Number Sets

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November 6, 2023

Contents

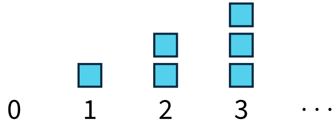
Natural Numbers

Natural Numbers – Intuition

Natural numbers are intuitively objects which represent a **quantity**.
They're the following set:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



Natural Numbers – Definition

There are many ways to define natural numbers.
One of the popular ones is using a **successor** function, denoted s. If n is a natural number, then $s(n)$ basically means 'add another block on top of n'.
One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet**! In fact, you need the successor function to define addition in the first place.

Natural Numbers – Definition

The following **five axioms** (often called *Peano axioms*) constitute the definition of natural numbers:

1. There exists the natural number 0.
2. Every natural number has a successor which is also natural.
3. The number 0 is not the successor of any natural number.
4. If $s(n) = s(m)$, then $n = m$.
5. (Induction Axiom) If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

1

Unpacking The Axioms

Natural Numbers – Axiom 1

There exists the natural number 0.

Hopefully obvious.

Natural Numbers – Axiom 2

Every natural number has a successor which is also natural.

Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

Natural Numbers – Axiom 3

The number 0 is not the successor of any natural number.

Basically means that the natural numbers are infinite only ‘in one direction’. There is a **first** natural number.

Natural Numbers – Axiom 4

$$\text{If } s(n) = s(m), \text{ then } n = m.$$

This means that the successor function is **injective** – each natural number has a different successor.

Natural Numbers – Axiom 5

If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

This means that any feature of the natural numbers 'propagates' via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it true for 1, it is true for 2 as well, etc.

2

Operations On Natural Numbers

What Is An Operation?

By **operation**, we mean a function which takes **one or multiple** natural numbers and produces **one** natural number.

For example, $+$ and $\cdot$ are operations because they take **two** natural numbers and produce **one**.

We don't often see them as functions because we don't write them as such. We write $n + m$ instead of $+(n, m)$ and $n \cdot m$ instead of $\cdot(n, m)$.

In this sense, subtraction and division **are not operations**! They take two natural numbers but they **do not produce a natural number**.

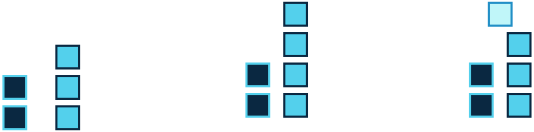
Addition

We define **addition** on natural numbers by the following two formulae:

- $n + 0 = n$,
- $n + s(m) = s(n + m)$.

We can imagine addition as ‘adding blocks *to the side*’ and the successor function as ‘adding one block *on top*’.
In this sense, $n + s(m) = s(n + m)$ only means that if you add one block atop m blocks and then n blocks to the side you have the same number of blocks as if you add n blocks next to m blocks and then another on top of that.

Addition



2 3 $\qquad 2 + s(3) \qquad s(2 + 3)$

Using the formula $n + s(m) = s(n + m)$, one calculates $n + m$ by taking the successor of n, m times. Like this:

$$
\begin{aligned}
n + 0 &= n,\\
n + 1 &= n + s(0) = s(n + 0) = s(n),\\
n + 2 &= n + s(1) = s(n + 1) = s(n + s(0)) = s(s(n + 0)) = s(s(n)),\\
&\vdots
\end{aligned}
$$

Addition – Properties

Addition of natural numbers satisfies these two properties:

- **Commutativity**:

$$n + m = m + n.$$

- **Associativity**:

$$n + (m + k) = (n + m) + k.$$

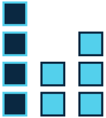
Addition – Properties

Using blocks, **commutativity** just means that putting n blocks next to m blocks is the same as putting m blocks next to n blocks.

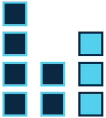
$2 + 3$ $\qquad$ $3 + 2$

Addition – Properties

Using blocks, **associativity** just means that putting *m* blocks next to *k* blocks and then *n* more blocks next to those is the same as putting *m* blocks next to *n* blocks and then *k* more blocks next to those.



$4 + (2 + 3)$



$(4 + 2) + 3$

Multiplication

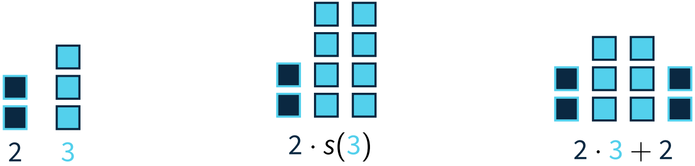
We define **multiplication** on natural numbers by the following formulae:

- $m \cdot 1 = m$,
- $m \cdot s(n) = m \cdot n + n$.

We can imagine multiplication $m \cdot n$ by adding a collections of n blocks for every one block in the collection of m blocks.
If we write $s(n) = n + 1$, then the second formula just means that

$$m \cdot s(n) = m \cdot (n + 1) = m \cdot n + n.$$

Multiplication



The formula $m \cdot s(n) = m \cdot n + n$ allows us to compute $m \cdot n$ by applying it n times. More precisely,

$$
\begin{aligned}
m \cdot 1 &= m \\
m \cdot 2 &= m \cdot s(1) = m \cdot 1 + m = m + m \\
m \cdot 3 &= m \cdot s(2) = m \cdot 2 + m = m \cdot s(1) + m = m \cdot 1 + m + m = m + m + m \\
&\vdots
\end{aligned}
$$

Multiplication – Properties

- **Commutativity:**

$$m \cdot n = n \cdot m.$$

- **Associativity:**

$$m \cdot (n \cdot k) = (m \cdot n) \cdot k.$$

- **Distributivity:**

$$m \cdot (n + k) = m \cdot n + m \cdot k.$$