Math Homework – PreIB 3.AB 2 & 3

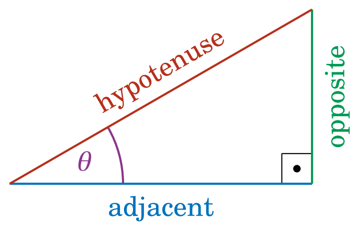
Trigonometric Functions

Áďa Klepáčů

June 7, 2023

Review of Trig Functions

Given a right triangle



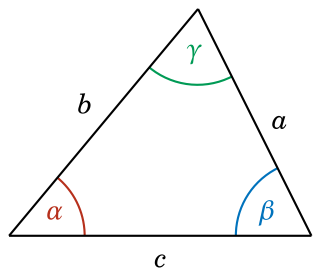
we define three **trigonometric functions**

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

The **inverse** trigonometric functions to these three are naturally

$$\sin^{-1}\left(\frac{\text{opposite}}{\text{hypotenuse}}\right) = \theta, \quad \cos^{-1}\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right) = \theta, \quad \tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right) = \theta.$$

Further, in **any** triangle



the **Law of Sines** holds. It says that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

We also have the **Law of Cosines** which says that

$$c^2 = a^2 + b^2 - 2ab \cdot \cos\gamma,$$
$$b^2 = a^2 + c^2 - 2ac \cdot \cos\beta,$$
$$a^2 = b^2 + c^2 - 2bc \cdot \cos\alpha.$$

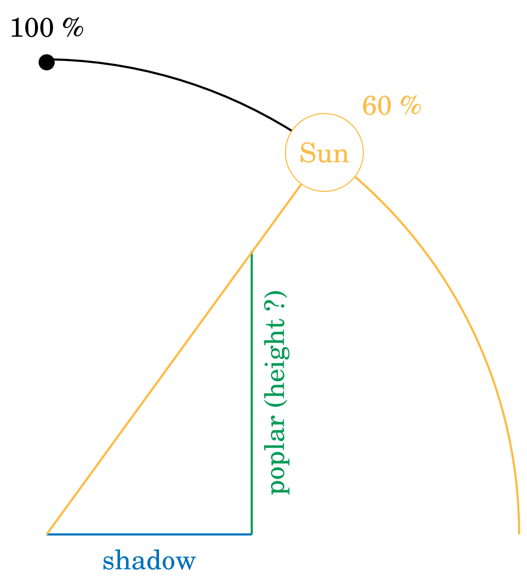
Exercises

1. Count Dolgorukov is a Russian landlord. He tends to lounge away his remaining days in his enormous garden. He is also very pedantic and meticulous about precision and symmetry. On one sunny noon, one of his poplar trees caught his unforgiving gaze. At first glance it seemed as though it was the same height as all the others but it cast a longer shadow.

   Count Dolgorukov, seething with rage, gathered his gardeners and ordered the tree shortened. Not knowing which it was, the gardeners had to wait until the next noon for the sun to be as high in the sky as it can be, which in Russia **is about 60 %**, with 100 % meaning right above the head, in order to tell the lengths of the shadows apart.

   The gardeners measured that all but one poplar cast a **shadow of approximately 21.7 meters**. The **highest poplar** cast a **shadow of approximately 22.3 meters**. How much should Count Dolgorukov's gardeners shorten the poplar so that it is as high as all the others?

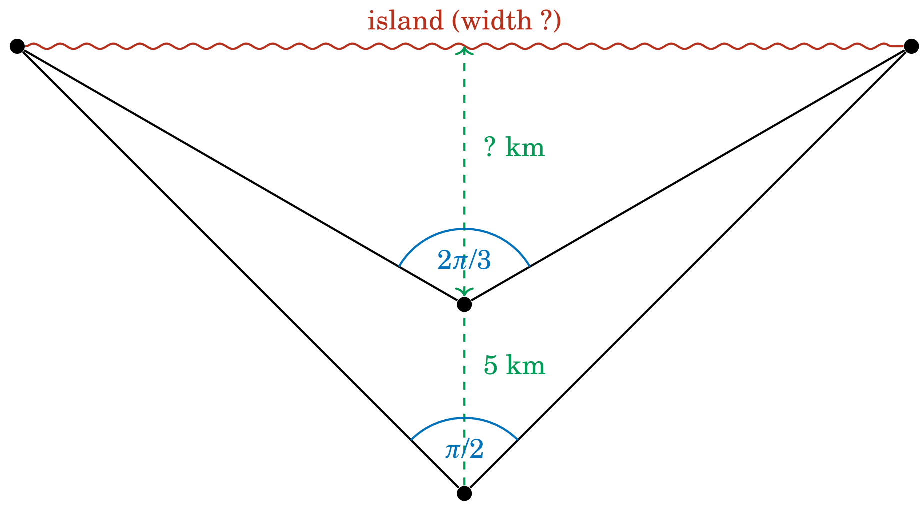
   You can assume for simplicity that the Sun follows a perfect arc.



