Name: ....

Question	1	2	3	Total
Points	4	6	9	19
Grade				

Throughout the exam, you're allowed to use any tools at your disposal. Write your answers thoroughly.

1. In each of the following groups, answer **YES** next to each statement if the statement is **always** true. Otherwise answer **NO**. Each group is worth 1 point if all statements in that group are evaluated correctly. (*4 points*)

Multiplying an equation of a linear system by a non-zero number doesn't change its solution set.	**YES**	**NO**
A system of n variables and m equations **can** have a unique solution only if $n \leq m$.	**YES**	**NO**
Every column of a linear system contains **exactly one** pivot.	**YES**	**NO**
A linear system has infinitely many solutions if and only if so does the corresponding homogeneous system.	**YES**	**NO**
The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined for **every** two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, where $n \geq 1$.	**YES**	**NO**
For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, where $n \geq 1$, if $\lVert\boldsymbol{u}\rVert + \lVert\boldsymbol{v}\rVert = \lVert\boldsymbol{u} + \boldsymbol{v}\rVert$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent.	**YES**	**NO**
The solution set of any linear system forms a vector space.	**YES**	**NO**
If V is a vector space with zero vector $\boldsymbol{0} \in V$, then $\{\boldsymbol{0}\} \leq V$.	**YES**	**NO**
$\operatorname{span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R}\right\}$.	**YES**	**NO**
Let V be a vector space and $S, T \subseteq V$. If $\operatorname{span} S \leq \operatorname{span} T$, then **necessarily** $S \subseteq T$.	**YES**	**NO**
If a set $S \subseteq V$, where V is a vector space, is linearly independent, then it **necessarily** has number of elements less or equal to $\dim V$.	**YES**	**NO**
If $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n)$ is a basis of V and $\boldsymbol{v} = 2 \cdot \boldsymbol{b}_3 - 7 \cdot \boldsymbol{b}_5$, then $(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{v}, \boldsymbol{b}_4, \ldots, \boldsymbol{b}_n)$ is also a basis of V.	**YES**	**NO**

2. Solve the following problems. Include important steps of your calculations. Each problem is worth 2 points.

(a) Write the solution set (in any form you wish) of the following linear system. (*2 points*)

$$\begin{aligned} -3x + 3y + z &= 1 \\ x \phantom{{}+3y} + 2z &= 2 \\ x + y + 3z &= 3 \end{aligned}$$

(b) Given the set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$, find a **linearly independent** set $T \subseteq S$ with $\operatorname{span} T = \operatorname{span} S$. (*2 points*)

(c) Prove that the quadruple $B = \left(\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix} \right)$ is a basis of $\mathbb{R}^{2\times 2}$, the vector space of 2×2 real matrices. (*2 points*)

3. Prove the following statements. If you base your proof upon another result, refer to the latter as precisely as you can. Of course, you may not refer to the given statement directly, or to propositions whose proofs use the statement.

(a) Prove that the linear system (*2 points*)

$$\begin{aligned} ax + y &= a^2 \\ x + ay &= 1 \end{aligned}$$

has a unique solution as long as $a \notin \{-1, 1\}$.

(b) The *generalised triangle inequality* states that (*3 points*)

$$\|\boldsymbol{v}_1 + \boldsymbol{v}_2 + \ldots + \boldsymbol{v}_k\| \leq \|\boldsymbol{v}_1\| + \|\boldsymbol{v}_2\| + \ldots + \|\boldsymbol{v}_k\|$$

for all $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ and $k, n \geq 1$. Prove it by induction on the number of vectors, k.

(c) Let $V, W \leq \mathbb{R}^n$. Prove that if $\dim V + \dim W > n$, then $\dim(V \cap W) > 0$. (*4 points*)