COMBINATORIAL PROPERTIES OF HIGHER CLUSTER CATEGORIES

ADAM KLEPÁČ

Current State of Knowledge

In [FZ01a], Fomin and Zelenivsky introduce *cluster algebras* to provide an algebraic-combinatorial description of the structure of dual canonical bases in coordinate rings of varieties associated to semisimple algebraic groups. Such descriptions were indeed found in several cases (see [BFZ03], [Sco03]). Applications of cluster algebras have since been discovered in, for instance, Poisson geometry ([GSV03]), discrete dynamical systems (e.g. [FZ01b]), higher Teichmüller spaces ([FG06]) or commutative and non-commutative algebraic geometry ([IR07], [KS08] or [GMN12]).

Of particular interest to me are cluster algebras of *finite type*. Those were fully classified in [FZ03] – a cluster algebra $\mathcal{A}(Q)$ is of finite type if and only if the underlying graph Δ of the associated quiver Q is a (simply-laced) Dynkin diagram (figure 1). It was shown in [MRZ03] that in this case, the combinatorics of $\mathcal{A}(Q)$ can be obtained from the category of *decorated representations* of Q. However, with the construction of a *cluster category* in [Bua+04], emerged a more symmetric way of describing the combinatorics of $\mathcal{A}(Q)$.

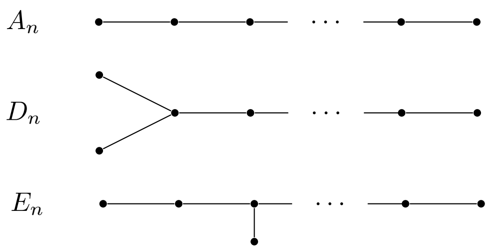

Figure 1. Simply-laced Dynkin diagrams of types A_n, D_n and E_n.

Concretely, let k be a field, let $\Lambda := kQ$ denote the path algebra of Q and $\mathcal{D} := D^b(\operatorname{mod}\Lambda)$ the bounded derived category of finitely generated Λ-modules with shift functor [1]. Let $G : \mathcal{D} \to \mathcal{D}$ be a triangle functor with some necessary additional properties (see [Kel05]). We denote by $\mathcal{D}/G$ the corresponding orbit category, that is, a category with objects the G-orbits of objects of $\mathcal{D}$ and with morphisms

$$\operatorname{Hom}_{\mathcal{D}/G}(\bar{X}, \bar{Y}) = \coprod_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(G^i X, Y)$$

where $X, Y \in \mathcal{D}$ and $\bar{X}, \bar{Y}$ are the corresponding objects of $\mathcal{D}/G$. In [Kel05], it is shown that $\mathcal{D}/G$ is triangulated, the shift functor in $\mathcal{D}/G$ is induced by [1] – we shall denote it the same – and that $\mathcal{D}/G$ is Krull-Schmidt. We write $\operatorname{Ext}^1(U, V) := \operatorname{Hom}(U, V[1])$ where the morphisms are considered in $\mathcal{D}$ or $\mathcal{D}/G$.

Henceforth, we focus on $\mathcal{D}/G$ with the special choice $G := \tau^{-1}[1]$ where τ is the AR translation in $\mathcal{D}$ (induced by $D\mathrm{Tr}$ on non-projective indecomposable modules in $\operatorname{mod}\Lambda$ and satisfying $\tau(P) = I[-1]$ for P indecomposable projective and I indecomposable injective with $\operatorname{soc} I \cong P/\operatorname{rad} P$). Because of its connection to the theory of cluster algebras, we call it the *cluster category* associated to Λ. When k is algebraically closed and Q of Dynkin type, the cluster category $\mathcal{C} := \mathcal{D}/G$ depends (as shown in [Bua+04]) only on the underlying graph Δ of Q and the choice of orientation is immaterial. We write $\mathcal{C} = \mathcal{C}(\Delta)$. Most importantly, in this case, the combinatorics of the cluster algebra $\mathcal{A}(Q)$ can be fully recovered in terms of Ext^1-groups of $\mathcal{C}$. In particular, the clusters of $\mathcal{A}(Q)$ are in 1-1 correspondence with the cluster tilting objects of $\mathcal{C}$ – objects $T \in \mathcal{C}$ such that $\operatorname{Ext}^1(T, T) = 0$ and T has the maximal number of non-isomorphic direct summands.

