Chapter II

Quivers and algebras

In this chapter, we show that to each finite dimensional algebra over an algebraically closed field K corresponds a graphical structure, called a quiver, and that, conversely, to each quiver corresponds an associative K-algebra, which has an identity and is finite dimensional under some conditions. Similarly, as will be seen in the next chapter, using the quiver associated to an algebra A, it will be possible to visualise a (finitely generated) A-module as a family of (finite dimensional) K-vector spaces connected by linear maps (see Examples (I.2.4)–(I.2.6)). The idea of such a graphical representation seems to go back to the late forties (see Gabriel [70], Grothendieck [82], and Thrall [167]) but it became widespread in the early seventies, mainly due to Gabriel [72], [73]. In an explicit form, the notions of quiver and linear representation of quiver were introduced by Gabriel in [72]. It was the starting point of the modern representation theory of associative algebras.

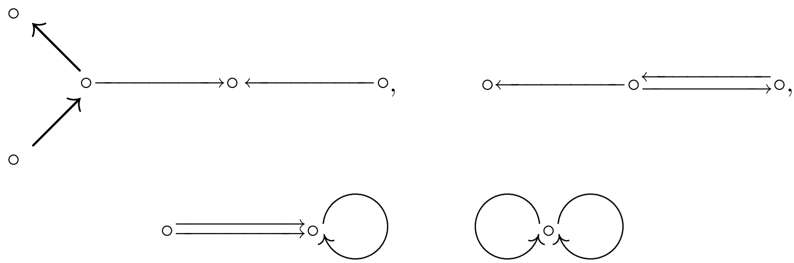
II.1. Quivers and path algebras

This first section is devoted to defining the graphical structures we are interested in and introducing the related terminology. We shall then be able to show how one can associate an algebra to each such graphical structure and study its properties.

1.1. Definition. A **quiver** $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0 (whose elements are called **points**, or **vertices**) and Q_1 (whose elements are called **arrows**), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its **source** $s(\alpha) \in Q_0$ and its **target** $t(\alpha) \in Q_0$, respectively.

An arrow $\alpha \in Q_1$ of source $a = s(\alpha)$ and target $b = t(\alpha)$ is usually denoted by $\alpha : a \to b$. A quiver $Q = (Q_0, Q_1, s, t)$ is usually denoted briefly by $Q = (Q_0, Q_1)$ or even simply by Q.

Thus, a quiver is nothing but an oriented graph without any restriction as to the number of arrows between two points, to the existence of loops or oriented cycles. There are two main reasons for using the term quiver rather than graph: the first one is that the former has become generally accepted by specialists; the second is that the latter is used in so many different contexts and even senses (a graph can be oriented or not, with or without multiple arrows or loops) that it may lead, for our purposes at least, to certain ambiguities. When drawing a quiver, we agree to represent each point by an open dot, and each arrow will be pointing towards its target. With these conventions, the following are examples of quivers:



A **subquiver** of a quiver $Q = (Q_0, Q_1, s, t)$ is a quiver $Q' = (Q'_0, Q'_1, s', t')$ such that $Q'_0 \subseteq Q_0, Q'_1 \subseteq Q_1$ and the restrictions $s\mid_{Q'_1}, t\mid_{Q'_1}$ of s, t to Q'_1 are respectively equal to s', t' (that is, if $\alpha : a \to b$ is an arrow in Q_1 such that $\alpha \in Q'_1$ and $a, b \in Q'_0$, then $s'(\alpha) = a$ and $t'(\alpha) = b$). Such a subquiver is called **full** if Q'_1 equals the set of all those arrows in Q_1 whose source and target both belong to Q'_0, that is,

$$Q'_1 = \{\alpha \in Q_1 \mid s(\alpha) \in Q'_0 \quad \text{and} \quad t(\alpha) \in Q'_0\}.$$

In particular, a full subquiver is uniquely determined by its set of points.

A quiver Q is said to be **finite** if Q_0 and Q_1 are finite sets. The **underlying graph** $\overline{Q}$ of a quiver Q is obtained from Q by forgetting the orientation of the arrows. The quiver Q is said to be **connected** if $\overline{Q}$ is a connected graph.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $a, b \in Q_0$. A **path** of **length** $\ell \geq 1$ with source a and target b (or, more briefly, from a to b) is a sequence

$$(a \mid \alpha_1, \alpha_2, \ldots, \alpha_\ell \mid b),$$

where $\alpha_k \in Q_1$ for all $1 \leq k \leq \ell$, and we have $s(\alpha_1) = a$, $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \leq k < \ell$, and finally $t(\alpha_\ell) = b$. Such a path is denoted briefly by $\alpha_1\alpha_2 \ldots \alpha_\ell$ and may be visualised as follows

$$a = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \longrightarrow \cdots \xrightarrow{\alpha_\ell} a_\ell = b.$$

We denote by Q_ℓ the set of all paths in Q of length ℓ. We also agree to associate with each point $a \in Q_0$ a path of length $\ell = 0$, called the **trivial** or **stationary path** at a, and denoted by

$$\varepsilon_a = (a \parallel a).$$

Thus the paths of lengths 0 and 1 are in bijective correspondence with the elements of Q_0 and Q_1, respectively. A path of length $\ell \geq 1$ is called a **cycle** whenever its source and target coincide. A cycle of length 1 is called a **loop**. A quiver is called **acyclic** if it contains no cycles.

We also need a notion of unoriented path, or a walk. To each arrow $\alpha : a \to b$ in a quiver Q, we associate a formal reverse $\alpha^{-1} : b \to a$, with the source $s(\alpha^{-1}) = b$ and the target $t(\alpha^{-1}) = a$. A **walk** of length $\ell \geq 1$ from a to b in Q is, by definition, a sequence $w = \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\ldots\alpha_\ell^{\varepsilon_\ell}$ with $\varepsilon_j \in \{-1, 1\}$, $s(\alpha_1^{\varepsilon_1}) = a$, $t(\alpha_\ell^{\varepsilon_\ell}) = b$ and $t(\alpha_j^{\varepsilon_j}) = s(\alpha_{j+1}^{\varepsilon_{j+1}})$, for all j such that $1 \leq j \leq \ell$.

If there exists in Q a path from a to b, then a is said to be a **predecessor** of b, and b is said to be a **successor** of a. In particular, if there exists an arrow $a \to b$, then a is said to be a **direct** (or **immediate**) **predecessor** of b, and b is said to be a **direct** (or **immediate**) **successor** of a. For $a \in Q_0$, we denote by a^- (or by a^+) the set of all direct predecessors (or successors, respectively) of a. The elements of $a^+ \cup a^-$ are called the **neighbours** of a.

Clearly, the composition of paths is a partially defined operation on the set of all paths in a quiver. We use it to define an algebra.

1.2. Definition. Let Q be a quiver. The **path algebra** KQ of Q is the K-algebra whose underlying K-vector space has as its basis the set of all paths $(a \mid \alpha_1, \ldots, \alpha_\ell \mid b)$ of length $\ell \geq 0$ in Q and such that the product of two basis vectors $(a \mid \alpha_1, \ldots, \alpha_\ell \mid b)$ and $(c \mid \beta_1, \ldots, \beta_k \mid d)$ of KQ is defined by

$$(a \mid \alpha_1, \ldots, \alpha_\ell \mid b)(c \mid \beta_1, \ldots, \beta_k \mid d) = \delta_{bc}(a \mid \alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_k \mid d),$$

where δ_{bc} denotes the Kronecker delta. In other words, the product of two paths $\alpha_1 \ldots \alpha_\ell$ and $\beta_1 \ldots \beta_k$ is equal to zero if $t(\alpha_\ell) \neq s(\beta_1)$ and is equal to the composed path $\alpha_1 \ldots \alpha_\ell \beta_1 \ldots \beta_k$ if $t(\alpha_\ell) = s(\beta_1)$. The product of basis elements is then extended to arbitrary elements of KQ by distributivity.

In other words, there is a direct sum decomposition

$$KQ \ = \ KQ_0 \oplus KQ_1 \oplus KQ_2 \oplus \ldots \oplus KQ_\ell \oplus \ldots$$

of the K-vector space KQ, where, for each $\ell \geq 0$, KQ_ℓ is the subspace of KQ generated by the set Q_ℓ of all paths of length ℓ. It is easy to see that $(KQ_n) \cdot (KQ_m) \subseteq KQ_{n+m}$ for all $n, m \geq 0$, because the product in KQ of a path of length n by a path of length m is either zero or a path of length $n+m$. This is expressed sometimes by saying that the decomposition defines a **grading** on KQ or that KQ is a **graded K-algebra**.

1.3. Examples. (a) Let Q be the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[loop right, "\alpha"] \end{tikzcd}$$

consisting of a single point and a single loop. The defining basis of the path algebra KQ is $\{\varepsilon_1, \alpha, \alpha^2, \ldots, \alpha^\ell, \ldots\}$ and the multiplication of basis vectors is given by

$$\begin{array}{lll} \varepsilon_1 \alpha^\ell = \alpha^\ell \varepsilon_1 = \alpha^\ell & \quad \text{for all} & \ell \geq 0, \text{ and} \\ \alpha^\ell \alpha^k = \alpha^{\ell+k} & \quad \text{for all} & \ell, k \geq 0, \end{array}$$

where $\alpha^0 = \varepsilon_1$. Thus KQ is isomorphic to the polynomial algebra $K[t]$ in one indeterminate t, the isomorphism being induced by the K-linear map such that

$$\varepsilon_1 \mapsto 1 \qquad \text{and} \quad \alpha \mapsto t.$$

(b) Let Q be the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[loop left, "\alpha"'] \arrow[loop right, "\beta"] \end{tikzcd}$$

consisting of a single point and two loops α and β. The defining basis of KQ is the set of all words on $\{\alpha, \beta\}$, with the empty word equal to ε_1: this is the identity of the path algebra KQ. Also, the multiplication of basis vectors reduces to the multiplication in the free monoid over $\{\alpha, \beta\}$. Thus KQ is isomorphic to the free associative algebra in two noncommuting indeterminates $K\langle t_1, t_2\rangle$, the isomorphism being the K-linear map such that

$$\varepsilon_1 \mapsto 1, \quad \alpha \mapsto t_1, \quad \text{and} \quad \beta \mapsto t_2.$$

More generally, let $Q = (Q_0, Q_1)$ be a quiver such that Q_0 has only one element, then each $\beta \in Q_1$ is a loop and we have similarly that KQ is isomorphic to the free associative algebra in the indeterminates $(X_\beta)_{\beta \in Q_1}$.

