \\ f(\mathbf{v}_2) &= c_1 \cdot \mathbf{w}_1 + \ldots + c_n \cdot \mathbf{w}_n. \end{aligned}$$

Finally, ad (3), assume that there is a homomorphism $\hat{f} \in \mathrm{Hom}(V, W)$ such that $\hat{f}(\mathbf{b}_1) = \mathbf{w}_1, \dots, \hat{f}(\mathbf{b}_n) = \mathbf{w}_n$. The, for every $\mathbf{v} \in V$ expressed as the linear combination

$$\mathbf{v} = t_1 \cdot \mathbf{b}_1 + \ldots + t_n \cdot \mathbf{b}_n,$$

we have

$$f(\mathbf{v}) = t_1 \cdot f(\mathbf{b}_1) + \ldots + t_n \cdot f(\mathbf{b}_n) = t_1 \cdot \mathbf{w}_1 + \ldots + t_n \cdot \mathbf{w}_n = t_1 \cdot \hat{f}(\mathbf{b}_1) + \ldots + t_n \cdot \hat{f}(\mathbf{b}_n) = \hat{f}(\mathbf{v})$$

and thus $f = \hat{f}$, as claimed. ■

In light and spirit of theorem 4.0.15, we *extend* any map $V \to W$ whose images of basis vectors we know, to a homomorphism. Such homomorphism is called its *linear extension*.

Definition 4.0.16 (Linear extension)

Let $f : V \to W$ be a map and $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$. By theorem 4.0.15, there exists a unique homomorphism $\hat{f} \in \mathrm{Hom}(V, W)$ such that $\hat{f}(\mathbf{b}_i) = f(\mathbf{b}_i)$ for every $i \leq n$. The homomorphism $\hat{f}$ is called the *linear extension* of f.

4.1 Image and kernel

A homomorphism $f \in \mathrm{Hom}(V, W)$ basically digs out for itself a subspace of W where it then lives. We called this subspace the image of f and denoted it $f(V)$ or $\mathrm{im}\, f$. The image carries on its back quite a few geometric properties of f. In particular, its *dimension* tells us how many direction of movement in the original vector space f moves over to its codomain space. This dimension is called its rank.

Definition 4.1.1 (Rank of a homomorphism)

Given $f \in \mathrm{Hom}(V, W)$, we define

$$\mathrm{rank}\, f := \dim(\mathrm{im}\, f)$$

and call it the *rank* of f.

Example 4.1.2

The derivative homomorphism

$$\frac{\partial}{\partial x} : \mathcal{P}_3 \to \mathcal{P}_3$$
$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mapsto a_1 + 2a_2 x + 3a_3 x^2$$

has rank 3 because its image is the three-dimensional subspace $\mathcal{P}_2$ of $\mathcal{P}_3$. In a sense, it 'forgets' the information of the constant coefficient of the polynomial, leaving only the linear parts and above recoverable. More explicitly, given a polynomial $a_0 + a_1 x + a_2 x^2 \in \frac{\partial}{\partial x}(\mathcal{P}_3)$, we can't know *precisely* where it came from. We only know that its original image lies somewhere in the set

$$\{c + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 \mid c \in \mathbb{R}\} \subseteq \mathcal{P}_3.$$

Example 4.1.3

The injection homomorphism

$$\begin{aligned} \iota : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \end{aligned}$$

has rank 2, the maximum possible. Clearly, the homomorphism can't possibly fill the entirety of $\mathbb{R}^3$ as the two-dimensional space $\mathbb{R}^2$ doesn't 'carry enough information' for that to be possible. What ι does is move the entire plane $\mathbb{R}^2$ into $\mathbb{R}^3$ as ... well ... a plane.

If rank of a homomorphism measures the dimension of the subspace that f creates, could we also somehow measure the number of directions of movement that f 'forgets'? Indeed, we could. Observe that a homomorphism sends two different vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ to a single vector $\mathbf{w} \in W$, it sends their difference to $\mathbf{0}$. We can compute

$$f(\mathbf{v}_2 - \mathbf{v}_1) = f(\mathbf{v}_2) - f(\mathbf{v}_1) = \mathbf{w} - \mathbf{w} = \mathbf{0}.$$

As $\{\mathbf{0}\}$ is a subspace of W, the preimage of $\{\mathbf{0}\}$ via f, that is $f^{-1}(\{\mathbf{0}\})$ is a subspace of V, by lemma 4.0.10. As a consequence of the observation made just above, the dimension of this subspace measure the number of directions of movement that f **doesn't** carry over to its image inside W.

Definition 4.1.4 (Kernel and nullity)

For any homomorphism $f \in \operatorname{Hom}(V, W)$, the preimage $f^{-1}(\{\mathbf{0}\})$ is called the *kernel* or *nullspace* of V. Its dimension is then called the *nullity* of f and denoted $\operatorname{null} f$.

Example 4.1.5

As already observed, the derivative homomorphism $\frac{\partial}{\partial x}$ 'forgets' all constant polynomials by sending them to $\mathbf{0}$. It follows that its *nullity* is 1 as the dimension of the subspace of constant polynomials is 1.

Example 4.1.6

The injection homomorphism $\iota : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ has nullity 0. This is easily seen from the fact that its injective, it cannot send any other vector than $\mathbf{0}$ to $\mathbf{0}$.

It stands to reason that if the nullity measure the number of 'forgotten directions' and the rank the number of 'remembered directions', their sum should equal the dimension of the original space. We shall prove that now.

