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Polygons

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Cryptography on Regular Polygons

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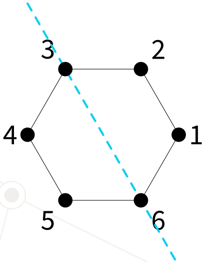
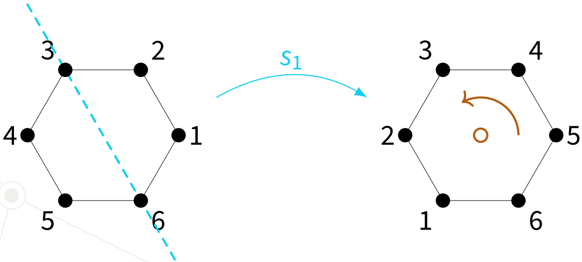
We’ll denote this composition simply by $s_1 s_2$.

Chaining Symmetries – Example



Example of chains of symmetries.

Chaining Symmetries – Example



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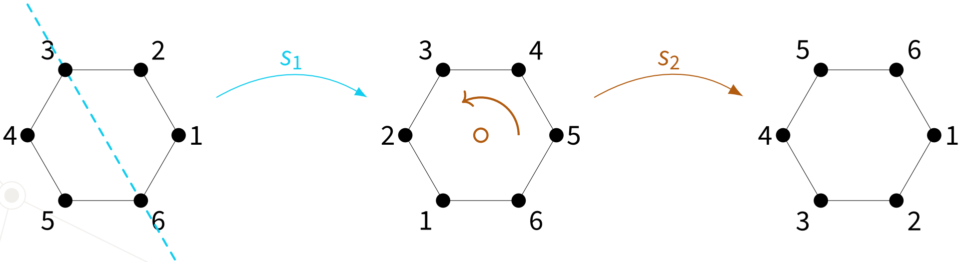

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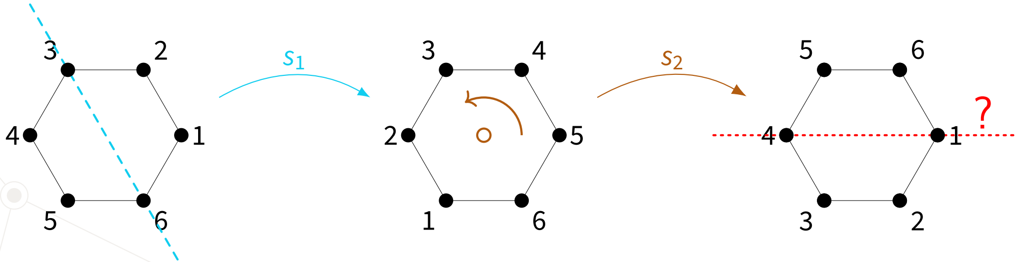

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Chaining Symmetries – How Many Do We Need?

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For example,

- if s_1 is any line symmetry and s_2 is a rotation by 60° counter-clockwise, then $s_2^3 s_1$ (s_2^3 means $s_2 s_2 s_2$) reflects a hexagon through a line perpendicular to the line of s_1.

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- if s_1 is a rotation by 120° clockwise and s_2 is a reflection through a vertical line passing through the top vertex, then $s_1 s_2$ is a reflection through the line given by the rotation of the line of s_2 60° clockwise.

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- any reflection (we'll denote it s).

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- The other two rotational symmetries are r^2 and r^3.
- The other two line symmetries are rs and r^2s.
- Therefore, all the symmetries of an equilateral triangle are

$$\{r, r^2, r^3, s, rs, r^2s\}.$$

Chaining Symmetries – General Algorithm

In general, to create all symmetries, one needs a rotation by an angle $k \cdot 360^\circ / n$ where k **doesn't share a prime factor** with n (in other words, the fraction $\frac{k}{n}$ cannot be simplified) and any one line symmetry.

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3. Then, find b such that $(r^a)^b = r^{ab}$ is your desired rotation.

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You're given a rotation r by $k \cdot 360^\circ/n$ such that k doesn't share factors with n and a line symmetry s.

If you need to calculate a reflection, then

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1. Find a such that r^a is the rotation by $360^\circ/n$.
2. Determine the angle **in any direction** between your given line of symmetry s and the reflection you want.

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3. Find b such that r^{ab} is a rotation **in the opposite direction** by **twice** the angle from the previous step.

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Voluntary HW: *Why* does this algorithm work?

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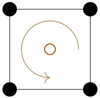
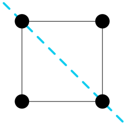
We're given two symmetries of the square:

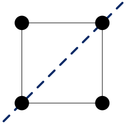


rotation r by 270°



reflection s over the first diagonal

and want to produce



reflection over the second diagonal

Chaining Symmetries – Algorithm Example

We're given two symmetries of the square: rotation r by 270° counter-clockwise and reflection s over the first diagonal.

How to produce reflection over the other diagonal?

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How to produce reflection over the other diagonal?

We use the algorithm.

1. Repeating r three times gives the rotation by 90° counter-clockwise, that is, $a = 3$.

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2. The angle between the two diagonals is 90° in any direction.
3. Repeating the rotation from step 1 two times (that is, $b = 2$) and then using s gives the desired symmetry – in this case it's $(r^3)^2 s = r^6 s$. Of course, r^4 is rotation by 360° which does nothing, so the final symmetry is $r^2 s$.