Chapter VII

Representation–finite hereditary algebras

As we saw in Chapter II, any basic and connected finite dimensional algebra A over an algebraically closed field K admits a presentation as a bound quiver algebra $A \cong KQ/\mathcal{I}$, where Q is a finite connected quiver and $\mathcal{I}$ is an admissible ideal of KQ. It is thus natural to study the representation theory of the algebras of the form $A \cong KQ$, that is, of the path algebras of finite, connected, and acyclic quivers. It turns out that an algebra A is of this form if and only if it is hereditary, that is, every submodule of a projective A-module is projective. We are thus interested in the representation theory of hereditary algebras. In [72], Gabriel showed that a connected hereditary algebra is representation–finite if and only if the underlying graph of its quiver is one of the Dynkin diagrams $\mathbb{A}_m$ with $m \geq 1$; $\mathbb{D}_n$ with $n \geq 4$; and $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, that appear also in Lie theory (see, for instance, [41]). Later, Bernstein, Gelfand, and Ponomarev [32] gave a very elegant and conceptual proof underlining the links between the two theories, by applying the nice concept of reflection functors. In this chapter, using reflection functors (which may now be thought of as tilting functors), we prove Gabriel's theorem and show how to compute all the (isomorphism classes of) indecomposable modules over a representation–finite hereditary algebra.

VII.1. Hereditary algebras

This introductory section is devoted to defining and giving various characterisations of hereditary algebras. In particular, we show that the hereditary algebras coincide with the path algebras of finite, connected, and acyclic quivers. Throughout, we let A denote a basic and connected finite dimensional algebra over an algebraically closed field K.

1.1. Definition. An algebra A is said to be **right hereditary** if any right ideal of A is projective as an A-module.

Left hereditary algebras are defined dually. It is not clear a priori whether a right hereditary algebra is also left hereditary, though we show in (1.4) that this is the case. The most obvious example of a right (and left) hereditary algebra is provided by the class of semisimple algebras; because any right (or left) module over a semisimple algebra is projective, then so is any right (or left, respectively) ideal of the algebra. On the other hand, let A be the full 2×2 lower triangular matrix algebra $A = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}$; see (I.2.4). Then, denoting by $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ the matrix idempotents, an immediate calculation shows that the only proper right ideals are $e_1 A$, $e_2 A$, and $e_{21} K = \left(\begin{smallmatrix} 0 & 0 \\ K & 0 \end{smallmatrix}\right) \cong e_1 A$, where $e_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Because $e_1 A$ and $e_2 A$ are direct summands of A_A, all these are projective A-modules and A is right hereditary.

The following theorem, due to Kaplansky [100], is fundamental. We warn the reader that, contrary to our custom, the modules we consider in (1.2)–(1.4) are not necessarily finitely generated.

1.2. Theorem. *Let A be a right hereditary algebra. Every submodule of a free A-module is isomorphic to a direct sum of right ideals of A.*

Proof. Let L be a free A-module with basis $(e_\lambda)_{\lambda \in \Lambda}$ and M be a submodule of L. We wish to show that M is isomorphic to a direct sum of right ideals of A. Without loss of generality, we may assume the index set Λ to be well-ordered. For each $\lambda \in \Lambda$, let $L_\lambda = \bigoplus_{\mu < \lambda} (e_\mu A)$. Then $L_0 = 0$ and $L_{\lambda+1} = \bigoplus_{\mu \leq \lambda} (e_\mu A) = L_\lambda \oplus (e_\lambda A)$. An element $x \in M \cap L_{\lambda+1}$ has a unique expression of the form $x = y + e_\lambda a$ with $y \in L_\lambda$ and $a \in A$. We may thus define an A-module homomorphism $f_\lambda : M \cap L_{\lambda+1} \to A$ by $x \mapsto a$, and hence we have a short exact sequence

$$0 \longrightarrow M \cap L_\lambda \longrightarrow M \cap L_{\lambda+1} \xrightarrow{f_\lambda} \operatorname{Im} f_\lambda \longrightarrow 0.$$

Because $\operatorname{Im} f_\lambda$ is a right ideal of the right hereditary algebra A, it is projective and the sequence splits. Hence there exists a submodule N_λ of $M \cap L_{\lambda+1}$, isomorphic to $\operatorname{Im} f_\lambda$ and such that $M \cap L_{\lambda+1} = (M \cap L_\lambda) \oplus N_\lambda$. To complete the proof, it suffices to show that $M \cong \bigoplus_{\lambda \in \lambda} N_\lambda$.

First, we show that M is equal to its submodule $N = \sum_{\lambda \in \Lambda} N_\lambda$. Because L equals the union of the increasing chain of submodules $(L_\lambda)_{\lambda \in \lambda}$, for each $x \in L$, there exists a least index $\lambda \in \Lambda$ such that $x \in L_{\lambda+1}$. Denote this index by μ_x. If $N \subsetneq M$, there exists $x \in M$ such that $x \notin N$. Let μ denote the least μ_x with $x \in M$, $x \notin N$ and take $y \in M$ such that $y \notin N$ and $\mu = \mu_y$. We have $y \in M \cap L_{\mu+1}$ hence $y = u + v$ with $u \in M \cap L_\mu$ and $v \in N_\mu$. Therefore $u = y - v \in M$ and $u \notin N$ (otherwise, $y \in N$, which is a contradiction). But, on the other hand, $u \in M \cap L_\mu$ gives $\mu_u < \mu$, and this contradicts the minimality of μ. Hence $M = \sum_{\lambda \in \Lambda} N_\lambda$.

There remains to show that the sum $\sum_{\lambda\in\Lambda} N_\lambda$ is direct. Assume that $x_1+\ldots+x_n=0$ with $x_i\in N_{\lambda_i}$, where we can suppose that $\lambda_1<\ldots<\lambda_n$. Then $x_1+\ldots+x_{n-1}=-x_n\in(M\cap L_{\lambda_n})\cap N_{\lambda_n}=0$ gives $x_n=0$. By descending induction, $x_i=0$ for each i. □

1.3. Corollary. *Let A be a right hereditary algebra. Every submodule of a projective A-module is projective.*

Proof. Indeed, any projective module is isomorphic to a direct summand of a free module. □

We are now able to state and prove our first characterisation of right hereditary algebras.

1.4. Theorem. *Let A be an algebra. The following conditions are equivalent:*

(a) *A is right hereditary.*
(b) *The global dimension of A is at most one.*
(c) *Every submodule of a projective right A-module is projective.*
(d) *Every quotient of an injective right A-module is injective.*
(e) *Every submodule of a finitely generated projective right A-module is projective.*
(f) *Every quotient of a finitely generated injective right A-module is injective.*
(g) *The radical of any indecomposable finitely generated projective right A-module is projective.*
(h) *The quotient of any indecomposable finitely generated injective right A-module by its socle is injective.*

Proof. (a) is equivalent to (c). Indeed, it follows from (1.3) that (a) implies (c). The converse is obvious.

(b) is equivalent to (c). If $\mathrm{gl.dim}\, A\leq 1$ and M_A is a submodule of a projective module P_A then, in the short exact sequence

$$0\longrightarrow M\longrightarrow P\longrightarrow P/M\longrightarrow 0,$$

we have $\mathrm{pd}\,(P/M)\leq 1$; hence, by (A.4.7) of the Appendix, M is projective. Conversely, if every submodule of a projective module is projective, let N be an arbitrary A-module. Then there exists a projective module P_A and an epimorphism $f: P\to N$. Because $\mathrm{Ker}\, f$ is a submodule of P, it is projective. Hence the exact sequence $0\longrightarrow \mathrm{Ker}\, f\longrightarrow P\xrightarrow{f} N\longrightarrow 0$ gives $\mathrm{pd}\, N\leq 1$. Consequently, $\mathrm{gl.dim}\, A\leq 1$.

Obviously, (c) implies (e) and (e) implies (a), because A_A is finitely generated as an A-module.

(e) is equivalent to (g). The necessity being obvious, let us show the sufficiency. Let P be a finitely generated projective A-module and M be a submodule of P. We prove that M is projective by induction on $d = \dim_K P$. If $d = 1$, there is nothing to show. Assume $d > 1$ and that the statement holds for every finitely generated projective A-module of dimension $< d$. The module P can be written in the form $P = P_1 \oplus P_2$, where P_1 is indecomposable and P_2 may be zero. Let $p : P \to P_1$ denote the canonical projection. If $p(M) = P_1$, then the composition of the injection $j : M \to P$ with $p : P \to P_1$ is an epimorphism and hence splits, because P_1 is projective. Therefore $M \cong P_1 \oplus M'$, where $M' \cong M \cap P_2 \subseteq P_2$. Because $\dim_K P_2 < d$, the induction hypothesis yields that M' is projective. Hence M is also projective. If $p(M) \neq P_1$, then $M \subseteq (\operatorname{rad} P_1) \oplus P_2$, where $\operatorname{rad} P_1$ is projective by hypothesis. Now $\dim_K [(\operatorname{rad} P_1) \oplus P_2] = d - 1$, because $\operatorname{rad} P_1$ is a maximal submodule of P_1. The induction hypothesis again implies that M is projective. The equivalence with the remaining conditions is proven similarly and left to the reader. □

Because condition (b) of the theorem is right-left symmetric (see (A.4.9) of the Appendix), it follows that a finite dimensional algebra is right hereditary if and only if it is left hereditary. Thus, from now on, we speak about hereditary algebras without further specification, and hereditary algebras also satisfy the "left-hand" analogues of the equivalent conditions of the theorem. On the other hand, conditions (e) to (h) show that we may revert to our custom of considering only finitely generated modules. From now on, the term *module* means, as usual, a finitely generated module.

1.5. Corollary. *Let A be a hereditary algebra.*

(a) *Any nonzero A-homomorphism between indecomposable projective A-modules is a monomorphism.*

(b) *If P is an indecomposable projective A-module, then* $\operatorname{End} P \cong K$.

Proof. Let $f : P \to P'$ be a nonzero homomorphism, with P and P' indecomposable projective. Because $\operatorname{Im} f \subseteq P'$ is projective, the short exact sequence $0 \to \operatorname{Ker} f \to P \to \operatorname{Im} f \to 0$ splits and $P \cong \operatorname{Im} f \oplus \operatorname{Ker} f$. Because the module P is indecomposable and $\operatorname{Im} f \neq 0$, $\operatorname{Ker} f = 0$ and f is a monomorphism, hence (a) follows. The statement (b) is an immediate consequence of (a). □

The following lemma is used repeatedly in the sequel. We first recall that if A is a K-algebra and M, N are indecomposable modules in $\operatorname{mod} A$, then $\operatorname{rad}_A(M,N)$ is the subspace of $\operatorname{Hom}_A(M,N)$ consisting of all nonisomorphisms, and the subspace $\operatorname{rad}^2_A(M,N)$ of $\operatorname{rad}_A(M,N)$ consists of the sums $f_1f'_1+\ldots+f_tf'_t$, where for each $i\in\{1,\ldots,,t\}$, $f'_i\in\operatorname{rad}_A(M,L_i)$ and $f_i\in\operatorname{rad}_A(L_i,N)$ for some indecomposable module L_i. The space of irreducible morphisms from M to N is then the K-vector space $\operatorname{Irr}(M,N)=\operatorname{rad}_A(M,N)/\operatorname{rad}^2_A(M,N)$. We use essentially the functorial isomorphism $\theta:\operatorname{Hom}_A(eA,M)\xrightarrow{\simeq}Me$, $f\mapsto f(e)$, established in (I.4.2).

1.6. Lemma. *Let A be a basic hereditary K-algebra and e, e' primitive idempotents of A. There exists an isomorphism of K-vector spaces*

$$\operatorname{Irr}(e'A,eA)\cong e(\operatorname{rad}A/\operatorname{rad}^2A)e'.$$

Proof. First we note that, because the canonical A-module projection $e(\operatorname{rad}A)e'\longrightarrow e(\operatorname{rad}A/\operatorname{rad}^2A)e'$ has kernel $e(\operatorname{rad}^2A)e'$, it induces a K-linear isomorphism $e(\operatorname{rad}A/\operatorname{rad}^2A)e'\cong e(\operatorname{rad}A)e'/e(\operatorname{rad}^2A)e'$.

We split the proof into two cases. Assume first that $e=e'$. By (1.5), any nonzero A-homomorphism $eA\to eA$ is injective, and hence is an isomorphism. Consequently, $\operatorname{Hom}_A(eA,eA)\cong K$ and $\operatorname{rad}_A(eA,eA)=0$ (so that $\operatorname{Irr}(eA,eA)=0$). On the other hand, $e(\operatorname{rad}A)e=\operatorname{rad}(eAe)=0$. This establishes the statement in this case.

Assume next that $e\neq e'$. Because A is basic, $eA\ncong e'A$ and therefore $\operatorname{rad}_A(e'A,eA)=\operatorname{Hom}_A(e'A,eA)\cong\operatorname{Hom}_A(e'A,\operatorname{rad}eA)$, because the idempotent e is primitive and $\operatorname{rad}eA$ is the unique maximal submodule of eA (by (I.4.5)). Because $\operatorname{rad}eA=eA(\operatorname{rad}A)$, it follows that the functorial isomorphism θ induces an isomorphism $\theta_1:\operatorname{rad}_A(e'A,eA)\longrightarrow(\operatorname{rad}eA)e'=e(\operatorname{rad}A)e'$. Similarly, the isomorphism θ induces another A-module isomorphism $\theta'_1:\operatorname{Hom}_A(e'A,e(\operatorname{rad}^2A))\longrightarrow e(\operatorname{rad}^2A)e'$. Denote by

$$e(\operatorname{rad}^2A)\xrightarrow{u}e(\operatorname{rad}A)\xrightarrow{v}eA$$

the inclusion homomorphisms. Then the functoriality of θ implies the commutativity of the following square

$$\begin{tikzcd}
\operatorname{rad}_A(e'A,eA) \arrow[r, "\theta_1", "\simeq"'] & e(\operatorname{rad}A)e' \\
\operatorname{Hom}_A(e'A,e(\operatorname{rad}^2A)) \arrow[u, hook, "j"] \arrow[r, "\theta'_1", "\simeq"'] & e(\operatorname{rad}^2A)e' \arrow[u, hook, "j'"']
\end{tikzcd}$$

where $j=\operatorname{Hom}_A(e'A,vu)$ and j' is the restriction of u to $e(\operatorname{rad}^2A)e'$.

We claim that the image of j is contained in $\operatorname{rad}^2_A(e'A,eA)$. Indeed, because A is hereditary, $\operatorname{rad}eA$ is projective. Because, clearly, no indecomposable summand of $\operatorname{rad}eA$ is isomorphic to eA, we have $v\in\operatorname{rad}_A(\operatorname{rad}eA,eA)$.

Similarly, $u \in \operatorname{rad}_A(\operatorname{rad}^2 eA, \operatorname{rad} eA)$. Consequently, $vu \in \operatorname{rad}_A^2(\operatorname{rad}^2 eA, eA)$ and, therefore, for any homomorphism $f \in \operatorname{Hom}_A(e'A, e(\operatorname{rad}^2 A))$ we have $vuf \in \operatorname{rad}_A^2(e'A, eA)$, because rad_A^2 defines a two-sided ideal in the category $\operatorname{mod} A$.

Next we claim that θ_1 maps the space $\operatorname{rad}_A^2(e'A, eA)$ into $e(\operatorname{rad}^2 A)e'$. Let $f \in \operatorname{rad}_A^2(e'A, eA)$. Then there exist indecomposable modules $L_1, \ldots, L_t$ in $\operatorname{mod} A$ and, for each $s \in \{1, \ldots, , t\}$, homomorphisms $f'_s \in \operatorname{rad}_A(e'A, L_s)$ and $f_s \in \operatorname{rad}_A(L_s, eA)$ such that $f = f_1 f'_1 + \ldots + f_t f'_t$. For any $s \in \{1, \ldots, , t\}$, the submodule $\operatorname{Im} f_s$ of the projective module eA is itself projective, because A is hereditary. Hence $\operatorname{Im} f_s$ is isomorphic to a direct summand of the indecomposable module L_s, so that $L_s \cong \operatorname{Im} f_s$ is projective. Therefore there exists a primitive idempotent e_s of A such that $L_s \cong e_s A$. Because θ induces isomorphisms $\operatorname{Hom}_A(e'A, e_s A) \cong e_s(\operatorname{rad} A)e'$ and $\operatorname{rad}_A(e'A, e_s A) \cong e_s(\operatorname{rad} A)e'$, we deduce that

$$\theta_1(f_s f'_s) \in e(\operatorname{rad} A)e_s \cdot e_s(\operatorname{rad} A)e' \subseteq e(\operatorname{rad}^2 A)e'.$$

This shows that $\theta(f) \in e(\operatorname{rad}^2 A)e'$ and, consequently, that θ_1 restricts to a linear map $\theta_2 : \operatorname{rad}_A^2(e'A, eA) \longrightarrow e(\operatorname{rad}^2 A)e'$. Therefore the previous square induces the following commutative diagram:

$$\begin{tikzcd}
\operatorname{rad}_A(e'A, eA) \arrow[r, "\theta_1", "\simeq"'] & e(\operatorname{rad} A)e' \\
\operatorname{rad}_A^2(e'A, eA) \arrow[u, hook] \arrow[r, "\theta_2"] & e(\operatorname{rad}^2 A)e' \arrow[u, hook, "j'"'] \\
\operatorname{Hom}_A(e'A, e(\operatorname{rad}^2 A)) \arrow[u, hook, "j"] \arrow[r, "\theta'_1", "\simeq"'] & e(\operatorname{rad}^2 A)e' \arrow[u, "1"']
\end{tikzcd}$$

It follows that θ_2 is bijective. Passing to the quotients yields

$$\operatorname{Irr}(e'A, eA) = \frac{\operatorname{rad}_A(e'A, eA)}{\operatorname{rad}_A^2(e'A, eA)} \cong \frac{e(\operatorname{rad} A)e'}{e(\operatorname{rad}^2 A)e'} \cong e\left(\frac{\operatorname{rad} A}{\operatorname{rad}^2 A}\right)e'.$$

The lemma is proved. □

Our next objective is to prove that an algebra is hereditary if and only if it is the path algebra of a finite, connected, and acyclic quiver.

1.7. Theorem. (a) *If Q is a finite, connected, and acyclic quiver, then the algebra $A = KQ$ is hereditary and $Q_A = Q$.*

(b) *If A is a basic, connected, hereditary algebra and $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A, then*

(i) *the quiver Q_A of A is finite, connected, and acyclic; and*

(ii) *there exists a K-algebra isomorphism $A \cong KQ_A$.*

Proof. (a) Let Q be a finite, connected, and acyclic quiver and let ε_a be the stationary path at $a \in Q_0$. To show that $A = KQ$ is hereditary, it suffices, by (1.4), to show that the radical $\operatorname{rad} P(a)$ of each indecomposable projective KQ-module $P(a) = \varepsilon_a KQ$ is itself projective. In view of (III.1.6), we identify modules X in $\operatorname{mod} KQ$ with K-linear representations $(X_b, \varphi_\beta)_{b \in Q_0, \beta \in Q_1}$ of Q.

Let $a \in Q_0$. By (III.2.4)(a), we have $P(a) = (P(a)_b, \varphi_\beta)$, where $P(a)_b = \varepsilon_a(KQ)\varepsilon_b$ has as a basis the set of all the paths from a to b, and for an arrow $\beta : b \to c$ in Q, the K-linear map $\varphi_\beta : P(a)_b \to P(a)_c$ is given by the right multiplication by β, hence it is injective. For $x, y \in Q_0$, let $w(x, y)$ denote the number of paths from x to y. We thus have $\dim_K P(a)_b = w(a, b)$. By (III.2.4)(b), $\operatorname{rad} P(a) = (J_b, \gamma_\beta)$ is a representation of Q with $J_b = P(a)_b$ for $b \neq a$, $J_a = 0$ and $\gamma_\beta = \varphi_\beta$ for any arrow β of source $b \neq a$.

Let $\{b_1, \dots, b_t\}$ be the set of all direct successors of a in Q, and n_i be the number of arrows from a to b_i (for $1 \leq i \leq t$). By (III.2.2)(d), the top of $\operatorname{rad} P(a)$ is isomorphic to $\bigoplus_{i=1}^t S(b_i)^{n_i}$; hence we have a projective cover $f : \bigoplus_{i=1}^t P(b_i)^{n_i} \longrightarrow \operatorname{rad} P(a)$. On the other hand, for $b \neq a$, there are K-linear isomorphisms

$$\begin{aligned} J_b = (\operatorname{rad} \varepsilon_a(KQ))\varepsilon_b &\cong \operatorname{Hom}_{KQ}(\varepsilon_b(KQ), \operatorname{rad} \varepsilon_a(KQ)) \\ &\cong \operatorname{Hom}_{KQ}(\varepsilon_b(KQ), \varepsilon_a(KQ)) \cong \varepsilon_a(KQ)\varepsilon_b = P(a)_b. \end{aligned}$$

Note that the existence of the isomorphism

$$\operatorname{Hom}_{KQ}(\varepsilon_b(KQ), \operatorname{rad} \varepsilon_a(KQ)) \cong \operatorname{Hom}_{KQ}(\varepsilon_b(KQ), \varepsilon_a(KQ))$$

is a consequence of the facts that $\varepsilon_a(KQ) \not\cong \varepsilon_b(KQ)$ and $\operatorname{rad} \varepsilon_a(KQ)$ is the unique maximal submodule of the right ideal $\varepsilon_a(KQ)$. Consequently, for any $b \neq a$ in Q, we have

$$\begin{aligned} \dim_K [\operatorname{rad} P(a)]_b &= \dim_K J_b = \dim_K P(a)_b = w(a, b) = \sum_{i=1}^t n_i w(b_i, b) \\ &= \sum_{i=1}^t n_i \dim_K P(b_i)_b = \dim_K \left[\bigoplus_{i=1}^t P(b_i)^{n_i} \right]_b . \end{aligned}$$

It follows that f is an isomorphism, and we are done.

Now we prove the statement (b).

(i) Because A is connected, its quiver Q_A of A is connected, by (II.3.4). We notice that to each arrow $\alpha : a \to b$ in Q_A corresponds an irreducible morphism $f_\alpha : e_b A \to e_a A$. By (1.5), f_α is a monomorphism and obviously $\operatorname{Im} f_\alpha \subseteq \operatorname{rad} e_a A$. To show that Q_A is acyclic, assume to the contrary that it is not and let $\alpha_1 \dots \alpha_t$ be a cycle in Q_A passing through a point a. Then $f = f_{\alpha_t} \dots f_{\alpha_1} : e_a A \to e_a A$ is a monomorphism, because each f_{α_i} is. But also $\operatorname{Im} f \subseteq \operatorname{rad} e_a A$. Hence $\dim_K e_a A = \dim_K \operatorname{Im} f \leq \dim_K \operatorname{rad} e_a A < \dim_K e_a A$, which is a contradiction.

(ii) By (II.3.7), there exists an admissible ideal $\mathcal{I}$ of KQ_A such that $A \cong KQ_A/\mathcal{I}$. We identify A with $KQ_A/\mathcal{I}$ and the idempotent $e_a \in A$ with the class $\overline{\varepsilon}_a = \varepsilon_a + \mathcal{I}$ of the stationary path ε_a at $a \in (Q_A)_0$. By (III.2.4), for each $a \in Q_0$, the corresponding indecomposable projective module $P(a) = e_a A$ is viewed as a representation of Q_A as follows: $P(a) = (P(a)_b, \varphi_\beta)$, $P(a)_b = P(a)e_b = e_a A e_b = e_a(KQ)e_b/e_a\mathcal{I}e_b$ is the K-vector space with basis the set of all $\overline{w} = w + \mathcal{I}$, where w is a path from a to b, and, for an arrow $\beta : b \to c$, the K-linear map $\varphi_\beta : P(a)_b \to P(a)_c$ is given by the right multiplication by $\overline{\beta} = \beta + \mathcal{I}$. Note that, because $\dim_K(\varepsilon_a KQ \varepsilon_b)$ equals the number $w(a,b)$ of paths from a to b in Q_A, $\dim_K P(a)e_b = w(a,b) - \dim_K \varepsilon_a \mathcal{I} \varepsilon_b$.

