Linear Equations
Scales, Lines & Functions

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We have seen three different ways of interpreting linear equations – as comparisons of functions, as crossings of lines and as pairs of scales. We'll review them here and then you'll have to do a few exercises.

Linear equations as functions

In this interpretation, it is useful to think of numbers as representing *prices* of the things, for instance in dollars. A linear functions is some 'box' which eats a thing of some price and gives back a multiple (possibly negative) of that thing and an amount (possibly negative) of dollars on top.

One can draw such a function for instance like this:

This function receives a $\triangle$ of some price and gives back two such $\triangle$'s and $4. A *linear equation* is a comparison of the total price of things two (typically different) functions give back when receiving **the same thing**. The *solution* is the **price of the thing** for which the total prices of what is given back by these two functions are equal. One can visualize it like this:

Of course, one typically calls BOX #1 and BOX #2 by letters, such as f and g and treats them as linear functions in one variable. Then, the picture above is simply written as $f(\triangle) = g(\triangle)$.

Exercises.

1. Sir Minkowski and Sir Riemann are pickpockets. Sir Minkowski boasts that he robs on average 5 people a day of their wallets and finds on average \$50 by picking coins from the ground. Sir Riemann takes a nobler approach. He only ever robs or cajoles people and deigns not pick money from the ground. On average, he robs 7 people a day of their wallets. How much money must an average person's wallet contain so that Sir Riemann and Sir Minkowski have gained the same amount of money by the end of each day?

   Express the total value both Sir Riemann and Sir Minkowski acquire each day as linear functions and compare their outputs. What does the **variable** of these functions represent? What does the **value/output** of these functions represent? Are they the same?

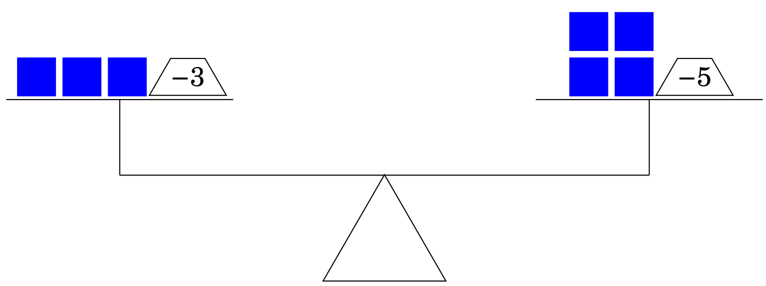
2. The cryptocurrencies Dogecoin and Freedomcoin are steadily losing their value (as expressed in US Dollars). At the time of writing, Dogecoin stands at \$0.094 apiece and Freedomcoin stands at \$0.0188 apiece. For every lost investor, the value of Dogecoin drops by 0.1 % of its current value. Assuming that Freedomcoin loses no investors, how many investors does Dogecoin have to lose so that its value falls to the value of Freedomcoin?

   Express the drop in value of Dogecoin based on the number of lost investors as a linear function. Compare it to the value of Freedomcoin. As the numbers are ugly, **you simply have to construct the equation, not solve it**.

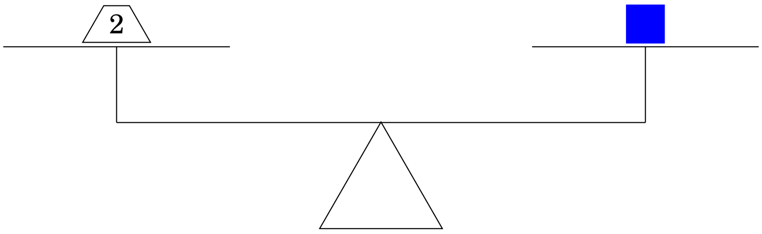
Linear equations as scales

This interpretation of linear equations is based on the comparison of *weights* of objects. Unlike in physics, we have no qualms about giving our objects negative weights. The bowls of the given pair of scales contain one type of object (whose weight we typically don't know) and then some absolute weights. The word *equation* is here expressed in the fact that the scales are always in balance. The *solution* to such an equation is a situation where in either bowl, there is only one object of unknown value and in the other are only absolute weights.

