



NUMBER SETS

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CONTENTS

Natural Numbers

Unpacking The Axioms

Operations On Natural Numbers

Integers

Digression



Mathematical Structures

Destructive & Symmetric Transformations

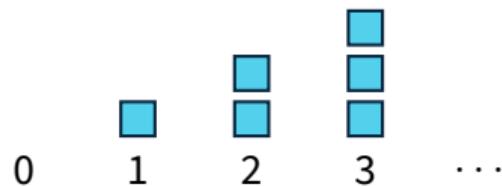
NATURAL NUMBERS

NATURAL NUMBERS – INTUITION

Natural numbers are intuitively objects which represent a **quantity**.
They're the following set:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



NATURAL NUMBERS – DEFINITION

There are many ways to define natural numbers.

One of the popular ones is using a **successor** function, denoted s . If n is a natural number, then $s(n)$ basically means ‘add another block on top of n ’.

One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet!** In fact, you need the successor function to define addition in the first place.

NATURAL NUMBERS – DEFINITION

The following **five axioms** (often called *Peano axioms*) constitute the definition of natural numbers:

1. There exists the natural number 0.
2. Every natural number has a successor which is also natural.
3. The number 0 is not the successor of any natural number.
4. If $s(n) = s(m)$, then $n = m$.
5. (Induction Axiom) If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

1

UNPACKING THE AXIOMS



NATURAL NUMBERS – AXIOM 1

There exists the natural number 0.

Hopefully obvious.

NATURAL NUMBERS – AXIOM 2

Every natural number has a successor which is also natural.

Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

NATURAL NUMBERS – AXIOM 3

The number 0 is not the successor of any natural number.

Basically means that the natural numbers are infinite only ‘in one direction’. There is a **first** natural number.

NATURAL NUMBERS – AXIOM 4

If $s(n) = s(m)$, then $n = m$.

This means that the successor function is **injective** – each natural number has a different successor.

NATURAL NUMBERS – AXIOM 5

If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

This means that any feature of the natural numbers ‘propagates’ via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it is true for 1, it is true for 2 as well, etc.

2

OPERATIONS ON NATURAL NUMBERS



WHAT IS AN OPERATION?

By **operation**, we mean a function which takes **one or multiple** natural numbers and produces **one** natural number.

For example, $+$ and \cdot are operations because they take **two** natural numbers and produce **one**.

We don't often see them as functions because we don't write them as such. We write $n + m$ instead of $+(n, m)$ and $n \cdot m$ instead of $\cdot(n, m)$.

In this sense, subtraction and division **are not operations!** They take two natural numbers but they **do not produce a natural number**.

ADDITION

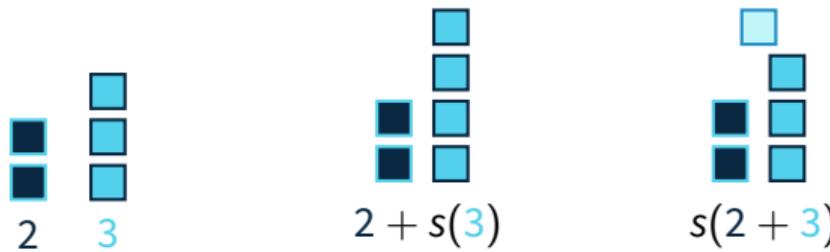
We define **addition** on natural numbers by the following two formulae:

- $n + 0 = n$,
- $n + s(m) = s(n + m)$.

We can imagine addition as ‘adding blocks *to the side*’ and the successor function as ‘adding one block *on top*’.

In this sense, $n + s(m) = s(n + m)$ only means that if you add one block atop m blocks and then n blocks to the side you have the same number of blocks as if you add n blocks next to m blocks and then another on top of that.

ADDITION



Using the formula $n + s(m) = s(n + m)$, one calculates $n + m$ by taking the successor of n , m times. Like this:

$$n + 0 = n,$$

$$n + 1 = n + s(0) = s(n + 0) = s(n),$$

$$n + 2 = n + s(1) = s(n + 1) = s(n + s(0)) = s(s(n + 0)) = s(s(n)),$$

⋮

ADDITION – PROPERTIES

Addition of natural numbers satisfies these two properties:

- **Commutativity:**

$$n + m = m + n.$$

- **Associativity:**

$$n + (m + k) = (n + m) + k.$$

ADDITION – PROPERTIES

Using blocks, **commutativity** just means that putting n blocks next to m blocks is the same as putting m blocks next to n blocks.



$$2 + 3$$



$$3 + 2$$

ADDITION – PROPERTIES

Using blocks, **associativity** just means that putting m blocks next to k blocks and then n more blocks next to those is the same as putting m blocks next to n blocks and then k more blocks next to those.


$$4 + (2 + 3)$$


$$(4 + 2) + 3$$

MULTIPLICATION

We define **multiplication** on natural numbers by the following formulae:

- $m \cdot 1 = m$,
- $m \cdot s(n) = m \cdot n + m$.

