# Probability

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The **probability distribution** of this random variable is a function $f : \{\text{heads}, \text{tails}\} \to [0, 1]$ which assigns to the element 'heads' the probability $P(X = \text{heads})$ and to 'tails' the probability $P(X = \text{tails})$.

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In other words, $f(\text{heads}) = f(\text{tails}) = 1/2$.

# Probability Distribution – Examples

- The **probability distribution** of a random variable representing the value of a dice roll is a function

$$f : \{1, 2, 3, 4, 5, 6\} \to [0, 1]$$

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$$f : \{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\} \to [0, 1]$$
such that $f(r) = 4/52$ where $r$ is a rank of a playing card.

# Visualizing Probability Distributions – Tables

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For example, the probability distribution of a dice roll is given simply by

| **Roll** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Probability** | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$. |

# Visualizing Probability Distributions – Tables

For a more abstract example, if $X$ can attain any of the four values $a, b, c, d$ with probabilities $P(X = a) = 3/10, P(X = b) = 5/10, P(X = c) = 1/10, P(X = d) = 1/10$, then its probability distribution is

| **Value** | a | b | c | d |
|---|---|---|---|---|
| **Probability** | $\frac{3}{10}$ | $\frac{5}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$. |

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**Probability distributions** (both *discrete* and *continuous*) can be represented as graphs. These are your typical function graphs which draw inputs on the $x$-axis and outputs on the $y$-axis.

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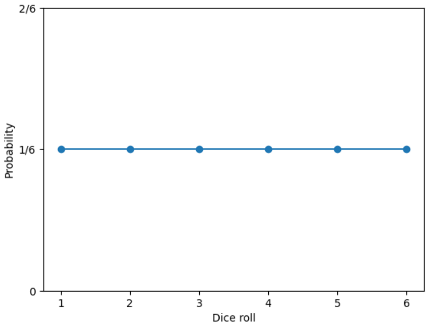
The probability distribution of a dice roll looks like this



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The probability distribution from this table

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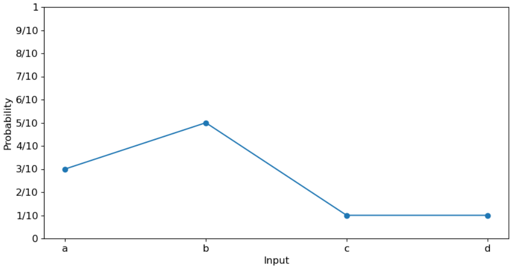

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- The **mean** of $X$ is defined as $E(X) = \sum_{a \in A} a \cdot P(X = a)$. It represents the 'expected' value of $X$.

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- The **cumulative distribution function** of $X$ gives the probability that a random variable is **less than a certain value**. It is defined as $F(a) = P(X \leq a)$ for $a \in A$.
- The **mean** of $X$ is defined as $E(X) = \sum_{a \in A} a \cdot P(X = a)$. It represents the 'expected' value of $X$.
- The **variance** (describing the *dispersion* of the distribution around the mean) of $X$ is defined as

$$\mathrm{Var}(X) = \sum_{a \in A} (a - E(X))^2 \cdot P(X = a).$$

# Discrete Probability Distribution – Example

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Suppose we measure the height of a randomly picked 20-year-old males. We might get something akin to the following table

| **Height** | 175 | 176 | 177 | 178 | 179 | 180 | 181 | 182 | 183 |
|---|---|---|---|---|---|---|---|---|---|
| **Count** | 13 | 20 | 11 | 17 | 11 | 8 | 10 | 7 | 3 |

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We can easily calculate the mean and standard deviation of this data.

# Discrete Probability Distribution – Example

| **Height** | 175 | 176 | 177 | 178 | 179 | 180 | 181 | 182 | 183 |
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Using the formula for the arithmetic mean, we get

$$\bar{x} = \frac{175 \cdot 13 + 176 \cdot 20 + \ldots + 183 \cdot 3}{13 + 20 + \ldots + 3} = 178.1$$

# Discrete Probability Distribution – Example

| Height | 175 | 176 | 177 | 178 | 179 | 180 | 181 | 182 | 183 |
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The standard deviation is then

$$\sigma = \sqrt{\frac{13 \cdot (175 - 178.1)^2 + 20 \cdot (176 - 178.1)^2 + \ldots + 3 \cdot (183 - 178.1)^2}{13 + 20 + \ldots + 3}} = 8.203.$$

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Let's now define a random variable $X$ which can be any of those heights in the table above.