For instance, one possible solution to the equation

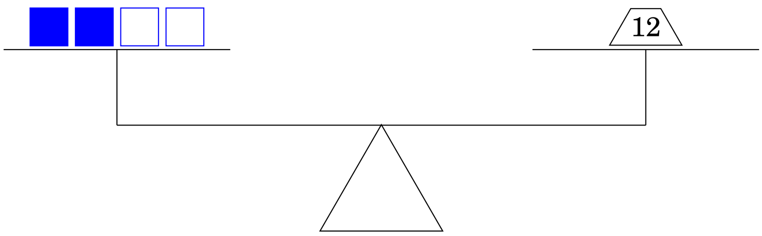


is the following state of the scales:



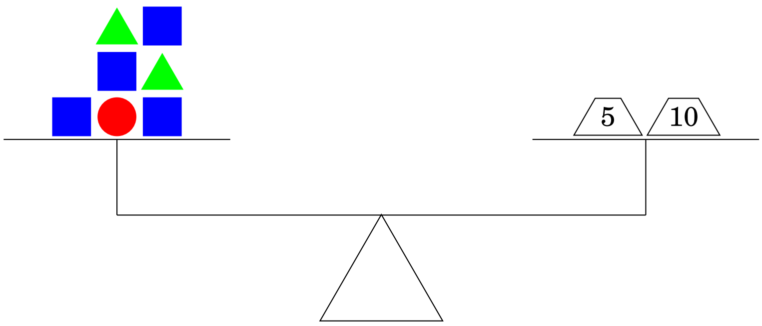
Exercises.

1. Pharaoh Amenhotep had been a notoriously known swindler. He would fill hollow objects with dirt to increase their weight and thus sell them for more gold. Knowing that the following scales are in balance

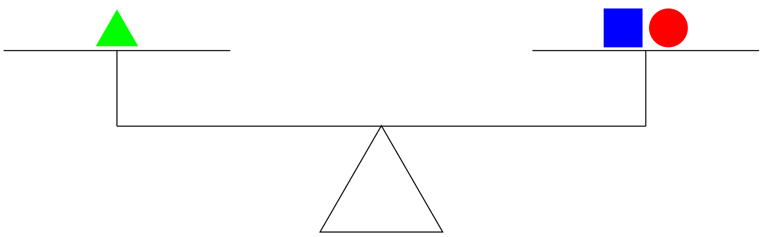


where □ are the hollow objects and ■ are those filled with dirt, determine the weight of □ if one ■ is 2 kg heavier than one □.

2. The following scales are in balance:



Find a way to remove all objects from both bowls **without ever breaking the balance** assuming that the scales



are also in balance. **You are not allowed to add objects to the scales or to split weights. You can only remove objects as they are and you must maintain the balance of the scales in the process**.