Proposition 4.1.7 (Rank and nullity theorem)

Let $f \in \operatorname{Hom}(V, W)$*. Then* $\operatorname{rank} f + \operatorname{null} f = \dim V$.

Proof. Assume $\operatorname{rank} f = k$ and let $B = (\mathbf{b}_1, \dots, \mathbf{b}_k)$ be the basis of $\ker f = f^{-1}(\{\mathbf{0}\})$. By corollary 3.2.30, as $\ker f$ is a subspace of V, the basis B can be extended to a basis

$$\hat{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{b}_{k+1}, \ldots, \mathbf{b}_n)$$

of the space V. We aim to show that $(f(\mathbf{b}_{k+1}), \ldots, f(\mathbf{b}_n))$ is a basis for $\operatorname{im} f = f(V)$. This will complete the proof as this sequence contains exactly $n - k$ vectors.

We first prove linear independence. Assume there exists a linear combination

$$t_{k+1} \cdot f(\mathbf{b}_{k+1}) + t_{k+2} \cdot f(\mathbf{b}_{k+2}) + \ldots + t_n \cdot f(\mathbf{b}_n) = \mathbf{0} \tag{4.1}$$

for $t_{k+1}, \ldots, t_n \in \mathbb{F}$. As f is a homomorphism, this equality is equivalent to

$$f(t_{k+1} \cdot \mathbf{b}_{k+1} + \ldots + t_n \cdot \mathbf{b}_n) = \mathbf{0}.$$

From this, it follows that

$$t_{k+1} \cdot \mathbf{b}_{k+1} + \ldots + t_n \cdot \mathbf{b}_n \in \ker f.$$

In particular, as $B = (\mathbf{b}_1, \ldots, \mathbf{b}_k)$ is a basis for $\ker f$, there exist coefficients $t_1, \ldots, t_k \in \mathbb{F}$ such that

$$t_1 \cdot \mathbf{b}_1 + \ldots + t_k \cdot \mathbf{b}_k = t_{k+1} \cdot \mathbf{b}_{k+1} + \ldots + t_n \cdot \mathbf{b}_n.$$

However, a simple rearrangement yields

$$t_1 \cdot \mathbf{b}_1 + \ldots + t_k \cdot \mathbf{b}_k - t_{k+1} \cdot \mathbf{b}_{k+1} - \ldots - t_n \cdot \mathbf{b}_n = \mathbf{0},$$

a representation of the zero vector in the extended basis $\hat{B}$. By theorem 3.2.20, all the coefficients are necessarily zero. Thus, also $t_{k+1} = t_{k+2} = \ldots = t_n = 0$ and equation (4.1) asserts, by virtue of the same theorem 3.2.20, that $(f(\mathbf{b}_{k+1}, \ldots, f(\mathbf{b}_n))$ is linearly independent.

We now prove that $(f(\mathbf{b}_{k+1}, \ldots, f(\mathbf{b}_n))$ spans $\operatorname{im} f$. Pick $\mathbf{w} \in \operatorname{im} f$. By definition, there exists $\mathbf{v} \in V$ such that $f(\mathbf{v}) = \mathbf{w}$. Represent $\mathbf{v}$ in the extended basis $\hat{B}$ like

$$\mathbf{v} = t_1 \cdot \mathbf{b}_1 + t_2 \cdot \mathbf{b}_2 + \ldots + t_k \cdot \mathbf{b}_k + t_{k+1} \cdot \mathbf{b}_{k+1} + \ldots + t_n \cdot \mathbf{b}_n.$$

Now applying f and using again the fact that its a homomorphism gives

$$f(\mathbf{v}) = t_1 \cdot f(\mathbf{b}_1) + \ldots + t_k \cdot f(\mathbf{b}_k) + t_{k+1} \cdot f(\mathbf{b}_{k+1}) + \ldots + t_n \cdot f(\mathbf{b}_n).$$

As $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \ker f$, we arrive at the conclusion that

$$t_1 \cdot f(\mathbf{b}_1) + \ldots + t_k \cdot f(\mathbf{b}_k) = \mathbf{0}.$$

Finally, this means that the vector $f(\mathbf{v})$ is expressed as

$$f(\mathbf{v}) = t_{k+1} \cdot f(\mathbf{b}_{k+1}) + \ldots + t_n \cdot f(\mathbf{b}_n),$$

i.e., as a linear combination of vectors from $(f(\mathbf{b}_{k+1}), \ldots, f(\mathbf{b}_n))$. This concludes the proof. ■

The preceding theorem offers as its corollary a useful characterisation of injective homomorphisms. It's going to become especially crucial as we delve back into the realm of matrices, in the next section.

Corollary 4.1.8 (Characterisation of injective homomorphisms)

Let $f \in \operatorname{Hom}(V, W)$. The following statements are equivalent.

(a) The homomorphism f is injective.

(b) There exists an inverse $f^{-1} \in \operatorname{Hom}(\operatorname{im} f, V)$.

(c) The nullity of f is 0.

(d) The rank of f is $\dim V$.

(e) If $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ is a basis of V, then $f(B) = (f(\mathbf{b}_1), \dots, f(\mathbf{b}_n))$ is a basis of $\operatorname{im} f$.

Proof. We first prove $(a) \Rightarrow (b)$. The injectivity of f implies the existence of an inverse **map** $f^{-1} : \operatorname{im} f \to V$. It remains to show that this map is a homomorphism. To this end, choose $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{im} f$. By definition, there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $f(\mathbf{v}_1) = \mathbf{w}_1$ and $f(\mathbf{v}_2) = \mathbf{w}_2$. Thus, also $\mathbf{v}_1 = f^{-1}(\mathbf{w}_1)$ and $\mathbf{v}_2 = f^{-1}(\mathbf{w}_2)$.

