Pro $x, y > 0$ platí

$$\lim_{p \to 0} \sqrt[p]{\frac{x^p + y^p}{2}} = \sqrt{xy}.$$

Důkaz. Spočteme nejprve limitu zprava. Položíme $r := 1/p$ a počítáme

$$\lim_{p \to 0^+} \sqrt[p]{\frac{x^p + y^p}{2}} = \lim_{r \to \infty} \left(\frac{\sqrt[r]{x} + \sqrt[r]{y}}{2} \right)^r.$$

Upravíme

$$\left(\frac{\sqrt[r]{x} + \sqrt[r]{y}}{2} \right)^r = \frac{x}{2^r} \left(1 + \sqrt[r]{\frac{y}{x}} \right)^r.$$

Položme $a := y/x$. Ukážeme, že

$$\lim_{r \to \infty} \frac{(1 + \sqrt[r]{a})^r}{2^r \sqrt{a}} = 1.$$

Protože log je spojitá funkce na $(0, \infty)$ a výraz v limitě je vždy kladný, je tato rovna 1, právě když

$$\lim_{r \to \infty} \log \frac{(1 + \sqrt[r]{a})^r}{2^r \sqrt{a}} = 0.$$

Opět upravíme

$$\log \frac{(1 + \sqrt[r]{a})^r}{2^r \sqrt{a}} = r \cdot \log \frac{1 + \sqrt[r]{a}}{2 \sqrt[2r]{a}} = \frac{\log \frac{1 + \exp(\frac{1}{r} \log a)}{2 \exp(\frac{1}{2r} \log a)}}{\frac{1}{r}}. \tag{$\heartsuit$}$$

Jelikož $\lim_{r \to \infty} 1/r = 0$ a exp je spojitá na $\mathbb{R}$, platí $\lim_{r \to \infty} \exp((1/r) \log a) = 1$. Potom též

$$\lim_{r \to \infty} \log \frac{1 + \exp\left(\frac{1}{r} \log a\right)}{2 \exp\left(\frac{1}{2r} \log a\right)} = \log \frac{1+1}{2} = \log 1 = 0.$$

Na limitu pro $r \to \infty$ výrazu ($\heartsuit$) lze proto použít l'Hospitalovo pravidlo. Máme

$$\left(\frac{1}{r} \right)' = -\frac{1}{r^2},$$
$$\exp'\left(\frac{1}{cr} \log a \right) = -\frac{1}{cr^2} \log a \cdot \exp\left(\frac{1}{cr} \log a \right) \quad \text{pro } c \neq 0,$$

čili

$$\begin{aligned}
\left(\frac{1 + \exp\left(\frac{1}{r} \log a\right)}{2 \exp\left(\frac{1}{2r} \log a\right)} \right)' &= \frac{\frac{-2 \log a}{r^2} \exp\left(\frac{1}{r} \log a + \frac{1}{2r} \log a\right) + \frac{\log a}{r^2} \exp\left(\frac{1}{2r} \log a\right) \cdot \left(1 + \exp\left(\frac{1}{r} \log a\right)\right)}{4 \exp\left(\frac{1}{r} \log a\right)} \\
&= \frac{\frac{\log a}{r^2} \left(-2a - 2 \exp\left(\frac{3}{2r}\right) + 2a + \exp\left(\frac{1}{2r}\right) + \exp\left(\frac{3}{2r}\right)\right)}{4 \exp\left(\frac{1}{r} \log a\right)} \\
&= \frac{\frac{\log a}{r^2} \left(\exp\left(\frac{1}{2r}\right) - \exp\left(\frac{3}{2r}\right)\right)}{4 \exp\left(\frac{1}{r} \log a\right)}.
\end{aligned}$$

Odtud

$$\begin{aligned}
\frac{\left(\log \frac{1 + \exp(\frac{1}{r} \log a)}{2 \exp(\frac{1}{2r} \log a)} \right)'}{\left(\frac{1}{r}\right)'} &= -r^2 \cdot \frac{2 \exp\left(\frac{1}{2r} \log a\right)}{1 + \exp\left(\frac{1}{r} \log a\right)} \cdot \frac{\frac{\log a}{r^2} \left(\exp\left(\frac{1}{2r}\right) - \exp\left(\frac{3}{2r}\right)\right)}{4 \exp\left(\frac{1}{r} \log a\right)} \\
&= -\log a \cdot \frac{2 \exp\left(\frac{1}{2r} \log a\right)}{1 + \exp\left(\frac{1}{r} \log a\right)} \cdot \frac{\exp\left(\frac{1}{2r}\right) - \exp\left(\frac{3}{2r}\right)}{4 \exp\left(\frac{1}{r} \log a\right)}.
\end{aligned}$$

Protože

$$\lim_{r\to\infty} \exp\left(\frac{1}{cr}\log a\right) = 1,$$

spočteme

$$\lim_{r\to\infty} -\log a \cdot \frac{2\exp\left(\frac{1}{2r}\log a\right)}{1+\exp\left(\frac{1}{r}\log a\right)} \cdot \frac{\exp\left(\frac{1}{2r}\right)-\exp\left(\frac{3}{2r}\right)}{4\exp\left(\frac{1}{r}\log a\right)} = -\log a \cdot \frac{2\cdot 1}{1+1} \cdot \frac{1-1}{4\cdot 1} = 0,$$

jak jsme chtěli.

Teď víme, že

$$\lim_{r\to\infty} \frac{(1+\sqrt[r]{a})^r}{2^r\sqrt{a}} = 1.$$

Takže

$$\lim_{p\to 0^+} \sqrt[p]{\frac{x^p+y^p}{2}} = \lim_{r\to\infty} \frac{x}{2^r}\left(1+\sqrt[r]{\frac{y}{x}}\right)^r = \lim_{r\to\infty} \frac{x}{2^r}\cdot 2^r\sqrt{\frac{y}{x}} = \sqrt{xy},$$

čímž je důkaz hotov pro limitu zprava.

Pro limitu zleva platí

$$\lim_{p\to 0^-} \sqrt[p]{\frac{x^p+y^p}{2}} = \lim_{p\to 0^+} \sqrt[-p]{\frac{x^{-p}+y^{-p}}{2}} = \lim_{p\to 0^+} \frac{1}{\sqrt[p]{\frac{x^{-p}+y^{-p}}{2}}}.$$

Po substituci $a := x^{-1}$ a $b := y^{-1}$ dostaneme

$$\lim_{p\to 0^+} \frac{1}{\sqrt[p]{\frac{a^p+b^p}{2}}} = \frac{1}{\sqrt{ab}} = \sqrt{xy},$$

kde první rovnost plyne z předchozího výpočtu. Důkaz je hotov. □