We can imagine multiplication $m \cdot n$ by adding a collection of n blocks for every one block in the collection of m blocks.

If we write $s(n) = n + 1$, then the second formula just means that

$$m \cdot s(n) = m \cdot (n + 1) = m \cdot n + m.$$

MULTIPLICATION

$$\begin{matrix} \text{■} \\ \text{■} \end{matrix} \quad \begin{matrix} \text{□} \\ \text{□} \\ \text{□} \end{matrix}$$

2 3

$$\begin{matrix} \text{□} & \text{□} \\ \text{□} & \text{□} \\ \text{□} & \text{□} \\ \text{□} & \text{□} \end{matrix}$$

$2 \cdot s(3)$

$$\begin{matrix} \text{□} & \text{□} \\ \text{□} & \text{□} \\ \text{□} & \text{□} \\ \text{□} & \text{□} \end{matrix} \quad \text{■}$$

$2 \cdot 3 + 2$

The formula $m \cdot s(n) = m \cdot n + m$ allows us to compute $m \cdot n$ by applying it n times. More precisely,

$$m \cdot 1 = m$$

$$m \cdot 2 = m \cdot s(1) = m \cdot 1 + m = m + m$$

$$m \cdot 3 = m \cdot s(2) = m \cdot 2 + m = m \cdot s(1) + m = m \cdot 1 + m + m = m + m + m$$

⋮

MULTIPLICATION – PROPERTIES

- Commutativity:

$$m \cdot n = n \cdot m.$$

- Associativity:

$$m \cdot (n \cdot k) = (m \cdot n) \cdot k.$$

- Distributivity:

$$m \cdot (n + k) = m \cdot n + m \cdot k.$$



INTEGERS

INTEGERS

Integers or whole numbers come about when we say we can not only add blocks on top of collections but we can also **remove** blocks.

In mathematical terms, this means introducing **inverses** to numbers **with respect to addition**.

We typically denote these inverses by the symbol $-$.

Beware! ‘Taking inverse’ is a **unary** operation (meaning it acts on **one element**), not a binary one.

1

DIGRESSION



MATHEMATICAL STRUCTURES

DESTRUCTIVE & SYMMETRIC TRANSFORMATIONS



MATHEMATICAL STRUCTURES

MATHEMATICAL STRUCTURE (ALSO ‘UNIVERSE’)

A **mathematical** structure is a **set** with **operations**. If X is the set and op_1, \dots, op_n the operations on X , we write the resulting structure as

$$(X, op_1, \dots, op_n).$$

The reason mathematical structures are called by some enthusiasts as mathematical ‘universes’ is that they really *are* universes in the broadest sense possible – a bunch of elements with prescribed rules of interaction.

OPERATION ON A SET

OPERATION

An **operation** on a set X is really just a **rule of interaction** between its elements. In symbols, it is a **function**

$$op : X^n \rightarrow X$$

where $X^n \rightarrow X$ just means ‘Take n elements of X and give me back one.’

OPERATION ON A SET

Examples:

- $(\mathbb{N}, +, \cdot)$ is a structure where $+$ and \cdot are **binary** operations (meaning they take two elements and return one). They can be seen as functions $\mathbb{N}^2 \rightarrow \mathbb{N}$.
- $(\{\text{orderings of vertices of a regular } n\text{-gon}\}, r, r^2, \dots, r^n, s_1, \dots, s_n)$ is a structure where r is the rotation by $360^\circ/n$ and s_1, \dots, s_n are all the reflections. They are all **unary** operations (they take one element and return one).

OPERATION AS A TRANSFORMATION

Operations basically describe interactions between set elements – two or more elements combine or **transform** into another one.

This interaction can be **destructive** or **symmetric**.

- The operations $+$ and \cdot on \mathbb{N} are **destructive** – they *destroy* the elements. When I multiply $3 \cdot 5 = 15$, I have no way to get back the 3 or the 5.
- The operation r on the vertices of a regular polygon is **symmetric** – it can be reversed or **inverted**. If, for example, vertex A is sent to G by this rotation, then rotation in the opposite direction sends G back to A .

OPERATION AS A TRANSFORMATION

All destructive operations on a set X **can be made symmetric** by making the set larger and introducing **inverses** and **identity elements** with respect to the given operation.

Let's pick an commutative and associative operation Δ on a set X and two elements $x, y \in X$.

- Suppose $x \Delta y$ is some element $z \in X$. For the operation Δ to be **symmetric**, we need two things:
 - A special element $e \in X$ (the **identity** element) which satisfies

$$x \Delta e = x$$

for all elements $x \in X$.

- Elements x^* and y^* (the **inverse** elements to x and y) such that

$$x \Delta x^* = y \Delta y^* = e.$$

OPERATION AS A TRANSFORMATION

Let's pick an (commutative) operation Δ on a set X and two elements $x, y \in X$.

- Suppose $x \Delta y$ is some element $z \in X$. For the operation Δ to be **symmetric**, we need two things:
 - A special element $e \in X$ (the **identity** element) which satisfies

$$x \Delta e = x$$

for all elements $x \in X$.

