Number Sets

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Rational Numbers

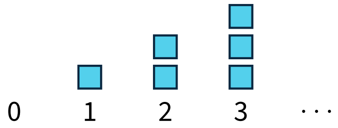
Natural Numbers

Natural Numbers – Intuition

Natural numbers are intuitively objects which represent a **quantity**.
They're the following set:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



Natural Numbers – Definition

There are many ways to define natural numbers.

One of the popular ones is using a **successor** function, denoted s. If n is a natural number, then $s(n)$ basically means 'add another block on top of n'.

One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet**! In fact, you need the successor function to define addition in the first place.

Natural Numbers – Definition

The following **five axioms** (often called *Peano axioms*) constitute the definition of natural numbers:

1. There exists the natural number 0.
2. Every natural number has a successor which is also natural.
3. The number 0 is not the successor of any natural number.
4. If $s(n) = s(m)$, then $n = m$.
5. (Induction Axiom) If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

1

Unpacking The Axioms

Natural Numbers – Axiom 1

There exists the natural number 0.

Hopefully obvious.

Natural Numbers – Axiom 2

Every natural number has a successor which is also natural.

Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

Natural Numbers – Axiom 3

The number 0 is not the successor of any natural number.

Basically means that the natural numbers are infinite only 'in one direction'. There is a **first** natural number.

Natural Numbers – Axiom 4

If $s(n) = s(m)$, then $n = m$.

This means that the successor function is **injective** – each natural number has a different successor.

Natural Numbers – Axiom 5

If a statement is true for 0 and it being true for n also implies that it is true for $s(n)$, then it is true for all natural numbers.

This means that any feature of the natural numbers 'propagates' via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it true for 1, it is true for 2 as well, etc.

2

Operations On Natural Numbers

What Is An Operation?

By **operation**, we mean a function which takes **one or multiple** natural numbers and produces **one** natural number.

For example, $+$ and $\cdot$ are operations because they take **two** natural numbers and produce **one**.

We don't often see them as functions because we don't write them as such. We write $n + m$ instead of $+(n, m)$ and $n \cdot m$ instead of $\cdot(n, m)$.

In this sense, subtraction and division **are not operations**! They take two natural numbers but they **do not produce a natural number**.

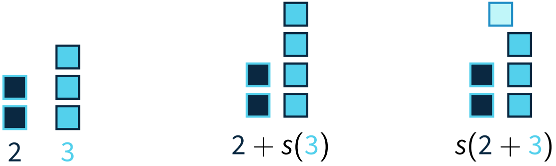
Addition

We define **addition** on natural numbers by the following two formulae:

- $n + 0 = n$,
- $n + s(m) = s(n + m)$.

We can imagine addition as ‘adding blocks *to the side*’ and the successor function as ‘adding one block *on top*’.
In this sense, $n + s(m) = s(n + m)$ only means that if you add one block atop m blocks and then n blocks to the side you have the same number of blocks as if you add n blocks next to m blocks and then another on top of that.

Addition



Using the formula $n + s(m) = s(n + m)$, one calculates $n + m$ by taking the successor of n, m times. Like this:

$$
\begin{aligned}
n + 0 &= n, \\
n + 1 &= n + s(0) = s(n + 0) = s(n), \\
n + 2 &= n + s(1) = s(n + 1) = s(n + s(0)) = s(s(n + 0)) = s(s(n)), \\
&\vdots
\end{aligned}
$$

Addition – Properties

Addition of natural numbers satisfies these two properties:

- **Commutativity**:

$$n + m = m + n.$$

- **Associativity**:

$$n + (m + k) = (n + m) + k.$$

Addition – Properties

Using blocks, **commutativity** just means that putting n blocks next to m blocks is the same as putting m blocks next to n blocks.

Addition – Properties

Using blocks, **associativity** just means that putting *m* blocks next to *k* blocks and then *n* more blocks next to those is the same as putting *m* blocks next to *n* blocks and then *k* more blocks next to those.