Linear equations as lines

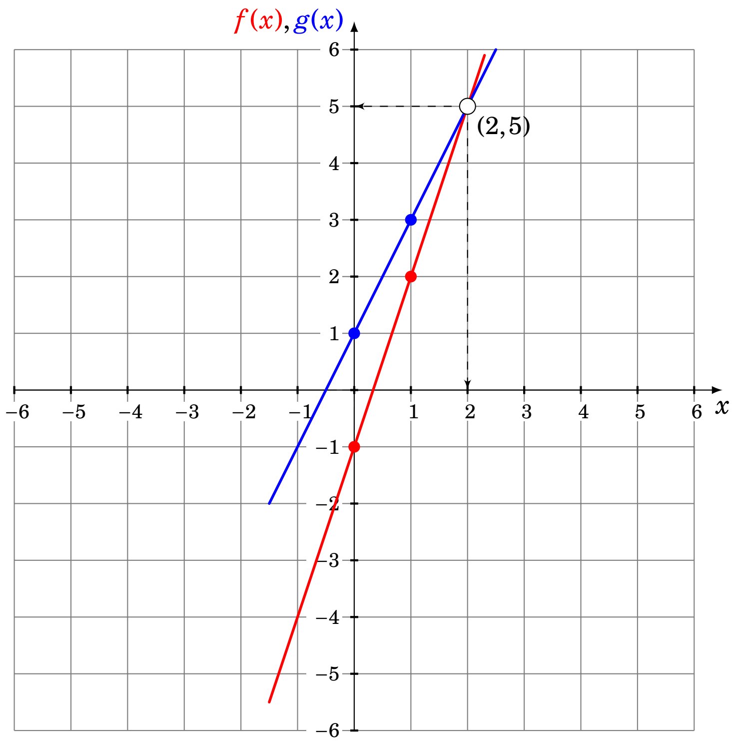
Given some linear equation, for example

$$3x - 1 = 2x + 1,$$

one can look at the left side and the right side separately as linear functions in one variable. We know that we can visualize linear functions in one variable as lines. Hence, the *equation* in this case is a 'drawing' of two lines in the plane. We let $f(x) = 3x - 1$ and $g(x) = 2x + 1$ and draw the values of x on the horizontal axis and the values of both f and g on the vertical. For each, we calculate two points (that is all we need to draw a line). We can put them into a table:

x	0	1
$f(x)$	-1	2
$g(x)$	1	3

Using these points to draw our two lines gives the following picture.



The *solution* to this equation is the point of intersection of the line representing f with the line representing g.

Exercises.

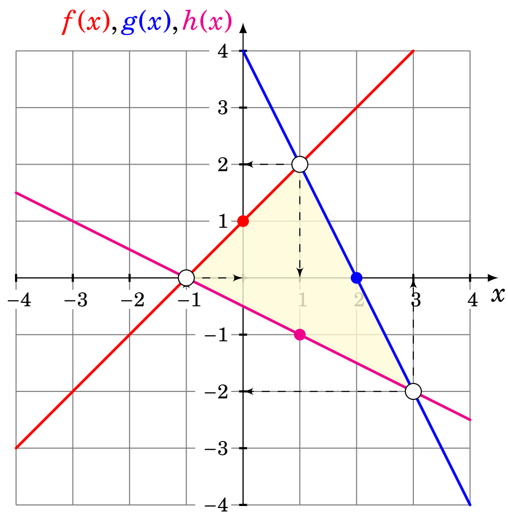
1. Two competing brothers, Lidl and Penny, decided to go on a vacation together. However, one cannot stand the idea of arriving to their chosen destination later than the other. They decided they're both going to travel on motorcycles this time around. Lidl is the poorer of the two and his motorcycle only reaches the velocity of 80 km/h on average. But, Lidl doesn't mind getting up early and is thus willing to set off already at 5 AM. Penny, on the other hand, owns a motorcycle that tops 120 km/h but is too lazy to get up so early in the morning. Assuming that their destination is 720 km away, when does Penny have to get up to arrive there **at the same time as** Lidl and prevent a quarrel?

   Represent time traveled on the horizontal axis and distance traveled on the vertical axis. Define linear functions $l(x)$ and $p(x)$ representing the distance traveled by Lidl and Penny based on x (which is the time traveled). Make sure to choose $l(x)$ so that the point of intersection of these two lines has y-coordinate equal to 720, which precisely means that they both traveled 720 km before meeting each other.

2. You're given three linear functions,

$$
\begin{aligned}
f(x) &= x + 1, \\
g(x) &= -2x + 4, \\
h(x) &= -\frac{1}{2}x - \frac{1}{2}.
\end{aligned}
$$

   Represented as lines, they look like this:



As you can see, the linear equation

$$f(x) = g(x) = h(x)$$

has no solution because the three lines don't intersect at a single point. Move any one of the lines (**without changing its slope/steepness**) so that all three *do* intersect at a single point. Change also the definition of the corresponding linear function. That is, if you choose to move for example the line representing h, then the definition of the function h as given above must change as well.

Look at the triangle (filled with yellow color) which is determined by the three intersection points. Observe that if the equation $f(x) = g(x) = h(x)$ *has no solution*, then the three lines always determine a triangle. Deduce the relation between the number of solutions of $f(x) = g(x) = h(x)$ (that is, one or no solution) and the area of this triangle. In other words, what is the area of this triangle if $f(x) = g(x) = h(x)$ **has** a solution?