



POLYGONS

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GENERAL POLYGONS

The background of the slide features three large, overlapping triangles. A yellow triangle is on the left, a cyan triangle is on the right, and a green triangle is at the bottom center where the other two overlap.

GENERAL POLYGONS – DEFINITION

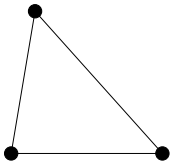
POLYGON

A **polygon** is a closed 2D shape made of only segments.

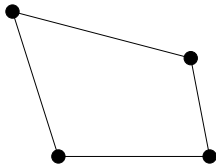
The endpoints of those segments are called **vertices**.

The segments themselves are called **edges**.

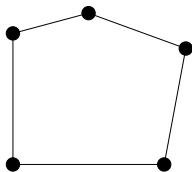
GENERAL POLYGONS – EXAMPLES



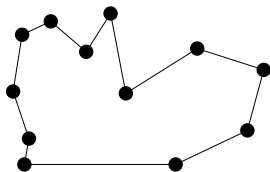
Triangle



Quadrilateral



Pentagon

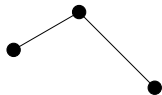


Dodecagon

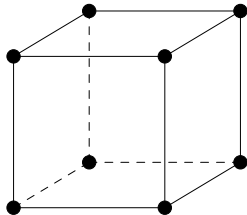
A polygon with $n \in \mathbb{N}$ sides is called an n -gon.

For example a polygon with 123456 sides is called a 123456-gon or decadismyriatrichilliatetrahectapentacontakaihexasagon.

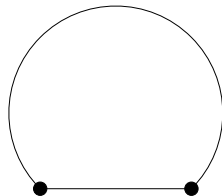
GENERAL POLYGONS – COUNTEREXAMPLES



Not closed



3D

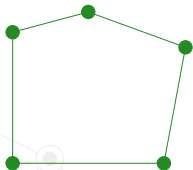


Not straight

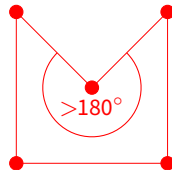
GENERAL POLYGONS – CONVEXITY

CONVEX POLYGON

A polygon is called **convex** if it has no internal angle greater than 180° .



Convex

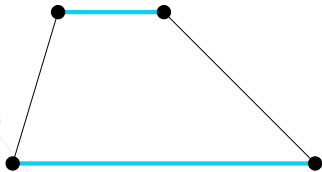


NOT convex

CONVEX POLYGONS

The background of the slide is composed of three large, solid-colored triangles that meet at a central point. A yellow triangle is on the left, a cyan triangle is on the right, and a green triangle is at the bottom. The top portion of the slide is white.

CONVEX POLYGONS – SPECIAL TYPES



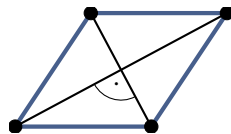
Trapezoid/Trapezium

A convex quadrilateral with at least two parallel sides.



Parallelogram

A convex quadrilateral with two pairs of parallel sides.



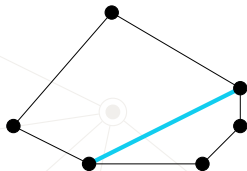
Rhombus

An **equilateral** (all sides of the same length) parallelogram.

CONVEX POLYGONS – DIAGONALS

DIAGONAL IN A CONVEX POLYGON

A **diagonal** of a **convex** polygon is a segment connecting two of its non-adjacent vertices.



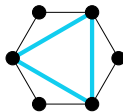
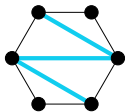
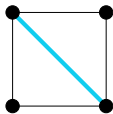
Diagonal in a convex hexagon.

Voluntary HW: How many different diagonals does a convex n -gon have?

CONVEX POLYGONS – TRIANGULATIONS

TRIANGULATION OF A CONVEX POLYGON

A **triangulation** of a **convex** polygon is its division into triangles by non-intersecting diagonals.



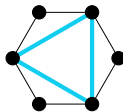
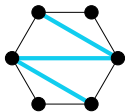
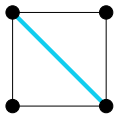
Examples of **triangulations**.