$4 + (2 + 3)$ $\qquad$ $(4 + 2) + 3$

Multiplication

We define **multiplication** on natural numbers by the following formulae:

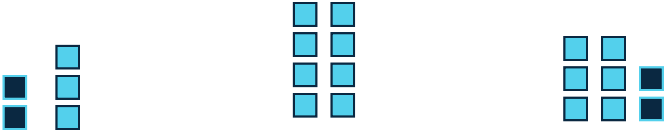
- $m \cdot 1 = m$,
- $m \cdot s(n) = m \cdot n + m$.

We can imagine multiplication $m \cdot n$ by adding a collections of n blocks for every one block in the collection of m blocks.

If we write $s(n) = n + 1$, then the second formula just means that

$$m \cdot s(n) = m \cdot (n + 1) = m \cdot n + m.$$

Multiplication



$2 \quad 3$ $\qquad 2 \cdot s(3)$ $\qquad 2 \cdot 3 + 2$

The formula $m \cdot s(n) = m \cdot n + m$ allows us to compute $m \cdot n$ by applying it n times. More precisely,

$$
\begin{aligned}
m \cdot 1 &= m \\
m \cdot 2 &= m \cdot s(1) = m \cdot 1 + m = m + m \\
m \cdot 3 &= m \cdot s(2) = m \cdot 2 + m = m \cdot s(1) + m = m \cdot 1 + m + m = m + m + m \\
&\vdots
\end{aligned}
$$

Multiplication – Properties

- **Commutativity:**

$$m \cdot n = n \cdot m.$$

- **Associativity:**

$$m \cdot (n \cdot k) = (m \cdot n) \cdot k.$$

- **Distributivity:**

$$m \cdot (n + k) = m \cdot n + m \cdot k.$$

Integers

INTEGERS

Integers or **whole numbers** come about when we say we can not only add blocks on top of collections but we can also **remove** blocks.

In mathematical terms, this means introducing **inverses** to numbers **with respect to addition**.

We typically denote these inverses by the symbol $-$.

Beware! 'Taking inverse' is a **unary** operation (meaning it acts on **one** element), not a binary one.

1

Digression

↓

Mathematical Structures

Destructive & Symmetric Transformations

Mathematical Structures

Mathematical Structure (also 'Universe')

A **mathematical** structure is a **set** with **operations**. If X is the set and $op_1, \ldots, op_n$ the operations on X, we write the resulting structure as

$$(X, op_1, \ldots, op_n).$$

The reason mathematical structures are called by some enthusiasts as mathematical 'universes' is that they really *are* universes in the broadest sense possible – a bunch of elements with prescribed rules of interaction.

Operation On A Set

Operation

An **operation** on a set X is really just a **rule of interaction** between its elements. In symbols, it is a **function**

$$op : X^n \to X$$

where $X^n \to X$ just means 'Take n elements of X and give me back one.'

Operation On A Set

Examples:

- $(\mathbb{N}, +, \cdot)$ is a structure where $+$ and $\cdot$ are **binary** operations (meaning they take two elements and return one). They can be seen as functions $\mathbb{N}^2 \to \mathbb{N}$.
- $(\{\text{orderings of vertices of a regular } n\text{-gon}\}, r, r^2, \ldots, r^n, s_1, \ldots, s_n)$ is a structure where r is the rotation by $360^\circ/n$ and $s_1, \ldots, s_n$ are all the reflections. They are all **unary** operations (they take one element and return one).

Operation As A Transformation

Operations basically describe interactions between set elements – two or more elements combine or **transform** into another one.
This interaction can be **destructive** or **symmetric**.

- The operations $+$ and $\cdot$ on $\mathbb{N}$ are **destructive** – they *destroy* the elements. When I multiply $3 \cdot 5 = 15$, I have no way to get back the 3 or the 5.
- The operation r on the vertices of a regular polygon is **symmetric** – it can be reversed or **inverted**. If, for example, vertex A is sent to G by this rotation, then rotation in the opposite direction sends G back to A.

Operation As A Transformation

Most destructive operations on a set X **can be made symmetric** by making the set larger and introducing **inverses** and **identity elements** with respect to the given operation.

Let's pick an commutative and associative **binary** operation $\triangle$ on a set X and two elements $x, y \in X$.

