Homework – PreIB 3.AB 3 & 4

Structures and Operations

Áďa Klepáčů

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DON'T FORGET TO EXPLAIN EVERYTHING EVEN IF YOU THINK IT'S OBVIOUS!

Natural Numbers

Exponentiation of two natural numbers $n, m \in \mathbb{N}$ is defined by the following two formulae:

- $n^0 = 1$,
- $n^{s(m)} = n^m \cdot n$.

Explain **very clearly** how to calculate n^m using **only** the two rules above.
Hint: This process is very similar to the definition of *addition* and *multiplication* on natural numbers.

Answer the following questions:

1. Is exponentiation *commutative*, that is, is it true that $n^m = m^n$ for all pairs of natural numbers $n, m \in \mathbb{N}$? If yes, explain why. If not, provide a counterexample.
2. Is exponentiation *associative*, that is, is it true that $n^{(m^k)} = (n^m)^k$ for all triples of natural numbers $n, m, k \in \mathbb{N}$? If yes, explain why. If not, provide a counterexample.
3. Is exponentiation an *operation* (by definition) on natural numbers? Explain.

Operations

On the set $X = \{a, b, c, d\}$, there are two operations given by the following picture.

(a) First operation. (b) Second operation.

Solve the following problems:

1. Change operations (a) and (b) **as little as possible** to make them *symmetric* (or *invertible*). By a 'change', I mean altering the source and target of a single arrow. **Explain** why your method requires the fewest changes.

2. Let's add another element e to the set X. Change operation (b) so that one arrow ends in e and one arrow starts at e so that the new operation is *symmetric*.

3. We label the operation from point 2 by $\sim$. Find an inverse to each element of X with respect to $\sim$. Recall that, in this case, an *inverse* to $x \in X$ is an element $y \in X$ such that $\tilde{y} = x$.

4. Does $\sim$ have an *identity element*? That is, is there an element $x \in X$ such that $\tilde{x} = x$?

We define a binary operation $\square$ on the set X by the following table.

$\square$	a	b	c	d
a	a	b	?	?
b	b	?	d	a
c	?	d	a	?
d	d	?	b	c

Operation $\square$.

Solve the following problems:

1. Substitute all ?'s in the table by an adequate element of X so that the resulting operation is *symmetric*.

2. Show that the operation you get in point 1 is indeed symmetric, that is, **find inverse with respect to $\square$ of each element in X** and **find the identity element**.