In [FST08], cluster algebras of *surface* type are introduced. These are cluster algebras whose underlying quiver arises from ideal triangulations of bordered surfaces with marked points – connected oriented 2-dimensional Riemann surfaces with a finite number of distinguished points in their closure. Both cluster algebras of finite type and those of surface type share a '2-dimensional quality'. In the case of cluster algebras of surface type, it is the dimension of the surface; the 2-dimensionality of cluster algebras of finite type is reflected in the fact that the associated cluster category is 2-Calabi-Yau (see [Bua+04, Proposition 1.7]). It seems natural to seek higher-dimensional analogues of these constructions.

The only cluster algebras of both finite and surface type are those with underlying quiver of Dynkin type A or D. Quivers of types not Dynkin fail to produce cluster algebras of finite type and quivers of Dynkin type E fail to correspond to any triangulation of any bordered surface with marked points. As the combinatorial properties of $\mathcal{A}(Q)$ depend only on the underlying graph of Q, the main focus shifts to the path algebra kQ (with Q being arbitrarily oriented) and the associated cluster category.

The paper [OT11] discusses the connection between d-Auslander algebras of kA_n (with A_n meaning the linearly-oriented quiver of Dynkin type A on n vertices) and cyclic polytopes of dimension $2d$ on $n+2d$ vertices.

The *cyclic polytope* $C(m,\delta)$ is the convex hull of m distinct points on the moment curve $t \mapsto (t, t^2, \dots, t^\delta)$ of dimension δ. It is the most natural higher-dimensional analogue of a disc with a finite number of distinguished points on its boundary whose triangulations give rise to quivers of Dynkin type A (see [FST08]).

The notions of n-*cluster tilting* modules and n-*representation-finite* algebras are introduced in [Iya10] and [IO10], respectively. Suppose Λ is any finite-dimensional algebra over a field k. A module $M \in \operatorname{mod}\Lambda$ is called n-cluster tilting if

$$\begin{aligned}\operatorname{add} M &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i(X,M) = 0 \ \forall i \in \{1,\dots,n-1\}\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i(M,X) = 0 \ \forall i \in \{1,\dots,n-1\}\}.\end{aligned}$$

If Λ has an n-cluster tilting module and also $\operatorname{gl.dim}\Lambda \le n$, then Λ is called n-representation-finite. In such case, the endomorphism algebra $\Gamma \coloneqq \operatorname{End}_\Lambda(M)$ is called the n-Auslander algebra of Λ and denoted $\Lambda^{(n)}$. Iyama ([Iya10]) shows that if Λ is n-representation-finite, then $\operatorname{End}_\Lambda(M)$ is $(n+1)$-representation-finite, allowing for an inductive construction of higher Auslander algebras of a 1-representation-finite (that is, representation-finite and hereditary) algebra. Since representation-finite hereditary algebras are precisely the path algebras of Dynkin quivers, the list of algebras to study narrows dramatically. Moreover, for such algebras, Iyama gives a complete description of their higher Auslander algebras in terms of their AR quivers and relations (see [Iya10, Section 6]).

In [OT11], Oppermann and Thomas focus chiefly on the path algebra $\Lambda \coloneqq kA_n$, its $(d-1)$-st higher Auslander algebras (denoted A_n^d) and the associated (generalized) cluster category. They discover a bijection

$$\left\{\begin{matrix}\text{triangulations of}\\ C(n+2d,2d)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{basic tilting modules of } A_n^d\\ \text{contained in add } {}_{A_n^d}M\end{matrix}\right\},$$

