Polygons

Adam Klepáč

September 18, 2023

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Polygon

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The endpoints of those segments are called vertices.

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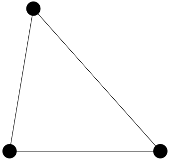
Polygon

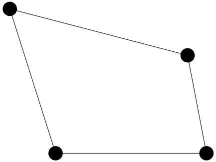
A polygon is a closed 2D shape made of only segments.

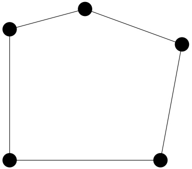
The endpoints of those segments are called vertices.

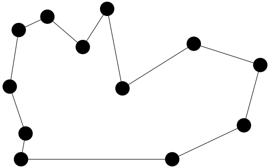
The segments themselves are called edges.

General Polygons – Examples

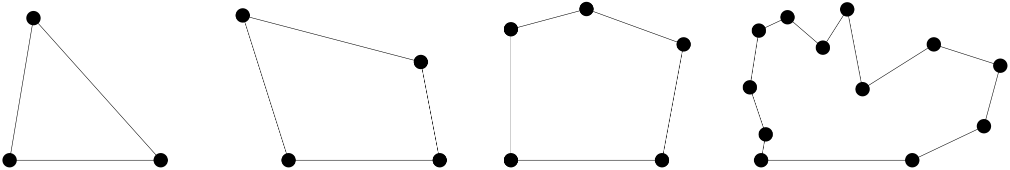

Triangle


Quadrilateral


Pentagon


Dodecagon

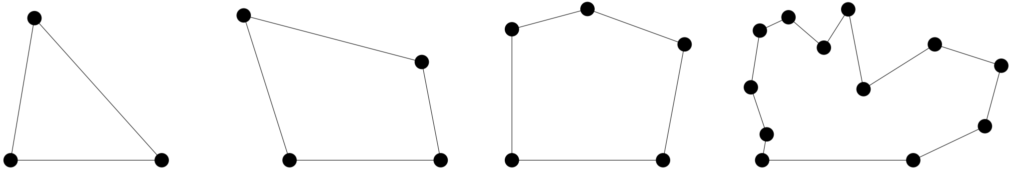
General Polygons – Examples



Triangle Quadrilateral Pentagon Dodecagon

A polygon with $n \in \mathbb{N}$ sides is called an n-gon.

General Polygons – Examples

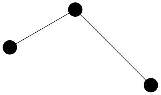


Triangle Quadrilateral Pentagon Dodecagon

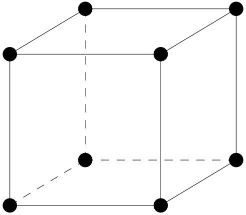
A polygon with $n \in \mathbb{N}$ sides is called an n-gon.

For example a polygon with 123456 sides is called a 123456-gon or decadismyriatrischilliatetrahectapentacontakaihexagon.

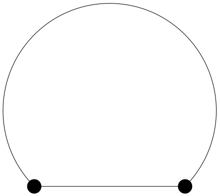
General Polygons – Counterexamples



Not closed



3D



Not straight

General Polygons – Convexity

Convex Polygon

A polygon is called **convex** if it has no internal angle greater than 180°.

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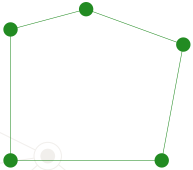
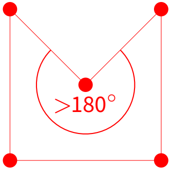


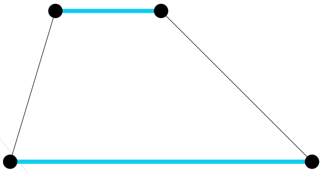
Convex



NOT convex

Convex Polygons

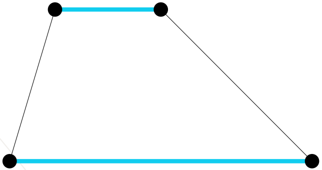
Convex Polygons – Special Types



Trapezoid/Trapezium

A convex quadrilateral with at least two parallel sides.

Convex Polygons – Special Types



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Parallelogram

A convex quadrilateral with two pairs of parallel sides.

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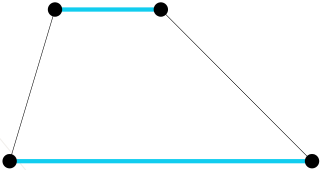

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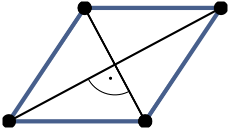


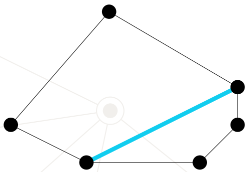
Rhombus

An **equilateral** (all sides of the same length) parallelogram.

Convex Polygons – Diagonals

Diagonal in a Convex Polygon

A **diagonal** of a **convex** polygon is a segment connecting two of its non-adjacent vertices.



Diagonal in a convex hexagon.

