Polygons & Transformations Cheatsheet

3.AB PreIB Math

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Polygons

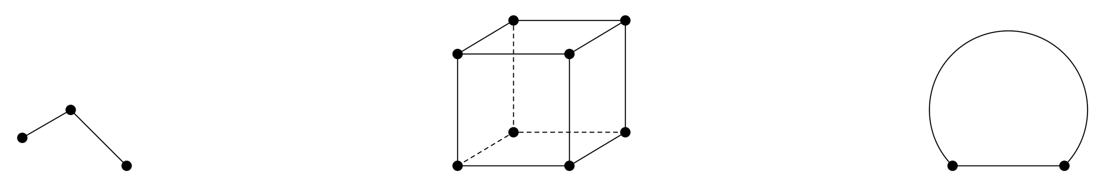
Polygon is a **closed** 2D shape **made only of segments**. We call the endpoints of those segments, **vertices**, and the segments themselves, **edges**.

Examples

Triangle Quadrilateral Pentagon Dodecagon

Polygons with n sides are called **n-gons**.

Counterexamples



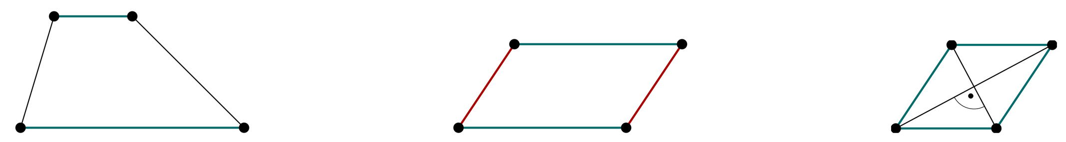
Not closed 3D Not straight

Convex Polygons

A polygon is called **convex** if it has no internal angle greater than 180°.

Convex NOT convex

Special types of convex polygons



Trapezoid/Trapezium
A convex quadrilateral with at least two parallel sides.

Parallelogram
A convex quadrilateral with two pairs of parallel sides.

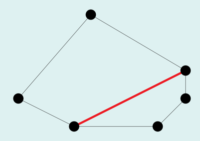
Rhombus
An **equilateral** (all sides of the same length) parallelogram.

In addition, if a **convex** polygon has **all sides of the same length** and **all angles of the same size**, it is called **regular**.

Equilateral triangle (regular trigon) Square (regular tetragon) Regular pentagon Regular hexagon

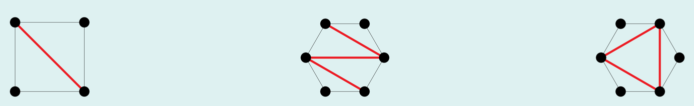
Diagonals & Triangulations

A **diagonal** in a **convex** polygon is a segment connecting two of its **non-adjacent** vertices.

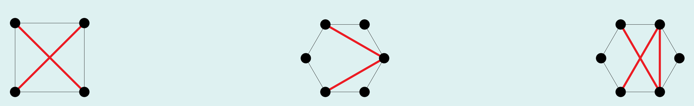


Diagonal in a convex hexagon.

A **triangulation** of a **convex** polygon is its division into triangles by **non-intersecting** diagonals.



Examples of triangulations.



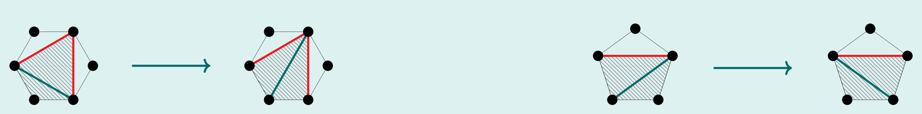
Counterexamples of triangulations.

The total number of different triangulations of a convex n-gon is

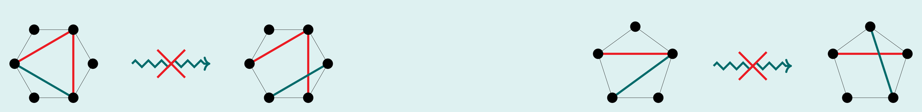
$$\frac{n \cdot (n+1) \cdot \ldots \cdot (2n-4)}{(n-2)!},$$

which you **of course don't have to remember**. Interestingly enough, every triangulation can be transformed into any other by a series of **flips**.