We show that $\mathcal{I} = 0$. Assume that this is not the case. Because, according to (i), the quiver Q_A is acyclic, we may number its points so that the existence of a path from x to y implies $x > y$. Then there is a least a such that there exists $b \in (Q_A)_0$ with $\varepsilon_a \mathcal{I} \varepsilon_b \neq 0$. In particular, a is not a sink, and so $\operatorname{rad} P(a) \neq 0$, by (III.2.4). Because A is hereditary, the nonzero module $\operatorname{rad} P(a)$ is projective, and therefore there exist $t \geq 1$, vertices $b_1, \ldots, b_t \in (Q_A)_0$, and positive integers $n_1, \ldots, n_t$ such that

$$\operatorname{rad} P(a) \cong P(b_1)^{n_1} \oplus \cdots \oplus P(b_t)^{n_t}.$$

It follows from (III.2.4), (IV.4.3), and (1.6) that $\{b_1, \ldots, b_t\}$ is the set of direct successors of a in Q_A and

$$n_i = \dim_K \operatorname{Irr}(P(b_i), P(a)) = \dim_K \varepsilon_a (\operatorname{rad} A/\operatorname{rad}^2 A)\varepsilon_b,$$

that is, n_i is the number of arrows from a to b_i in Q_A for i such that $1 \leq i \leq t$. The minimality of a implies that $\varepsilon_{b_i} \mathcal{I} \varepsilon_b = 0$ and $\dim_K P(b_i)\varepsilon_b = \dim_K \varepsilon_{b_i} A \varepsilon_b = w(b_i, b)$ for each b and each i. It follows that

$$\begin{aligned}\dim_K(\operatorname{rad} P(a))\varepsilon_b &= \sum_{i=1}^{t} n_i \dim_K P(b_i)\varepsilon_b = \sum_{i=1}^{t} n_i w(b_i, b) = w(a,b) \\ &> w(a,b) - \dim_K \varepsilon_a \mathcal{I} \varepsilon_b = \dim_K P(a)\varepsilon_b,\end{aligned}$$

and this is clearly a contradiction. The proof is complete. $\square$

We end this section with some remarks on the Auslander–Reiten translation and the Auslander–Reiten quiver of a hereditary algebra.

1.8. Lemma. *Let A be a hereditary algebra and M be an A-module. There exists a functorial isomorphism* $\operatorname{Tr} M \cong \operatorname{Ext}_A^1(M, A)$.

Proof. Because $\operatorname{gl.dim} A \leq 1$, a minimal projective resolution of the A-module M is of the form $0 \longrightarrow P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$. Applying the functor $(-)^t = \operatorname{Hom}_A(-, A)$, we obtain an exact sequence of left A-modules

$$0 \longrightarrow M^t \longrightarrow P_0^t \xrightarrow{f^t} P_1^t \longrightarrow \operatorname{Ext}_A^1(M, A) \longrightarrow 0.$$

The statement follows at once. □

Actually, the proof shows that the isomorphism $\operatorname{Tr} M \cong \operatorname{Ext}_A^1(M, A)$ holds whenever $\operatorname{pd} M \leq 1$. One consequence of this lemma is that the Auslander–Reiten translations $\tau = D\mathrm{Tr}$ and $\tau^{-1} = \operatorname{Tr} D$ are endofunctors of the module category $\operatorname{mod} A$ of a hereditary algebra A.

1.9. Corollary. *Let A be a hereditary algebra, and M be an A-module. There exist functorial isomorphisms*

$$\tau M \cong D\operatorname{Ext}_A^1(M, A) \quad \textit{and } \tau^{-1} M \cong \operatorname{Ext}_A^1(DM, A). \qquad \square$$

We also have the following easy characterisation of hereditary algebras by means of the Auslander–Reiten quiver.

1.10. Proposition. *Let A be an algebra and $\Gamma(\operatorname{mod} A)$ be its Auslander–Reiten quiver. The following conditions are equivalent:*

(a) *A is hereditary.*

(b) *The predecessors of the points in $\Gamma(\operatorname{mod} A)$ corresponding to the indecomposable projective modules correspond to indecomposable projective modules.*

(c) *The successors of the points in $\Gamma(\operatorname{mod} A)$ corresponding to the indecomposable injective modules correspond to indecomposable injective modules.*

Proof. We prove the equivalence of (a) and (b); the proof of the equivalence of (a) and (c) is similar.

For the necessity, let M be an immediate predecessor of an indecomposable projective P in $\Gamma(\operatorname{mod} A)$. Then there exists an irreducible morphism $f : M \longrightarrow P$. By (IV.1.10) and (IV.3.5), there exist a module N, an A-module isomorphism $h : M \oplus N \xrightarrow{\simeq} \operatorname{rad} P$, and a homomorphism $f' : N \longrightarrow P$ such that $[f \ f'] = jh$, where $j : \operatorname{rad} P \to P$ denotes the inclusion. Because A is hereditary, $\operatorname{rad} P$ is projective, hence the module M is projective. Consequently, every immediate predecessor of an indecomposable projective is an indecomposable projective module. The statement follows from an obvious induction. Note that, because $\Gamma(\operatorname{mod} A)$ contains only finitely many projectives, any indecomposable projective has only finitely many predecessors.

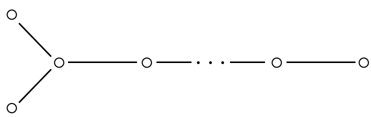
The sufficiency follows from the fact that the given condition implies that the radical of any indecomposable projective module is projective. □

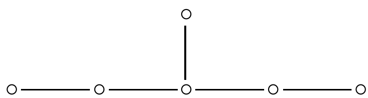
VII.2. The Dynkin and Euclidean graphs

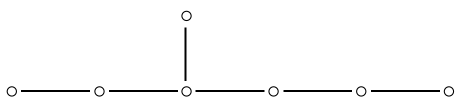
Certain graphs are of particular interest in this chapter (and the following ones).

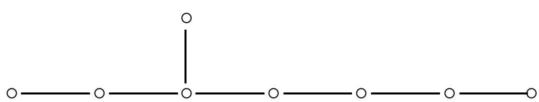
(a) The Dynkin graphs

$\mathbb{A}_m$:  $m \geq 1$

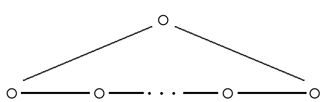
$\mathbb{D}_n$:  $n \geq 4$

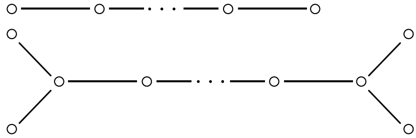
$\mathbb{E}_6$: 

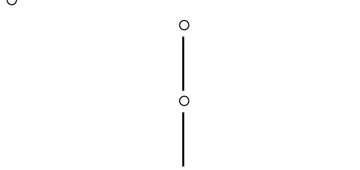
$\mathbb{E}_7$: 

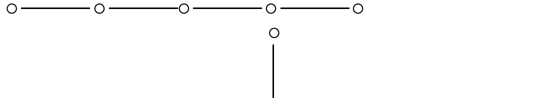
$\mathbb{E}_8$: 

(b) The Euclidean graphs

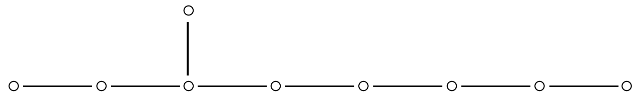
$\widetilde{\mathbb{A}}_m$:  $m \geq 1$

$\widetilde{\mathbb{D}}_n$:  $n \geq 4$

$\widetilde{\mathbb{E}}_6$: 

$\widetilde{\mathbb{E}}_7$: 

$\widetilde{\mathbb{E}}_8$:



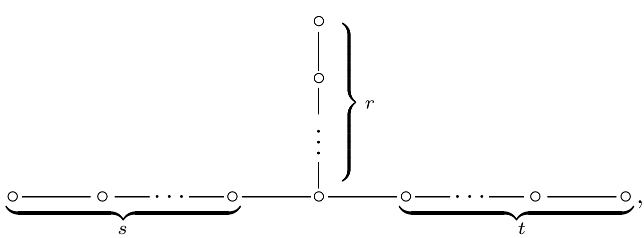
The index in the Dynkin graphs always refers to the number of points in the graph, whereas in the Euclidean, it refers to the number of points minus one (thus, $\mathbb{A}_m$ has m points while $\widetilde{\mathbb{A}}_m$ has $m+1$ points). In fact, a Euclidean graph can be constructed from the corresponding Dynkin graph by adding one point. Dynkin graphs and Euclidean graphs are also called *Dynkin diagrams* and *Euclidean diagrams*, respectively (see [41] and [72]).

We are interested in the path algebras of quivers having one of the preceding as underlying graph, that is, of quivers arising from arbitrary orientations of these graphs (excluding the orientation making $\widetilde{\mathbb{A}}_m$ an oriented cycle; this orientation gives an infinite dimensional path algebra). As pointed out in the introduction, the main result of this chapter says that the path algebra of a quiver Q is representation–finite if and only if the underlying graph $\overline{Q}$ of Q is a Dynkin graph.

We start with a purely combinatorial lemma.

2.1. Lemma. *Let Q be a finite, connected, and acyclic quiver. If the underlying graph $\overline{Q}$ of Q is not a Dynkin graph, then $\overline{Q}$ contains a Euclidean graph as a subgraph.*

Proof. We show that if $\overline{Q}$ contains no Euclidean subgraph, then $\overline{Q}$ is a Dynkin graph. The exclusion of $\widetilde{\mathbb{A}}_m$ implies that $\overline{Q}$ is a tree. The exclusion of $\widetilde{\mathbb{D}}_4$ implies that no point in $\overline{Q}$ has more than three neighbours, and the exclusion of $\widetilde{\mathbb{D}}_n$ with $n \geq 5$ implies that at most one point has three neighbours. Hence $\overline{Q}$ is of the following form



where we may assume without loss of generality that $r \leq s \leq t$. The exclusion of $\widetilde{\mathbb{E}}_6$ gives $r \leq 1$. If $r = 0$, then $\overline{Q} = \mathbb{A}_{s+t+1}$. If $r = 1$, the exclusion of $\widetilde{\mathbb{E}}_7$ gives $1 \leq s \leq 2$. If $s = 1$, then $\overline{Q} = \mathbb{D}_{t+3}$. Finally, if $s = 2$, the exclusion of $\widetilde{\mathbb{E}}_8$ gives $2 \leq t \leq 4$, so that $\overline{Q}$ is equal to $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$. $\square$

We use this lemma to show that if $A \cong KQ$ is representation–finite, then $\overline{Q}$ is a Dynkin graph. To do so, we start by showing that if Q' is a subquiver of Q such that KQ' is representation–infinite, then KQ itself is representation–infinite. It will then remain to show that if $\overline{Q'}$ is Euclidean, then KQ' is representation–infinite.

2.2. Lemma. *Let Q be a finite, connected, and acyclic quiver. If Q' is a subquiver of Q such that KQ' is representation–infinite, then KQ is representation–infinite.*

Proof. We must show that Q has at least as many nonisomorphic indecomposable representations as Q'. Let $M' = (M'_a, \varphi'_\alpha)$ be a representation of Q'. We define its extension $E(M')$ to be the representation (M_a, φ_α) of Q defined by

$$M_a = \begin{cases} M'_a & \text{if } a \in Q'_0, \\ 0 & \text{if } a \notin Q'_0, \end{cases} \qquad \text{and} \qquad \varphi_\alpha = \begin{cases} \varphi'_\alpha & \text{if } \alpha \in Q'_1, \\ 0 & \text{if } \alpha \notin Q'_1. \end{cases}$$

Given a morphism $f' : M' \to N'$ of representations of Q', where $M' = (M'_a, \varphi'_\alpha)$ and $N' = (N'_a, \psi'_\alpha)$, we define $f = E(f') : E(M') \to E(N')$ to be the morphism of representations of Q given by

$$f_a = \begin{cases} f'_a & \text{if } a \in Q'_0, \\ 0 & \text{if } a \notin Q'_0. \end{cases}$$

Clearly, E induces a full and faithful functor $\operatorname{mod} KQ' \to \operatorname{mod} KQ$ so that $\operatorname{End}_{KQ} E(M') \cong \operatorname{End}_{KQ'} M'$. In particular, $E(M')$ is indecomposable if and only if M' is indecomposable (see (I.4.8)), and we have $M' \cong N'$ if and only if $E(M') \cong E(N')$. $\square$

We now want to show that if Q is a quiver whose underlying graph is Euclidean, then KQ is representation–infinite. The first step in this direction is the following proposition.

2.3. Proposition. *Let Q be a finite, connected, and acyclic quiver. If KQ is representation–finite, then Q is a tree.*

Proof. Because Q has no loops, that is, it is not a tree, is equivalent to saying that Q contains a subquiver Q' with $\overline{Q'} = \widetilde{\mathbb{A}}_m$ for some $m \geq 1$. We show that, in this case, KQ' is representation–infinite. We may suppose that the points of Q' are numbered from 1 to $m+1$ and that there exists an arrow $\alpha : 1 \to 2$. For each scalar $\lambda \in K$, let $M(\lambda) = (M_i^{(\lambda)}, \varphi_\beta^{(\lambda)})$ be the representation of Q' defined as follows

$$M_i^{(\lambda)} = K \quad \text{for each } 1 \leq i \leq m+1$$

and

$$\varphi_\beta^{(\lambda)}(x) = \begin{cases} \lambda x & \text{if } \beta = \alpha, \\ x & \text{if } \beta \neq \alpha, \end{cases}$$

(that is, $\varphi_\beta^{(\lambda)}$ is the identity map for each arrow $\beta \neq \alpha$, and $\varphi_\alpha^{(\lambda)}$ is the multiplication by λ). Let $\lambda, \mu \in K$. We claim that each nonzero homomorphism $f : M(\lambda) \to M(\mu)$ is an isomorphism and, if this is the case, then $\lambda = \mu$ and $\operatorname{End} M(\lambda) \cong K$. Indeed, if $f : M(\lambda) \to M(\mu)$ is a nonzero homomorphism, then the commutativity relations

$$\begin{array}{ccc} M_i^{(\lambda)} & \xrightarrow{\varphi_\beta^{(\lambda)}} & M_j^{(\lambda)} \\ {\scriptstyle f_i}\Big\downarrow & & \Big\downarrow{\scriptstyle f_j} \\ M_i^{(\mu)} & \xrightarrow{\varphi_\beta^{(\mu)}} & M_j^{(\mu)} \end{array}$$

corresponding to all arrows $\beta : i \to j$ with $\beta \neq \alpha$ give $f_1 = \ldots = f_{m+1}$. In particular, $f \neq 0$ implies $f_i \neq 0$ for each i. Therefore, the map f_i, being a nonzero K-linear endomorphism of K is an isomorphism (and actually is the multiplication by a nonzero scalar). Finally, the commutativity condition corresponding to $\alpha : 1 \to 2$ gives

$$\mu f_1(1) = \varphi_\alpha^{(\mu)} f_1(1) = f_2 \varphi_\alpha^{(\lambda)}(1) = f_2(\lambda) = \lambda f_2(1).$$

Because $f_1 = f_2$ and both are nonzero, we have $\lambda = \mu$. On the other hand, f is entirely determined by $f_1(1)$. Because f_1 is the multiplication by a nonzero scalar ν (say), we deduce that $f : M(\lambda) \to M(\mu)$ is the map $\nu 1_{M(\lambda)}$. Thus $\operatorname{End} M(\lambda) \cong K$ and $M(\lambda)$ is indecomposable.

We have shown that the family $(M(\lambda))_{\lambda \in K}$ consists of pairwise nonisomorphic indecomposable representations. Because K is an algebraically closed (hence infinite) field, this gives an infinite family of pairwise nonisomorphic indecomposable representations of Q'. Therefore KQ' is representation–infinite. By (2.2), KQ is also representation–infinite. □

We have considered, in the preceding proof, representations M having the property that $\operatorname{End} M \cong K$. Such a representation carries a name.

2.4. Definition. Let A be a finite dimensional K-algebra. An A-module M such that $\operatorname{End} M \cong K$ is called a **brick**.

Clearly, each brick is an indecomposable module. On the other hand, there exist indecomposables that are not bricks. Let, for instance, A be a nonsimple local algebra (we may, for example, take $A = K[t]/\langle t^n \rangle$, with $n \geq 2$); then A_A is an indecomposable module that is not a brick, because $\operatorname{End} A_A \cong A \ncong K$. We showed in the proof of (2.3) that if Q' is a quiver with underlying graph $\widetilde{\mathbb{A}}_m$, with $m \geq 1$, then KQ' admits an infinite family of pairwise nonisomorphic bricks.

2.5. Proposition. *Let Q be a finite, connected, and acyclic quiver and M_{KQ} be a brick such that there exists $a \in Q_0$ with $\dim_K M_a > 1$. Let Q' be the quiver defined as follows: $Q' = (Q'_0, Q'_1)$, where $Q'_0 = Q_0 \cup \{b\}$; $Q'_1 = Q_1 \cup \{\alpha\}$; and $\alpha : b \to a$. Then KQ' is representation–infinite.*

Proof. Let $\psi : K \to M_a$ be a nonzero K-linear map. We define $M(\psi)$ to be the representation (M'_c, φ'_γ) of Q' given by the formulas:

$$M'_c = \begin{cases} M_c & \text{if } c \in Q_0, \\ K & \text{if } c = b \end{cases} \qquad \text{and} \qquad \varphi'_\gamma = \begin{cases} \varphi_\gamma & \text{if } \gamma \in Q_1, \\ \psi & \text{if } \gamma = \alpha. \end{cases}$$

Let $\psi, \eta : K \to M_a$ be nonzero K-linear maps and $f : M(\psi) \to M(\eta)$ be a nonzero morphism. Because the restriction $f|_M$ of f to M is an endomorphism of the brick M, $f|_M$ equals the multiplication by some scalar $\lambda \in K$. On the other hand, $f_b : M(\psi)_b \to M(\eta)_b$ is a K-linear endomorphism of K and hence it equals the multiplication by a scalar $\mu \in K$. Note that, because $f \neq 0$ and $\psi, \eta \neq 0$, we have $\lambda, \mu \neq 0$. Consider $x \in M(\eta)_b$ and the commutativity condition corresponding to the arrow α

$$\eta(x) = \eta f_b(x\mu^{-1}) = f_a \psi(x\mu^{-1}) = \psi(x) \cdot (\mu^{-1}\lambda)$$

$$\begin{tikzcd} M(\psi)_b \arrow[r, "\psi"] \arrow[d, "f_b"'] & M(\psi)_a \arrow[d, "f_a"] \\ M(\eta)_b \arrow[r, "\eta"] & M(\eta)_a \end{tikzcd}$$

Thus $\eta = \psi \cdot (\mu^{-1}\lambda)$.

This relation implies that each $M(\psi)$ is a brick. Indeed, setting $\psi = \eta$, we see that each endomorphism f of $M(\psi)$ equals the multiplication by a scalar: the preceding relation gives $\mu^{-1}\lambda = 1$; hence $\lambda = \mu$ and f is the multiplication by λ (or μ).

Assume now $f : M(\psi) \to M(\eta)$ is an isomorphism. The maps ψ and η are given by column matrices with $d = \dim_K M_a$ coefficients (and $d \geq 2$ by hypothesis), that is, $\psi = [\psi_1 \ldots \psi_d]^t$ and $\eta = [\eta_1 \ldots \eta_d]^t$. Hence $\eta = \psi \cdot (\mu^{-1}\lambda)$ yields $\eta_i = \psi_i \cdot (\mu^{-1}\lambda)$ for each $1 \leq i \leq d$. This can be expressed by saying that $(\psi_1, \ldots, \psi_d)$ and $(\eta_1, \ldots, \eta_d)$ correspond to the same point of the projective space $\mathbb{P}_{d-1}(K)$. Because K is an algebraically closed (hence infinite) field, $\mathbb{P}_{d-1}(K)$ has infinitely many points. We have thus shown the existence of infinitely many pairwise nonisomorphic bricks of the form $M(\psi)$. □

We apply this proposition as follows: For each of the Dynkin graphs $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$, we consider a quiver Q having it as underlying graph, and we show that there exists a brick M over KQ and a point $a \in Q_0$ such that $\dim_K M_a > 1$; applying the construction of the proposition yields that the path algebra of the corresponding enlarged quiver (whose underlying graph is Euclidean) is representation–infinite.

2.6. Lemma. *Let Q be one of the following quivers with underlying graph a Dynkin diagram:*

(i)
$$\begin{tikzcd}[column sep=small] & & & & & n-2 \arrow[dl] \\ n \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & n-3 \\ & & & & & n-1 \arrow[ul] \end{tikzcd} \qquad n \geq 4,$$

(ii)
$$\begin{tikzcd} & & 6 \arrow[d] & & \\ 5 \arrow[r] & 4 \arrow[r] & 3 & 2 \arrow[l] & 1 \arrow[l] \end{tikzcd}$$

(iii)
$$\begin{tikzcd} & & 5 \arrow[d] & & & \\ 7 \arrow[r] & 6 \arrow[r] & 4 & 3 \arrow[l] & 2 \arrow[l] & 1 \arrow[l] \end{tikzcd}$$

(iv)
$$\begin{tikzcd} & & 2 \arrow[d] & & & & \\ 4 \arrow[r] & 3 \arrow[r] & 1 & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l] \end{tikzcd}$$

Then there exists a brick M_{KQ} in mod KQ *such that* $\dim_K M_a > 1$, *where* $a \in Q_0$ *is the point* 1, 6, 7, *and* 8 *in cases* (i), (ii), (iii), *and* (iv), *respectively.*

Proof. We exhibit in each case the wanted brick $M = (M_b, \varphi_\beta)$ such that $\dim_K M_a > 1$.

(i) $M_1 = \ldots = M_{n-3} = K^2$, where K^2 is given its canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, $M_{n-1} = \mathbf{e}_1 K$, $M_{n-2} = \mathbf{e}_2 K$, and $M_n = (\mathbf{e}_1 + \mathbf{e}_2)K$. All the φ_β are taken to be the canonical inclusions. We claim that M is a brick with $\dim_K M_1 > 1$. Let $\overline{f} \in \operatorname{End} M_{KQ}$. The commutativity conditions give $\overline{f}_1 = \ldots = \overline{f}_{n-3} = f$ (say) and $\overline{f}_i = f|_{M_i}$ for $i = n-2, n-1, n$. Therefore $f(\mathbf{e}_1) \in \mathbf{e}_1 K$, $f(\mathbf{e}_2) \in \mathbf{e}_2 K$, and $f(\mathbf{e}_1 + \mathbf{e}_2) \in (\mathbf{e}_1 + \mathbf{e}_2)K$. Letting $f(\mathbf{e}_1) = \mathbf{e}_1 \lambda_1$, $f(\mathbf{e}_2) = \mathbf{e}_2 \lambda_2$ where $\lambda_1, \lambda_2 \in K$, we have

$$f(\mathbf{e}_1 + \mathbf{e}_2) = f(\mathbf{e}_1) + f(\mathbf{e}_2) = \mathbf{e}_1 \lambda_1 + \mathbf{e}_2 \lambda_2 \in (e_1 + e_2)K;$$

hence $\lambda_1 = \lambda_2$ and therefore f is a multiplication by the scalar λ_1. This shows that M is indeed a brick with $\dim_K M_1 \geq 2$.

