Chapter I

Algebras and modules

We introduce here the notations and terminology we use on algebras and modules, and we briefly recall some of the basic facts from module theory. Examples of algebras, modules, and functors are presented. We introduce the notions of the (Jacobson) radical of an algebra and of a module; the notions of semisimple module, projective cover, injective envelope, the socle and the top of a module, local algebra, and primitive idempotent. We also collect basic facts from the module theory of finite dimensional K-algebras. In this chapter we present complete proofs of most of the results, except for a few classical theorems. In these cases the reader is referred to the following textbooks on this subject [2], [6], [49], [61], [131], and [165].

Throughout, we freely use the basic notation and facts on categories and functors introduced in the Appendix.

The reader interested mainly in linear representations of quivers and path algebras or familiar with elementary facts on rings and modules can skip this chapter and begin with Chapter II.

For the sake of simplicity of presentation, we always suppose that K is an algebraically closed field and that an algebra means a finite dimensional K-algebra, unless otherwise specified.

I.1. Algebras

By a **ring**, we mean a triple $(A, +, \cdot)$ consisting of a set A, two binary operations: addition $+ : A \times A \to A$, $(a, b) \mapsto a + b$; multiplication $\cdot : A \times A \to A$, $(a, b) \mapsto ab$, such that $(A, +)$ is an abelian group, with zero element $0 \in A$, and the following conditions are satisfied:

(i) $(ab)c = a(bc)$,

(ii) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

for all $a, b, c \in A$. In other words, the multiplication is associative and both left and right distributive over the addition. A ring A is **commutative** if $ab = ba$ for all $a, b \in A$.

We only consider rings such that there is an element $1 \in A$ where $1 \neq 0$ and $1a = a1 = a$ for all $a \in A$. Such an element is unique with respect to this property; we call it the **identity** of the ring A. In this case the ring is a quadruple $(A, +, \cdot, 1)$. Throughout, we identify the ring $(A, +, \cdot, 1)$ with its underlying set A.

A ring K is a **skew field** (or division ring) if every nonzero element a in K is invertible, that is, there exists $b \in K$ such that $ab = 1$ and $ba = 1$. A skew field K is said to be a **field** if K is commutative.

A field K is **algebraically closed** if any nonconstant polynomial $h(t)$ in one indeterminate t with coefficients in K has a root in K.

If A and B are rings, a map $f : A \to B$ is called a **ring homomorphism** if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in A$. If, in addition, A and B are rings with identity elements we assume that the ring homomorphism f preserves the identities, that is, that $f(1) = 1$.

Let K be a field. A K-**algebra** is a ring A with an identity element (denoted by 1) such that A has a K-vector space structure compatible with the multiplication of the ring, that is, such that

$$\lambda(ab) = (a\lambda)b = a(\lambda b) = (ab)\lambda$$

for all $\lambda \in K$ and all $a, b \in A$. A K-algebra A is said to be **finite dimensional** if the dimension $\dim_K A$ of the K-vector space A is finite.

A K-vector subspace B of a K-algebra A is a K-**subalgebra** of A if the identity of A belongs to B and $bb' \in B$ for all $b, b' \in B$. A K-vector subspace I of a K-algebra A is a **right ideal** of A (or **left ideal** of A) if $xa \in I$ (or $ax \in I$, respectively) for all $x \in I$ and $a \in A$. A two-sided ideal of A (or simply an ideal of A) is a K-vector subspace I of A that is both a left ideal and a right ideal of A.

It is easy to see that if I is a two-sided ideal of a K-algebra A, then the quotient K-vector space A/I has a unique K-algebra structure such that the canonical surjective linear map $\pi : A \to A/I$, $a \mapsto \overline{a} = a + I$, becomes a K-algebra homomorphism.

If I is a two-sided ideal of A and $m \geq 1$ is an integer, we denote by I^m the two-sided ideal of A generated by all elements $x_1 x_2 \ldots x_m$, where $x_1, x_2, \ldots, x_m \in I$, that is, I^m consists of all finite sums of elements of the form $x_1 x_2 \ldots x_m$, where $x_1, x_2, \ldots, x_m \in I$. We set $I^0 = A$. The ideal I is said to be **nilpotent** if $I^m = 0$ for some $m \geq 1$.

If A and B are K-algebras, then a ring homomorphism $f : A \to B$ is called a K-**algebra homomorphism** if f is a K-linear map. Two K-algebras A and B are called isomorphic if there is a K-algebra isomorphism $f : A \to B$, that is, a bijective K-algebra homomorphism. In this case we write $A \cong B$.

Throughout this book, K denotes an algebraically closed field.

1.1. Examples. (a) The ring $K[t]$ of all polynomials in the indeterminate t with coefficients in K and the ring $K[t_1, \ldots, t_n]$ of all polynomials in commuting indeterminates $t_1, \dots, t_n$ with coefficients in K are infinite dimensional K-algebras.

