



NUMBER SETS

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NATURAL NUMBERS

NATURAL NUMBERS – INTUITION

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They're the following set:

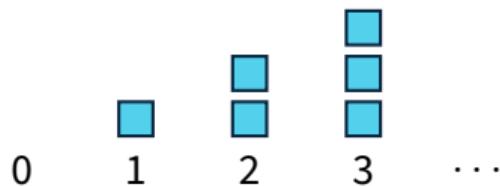
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NATURAL NUMBERS – INTUITION

Natural numbers are intuitively objects which represent a **quantity**.
They're the following set:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



NATURAL NUMBERS – DEFINITION

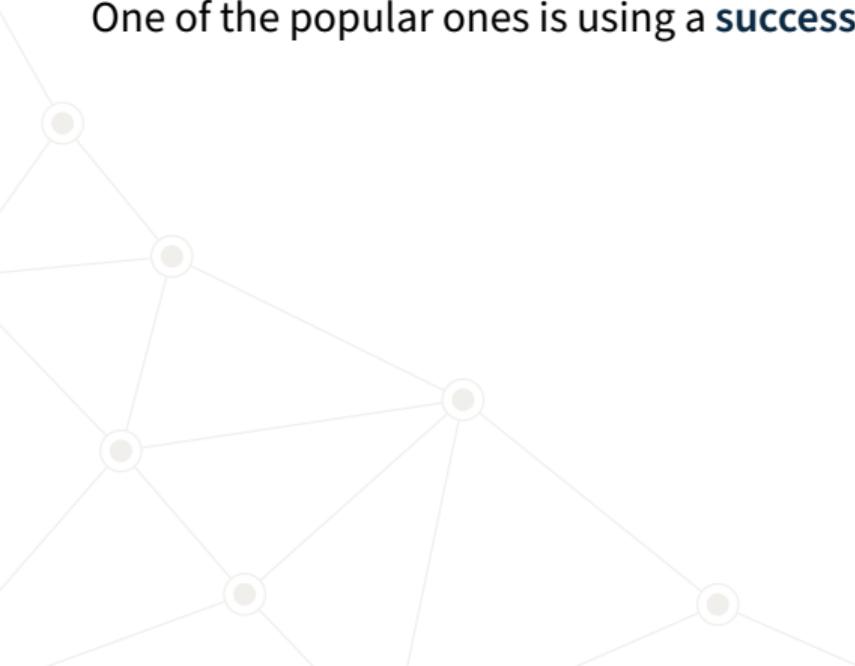
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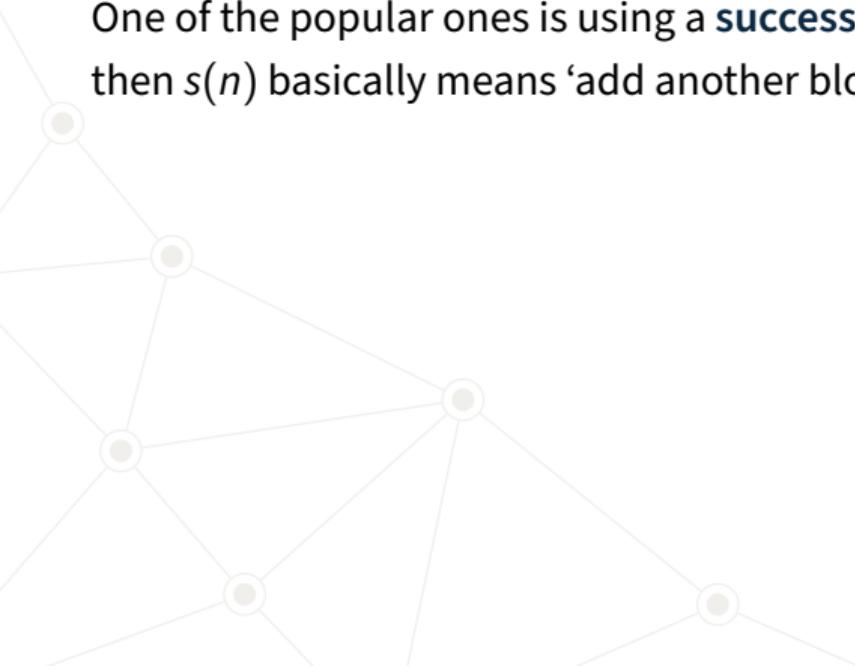
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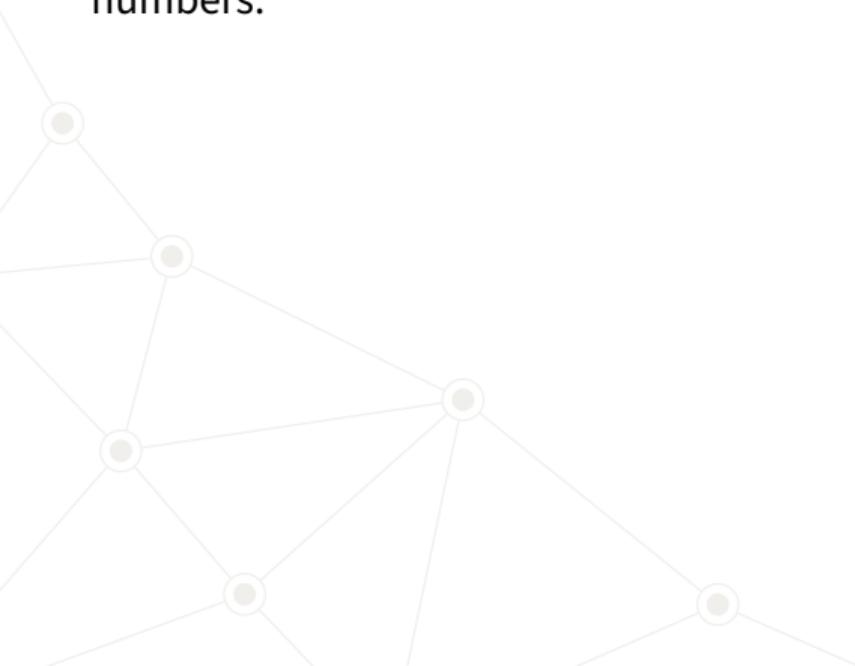
One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet!** In fact, you need the successor function to define addition in the first place.