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Let's now define a random variable $X$ which can be any of those heights in the table above. We define the probabilities that $X$ is a particular height based on the counts above. That gives the following table

| Height | 175 | 176 | 177 | 178 | 179 | 180 | 181 | 182 | 183 |
|---|---|---|---|---|---|---|---|---|---|
| Probability | $\frac{13}{100}$ | $\frac{20}{100}$ | $\frac{11}{100}$ | $\frac{17}{100}$ | $\frac{11}{100}$ | $\frac{8}{100}$ | $\frac{10}{100}$ | $\frac{7}{100}$ | $\frac{3}{100}$ |

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In other words, this gives a **distribution function** $f$ of $X$ where the set $A = \{175, 176, 177, 178, 179, 180, 181, 182, 183\}$ and the outputs of $f$ on each of these numbers are given by the table above.

# Discrete Probability Distribution – Example

| **Height** | 175 | 176 | 177 | 178 | 179 | 180 | 181 | 182 | 183 |
|---|---|---|---|---|---|---|---|---|---|
| **Probability** | $\frac{13}{100}$ | $\frac{20}{100}$ | $\frac{11}{100}$ | $\frac{17}{100}$ | $\frac{11}{100}$ | $\frac{8}{100}$ | $\frac{10}{100}$ | $\frac{7}{100}$ | $\frac{3}{100}$ |

- The **cumulative distribution function** $F$ describes the probability that a randomly chosen person from the group has height *less than* a particular number.

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- The **cumulative distribution function** $F$ describes the probability that a randomly chosen person from the group has height *less than* a particular number. For example,

$$F(178) = P(X \leq 178) = P(X = 175) + P(X = 176) + P(X = 177) + P(X = 178)$$
$$= \frac{13}{100} + \frac{20}{100} + \frac{11}{100} + \frac{17}{100} = \frac{61}{100}.$$

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- The **mean** of $X$ is the same as the arithmetic mean of the data. Indeed,

$$\begin{aligned} E(X) &= \sum_{a \in A} a \cdot P(X = a) \\ &= 175 \cdot P(X = 175) + 176 \cdot P(X = 176) + \ldots + 183 \cdot P(X = 183) \\ &= 175 \cdot \frac{13}{100} + 176 \cdot \frac{20}{100} + \ldots + 183 \cdot \frac{3}{100} = 178.1. \end{aligned}$$

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- The **variance** of $X$ is the same as the standard deviation *squared* (that is, $\mathrm{Var}(X) = \sigma^2$). Indeed,

$$
\begin{aligned}
\mathrm{Var}(X) &= \sum_{a \in A} (a - E(X))^2 \cdot P(X = a) \\
&= (175 - 178.1)^2 \cdot P(X = 175) + \ldots + (183 - 178.1)^2 \cdot P(X = 183) \\
&= 67.29 = 8.203^2.
\end{aligned}
$$

# Some Important Discrete Distributions

# The Bernoulli Distribution

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If we denote these values as 0 and 1, then the Bernoulli distribution is the function

$$f(x) = \begin{cases} p, & \text{if } x = 1, \\ 1 - p, & \text{if } x = 0, \end{cases}$$

where $p \in [0, 1]$ is a **fixed** probability.

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A **coin toss** is a perfect example of a Bernoulli distribution with $p = 1/2$.

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A **coin toss** is a perfect example of a Bernoulli distribution with $p = 1/2$. Indeed, if $f$ is the probability distribution of the result of a coin toss, then

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x = \text{heads}, \\ \frac{1}{2}, & \text{if } x = \text{tails}. \end{cases}$$

# The Bernoulli Distribution – Properties

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- Since we have only two values, $F(0) = P(X \leq 0) = P(X = 0) = f(0) = 1 - p$ and $F(1) = P(X \leq 1) = f(0) + f(1) = 1$.

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- We calculate,

$$E(X) = \sum_{a \in \{0,1\}} a \cdot f(a) = 0 \cdot f(0) + 1 \cdot f(1) = 0 \cdot (1 - p) + 1 \cdot p = p.$$

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- We calculate,

$$E(X) = \sum_{a \in \{0,1\}} a \cdot f(a) = 0 \cdot f(0) + 1 \cdot f(1) = 0 \cdot (1 - p) + 1 \cdot p = p.$$

- And also

$$\mathrm{Var}(X) = \sum_{a \in \{0,1\}} (a - E(X))^2 \cdot f(a) = (0 - p)^2 \cdot (1 - p) + (1 - p)^2 \cdot p = p(1 - p).$$

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## Digression

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## Variations & Combinations

# How To Count The Number Of Repetitions?

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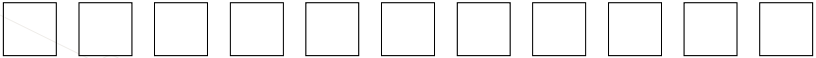
Imagine 11 empty boxes and count how many ways can you distribute 11 people into them.