(b) If A is a K-algebra and $n \in \mathbb{N}$, then the set $\mathbb{M}_n(A)$ of all $n \times n$ square matrices with coefficients in A is a K-algebra with respect to the usual matrix addition and multiplication. The identity of $\mathbb{M}_n(A)$ is the matrix $E = \mathrm{diag}(1, \dots, 1) \in \mathbb{M}_n(A)$ with 1 on the main diagonal and zeros elsewhere. In particular $\mathbb{M}_n(K)$ is a K-algebra of dimension n^2. A K-basis of $\mathbb{M}_n(K)$ is the set of matrices e_{ij}, $1 \le i, j \le n$, where e_{ij} has the coefficient 1 in the position (i, j) and the coefficient 0 elsewhere.

(c) The subset

$$\mathbb{T}_n(K) = \begin{bmatrix} K & 0 & \dots & 0 \\ K & K & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K & K & \dots & K \end{bmatrix}$$

of $\mathbb{M}_n(K)$ consisting of all triangular matrices $[a_{ij}]$ in $\mathbb{M}_n(K)$ with zeros over the main diagonal is a K-subalgebra of $\mathbb{M}_n(K)$. If $n = 3$ then the subset

$$A = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ K & K & K \end{bmatrix}$$

of $\mathbb{M}_3(K)$ consisting of all lower triangular matrices $\lambda = [\lambda_{ij}] \in \mathbb{T}_3(K)$ with $\lambda_{21} = 0$ is a K-subalgebra of $\mathbb{M}_3(K)$, and also of $\mathbb{T}_3(K)$.

(d) Suppose that $(I; \preceq)$ is a finite **poset** (partially ordered set), where $I = \{a_1, \dots, a_n\}$ and $\preceq$ is a partial order relation on I. The subset

$$KI = \left\{ \lambda = [\lambda_{ij}] \in \mathbb{M}_n(K);\ \lambda_{st} = 0 \text{ if } a_s \not\preceq a_t \right\}$$

of $\mathbb{M}_n(K)$ consisting of all matrices $\lambda = [\lambda_{ij}]$ such that $\lambda_{ij} = 0$ if the relation $a_i \preceq a_j$ does not hold in I is a K-subalgebra of $\mathbb{M}_n(K)$. We call KI the **incidence algebra** of the poset $(I; \preceq)$ with coefficients in K. The matrices $\{e_{ij}\}$ with $a_i \preceq a_j$ form a basis of the K-vector space KI.

Without loss of generality, we may suppose that $I = \{1, \dots, n\}$ and that $i \preceq j$ implies that $i \ge j$ in the natural order. This can easily be done by a suitable renumbering of the elements in I. In this case, KI takes the form of the lower triangular matrix algebra

$$KI = \begin{bmatrix} K & 0 & \dots & 0 \\ K_{21} & K & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \dots & K \end{bmatrix},$$

where $K_{ij} = K$ if $i \preceq j$ and $K_{ij} = 0$ otherwise. For example, if $(I; \preceq)$ is the poset $\{1 \succ 2 \succ 3 \succ \cdots \succ n\}$ then the algebra KI is isomorphic to the algebra $\mathbb{T}_n(K)$ in Example 1.1 (c). If $(I; \preceq)$ is the poset $\{1 \succ 3 \prec 2\}$ then the incidence algebra KI is isomorphic to the five-dimensional algebra A in Example 1.1 (c). If the poset $(I; \preceq)$ is given by $I = \{1, 2, 3, 4\}$ and the relations $\{3 \succ 4 \prec 2 \prec 1 \succ 3\}$ then

$$KI = \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & K & K & K \end{bmatrix}.$$

(e) The associative ring $K\langle t_1, t_2\rangle$ of all polynomials in two noncommuting indeterminates t_1 and t_2 with coefficients in K is an infinite dimensional K-algebra. Note that, if I is the two-sided ideal in $K\langle t_1, t_2\rangle$ generated by the element $t_1t_2 - t_2t_1$, then the K-algebra $K\langle t_1, t_2\rangle/I$ is isomorphic to $K[t_1, t_2]$.

(f) Let $(G, \cdot)$ be a finite group with identity element e and let A be a K-algebra. The **group algebra** of G with coefficients in A is the K-vector space AG consisting of all the formal sums $\sum_{g\in G} g\,\lambda_g$, where $\lambda_g \in A$ and $g \in G$, with the multiplication defined by the formula

$$(\sum_{g\in G} g\,\lambda_g) \cdot (\sum_{h\in G} h\,\mu_h) = \sum_{f=gh\in G} f\lambda_g\mu_h.$$

Then AG is a K-algebra of dimension $|G| \cdot \dim_K A$ (here $|G|$ denotes the order of G) and the element $e = e1$ is the identity of AG. If $A = K$, then the elements $g \in G$ form a basis of KG over K.

For example, if G is a cyclic group of order m, then $KG \cong K[t]/(t^m - 1)$.

