Probability

Adam Klepáč

November 30, 2023

Probabilistic Intuition

What Is Chance?

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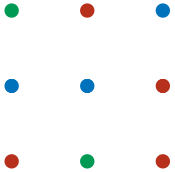

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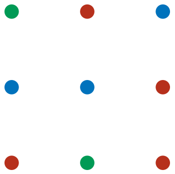

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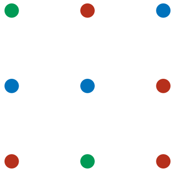

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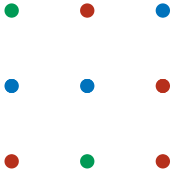

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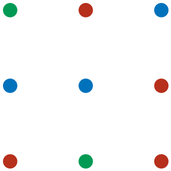
Quantifying Probability

Probability

A **probability** is a number between 0 and 1 measuring how **likely** is something to happen.

Quantifying Probability – Example

In our example of 9 balls



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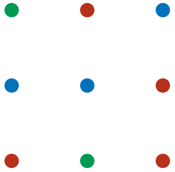

what is the probability of picking a ball of a specific colour?

- For red, it’s 4/9.
- For blue, it’s 3/9.
- For green, it’s 2/9.

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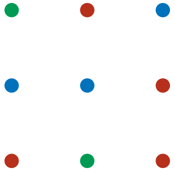

what is the probability of picking a ball of a specific colour?

- For red, it's $4/9$.
- For blue, it's $3/9$.
- For green, it's $2/9$.

The probabilities above **sum up to 1** because I am certain to pick *some* ball.

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So, for the 9-ball example from before, we would have

$$P(X = \text{red}) = \frac{4}{9}, \quad P(X = \text{blue}) = \frac{3}{9}, \quad P(X = \text{green}) = \frac{2}{9}.$$

Calculating Probability

In the case the set of outcomes is **finite**, the probability of X being one of the possible outcomes is always

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$$P(X \in S) = \frac{|S|}{|O|},$$

where S is a certain subset of O – all the possible outcomes.

Calculating Probability – Example

We'll describe our 9-ball example more formally.

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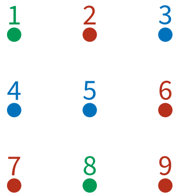
We'll assign the balls number from 1 to 9. The set of all possible outcomes of picking a random ball is then

$$O = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$



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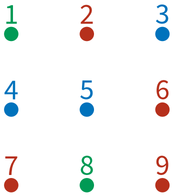

We'll form three subsets of O:

$$R = \{2, 6, 7, 9\},$$
$$B = \{3, 4, 5\},$$
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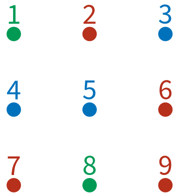

We'll form three subsets of O:

$$R = \{2, 6, 7, 9\},$$
$$B = \{3, 4, 5\},$$
$$G = \{1, 8\}.$$

We can use the formula from before to calculate the probability that X will be a green ball:

$$P(X \in G) = \frac{|G|}{|O|} = \frac{2}{9}.$$

Probability Equations

Sums of Probabilities

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However, this example cannot be easily generalized. We'll see why.

Sums of Probabilities – Counterexample

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$$E = \{2, 4, 6, \ldots, 20\},$$
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and we want to figure out the probability $P(X \in E \cup F)$.

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Let's try to use the same formula as before:

$$P(X \in E \cup F) = \frac{|E \cup F|}{|O|} \overset{??}{=} \frac{|E| + |F|}{|O|} = \frac{10 + 4}{20} = \frac{14}{20}.$$

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So, to get the size of $E \cup F$, we cannot just add the size of E to the size of F but we also have to subtract the elements that appear twice – the size of $E \cap F$.

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where $A_1, A_2, \ldots, A_n$ are any sets.

Such a formula is widely known as the **principle of inclusion and exclusion**.

Principle of Inclusion and Exclusion – Example

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How many people speak at least one language?

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Label the three language groups E, F and G. The setup from the previous slide can be summarized as

$\lvert E\rvert$	$\lvert F\rvert$	$\lvert G\rvert$	$\lvert E \cap F\rvert$	$\lvert E \cap G\rvert$	$\lvert F \cap G\rvert$	$\lvert E \cap F \cap G\rvert$
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We're trying to calculate $\lvert E \cup F \cup G\rvert$.