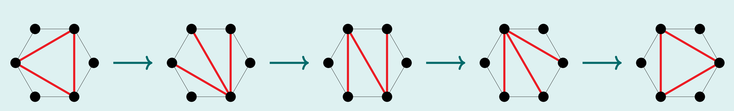
A **flip** is a swap of one diagonal for the other in a chosen quadrilateral so that the **result is again a triangulation**.



Examples of flips.



Counterexamples of flips.



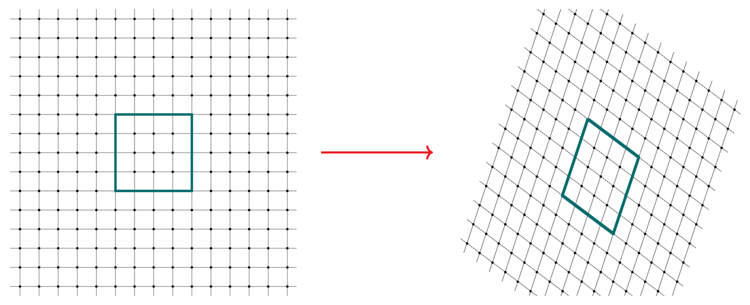
Passage from one triangulation to another through a series of flips.

I encourage you to think about how to determine the number of flips necessary to pass from one triangulation to another. Can I have made the passage above in fewer flips?

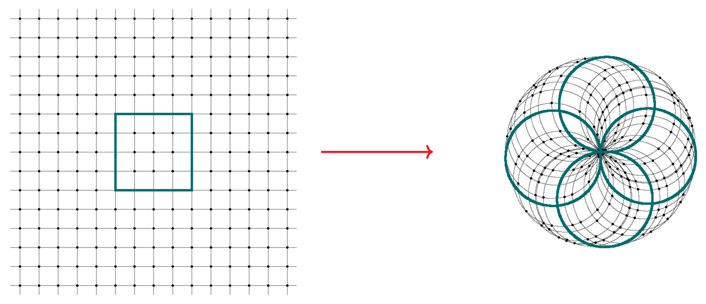
Plane Transformations

The **plane** is basically just the set $\mathbb{R}^2$ of all **pairs of real numbers**. A pair $(x, y) \in \mathbb{R}^2$ is typically called a **point**. Then, a plane **transformation** is a **function** which maps points to points. In symbols, it's a function $\mathbb{R}^2 \to \mathbb{R}^2$.

We can visualise what a transformation does for example by look at the image of a square (or an entire grid).



The transformation $(x, y) \mapsto (\frac{1}{3}(2x - y), \frac{1}{2}(x + 2y))$.

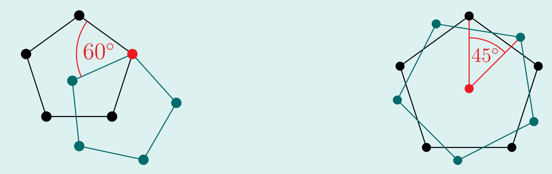


The transformation $(x, y) \mapsto (100(\sin x + \cos y), 100(\cos x + \sin y))$.

Rotations & Reflections

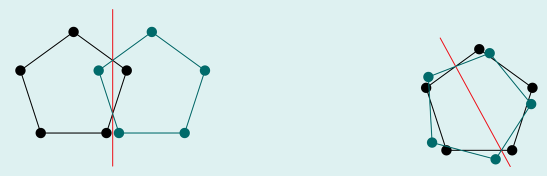
We shall be interested in two specific plane transformations – **rotations** and **reflections**.

Rotations are plane transformations that, well ..., rotates the entire plane around a fixed point called **anchor**. Applied to polygons, rotations may look like this:



Examples of rotations around a given anchor.

