Polygons

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General Polygons

General Polygons – Definition

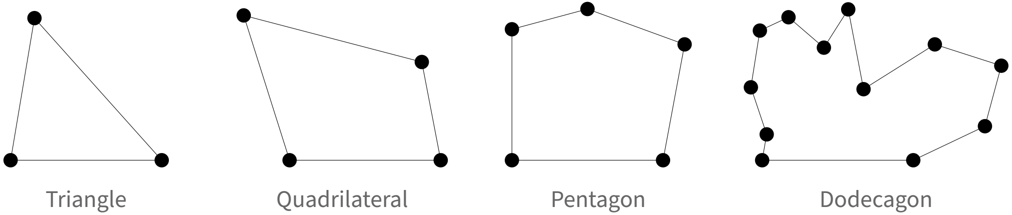
Polygon

A polygon is a closed 2D shape made of only segments.

The endpoints of those segments are called vertices.

The segments themselves are called edges.

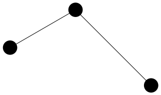
General Polygons – Examples



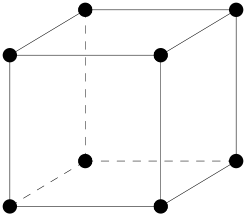
A polygon with $n \in \mathbb{N}$ sides is called an n-gon.

For example a polygon with 123456 sides is called a 123456-gon or decadismyriatrischilliatetrahectapentacontakaihexagon.

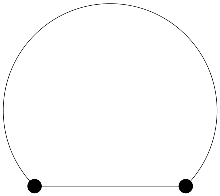
General Polygons – Counterexamples



Not closed



3D

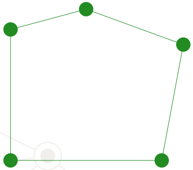


Not straight

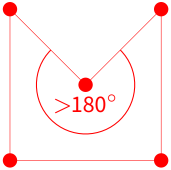
General Polygons – Convexity

Convex Polygon

A polygon is called **convex** if it has no internal angle greater than 180°.



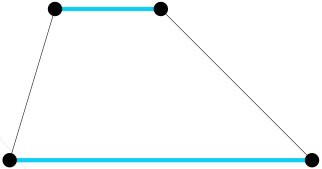
Convex



NOT convex

Convex Polygons

Convex Polygons – Special Types

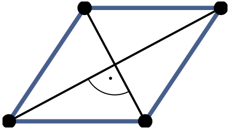


Trapezoid/Trapezium

A convex quadrilateral with at least two parallel sides.

Parallelogram

A convex quadrilateral with two pairs of parallel sides.



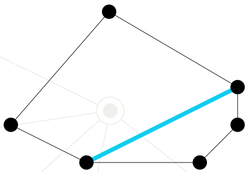
Rhombus

An **equilateral** (all sides of the same length) parallelogram.

Convex Polygons – Diagonals

Diagonal in a Convex Polygon

A diagonal of a **convex** polygon is a segment connecting two of its non-adjacent vertices.



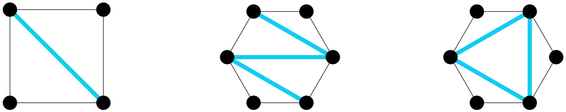
Diagonal in a convex hexagon.

Voluntary HW: How many different diagonals does a convex n-gon have?

Convex Polygons – Triangulations

Triangulation of a Convex Polygon

A triangulation of a **convex** polygon is its division into triangles by non-intersecting diagonals.



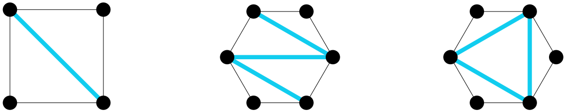
Examples of triangulations.

Voluntary HW: How many different triangulations of an n-gon are there?

Convex Polygons – Triangulations

Triangulation of a Convex Polygon

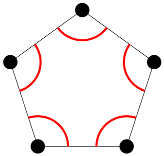
A triangulation of a **convex** polygon is its division into triangles by non-intersecting diagonals.



Examples of triangulations.

Voluntary HW: Find a **non-convex** polygon which **cannot** be triangulated.

Convex Polygons – Internal Angles



Internal angles of a pentagon.

Question: What is the sum of internal angles of a convex n-gon?

- For a triangle, it's 180°.
- For a square, it's 360°.
- For a pentagon, it's 540°.

Convex Polygons – Internal Angles

We can count internal angles using triangulations.

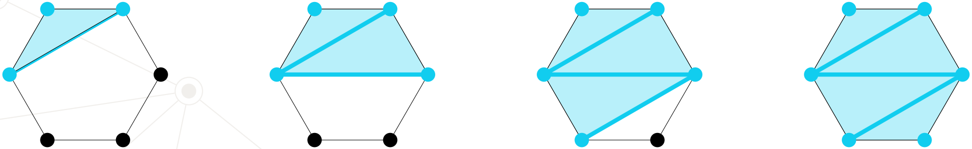
Into how many triangles is a convex n-gon divided?

