Probability

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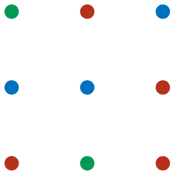
Probabilistic Intuition

WHAT IS CHANCE?

Imagine you have 9 balls of different colours.



- If you pick a ball **at random**, what colour is it most likely to be?
- How many times more likely is picking a red ball than picking a green ball?

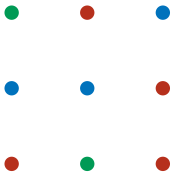
Quantifying Probability

Probability

A **probability** is a number between 0 and 1 measuring how **likely** is something to happen.

Quantifying Probability – Example

In our example of 9 balls



what is the probability of picking a ball of a specific colour?

- For red, it's 4/9.
- For blue, it's 3/9.
- For green, it's 2/9.

The probabilities above **sum up to 1** because I am certain to pick *some* ball.

Quantifying Probability – Example

We'll all the outcome of a random choice, a **random variable** and typically write it as X.

A random variable always lies in the set of all possible outcomes.

In this case, the variable X must lie in the set of possible colours, $\{\text{red}, \text{blue}, \text{green}\}$.

We'll write the probability that X is equal to one of the elements in the set as $P(X = \text{colour})$.

So, for the 9-ball example from before, we would have

$$P(X = \text{red}) = \frac{4}{9}, \quad P(X = \text{blue}) = \frac{3}{9}, \quad P(X = \text{green}) = \frac{2}{9}.$$

Calculating Probability

In the case the set of outcomes is **finite**, the probability of X being one of the possible outcomes is always

$$P(X \in S) = \frac{|S|}{|O|},$$

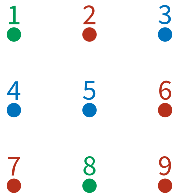
where S is a certain subset of O – all the possible outcomes.

Calculating Probability – Example

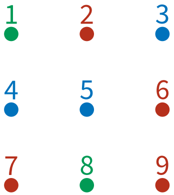
We'll describe our 9-ball example more formally.

We'll assign the balls number from 1 to 9. The set of all possible outcomes of picking a random ball is then

$$O = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$



Calculating Probability – Example



We'll form three subsets of O:

$$R = \{2, 6, 7, 9\},$$
$$B = \{3, 4, 5\},$$
$$G = \{1, 8\}.$$

We can use the formula from before to calculate the probability that X will be a green ball:

$$P(X \in G) = \frac{|G|}{|O|} = \frac{2}{9}.$$

Probability Equations

Sums of Probabilities

What if I asked about the probability that the ball I pick is red or blue?

We can literally use the same formula as before. Now, the set of outcomes we're interested in is $R \cup B$ and so

$$P(X \in R \cup B) = \frac{|R \cup B|}{|O|} = \frac{|R| + |B|}{|O|} = \frac{4+3}{9} = \frac{7}{9}.$$

However, this example cannot be easily generalized. We'll see why.

Sums of Probabilities – Counterexample

Suppose we're instead choosing from a set of numbers between 1 and 20.
We want to calculate the probability that a randomly picked number is **even or divisible by** 5.
So, we have

$$O = \{1, 2, \dots, 20\},$$
$$E = \{2, 4, 6, \dots, 20\},$$
$$F = \{5, 10, 15, 20\}.$$

and we want to figure out the probability $P(X \in E \cup F)$.

Sums of Probabilities – Counterexample

Let's try to use the same formula as before:

$$P(X \in E \cup F) = \frac{|E \cup F|}{|O|} \stackrel{??}{=} \frac{|E| + |F|}{|O|} = \frac{10 + 4}{20} = \frac{14}{20}.$$

This doesn't quite add up.

If we count such numbers by hand, we get the set

$$\{2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18, 20\}.$$

There's **only 12 of them.**

The problem is that **we counted the numbers 10 and 20 twice**!

So, to get the size of $E \cup F$, we cannot just add the size of E to the size of F but we also have to subtract the elements that appear twice – the size of $E \cap F$.

Sums of Probabilities – Formula

The previous example applies in general. If A, B are two subsets of the set of outcomes, O, then

$$P(X \in A \cup B) = \frac{|A \cup B|}{|O|} = \frac{|A| + |B| - |A \cap B|}{|O|}.$$

Sums of Probabilities – Formula

We have a formula for two sets but how about three sets? Four sets? Million sets?

We need a **general formula** to calculate the size

$$|A_1 \cup A_2 \cup \ldots \cup A_n|$$

where $A_1, A_2, \ldots, A_n$ are any sets.

Such a formula is widely known as the **principle of inclusion and exclusion**.

Principle of Inclusion and Exclusion – Example

Let's consider the following setup: There are three language groups – English, French and German.

- 40 people speak English, 23 speak German and 11 speak French.
- 10 people speak both English and German, 5 speak both English and French and only 3 speak both German and French.
- Finally, just one person speaks all three languages.

How many people speak at least one language?

Principle of Inclusion and Exclusion – Example

Let's tackle this formally.

Label the three language groups E, F and G. The setup from the previous slide can be summarized as

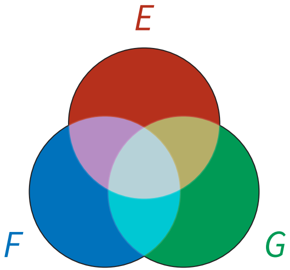
$\lvert E\rvert$	$\lvert F\rvert$	$\lvert G\rvert$	$\lvert E \cap F\rvert$	$\lvert E \cap G\rvert$	$\lvert F \cap G\rvert$	$\lvert E \cap F \cap G\rvert$
40	11	23	5	10	3	1

We're trying to calculate $\lvert E \cup F \cup G\rvert$.

PRINCIPLE OF INCLUSION AND EXCLUSION – EXAMPLE

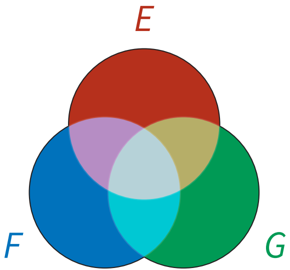
Let's picture the problem first.

When working with sets, Venn diagrams are often a great choice.



There are 7 regions in total (differentiated by colour) in this picture, corresponding to the 7 sets in the previous slide.

Principle of Inclusion and Exclusion – Example



What we need to count is the total number of elements inside this entire shape. Let's start by counting the number of elements in each of the regions separately and assign numbers to regions corresponding to **how many times we've counted all the elements in that region.**

Principle of Inclusion and Exclusion – Example