Logic & Set Theory Cheatsheet

3.AB PreIB Math

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Logic

Logic is the language of mathematics. It uses propositions to talk about sets.

Propositions are sentences which can be either true or false. For example

- '**Cats are black.**' is a proposition;
- '**How are you?**' is *not* a proposition;
- '**We will have colonised Mars by 2500.**' is also a proposition.

As the third example suggests, we need not necessarily know whether a proposition is true or false – it remains a proposition anyway.

Logical Conjunctions

Propositions can be joined together using logical conjunctions. They pretty much correspond to the conjunctions of natural language. Let us consider two propositions:

$$p = \text{'It's raining outside.'}$$
$$q = \text{'I'll stay at home.'}$$

($\wedge$) Logical and forms a proposition that is only true if both of its constituents are also true. In natural language, the proposition $p \wedge q$ can be expressed as

$$p \wedge q = \text{'It's raining outside and I'll stay at home.'}$$

($\vee$) Logical or forms a proposition that is true if at least one of its constituents is true. In natural language, the proposition $p \vee q$ can be expressed as

$$p \vee q = \text{'It's raining outside or I'll stay at home.'}$$

In mathematical logic, or is **not exclusive**! This means that $p \vee q$ is true even if both p and q are true.

($\neg$) Logical not isn't strictly speaking a conjunction but I include it anyway. It reverses the truth value of a proposition. For example, the proposition $\neg p$ can be read as

$$\neg p = \text{'It's not raining outside.'}$$

It follows that $\neg p$ is true exactly when p is false and vice versa.

($\Rightarrow$) Logical implication is a conjunction that makes the first proposition into an *assumption* or *premise* and the second one into a *conclusion*. The proposition $p \Rightarrow q$ is read in multiple ways, to list a few:

$$p \Rightarrow q = \text{'If it's raining outside, then I'll stay at home.'}$$
$$p \Rightarrow q = \text{'It raining outside implies that I'll stay at home.'}$$
$$p \Rightarrow q = \text{'Assuming it's raining outside, I'll stay at home.'}$$

The implication is tricky. It's true if both p and q are true and false if p is true but q is false. However, it is always true if p is false. That is because, in mathematical logic, whatever follows from a lie is automatically true.

($\Leftrightarrow$) Logical equivalence is true only if both propositions have the same truth value – they're both true or both false. In natural language, it is typically read like this:

$$p \Leftrightarrow q = \text{'It's raining if and only if I stay at home.'}$$

Equivalence is basically just a two-way implication. The proposition p is both a premise and a conclusion to q and q is both a premise and a conclusion to p. If it's raining outside, I stay at home and if I stay at home, then it's raining outside.

Truth Tables

A conjunction of propositions being true or false based on whether its constituent propositions are true or false can be summarized using so-called truth table. It is basically just a table that lists all the possibilities of p and q being true or false and the resulting truth value of their conjunctions.

For the basic logical conjunctions from above, it can look like this (we represent true by 1 and false by 0):

p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	1	1	0	0	1	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	0	0
1	1	0	0	1	1	1	1

Sets

Sets are the 'stuff' that makes up the world of mathematics. Their basic characteristics and properties are described using logic.

Sets cannot be defined inside set theory but we interpret them as *groups of things*.