



PRIME FACTORIZATIONS GCD & LCM

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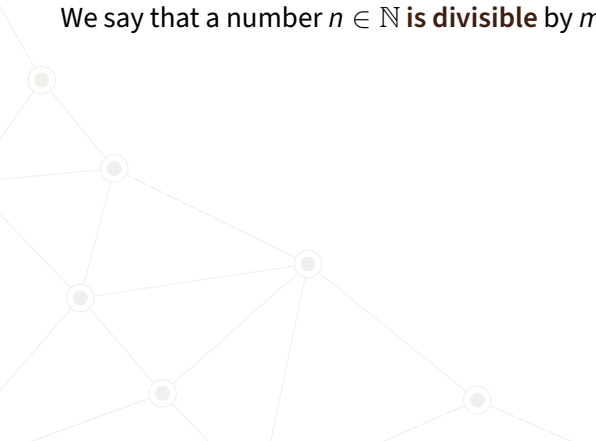
Chinese Remainder Theorem

DIVISIBILITY

The background of the slide features a minimalist design with two large triangles meeting at a point in the center. The triangle on the left is a dark, rich brown, and the triangle on the right is a warm, golden-brown color. The top half of the slide is a solid, very light cream or off-white color, creating a clean, modern aesthetic.

DIVISIBILITY

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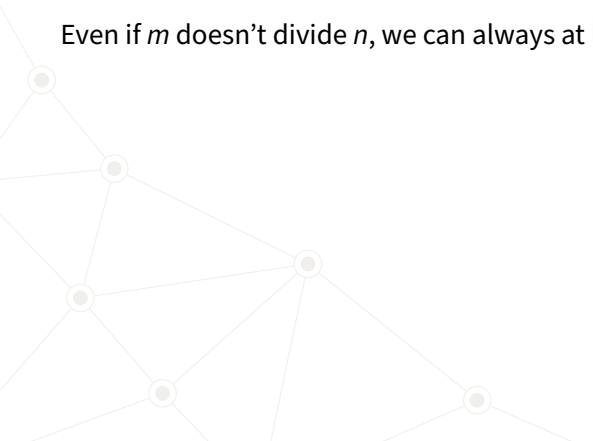
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We write the fact that m divides n as $m \mid n$.

DIVISIBILITY

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Formally, for every two numbers $n, m \in \mathbb{N}$, there always exist number $q \in \mathbb{N}$ and $r \leq n$ (called the **remainder**) such that

$$n = m \cdot q + r.$$

INTEGER DIVISION & MODULUS

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- The operation which to a pair (n, m) assigns the **remainder** of the division of n by m is called **modulus** and denoted mod .
 - For example, $7 \text{ mod } 2 = 1$ and $5 \text{ mod } 9 = 5$.

INTEGER DIVISION & MODULUS

Using integer division and modulus, we can write the operation of division with remainder like this:

$$n = m \cdot (n \operatorname{div} m) + n \operatorname{mod} m.$$

CHINESE REMAINDER THEOREM

The slide features a white background with two large, overlapping geometric shapes at the bottom. On the left, a dark brown triangle points upwards towards the center. On the right, a tan triangle points downwards towards the center. The two triangles overlap in the middle, creating a darker brown triangular region at the bottom center.

COUNTING SOLDIERS

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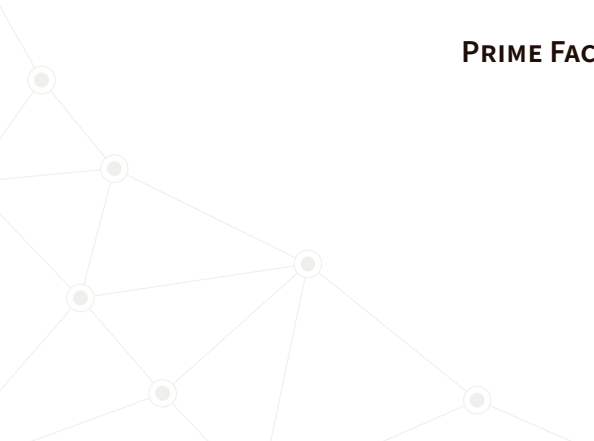
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- 21 soldiers in each row. He counts the number of soldiers remaining in the last row to be 3.
- 25 soldiers in each row. He counts the number of soldiers remaining in the last row to be 6.
- 32 soldiers in each row. He counts the number of soldiers remaining in the last row to be 23.