Voluntary HW: How many different triangulations of an n -gon are there?

CONVEX POLYGONS – TRIANGULATIONS

TRIANGULATION OF A CONVEX POLYGON

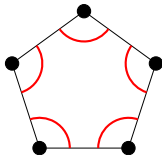
A **triangulation** of a **convex** polygon is its division into triangles by non-intersecting diagonals.



Examples of **triangulations**.

Voluntary HW: Find a **non-convex** polygon which **cannot** be triangulated.

CONVEX POLYGONS – INTERNAL ANGLES



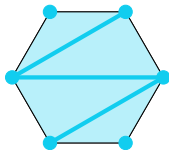
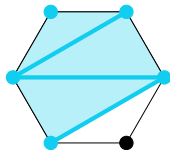
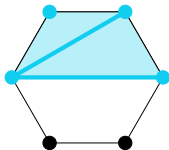
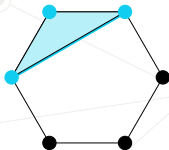
Internal angles of a pentagon.

Question: What is the sum of internal angles of a convex n -gon?

- For a triangle, it's 180° .
- For a square, it's 360° .
- For a pentagon, it's 540° .

CONVEX POLYGONS – INTERNAL ANGLES

We can count internal angles using triangulations.
 Into how many triangles is a convex n -gon divided?
 Each triangle shares two vertices with an adjacent one.
 We choose the first triangle – it covers 3 vertices.
 After that, each triangle covers only one more vertex.
 This means, that an n -gon is divided into $n - 2$ triangles.



Construction of a **triangulation** of a hexagon.

CONVEX POLYGONS – INTERNAL ANGLES

A convex n -gon is divided into $n - 2$ triangles.

The sum of all internal angles in a triangle is 180° .

SUM OF INTERNAL ANGLES IN A CONVEX POLYGON

The sum of all internal angles of a convex n -gon is $(n - 2) \cdot 180^\circ$.

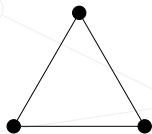
REGULAR POLYGONS

The background of the slide features three large, solid-colored triangles that meet at a central point. A yellow triangle is on the left, a cyan triangle is on the right, and a green triangle is at the bottom. The top portion of the slide is white.

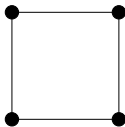
DEFINITION

REGULAR POLYGON

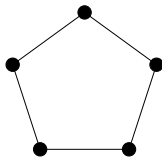
A **regular polygon** is a convex polygon whose sides all have the same length and whose internal angles all have the same size.



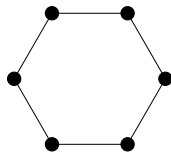
Equilateral triangle
(regular trigon)



Square (regular tetragon)



Regular pentagon

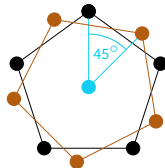
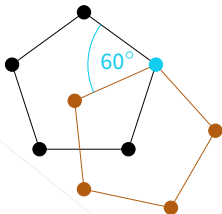


Regular hexagon

REVIEW – PLANE TRANSFORMATIONS

ROTATION

Rotation of a polygon consists of well ... rotating each of its points by a fixed angle around a fixed point (called *anchor*).

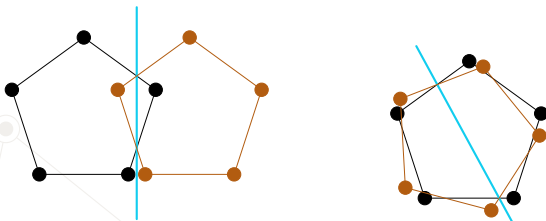


Examples of rotations.

REVIEW – PLANE TRANSFORMATIONS

REFLECTION

Reflection of a polygon consists of ‘mirroring’ each of its points through a given line (called *axis of reflection*).

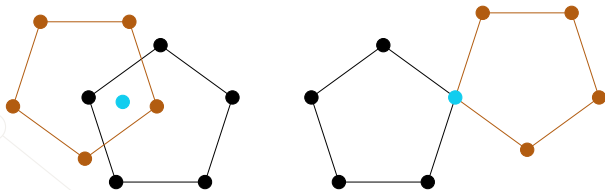


Examples of reflections.

