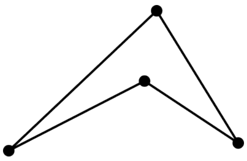
Convex Polygons and Their Symmetries

3.AB PreIB Maths – Exam A

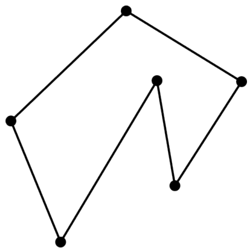
Unless specified otherwise, you are to **always** (at least briefly) explain your reasoning. Even in closed questions.

1. Definition of a polygon.

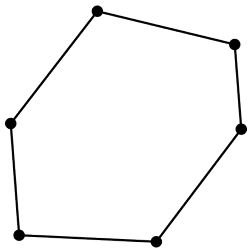
   (a) Which of these polygons *are not* convex? **Explain**. [10 %]



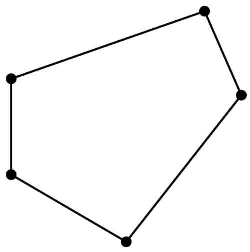
Option A.



Option B.



Option C.

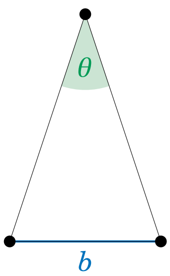


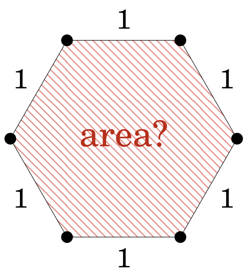
Option D.

   (b) The area of an isosceles (rovnoramenný) triangle with base b and the opposite angle θ (see the picture below) is equal to [10 %]

$$A = \frac{b^2 \cdot \cos^2\left(\frac{\theta}{2}\right)}{2 \cdot \sin\theta}.$$

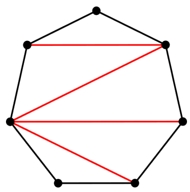
   Use this fact to calculate the *area* of the regular hexagon with *side length* 1. *Hint:* $\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$.



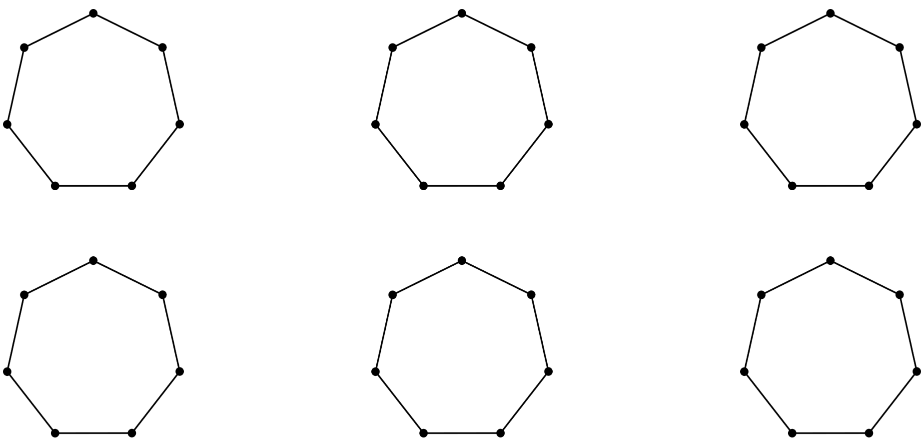


2. Triangulations of convex polygons.

   (a) Draw all triangulations of the heptagon *that can be reached in one flip* from the one shown below. Use the provided shapes (not all of them necessarily). **No explanation required**. [10 %]

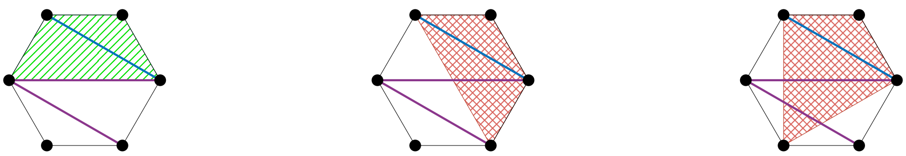


The initial triangulation.



Shapes to draw diagonals into.

   (b) The 'formal' definition of a *flip* I gave goes like this: *To flip a diagonal, choose a quadrilateral containing this diagonal and such that it does not intersect any other diagonals of the triangulation. Swap this diagonal for the other one of this quadrilateral*. There is always *exactly one* correct choice of this quadrilateral. However, *how many* choices of such a quadrilateral *are wrong* in a convex polygon on n vertices? Is the number the same for *every* diagonal? [10 %]

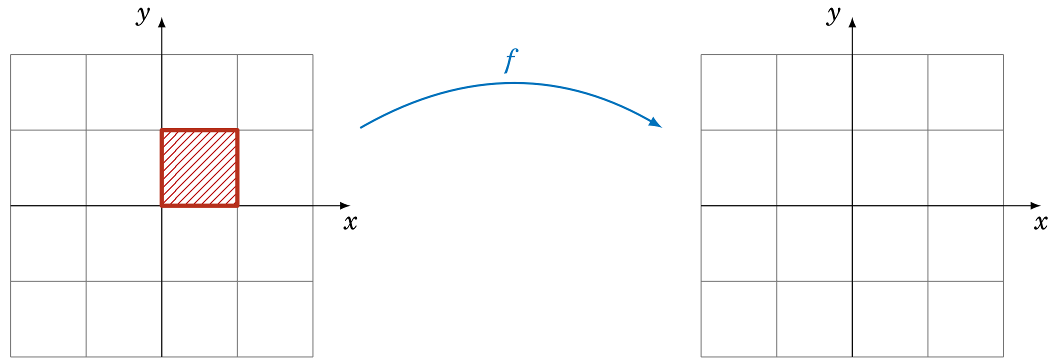


Correct quadrilateral. Incorrect quadrilateral. Incorrect quadrilateral.