Each triangle shares two vertices with an adjacent one.

We choose the first triangle – it covers 3 vertices.

After that, each triangle covers only one more vertex.

This means, that an n-gon is divided into $n - 2$ triangles.



Construction of a triangulation of a hexagon.

Convex Polygons – Internal Angles

A convex n-gon is divided into $n-2$ triangles.

The sum of all internal angles in a triangle is $180°$.

Sum of Internal Angles in a Convex Polygon

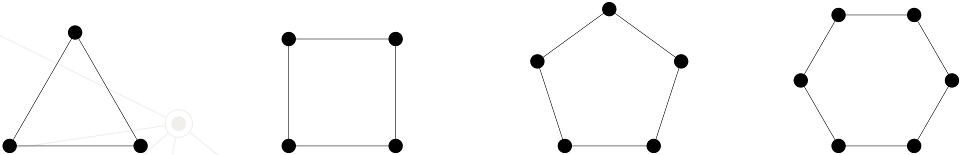
The sum of all internal angles of a convex n-gon is $(n-2) \cdot 180°$.

Regular Polygons

Definition

Regular Polygon

A **regular polygon** is a convex polygon whose sides all have the same length and whose internal angles all have the same size.

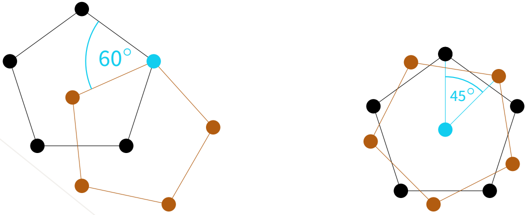


Equilateral triangle (regular trigon) Square (regular tetragon) Regular pentagon Regular hexagon

Review – Plane Transformations

Rotation

Rotation of a polygon consists of well … rotating each of its points by a fixed angle around a fixed point (called *anchor*).

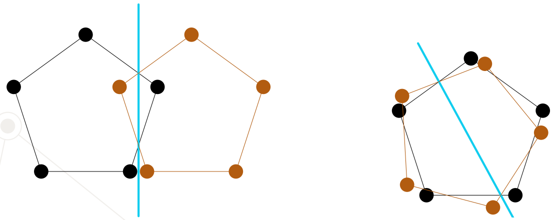


Examples of rotations.

Review – Plane Transformations

Reflection

Reflection of a polygon consists of 'mirroring' each of its points through a given line (called *axis of reflection*).

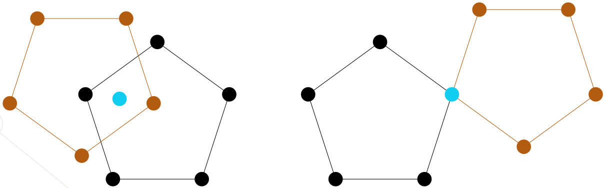


Examples of reflections.

Review – Plane Transformations

Point Symmetry

Point symmetry of a polygon consists of ‘mirroring’ each of its points through a given point (called *center of symmetry*).



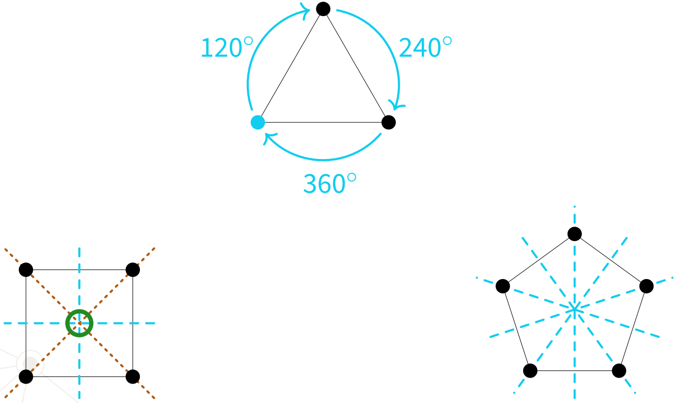
Examples of point symmetries.

Symmetries of Regular Polygons

Question: What are the transformations that don't change regular polygons in any way?

