



NUMBER SETS

Adam Klepáč

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They're the following set:

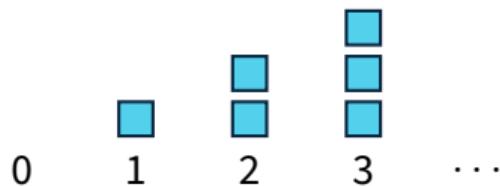
$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

NATURAL NUMBERS – INTUITION

Natural numbers are intuitively objects which represent a **quantity**.
They're the following set:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



NATURAL NUMBERS – DEFINITION

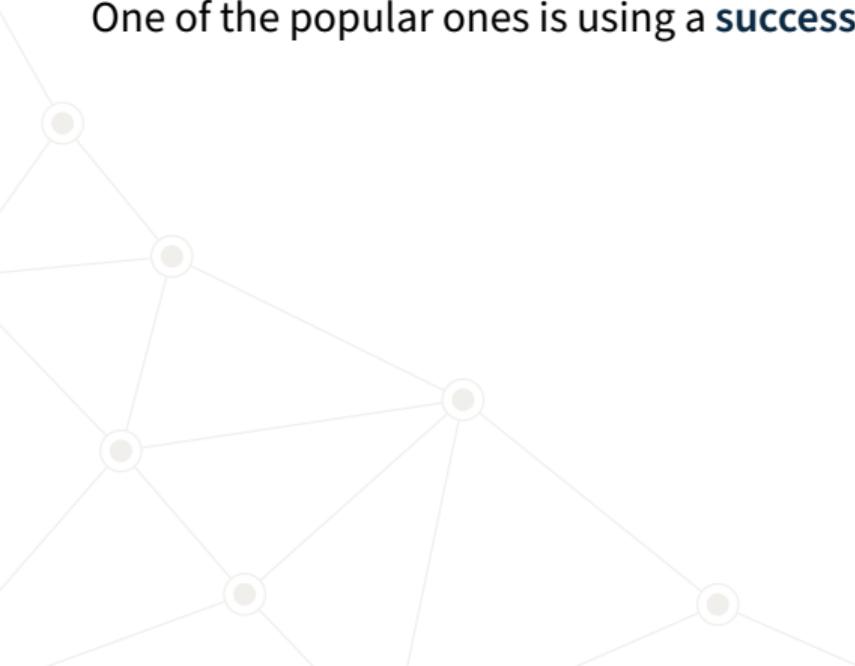
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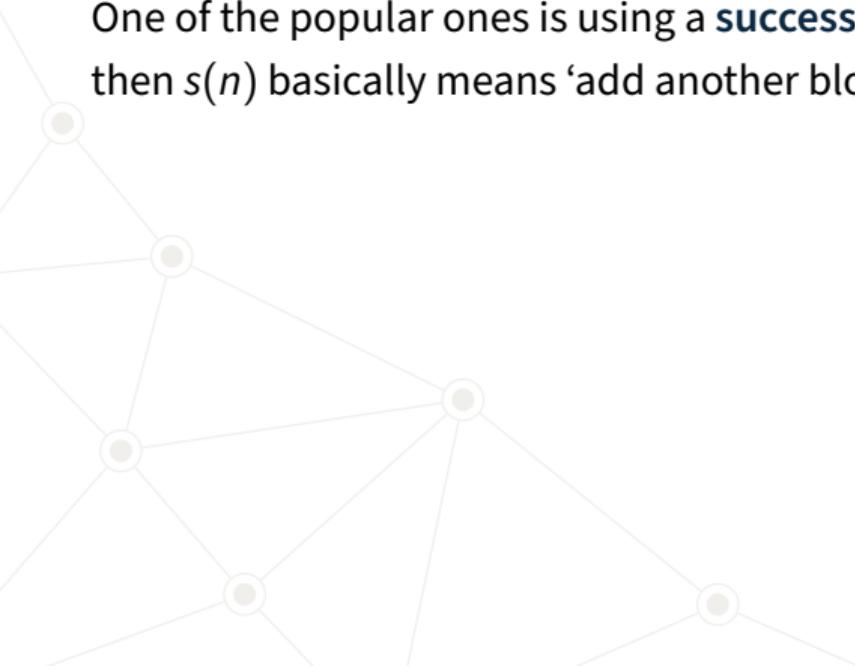
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One of the popular ones is using a **successor** function, denoted s . If n is a natural number, then $s(n)$ basically means ‘add another block on top of n ’.

