Probability

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December 12, 2023

Probabilistic Intuition

What Is Chance?

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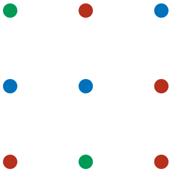

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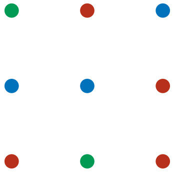

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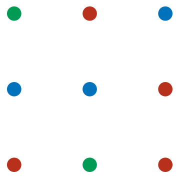

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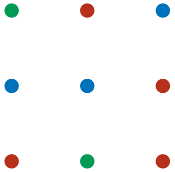

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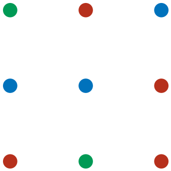
Quantifying Probability

Probability

A **probability** is a number between 0 and 1 measuring how **likely** is something to happen.

Quantifying Probability – Example

In our example of 9 balls



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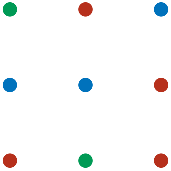

what is the probability of picking a ball of a specific colour?

- For red, it’s 4/9.
- For blue, it’s 3/9.
- For green, it’s 2/9.

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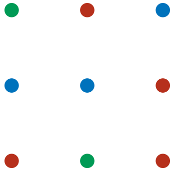

what is the probability of picking a ball of a specific colour?

- For red, it's 4/9.
- For blue, it's 3/9.
- For green, it's 2/9.

The probabilities above **sum up to 1** because I am certain to pick *some* ball.

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So, for the 9-ball example from before, we would have

$$P(X = \text{red}) = \frac{4}{9}, \quad P(X = \text{blue}) = \frac{3}{9}, \quad P(X = \text{green}) = \frac{2}{9}.$$

Calculating Probability

In the case the set of outcomes is **finite**, the probability of X being one of the possible outcomes is always

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$$P(X \in S) = \frac{|S|}{|O|},$$

where S is a certain subset of O – all the possible outcomes.

Calculating Probability – Example

We'll describe our 9-ball example more formally.

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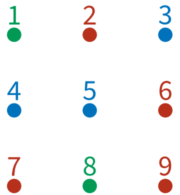
We’ll assign the balls number from 1 to 9. The set of all possible outcomes of picking a random ball is then

$$O = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$



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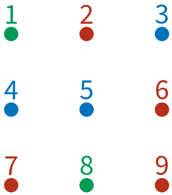

We'll form three subsets of O:

$$R = \{2, 6, 7, 9\},$$

$$B = \{3, 4, 5\},$$

$$G = \{1, 8\}.$$

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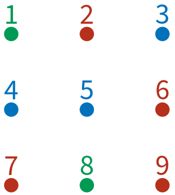

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$$R = \{2, 6, 7, 9\},$$

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$$G = \{1, 8\}.$$

We can use the formula from before to calculate the probability that X will be a green ball:

$$P(X \in G) = \frac{|G|}{|O|} = \frac{2}{9}.$$

Probability Equations

Sums of Probabilities

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However, this example cannot be easily generalized. We'll see why.

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$$E = \{2, 4, 6, \ldots, 20\},$$
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and we want to figure out the probability $P(X \in E \cup F)$.

Sums of Probabilities – Counterexample

Let's try to use the same formula as before:

$$P(X \in E \cup F) = \frac{|E \cup F|}{|O|} \stackrel{??}{=} \frac{|E| + |F|}{|O|} = \frac{10 + 4}{20} = \frac{14}{20}.$$

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So, to get the size of $E \cup F$, we cannot just add the size of E to the size of F but we also have to subtract the elements that appear twice – the size of $E \cap F$.

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where $A_1, A_2, \ldots, A_n$ are any sets.

Such a formula is widely known as the **principle of inclusion and exclusion**.

Principle of Inclusion and Exclusion – Example

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How many people speak at least one language?

