Systems of Linear Equations

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Functions

What Is A Function?

Intuitively, a function is a box which receives data and gives some data back.

input $\longrightarrow$ **function** *'do something with input'* $\longrightarrow$ output

We'll call the data that **a function receives**, inputs and the **data it gives back**, outputs. Inputs and outputs need not necessarily be just 'one object', they can be for example lists of numbers.

Functions – Example

A function which returns the **average** of a given set of numbers receives the numbers and also their count as input and returns the **average** as output.

$$x_1, x_2, \ldots, x_n \quad n \longrightarrow \boxed{\begin{array}{c}\text{average} \\ \textit{`sum all the numbers} \\ \textit{and divide them by n'}\end{array}} \longrightarrow \frac{x_1 + x_2 + \ldots + x_n}{n}$$

Functions – Example

We can also consider 'non-mathematical' functions. Like a function which receives a type of meal and returns the ingredients.

$$\text{omelette} \longrightarrow \boxed{\text{ingredients}} \longrightarrow \begin{matrix}\text{eggs}\\ \text{salt}\\ \text{butter}\end{matrix}$$

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Function Composition

Function Composition

If we have two functions, we can in certain cases 'compose' them.

Composition simply means that one function follows the other – in other words, the output of the first function is the input of the second.

Of course, composition is only possible if the output of the first function is a valid input for the second.

For instance, you could hardly compose the ingredients function with the average function.

Function Composition

Considering two functions

$$a, b \longrightarrow \boxed{\text{sum}} \longrightarrow a + b$$

$$a \longrightarrow \boxed{\text{times two}} \longrightarrow 2 \cdot a$$

their composition can look like this

$$a, b \longrightarrow \boxed{\text{sum}} \longrightarrow \boxed{\text{times two}} \longrightarrow 2 \cdot (a + b)$$

What would the output of this composition look like

$$a \longrightarrow \boxed{\text{times two}} \longrightarrow \boxed{\text{sum}} \longrightarrow ?$$

Function Composition

So, the **order of the composition matters**! Here are a few examples of compositions which make sense:

$$\text{omelette} \longrightarrow \boxed{\text{ingredients}} \longrightarrow \boxed{\text{total price}} \longrightarrow \$4.37$$

$$\begin{matrix} x_1, x_2, \ldots, x_n \\ n \end{matrix} \longrightarrow \boxed{\text{average}} \longrightarrow \boxed{\text{times two}} \longrightarrow 2 \cdot \frac{x_1 + x_2 + \ldots + x_n}{n}$$

We can of course compose **as many functions** as we like. An example of this:

$$\text{omelette} \longrightarrow \boxed{\text{ingredients}} \longrightarrow \boxed{\text{total price}} \longrightarrow \boxed{\text{times two}} \longrightarrow \$8.74$$

Functions – Notation

Drawing pictures like this would be cumbersome. Instead of

$$\text{input} \longrightarrow \boxed{\text{function}} \longrightarrow \text{output}$$

we simply write **function**(input) = output.

You're probably used to seeing function written like $f(x) = y$. The picture corresponding to this is

$$x \longrightarrow \boxed{f} \longrightarrow y$$

Functions – Notation

The symbol used for function composition is $\circ$. It is however a little confusing because it is written in order 'from right to left'.

For example, if f and g are two functions, their composition $f \circ g$ corresponds to this picture

$$x \longrightarrow \boxed{g} \longrightarrow \boxed{f} \longrightarrow y$$

that is, **first g, then f**.

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REAL FUNCTIONS

Real Function

A **real function** is simply a function whose **input and output are both real numbers**. Examples of such functions are

- $f(x) = 0$,
- $g(x) = \tan^6(\log^{\sin(x^2+4)}(\frac{5x^3-2}{9x^7}))$,

where $x \in \mathbb{R}$. Or, using pictures,

$$x \longrightarrow \boxed{f} \longrightarrow 0$$

$$x \longrightarrow \boxed{g} \longrightarrow \tan^6(\log^{\sin(x^2+4)}(\frac{5x^3-2}{9x^7}))$$

Operations On Real Functions

As both the input and the output of a real function is a real number, **we can always compose real functions**.
However, that **doesn't mean that the order doesn't matter!** Different order of composition gives different functions.
For example, take $f(x) = 2x^2 + 7$ and $g(x) = \frac{1}{1+x}$. Then,

- $(f \circ g)(x) = 2\left(\frac{1}{1+x}\right)^2 + 7$ and
- $(g \circ f)(x) = \frac{1}{1+(2x^2+7)}$.

Operations On Real Functions

Real functions can also be **added** and **multiplied**, just like real numbers.