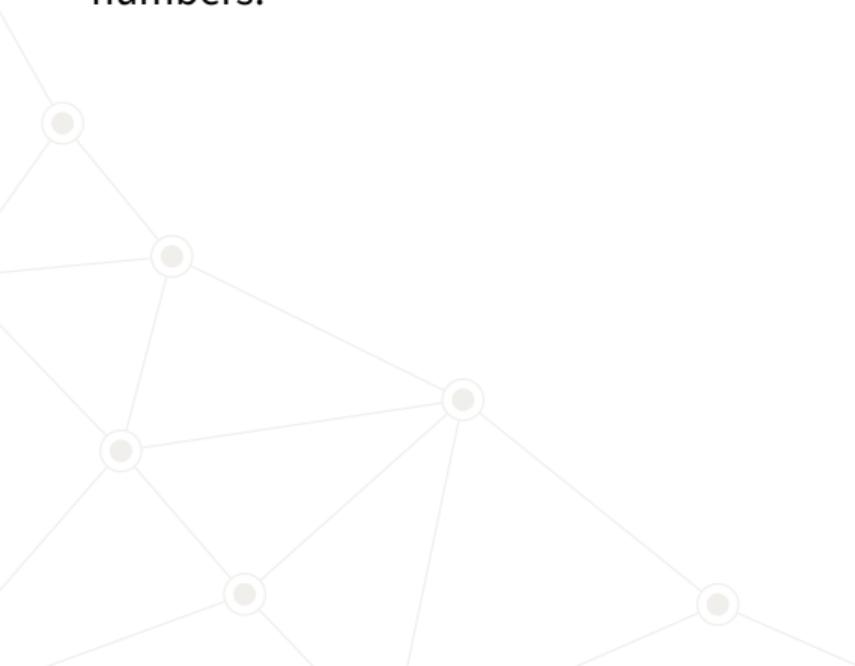
One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet!** In fact, you need the successor function to define addition in the first place.

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4. If $s(n) = s(m)$, then $n = m$.
5. (Induction Axiom) If a statement is true for 0 and it being true for n also implies that it is true for $n + 1$, then it is true for all natural numbers.

1

UNPACKING THE AXIOMS



NATURAL NUMBERS – AXIOM 1

There exists the natural number 0.

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There exists the natural number 0.

Hopefully obvious.

NATURAL NUMBERS – AXIOM 2

Every natural number has a successor which is also natural.

NATURAL NUMBERS – AXIOM 2

Every natural number has a successor which is also natural.

Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

NATURAL NUMBERS – AXIOM 3

The number 0 is not the successor of any natural number.

NATURAL NUMBERS – AXIOM 3

The number 0 is not the successor of any natural number.

Basically means that the natural numbers are infinite only ‘in one direction’. There is a **first** natural number.

NATURAL NUMBERS – AXIOM 4

If $s(n) = s(m)$, then $n = m$.

NATURAL NUMBERS – AXIOM 4

If $s(n) = s(m)$, then $n = m$.

This means that the successor function is **injective** – each natural number has a different successor.

NATURAL NUMBERS – AXIOM 5

If a statement is true for 0 and it being true for n also implies that it is true for $n + 1$, then it is true for all natural numbers.

NATURAL NUMBERS – AXIOM 5

If a statement is true for 0 and it being true for n also implies that it is true for $n + 1$, then it is true for all natural numbers.

This means that any feature of the natural numbers ‘propagates’ via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it is true for 1, it is true for 2 as well, etc.

2

OPERATIONS ON NATURAL NUMBERS



WHAT IS AN OPERATION?

By **operation**, we mean a function which takes **one or multiple** natural numbers and produces **one** natural number.



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For example, $+$ and \cdot are operations because they take **two** natural numbers and produce **one**.

We don't often see them as functions because we don't write them as such. We write $n + m$ instead of $+(n, m)$ and $n \cdot m$ instead of $\cdot(n, m)$.

In this sense, subtraction and division **are not operations!** They take two natural numbers but they **do not produce a natural number**.

ADDITION

We define **addition** on natural numbers by the following two formulae:

- $n + 0 = n$,
- $n + s(m) = s(n + m)$.

