



# NUMBER SETS

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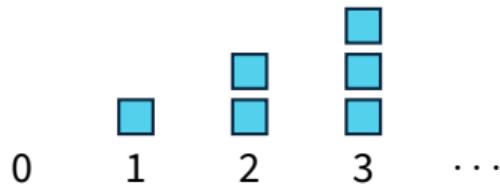
# NATURAL NUMBERS

## NATURAL NUMBERS – INTUITION

**Natural numbers** are intuitively objects which represent a **quantity**.  
They're the following set:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

A good way to think about them is to view them as '*collections of blocks*'. You get the next natural number by adding another block on top of the previous collection.



## NATURAL NUMBERS – DEFINITION

There are many ways to define natural numbers.

One of the popular ones is using a **successor** function, denoted  $s$ . If  $n$  is a natural number, then  $s(n)$  basically means ‘add another block on top of  $n$ ’.

One would be of course tempted to write

$$s(n) = n + 1$$

but that **doesn't make any sense**. We **don't have addition yet!** In fact, you need the successor function to define addition in the first place.

## NATURAL NUMBERS – DEFINITION

The following **five axioms** (often called *Peano axioms*) constitute the definition of natural numbers:

1. There exists the natural number 0.
2. Every natural number has a successor which is also natural.
3. The number 0 is not the successor of any natural number.
4. If  $s(n) = s(m)$ , then  $n = m$ .
5. (Induction Axiom) If a statement is true for 0 and it being true for  $n$  also implies that it is true for  $s(n)$ , then it is true for all natural numbers.

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## UNPACKING THE AXIOMS



## NATURAL NUMBERS – AXIOM 1

There exists the natural number 0.

Hopefully obvious.

## NATURAL NUMBERS – AXIOM 2

**Every natural number has a successor which is also natural.**

Basically means that the natural numbers are an infinite set. You can add another block atop any collection of blocks.

## NATURAL NUMBERS – AXIOM 3

The number 0 is not the successor of any natural number.

Basically means that the natural numbers are infinite only ‘in one direction’. There is a **first** natural number.

## NATURAL NUMBERS – AXIOM 4

If  $s(n) = s(m)$ , then  $n = m$ .

This means that the successor function is **injective** – each natural number has a different successor.

## NATURAL NUMBERS – AXIOM 5

If a statement is true for 0 and it being true for  $n$  also implies that it is true for  $s(n)$ , then it is true for all natural numbers.

This means that any feature of the natural numbers ‘propagates’ via the successor function. Basically, if something is true for 0 and we know that it is true for the next natural number if it is true for the previous one, then it is true for 1 as well. Because it is true for 1, it is true for 2 as well, etc.

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## OPERATIONS ON NATURAL NUMBERS



# WHAT IS AN OPERATION?

By **operation**, we mean a function which takes **one or multiple** natural numbers and produces **one** natural number.

For example,  $+$  and  $\cdot$  are operations because they take **two** natural numbers and produce **one**.

We don't often see them as functions because we don't write them as such. We write  $n + m$  instead of  $+(n, m)$  and  $n \cdot m$  instead of  $\cdot(n, m)$ .

In this sense, subtraction and division **are not operations!** They take two natural numbers but they **do not produce a natural number**.

## ADDITION

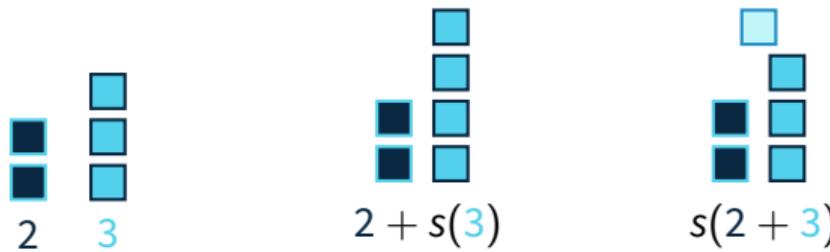
We define **addition** on natural numbers by the following two formulae:

- $n + 0 = n$ ,
- $n + s(m) = s(n + m)$ .