We calculate

$$f(t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2) = t_1 \cdot f(\mathbf{v}_1) + t_2 \cdot f(\mathbf{v}_2) = t_1 \cdot \mathbf{w}_1 + t_2 \cdot \mathbf{w}_2.$$

Applying f^{-1} to both sides yields

$$t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2 = f^{-1}(t_1 \cdot \mathbf{w}_1 + t_2 \cdot \mathbf{w}_2).$$

This asserts the claim as

$$t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2 = t_1 \cdot f^{-1}(\mathbf{w}_1) + t_2 \cdot f^{-1}(\mathbf{w}_2).$$

The implication $(b) \Rightarrow (a)$ follows similarly. If there exists an inverse homomorphism f^{-1}, then this must be injective. By what we've just proven this asserts the existence of an **injective** (it must be, as it has an inverse) homomorphism $(f^{-1})^{-1} = f \in \operatorname{Hom}(V, W)$.

Ad $(a) \Rightarrow (c)$. The map f, being a homomorphism, sends $\mathbf{0}_V$ to $\mathbf{0}_W$. As it is injective, it cannot send any other vector from V to $\mathbf{0}_W$. In particular, $f^{-1}(\{\mathbf{0}_W\}) = \{\mathbf{0}_V\}$ and the dimension of the latter is 0.

The equivalence $(c) \Leftrightarrow (d)$ is the content of proposition 4.1.7.

The implication $(d) \Rightarrow (e)$ is immediate because $\operatorname{im} f$, having dimension $n = \dim V$, is spanned by n linearly independent images of vectors from V. Any n linearly independent vectors from V form a basis of V.

Finally, we prove $(e) \Rightarrow (a)$ by asserting $\neg(a) \Rightarrow \neg(e)$. Assume f is not injective. Thus, there exist two different vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $f(\mathbf{v}_1) = f(\mathbf{v}_2)$. Consequently, as f is a homomorphism, $f(\mathbf{v}_2 - \mathbf{v}_1) = \mathbf{0}$. So, there exists a non-zero vector that f sends to $\mathbf{0}$. By corollary 3.2.30, the vector $\mathbf{v}_2 - \mathbf{v}_1$ can be expanded into a basis

$$B = (\mathbf{v}_2 - \mathbf{v}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$$

of V. Hence, $f(B) = (f(\mathbf{v}_2 - \mathbf{v}_1), f(\mathbf{b}_2), \dots, f(\mathbf{b}_n))$ is **not** a basis of $\operatorname{im} f$ for it contains the vector $\mathbf{0}$. This means that (e) doesn't hold. ■

4.2 Homomorphisms As Matrices

Homomorphisms of vector spaces are even more special (as maps between sets) than the preceding text might have grown to reflect. They are one of those very few maps that, even though infinite in nature (in the sense that they have well-defined image of every vector in a usually infinite vector space), can be represented by finite sets of numbers. Let us derive this 'representation' using an example.

Assume we're given a homomorphism $f : \mathbb{R}^3 \to R^3$ and wish to determine its image on any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$. The fact that f is a homomorphism enables the following calculation.

$$\begin{aligned} f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) &= f\left(\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}\right) = f\left(x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= x \cdot f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + y \cdot f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + z \cdot f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right). \end{aligned} \tag{4.2}$$

It shows that it is in fact enough to know the image of the standard basis vectors to know the image of *every* vector via the homomorphism f, as witnessed by theorem 4.0.15. Perhaps even more importantly, the final expression reeks of dot product of two vectors. Imagine for a second that f is actually a homomorphism $f : \mathbb{R}^3 \to \mathbb{R}$ so that each of the vectors

$$f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right), f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right), f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

is just a real number, say $r_1, r_2, r_3 \in \mathbb{R}$ in order, for the sake of concreteness. Hence, the expression (4.2) becomes

$$x \cdot f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + y \cdot f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + z \cdot f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = xr_1 + yr_2 + yr_3 = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We may generalise this idea to homomorphisms whose codomain is a multi-dimensional space by passing from vectors to matrices. Let us perform said generalisation one step at a time. Starting with a homomorphism $f \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, such map is entirely determined by the images $f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)$, where, we recall, $\mathcal{E}_n = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is the standard basis of $\mathbb{R}^n$. Each of the $f(\mathbf{e}_i)$ is a vector in $\mathbb{R}^m$. Then, a more general version of the computation (4.2) for a vector $\mathbf{v} = v_1 \cdot \mathbf{e}_1 + v_2 \cdot \mathbf{e}_2 + \ldots + v_n \cdot \mathbf{e}_n \in \mathbb{R}^n$ is

$$\begin{aligned} f(\mathbf{v}) &= f(v_1 \cdot \mathbf{e}_1 + v_2 \cdot \mathbf{e}_2 + \ldots + v_n \cdot \mathbf{e}_n) \\ &= v_1 \cdot f(\mathbf{e}_1) + v_2 \cdot f(\mathbf{e}_2) + \ldots + v_n \cdot f(\mathbf{e}_n). \end{aligned} \tag{4.3}$$

If each of the $f(\mathbf{e}_i)$ where a number and not a vector, the last expression would be the dot product of the vector $\begin{pmatrix} f(\mathbf{e}_1) \\ \vdots \\ f(\mathbf{e}_n) \end{pmatrix}$ with $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. As stated before, when passing from homomorphisms with one-dimensional codomain to homomorphisms with a multi-dimensional one, it is needed to replace vectors by matrices. As each of the $f(\mathbf{e}_i)$ is a column vector, it is customary to assemble them into columns of a matrix

$$A = (f(\mathbf{e}_1) \mid f(\mathbf{e}_2) \mid \cdots \mid f(\mathbf{e}_n)) \in \mathbb{R}^{m \times n}.$$

The last expression in (4.3) is then just a **dot product** of the matrix A with the vector $\mathbf{v}$, as defined below.