(g) Assume that A_1 and A_2 are K-algebras. The **product of the algebras** A_1 and A_2 is the algebra $A = A_1 \times A_2$ with the addition and the multiplication given by the formulas $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ and $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$, where $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. The identity of A is the element $1 = (1, 1) = e_1 + e_2 \in A_1 \times A_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

(h) For any K-algebra A we define the **opposite algebra** A^{op} of A to be the K-algebra whose underlying set and vector space structure are just those of A, but the multiplication $*$ in A^{op} is defined by formula $a * b = ba$.

1.2. Definition. The (Jacobson) **radical** $\operatorname{rad} A$ of a K-algebra A is the intersection of all the maximal right ideals in A.

It follows from (1.3) that $\operatorname{rad} A$ is the intersection of all the maximal left ideals in A. In particular, $\operatorname{rad} A$ is a two-sided ideal.

1.3. Lemma. *Let A be a K-algebra and let $a \in A$. The following conditions are equivalent:*

(a) $a \in \operatorname{rad} A$;

(a′) *a belongs to the intersection of all maximal left ideals of* A*;*
(b) *for any* $b \in A$*, the element* $1 - ab$ *has a two-sided inverse;*
(b′) *for any* $b \in A$*, the element* $1 - ab$ *has a right inverse;*
(c) *for any* $b \in A$*, the element* $1 - ba$ *has a two-sided inverse;*
(c′) *for any* $b \in A$*, the element* $1 - ba$ *has a left inverse.*

Proof. (a) implies (b′). Let $b \in A$ and assume to the contrary that $1 - ab$ has no right inverse. Then there exists a maximal right ideal I of A such that $1 - ab \in I$. Because $a \in \operatorname{rad} A \subseteq I$, $ab \in I$ and $1 \in I$; this is a contradiction. This shows that $1 - ab$ has a right inverse.

(b′) implies (a). Assume to the contrary that $a \notin \operatorname{rad} A$ and let I be a maximal right ideal of A such that $a \notin I$. Then $A = I + aA$ and therefore there exist $x \in I$ and $b \in A$ such that $1 = x + ab$. It follows that $x = 1 - ab \in I$ has no right inverse, contrary to our assumption. The equivalence of (a′) and (c′) can be proved in a similar way.

The equivalence of (b) and (c) is a consequence of the following two simple implications:

(i) If $(1 - cd)x = 1$, then $(1 - dc)(1 + dxc) = 1$.
(ii) If $y(1 - cd) = 1$, then $(1 + dyc)(1 - dc) = 1$.

(b′) implies (b). Fix an element $b \in A$. By (b′), there exists an element $c \in A$ such that $(1 - ab)c = 1$. Hence $c = 1 - a(-bc)$ and, according to (b′), there exists $d \in A$ such that $1 = cd = d + abcd = d + ab$. It follows that $d = 1 - ab$, c is the left inverse of $1 - ab$ and (b) follows. That (c′) implies (c) follows in a similar way. Because (b) implies (b′) and (c) implies (c′) obviously, the lemma is proved. □

1.4. Corollary. *Let* $\operatorname{rad} A$ *be the radical of an algebra* A*.*

(a) $\operatorname{rad} A$ *is the intersection of all the maximal left ideals of* A*.*
(b) $\operatorname{rad} A$ *is a two-sided ideal and* $\operatorname{rad}(A/\operatorname{rad} A) = 0$*.*
(c) *If* I *is a two-sided nilpotent ideal of* A*, then* $I \subseteq \operatorname{rad} A$*. If, in addition, the algebra* A/I *is isomorphic to a product* $K \times \cdots \times K$ *of copies of* K*, then* $I = \operatorname{rad} A$*.*

Proof. The statements (a) and (b) easily follow from (1.3).

(c) Assume that $I^m = 0$ for some $m > 0$. Let $x \in I$ and let a be an element of A. Then $ax \in I$ and therefore $(ax)^r = 0$ for some $r > 0$. It follows that the equality $(1 + ax + (ax)^2 + \cdots + (ax)^{r-1})(1 - ax) = 1$ holds for any element $a \in A$, and, according to (1.3), the element x belongs to $\operatorname{rad} A$. Consequently, $I \subseteq \operatorname{rad} A$. To prove the reverse inclusion, assume that the algebra A/I is isomorphic to a product of copies of K. It follows that $\operatorname{rad}(A/I) = 0$. Next, the canonical surjective algebra homomorphism $\pi : A \to A/I$ carries $\operatorname{rad} A$ to $\operatorname{rad}(A/I) = 0$. Indeed, if $a \in \operatorname{rad} A$ and $\pi(b) = b + I$, with $b \in A$, is any element of A/I then, by (1.3), $1 - ba$ is invertible in A and therefore the element $\pi(1-ba) = 1-\pi(b)\pi(a)$ is invertible in A/I; thus $\pi(a) \in \operatorname{rad} A/I = 0$, by (1.3). This yields $\operatorname{rad} A \subseteq \operatorname{Ker} \pi = I$ and finishes the proof. □

1.5. Examples. (a) Let $s_1, \ldots, s_n$ be positive integers and let $A = K[t_1, \ldots, t_n]/(t_1^{s_1}, \ldots, t_n^{s_n})$. Because the ideal $I = (\overline{t}_1, \ldots, \overline{t}_n)$ of A generated by the cosets $\overline{t}_1, \ldots, \overline{t}_n$ of the indeterminates $t_1, \ldots, t_n$ modulo the ideal $(t_1^{s_1}, \ldots, t_n^{s_n})$ is nilpotent, then (1.4) yields $I \subseteq \operatorname{rad} A$. On the other hand, there is a K-algebra isomorphism $A/I \cong K$. It follows that I is a maximal ideal and therefore $\operatorname{rad} A = I$.