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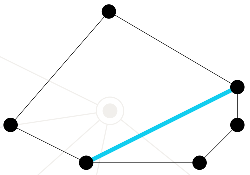

Diagonal in a convex hexagon.

Voluntary HW: How many different diagonals does a convex n-gon have?

Convex Polygons – Triangulations

Triangulation of a Convex Polygon

A triangulation of a **convex** polygon is its division into triangles by non-intersecting diagonals.

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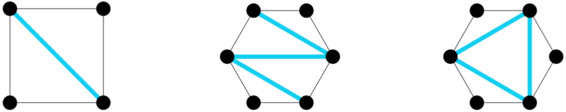

Examples of triangulations.

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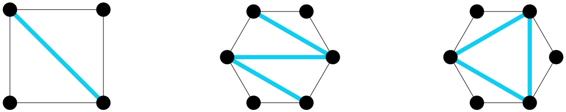

Examples of triangulations.

Voluntary HW: How many different triangulations of an n-gon are there?

Convex Polygons – Triangulations

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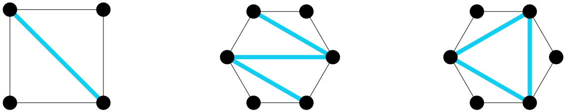
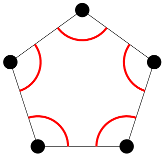


Examples of triangulations.

Voluntary HW: Find a **non-convex** polygon which **cannot** be triangulated.

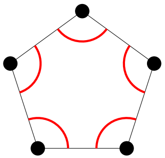
Convex Polygons – Internal Angles



Internal angles of a pentagon.

Question: What is the sum of internal angles of a convex *n*-gon?

Convex Polygons – Internal Angles



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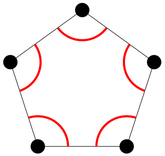
CONVEX POLYGONS – INTERNAL ANGLES



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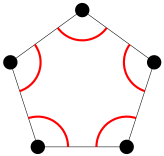

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- For a triangle, it's 180°.
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- For a pentagon, it's 540°.

Convex Polygons – Internal Angles

We can count internal angles using triangulations.

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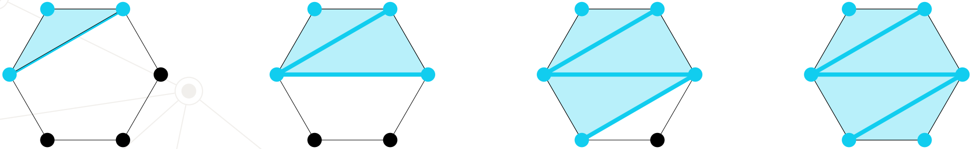

Construction of a triangulation of a hexagon.

Convex Polygons – Internal Angles

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The sum of all internal angles in a triangle is 180°.

Convex Polygons – Internal Angles

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The sum of all internal angles in a triangle is 180°.

Sum of Internal Angles in a Convex Polygon

The sum of all internal angles of a convex n-gon is $(n-2)\cdot 180^\circ$.

Regular Polygons

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Regular Polygon

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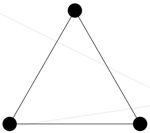
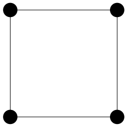
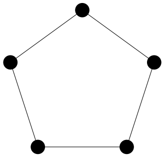


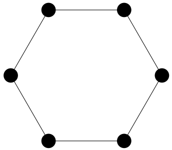
Equilateral triangle (regular trigon)



Square (regular tetragon)



Regular pentagon



Regular hexagon

Review – Plane Transformations

Rotation

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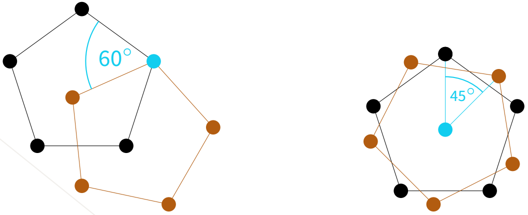

Examples of rotations.

Review – Plane Transformations

Reflection

Reflection of a polygon consists of ‘mirroring’ each of its points through a given line (called *axis of reflection*).

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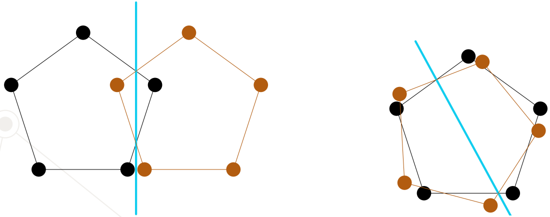

Examples of reflections.

Review – Plane Transformations

Point Symmetry

Point symmetry of a polygon consists of ‘mirroring’ each of its points through a given point (called *center of symmetry*).

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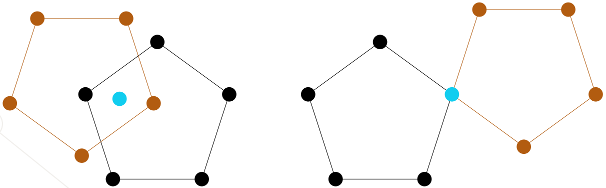

Examples of point symmetries.

