Logic & Set Theory Cheatsheet

2.AB PreIB Math

Adam Klepáč

Logic

Logic is the language of mathematics. It uses propositions to talk about sets.

Propositions are sentences which can be either true or false. For example

- '**Cats are black.**' is a proposition;
- '**How are you?**' is *not* a proposition;
- '**We will have colonised Mars by 2500.**' is also a proposition.

As the third example suggests, we need not necessarily know whether a proposition is true or false – it remains a proposition anyway.

Logical Operators

Propositions can be transformed using logical operators. They pretty much correspond to the conjunctions of natural language. Let us consider two propositions:

$$p = \text{'It's raining outside.'}$$
$$q = \text{'I'll stay at home.'}$$

($\wedge$) Logical and forms a proposition that is only true if both of its constituents are also true. In natural language, the proposition $p \wedge q$ can be expressed as

$$p \wedge q = \text{'It's raining outside and I'll stay at home.'}$$

($\vee$) Logical or forms a proposition that is true if at least one of its constituents is true. In natural language, the proposition $p \vee q$ can be expressed as

$$p \vee q = \text{'It's raining outside or I'll stay at home.'}$$

In mathematical logic, or is **not exclusive**! This means that $p \vee q$ is true even if both p and q are true.

($\neg$) Logical not reverses the truth value of a proposition. For example, the proposition $\neg p$ can be read as

$$\neg p = \text{'It's not raining outside.'}$$

It follows that $\neg p$ is true exactly when p is false and vice versa.

($\Rightarrow$) Logical implication is an operator that makes the first proposition into an *assumption* or *premise* and the second one into a *conclusion*. The proposition $p \Rightarrow q$ is read in multiple ways, to list a few:

$$p \Rightarrow q = \text{'If it's raining outside, then I'll stay at home.'}$$
$$p \Rightarrow q = \text{'It raining outside implies that I'll stay at home.'}$$
$$p \Rightarrow q = \text{'Assuming it's raining outside, I'll stay at home.'}$$

The implication is tricky. It's true if both p and q are true and false if p is true but q is false. However, it is always true if p is false. That is because, in mathematical logic, whatever follows from a lie is automatically true.

($\Leftrightarrow$) Logical equivalence is true only if both propositions have the same truth value – they're both true or both false. In natural language, it is typically read like this:

$$p \Leftrightarrow q = \text{'It's raining if and only if I stay at home.'}$$

Equivalence is basically just a two-way implication. The proposition p is both a premise and a conclusion to q and q is both a premise and a conclusion to p. If it's raining outside, I stay at home and if I stay at home, then it's raining outside.

Truth Tables

A proposition made up of smaller propositions is true or false based on whether its constituent propositions are true or false. All possible scenarios can be summarized using so-called truth table. It is basically just a table that lists all the possibilities of p and q (or any number of propositions, really) being true or false and the resulting truth value of their combinations.

For the basic logical conjunctions from above, it can look like this (we represent true by 1 and false by 0):

p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	1	1	0	0	1	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	0	0
1	1	0	0	1	1	1	1

Sets

Sets are the 'stuff' that makes up the world of mathematics. Their basic characteristics and properties are described using logic.

Sets cannot be defined inside set theory but we interpret them as *groups of things*.

There's only one foundational *proposition* related to set theory – the proposition 'An object is an element of a set.' If we label the object in question, x, and the set, A, this proposition is written as $x \in A$ (the symbol $\in$ is just the letter 'e' in 'element'). Combining these propositions using logical conjunctions allows for various set-theoretic constructions.

