Number Sets

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Rational Numbers

Natural Numbers

Natural Numbers – Intuition

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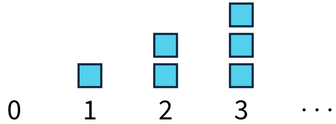
A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



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One of the popular ones is using a **successor** function, denoted s. If n is a natural number, then $s(n)$ basically means 'add another block on top of n'.

One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet**! In fact, you need the successor function to define addition in the first place.

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1. There exists the natural number 0.
2. Every natural number has a successor which is also natural.
3. The number 0 is not the successor of any natural number.
4. If $s(n) = s(m)$, then $n = m$.
5. (Induction Axiom) If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

1

Unpacking The Axioms

Natural Numbers – Axiom 1

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Hopefully obvious.

Natural Numbers – Axiom 2

Every natural number has a successor which is also natural.

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Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

Natural Numbers – Axiom 3

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Basically means that the natural numbers are infinite only ‘in one direction’. There is a **first** natural number.

Natural Numbers – Axiom 4

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$$\text{If } s(n) = s(m), \text{ then } n = m.$$

This means that the successor function is **injective** – each natural number has a different successor.

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If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

This means that any feature of the natural numbers 'propagates' via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it true for 1, it is true for 2 as well, etc.

2

Operations On Natural Numbers

What Is An Operation?

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We don't often see them as functions because we don't write them as such. We write $n + m$ instead of $+(n, m)$ and $n \cdot m$ instead of $\cdot(n, m)$.

In this sense, subtraction and division **are not operations**! They take two natural numbers but they **do not produce a natural number**.

Addition

We define **addition** on natural numbers by the following two formulae:

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- $n + s(m) = s(n + m)$.

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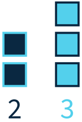
We can imagine addition as 'adding blocks *to the side*' and the successor function as 'adding one block *on top*'.
In this sense, $n + s(m) = s(n + m)$ only means that if you add one block atop m blocks and then n blocks to the side you have the same number of blocks as if you add n blocks next to m blocks and then another on top of that.

Addition



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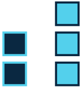
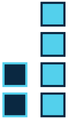
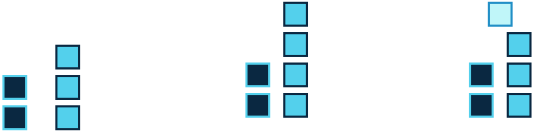


2 3



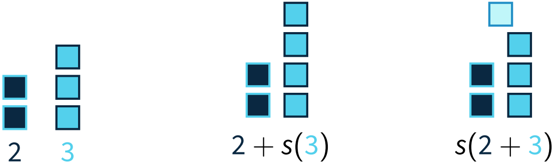
$2 + s(3)$

Addition



$2 \quad 3 \qquad 2 + s(3) \qquad s(2+3)$

Addition



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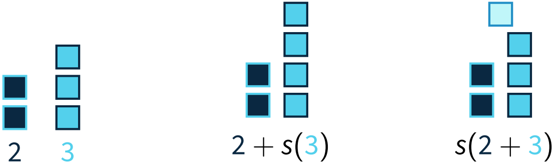

Using the formula $n + s(m) = s(n + m)$, one calculates $n + m$ by taking the successor of n, m times. Like this:

$$
\begin{aligned}
n + 0 &= n, \\
n + 1 &= n + s(0) = s(n + 0) = s(n), \\
n + 2 &= n + s(1) = s(n + 1) = s(n + s(0)) = s(s(n + 0)) = s(s(n)), \\
&\vdots
\end{aligned}
$$

Addition – Properties

Addition of natural numbers satisfies these two properties:

- **Commutativity**:

$$n + m = m + n.$$

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$$n + (m + k) = (n + m) + k.$$

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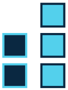


$2 + 3$

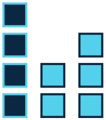
Addition – Properties

Using blocks, **commutativity** just means that putting n blocks next to m blocks is the same as putting m blocks next to n blocks.

$2 + 3$ $\qquad\qquad$ $3 + 2$

Addition – Properties

Using blocks, **associativity** just means that putting *m* blocks next to *k* blocks and then *n* more blocks next to those is the same as putting *m* blocks next to *n* blocks and then *k* more blocks next to those.



$$4 + (2 + 3)$$

Addition – Properties

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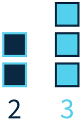
- $m \cdot s(n) = m \cdot n + m$.

We can imagine multiplication $m \cdot n$ by adding a collections of n blocks for every one block in the collection of m blocks.

If we write $s(n) = n + 1$, then the second formula just means that

$$m \cdot s(n) = m \cdot (n + 1) = m \cdot n + m.$$

Multiplication



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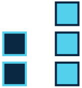
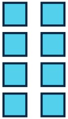
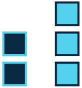


2 3

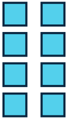


$2 \cdot s(3)$

Multiplication



$2 \quad 3$



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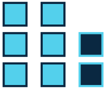
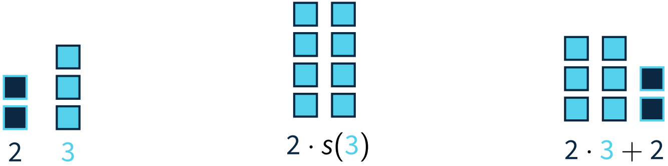


$2 \cdot 3 + 2$

Multiplication



The formula $m \cdot s(n) = m \cdot n + m$ allows us to compute $m \cdot n$ by applying it n times.