- rotational symmetries
  - rotation by $\frac{k \cdot 360^\circ}{n}$ where k is any number between 1 and n
- reflectional (line) symmetries
  - for n even reflections over lines passing through centres of opposite sides
  - for n even over lines passing through opposite vertices
  - for n odd over lines passing through a centre of a side and the opposite vertex
- point symmetries
  - only through the 'centre' – the point where its axes of symmetry intersect – in case n is even

Symmetries of Regular Polygons



Examples of regular polygon symmetries

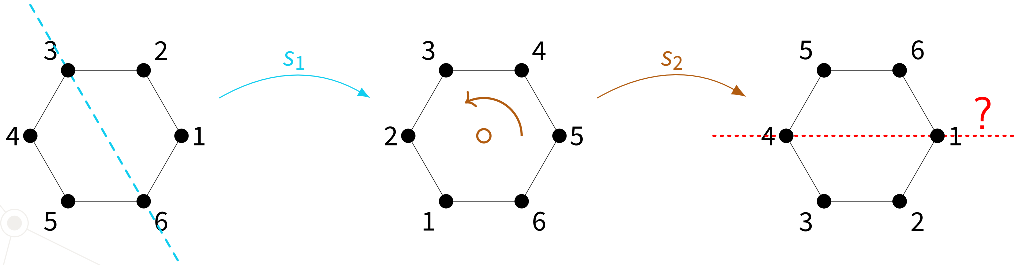
Cryptography on Regular Polygons

Chaining Symmetries

Given two symmetries, s_1 and s_2 of a regular polygon, one can apply them one after the other ('compose' them, like functions).

We'll denote this composition simply by $s_1 s_2$.

Chaining Symmetries – Example



Example of a chain of symmetries.

Chaining Symmetries – How Many Do We Need?

Discounting point symmetry, an n-gon has $2n$ symmetries.

Two symmetries can 'combine' to create a different symmetry.

Natural question: How many (and which) symmetries of a regular polygon do I need to get all the others?

For example,

- if s_1 is any reflectional symmetry and s_2 is a rotation by 60° counter-clockwise, then $s_2^3 s_1$ (s_2^3 means $s_2 s_2 s_2$) reflects a hexagon through a line perpendicular to the line of s_1.
- if s_1 is a rotation by 120° clockwise and s_2 is a reflection through a vertical line passing through the top vertex, then $s_1 s_2$ is a reflection through the line given by the rotation of the line of s_2 60° clockwise.

Chaining Symmetries – How Many Do We Need?

Actually, for a general n-gon, we need only **two**:

- rotation by $360^\circ/n$ in any direction (we'll denote it r),
- any reflection (we'll denote it s).

Chaining Symmetries – Triangle

Let r be the rotation by 120° and s any reflectional symmetry.

- The other two rotational symmetries are r^2 and r^3.
- The other two reflectional symmetries are rs and r^2s.
- Therefore, all the symmetries of an equilateral triangle are

$$\{r, r^2, r^3, s, rs, r^2s\}.$$

Chaining Symmetries – General Algorithm

In general, to create all symmetries, one needs a rotation by an angle $k \cdot 360^\circ/n$ where k **doesn't share a prime factor** with n (in other words, the fraction $\frac{k}{n}$ cannot be simplified) and any one reflectional symmetry.

Why?

- If k shares factors with n, then you can never get rotation by $360^\circ/n$.
- Two symmetries cannot in general produce every rotation.
- Two rotations can never produce a symmetry.

Chaining Symmetries – General Algorithm

You're given a rotation r by $k \cdot 360^\circ/n$ such that k doesn't share factors with n and a reflectional symmetry s.

If you need to calculate a rotation, then

1. First measure the angle **counter-clockwise**.
2. Find a such that r^a is the rotation by $360^\circ/n$.
3. Then, find b such that $(r^a)^b = r^{ab}$ is your desired rotation.

Chaining Symmetries – General Algorithm

You're given a rotation r by $k \cdot 360^\circ/n$ such that k doesn't share factors with n and a line symmetry s.

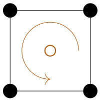
If you need to calculate a reflection, then

1. Find a such that r^a is the rotation by $360^\circ/n$.
2. Determine the angle **in any direction** between the lines of your given reflection s and the reflection you want.
3. Find b such that r^{ab} is a rotation **in the opposite direction** by **twice** the angle from the previous step.
4. $r^{ab}s$ is your desired reflection.

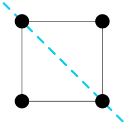
Voluntary HW: *Why* does this algorithm work?

Chaining Symmetries – Algorithm Example

We're given two symmetries of the square:

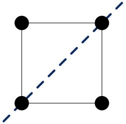


rotation r by 270°



reflection s over the first diagonal

and want to produce



reflection over the second diagonal

Chaining Symmetries – Algorithm Example

We're given two symmetries of the square: rotation r by 270° counter-clockwise and reflection s over the first diagonal.
How to produce reflection over the other diagonal?

We use the algorithm.

1. Repeating r three times gives the rotation by 90° counter-clockwise, that is, $a = 3$.
2. The angle between the two diagonals is 90° in any direction.
3. Repeating the rotation from step 1 two times (that is, $b = 2$) and then using s gives the desired symmetry – in this case it's $(r^3)^2 s = r^6 s$. Of course, r^4 is rotation by 360° which does nothing, so the final symmetry is $r^2 s$.