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4. If $s(n) = s(m)$, then $n = m$.
5. (Induction Axiom) If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

1

UNPACKING THE AXIOMS



NATURAL NUMBERS – AXIOM 1

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There exists the natural number 0.

Hopefully obvious.

NATURAL NUMBERS – AXIOM 2

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Every natural number has a successor which is also natural.

Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

NATURAL NUMBERS – AXIOM 3

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The number 0 is not the successor of any natural number.

Basically means that the natural numbers are infinite only ‘in one direction’. There is a **first** natural number.

NATURAL NUMBERS – AXIOM 4

If $s(n) = s(m)$, then $n = m$.

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If $s(n) = s(m)$, then $n = m$.

This means that the successor function is **injective** – each natural number has a different successor.

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NATURAL NUMBERS – AXIOM 5

If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

This means that any feature of the natural numbers ‘propagates’ via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it is true for 1, it is true for 2 as well, etc.

2

OPERATIONS ON NATURAL NUMBERS



WHAT IS AN OPERATION?

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We don't often see them as functions because we don't write them as such. We write $n + m$ instead of $+(n, m)$ and $n \cdot m$ instead of $\cdot(n, m)$.

In this sense, subtraction and division **are not operations!** They take two natural numbers but they **do not produce a natural number**.

ADDITION

We define **addition** on natural numbers by the following two formulae:

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- $n + s(m) = s(n + m)$.

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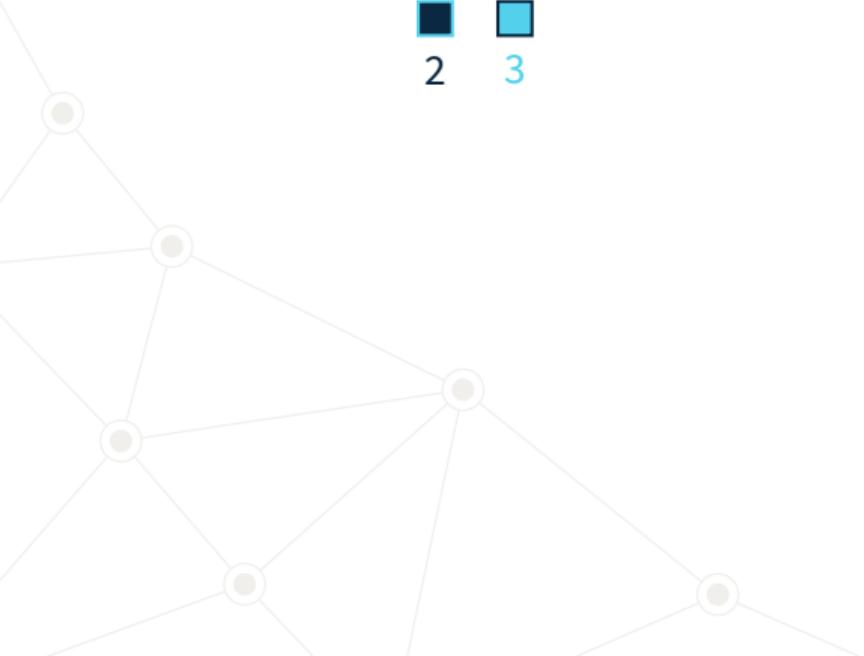
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We can imagine addition as ‘adding blocks *to the side*’ and the successor function as ‘adding one block *on top*’.