Definition 4.2.1 (Matrix dot product)

Let $A = (\mathbf{a}_1 \mid \cdots \mid \mathbf{a}_n) \in \mathbb{R}^{m\times n}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$. We define the product $A \cdot \mathbf{v}$ by the formula

$$A \cdot \mathbf{v} = v_1 \cdot \mathbf{a}_1 + v_2 \cdot \mathbf{a}_2 + \ldots + v_n \cdot \mathbf{a}_n.$$

Warning 4.2.2

Notice that for the product $A\cdot\mathbf{v}$ to be defined, it is key that the number of columns of A matches the number of coordinates of $\mathbf{v}$. Should these numbers be different, the product would remain undefined.

It is now clear that a homomorphism $f \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ may be *entirely* represented by the matrix

$$A = (f(\mathbf{e}_1) \mid f(\mathbf{e}_2) \mid \cdots \mid f(\mathbf{e}_n)) \in \mathbb{R}^{m\times n}$$

since the image $f(\mathbf{v})$ of any vector $\mathbf{v} \in \mathbb{R}^n$ may be computed as $A \cdot \mathbf{v}$.

The last step in our generalisation of homomorphism representation is the passage from $\mathbb{R}^n$ and $\mathbb{R}^m$ to abstract vector spaces over any field of dimensions n and m, respectively. We are doing that now.

First, an observation. It is clear that any matrix representation of a fixed homomorphism $f \in \mathrm{Hom}(V, W)$ (where V, W are vector spaces over $\mathbb{F}$ of dimensions n and m) is wholly dependent on a particular choice of basis for V and the basis for W. This is easily seen from the examples given previously. Had we chosen to represent the vector $\mathbf{v} \in \mathbb{R}^n$ in a basis distinct from the standard one, say $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, we would have instead ended up with the matrix

$$(f(\mathbf{b}_1) \mid f(\mathbf{b}_2) \mid \cdots \mid f(\mathbf{b}_n))$$

representing f **with respect to** the basis B.

Thus, we first have to fix bases $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ for V and $C = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ for W. To represent f, choose a vector $\mathbf{v}$ and represent it with respect to the basis B, e.g.

$$\mathbf{v} = v_1 \cdot \mathbf{b}_1 + v_2 \cdot \mathbf{b}_2 + \ldots + v_n \cdot \mathbf{b}_n,$$

or, in the nomenclature introduced in subsection 3.2.2,

$$[\mathbf{v}]_B = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

The vector $f(\mathbf{v})$ is thus equal to

$$f(\mathbf{v}) = v_1 \cdot f(\mathbf{b}_1) + \ldots + v_n \cdot f(\mathbf{b}_n).$$

The vectors $f(\mathbf{b}_1), \ldots, f(\mathbf{b}_n)$ lie in W so they are represented with respect to the basis C, say, as

$$[f(\mathbf{b}_1)]_C = \begin{pmatrix} f_{1,1} \\ f_{2,1} \\ \vdots \\ f_{m,1} \end{pmatrix}, [f(\mathbf{b}_2)]_C = \begin{pmatrix} f_{1,2} \\ f_{2,2} \\ \vdots \\ f_{m,2} \end{pmatrix}, \ldots, [f(\mathbf{b}_n)]_C = \begin{pmatrix} f_{1,n} \\ f_{2,n} \\ \vdots \\ f_{m,n} \end{pmatrix}.$$

By representing also the vector $f(\mathbf{v})$ with respect to C, we get

$$[f(\mathbf{v})]_C = [v_1 \cdot f(\mathbf{b}_1) + \ldots + v_n \cdot f(\mathbf{b}_n)]_C = v_1 \cdot [f(\mathbf{b}_1)]_C + \ldots + v_n \cdot [f(\mathbf{b}_n)]_C.$$

In other words, the representation of $f(\mathbf{v})$ with respect to C is just the product of the matrix

$$A = ([f(\mathbf{b}_1)]_C \mid [f(\mathbf{b}_2)]_C \mid \ldots \mid [f(\mathbf{b}_n)]_C) = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,n} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1} & f_{m,2} & \cdots & f_{m,n} \end{pmatrix}$$

with the vector

$$[\mathbf{v}]_B = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

We have just motivated the ensuing definition.

Definition 4.2.3 (Matrix of a homomorphism)

Assume V, W are vector spaces over $\mathbb{F}$ and $f \in \operatorname{Hom}(V, W)$. Given bases $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of V and $C = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ of W, we call the matrix

$$[f]_C^B := ([f(\mathbf{b}_1)]_C \mid \cdots \mid [f(\mathbf{b}_n)]_C)$$

the *matrix of f with respect to the bases B and C.*

Proposition 4.2.4

For every $f \in \operatorname{Hom}(V, W)$ and any bases B of V and C of W, the matrix $[f]_C^B$ exists, is determined uniquely and for every $\mathbf{v} \in V$, the equality

$$[f]_C^B \cdot [\mathbf{v}]_B = [f(\mathbf{v})]_C$$

holds.