(b) Let I be a finite poset and $A = KI$ be its incidence K-algebra viewed, as in (1.1)(d), as a subalgebra of the full matrix algebra $\mathbb{M}_n(K)$. Then $\operatorname{rad} A$ is the set U of all matrices $\lambda = [\lambda_{ij}] \in KI$ with $\lambda_{ii} = 0$ for $i = 1, 2, \ldots, n$, and the algebra $A/\operatorname{rad} A$ is isomorphic to the product $K \times \cdots \times K$ of n copies of K. Indeed, we note that the set U is clearly a two-sided ideal of KI, it is easily seen that $U^n = 0$ and finally the algebra A/U is isomorphic to the product of n copies of K, thus (1.4)(c) applies.

(c) By applying the preceding arguments, one also shows that the radical $\operatorname{rad} A$ of the lower triangular matrix algebra $A = \mathbb{T}_n(K)$ of (1.1)(c) consists of all matrices in A with zeros on the main diagonal. It follows that $(\operatorname{rad} A)^n = 0$.

In the study of modules over finite dimensional K-algebras over an algebraically closed field K an important rôle is played by the following theorem, known as the Wedderburn–Malcev theorem.

1.6. Theorem. *Let A be a finite dimensional K-algebra. If the field K is algebraically closed, then there exists a K-subalgebra B of A such that there is a K-vector space decomposition $A = B \oplus \operatorname{rad} A$ and the restriction of the canonical surjective algebra homomorphism $\pi : A \to A/\operatorname{rad} A$ to B is a K-algebra isomorphism.*

Proof. See [61, section VI.2] and [131, section 11.6]. □

I.2 Modules

2.1. Definition. Let A be a K-algebra. A **right A-module** (or a right module over A) is a pair $(M, \cdot)$, where M is a K-vector space and $\cdot : M \times A \to M$, $(m, a) \mapsto ma$, is a binary operation satisfying the following conditions:

(a) $(x + y)a = xa + ya$;
(b) $x(a + b) = xa + xb$;
(c) $x(ab) = (xa)b$;
(d) $x1 = x$;

(e) $(x\lambda)a = x(a\lambda) = (xa)\lambda$
for all $x, y \in M$, $a, b \in A$ and $\lambda \in K$.

The definition of a left A-module is analogous. Throughout, we write M or M_A instead of $(M, \cdot)$. We write A_A and ${}_AA$ whenever we view the algebra A as a right or left A-module, respectively.

A module M is said to be **finite dimensional** if the dimension $\dim_K M$ of the underlying K-vector space of M is finite.

A K-subspace M' of a right A-module M is said to be an **A-submodule** of M if $ma \in M'$ for all $m \in M'$ and all $a \in A$. In this case the K-vector space M/M' has a natural A-module structure such that the canonical epimorphism $\pi : M \to M/M'$ is an A-module homomorphism.

Let M be a right A-module and let I be a right ideal of A. It is easy to see that the set MI consisting of all sums $m_1a_1 + \ldots + m_sa_s$, where $s \geq 1$, $m_1, \ldots, m_s \in M$ and $a_1, \ldots, a_s \in I$, is a submodule of M.

A right A-module M is said to be **generated** by the elements $m_1, \ldots, m_s$ of M if any element $m \in M$ has the form $m = m_1a_1 + \cdots + m_sa_s$ for some $a_1, \ldots, a_s \in A$. In this case, we write $M = m_1A + \ldots + m_sA$. A module M is said to be **finitely generated** if it is generated by a finite subset of elements of M.

Let $M_1, \ldots, M_s$ be submodules of a right A-module M. We define $M_1 + \ \ldots\ + M_s$ to be the submodule of M consisting of all sums $m_1 + \cdots + m_s$, where $m_1 \in M_1, \cdots, m_s \in M_s$, and we call it the submodule generated by $M_1, \ldots, M_s$, or the sum of $M_1, \ldots, M_s$.

Note that a right module M over a finite dimensional K-algebra A is finitely generated if and only if M is finite dimensional. Indeed, if $x_1, \ldots, x_m$ is a K-basis of M, then it is obviously a set of A-generators of M. Conversely, if the A-module M is generated by the elements $m_1, \ldots, m_n$ over A and $\xi_1, \ldots \xi_s$ is a K-basis of A then the set $\{m_j\xi_i; j = 1, \ldots, n, i = 1, \ldots, s\}$ generates the K-vector space M.

Throughout, we frequently use the following lemma, known as Nakayama's lemma.