We can imagine addition as ‘adding blocks *to the side*’ and the successor function as ‘adding one block *on top*’.

In this sense,  $n + s(m) = s(n + m)$  only means that if you add one block atop  $m$  blocks and then  $n$  blocks to the side you have the same number of blocks as if you add  $n$  blocks next to  $m$  blocks and then another on top of that.

# ADDITION



Using the formula  $n + s(m) = s(n + m)$ , one calculates  $n + m$  by taking the successor of  $n$ ,  $m$  times. Like this:

$$n + 0 = n,$$

$$n + 1 = n + s(0) = s(n + 0) = s(n),$$

$$n + 2 = n + s(1) = s(n + 1) = s(n + s(0)) = s(s(n + 0)) = s(s(n)),$$

⋮

## ADDITION – PROPERTIES

Addition of natural numbers satisfies these two properties:

- **Commutativity:**

$$n + m = m + n.$$

- **Associativity:**

$$n + (m + k) = (n + m) + k.$$

## ADDITION – PROPERTIES

Using blocks, **commutativity** just means that putting  $n$  blocks next to  $m$  blocks is the same as putting  $m$  blocks next to  $n$  blocks.



$$2 + 3$$



$$3 + 2$$

## ADDITION – PROPERTIES

Using blocks, **associativity** just means that putting  $m$  blocks next to  $k$  blocks and then  $n$  more blocks next to those is the same as putting  $m$  blocks next to  $n$  blocks and then  $k$  more blocks next to those.


$$4 + (2 + 3)$$


$$(4 + 2) + 3$$

# MULTIPLICATION

We define **multiplication** on natural numbers by the following formulae:

- $m \cdot 1 = m$ ,
- $m \cdot s(n) = m \cdot n + m$ .

We can imagine multiplication  $m \cdot n$  by adding a collection of  $n$  blocks for every one block in the collection of  $m$  blocks.

If we write  $s(n) = n + 1$ , then the second formula just means that

$$m \cdot s(n) = m \cdot (n + 1) = m \cdot n + m.$$

# MULTIPLICATION

$$\begin{matrix} \text{■} \\ \text{■} \end{matrix} \quad \begin{matrix} \text{□} \\ \text{□} \\ \text{□} \end{matrix}$$

2      3

$$\begin{matrix} \text{□} & \text{□} \\ \text{□} & \text{□} \\ \text{□} & \text{□} \\ \text{□} & \text{□} \end{matrix}$$

$2 \cdot s(3)$

$$\begin{matrix} \text{□} & \text{□} \\ \text{□} & \text{□} \\ \text{□} & \text{□} \\ \text{□} & \text{□} \end{matrix} \quad \text{■}$$

$2 \cdot 3 + 2$

The formula  $m \cdot s(n) = m \cdot n + m$  allows us to compute  $m \cdot n$  by applying it  $n$  times. More precisely,

$$m \cdot 1 = m$$

$$m \cdot 2 = m \cdot s(1) = m \cdot 1 + m = m + m$$

$$m \cdot 3 = m \cdot s(2) = m \cdot 2 + m = m \cdot s(1) + m = m \cdot 1 + m + m = m + m + m$$

⋮

# MULTIPLICATION – PROPERTIES

- Commutativity:

$$m \cdot n = n \cdot m.$$

- Associativity:

$$m \cdot (n \cdot k) = (m \cdot n) \cdot k.$$

- Distributivity:

$$m \cdot (n + k) = m \cdot n + m \cdot k.$$



INTEGERS

# INTEGERS

**Integers or whole numbers** come about when we say we can not only add blocks on top of collections but we can also **remove** blocks.

In mathematical terms, this means introducing **inverses** to numbers **with respect to addition**.

We typically denote these inverses by the symbol  $-$ .

