COMBINATORIAL PROPERTIES OF HIGHER CLUSTER CATEGORIES

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METHODS OF RESEARCH

The majority of the first year will be spent on research objective 1 – the combinatorial description of higher Auslander algebras of path algebras of Dynkin type D and E and their associated cluster categories – followed by a thorough study of the theory of generalized cluster categories (mainly [Ami09, Sections 3 and 4], [Ami08, Chapters 5, 6 and 7] and [Guo12, Chapters 2 and 3]), of n-representation-infinite algebras ([HIO14, Sections 1-5]) and n-angulated categories (the entirety of [GKO13]).

The correspondence between quivers of Dynkin type D and triangulations of discs with marked points on their boundaries and one in their interior is bound to prove crucial for this endeavour. Unlike in the case of triangulations of discs with marked points lying only on their boundaries (giving rise to quivers of type A), the existence of an inner marked point naturally leads to the concept of a self-folded triangle (a triangle with only two distinct sides). The first non-trivial task will entail generalizing this concept to higher dimensions and understanding the triangulations of a cyclic polytope with a distinguished point in its interior.

In [OT11, Section 3], an orientation of A_n is chosen that guarantees the equality $\mathrm{Hom}_{A_n^1}(P_i, P_j) = 0$ if and only if $i > j$ (here P_i denotes the projective cover of the simple module S_i concentrated in vertex i). With the right approach, this choice makes the study of partial tilting modules of A_n^d contained in $\mathrm{add}_{A_n^d} M$ (here $_{A_n^d} M$ denotes the cluster tilting module of A_n^d) – and also the indecomposable objects of the associated cluster category – smooth and elegant. Such an orientation cannot be chosen for the quivers of type D or E and thus irregularities will inevitably occur and need to be dealt with. More irregularities lurk within the connection between bistellar flips of the triangulations of cyclic polytopes and mutations of basic tilting modules of A_n^d – as well as mutations of the cluster tilting objects of the associated cluster category – detailed in [OT11, Sections 4 and 6]. The issue lies in the fact that the folded edges in self-folded triangles cannot be flipped. This situation does not arise in the study of triangulations of discs without internal marked points and is likely to propagate to higher dimensions. It is already a predictor of irregular behaviour even in the low-dimensional case. The flip-mutation correspondence would imply that the indecomposable direct summand (of the tilting module – or the cluster tilting object – determined by the given triangulation) corresponding to an unflippable edge (or facet in higher dimensions) cannot be substituted for another. Nonetheless, this statement is false for the (classical) cluster categories of acyclic quivers where any indecomposable summand of a cluster tilting object can be substituted. This points to the possibility that the correspondence between triangulations and tilting modules (or cluster tilting objects) will not be entirely bijective or the triangulations will have to be considered up to a certain kind of equivalence.

Finally, as quivers of type E do not arise from triangulations of bordered surfaces with marked points, a combinatorial structure of not necessarily geometric nature needs to be found to deal with this case. Hopefully, the same structure can be used for all three such quivers.

The next two years will be dedicated to the generalization of the theory of higher Auslander algebras of representation-finite hereditary algebras to the path algebras of affine Dynkin quivers, the study of their associated generalized cluster categories and the combinatorial properties of both.

First, I need to conceive a suitable notion of a generalized higher Auslander algebra, that is, work on research objective 2. The construction in [HIO14, Section 5] of the n-representation-infinite algebras of type $\tilde{A}$ can easily be generalized to types $\tilde{D}$ or $\tilde{E}$ by simply considering root systems of type D_n or E_n at its very beginning. In the case of type $\tilde{A}$, these graded algebras are bimodule $(n+1)$-Calabi-Yau and for $n = 1$ their degree 0 part are precisely the path algebras of quivers of type $\tilde{A}$ for non-cyclic orientations. It remains to prove that this is also the case if the initial root system is different. However, it may well happen that a complete understanding of the combinatorics of their tilting modules proves unrealistic or exceedingly difficult. In such a case, I will strive to reach a simpler construction which exploits the apparent proximity of the path algebras of quivers of (pure) Dynkin type to those of type affine Dynkin.

This would allow me to successfully put to use the knowledge gained from [OT11] and research objective 1 at the cost of generality.

The next step is to understand the combinatorics of tilting modules in the algebras from the previous paragraph. The most natural place to start are the Auslander-Reiten quivers of the path algebras of affine Dynkin quivers which have already been described and studied. I trust a reasonable approach is to find a combinatorial counterpart in this low-dimensional case before moving on to higher dimensions. The reason for that is the following: a combinatorial model for the tilting modules of path algebras of affine Dynkin quivers may serve as a sort of 'benchmark' for the naturality of the definition of their generalized higher Auslander algebras. Should the model be too simple or just way off the mark in higher dimensions, it will call into question not only the correctness of the model itself but also the method wherewith said algebras were conceived.

This step also naturally includes the computation of AR quivers of the generalized higher Auslander algebras. Should it prove hard to achieve, I intend to first study the AR quivers of their n-preprojective, n-preinjective and n-regular subcategories which form disjoint parts of the AR quiver of the entire n-representation-infinite algebra in question (see [HIO14, Theorem 4.18]).

Research objective 4 is quite likely the most difficult to accomplish. In order to define a suitable generalized cluster category associated to an n-representation-infinite algebra Λ defined in objective 2, I shall first try to mimic (with necessary adjustments) the construction from [OT11, Section 5] of a $(d+2)$-angulated d-cluster tilting subcategory of $\mathcal{C}_\Lambda^{2d}$ (in the sense of [Ami09]). It would be a remarkable display of symmetry should their cluster tilting objects be modelled by the structure discovered in objective 3. The pessimist in me is inclined to think this will not be the case and a different approach will have to be taken, quite possibly utilizing the further generalizations of the theory of higher cluster categories depicted in [Guo12].