Addition & Multiplication

Some 'fun' with basic operations

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Addition (I denote by $+$) and multiplication (I denote by $\cdot$ or by nothing) are operations on, let's say real numbers, satisfying the following properties.

Addition	**Multiplication**
Commutativity ($C+$) $a+b=b+a$	Commutativity ($C\cdot$) $a\cdot b=b\cdot a$
Associativity ($A+$) $a+(b+c)=(a+b)+c$	Associativity ($A\cdot$) $a\cdot(b\cdot c)=(a\cdot b)\cdot c$
Distributivity ($D+\cdot$) $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$	

We also automatically assume that multiplication has priority over addition and that there exist numbers 0 and 1 such that $x+0=x$ and $x\cdot 1=x$ for every number x.

We have seen that it's not too easy to deduce the basic bracket expansion rule, that is, for instance that

$$(2\cdot x+3)\cdot(3\cdot x+4)=6\cdot x^2+17\cdot x+12$$

from only these properties. As an exercise, I'd like you to think about various other things concerning addition and multiplication which might seem obvious but aren't necessarily so.

Exercises (the ones labeled by an * are probably hard):

1. * Invent an operation $\triangle$ on real numbers which is commutative but **not** associative. This means that

   $$a\triangle b=b\triangle a \quad \text{but} \quad a\triangle(b\triangle c)\neq(a\triangle b)\triangle c.$$

   You'll have to think quite a bit about this one.
   Hint: linear functions in two variables, something like $x\triangle y=f(x,y)=2\cdot x+3\cdot y$, are good candidates for such an operation. Play with them.

2. Invent an operation $\blacksquare$ on real numbers which is associative but **not** commutative. This means that

$$a \blacksquare (b \blacksquare c) = (a \blacksquare b) \blacksquare c \quad \text{but} \quad a \blacksquare b \neq b \blacksquare a.$$

   This one is actually much easier than 1.

3. * Invent two operations $\spadesuit$ and $\clubsuit$ that are both commutative **and** associative but they are **not** distributive in any direction. This means that

$$\begin{aligned} a \spadesuit b = b \spadesuit a \quad &\text{and} \quad a \spadesuit (b \spadesuit c) = (a \spadesuit b) \spadesuit c, \\ a \clubsuit b = b \clubsuit a \quad &\text{and} \quad a \clubsuit (b \clubsuit c) = (a \clubsuit b) \clubsuit c \end{aligned}$$

   but

$$a \spadesuit (b \clubsuit c) \neq (a \spadesuit b) \clubsuit (a \spadesuit c) \quad \text{and} \quad a \clubsuit (b \spadesuit c) \neq (a \clubsuit b) \spadesuit (a \clubsuit c).$$

4. Explain how to get the equality

$$(1 + x \cdot 2) + (y \cdot (3 \cdot 5)) = (15 \cdot y) + (2 \cdot x) + 1$$

   using only the rules of addition and multiplication in the table.

5. Where would the parentheses be in the expression

$$(1 \cdot 2 + 3) \cdot (4 + 5 \cdot 6)$$

   if **addition had priority over multiplication**. I mean, write the expression which has the same numerical value as this one assuming that we first add and then we multiply.