Logic & Set Theory Cheatsheet

3.AB PreIB Math

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Logic

Logic is the language of mathematics. It uses propositions to talk about sets.

Propositions are sentences which can be either true or false. For example

- '**Cats are black.**' is a proposition;
- '**How are you?**' is *not* a proposition;
- '**We will have colonised Mars by 2500.** is also a proposition.

As the third example suggests, we need not necessarily know whether a proposition is true or false – it remains a proposition anyway.

Logical Conjunctions

Propositions can be joined together using logical conjunctions. They pretty much correspond to the conjunctions of natural language. Let us consider two propositions:

$$p = \text{'It's raining outside.'}$$
$$q = \text{'I'll stay at home.'}$$

($\wedge$) Logical and forms a proposition that is only true if both of its constituents are also true. In natural language, the proposition $p \wedge q$ can be expressed as

$$p \wedge q = \text{'It's raining outside and I'll stay at home.'}$$

($\vee$) Logical or forms a proposition that is true if at least one of its constituents is true. In natural language, the proposition $p \vee q$ can be expressed as

$$p \vee q = \text{'It's raining outside or I'll stay at home.'}$$

In mathematical logic, or is **not exclusive**! This means that $p \vee q$ is true even if both p and q are true.

($\neg$) Logical not isn't strictly speaking a conjunction but I include it anyway. It reverses the truth value of a proposition. For example, the proposition $\neg p$ can be read as

$$\neg p = \text{'It's not raining outside.'}$$

It follows that $\neg p$ is true exactly when p is false and vice versa.

($\Rightarrow$) Logical implication is a conjunction that makes the first proposition into an *assumption* or *premise* and the second one into a *conclusion*. The proposition $p \Rightarrow q$ is read in multiple ways, to list a few:

$$p \Rightarrow q = \text{'If it's raining outside, then I'll stay at home.'}$$
$$p \Rightarrow q = \text{'It raining outside implies that I'll stay at home.'}$$
$$p \Rightarrow q = \text{'Assuming it's raining outside, I'll stay at home.'}$$

The implication is tricky. It's true if both p and q are true and false if p is true but q is false. However, it is always true if p is false. That is because, in mathematical logic, whatever follows from a lie is automatically true.

($\Leftrightarrow$) Logical equivalence is true only if both propositions have the same truth value – they're both true or both false. In natural language, it is typically read like this:

$$p \Leftrightarrow q = \text{'It's raining if and only if I stay at home.'}$$

Equivalence is basically just a two-way implication. The proposition p is both a premise and a conclusion to q and q is both a premise and a conclusion to p. If it's raining outside, I stay at home and if I stay at home, then it's raining outside.