Math Homework – PreIB 3.AB 2 & 3

Trigonometric Functions

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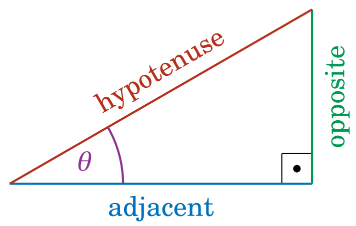
Review of Trig Functions

Given a right triangle



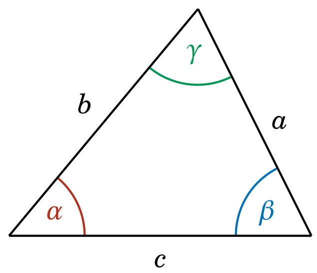
we define three **trigonometric functions**

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

The **inverse** trigonometric functions to these three are naturally

$$\sin^{-1}\left(\frac{\text{opposite}}{\text{hypotenuse}}\right) = \theta, \quad \cos^{-1}\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right) = \theta, \quad \tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right) = \theta.$$

Further, in **any** triangle



the **Law of Sines** holds. It says that

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

We also have the **Law of Cosines**, the following equalities:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos \gamma,$$
$$b^2 = a^2 + c^2 - 2ac \cdot \cos \beta,$$
$$a^2 = b^2 + c^2 - 2bc \cdot \cos \alpha.$$

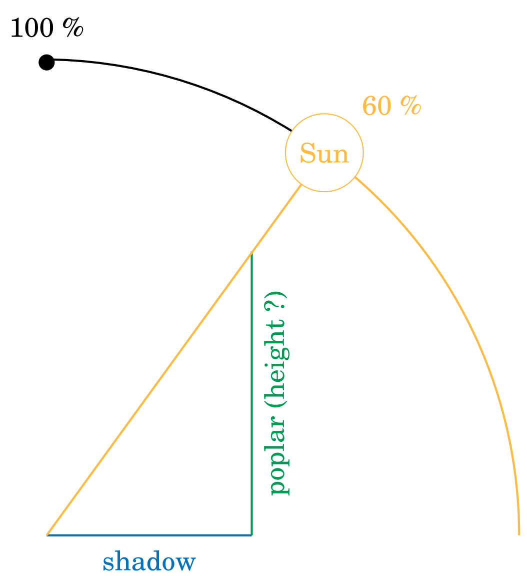
Exercises

1. Count Dolgorukov is a Russian landlord. He tends to lounge away his remaining days in his enormous garden. He is also very pedantic and meticulous about precision and symmetry. On one sunny noon, one of his poplar trees caught his unforgiving gaze. At first glance it seemed as though it was the same height as all the others but it cast a longer shadow.

   Count Dolgorukov, seething with rage, gathered his gardeners and ordered the tree shortened. Not knowing which it was, the gardeners had to wait until the next noon for the sun to be as high in the sky as it can be, which in Russia **is about 60 %**, with 100 % meaning right above the head, in order to tell the lengths of the shadows apart.

   The gardeners measured that all but one poplar cast a **shadow of approximately 21.7 meters**. The **highest poplar** cast a **shadow of approximately 22.3 meters**. How much should Count Dolgorukov's gardeners shorten the poplar so that it is as high as all the others?

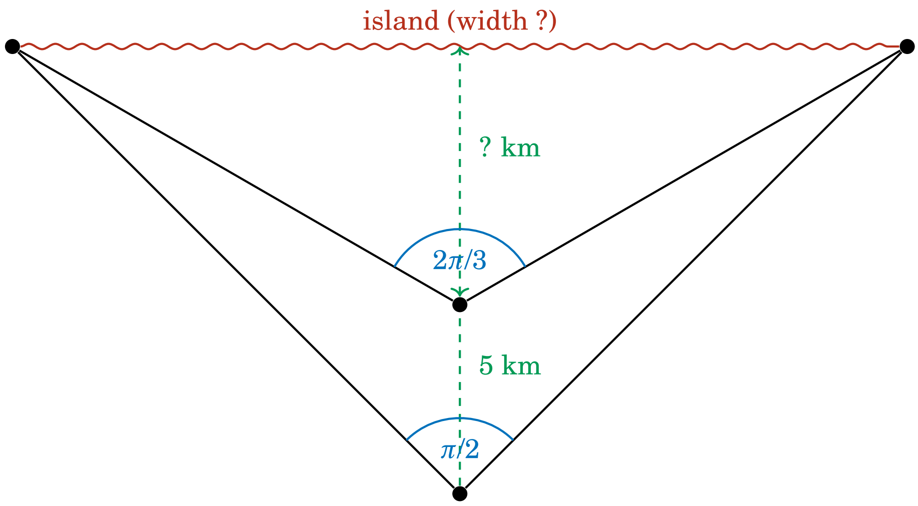
   You can assume for simplicity that the Sun follows a perfect arc.