Reflections are basically 'mirrors'. They mirror each point in the plane through a given line called **axis** (of reflection).



Examples of reflections over a given axis.

Symmetries of Regular Polygons

Some **rotations** and **reflections** get along nicely with **regular polygons**. By this, we mean that they keep them intact. We call them the **symmetries** of the polygon.

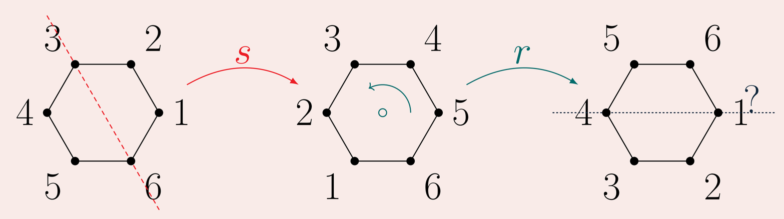
Each regular n-gon has multiple symmetries:

(r) **rotation** by $k \cdot 360°/n$ for any k between 1 and n.
(s) **reflection**
- over lines passing through centres of opposite sides or through opposite vertices if n is **even**;
- over lines passing through a centre of a side and the opposite vertex if n is **odd**.

Therefore, an n-gon has n rotational and n reflectional symmetries.

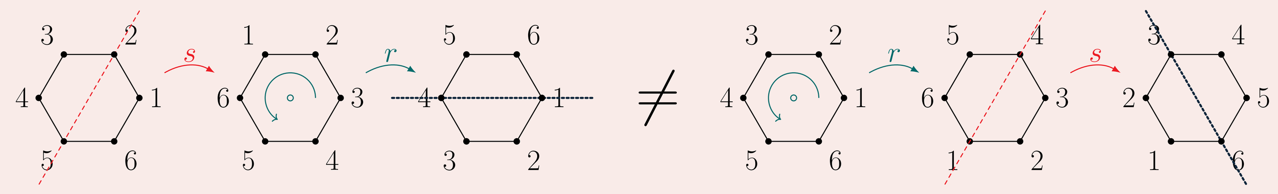
Examples of regular polygon symmetries.

Moreover, symmetries (being functions) can be **chained** or **composed**, creating new symmetries. We'll label rotations by the letter r and reflections by s. A **chain** or **composition** is read left-to-right, that is, sr means 'apply s first, then r'.



Example of the composition $\mathbf{sr}$ of a reflection s and a rotation r.

The order of composition matters!