where ${}_{A_n^d}M$ is the d-cluster tilting module of A_n^d which is unique up to multiplicity ([Iya10]). Furthermore, in [OT11, Section 5], the authors generalize the concept of a triangulated cluster category to a $(d+2)$-angulated cluster category for any d-represention-finite algebra Λ. See [GKO13] for a definition of an n-angulated category. Specifically, a full subcategory $\mathcal{A}$ of a triangulated category $\mathcal{B}$ is called δ-cluster tilting if $\mathcal{A}$ is functorially finite in $\mathcal{B}$ (see [AS81]) and $\mathcal{A}$ coincides with both

$$\{T \in \mathcal{B} \mid \operatorname{Hom}_{\mathcal{B}}(\mathcal{A}, T[i]) = 0 \ \forall i \in \{1,\dots,\delta-1\}\} \quad \text{and} \quad \{T \in \mathcal{B} \mid \operatorname{Hom}_{\mathcal{B}}(T, \mathcal{A}[i]) = 0 \ \forall i \in \{1,\dots,\delta-1\}\}.$$

If $\operatorname{add} T$ is a δ-cluster tilting subcategory of $\mathcal{B}$, the object T is called δ-cluster tilting. If Λ is d-representation-finite, S denotes the Serre functor on $\mathcal{D} = D^b(\operatorname{mod}\Lambda)$ and S_d its d-th desuspension $S[-d]$, then the category

$$\mathcal{U} \coloneqq \operatorname{add}\{S_d^i\Lambda \mid i \in \mathbb{Z}\} = \operatorname{add}\{M[id] \mid i \in \mathbb{Z}\},$$

where M is a d-cluster tilting Λ-module, is a d-cluster tilting subcategory of $\mathcal{D}$ (see [Iya10, Theorem 1.23]). The $(d+2)$-angulated cluster category of Λ is then defined as the orbit category

$$\mathcal{O}_\Lambda \coloneqq \mathcal{U}/S_{2d}.$$

It is shown in [OT11, Theorem 5.2] that this category is Krull-Schmidt and $2d$-Calabi-Yau and the isomorphism classes of indecomposable objects of $\mathcal{O}_\Lambda$ are in 1-1 correspondence with the indecomposable summands of $M \oplus \Lambda[d]$. Finally, in the case of $\Lambda = kA_n$, the authors prove there is a correspondence

$$\left\{\begin{matrix}\text{internal } d\text{-simplices}\\ \text{of } C(n+2d+1,2d)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{indecomposable}\\ \text{objects of } \mathcal{O}_{A_n^d}\end{matrix}\right\},$$

which induces the correspondence

$$\left\{\begin{matrix}\text{triangulations of}\\ C(n+2d+1,2d)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{basic cluster tilting}\\ \text{objects of } \mathcal{O}_{A_n^d}\end{matrix}\right\}.$$

In recent strongly related papers [DJW19] and [DJL21], an alternative combinatorial description of (perfect derived categories of) higher Auslander algebras of type A is given from the viewpoint of symplectic geometry and simplicial theory, respectively.

It is my aim to reproduce these results (namely those in [OT11]) for the path algebras of quivers of Dynkin type D and E and their associated cluster categories. In the case of type D, such quivers correspond to ideal triangulations of a disc with finite number of marked points on its boundary and a single marked point in its interior ([FST08]). The obvious higher-dimensional analogue is again the cyclic polytope with one distinguished point in its interior. Unfortunately, quivers of Dynkin type E do not arise from triangulations of bordered surfaces with marked points and thus one is required to search for a combinatorial counterpart which is not necessarily 'geometric'.

It should be noted that the 2-dimensional (or classical) cluster categories of type D were already studied from a geometric point of view in [Sch06].

Next, I intend to focus on higher-dimensional analogues and the cluster categories of path algebras kQ which are not representation-finite and hereditary, that is, Q is not Dynkin. Since full combinatorial description doesn't seem at all plausible in the current state of research, I hope to find hints towards such a goal by studying path algebras of quivers which are 'close enough' to Dynkin quivers so that the theory doesn't get completely out of hand. Concretely, I shall focus on quivers whose underlying graphs are so-called *affine* Dynkin diagrams (figure 2).