2. Mr. Trump is a sailor. A brutal sea storm has completely thrown him off course. After a while, he's found himself approaching an unknown island. The sea is still wild and it's raining making Mr. Trump unable to estimate his distance to the island. Using his old astrolabe (an angle-measuring device), he determines **he's viewing the island under an angle of about** $\pi/2$. Thirty minutes later, he repeats the measurement and **measures an angle of** $2\pi/3$. Basing himself upon a lifetime's worth of experience, Mr. Trump thinks he might have closed the distance to the island **by about 5 km**.

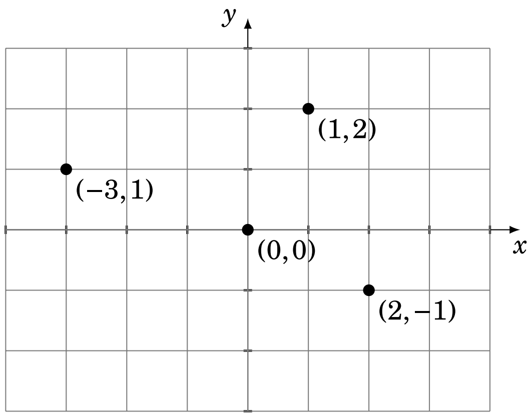
   Using these three pieces of data, determine **how long is reaching the island going to take Mr. Trump** and also **how wide the island approximately is**.

   Hint: Use the Law of Sines.



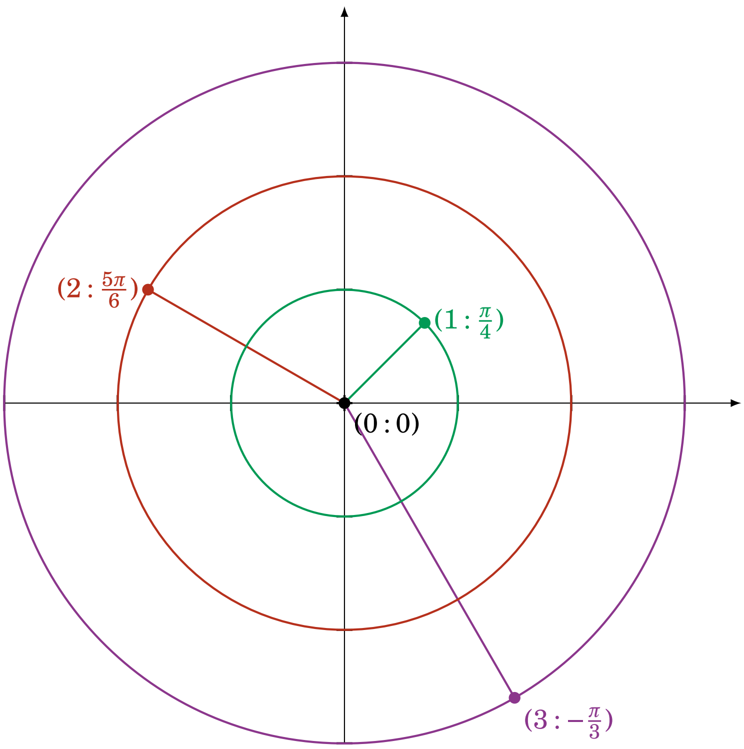
Polar Coordinates

The Cartesian coordinate system, which you're all used to, determines points in the plane by their pair of coordinates – (x, y) – which tells one how many steps to the right (x) and how many steps upward (y) you need to make to get to this point from the origin $(0, 0)$. It's basically a grid.



As you can see, the point $(1,2)$ is 1 step to the right and 2 steps upward from $(0,0)$, the point $(-3,1)$ is 3 steps to the left and 1 step upward and the point $(2,-1)$ is located 2 steps to the right and 1 step downward.

There is another system how to describe points, which is more natural to humans, called **polar coordinates**. Besides other things, trigonometry tells us that every point is determined by its angle 'of elevation' and distance from the origin. We'll write these coordinates as $(r:\theta)$ where r is the distance and θ the angle. For example, in order to see the point $(3:\pi/3)$ I have to turn my head $\pi/3$ radians (60°) from the ground and look 3 meters ahead. The plane in these coordinates looks like a lot of concentric circles.



To see the point $(1 : \pi/4)$, I need to tilt my head $\pi/4$ radians counter-clockwise and look 1 meter ahead and to see for instance $(3 : -\pi/3)$ I tilt my head $\pi/3$ radians clockwise and look 3 meters ahead.

We'll learn how to convert coordinates between Cartesian and polar coordinates. Let's start with the direction polar $\to$ Cartesian. Suppose I have a point $(2 : \pi/4)$ in polar coordinates. Let's draw it in **Cartesian** coordinates instead.