REVIEW – PLANE TRANSFORMATIONS

POINT SYMMETRY

Point symmetry of a polygon consists of ‘mirroring’ each of its points through a given point (called *center of symmetry*).



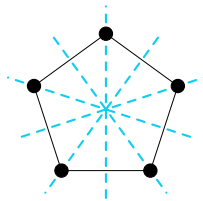
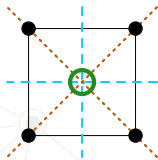
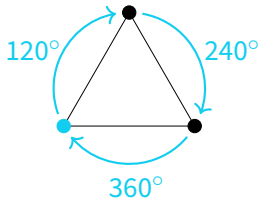
Examples of point symmetries.

SYMMETRIES OF REGULAR POLYGONS

Question: What are the transformations that don't change regular polygons in any way?

- rotational symmetries
 - rotation by $\frac{k \cdot 360^\circ}{n}$ where k is any number between 1 and n
- reflectional (line) symmetries
 - for n even reflections over lines passing through centres of opposite sides
 - for n even over lines passing through opposite vertices
 - for n odd over lines passing through a centre of a side and the opposite vertex
- point symmetries
 - only through the 'centre' – the point where its axes of symmetry intersect – in case n is even

SYMMETRIES OF REGULAR POLYGONS



Examples of regular polygon symmetries

CRYPTOGRAPHY ON REGULAR POLYGONS

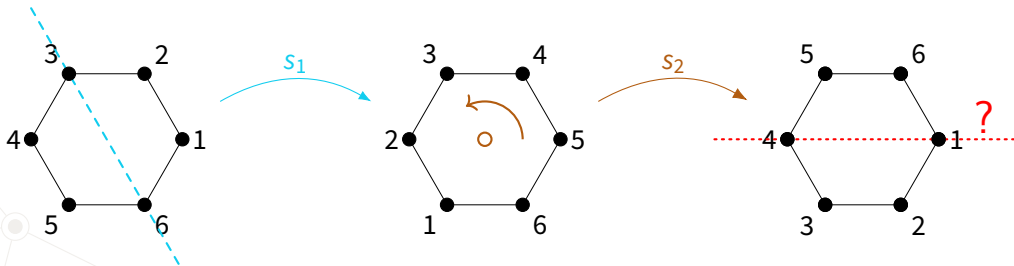
The background of the slide features a minimalist geometric design. It consists of three large triangular regions meeting at a central point. The region on the left is a solid yellow triangle. The region on the right is a solid cyan triangle. The region at the bottom, which is the intersection of the other two, is a solid green triangle. The overall composition is clean and modern, with the title text centered in the white space above the triangles.

CHAINING SYMMETRIES

Given two symmetries, s_1 and s_2 of a regular polygon, one can apply them one after the other ('compose' them, like functions).

We'll denote this composition simply by s_1s_2 .

CHAINING SYMMETRIES – EXAMPLE



Example of a chain of symmetries.

CHAINING SYMMETRIES – HOW MANY DO WE NEED?



Discounting point symmetry, an n -gon has $2n$ symmetries.

Two symmetries can 'combine' to create a different symmetry.

Natural question: How many (and which) symmetries of a regular polygon do I need to get all the others?

For example,

- if s_1 is any reflectional symmetry and s_2 is a rotation by 60° counter-clockwise, then $s_2^3 s_1$ (s_2^3 means $s_2 s_2 s_2$) reflects a hexagon through a line perpendicular to the line of s_1 .
- if s_1 is a rotation by 120° clockwise and s_2 is a reflection through a vertical line passing through the top vertex, then $s_1 s_2$ is a reflection through the line given by the rotation of the line of s_2 60° clockwise.

CHAINING SYMMETRIES – HOW MANY DO WE NEED?



Actually, for a general n -gon, we need only **two**:

- rotation by $360^\circ/n$ in any direction (we'll denote it r),
- any reflection (we'll denote it s).

CHAINING SYMMETRIES – TRIANGLE

Let r be the rotation by 120° and s any reflectional symmetry.