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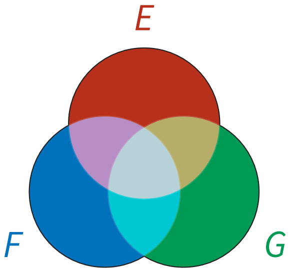
When working with sets, Venn diagrams are often a great choice.



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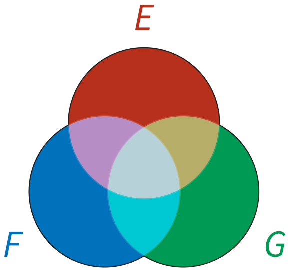

There are 7 regions in total (differentiated by colour) in this picture, corresponding to the 7 sets in the previous slide.

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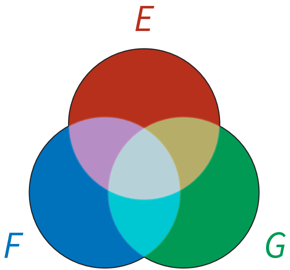

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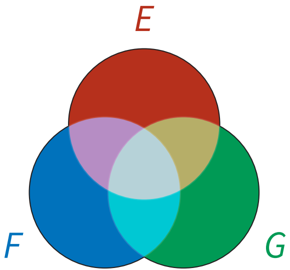

What we need to count is the total number of elements inside this entire shape. Let's start by counting the number of elements in each of the regions separately and assign numbers to regions corresponding to **how many times we've counted all the elements in that region.**

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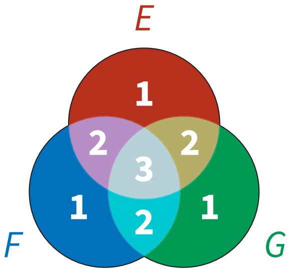

$$|E \cup F \cup G| = |E| + |F| + |G| \dots$$

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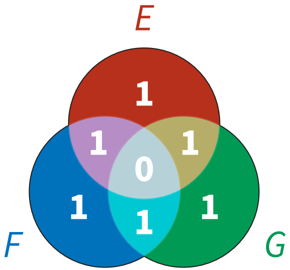

$$|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G|$$

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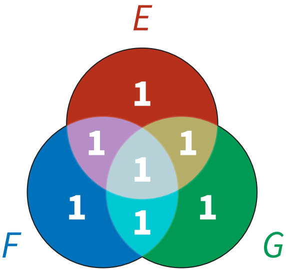

$$|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G| + |E \cap F \cap G|.$$

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$$|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G| + |E \cap F \cap G|.$$

Apply this formula to our example with language groups gives

$$|E \cup F \cup G| = 40 + 11 + 23 - 5 - 10 - 3 + 1 = 57.$$

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Apply this formula to our example with language groups gives

$$|E \cup F \cup G| = 40 + 11 + 23 - 5 - 10 - 3 + 1 = 57.$$

So, 57 people speak at least one language.

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4. Subtract the sizes of all four-set intersections.

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5. ...

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If $A_1, A_2, \ldots, A_n$ are sets with $n \in \mathbb{N}$, then

Principle of Inclusion and Exclusion

$$\begin{aligned}
|A_1 \cup A_2 \cup \ldots A_n| = {} & |A_1| + |A_2| + |A_3| + \ldots + |A_n| \\
& - |A_1 \cap A_2| - \ldots - |A_1 \cap A_n| - |A_2 \cap A_3| - \ldots - |A_{n-1} \cap A_n| \\
& + |A_1 \cap A_2 \cap A_3| + \ldots + |A_1 \cap A_2 \cap A_n| + \ldots |A_{n-2} \cap A_{n-1} \cap A_n| \\
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The $(-1)^n$ only means that if n is odd, then I subtract the last term, and I add it if n is even.

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and

$$O = \{1, 2, \ldots, 100\}.$$

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Using the **inclusion-exclusion principle**, we count

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$$= 50 + 33 + 14 - 16 - 7 - 4 + 2 = 72.$$

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So,

$$P(X \in E \cup T \cup S) = \frac{72}{100}.$$