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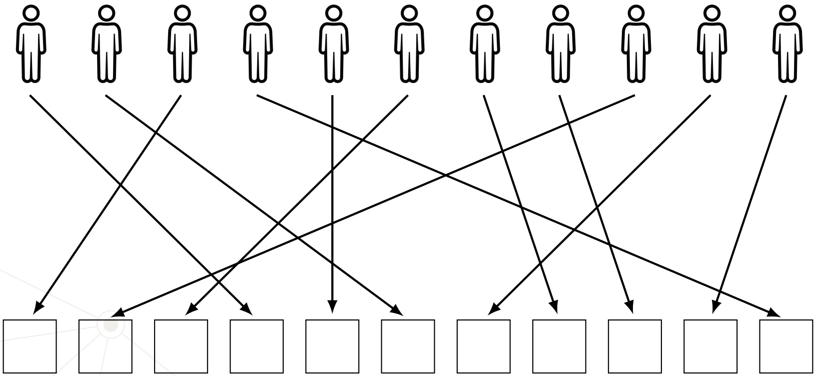

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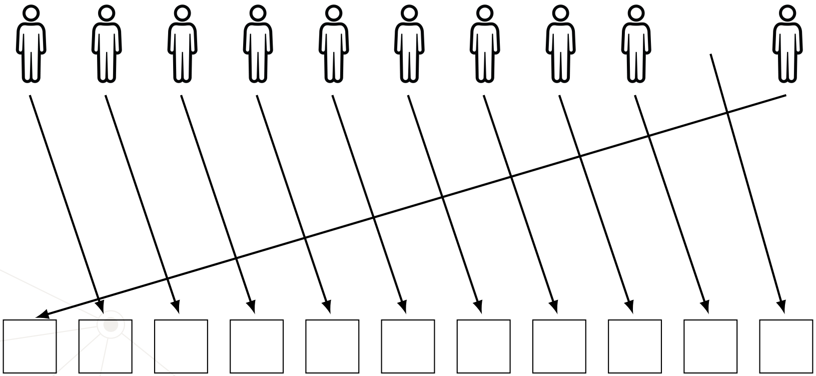

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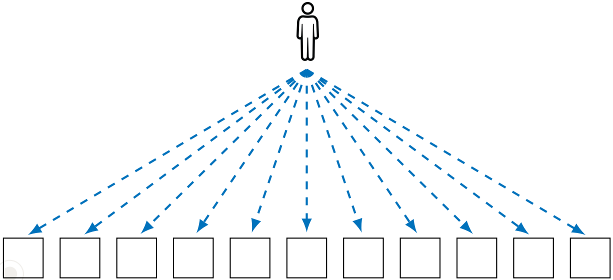

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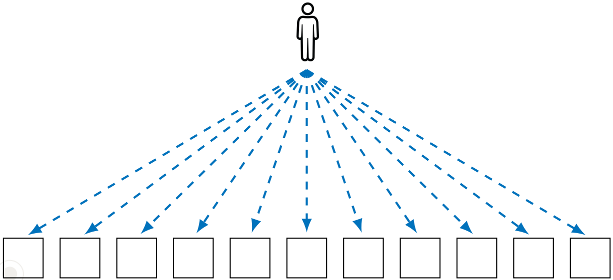

I'd have exactly 11 options where to put him.

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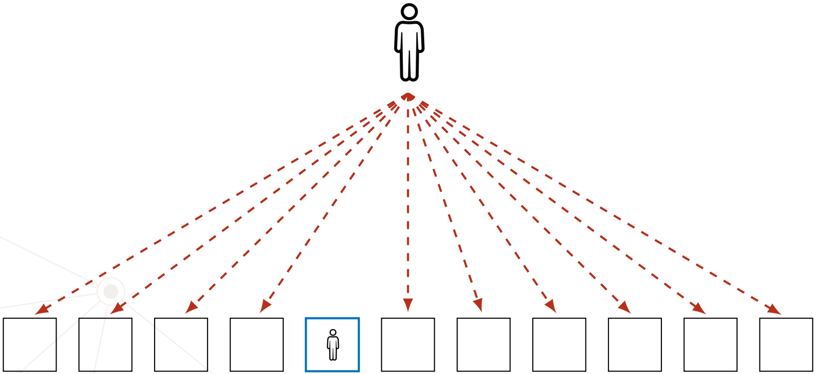