- The other two rotational symmetries are r^2 and r^3 .
- The other two reflectional symmetries are rs and r^2s .
- Therefore, all the symmetries of an equilateral triangle are

$$\{r, r^2, r^3, s, rs, r^2s\}.$$

CHAINING SYMMETRIES – GENERAL ALGORITHM

In general, to create all symmetries, one needs a rotation by an angle $k \cdot 360^\circ / n$ where k **doesn't share a prime factor** with n (in other words, the fraction $\frac{k}{n}$ cannot be simplified) and any one reflectional symmetry.

Why?

- If k shares factors with n , then you can never get rotation by $360^\circ / n$.
- Two symmetries cannot in general produce every rotation.
- Two rotations can never produce a symmetry.

CHAINING SYMMETRIES – GENERAL ALGORITHM

You're given a rotation r by $k \cdot 360^\circ / n$ such that k doesn't share factors with n and a reflectional symmetry s .

If you need to calculate a rotation, then

1. First measure the angle **counter-clockwise**.
2. Find a such that r^a is the rotation by $360^\circ / n$.
3. Then, find b such that $(r^a)^b = r^{ab}$ is your desired rotation.

CHAINING SYMMETRIES – GENERAL ALGORITHM

You're given a rotation r by $k \cdot 360^\circ/n$ such that k doesn't share factors with n and a line symmetry s .

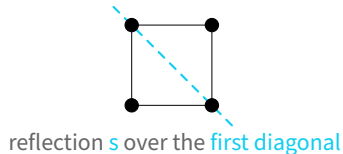
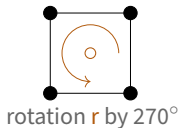
If you need to calculate a reflection, then

1. Find a such that r^a is the rotation by $360^\circ/n$.
2. Determine the angle **in any direction** between the lines of your given reflection s and the reflection you want.
3. Find b such that r^{ab} is a rotation **in the opposite direction** by **twice** the angle from the previous step.
4. $r^{ab}s$ is your desired reflection.

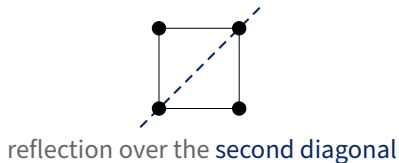
Voluntary HW: *Why does this algorithm work?*

CHAINING SYMMETRIES – ALGORITHM EXAMPLE

We're given two symmetries of the square:



and want to produce



CHAINING SYMMETRIES – ALGORITHM EXAMPLE

We're given two symmetries of the square: rotation r by 270° counter-clockwise and reflection s over the first diagonal.

How to produce reflection over the other diagonal?

We use the algorithm.

1. Repeating r three times gives the rotation by 90° counter-clockwise, that is, $a = 3$.
2. The angle between the two diagonals is 90° in any direction.
3. Repeating the rotation from step 1 two times (that is, $b = 2$) and then using s gives the desired symmetry – in this case it's $(r^3)^2s = r^6s$. Of course, r^4 is rotation by 360° which does nothing, so the final symmetry is r^2s .

1

ENCODING MESSAGES USING SYMMETRIES

ENCODING MESSAGES – GENERAL IDEA

- Send a message as a sequence of strings which **can be decoded on the other side**.
- The decoding should require a 'key' which is **statistically impossible to determine quickly**.
- The encoding and decoding must be done procedurally – requires a system with concrete rules and a **limited** (but huge) **number of combinations**.

ENCODING MESSAGES – REGULAR POLYGONS

How can one use regular polygons to encode messages?

Quite easily, assign symbols to vertices and choose one **main** vertex.

The symbol in the **main** vertex is what is sent during each tick (time interval during which symbols are sent).

Rotations and reflections ‘move symbols between vertices’.

This means that after applying a rotation or reflection, a (in most cases) different symbol will appear in the main vertex.

ENCODING MESSAGES – MORSE CODE

The Morse Code has two symbols: — and ·.

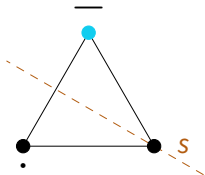
We also need to be able to determine spaces between words – we can assign one vertex a ‘blank’, meaning space.