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We can imagine addition as ‘adding blocks *to the side*’ and the successor function as ‘adding one block *on top*’.

In this sense, $n + s(m) = s(n + m)$ only means that if you add one block atop m blocks and then n blocks to the side you have the same number of blocks as if you add n blocks next to m blocks and then another on top of that.

ADDITION



ADDITION


$$2 + s(3)$$

ADDITION

$$\begin{matrix} \text{dark blue} \\ \text{dark blue} \end{matrix} \quad \begin{matrix} \text{light blue} \\ \text{light blue} \\ \text{light blue} \end{matrix}$$

2 3

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$2 + s(3)$

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$s(2 + 3)$

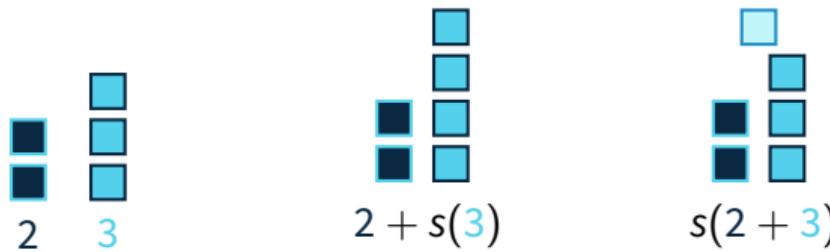
ADDITION

The diagram shows three stages of adding two sets of blocks. Stage 1: Two groups of blocks, labeled 2 and 3. Group 2 has 2 dark blue blocks. Group 3 has 3 light blue blocks. Stage 2: The expression $2 + s(3)$. Group 2 has 2 dark blue blocks. Group 3 has 3 light blue blocks, with one additional light blue block placed above the others. Stage 3: The expression $s(2 + 3)$. Group 2 has 2 dark blue blocks. Group 3 has 3 light blue blocks, with one additional light blue block placed above the others, and one additional light blue block placed to the right of the group.

$$\begin{array}{cc} \begin{array}{c} \text{dark blue} \\ \text{light blue} \end{array} & \begin{array}{c} \text{light blue} \\ \text{light blue} \\ \text{light blue} \end{array} \\ 2 & 3 \end{array}$$
$$2 + s(3)$$
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Using the formula $n + s(m) = s(n + m)$, one calculates $n + m$ by taking the successor of n , m times.

ADDITION



Using the formula $n + s(m) = s(n + m)$, one calculates $n + m$ by taking the successor of n , m times. Like this:

$$n + 0 = n,$$

$$n + 1 = n + s(0) = s(n + 0) = s(n),$$

$$n + 2 = n + s(1) = s(n + 1) = s(n + s(0)) = s(s(n + 0)) = s(s(n)),$$

⋮

ADDITION – PROPERTIES

Addition of natural numbers satisfies these two properties:

- **Commutativity:**

$$n + m = m + n.$$

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- **Commutativity:**

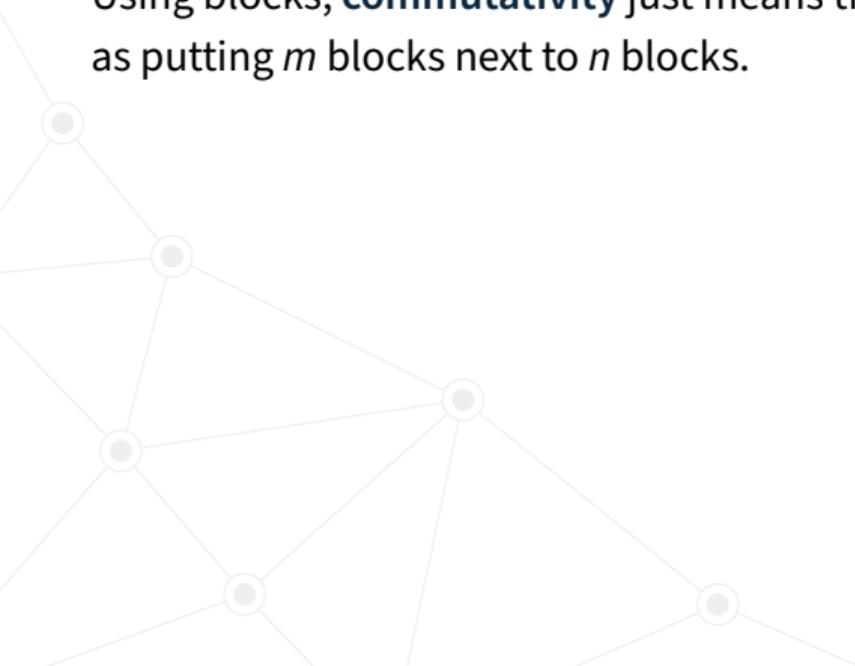
$$n + m = m + n.$$

- **Associativity:**

$$n + (m + k) = (n + m) + k.$$

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Using blocks, **commutativity** just means that putting n blocks next to m blocks is the same as putting m blocks next to n blocks.



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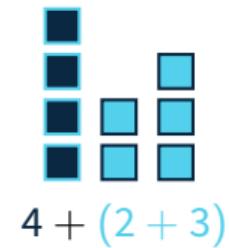
$$2 + 3$$



$$3 + 2$$

ADDITION – PROPERTIES

Using blocks, **associativity** just means that putting m blocks next to k blocks and then n more blocks next to those is the same as putting m blocks next to n blocks and then k more blocks next to those.


$$4 + (2 + 3)$$

ADDITION – PROPERTIES

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$$4 + (2 + 3)$$


$$(4 + 2) + 3$$

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- $m \cdot 1 = m$,
- $m \cdot s(n) = m \cdot n + n$.

We can imagine multiplication $m \cdot n$ by adding a collections of n blocks for every one block in the collection of m blocks.

If we write $s(n) = n + 1$, then the second formula just means that

$$m \cdot s(n) = m \cdot (n + 1) = m \cdot n + n.$$

MULTIPLICATION

$$\begin{array}{c} \blacksquare \\ \blacksquare \end{array} \quad \begin{array}{c} \square \\ \square \\ \square \end{array}$$

2 3

A diagram illustrating multiplication. On the left, there is a network of five nodes connected by lines. To the right of this network, two sets of colored squares are shown vertically. The first set, labeled '2', contains two dark blue squares. The second set, labeled '3', contains three light blue squares. This visual representation likely corresponds to the multiplication problem $2 \times 3 = 6$.

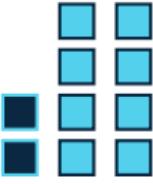
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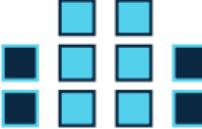
$$\begin{matrix} & \text{blue} \\ & \text{blue} \\ 2 & \end{matrix} \quad \begin{matrix} & \text{blue} \\ & \text{blue} \\ & \text{blue} \\ 3 & \end{matrix}$$

$$\begin{matrix} & & \text{blue} & \text{blue} \\ 2 \cdot s(3) & & \text{blue} & \text{blue} \end{matrix}$$

MULTIPLICATION


$$2 \quad 3$$


$$2 \cdot s(3)$$


$$2 \cdot 3 + 2$$

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$2 \cdot s(3)$

$$\begin{matrix} \text{■} & \text{□} & \text{□} \\ \text{■} & \text{□} & \text{□} \\ \text{□} & \text{□} & \text{□} \end{matrix}$$

$2 \cdot 3 + 2$

The formula $m \cdot s(n) = m \cdot n + n$ allows us to compute $m \cdot n$ by applying it n times.

MULTIPLICATION

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$$2 \cdot s(3)$$


$$2 \cdot 3 + 2$$


The formula $m \cdot s(n) = m \cdot n + n$ allows us to compute $m \cdot n$ by applying it n times. More precisely,

$$m \cdot 1 = m$$

$$m \cdot 2 = m \cdot s(1) = m \cdot 1 + m = m + m$$

$$m \cdot 3 = m \cdot s(2) = m \cdot 2 + m = m \cdot s(1) + m = m \cdot 1 + m + m = m + m + m$$

⋮

MULTIPLICATION – PROPERTIES

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$$m \cdot n = n \cdot m.$$



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- Commutativity:

$$m \cdot n = n \cdot m.$$

- Associativity:

$$m \cdot (n \cdot k) = (m \cdot n) \cdot k.$$

- Distributivity:

$$m \cdot (n + k) = m \cdot n + m \cdot k.$$