If a set A has, for example, exactly three elements – $\square$, $\triangle$ and $\bigcirc$, I can write it as a list of these three elements inside curly brackets $\{\}$. In this case,

$$A = \{\square, \triangle, \bigcirc\}.$$

Two warnings about sets:

- **Sets are not ordered**. There is nothing like a 'first', 'second' or 'last' element of a set. Either an object **is** inside a set or it **isn't**. Nothing else. For example, the three sets below are exactly the same, only written differently.

$$\{\square, \triangle, \bigcirc\} = \{\bigcirc, \triangle, \square\} = \{\triangle, \square, \bigcirc\}$$

- **Elements of sets have no frequency**. Again, an element either is inside a set or not. It cannot be twice in a set, for example. The three sets below are exactly the same.

$$\{\square, \triangle, \bigcirc\} = \{\square, \triangle, \bigcirc, \triangle, \bigcirc\} = \{\triangle, \square, \square, \triangle, \bigcirc, \triangle\}$$

There are various ways of constructing sets. We shall take a look at two: *enumeration* and *condition*.

By *enumeration*, we simply mean that a set is defined by listing all its elements. We have already seen this before. The equality $A = \{\square, \triangle, \bigcirc\}$ is an example of defining a set by enumeration.

A more potent way of creating sets entails using logical propositions. Assume x is an object and $p(x)$ is any logical proposition involving x. For example,

$$p(x) = \text{'}x\text{ is beautiful.'}$$
$$p(x) = \text{'}x\text{ is a number.'}$$

The set $\{x \mid p(x)\}$ is the set of all objects x for which $p(x)$ is true. Imagine

$$p(x) = \text{'}x\text{ is natural number and }x\text{ is smaller than five.'}.$$

Then,

$$\{x \mid p(x)\} = \{0, 1, 2, 3, 4\}.$$

Set Operations

Using logical operators, we can form new sets from existing ones or establish relations between sets. Consider two sets – A and B.

($\cap$) We can form the set $\{x \mid x \in A \wedge x \in B\}$, that is, the set of all objects that lie in both A and B. This set is called the intersection of A and B and written $A \cap B$. For example,

$$\{\bigcirc, \triangle, \square\} \cap \{\times, \bigcirc, \square, \sim\} = \{\bigcirc, \square\}.$$

($\cup$) We can form the set $\{x \mid x \in A \vee x \in B\}$, i.e. the set of all objects that lie in A or in B. It is called the union of A and B and denoted $A \cup B$. All elements of $A \cup B$ can be found *only* in A, *only* in B or in *both* A and B. For example,

$$\{\bigcirc, \triangle, \square\} \cup \{\times, \bigcirc, \square, \sim\} = \{\bigcirc, \triangle, \square, \times, \sim\}.$$

($\neg$) Negation itself doesn't really do much but we can combine it with conjunction ($\wedge$) to form the difference of two sets. The set $\{x \mid x \in A \wedge \neg(x \in B)\} = \{x \mid x \in A \wedge x \notin B\}$ is the set of all elements that belong to A but do not belong to B and is denoted $A \setminus B$. For example,

$$\{\bigcirc, \triangle, \square\} \setminus \{\times, \bigcirc, \square, \sim\} = \{\triangle\}.$$

Beware: the difference (unlike union and intersection) is not commutative, meaning that $A \setminus B \neq B \setminus A$. In the example above,

$$\{\times, \bigcirc, \square, \sim\} \setminus \{\bigcirc, \triangle, \square\} = \{\times, \sim\}.$$

($\Rightarrow$) Implication is a little different from intersection and union. It describes a lot of different sets with one logical proposition. I ask: 'Which sets A satisfy the proposition $x \in A \Rightarrow x \in B$?' In other words, which sets A have all their elements contained in the set B? The answer is that A must be a subset of B and we denote that fact by $A \subseteq B$. The set A is only allowed to have elements which also lie in B but not necessarily all of them. All the subsets of $B = \{\triangle, \bigcirc\}$ are listed below.

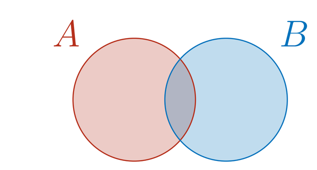
$$\{\}, \{\triangle\}, \{\bigcirc\}, \{\triangle, \bigcirc\},$$

where $\{\}$ is the empty set, a set containing no elements.