**Beware!** ‘Taking inverse’ is a **unary** operation (meaning it acts on **one element**), not a binary one.

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DIGRESSION



MATHEMATICAL STRUCTURES

DESTRUCTIVE & SYMMETRIC TRANSFORMATIONS



# MATHEMATICAL STRUCTURES

## MATHEMATICAL STRUCTURE (ALSO ‘UNIVERSE’)

A **mathematical** structure is a **set** with **operations**. If  $X$  is the set and  $op_1, \dots, op_n$  the operations on  $X$ , we write the resulting structure as

$$(X, op_1, \dots, op_n).$$

The reason mathematical structures are called by some enthusiasts as mathematical ‘universes’ is that they really *are* universes in the broadest sense possible – a bunch of elements with prescribed rules of interaction.

# OPERATION ON A SET

## OPERATION

An **operation** on a set  $X$  is really just a **rule of interaction** between its elements. In symbols, it is a **function**

$$op : X^n \rightarrow X$$

where  $X^n \rightarrow X$  just means ‘Take  $n$  elements of  $X$  and give me back one.’

# OPERATION ON A SET

## Examples:

- $(\mathbb{N}, +, \cdot)$  is a structure where  $+$  and  $\cdot$  are **binary** operations (meaning they take two elements and return one). They can be seen as functions  $\mathbb{N}^2 \rightarrow \mathbb{N}$ .
- $(\{\text{orderings of vertices of a regular } n\text{-gon}\}, r, r^2, \dots, r^n, s_1, \dots, s_n)$  is a structure where  $r$  is the rotation by  $360^\circ/n$  and  $s_1, \dots, s_n$  are all the reflections. They are all **unary** operations (they take one element and return one).

# OPERATION AS A TRANSFORMATION

**Operations** basically describe interactions between set elements – two or more elements combine or **transform** into another one.

This interaction can be **destructive** or **symmetric**.

- The operations  $+$  and  $\cdot$  on  $\mathbb{N}$  are **destructive** – they *destroy* the elements. When I multiply  $3 \cdot 5 = 15$ , I have no way to get back the 3 or the 5.
- The operation  $r$  on the vertices of a regular polygon is **symmetric** – it can be reversed or **inverted**. If, for example, vertex  $A$  is sent to  $G$  by this rotation, then rotation in the opposite direction sends  $G$  back to  $A$ .

## OPERATION AS A TRANSFORMATION

Most destructive operations on a set  $X$  can be made symmetric by making the set larger and introducing **inverses** and **identity elements** with respect to the given operation.

Let's pick an commutative and associative **binary** operation  $\Delta$  on a set  $X$  and two elements  $x, y \in X$ .

- For the operation  $\Delta$  to be **symmetric**, we need two things:
  - A special element  $e \in X$  (the **identity** element) which satisfies

$$x \Delta e = x$$

for all elements  $x \in X$ .

- Elements  $x^*$  and  $y^*$  (the **inverse** elements to  $x$  and  $y$ ) such that

$$x \Delta x^* = y \Delta y^* = e.$$

# OPERATION AS A TRANSFORMATION

Let's pick an (commutative) operation  $\Delta$  on a set  $X$  and two elements  $x, y \in X$ .

- Suppose  $x \Delta y$  is some element  $z \in X$ . For the operation  $\Delta$  to be **symmetric**, we need two things:
  - A special element  $e \in X$  (the **identity** element) which satisfies

$$x \Delta e = x$$

for all elements  $x \in X$ .

- Elements  $x^*$  and  $y^*$  (the **inverse** elements to  $x$  and  $y$ ) such that

$$x \Delta x^* = y \Delta y^* = e.$$

- It's possible  $X$  doesn't have those things – which means the operation  $\Delta$  cannot be symmetric **on  $X$** , but it can be symmetric on a larger set which does contain those elements.

## OPERATION AS A TRANSFORMATION

For an operation to be invertible, it really just means its **effects can be reversed**, that there's always a way back.