(c) Let Q be the quiver

$$\underset{1}{\circ} \xleftarrow{\quad\alpha\quad} \underset{2}{\circ}.$$

The path algebra KQ has as its defining basis the set $\{\varepsilon_1, \varepsilon_2, \alpha\}$ with the multiplication table

	ε_1	ε_2	α
ε_1	ε_1	0	0
ε_2	0	ε_2	α
α	α	0	0

Clearly, KQ is isomorphic to the 2×2 lower triangular matrix algebra

$$\mathbb{T}_2(K) = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix} = \left\{ \left[\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right] \mid a, b, c \in K \right\}$$

where the isomorphism is induced by the K-linear map such that

$$\varepsilon_1 \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \quad \varepsilon_2 \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \quad \alpha \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right].$$

(d) Let Q be the quiver

$$\begin{tikzcd}[row sep=small]
 & \circ 2 \arrow[dl, "\alpha"'] \\
1 \circ & \circ 3 \arrow[l, "\beta"'] \\
 & \circ 4 \arrow[ul, "\gamma"]
\end{tikzcd}$$

One can easily show, as above, that there is a K-algebra isomorphism

$$KQ \cong \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & 0 & 0 & K \end{bmatrix}.$$

1.4. Lemma. *Let Q be a quiver and KQ be its path algebra. Then*
(a) *KQ is an associative algebra,*
(b) *KQ has an identity element if and only if Q_0 is finite, and*
(c) *KQ is finite dimensional if and only if Q is finite and acyclic.*

Proof. (a) This follows directly from the definition of multiplication because the product of basis vectors is the composition of paths, which is associative.

(b) Clearly, each stationary path $\varepsilon_a = (a \parallel a)$ is an idempotent of KQ. Thus, if Q_0 is finite, $\sum\limits_{a \in Q_0} \varepsilon_a$ is an identity for KQ. Conversely, suppose that Q_0 is infinite, and suppose to the contrary that $1 = \sum\limits_{i=1}^{m} \lambda_i w_i$ is an identity element of KQ (where the λ_i are nonzero scalars and the w_i are paths in Q). The set Q'_0 of the sources of the w_i has at most m elements and in particular is finite. Let thus $a \in Q_0 \backslash Q'_0$, then $\varepsilon_a \cdot 1 = 0$, a contradiction.

(c) If Q is infinite, then so is the basis of KQ, which is therefore infinite dimensional. If $w = \alpha_1 \alpha_2 \ldots \alpha_\ell$ is a cycle in Q then, for each $t \geq 0$, we have a basis vector $w^t = (\alpha_1\alpha_2 \dots \alpha_\ell)^t$, so that KQ is again infinite dimensional. Conversely, if Q is finite and acyclic, it contains only finitely many paths and so KQ is finite dimensional. □

1.5. Corollary. *Let Q be a finite quiver. The element $1 = \sum_{a \in Q_0} \varepsilon_a$ is the identity of KQ and the set $\{\varepsilon_a \mid a \in Q_0\}$ of all the stationary paths $\varepsilon_a = (a \| a)$ is a complete set of primitive orthogonal idempotents for KQ.*

Proof. It follows from the definition of multiplication that the ε_a are orthogonal idempotents for KQ. Because the set Q_0 is finite, the element $1 = \sum_{a \in Q_0} \varepsilon_a$ is the identity of KQ. There remains to show that the ε_a are primitive or, what amounts to the same, that the only idempotents of the algebra $\varepsilon_a(KQ)\varepsilon_a$ are 0 and ε_a; see (I.4.7). Indeed, any idempotent ε of $\varepsilon_a(KQ)\varepsilon_a$ can be written in the form $\varepsilon = \lambda\varepsilon_a + w$, where $\lambda \in K$ and w is a linear combination of cycles through a of length ≥ 1. The equality

$$0 = \varepsilon^2 - \varepsilon = (\lambda^2 - \lambda)\varepsilon_a + (2\lambda - 1)w + w^2$$

gives $w = 0$ and $\lambda^2 = \lambda$, thus $\lambda = 0$ or $\lambda = 1$. In the former case, $\varepsilon = 0$ and in the latter $\varepsilon = \varepsilon_a$. □

Clearly, the set $\{\varepsilon_a \mid a \in Q_0\}$ is usually not the unique complete set of primitive orthogonal idempotents for KQ. For instance, in Example 1.3 (c), besides the set $\{\varepsilon_1, \varepsilon_2\}$, the set $\{\varepsilon_1 + \alpha, \varepsilon_2 - \alpha\}$ is also a complete set of primitive orthogonal idempotents for KQ.

The following lemma reduces the connectedness of an algebra to a partition of a complete set of primitive orthogonal idempotents for this algebra. It will allow us to characterise connected path algebras, then, in Section 2, connected quotients of path algebras.

1.6. Lemma. *Let A be an associative algebra with an identity and assume that $\{e_1, \dots, e_n\}$ is a (finite) complete set of primitive orthogonal idempotents. Then A is a connected algebra if and only if there does not exist a nontrivial partition $I \dot\cup J$ of the set $\{1, 2, \dots, n\}$ such that $i \in I$ and $j \in J$ imply $e_iAe_j = 0 = e_jAe_i$.*

Proof. Assume that there exists such a partition and let $c = \sum_{j \in J} e_j$. Because the partition is nontrivial, $c \neq 0, 1$. Because the e_j are orthogonal idempotents, c is an idempotent. Moreover, $ce_i = e_ic = 0$ for each $i \in I$, and $ce_j = e_jc = e_j$ for each $j \in J$. Let now $a \in A$ be arbitrary. By hypothesis, $e_i a e_j = 0 = e_j a e_i$ whenever $i \in I$ and $j \in J$. Consequently

$$\begin{array}{rclcl} ca & = & (\sum_{j\in J} e_j)a & = & (\sum_{j\in J} e_j a)\cdot 1 = (\sum_{j\in J} e_j a)(\sum_{i\in I} e_i + \sum_{k\in J} e_k) \\ & = & \sum_{j,k\in J} e_j a e_k & = & (\sum_{j\in J} e_j + \sum_{i\in I} e_i)a(\sum_{k\in J} e_k) = ac. \end{array}$$

Thus c is a central idempotent, and $A = cA \times (1-c)A$ is a nontrivial product decomposition of A. Conversely, if A is not connected, it contains a central idempotent $c \neq 0, 1$. We have

$$c = 1 \cdot c \cdot 1 = (\sum_{i=1}^{n} e_i)c(\sum_{j=1}^{n} e_j) = \sum_{i,j=1}^{n} e_i c e_j = \sum_{i=1}^{n} e_i c e_i,$$

because c is central. Let $c_i = e_i c e_i \in e_i A e_i$. Then $c_i^2 = (e_i c e_i)(e_i c e_i) = e_i c^2 e_i = c_i$, so that c_i is an idempotent of $e_i A e_i$. Because e_i is primitive, $c_i = 0$ or $c_i = e_i$. Let $I = \{i \mid c_i = 0\}$ and $J = \{j \mid c_j = e_j\}$. Because $c \neq 0, 1$, this is indeed a nontrivial partition of $\{1, 2, \ldots, n\}$. Moreover, if $i \in I$, we have $e_i c = c e_i = 0$ and, if $j \in J$, we have $e_j c = c e_j = e_j$. Therefore, if $i \in I$ and $j \in J$, we have $e_i A e_j = e_i A c e_j = e_i c A e_j = 0$ and similarly $e_j A e_i = 0$. $\square$

1.7. Lemma. *Let Q be a finite quiver. The path algebra KQ is connected if and only if Q is a connected quiver.*

Proof. Assume that Q is not connected and let Q' be a connected component of Q. Denote by Q'' the full subquiver of Q having as set of points $Q''_0 = Q_0 \backslash Q'_0$. By hypothesis, neither Q' nor Q'' is empty. Let $a \in Q'_0$ and $b \in Q''_0$. Because Q is not connected, an arbitrary path w in Q is entirely contained in either Q' or (a connected component of) Q''. In the former case, we have $w\varepsilon_b = 0$ and hence $\varepsilon_a w \varepsilon_b = 0$. In the latter case, we have $\varepsilon_a w = 0$ and hence again $\varepsilon_a w \varepsilon_b = 0$. This shows that $\varepsilon_a(KQ)\varepsilon_b = 0$. Similarly, $\varepsilon_b(KQ)\varepsilon_a = 0$. By (1.6), KQ is not connected.

Suppose now that Q is connected but KQ is not. By (1.6), there exists a disjoint union partition $Q_0 = Q'_0 \dot{\cup} Q''_0$ such that, if $x \in Q'_0$ and $y \in Q''_0$, then $\varepsilon_x(KQ)\varepsilon_y = 0 = \varepsilon_y(KQ)\varepsilon_x$. Because Q is connected, there exist $a \in Q'_0$ and $b \in Q''_0$ that are neighbours. Without loss of generality, we may suppose that there exists an arrow $\alpha : a \to b$. But then we have

$$\alpha = \varepsilon_a \alpha \varepsilon_b \in \varepsilon_a(KQ)\varepsilon_b = 0,$$

a contradiction that completes the proof of the lemma. $\square$

To summarise, we have shown that if Q is a finite connected quiver, the path algebra KQ of Q is a connected associative K-algebra with an identity, which admits $\{\varepsilon_a = (a \parallel a) \mid a \in Q_0\}$ as a complete set of primitive orthogonal idempotents. We shall now characterise it by a universal property.

1.8. Theorem. *Let Q be a finite connected quiver and A be an associative K-algebra with an identity. For any pair of maps $\varphi_0 : Q_0 \to A$ and $\varphi_1 : Q_1 \to A$ satisfying the folowing conditions:*

(i) $1 = \sum_{a \in Q_0} \varphi_0(a)$, $\varphi_0(a)^2 = \varphi_0(a)$, *and* $\varphi_0(a)\cdot\varphi_0(b) = 0$, *for all* $a \neq b$,

(ii) *if* $\alpha : a \to b$ *then* $\varphi_1(\alpha) = \varphi_0(a)\varphi_1(\alpha)\varphi_0(b)$,

there exists a unique K-algebra homomorphism $\varphi : KQ \to A$ such that $\varphi(\varepsilon_a) = \varphi_0(a)$ for any $a \in Q_0$ and $\varphi(\alpha) = \varphi_1(\alpha)$ for any $\alpha \in Q_1$.