Proof. The existence and uniqueness of $[f]_C^B$ follow immediately from its definition as C is a basis of W and thus each vector $f(\mathbf{b}_i)$ has a *unique* representation with respect to C.

As for the last claim, assume

$$[\mathbf{v}]_B = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

that is $\mathbf{v} = v_1 \cdot \mathbf{b}_1 + \ldots + v_n \cdot \mathbf{b}_n$. By definition of $[f]_C^B$, we calculate

$$[f]_C^B \cdot [\mathbf{v}]_B = v_1 \cdot [f(\mathbf{b}_1)]_C + \ldots + v_n \cdot [f(\mathbf{b}_n)]_C = [f(v_1 \cdot \mathbf{b}_1 + \ldots + v_n \cdot \mathbf{b}_n)]_C = [f(\mathbf{v})]_C$$

and we're done. ■

Remark 4.2.5

The matrix $[f]_C^B$ of a homomorphism $f \in \mathrm{Hom}(V, W)$, where V, W are vector spaces over $\mathbb{F}$ of dimensions n and m, is always a matrix with entries in $\mathbb{F}$, m rows and n columns, that is, a matrix in $\mathbb{F}^{m \times n}$. This is clear from the fact that the representation $[\mathbf{v}]_B$ of $\mathbf{v} \in V$ has n entries (as $\dim V = n$) and thus $[f]_C^B$ must have n columns. Furthermore, the representation $[f(\mathbf{v})]_C$ has m entries (as $\dim W = m$) and thus each column of $[f]_C^B$ must also have m entries; in other words, $[f]_C^B$ has m rows.

Example 4.2.6

The matrix of the projection homomorphism $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\pi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$$

with respect to the standard bases $\mathcal{E}_3$ and $\mathcal{E}_2$ is

$$[\pi]_{\mathcal{E}_2}^{\mathcal{E}_3} = \left(\pi\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \middle| \pi\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \middle| \pi\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Indeed, we can calculate

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Notice, referring to remark 4.2.5, that $[\pi]_{\mathcal{E}_2}^{\mathcal{E}_3} \in \mathbb{R}^{2\times 3}$.

Example 4.2.7

Let us compute the matrix of the 'derivative' homomorphism

$$\frac{\partial}{\partial x} : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}),$$
$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mapsto a_1 + 2a_2 x + 3a_3 x^2.$$

The standard bases for $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$ are $(1, x, x^2, x^3)$ and $(1, x, x^2)$, respectively. Let us denote them P_3 and P_2. With respect to these bases, the matrix of $\partial/\partial x$ is

$$\left[\frac{\partial}{\partial x}\right]_{P_2}^{P_3} = \left(\left[\tfrac{\partial}{\partial x}(1)\right]_{P_2} \mid \left[\tfrac{\partial}{\partial x}(x)\right]_{P_2} \mid \left[\tfrac{\partial}{\partial x}(x^2)\right]_{P_2} \mid \left[\tfrac{\partial}{\partial x}(x^3)\right]_{P_2}\right)$$
$$= \left([0]_{P_2} \mid [1]_{P_2} \mid [2x]_{P_2} \mid [3x^2]_{P_2}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Indeed, representing the polynomial e.g. $4 + 2x - 3x^2 + x^3$ with respect to P_3 gives

$$[4 + 2x - 3x^2 + x^3]_{P_3} = \begin{pmatrix} 4 \\ 2 \\ -3 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -3 \\ 1 \end{pmatrix} = 4 \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3 \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$$

Since

$$[2 - 6x + 3x^2]_{P_2} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix},$$

we've just computed that

$$\frac{\partial}{\partial x}(4 + 2x - 3x^2 + x^3) = 2 - 6x + 3x^2$$

using vectors and matrices.

Example 4.2.8

For the representation of the integral homomorphism

$$\int : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_3(\mathbb{R}),$$
$$a_0 + a_1 x + a_2 x^2 \longmapsto a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3$$

we might wish to use a different basis for $\mathcal{P}_3(\mathbb{R})$. The reason being, that with respect to P_2 and P_3 its matrix would look like this (**check!**)

$$\left[\int\right]_{P_3}^{P_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

and we might not wish to deal with fractions. Let us instead choose the basis $\hat{P}_3 := (1, x, \frac{1}{2}x^2, \frac{1}{3}x^3)$

of $\mathcal{P}_3(\mathbb{R})$. With respect to P_2 and $\hat{P}_3$, the matrix of $\int$ instead looks like this:

$$\left[\int\right]_{\hat{P}_3}^{P_2} = \left([\int 1]_{\hat{P}_3} \mid [\int x]_{\hat{P}_3} \mid [\int x^2]_{\hat{P}_3}\right) = \left([x]_{\hat{P}_3} \mid \left[\tfrac{x^2}{2}\right]_{\hat{P}_3} \mid \left[\tfrac{x^3}{3}\right]_{\hat{P}_3}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Of course, the integrated polynomial is then also represented in the basis $\hat{P}_3$ and *not* in the canonical basis P_3. For example,

$$\left[\int\right]_{\hat{P}_3}^{P_2} \cdot [3 + x - 2x^2]_{P_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \\ -2 \end{pmatrix} = \left[3x + \frac{1}{2}x^2 - \frac{2}{3}x^3\right]_{\hat{P}_3}.$$

To every vector space V of dimension n and a chosen basis B, there is associated a special homomorphism $[\cdot]_B : V \to \mathbb{F}^n$ which assigns to each vector $\mathbf{v} \in V$ its representation $[\mathbf{v}]_B$ with respect to b (**check** that this is indeed a homomorphism!).

