Systems of Linear Equations

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input ⟶ function ‘*do something with input*’ ⟶ output

We’ll call the data that **a function receives**, inputs and the **data it gives back**, outputs. Inputs and outputs need not necessarily be just ‘one object’, they can be for example lists of numbers.

Functions – Example

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$x_1, x_2, \ldots, x_n$

n

$\longrightarrow$

average

'sum all the numbers and divide them by n'

$\longrightarrow$

$$\frac{x_1 + x_2 + \ldots + x_n}{n}$$

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omelette $\longrightarrow$ $\boxed{\text{ingredients}}$ $\longrightarrow$ eggs
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Function Composition

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For instance, you could hardly compose the ingredients function with the average function.

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Considering two functions

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What would the output of this composition look like

$$a \longrightarrow \boxed{\text{times two}} \longrightarrow \boxed{\text{sum}} \longrightarrow ?$$

Function Composition

So, the **order of the composition matters**! Here are a few examples of compositions which make sense:

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We can of course compose **as many functions** as we like. An example of this:

$$\text{omelette} \longrightarrow \boxed{\text{ingredients}} \longrightarrow \boxed{\text{total price}} \longrightarrow \boxed{\text{times two}} \longrightarrow \$8.74$$

Functions – Notation

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You're probably used to seeing function written like $f(x) = y$. The picture corresponding to this is

$$x \longrightarrow \boxed{f} \longrightarrow y$$

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For example, if f and g are two functions, their composition $f \circ g$ corresponds to this picture

$$x \longrightarrow \boxed{g} \longrightarrow \boxed{f} \longrightarrow y$$

that is, **first g, then f**.

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Real Functions

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where $x \in \mathbb{R}$. Or, using pictures,

$$x \longrightarrow \boxed{f} \longrightarrow 0$$

$$x \longrightarrow \boxed{g} \longrightarrow \tan^6(\log^{\sin(x^2+4)}(\frac{5x^3-2}{9x^7}))$$

Operations On Real Functions

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- $(f \circ g)(x) = 2\left(\frac{1}{1+x}\right)^2 + 7$ and
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- **product** is the function with output

$$(f \cdot g)(x) = f(x) \cdot g(x) = (2x^2 + 7) \cdot \left(\frac{1}{1+x}\right) = \frac{2x^2+7}{1+x}.$$

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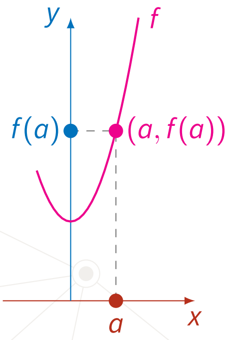
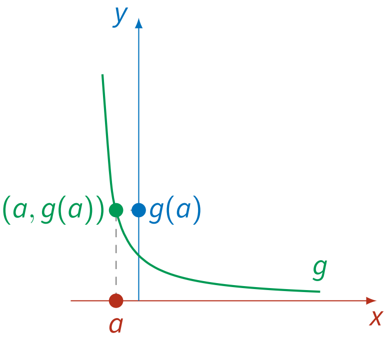
However, later, we'll also use the **polar** coordinate system where every point is instead determined by its angle and distance from the origin of the system.

Graphs

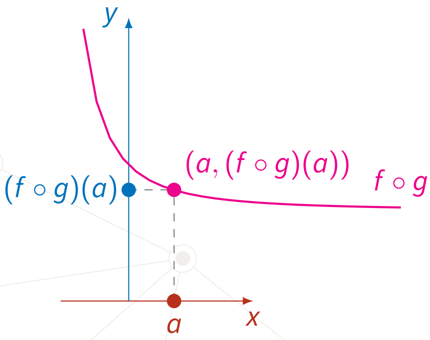
The functions $f(x) = 2x^2 + 7$ and $g(x) = \frac{1}{1+x}$ have the following (parts of) graphs:

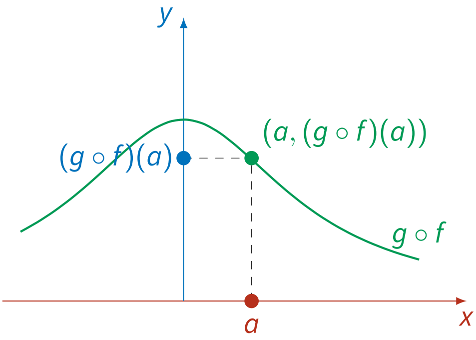




Graphs

Just to better drive home the idea that the order of function composition is important, look at the graphs of $f \circ g$ and $g \circ f$.