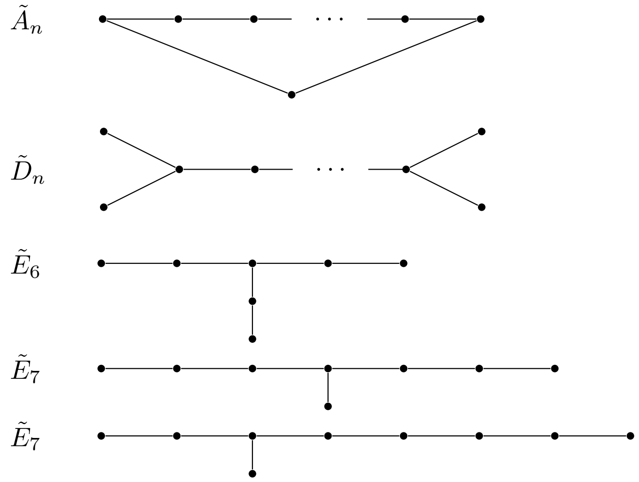

FIGURE 2. Affine Dynkin diagrams of types $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

Several required generalizations of the aforewritten theory for n-representation-finite algebras have already been constructed. In [HIO14], *n-representation-infinite* algebras are defined in general and, particularly, n-representation-infinite algebras of type $\tilde{A}$ are constructed as certain graded orbit algebras. The authors then proceed to show that the degree 0 parts of these algebras are indeed the path algebras of quivers of type $\tilde{A}$. Whether such construction proves significant in the study of combinatorial properties of these algebras remains to be seen. Moreover, by [HIO14, Theorem 4.18], the category of finitely generated modules over any n-representation-infinite algebra splits into its n-preprojective, n-regular and n-preinjective components. This fact has been successfully used to, for example, study the combinatorics of AR quivers of n-representation-infinite path algebras kQ through the truncations of the (generalized) *repetition quiver* $\mathbb{Z}Q$ – a quiver whose vertices are elements of $\mathbb{Z} \times Q_0$ and for each arrow $i \to j$ in Q_1 there are arrows $(p, i) \to (p, j)$ and $(p-1, j) \to (p, i)$ where p ranges over $\mathbb{Z}$ (see [GLL22]).

To the best of my knowledge, there has thus far not been developed a notion of a generalized n-Auslander algebra for algebras where the existence of an n-cluster tilting module is not guaranteed which would allow for an iterative construction similar in nature to the one for algebras n-representation-finite. I do not expect to be able to develop such theory in full generality, but I expect the path algebras of affine Dynkin quivers will provide a more hospitable place to such endeavour.

There exist multiple definitions of 'higher-dimensional' cluster categories. One is due to C. Amiot ([Ami08] and [Ami09]) and stays true to the original definition in [Kel05]. She defines the triangulated category $\mathcal{C}_\Lambda^\delta$ of any algebra Λ with $\operatorname{gl.dim} \Lambda \leq \delta$ as the *triangulated hull* of the orbit category $D^b(\operatorname{mod} \Lambda)/S_\delta$ where, again, S is the Serre functor on $D^b(\operatorname{mod} \Lambda)$ and $S_\delta := S[-\delta]$. She then proceeds to prove that as long as $\mathcal{C}_\Lambda^\delta$ is Hom-finite, it is δ-Calabi-Yau and admits a δ-cluster tilting object. The $(d+2)$-angulated cluster category $\mathcal{O}_\Lambda$ for Λ n-representation-finite defined in [OT11] and discussed above is a d-cluster tilting subcategory of $\mathcal{C}_\Lambda^{2d}$.