(ii) $M_3 = K^3$, where K^3 is given its canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $M_1 = \mathbf{e}_1 K$, $M_2 = \mathbf{e}_1 K \oplus \mathbf{e}_2 K$, $M_4 = \mathbf{e}_2 K \oplus \mathbf{e}_3 K$, $M_6 = (\mathbf{e}_1 + \mathbf{e}_2)K \oplus (\mathbf{e}_2 + \mathbf{e}_3)K$,

$M_5 = \mathbf{e}_3K$. All the φ_β are taken to be the canonical inclusions. We observe that $M_2 \cap M_4 = \mathbf{e}_2K$, $M_4 \cap M_6 = (\mathbf{e}_2 + \mathbf{e}_3)K$, $M_2 \cap M_6 = (\mathbf{e}_1 + \mathbf{e}_2)K$. We claim that M is a brick with $\dim_K M_6 > 1$. Let $\overline{f} \in \operatorname{End} M_{KQ}$. Then $\overline{f}_i = f|_{M_i}$ where $f = \overline{f}_3 \in \operatorname{End}_K M_3$. Because $f(M_i) \subseteq M_i$ for $1 \le i \le 6$, $f(M_2 \cap M_4) \subseteq f(M_2) \cap f(M_4) \subseteq M_2 \cap M_4$. Similarly, $f(M_4 \cap M_6) \subseteq M_4 \cap M_6$ and $f(M_2 \cap M_6) \subseteq M_2 \cap M_6$. Thus, there exist $\lambda_1, \lambda_2, \lambda_3, \mu, \nu \in K$ such that

$$f(\mathbf{e}_1) = \mathbf{e}_1\lambda_1, \quad f(\mathbf{e}_2) = \mathbf{e}_2\lambda_2, \quad f(\mathbf{e}_3) = \mathbf{e}_3\lambda_3,$$
$$f(\mathbf{e}_1 + \mathbf{e}_2) = (\mathbf{e}_1 + \mathbf{e}_2)\mu, \quad f(\mathbf{e}_2 + \mathbf{e}_3) = (\mathbf{e}_2 + \mathbf{e}_3)\nu.$$

Hence $\lambda_1 = \mu = \lambda_2 = \nu = \lambda_3$ and f equals the multiplication by their common value. This shows that M is a brick such that $\dim_K M_6 \ge 2$.

(iii) $M_4 = K^4$, where K^4 is given its canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, $M_1 = \mathbf{e}_1K$, $M_2 = \mathbf{e}_1K \oplus \mathbf{e}_2K$, $M_3 = \mathbf{e}_1K \oplus \mathbf{e}_2K \oplus \mathbf{e}_3K$, $M_7 = (\mathbf{e}_2 - \mathbf{e}_3)K \oplus (\mathbf{e}_1 + \mathbf{e}_4)K$, $M_6 = (\mathbf{e}_1 + \mathbf{e}_2)K \oplus (\mathbf{e}_1 + \mathbf{e}_3)K \oplus (\mathbf{e}_1 + \mathbf{e}_4)K$, $M_5 = \mathbf{e}_3K \oplus \mathbf{e}_4K$. All the φ_β are taken to be the canonical inclusions. We observe that $M_3 \cap M_5 = \mathbf{e}_3K$, $M_2 \cap M_6 = (\mathbf{e}_1 + \mathbf{e}_2)K$, $M_5 \cap M_6 = (\mathbf{e}_3 - \mathbf{e}_4)K$, $M_3 \cap M_7 = (\mathbf{e}_2 - \mathbf{e}_3)K$, $M_7 \cap (M_1 + M_5) = (\mathbf{e}_1 + \mathbf{e}_4)K$, $M_6 \cap [M_1 + (M_3 \cap M_5)] = (\mathbf{e}_1 + \mathbf{e}_3)K$. We claim that M is a brick with $\dim_K M_7 > 1$. Let $\overline{f} \in \operatorname{End} M_{KQ}$. As earlier, we show that $\overline{f}_i = f|_{M_i}$ for $1 \le i \le 7$, where $f \in \operatorname{End}_K M_4$ is such that there exist $\lambda_1, \lambda_3, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \in K$ satisfying the following conditions:

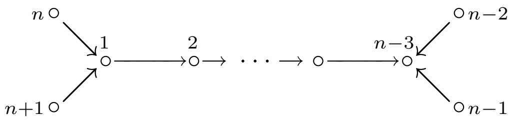
$$\begin{aligned} f(\mathbf{e}_1) &= \mathbf{e}_1\lambda_1, & f(\mathbf{e}_3) &= \mathbf{e}_3\lambda_3, \\ f(\mathbf{e}_3 - \mathbf{e}_4) &= (\mathbf{e}_3 - \mathbf{e}_4)\mu_1, & f(\mathbf{e}_1 + \mathbf{e}_2) &= (\mathbf{e}_1 + \mathbf{e}_2)\mu_2, \\ f(\mathbf{e}_2 - \mathbf{e}_3) &= (\mathbf{e}_2 - \mathbf{e}_3)\mu_3, & f(\mathbf{e}_1 + \mathbf{e}_4) &= (\mathbf{e}_1 + \mathbf{e}_4)\mu_4, \\ f(\mathbf{e}_1 + \mathbf{e}_3) &= (\mathbf{e}_1 + \mathbf{e}_3)\mu_5. \end{aligned}$$

A straightforward calculation shows that f is indeed the multiplication by a scalar. Hence M is a brick such that $\dim_K M_7 \ge 2$.

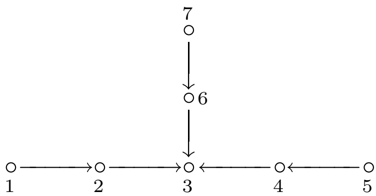
(iv) $M_1 = K^6$, where K^6 is given its canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_6\}$, $M_2 = (\mathbf{e}_4 + \mathbf{e}_6)K \oplus (\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_5)K \oplus (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4)K$, $M_3 = \mathbf{e}_1K \oplus \mathbf{e}_2K \oplus \mathbf{e}_3K \oplus \mathbf{e}_6K$, $M_4 = \mathbf{e}_1K \oplus \mathbf{e}_6K$, $M_5 = \mathbf{e}_1K \oplus \mathbf{e}_2K \oplus \mathbf{e}_3K \oplus \mathbf{e}_4K \oplus \mathbf{e}_5K$, $M_6 = \mathbf{e}_2K \oplus \mathbf{e}_3K \oplus \mathbf{e}_4K \oplus \mathbf{e}_5K$, $M_7 = \mathbf{e}_3K \oplus \mathbf{e}_4K \oplus \mathbf{e}_5K$, $M_8 = \mathbf{e}_4K \oplus \mathbf{e}_5K$. All the φ_β are taken to be the canonical inclusions. We observe that $M_4 \cap M_5 = \mathbf{e}_1K$, $M_3 \cap M_7 = \mathbf{e}_3K$, $M_3 \cap M_6 = \mathbf{e}_2K \oplus \mathbf{e}_3K$, $M_2 \cap M_3 = (\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_6)K$, $(M_4 + M_8) \cap M_2 = (\mathbf{e}_4 + \mathbf{e}_6)K$ and $M_2 \cap M_6 = (\mathbf{e}_2 - \mathbf{e}_3 + \mathbf{e}_4 - \mathbf{e}_5)K$. Let $\overline{f} \in \operatorname{End} M_{KQ}$. As earlier, we show that $\overline{f}_i = f|_{M_i}$ for $1 \le i \le 8$, where $f \in \operatorname{End}_K M_1$. Moreover, the subspaces $M_4 \cap M_5$, $M_3 \cap M_7$, $M_3 \cap M_6$, $(M_4 + M_8) \cap M_2$, and $M_2 \cap M_6$ of K^6 are invariant under f. A straightforward calculation shows that if f is given in the canonical basis $\mathbf{e}_1, \dots, \mathbf{e}_6$ by a

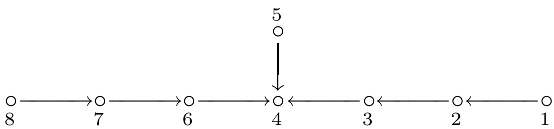
6×6 matrix $[a_{ij}]$, then $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66}$ and $a_{ij} = 0$ for any $i \neq j$, and so f is a multiplication by the scalar a_{11}. Therefore M is a brick with $\dim_K M_8 \geq 2$. □

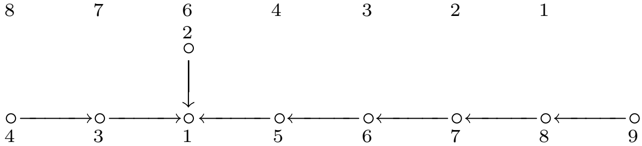
2.7. Corollary. *The path algebra of each of the following quivers is representation-infinite:*

(i) 

$n \geq 4$

(ii) 

(iii) 

(iv) 

Proof. This follows at once from (2.5) and (2.6). □

We have shown in this section that if KQ is representation–finite, then Q is a tree (that is, $\overline{Q}$ contains no subgraph of the form $\widetilde{\mathbb{A}}_m$, for some $m \geq 1$) and contains no subquiver of one of the forms listed in (2.7). This does **not** yet imply that Q contains no subquivers whose underlying graph is Euclidean. Indeed, there remains to show that if Q is a tree, KQ is representation–infinite and Q' is a quiver such that $\overline{Q'} = \overline{Q}$ (that is, Q' has the same underlying graph as Q, but perhaps a different orientation), then KQ' is also representation–infinite. To prove this, we need to develop some new concepts.

VII.3. Integral quadratic forms

When studying hereditary algebras, it turns out that the Euler quadratic form, that is, the quadratic form arising from the Euler characteristic (see (III.3.11)) plays a prominent rôle. This quadratic form is an integral quadratic form, and this section is devoted to studying integral quadratic forms in general. Throughout, we denote by $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ the canonical basis of the free abelian group $\mathbb{Z}^n$ on n generators. As usual, elements in $\mathbb{Z}^n$ are written as column vectors.

3.1. Definition. A quadratic form $q = q(x_1, \dots, x_n)$ on $\mathbb{Z}^n$ in n indeterminates $x_1, \dots, x_n$ is said to be an **integral quadratic form** if it is of the form

$$q(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} a_{ij} x_i x_j$$

where $a_{ij} \in \mathbb{Z}$ for all i, j.

Evaluating an integral quadratic form q on the vectors $\mathbf{x} = [x_1 \, \dots \, x_n]^t$ in $\mathbb{Z}^n$, we obtain a mapping from $\mathbb{Z}^n$ to $\mathbb{Z}$, also denoted by q. We may endow $\mathbb{Z}^n$ with a partial order defined componentwise: a vector $\mathbf{x} = [x_1 \, \dots \, x_n]^t \in \mathbb{Z}^n$ is called **positive** if $\mathbf{x} \neq 0$ and $x_j \geq 0$, for all j such that $1 \leq j \leq n$. We denote the positivity of a vector $\mathbf{x}$ as $\mathbf{x} > 0$. An integral quadratic form q is called **weakly positive** if $q(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$; it is called **positive semidefinite** if $q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{Z}^n$, and **positive definite** if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$; finally, it is called **indefinite** if there exists a nonzero vector $\mathbf{x}$ such that $q(\mathbf{x}) < 0$. For a positive semidefinite form q, the set

$$\operatorname{rad} q = \{\mathbf{x} \in \mathbb{Z}^n \mid q(\mathbf{x}) = 0\}$$

is called the **radical** of q, and its elements are called **radical vectors**. It is a subgroup of $\mathbb{Z}^n$. Indeed, if $q(\mathbf{x}) = 0 = q(\mathbf{y})$, then

$$q(\mathbf{x}+\mathbf{y}) + q(\mathbf{x}-\mathbf{y}) = 2[q(\mathbf{x}) + q(\mathbf{y})] = 0 \text{ gives } q(\mathbf{x}+\mathbf{y}) = q(\mathbf{x}-\mathbf{y}) = 0,$$

by the positive semidefiniteness of q, and hence $\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y} \in \operatorname{rad} q$.

The rank of the subgroup $\operatorname{rad} q$ is called the **corank** of q. Clearly, q is positive definite if and only if its corank is zero.

3.2. Examples. (a) The integral quadratic form

$$q(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3 + x_2x_3$$

on $\mathbb{Z}^3$ is weakly positive, positive semidefinite of corank 1 (hence is not positive definite). Indeed, $q(\mathbf{x}) = (x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3)^2 + \frac{3}{4}(x_2 + x_3)^2$ so that $\operatorname{rad} q$ is generated by the vector $[1 \; 1 \; -1]^t$. This implies our claim.

(b) The integral quadratic form $q(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1x_2 = (x_1 - x_2)^2$ on $\mathbb{Z}^2$ is positive semidefinite of corank 1 and $\operatorname{rad} q$ is generated by the vector $[1 \; 1]^t$. In particular, q is not weakly positive.

We denote by $(-,-)$ the **symmetric bilinear form** on $\mathbb{Z}^n$ corresponding to q, that is, for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$, we have

$$(\mathbf{x}, \mathbf{y}) = \frac{1}{4}[q(\mathbf{x}+\mathbf{y}) - q(\mathbf{x}-\mathbf{y})].$$

For instance, if q is as in Example 3.2 (b), we have

$$(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1.$$

It is easily seen that the following relations hold:

(a) $q(\mathbf{x}) = (\mathbf{x}, \mathbf{x})$ for all $\mathbf{x} \in \mathbb{Z}^n$;
(b) $a_{ij} = 2(\mathbf{e}_i, \mathbf{e}_j)$ for all i, j such that $1 \le i < j \le n$; and $a_{ji} = 2(\mathbf{e}_i, \mathbf{e}_j)$ for all i, j such that $1 \le j < i \le n$;
(c) $q(\mathbf{x}+\mathbf{y}) = q(\mathbf{x}) + q(\mathbf{y}) + 2(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$.

We also define the n **partial derivatives** of the quadratic form q to be the group homomorphisms from $\mathbb{Z}^n$ to $\mathbb{Z}$ defined by:

$$D_i q(x) = \frac{\partial q}{\partial x_i}(x) = 2(\mathbf{e}_i, \mathbf{x}) = 2x_i + \sum_{i<t} a_{it}x_t + \sum_{t<i} a_{ti}x_t$$

for each i such that $1 \le i \le n$.

3.3. Lemma. *Let q be a positive semidefinite quadratic form on $\mathbb{Z}^n$. Then $q(\mathbf{x}) = 0$ if and only if $D_i q(\mathbf{x}) = 0$ for all i such that $1 \le i \le n$.*

Proof. If $D_i q(\mathbf{x}) = 0$ for all i, then $(\mathbf{e}_i, \mathbf{x}) = 0$ for all i. Consequently, $q(\mathbf{x}) = (\mathbf{x}, \mathbf{x}) = \sum_{i=1}^n x_i(\mathbf{e}_i, \mathbf{x}) = 0$.

Conversely, assume that $q(\mathbf{x}) = 0$. For all $\lambda \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^n$, we have $q(\lambda\mathbf{y}) = \lambda^2 q(\mathbf{y})$. Because, by hypothesis, $q(\mathbf{y}) \ge 0$ for all $\mathbf{y} \in \mathbb{Z}^n$, we have $q(\mathbf{y}) \ge 0$ for all $\mathbf{y} \in \mathbb{Q}^n$. The continuity of q and the density of $\mathbb{Q}^n$ in $\mathbb{R}^n$ imply that $q(\mathbf{y}) \ge 0$ for all $\mathbf{y} \in \mathbb{R}^n$. Thus $q(\mathbf{x}) = 0$ if and only if the function $q : \mathbb{R}^n \to \mathbb{R}$ admits a global minimum at $\mathbf{x}$: the partial derivatives must then vanish at this point. $\square$

Let q be an integral quadratic form on $\mathbb{Z}^n$. A vector $\mathbf{x} \in \mathbb{Z}^n$ such that $q(\mathbf{x}) = 1$ is called a **root** of q. All the vectors of the canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbb{Z}^n$ are clearly roots of q. The reason for studying roots is that, as we shall see, over a representation–finite hereditary algebra, there exists a bijection between the positive roots of the Euler quadratic form and the isomorphism classes of indecomposable modules. The following fundamental result, due to Drozd [59], shows that weakly positive quadratic forms have only finitely many roots that are positive vectors of $\mathbb{Z}^n$.

3.4. Proposition. *Let q be a weakly positive integral quadratic form on $\mathbb{Z}^n$. Then q has only finitely many positive roots.*

Proof. We consider q as a function from $\mathbb{R}^n$ to $\mathbb{R}$. As in the proof of (3.3), we see that $q(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$ in $\mathbb{Q}^n$ and hence $q(\mathbf{x}) \geq 0$ for all $\mathbf{x} > 0$ in $\mathbb{R}^n$. We show by induction on n that in fact $q(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$ in $\mathbb{R}^n$.

This is trivial if $n = 1$ because if $\lambda \in \mathbb{R}$, $\lambda \neq 0$, then $q(\lambda) = \lambda^2 q(1) > 0$. Assume that there exists a weakly positive quadratic form q in n indeterminates (with $n \geq 2$) and a positive vector $\mathbf{x} \in \mathbb{R}^n$ such that $q(\mathbf{x}) = 0$. It follows from the induction hypothesis that we can assume all the components x_i of $\mathbf{x}$ to be strictly positive. Then $\mathbf{x}$ lies in the positive cone of $\mathbb{R}^n$ and q attains a local minimum at $\mathbf{x}$. Consequently, we have $D_1 q(\mathbf{x}) = \ldots = D_n q(\mathbf{x}) = 0$. The linear forms $D_i q$ have integral, hence rational, coefficients, and $\mathbf{x} \in \bigcap_{i=1}^{n} \operatorname{Ker} D_i q$ implies that the real vector space

$$V = \{\mathbf{z} \in \mathbb{R}^n \mid D_1 q(\mathbf{z}) = \ldots = D_n q(\mathbf{z}) = 0\}$$

is nonzero . Hence the rank of the $n \times n$ matrix (with rational coefficients) determining this system of linear equations is smaller than n. Thus the rational vector space

$$U = \{\mathbf{y} \in \mathbb{Q}^n \mid D_1 q(\mathbf{y}) = \ldots = D_n q(\mathbf{y}) = 0\}$$

is nonzero, and V has a basis contained in U. In particular, V is the closure of U, because $\mathbb{Q}$ is dense in $\mathbb{R}$. Therefore, there exists a positive vector $\mathbf{x}'$ with rational coefficients lying in $\bigcap\limits_{i=1}^{n} \operatorname{Ker} D_i q$. But then $q(\mathbf{x}') = 0$ because of (3.3) and the fact that $D_i q(\mathbf{x}') = 0$ for all $1 \leq i \leq n$, and this contradicts the fact that $q(\mathbf{x}') > 0$ because $\mathbf{x}' \in \mathbb{Q}^n$ is a positive vector. This completes the proof of our claim that $q(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$ in $\mathbb{R}^n$.

Let now $\| - \| : \mathbb{R}^n \to \mathbb{R}$ denote the Euclidean norm. Because the set $C = \{x \in \mathbb{R}^n \mid \mathbf{x} > 0,\ \|\mathbf{x}\| = 1\}$ is compact in $\mathbb{R}^n$, $q|_C$ attains its minimum μ on a point of C. It follows from the preceding discussion that $\mu > 0$. For each $\mathbf{x} > 0$ in $\mathbb{R}^n$, we have

$$\mu \leq q\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = \frac{1}{\|\mathbf{x}\|^2} q(\mathbf{x}).$$

Consequently, $\|\mathbf{y}\| \leq \frac{1}{\sqrt{\mu}}$ for each positive root $\mathbf{y}$ of q. Thus, q has only finitely many positive roots. $\square$

3.5. Corollary. *A weakly positive integral quadratic form always admits maximal positive roots.* □

Let $\mathbf{x} = \sum_{i=1}^{n} x_i e_i$ be a vector in $\mathbb{Z}^n$. Its **support** is the subset of $\{1, \ldots, n\}$ defined by $\operatorname{supp}\mathbf{x} = \{i \mid 1 \leq i \leq n,\ x_i \neq 0\}$.

3.6. Lemma. *Let q be a weakly positive integral quadratic form on $\mathbb{Z}^n$ and $\mathbf{x}$ be a positive root of q such that $\mathbf{x} \neq \mathbf{e}_i$ for all i. Then there exists $i \in \operatorname{supp}\mathbf{x}$ such that $D_i q(\mathbf{x}) = 1$.*

Proof. We have $\sum_{i=1}^{n} x_i D_i q(\mathbf{x}) = 2\sum_{i=1}^{n} x_i(\mathbf{e}_i, \mathbf{x}) = 2(\mathbf{x}, \mathbf{x}) = 2$; hence there exists i such that $x_i D_i q(\mathbf{x}) \geq 1$. Because $\mathbf{x} > 0$, we have $x_i \geq 1$ and $D_i q(\mathbf{x}) \geq 1$. Therefore, $i \in \operatorname{supp}\mathbf{x}$. Because $\mathbf{x} \neq \mathbf{e}_i$ by hypothesis, $\mathbf{x} - \mathbf{e}_i > 0$ and

$$0 < q(\mathbf{x} - \mathbf{e}_i) = q(\mathbf{x}) + q(\mathbf{e}_i) - 2(\mathbf{e}_i, \mathbf{x}) = 2 - D_i q(\mathbf{x})$$

gives $D_i q(\mathbf{x}) < 2$. Consequently, $D_i q(\mathbf{x}) = 1$. □

Let q be an integral quadratic form on $\mathbb{Z}^n$ and let $(-,-)$ be the corresponding symmetric bilinear form on $\mathbb{Z}^n$. For each i with $1 \leq i \leq n$, we define a mapping $s_i : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$s_i(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}, \mathbf{e}_i)\mathbf{e}_i.$$

Such a mapping is called a **reflection** at i. Note that $s_i(\mathbf{e}_i) = -\mathbf{e}_i$: that is, s_i transforms $\mathbf{e}_i$ to its negative. The properties of reflections are summarised in the following lemma.

3.7. Lemma. *Let $s_i : \mathbb{Z}^n \to \mathbb{Z}^n$ be a reflection. Then*

(a) *s_i is a group homomorphism;*
(b) *$(s_i(\mathbf{x}), s_i(\mathbf{y})) = (\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$; and*
(c) *$s_i^2 = 1$, thus s_i is an automorphism of $\mathbb{Z}^n$.*

Proof. (a) This is evident.

