Logic & Set Theory Cheatsheet

3.AB PreIB Math

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Logic

Logic is the language of mathematics. It uses propositions to talk about sets.

Propositions are sentences which can be either true or false. For example

- 'Cats are black.' is a proposition;
- 'How are you?' is *not* a proposition;
- 'We will have colonised Mars by 2500.' is also a proposition.

As the third example suggests, we need not necessarily know whether a proposition is true or false – it remains a proposition anyway.

Logical Conjunctions

Propositions can be joined together using logical conjunctions. They pretty much correspond to the conjunctions of natural language. Let us consider two propositions:

$$p = \text{'It's raining outside.'}$$
$$q = \text{'I'll stay at home.'}$$

($\wedge$) Logical and forms a proposition that is only true if both of its constituents are also true. In natural language, the proposition $p \wedge q$ can be expressed as

$$p \wedge q = \text{'It's raining outside and I'll stay at home.'}$$

($\vee$) Logical or forms a proposition that is true if at least one of its constituents is true. In natural language, the proposition $p \vee q$ can be expressed as

$$p \vee q = \text{'It's raining outside or I'll stay at home.'}$$

In mathematical logic, or is **not exclusive**! This means that $p \vee q$ is true even if both p and q are true.

($\neg$) Logical not isn't strictly speaking a conjunction but I include it anyway. It reverses the truth value of a proposition. For example, the proposition $\neg p$ can be read as

$$\neg p = \text{'It's not raining outside.'}$$

It follows that $\neg p$ is true exactly when p is false and vice versa.

($\Rightarrow$) Logical implication is a conjunction that makes the first proposition into an *assumption* or *premise* and the second one into a *conclusion*. The proposition $p \Rightarrow q$ is read in multiple ways, to list a few:

$$p \Rightarrow q = \text{'If it's raining outside, then I'll stay at home.'}$$
$$p \Rightarrow q = \text{'It raining outside implies that I'll stay at home.'}$$
$$p \Rightarrow q = \text{'Assuming it's raining outside, I'll stay at home.'}$$

The implication is tricky. It's true if both p and q are true and false if p is true but q is false. However, it is always true if p is false. That is because, in mathematical logic, whatever follows from a lie is automatically true.

($\Leftrightarrow$) Logical equivalence is true only if both propositions have the same truth value – they're both true or both false. In natural language, it is typically read like this:

$$p \Leftrightarrow q = \text{'It's raining if and only if I stay at home.'}$$

Equivalence is basically just a two-way implication. The proposition p is both a premise and a conclusion to q and q is both a premise and a conclusion to p. If it's raining outside, I stay at home and if I stay at home, then it's raining outside.

Truth Tables

A conjunction of propositions being true or false based on whether its constituent propositions are true or false can be summarized using so-called truth table. It is basically just a table that lists all the possibilities of p and q being true or false and the resulting truth value of their conjunctions.

For the basic logical conjunctions from above, it can look like this (we represent true by 1 and false by 0):

p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	1	1	0	0	1	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	0	0
1	1	0	0	1	1	1	1

Sets

Sets are the 'stuff' that makes up the world of mathematics. Their basic characteristics and properties are described using logic.

Sets cannot be defined inside set theory but we interpret them as *groups of things*.

There's only one foundational *proposition* related to set theory – the proposition 'An object is an element of a set.' If we label the object in question x and the set A, this proposition is written as $x \in A$ (the symbol $\in$ is just the letter 'e' in 'element'). Combining these propositions using logical conjunctions allows for various set-theoretic constructions.

If a set A has, for example, exactly three elements – $\square$, $\triangle$ and $\bigcirc$, I can write it as a list of these three elements inside curly brackets $\{\}$. In this case,

$$A = \{\square, \triangle, \bigcirc\}.$$

A few warnings about sets:

- **Sets are not ordered**. There is nothing like a 'first', 'second' or 'last' element of a set. Either an object **is** inside a set or it **isn't**. Nothing else. For example, the three sets below are exactly the same, only written differently.

$$\{\square, \triangle, \bigcirc\} = \{\bigcirc, \triangle, \square\} = \{\triangle, \square, \bigcirc\}$$

- **Elements of sets have no frequency**. Again, an element either is inside a set or not. It cannot be twice in a set, for example. The three sets below are exactly the same.

$$\{\square, \triangle, \bigcirc\} = \{\square, \triangle, \bigcirc, \triangle, \bigcirc\} = \{\triangle, \square, \square, \triangle, \bigcirc, \triangle\}$$

Set Operations

Using logical conjunctions, we form new sets from existing ones. Consider two sets – A and B.