Another approach is due to Guo ([Guo12]). Given a differential-graded k-algebra Λ, we denote by $\operatorname{per}\Lambda$ its *perfect derived category*, that is, the smallest triangulated subcategory of $D(\Lambda)$ – the *unbounded* derived category of Λ – containing Λ which is closed under forming direct summands. Supposing Λ satisfies certain additional properties (laid out in [Guo12, Section 3.2]), in particular, assuming Λ is $(m+2)$-Calabi-Yau as a bimodule for an integer m, the *generalized m-cluster category* associated with Λ is defined as the triangulated quotient $\operatorname{per}\Lambda/D^b(\operatorname{mod}\Lambda)$. It is $(m+1)$-Calabi-Yau and hosts an m-cluster tilting object. This generalized notion of a cluster category can be related to path algebras of finite quivers by works of Ginzburg and Keller. For a finite quiver Q with potential W, Ginzburg defines ([Gin07, Section 4.2]) certain differential-graded k-algebra $\Gamma(Q,W)$ whose underlying graded algebra is the path algebra $k\hat{Q}$ with $\hat{Q}$ being a graded quiver constructed from Q. The algebra $\Gamma(Q,W)$ is later shown by Keller ([Kel09]) to satisfy the criteria laid out in [Guo12, Section 3.2]. In particular, this algebra is always 3-Calabi-Yau, suggesting possibilities of constructions of this nature for higher-dimensional analogues of the path algebra kQ.

No parallels between n-representation-infinite algebras or generalized cluster categories and purely combinatorial structures have thus far been drawn. While focusing on the path algebras of affine Dynkin quivers, it seems necessary to generalize the construction of higher Auslander algebras from [Iya10] to account for the loss of a guaranteed existence of an n-cluster tilting module. Similarly, an alternative construction of the generalized cluster categories associated to such path algebras might be required to remedy the fact that the orientation of the quiver has now influence on the overall structure. I hope to prove that, at least in this 'almost-Dynkin' case, there still exist natural combinatorial descriptions of the tilting modules of these path algebras and the tilting objects of their associated generalized cluster categories which are mutually compatible.

References

[AS81] M. Auslander and Sverre O. Smalø. "Almost split sequences in subcategories". In: *Journal of Algebra* 69.2 (1981), pp. 426–454. ISSN: 0021-8693. URL: https://www.sciencedirect.com/science/article/pii/0021869381902143.

[FZ01a] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras I: Foundations". In: (2001). URL: https://arxiv.org/abs/math/0104151v1.

[FZ01b] Sergey Fomin and Andrei Zelevinsky. "Y-systems and generalized associahedra". In: (2001). URL: https://arxiv.org/abs/hep-th/0111053v1.

[BFZ03] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. "Cluster algebras III: Upper bounds and double Bruhat cells". In: (2003). URL: https://arxiv.org/abs/math/0305434v3.

[FZ03] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras II: Finite type classification". In: (2003). URL: https://arxiv.org/abs/math/0208229v2.

[GSV03] M. Gekhtman, M. Shapiro, and A. Vainshtein. "Cluster algebras and Poisson geometry". In: (2003). URL: https://arxiv.org/abs/math/0208033v2.

[MRZ03] Bethany Marsh, Markus Reineke, and Andrei Zelevinsky. "Generalized associahedra via quiver representations". In: (2003). URL: https://arxiv.org/abs/math/0205152v2.

[Sco03] Joshua S. Scott. "Grassmannians and Cluster Algebras". In: (2003). URL: https://arxiv.org/abs/math/0311148v1.

[Bua+04] Aslak Bakke Buan et al. "Tilting theory and cluster combinatorics". In: (2004). URL: https://arxiv.org/abs/math/0402054v1.

[Kel05] Bernhard Keller. "On triangulated orbit categories". In: (2005). URL: https://arxiv.org/abs/math/0503240v4.

[FG06] V. V. Fock and A. B. Goncharov. *Moduli spaces of local systems and higher Teichmuller theory*. 2006. arXiv: math/0311149 [math.AG]. URL: https://arxiv.org/abs/math/0311149v4.

[Sch06] Ralf Schiffler. *A geometric model for cluster categories of type D_n*. 2006. arXiv: math/0608264 [math.RT]. URL: https://arxiv.org/abs/math/0608264v3.

[Gin07] Victor Ginzburg. *Calabi-Yau algebras*. 2007. arXiv: math/0612139 [math.AG]. URL: https://arxiv.org/abs/math/0612139v3.