(b) $$(s_i(\mathbf{x}), s_i(\mathbf{y})) = (\mathbf{x}, \mathbf{y}) - 2(\mathbf{x}, \mathbf{e}_i)(\mathbf{y}, \mathbf{e}_i) - 2(\mathbf{y}, \mathbf{e}_i)(\mathbf{x}, \mathbf{e}_i) + 4(\mathbf{x}, \mathbf{e}_i)(\mathbf{y}, \mathbf{e}_i) = (\mathbf{x}, \mathbf{y}).$$

(c) $s_i(s_i(\mathbf{x})) = s_i(\mathbf{x} - 2(\mathbf{x}, \mathbf{e}_i)\mathbf{e}_i) = \mathbf{x} - 2(\mathbf{x}, \mathbf{e}_i)\mathbf{e}_i + 2(\mathbf{x}, \mathbf{e}_i)\mathbf{e}_i = \mathbf{x}$. □

3.8. Lemma. *Let q be a weakly positive integral quadratic form on $\mathbb{Z}^n$ and $\mathbf{x}$ be a positive root of q such that $\mathbf{x} \neq \mathbf{e}_i$ for all i. Then there exists $i \in \operatorname{supp}\mathbf{x}$ such that $s_i(\mathbf{x}) = \mathbf{x} - \mathbf{e}_i$ is still a positive root.*

Proof. By (3.6), there exists $i \in \operatorname{supp}\mathbf{x}$ such that $D_i q(\mathbf{x}) = 1$. Now $D_i q(\mathbf{x}) = 2(\mathbf{x}, \mathbf{e}_i)$ so that $s_i(\mathbf{x}) = \mathbf{x} - \mathbf{e}_i > 0$. □

3.9. Corollary. *Let q be a weakly positive integral quadratic form on $\mathbb{Z}^n$ and $\mathbf{x}$ be a positive root of q. There exists a sequence $i_1, \ldots, i_t$, j of elements of $\{1, \ldots, n\}$ such that*

$$\mathbf{x} > s_{i_1}(\mathbf{x}) > s_{i_2}s_{i_1}(\mathbf{x}) > \ldots > s_{i_t} \ldots s_{i_1}(\mathbf{x}) = \mathbf{e}_j.$$

Proof. This follows at once from (3.8) and induction. □

3.10. Definition. Let q be a weakly positive integral quadratic form on $\mathbb{Z}^n$. The subgroup W_q of the automorphism group of $\mathbb{Z}^n$ generated by the reflections $s_1, \ldots, s_n$ is called the **Weyl group** of q. A root $\mathbf{x}$ of q is called a **Weyl root** if there exist $w \in W_q$ and i with $1 \leq i \leq n$ such that $\mathbf{x} = w\mathbf{e}_i$.

It follows from (3.9) and (3.7)(c) that every positive root $\mathbf{x}$ of a weakly positive integral quadratic form q can be written as $\mathbf{x} = s_{i_1} \ldots s_{i_t}\mathbf{e}_j$: that is, every positive root of a weakly positive form is a Weyl root.

As is shown later, this applies to the Euler quadratic form for the representation–finite hereditary algebras; in this case, the form is positive definite, hence weakly positive, and therefore all positive roots are Weyl roots.

We end this section with an observation due to Happel [86] showing that the converse to (3.4) also holds.

3.11. Proposition. *Let q be an integral quadratic form having only finitely many positive roots. Then q is weakly positive.*

Proof. Let q be an integral quadratic form on $\mathbb{Z}^n$. Suppose that q is not weakly positive. Then $n \geq 2$ and there exists a positive vector $\mathbf{x} = [x_1 \ldots x_n]^t \in \mathbb{Z}^n$ such that $q(\mathbf{x}) \leq 0$. Because any restriction of q to a smaller number of indeterminates has also finitely many positive roots, we may assume that $x_i > 0$ for all i with $1 \leq i \leq n$. Clearly, we may also assume that $q(\mathbf{x}') > 0$ for any vector $\mathbf{x}' \in \mathbb{Z}^n$ with $0 < \mathbf{x}' < \mathbf{x}$. By our assumption on q, we may also choose a maximal positive root $\mathbf{y}$ of q. Then $(\mathbf{y}, \mathbf{e}_i) \geq 0$ for all i with $1 \leq i \leq n$, because, by (3.7), the reflections $s_i(\mathbf{y}) = \mathbf{y} - 2(\mathbf{y}, \mathbf{e}_i)\mathbf{e}_i$ are also roots of q. We claim that $(\mathbf{x}, \mathbf{y}) > 0$. Indeed, if $(\mathbf{x}, \mathbf{y}) \leq 0$ then $\sum_{i=1}^n x_i(\mathbf{e}_i, \mathbf{y}) \leq 0$, and hence $(\mathbf{e}_i, \mathbf{y}) = (\mathbf{y}, \mathbf{e}_i) = 0$ for all i with $1 \leq i \leq n$. But then we get $1 = q(\mathbf{y}) = (\mathbf{y}, \mathbf{y}) = \sum_{i=1}^n y_i(\mathbf{e}_i, \mathbf{y}) = 0$, a contradiction. Therefore, $\sum_{i=1}^n y_i(\mathbf{x}, \mathbf{e}_i) = (\mathbf{x}, \mathbf{y}) > 0$ and there exists i with $1 \leq i \leq n$ such that $(\mathbf{x}, \mathbf{e}_i) > 0$, because $\mathbf{y} > 0$. Take now $\mathbf{z} = \mathbf{x} - \mathbf{e}_i$. Then $\mathbf{z} > 0$ and $q(\mathbf{z}) = q(\mathbf{x} - \mathbf{e}_i) = q(\mathbf{x}) + q(\mathbf{e}_i) - 2(\mathbf{x}, \mathbf{e}_i) = 2 - 2(\mathbf{x}, \mathbf{e}_i) \leq 0$. This contradicts our choice of $\mathbf{x}$. Thus, q is weakly positive. □

VII.4. The quadratic form of a quiver

Throughout this section, we let Q denote a finite, connected, and acyclic quiver. If we let $n = |Q_0|$ denote the number of points in Q, it follows from (III.3.5) that the Grothendieck group $K_0(KQ)$ of the path algebra KQ is isomorphic to $\mathbb{Z}^n$. We denote, as usual, by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the canonical basis of $\mathbb{Z}^n$. It is sometimes convenient to work in a $\mathbb{Q}$-vector space rather than in the abelian group $\mathbb{Z}^n$. For this purpose, we denote by E the $\mathbb{Q}$-vector space

$$E = K_0(KQ) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^n$$

and by F the subgroup of E consisting of the vectors having only integral coordinates, that is,

$$F = \bigoplus_{i=1}^{n} \mathbf{e}_i\mathbb{Z} \cong \mathbb{Z}^n \cong K_0(KQ).$$

The **quadratic form of a quiver** Q is defined to be the form

$$q_Q(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)},$$

where $\mathbf{x} = [x_1 \ \ldots \ x_n]^t \in \mathbb{Z}^n$.

Our first objective is to describe the Euler quadratic form of KQ by means of the quadratic form q_Q.

A first, but important, observation is that q_Q depends only on the underlying graph $\overline{Q}$ of Q, not on the particular orientation of the arrows in Q.

4.1. Lemma. *Let Q be a finite, connected, and acyclic quiver. Then the Euler quadratic form q_A of the path algebra $A = KQ$ and the quadratic form q_Q of the quiver Q coincide. Moreover,*

$$q_A(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{i,j \in Q_0} a_{ij} x_i x_j,$$

where $a_{ij} = \dim_K \mathrm{Ext}^1_A(S(i), S(j))$.

Proof. By (III.3.13), the Euler characteristic is the bilinear form defined on the dimension vectors of the simple KQ-modules $S(i)$ by:

$$\begin{aligned}\langle \mathbf{dim}\, S(i), \mathbf{dim}\, S(j) \rangle &= \sum_{l \geq 0} (-1)^l \dim_K \mathrm{Ext}^l_{KQ}(S(i), S(j)) \\ &= \dim_K \mathrm{Hom}_{KQ}(S(i), S(j)) - \dim_K \mathrm{Ext}^1_{KQ}(S(i), S(j)),\end{aligned}$$

because, by (1.4) and (1.7), $\mathrm{gl.dim}\, KQ \leq 1$. Because there are no loops in Q at i, by (III.2.12), $\dim_K \mathrm{Ext}^1_{KQ}(S(i), S(j))$ equals the number a_{ij} of arrows from i to j.

Taking $i = j$, we get $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = \langle \mathbf{dim}\, S(i), \mathbf{dim}\, S(i) \rangle = 1$. On the other hand, if $i \neq j$, we get

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle \mathbf{dim}\, S(i), \mathbf{dim}\, S(j) \rangle = -\dim_K \mathrm{Ext}^1_{KQ}(S(i), S(j)) = -a_{ij}.$$

Hence, for two arbitrary vectors $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_{i=1}^n y_i \mathbf{e}_i$, we get

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \sum_{i,j=1}^{n} x_i y_j \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{i,j \in Q_0} a_{ij} x_i y_j \\ &= \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}. \end{aligned}$$

The result follows at once. □

The Euler quadratic form of the algebra KQ will be simply referred to as the quadratic form of the quiver Q.

We denote by $(-,-)$ the symmetric bilinear form corresponding to q_Q, that is, the symmetrisation of the Euler characteristic. Thus,

$$(\mathbf{x}, \mathbf{y}) = \sum_{i \in Q_0} x_i y_i - \frac{1}{2} \sum_{\alpha \in Q_1} \{ x_{s(\alpha)} y_{t(\alpha)} + x_{t(\alpha)} y_{s(\alpha)} \}.$$

This can also be expressed in terms of the Cartan matrix $\mathbf{C}_{KQ}$; indeed, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t (\mathbf{C}_{KQ}^{-1})^t \mathbf{y}$, hence

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t \left[\frac{1}{2} (\mathbf{C}_{KQ}^{-1} + (\mathbf{C}_{KQ}^{-1})^t) \right] \mathbf{y}.$$

Clearly, $(\mathbf{x}, \mathbf{x}) = q_Q(\mathbf{x})$ for all $\mathbf{x}$, and $(\mathbf{x}, \mathbf{y}) = \frac{1}{4}[q_Q(\mathbf{x}+\mathbf{y}) - q_Q(\mathbf{x}-\mathbf{y})]$ for all $\mathbf{x}$, $\mathbf{y}$.

For example, if Q is the quiver

$$\begin{tikzcd}[column sep=huge] 1 \circ & \circ\, 2 \arrow[l, bend right=40, "\alpha_1"'] \arrow[l, "\alpha_2"', "\vdots"] \arrow[l, bend left=40, "\alpha_m"] \end{tikzcd}$$

then $q_Q(\mathbf{x}) = x_1^2 + x_2^2 - m x_1 x_2 = (x_1 - \frac{m}{2} x_2)^2 + (1 - \frac{m^2}{4}) x_2^2$. Consequently, q_Q is positive definite if $m = 1$, semidefinite of corank 1 if $m = 2$, and indefinite if $m \geq 3$. Observe also that for $m \geq 2$ and $\mathbf{x} = (m, m)^t$ we have $q_Q(\mathbf{x}) \leq 0$, and hence q_Q is not weakly positive.

We saw in Section 3 that if q_Q is positive semidefinite then its radical $\operatorname{rad} q_Q = \{x \in F;\ q_Q(x) = 0\}$ is a subgroup of $F \cong \mathbb{Z}^n$. After tensoring by ${}_{\mathbb{Z}}\mathbb{Q}$, it yields a subspace of the $\mathbb{Q}$-vector space

$$E = K_0(KQ) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^n,$$

denoted by $(\operatorname{rad} q_Q)\mathbb{Q}$. The dimension of this subspace $(\operatorname{rad} q_Q)\mathbb{Q}$ equals the corank of q_Q. The following purely computational lemma provides many examples of quivers with positive semidefinite form.

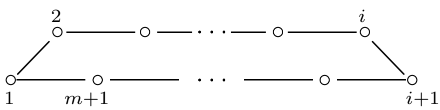
4.2. Lemma. *Let Q be a quiver whose underlying graph $\overline{Q}$ is Euclidean. Then q_Q is positive semidefinite of corank one and* $\operatorname{rad} q_Q = \mathbb{Z}\mathbf{h}_Q$, *where* $\mathbf{h}_Q$ *is the vector*

$$\begin{smallmatrix} & 1 & \cdots & 1 & \\ 1 & 1 & \cdots & 1 & 1 \end{smallmatrix},\quad \begin{smallmatrix} 1 & & & & 1 \\ & 2 & \cdots & 2 & \\ 1 & & & & 1 \end{smallmatrix},\quad \begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix},\quad \begin{smallmatrix} & & & 2 & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{smallmatrix},\quad \text{and}\quad \begin{smallmatrix} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{smallmatrix}$$

in case $\overline{Q}$ is the graph $\widetilde{\mathbb{A}}_m$, $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, and $\widetilde{\mathbb{E}}_8$, respectively.

Proof. (i) Assume that

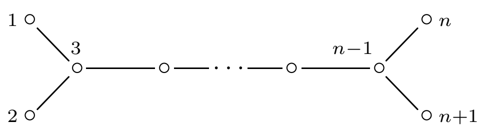
$\overline{Q} = \widetilde{\mathbb{A}}_m$:



for some $m \geq 1$. Then $2q_Q(\mathbf{x}) = \sum_{i-j} (x_i - x_j)^2$, where the sum is taken over all edges i—j in $\overline{Q}$. It follows that q_Q is positive semidefinite of corank 1 and a generator of $\operatorname{rad} q_Q$ is given by $\begin{smallmatrix} & 1 & \cdots & 1 & \\ 1 & 1 & \cdots & 1 & 1 \end{smallmatrix}$.

(ii) Assume that

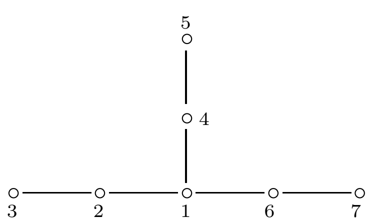
$\overline{Q} = \widetilde{\mathbb{D}}_n$:



for some $n \geq 4$. Then $4q_Q(\mathbf{x}) = (2x_1 - x_3)^2 + (2x_2 - x_3)^2 + (x_{n-1} - 2x_n)^2 + (x_{n-1} - 2x_{n+1})^2 + 2\sum_{i=3}^{n-2}(x_i - x_{i+1})^2$. It follows that q_Q is positive semidefinite of corank 1 and a generator of $\operatorname{rad} q_Q$ is given by $\begin{smallmatrix} 1 & & & & 1 \\ & 2 & \cdots & 2 & \\ 1 & & & & 1 \end{smallmatrix}$.

(iii) Assume that

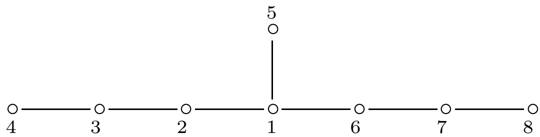
$\overline{Q} = \widetilde{\mathbb{E}}_6$:



Then $36q_Q(\mathbf{x}) = (6x_3-3x_2)^2+(6x_7-3x_6)^2+(6x_5-3x_4)^2+3[(3x_2-2x_1)^2+(3x_6-2x_1)^2+(3x_4-2x_1)^2]$. It follows that q_Q is positive semidefinite of corank 1 and a generator of $\operatorname{rad} q_Q$ is given by $\begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$.

(iv) Assume that

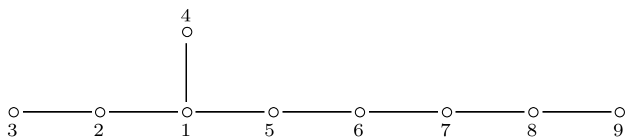
$\overline{Q} = \widetilde{\mathbb{E}}_7$:



Then $24q_Q(\mathbf{x}) = 6[(2x_4-x_3)^2+(2x_8-x_7)^2]+2[(3x_3-2x_2)^2+(3x_7-2x_6)^2]+(4x_2-3x_1)^2+(4x_6-3x_1)^2+6(2x_5-x_1)^2$. Here, q_Q is positive semidefinite of corank 1, a generator of $\operatorname{rad} q_Q$ is given by $\begin{smallmatrix} & & & 2 & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{smallmatrix}$.

(v) Assume that

$\overline{Q} = \widetilde{\mathbb{E}}_8$:



Then $120q_Q(\mathbf{x}) = 30(2x_9-x_8)^2+10(3x_8-2x_7)^2+5(4x_7-3x_6)^2+3(5x_6-4x_5)^2+30(2x_3-x_2)^2+2(6x_5-5x_1)^2+10(3x_2-2x_1)^2+30(2x_4-x_1)^2$. It follows that q_Q is positive semidefinite of corank 1 and a generator of $\operatorname{rad} q_Q$ is given by $\begin{smallmatrix} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{smallmatrix}$. $\square$

We show later that the Dynkin and Euclidean graphs can in fact be characterised by the positivity of their quadratic forms. We need the following lemma.

4.3. Lemma. *Let Q be a connected quiver such that q_Q is positive semidefinite and Q' be a proper full subquiver of Q. Then the restriction $q_{Q'}$ of q_Q to Q' is positive definite.*

Proof. The form $q_{Q'}$ is certainly positive semidefinite, for every full subquiver Q' of Q. Let then Q' be a proper full subquiver of Q such that $q_{Q'}$ is not positive definite. We may, without loss of generality, assume Q' to be minimal with this property. Let $\mathbf{x}' = \sum x'_i \mathbf{e}_i$ be a nonzero vector such that $q_{Q'}(\mathbf{x}') = 0$. The minimality of Q' implies that $x'_i \neq 0$ for each $i \in Q'_0$. Actually, because $q_{Q'}$ is positive semidefinite, we may suppose that $x'_i > 0$ for each $i \in Q'_0$; indeed, the vector $\mathbf{x}'' = \sum |x'_i| \mathbf{e}_i$ satisfies $q_{Q'}(\mathbf{x}'') \leq q_{Q'}(\mathbf{x}')$.

Let $j \in Q_0 \setminus Q'_0$ be a neighbour of $k \in Q'_0$ (such points j, k certainly exist, because Q' is a proper full subquiver of the connected quiver Q). We define a vector $\mathbf{x} = \sum x_i \mathbf{e}_i$ in $E = \mathbb{Q}^n$ by the formula

$$x_i = \begin{cases} x'_i & \text{if } i \in Q'_0, \\ \frac{1}{2}x'_k & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $q_Q(\mathbf{x}) = q_Q(\mathbf{x}' + x_j\mathbf{e}_j) = q_{Q'}(\mathbf{x}') + x_j^2 - \sum_{l - j} x'_l x_j = x_j^2 - \sum_{l - j} x'_l x_j \leq x_j^2 - x'_k x_j = \frac{1}{4}x'^2_k - \frac{1}{2}x'^2_k = -\frac{1}{4}x'^2_k < 0$, which is a contradiction. □

4.4. Corollary. *Let Q be a quiver whose underlying graph is Dynkin. Then q_Q is positive definite.*

Proof. This follows from (4.2), (4.3), and the observation that each quiver whose underlying graph is Dynkin is a proper full subquiver of a quiver whose underlying graph is Euclidean. □

We are now able to prove the characterisation of the Dynkin and Euclidean graphs by means of their quadratic forms.

4.5. Proposition. *Let Q be a finite, connected, and acyclic quiver and let $\overline{Q}$ be the underlying graph of Q.*

(a) *$\overline{Q}$ is a Dynkin graph if and only if q_Q is positive definite.*
(b) *$\overline{Q}$ is a Euclidean graph if and only if q_Q is positive semidefinite but not positive definite.*
(c) *$\overline{Q}$ is neither a Dynkin nor a Euclidean graph if and only if q_Q is indefinite.*

Proof. The necessity of (a) follows from (4.4) and the necessity of (b) follows from (4.2). Conversely, assume q_Q to be positive semidefinite. Then it follows from the example preceding (4.2) that $\overline{Q}$ does not contain a full subgraph consisting of two points connected by more than two edges. Hence, if $\overline{Q}$ is not Dynkin, then, by (2.1), $\overline{Q}$ contains a Euclidean graph as a full subgraph. By (4.3), this Euclidean subgraph cannot be proper. Hence $\overline{Q}$ is Euclidean. This shows (a) and (b).

Let Q be such that $\overline{Q}$ is neither a Dynkin nor a Euclidean graph. By (a) and (b), q_Q is not positive semidefinite. Consequently, it is indefinite. The converse follows clearly from the sufficiency parts of (a) and (b). □

We may clearly strengthen condition (b) as follows: $\overline{Q}$ is a Euclidean graph if and only if q_Q is positive semidefinite of corank one.

4.6. Corollary. *Let Q be a finite, connected, and acyclic quiver. The following conditions are equivalent:*

(a) *q_Q is weakly positive.*

(b) *q_Q is positive definite.*
(c) *The underlying graph $\overline{Q}$ of Q is a Dynkin graph.*

Proof. We have seen that (b) and (c) are equivalent, and (b) implies (a) trivially. Assume that q_Q is weakly positive. Then again $\overline{Q}$ does not contain a full subgraph consisting of two vertices connected by at least two edges. Hence, if q_Q is not positive definite, then $\overline{Q}$ is not Dynkin so that, by (2.1), Q contains a full subquiver Q' whose underlying graph is Euclidean. We computed in (4.2) generators for the (one-dimensional) radical subspaces of the forms arising from Euclidean graphs. Let $\mathbf{x}'$ be the generator of the radical subspace of $q_{Q'}$. As seen in (4.2), $\mathbf{x}'$ is positive. Consider the vector $\mathbf{x}$ defined by

$$\mathbf{x}_i = \begin{cases} x'_i & \text{if } i \in Q'_0, \\ 0 & \text{if } i \notin Q'_0. \end{cases}$$

Clearly, $\mathbf{x}$ is positive and $q_Q(\mathbf{x}) = 0$. Thus q_Q is not weakly positive. $\square$

A consequence of this corollary and the results of Section 3 is that if $\overline{Q}$ is a Dynkin graph, then the positive roots of q_Q are Weyl roots and there are only finitely many such positive roots. We thus proceed to define reflections and the Weyl roots for the quadratic form q_Q of a finite, connected, and acyclic quiver Q. We recall that $E = \mathbb{Q}^n$ and $F = \mathbb{Z}^n$. For each point $i \in Q_0$, we define the **reflection** $s_i : E \to E$ at i to be the $\mathbb{Q}$-linear map given by

$$s_i(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}, \mathbf{e}_i)\mathbf{e}_i$$

for $\mathbf{x} \in E$. In terms of the coordinates x_i of $\mathbf{x}$ in the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of E, we see that $\mathbf{y} = s_i(\mathbf{x})$ has coordinates

$$y_j = \begin{cases} x_j & \text{if } j \neq i, \\ -x_i + \sum\limits_{k - i} x_k & \text{if } j = i, \end{cases}$$

where the sum is taken over all edges k—i. Because $s_i(F) \subseteq F$, we see that s_i is indeed a reflection in the sense of Section 3.

For example, if Q is the quiver

$$\overset{1}{\circ} \longleftarrow \overset{3}{\circ} \longrightarrow \overset{2}{\circ}$$

whose underlying graph is the Dynkin graph $\mathbb{A}_3$, then $E \cong \mathbb{Q}^3$ and the reflections s_1, s_2, s_3 are expressed by their matrices in the canonical basis as

$$\mathbf{s}_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{s}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{s}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

The **Weyl group** W_Q of Q is the Weyl group of the quadratic form q_Q, that is, the group of automorphisms of $E = \mathbb{Q}^n$ generated by the set of reflections $\{s_i\}_{i \in Q_0}$.

Because, by hypothesis, Q is acyclic, there exists a bijection between Q_0 and the set $\{1, \dots, n\}$ such that if we have an arrow $j \to i$, then $j > i$; indeed, such a bijection is constructed as follows. Let 1 be any sink in Q, then consider the full subquiver $Q(1)$ of Q having as set of points $Q_0 \setminus \{1\}$; let 2 be a sink of $Q(1)$, and continue by induction. Such a numbering of the points of Q is called an **admissible numbering**. For instance, in the preceding example, the shown numbering of the points is admissible. Clearly, a given quiver Q usually admits many possible admissible numberings of the set of points.