Proof. Indeed, assume there exists a homomorphism $\varphi : KQ \to A$ of K-algebras extending φ_0 and φ_1, and let $\alpha_1\alpha_2 \ldots \alpha_\ell$ be a path in Q. Because φ is a K-algebra homomorphism, we have

$$\begin{aligned} \varphi(\alpha_1\alpha_2 \ldots \alpha_\ell) &= \varphi(\alpha_1)\varphi(\alpha_2) \ldots \varphi(\alpha_\ell) \\ &= \varphi_1(\alpha_1)\varphi_1(\alpha_2) \ldots \varphi_1(\alpha_\ell). \end{aligned}$$

This shows uniqueness. On the other hand, this formula clearly defines a K-linear mapping from KQ to A that is compatible with the composition of paths (thus preserves the product) and is such that

$$\varphi(1) = \varphi(\sum_{a \in Q_0} \varepsilon_a) = \sum_{a \in Q_0} \varphi(\varepsilon_a) = \sum_{a \in Q_0} \varphi_0(a) = 1,$$

that is, it preserves the identity. It is therefore a K-algebra homomorphism. $\square$

We now calculate the radical of the path algebra of a finite, connected, and acyclic quiver. We need the following definition.

1.9. Definition. Let Q be a finite and connected quiver. The two-sided ideal of the path algebra KQ generated (as an ideal) by the arrows of Q is called the **arrow ideal** of KQ and is denoted by R_Q. Whenever this can be done without ambiguity we shall use the notation R instead of R_Q.

Note that there is a direct sum decomposition

$$R_Q \; = \; KQ_1 \oplus KQ_2 \oplus \ldots \oplus KQ_\ell \oplus \ldots$$

of the K-vector space R_Q, where KQ_ℓ is the subspace of KQ generated by the set Q_ℓ of all paths of length ℓ. In particular, the underlying K-vector space of R_Q is generated by all paths in Q of length $\ell \geq 1$. This implies that, for each $\ell \geq 1$,

$$R_Q^\ell = \bigoplus_{m \geq \ell} KQ_m$$

and therefore R_Q^ℓ is the ideal of KQ generated, as a K-vector space, by the set of all paths of length $\geq \ell$. Consequently, the K-vector space $R_Q^\ell / R_Q^{\ell+1}$ is generated by the residual classes of all paths in Q of length (exactly) equal to ℓ and there is an isomorphism of K-vector spaces $R_Q^\ell / R_Q^{\ell+1} \cong KQ_\ell$.

1.10. Proposition. *Let Q be a finite connected quiver, R be the arrow ideal of KQ and $\varepsilon_a = (a \parallel a)$ for $a \in Q_0$. The set $\{\overline{\varepsilon}_a = \varepsilon_a + R \mid a \in Q_0\}$ is a complete set of primitive orthogonal idempotents for KQ/R, and the latter is isomorphic to a product of copies of K. If, in addition, Q is acyclic, then* $\operatorname{rad} KQ = R$ *and KQ is a finite dimensional basic algebra.*

Proof. Clearly, there is a direct sum decomposition

$$KQ/R = \bigoplus_{a,b \in Q_0} \overline{\varepsilon}_a (KQ/R) \overline{\varepsilon}_b$$

as a K-vector space. Because R contains all paths of length ≥ 1, this becomes

$$KQ/R = \bigoplus_{a \in Q_0} \overline{\varepsilon}_a (KQ/R) \overline{\varepsilon}_a.$$

Then KQ/R is generated, as a K-vector space, by the residual classes of the paths of length zero, that is, by the set $\{\overline{\varepsilon}_a = \varepsilon_a + R \mid a \in Q_0\}$. Clearly, this set is a complete set of primitive orthogonal idempotents of the quotient algebra KQ/R. Moreover, for each $a \in Q_0$, the algebra $\overline{\varepsilon}_a (KQ/R) \overline{\varepsilon}_a$ is generated, as a K-vector space, by $\overline{\varepsilon_a}$ and consequently is isomorphic, as a K-algebra, to K. This shows that the quotient algebra KQ/R is isomorphic to a product of $|Q_0|$ copies of K.

Assume now that Q is acyclic (so that, by (1.4), KQ is a finite dimensional algebra). There exists a largest $\ell \geq 1$ such that Q contains a path of length ℓ. But this implies that any product of $\ell + 1$ arrows is zero, that is, $R^{\ell+1} = 0$. Consequently, the ideal R is nilpotent and hence, by (I.1.4), $R \subseteq \operatorname{rad} KQ$. Because KQ/R is isomorphic to a product of copies of K, it follows from (I.1.4) and (I.6.2) that $\operatorname{rad} KQ = R$ and the algebra KQ is basic. $\square$

We remark that if Q is not acyclic, it is generally not true that $\operatorname{rad} KQ = R_Q$. For instance, let Q be the quiver

$$\begin{tikzcd} 1 \circ \arrow[loop right, "\alpha"] \end{tikzcd}$$

As we have seen before, $KQ \cong K[t]$. Thus $\operatorname{rad} KQ = 0$, because the field K is algebraically closed (and hence infinite); then the set $\{t - \lambda \mid \lambda \in K\}$ is an infinite set of irreducible polynomials, which generates an infinite set of maximal ideals with zero intersection. On the other hand, $R_Q = \bigoplus_{\ell>0} K\alpha^\ell$ as a K-vector space and thus is certainly nonzero.

We summarise our findings in the following corollary.

1.11. Corollary. *Let Q be a finite, connected, and acyclic quiver. The path algebra KQ is a basic and connected associative finite dimensional K-algebra with an identity, having the arrow ideal as radical, and the set $\{\varepsilon_a = (a \parallel a) \mid a \in Q_0\}$ as a complete set of primitive orthogonal idempotents.*

Proof. The statement collects results from (1.4), (1.5), (1.7), and (1.10). $\square$

We now give a construction showing that an algebra as in (1.11) can always be realised as an algebra of lower triangular matrices. We start by recalling a classical construction for generalised matrix algebras. Let $(A_i)_{1\leq i\leq n}$ be a family of K-algebras and $(M_{ij})_{1\leq i,j\leq n}$ be a family of A_i-A_j-bimodules such that $M_{ii} = A_i$, for each i. Moreover, assume that we have for each triple (i, j, k) an A_i-A_k-bimodule homomorphism

$$\varphi_{ik}^{j} : M_{ij} \otimes M_{jk} \to M_{ik}$$

satisfying, for each quadruple (i, j, k, ℓ), the "associativity" condition

$$\varphi_{i\ell}^{k}\left(\varphi_{ik}^{j} \otimes 1\right) = \varphi_{i\ell}^{j}(1 \otimes \varphi_{j\ell}^{k}),$$

that is, the following square is commutative:

$$\begin{tikzcd}
M_{ij} \otimes M_{jk} \otimes M_{kl} \arrow[r, "1\otimes\varphi_{jl}^{k}"] \arrow[d, "\varphi_{ik}^{j}\otimes 1"] & M_{ij} \otimes M_{jl} \arrow[d, "\varphi_{il}^{j}"] \\
M_{ik} \otimes M_{kl} \arrow[r, "\varphi_{il}^{k}"] & M_{il}
\end{tikzcd}$$

Then it is easily verified that the K-vector space of $n \times n$ matrices

$$A = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{bmatrix} = \left\{[x_{ij}] \mid x_{ij} \in M_{ij} \quad \text{for all} \quad 1 \leq i, j \leq n\right\}$$

becomes a K-algebra if we define its multiplication by the formula

$$[x_{ij}] \cdot [y_{ij}] = \left[\sum_{k=1}^{n} \varphi_{ij}^{k}(x_{ik} \otimes y_{kj})\right].$$

Assume that Q is a finite and acyclic quiver. Let $n = |Q_0|$ be the number of points in Q. It is easy to see that we may number the points of Q from 1 to n such that, if there exists a path from i to j, then $j \leq i$.

1.12. Lemma. *Let Q be a connected, finite, and acyclic quiver with $Q_0 = \{1, 2, \ldots, n\}$ such that, for each $i, j \in Q_0$, $j \leq i$ whenever there exists a path from i to j in Q. Then the path algebra KQ is isomorphic to the triangular matrix algebra*

$$A = \begin{bmatrix} \varepsilon_1(KQ)\varepsilon_1 & 0 & \ldots & 0 \\ \varepsilon_2(KQ)\varepsilon_1 & \varepsilon_2(KQ)\varepsilon_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \varepsilon_n(KQ)\varepsilon_1 & \varepsilon_n(KQ)\varepsilon_2 & \ldots & \varepsilon_n(KQ)\varepsilon_n \end{bmatrix},$$

where $\varepsilon_a = (a \| a)$ for any $a \in Q_0$, the addition is the obvious one, and the multiplication is induced from the multiplication of KQ.

Proof. Because $\{\varepsilon_a = (a \| a) \mid a \in Q_0\}$ is a complete set of primitive orthogonal idempotents for KQ (by (1.11)), we have a K-vector space decomposition of KQ

$$KQ = \bigoplus_{a,b \in Q_0} \varepsilon_a(KQ)\varepsilon_b.$$

It follows from the hypothesis that if $\varepsilon_i(KQ)\varepsilon_j \neq 0$, then $j \leq i$. For each point $i \in Q_0$, the absence of cycles through i implies that the algebra $\varepsilon_i(KQ)\varepsilon_i$ is isomorphic to K. The definition of the multiplication in KQ implies that, for each pair (j, i) such that $j \leq i$, $\varepsilon_i(KQ)\varepsilon_j$ is an $\varepsilon_i(KQ)\varepsilon_i$-$\varepsilon_j(KQ)\varepsilon_j$-bimodule and, for each triple (k, j, i) such that $k \leq j \leq i$, there exists a K-linear map

$$\varphi_{ik}^j : \varepsilon_i(KQ)\varepsilon_j \otimes \varepsilon_j(KQ)\varepsilon_k \to \varepsilon_i(KQ)\varepsilon_k,$$

where the tensor product is taken over $\varepsilon_j(KQ)\varepsilon_j$. It is easily seen that the φ_{ik}^j are actually $\varepsilon_i(KQ)\varepsilon_i$–$\varepsilon_k(KQ)\varepsilon_k$-bimodule homomorphisms satisfying the "associativity" conditions $\varphi_{i\ell}^k(\varphi_{ik}^j \otimes 1) = \varphi_{i\ell}^j(1 \otimes \varphi_{j\ell}^k)$ whenever $i \leq j \leq k \leq \ell$. We may thus construct a generalised matrix algebra as done earlier. Now, by associating to each path from i to j in KQ the corresponding element of A (that is, basis element of the bimodule $\varepsilon_i(KQ)\varepsilon_j$), we get a K-algebra isomorphism $KQ \cong A$. Indeed, the algebras A and KQ are clearly isomorphic as K-vector spaces and the bijection between the basis vectors is compatible with the algebra multiplications (by definition of the φ_{ik}^j), thus this vector space isomorphism is a K-algebra isomorphism. □

In particular, if Q has no multiple arrows and its underlying graph is a tree, then there is at most one path between two given points of Q so that, for all $j \leq i$, we have $\dim_K(\varepsilon_i(KQ)\varepsilon_j) \leq 1$. Consequently, KQ is isomorphic to a subalgebra of the full lower triangular matrix algebra

$$\mathbb{T}_n(K) = \begin{bmatrix} K & 0 & \dots & 0 \\ K & K & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K & K & \dots & K \end{bmatrix}.$$

1.13. Examples. (a) Let Q be the quiver

$$\underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ} \longleftarrow \cdots \longleftarrow \underset{n-1}{\circ} \longleftarrow \underset{n}{\circ}$$

This construction gives the algebra isomorphism $KQ \cong \mathbb{T}_n(K)$.