We shall now determine its matrix: first with respect to B and $\mathcal{E}_n$ and then with respect to B and C where C is any basis of the space $\mathbb{F}^n$.

By definition, the matrix $[[\cdot]_B]_{\mathcal{E}_n}^B$ should send the representation $[\mathbf{v}]_B$ to ... well ... $[\mathbf{v}]_B$ (since the representation of any vector with respect to $\mathcal{E}_n$ is just the vector itself). In other words, its a matrix which keeps the multiplied vector intact. Such matrix is called the *identity* matrix and it stands to reason that it should have 1's on its diagonal and 0's elsewhere.

Definition 4.2.9 (Identity matrix)

The matrix

$$I_n := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathbb{F}^{n \times n}$$

is called the *identity* matrix of size n. It is easily seen that for any vector $\mathbf{v} \in \mathbb{F}^n$, we have $I_n \cdot \mathbf{v} = \mathbf{v}$.

Now, the matrix of the representation homomorphism with respect to a different basis than $\mathcal{E}_n$ is more interesting. Let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$. By definition,

$$[[\cdot]_B]_C^B = ([[\mathbf{b}_1]_B]_C \mid [[\mathbf{b}_2]_B]_C \mid \cdots \mid [[\mathbf{b}_n]_B]_C).$$

However, clearly $[\mathbf{b}_i]_B = \mathbf{e}_i$ for every i. Therefore, the columns of the matrix $[[\cdot]_B]_C^B$ are simply the standard vectors $\mathbf{e}_i$ represented with respect to the basis C.

The last paragraph begs a question: couldn't the representation homomorphism be used to 'travel' between two different bases of the same space? As you may have guessed, it could. First, let us consider the trivial homomorphism

$$[\cdot]_{\mathcal{E}_n} : \mathbb{F}^n \to \mathbb{F}^n$$

which simply sends each vector $\mathbf{v} \in \mathbb{F}^n$ to its representation with respect to the standard basis, i.e. to itself. Its matrix with respect to the standard bases is naturally the identity matrix, however, we may choose to represent it with respect to non-standard bases, as well. Concretely, fix bases $X = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ and $Y = (\mathbf{y}_1, \dots, \mathbf{y}_n)$ of $\mathbb{F}^n$. We have

$$[[\cdot]_{\mathcal{E}_n}]_Y^X = \left([[\mathbf{x}_1]_{\mathcal{E}_n}]_Y \mid [[\mathbf{x}_2]_{\mathcal{E}_n}]_Y \mid \cdots \mid [[\mathbf{x}_n]_{\mathcal{E}_n}]_Y\right) = \left([\mathbf{x}_1]_Y \mid [\mathbf{x}_2]_Y \mid \cdots \mid [\mathbf{x}_n]_Y\right).$$

The matrix $[[\cdot]_{\mathcal{E}_n}]_Y^X$ thus assigns transforms the representation $[\mathbf{v}]_X$ of any vector $\mathbf{v} \in \mathbb{F}^n$ to $[\mathbf{v}]_Y$ to its representation with respect to Y. Indeed, one may easily see this fact from the equality

$$[[\cdot]_{\mathcal{E}_n}]_Y^X \cdot [\mathbf{v}]_X = [[\mathbf{v}]_{\mathcal{E}_n}]_Y = [\mathbf{v}]_Y.$$

Matrices that capacitate transporting vectors from one basis to another are extremely important tools and have thus won for themselves a title – *transition* matrices.

Before reading the following definition, kind readers are encouraged to devise a generalisation of the above construction for any vector space V of dimension n, not just $\mathbb{F}^n$. We shan't do that here explicitly.

Definition 4.2.10 (Change of basis matrix)

Let V be a vector space over $\mathbb{F}$ of dimension n and B, C two bases of V. The *matrix of transition* from B to C is the matrix

$$[\mathrm{id}]_C^B = \left([\mathbf{b}_1]_C \mid [\mathbf{b}_2]_C \mid \cdots \mid [\mathbf{b}_n]_C\right).$$

Example 4.2.11

Consider the bases

$$B = \left(\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) \quad \text{and} \quad C = \left(\begin{pmatrix} 2 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}\right)$$

of the space $\mathbb{R}^3$. In order to determine $[\mathrm{id}]_C^B$, we must represent the vectors from B with respect to C. This involves solving three linear systems with left side the vectors from B and right side one of the vectors from C. Fortunately, we have to go through only one process of Gauss-Jordan elimination since we can actually put *all* the vectors from C to the right side at once.

We thus need to eliminate the matrix

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 1 & 2 & 1 & 1 \\ -1 & 0 & 0 & 8 & 0 & 3 \\ 1 & -2 & 1 & 4 & -5 & 4 \end{array}\right).$$

Gauss-Jordan elimination (with some swapping of rows) results in

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 1 & 2 & 1 & 1 \\ -1 & 0 & 0 & 8 & 0 & 3 \\ 1 & -2 & 1 & 4 & -5 & 4 \end{array}\right).$$

List Of Exercises

Linear Systems

(1) Solve each of the systems below using matrix notation. Write the solution in the form of theorem 1.3.5.

$$\begin{array}{rcrcl} 3x & + & 6y & = & 18 \\ x & + & 2y & = & 6 \end{array} \qquad \begin{array}{rcrcr} x & + & y & = & 1 \\ x & - & y & = & -1 \end{array} \qquad \begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & & & = & 3 \\ 2x_1 & + & x_2 & & & + & x_4 & = & 4 \\ x_1 & - & x_2 & + & x_3 & + & x_4 & = & 1 \end{array}$$

(2) Show that any five points in the plane $\mathbb{R}^2$ lie on a common *conic section*, that is, they all satisfy an equation of the form

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

for some $a, \dots, f \in \mathbb{R}$.