Let $(a_1, \dots, a_n)$ be an admissible numbering of the points of Q and let $E = \mathbb{Q}^n$. The element

$$c = s_{a_n} \dots s_{a_2} s_{a_1} : E \longrightarrow E$$

of the Weyl group W_Q of Q is called the **Coxeter transformation** of Q (corresponding to the given admissible numbering). Because, for each i, we have $s_{a_i}^2 = 1$, clearly, $c^{-1} = s_{a_1} s_{a_2} \dots s_{a_n}$. For instance, in the example, the matrices of c and c^{-1} in the canonical basis are

$$\mathbf{c} = \mathbf{s}_3 \mathbf{s}_2 \mathbf{s}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

and

$$\mathbf{c}^{-1} = \mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_3 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

It turns out that the Coxeter transformation only depends on the quiver Q, not on the admissible numbering chosen. Indeed, if $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are two admissible numberings of the points of Q, then there exists an i with $1 \leq i \leq n$ such that $b_1 = a_i$; because b_1 is a sink, there exists no edge $a_j — a_i$ with $j < i$ and, because it is easily seen that reflections corresponding to non-neighbours commute, we have $s_{a_j} s_{a_i} = s_{a_i} s_{a_j}$ for all $j < i$. The numbering $(a_i, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ is admissible and an obvious induction implies that $s_{a_n} \dots s_{a_1} = s_{b_n} \dots s_{b_1}$. We thus refer to c as being the Coxeter transformation of the quiver Q.

The matrix of the Coxeter transformation c, as defined earlier, is just the Coxeter matrix $\mathbf{\Phi}_{KQ}$ of KQ, as defined in (III.3.14).

4.7. Proposition. *The matrix of the Coxeter transformation $c : E \to E$ of a quiver Q in the canonical basis of E is equal to the Coxeter matrix $\mathbf{\Phi}_{KQ}$ of KQ.*

Proof. We recall that $\mathbf{\Phi}_{KQ} = -\mathbf{C}_{KQ}^t \mathbf{C}_{KQ}^{-1}$, where $\mathbf{C}_{KQ}$ denotes the Cartan matrix of KQ. Assume that $(1, \ldots, n)$ is an admissible numbering of Q_0. Identifying the reflections s_i and the Coxeter transformation c to their matrices in the canonical basis of the $\mathbb{Q}$-vector space $E = \mathbb{Q}^n$, we must show that $-\mathbf{C}_{KQ}^t \mathbf{C}_{KQ}^{-1} = s_n \ldots s_1$. For this purpose, it suffices to show that $-\mathbf{C}_{KQ}^t = s_n \ldots s_1 \mathbf{C}_{KQ}$, or, equivalently, that

$$\mathbf{C}_{KQ}^t s_1^t \ldots s_n^t = -\mathbf{C}_{KQ}.$$

We show by induction on k that

$$\mathbf{C}_{KQ}^t s_1^t \ldots s_k^t = [-\mathbf{C}_k \mid \mathbf{C}_{n-k}^t],$$

where $\mathbf{C}_k$ (or $\mathbf{C}_{n-k}^t$) is the matrix formed by the k first columns of $\mathbf{C}_{KQ}$ (or of the $(n-k)$ last columns of $\mathbf{C}_{KQ}^t$, respectively). Recall that $c_{ij} = \dim_K \varepsilon_j (KQ) \varepsilon_i$ is the (i,j)-coefficient of $\mathbf{C}_{KQ}$. Moreover, let a_{ij} be the number of arrows from j to i. It is easily seen that:

(1) $a_{ij} = 0$ for $i \geq j$ (because $(1, \ldots, n)$ is an admissible ordering of Q_0);
(2) $c_{i,i+1} = a_{i,i+1}$, for each i;
(3) $c_{ii} = 1$, for each i; and
(4) $c_{ij} = \sum_{i \leq k \leq j} a_{ik} c_{kj}$, for $i < j$.

For $k = 1$, we then have

$$\begin{aligned} \mathbf{C}_{KQ}^t s_1^t &= \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ c_{12} & 1 & 0 & \ldots & 0 \\ c_{13} & c_{23} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & c_{3n} & \ldots & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & \ldots & 0 \\ a_{12} & 1 & 0 & \ldots & 0 \\ a_{13} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & 0 & 0 & \ldots & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 & \ldots & 0 \\ -c_{12} + a_{12} & 1 & 0 & \ldots & 0 \\ -c_{13} + a_{12}c_{23} + a_{13} & c_{23} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{1n} + \sum_{1 \leq i \leq n} a_{1i} c_{in} & c_{2n} & c_{3n} & \ldots & 1 \end{bmatrix}. \end{aligned}$$

Using (2), (3) and (4), we get $\mathbf{C}_{KQ}^t s_1^t = [-\mathbf{C}_1 \mid \mathbf{C}_{n-1}^t]$.

Assume the result to hold for $k-1$. Then

$$\mathbf{C}_{KQ}^t s_1^t \dots s_k^t = [-\mathbf{C}_{k-1} \mid \mathbf{C}_{n-k+1}^t] s_k^t$$

$$= \begin{bmatrix} -1 & -c_{12} & \dots & -c_{1,k-2} & -c_{1,k-1} & 0 & 0 & \dots & 0 \\ 0 & -1 & \dots & -c_{2,k-2} & -c_{2,k-1} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -1 & -c_{k-2,k-1} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & c_{k,k+1} & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & c_{k,n} & c_{k+1,n} & \dots & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 & a_{1k} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & a_{2k} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & a_{k-1,k} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} & -\sum_{1\le i\le k} a_{ik}c_{1i} & \\ & -\sum_{2\le i\le k} a_{ik}c_{2i} & \\ & \vdots & \\ -\mathbf{C}_{k-1} & -\sum_{k-2\le i\le k} a_{ik}c_{k-2,i} & \mathbf{C}_{n-k}^t \\ & -a_{k-1,k} & \\ & -1 & \\ & 0 & \\ & \vdots & \\ & 0 & \end{array} \right].$$

The conclusion follows from (2), (3), and (4). $\square$

For the rest of this section, we assume that Q is a quiver whose underlying graph $\overline{Q}$ is Dynkin. Then q_Q is positive definite and hence weakly positive. We denote by R, R^+, R^-, $R(W_Q)$, respectively, the sets of all roots, all positive roots, all negative roots, and all Weyl roots of q_Q. It follows from (3.4) that R^+ is a finite set and, from (3.9), that $R^+ \subseteq R(W_Q)$. We note that, if $\mathbf{x} \in F = \mathbb{Z}^n$ is a root, the vector $-\mathbf{x}$ is also a root, because $q_Q(-\mathbf{x}) = q_Q(\mathbf{x})$. In particular, the assignment $\mathbf{x} \mapsto -\mathbf{x}$ induces a bijection between R^+ and R^- (so that R^- is also finite).

4.8. Lemma. *Let Q be a quiver whose underlying graph is Dynkin. Then $R = R^+ \cup R^- = R(W_Q)$.*

Proof. To show that $R = R^+ \cup R^-$, it suffices to show that every root $\mathbf{x}$ of q_Q is either positive or negative. We may write $\mathbf{x} = \mathbf{x}^+ + \mathbf{x}^-$, where $\mathbf{x}^+$ is a vector all of whose nonzero coordinates are positive, while $\mathbf{x}^-$ is a vector all of whose nonzero coordinates are negative. Put $|\mathbf{x}| = \mathbf{x}^+ - \mathbf{x}^-$. Because $\mathbf{x}$ is a root, we have $\mathbf{x} \neq 0$. Hence $|\mathbf{x}| \neq 0$ and therefore, $|\mathbf{x}| > 0$. The inequalities $|\mathbf{x}|_j \geq \mathbf{x}_j$ and the equalities $|\mathbf{x}|_j^2 = \mathbf{x}_j^2$ for all $j \in Q_0$ yield

$$\begin{aligned} 0 < q_Q(|\mathbf{x}|) &= \sum_{i\in Q_0} |\mathbf{x}|_i^2 - \sum_{\alpha\in Q_1} |\mathbf{x}|_{s(\alpha)}|\mathbf{x}|_{t(\alpha)} \\ &\leq \sum_{i\in Q_0} \mathbf{x}_i^2 - \sum_{\alpha\in Q_1} \mathbf{x}_{s(\alpha)}\mathbf{x}_{t(\alpha)} = q_Q(\mathbf{x}) = 1, \end{aligned}$$

and therefore $q_Q(|\mathbf{x}|) = 1$, that is, $|\mathbf{x}|$ is a root. Consequently the equalities

$$2 = q_Q(\mathbf{x}) + q_Q(|\mathbf{x}|) = q_Q(\mathbf{x}^+ + \mathbf{x}^-) + q_Q(\mathbf{x}^+ - \mathbf{x}^-) = 2[q_Q(\mathbf{x}^+) + q_Q(\mathbf{x}^-)]$$

yield $q_Q(\mathbf{x}^+) + q_Q(\mathbf{x}^-) = 1$. Because q_Q is positive definite, we have either $q_Q(\mathbf{x}^+) = 1$ and $q_Q(\mathbf{x}^-) = 0$ (hence $\mathbf{x} = \mathbf{x}^+ \in R^+$) or $q_Q(\mathbf{x}^-) = 1$ and $q_Q(\mathbf{x}^+) = 0$ (hence $\mathbf{x} = \mathbf{x}^- \in R^-$). This completes the proof that $R = R^+ \cup R^-$.

We have $R^+ \subseteq R(W_Q)$. Similarly, if $\mathbf{x} \in R^-$, then $\mathbf{x} \in R(W_Q)$; indeed, $-\mathbf{x} \in R^+$ gives $-\mathbf{x} = w\mathbf{e}_i$, for some $w \in W_Q$ and $i \in Q_0$, hence $\mathbf{x} = w(-\mathbf{e}_i) = ws_i(\mathbf{e}_i) \in R(W_Q)$. Thus $R^- \subseteq R(W_Q)$ and $R = R^+ \cup R^- \subseteq R(W_Q)$. Because, trivially, $R(W_Q) \subseteq R$, we have indeed $R = R(W_Q)$. $\square$

4.9. Proposition. *Let Q be a quiver whose underlying graph is Dynkin. Then the Weyl group W_Q of Q is finite.*

Proof. We show that W_Q is isomorphic to a subgroup of the group of permutations of R. Because, by (4.8), $R = R^+ \cup R^-$ is finite, this implies the statement.

We first observe that W_Q permutes the roots of q_Q because $q_Q(\mathbf{x}) = 1$ implies $q_Q(w\mathbf{x}) = 1$ for every $w \in W_Q$ (by (3.7)(b)). On the other hand, the action of W_Q on R is faithful, that is, the mapping $w \mapsto (\sigma_w : \mathbf{x} \mapsto w\mathbf{x})$, from W_Q into the group of permutations of R is injective; indeed, $\sigma_w = \sigma_v$ (for $w, v \in W_Q$) implies $w\mathbf{x} = v\mathbf{x}$ and hence $w^{-1}v\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in R$. In particular, $w^{-1}v\mathbf{e}_i = \mathbf{e}_i$ for every $i \in Q_0$, which implies, by linearity, $w^{-1}v\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in E$, that is, $w^{-1}v = 1$ and $w = v$. This proves our claim. $\square$

We need the following lemma.

4.10. Lemma. *Let Q be a quiver whose underlying graph is Dynkin, $\mathbf{x}$ be a positive root of q_Q, and i be a vertex of Q. Then either $s_i(\mathbf{x})$ is positive or $\mathbf{x} = \mathbf{e}_i$.*

Proof. From (3.7)(b), we know that $s_i(\mathbf{x})$ is a root of q_Q. Because q_Q is positive definite, we get the following:

$$0 \leq q_Q(\mathbf{x} \pm \mathbf{e}_i) = (\mathbf{x} \pm e_i, \mathbf{x} \pm e_i) = q_Q(\mathbf{x}) + q_Q(\mathbf{e}_i) \pm 2(\mathbf{x}, \mathbf{e}_i) = 2(1 \pm (\mathbf{x}, \mathbf{e}_i)).$$

Hence $-1 \leq (\mathbf{x}, \mathbf{e}_i) \leq 1$. If $(\mathbf{x}, \mathbf{e}_i) = 1$, then $q_Q(\mathbf{x} - \mathbf{e}_i) = 0$ and consequently $\mathbf{x} = \mathbf{e}_i$. On the other hand, if $(\mathbf{x}, \mathbf{e}_i) \leq 0$, then $s_i(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}, \mathbf{e}_i) > 0$, because $\mathbf{x} > 0$. This proves our claim. $\square$

4.11. Lemma. *Let Q be a finite, connected, and acyclic quiver; c be its Coxeter transformation; s_i be the reflection at i; and $\mathbf{x} \in E = \mathbb{Q}^n$. The following conditions are equivalent:*

(a) $c\mathbf{x} = \mathbf{x}$,

(b) $s_i\mathbf{x} = \mathbf{x}$ *for each point* $i \in Q_0$, *and*

(c) $(\mathbf{x}, \mathbf{y}) = 0$ *for each vector* $\mathbf{y} \in E$.

If, moreover, the underlying graph $\overline{Q}$ of Q is Dynkin or Euclidean, then the preceding conditions are equivalent to the following one:

(d) $q_Q(\mathbf{x}) = 0$.

Proof. Clearly, (b) implies (a). Conversely, if $(1, \ldots, n)$ is an admissible numbering of the points of Q, $c = s_n \ldots s_1$ and $c\mathbf{x} = \mathbf{x}$ holds, then, for any $i \in \{1, \ldots, n\}$, we have $x_i = (c\mathbf{x})_i = (s_n \ldots s_i\mathbf{x})_i$. Hence, by descending induction on i, we get $s_1\mathbf{x} = \ldots = s_n\mathbf{x} = \mathbf{x}$. The equivalence of (b) and (c) follows from the fact that $s_i\mathbf{x} = \mathbf{x}$ for each point $i \in Q_0$ is equivalent to $(\mathbf{x}, \mathbf{e}_i) = 0$ for each point $i \in Q_0$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis of E.

If $\overline{Q}$ is Dynkin or Euclidean, then, by (4.5), the quadratic form q_Q is positive semidefinite. Therefore $|(\mathbf{x}, \mathbf{y})|^2 \leq q_Q(\mathbf{x})q_Q(\mathbf{y})$ for each vector $\mathbf{y} \in E$, so that (d) implies (c). The converse implication follows from the equality $q_Q(\mathbf{x}) = (\mathbf{x}, \mathbf{x})$. $\square$

4.12. Corollary. *Let Q be a quiver whose underlying graph is Dynkin and c be its Coxeter transformation.*

(a) *If $c\mathbf{x} = \mathbf{x}$ for a vector $\mathbf{x} \in E$, then $\mathbf{x} = 0$.*

(b) *For every positive vector $\mathbf{x}$, there exist $s \geq 0$ such that $c^s\mathbf{x} > 0$ but $c^{s+1}\mathbf{x} \not> 0$, and $t \geq 0$ such that $c^{-t}\mathbf{x} > 0$ but $c^{-t-1}\mathbf{x} \not> 0$.*

Proof. (a) If $c\mathbf{x} = \mathbf{x}$ then, by (4.11), we get $q_Q(\mathbf{x}) = 0$. Because, by (4.5), q_Q is positive definite, this implies $\mathbf{x} = 0$.

(b) Because W_Q is a finite group, c has finite order m (say). Consider the vector $\mathbf{y} = \mathbf{x} + c\mathbf{x} + \ldots + c^{m-1}\mathbf{x}$. Then $c\mathbf{y} = \mathbf{y}$. By (a), $\mathbf{y} = 0$. Therefore, there exists a least integer $s \geq 0$ such that $c^{s+1}\mathbf{x} \not> 0$ (and then $c^s\mathbf{x} > 0$). Similarly, one finds t as required. $\square$

The preceding corollary implies that one should look at those positive roots that become nonpositive after application of the Coxeter transformation.

4.13. Lemma. *Let Q be a quiver whose underlying graph is Dynkin and c be its Coxeter transformation. For a positive root $\mathbf{x}$, we have*

(a) *$c\mathbf{x} \not> 0$ if and only if $\mathbf{x} = \mathbf{p}_i$ for some i such that $1 \leq i \leq n$, where $\mathbf{p}_i = s_1 \ldots s_{i-1}\mathbf{e}_i$.*

(b) $c^{-1}\mathbf{x} \not> 0$ *if and only if* $\mathbf{x} = \mathbf{q}_i$ *for some* i *such that* $1 \le i \le n$, *where* $\mathbf{q}_i = s_n \dots s_{i+1}\mathbf{e}_i$.

Proof. We only prove part (a); the proof of (b) is similar. If $c\mathbf{x} = s_n \dots s_1\mathbf{x} \not> 0$, there exists a least integer $i \le n$ such that $s_{i-1} \dots s_1\mathbf{x} > 0$ and $s_i \dots s_1\mathbf{x} \not> 0$. Then, invoking (4.10), we get $s_{i-1} \dots s_1\mathbf{x} = \mathbf{e}_i$ and so $\mathbf{x} = (s_{i-1} \dots s_1)^{-1}\mathbf{e}_i = s_1 \dots s_{i-1}\mathbf{e}_i = \mathbf{p}_i$. Conversely, it is clear that $c\mathbf{p}_i \not> 0$. □

The last two results yield an algorithm allowing us to compute all the positive roots of the quadratic form of a quiver whose underlying graph is Dynkin.

4.14. Proposition. *Let Q be a quiver whose underlying graph is Dynkin and c be the Coxeter transformation of Q.*

(a) *If m_i is the least integer such that $c^{-m_i-1}\mathbf{p}_i \not> 0$, then the set*

$$\{c^{-s}\mathbf{p}_i \mid 1 \le i \le n,\ 0 \le s \le m_i\}$$

equals the set of all the positive roots of q_Q.

(b) *If n_i is the least integer such that $c^{n_i+1}\mathbf{q}_i \not> 0$, then the set*

$$\{c^{t}\mathbf{q}_i \mid 1 \le i \le n,\ 0 \le t \le n_i\}$$

equals the set of all the positive roots of q_Q.

Proof. We only prove (a). The proof of (b) is similar. Because it is clear that each $c^{-s}\mathbf{p}_i$, with $1 \le i \le n$, $0 \le s \le m_i$ is a positive root, it remains to show that each positive root is of this form. Let $\mathbf{x}$ be a positive root. By (4.12), there exists $s \ge 0$ such that $c^s\mathbf{x} > 0$ but $c^{s+1}\mathbf{x} \not> 0$. By (4.13), we have $c^s\mathbf{x} = \mathbf{p}_i$ for some $1 \le i \le n$. Therefore $\mathbf{x} = c^{-s}\mathbf{p}_i$ and clearly $s \le m_i$. □

4.15. Examples. (a) Let Q be the quiver $\overset{1}{\circ} \longleftarrow \overset{3}{\circ} \longrightarrow \overset{2}{\circ}$ whose underlying graph is the Dynkin graph $\mathbb{A}_3$. Then $E \cong \mathbb{Q}^3$ and, as before,

$$\mathbf{s}_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \quad \mathbf{c}^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

We have thus

$$\mathbf{p}_1 = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{p}_2 = \mathbf{s}_1\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{p}_3 = \mathbf{s}_1\mathbf{s}_2\mathbf{e}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

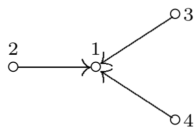
Consequently,

$$\mathbf{c}^{-1}\mathbf{p}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \qquad \mathbf{c}^{-1}\mathbf{p}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad \mathbf{c}^{-1}\mathbf{p}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$\mathbf{c}^{-2}\mathbf{p}_1 = \begin{bmatrix}0\\-1\\0\end{bmatrix} \not> 0, \qquad \mathbf{c}^{-2}\mathbf{p}_2 = \begin{bmatrix}-1\\0\\0\end{bmatrix} \not> 0, \qquad \mathbf{c}^{-2}\mathbf{p}_3 = \begin{bmatrix}-1\\-1\\-1\end{bmatrix} \not> 0.$$

Hence all the positive roots are $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

(b) Let Q be the quiver



whose underlying graph is the Dynkin graph $\mathbb{D}_4$. Then $E \cong \mathbb{Q}^4$ and the reflections are expressed by the following matrices (in the canonical basis):

$$\mathbf{s}_1 = \begin{bmatrix}-1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \qquad \mathbf{s}_2 = \begin{bmatrix}1&0&0&0\\1&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$\mathbf{s}_3 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&-1&0\\0&0&0&1\end{bmatrix}, \qquad \mathbf{s}_4 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&-1\end{bmatrix}.$$

Then

$$\mathbf{c}^{-1} = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4 = \begin{bmatrix}2&-1&-1&-1\\1&-1&0&0\\1&0&-1&0\\1&0&0&-1\end{bmatrix}.$$

We have

$$\mathbf{p}_1 = \mathbf{e}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \qquad \mathbf{p}_2 = \mathbf{s}_1\mathbf{e}_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix},$$

$$\mathbf{p}_3 = \mathbf{s}_1\mathbf{s}_2\mathbf{e}_3 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \qquad \mathbf{p}_4 = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{e}_4 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}.$$

Hence the complete list of the positive roots, given by the action of $\mathbf{c}^{-1}$ on the $\mathbf{p}_i$:

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \xrightarrow{c^{-1}} \begin{bmatrix}2\\1\\1\\1\end{bmatrix} \xrightarrow{c^{-1}} \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \xrightarrow{c^{-1}} \not> 0,$$

$$\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \xrightarrow{c^{-1}} \begin{bmatrix}1\\0\\1\\1\end{bmatrix} \xrightarrow{c^{-1}} \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \xrightarrow{c^{-1}} \not> 0,$$
$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \xrightarrow{c^{-1}} \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \xrightarrow{c^{-1}} \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \xrightarrow{c^{-1}} \not> 0,$$
$$\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \xrightarrow{c^{-1}} \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \xrightarrow{c^{-1}} \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \xrightarrow{c^{-1}} \not> 0.$$

VII.5. Reflection functors and Gabriel's theorem

We now return to the proof of Gabriel's theorem. As said before, the latter states that the path algebra of a connected quiver is representation–finite if and only if the underlying graph of this quiver is a Dynkin diagram. In particular, the representation–finiteness of a path algebra is independent of the orientation of its quiver. This remark led to the definition of reflection functors [32], which are now understood as APR-tilts (see [18]). Before introducing these, we need some combinatorial considerations meant to make more precise the idea of a change of orientation.

Let $Q = (Q_0, Q_1, s, t)$ be a finite, connected, and acyclic quiver and let $n = |Q_0|$. For every point $a \in Q_0$, we define a new quiver

$$\sigma_a Q = (Q'_0, Q'_1, s', t')$$

as follows: All the arrows of Q having a as source or as target are reversed, all other arrows remain unchanged. More precisely, $Q'_0 = Q_0$ and there exists a bijection $Q_1 \to Q'_1$ such that if $\alpha' \in Q'_1$ denotes the arrow corresponding to $\alpha \in Q_1$ under this bijection, then:

(i) if $s(\alpha) \neq a$ and $t(\alpha) \neq a$, then $s'(\alpha') = s(\alpha)$ and $t'(\alpha') = t(\alpha)$; whereas

(ii) if $s(\alpha) = a$ or $t(\alpha) = a$, then $s'(\alpha') = t(\alpha)$ and $t'(\alpha') = s(\alpha)$.

For instance, if Q is the quiver

$$\begin{tikzcd} & & & \overset{4}{\circ} \\ \overset{1}{\circ} & \overset{2}{\circ} \arrow[l] & \overset{3}{\circ} \arrow[l] \arrow[ur] & \\ & & & \overset{5}{\circ} \arrow[ul] \end{tikzcd}$$

then $\sigma_3 Q$ is the quiver

$$\begin{tikzcd}[row sep=small] & & & \overset{4}{\circ} \arrow[dl] \\ \overset{1}{\circ} & \overset{2}{\circ} \arrow[l] \arrow[r] & \overset{3}{\circ} \arrow[dr] & \\ & & & \underset{5}{\circ} \end{tikzcd}$$

We defined, in the previous section, the notion of an admissible numbering of the points of a quiver. We now need a reformulation of this concept.

An **admissible sequence of sinks** in a quiver Q is defined to be a total ordering $(a_1, \dots, a_n)$ of all the points in Q such that:

(i) a_1 is a sink in Q; and

(ii) a_i is a sink in $\sigma_{a_{i-1}} \dots \sigma_{a_1} Q$, for every $2 \leq i \leq n$.

