Probability

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November 28, 2023

Probabilistic Intuition

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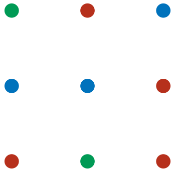

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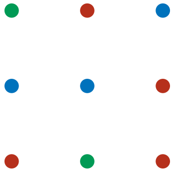

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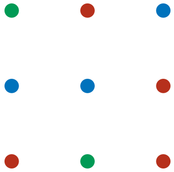

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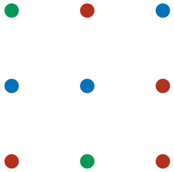

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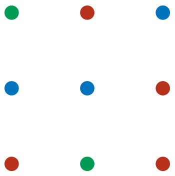
Quantifying Probability

Probability

A **probability** is a number between 0 and 1 measuring how **likely** is something to happen.

Quantifying Probability – Example

In our example of 9 balls



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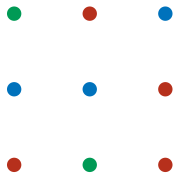

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- For blue, it's 3/9.
- For green, it's 2/9.

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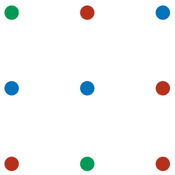

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- For red, it's $4/9$.
- For blue, it's $3/9$.
- For green, it's $2/9$.

The probabilities above **sum up to 1** because I am certain to pick *some* ball.

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So, for the 9-ball example from before, we would have

$$P(X = \text{red}) = \frac{4}{9}, \quad P(X = \text{blue}) = \frac{3}{9}, \quad P(X = \text{green}) = \frac{2}{9}.$$

Calculating Probability

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$$P(X \in S) = \frac{|S|}{|O|},$$

where S is a certain subset of O – all the possible outcomes.

Calculating Probability – Example

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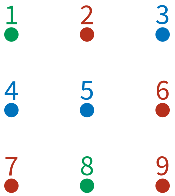
We'll assign the balls number from 1 to 9. The set of all possible outcomes of picking a random ball is then

$$O = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$



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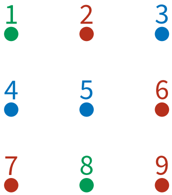

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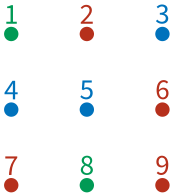

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We can use the formula from before to calculate the probability that X will be a green ball:

$$P(X \in G) = \frac{|G|}{|O|} = \frac{2}{9}.$$

Probability Equations

Sums of Probabilities

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However, this example cannot be easily generalized. We'll see why.

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$$E = \{2, 4, 6, \dots, 20\},$$
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and we want to figure out the probability $P(X \in E \cup F)$.

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Let's try to use the same formula as before:

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So, to get the size of $E \cup F$, we cannot just add the size of E to the size of F but we also have to subtract the elements that appear twice – the size of $E \cap F$.

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$$P(X \in A \cup B) = \frac{|A \cup B|}{|O|} = \frac{|A| + |B| - |A \cap B|}{|O|}.$$