In general, a composition of

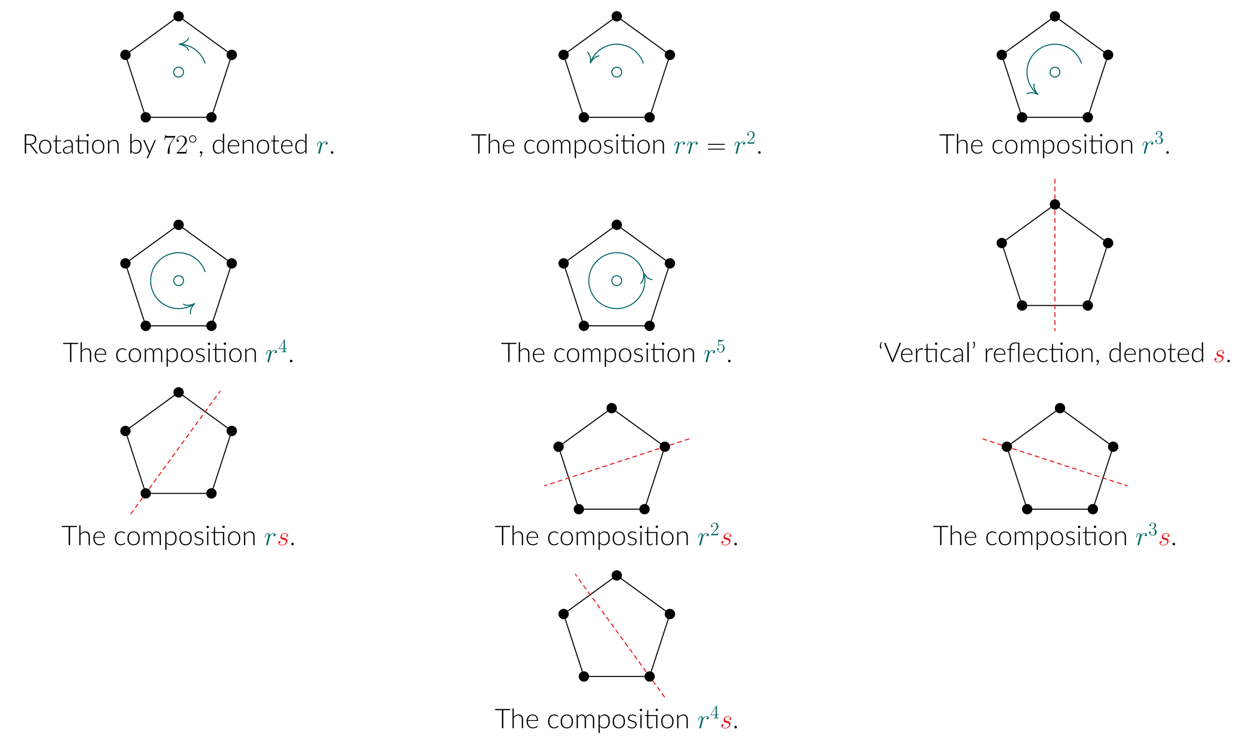
- a **rotation** and a **rotation** is again a **rotation**,
- a **rotation** and a **reflection** (in any order) is a **reflection**,
- a **reflection** and a **reflection** is a **rotation**.

Generating Symmetries

Question: If I can compose symmetries, how many (and which) symmetries are enough to generate (create by composition) all the others?

Answer: Just two are enough. Basically, I need to be able to generate the smallest rotation (that is, rotation by $360°/n$) and any reflection. Composing these two gives me all the other rotations and reflections.

Let's see this on a regular pentagon. Here, r denotes the rotation by $360°/5 = 72°$ and s denotes the 'vertical' reflection.



Rotation by 72°, denoted r. The composition $rr = r^2$. The composition r^3.

The composition r^4. The composition r^5. 'Vertical' reflection, denoted s.

The composition rs. The composition r^2s. The composition r^3s.

The composition r^4s.

All symmetries of the regular pentagon written as compositions of r and s.

Question: Given two symmetries, how do I know I can generate the rest using only these two?

Answer: It of course depends on their type.

- If I'm given 2 rotations, I can never generate all symmetries because composing rotations doesn't yield a reflection.
- If I'm given a rotation and a reflection, then the rotation must be by $k \cdot 360°/n$ with k **coprime to** n. Otherwise I can never generate the rotation by $360°/n$. The reflection can be of any kind.
- If I'm given two reflections, their composition must be a rotation by $k \cdot 360°/n$ with k **coprime to** n for the same reason as above.

Vocabulary

English	Czech	Definition
Segment	Úsečka	Straight finite line connecting two points.
Polygon	Mnohoúhelník	Closed 2D shape made of segments.
Vertex	Vrchol	Any of the endpoints of the segments forming a polygon.
Edge	Hrana	Any of the segments forming a polygon.
Triangle	Trojúhelník	A polygon with 3 vertices.
Quadrilateral	Čtyřúhelník	A polygon with 4 vertices.
Internal/External angle	Vnitřní/Vnější úhel	The angle on the inside/outside of a polygon formed by its edges.
Convex	Konvexní	A polygon whose internal angles don't exceed 180°.
Equilateral	Rovnostranný	Describes a polygon with all sides of equal length.
Trapezoid	Lichoběžník	A quadrilateral with a pair of parallel sides.
Parallelogram	Rovnoběžník	A quadrilateral with two pairs of parallel sides.
Rhombus	Kosočtverec	An equilateral parallelogram.
Regular	Pravidelný	Describes an equilateral polygon with all angles of the same size.
Square	Čtverec	A regular quadrilateral.