2.2. Lemma. *Let A be a K-algebra, M be a finitely generated right A-module, and $I \subseteq \operatorname{rad} A$ be a two-sided ideal of A. If $MI = M$, then $M = 0$.*

Proof. Suppose that $M = MI$ and $M = m_1A + \cdots + m_sA$, that is, M is generated by the elements $m_1, \ldots, m_s$. We proceed by induction on s. If $s = 1$, then the equality $m_1A = m_1I$ implies that $m_1 = m_1x_1$ for some $x_1 \in I$. Hence $m_1(1 - x_1) = 0$ and therefore $m_1 = 0$, because $1 - x_1$ is invertible. Consequently $M = 0$, as required.

Assume that $s \geq 2$. The equality $M = MI$ implies that there are elements $x_1, \ldots, x_s \in I$ such that $m_1 = m_1x_1 + m_2x_2 + \cdots + m_sx_s$. Hence $m_1(1 - x_1) = m_2x_2 + \cdots + m_sx_s$ and therefore $m_1 \in m_2A + \cdots + m_sA$ because $1 - x_1$ is invertible. This shows that $M = m_2A + \cdots + m_sA$ and the inductive hypothesis yields $M = 0$. □

2.3. Corollary. *If A is a finite dimensional K-algebra, then* $\operatorname{rad} A$ *is nilpotent.*

Proof. Because $\dim_K A < \infty$, the chain

$$A \supseteq \operatorname{rad} A \supseteq (\operatorname{rad} A)^2 \supseteq \cdots \supseteq (\operatorname{rad} A)^m \supseteq (\operatorname{rad} A)^{m+1} \supseteq \cdots$$

becomes stationary. It follows that $(\operatorname{rad} A)^m = (\operatorname{rad} A)^m \operatorname{rad} A$ for some m, and Nakayama's lemma (2.2) yields $(\operatorname{rad} A)^m = 0$. □

Let M and N be right A-modules. A K-linear map $h : M \to N$ is said to be an **A-module homomorphism** (or simply an A-homomorphism) if $h(ma) = h(m)a$ for all $m \in M$ and $a \in A$. An A-module homomorphism $h : M \to N$ is said to be a **monomorphism** (or an **epimorphism**) if it is injective (or surjective, respectively). A bijective A-module homomorphism is called an **isomorphism**. The right A-modules M and N are said to be **isomorphic** if there exists an A-module isomorphism $h : M \to N$. In this case, we write $M \cong N$. An A-module homomorphism $h : M \to M$ is said to be an **endomorphism** of M.

The set $\operatorname{Hom}_A(M, N)$ of all A-module homomorphisms from M to N is a K-vector space with respect to the scalar multiplication $(f, \lambda) \mapsto f\lambda$ given by $(f\lambda)(m) = f(m\lambda)$ for $f \in \operatorname{Hom}_A(M, N)$, $\lambda \in K$ and $m \in M$. If the modules M and N are finite dimensional, then the K-vector space $\operatorname{Hom}_A(M, N)$ is finite dimensional. The K-vector space

$$\operatorname{End} M = \operatorname{Hom}_A(M, M)$$

of all A-module endomorphisms of any right A-module M is an associative K-algebra with respect to the composition of maps. The identity map 1_M on M is the identity of $\operatorname{End} M$.

It is easy to check that for any triple L, M, N of right A-modules the composition mapping $\cdot : \operatorname{Hom}_A(M, N) \times \operatorname{Hom}_A(L, M) \longrightarrow \operatorname{Hom}_A(L, N)$, $(h, g) \mapsto hg$, is K-bilinear.

It is clear that the **kernel** $\operatorname{Ker} h = \{m \in M \mid h(m) = 0\}$, the **image** $\operatorname{Im} h = \{h(m) \mid m \in M\}$, and the **cokernel** $\operatorname{Coker} h = N/\operatorname{Im} h$ of an A-module homomorphism $h : M \to N$ have natural A-module structures.

The **direct sum** of the right A-modules $M_1, \ldots, M_s$ is defined to be the K-vector space direct sum $M_1 \oplus \cdots \oplus M_s$ equipped with an A-module structure defined by $(m_1, \ldots, m_s)a = (m_1a, \ldots, m_sa)$ for $m_1 \in M_1, \ldots, m_s \in M_s$

and $a \in A$. We set

$$M^s = M \oplus \cdots \oplus M, \quad (s \text{ copies}).$$

A right A-module M is said to be **indecomposable** if M is nonzero and M has no direct sum decomposition $M \cong N \oplus L$, where L and N are nonzero A-modules.

We denote by $\operatorname{Mod} A$ the abelian category of all right A-modules, that is, the category whose objects are right A-modules, the morphisms are A-module homomorphisms, and the composition of morphisms is the usual composition of maps. The reader is referred to Sections 1 and 2 of the Appendix for basic facts on categories and functors. Throughout, we freely use the notation introduced there.

We note that any left A-module can be viewed as a right A^{op}-module and conversely. Therefore, throughout the text, the category $\operatorname{Mod} A^{\mathrm{op}}$ is identified with the category of all left A-modules.