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Label the three language groups E, F and G. The setup from the previous slide can be summarized as

$\lvert E\rvert$	$\lvert F\rvert$	$\lvert G\rvert$	$\lvert E \cap F\rvert$	$\lvert E \cap G\rvert$	$\lvert F \cap G\rvert$	$\lvert E \cap F \cap G\rvert$
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We're trying to calculate $\lvert E\cup F\cup G\rvert$.

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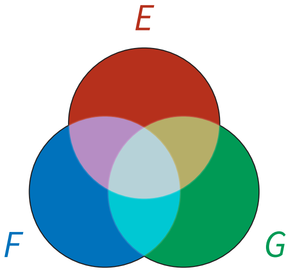
When working with sets, Venn diagrams are often a great choice.



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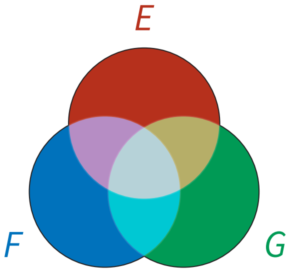

There are 7 regions in total (differentiated by colour) in this picture, corresponding to the 7 sets in the previous slide.

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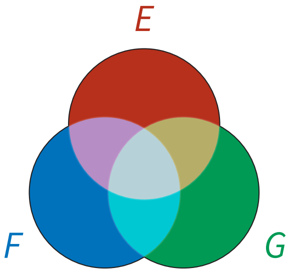

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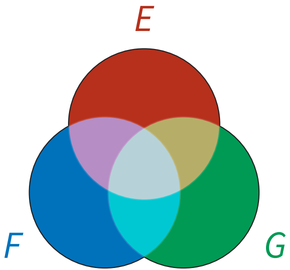

What we need to count is the total number of elements inside this entire shape. Let's start by counting the number of elements in each of the regions separately and assign numbers to regions corresponding to **how many times we've counted all the elements in that region.**

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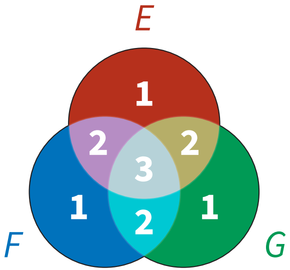

$$|E \cup F \cup G| = |E| + |F| + |G| \ldots$$

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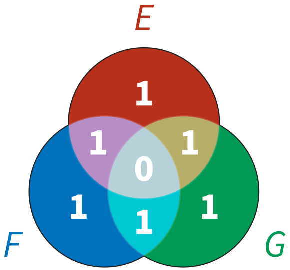

$$|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G|$$

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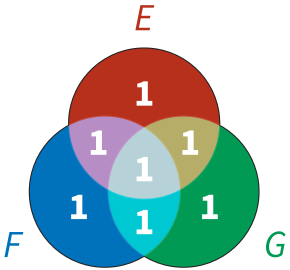

$$|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G| + |E \cap F \cap G|.$$

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$$|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G| + |E \cap F \cap G|.$$

Apply this formula to our example with language groups gives

$$|E \cup F \cup G| = 40 + 11 + 23 - 5 - 10 - 3 + 1 = 57.$$

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$$|E \cup F \cup G| = 40 + 11 + 23 - 5 - 10 - 3 + 1 = 57.$$

So, 57 people speak at least one language.

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5. ...

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If $A_1, A_2, \ldots, A_n$ are sets with $n \in \mathbb{N}$, then

Principle of Inclusion and Exclusion

$$\begin{aligned}
|A_1 \cup A_2 \cup \ldots A_n| = {} & |A_1| + |A_2| + |A_3| + \ldots + |A_n| \\
& - |A_1 \cap A_2| - \ldots - |A_1 \cap A_n| - |A_2 \cap A_3| - \ldots - |A_{n-1} \cap A_n| \\
& + |A_1 \cap A_2 \cap A_3| + \ldots + |A_1 \cap A_2 \cap A_n| + \ldots |A_{n-2} \cap A_{n-1} \cap A_n| \\
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& + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|.
\end{aligned}$$

The $(-1)^n$ only means that if n is odd, then I subtract the last term, and I add it if n is even.