This involves simply adding or multiplying their respective outputs. For the functions $f(x) = 2x^2 + 7$ and $g(x) = \frac{1}{1+x}$, their

- **sum** is the function with output

$$(f+g)(x) = f(x) + g(x) = 2x^2 + 7 + \frac{1}{1+x}.$$

- **product** is the function with output

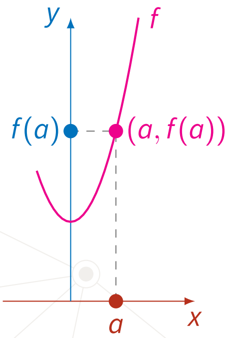
$$(f \cdot g)(x) = f(x) \cdot g(x) = (2x^2 + 7) \cdot \left(\frac{1}{1+x} \right) = \frac{2x^2+7}{1+x}.$$

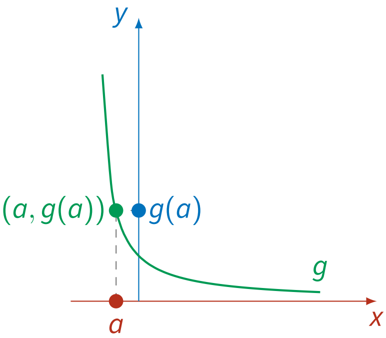
Graphs

As real functions have real numbers as inputs and outputs, they can be easily **graphed**.
Graphing a real function f simply means drawing the points $(x, f(x))$ or $(input, output)$ in some chosen coordinate system.
We typically use the **Cartesian** coordinate system with two axes (one for input and one for output) that are mutually perpendicular. These are often called the x-axis and the y-axis.
However, later, we'll also use the **polar** coordinate system where every point is instead determined by its angle and distance from the origin of the system.

Graphs

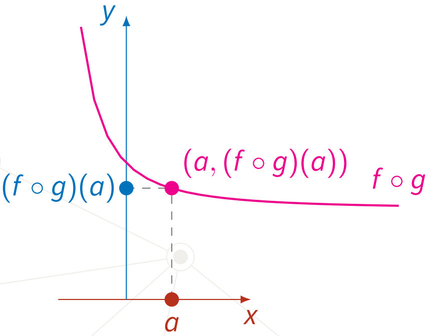
The functions $f(x) = 2x^2 + 7$ and $g(x) = \frac{1}{1+x}$ have the following (parts of) graphs:

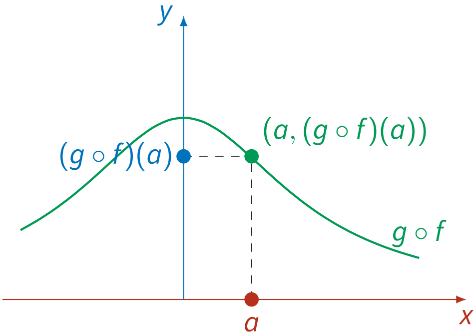




GRAPHS

Just to better drive home the idea that the order of function composition is important, look at the graphs of $f \circ g$ and $g \circ f$.





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Linear Functions

Linear Function

Linear functions are a special type of real functions whose output is always of the form $a \cdot (input) + b$ for fixed numbers $a, b \in \mathbb{R}$.
In other words, linear functions are those real functions that only scale the input and add something to it.

Linear Function

A real function f is linear if

$$f(x) = ax + b$$

for some $a, b \in \mathbb{R}$.

Linear Functions – Properties

Linear functions have some unique properties:

- If f and g are linear, so are $f \circ g$ and $g \circ f$. Indeed, we can see this easily. Suppose $f(x) = ax + b$ and $g(x) = cx + d$, then

$$(f \circ g)(x) = a(cx + d) + b = (ac)x + (ad + b),$$
$$(g \circ f)(x) = c(ax + b) + d = (ac)x + (cb + d).$$

- If f and g are linear, so is $f + g$. If we just compute the sum, we get

$$(f + g)(x) = (ax + b) + (cx + d) = (a + c)x + (b + d).$$

Linear Functions – Graphs

Graphs of linear functions are straight lines.

Let us see why this is true. Choose a linear function $f(x) = ax + b$.

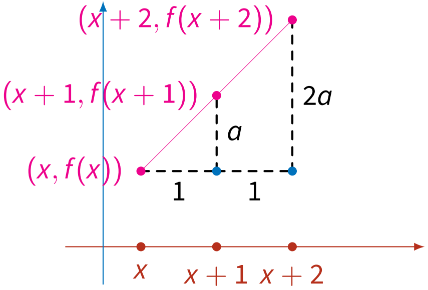
Pick a number – x. One way to show that the graph of f is a line is to move by two different distances from x and see that we get two similar triangles.

If we move by 1, from x to $x+1$, then on the y-axis we move from $ax+b$ to $a(x+1)+b$, that is, we move by