Dually, an **admissible sequence of sources** in Q is a total ordering $(b_1, \dots, b_n)$ of all the points in Q such that:

(i) b_1 is a source in Q; and

(ii) b_i is a source in $\sigma_{b_{i-1}} \dots \sigma_{b_1} Q$, for every $2 \leq i \leq n$.

It is clear that if $(a_1, \dots, a_n)$ is an admissible sequence of sinks, then $(a_n, \dots, a_1)$ is an admissible sequence of sources, and conversely. Because, by hypothesis, Q is acyclic, there exists an admissible numbering $(1, \dots, n)$ of its points. Such an admissible numbering is always an admissible sequence of sinks and, conversely, if $(a_1, \dots, a_n)$ is an admissible sequence of sinks, then an admissible numbering of the points in Q is given by the mapping $a_i \mapsto i$. In general, a given quiver admits many admissible sequences of sinks.

5.1. Lemma. *Let Q be a finite, connected, and acyclic quiver whose n points are admissibly numbered as $(a_1, \dots, a_n)$.*

(a) *If $1 \leq i \leq n$, then a_i is a source and a_{i+1} is a sink in $\sigma_{a_i} \dots \sigma_{a_1} Q$.*

(b) *If $1 \leq i \leq n$, then a_i is a sink and a_{i-1} is a source in $\sigma_{a_i} \dots \sigma_{a_n} Q$.*

(c) $\sigma_{a_n} \dots \sigma_{a_1} Q = Q = \sigma_{a_1} \dots \sigma_{a_n} Q$.

Proof. For (a) and (b), an obvious induction on i yields the result. For (c), we need only observe that each arrow in Q is reversed exactly twice. $\square$

5.2. Lemma. *Let Q and Q' be two trees having the same underlying graph. There exists a sequence $i_1, \dots, i_t$ of points of Q such that*

(a) *for each s such that $1 \leq s \leq t$, i_s is a sink in $\sigma_{i_{s-1}} \dots \sigma_{i_1} Q$; and*

(b) $\sigma_{i_t} \dots \sigma_{i_1} Q = Q'$.

Proof. It suffices to prove the result if Q and Q' differ in the orientation of exactly one arrow. Let thus $\alpha : i \to j$ be an arrow in Q_1 such that the corresponding arrow in Q'_1 is $\alpha' : j \to i$ whereas if $\beta \in Q_1$, $\beta \neq \alpha$, then the corresponding arrow $\beta' \in Q'_1$ has the same source and target, respectively, as β. Let $Q'' = (Q_0, Q_1 \setminus \{\alpha\})$; then Q'' is a (common) subquiver of (both of the trees) Q and Q' and it is not connected. Indeed, i and j belong to distinct connected components of Q''. We may thus write $Q'' = Q^i \cup Q^j$, where Q^i and Q^j are connected subquivers of Q'' containing i and j, respectively. Because Q^i and Q^j are trees, we may assume both to be admissibly numbered with $Q^i_0 = \{1, \ldots, m\}$ and $Q^j_0 = \{m+1, \ldots, n\}$. Because, by (5.1), for each k such that $1 \leq k \leq m$, k is a sink in $\sigma_{k-1} \ldots \sigma_1 Q^i$, hence a sink in $\sigma_{k-1} \ldots \sigma_1 Q$, and moreover we have $\sigma_m \ldots \sigma_1 Q = Q'$, the statement follows. $\square$

We now come to the definition of reflection functors. Let A be a hereditary algebra, which we can assume to be nonsimple. By (1.7), there exists an algebra isomorphism $A \cong KQ_A$, where Q_A is a finite, connected, and acyclic quiver, with $n = |(Q_A)_0| > 1$. Then there exists a sink $a \in (Q_A)_0$ that is not a source, so that the simple A-module $S(a)_A$ is projective and noninjective. Let

$$T[a]_A = \tau^{-1} S(a) \oplus \left(\bigoplus_{b \neq a} P(b) \right)$$

denote the APR-tilting module at a (see (VI.2.8)(c)) and $B = \operatorname{End} T[a]_A$.

It also follows from the tilting theorem (VI.3.8) that the left B-module ${}_B T[a]$ is a tilting module and that $A \cong \operatorname{End}_B(T[a])^{\mathrm{op}}$. We will show that $Q_B = \sigma_a Q_A$, and therefore a is a source in Q_B. The functors

$$\operatorname{mod} A \underset{S_a^-}{\overset{S_a^+}{\rightleftarrows}} \operatorname{mod} B$$

defined by the formulas $S_a^+ = \operatorname{Hom}_A(T[a], -)$ and $S_a^- = (-) \otimes_B T[a]$ are called, respectively, the **reflection functor** at the sink $a \in (Q_A)_0$ and the **reflection functor** at the source $a \in (Q_B)_0$. The following theorem shows that passing from A to B amounts to passing from Q_A to $\sigma_a Q_A$; hence the reflection functors correspond to changes of orientation in the quiver Q_A.

5.3. Theorem. *Let A be a basic hereditary and nonsimple algebra, a be a sink in its quiver Q_A, and $T[a]$ be the APR-tilting A-module at a.*

(a) *The algebra $B = \operatorname{End} T[a]_A$ is isomorphic to $K(\sigma_a Q_A)$, a is a source in Q_B, the simple B-module $S(a)_B$ is injective and isomorphic to*

$\mathrm{Ext}^1_A(T[a], S(a))$, *the left* B*-module* ${}_BT[a]$ *is a tilting module, and* $A \cong \mathrm{End}_B(T[a])^{\mathrm{op}}$.

(b) *The reflection functor* $S_a^+ : \mathrm{mod}\, A \to \mathrm{mod}\, B$ *induces an equivalence between the* K*-linear full subcategory of* $\mathrm{mod}\, A$ *of all* A*-modules without direct summand isomorphic to the simple projective module* $S(a)_A$ *and the* K*-linear full subcategory of* $\mathrm{mod}\, B$ *of all* B*-modules without direct summand isomorphic to the simple injective* B*-module* $S(a)_B$. *The quasi-inverse equivalence is induced by the reflection functor* $S_a^- : \mathrm{mod}\, B \to \mathrm{mod}\, A$.

Proof. Throughout this proof, we denote the APR-tilting A-module $T[a]$ briefly by T, and we use the notation introduced in (VI.3.10).

By our assumption and (1.7), the quiver Q_A of A is finite, connected, and acyclic; $|(Q_A)_0| \geq 2$; and we may suppose, without loss of generality, that $A = KQ_A$. Note that $S(a) = P(a) = \varepsilon_a A$, where ε_c is the stationary path at c in Q_A.

By (VI.2.8)(c), we have $T = \bigoplus_{c \in (Q_A)_0} T_c$, where $T_a = \tau^{-1}\varepsilon_a A = \tau^{-1}P(a)$ and $T_c = \varepsilon_c A$ for $c \neq a$. By (VI.3.1)(b), the right B-modules $\mathrm{Hom}_A(T, T_a)$ and $\mathrm{Hom}_A(T, T_b)$, for $b \neq a$, form a complete set of pairwise nonisomorphic indecomposable projective modules. For each $c \in (Q_A)_0$, denote by $e_c \in \mathrm{End}\, T_A$ the composition of the canonical projection $p_c : T \to T_c$ with the canonical injection $u_c : T_c \to T$. According to (3.10), we have $e_c B \cong \mathrm{Hom}_A(T, T_c)$ for all $c \in (Q_A)_0$ and the elements e_c are primitive orthogonal idempotents of $B = \mathrm{End}\, T_A$ such that

$$B = \bigoplus_{c \in (Q_A)_0} e_c B.$$

It follows directly from the tilting theorem (VI.3.8) that the left B-module ${}_BT$ is a tilting module and that $A \cong \mathrm{End}_B(T)^{\mathrm{op}}$.

We claim that the simple B-module $S(a)_B = e_a B / \mathrm{rad}\, e_a B$ is isomorphic to $\mathrm{Ext}^1_A(T, S(a))$. For this, we notice first that

$$\mathrm{Ext}^1_A(T, S(a)) \cong D\mathrm{Hom}_A(S(a), \tau T) \cong D\mathrm{Hom}_A(S(a), S(a)) \cong K.$$

Hence $\mathrm{Ext}^1_A(T, S(a))$ is a one-dimensional K-vector space and is therefore simple as a B-module. On the other hand, (VI.3.10)(a) yields

$$\begin{aligned}\mathrm{Ext}^1_A(T, S(a))e_a &\cong \mathrm{Ext}^1_A(e_a T, S(a)) \\ &\cong \mathrm{Ext}^1_A(\tau^{-1}S(a), S(a)) \cong D\mathrm{Hom}_A(S(a), S(a)) \cong K.\end{aligned}$$

This establishes our claim.

By (VI.2.8)(c), the tilting module T_A is separating, and

$$\mathcal{F}(T) = \operatorname{add} S(a)_A,$$

whereas $\mathcal{T}(T)$ is the full subcategory of $\operatorname{mod} A$ generated by the remaining indecomposable modules. On the other hand, by (VI.5.6)(b), T_A is also splitting, so that $\mathcal{X}(T_A) = \operatorname{add} S(a)_B$, whereas $\mathcal{Y}(T)$ is the full subcategory $\operatorname{mod} B$ generated by the remaining indecomposable modules. Then (b) follows at once from the tilting theorem (VI.3.8).

To prove that B is hereditary it suffices, by (1.4), to show that, for each simple B-module S_B, we have $\operatorname{pd} S_B \leq 1$. If $S_B \not\cong S(a)_B$, then $S_B \in \mathcal{Y}(T)$; hence there exists $M \in \mathcal{T}(T)$ such that $S_B \cong \operatorname{Hom}_A(T, M)$. By (VI.4.1), we have $\operatorname{pd} S_B \leq \operatorname{pd} M_A \leq 1$, because A is hereditary. On the other hand, we know from (IV.3.9) and (IV.4.4) that the almost split sequence in $\operatorname{mod} A$ starting with $S(a)_A = P(a)$ is of the form

$$0 \longrightarrow S(a) \longrightarrow \bigoplus_{c \neq a} P(c)^{m_c} \longrightarrow \tau^{-1} S(a) \longrightarrow 0,$$

where $P(c) = \varepsilon_c A$ and $m_c = \dim_K \operatorname{Irr}(S(a), P(c)) = \dim_K \varepsilon_c (\operatorname{rad} A / \operatorname{rad}^2 A) \varepsilon_a$, by (1.6). In particular, m_c equals the number of arrows from c to a in Q_A. Thus the direct sum in the almost split sequence is taken over all $c \in (Q_A)_0$ that are neighbours of the sink a. Applying the functor $S_a^+ = \operatorname{Hom}_A(T, -)$ to this almost split sequence yields a short exact sequence

$$0 \to \operatorname{Hom}_A(T, \bigoplus_{c \to a} P(c)^{m_c}) \longrightarrow \operatorname{Hom}_A(T, \tau^{-1} S(a)) \longrightarrow S(a)_B \to 0$$

in $\operatorname{mod} B$, because $\operatorname{Hom}_A(T, S(a)) = 0$, $\operatorname{Ext}_A^1(T, S(a)) \cong S(a)_B$ and $\operatorname{Ext}_A^1(T, P(c)) \cong D\operatorname{Hom}_A(P(c), S(a)) = 0$ for any $c \neq a$. Because the B-modules $\operatorname{Hom}_A(T, \tau^{-1} S(a))$ and $\operatorname{Hom}_A(T, P(c)) \cong e_c B$ for $c \neq a$ are projective, we infer that $\operatorname{pd} S(a)_B \leq 1$.

It remains to show that $Q_B = \sigma_a Q$. Clearly, $(Q_B)_0 = (Q_A)_0 = (\sigma_a Q_A)_0$. On the other hand, it follows from the tilting theorem (VI.3.8) that the functor $S_a^+ = \operatorname{Hom}_A(T, -) : \operatorname{mod} A \longrightarrow \operatorname{mod} B$ induces isomorphisms of K-vector spaces

$$\begin{aligned} \operatorname{Hom}_A(\varepsilon_c A, \tau^{-1} S(a)) &\cong \operatorname{Hom}_B(e_c B, e_a B), \text{and} \\ \operatorname{Hom}_A(\tau^{-1} S(a), \varepsilon_c A) &\cong \operatorname{Hom}_B(e_a B, e_c B). \end{aligned}$$

Also, $\operatorname{Hom}_B(e_a B, e_b B) = 0$ for all $b \neq a$. Indeed, there are isomorphisms

$$\begin{aligned} \operatorname{Hom}_B(e_a B, e_b B) &\cong \operatorname{Hom}_B(\operatorname{Hom}_A(T, \tau^{-1} S(a)), \operatorname{Hom}_A(T, P(b))) \\ &\cong \operatorname{Hom}_A(\tau^{-1} S(a), P(b)), \end{aligned}$$

and there is no nonzero homomorphism $h : \tau^{-1}S(a) \to P(b)$, because otherwise, by (1.4), the A-module $\mathrm{Im}\, h$ is projective; hence $\tau^{-1}S(a)$ is projective, and we get a contradiction. This shows our claim, which implies that $\mathrm{Irr}(e_aB, e_bB) = 0$ for all $b \neq a$. Then, by (1.6), a is a source in Q_B.

We now show that $S_a^+ = \mathrm{Hom}_A(T, -)$ induces, for all $b \neq a$ and $c \neq a$, an isomorphism of K-vector spaces $\mathrm{Irr}(\varepsilon_bA, \varepsilon_cA) \cong \mathrm{Irr}(e_bB, e_cB)$. Because, by (1.7), the quivers Q_A and Q_B are acyclic, we may suppose that $b \neq c$. Then $\varepsilon_bA \not\cong \varepsilon_cA$ and (consequently) $e_bB \not\cong e_cB$. Therefore, $\mathrm{rad}_A(\varepsilon_bA, \varepsilon_cA) = \mathrm{Hom}_A(\varepsilon_bA, \varepsilon_cA)$ and $\mathrm{rad}_B(e_bB, e_cB) = \mathrm{Hom}_B(e_bB, e_cB)$, so that the functor $\mathrm{Hom}_A(T, -)$ induces an isomorphism $\mathrm{rad}_A(\varepsilon_bA, \varepsilon_cA) \cong \mathrm{rad}_B(e_bB, e_cB)$.

We claim that it also induces an isomorphism between the subspaces $\mathrm{rad}_A^2(\varepsilon_bA, \varepsilon_cA)$ and $\mathrm{rad}_B^2(e_bB, e_cB)$. Indeed, assume that f belongs to $\mathrm{rad}_A^2(\varepsilon_bA, \varepsilon_cA)$. Then there exist indecomposable A-modules $M_1, \ldots, M_t$ and homomorphisms $f_j' \in \mathrm{rad}_A(\varepsilon_bA, M_j)$, $f_j \in \mathrm{rad}_A(M_j, \varepsilon_cA)$ such that $f = f_1f_1' + \ldots + f_rf_t'$. For any $j \in \{1, \ldots, t\}$, $\mathrm{Im}\, f_j$ is a submodule of the projective module ε_cA and hence is projective by (1.4). Then $\mathrm{Im}\, f_j$ is isomorphic to a direct summand of the indecomposable module M_j and therefore $M_j \cong \mathrm{Im}\, f_j$. Consequently, M_j is projective and, by (I.5.17), there exists $a_j \in (Q_A)_0$ such that $M_j \cong \varepsilon_{a_j}A$. Note that $a_j \neq c$, because f_j' is a nonisomorphism.

The additivity of $\mathrm{Hom}_A(T, -)$ yields

$$\mathrm{Hom}_A(T, f) = \mathrm{Hom}_A(T, \sum_{j=1}^{t} f_jf_j') = \sum_{j=1}^{t} \mathrm{Hom}_A(T, f_j)\mathrm{Hom}_A(T, f_j').$$

Now $f_j \in \mathrm{rad}_A(M_j, \varepsilon_cA)$ implies that $\mathrm{Hom}_A(T, f_j) \in \mathrm{rad}_B(e_{a_j}B, e_cB)$, by the observation. Similarly, $\mathrm{Hom}_A(T, f_j') \in \mathrm{rad}_B(e_bB, e_{a_j}B)$, and consequently, $\mathrm{Hom}_A(T, f) \in \mathrm{rad}_B^2(e_bB, e_cB)$. Similarly, one shows that the reflection functor $S_a^- = - \otimes_B T : \mathrm{mod}\, B \longrightarrow \mathrm{mod}\, A$ applies $\mathrm{rad}_B^2(e_bB, e_cB)$ into $\mathrm{rad}_A^2(\varepsilon_bA, \varepsilon_cA)$. This shows our claim.

Applying (1.6) yields

$$\varepsilon_c(\mathrm{rad}\, A/\mathrm{rad}^2 A)\varepsilon_b \cong \mathrm{Irr}(\varepsilon_bA, \varepsilon_cA) \cong \mathrm{Irr}(e_bB, e_cB) \cong e_c(\mathrm{rad}\, B/\mathrm{rad}^2 B)e_b.$$

Therefore, if $b, c \neq a$, then there is a bijection between the set of arrows from c to b in Q_A and in Q_B.

The same arguments as earlier show the existence of an isomorphism of K-vector spaces $\mathrm{Irr}(\varepsilon_bA, \tau^{-1}S(a)) \cong \mathrm{Irr}(e_bB, e_aB)$ for all $b \neq a$. Applying (1.6) and (IV. 4.4), we get

$$\begin{aligned}\varepsilon_b(\mathrm{rad}\, A/\mathrm{rad}^2 A)\varepsilon_a &\cong \mathrm{Irr}(\varepsilon_aA, \varepsilon_bA) \cong \mathrm{Irr}(S(a), \varepsilon_bA) \cong \mathrm{Irr}(\varepsilon_bA, \tau^{-1}S(a)) \\ &\cong \mathrm{Irr}(e_bB, e_aB) \cong e_a(\mathrm{rad}\, B/\mathrm{rad}^2 B)e_b.\end{aligned}$$

This defines a bijection between the set of arrows from a to b in Q_A and the set of arrows from b to a in Q_B, and it finishes the proof of the equality $\sigma_a Q = Q_B$.

In particular, while $S(a)_A$ is a simple projective noninjective module, we have that $S(a)_B$ is a simple injective nonprojective module (because a becomes a source in Q_B). □

Now we show that the reflection functors S_a^+ and S_a^-, when applied to indecomposable modules M, correspond to the reflection homomorphism $s_a : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ (as defined in Section 4) applied to their dimension vectors $\mathbf{dim}\, M$, where $n = |Q_0|$.

5.4. Proposition. *Let A be a basic hereditary and nonsimple algebra, a be a sink in its quiver Q_A, and $n = |Q_0|$. Let $T[a]$ be the APR-tilting A-module at a, $B = \operatorname{End} T[a]$, S_a^+, S_a^- the reflection functors at a, and $s_a : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ the reflection at a.*

(a) *Let M be an indecomposable A-module. Then M is isomorphic to $S(a)_A$ if and only if $S_a^+ M = 0$ (or equivalently, $s_a(\mathbf{dim}\, M) \not> 0$). If $M \not\cong S(a)_A$, then $S_a^+ M$ is an indecomposable B-module and $\mathbf{dim}\,(S_a^+ M) = s_a(\mathbf{dim}\, M)$.*

(b) *Let N be an indecomposable B-module. Then N is isomorphic to $S(a)_B$ if and only if $S_a^- N = 0$ (or equivalently, $s_a(\mathbf{dim}\, N) \not> 0$). If $N \not\cong S(a)_B$, then $S_a^- N$ is an indecomposable A-module and $\mathbf{dim}\,(S_a^- N) = s_a(\mathbf{dim}\, N)$.*

Proof. We only prove (a); the proof of (b) is similar. We denote the APR-tilting A-module $T[a]$ by T. Because T_A is an APR-tilting module, $\mathcal{F}(T) = \operatorname{add} S(a)_A$, by (VI.2.8)(c). It follows from (VI.2.3) that if M is an indecomposable A-module, then $S_a^+ M = \operatorname{Hom}_A(T, M) = 0$ if and only if M is isomorphic to $S(a)_A$.

Assume that M is an indecomposable module nonisomorphic to $S(a)_A$. By (5.3), the B-module $S_a^+ M = \operatorname{Hom}_A(T, M)$ is indecomposable. Let $b \neq a$ be a point in $Q = Q_A$. By (VI.3.10), the fact that $M \in \mathcal{T}(T)$ implies that

$$
\begin{aligned}
(\mathbf{dim}\, S_a^+ M)_b &= \dim_K \operatorname{Hom}_A(\operatorname{Hom}_A(T, \varepsilon_b A), \operatorname{Hom}_A(T, M)) \\
&= \dim_K \operatorname{Hom}_A(\varepsilon_b A, M) \\
&= \dim_K M \varepsilon_b = (\mathbf{dim}\, M)_b = (s_a(\mathbf{dim}\, M))_b.
\end{aligned}
$$

On the other hand, if $b = a$, we have isomorphisms

$$\begin{aligned}(S_a^+M)e_a &\cong \mathrm{Hom}_B(e_aB, S_a^+M)\\ &\cong \mathrm{Hom}_B(\mathrm{Hom}_A(T,\tau^{-1}S(a)), \mathrm{Hom}_A(T, S_a^+M))\\ &\cong \mathrm{Hom}_A(\tau^{-1}S(a), M).\end{aligned}$$

Consider the almost split sequence

$$0 \longrightarrow S(a) \longrightarrow \bigoplus_{c\to a} P(c)^{m_c} \longrightarrow \tau^{-1}S(a) \longrightarrow 0$$

constructed in the proof of (5.3), where m_c equals the number of arrows from c to a in Q_A. Because M is indecomposable, $S(a)$ is projective, and $M \not\cong S(a)$, there is no nonzero homomorphism $M \to S(a)_A$ and therefore $\mathrm{Ext}^1_A(\tau^{-1}S(a), M) \cong D\mathrm{Hom}_A(M, S(a)) = 0$. It follows that applying $\mathrm{Hom}_A(-, M)$ to the almost split sequence yields the exact sequence

$$0 \to \mathrm{Hom}_A(\tau^{-1}S(a), M) \to \mathrm{Hom}_A(\bigoplus_{c\to a} P(c)^{m_c}; M) \to \mathrm{Hom}_A(S(a), M) \to 0.$$

Therefore

$$\begin{aligned}(\mathbf{dim}\, S_a^+M)_a &= \dim_K (S_a^+M)e_a = \dim_K \mathrm{Hom}_A(\tau^{-1}S(a), M)\\ &= -\dim_K \mathrm{Hom}_A(S(a), M) + \sum_{c\to a} m_c \dim_K \mathrm{Hom}_A(P(c), M)\\ &= -\dim_K M\varepsilon_a + \sum_{c\to a} m_c(\dim_K M\varepsilon_c) = (s_a(\mathbf{dim}\, M))_a.\end{aligned}$$

We have thus shown that $\mathbf{dim}\, S_a^+M = s_a(\mathbf{dim}\, M)$.

It remains to show that there is an isomorphism $M \cong S(a)_A$ if and only if the vector $s_a(\mathbf{dim}\, M)$ is not positive. If $M \cong S(a)_A$, then the ath coordinate of $s_a(\mathbf{dim}\, M) = s_a(\mathbf{e}_a)$ equals -1. Conversely, if $M \not\cong S(a)_A$, then $s_a(\mathbf{dim}\, M) = \mathbf{dim}\, S_a^+M > 0$, and we are done. $\square$

As shown in (III.1.7), a module over a path K-algebra KQ can be thought of as a K-linear representation of the quiver Q. We now present the original construction of reflection functors given by Bernstein, Gelfand, and Ponomarev [32] for linear representations of quivers. Here we get it by translating, in terms of representations of the quivers Q_A and $Q_B = \sigma_a Q_A$, the effect of the tilting functors S_a^+, S_a^- between the categories of A-modules and B-modules.