孙子 then determines, that there are 1431 soldiers standing before him.

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PRIME FACTORIZATION



PRIME NUMBER

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A number $n \in \mathbb{N}$ is **prime** if it has **exactly two divisors** – 1 and itself.
We denote the set of all prime numbers by \mathbb{P} .

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PRIME FACTORIZATION

For each number $n \in \mathbb{N}$, there exist prime numbers $p_1, \dots, p_m \in \mathbb{P}$ and powers $k_1, \dots, k_m \in \mathbb{N}$ such that

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_m^{k_m}.$$

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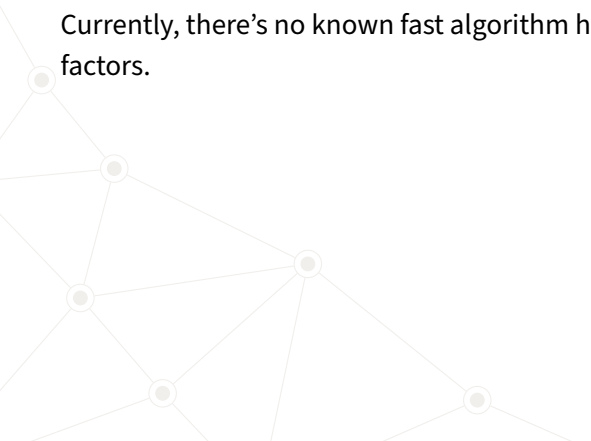
For each number $n \in \mathbb{N}$, there exist prime numbers $p_1, \dots, p_m \in \mathbb{P}$ and powers $k_1, \dots, k_m \in \mathbb{N}$ such that

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_m^{k_m}.$$

The numbers p_1, \dots, p_m are called the **prime factors** of n .

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The best we can do is to try each prime number from 2 up to \sqrt{n} .

We divide the number in question as many times as possible and proceed to find the next prime factor of the resulting quotient.

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Prime Factors	Quotients
3	5445
3	1815
3	605

DECOMPOSITION INTO PRIMES – EXAMPLE

Let's decompose the number 16335 into its prime factors.

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Prime Factors	Quotients
3	5445
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This means that $16335 = 3^3 \cdot 5 \cdot 11^2$.

GREATEST COMMON DIVISOR

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Given two numbers $n, m \in \mathbb{N}$, their **Greatest Common Divisor** (GCD) or Highest Common Factor (HCF) is the largest natural number $k \in \mathbb{N}$ such that $k \mid n$ and $k \mid m$.

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For example, let's take a look at the decompositions of 16335 and 17325.

Prime Factors	Quotients
3	5445
3	1815
3	605
5	121
11	11
11	1

Prime Factors	Quotients
3	5775
3	1925
5	385
5	77
7	11
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GREATEST COMMON DIVISOR

If we highlight the prime factors the two numbers share

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We'll denote the GCD of n and m as $\gcd(n, m)$ or simply as (n, m) .

LEAST COMMON MULTIPLE

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Given $n, m \in \mathbb{N}$ their **Least Common Multiple** (LCM) is the **smallest** number $k \in \mathbb{N}$ such that $n \mid k$ and $m \mid k$.

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Given $n, m \in \mathbb{N}$ their **Least Common Multiple** (LCM) is the **smallest** number $k \in \mathbb{N}$ such that $n \mid k$ and $m \mid k$. It is given by the formula

$$\text{lcm}(n, m) = \frac{n \cdot m}{\text{gcd}(n, m)}.$$

LEAST COMMON MULTIPLE

The other option how to calculate the least common multiple of two numbers is to take their prime factors elevated on the highest power that appears in at least one of the numbers.

