Interesting Combinatorics In Higher Auslander Theory

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10th Day of Doctoral Students of the School of Mathematics

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Outline

Fundamentals

Algebras, Modules, Quivers

Auslander-Reiten Theory

Path Algebras, Representations, AR Quivers

Elements

Fundamentals

Algebras, Modules, Quivers

Algebras

k-algebra

An algebra over a field k is a k-vector space equipped with a bilinear product.

Motivating examples

- Complex numbers as the vector space $\mathbb{R}^2$ with the typical product of complex numbers.
- Ring of polynomials (over k) with polynomial multiplication.
- Ring of square matrices with matrix multiplication.

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Modules

Λ-module

Let Λ be a k-algebra. A right Λ-module is a pair $(M, \cdot)$ where M is a k-vector space and $\cdot : M \times A \to M$ is a binary operation satisfying natural commutativity and associativity rules.

Examples

- Each algebra is a module (left or right) over itself.
- $k[x,y] = (k[x])[y]$ is a module (left or right) over $k[x]$.

Indecomposability ('prime' modules)

A (right) Λ-module M is indecomposable if $M \neq 0$ and $M = M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$.

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Λ-module homomorphism

A map $f : M \to N$ between two (right) Λ-modules M and N is a Λ-module homomorphism if it's k-linear and respects $\cdot$, that is

$$f(m \cdot \lambda) = f(m) \cdot \lambda \text{ for } \lambda \in \Lambda, m \in M.$$

Section/retraction

A Λ-module homomorphism $f : M \to N$ is

- a section if $\exists g : N \to M$ such that $g \circ f = 1_N$.
- a retraction if $\exists h : N \to M$ such that $f \circ h = 1_M$.

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Irreducibility ('prime' homomorphisms)

A Λ-module homomorphism $f : M \to N$ is irreducible if

- f is neither a **section** nor a **retraction**;
- whenever $f = f_2 \circ f_1$, then f_2 is a retraction or f_1 is a section.

We denote the k-vector space of irreducible homomorphisms $M \to N$ as $\mathrm{Irr}(M, N)$.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "f_1"'] & & N \\ & X \arrow[ur, "f_2"'] & \end{tikzcd}$$

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Quivers

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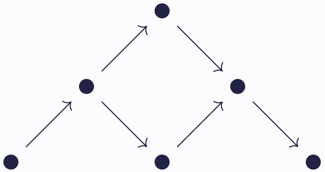

Auslander-Reiten Theory

Path Algebras, Representations, AR Quivers

Path algebras

The path algebra of a quiver

Let Q be a quiver. The path algebra kQ of Q is the k-algebra whose k-vector space has as its basis all paths of length ≥ 0 in Q and the product of two basis elements is the concatenation of paths.

Path algebras – Example

Consider the quiver

$$\underset{1}{\bullet} \xleftarrow{\;a\;} \underset{2}{\bullet}$$

The basis of the path algebra kQ is the triple (e_1, e_2, a) (where e_i means 'stay at i') and its multiplication table is

	e_1	e_2	a
e_1	e_1	0	0
e_2	0	e_2	a
a	a	0	0

It's actually isomorphic to the k-algebra of lower triangular 2×2 matrices over k.

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Every algebra is a path algebra

Theorem

Let k be an algebraically closed field and Λ a basic, connected and finite-dimensional algebra over k. Then there exists a finite connected quiver Q such that $\Lambda = kQ/I$ for some admissible ideal I of kQ.

Integrals and Other Expressions

$$\iint_{\partial\Omega} f(x)\mathrm{d}x \in \mathbb{C} \tag{1}$$

$$E = mc^2 \tag{2}$$

$$F = ma \tag{3}$$

m	Mass
c	Speed of light

Theorem

The following statement is correct

$$\frac{\partial f(\vec{x})}{\partial x_i} = \sum_{l=1}^{L} \cos\left(l \frac{2\pi}{L} + 0\right) \tag{4}$$

Elements

Typography

```
The theme provides sensible defaults to
  \emph{emphasize} text, \alert{accent} parts
  or show \textbf{bold} results.
```

becomes

The theme provides sensible defaults to *emphasize* text, accent parts or show **bold** results.

Font feature test

- Regular
- *Italic*
- Small Caps
- **Bold**
- ***Bold Italic***
- **Bold Small Caps**
- `Monospace`
- *`Monospace Italic`*
- **`Monospace Bold`**
- ***`Monospace Bold Italic`***

Lists

Items

- Milk
- Eggs
- Potatoes

Enumerations

1. First,
2. Second and
3. Last.

Descriptions

PowerPoint Meeh.

Beamer Yeeeha.

Tables

Table 1: Largest cities in the world (source: Wikipedia)

City	Population
Mexico City	20,116,842
Shanghai	19,210,000
Peking	15,796,450
Istanbul	14,160,467

Blocks

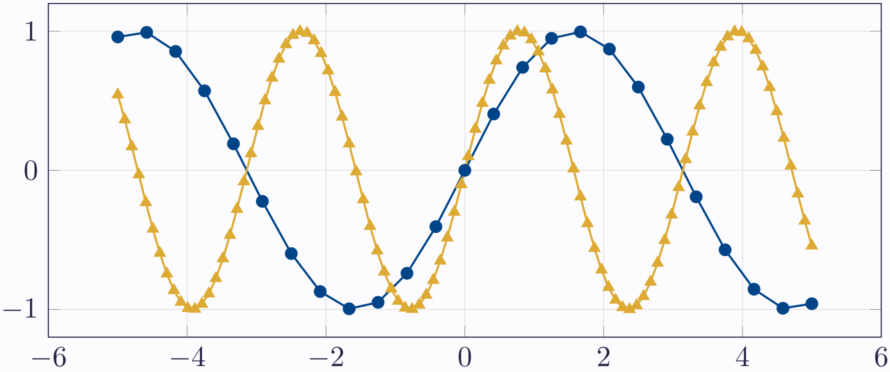
Three different block environments are pre-defined and may be styled with an optional background color.

Default	Default
Block content.	Block content.

Alert	Alert
Block content.	Block content.

Example	Example
Block content.	Block content.

Line plots



Standout Frame!

Backup slides

Sometimes, it is useful to add slides at the end of your presentation to refer to during audience questions.

The best way to do this is to include the `appendixnumberbeamer` package in your preamble and call `\appendix` before your backup slides.

The theme will automatically turn off slide numbering and progress bars for slides in the appendix.