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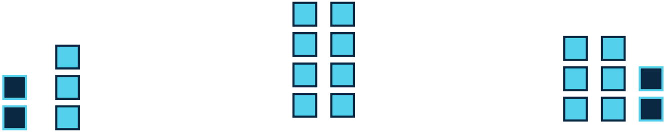

$2 \quad 3$ $\qquad 2 \cdot s(3)$ $\qquad 2 \cdot 3 + 2$

The formula $m \cdot s(n) = m \cdot n + m$ allows us to compute $m \cdot n$ by applying it n times. More precisely,

$$\begin{aligned}
m \cdot 1 &= m \\
m \cdot 2 &= m \cdot s(1) = m \cdot 1 + m = m + m \\
m \cdot 3 &= m \cdot s(2) = m \cdot 2 + m = m \cdot s(1) + m = m \cdot 1 + m + m = m + m + m \\
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\end{aligned}$$

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- **Distributivity:**

$$m \cdot (n + k) = m \cdot n + m \cdot k.$$

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Beware! 'Taking inverse' is a **unary** operation (meaning it acts on **one** element), not a binary one.

1

Digression

↓

Mathematical Structures

Destructive & Symmetric Transformations

Mathematical Structures

Mathematical Structure (also 'Universe')

A **mathematical** structure is a **set** with **operations**. If X is the set and $op_1, \ldots, op_n$ the operations on X, we write the resulting structure as

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The reason mathematical structures are called by some enthusiasts as mathematical 'universes' is that they really *are* universes in the broadest sense possible – a bunch of elements with prescribed rules of interaction.

Operation On A Set

Operation

An **operation** on a set X is really just a **rule of interaction** between its elements. In symbols, it is a **function**

$$op : X^n \to X$$

where $X^n \to X$ just means 'Take n elements of X and give me back one.'

Operation On A Set

Examples:

- $(\mathbb{N}, +, \cdot)$ is a structure where $+$ and $\cdot$ are **binary** operations (meaning they take two elements and return one). They can be seen as functions $\mathbb{N}^2 \to \mathbb{N}$.

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- $(\{\text{orderings of vertices of a regular } n\text{-gon}\}, r, r^2, \ldots, r^n, s_1, \ldots, s_n)$ is a structure where r is the rotation by $360^\circ/n$ and $s_1, \ldots, s_n$ are all the reflections. They are all **unary** operations (they take one element and return one).

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- The operation r on the vertices of a regular polygon is **symmetric** – it can be reversed or **inverted**. If, for example, vertex A is sent to G by this rotation, then rotation in the opposite direction sends G back to A.

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    $$x \triangle x^* = y \triangle y^* = e.$$

Operation As A Transformation

Let's pick an (commutative) operation $\triangle$ on a set X and two elements $x, y \in X$.

- Suppose $x \triangle y$ is some element $z \in X$. For the operation $\triangle$ to be **symmetric**, we need two things:
  - A special element $e \in X$ (the **identity** element) which satisfies
    $$x \triangle e = x$$
    for all elements $x \in X$.
  - Elements x^* and y^* (the **inverse** elements to x and y) such that
    $$x \triangle x^* = y \triangle y^* = e.$$
- It's possible X doesn't have those things – which means the operation $\triangle$ cannot be symmetric **on** X, but it can be symmetric on a larger set which does contain those elements.

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Why? We know that $x \triangle x^* = e = y \triangle y^*$. Therefore, for example,

$$z \triangle y^* = (x \triangle y) \triangle y^* = x \triangle (y \triangle y^*) = x \triangle e = x.$$

Inverse As A Function

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Clearly, the inverse to x^* is x, again.

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$\hat{a}$	$\hat{b}$	$\hat{c}$
c	a	b

Drawing Structures

The table

$$\begin{array}{c|c|c} \color{darkred}{\hat{a}} & \color{blue}{\hat{b}} & \color{green}{\hat{c}} \\ \hline \color{green}{c} & \color{darkred}{a} & \color{blue}{b} \end{array}$$

from the previous slide can be also drawn using dots and arrows for example like this:

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$\hat{a}$	$\hat{b}$	$\hat{c}$
c	a	b

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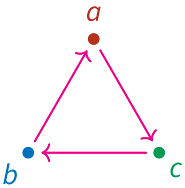
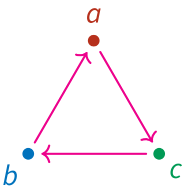


where the magenta arrows denote the operation $\hat{}$.

Drawing Structures

Let's think what could be an **inverse** to the operation ^.