In this sense, $n + s(m) = s(n + m)$ only means that if you add one block atop m blocks and then n blocks to the side you have the same number of blocks as if you add n blocks next to m blocks and then another on top of that.

ADDITION



ADDITION


$$2 + s(3)$$

ADDITION

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2 3

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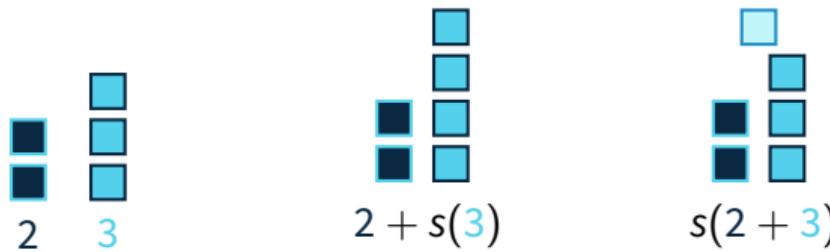
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$$n + 0 = n,$$

$$n + 1 = n + s(0) = s(n + 0) = s(n),$$

$$n + 2 = n + s(1) = s(n + 1) = s(n + s(0)) = s(s(n + 0)) = s(s(n)),$$

⋮

ADDITION – PROPERTIES

Addition of natural numbers satisfies these two properties:

- **Commutativity:**

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- **Commutativity:**

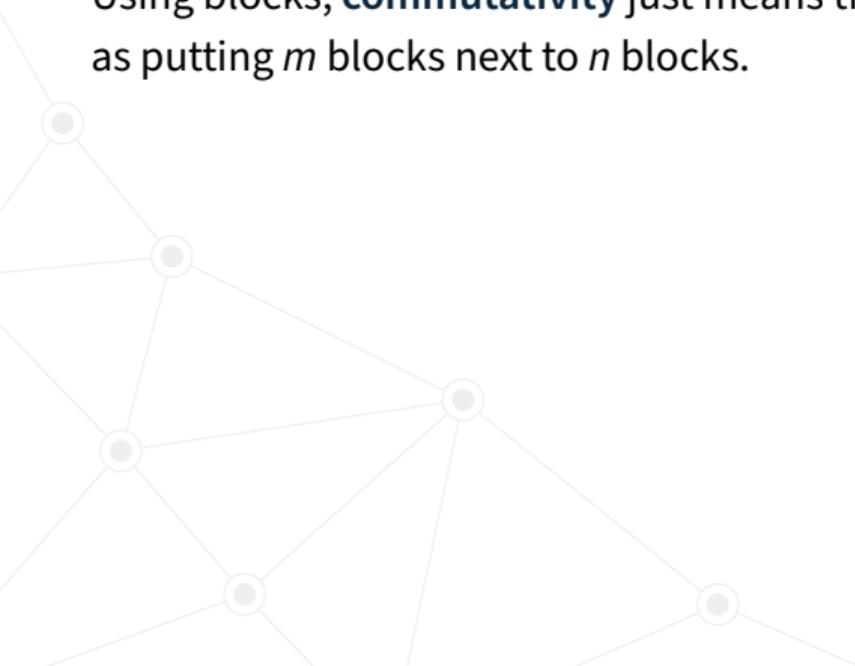
$$n + m = m + n.$$

- **Associativity:**

$$n + (m + k) = (n + m) + k.$$

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Using blocks, **commutativity** just means that putting n blocks next to m blocks is the same as putting m blocks next to n blocks.



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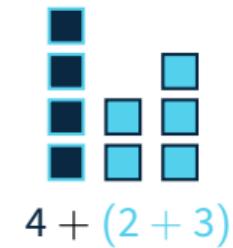
$$2 + 3$$



$$3 + 2$$

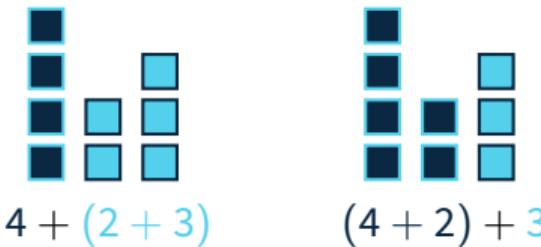
ADDITION – PROPERTIES

Using blocks, **associativity** just means that putting m blocks next to k blocks and then n more blocks next to those is the same as putting m blocks next to n blocks and then k more blocks next to those.


$$4 + (2 + 3)$$

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MULTIPLICATION

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- $m \cdot s(n) = m \cdot n + m$.