If  $x^*, y^*$  are the inverse to  $x, y$  with respect to  $\Delta$ , then we can **recover**  $x$  and  $y$  after performing the operation  $\Delta$ .

Suppose  $x \Delta y = z$ . Then,  $z \Delta y^* = x$  and  $z \Delta x^* = y$ .

Why? We know that  $x \Delta x^* = e = y \Delta y^*$ . Therefore, for example,

$$z \Delta y^* = (x \Delta y) \Delta y^* = x \Delta (y \Delta y^*) = x \Delta e = x.$$

## INVERSE AS A FUNCTION

Inverse can be thought of as a **unary** operation – it takes an element  $x$  and gives back its inverse  $x^*$ .

Clearly, the inverse to  $x^*$  is  $x$ , again.

## DRAWING STRUCTURES

Let's draw some mathematical structures. Remember that a mathematical structure is just a set with a bunch of operations. If our set  $X$  is finite, we can draw its elements as dots. The simplest way to 'draw' a mathematical structure is a table.

Imagine our set has three points  $a$ ,  $b$  and  $c$ . We can define a **unary** operation  ${}^{\wedge} : X \rightarrow X$  (meaning it takes one element) on  $X$  by writing what it does into a table:

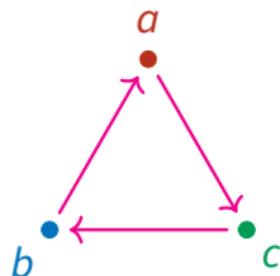
$\hat{a}$	$\hat{b}$	$\hat{c}$
<hr/>		
$c$	$a$	$b$

# DRAWING STRUCTURES

The table

$\hat{a}$	$\hat{b}$	$\hat{c}$
<hr/>		
$c$	$a$	$b$

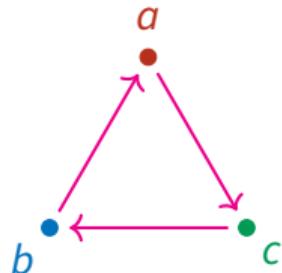
from the previous slide can be also drawn using dots and arrows for example like this:



where the magenta arrows denote the operation  $\wedge$ .

## DRAWING STRUCTURES

Let's think what could be an **inverse** to the operation  $\wedge$ .



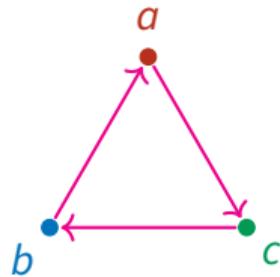
What the operation  $\wedge$  does is basically changing a dot into another dot following the **arrows**.

As I have three dots, this means, that after traversing one **arrow**, I have to **traverse two arrows** to get back to where I started.

However, traversing two **arrows** just means **applying the operation  $\wedge$  twice**.

## DRAWING STRUCTURES

Let's think what could be an **inverse** to the operation  $\wedge$ .



What this means in symbols, is that if  $\hat{c} = b$ , then  $\hat{\hat{b}} = c$ .

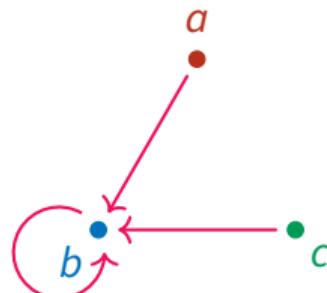
So, to get an inverse element with respect to  $\wedge$ , we just need to apply  $\wedge$  twice.

## DRAWING STRUCTURES – EXAMPLES

Imagine a similar unary operation  $\tilde{\cdot}$ . It is given by the following table

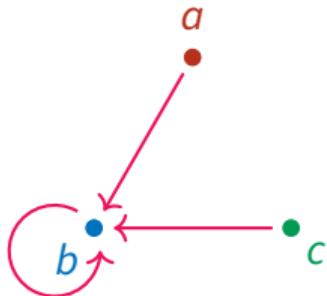
$\tilde{a}$	$\tilde{b}$	$\tilde{c}$
<hr/>		
$b$	$b$	$b$

and can be pictured like this



How many elements do we need to add to make  $\tilde{\cdot}$  symmetric?