($\cap$) I can form the set of all objects x that satisfy the proposition $x \in A \wedge x \in B$, that is all objects that lie in both A and B. This set is called the intersection of A and B and written $A \cap B$. For example,

$$\{\bigcirc, \triangle, \square\} \cap \{\times, \bigcirc, \square, \sim\} = \{\bigcirc, \square\}.$$

($\cup$) I can form the set of all objects that satisfy the proposition $x \in A \vee x \in B$, the set of all objects that lie in A or in B. It is called the union of A and B and denoted $A \cup B$. All elements of $A \cup B$ can be found *only* in A, *only* in B or in *both* A and B. For example,

$$\{\bigcirc, \triangle, \square\} \cup \{\times, \bigcirc, \square, \sim\} = \{\bigcirc, \triangle, \square, \times, \sim\}.$$

($\Rightarrow$) Implication is a little different from intersection and union. It describes a lot of different sets with one logical proposition. I ask: 'Which sets A satisfy the proposition $x \in A \Rightarrow x \in B$?' In other words, which sets A have all their elements contained in the set B? The answer is that A must be a subset of B and we denote that fact by $A \subseteq B$. The set A is only allowed to have elements which also lie in B but not necessarily all of them. All the subsets of $B = \{\triangle, \bigcirc\}$ are listed below.

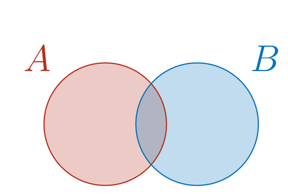
$$\emptyset, \{\triangle\}, \{\bigcirc\}, \{\triangle, \bigcirc\},$$

where $\emptyset$ is the empty set, a set containing no elements.

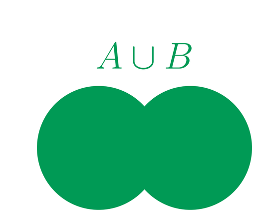
($\Leftrightarrow$) Equivalence defines equality on sets. If sets A and B must satisfy the proposition $x \in A \Leftrightarrow x \in B$, then they must be equal because all the elements of A lie in B and all elements of B lie in A. That is, $A = B$.

Drawing Sets

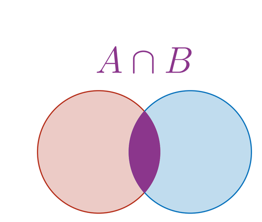
Set operations can be visualized using so-called *Venn diagrams*. This just means using circles to represent the sets in questions. For example, two sets – A and B – can be drawn like this:



In these pictures, one can easily visualize the operations of union and intersection. The union $A \cup B$ is the entire area covered by A and B. It looks like this:



The intersection $A \cap B$ is the 'strip' in the middle, the area which is shared between both A and B. It can be depicted like this:



Products of Sets & Relations

Before introducing *products* of sets, we must define a *pair*. Simply said, a pair of objects (a, b) is just a set containing a and b **with ordering**, that is, a is the **first** element of (a, b) and b is **second**. This means that $(a, b) \neq (b, a)$ because the order is not the same.

Now, the product of sets A and B, denoted $A \times B$, is the set of all pairs (a, b) where $a \in A$ and $b \in B$. For example, if

$$A = \{\bigcirc, \triangle\} \quad \text{and} \quad B = \{\triangle, \times, \sim\},$$

then

$$A \times B = \{(\bigcirc, \triangle), (\bigcirc, \times), (\bigcirc, \sim), (\triangle, \triangle), (\triangle, \times), (\triangle, \sim)\}.$$

Notice that $A \times B \neq B \times A$ because the **order** of elements in a pair **matters**. In this case,

$$B \times A = \{(\triangle, \bigcirc), (\triangle, \triangle), (\times, \bigcirc), (\times, \triangle), (\sim, \bigcirc), (\sim, \triangle)\}.$$

As another example, consider the *real plane* – the set of all points with two coordinates. That is simply the set of pairs of real numbers, $\mathbb{R} \times \mathbb{R}$.