A triangulation of the hexagon. The blue diagonal belongs to three different quadrilaterals. However, only the green one leads to a correct flip.

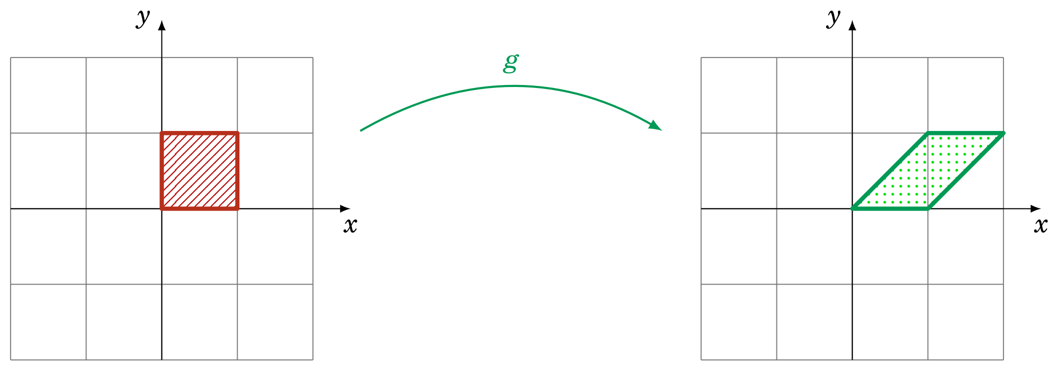
3. Plane transformations.

   (a) Find out the *image* (the resulting shape when transformed) of a square (depicted below) under the plane transformation $f(x, y) = (2x, y)$. **Provide a short explanation**. [10 %]



The initial square. Draw the resulting shape here.

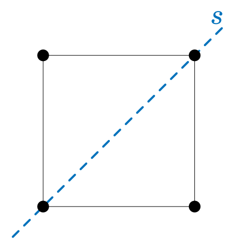
   (b) Below, you see a unit square transformed by the plane transformation f defined as $f(x, y) = (2 - x - y, 1 - y)$. Write this transformation as a composition $g \circ s$ (that is, first goes s, then goes g) where s is a symmetry of the square (applying it keeps the square intact). **Explain**. [10 %]



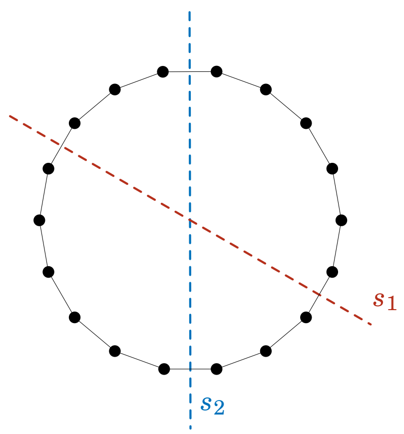
The initial square. The square transformed by f.

4. Symmetries of regular polygons.

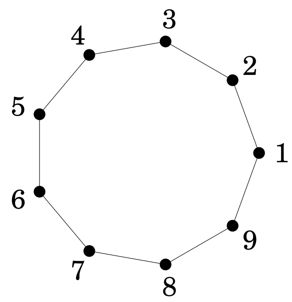
   (a) Given two symmetries of the *square* – the rotation $r = ↺ 270°$ by 270° counter-clockwise and the reflection s drawn below – determine (using any method you wish) the composition sr. **Explain**. [10 %]



   (b) Given two symmetries of the octakaidecagon (18 vertices) – the reflections s_1 and s_2 depicted below – compute (using any method you wish) the composition $s_1 s_2$. **Explain**. [10 %]



(c) Select those of the following four pairs of symmetries of the regular enneagon (9 vertices) that *generate all* of its symmetries. **No explanation necessary**. [10 %]



Picture of the enneagon for reference.

- ◯ the rotation $r_1 = \circlearrowleft 4\cdot 360°/9$ and the rotation $r_2 = \circlearrowleft 8\cdot 360°/9$,
- ◯ the reflection s_1 over the line passing through vertex 6 and the midpoint of 12 and the reflection s_2 over the line passing through vertex 3 and the midpoint of 78.
- ◯ the reflection s over the line passing through vertex 1 and the midpoint of 56 and the rotation $r = \circlearrowleft 6\cdot 360°/9$,
- ◯ the rotation $r = \circlearrowleft 7\cdot 360°/9$ and the reflection s over the line passing through vertex 8 and the midpoint of 34.

(d) Given reflections s_1 and s_2 of the heptagon (7 vertices), compose them (and *only* them) to create the reflection s_3 illustrated below. **Explain**. [10 %]