## DRAWING STRUCTURES – EXAMPLES



Can we make  $\sim$  symmetric?

The answer is **no** because I can never get back after reaching  $b$ . This happens since all the arrows end in  $b$ .

## DRAWING STRUCTURES – EXAMPLES

Let's now imagine a structure with a binary operation  $\triangle$  instead. Let's upgrade the number of elements to 4 and label them  $a$ ,  $b$ ,  $c$  and  $d$ .

To represent a binary operation using a table, we need to write each element to each row and each column.

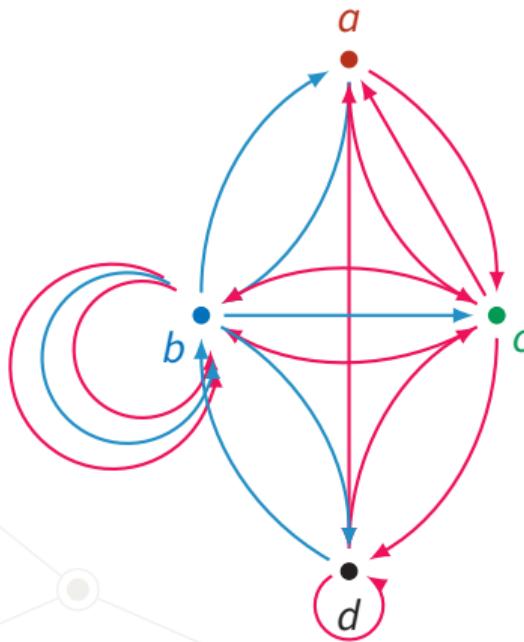
Suppose our operation  $\triangle$  acts like this:

$\triangle$	$a$	$b$	$c$	$d$
$a$	$c$	$a$	$d$	$b$
$b$	$a$	$b$	$c$	$d$
$c$	$d$	$c$	$b$	$a$
$d$	$b$	$d$	$a$	$c$

Is this operation **symmetric** or **destructive**?

## DRAWING STRUCTURES – EXAMPLES

Drawing binary operations as arrows between elements would be a nightmare. Just watch what such a diagram would look like for the operation from the previous slide.



## DRAWING STRUCTURES – EXAMPLES

Let's change the operation  $\triangle$  a little.

$\triangle$	a	b	c	d
a	c	d	d	b
b	d	b	c	d
c	d	c	b	d
d	b	d	d	c

Is it still **symmetric**?

# OPERATION AS A TRANSFORMATION

## Motivating examples:

- The operation  $+$  on  $\mathbb{N}$  **cannot be made symmetric**. Natural numbers do not contain inverses to  $+$ . We fix this by introducing **integers**, basically forcefully adding to each number  $n$  its inverse with respect to  $+$ , that is, the negative number  $-n$ . The new set is called the **integers**, denoted  $\mathbb{Z}$ . The identity element is 0 in this case.
- The operation  $\cdot$  on  $\mathbb{Z}$  cannot be made symmetric. We fix this by introducing **fractions/reciprocals** to elements. The inverse with respect to  $\cdot$  of an element  $z$  is  $\frac{1}{z}$ . The identity element is 1.

## DEFINING INTEGERS

We define the set of integers (or whole numbers)  $\mathbb{Z}$  as the set of natural numbers  $\mathbb{N}$  plus the inverses to all natural numbers with respect to addition.

We define addition on integers by the formulae

- $n + (-m) = x$  where  $x + m = n$  for  $n \geq m$  and
- $n + (-m) = -(m + (-n))$  for  $n \leq m$ .

and multiplication simply by

$$m \cdot (-n) = -(n \cdot m).$$

The operation  $+$  on  $\mathbb{Z}$  is symmetric, while  $\cdot$  is still **destructive**. This leads us to the rationals.

# RATIONAL NUMBERS