



SYSTEMS OF LINEAR EQUATIONS

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FUNCTIONS

The bottom of the slide features two large, overlapping triangular shapes. The triangle on the left is a light blue color, and the triangle on the right is a darker blue color. They meet at a point in the center, creating a V-shape that points downwards.

WHAT IS A FUNCTION?

Intuitively, a function is a **box** which receives data and gives some data back.



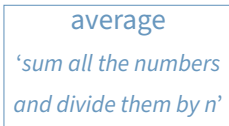
We'll call the data that **a function receives**, **inputs** and the **data it gives back**, **outputs**.

Inputs and **outputs** need not necessarily be just 'one object', they can be for example lists of numbers.

FUNCTIONS – EXAMPLE

A function which returns the **average** of a given set of numbers receives the numbers and also their count as **input** and returns the **average** as **output**.

x_1, x_2, \dots, x_n
 n



$\frac{x_1 + x_2 + \dots + x_n}{n}$

FUNCTIONS – EXAMPLE

We can also consider ‘non-mathematical’ functions. Like a function which receives a type of meal and returns the ingredients.



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FUNCTION COMPOSITION



FUNCTION COMPOSITION

If we have **two functions**, we can **in certain cases** 'compose' them.

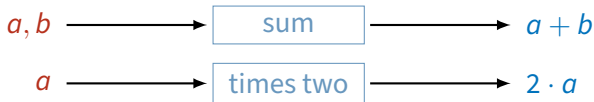
Composition simply means that **one function follows the other** – in other words, the **output** of the first function is the **input** of the second.

Of course, **composition** is only possible if the **output** of the first function is a valid **input** for the second.

For instance, you could hardly compose the **ingredients** function with the **average** function.

FUNCTION COMPOSITION

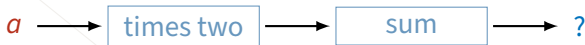
Considering two functions



their composition can look like this

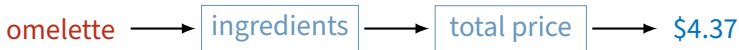


What would the output of this composition look like



FUNCTION COMPOSITION

So, the **order of the composition matters!** Here are a few examples of compositions which make sense:



We can of course compose **as many functions** as we like. An example of this:



FUNCTIONS – NOTATION

Drawing pictures like this would be cumbersome. Instead of



we simply write **function(input) = output**.

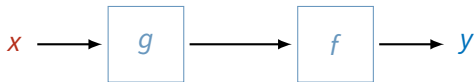
You're probably used to seeing function written like $f(x) = y$. The picture corresponding to this is



FUNCTIONS – NOTATION

The symbol used for function composition is \circ . It is however a little confusing because it is written in order ‘from right to left’.

For example, if f and g are two functions, their composition $f \circ g$ corresponds to this picture



that is, **first g , then f** .

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REAL FUNCTIONS



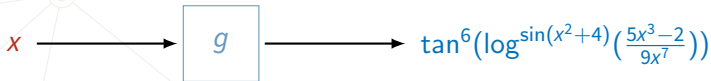
REAL FUNCTION

A **real function** is simply a function whose **input** and **output** are both real numbers.

Examples of such functions are

- $f(x) = 0$,
- $g(x) = \tan^6(\log^{\sin(x^2+4)}(\frac{5x^3-2}{9x^7}))$,

where $x \in \mathbb{R}$. Or, using pictures,



OPERATIONS ON REAL FUNCTIONS

As both the **input** and the **output** of a real function is a real number, **we can always compose real functions.**

However, that **doesn't mean that the order doesn't matter!** Different order of composition gives different functions.

For example, take $f(x) = 2x^2 + 7$ and $g(x) = \frac{1}{1+x}$. Then,

- $(f \circ g)(x) = 2 \left(\frac{1}{1+x} \right)^2 + 7$ and
- $(g \circ f)(x) = \frac{1}{1+(2x^2+7)}.$

OPERATIONS ON REAL FUNCTIONS

Real functions can also be **added** and **multiplied**, just like real numbers.