We denote by $\operatorname{mod} A$ the full subcategory of $\operatorname{Mod} A$ whose objects are the finitely generated modules. It follows that if A is a finite dimensional K-algebra, then all modules in $\operatorname{mod} A$ are finite dimensional.

An important idea in the study of A-modules is to view them as sets of K-vector spaces connected by K-linear maps. This is illustrated by the following three examples.

2.4. Example. Let A be the lower triangular matrix K-subalgebra

$$A = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}$$

of the matrix algebra $\mathbb{M}_2(K)$. We note that the matrices $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ form a K-basis of A over K, $1_R = e_1 + e_2$, and $e_1 e_2 = e_2 e_1 = 0$.

It follows that every module X in $\operatorname{mod} A$, viewed as a K-vector space, has a direct sum decomposition $X = X_1 \oplus X_2$, where X_1, X_2 are the vector spaces Xe_1, Xe_2 over K. Note that given $a = \left(\begin{smallmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{smallmatrix}\right) \in A$ and $x = (x_1, x_2) \in X$ with $x_1 \in X_1$ and $x_2 \in X_2$ we have

$$xa = (x_1 a_{11} + x_2 a_{21}, x_2 a_{22}) = (x_1 a_{11} + f_X(x_2) a_{21}, x_2 a_{22}),$$

where $f_X : X_2 \to X_1$ is the K-linear map given by the formula $f_X(x_2) = x_2 e_{21} = x_2 e_{21} e_{11}$. It follows that X, viewed as a right A-module, can be identified with the triple $(X_1 \xleftarrow{f_X} X_2)$. Moreover, any A-module homomorphism $h : X \to Y$ can be identified with the pair (h_1, h_2) of K-linear maps $h_1 : X_1 \to Y_1$, $h_2 : X_2 \to Y_2$ that are the restrictions of h to, respectively, X_1 and X_2. These satisfy the equation $h_1 f_X = f_Y h_2$.

The converse correspondence to $X \mapsto (X_1 \xleftarrow{f_X} X_2)$ is defined by associating to any triple $(X_1 \xleftarrow{f} X_2)$ with K-vector spaces X_1, X_2 and

$f \in \mathrm{Hom}_K(X_2, X_1)$, the K-vector space $X = X_1 \oplus X_2$ endowed with the right action $\cdot : X \times A \to X$ of A on X defined by the formula $(x_1, x_2)\left(\begin{smallmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{smallmatrix}\right) = (x_1 a_{11} + f(x_2)a_{21}, x_2 a_{22})$, where $x_1 \in X_1$, $x_2 \in X_2$, and $\left(\begin{smallmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{smallmatrix}\right) \in A$.

2.5. Example. Let A be the **Kronecker algebra**

$$A = \begin{bmatrix} K & 0 \\ K^2 & K \end{bmatrix}$$

whose elements are 2×2 matrices of the form $\left(\begin{smallmatrix} \lambda & 0 \\ (u_1,u_2) & \mu \end{smallmatrix}\right)$ with $\lambda, \mu \in K$, $(u_1, u_2) \in K^2$, and the multiplication in A is defined by the formula

$$\begin{pmatrix} d & 0 \\ (u_1,u_2) & c \end{pmatrix} \begin{pmatrix} f & 0 \\ (v_1,v_2) & e \end{pmatrix} = \begin{pmatrix} df & 0 \\ (u_1 f + v_1 c, u_2 f + v_2 c) & ce \end{pmatrix}.$$

Finite dimensional right A-modules are called **Kronecker modules**. Every such A-module X can be identified with a quadruple

$$\Big(X_1 \underset{\varphi_2}{\overset{\varphi_1}{\leftleftarrows}} X_2 \Big),$$

where X_1, X_2 are the K-vector spaces Xe_1, Xe_2, respectively, $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, φ_1, φ_2 are the K-linear maps defined by the formulas

$$\varphi_1(x) = x \cdot \left(\begin{smallmatrix} 0 & 0 \\ \xi_1 & 0 \end{smallmatrix}\right) = x \cdot \left(\begin{smallmatrix} 0 & 0 \\ \xi_1 & 0 \end{smallmatrix}\right) \cdot e_1, \ \ \varphi_2(x) = x \cdot \left(\begin{smallmatrix} 0 & 0 \\ \xi_2 & 0 \end{smallmatrix}\right) = x \cdot \left(\begin{smallmatrix} 0 & 0 \\ \xi_2 & 0 \end{smallmatrix}\right) \cdot e_1,$$

for $x \in X_2$, where $\xi_1 = (1, 0)$ and $\xi_2 = (0, 1)$ are the standard basis vectors of K^2. Any A-module homomorphism $c : X' \to X$ can be identified with a pair (c_1, c_2) of K-linear maps $c_1 : X'_1 \to X_1$ and $c_2 : X'_2 \to X_2$ such that $c_1\varphi'_1 = \varphi_1 c_2$ and $c_1\varphi'_2 = \varphi_2 c_2$.