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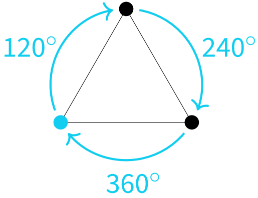
  - only through the 'centre' – the point where its axes of symmetry intersect – in case n is even

Symmetries of Regular Polygons



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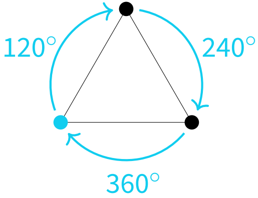

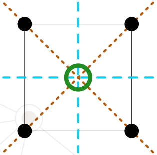

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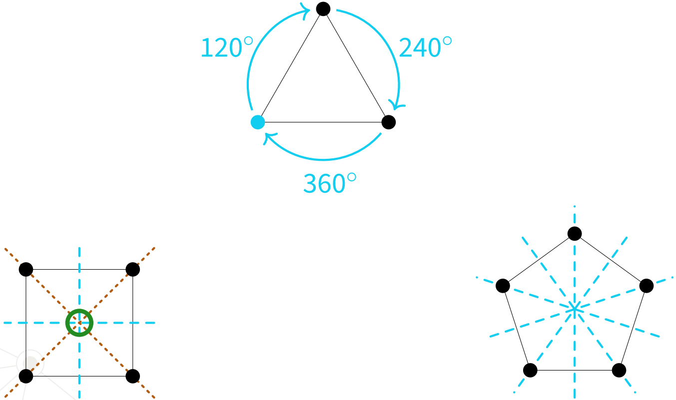

Examples of regular polygon symmetries

Cryptography on Regular Polygons

Chaining Symmetries

Given two symmetries, s_1 and s_2 of a regular polygon, one can apply them one after the other ('compose' them, like functions).

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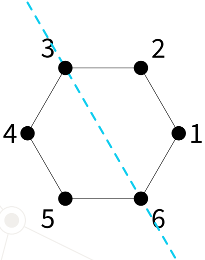
We'll denote this composition simply by $s_1 s_2$.

Chaining Symmetries – Example



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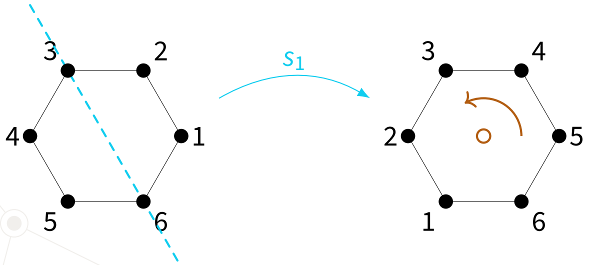

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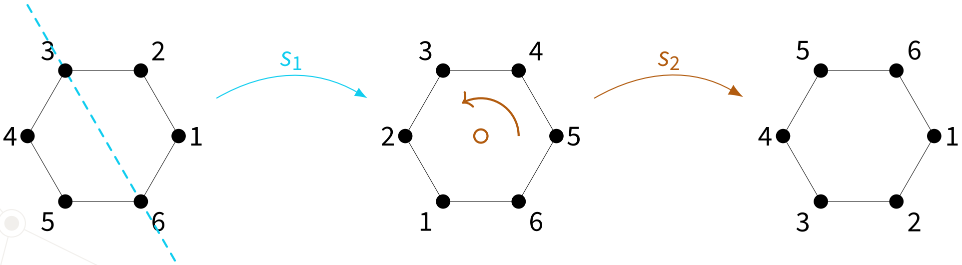

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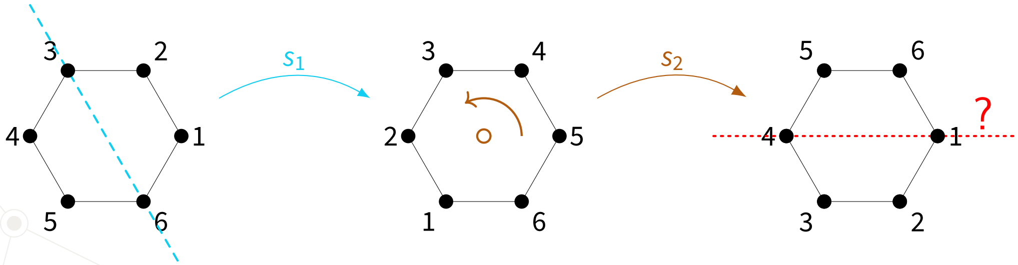

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For example,

- if s_1 is any line symmetry and s_2 is a rotation by 60° counter-clockwise, then $s_2^3 s_1$ (s_2^3 means $s_2 s_2 s_2$) reflects a hexagon through a line perpendicular to the line of s_1.

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- if s_1 is a rotation by 120° clockwise and s_2 is a reflection through a vertical line passing through the top vertex, then $s_1 s_2$ is a reflection through the line given by the rotation of the line of s_2 60° clockwise.

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