(3) Prove that if $\mathbf{s}$ and $\mathbf{t}$ are solutions of a homogeneous linear system, then so are

(a) $\mathbf{s} + \mathbf{t}$,

(b) $3\mathbf{s}$,

(c) $k\mathbf{s} + m\mathbf{t}$ for any numbers k, m.

What is wrong with the following argument: 'These three show that if a homogeneous system has one solution, then it has many solutions – any multiple of a solution is another solution, and any sum of solutions is also a solution – so there are no homogeneous linear systems with exactly one solution.'?

(4) Find examples of linear systems of three equations in two variables that correspond to parts (b) and (d) of figure 1.4.

(5) Draw the following linear systems.

$$\begin{array}{rcrcl} 2x & + & y & = & 1 \\ 3x & + & 2y & = & 3 \end{array} \qquad \begin{array}{rcrcl} -x & + & y & = & 2 \\ 2x & - & 2y & = & 5 \end{array} \qquad \begin{array}{rcrcl} -x & - & y & = & 1 \\ 3x & + & 2y & = & 0 \end{array}$$

(6) Without depicting them visually, determine the arrangement of planes corresponding to the linear system below.

$$\begin{array}{rcrcrcl} 2x & - & y & + & z & = & 3 \\ x & - & 3y & + & 4z & = & 1 \\ x & + & 2y & - & 3z & = & 2 \end{array}$$

(7) Find linear systems in three variables and three equations corresponding to cases (1), (4) and (5) in the text above.

(8) Show that transformations (1) and (2) also don't change the set of solutions of the transformed linear system.

(9) Using *Gauss-Jordan elimination* solve the systems from examples 1.0.8 and 1.0.9.

(10) Use Gauss-Jordan elimination to solve the following system.

$$\begin{aligned} x_1 \quad\quad - x_3 &= 0 \\ 3x_1 + x_2 \quad\quad &= 1 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$$

(11) Each of the following systems is in echelon form. Determine their number of solutions (without calculation).

$$\begin{aligned} -3x_1 + 2x_2 &= 0 \\ -2x_2 &= 0 \end{aligned} \qquad \begin{aligned} 2x_1 + 2x_2 &= 4 \\ x_2 &= 1 \\ 0 &= 4 \end{aligned} \qquad 2x_1 + x_2 = 4$$

(12) Find the values of a, b and c that cause the graph of $f(x) = ax^2 + bx + c$ to pass through the points $(1, 2)$, $(-1, 6)$ and $(2, 3)$.

(13) Show that for all numbers a, b, c, d, j, k such that $ad - bc \neq 0$, the system

$$\begin{aligned} ax_1 + bx_2 &= j \\ cx_1 + dx_2 &= k \end{aligned}$$

has a *unique* solution.

Linear Geometry

(1) Describe the plane passing through points $(1, 1, 5, -1)$, $(2, 2, 2, 0)$ and $(3, 1, 0, 4)$ as
   (a) a set of points,
   (b) a set of vectors.

   Does the origin $(0, 0, 0, 0)$ lie in the plane?

(2) Describe the plane (as a set of points or vectors, as you wish) that contains

$$\text{the point } \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \text{ and the line } \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

(3) A person travelling eastward at a rate of 3 miles per hour finds that the wind appears to blow directly from the north. On doubling his speed it appears to come from the north east. What was the wind's velocity?

(4) Find the length of each of the vectors

$$\begin{pmatrix}3\\1\end{pmatrix}, \quad \begin{pmatrix}-1\\2\end{pmatrix}, \quad \begin{pmatrix}4\\1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}.$$

(5) Find the angle between each two of these vectors, if it is defined.

(a) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}$ (b) $\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\4\\1\end{pmatrix}$ (c) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\4\\-1\end{pmatrix}$

(6) Suppose that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ for some $\mathbf{u} \neq \mathbf{0}$. Is it necessarily true that $\mathbf{v} = \mathbf{w}$? Prove or provide a counterexample.

(7) Find the midpoint of the line segment connecting (x_1, y_1) to (x_2, y_2). Generalize to $\mathbb{R}^n$.

(8) Generalize the Pythagorean Theorem: if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are perpendicular, then

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

(9) Show that the dot product is *linear*, that is, given $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $k, m \in \mathbb{R}$, the equality

$$\mathbf{u} \cdot (k\mathbf{v} + m\mathbf{w}) = k(\mathbf{u} \cdot \mathbf{v}) + m(\mathbf{u} \cdot \mathbf{w})$$

holds. You may use the properties of dot product from the following exercise (10).

(10) Prove that for any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, the following equalities hold:

- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

Abstract Vector Spaces

(1) Prove (by checking the axioms) that the sets of functions mentioned in examples 3.0.10 and 3.0.11 are indeed vector spaces by definition.

(2) Decide which of the following sets are linearly independent.

(a) $\left\{\begin{pmatrix}1\\-3\\5\end{pmatrix}, \begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}4\\-4\\14\end{pmatrix}\right\}$;

(b) $\left\{\begin{pmatrix}1\\7\\7\end{pmatrix}, \begin{pmatrix}2\\7\\7\end{pmatrix}, \begin{pmatrix}3\\7\\7\end{pmatrix}\right\}$;

(c) $\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\4\end{pmatrix}\right\}$;

(d) $\left\{ \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \\ -1 \end{pmatrix} \right\}$.