This means that to send messages written Morse Code using regular polygons, we need three vertices – a triangle.

ENCODING MESSAGES – MORSE CODE

Let the top vertex be main and let's choose a rotation $r = \curvearrowright 120^\circ$ and a reflection s depicted below.

We should also assign Morse Code symbols to two of the vertices. We can do so randomly.

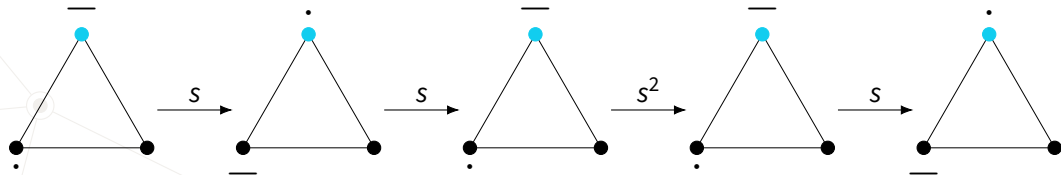


The **top vertex** is the main one, which means that the symbol above it will get sent.

ENCODING MESSAGES – MORSE CODE

The important property of symmetries is that two different sequences can give the same symmetry. Meaning, we can encode one message **MANY** different ways.

For example, we can send the letter P ($\cdot - - \cdot$) like this:



Or, as a sequence

$s, s, s^2, s.$

ENCODING MESSAGES – MORSE CODE

Notice that we didn't need the rotation r at all in this example.

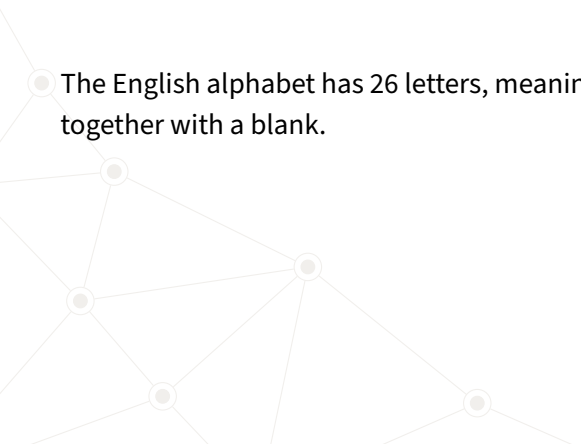
However, that doesn't mean we **can't** use it.

Actually, it's often beneficial to encode messages using different combinations of symmetries for security.

The letter $\cdot - - \cdot$ can also be sent for example like this

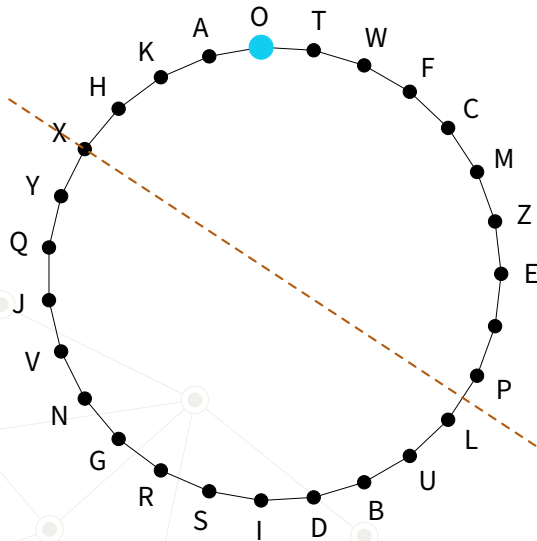
$$r^2, sr, sr^2, rsr.$$

ENCODING MESSAGES – ALPHABET

A decorative geometric pattern in the bottom-left corner, consisting of several interconnected triangles and lines, with small circles at the vertices.

The English alphabet has 26 letters, meaning we need a 27-gon to represent all of them together with a blank.

ENCODING MESSAGES – ALPHABET



Let's start with $r = \odot \frac{2}{27} \cdot 360^\circ$ and s as on the picture.

One way to encode the word POLY-GONS would be

$$sr^5, r^{10}sr^{15}, r^5, sr, r^7s, r^5sr^{13}, sr, r^3s$$