We can imagine multiplication $m \cdot n$ by adding a collection of n blocks for every one block in the collection of m blocks.

If we write $s(n) = n + 1$, then the second formula just means that

$$m \cdot s(n) = m \cdot (n + 1) = m \cdot n + m.$$

MULTIPLICATION

$$\begin{array}{c} \blacksquare \\ \blacksquare \end{array} \quad \begin{array}{c} \square \\ \square \\ \square \end{array}$$

2 3

A diagram illustrating multiplication. On the left, there is a network of five nodes connected by lines. To the right of this network, two sets of colored squares are shown vertically. The first set, labeled '2', contains two dark blue squares. The second set, labeled '3', contains three light blue squares. This visual representation likely corresponds to the multiplication problem $2 \times 3 = 6$.

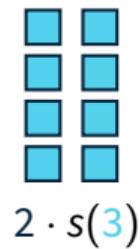
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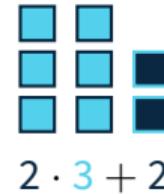
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$$\begin{matrix} & \text{□} & \text{□} \\ 2 \cdot s(3) & \end{matrix}$$

MULTIPLICATION


$$\begin{matrix} & \\ \text{2} & \text{3} \end{matrix}$$


$$2 \cdot s(3)$$


$$2 \cdot 3 + 2$$

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$2 \cdot 3 + 2$

The formula $m \cdot s(n) = m \cdot n + m$ allows us to compute $m \cdot n$ by applying it n times. More precisely,

$$m \cdot 1 = m$$

$$m \cdot 2 = m \cdot s(1) = m \cdot 1 + m = m + m$$

$$m \cdot 3 = m \cdot s(2) = m \cdot 2 + m = m \cdot s(1) + m = m \cdot 1 + m + m = m + m + m$$

⋮

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- Distributivity:

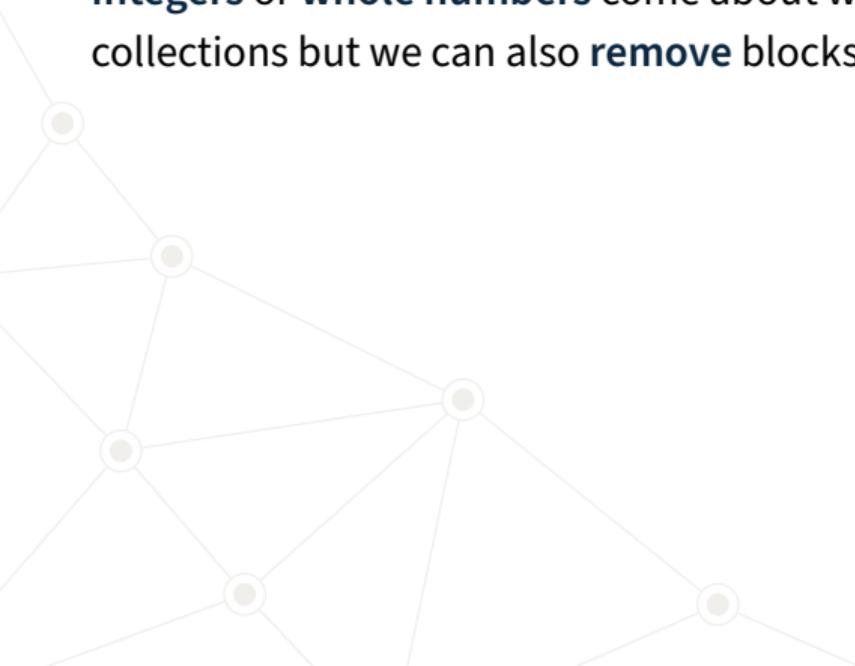
$$m \cdot (n + k) = m \cdot n + m \cdot k.$$



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Beware! ‘Taking inverse’ is a **unary** operation (meaning it acts on **one element**), not a binary one.

1

DIGRESSION



MATHEMATICAL STRUCTURES

DESTRUCTIVE & SYMMETRIC TRANSFORMATIONS



MATHEMATICAL STRUCTURES

MATHEMATICAL STRUCTURE (ALSO ‘UNIVERSE’)

A **mathematical** structure is a **set** with **operations**. If X is the set and op_1, \dots, op_n the operations on X , we write the resulting structure as

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The reason mathematical structures are called by some enthusiasts as mathematical ‘universes’ is that they really *are* universes in the broadest sense possible – a bunch of elements with prescribed rules of interaction.