DRAWING STRUCTURES

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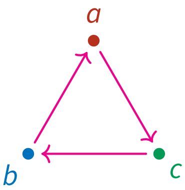

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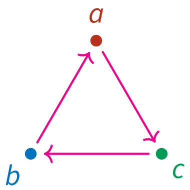

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As I have three dots, this means, that after traversing one arrow, I have to **traverse two arrows** to get back to where I started.

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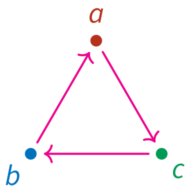

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As I have three dots, this means, that after traversing one arrow, I have to **traverse two arrows** to get back to where I started.

However, traversing two arrows just means **applying the operation ^ twice**.

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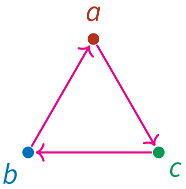

What this means in symbols, is that if $\hat{c} = b$, then $\hat{\hat{b}} = c$.

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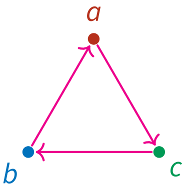

What this means in symbols, is that if $\hat{c} = b$, then $\hat{\hat{b}} = c$.
So, to get an inverse element with respect to $\hat{}$, we just need to apply $\hat{}$ twice.

Drawing Structures – Examples

Imagine a similar unary operation $\tilde{}$. It is given by the following table

$\tilde{a}$	$\tilde{b}$	$\tilde{c}$
b	b	b

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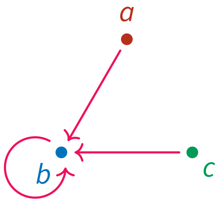

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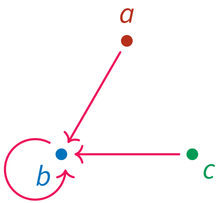
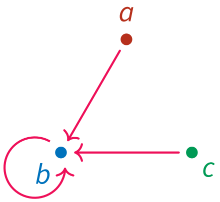


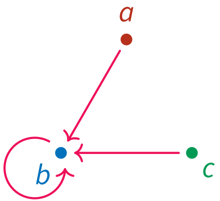
How many elements do we need to add to make $\tilde{}$ symmetric?

Drawing Structures – Examples



Can we make $\sim$ symmetric?

Drawing Structures – Examples



Can we make $\sim$ symmetric?

The answer is **no** because I can never get back after reaching *b*. This happens since all the arrows end in b.

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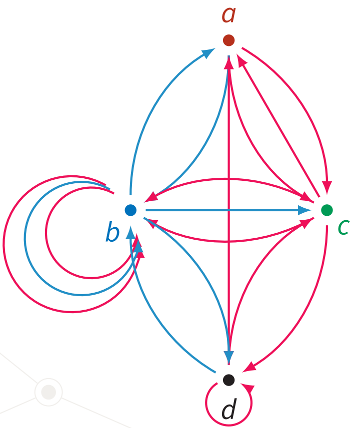
Is this operation **symmetric** or **destructive**?

Drawing Structures – Examples

Drawing binary operations as arrows between elements would be a nightmare. Just watch what such a diagram would look like for the operation from the previous slide.



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$\triangle$	a	b	c	d
a	c	d	d	b
b	d	b	c	d
c	d	c	b	d
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Is it still **symmetric**?

Operation As A Transformation

Motivating examples:

- The operation $+$ on $\mathbb{N}$ **cannot be made symmetric.** Natural numbers do not contain inverses to $+$.

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- The operation $+$ on $\mathbb{N}$ **cannot be made symmetric**. Natural numbers do not contain inverses to $+$.We fix this by introducing **integers**, basically forcefully adding to each number n its inverse with respect to $+$, that is, the negative number $-n$. The new set is called the **integers**, denoted $\mathbb{Z}$. The identity element is 0 in this case.

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- The operation $+$ on $\mathbb{N}$ **cannot be made symmetric**. Natural numbers do not contain inverses to $+$.We fix this by introducing **integers**, basically forcefully adding to each number n its inverse with respect to $+$, that is, the negative number $-n$. The new set is called the **integers**, denoted $\mathbb{Z}$. The identity element is 0 in this case.
- The operation $\cdot$ on $\mathbb{Z}$ cannot be made symmetric. We fix this by introducing **fractions/reciprocals** to elements. The inverse with respect to $\cdot$ of an element z is $\frac{1}{z}$. The identity element is 1.

Defining Integers

We define the set of integers (or whole numbers) $\mathbb{Z}$ as the set of natural numbers $\mathbb{N}$ plus the inverses to all natural numbers with respect to addition.

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The operation $+$ on $\mathbb{Z}$ is symmetric, while $\cdot$ is still **destructive**. This leads us to the rationals.

Rational Numbers

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because both fractions represent the same amount, essentially.

Nonetheless, as *numbers*, $3 \neq 1$ and $6 \neq 2$. We would like to formalize the notion of *sameness* for two fractions.

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Examples:

- $3/4 = 6/8$ because $3 \cdot 8 = 4 \cdot 6$.
- $1/2 = 10/20$ because $1 \cdot 20 = 2 \cdot 10$.

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$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

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The result is of course the formula

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}.$$