($\Leftrightarrow$) Equivalence defines equality on sets. If sets A and B must satisfy the proposition $x \in A \Leftrightarrow x \in B$, then they must be equal because all the elements of A lie in B and all elements of B lie in A. That is, $A = B$.

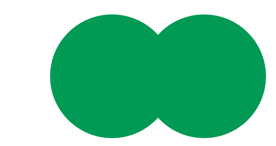
Drawing Sets

Set operations can be visualized using so-called *Venn diagrams*. This just means using circles to represent the sets in question. For example, two sets – A and B – can be drawn like this:



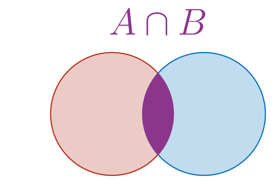
In these pictures, one can easily visualize the operations of union, intersection and difference. The union $A \cup B$ is the entire area covered by A and B. It looks like this:

$A \cup B$



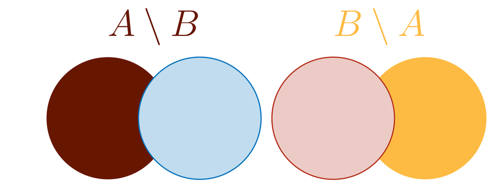
The intersection $A \cap B$ is the 'strip' in the middle, the area which is shared between both A and B. It can be depicted like this:

$A \cap B$

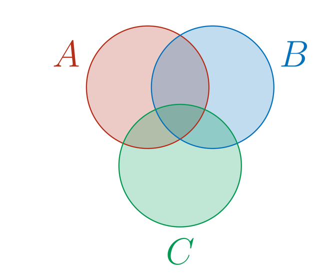


The difference $A \setminus B$ is the area of the red circle which doesn't lie in the blue circle. The difference $B \setminus A$ is the mirror version of that.

$A \setminus B$ $\quad$ $B \setminus A$



Adding a third set C results in a picture like this.

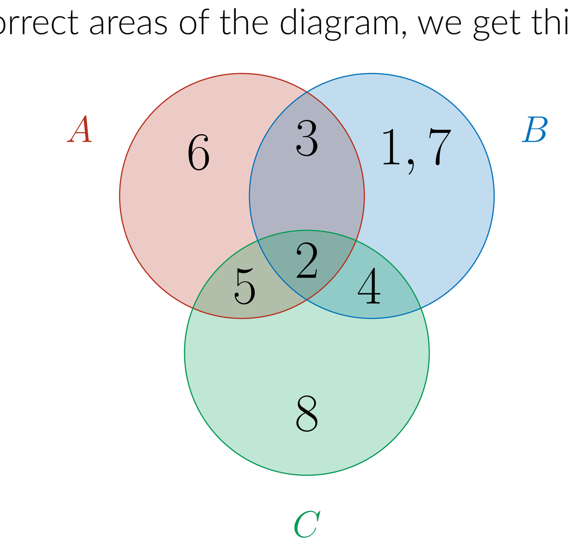


Every area in the picture corresponds to a different combination of set operations. For example, the isolated part of the red circle consists of elements found in A but not in B or C. This set is the result of the operation $(A \setminus B) \setminus C$. The area in the middle where all circles intersect represents the set $A \cap B \cap C$.

Let us see this precisely. We choose the sets

$$A = \{2, 3, 5, 6\}, \quad B = \{1, 2, 3, 4, 7\}, \quad C = \{2, 4, 5, 8\}.$$

If we place the numbers into correct areas of the diagram, we get this picture.



As our final example, we draw two areas defined by two different combinations of set operations.

$(A \cap B) \cup (B \cap C)$ $\quad$ $(A \cup B) \setminus C$