The converse correspondence to $X \mapsto (X_1 \underset{\varphi_2}{\overset{\varphi_1}{\leftleftarrows}} X_2)$ is defined by associating to any quadruple $(X_1 \underset{\varphi_2}{\overset{\varphi_1}{\leftleftarrows}} X_2)$ with finite dimensional K-vector spaces X_1, X_2 and $\varphi_1, \varphi_2 \in \mathrm{Hom}_K(X_2, X_1)$, the K-vector space $X = X_1 \oplus X_2$ endowed with the right action $\cdot : X \times A \to X$ of A on X defined by the formula

$$(x_1, x_2)\begin{pmatrix} \lambda & 0 \\ (u_1,u_2) & \mu \end{pmatrix} = (x_1\lambda + \varphi_1(x_2)u_1 + \varphi_1(x_2)u_2,\ x_2\mu),$$

where $x_1 \in X_1$, $x_2 \in X_2$ and $\begin{pmatrix} \lambda & 0 \\ (u_1,u_2) & \mu \end{pmatrix} \in A$.

It follows that the category of Kronecker modules is equivalent to the category of pairs $[\Phi_1, \Phi_2]$ of matrices Φ_1, Φ_2 over K of the same size, where the map from $[\Phi'_1, \Phi'_2]$ to $[\Phi_1, \Phi_2]$ is a pair (C_1, C_2) of matrices with coefficients in K such that $C_1\Phi'_1 = \Phi_1 C_2$ and $C_1\Phi'_2 = \Phi_2 C_2$.

2.6. Example. Let $K[t]$ be the K-algebra of all polynomials in the indeterminate t with coefficients in K. Note that every module V in $\mathrm{Mod}\, K[t]$

may be viewed as a pair (V, h), where V is the underlying K-vector space and $h : V \to V$ is the K-linear endomorphism $v \mapsto vt$. Every $K[t]$-module homomorphism $f : V \to V'$ may be viewed as a K-linear map such that $fh = h'f$.

The converse correspondence to $V \mapsto (V, h)$ is given by attaching to any pair (V, h), with a K-vector space V and $h \in \mathrm{End}_K V$, the K-vector space V endowed with the right action $\cdot : V \times K[t] \longrightarrow V$ of $K[t]$ on V given by the formula

$$v \cdot (\lambda_0 + t\lambda_1 + \cdots + t^m \lambda_m) = v\lambda_0 + h(v)\lambda_1 + \cdots + h^m(v)\lambda_m,$$

where $v \in V$ and $\lambda_0, \ldots, \lambda_m \in K$. The reader is referred to [49] for details.

2.7. Example. Assume that $A = A_1 \times A_2$ is the product of two K-algebras A_1 and A_2. The identity of A is the element $1 = (1, 1) = e_1 + e_2 \in A_1 \times A_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that $e_1 e_2 = e_2 e_1 = 0$. If X_A is a right A-module, then Xe_1 is a right A_1-module, Xe_2 is a right A_2-module and there is an A-module direct sum decomposition $X = Xe_1 \oplus Xe_2$, where Xe_j is viewed as a right A-module via the algebra projection $A \to A_j$ for $j = 1, 2$. Then the same type of arguments as in the previous examples shows that the correspondence $X_A \mapsto (Xe_1, Xe_2)$ defines an equivalence of categories $\mathrm{Mod}(A_1 \times A_2) \cong \mathrm{Mod}\, A_1 \times \mathrm{Mod}\, A_2$, which we use throughout as an identification.

2.8. A matrix notation. In presenting homomorphisms between direct sums of A-modules, we use the following matrix notation. Given a set of A-module homomorphisms $f_1 : X_1 \to Y, \ldots, f_n : X_n \to Y$ and $g_1 : Y \to Z_1, \ldots, g_m : Y \to Z_m$ in $\mathrm{Mod}\, A$ we define two A-module homomorphisms

$$f = [f_1 \ \ldots \ f_n] : X_1 \oplus \cdots \oplus X_n \longrightarrow Y, \quad g = \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix} : Y \longrightarrow Z_1 \oplus \cdots \oplus Z_m$$

by the following formulas $f(x_1, \ldots, x_n) = f_1(x_1) + \ldots + f_n(x_n)$ and $g(y) = (g_1(y), \ldots, g_m(y))$ for $x_j \in X_j$ and $y \in Y$. It is easy to see that f and g are the unique A-module homomorphisms in $\mathrm{Mod}\, A$ such that $fu_j = f_j$ for $j = 1, \ldots, n$ and $p_i g = g_i$ for $i = 1, \ldots, m$, where $u_j : X_j \to X_1 \oplus \cdots \oplus X_n$ is the jth summand embedding $x_j \mapsto (0, \ldots, 0, x_j, 0, \ldots, 0)$ and $p_i : Z_1 \oplus \cdots \oplus Z_m \to Z_i$ is the ith summand projection $(z_1, \ldots, z_m) \mapsto z_i$. If $X = X_1 \oplus \cdots \oplus X_n$ and $Z = Z_1 \oplus \cdots \oplus Z_m$, then any A-module homomorphism $h : X \to Z$ in $\mathrm{Mod}\, A$ can be written in the form of an $m \times n$ matrix

$$h = [h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1n} \\ h_{21} & h_{22} & \ldots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1} & h_{m2} & \ldots & h_{mn} \end{bmatrix},$$

where $h_{ij} = p_i h u_j \in \mathrm{Hom}_A(X_j, Z_i)$.