(3) Determine which of the sets are linearly independent in the space of quadratic polynomials.

(a) $\{3 - x + 9x^2, 5 - 6x + 3x^2, 1 + 1x - 5x^2\}$;

(b) $\{-x^2, 1 + 4x^2\}$;

(c) $\{2 + x + 7x^2, 3 - x + 2x^2, 4 - 3x^2\}$.

(4) Prove that each of the following sets is linearly independent in the vector space of all functions $f : (0, \infty) \to \mathbb{R}$.

(a) $\{x \mapsto x, x \mapsto \frac{1}{x}\}$;

(b) $\{x \mapsto \cos x, x \mapsto \sin x\}$;

(c) $\{x \mapsto \exp x, x \mapsto \log x\}$.

(5) Prove that the rows of a real-valued matrix in echelon form are a linearly independent set.

(6) Prove that if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly independent set, then so are all its proper subsets: $\{\mathbf{x}, \mathbf{y}\}$, $\{\mathbf{x}, \mathbf{z}\}$, $\{\mathbf{y}, \mathbf{z}\}$, $\{\mathbf{x}\}$, $\{\mathbf{y}\}$, $\{\mathbf{z}\}$ and $\emptyset$. Is the converse also true?

(7) Is there a set of four vectors in $\mathbb{R}^3$ such that any three of them form a linearly independent set?

(8) Prove that a set of two perpendicular non-zero vectors in $\mathbb{R}^n$ is always linearly independent as long as $n > 1$. Generalise the result to more than two vectors.

(9) Decide whether $\{x^2 - x + 1, 2x + 1, 2x - 1\}$ and $\{x + x^2, x - x^2\}$ are bases of the space of quadratic polynomials.

(10) Find a basis for the solution set of the linear system

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0. \end{aligned}$$

(11) Find a basis for $\mathbb{R}^{2\times 2}$, the space of 2×2 real matrices.

(12) Let $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ be a basis.

(a) Show that $(r_1 \cdot \mathbf{b}_1, r_2 \cdot \mathbf{b}_2, r_3 \cdot \mathbf{b}_3)$ is also a basis as long $r_1, r_2, r_3 \neq 0$. What happens if at least one of r_i **is** zero?

(b) Prove that $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is also a basis where $\mathbf{a}_i = \mathbf{b}_1 + \mathbf{b}_i$, $i \in \{1, 2, 3\}$.

(13) Theorem 3.2.20 shows that, with respect to a basis, every linear combination is unique. If a subset is not a basis, can linear combinations be not unique? If so, must they be?

(14) Represent the polynomials

a) $2 + 4x^2$ b) $1 + 3x^2$ c) $1 + 5x^2$

with respect to the basis $B = (1 - x, 1 + x, x^2)$ of the space of quadratic polynomials. Use these representations to show that the three featured polynomials are linearly dependent.

(15) Represent the vector

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

with respect to the basis

$$B = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

of $\mathbb{R}^2$.

(16) Decide if the vector

(a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ lies in the row space of the matrix $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ lies in the row space of the matrix $\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}$.

(17) Decide if the vector

(a) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ lies in the column space of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ lies in the column space of the matrix $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ 1 & -3 & -3 \end{pmatrix}$.

(18) Find the basis of both the row space and column space of the matrix

$$\begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix}.$$

(19) Given $a, b, c \in \mathbb{R}$ what choice of $d \in \mathbb{R}$ will cause the following matrix to have rank one?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(20) Find the column rank of the matrix

$$\begin{pmatrix} 1 & 3 & -1 & 5 & 0 & 4 \\ 2 & 0 & 1 & 0 & 4 & 1 \end{pmatrix}$$

(21) An $m \times n$ has *full row rank* if its row rank is m and has *full column rank* if its column rank is n.

(a) Show that a matrix can have both full row rank and full column rank only if it is square (that is, $m = n$).

(b) Prove that a linear system with matrix of coefficients A has a solution for **any** right side if and only if A has full row rank.

(c) Prove that a homogeneous system has a unique solution if and only if its matrix of coefficients A has full column rank.

(d) Prove that the statement '*if a system with matrix of coefficients A has any solution, then it has a unique solution*' holds if and only if A has full column rank.

(22) What is the relationship (if any) between

(a) $\operatorname{rank} A$ and $\operatorname{rank}(-A)$?

(b) $\operatorname{rank} A$ and $\operatorname{rank}(kA)$ for $k \neq 0$?

(c) $\operatorname{rank} A + \operatorname{rank} B$ and $\operatorname{rank}(A + B)$?

Homomorphisms (chybějí obrázky)

(1) Prove that the map $f : \mathcal{P}_3(\mathbb{F}) \to \mathbb{F}^4$ from example 4.0.2 is a homomorphism.

(2) Prove that the map $f : \mathbb{R}^{2\times 2} \to \mathbb{R}^2$ from example 4.0.2 is **not** an isomorphism.

(3) Find an example of a homomorphism $f : V \to W$ which is

(a) an *isomorphism* but **not** an automorphism,

(b) an *endomorphism* but **not** an automorphism,

(c) an *automorphism*.

(4) Prove that for two homomorphisms $f, g \in \operatorname{Hom}(V, W)$, $a, b \in \mathbb{F}$, $\mathbf{v} \in V$ and $t \in \mathbb{F}$, we have

$$(a \cdot f + b \cdot g)(t \cdot \mathbf{v}) = t \cdot (a \cdot f + b \cdot g)(\mathbf{v}).$$