(b) Let Q be the Kronecker quiver

$$1 \ \circ \ \leftleftarrows \ \circ \ 2$$

Then there is an algebra isomorphism

$$KQ \cong \begin{bmatrix} K & 0 \\ K^2 & K \end{bmatrix},$$

where K^2 is considered as a K-K-bimodule in the obvious way

$$a \cdot (x, y) \;=\; (ax, ay), \qquad (x, y) \cdot b \;=\; (xb, yb)$$

for all $a, b, x, y \in K$. The path algebra of the Kronecker quiver is called the **Kronecker algebra**. Its module category is studied in detail later (see also (I.2.5)).

We remark that the expression of KQ as an algebra of lower triangular matrices (1.12) is not unique. For instance, the Kronecker algebra is isomorphic to the subalgebra

$$A = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & 0 & d \end{bmatrix} \;\middle|\; a, b, c, d \in K \right\}$$

of $\mathbb{T}_3(K)$. An algebra isomorphism between A and the Kronecker algebra is given by

$$\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & 0 & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ (b, c) & d \end{bmatrix}.$$

(c) Let Q be the quiver $\underset{1}{\circ} \longleftarrow \underset{2}{\circ} \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \underset{4}{\circ} \longrightarrow \underset{3}{\circ}$. Then

$$KQ \cong \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ 0 & 0 & K & 0 \\ K^3 & K^3 & K & K \end{bmatrix},$$

where the multiplication is defined in a manner analogous to the one used in example (b).

II.2. Admissible ideals and quotients of the path algebra

Let Q be a finite quiver. By (1.4), the path algebra KQ of Q is an associative algebra with an identity and is finite dimensional if and only if Q is acyclic. Our objective in this section is to study the finite dimensional quotients of a not necessarily finite dimensional path algebra. We see in particular that they correspond to certain ideals we call admissible.

2.1. Definition. Let Q be a finite quiver and R_Q be the arrow ideal of the path algebra KQ. A two-sided ideal $\mathcal{I}$ of KQ is said to be **admissible** if there exists $m \geq 2$ such that

$$R_Q^m \subseteq \mathcal{I} \subseteq R_Q^2.$$

If $\mathcal{I}$ is an admissible ideal of KQ, the pair $(Q, \mathcal{I})$ is said to be a **bound quiver**. The quotient algebra $KQ/\mathcal{I}$ is said to be the algebra of the bound quiver $(Q, \mathcal{I})$ or, simply, a **bound quiver algebra**.

It follows directly from the definition that an ideal $\mathcal{I}$ of KQ, contained in R_Q^2, is admissible if and only if it contains all paths whose length is large enough. It can be shown that this is the case if and only if, for each cycle σ in Q, there exists $s \geq 1$ such that $\sigma^s \in \mathcal{I}$.

If, in particular, Q is acyclic, any ideal contained in R_Q^2 is admissible.

2.2. Examples. (a) For any finite quiver Q and any $m \geq 2$, the ideal R_Q^m is admissible.

(b) The zero ideal is admissible in KQ if and only if Q is acyclic. Indeed, the zero ideal is admissible if and only if there exists $m \geq 2$ such that $R_Q^m = 0$, that is, any product of m arrows in KQ is zero. This is the case if and only if Q is acyclic.

(c) Let Q be the quiver

$$\begin{tikzcd}
& \overset{2}{\circ} \arrow[dl, "\beta"'] & \\
1\circ & & \circ 4 \arrow[ul, "\alpha"'] \arrow[ll, "\lambda"] \arrow[dl, "\gamma"] \\
& \underset{3}{\circ} \arrow[ul, "\delta"] &
\end{tikzcd}$$

The ideal $\mathcal{I}_1 = \langle \alpha\beta - \gamma\delta \rangle$ of the K-algebra KQ is admissible, but $\mathcal{I}_2 = \langle \alpha\beta - \lambda \rangle$ is not; indeed, $\alpha\beta - \lambda \notin R_Q^2$.

(d) Let Q be the quiver

$$\begin{tikzcd}
 & 2 \arrow[dl, "\beta"'] & \\
1 \arrow[loop left, "\lambda"] & & 4 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
 & 3 \arrow[ul, "\delta"] &
\end{tikzcd}$$

The ideal $\mathcal{I} = \langle \alpha\beta - \gamma\delta, \beta\lambda, \lambda^3 \rangle$ is admissible. Indeed, it is clear that $\mathcal{I} \subseteq R_Q^2$. We show that $R_Q^4 \subseteq \mathcal{I}$. Every path of length ≥ 4 and source 1, 2, or 3 contains the product λ^3 and hence lies in $\mathcal{I}$. The paths of length ≥ 4 and source 4 contain a path of the form $\alpha\beta\lambda^2$ or $\gamma\delta\lambda^2$ and hence lie in $\mathcal{I}$, in the first case, because $\beta\lambda \in \mathcal{I}$, and in the second, because $\gamma\delta\lambda^2 = (\gamma\delta - \alpha\beta)\lambda^2 + \alpha\beta\lambda^2 \in \mathcal{I}$. This completes the proof that $\mathcal{I} = \langle \alpha\beta - \gamma\delta, \beta\lambda, \lambda^3 \rangle$ is admissible. Another example of an admissible ideal is $\langle \lambda^5 \rangle$. On the other hand, $\langle \beta\lambda, \alpha\beta - \gamma\delta \rangle$ is not admissible.

(e) Let Q be the quiver $\underset{1}{\circ} \overset{\beta}{\underset{\gamma}{\leftleftarrows}} \underset{2}{\circ} \xleftarrow{\alpha} \underset{3}{\circ}$. Each of the ideals $\mathcal{I}_1 = \langle \alpha\beta \rangle$ and $\mathcal{I}_2 = \langle \alpha\beta - \alpha\gamma \rangle$ is clearly admissible. The bound quiver algebras $KQ/\mathcal{I}_1$ and $KQ/\mathcal{I}_2$ are isomorphic under the isomorphism $KQ/\mathcal{I}_1 \to KQ/\mathcal{I}_2$ induced by the correspondence $\varepsilon_i \mapsto \varepsilon_i$ for $i = 1, 2, 3$; $\alpha \mapsto \alpha$, $\beta \mapsto \beta - \gamma$, and $\gamma \mapsto \gamma$.

The preceding examples show that it is convenient to define an admissible ideal in terms of its generators. These are called relations.

2.3. Definition. Let Q be a quiver. A **relation** in Q with coefficients in K is a K-linear combination of paths of length at least two having the same source and target. Thus, a relation ρ is an element of KQ such that

$$\rho = \sum_{i=1}^{m} \lambda_i w_i,$$

where the λ_i are scalars (not all zero) and the w_i are paths in Q of length at least 2 such that, if $i \neq j$, then the source (or the target, respectively) of w_i coincides with that of w_j.

If $m = 1$, the preceding relation is called a **zero relation** or a **monomial relation**. If it is of the form $w_1 - w_2$ (where w_1, w_2 are two paths), it is called a **commutativity relation**.

If $(\rho_j)_{j \in J}$ is a set of relations for a quiver Q such that the ideal they generate $\langle \rho_j \mid j \in J \rangle$ is admissible, we say that the quiver Q is **bound by the relations** $(\rho_j)_{j \in J}$ or by the relations $\rho_j = 0$ for all $j \in J$.

For instance, in Example 2.2 (d), the ideal $\mathcal{I}$ is generated by one commutativity relation $\rho_1 = \alpha\beta - \gamma\delta$ and two zero relations $\rho_2 = \beta\lambda$ and $\rho_3 = \lambda^3$;

we thus say that Q is bound by the relations $\alpha\beta = \gamma\delta$, $\beta\lambda = 0$, and $\lambda^3 = 0$.

2.4. Lemma. *Let Q be a finite quiver and $\mathcal{I}$ be an admissible ideal of KQ. The set $\{e_a = \varepsilon_a + \mathcal{I} \mid a \in Q_0\}$ is a complete set of primitive orthogonal idempotents of the bound quiver algebra $KQ/\mathcal{I}$.*

Proof. Because e_a is the image of ε_a under the canonical homomorphism $KQ \to KQ/\mathcal{I}$, it follows from (1.5) that the given set is indeed a complete set of orthogonal idempotents. There remains to check that each e_a is primitive, that is, the only idempotents of $e_a(KQ/\mathcal{I})e_a$ are 0 and e_a. Indeed, any idempotent e of $e_a(KQ/\mathcal{I})e_a$ can be written in the form $e = \lambda\varepsilon_a + w + \mathcal{I}$, where $\lambda \in K$ and w is a linear combination of cycles through a of length ≥ 1. The equality $e^2 = e$ gives

$$(\lambda^2 - \lambda)\varepsilon_a + (2\lambda - 1)w + w^2 \in \mathcal{I}.$$

Let R_Q be the arrow ideal of KQ. Because $\mathcal{I} \subseteq R_Q^2$, we must have $\lambda^2 - \lambda = 0$, so that $\lambda = 0$ or $\lambda = 1$. Assume that $\lambda = 0$, then $e = w + \mathcal{I}$, where w is idempotent modulo $\mathcal{I}$. On the other hand, because $R_Q^m \subseteq \mathcal{I}$ for some $m \geq 2$, we must have $w^m \in \mathcal{I}$, that is, w is also nilpotent modulo $\mathcal{I}$. Consequently, $w \in \mathcal{I}$ and e is zero. On the other hand, if $\lambda = 1$, then $e_a - e = -w + \mathcal{I}$ is also an idempotent in $e_a(KQ/\mathcal{I})e_a$ so that w is again idempotent modulo $\mathcal{I}$. Because, as before, it is also nilpotent modulo $\mathcal{I}$, it must belong to $\mathcal{I}$. Consequently, $e_a = e$. $\square$

2.5. Lemma. *Let Q be a finite quiver and $\mathcal{I}$ be an admissible ideal of KQ. The bound quiver algebra $KQ/\mathcal{I}$ is connected if and only if Q is a connected quiver.*

Proof. If Q is not a connected quiver, KQ is not a connected algebra (by (1.7)). Hence KQ contains a central idempotent γ not equal to 0 or 1 that may, by the proof of (1.6), be chosen to be a sum of paths of length zero, that is, of points. But then $c = \gamma + \mathcal{I}$ is not equal to $\mathcal{I}$. On the other hand, $c = 1 + \mathcal{I}$ implies $1 - \gamma \in \mathcal{I}$, which is also impossible (because $\mathcal{I} \subseteq R_Q^2$). Because it is clear that c is a central idempotent of $KQ/\mathcal{I}$, we infer that the latter is not a connected algebra.

