Homework – PreIB 3.AB 3

Triangulations and Symmetries of Regular Polygons

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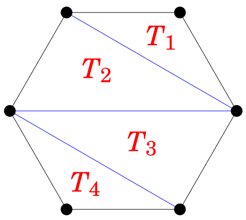
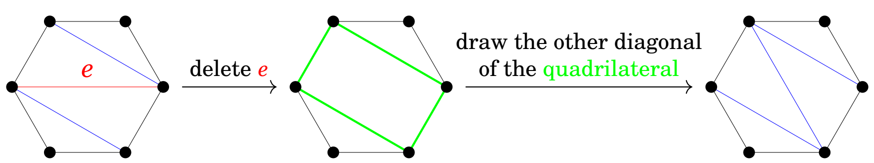
DON'T FORGET TO EXPLAIN EVERYTHING EVEN IF YOU THINK IT'S OBVIOUS!

Triangulations

Recall that a **triangulation** of a regular (or generally convex) polygon is its division into triangles by non-intersecting diagonals. For example, here is a triangulation of a regular hexagon intro triangles T_1, T_2, T_3 and T_4.



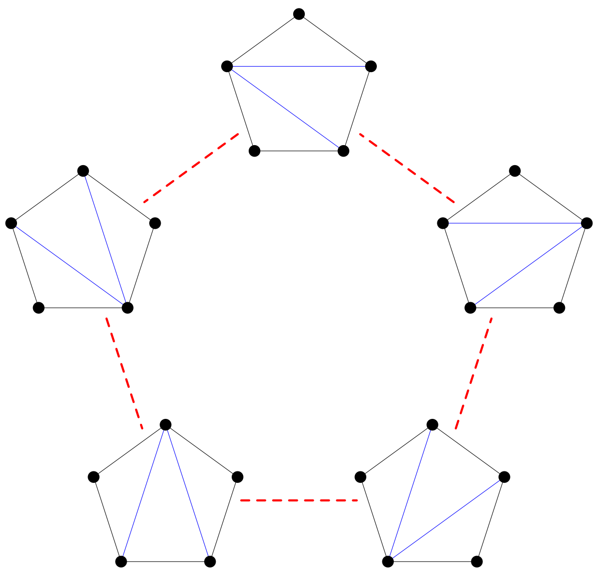
Two triangulations of a regular polygon are related by a **flip** if you can get one from the other by deleting one edge and replacing it by the other diagonal in the resulting quadrilateral. See the picture below.



Flip of a triangulation.

Actually, any triangulation of a regular polygon can be transformed into any other triangulation just by flipping (often more than once). We can draw a graph where we connect two different triangulations of a regular polygon if they are related by a single flip.

The pentagon has 5 different triangulations. Their graph can look like this:



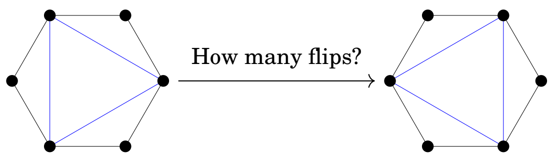
Triangulations of the pentagon. Blue segments are diagonals in a triangulation and red lines are flips.

Problems

1. In the graph above, each triangulation is connected to two other by a flip because a triangulation of a pentagon has two diagonals. If we drew the same graph of triangulations for the hexagon, how many other triangulations would a fixed triangulation be connected to? Why?

2. A regular hexagon has 14 distinct triangulations. Draw its graph of triangulations. It's going to take you a while...

3. Using the graph find the *smallest* possible number of flips to get from the triangulation on the left to the triangulation on the right.



4. Find at least three other ways how to get from the left triangulation to the right one using the same number of flips.

5. (**OPTIONAL**) How many ways are there using the same number of flips?

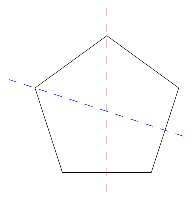
6. Can flips be achieved using symmetries of regular polygons (meaning rotations and reflections)? If yes, show how on the triangulation graph of the pentagon. If not, explain why.

Symmetries

Recall the each regular n-gon has n rotational symmetries and n reflectional symmetries. The rotational symmetries are rotations by $k \cdot 360°/n$ where k ranges from 1 to n. If n is odd, then all the reflectional symmetries are given by lines passing through one vertex and the midpoint of the opposite side. If n is even, then $n/2$ reflectional symmetries are over lines connecting opposite vertices and the other $n/2$ over lines passing through midpoints of opposite sides.

We saw that we can apply symmetries one after another to get new symmetries. We also determined which types of symmetries we need to create all the other symmetries. It was the rotation by $k \cdot 360°/n$ if k/n cannot be simplified and any reflection.

But, not all polygons are created equal. For example, in the pentagon one can create all symmetries just by using two reflections. Take s_1 and s_2 from the picture below.



We'll denote rotation by $n°$ in the counter-clockwise direction as $\circlearrowleft n°$. You can check that $s_1 s_2 = \circlearrowleft 144°$. But, $144 = 2 \cdot 360/5$ and $2/5$ can't be simplified! This means that we can get all other rotations and reflections using just s_1 and s_2.

Finally, I want you to think about $s_1 s_2$ as 'multiplication' of symmetries and of

$$s_1 s_2 = \circlearrowleft 144° \tag{$*$}$$

as an 'equation'. You can multiply both sides by any symmetry and it's still going to hold true. But careful! Symmetries **do not commute**, so it matters if you multiply from the left or from the right. The angle from s_2 to s_1 is $72°$ in the clockwise direction. Which means that to get s_1 from s_2, we need to rotate by $144°$ counter-clockwise and then use s_2. Indeed, look what happens if we multiply ($*$) by s_2 **from the right**. We get

$$s_1 s_2 s_2 = \circlearrowleft 144° s_2.$$

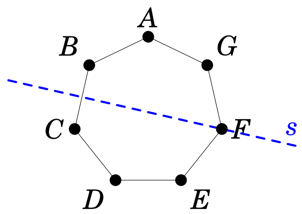
But $s_2 s_2$ does nothing because it's the same reflection repeated twice. In the end, we get

$$s_1 = \circlearrowleft 144° s_2.$$

Which is precisely what we just said. Rotate by $144°$ counter-clockwise and use s_2.

Problems

1. In the heptagon



you're given the reflection s and the rotation $r = ⟲ 3 \cdot 360°/7$.

   (a) Is it possible to construct all other symmetries of the regular heptagon using only r and s? Why?

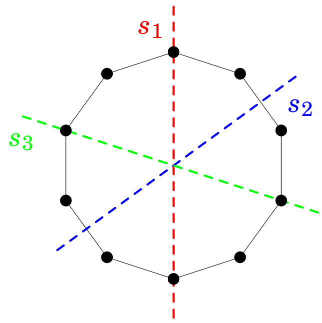
   (b) Using only r and s find

      i. the rotation $⟲ 4 \cdot 360°/7$;

      ii. the rotation $⟳ 360°/7$, where $⟳$ means 'rotate clockwise';

      iii. the reflection over the line passing through B and the midpoint of EF.

2. In a decagon (10 vertices), you're given the reflections s_1, s_2 and s_3.



   (a) Can you generate all the symmetries of the regular decagon using only s_1 and s_2? Explain.

   (b) How about using only s_1 and s_3? Explain again.