

# Homework – PreIB 3.AB 3 & 4

## Structures and Operations

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**DON'T FORGET TO EXPLAIN EVERYTHING EVEN IF YOU THINK IT'S OBVIOUS!**

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### Natural Numbers

**Exponentiation** of two natural numbers  $n, m \in \mathbb{N}$  is defined by the following two formulae:

- $n^0 = 1$ ,
- $n^{s(m)} = n^m \cdot n$ .

Explain **very clearly** how to calculate  $n^m$  using **only** the two rules above.

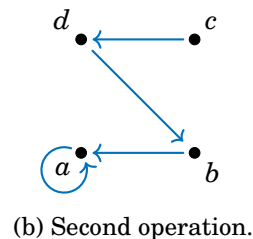
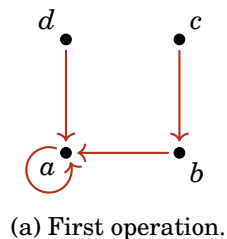
**Hint:** This process is very similar to the definition of *addition* and *multiplication* on natural numbers.

Answer the following questions:

1. Is exponentiation *commutative*, that is, is it true that  $n^m = m^n$  for all pairs of natural numbers  $n, m \in \mathbb{N}$ ? If yes, explain why. If not, provide a counterexample.
2. Is exponentiation *associative*, that is, is it true that  $n^{(m^k)} = (n^m)^k$  for all triples of natural numbers  $n, m, k \in \mathbb{N}$ ? If yes, explain why. If not, provide a counterexample.
3. Is exponentiation an *operation* (by definition) on natural numbers? Explain.

## Operations

On the set  $X = \{a, b, c, d\}$ , there are two operations given by the following picture.



Solve the following problems:

1. Change operations (a) and (b) **as little as possible** to make them *symmetric* (or *invertible*). By a ‘change’, I mean altering the source and target of a single arrow. **Explain** why your method requires the fewest changes.
2. Let’s add another element  $e$  to the set  $X$ . Change operation (b) so that one arrow ends in  $e$  and one arrow starts at  $e$  so that the new operation is *symmetric*.
3. We label the operation from point 2 by  $\sim$ . Find an inverse to each element of  $X$  with respect to  $\sim$ . Recall that, in this case, an *inverse* to  $x \in X$  is an element  $y \in X$  such that  $\tilde{y} = x$ .
4. Does  $\sim$  have an *identity element*? That is, is there an element  $x \in X$  such that  $\tilde{x} = x$ ?

We define a binary operation  $\square$  on the set  $X$  by the following table.

$\square$	$a$	$b$	$c$	$d$
$a$	$a$	$b$	?	?
$b$	$b$	?	$d$	$a$
$c$	?	$d$	$a$	?
$d$	$d$	?	$b$	$c$

Operation  $\square$ .

Solve the following problems:

1. Substitute all ?’s in the table by an adequate element of  $X$  so that the resulting operation is *symmetric*.
2. Show that the operation you get in point 1 is indeed symmetric, that is, **find inverse with respect to  $\square$  of each element in  $X$  and find the identity element.**