[IR07] Osamu Iyama and Idun Reiten. *Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras*. 2007. arXiv: math/0605136 [math.RT]. URL: https://arxiv.org/abs/math/0605136v3.

[Ami08] Claire Amiot. "Sur les petites catégories triangulées". In: (2008). PhD thesis, Université Denis Diderot - Paris 7.

[FST08] Sergey Fomin, Michael Shapiro, and Dylan Thurston. "Cluster algebras and triangulated surfaces. Part I: Cluster complexes". In: *Acta Mathematica, 201:83-146* (2008). URL: `https://arxiv.org/abs/math/0608367v3`.

[KS08] Maxim Kontsevich and Yan Soibelman. *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations.* 2008. arXiv: `0811.2435 [math.AG]`. URL: `https://arxiv.org/abs/0811.2435v1`.

[Ami09] Claire Amiot. *Cluster categories for algebras of global dimension 2 and quivers with potential.* 2009. arXiv: `0805.1035 [math.RT]`. URL: `https://arxiv.org/abs/0805.1035v2`.

[Kel09] Bernhard Keller. *Deformed Calabi-Yau Completions.* 2009. arXiv: `0908.3499 [math.RT]`. URL: `https://arxiv.org/abs/0908.3499v5`.

[Iya10] Osamu Iyama. "Cluster tilting for higher Auslander algebras". In: (2010). URL: `https://arxiv.org/abs/0809.4897v3`.

[IO10] Osamu Iyama and Steffen Oppermann. "n-representation-finite algebras and n-APR tilting". In: (2010). URL: `https://arxiv.org/abs/0909.0593v2`.

[OT11] Steffen Oppermann and Hugh Thomas. "Higher dimensional cluster combinatorics and representation theory". In: (2011). URL: `https://arxiv.org/abs/1001.5437v4`.

[GMN12] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. *Framed BPS States.* 2012. arXiv: `1006.0146 [hep-th]`. URL: `https://arxiv.org/abs/1006.0146v2`.

[Guo12] Lingyan Guo. "Catégories amassées supérieureset frises tropicales". In: (2012). PhD thesis, Université Paris Diderot - Paris 7.

[GKO13] Christof Geiss, Bernhard Keller, and Steffen Oppermann. "n-angulated categories". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2013.675 (Jan. 2013). URL: `https://doi.org/10.1515%2Fcrelle.2011.177`.

[HIO14] Martin Herschend, Osamu Iyama, and Steffen Oppermann. "n-representation infinite algebras". In: *Advances in Mathematics* 252 (2014), pp. 292–342. ISSN: 0001-8708. URL: `https://www.sciencedirect.com/science/article/pii/S0001870813003642`.

[DJW19] Tobias Dyckerhoff, Gustavo Jasso, and Tashi Walde. "Simplicial structures in higher Auslander–Reiten theory". In: *Advances in Mathematics* 355 (2019). Cited by: 12; All Open Access, Bronze Open Access, Green Open Access. URL: `https://www.scopus.com/inward/record.uri?eid=2-s2.0-85070678684&doi=10.1016%2fj.aim.2019.106762&partnerID=40&md5=70875d157ac34398c1f304716cf2121c`.

[DJL21] Tobias Dyckerhoff, Gustavo Jasso, and Yank Lekili. "The symplectic geometry of higher Auslander algebras: Symmetric products of disks". In: *Forum of Mathematics, Sigma* 9 (2021). Cited by: 8; All Open Access, Gold Open Access, Green Open Access. URL: `https://www.scopus.com/inward/record.uri?eid=2-s2.0-85102142135&doi=10.1017%2ffms.2021.2&partnerID=40&md5=843f2df0508cc166084e4d9f78c5e5a8`.

[GLL22] Jin Yun Guo, Xiaojian Lu, and Deren Luo. *$\mathbb{Z}Q$ type constructions in higher representation theory.* 2022. arXiv: `1908.06546 [math.RT]`. URL: `https://arxiv.org/abs/1908.06546v3`.