Polygons & Transformations Cheatsheet

3.AB PreIB Math

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Polygons

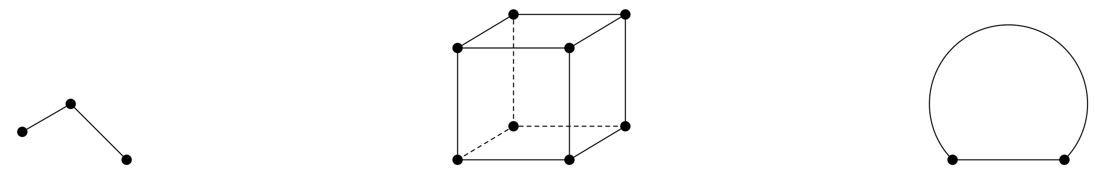
Polygon is a **closed** 2D shape **made only of segments**. We call the endpoints of those segments, **vertices**, and the segments themselves, **edges**.

Examples

Triangle Quadrilateral Pentagon Dodecagon

Polygons with n sides are called n**-gons**.

Counterexamples



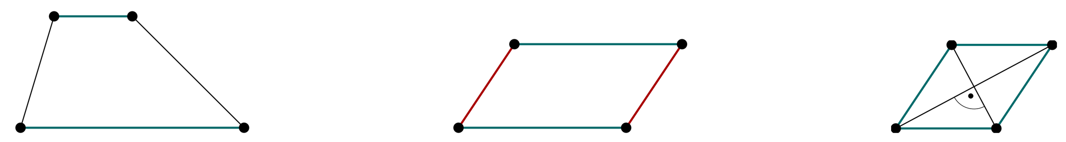
Not closed 3D Not straight

Convex Polygons

A polygon is called **convex** if it has no internal angle greater than 180°.

Convex NOT convex

Special types of convex polygons



Trapezoid/Trapezium
A convex quadrilateral with at least two parallel sides.

Parallelogram
A convex quadrilateral with two pairs of parallel sides.

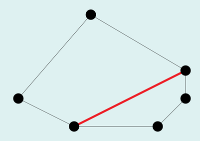
Rhombus
An **equilateral** (all sides of the same length) parallelogram.

In addition, if a **convex** polygon has **all sides of the same length** and **all angles of the same size**, it is called **regular**.

Equilateral triangle (regular trigon) Square (regular tetragon) Regular pentagon Regular hexagon

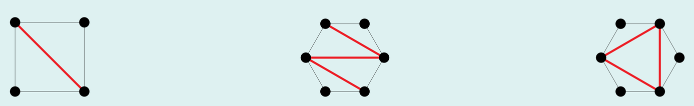
Diagonals & Triangulations

A **diagonal** in a **convex** polygon is a segment connecting two of its **non-adjacent** vertices.

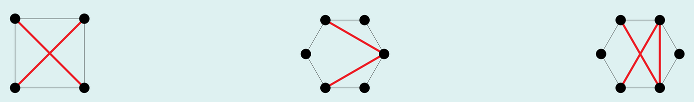


Diagonal in a convex hexagon.

A **triangulation** of a **convex** polygon is its division into triangles by **non-intersecting** diagonals.



Examples of triangulations.



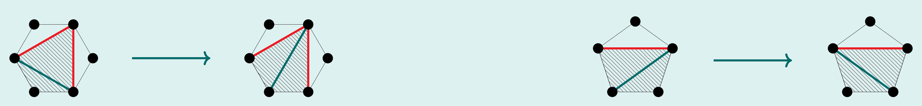
Counterexamples of triangulations.

The total number of different triangulations of a convex n-gon is

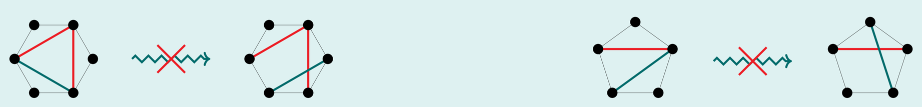
$$\frac{n \cdot (n+1) \cdot \ldots \cdot (2n-4)}{(n-2)!},$$

which you **of course don't have to remember**. Interestingly enough, every triangulation can be transformed into any other by a series of **flips**.

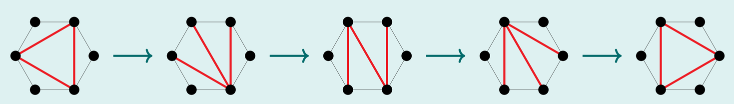
A **flip** is a swap of one diagonal for the other in a chosen quadrilateral so that the **result is again a triangulation**.



Examples of flips.



Counterexamples of flips.



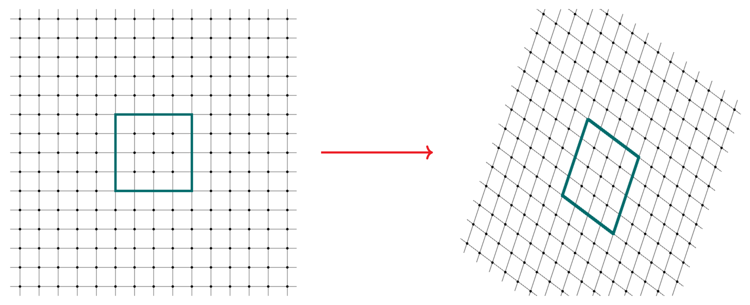
Passage from one triangulation to another through a series of flips.

I encourage you to think about how to determine the number of flips necessary to pass from one triangulation to another. Can I have made the passage above in fewer flips?