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Out of the numbers 1 to 100, what is the probability that a randomly picked number is a multiple of 2, 3 or 7?

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and

$$O = \{1, 2, \dots, 100\}.$$

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So,

$$P(X \in E \cup T \cup S) = \frac{72}{100}.$$

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In other words, we want to determine the maximum size of

$$|(C_1 \cap C_2) \cup (C_1 \cap T) \cup (C_2 \cap T)|.$$

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- circle and a triangle can share is 6. So $|C_1 \cap T| = |C_2 \cap T| = 6$.
- all three objects share is zero if the number of intersections is maximized. So $|C_1 \cap C_2 \cap T| = 0$.

Principle of Inclusion and Exclusion – Problems

Let's apply the inclusion-exclusion principle. We get

$$
\begin{aligned}
&|(C_1 \cap C_2) \cup (C_1 \cap T) \cup (C_2 \cup T)| \\
&= |C_1 \cap C_2| + |C_1 \cap T| + |C_2 \cap T| \\
&- |(C_1 \cap C_2) \cap (C_1 \cap T)| - |(C_1 \cap C_2) \cap (C_2 \cap T)| - |(C_1 \cap T) \cap (C_2 \cap T)| \\
&+ |(C_1 \cap C_2) \cap (C_1 \cap T) \cap (C_2 \cap T)|.
\end{aligned}
$$

Principle of Inclusion and Exclusion – Problems

This is less scary than it looks. Actually, most of the intersections there are one and the same. Really,

$$\begin{aligned}
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So, the previous expression just ends up being

$$|C_1 \cap C_2| + |C_1 \cap T| + |C_2 \cap T| - 2 \cdot |C_1 \cap C \cap T| = 2 + 6 + 6 - 2 \cdot 0 = 14.$$

Events

What Is An Event?

Formally, an **event** is just an element which **has some probability**.

However, we typically think of events as **things that have some chance of happening**.

For example

- the fact that a random variable X lies in some set is an event.

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For example

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- the fact that a randomly chosen ball has a specific colour is an event.
- the fact that the universe ends today at midnight is an event.

Operations On Events

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- **or** (written as $\vee$).

We want to understand how to calculate $P(\neg A), P(A \wedge B), P(A \vee B)$ for two events A, B whose probabilities we know.

Negating Events

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Quite naturally, it's 0.8 because it either is going to rain or it's not. The total of the probabilities of those two events has to be 1.

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Quite naturally, it's 0.8 because it either is going to rain or it's not. The total of the probabilities of those two events has to be 1.

Negation Formula

If A is an event with probability $P(A) = p$, then

$$P(\neg A) = 1 - p.$$

Independent and Incompatible Events

Two events are called **independent** if the result of one event doesn't at all influence the result of the other.

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For example, the events 'I'm going to spend vacation in Florida.' and 'I'm going to spend vacation in Spain.' are incompatible.
An event is always incompatible with its own negation.

Conjunction of Events

Conjunction Formula

If A, B are two **independent** events, then

$$P(A \wedge B) = P(A) \cdot P(B).$$

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If the two events are **dependent**, then calculating the probability of their conjunction is much more difficult. We'll need *conditional probability* for that.

Disjunction of Events

The disjunction formula for events is really just the inclusion-exclusion principle.

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Disjunction Formula

If A, B are any events, then

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B).$$

If A, B are **incompatible**, then $P(A \wedge B) = 0$ and the formula above becomes $P(A \vee B) = P(A) + P(B)$.

Conditional Probability

Conditional is the probability of an event happening **given another event has already happened**.

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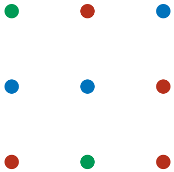
Conditional Probability

If A, B are events, then

$$P(A \mid B)$$

is the probability that A happens supposing B has already happened.

