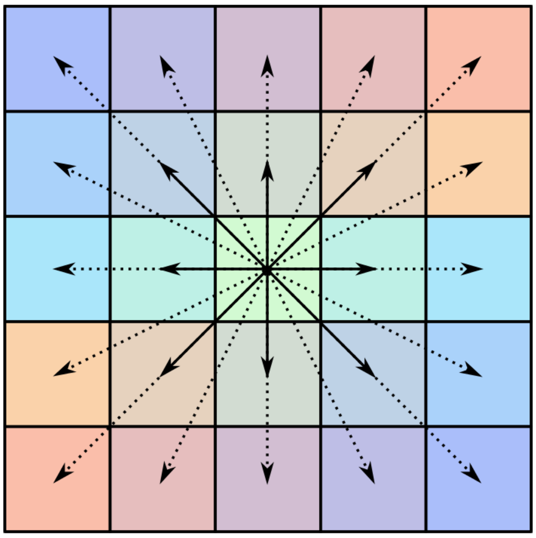
Gymnázium Evolution Jižní Město



Intro-ish To Linear Algebra

Madam Pekáč

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Preface

This text covers selected topics from the curriculum of a typical undergraduate linear algebra course. Almost no pre-existing knowledge is strictly required save a superficial understanding of propositional logic and set theory. A reasonably good ability to manipulate algebraic expressions should prove advantageous, too.

Mathematics is an exact and rigorous language. Words and symbols have singular, precisely defined, meaning. Many students fail to grasp that intuition and imagination are paramount, but they serve as a *starting point*, with formal logical expression being the end. For example, an intuitive understanding of a *line* as an infinite flat 1D object is pretty much correct but not *formal.* It is indeed the formality of mathematics which puts many students off. Whereas high school mathematics is mostly algorithmic and non-argumentative, higher level maths tends to be the exact opposite – full of concepts and relations between those which one is expected to be capable of grasping and formally describing. Owing to this, I wish this text would be a kind of synthesis of the formal and the conceptual. On one hand, rigorous definitions and proofs are given; on the other, illustrations, examples and applications serve as hopefully efficient conveyors of the former's geometric nature.

Linear algebra is a mathematical discipline which studies – as its name rightly suggests – the *linear.* Nevertheless, the word *linear* (as in 'line-like') is slightly misplaced. The correct term would perhaps be *flat* or, nigh equivalently, *not curved.* It isn't hard to imagine why curved objects (as in *geometric* objects, say) are more difficult to describe and manipulate than objects flat. For instance, the formula for the volume of a cube is just the product of the lengths of its sides. Contrast this with the volume of a still 'simple', yet curved, object – the ball. Its volume cannot even be *precisely* determined; its calculation involves approximating an irrational constant and the derivation of its formula is starkly unintuitive without basic knowledge of measure theory.

As such, linear algebra is a highly 'geometric' discipline and opportunities for visual interpretations abound. This is also a drawback in a certain sense. One should not dwell on visualisations alone as they tend to lead astray when imagination falls short. Symbolic representation of the geometry at hand is key.

The word *linear* however dons a broader sense in modern mathematics. It can be rephrased as reading, 'related by addition and multiplication by a scalar'. We trust kind readers have been acquainted with the notion of a *linear function.* A linear function is (rightly) called *linear* for it receives a number as input and outputs its *constant* multiple plus another *constant* number. Therefore, the output is in a *linear* relation to the input – it is multiplied by some fixed number and added to another. This understanding of the word is going to prove crucial already in the first chapter, where we study *linear systems.* Following are *vector spaces* and *linear maps*, concepts whose depth shall occupy the span of this text. Each chapter is further endowed with an *applications* section where I try to draw a simile between mathematics and common sense.

Contents

Chapter 1

Linear Systems

Linear systems are by definition sets of linear equations, that is, of equations which relate present variables in a *linear* way. It is important to understand what this means. Spelled out, an expression on either side of any of the equations is formed *solely* by

(1) multiplying the variables by a given number (**not another variable**),

(2) adding these multiples together.

Any such combination where variables are only allowed to be multiplied by a constant and added is called a *linear combination*. This term is extremely important and ubiquitous throughout the text; hence, it warrants an isolated definition.

Definition 1.0.1 (Linear combination)

Let $x_1, \dots, x_n$ with $n \in \mathbb{N}$ be variables. Their *linear combination* is any expression of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where $a_1, \dots, a_n$ are numbers.