The reverse implication is shown exactly as in (1.7). Assume that Q is a connected quiver but that $KQ/\mathcal{I}$ is not a connected algebra. By (1.6) (and (2.4)), there exists a nontrivial partition $Q_0 = Q_0' \dot{\cup} Q_0''$ such that $x \in Q_0'$ and $y \in Q_0''$ imply $e_x(KQ/\mathcal{I})e_y = 0 = e_y(KQ/\mathcal{I})e_x$. Because Q is connected, there exist $a \in Q_0'$ and $b \in Q_0''$ that are neighbours. Without loss of generality, we may suppose that there exists an arrow $\alpha : a \to b$. But then $\alpha = \varepsilon_a \alpha \varepsilon_b$ implies that $\overline{\alpha} = \alpha + \mathcal{I}$ satisfies $\overline{\alpha} = e_a \overline{\alpha} e_b \in e_a(KQ/\mathcal{I})e_b = 0$.

As $\overline{\alpha} \neq \mathcal{I}$ (because $\mathcal{I} \subseteq R_Q^2$), we have reached a contradiction. □

2.6. Proposition. *Let Q be a finite quiver and $\mathcal{I}$ be an admissible ideal of KQ. The bound quiver algebra $KQ/\mathcal{I}$ is finite dimensional.*

Proof. Because $\mathcal{I}$ is admissible, there exists $m \geq 2$ such that $R^m \subseteq I$, where R is the arrow ideal R_Q of KQ. But then there exists a surjective algebra homomorphism $KQ/R^m \to KQ/\mathcal{I}$. Thus it suffices to prove that KQ/R^m is finite dimensional. Now the residual classes of the paths of length less than m form a basis of KQ/R^m as a K-vector space. Because there are only finitely many such paths, our statement follows. □

If $\mathcal{I}$ is not admissible, the algebra $KQ/\mathcal{I}$ is generally not finite dimensional or even not right noetherian, that is, it may contain a right ideal that is not finitely generated. The following classical example, due to J. Dieudonné (see [48], p. 16) shows a finitely generated (even cyclic) module that has a submodule that is not finitely generated.

2.7. Example. Let Q be the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

and $\mathcal{I} = \langle \beta\alpha, \beta^2 \rangle$. It is clear that $\mathcal{I}$ is not admissible, because $\alpha^m \notin \mathcal{I}$ for any $m \geq 1$. Let $A = KQ/\mathcal{I}$ and J be the subspace of A (considered as a K-vector space) generated by the elements of the form $\overline{\alpha}^n\overline{\beta}$, for all $n \geq 1$ (where, as usual, $\overline{\alpha} = \alpha + \mathcal{I}, \overline{\beta} = \beta + \mathcal{I}$). Then J is a right ideal of A. Indeed, it suffices to show that $J\overline{\alpha} \subseteq J$ and $J\overline{\beta} \subseteq J$, and this follows from the equalities $\overline{\alpha}^n\overline{\beta}\overline{\alpha} = 0$ and $\overline{\alpha}^n\overline{\beta}^2 = 0$ for all $n \geq 1$. In particular, J_A is a submodule of the cyclic module A_A but is not finitely generated (indeed, let m be the largest exponent of $\overline{\alpha}$ among the elements of a finite set $\mathcal{J}$ of generators of J, then $\overline{\alpha}^{m+1}\overline{\beta} \in J$ cannot be a K-linear combination of elements from $\mathcal{J}$).

2.8. Lemma. *Let Q be a finite quiver. Every admissible ideal $\mathcal{I}$ of KQ is finitely generated.*

Proof. Let R be the arrow ideal of KQ and $m \geq 2$ be an integer such that $R^m \subseteq \mathcal{I}$. We have a short exact sequence $0 \to R^m \to \mathcal{I} \to \mathcal{I}/R^m \to 0$ of KQ-modules.

It thus suffices to show that R^m and $\mathcal{I}/R^m$ are finitely generated as KQ-modules. Obviously, R^m is the KQ-module generated by the paths of length m. Because there are only finitely many such paths, R^m is finitely generated. On the other hand, $\mathcal{I}/R^m$ is an ideal of the finite dimensional algebra KQ/R^m (see(2.6)). Therefore $\mathcal{I}/R^m$ is a finite dimensional K-vector space, hence a finitely generated KQ-module. $\square$

2.9. Corollary. *Let Q be a finite quiver and $\mathcal{I}$ be an admissible ideal of KQ. There exists a finite set of relations $\{\rho_1, \ldots, \rho_m\}$ such that $\mathcal{I} = \langle \rho_1, \ldots, \rho_m \rangle$.*

Proof. By (2.8), an admissible ideal $\mathcal{I}$ of KQ always has a finite generating set $\{\sigma_1, \ldots, \sigma_t\}$. The elements σ_i of such a set are generally not relations, because the paths composing σ_i do not necessarily have the same sources and targets. On the other hand, for any i such that $1 \leq i \leq t$ and $a, b \in Q_0$, the term $\varepsilon_a \sigma_i \varepsilon_b$ is either zero or a relation. Because $\sigma_i = \sum_{a,b \in Q_0} \varepsilon_a \sigma_i \varepsilon_b$, for $i \leq t$, the nonzero elements among the set $\{\varepsilon_a \sigma_i \varepsilon_b \mid 1 \leq i \leq t; a, b \in Q_0\}$ form a finite set of relations generating $\mathcal{I}$. $\square$

2.10. Lemma. *Let Q be a finite quiver, R_Q be the arrow ideal of KQ, and $\mathcal{I}$ be an admissible ideal of KQ. Then* $\operatorname{rad}(KQ/\mathcal{I}) = R_Q/\mathcal{I}$. *Moreover, the bound quiver algebra $KQ/\mathcal{I}$ is basic.*

Proof. Because $\mathcal{I}$ is an admissible ideal of KQ, there exists $m \geq 2$ such that $R^m \subseteq \mathcal{I}$, where $R = R_Q$. Consequently, $(R/\mathcal{I})^m = 0$ and $R/\mathcal{I}$ is a nilpotent ideal of $KQ/\mathcal{I}$. On the other hand, the algebra $(KQ/\mathcal{I})/(R/\mathcal{I}) \cong KQ/R$ is isomorphic to a direct product of copies of K, by (1.10). This implies both assertions, by (I.1.4). $\square$

2.11. Corollary. *For each $\ell \geq 1$, we have* $\operatorname{rad}^\ell(KQ/\mathcal{I}) = (R_Q/\mathcal{I})^\ell$. $\square$

It follows from Lemma 2.10 and Corollary 2.11 that the K-vector space

$$\operatorname{rad}(KQ/\mathcal{I})/\operatorname{rad}^2(KQ/\mathcal{I}) = (R_Q/\mathcal{I})/(R_Q/\mathcal{I})^2 \cong R_Q/R_Q^2$$

admits as basis the set $\overline{\alpha} + \operatorname{rad}^2(KQ/\mathcal{I})$, where $\overline{\alpha} = \alpha + KQ/\mathcal{I}$ and $\alpha \in Q_1$. This remark is crucial for the understanding of Section 3.

We summarise our findings in the following corollary.

2.12. Corollary. *Let Q be a finite connected quiver, R_Q be the arrow ideal of KQ, and $\mathcal{I}$ be an admissible ideal of KQ. The bound quiver algebra $KQ/\mathcal{I}$ is a basic and connected finite dimensional algebra with an identity, having $R_Q/\mathcal{I}$ as radical and $\{e_a \mid a \in Q_0\}$ as complete set of primitive orthogonal idempotents.*

Proof. The statement collects results from (2.4), (2.5), (2.6), and (2.10). $\square$

2.13. Examples. (a) Let Q be the quiver $\underset{1}{\circ} \xleftarrow{\ \beta\ } \underset{2}{\circ} \xleftarrow{\ \alpha\ } \underset{3}{\circ}$.

We have seen in (1.13)(a) that

$$KQ \cong \mathbb{T}_3(K) = \begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & K & K \end{bmatrix}.$$

The ideal $\mathcal{I} = \langle \alpha\beta \rangle$ is admissible and actually equal to R_Q^2, that is,

$$\mathcal{I} \cong \operatorname{rad}^2 \mathbb{T}_3(K) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K & 0 & 0 \end{bmatrix}.$$

Thus $KQ/\mathcal{I}$ is isomorphic to the quotient of $\mathbb{T}_3(K)$ by the square of its radical.

(b) Let Q be the quiver

$$\begin{tikzcd} & \overset{2}{\circ} \arrow[dl, "\beta"'] & \\ 1\circ & & \circ 4 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & \underset{3}{\circ} \arrow[ul, "\delta"] & \end{tikzcd}$$

The ideal $\mathcal{I}$ of KQ generated by the commutativity relation $\alpha\beta - \gamma\delta$ is admissible. Thus $KQ/\mathcal{I}$ is a finite dimensional K-algebra, and $\{e_1, e_2, e_3, e_4, \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}, \overline{\alpha\beta}\}$ is its K-vector space basis. Using the construction in (1.12), we see that

$$KQ/\mathcal{I} \cong \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & K & K & K \end{bmatrix}$$

under the isomorphism defined by

$$e_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ e_2 \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ e_3 \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_4 \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \overline{\alpha} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ \overline{\beta} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{\gamma} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ \overline{\delta} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \overline{\alpha\beta} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(c) Let Q be the quiver

$$\begin{tikzcd} 1\circ \arrow[loop right, "\alpha"] \end{tikzcd}$$

We have seen in (1.3)(a) that $KQ \cong K[t]$ (which is infinite dimensional). For each $m \geq 2$, the ideal $\langle \alpha^m \rangle$ is admissible (and actually any admissible ideal of KQ is of this form). Thus $KQ/\mathcal{I} \cong K[t]/\langle t^m \rangle$ is m-dimensional.