5.5. Definition. Let Q be a finite connected quiver, a a sink in Q, and $Q' = \sigma_a Q$. We define the **reflection functor**

$$\mathcal{S}_a^+ : \mathrm{rep}_K(Q) \longrightarrow \mathrm{rep}_K(Q')$$

between the categories of finite dimensional K-linear representations of the quivers Q and Q' as follows. Let $M = (M_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ be an object in

$\operatorname{rep}_K(Q)$. We define the object $\mathcal{S}_a^+M = (M'_i, \varphi'_\alpha)_{i\in Q'_0,\alpha\in Q'_1}$ in $\operatorname{rep}_K(Q')$ as follows:

(a) $M'_i = M_i$ for $i \neq a$, whereas M'_a is the kernel of the K-linear map $(\varphi_\alpha)_\alpha : \bigoplus\limits_{\alpha:s(\alpha)\to a} M_{s(\alpha)} \longrightarrow M_a$ (the direct sum is being taken over all arrows α in Q with target a);

(b) $\varphi'_\alpha = \varphi_\alpha$ for all arrows $\alpha : i \to j$ in Q with $j \neq a$, whereas, if $\alpha : i \to a$ is an arrow in Q, then $\varphi'_\alpha : M'_a \to M'_i = M_i$ is the composition of the inclusion of M'_a into $\bigoplus\limits_{\alpha:s(\beta)\to a} M_{s(\beta)}$ with the projection onto the direct summand M_i.

Let $f = (f_i)_{i\in Q_0} : M \longrightarrow N$ be a morphism in $\operatorname{rep}_K(Q)$, where $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \psi_\alpha)$. We define the morphism

$$\mathcal{S}_a^+ f = f' = (f'_i)_{i\in Q'_0} : \mathcal{S}_a^+M \to \mathcal{S}_a^+N$$

in $\operatorname{rep}_K(Q')$ as follows. For all $i \neq a$, we let $f'_i = f_i$, whereas f'_a is the unique K-linear map, making the following diagram commutative

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\mathcal{S}_a^+M)_a & \longrightarrow & \bigoplus\limits_{\alpha:s(\alpha)\to a} M_{s(\alpha)} & \xrightarrow{(\varphi_\alpha)_\alpha} & M_a \\
 & & \Big\downarrow{\scriptstyle f'_a} & & \Big\downarrow{\scriptstyle \bigoplus_\alpha f_{s(\alpha)}} & & \Big\downarrow{\scriptstyle f_a} \\
0 & \longrightarrow & (\mathcal{S}_a^+N)_a & \longrightarrow & \bigoplus\limits_{\alpha:s(\alpha)\to a} N_{s(\alpha)} & \xrightarrow{(\psi_\alpha)_\alpha} & N_a
\end{array}$$

Now we define the reflection functor attached to a source.

Let Q' be a finite connected quiver, a be a source in Q', and $Q = \sigma_a Q'$. We define a **reflection functor**

$$\mathcal{S}_a^- : \operatorname{rep}_K(Q') \longrightarrow \operatorname{rep}_K(Q)$$

between the categories of finite dimensional K-linear representations of the quivers Q' and Q as follows. Let $M' = (M'_i, \varphi'_\alpha)_{i\in Q'_0,\alpha\in Q'_1}$ be an object in $\operatorname{rep}_K(Q')$. We define the object $\mathcal{S}_a^-M' = (M_i, \varphi_\alpha)_{i\in Q_0,\alpha\in Q_1}$ in $\operatorname{rep}_K(Q')$ as follows:

(a$'$) $M_i = M'_i$ for all $i \neq a$, whereas M_a is the cokernel of the K-linear map $(\varphi'_\alpha)_\alpha : M'_a \longrightarrow \bigoplus\limits_{\alpha:a\to t(\alpha)} M'_{t(\alpha)}$ (the direct sum is being taken over all arrows α in Q' with source a);

(b$'$) $\varphi_\alpha = \varphi'_\alpha$ for all arrows $\alpha : i \to j$ in Q' with $i \neq a$, whereas, if $\alpha : a \to j$ is an arrow in Q', then $\varphi_\alpha : M_j = M'_j \to M_a$ is the composition of the inclusion of M'_j into $\bigoplus\limits_{\alpha:a\to t(\beta)} M'_{t(\beta)}$ with the cokernel projection onto M_a.

Let $f' = (f'_i)_{i\in Q'_0} : M' \longrightarrow N'$ be a morphism in $\operatorname{rep}_K(Q')$, where $M' = (M'_i, \varphi'_\alpha)$ and $N' = (N'_i, \psi'_\alpha)$. We define the morphism $\mathcal{S}_a^- f' =$

$f = (f_i)_{i \in Q_0} : \mathcal{S}_a^- M' \to \mathcal{S}_a^- N'$ in $\operatorname{rep}_K(Q)$ as follows. For all $i \neq a$, we let $f_i = f_i'$, whereas f_a is the unique K-linear map, making the following diagram commutative

$$\begin{array}{ccccccc}
M_a' & \longrightarrow & \bigoplus\limits_{\alpha:a\to t(\alpha)} M'_{t(\alpha)} & \xrightarrow{(\varphi_\alpha)_\alpha} & (\mathcal{S}_a^- M')_a & \longrightarrow & 0 \\
\Big\downarrow f_a' & & \Big\downarrow \bigoplus_\alpha f'_{t(\alpha)} & & \Big\downarrow f_a & & \\
N_a' & \longrightarrow & \bigoplus\limits_{\alpha:a\to t(\alpha)} N'_{t(\alpha)} & \xrightarrow{(\psi_\alpha)_\alpha} & (\mathcal{S}_a^- N')_a & \longrightarrow & 0
\end{array}$$

The following proposition shows that, up to the equivalences of categories (constructed in (III.1.6)) between modules over a path algebra and representations of its quiver, the reflection functors $\mathcal{S}_a^+$ and $\mathcal{S}_a^-$ coincide respectively with the reflection functors S_a^+ and S_a^- defined earlier.

5.6. Proposition. *Let Q be a finite, connected, and acyclic quiver; a be a sink in Q; and $Q' = \sigma_a Q$. Then the following diagram is commutative*

$$\begin{array}{ccc}
\operatorname{mod} KQ & \underset{S_a^-}{\overset{S_a^+}{\rightleftarrows}} & \operatorname{mod} KQ' \\
F \Big\downarrow \cong & & F' \Big\downarrow \cong \\
\operatorname{rep}_K(Q) & \underset{\mathcal{S}_a^-}{\overset{\mathcal{S}_a^+}{\rightleftarrows}} & \operatorname{rep}_K(Q')
\end{array}$$

that is, $\mathcal{S}_a^+ F \cong F' S_a^-$ and $\mathcal{S}_a^- F' \cong F S_a^+$, where F and F' are the category equivalences defined in (III.1.6) *for KQ and KQ', respectively.*

Proof. We only prove that $\mathcal{S}_a^+ F \cong F' S_a^+$; the proof of the second statement is similar. We let $A = KQ$ and $B = KQ'$, and we use freely the notation of (5.1)–(5.5). We recall from (III.1.6) that the functor F associates with any module M in $\operatorname{mod} A$ the representation $FM = ((FM)_i, \varphi_\alpha)$ in $\operatorname{rep}_K(Q)$, where $(FM)_i = M\varepsilon_i$ and, for an arrow $\alpha : i \to j$ in Q, the K-linear map $\varphi_\alpha : M\varepsilon_i \to M\varepsilon_j$ is defined by $x \mapsto x\alpha = x\alpha\varepsilon_j$. The functor F' is defined analogously, with ε_i and e_i interchanged.

Let $b \neq a$ be a point in Q. It follows from (5.3) and (I.4.2), that

$$\begin{aligned}
(F' S_a^+ M)_b = (S_a^+ M)e_b &\cong \operatorname{Hom}_B(e_b B, S_a^+ M) \cong \operatorname{Hom}_B(S_a^+(e_b A), S_a^+ M) \\
&\cong \operatorname{Hom}_A(\varepsilon_b A, M) \cong M\varepsilon_b = (\mathcal{S}_a^+ FM)_b,
\end{aligned}$$

and the composed isomorphism $(F'S_a^+M)_b \cong (\mathcal{S}_a^+FM)_b$ is obviously functorial. On the other hand, if $b = a$, we have vector space isomorphisms

$$\begin{aligned}(F'S_a^+M)_a = (S_a^+M)e_a &\cong \mathrm{Hom}_B(e_aB, S_a^+M)\\ &\cong \mathrm{Hom}_B(S_a^+(\tau^{-1}S(a)), S_a^+M)) \cong \mathrm{Hom}_A(\tau^{-1}S(a), M).\end{aligned}$$

We recall that the almost split sequence in $\mathrm{mod}\,A$ starting from the simple projective module $S(a) = P(a)$ is of the form

$$0 \longrightarrow S(a) \xrightarrow{u} \bigoplus_{c\neq a} P(c)^{m_c} \longrightarrow \tau^{-1}S(a) \longrightarrow 0,$$

where $P(c) = \varepsilon_c A$, $m_c = \dim_K \mathrm{Irr}(S(a), P(c)) = \dim_K \varepsilon_c(\mathrm{rad}\,A/\mathrm{rad}^2 A)\varepsilon_a$ is the number of arrows $\alpha : c \to a$ in Q. Hence, there are K-linear isomorphisms $\mathrm{Irr}(S(a), P(c)) \cong \varepsilon_c(\mathrm{rad}\,A/\mathrm{rad}^2 A)\varepsilon_a \cong \bigoplus_{\alpha:c\to a} K\alpha$, because the set of all arrows $\alpha : c \to a$ in $Q_A = Q$ gives (by definition) a basis of the K-vector space $\varepsilon_c(\mathrm{rad}\,A/\mathrm{rad}^2 A)\varepsilon_a$. The left minimal almost split morphism $u = (u_c)_c : S(a) \longrightarrow \bigoplus_{c\neq a} P(c)^{m_c}$ is such that, for each c, the homomorphism $u_c = [u_{c_1} \ldots u_{c_{m_c}}]^t : S(a) \longrightarrow P(c)^{m_c}$ is given by a basis $\{u_{c_1} \ldots u_{c_{m_c}}\}$ of the K-vector space $\mathrm{Irr}(S(a), P(c))$. We may therefore rewrite u_c as (u_α), where α runs over all arrows $c \to a$, so that the almost split sequence becomes

$$0 \longrightarrow S(a) \xrightarrow{u=(u_\alpha)_\alpha} \bigoplus_{\alpha:s(\alpha)\to a} P(s(\alpha)) \xrightarrow{\quad v \quad} \tau^{-1}S(a) \longrightarrow 0$$

where the direct sum is being taken over all arrows α in $Q_A = Q$ having a as a target. Applying $\mathrm{Hom}_A(-, M)$ yields the top left exact sequence in the commutative diagram

$$\begin{array}{ccccccc} 0 \to \mathrm{Hom}_A(\tau^{-1}S(a), M) & \to & \mathrm{Hom}_A\big(\bigoplus_{\alpha:s(\alpha)\to a} P(s(\alpha)), M\big) & \xrightarrow{\mathrm{Hom}_A(u,M)} & \mathrm{Hom}_A(S(a), M) \\ & & \Big\downarrow{\cong} & & \Big\downarrow{\cong} \\ 0 \longrightarrow (S_a^+FM)_a & \longrightarrow & \bigoplus_{\alpha:s(\alpha)\to a} (FM)_{s(\alpha)} & \xrightarrow{(\varphi_\alpha)_\alpha} & (FM)_a \end{array}$$

where $(FM)_j = M\varepsilon_j$, $\mathrm{Hom}_A(u, M) = (\mathrm{Hom}_A(u, M)_\alpha)_{\alpha:s(\alpha)\to a}$, and the vertical isomorphisms are induced by the isomorphism $\mathrm{Hom}_A(eA, L) \cong Le$ of (I.4.2), where L is an A-module and e is an idempotent of A. The lower row is (left) exact by definition of S_a^+. Therefore there exists a K-vector space isomorphism $\mathrm{Hom}_A(\tau^{-1}S(a), M) \cong (\mathcal{S}_a^+FM)_a$ making the left-hand square commutative. Hence $(\mathcal{S}_a^+ FM)_a \cong (F' S_a^+ M)_a$. A simple calculation (left as an exercise) shows that the vector space isomorphisms $(\mathcal{S}_a^+ FM)_c \cong (F' S_a^+ M)_c$ for $c \in Q_0$ induce an isomorphism of representations $\mathcal{S}_a^+ FM \cong F' S_a^+ M$ in $\operatorname{rep}_K(Q')$. It is also easy to verify that this isomorphism is functorial, so that we have $F' S_a^+ \cong \mathcal{S}_a^+ F$. $\square$

The following corollary summarises the properties of the functors $\mathcal{S}_a^+$, $\mathcal{S}_a^-$ that translate those of the functors S_a^+, S_a^- into the language of representations of a quiver. The proof is easy and left as an exercise to the reader.

5.7. Corollary. *Let Q be a finite, connected, and acyclic quiver with at least two points; a a sink in Q; and $Q' = \sigma_a Q$. The reflection functors $\mathcal{S}_a^+ : \operatorname{rep}_K(Q) \to \operatorname{rep}_K(Q')$ and $\mathcal{S}_a^- : \operatorname{rep}_K(Q') \to \operatorname{rep}_K(Q)$ satisfy the following properties:*

(a) *The functor $\mathcal{S}_a^-$ is left adjoint to $\mathcal{S}_a^+$.*

(b) *If M is indecomposable in $\operatorname{rep}_K(Q)$, then the following three conditions are equivalent:*

(i) $\mathcal{S}_a^+ M \neq 0$,

(ii) $M \ncong S(a)$,

(iii) $s_a(\mathbf{dim}\, M) > 0$.

Moreover, if this is the case, then $\mathbf{dim}\, \mathcal{S}_a^+ M = s_a(\mathbf{dim}\, M)$, $\mathcal{S}_a^- \mathcal{S}_a^+ M \cong M$ and $\mathcal{S}_a^+$ induces an algebra isomorphism $\operatorname{End} M \cong \operatorname{End}(\mathcal{S}_a^+ M)$.

(c) *If M' is indecomposable in $\operatorname{rep}_K(Q')$, then the following three conditions are equivalent:*

(i) $\mathcal{S}_a^- M' \neq 0$,

(ii) $M' \ncong S(a)$,

(iii) $s_a(\mathbf{dim}\, M') > 0$.

Moreover, if this is the case, then $\mathbf{dim}\, \mathcal{S}_a^- M' = s_a(\mathbf{dim}\, M')$, $\mathcal{S}_a^+ \mathcal{S}_a^- M' \cong M'$, and $\mathcal{S}_a^-$ induces an algebra isomorphism $\operatorname{End} M' \cong \operatorname{End}(\mathcal{S}_a^- M')$.

(d) *The functors $\mathcal{S}_a^+$ and $\mathcal{S}_a^-$ induce quasi-inverse equivalences between the K-linear full subcategory of $\operatorname{rep}_K(Q)$ of the representations having no direct summand isomorphic to the simple projective representation $S(a)$, and the K-linear full subcategory of $\operatorname{rep}_K(Q')$ of the representations having no direct summand isomorphic to the simple injective representation $S(a)$.* $\square$

Let A be a hereditary nonsimple algebra and $(j_1, \ldots, j_n)$ be an admissible numbering of the points of Q_A. It follows from (5.1)–(5.4) that the functors

$$C^+ = S_{j_n}^+ \ldots S_{j_1}^+ \quad \text{and} \quad C^- = S_{j_1}^- \ldots S_{j_n}^-$$

are endofunctors of mod A. They are called the **Coxeter functors**. The definition of C^+ and C^- does not depend on the choice of the admissible numbering $(j_1, \ldots, j_n)$ of the points of Q_A, because of the following interpretation of the Coxeter functors in terms of the Auslander–Reiten translation.

5.8. Lemma. *Let A be a hereditary and nonsimple K-algebra, and let $(j_1,\ldots,j_n)$ be an admissible numbering of the points of Q_A.*

(a) *If M is an indecomposable nonprojective A-module, then there are A-module isomorphisms $C^+M \cong \tau M$ and $C^-C^+M \cong M$.*
(b) *If N is an indecomposable noninjective A-module, then there are A-module isomorphisms $C^-N \cong \tau^{-1}N$ and $C^+C^-N \cong N$.*

Proof. In view of (IV.2.10), it suffices to prove the first statements in (a) and (b). We only prove (a); the proof of (b) is similar. We may assume the points of Q_A to be admissibly numbered as $(1, \ldots, n)$. Applying repeatedly (5.3) to the admissible sequence of sinks $(1, \ldots, n)$, we see that for each i such that $1 \leq i \leq n$, the module $P(i)$ is simple projective over $K(\sigma_{i-1} \ldots \sigma_1 Q_A)$ and that, for every indecomposable nonprojective A-module M, we have

$$\mathrm{Hom}_A\Big(\tau^{-1}(\bigoplus_{k=1}^{i} P(k)) \oplus (\bigoplus_{l=i+1}^{n} P(l)),\ M\Big) \cong S_i^+ \ldots S_1^+ M.$$

Therefore $C^+M = S_n^+ \ldots S_1^+ M \cong \mathrm{Hom}_A(\tau^{-1}A, M)$. Because the algebra A is hereditary, (IV.2.14) applies to A and M, and we get A-module isomorphisms $C^+M \cong \mathrm{Hom}_A(\tau^{-1}A, M) \cong \mathrm{Hom}_A(A, \tau M) \cong \tau M$. $\square$

We also need the following technical result.

5.9. Lemma. *Let A be a hereditary and nonsimple algebra, $(j_1, \ldots, j_n)$ be an admissible numbering of the points of Q_A, and M be an indecomposable module in* mod A.

(a) *If $b \leq a \leq n$ and $s_{j_a} \ldots s_{j_1}(\mathbf{dim}\, M) > 0$, then $s_{j_b} \ldots s_{j_1}(\mathbf{dim}\, M) > 0$, the module $S_{j_b}^+ \ldots S_{j_1}^+ M$ over the algebra $K(\sigma_{j_b} \ldots \sigma_{j_1} Q_A)$ is indecomposable, and $\mathbf{dim}\, S_{j_b}^+ \ldots S_{j_1}^+ M = s_{j_b} \ldots s_{j_1}(\mathbf{dim}\, M)$.*
(b) *If $c(\mathbf{dim}\, M) > 0$, then the module C^+M is indecomposable and $\mathbf{dim}\, C^+M = c(\mathbf{dim}\, M)$.*

Proof. We assume for simplicity that the points of Q_A are admissibly numbered as $(1, \ldots, n)$. Assume to the contrary that there exists $b \leq a$ such that $s_b \ldots s_1(\mathbf{dim}\, M) \not> 0$. We clearly may suppose that b is minimal with this property, that is, that $s_c \dots s_1(\mathbf{dim}\, M) > 0$ for all $c \le b-1$. It follows from (5.4)(a) and an obvious induction, that for any $c \le b-1$, the module $S_c^+ \dots S_1^+ M$ over the algebra $K(\sigma_c \dots \sigma_1 Q_A)$ is indecomposable and $\mathbf{dim}\,(S_c^+ \dots S_1^+ M) = s_c \dots s_1(\mathbf{dim}\, M)$. Furthermore, the module $S_{b-1}^+ \dots S_1^+ M \cong S(b)$ is simple projective over the algebra $K(\sigma_b \dots \sigma_1 Q_A)$. Therefore $\mathbf{dim}\,(S_{b-1}^+ \dots S_1^+ M)$ is the canonical basis vector $\mathbf{e}_b$ of $\mathbb{Z}^n$ so that $s_a \dots s_1(\mathbf{dim}\, M) = s_a \dots s_b(\mathbf{e}_b) = s_a \dots s_{b+1}(-\mathbf{e}_b) = -\mathbf{e}_b \not> 0$, which is a contradiction.

This shows indeed that $s_b \dots s_1(\mathbf{dim}\, M) > 0$ for all $b \le a$, but also that, for any $b \le a$, the module $S_b^+ \dots S_1^+ M$ over the algebra $K(\sigma_b \dots \sigma_1 Q_A)$ is indecomposable and $\mathbf{dim}\,(S_b^+ \dots S_1^+ M) = s_b \dots s_1(\mathbf{dim}\, M)$. This completes the proof of (a). To prove (b), we apply (a) to the case where $a = n$. $\square$

We are now able to prove Gabriel's theorem.

5.10. Theorem. *Let Q be a finite, connected, and acyclic quiver; K be an algebraically closed field; and $A = KQ$ be the path K-algebra of Q.*

(a) *The algebra A is representation-finite if and only if the underlying graph $\overline{Q}$ of Q is one of the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, with $n \ge 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$.*

(b) *If $\overline{Q}$ is a Dynkin graph, then the mapping* $\mathbf{dim} : M \mapsto \mathbf{dim}\, M$ *induces a bijection between the set of isomorphism classes of indecomposable A-modules and the set $\{\mathbf{x} \in \mathbb{N}^n;\ q_Q(\mathbf{x}) = 1\}$ of positive roots of the quadratic form q_Q of Q.*

(c) *The number of the isomorphism classes of indecomposable A-modules equals $\frac{1}{2}n(n+1)$, $n^2 - n$, 36, 63, and 120, if $\overline{Q}$ is the Dynkin graph $\mathbb{A}_n$, $\mathbb{D}_n$, with $n \ge 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$, respectively.*

Proof. Necessity of (a). Assume that $\overline{Q}$ is not a Dynkin diagram. By (2.1), $\overline{Q}$ contains a Euclidean graph as a subgraph. By (2.2), we may assume that $\overline{Q}$ is itself Euclidean. If $\overline{Q} = \widetilde{\mathbb{A}}_m$ for some $m \ge 1$, then (2.3) gives that KQ is representation-infinite. Otherwise, we observe that, according to (5.3), the algebra KQ is representation–infinite if and only if $K(\sigma_a Q)$ is representation–infinite for each sink (or source) a of Q. Thus, if $\overline{Q}$ is Euclidean of type $\widetilde{\mathbb{D}}_n$ (for some $n \ge 4$) or $\widetilde{\mathbb{E}}_p$ (for $p = 6$, 7, or 8), it follows from (2.7) and (5.2) that KQ is representation–infinite. We have thus shown that if KQ is representation–finite, then $\overline{Q}$ is a Dynkin graph.

Sufficiency of (a). Assume that Q is a quiver whose underlying graph is a Dynkin graph. We must show that $A = KQ$ is representation–finite. We may assume the points of Q to be admissibly numbered as $(1, \dots, n)$. Let

M be an indecomposable A-module. We claim that the vector $\mathbf{x} = \mathbf{dim}\, M$ is a positive root of the quadratic form q_Q of the quiver Q.

Let $c = s_n \dots s_1$ denote the Coxeter transformation of Q and $C^+ = S_n^+ \dots S_1^+$, $C^- = S_1^- \dots S_n^-$ be the Coxeter functors defined with respect to the admissible numbering $(1, \dots, n)$ of points of Q. By (4.12), there exists a least $t \geq 0$ such that $c^t\mathbf{x} > 0$ but $c^{t+1}\mathbf{x} \not> 0$. Because $c = s_n \dots s_1$, there also exists a least i such that $0 \leq i \leq n-1$, $s_i \dots s_1 c^t \mathbf{x} > 0$, but $s_{i+1} s_i \dots s_1 c^t \mathbf{x} \not> 0$.