This involves simply adding or multiplying their respective outputs. For the functions

$f(x) = 2x^2 + 7$ and $g(x) = \frac{1}{1+x}$, their

- **sum** is the function with output

$$(f + g)(x) = f(x) + g(x) = 2x^2 + 7 + \frac{1}{1+x}.$$

- **product** is the function with output

$$(f \cdot g)(x) = f(x) \cdot g(x) = (2x^2 + 7) \cdot \left(\frac{1}{1+x} \right) = \frac{2x^2 + 7}{1+x}.$$

GRAPHS

As real functions have real numbers as inputs and outputs, they can be easily **graphed**.

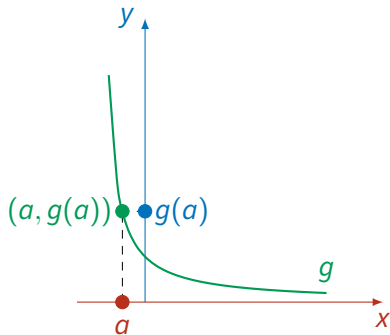
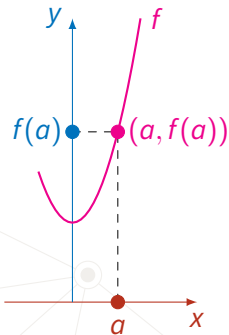
Graphing a real function f simply means drawing the points $(x, f(x))$ or (*input*, *output*) in some chosen coordinate system.

We typically use the **Cartesian** coordinate system with two axes (one for *input* and one for *output*) that are mutually perpendicular. These are often called the *x*-axis and the *y*-axis.

However, later, we'll also use the **polar** coordinate system where every point is instead determined by its angle and distance from the origin of the system.

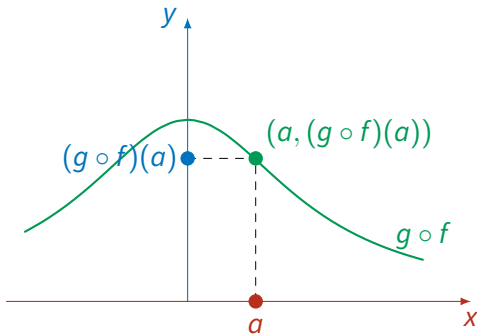
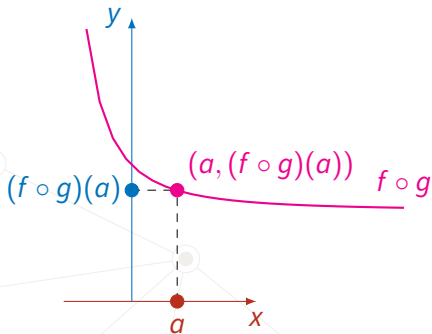
GRAPHS

The functions $f(x) = 2x^2 + 7$ and $g(x) = \frac{1}{1+x}$ have the following (parts of) graphs:



GRAPHS

Just to better drive home the idea that the order of function composition is important, look at the graphs of $f \circ g$ and $g \circ f$.



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LINEAR FUNCTIONS



LINEAR FUNCTION

Linear functions are a special type of real functions **whose output** is always **of the form** $a \cdot (\text{input}) + b$ for fixed numbers $a, b \in \mathbb{R}$.

In other words, linear functions are those real functions **that only scale the input and add something to it**.

LINEAR FUNCTION

A real function f is **linear** if

$$f(x) = ax + b$$

for some $a, b \in \mathbb{R}$.