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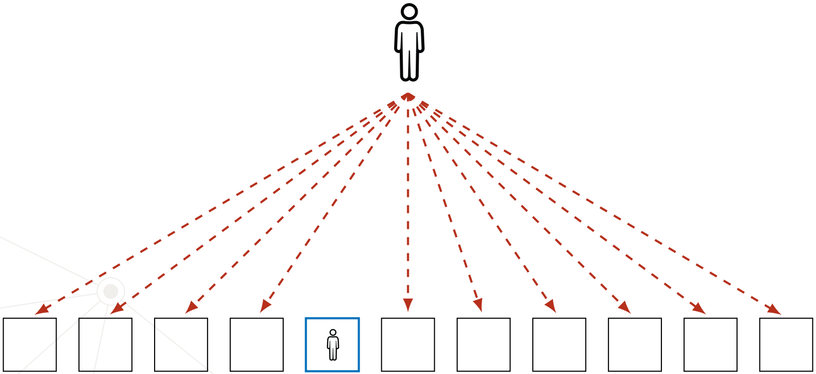
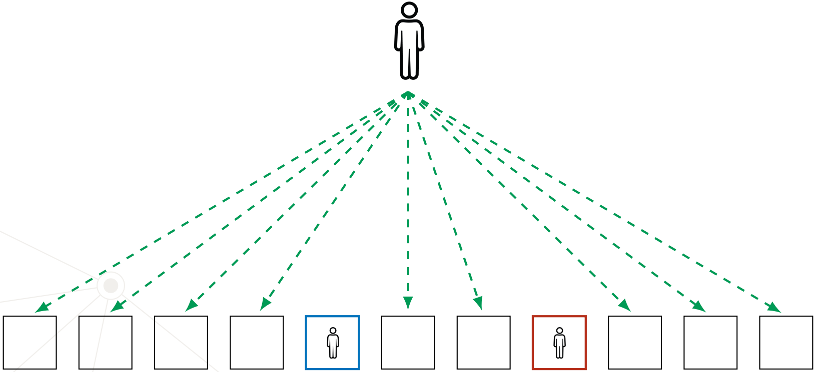


I'd have only 10 boxes left to place him into.

# How To Count The Number Of Repetitions?

For the third human, only 9 boxes are left, etc.



# Factorial

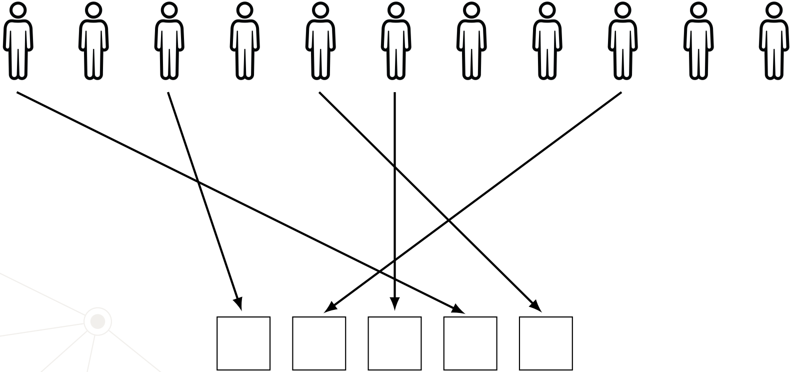
## Factorial

Overall, given $n$ objects, I have

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

ways how to order them. This number is written as $n!$ and read $n$ **factorial**.

# What If I Have Fewer Boxes?



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If I have only 5 boxes and 11 humans, I have

$$11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$$

ways to put 5 of those humans into all the boxes.

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## Variations

If I have $n$ elements, the number of ways how to order any $k$ of them is

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

This number is sometimes called the number of **variations** of $k$ elements out of $n$.

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Finally, suppose I'm not interested in the differences between individual humans and I just want to place some 5 of them into boxes.

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In other words, I don’t care about the order I put them into those boxes. It doesn’t matter to me which human goes to which box as long as all the boxes are full.

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In other words, I don't care about the order I put them into those boxes. It doesn't matter to me which human goes to which box as long as all the boxes are full.

This is a similar problem. The only difference is that I disregard all the ways I can order those $k$ elements inside the boxes.

# Combinations

## Combinations

If I have $n$ elements, the number of ways I can choose any $k$ of them regardless of order, is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

This number is typically written as $\binom{n}{k}$ and read '$n$ **choose** $k$'.