2. Mr. Trump is a sailor. A brutal sea storm has completely thrown him off course. After a while, he's found himself approaching an unknown island. The still wild sea and bitter rain render Mr. Trump unable to estimate his distance to the island. Using his old astrolabe (an angle-measuring device), he determines **he's viewing the island under an angle of about** $\pi/2$. Thirty minutes later, he repeats the measurement and **measures an angle of** $2\pi/3$. Basing himself upon a lifetime's worth of experience, Mr. Trump thinks he might have closed the distance to the island **by about 5 km** in these last 30 minutes.

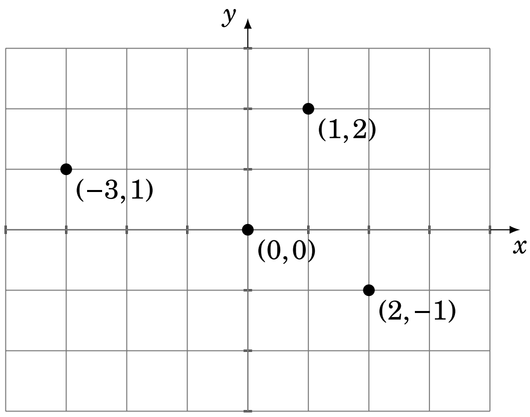
   Using these three pieces of data, determine **how long is reaching the island going to take Mr. Trump** and also **how wide the island approximately is**.

   Hint: Use the Law of Sines.



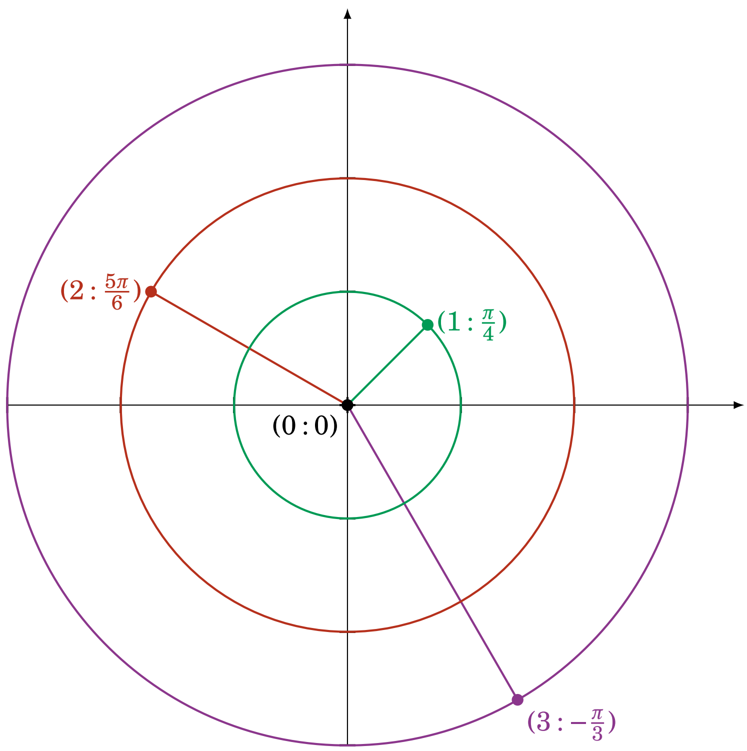
Polar Coordinates

The Cartesian coordinate system, which you're all used to, determines points in the plane by their pair of coordinates – (x, y) – which tell one how many steps to the right (x) and how many steps upward (y) he is to make to reach this point from the origin $(0, 0)$. It's basically a grid.



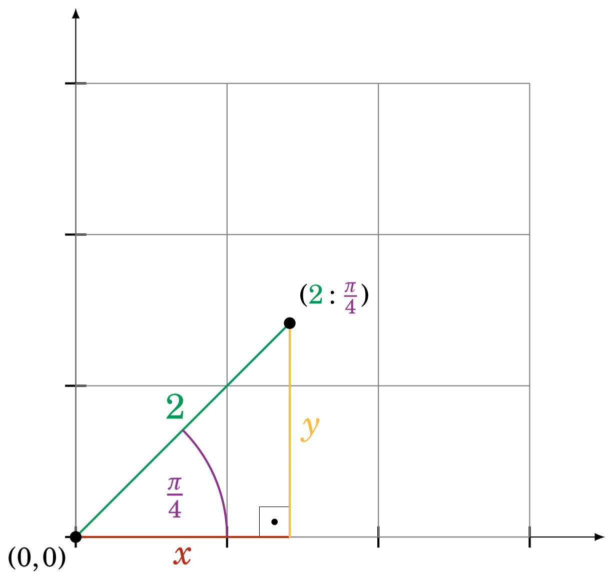
As you can see, the point $(1,2)$ is 1 step to the right and 2 steps upward from $(0,0)$, the point $(-3,1)$ is 3 steps to the left and 1 step upward and the point $(2,-1)$ is located 2 steps to the right and 1 step downward.

