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Assignment I

Dualities In Triangulated Categories

(1) Consider the arrow $X \xrightarrow{f} Y$. By the axiom (TR1), there exists a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{i} W \xrightarrow{j} \Sigma X.$$

Also, by the axiom (TR3), there exists an arrow $\alpha : W \to Z \oplus Z'$, such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\binom{1_X}{0}"] & Y \arrow[r, "i"] \arrow[d, "\binom{1_Y}{0}"] & W \arrow[r, "j"] \arrow[d, dashed, "\exists\alpha"] & \Sigma X \arrow[d, "\binom{1_{\Sigma X}}{0}"] \\
X \oplus X' \arrow[r, "f\oplus f'"] & Y \oplus Y' \arrow[r, "g\oplus g'"] & Z \oplus Z' \arrow[r, "h\oplus h'"] & \Sigma X \oplus \Sigma X'
\end{tikzcd}$$

commutes, where 1_A denotes the identity arrow of A and $\binom{1_A}{0}$ the (unique) inclusion arrow $A \hookrightarrow A \oplus A'$ for $A \in \{X, Y, \Sigma X\}$.

Finally, consider the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\binom{1_X}{0}"] & Y \arrow[r, "i"] \arrow[d, "\binom{1_Y}{0}"] & W \arrow[r, "j"] \arrow[d, dashed, "\exists\alpha"] & \Sigma X \arrow[d, "\binom{1_{\Sigma X}}{0}"] \\
X \oplus X' \arrow[r, "f\oplus f'"] \arrow[d, "(1_X\ 0)"] & Y \oplus Y' \arrow[r, "g\oplus g'"] \arrow[d, "(1_Y\ 0)"] & Z \oplus Z' \arrow[r, "h\oplus h'"] \arrow[d, "(1_Z\ 0)"] & \Sigma X \oplus \Sigma X' \arrow[d, "(1_{\Sigma X}\ 0)"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X,
\end{tikzcd}$$

where $(1_A\ 0)$ denote the projection arrows. The top rectangle is commutative by (TR3) and the commutativity of the bottom rectangle is obvious. Thus, the diagram is a morphism of triangles. Moreover, the maps $(1_A\ 0) \circ \binom{1_A}{0} = 1_A$ are all isomorphisms so **Lemma 2.14.** asserts that also $(1_Z\ 0) \circ \alpha$ is an isomorphism which in turn means that T is isomorphic to the distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{i} W \xrightarrow{j} \Sigma X$$

making it a distinguished triangle as well, by (TR0).

The argument for the distinction of T' is nearly identical.

(2) (i) By definition of $\mathsf{K}(\mathbb{Z})$, there exists a distinguished triangle

$$\mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{i} C(\cdot 2) \xrightarrow{c_{\cdot 2}} \Sigma(\mathbb{Z}/2),$$

where $i : \mathbb{Z}/4 \hookrightarrow C(\cdot 2)$ is the canonical inclusion for $C(\cdot 2)$ is the complex $(\mathbb{Z}/2 \xrightarrow{-(\cdot 2)} \mathbb{Z}/4)$ concentrated in degrees 1 and 0.
Assume for contradiction that there also exists a map $h : \mathbb{Z}/2 \to \Sigma(\mathbb{Z}/2)$ such that

$$\mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\bmod 2} \mathbb{Z}/2 \xrightarrow{h} \Sigma(\mathbb{Z}/2)$$

is a distinguished triangle. By (TR3), there exists a map $\alpha : C(\cdot 2) \to \mathbb{Z}/2$ making the diagram

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r, "\cdot 2"] \arrow[d, "1"] & \mathbb{Z}/4 \arrow[r, "i"] \arrow[d, "1"] & C(\cdot 2) \arrow[r, "c_{\cdot 2}"] \arrow[d, dashed, "\exists\alpha"] & \Sigma(\mathbb{Z}/2) \arrow[d, "1"] \\
\mathbb{Z}/2 \arrow[r, "\cdot 2"] & \mathbb{Z}/4 \arrow[r, "\bmod 2"] & \mathbb{Z}/2 \arrow[r, "h"] & \Sigma(\mathbb{Z}/2)
\end{tikzcd}$$

commute. By commutativity of the central square, we get that the map $\alpha \circ i - \bmod 2$ is null-homotopic, which in the case of complexes concentrated in degree 0 simply means that $\alpha \circ i = \bmod 2$. As i is just $\mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/4$ in degree 0, it follows that $\alpha = \bmod 2$. By definition of $C(\cdot 2)$, the map $c_{\cdot 2} : C(\cdot 2) \to \Sigma(\mathbb{Z}/2)$ is the map $\mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2$ in degree 1. The commutativity of the rightmost square implies that $c_{\cdot 2} = h \circ \alpha = h \circ \bmod 2$. Nonetheless, this is a contradiction as $h \circ \bmod 2 = 0$ because

$$\mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\bmod 2} \mathbb{Z}/2 \xrightarrow{h} \Sigma(\mathbb{Z}/2)$$

is a triangle. It follows that there indeed doesn't exist a distinguished triangle of the above form.

(ii)