The mathematical way to define a relation between two sets is to simply **list all the elements that are related**. Said formally, a relation R is a subset $R \subseteq A \times B$. For example,

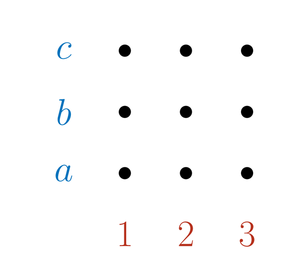
$$\{(\triangle, \times), (\triangle, \sim)\} \subseteq \{\bigcirc, \triangle\} \times \{\triangle, \times, \sim\}$$

is a relation between the sets A and B from above. It literally says that $\triangle$ is related to $\times$ and $\sim$. Instead of writing $(\triangle, \times) \in R$, we write $\triangle R \times$ because it's more natural. Typical examples of relations include $\leq$ and $=$ and we don't write $(2, 3) \in \leq$ but $2 \leq 3$.

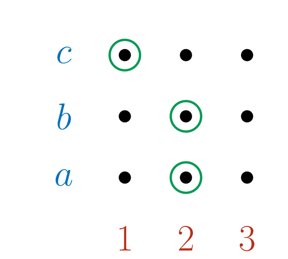
Drawing Products And Relations

One can draw the product $A \times B$ similarly to the way we draw Cartesian coordinates – by distributing the elements of A horizontally and those of B vertically. Each point in the resulting grid represents an element of $A \times B$.

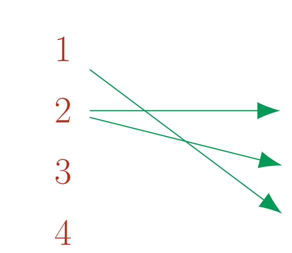
Take, for example, $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. We can depict the set $A \times B$ like this:



Any relation $R \subseteq A \times B$ can now be easily drawn into the grid just by marking certain dots. For example, the relation $R = \{(1, c), (2, b), (2, a)\} \subseteq A \times B$ looks like this:



There is another way to draw relations – as arrows from A to B. This style of drawing emphasises the 'one-way' nature of relations. The same relation $R \subseteq A \times B$ is drawn using arrows like this:



Equivalence & Equivalence Classes

A relation $E \subseteq A \times A$ (do note that it's a relation **on a set**) is called an equivalence if it behaves **something like 'equals'**, that is,

(R) it's reflexive, i.e. aEa for every $a \in A$ (every element is equivalent to itself);
(S) it's symmetric, i.e. if $a_1 E a_2$, then also $a_2 E a_1$ (all elements are **mutually** equivalent);
(T) it's transitive, i.e. if $a_1 E a_2$ and $a_2 E a_3$, then $a_1 E a_3$ (it **propagates** through a middle element).

A good example of equivalence to keep in mind is the relation '**being the same age**' on the set of all people. It's reflexive because I'm as old as me. It's symmetric because if I'm as old as you, then you are as old as me. And finally, it's transitive because if I'm as old as you and you are as old as someone else, then I'm as old as that someone. This example also drives home the idea of equivalence 'being like equals'. It's not that two people of the same age are *equal*, they are *equal by some criterion*. This is exactly what equivalence is, the relation of being equal by some criterion.

With an equivalence $E \subseteq A \times A$, we can divide the set A into 'packets' of elements – each packet consisting of elements which are **equivalent** by E. For example, I can divide the set of all people by putting equally old people to the same packet. Formally, we write

$$[a]_E = \{b \in A \mid aEb\},$$

in other words, $[a]_E$ is the set of all elements from A that are $equivalent$ to a. We call it an equivalence class. The element a is then called a representative.