LINEAR FUNCTIONS – PROPERTIES

Linear functions have some unique properties:

- If f and g are linear, so are $f \circ g$ and $g \circ f$. Indeed, we can see this easily. Suppose $f(x) = ax + b$ and $g(x) = cx + d$, then

$$(f \circ g)(x) = a(cx + d) + b = (ac)x + (ad + b),$$

$$(g \circ f)(x) = c(ax + b) + d = (ac)x + (cb + d).$$

- If f and g are linear, so is $f + g$. If we just compute the sum, we get

$$(f + g)(x) = (ax + b) + (cx + d) = (a + c)x + (b + d).$$

LINEAR FUNCTIONS – GRAPHS

Graphs of linear functions **are straight lines**.

Let us see why this is true. Choose a linear function

$$f(x) = ax + b.$$

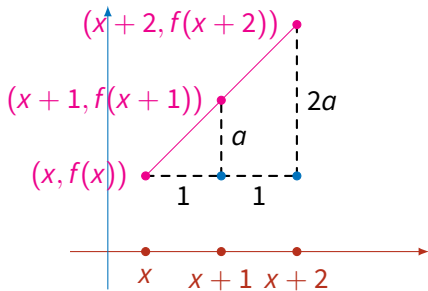
Pick a number – x . One way to show that the graph of f is a line is to move by two different distances from x and see that we get two similar triangles.

If we move by 1, from x to $x + 1$, then on the y -axis we move from $ax + b$ to $a(x + 1) + b$, that is, we move by

$$a(x + 1) + b - (ax + b) = a.$$

If we move by 2, from x to $x + 2$, on the y -axis, we move by

$$a(x + 2) + b - (ax + b) = 2a.$$



LINEAR FUNCTIONS – INTERSECTIONS

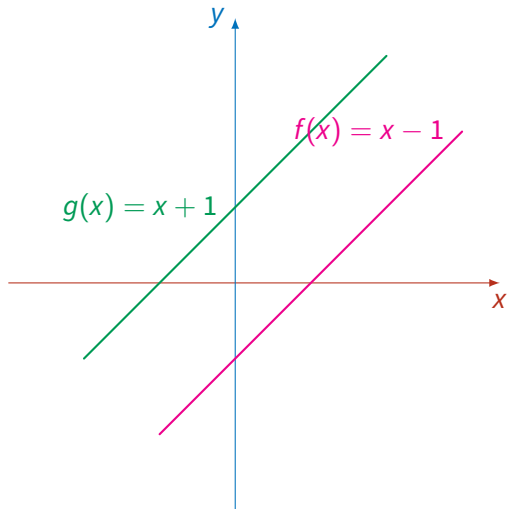
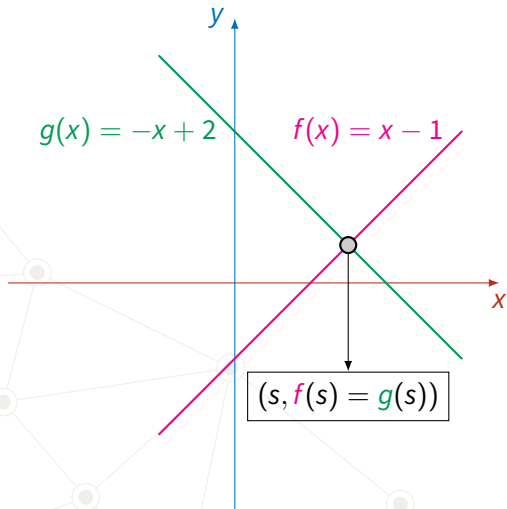
As two non-parallel lines **always intersect**, we should expect the equation $f(x) = g(x)$ to always **have a solution** assuming the graphs of f and g are not parallel.

The question is, when are the graphs of f and g parallel?

That happens exactly when their **rates growth** are identical.

In symbols, if $f(x) = ax + b$ and $g(x) = cx + d$, then the graphs of f and g are parallel if $a = c$.