OPERATION ON A SET

OPERATION

An **operation** on a set X is really just a **rule of interaction** between its elements. In symbols, it is a **function**

$$op : X^n \rightarrow X$$

where $X^n \rightarrow X$ just means ‘Take n elements of X and give me back one.’

OPERATION ON A SET

Examples:

- $(\mathbb{N}, +, \cdot)$ is a structure where $+$ and \cdot are **binary** operations (meaning they take two elements and return one). They can be seen as functions $\mathbb{N}^2 \rightarrow \mathbb{N}$.

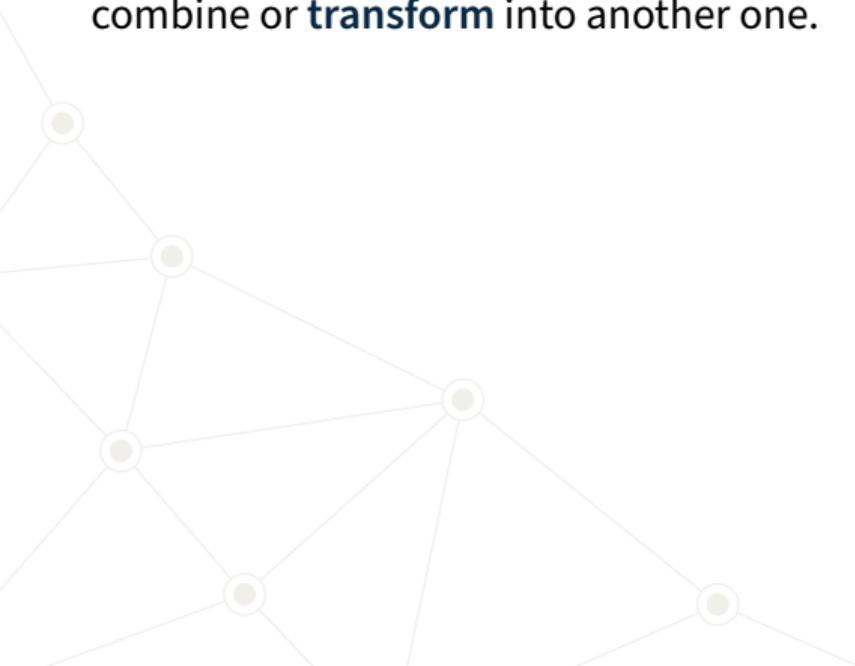
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- $(\{\text{orderings of vertices of a regular } n\text{-gon}\}, r, r^2, \dots, r^n, s_1, \dots, s_n)$ is a structure where r is the rotation by $360^\circ/n$ and s_1, \dots, s_n are all the reflections. They are all **unary** operations (they take one element and return one).

OPERATION AS A TRANSFORMATION

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- The operation r on the vertices of a regular polygon is **symmetric** – it can be reversed or **inverted**. If, for example, vertex A is sent to G by this rotation, then rotation in the opposite direction sends G back to A .

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$$x \Delta e = x$$

for all elements $x \in X$.

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for all elements $x \in X$.

- Elements x^* and y^* (the **inverse** elements to x and y) such that

$$x \Delta x^* = y \Delta y^* = e.$$

OPERATION AS A TRANSFORMATION

Let's pick an (commutative) operation Δ on a set X and two elements $x, y \in X$.

- Suppose $x \Delta y$ is some element $z \in X$. For the operation Δ to be **symmetric**, we need two things:
 - A special element $e \in X$ (the **identity** element) which satisfies

$$x \Delta e = x$$

for all elements $x \in X$.

- Elements x^* and y^* (the **inverse** elements to x and y) such that

$$x \Delta x^* = y \Delta y^* = e.$$

- It's possible X doesn't have those things – which means the operation Δ cannot be symmetric **on X** , but it can be symmetric on a larger set which does contain those elements.

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Suppose $x \Delta y = z$. Then, $z \Delta y^* = x$ and $z \Delta x^* = y$.

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Suppose $x \Delta y = z$. Then, $z \Delta y^* = x$ and $z \Delta x^* = y$.

Why? We know that $x \Delta x^* = e = y \Delta y^*$. Therefore, for example,

$$z \Delta y^* = (x \Delta y) \Delta y^* = x \Delta (y \Delta y^*) = x \Delta e = x.$$

INVERSE AS A FUNCTION

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Clearly, the inverse to x^* is x , again.

DRAWING STRUCTURES

Let's draw some mathematical structures.