LEAST COMMON MULTIPLE

Explicitly, from the prime decompositions of 16335 and 17325

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we see that their LCM is

$$3^3 \cdot 5^2 \cdot 7 \cdot 11^2 = 571725.$$

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CONGRUENCE



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Congruence is a relation between two natural numbers $n, m \in \mathbb{N}$ that says they have the same remainder when divided by some third number $k \in \mathbb{N}$.

Formally, we write that

$$n \equiv m \pmod{k}$$

and read it as ‘ n is congruent to m modulo k ’ if

$$n \bmod k = m \bmod k.$$

CONGRUENCE – EXAMPLES

For example,

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For example,

- $14 \equiv 9 \pmod{5}$ because $9 \bmod 5 = 14 \bmod 5$;
- $x \equiv 3 \pmod{6}$ is the same as saying $x \bmod 6 = 3$;
- if $x \equiv 1 \pmod{3}$, then the remainder of x after division by 3 is 1. This also means that $x = 3k + 1$ for some number $k \in \mathbb{N}$.

SYSTEMS OF LINEAR CONGRUENCES

If we require that a number x satisfies more than one congruence, we call the resulting set of congruences a **system of linear congruences**.

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If $k_1, \dots, k_n, l_1, \dots, l_n \in \mathbb{N}$, then the set of congruences

$$x \equiv k_1 \pmod{l_1}$$

$$x \equiv k_2 \pmod{l_2}$$

$$\vdots$$

$$x \equiv k_n \pmod{l_n}$$

where $x \in \mathbb{N}$ is called a **system of linear congruences**.

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EUCLID'S ALGORITHM



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Concretely, let's say we want to calculate (x, y) and $x > y$. Then,

$$(x, y) = (x \bmod y, y).$$

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Let's denote $g = (x, y)$ and $r = x \bmod y$. We need to make sure that $g \mid r$ and g is the largest number that divides both r and y .

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But why?

Let's denote $g = (x, y)$ and $r = x \bmod y$. We need to make sure that $g \mid r$ and g is the largest number that divides both r and y .

When we divide x by y , we get $x = q \cdot y + r$ for some $q \in \mathbb{N}$. We know that $g \mid x$ and so I can divide the whole equation by g and get natural numbers:

$$\frac{x}{g} = \frac{q \cdot y + r}{g}.$$

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$$\frac{x}{g} = \frac{q \cdot y + r}{g}.$$

We also know that $g \mid y$ and so $q \cdot y/g \in \mathbb{N}$. We can write the right fraction like this:

$$\frac{q \cdot y + r}{g} = \frac{q \cdot y}{g} + \frac{r}{g}.$$

Because the left side is a natural number, so is the right. This means that $r/g \in \mathbb{N}$ and thus $g \mid r$.

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Because the left side is a natural number, so is the right. This means that $r/g \in \mathbb{N}$ and thus $g \mid r$.

The fact that g is the **largest** common divisor of r and y is clear because if there was a larger number $g' > g$ that divided both y and r , then it would also divide $q \cdot y + r = x$.

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It goes like this:

1. If $x \geq y$, substitute $x := x \bmod y$. Otherwise, substitute $y := y \bmod x$.
2. If $x = 0$, the answer is y . If $y = 0$, the answer is x . If $x, y \neq 0$, repeat step 1.

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CHINESE REMAINDER THEOREM

COPRIME NUMBERS

COPRIME NUMBERS

We say that two numbers $n, m \in \mathbb{N}$ are **coprime** if $(n, m) = 1$, that is, if n and n have no common divisor besides 1.

CHINESE REMAINDER THEOREM

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If $l_1, \dots, l_n \in \mathbb{N}$ are **mutually coprime** numbers and $k_1, \dots, k_n \in \mathbb{N}$ are any natural numbers, then every system

$$x \equiv k_1 \pmod{l_1}$$

$$x \equiv k_2 \pmod{l_2}$$

$$\vdots$$

$$x \equiv k_n \pmod{l_n}$$

of linear congruences has **exactly one solution** between 0 and $l_1 \cdot l_2 \cdot \dots \cdot l_n$.