- For the operation $\triangle$ to be **symmetric**, we need two things:
  - A special element $e \in X$ (the **identity** element) which satisfies
$$x \triangle e = x$$
    for all elements $x \in X$.
  - Elements x^* and y^* (the **inverse** elements to x and y) such that
$$x \triangle x^* = y \triangle y^* = e.$$

Operation As A Transformation

Let's pick an (commutative) operation $\triangle$ on a set X and two elements $x, y \in X$.

- Suppose $x \triangle y$ is some element $z \in X$. For the operation $\triangle$ to be **symmetric**, we need two things:
  - A special element $e \in X$ (the **identity** element) which satisfies
    $$x \triangle e = x$$
    for all elements $x \in X$.
  - Elements x^* and y^* (the **inverse** elements to x and y) such that
    $$x \triangle x^* = y \triangle y^* = e.$$
- It's possible X doesn't have those things – which means the operation $\triangle$ cannot be symmetric **on** X, but it can be symmetric on a larger set which does contain those elements.

Operation As A Transformation

For an operation to be invertible, it really just means its **effects can be reversed**, that there's always a way back.

If x^*, y^* are the inverse to x, y with respect to $\triangle$, then we can **recover** x and y after performing the operation $\triangle$.

Suppose $x \triangle y = z$. Then, $z \triangle y^* = x$ and $z \triangle x^* = y$.

Why? We know that $x \triangle x^* = e = y \triangle y^*$. Therefore, for example,

$$z \triangle y^* = (x \triangle y) \triangle y^* = x \triangle (y \triangle y^*) = x \triangle e = x.$$

Inverse As A Function

Inverse can be thought of as a **unary** operation – it takes an element x and gives back its inverse x^*.

Clearly, the inverse to x^* is x, again.

Drawing Structures

Let's draw some mathematical structures. Remember that a mathematical structure is just a set with a bunch of operations. If our set X is finite, we can draw its elements as dots.

The simplest way to 'draw' a mathematical structure is a table.

Imagine our set has three points a, b and c. We can define a **unary** operation $\hat{\ }: X \to X$ (meaning it takes one element) on X by writing what it does into a table:

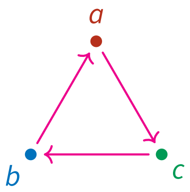
$\hat{a}$	$\hat{b}$	$\hat{c}$
c	a	b

Drawing Structures

The table

$\hat{a}$	$\hat{b}$	$\hat{c}$
c	a	b

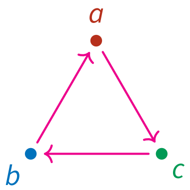
from the previous slide can be also drawn using dots and arrows for example like this:



where the magenta arrows denote the operation $\hat{}$.

Drawing Structures

Let's think what could be an **inverse** to the operation ^.



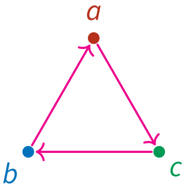
What the operation ^ does is basically changing a dot into another dot following the arrows.

As I have three dots, this means, that after traversing one arrow, I have to **traverse two arrows** to get back to where I started.

However, traversing two arrows just means **applying the operation ^ twice**.

DRAWING STRUCTURES

Let's think what could be an **inverse** to the operation $\hat{}$.



What this means in symbols, is that if $\hat{c} = b$, then $\hat{\hat{b}} = c$.

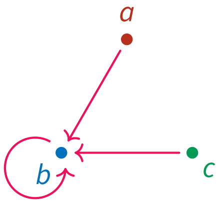
So, to get an inverse element with respect to $\hat{}$, we just need to apply $\hat{}$ twice.

Drawing Structures – Examples

Imagine a similar unary operation $\tilde{\ }$. It is given by the following table

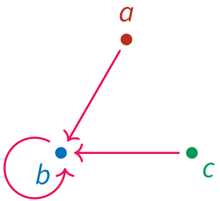
$\tilde{a}$	$\tilde{b}$	$\tilde{c}$
b	b	b

and can be pictured like this



How many elements do we need to add to make $\tilde{\ }$ symmetric?

Drawing Structures – Examples



Can we make $\sim$ symmetric?