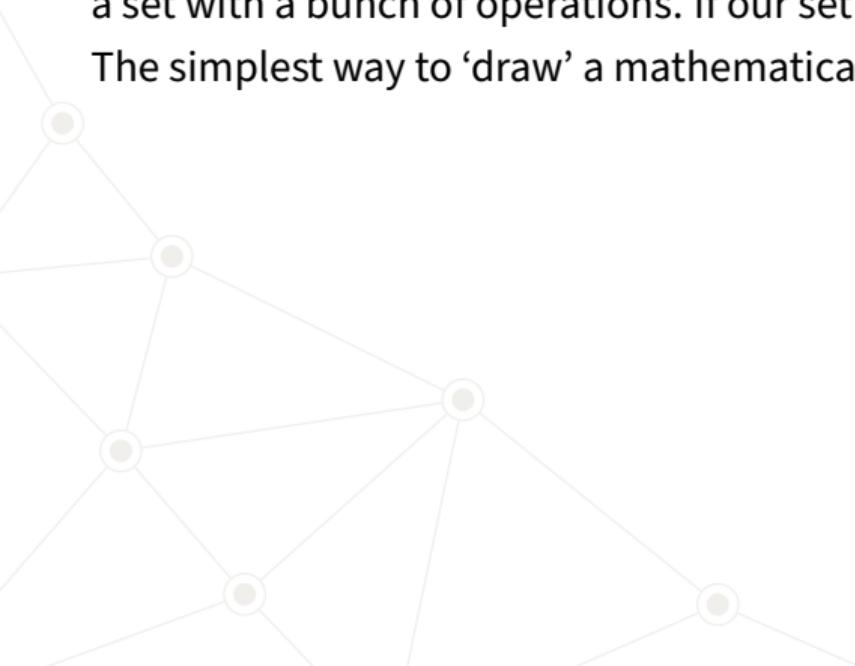
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DRAWING STRUCTURES

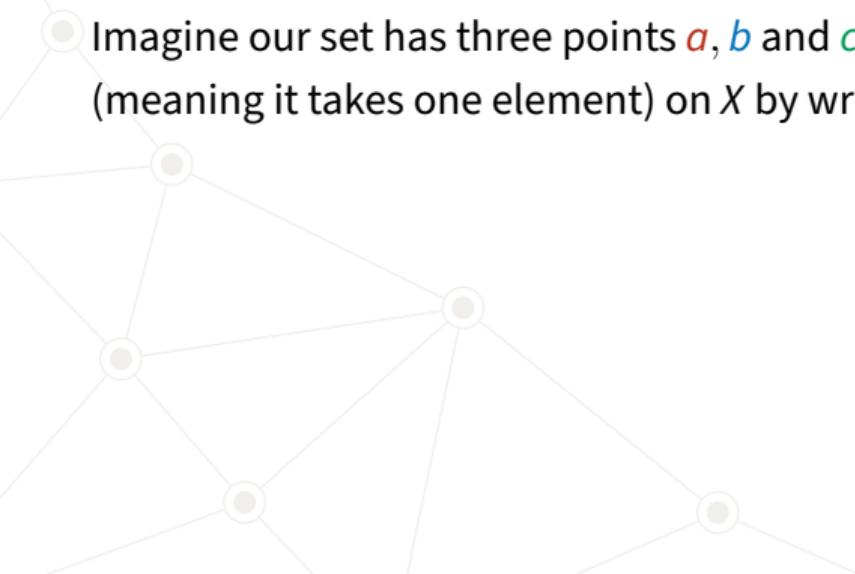
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<hr/>		
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DRAWING STRUCTURES

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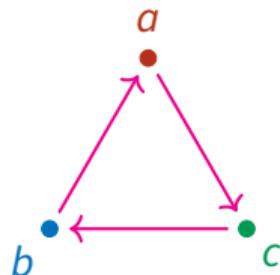
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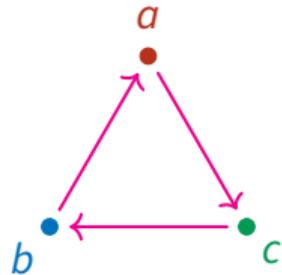
from the previous slide can be also drawn using dots and arrows for example like this:



where the magenta arrows denote the operation \wedge .

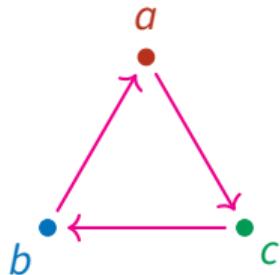
DRAWING STRUCTURES

Let's think what could be an **inverse** to the operation \wedge .