(d) Let Q be the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

We have seen in (1.3)(b) that $KQ \cong K\langle t_1, t_2 \rangle$. The ideal $\mathcal{I}$ generated by $\alpha\beta - \beta\alpha, \beta^2, \alpha^2$ is admissible. Indeed, it is clear that $\mathcal{I} \subseteq R_Q^2$. On the other hand, any path of length 3 belongs to $\mathcal{I}$ (and consequently $R_Q^3 \subseteq \mathcal{I}$). Indeed, such a path either contains a term of the form α^2 or β^2 or is of one of the forms $\alpha\beta\alpha$ or $\beta\alpha\beta$; because $\alpha\beta\alpha = (\alpha\beta - \beta\alpha)\alpha + \beta\alpha^2 \in \mathcal{I}$ and $\beta\alpha\beta = (\beta\alpha - \alpha\beta)\beta + \alpha\beta^2 \in \mathcal{I}$, we are done. The bound quiver algebra $KQ/\mathcal{I}$ is four-dimensional, with basis given by $\{e_1, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}\}$. In fact, $KQ/\mathcal{I} \cong K[t_1, t_2]/\langle t_1^2, t_2^2 \rangle$, under the isomorphism defined by the formulas

$$e_1 \mapsto 1 + \langle t_1^2, t_2^2 \rangle,\ \overline{\alpha} \mapsto t_1 + \langle t_1^2, t_2^2 \rangle,\ \overline{\beta} \mapsto t_2 + \langle t_1^2, t_2^2 \rangle,\ \overline{\alpha\beta} \mapsto t_1 t_2 + \langle t_1^2, t_2^2 \rangle.$$

II.3. The quiver of a finite dimensional algebra

Let A be a finite dimensional (associative) algebra (with an identity) over an algebraically closed field K. As seen in (I.6.10), it may be assumed, from the point of view of studying the representation theory of A, that A is basic and connected. We now show that, under these hypotheses, A is isomorphic to a bound quiver algebra $KQ/\mathcal{I}$, where Q is a finite connected quiver and $\mathcal{I}$ is an admissible ideal of KQ. We start by associating, in a natural manner, a finite quiver to each basic and connected finite dimensional algebra A.

3.1. Definition. Let A be a basic and connected finite dimensional K-algebra and $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. The **(ordinary) quiver** of A, denoted by Q_A, is defined as follows:

(a) The points of Q_A are the numbers $1, 2, \ldots, n$, which are in bijective correspondence with the idempotents $e_1, e_2, \ldots, e_n$.

(b) Given two points $a, b \in (Q_A)_0$, the arrows $\alpha : a \to b$ are in bijective correspondence with the vectors in a basis of the K-vector space $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$.

Because A is finite dimensional, so is every vector space of the form $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$ (with $a, b \in (Q_A)_0$). Consequently, Q_A is finite. The term "ordinary quiver," sometimes used for Q_A, comes from the fact that other quivers are also used to study A, as will be seen later. Now, Q_A

is constructed starting from a given complete set of primitive orthogonal idempotents. We must thus show that it does not depend on the particular set we have chosen.

3.2. Lemma. *Let A be a finite dimensional, basic, and connected algebra.*

(a) *The quiver Q_A of A does not depend on the choice of a complete set of primitive orthogonal idempotents in A.*

(b) *For any pair e_a, e_b of primitive orthogonal idempotents of A the K-linear map $\psi : e_a(\operatorname{rad} A)e_b/e_a(\operatorname{rad}^2 A)e_b \longrightarrow e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$, defined by the formula $e_a x e_b + e_a(\operatorname{rad}^2 A)e_b \mapsto e_a(x + \operatorname{rad}^2 A)e_b$, is an isomorphism.*

Proof. (a) The number of points in Q_A is uniquely determined, because it equals the number of indecomposable direct summands of A_A, and the latter is unique by the unique decomposition theorem (I.4.10). On the other hand, the same theorem says that the factors of this decomposition are uniquely determined up to isomorphism, that is, if

$$A_A = \bigoplus_{a=1}^{n} e_a A = \bigoplus_{b=1}^{n} e'_b A$$

then we can renumber the factors so that $e_a A \cong e'_a A$, for each a with $1 \leq a \leq n$. We must show that this implies $\dim_K e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b = \dim_K e'_a(\operatorname{rad} A/\operatorname{rad}^2 A)e'_b$, for every pair (a, b). A routine calculation shows that the A-module homomorphism $\varphi : e_a(\operatorname{rad} A) \to e_a(\operatorname{rad} A/\operatorname{rad}^2 A)$ given by $e_a x \mapsto e_a(x + \operatorname{rad}^2 A)$ admits $e_a(\operatorname{rad}^2 A)$ as a kernel. Consequently

$$e_a(\operatorname{rad} A/\operatorname{rad}^2 A) \;\cong\; e_a(\operatorname{rad} A)/e_a(\operatorname{rad}^2 A) \;\cong\; \operatorname{rad}(e_a A)/\operatorname{rad}^2(e_a A).$$

We thus have a sequence of K-vector space isomorphisms

$$\begin{aligned}
e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b &\cong [\operatorname{rad}(e_a A)/\operatorname{rad}^2(e_a A)]e_b \\
&\cong \operatorname{Hom}_A(e_b A, \operatorname{rad}(e_a A)/\operatorname{rad}^2(e_a A)) \\
&\cong \operatorname{Hom}_A(e'_b A, \operatorname{rad}(e'_a A)/\operatorname{rad}^2(e'_a A)] \\
&\cong [\operatorname{rad}(e'_a A)/\operatorname{rad}^2(e'_a A)]e'_b \\
&\cong e'_a(\operatorname{rad} A/\operatorname{rad}^2 A)e'_b.
\end{aligned}$$

(b) It is obvious that the K-linear map $e_a(\operatorname{rad} A)e_b \to e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$ defined by the formula $e_a x e_b \mapsto e_a(x + \operatorname{rad}^2 A)e_b$ admits $e_a(\operatorname{rad}^2 A)e_b$ as a kernel. Hence we conclude that the map ψ defined in the statement is an isomorphism. This finishes the proof. $\square$

We now show that the connectedness of the algebra A implies that of its quiver Q_A. By definition, there exists a basis $\{\overline{x}_\alpha\}_\alpha$ of $\operatorname{rad} A/\operatorname{rad}^2 A$, where α ranges over the set $(Q_A)_1$ of arrows of Q_A. For each $\alpha \in (Q_A)_1$, let $x_\alpha \in \operatorname{rad} A$ be such that $\overline{x}_\alpha = x_\alpha + \operatorname{rad}^2 A$. We show that we can express all the elements of $\operatorname{rad} A$ in terms of the x_α and the paths in Q_A.

3.3. Lemma. *For each arrow $\alpha : i \to j$ in $(Q_A)_1$, let $x_\alpha \in e_i(\operatorname{rad} A)e_j$ be such that* the set $\{x_\alpha + \operatorname{rad}^2 A \mid \alpha : i \to j\}$ *is a basis of $e_i(\operatorname{rad} A/\operatorname{rad}^2 A)e_j$ (see (3.2)(a)). Then*

(a) *for any two points $a, b \in (Q_A)_0$, every element $x \in e_a(\operatorname{rad} A)e_b$ can be written in the form: $x = \sum x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_\ell} \lambda_{\alpha_1\alpha_2\dots\alpha_\ell}$, where $\lambda_{\alpha_1\alpha_2\dots\alpha_\ell} \in K$ and the sum is taken over all paths $\alpha_1\alpha_2\dots\alpha_\ell$ in Q_A from a to b; and*

(b) *for each arrow $\alpha : i \to j$, the element x_α uniquely determines a nonzero nonisomorphism $\widetilde{x}_\alpha \in \operatorname{Hom}_A(e_jA, e_iA)$ such that $\widetilde{x}_\alpha(e_j) = x_\alpha$, $\operatorname{Im} \widetilde{x}_\alpha \subseteq e_i(\operatorname{rad} A)$ and $\operatorname{Im} \widetilde{x}_\alpha \not\subseteq e_i(\operatorname{rad}^2 A)$.*

Proof. (a) Because, as a K-vector space, $\operatorname{rad} A \cong (\operatorname{rad} A/\operatorname{rad}^2 A) \oplus \operatorname{rad}^2 A$, we have $e_a(\operatorname{rad} A)e_b \cong e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b \oplus e_a(\operatorname{rad}^2 A)e_b$. Thus x can be written in the form

$$x = \sum_{\alpha: a \to b} x_\alpha \lambda_\alpha \quad \text{modulo} \quad e_a(\operatorname{rad}^2 A)e_b$$

(where $\lambda_\alpha \in K$ for every arrow α from a to b) or, more formally,

$$x' = x - \sum_{\alpha: a \to b} x_\alpha \lambda_\alpha \in e_a(\operatorname{rad}^2 A)e_b.$$

The decomposition $\operatorname{rad} A = \bigoplus_{i,j} e_i(\operatorname{rad} A)e_j$ implies that

$$e_a(\operatorname{rad}^2 A)e_b = \sum_{c \in (Q_A)_0} [e_a(\operatorname{rad} A)e_c][e_c(\operatorname{rad} A)e_b]$$

so that $x' = \sum_{c \in (Q_A)_0} x'_c y'_c$ where $x'_c \in e_a(\operatorname{rad} A)e_c$ and $y'_c \in e_c(\operatorname{rad} A)e_b$. By the preceding discussion, we have expressions of the form $x'_c = \sum_{\beta: a \to c} x_\beta \lambda_\beta$ and $y'_c = \sum_{\gamma: c \to b} x_\gamma \lambda_\gamma$ modulo $\operatorname{rad}^2 A$, where $\lambda_\beta, \lambda_\gamma \in K$. Hence

$$x = \sum_{\alpha: a \to b} x_\alpha \lambda_\alpha + \sum_{\beta: a \to c} \sum_{\gamma: c \to b} x_\beta x_\gamma \lambda_\beta \lambda_\gamma \quad \text{modulo} \quad e_a(\operatorname{rad}^3 A)e_b.$$

We complete the proof by an obvious induction using the fact that $\operatorname{rad} A$ is nilpotent.