By applying (5.9)(b) repeatedly, we prove that the right A-modules $C^+M, C^{+2}M, \dots, C^{+t}M$ are indecomposable and that

$$\mathbf{dim}\, C^{+j}M = c^j(\mathbf{dim}\, M)$$

for all $j \leq t$. Then applying (5.9)(a) to $C^{+t}M$ we conclude that $M' = S_i^+ \dots S_1^+ C^{+t}M$ is an indecomposable module over $K(\sigma_i \dots \sigma_1 Q)$ and

$$\mathbf{dim}\,(S_i^+ \dots S_1^+ C^{+t}M) = s_i \dots s_1 c^t(\mathbf{dim}\, M) = s_i \dots s_1 c^t \mathbf{x}.$$

Because $s_{i+1}(\mathbf{dim}\, M') \not> 0$, there is an isomorphism $M' \cong S(i+1)$, by (5.4)(a). But then $s_i \dots s_1 c^t \mathbf{x} = \mathbf{e}_{i+1}$, and according to (4.14) the vector $\mathbf{x} = c^{-t} s_1 \dots s_i \mathbf{e}_{i+1} = c^{-t}\mathbf{p}_{i+1}$ (in the notation of (4.13)) is a positive root of q_Q. Furthermore, in view of (5.8) and (5.3)(b), the isomorphism $S_i^+ \dots S_1^+ C^{+t}M \cong S(i+1)$ yields $M \cong C^{-t} S_1^- \dots S_i^- S(i+1)$.

We have shown that the mapping $\mathbf{dim} : M \mapsto \mathbf{dim}\, M$ takes an indecomposable A-module to a positive root of q_Q. Moreover, the integers i and t as defined earlier, only depend on the vector $\mathbf{x} = \mathbf{dim}\, M$. Thus, if M, N are two indecomposable A-modules such that $\mathbf{dim}\, M = \mathbf{x} = \mathbf{dim}\, N$, we have, as earlier $S_i^+ \dots S_1^+ C^{+t}M \cong S(i+1) \cong S_i^+ \dots S_1^+ C^{+t}N$ so that $M \cong C^{-t} S_1^- \dots S_i^- S(i+1) \cong N$. Thus $\mathbf{dim}$ is an injective mapping from the set of isomorphism classes of indecomposable A-modules to the set of positive roots of q_Q.

Finally, the mapping is surjective because, by (4.14), every positive root $\mathbf{x}$ of q_Q is of the form $\mathbf{x} = c^{-t}\mathbf{p}_{i+1} = c^{-t} s_1 \dots s_i \mathbf{e}_{i+1}$, for some i and t. But then the indecomposable module $M = C^{-t} S_1^- \dots S_i^- S(i+1)$ satisfies $\mathbf{x} = \mathbf{dim}\, M$. Because q_Q has only finitely many positive roots, by (3.4) and (4.6), A has only finitely many nonisomorphic indecomposable modules. This finishes the proof of (a) and (b).

The statement (c) follows from (b) and the fact that the number of positive roots of q_Q equals $\frac{1}{2}n(n+1)$, $n^2 - n$, 36, 63, and 120 if $\overline{Q}$ is the Dynkin graph $\mathbb{A}_n$, $\mathbb{D}_n$, with $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$, respectively (see [41], [95], and Exercises 10, 11, and 12). □

The reader may have observed that we have shown in the course of the proof (with the preceding notation) the following useful fact.

5.11. Corollary. *For any indecomposable module M over a representation–finite hereditary algebra A, there exist integers $t \geq 0$ and i with $0 \leq i \leq n-1$ (depending only on the vector $\mathbf{dim}\, M$) such that*

$$M \cong C^{-t} S_1^- \dots S_i^- S(i+1). \qquad \square$$

We can deduce from Gabriel's theorem the shape of the Auslander–Reiten quiver of a representation–finite hereditary algebra. We first obtain an expression of the indecomposable projective and injective modules by means of the reflection functors.

5.12. Corollary. *Let Q be a Dynkin quiver with n points admissibly numbered as $(1, \dots, n)$ and let i be such that $1 \leq i \leq n$. Denote by $P(i)$ and $I(i)$, respectively, the corresponding indecomposable projective and injective KQ-modules corresponding to the point $i \in Q_0$.*

(a) *If $S(i)$ denotes the simple $K(\sigma_i \dots \sigma_n Q)$-module corresponding to i in $\sigma_i \dots \sigma_n Q$, then $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$ and $\mathbf{p}_i = \mathbf{dim}\, P(i)$.*

(b) *If $S(i)$ denotes the simple $K(\sigma_i \dots \sigma_n Q)$-module corresponding to i in $\sigma_i \dots \sigma_1 Q$, then $I(i) \cong S_n^+ \dots S_{i+1}^+ S(i)$ and $\mathbf{q}_i = \mathbf{dim}\, I(i)$.*

Proof. We only prove (a); the proof of (b) is similar. By Gabriel's theorem (5.10), the indecomposable KQ-modules are uniquely determined up to isomorphism by their dimension vectors; hence it suffices to show that

$$\mathbf{p}_i = s_1 \dots s_{i-1}(\mathbf{e}_i) = \mathbf{dim}\, P(i).$$

We show by descending induction on k with $1 \leq k \leq i$ that $s_k \dots s_{i-1}(\mathbf{e}_i)_j$ equals 1 if $k \leq j \leq i$ and there exists a path from i to k through j, and equals 0 otherwise. There is nothing to show if $k = i$. Assume $k < i$ and that the statement holds for all $k < j \leq i$. There is at most one point j in Q such that $k < j \leq i$ and there is an arrow $j \to k$ and a path from i to j. Indeed, the existence of two such points j would contradict the fact that Q is a tree. Hence it follows from the definition of s_k that $s_k \dots s_{i-1}(\mathbf{e}_i)_k = 1$ if there exists $k < j \leq i$ such that there is an arrow $j \to k$ and a path from i to j (that is, if there exists a path from i to k), and $s_k \dots s_{i-1}(\mathbf{e}_i)_k = 0$ otherwise. Because, by our inductive assumption, $[s_k s_{k+1} \dots s_{i-1}(\mathbf{e}_i)]_j = [s_{k+1} \dots s_{i-1}(\mathbf{e}_i)]_j$ for all $j \neq k$, this shows our claim. The result follows after setting $k = 1$. $\square$

5.13. Proposition. *Let A be a representation–finite hereditary algebra.*

(a) *For every indecomposable A-module M, there exist $t \geq 0$ and an indecomposable projective A-module P such that $M \cong \tau^{-t}P$.*

(b) *The Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A is acyclic.*

Proof. We assume for simplicity that the points of Q_A are admissibly numbered as $(1, \ldots, n)$. Let $C^- = S_1^- \ldots S_n^-$ be the Coxeter functor.

(a) By (5.11), there exists a pair of integers $t \geq 0$ and $0 \leq i \leq n-1$ such that $M \cong C^{-t} S_1^- \ldots S_i^- S(i+1)$. The result follows from (5.8) and (5.12).

(b) Assume that

$$M_0 \to M_1 \to \ldots \to M_s = M_0$$

is a cycle in $\Gamma(\operatorname{mod} A)$. By (a), for each i with $0 \leq i < s$, there exist $t_i \geq 0$ and $a_i \in (Q_A)_0$ such that $M_i \cong \tau^{-t_i} P(a_i)$. Let $t = \min\{t_i \mid 0 \leq i < s\}$. Then the previous cycle induces a cycle

$$\tau^t M_0 \to \tau^t M_1 \to \ldots \to \tau^t M_s = \tau^t M_0$$

in $\Gamma(\operatorname{mod} A)$, because it follows from (IV.2.15) that $\operatorname{Irr}(X, Y) \cong \operatorname{Irr}(\tau X, \tau Y)$ for any pair of indecomposable nonprojective modules X and Y. Moreover, by definition of t, this cycle passes through a projective A-module. Because A is hereditary, by (1.10), the cycle consists of indecomposable projective modules connected by irreducible monomorphisms, which is a contradiction. $\square$

5.14. Corollary. *Let M be an indecomposable module over a representation–finite hereditary algebra A. Then $\operatorname{End}_A M \cong K$ and $\operatorname{Ext}_A^1(M, M) = 0$.*

Proof. By (5.13)(a), there exist $t \geq 0$ and an indecomposable projective A-module P such that $M \cong \tau^t P$. Applying (IV.2.14) and (IV.2.15) we get a sequence of isomorphisms $\operatorname{Hom}_A(M, M) \cong \operatorname{Hom}_A(\tau^t P, \tau^t P) \cong \operatorname{Hom}_A(P, P) \cong K$ (by (1.5)) and $\operatorname{Ext}_A^1(M, M) \cong D\operatorname{Hom}_A(M, \tau M) \cong D\operatorname{Hom}_A(\tau^t P, \tau^{t+1} P) \cong D\operatorname{Hom}_A(P, \tau P) \cong \operatorname{Ext}_A^1(P, P) = 0$. $\square$

By (IV.2.14), the fact that each indecomposable module over a representation–finite hereditary algebra A is a brick implies that $\operatorname{Ext}_A^1(M, \tau M)$ is one-dimensional for each indecomposable nonprojective module M_A and, hence, any nonsplit short exact sequence $0 \to \tau M \to L \to M \to 0$ is almost split.

We also note that it follows from (1.10) and (5.14) that the combinatorial method of constructing the Auslander–Reiten quiver explained in Examples (IV.4.10)–(IV.4.14) works perfectly well for representation–finite hereditary algebras.

5.15. Examples. (a) Let Q be the quiver $\overset{1}{\circ} \longleftarrow \overset{3}{\circ} \longrightarrow \overset{2}{\circ}$ whose underlying graph is the Dynkin graph $\mathbb{A}_3$. We wish to construct a complete list of the nonisomorphic indecomposable KQ-modules.
The simple representations are:

$$S(1) = (K \longleftarrow 0 \longrightarrow 0),\ S(2) = (0 \longleftarrow 0 \longrightarrow K), \text{ and } S(3) = (0 \longleftarrow K \longrightarrow 0).$$

The indecomposable projective representations are:

$$P(1) = S(1),\ P(2) = S(2), \text{ and } P(3) = (K \xleftarrow{1} K \xrightarrow{1} K).$$

The indecomposable injective representations are: $I(3) = S(3)$,

$$I(1) = (K \xleftarrow{1} K \longrightarrow 0), \text{ and } I(2) = (0 \longleftarrow K \xrightarrow{1} K).$$

The positive roots of q_Q have been computed in (4.15)(a). We see in particular that every indecomposable KQ-module is either projective or injective. To construct $\Gamma(\operatorname{mod} KQ)$ as in (IV.4.10), it suffices to observe that $\operatorname{rad} P(3) \cong P(1) \oplus P(2)$. The construction proceeds easily:

$$\begin{tikzcd}
100 \arrow[dr] & & 011 \arrow[dr] & \\
& 111 \arrow[ur] \arrow[dr] & & 010 \\
001 \arrow[ur] & & 110 \arrow[ur] &
\end{tikzcd}$$

(b) Let Q be the quiver

$$\begin{tikzcd}
\overset{2}{\circ} \arrow[dr] & & \\
& \overset{1}{\circ} & \overset{4}{\circ} \arrow[l] \\
\overset{3}{\circ} \arrow[ur] & &
\end{tikzcd}$$

whose underlying graph is the Dynkin graph $\mathbb{D}_4$. We wish to construct a complete list of the nonisomorphic indecomposable KQ-modules.

The simple representations are:

$$S(1) = \left(\begin{tikzcd} 0 \arrow[dr] & & \\ & K & 0 \arrow[l] \\ 0 \arrow[ur] & & \end{tikzcd}\right) \qquad S(3) = \left(\begin{tikzcd} 0 \arrow[dr] & & \\ & 0 & 0 \arrow[l] \\ K \arrow[ur] & & \end{tikzcd}\right)$$

$$S(2) = \left(\begin{tikzcd} K \arrow[dr] & & \\ & 0 & 0 \arrow[l] \\ 0 \arrow[ur] & & \end{tikzcd}\right) \qquad S(4) = \left(\begin{tikzcd} 0 \arrow[dr] & & \\ & 0 & K \arrow[l] \\ 0 \arrow[ur] & & \end{tikzcd}\right)$$

The indecomposable projective representations are: $P(1) = S(1)$ and

$$P(2) = \left(\begin{tikzcd} K \arrow[dr, "1"] & & \\ & K & 0 \arrow[l] \\ 0 \arrow[ur] & & \end{tikzcd}\right) \quad P(3) = \left(\begin{tikzcd} 0 \arrow[dr] & & \\ & K & 0 \arrow[l] \\ K \arrow[ur, "1"'] & & \end{tikzcd}\right) \quad P(4) = \left(\begin{tikzcd} 0 \arrow[dr] & & \\ & K & K \arrow[l, "1"'] \\ 0 \arrow[ur] & & \end{tikzcd}\right)$$

The indecomposable injective representations are:

$$I(1) = \left(\begin{tikzcd}[row sep=small, column sep=small] K \arrow[dr, "1"] & & \\ & K & K \arrow[l, "1"'] \\ K \arrow[ur, "1"'] & & \end{tikzcd}\right),$$

$I(2) = S(2)$, $I(3) = S(3)$, and $I(4) = S(4)$.

The positive roots of q_Q have been computed in (4.15)(b). To obtain the remaining indecomposable representations, it suffices, by Gabriel's theorem (5.10), to exhibit, for each positive root $\mathbf{x}$, an indecomposable representation having $\mathbf{x}$ as dimension vector. We thus have four other indecomposable representations, given respectively by:

(1) $\mathbf{dim}\, M_1 = \left(\begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix}\right)$, then $M_1 = \left(\begin{tikzcd}[row sep=small, column sep=small] K \arrow[dr, "1"] & & \\ & K^2 & K \arrow[l, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"'] \\ K \arrow[ur, "1"'] & & \end{tikzcd}\right)$ (this is indeed an indecomposable representation, by the proof of (2.6));

(2) $\mathbf{dim}\, M_2 = \left(\begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix}\right)$, then $M_2 = \left(\begin{tikzcd}[row sep=small, column sep=small] 0 \arrow[dr] & & \\ & K & K \arrow[l, "1"'] \\ K \arrow[ur, "1"'] & & \end{tikzcd}\right)$

(3) $\mathbf{dim}\, M_3 = \left(\begin{smallmatrix}1&&\\&1&1\\0&&\end{smallmatrix}\right)$, then $M_3 = \left(\begin{tikzcd}[row sep=small, column sep=small] K \arrow[dr, "1"] & & \\ & K & K \arrow[l, "1"'] \\ 0 \arrow[ur, "0"'] & & \end{tikzcd}\right)$

(4) $\mathbf{dim}\, M_4 = \left(\begin{smallmatrix}1&&\\&1&0\\1&&\end{smallmatrix}\right)$, then $M_4 = \left(\begin{tikzcd}[row sep=small, column sep=small] K \arrow[dr, "1"] & & \\ & K & 0 \arrow[l] \\ K \arrow[ur, "1"'] & & \end{tikzcd}\right)$

(indeed, M_2, M_3, and M_4 are indecomposable, because each has a simple socle isomorphic to $S(1)$).

To construct $\Gamma(\operatorname{mod} KQ)$, we note that there are isomorphisms

$$\operatorname{rad} P(2) \cong \operatorname{rad} P(3) \cong \operatorname{rad} P(4) \cong P(1).$$

The construction then proceeds easily, as in (IV.4.10)–(IV.4.14):

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \begin{smallmatrix}1&&\\&1&0\\0&&\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1&&\\&0&0\\0&&\end{smallmatrix} \\
\begin{smallmatrix}0&&\\&1&0\\0&&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0&&\\&1&1\\0&&\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1&&\\&1&0\\1&&\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1&&\\&1&1\\1&&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0&&\\&0&1\\0&&\end{smallmatrix} \\
 & \begin{smallmatrix}0&&\\&1&0\\1&&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1&&\\&1&1\\0&&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0&&\\&0&0\\1&&\end{smallmatrix}
\end{tikzcd}$$

(c) Let Q be the quiver

$$
\begin{tikzcd}
& & \overset{3}{\circ} & & \\
\underset{1}{\circ} & \underset{2}{\circ} \arrow[l] & \underset{6}{\circ} \arrow[l] \arrow[u] \arrow[r] & \underset{5}{\circ} \arrow[r] & \underset{4}{\circ}
\end{tikzcd}
$$

with underlying graph $\mathbb{E}_6$. Then $\Gamma(\operatorname{mod} KQ)$ is the quiver

$$
\begin{tikzcd}[column sep=small, row sep=small]
\overset{0}{00001} \arrow[dr] & & & & \overset{0}{10000} \arrow[dl] \\
& \overset{0}{00011} \arrow[dl] \arrow[dr] & \overset{1}{00000} \arrow[d] & \overset{0}{11000} \arrow[dl] \arrow[dr] & \\
\overset{0}{00010} \arrow[dr] & & \overset{1}{11111} \arrow[dl] \arrow[d] \arrow[dr] & & \overset{0}{01000} \arrow[dl] \\
& \overset{1}{11110} \arrow[dl] \arrow[dr] & \overset{0}{11111} \arrow[d] & \overset{1}{01111} \arrow[dl] \arrow[dr] & \\
\overset{1}{11100} \arrow[dr] & & \overset{1}{12221} \arrow[dl] \arrow[d] \arrow[dr] & & \overset{1}{00111} \arrow[dl] \\
& \overset{1}{12211} \arrow[dl] \arrow[dr] & \overset{1}{01110} \arrow[d] & \overset{1}{11221} \arrow[dl] \arrow[dr] & \\
\overset{0}{01111} \arrow[dr] & & \overset{2}{12321} \arrow[dl] \arrow[d] \arrow[dr] & & \overset{0}{11110} \arrow[dl] \\
& \overset{1}{01221} \arrow[dl] \arrow[dr] & \overset{1}{11211} \arrow[d] & \overset{1}{12210} \arrow[dl] \arrow[dr] & \\
\overset{1}{00110} \arrow[dr] & & \overset{1}{12321} \arrow[dl] \arrow[d] \arrow[dr] & & \overset{1}{01100} \arrow[dl] \\
& \overset{1}{11210} \arrow[dl] \arrow[dr] & \overset{0}{01110} \arrow[d] & \overset{1}{01211} \arrow[dl] \arrow[dr] & \\
\overset{0}{11100} \arrow[dr] & & \overset{1}{01210} \arrow[dl] \arrow[d] \arrow[dr] & & \overset{0}{00111} \arrow[dl] \\
& \overset{0}{01100} \arrow[dr] & \overset{1}{00100} \arrow[d] & \overset{0}{00110} \arrow[dl] & \\
& & \overset{0}{00100} & &
\end{tikzcd}
$$

We leave to the reader as an exercise to describe explicitly each of the indecomposable KQ-modules as a representation. Notice that the largest root ${}_{12\overset{2}{3}21}$ has already been described in the proof of (2.6).

VII.6. Exercises

1. Show that each of the following matrix algebras is hereditary:

$$\text{(a)}\quad \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & K & K & 0 \\ K & K & 0 & K \end{bmatrix} \qquad \text{(b)}\quad \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & 0 & 0 & K \end{bmatrix} \qquad \text{(c)}\quad \begin{bmatrix} K & 0 & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 & 0 \\ K & 0 & K & K & 0 & 0 \\ K & 0 & K & 0 & K & 0 \\ K & 0 & K & 0 & K & K \end{bmatrix}.$$

In each case, give the ordinary quiver, then describe the indecomposable projective and the indecomposable injective modules.

2. Construct, as a matrix algebra, a hereditary algebra whose ordinary quiver is one of the following:

$$\text{(a)}\quad \begin{tikzcd} \circ & & \circ \arrow[dl] \\ & \circ \arrow[ul] \arrow[dl] & \\ \circ & & \circ \arrow[ul] \end{tikzcd} \qquad \text{(b)}\quad \begin{tikzcd} & & & \circ \arrow[dl] \\ \circ & \circ \arrow[l] & \circ \arrow[l] & \\ & & & \circ \arrow[ul] \end{tikzcd} \qquad \text{(c)}\quad \begin{tikzcd} \circ & \circ \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

3. Let A be an algebra. Show that the following conditions are equivalent:

(a) A is hereditary.
(b) For each module M_A, the functor $\mathrm{Ext}_A^1(M,-)$ is right exact.
(b) For each module ${}_AN$, the functor $\mathrm{Tor}_1^A(-,N)$ is left exact.

4. Let A be a finite dimensional basic connected hereditary algebra. Show that the following conditions are equivalent:

(a) A is a Nakayama algebra.
(b) $A \cong \mathbb{T}_n(K)$ for some $n \geq 1$.
(c) A admits a projective-injective indecomposable module.

5. An algebra A is called triangular if there exists a hereditary algebra H and a surjective algebra morphism $\varphi : H \to A$ such that $\operatorname{Ker}\varphi \subseteq \operatorname{rad}^2 H$. Show that A is triangular if and only if Q_A is acyclic.

6. Let Q be the quiver

$$\begin{tikzcd} & & \circ \arrow[d] & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] \end{tikzcd}$$

Construct bricks having as dimension vectors $\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 1 & 0 \end{smallmatrix}$, $\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$, and $\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 2 & 1 \end{smallmatrix}$, respectively.

7. Show that each of the following integral quadratic forms is positive definite:

(a) $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_1x_3 - x_2x_4 - x_3x_4 + x_1x_4$.
(b) $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 + x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 - x_3x_4$.

8. Show that each of the following integral quadratic forms is weakly positive but not positive definite.

(a) $x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3 + x_2x_3$.
(b) $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_1x_3 - x_2x_4 - x_3x_4 + 2x_1x_4$.
(c) $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 - x_1x_4 - x_2x_4 - x_3x_4 - x_4x_5 - x_4x_6$
$+x_1x_5 + x_1x_6 + x_2x_5 + x_2x_6 + x_3x_5 + x_3x_6$.

Show that the quadratic form (c) is not positive semidefinite.

9. A vector $\mathbf{x} \in \mathbb{Z}^n$ is called **sincere** if all its coordinates are nonzero . Let $\mathbf{x}$ be a sincere positive root of a weakly positive integral quadratic form q. Show that the following conditions are equivalent:

(a) $\mathbf{x}$ is a maximal root.
(b) $s_i(\mathbf{x}) \leq \mathbf{x}$ for each i.
(c) $D_iq(\mathbf{x}) \geq 0$ for each i.

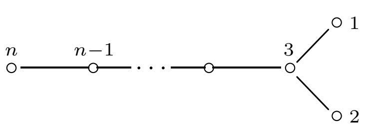
10. Let Q be a quiver with underlying graph

$\mathbb{A}_m$: 

$(m \geq 1)$.

Show that the positive roots of q_Q in $F = \bigoplus_{i=1}^m \mathbf{e}_i\mathbb{Z}$ are just the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\mathbf{e}_i + \mathbf{e}_{i+1} + \dots + \mathbf{e}_j$, where $1 \leq i < j \leq m$. Thus Q affords $\frac{m(m+1)}{2}$ positive roots.

11. Let Q be the quiver with underlying graph

$\widetilde{\mathbb{D}}_n$: 

$(n \geq 4)$.

Show that the positive roots of q_Q in $F = \bigoplus_{i=1}^n \mathbf{e}_i\mathbb{Z}$ are just the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$, $\mathbf{e}_i + \mathbf{e}_{i+1} + \dots + \mathbf{e}_j$, where $1 \leq i < j \leq n$, and $j \geq 3$, $\mathbf{e}_1 + \mathbf{e}_3 + \dots + \mathbf{e}_j$, where $j \geq 3$, $\mathbf{e}_1 + \mathbf{e}_2 + 2(\mathbf{e}_3 + \dots + \mathbf{e}_i) + \mathbf{e}_{i+1} + \dots + \mathbf{e}_j$, where $3 \leq i < j \leq n$. Thus Q affords $n(n-1)$ positive roots.

12. Compute all the positive roots for $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$ (one finds, respectively, 36, 63, and 120 positive roots).