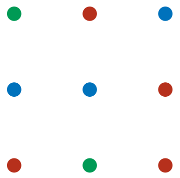
Conditional Probability – Example



In our balls example, suppose

- *A* is the event that the **second** randomly chosen ball is red.
- *B* is the event that the **first** randomly chosen ball is red.

Conditional Probability – Example



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- B is the event that the **first** randomly chosen ball is red.

What is the probability $P(A \mid B)$?

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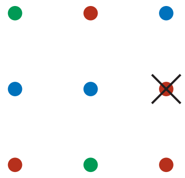
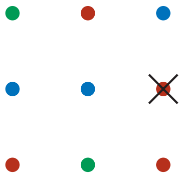


If B has happened, then there are only 3 red balls left in the set of 8 balls.

Conditional Probability – Example



If B has happened, then there are only 3 red balls left in the set of 8 balls. Therefore $P(A \mid B) = 3/8$.

Conditional Probability & Conjunction

We can use conditional probability to compute $P(A \wedge B)$ for **any** events A and B, not necessarily independent.

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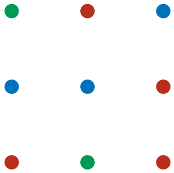
Event Conjunction Formula

If A, B are **any** events, then

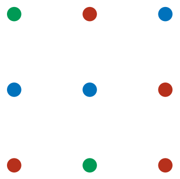
$$P(A \wedge B) = P(B) \cdot P(A \mid B) = P(A) \cdot P(B \mid A).$$

Conditional Probability & Conjunction



The event $A \wedge B$ in the ball example means that the first two randomly chosen balls are red.

Conditional Probability & Conjunction



The event $A \wedge B$ in the ball example means that the first two randomly chosen balls are red. We know that $P(B) = 4/9$ and $P(A \mid B) = 3/8$. Therefore,

$$P(A \wedge B) = P(B) \cdot P(A \mid B) = \frac{4}{9} \cdot \frac{3}{8} = \frac{1}{6}.$$

Conditional Probability – Examples

Suppose the probability that a woman will live to at least 70 years is 0.7 and that she will live to at least 80 years is 0.55. What is the probability that she will live to 80 supposing she has already turned 70?

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- S is the event that she will live to at least 70 and E is the event that she will live to at least 80.

Conditional Probability – Examples

Suppose the probability that a woman will live to at least 70 years is 0.7 and that she will live to at least 80 years is 0.55. What is the probability that she will live to 80 supposing she has already turned 70?

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- S is the event that she will live to at least 70 and E is the event that she will live to at least 80.
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- We know that $P(S) = 0.7$ and $P(E) = 0.55$.
- We want to know $P(E \mid S)$.

Conditional Probability – Examples

Suppose the probability that a woman will live to at least 70 years is 0.7 and that she will live to at least 80 years is 0.55. What is the probability that she will live to 80 supposing she has already turned 70?

Using the formula $P(E \wedge S) = P(S) \cdot P(E \mid S)$, we can calculate

$$P(E \mid S) = \frac{P(E \wedge S)}{P(S)}.$$

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Quite clearly, $P(E \wedge S) = P(E)$, so the above becomes $P(E)/P(S)$. This means that

$$P(E \mid S) = \frac{P(E)}{P(S)} = \frac{0.55}{0.7} = 0.786.$$

Conditional Probability – Examples

From a deck of 32 cards (8 ranks and 4 suits) two cards are drawn. What's the probability that the first is of diamonds and the second is of a different suit?

Let's formalize, again. Denote by D the event that the first card is of diamonds and by S the event that the second card is of a different suit. We want to know $P(D \wedge S)$.

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It's easy to see that $P(D) = 1/4$. If D has already happened, there are only 31 cards left in the deck, 24 of them being of a different suit than diamonds. This means that $P(S \mid D) = 24/31$.

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Multiplying these two values, we get our result:

$$P(D \wedge S) = P(D) \cdot P(S \mid D) = \frac{1}{4} \cdot \frac{24}{31} = \frac{6}{31}.$$