LINEAR FUNCTIONS – INTERSECTIONS



LINEAR EQUATIONS

The bottom of the slide features two large, overlapping triangular shapes. The shape on the left is a light blue triangle pointing downwards. The shape on the right is a darker blue triangle pointing upwards. They overlap in the center, creating a darker blue triangular region at the bottom.

LINEAR EQUATION

LINEAR EQUATION

If f, g are linear functions, the equation

$$f(x) = g(x)$$

is called a **linear equation** (in one variable).

LINEAR EQUATION – SOLUTION

Suppose $f(x) = ax + b$ and $g(x) = cx + d$. What are the possible solutions to $f(x) = g(x)$?

Three things can happen:

- $a \neq c$. In this case, we can divide the equation by $a - c$ and get

$$ax + b = cx + d$$

$$(a - c)x = d - b$$

$$x = \frac{d - b}{a - c}.$$

- $a = c$ and $b \neq d$. In this case the graphs of the two functions are parallel lines – there is no solution.
- $a = c$ and $b = d$. In this case, the functions are one and the same and every number is a solution.

LINEAR EQUATIONS & OPERATIONS

For two linear functions f, g , when does

$$(f \circ g)(x) = (g \circ f)(x)$$

have a solution?

We can calculate that easily. If $f(x) = ax + b$ and $g(x) = cx + d$, then

$$(f \circ g)(x) = a(cx + d) + b = (ac)x + (ad + b),$$

$$(g \circ f)(x) = c(ax + b) + d = (ac)x + (bc + d).$$

We see that the graphs of $f \circ g$ and $g \circ f$ are parallel, so this equation has a solution only in the case that $ad + b = bc + d$.

1

LINEAR EQUATIONS IN TWO VARIABLES

LINEAR EQUATIONS IN TWO VARIABLES

SYSTEM OF LINEAR EQUATIONS

A pair of equations

$$ax + by = c,$$

$$dx + ey = f,$$

that have to be **simultaneously** satisfied, is called a **system of (two) linear equations** (in two variables).

LINEAR EQUATIONS IN TWO VARIABLES

Linear equations in two variables can be **reduced to** linear equations **in one variable**. This is easily done. We can simply **isolate** one of the variables and make the other variable **into a function**.

For example, imagine the equation

$$4x - y = 2.$$

We can rewrite this as

$$y = 4x - 2,$$

basically making y into a linear function $f(x) = 4x - 2$.

LINEAR EQUATIONS IN TWO VARIABLES

Linear equations in two variables can be **reduced to** linear equations **in one variable**. This is easily done. We can simply **isolate** one of the variables and make the other variable **into a function**.

For example, imagine the equation

$$4x - y = 2.$$

Similarly, we can write

$$x = \frac{1}{4}y + \frac{2}{4}$$

and turn x into a linear function $g(y) = \frac{1}{4}y + \frac{2}{4}$.

LINEAR EQUATIONS IN TWO VARIABLES

You probably know this reduction under the name of **substitution**.

Let us see it in practice.

Suppose we want to solve the system

$$-3x + y = 6,$$

$$x - 2y = 6.$$

From the first equation, we see that

$$y = 3x + 6,$$

and from the second that

$$y = \frac{1}{2}x - 3.$$

LINEAR EQUATIONS IN TWO VARIABLES

From the first equation, we see that

$$y = 3x + 6,$$

and from the second that

$$y = \frac{1}{2}x - 3.$$

Therefore, we get the linear equation **in one variable**

$$3x + 6 = \frac{1}{2}x - 3.$$

whose solution is $x = -\frac{18}{5}$.

LINEAR EQUATIONS IN TWO VARIABLES

From the first equation, we see that

$$y = 3x + 6,$$

and from the second that

$$y = \frac{1}{2}x - 3.$$

The functions $f(x) = 3x + 6$ and $g(x) = \frac{1}{2}x - 3$ are linear, so we can draw this equation as an intersection of two lines.

LINEAR EQUATIONS IN TWO VARIABLES