- Elements x^* and y^* (the **inverse** elements to x and y) such that

$$x \Delta x^* = y \Delta y^* = e.$$

- It's possible X doesn't have those things – which means the operation Δ cannot be symmetric **on X** , but it can be symmetric on a larger set which does contain those elements.

OPERATION AS A TRANSFORMATION

For an operation to be invertible, it really just means its **effects can be reversed**, that there's always a way back.

If x^*, y^* are the inverse to x, y with respect to Δ , then we can **recover** x and y after performing the operation Δ .

Suppose $x \Delta y = z$. Then, $z \Delta y^* = x$ and $z \Delta x^* = y$.

Why? We know that $x \Delta x^* = e = y \Delta y^*$. Therefore, for example,

$$z \Delta y^* = (x \Delta y) \Delta y^* = x \Delta (y \Delta y^*) = x \Delta e = x.$$

INVERSE AS A FUNCTION

Inverse can be thought of as a **unary** operation – it takes an element x and gives back its inverse x^* .

Clearly, the inverse to x^* is x , again.

DRAWING STRUCTURES

Let's draw some mathematical structures. Remember that a mathematical structure is just a set with a bunch of operations. If our set X is finite, we can draw its elements as dots. The simplest way to 'draw' a mathematical structure is using a table.

Imagine our set has three points a , b and c . We can define a **unary** operation ${}^{\wedge} : X \rightarrow X$ (meaning it takes one element) on X by writing what it does into a table:

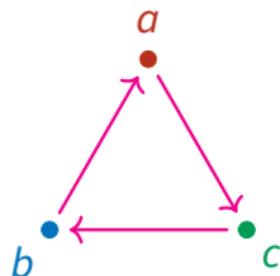
\hat{a}	\hat{b}	\hat{c}
<hr/>		
c	a	b

DRAWING STRUCTURES

The table

\hat{a}	\hat{b}	\hat{c}
<hr/>		
c	a	b

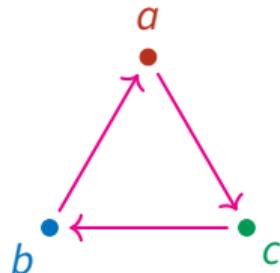
from the previous slide can be also drawn using dots and arrows for example like this:



where the magenta arrows denote the operation \wedge .

DRAWING STRUCTURES

Let's think what could be an **inverse** to the operation \wedge .



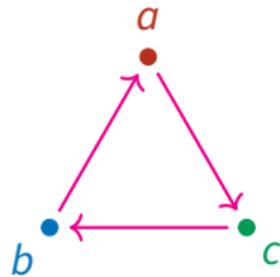
What the operation \wedge does is basically changing a dot into another dot following the **arrows**.

As I have three dots, this means, that after traversing one **arrow**, I have to **traverse two arrows** to get to where I started.

However, traversing two **arrows** just means **applying the operation \wedge twice**.

DRAWING STRUCTURES

Let's think what could be an **inverse** to the operation \wedge .



What this means in symbols is that if $\hat{c} = b$, then $\hat{b} = c$.

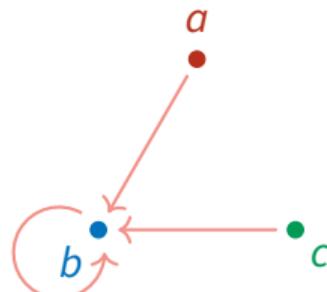
So, to get an inverse element with respect to \wedge , we just need to apply \wedge twice.

DRAWING STRUCTURES – EXAMPLES

Imagine a similar unary operation $\tilde{\cdot}$. It is given by the following table

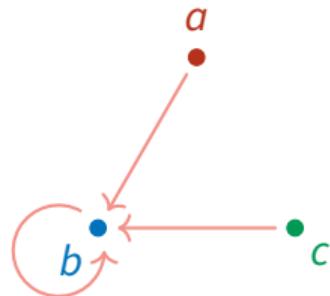
\tilde{a}	\tilde{b}	\tilde{c}
<hr/>		
b	b	b

and can be pictured like this



How many elements do we need to add to make \sim symmetric?

DRAWING STRUCTURES – EXAMPLES



How many elements do we need to add to make \sim symmetric?

The answer is **two elements** – that is because we need to be able to return from **b** to **a** and to **c** and we cannot do it using only one element because one element cannot be transformed into two different.

OPERATION AS A TRANSFORMATION

Motivating examples:

- The operation $+$ on \mathbb{N} **cannot be made symmetric**. Natural numbers do not contain inverses to $+$. We fix this by introducing **integers**, basically forcefully adding to each number n its inverse with respect to $+$, that is, the negative number $-n$. The new set is called the **integers**, denoted \mathbb{Z} . The identity element is 0 in this case.
- The operation \cdot on \mathbb{Z} cannot be made symmetric. We fix this by introducing **fractions/reciprocals** to elements. The inverse with respect to \cdot of an element z is $\frac{1}{z}$. The identity element is 1.