DRAWING STRUCTURES

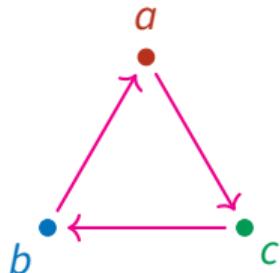
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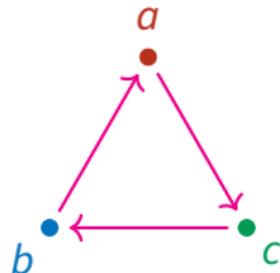


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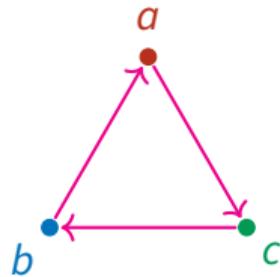
What the operation \wedge does is basically changing a dot into another dot following the **arrows**.

As I have three dots, this means, that after traversing one **arrow**, I have to **traverse two arrows** to get back to where I started.

However, traversing two **arrows** just means **applying the operation \wedge twice**.

DRAWING STRUCTURES

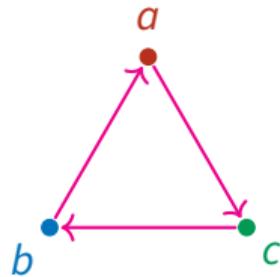
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What this means in symbols, is that if $\hat{c} = b$, then $\hat{\hat{b}} = c$.

So, to get an inverse element with respect to \wedge , we just need to apply \wedge twice.

DRAWING STRUCTURES – EXAMPLES

Imagine a similar unary operation $\tilde{\cdot}$. It is given by the following table

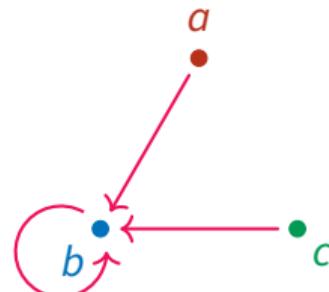
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and can be pictured like this

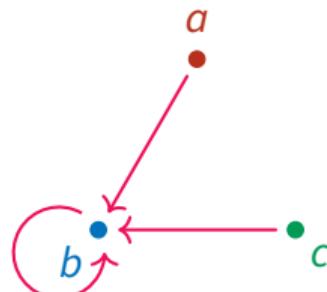


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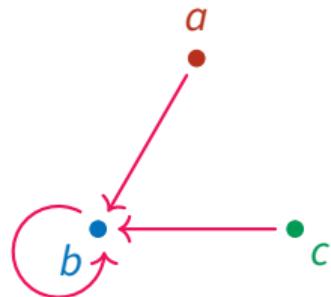
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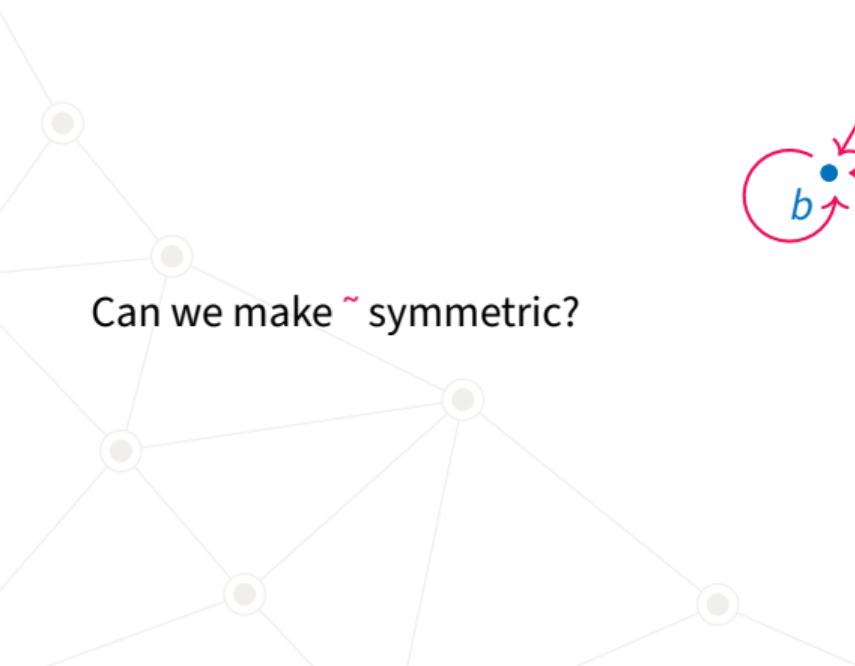


How many elements do we need to add to make $\tilde{\cdot}$ symmetric?

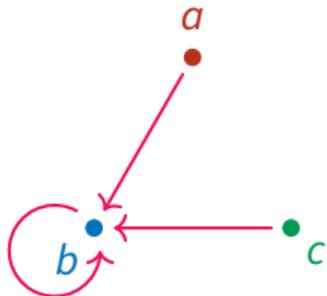
DRAWING STRUCTURES – EXAMPLES



Can we make \sim symmetric?