The answer is **no** because I can never get back after reaching b. This happens since all the arrows end in b.

Drawing Structures – Examples

Let's now imagine a structure with a binary operation $\triangle$ instead. Let's upgrade the number of elements to 4 and label them a, b, c and d.

To represent a binary operation using a table, we need to write each element to each row and each column.

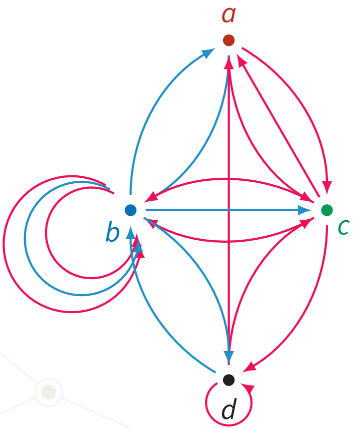
Suppose our operation $\triangle$ acts like this:

$\triangle$	a	b	c	d
a	c	a	d	b
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

Is this operation **symmetric** or **destructive**?

Drawing Structures – Examples

Drawing binary operations as arrows between elements would be a nightmare. Just watch what such a diagram would look like for the operation from the previous slide.



Drawing Structures – Examples

Let's change the operation $\triangle$ a little.

$\triangle$	a	b	c	d
a	c	d	d	b
b	d	b	c	d
c	d	c	b	d
d	b	d	d	c

Is it still **symmetric**?

Operation As A Transformation

Motivating examples:

- The operation $+$ on $\mathbb{N}$ **cannot be made symmetric**. Natural numbers do not contain inverses to $+$.We fix this by introducing **integers**, basically forcefully adding to each number n its inverse with respect to $+$, that is, the negative number $-n$. The new set is called the **integers**, denoted $\mathbb{Z}$. The identity element is 0 in this case.
- The operation $\cdot$ on $\mathbb{Z}$ cannot be made symmetric. We fix this by introducing **fractions/reciprocals** to elements. The inverse with respect to $\cdot$ of an element z is $\frac{1}{z}$. The identity element is 1.

DEFINING INTEGERS

We define the set of integers (or whole numbers) $\mathbb{Z}$ as the set of natural numbers $\mathbb{N}$ plus the inverses to all natural numbers with respect to addition.
We define addition on integers by the formulae

- $n + (-m) = x$ where $x + m = n$ for $n \geq m$ and
- $n + (-m) = -(m + (-n))$ for $n \leq m$.

and multiplication simply by

$$m \cdot (-n) = -(n \cdot m).$$

The operation $+$ on $\mathbb{Z}$ is symmetric, while $\cdot$ is still **destructive**. This leads us to the rationals.

Rational Numbers

Defining Rational Numbers

The rational numbers are an extension of the integers when we add all the **reciprocals** – fractions $1/z$ for $z \in \mathbb{Z}$.

However, rational numbers also have a strong *intuitive* meaning – they represent **parts of a whole**.

In this sense, it's logical to write things like

$$\frac{3}{6} = \frac{1}{2}$$

because both fractions represent the same amount, essentially.

Nonetheless, as *numbers*, $3 \neq 1$ and $6 \neq 2$. We would like to formalize the notion of *sameness* for two fractions.

Defining Rational Numbers

Let's try to figure out when fractions a/b and c/d are 'the same'.
This happens if the two fractions represent the same ratio. Notice that this is true **exactly when**

$$a \cdot d = b \cdot c.$$

Examples:

- $3/4 = 6/8$ because $3 \cdot 8 = 4 \cdot 6$.
- $1/2 = 10/20$ because $1 \cdot 20 = 2 \cdot 10$.

Multiplication of Rational Numbers

Defining multiplication on rational numbers is easy. We just set

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Addition of Rational Numbers

Addition is a little trickier.

The trick is to use the 'ratio representation' intuitive view of rational numbers.

If we want to add a/b to c/d, we choose a common denominator (a common 'part of a whole'), that is $b \cdot d$, then write a/b as $a \cdot d/b \cdot d$ and c/d as $b \cdot c/b \cdot d$ and add those.

The result is of course the formula

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}.$$