Gymnázium Evolution
Academic year 2024/2025
Maths Seminar, 6.AB/4.C

Linear Algebra
Second Written Exam
Vector Spaces & Homomorphisms

Name: ....

Question	1	2	3	Total
Points	4	7	10	21
Grade				

Throughout the exam, you're allowed to use any tools at your disposal. Write your answers thoroughly.

1. In each of the following groups, answer **YES** next to each statement if the statement is **always** true. Otherwise answer **NO**. Each group is worth 1 point if all statements in that group are evaluated correctly. (*4 points*)

Representation of a vector with respect to a linearly independent set is **always** unique.	**YES**	**NO**
Representation of a vector with respect to a spanning set is **always** unique.	**YES**	**NO**
Representation of a vector with respect to a basis is **always** unique.	**YES**	**NO**
The maximum number of linearly independent columns of a matrix of rank n is n.	**YES**	**NO**
If a matrix $A \in \mathbb{R}^{m \times n}$ has rank n, then every linear system with left side A has a **unique** solution.	**YES**	**NO**
If the row space of a matrix $A \in \mathbb{R}^{m \times n}$ has dimension n and $m \geq n$, then any linear system with left side A has $m - n$ free variables.	**YES**	**NO**
The matrix $\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ can have rank 1 or 2 depending on the choice of a.	**YES**	**NO**
For every $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{v} \in \mathbb{R}^n$, the vector $A \cdot \boldsymbol{v}$ lies in the row space of A.	**YES**	**NO**
If $A \in \mathbb{R}^{3 \times 3}$ has rank 2, then $(A \cdot \boldsymbol{e}_1, A \cdot \boldsymbol{e}_2)$ is a basis of the column space of A.	**YES**	**NO**
In $\mathbb{R}^2$, the reflection over any line is a homomorphism $\mathbb{R}^2 \to \mathbb{R}^2$.	**YES**	**NO**
If $f, g \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ and $f(\boldsymbol{e}_i) = g(\boldsymbol{e}_i)$ for every $i \leq n$, then $f = g$.	**YES**	**NO**
Given $f \in \mathrm{Hom}(V, W)$ and any subspace $U \leq V$, the image $f(U)$ is a subspace of W.	**YES**	**NO**

2. Solve the following problems. Include important steps of your calculations.

(a) Compute some basis of the row space of the matrix (*2 points*)

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & -2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

in $\mathbb{R}^{3\times 3}$ and determine the matrix's rank.

(b) Verify (by definition or using any of the proven statements) that the map $f : \mathscr{P}_2 \to \mathbb{R}^2$ given by (*2 points*)

$$f\left(ax^2 + bx + c\right) = \begin{pmatrix} a+b \\ a+c \end{pmatrix}$$

is a homomorphism.

(c) The homomorphism $f : (\mathbb{Z}/7)^3 \to (\mathbb{Z}/7)^3$ satisfies (*3 points*)

$$f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix}.$$

Determine $[f]_B^{\mathscr{E}_3}$, that is, the matrix of f with respect to the standard basis $\mathscr{E}_3$ of $(\mathbb{Z}/7)^3$ and the basis B, where

$$B = \left(\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

3. Prove the following statements. If you base your proof upon another result, refer to the latter as precisely as you can. Of course, you may not refer to the given statement directly, or to propositions whose proofs use the statement.

(a) Assume that f is an endomorphism of V (a vector space over $\mathbb{R}$) with basis $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n)$. Prove the following: (*3 points*)

- If $f(\boldsymbol{b}_i) = \mathbf{0}$ for every i, then $f(\boldsymbol{v}) = \mathbf{0}$ for every $\boldsymbol{v} \in V$.
- If $f(\boldsymbol{b}_i) = \boldsymbol{b}_i$ for every i, then $f(\boldsymbol{v}) = \boldsymbol{v}$ for every $\boldsymbol{v} \in V$.
- If $f(\boldsymbol{b}_i) = r \cdot \boldsymbol{b}_i$ for some $r \in \mathbb{R}$ and every i, then $f(\boldsymbol{v}) = r \cdot \boldsymbol{v}$ for every $\boldsymbol{v} \in V$.

(b) Show that the *transpose* operation is linear, that is, (*3 points*)

$$(r \cdot A + s \cdot B)^T = r \cdot A^T + s \cdot B^T$$

for every $r, s \in \mathbb{R}$ and $A, B \in \mathbb{R}^{m \times n}$.

(c) Assume that f is an **injective** homomorphism $V \to W$. This implies that $f(B)$ is a basis of $f(V)$ for a given basis $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n)$ of V. (*4 points*)

- Determine $[f]^B_{f(B)}$, the matrix of f with respect to the bases B and $f(B)$.
- Given $\boldsymbol{v} \in V$ with

$$[\boldsymbol{v}]_B = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix},$$

determine $[f(\boldsymbol{v})]_{f(B)}$, the representation of $f(\boldsymbol{v})$ with respect to the basis $f(B)$.