Mock Exam
Quadratic Functions & Equations

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Meet Mr. Newton, the innkeeper. Mr. Newton regularly supplies his inn with high quality beer he can however only order in huge barrels. Every time he orders a batch, some litres of beer go to waste.

Mr. Newton is not a maths guy. Sad that such a good beer is being wasted by his own inadequacy, he's started to experiment. He knows that **he needs at least two huge barrels** of beer a day to satisfy his thirsty customers. Of course, he doesn't need to order new barrels every day but to meet his quality standards, he mustn't keep the barrelled beer stocked for too long, either.

He measured that when he ordered

- **three barrels** of beer, **14 litres** of it went to waste;
- **seven barrels** of beer, **18 litres** of it went to waste;
- **nine barrels** of beer, **8 litres** of it went to waste.

Help make Mr. Newton happy by minimizing the amount of beer that go to waste, or, at least, tell him what's the worst possible order he could make.

1. Given only three pieces of data, the best we can do is model the situation using a quadratic function having as input the number of ordered barrels and the losses in litres as output. We know that it should match the provided data precisely. That is, our function, which we label f, must satisfy

$$\begin{aligned} f(3) &= 14 \\ f(7) &= 18 \\ f(9) &= 8. \end{aligned}$$

   Determine the definition of such a quadratic function.

2. Mr. Newton is most afraid of losing even more litres of beer than he already did. He demands you first tell him, what would be the **worst** order he could make. That is, calculate the number of barrels whose order would make him lose the most litres of beer.

3. Now that Mr. Newton has calmed down, knowing he won't lose too much beer, he also asks you to help him determine how many barrels of beer he should order **to lose no beer at all**. Remember, that he always needs to order at least two barrels. **USE THE VIÈTE FORMULAE TO SOLVE THIS!**

4. Use the results from 3 to factor f as a product of two linear functions.

5. Mr. Newton, happy with how his inn is now coming along, opens another. But, he's immediately stricken with grief, when his new inn starts wasting beer like crazy. He promptly starts to measure how much beer he's losing and, using the same method as in 1, you calculate that the beer losses of his second inn can be modelled by the quadratic function

$$g(x) = -x^2 + 10x - 21$$

where x is the number of ordered barrels. **How many barrels** should Mr. Newton order **for both of his inns**, so that the **total losses** (that is, the losses of the first inn + losses of the second inn) **are as small as possible**?

Possible solution.

1. *Since f is supposed to be a quadratic function, we know that its general formula is $f(x) = ax^2 + bx + c$ for some real numbers a, b and c. Plugging the known inputs and outputs into it gives the system of linear equations*

$$\begin{aligned} f(3) &= a \cdot 3^2 + b \cdot 3 + c = 14 \\ f(7) &= a \cdot 7^2 + b \cdot 7 + c = 18 \\ f(9) &= a \cdot 9^2 + b \cdot 9 + c = 8. \end{aligned}$$

If we subtract the first equation from the second and the second equation from the third, we get

$$\begin{aligned} 40a + 4b &= 4 \\ 32a + 2b &= -10. \end{aligned}$$

We can divide the first equation by 4 *and the second by* 2*, to further get*

$$\begin{aligned} 10a + b &= 1 \\ 16a + b &= -5. \end{aligned}$$

We can then subtract the first equation from the second to arrive at an equation with only one variable

$$6a = -6,$$

whose solution is $a = -1$. Plugging this into the first equation above gives

$$-10 + b = 1,$$

that is, $b = 11$. Finally, substituting a and b into, for example, the first original equation yields

$$-9+33+c=14,$$

from which we calculate that $c = 10$. It follows that we can model Mr. Newton's losses by the function $f(x) = -x^2 + 11x - 10$.

2. *Since the function f describes Mr. Newton's losses, the higher it is, the more litres of beer he loses. Hence, to find the number of barrels which causes the most number of litres of beer to go to waste, we're looking for the maximal value of the function f. The graph of f is a parabola which looks like a 'hill' in this case because $a = -1 < 0$. This means that f does have a maximum at its vertex, whose first coordinate we know is $x = -b/2a$. In our case, we get that $x = -11/(-2) = 5.5$ so Mr. Newton suffers the most losses if he orders either 5 or 6 barrels of beer.*

3. *In order to compute the number of barrels whose order would cause no losses to Mr. Newton, we need to solve the equation*

$$f(x) = 0.$$

Using Viète's formulae, we know that if x_1 and x_2 are the solutions to this equation, then

$$x_1 + x_2 = -\frac{b}{a} = \frac{-11}{-1} = 11,$$
$$x_1 \cdot x_2 = \frac{c}{a} = \frac{-10}{-1} = 10.$$

An immediate solution that comes to mind is $x_1 = 1$ and $x_2 = 10$. We know that Mr. Newton needs to order at least 2 barrels to keep his customers satisfied, so the only relevant solution to his problem is the order of 10 barrels.

4. *We know that if x_1 and x_2 are the roots of a quadratic function $f(x) = ax^2 + bx + c$, then this function decomposes as $f(x) = a(x - x_1)(x - x_2)$. In our case, this means that*

$$f(x) = -(x-1)(x-10).$$

5. *As the function f describes the losses of Mr. Newton's first inn and g describes the losses of his second inn, the total losses are given by the quadratic function $f(x) + g(x)$. We want the total losses to be as close to 0 as possible, which entails solving the quadratic equation*

$$f(x) + g(x) = 0.$$

Using the definitions of these functions, we get

$$-x^2 + 11x - 10 - x^2 + 10x - 21 = -2x^2 + 21x - 31 = 0.$$

This equation would not be trivial to solve using Viète's formulae, so we use the general formula for the solutions to a quadratic equation instead. We get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-21 \pm \sqrt{441 - 248}}{-4} = \frac{-21 \pm \sqrt{193}}{-4}.$$

As $\sqrt{193}$ *is approximately* 14 *(because* $14^2 = 196$*), the two solutions are approximately*

$$x_1 = \frac{-21 + 14}{-4} = \frac{7}{4} \quad \textit{and} \quad x_2 = \frac{-21 - 14}{-4} = \frac{35}{4}.$$

As Mr. Newton needs to order at least 4 barrels for both of his inns combined, the first solution is useless. Unfortunately, this time, Mr. Newton cannot avoid losing beer completely but the number of barrels whose order minimizes the number of wasted beer is 9 *because* 35/4 *is just* 1/4 *short of* 9.
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