There exists a different system of point description, which is more natural to humans, called **polar coordinates**. Besides other things, trigonometry tells us that every point is determined by its angle 'of elevation' and distance from the origin. We'll write these coordinates as $(r:\theta)$ where r is the distance and θ the angle. For example, in order to see the point $(3:\pi/3)$ I have to tilt my head $\pi/3$ radians (60°) toward the sky and look 3 meters ahead. The plane in these coordinates resembles a bunch of concentric circles.



To see the point $(1:\pi/4)$, I need to rotate my head $\pi/4$ radians counter-clockwise and look 1 meter ahead and to see for instance $(3:-\pi/3)$ I rotate it $\pi/3$ radians clockwise and look 3 meters ahead.

We'll learn how to convert coordinates between Cartesian and polar coordinates. Let's start with the direction polar → Cartesian. Suppose I have a point $(2:\pi/4)$ in polar coordinates. Let's draw it in **Cartesian** coordinates instead.



We **do** know how to calculate x and y, don't we? Using the definitions of sin and cos, we can compute

$$\cos\left(\frac{\pi}{4}\right) = \frac{x}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \frac{y}{2}.$$

This means that $x = 2 \cdot \cos(\pi/4)$ and $y = 2 \cdot \sin(\pi/4)$. In this particular case, the coordinates end up being $x = \sqrt{2}$ and $y = \sqrt{2}$. In general, a point given in polar coordinates as (r,θ) is expressed in Cartesian coordinates as $(r \cdot \cos\theta, r \cdot \sin\theta)$.

Let's also discuss the opposite direction – Cartestion → polar. What if I know the coordinates of a point in Cartesian coordinates and I want to know what they are in polar? It's actually very easy. We need to determine two variables – r and θ. Looking at the triangle above, it seems that r can be calculated via the Pythagoras' Theorem. Indeed, we know that

$$r^2 = x^2 + y^2$$

and so

$$r = \sqrt{x^2 + y^2}$$

because r is **always positive** (remember, it's a *distance*).

To compute θ we can use the $\tan^{-1}$ trig. function. By definition, we have

$$\tan\theta = \frac{y}{x}$$

and so

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Hence, a point (x, y) in Cartesian coordinates is actually the point

$$\left(\sqrt{x^2+y^2}, \tan^{-1}(y/x)\right)$$

in polar. Going back to our original example, if $(x, y) = (\sqrt{2}, \sqrt{2})$, then

$$r = \sqrt{x^2+y^2} = \sqrt{(\sqrt{2})^2+(\sqrt{2})^2} = \sqrt{2+2} = \sqrt{4} = 2$$

and

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

See, everything works.

Exercises

1. One of the reasons you don't hear of polar coordinates before Cartesian is that they're naturally **circular** coordinates. In essence, the simplest objects are curved and flat objects like straight lines are functions with difficult definitions. To see this, **determine the equation of a straight line in polar coordinates**.

   Hint: Start with the equation of a line in Cartesian coordinates, $y = ax + b$, convert x and y to polar coordinates and isolate either r or θ.

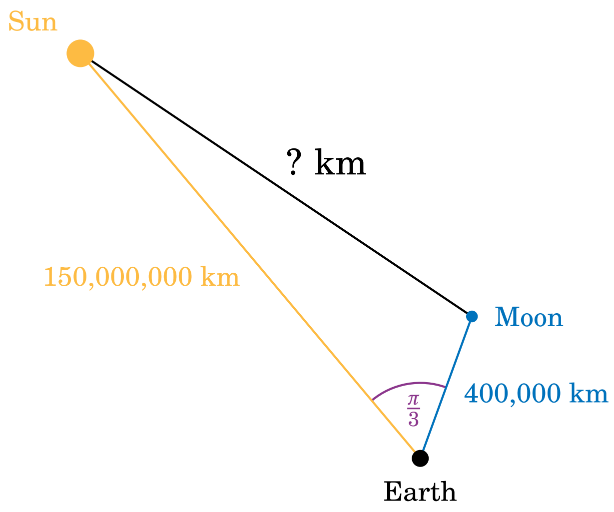
2. On the other hand, circles (centred at the origin) have very simple equations in polar coordinates. More specifically, the equation $r = R$ describes a circle with centre $(0,0)$ and radius R in polar coordinates. **Use this knowledge to find the equation of a circle in Cartesian coordinates**.

   Hint: Just convert r to Cartesian coordinates. Notice that some shapes are given by **functions** in polar coordinates that **cannot be described by a function** in the Cartesian (the circle **is not** a function in Cartesian coordinates, but it **is** in polar).

3. It's early morning and the Sun and Moon are both still visible on the sky. Say you're looking directly at the Moon and have to turn your head about $\pi/3$ radians to the left to look directly at the Sun. **How far away are the Moon and the Sun from one another at this exact moment**?

   Data: Assume the Sun is 150,000,000 km away and the Moon is 400,000 km away from the Earth.

   Hint: Use the **Law of Cosines**.



4. (**OPTIONAL**) Can you generalize the previous problem and **find a formula for the distance between two points** (r_1,θ_1) and (r_2,θ_2) in **polar** coordinates?