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Linear Function

A real function f is linear if

$$f(x) = ax + b$$

for some $a, b \in \mathbb{R}$.

Linear Functions – Properties

Linear functions have some unique properties:

- If f and g are linear, so are $f \circ g$ and $g \circ f$.

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$$(f \circ g)(x) = a(cx + d) + b = (ac)x + (ad + b),$$
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- If f and g are linear, so is $f + g$. If we just compute the sum, we get

$$(f + g)(x) = (ax + b) + (cx + d) = (a + c)x + (b + d).$$

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If we move by 1, from x to $x + 1$, then on the y-axis we move from $ax + b$ to $a(x + 1) + b$, that is, we move by

$$a(x + 1) + b - (ax + b) = a.$$

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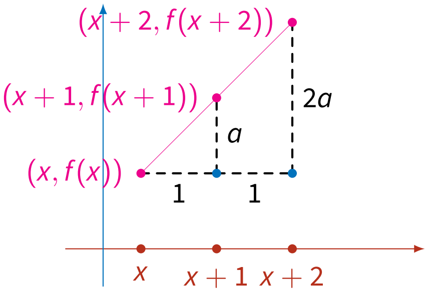
If we move by 2, from x to $x+2$, on the y-axis, we move by

$$a(x+2)+b-(ax+b)=2a.$$



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That happens exactly when their **rates growth** are identical.

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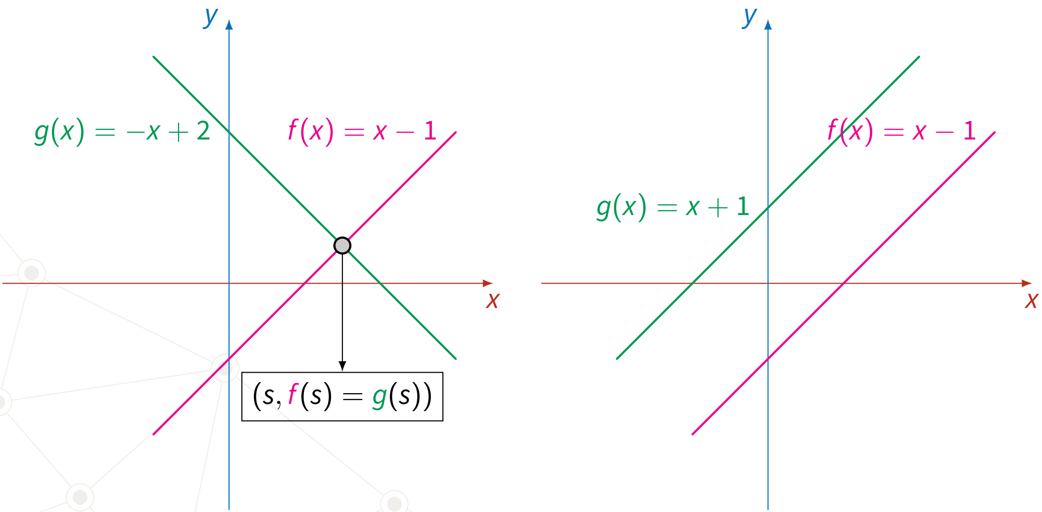
The question is, when are the graphs of f and g parallel?

That happens exactly when their **rates growth** are identical.

In symbols, if $f(x) = ax + b$ and $g(x) = cx + d$, then the graphs of f and g are parallel if $a = c$.

Linear Functions – Intersections



Linear Equations

Linear Equation

Linear Equation

If f, g are linear functions, the equation

$$f(x) = g(x)$$

is called a **linear equation** (in one variable).

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$$\begin{aligned} ax + b &= cx + d \\ (a - c)x &= d - b \\ x &= \frac{d - b}{a - c}. \end{aligned}$$

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- $a = c$ and $b \neq d$. In this case the graphs of the two functions are parallel lines – there is no solution.
- $a = c$ and $b = d$. In this case, the functions are one and the same and every number is a solution.

Linear Equations & Operations

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We see that the graphs of $f \circ g$ and $g \circ f$ are parallel, so this equation has a solution only in the case that $f = g$.