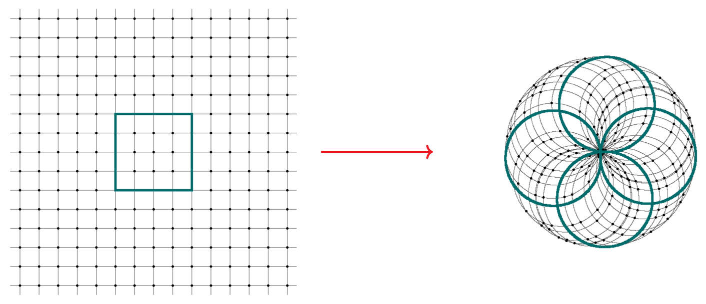
Plane Transformations

The **plane** is basically just the set $\mathbb{R}^2$ of all **pairs of real numbers**. A pair $(x, y) \in \mathbb{R}^2$ is typically called a **point**. Then, a plane **transformation** is a **function** which maps points to points. In symbols, it's a function $\mathbb{R}^2 \to \mathbb{R}^2$.

We can visualise what a transformation does for example by look at the image of a square (or an entire grid).



The transformation $(x, y) \mapsto (\frac{1}{3}(2x - y), \frac{1}{2}(x + 2y))$.

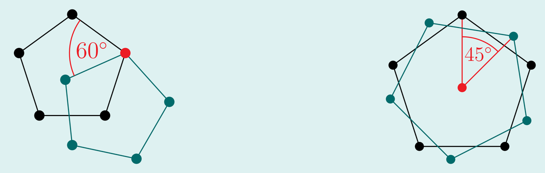


The transformation $(x, y) \mapsto (100(\sin x + \cos y), 100(\cos x + \sin y))$.

Rotations & Reflections

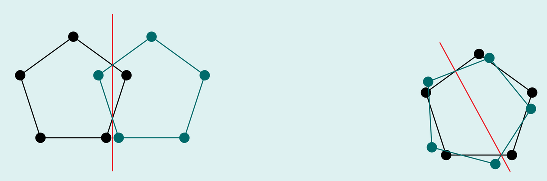
We shall be interested in two specific plane transformations – **rotations** and **reflections**.

Rotations are plane transformations that, well ..., rotates the entire plane around a fixed point called **anchor**. Applied to polygons, rotations may look like this:



Examples of rotations around a given anchor.

Reflections are basically 'mirrors'. They mirror each point in the plane through a given line called **axis** (of reflection).



Examples of reflections over a given axis.

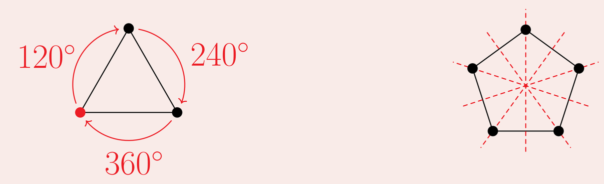
Symmetries of Regular Polygons

Some **rotations** and **reflections** get along nicely with **regular polygons**. By this, we mean that they keep them intact. We call them the **symmetries** of the polygon.

Each regular n-gon has multiple symmetries:

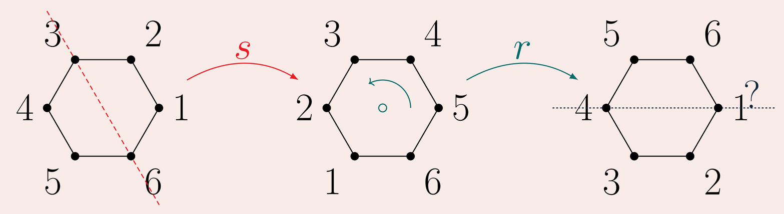
(r) **rotation** by $k \cdot 360°/n$ for any k between 1 and n.
(s) **reflection**
- over lines passing through centres of opposite sides or through opposite vertices if n is **even**;
- over lines passing through a centre of a side and the opposite vertex if n is **odd**.

Therefore, an n-gon has n rotational and n reflectional symmetries.



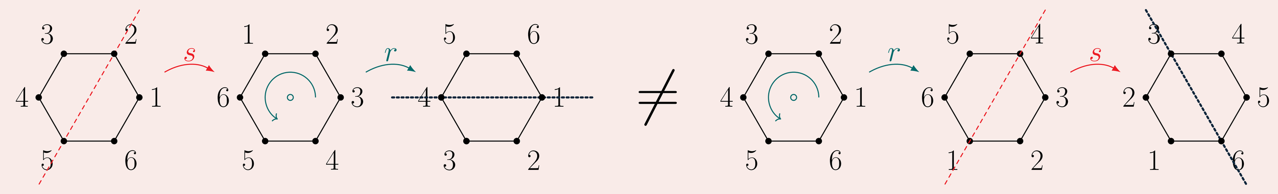
Examples of regular polygon symmetries.

Moreover, symmetries (being functions) can be **chained** or **composed**, creating new symmetries. We'll label rotations by the letter r and reflections by s. A **chain** or **composition** is read left-to-right, that is, sr means 'apply s first, then r'.



Example of the composition $\mathbf{sr}$ of a reflection s and a rotation r.

The order of composition matters!



In general, a composition of

- a **rotation** and a **rotation** is again a **rotation**,
- a **rotation** and a **reflection** (in any order) is a **reflection**,
- a **reflection** and a **reflection** is a **rotation**.

References (opcional)

[1] Claude E. Shannon.
A mathematical theory of communication.
Bell System Technical Journal, 27(3):379–423, 1948.