Number Sets

Adam Klepáč

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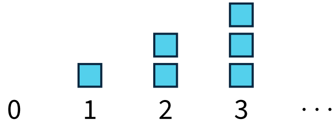
Natural Numbers

Natural Numbers – Intuition

Natural numbers are intuitively objects which represent a **quantity**.
They're the following set:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



Natural Numbers – Definition

There are many ways to define natural numbers.
One of the popular ones is using a **successor** function, denoted s. If n is a natural number, then $s(n)$ basically means 'add another block on top of n'.
One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet**! In fact, you need the successor function to define addition in the first place.

Natural Numbers – Definition

The following **five axioms** (often called *Peano axioms*) constitute the definition of natural numbers:

1. There exists the natural number 0.
2. Every natural number has a successor which is also natural.
3. The number 0 is not the successor of any natural number.
4. If $s(x) = s(y)$, then $x = y$.
5. (Induction Axiom) If a statement is true for 0 and it being true for n also implies that it is true for $n + 1$, then it is true for all natural numbers.

1

Unpacking The Axioms

Natural Numbers – Axiom 1

There exists the natural number 0.

Hopefully obvious.

Natural Numbers – Axiom 2

Every natural number has a successor which is also natural.

Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

Natural Numbers – Axiom 3

The number 0 is not the successor of any natural number.

Basically means that the natural numbers are infinite only 'in one direction'. There is a **first** natural number.

Natural Numbers – Axiom 4

$$\text{If } s(x) = s(y), \text{ then } x = y.$$

This means that the successor function is **injective** – each natural number has a different successor.

Natural Numbers – Axiom 5

If a statement is true for 0 and it being true for n also implies that it is true for $n + 1$, then it is true for all natural numbers.

This means that any feature of the natural numbers 'propagates' via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it true for 1, it is true for 2 as well, etc.

2

Operations On Natural Numbers

What Is An Operation?

By **operation**, we mean a function which takes **one or multiple** natural numbers and produces **one** natural number.

For example, $+$ and $\cdot$ are operations because they take **two** natural numbers and produce **one**.

We don't often see them as functions because we don't write them as such. We write $a + b$ instead of $+(a, b)$ and $a \cdot b$ instead of $\cdot(a, b)$.

In this sense, subtraction and division **are not operations**! They take two natural numbers but they **do not produce a natural number**.

Addition

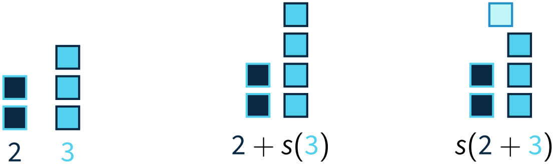
We define **addition** on natural numbers by the following two formulae:

- $n + 0 = n$,
- $n + s(m) = s(n + m)$.

We can imagine addition as 'adding blocks *to the side*' and the successor function as 'adding one block *on top*'.

In this sense, $n + s(m) = s(n + m)$ only means that if you add one block atop m blocks and then n blocks to the side you have the same number of blocks as if you add n blocks next to m blocks and then another on top of that.

Addition



Using the formula $n + s(m) = s(n + m)$, one calculates $a + b$ by taking the successor of a b times. Like this:

$$
\begin{aligned}
a + 0 &= a, \\
a + 1 &= a + s(0) = s(a + 0) = s(a), \\
a + 2 &= a + s(1) = s(a + 1) = s(a + s(0)) = s(s(a + 0)) = s(s(a)), \\
&\vdots
\end{aligned}
$$

Addition – Properties

Addition of natural numbers satisfies these two properties:

- **Commutativity:**

$$a + b = b + a.$$

- **Associativity:**

$$a + (b + c) = (a + b) + c.$$

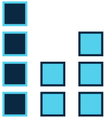
Addition – Properties

Using blocks, **commutativity** just means that putting *a* blocks next to *b* blocks is the same as putting *b* blocks next to *a* blocks.

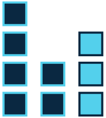
$2+3$ $\qquad$ $3+2$

Addition – Properties

Using blocks, **associativity** just means that putting *b* blocks next to *c* blocks and then *a* more blocks next to those is the same as putting *b* blocks next to *a* blocks and then *c* more blocks next to those.



$4 + (2 + 3)$



$(4 + 2) + 3$