(b) By our assumption, the element $x_\alpha \in e_i(\operatorname{rad} A)e_j$ is nonzero and maps to a nonzero element $\widetilde{x}_\alpha$ by the K-linear isomorphism $e_i(\operatorname{rad} A)e_j \cong$

$\mathrm{Hom}_A(e_jA, e_i(\mathrm{rad}A))$ (I.4.3). It follows that $\widetilde{x}_\alpha(e_j) = x_\alpha$, $\mathrm{Im}\,\widetilde{x}_\alpha \subseteq e_i(\mathrm{rad}A)$, and $\mathrm{Im}\,\widetilde{x}_\alpha \not\subseteq e_i(\mathrm{rad}^2 A)$. This finishes the proof. $\square$

3.4. Corollary. *If A is a basic and connected finite dimensional algebra, then the quiver Q_A of A is connected.*

Proof. If this is not the case, then the set $(Q_A)_0$ of points of Q_A can be written as the disjoint union of two nonempty sets Q_0' and Q_0'' such that the points of Q_0' are not connected to those of Q_0''. We show that, if $i \in Q_0'$ and $j \in Q_0''$, we have $e_iAe_j = 0$ and $e_jAe_i = 0$. Then (1.6) will imply that A is not connected, a contradiction. Because $i \neq j$, (I.4.2) yields

$$\begin{aligned} e_iAe_j &\cong \mathrm{Hom}_A(e_jA, e_iA) &&\cong \mathrm{Hom}_A(e_jA, \mathrm{rad}\, e_iA) \\ &\cong (\mathrm{rad}\, e_iA)e_j &&\cong e_i(\mathrm{rad}\, A)e_j. \end{aligned}$$

The conclusion follows at once from (3.3). $\square$

3.5. Examples. (a) If $A = K[t]/\langle t^m \rangle$, where $m \geq 1$, then Q_A has only one point, because the only nonzero idempotent of A is its identity. We have $\mathrm{rad}\, A = \langle \overline{t} \rangle$, where $\overline{t} = t + \langle t^m \rangle$; indeed, $\langle \overline{t} \rangle^m = 0$ and $A/\langle \overline{t} \rangle \cong K$. Consequently, $\mathrm{rad}^2 A = \langle \overline{t}^2 \rangle$ and $\dim_K(\mathrm{rad}\, A/\mathrm{rad}^2 A) = 1$. A basis of $\mathrm{rad}\, A/\mathrm{rad}^2 A$ is given by the class of $\overline{t}$ in the quotient $\langle \overline{t} \rangle / \langle \overline{t}^2 \rangle$. Thus Q_A is the quiver

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

(b) Let $A = \begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{bmatrix}$ be the algebra of the lower triangular matrices $[\lambda_{ij}] \in \mathbb{M}_3(K)$, with $\lambda_{32} = 0$ and $\lambda_{pq} = 0$, for $p > q$. An obvious complete set of primitive orthogonal idempotents of A is given by the three matrix idempotents:

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As in Example 3.5 (a), we show that $\mathrm{rad}\, A = \begin{bmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & 0 & 0 \end{bmatrix}$ and $\mathrm{rad}^2 A = 0$.

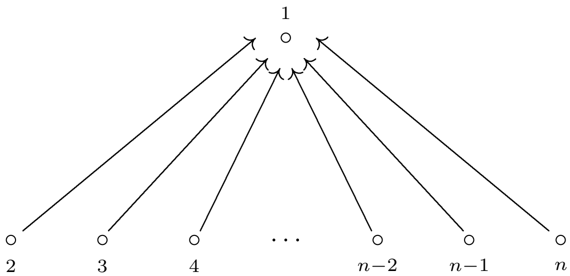
A straightforward calculation shows that $e_2(\mathrm{rad}\, A)e_1$ and $e_3(\mathrm{rad}\, A)e_1$ are one-dimensional and all remaining spaces of the form $e_i(\mathrm{rad}\, A)e_j$ are zero (because $\dim_K(\mathrm{rad}\, A) = 2$). Therefore Q_A is the quiver

$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur, "\alpha"] & & 3 \arrow[ul, "\beta"'] \end{tikzcd}$$

(c) An obvious generalisation of (b) is as follows. Let A be the algebra of $n \times n$ lower triangular matrices

$$A = \begin{bmatrix} K & 0 & 0 & \dots & 0 \\ K & K & 0 & \dots & 0 \\ K & 0 & K & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ K & 0 & 0 & \dots & K \end{bmatrix},$$

that is, an element of A might have a nonzero coefficient only in the first column or the main diagonal and has zero everywhere else. Then Q_A is the quiver



(d) Let A be the algebra of 3×3 lower triangular matrices

$$A = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & b & 0 \\ e & d & a \end{bmatrix} \quad \middle| \; a, b, c, d, e \in K \right\}$$

and $\mathcal{I}$ be the ideal

$$\mathcal{I} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e & 0 & 0 \end{bmatrix} \quad \middle| \; e \in K \right\}.$$

A complete set of primitive orthogonal idempotents for the algebra $B = A/\mathcal{I}$ consists of two elements

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mathcal{I} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{I}.$$

Also, $\operatorname{rad} B = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix} + \mathcal{I} \mid c, d \in K \right\}$ and $\operatorname{rad}^2 B = 0$. Thus the K-vector spaces $e_2(\operatorname{rad} B)e_1$ and $e_1(\operatorname{rad} B)e_2$ are both one-dimensional and Q_B is the quiver $1 \circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ 2$.

3.6. Lemma. *Let Q be a finite connected quiver, $\mathcal{I}$ be an admissible ideal of KQ, and $A = KQ/\mathcal{I}$. Then $Q_A = Q$.*

Proof. By (2.4), the set $\{e_a = \varepsilon_a + \mathcal{I} \mid a \in Q_0\}$ is a complete set of primitive orthogonal idempotents of $A = KQ/\mathcal{I}$. Thus the points of

Q_A are in bijective correspondence with those of Q. On the other hand, by (2.11) and the remark following it, the arrows from a to b in Q are in bijective correspondence with the vectors in a basis of the K-vector space $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$, thus with the arrows from a to b in Q_A. □

3.7. Theorem. *Let A be a basic and connected finite dimensional K-algebra. There exists an admissible ideal $\mathcal{I}$ of KQ_A such that $A \cong KQ_A/\mathcal{I}$.*

Proof. We first construct an algebra homomorphism $\varphi : KQ_A \to A$, then we show that φ is surjective and its kernel $\mathcal{I} = \operatorname{Ker} \varphi$ is an admissible ideal of KQ_A.

For each arrow $\alpha : i \to j$ in $(Q_A)_1$, let $x_\alpha \in \operatorname{rad} A$ be chosen so that $\{x_\alpha + \operatorname{rad}^2 A \mid \alpha : i \to j\}$ forms a basis of $e_i(\operatorname{rad} A/\operatorname{rad}^2 A)e_j$. Let $\varphi_0 : (Q_A)_0 \to A$ be the map defined by $\varphi_0(a) = e_a$ for $a \in (Q_A)_0$, and $\varphi_1 : (Q_A)_1 \to A$ be the map defined by $\varphi_1(\alpha) = x_\alpha$ for $\alpha \in (Q_A)_1$. Thus the elements $\varphi_0(a)$ form a complete set of primitive orthogonal idempotents in A, and if $\alpha : a \to b$, we have $\varphi_0(a)\varphi_1(\alpha)\varphi_0(b) = e_a x_\alpha e_b = x_\alpha = \varphi_1(\alpha)$.

By the universal property of path algebras (1.8), there exists a unique K-algebra homomorphism $\varphi : KQ_A \to A$ that extends φ_0 and φ_1.

We claim that φ is surjective. Because its image is clearly generated by the elements e_a (for $a \in (Q_A)_0$) and x_α (for $\alpha \in (Q_A)_1$), it suffices to show that these same elements generate A. Because K is algebraically closed, it follows from the Wedderburn–Malcev theorem (I.1.6) that the canonical homomorphism $A \to A/\operatorname{rad} A$ splits, that is, A is a split extension of the semisimple algebra $A/\operatorname{rad} A$ by $\operatorname{rad} A$. Because the former is clearly generated by the e_a, it suffices to show that each element of $\operatorname{rad} A$ can be written as a polynomial in the x_α and this follows from (3.3).

There remains to show that $\mathcal{I} = \operatorname{Ker} \varphi$ is admissible. Let R denote the arrow ideal of the algebra KQ_A. By definition of φ, we have $\varphi(R) \subseteq \operatorname{rad} A$ and hence $\varphi(R^\ell) \subseteq \operatorname{rad}^\ell A$ for each $\ell \geq 1$. Because $\operatorname{rad} A$ is nipotent, there exists $m \geq 1$ such that $\operatorname{rad}^m A = 0$ and consequently $R^m \subseteq \operatorname{Ker} \varphi = \mathcal{I}$. We now prove that $\mathcal{I} \subseteq R^2$. If $x \in \mathcal{I}$, then we can write

$$x = \sum_{a \in (Q_A)_0} \varepsilon_a \lambda_a + \sum_{\alpha \in (Q_A)_1} \alpha \mu_\alpha + y,$$

where $\lambda_a, \mu_\alpha \in K$ and $y \in R^2$. Now $\varphi(x) = 0$ gives

$$0 = \sum_{a \in (Q_A)_0} e_a \lambda_a + \sum_{\alpha \in (Q_A)_1} x_\alpha \mu_\alpha + \varphi(y).$$

Hence $\sum_{a \in (Q_A)_0} e_a \lambda_a = - \sum_{\alpha \in (Q_A)_1} x_\alpha \mu_\alpha - \varphi(y) \in \operatorname{rad} A$. Because $\operatorname{rad} A$ is nilpotent, and the e_a are orthogonal idempotents, we infer that $\lambda_a = 0$, for any $a \in (Q_A)_0$. Similarly $\sum\limits_{\alpha\in(Q_A)_1} x_\alpha\mu_\alpha = -\varphi(y) \in \operatorname{rad}^2 A$. Hence the equality $\sum\limits_{\alpha\in(Q_A)_1} (x_\alpha + \operatorname{rad}^2 A)\mu_\alpha = 0$ holds in $\operatorname{rad} A/\operatorname{rad}^2 A$. But the set $\{x_\alpha + \operatorname{rad}^2 A \mid \alpha \in (Q_A)_1\}$ is, by construction, a basis of $\operatorname{rad} A/\operatorname{rad}^2 A$. Therefore $\mu_\alpha = 0$ for each $\alpha \in (Q_A)_1$ and so $x = y \in R^2$. $\square$

3.8. Definition. Let A be a basic and connected finite dimensional K-algebra. An isomorphism $A \cong KQ_A/\mathcal{I}$, where $\mathcal{I}$ is an admissible ideal of KQ_A (such as the one constructed in Theorem 3.7) is called a **presentation** of the algebra A (as a bound quiver algebra).