DRAWING STRUCTURES – EXAMPLES



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The answer is **no** because I can never get back after reaching b . This happens since all the arrows end in b .

DRAWING STRUCTURES – EXAMPLES

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Let's now imagine a structure with a binary operation \triangle instead. Let's upgrade the number of elements to 4 and label them a , b , c and d .

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Suppose our operation \triangle acts like this:

\triangle	a	b	c	d
a	c	a	d	b
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

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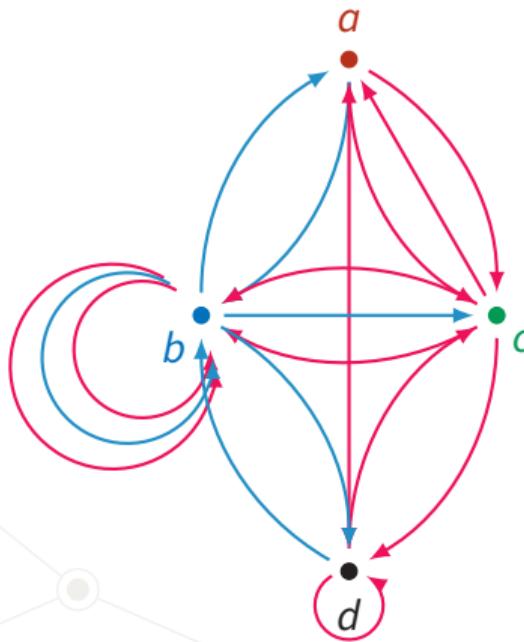
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\triangle	a	b	c	d
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b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

Is this operation **symmetric** or **destructive**?

DRAWING STRUCTURES – EXAMPLES

Drawing binary operations as arrows between elements would be a nightmare. Just watch what such a diagram would look like for the operation from the previous slide.



DRAWING STRUCTURES – EXAMPLES

Let's change the operation Δ a little.



DRAWING STRUCTURES – EXAMPLES

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Is it still **symmetric**?

OPERATION AS A TRANSFORMATION

Motivating examples:

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- The operation \cdot on \mathbb{Z} cannot be made symmetric. We fix this by introducing **fractions/reciprocals** to elements. The inverse with respect to \cdot of an element z is $\frac{1}{z}$. The identity element is 1.

DEFINING INTEGERS

We define the set of integers (or whole numbers) \mathbb{Z} as the set of natural numbers \mathbb{N} plus the inverses to all natural numbers with respect to addition.



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The operation $+$ on \mathbb{Z} is symmetric, while \cdot is still **destructive**. This leads us to the rationals.

RATIONAL NUMBERS

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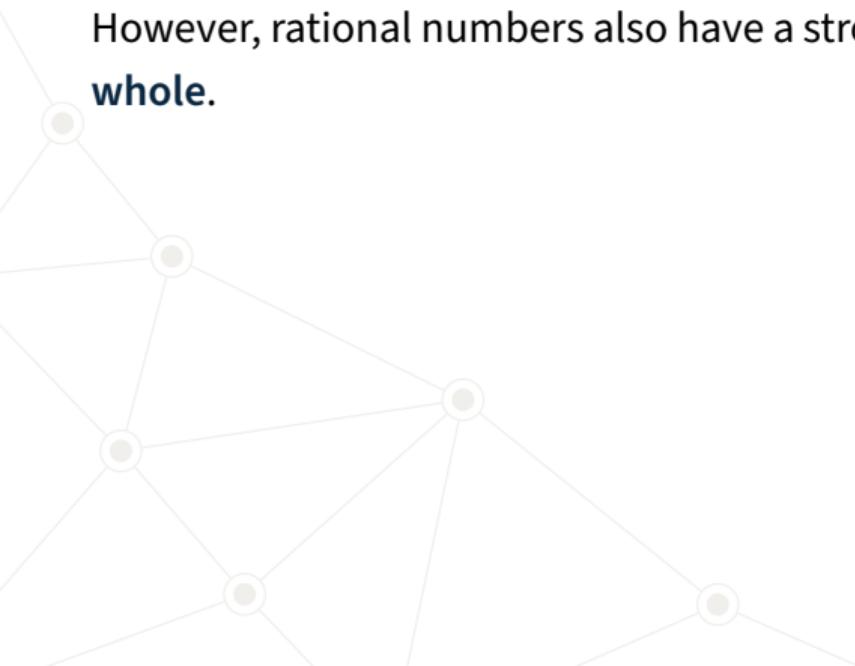
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because both fractions represent the same amount, essentially.

Nonetheless, as *numbers*, $3 \neq 1$ and $6 \neq 2$. We would like to formalize the notion of *sameness* for two fractions.