$$a(x+1)+b-(ax+b)=a.$$

If we move by 2, from x to $x+2$, on the y-axis, we move by

$$a(x+2)+b-(ax+b)=2a.$$



Linear Functions – Intersections

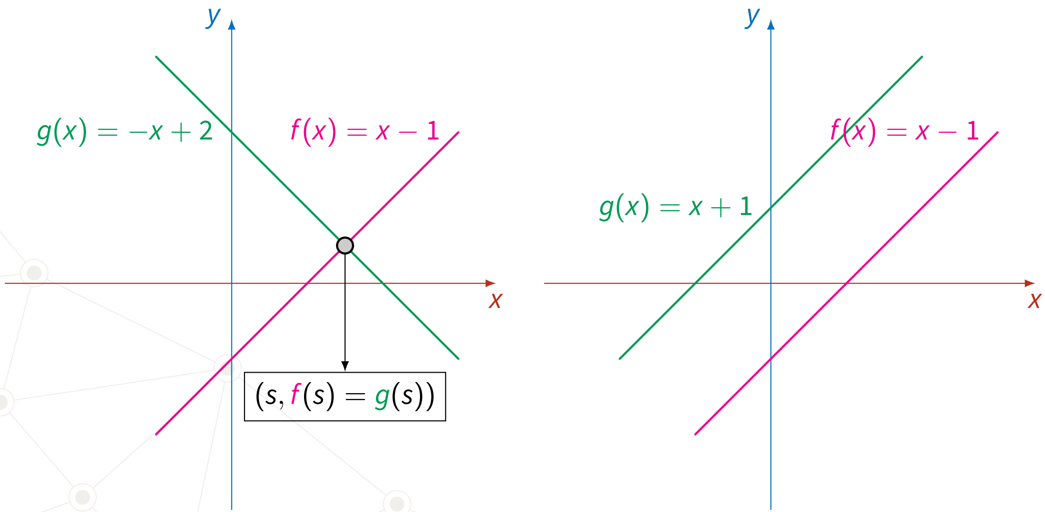
As two non-parallel lines **always intersect**, we should expect the equation $f(x) = g(x)$ to always **have a solution** assuming the graphs of f and g are not parallel.

The question is, when are the graphs of f and g parallel?

That happens exactly when their **rates growth** are identical.

In symbols, if $f(x) = ax + b$ and $g(x) = cx + d$, then the graphs of f and g are parallel if $a = c$.

Linear Functions – Intersections



Linear Equations

Linear Equation

Linear Equation

If f, g are linear functions, the equation

$$f(x) = g(x)$$

is called a **linear equation** (in one variable).

Linear Equation – Solution

Suppose $f(x) = ax + b$ and $g(x) = cx + d$. What are the possible solutions to $f(x) = g(x)$? Three things can happen:

- $a \neq c$. In this case, we can divide the equation by $a - c$ and get

$$\begin{aligned} ax + b &= cx + d \\ (a-c)x &= d - b \\ x &= \frac{d-b}{a-c}. \end{aligned}$$

- $a = c$ and $b \neq d$. In this case the graphs of the two functions are parallel lines – there is no solution.
- $a = c$ and $b = d$. In this case, the functions are one and the same and every number is a solution.

Linear Equations & Operations

For two linear functions f, g, when does

$$(f \circ g)(x) = (g \circ f)(x)$$

have a solution?

We can calculate that easily. If $f(x) = ax + b$ and $g(x) = cx + d$, then

$$(f \circ g)(x) = a(cx + d) + b = (ac)x + (ad + b),$$
$$(g \circ f)(x) = c(ax + b) + d = (ac)x + (bc + d).$$

We see that the graphs of $f \circ g$ and $g \circ f$ are parallel, so this equation has a solution only in the case that $ad + b = bc + d$.

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Linear equations in two variables

Linear Equations In Two Variables

System of linear equations

A pair of equations

$$\begin{aligned} ax + by &= c, \\ dx + ey &= f, \end{aligned}$$

that have to be **simultaneously** satisfied, is called a **system of (two) linear equations** (in two variables).

Linear Equations In Two Variables

Linear equations in two variables can be **reduced to** linear equations **in one variable**. This is easily done. We can simply **isolate** one of the variables and make the other variable **into a function**.

For example, imagine the equation

$$4x - y = 2.$$

We can rewrite this as

$$y = 4x - 2,$$

basically making y into a linear function $f(x) = 4x - 2$.

Linear Equations In Two Variables

Linear equations in two variables can be **reduced to** linear equations **in one variable**.
This is easily done. We can simply **isolate** one of the variables and make the other variable **into a function**.

For example, imagine the equation

$$4x - y = 2.$$

Similarly, we can write

$$x = \frac{1}{4}y + \frac{2}{4}$$

and turn x into a linear function $g(y) = \frac{1}{4}y + \frac{2}{4}$.

Linear Equations In Two Variables

You probably know this reduction under the name of **substitution**.

Let us see it in practice.

Suppose we want to solve the system

$$\begin{aligned} 3x + y &= 4, \\ x - 2y &= 6. \end{aligned}$$

From the first equation, we see that

$$y = 3x + 6,$$

and from the second that

$$y = \frac{1}{2}x - 3.$$

Linear Equations In Two Variables

From the first equation, we see that

$$y = 3x + 6,$$

and from the second that

$$y = \frac{1}{2}x - 3.$$

Therefore, we get the linear equation in one variable

$$3x + 6 = \frac{1}{2}x - 3.$$

whose solution is $x = -\frac{18}{5}$.

Linear Equations In Two Variables

From the first equation, we see that

$$y = 3x + 6,$$

and from the second that

$$y = \frac{1}{2}x - 3.$$

The functions $f(x) = 3x + 6$ and $g(x) = \frac{1}{2}x - 3$ are linear, so we can draw this equation as an intersection of two lines.

Linear Equations In Two Variables