Remark 1.0.2

In the definition above, we have deliberately not specified what type of *numbers* we mean. In the future, we shall work extensively with real and complex numbers as well as elements of other fields, which dear readers might not have even recognised as 'numbers' thus far. The only important concept in this regard is the clear distinction between a *number* (later *scalar*) and a *variable* (later *vector*).

Example 1.0.3

Consider the variables x, y and z. The expression

$$3x + 2y - 0.5z$$

is their linear combination whereas

$$5x + 3y - yz + 7z^2$$

is not.

To reiterate, a *linear system* is any set of equations featuring only linear combinations of variables; these equations are consequently called *linear* as well. A *solution* of a linear system is the set of all possible substitutions of numbers (in place of variables) which make the equations true.

It is clear that every linear equation can be rearranged to

$$a_1x_1 + \cdots + a_nx_n = c$$

for some variables $x_1, \ldots, x_n$ and numbers $a_1, \ldots, a_n, c$, by simple subtraction. This is how we shall define it, for simplicity.

Definition 1.0.4 (Linear equation)

Any equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c \tag{1.1}$$

where $x_1, \ldots, x_n$ are variables and $a_1, \ldots, a_n, c$ are numbers, is called *linear*. A *solution* of a linear equation is an n-tuple $(b_1, \ldots, b_n)$ of numbers such that under the substitutions $x_i \coloneqq b_i$, for $i \in \{1, \ldots, n\}$, the equation (1.1) is satisfied.

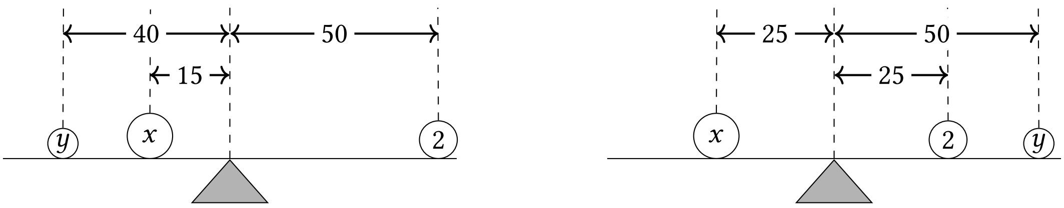
Definition 1.0.5 (Linear system)

Any set of linear equations in the given variables $x_1, \ldots, x_n$ is called a *linear system*. A *solution* of a linear system is an n-tuple $(b_1, \ldots, b_n)$ which solves every linear equation in the set.

We proceed to discuss two trivial examples, which readers might have discussed in high school, that naturally lead to linear systems. More sophisticated examples are presented in the applications section.

Example 1.0.6 (Static equations)

Suppose we have three objects – one with a mass of 2 and the other two with masses unknown. Experimentation produces these two balances.



For the weights to be in balance, the sum of *moments* on either side of the scales must be identical. A *moment* of an object is its distance from the centre of the scales times its mass.

This condition yields a system of two linear equations

$$\begin{array}{rcrcrcr} 15x & + & 40y & = & 50 \cdot 2, & & \\ & & 25x & = & 25 \cdot 2 & + & 50y. \end{array}$$

Or, after rearrangement (to stay true to our definition of linear equation),

$$\begin{array}{rcrcl} 15x & + & 40y & = & 50 \cdot 2, \\ 25x & - & 50y & = & 25 \cdot 2. \end{array}$$

Example 1.0.7 (Chemical reactions)

Toluene, $\mathtt{C_7H_8}$, mixes (under right conditions) with nitric acid, $\mathtt{HNO_3}$, to produce trinitrotoluene (widely known as TNT), $\mathtt{C_7H_5O_6N_3}$, along with dihydrogen monoxide, $\mathtt{H_2O}$. If we want this chemical reaction to occur successfully, we must (among other things) ascertain we mix the constituents in the right proportion. In pseudo-chemical notation, the reaction to take place can be written as

$$x \cdot \mathtt{C_7H_8} + y \cdot \mathtt{HNO_3} \longrightarrow z \cdot \mathtt{C_7H_5O_6N_3} + w \cdot \mathtt{H_2O}.$$

Comparing the number of atoms of each element before the reaction and afterwards (which must remain identical owing to the conservation of energy) yields the system

$$\begin{array}{lrcrcrcr} \mathtt{H}: & 8x & + & 1y & = & 5z & + & 2w, \\ \mathtt{C}: & 7x & & & = & 7z, & & \\ \mathtt{N}: & & & 1y & = & 3z, & & \\ \mathtt{O}: & & & 3y & = & 6z & + & 1w. \end{array}$$

In the next section, we devise an algorithm to solve any system of linear equations.