3.9. Examples. (a) In Example 3.5 (a), the K-algebra homomorphism $\varphi : KQ_A \to A$ is defined by $\varphi(\varepsilon_1) = 1, \varphi(\alpha) = \overline{t}$. Clearly, φ is surjective, and Ker $\varphi = \langle\alpha^m\rangle$.

(b) In Example 3.5 (b), the K-algebra homomorphism $\varphi : KQ_A \to A$ is defined by

$$\varphi(\varepsilon_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \varphi(\varepsilon_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \varphi(\varepsilon_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\varphi(\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \varphi(\beta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Here, φ is an isomorphism so that $A \cong KQ_A$. Later we characterise the algebras (such as A) that are isomorphic to the path algebras of their ordinary quivers.

(c) In Example 3.5 (d), the K-algebra homomorphism $\varphi : KQ_B \to B$ is defined by

$$\varphi(\varepsilon_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mathcal{I}, \quad \varphi(\varepsilon_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{I},$$
$$\varphi(\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{I}, \quad \varphi(\beta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \mathcal{I}.$$

We see that Ker $\varphi = \langle\alpha\beta, \beta\alpha\rangle = R_Q^2$ and hence $B \cong KQ_B/R_Q^2$, where $Q = Q_B$.

3.10. Remark. Usually, an algebra has more than one presentation as a bound quiver algebra; see, for instance, Example 2.2 (e).

II.4. Exercises

1. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. The opposite quiver Q^{op} is the quiver $Q^{\mathrm{op}} = (Q_0, Q_1, s', t')$ where, for $\alpha \in Q_1$, $s'(\alpha) = t(\alpha)$ and $t'(\alpha) = s(\alpha)$. Show that $(KQ)^{\mathrm{op}} \cong KQ^{\mathrm{op}}$.

2. Let Q be a finite quiver. Show that:

(a) KQ is semisimple if and only if $|Q_1| = 0$,

(b) KQ is simple if and only if $|Q_0| = 1$ and $|Q_1| = 0$.

If, moreover, Q is connected, show that:

(c) KQ is local only if $|Q_0| = 1$ and $|Q_1| = 0$,

(d) KQ is commutative if and only if $|Q_0| = 1$ and $|Q_1| \leq 1$.

3. For each of the following quivers, give a basis of the path algebra, then write the multiplication table of this basis, and finally write the path algebra as a triangular matrix algebra:

(a) $\circ \longleftarrow \circ \longrightarrow \circ$

(b) $\circ \longrightarrow \circ \longleftarrow \circ$

(c) $\circ \leftleftarrows \circ \longleftarrow \circ$

(d) $\circ \longrightarrow \circ \longleftarrow \circ \longleftarrow \circ$

(e)
$$\begin{tikzcd}[row sep=small] & & \circ \arrow[dl] \\ \circ & \circ \arrow[l] \arrow[dr] & \\ & & \circ \end{tikzcd}$$

(f)
$$\begin{tikzcd}[row sep=small] & & & \circ \arrow[dl] \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[dr] & \\ & & & \circ \end{tikzcd}$$

(g)
$$\begin{tikzcd}[row sep=small, column sep=small] \circ \arrow[dr] & & \circ \arrow[dl] \\ & \circ & \\ \circ \arrow[ur] & & \circ \arrow[ul] \end{tikzcd}$$

(h)
$$\begin{tikzcd} \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] \\ & & \circ \arrow[u] & & \end{tikzcd}$$

4. Let $E = \{1, 2, \ldots, n\}$ be partially ordered as follows: $1 \preceq i$ for all $1 \leq i \leq n$, and for each pair (i, j) with $2 \leq i$, $j \leq n$, we have $i \preceq j$ if and only if $i = j$. Show that the incidence K-algebra of $(E, \preceq)$ is isomorphic to the path algebra KQ of a quiver Q (to be determined).

5. Let $Q = (Q_0, Q_1)$ be a finite and acyclic quiver. Show that KQ is connected if and only if KQ/R^2 is connected, where R is the arrow ideal of KQ.

6. Let Q be the quiver

$$\begin{tikzcd} 1 \circ \arrow[loop right, "\alpha"] \end{tikzcd}$$

Show that the arrow ideal R_Q of the path K-algebra KQ is infinite dimensional, and $\operatorname{rad} KQ = 0$.

7. Let $Q = (Q_0, Q_1)$ be the quiver

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \circ \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

Show that each of the following ideals of KQ is admissible:
(a) $\mathcal{I}_1 = \langle \alpha^2 - \beta\gamma, \gamma\beta - \gamma\alpha\beta, \alpha^4 \rangle$,
(b) $\mathcal{I}_2 = \langle \alpha^2 - \beta\gamma, \gamma\beta, \alpha^4 \rangle$.

8. Let Q be a quiver and $\mathcal{I}$ an admissible ideal in KQ. Construct an admissible ideal $\mathcal{I}^{\mathrm{op}}$ of KQ^{op} such that $KQ^{\mathrm{op}}/\mathcal{I}^{\mathrm{op}} \cong (KQ/\mathcal{I})^{\mathrm{op}}$.

9. Let $Q' = (Q'_0, Q'_1)$ be a full subquiver of $Q = (Q_0, Q_1)$ such that if $\alpha : a \to b$ is an arrow in Q with $a \in Q'_0$, then $b \in Q'_0$ and $\alpha \in Q'_1$. Let $\mathcal{I}$ be an admissible ideal of KQ and $\varepsilon = \sum_{a \in Q'_0} \varepsilon_a$.

(a) Show that $KQ' = \varepsilon(KQ)\varepsilon$ and that $\mathcal{I}' = \varepsilon I \varepsilon$ is an admissible ideal of KQ'.

(b) Show that $A' = KQ'/\mathcal{I}'$ is isomorphic to the quotient of $A = KQ/\mathcal{I}$ by $J = \langle \varepsilon_a + \mathcal{I} \,|\, a \notin Q'_0 \rangle$.

10. Let A be an algebra such that $\mathrm{rad}^2 A = 0$. Show that if $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents, then $e_i A e_j \neq 0$ for $i \neq j$ if and only if there exists an arrow $i \to j$ in Q_A.

11. Describe, up to isomorphism, all basic three-dimensional algebras.

12. Let $A = \begin{bmatrix} K[t]/(t^2) & 0 \\ K[t]/(t^2) & K \end{bmatrix}$ and view A as a K-algebra with the usual matrix multiplication. Show that $A \cong KQ/\mathcal{I}$, where Q is the quiver

$$\begin{tikzcd} \circ \arrow[loop left, "\beta"] & \circ \arrow[l, "\gamma"'] \end{tikzcd}$$

and $\mathcal{I}$ is the ideal of KQ generated by one zero relation β^2.

13. Let $A = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ K & K & K \end{bmatrix}$ be the K-subalgebra of $\mathbb{M}_3(K)$ defined in (I.1.1)(c) and let B be the subalgebra of A consisting of all matrices $\lambda = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{22} & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$ in A such that $\lambda_{11} = \lambda_{22} = \lambda_{33}$. Show that the algebra B is commutative and local and that $\mathrm{rad}\, B$ consists of all matrices $\lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_{31} & \lambda_{32} & 0 \end{bmatrix}$ in B. Prove that there are K-algebra isomorphisms $B \cong K[t_1, t_2]/(t_1, t_2)^2 \cong KQ/\mathcal{I}$, where Q is the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

and $\mathcal{I} = \langle \alpha^2, \beta^2, \alpha\beta, \beta\alpha \rangle$.

14. Let $A = \mathbb{T}_3(K) = \begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & K & K \end{bmatrix}$ be as in (I.1.1) and let C be the subalgebra of A consisting of all matrices $\lambda = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$ in A such that $\lambda_{11} = \lambda_{22} = \lambda_{33}$. Show that the algebra C is noncommutative and local and that there are K-algebra isomorphisms $C \cong K\langle t_1, t_2 \rangle / (t_1^2, t_2^2, t_2 t_1) = KQ/\mathcal{I}$, where Q is the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

and $\mathcal{I} = (\alpha^2, \beta^2, \beta\alpha)$.

15. Write a bound quiver presentation of each of the following algebras:

$$\begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & K & K & 0 \\ K & K & K & K & K \end{bmatrix}, \quad \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix}, \quad \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ K & K & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix}.$$

16. The hypothesis that the base field is algebraically closed is necessary for Theorem 3.7 to be valid. **Hint:** Show that the $\mathbb{R}$-algebra $\mathbb{C}$ is two-dimensional, basic, and connected but that there is no quiver Q such that $\mathbb{C} \cong \mathbb{R}Q/\mathcal{I}$ with $\mathcal{I}$ an admissible ideal of $\mathbb{R}Q$.

17. The following three examples show that generators of an admissible ideal are not uniquely determined in general:

(a) Let $Q = (Q_0, Q_1)$ be the quiver

$$\begin{tikzcd} & \circ \arrow[dl, "\beta"'] & \\ \circ & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & \circ \arrow[ul, "\delta"] & \end{tikzcd}$$

and $\mathcal{I}_1 = \langle \alpha\beta + \gamma\delta \rangle$, $\mathcal{I}_2 = \langle \alpha\beta - \gamma\delta \rangle$ two-sided ideal of KQ. Show that $\mathcal{I}_1$ and $\mathcal{I}_2$ are admissible and distinct and that there is a K-algebra isomorphism $KQ/\mathcal{I}_1 \cong KQ/\mathcal{I}_2$, if $\operatorname{char} K \neq 2$.

(b) Same exercise with $Q = (Q_0, Q_1)$, $\mathcal{I}_1$, $\mathcal{I}_2$ as in Exercise 7, $\operatorname{char} K \neq 2$.

(c) Same exercise with $Q = (Q_0, Q_1)$ of the form $\circ \xleftarrow{\ \gamma\ } \circ \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \circ$, $\mathcal{I}_1 = \langle \alpha\gamma - \beta\gamma \rangle$, $\mathcal{I}_2 = \langle \alpha\gamma \rangle$, but the characteristic of K is arbitrary.

18. Let A be a finite dimensional commutative algebra. Show that A is a finite product of commutative local algebras.

19. Let A be a finite dimensional basic and connected algebra. Show that $Q_{A^{\mathrm{op}}} = (Q_A)^{\mathrm{op}}$ and that there exists an admissible ideal $\mathcal{I}^{\mathrm{op}}$ of $KQ_{A^{\mathrm{op}}}$ such that $A^{\mathrm{op}} \cong (KQ_{A^{\mathrm{op}}})/\mathcal{I}^{\mathrm{op}}$.