2.9. Standard dualities. Let A be a finite dimensional K-algebra. We define the functor

$$D : \operatorname{mod} A \longrightarrow \operatorname{mod} A^{\mathrm{op}}$$

by assigning to each right module M in $\operatorname{mod} A$ the dual K-vector space

$$M^* = \operatorname{Hom}_K(M, K)$$

endowed with the left A-module structure given by the formula $(a\varphi)(m) = \varphi(ma)$ for $\varphi \in \operatorname{Hom}_K(M, K)$, $a \in A$ and $m \in M$, and to each A-module homomorphism $h : M \to N$ the dual K-homomorphism $D(h) = \operatorname{Hom}_K(h, K) : D(N) \longrightarrow D(M)$, $\varphi \mapsto \varphi h$, of left A-modules. One shows that D is a duality of categories, called the **standard K-duality**. The quasi-inverse to the duality D is also denoted by

$$D : \operatorname{mod} A^{\mathrm{op}} \longrightarrow \operatorname{mod} A$$

and is defined by attaching to each left A-module Y the dual K-vector space $D(Y) = Y^* = \operatorname{Hom}_K(Y, K)$ endowed with the right A-module structure given by the formula $(\varphi a)(y) = \varphi(ay)$ for $\varphi \in \operatorname{Hom}_K(Y, K)$, $a \in A$ and $y \in Y$. A straightforward calculation shows that the evaluation K-linear map $\mathrm{ev} : M \to M^{**}$ given by the formula $\mathrm{ev}(m)(f) = f(m)$, where $m \in M$ and $f \in D(M)$, defines natural equivalences of functors $1_{\operatorname{mod} A} \cong D \circ D$ and $1_{\operatorname{mod} A^{\mathrm{op}}} \cong D \circ D$.

Any right A-module M is a left module over the algebra $\operatorname{End} M$ with respect to the left multiplication $(\operatorname{End} M) \times M \to M$, $(\varphi, m) \mapsto \varphi m = \varphi(m)$. It is easy to check that M is an $(\operatorname{End} M)$–A-bimodule in the following sense.

2.10. Definition. Let A and B be two K-algebras. An **A-B-bimodule** is a triple ${}_AM_B = (M, *, \cdot)$, where ${}_AM = (M, *)$ is a left A-module, $M_B = (M, \cdot)$ is a right B-module, and $(a * m) \cdot b = a * (m \cdot b)$ for all $m \in M$, $a \in A$, $b \in B$. Throughout, we write simply am and mb instead of $a * m$ and $m \cdot b$, respectively.

For any A-B-bimodule ${}_AM_B$ and for any right B-module X_B, the K-vector space $\operatorname{Hom}_B({}_AM_B, X_B)$ of all B-module homomorphisms from M_B to X_B is a right A-module with respect to the A-scalar multiplication $(f, a) \mapsto fa$ given by $(fa)(m) = f(am)$ for $f \in \operatorname{Hom}_B(M_B, X_B)$, $a \in A$ and $m \in M$. If M and X are finite dimensional over K, then so is $\operatorname{Hom}_B({}_AM_B, X_B)$.

Important examples of functors are the Hom-functors $\operatorname{Hom}_B({}_AM_B, -)$ and $\operatorname{Hom}_B(-, {}_AM_B)$. We define the covariant Hom-functor

$$\operatorname{Hom}_B({}_AM_B, -) : \operatorname{Mod} B \longrightarrow \operatorname{Mod} A$$

by associating to X_B in $\operatorname{Mod} B$ the K-vector space $\operatorname{Hom}_B({}_AM_B, X_B)$ endowed with the right A-module structure defined earlier. If $\varphi : X_B \to Y_B$ is a homomorphism of B-modules, we define the induced homomorphism $\operatorname{Hom}_B({}_AM_B, \varphi) : \operatorname{Hom}_B({}_AM_B, X_B) \to \operatorname{Hom}_B({}_AM_B, Y_B)$ of right A-modules by the formula $f \mapsto \varphi f$. The contravariant Hom-functor

$$\operatorname{Hom}_B(-, {}_AM_B) : \operatorname{Mod} B \longrightarrow \operatorname{Mod} A^{\mathrm{op}}$$

is defined by $X_B \mapsto \operatorname{Hom}_B(X_B, {}_AM_B)$ and by assigning to any homomorphism $\psi : X_B \longrightarrow Y_B$ of right B-modules the induced homomorphism $\operatorname{Hom}_B(\psi, {}_AM_B) : \operatorname{Hom}_B(Y_B, {}_AM_B) \to \operatorname{Hom}_B(X_B, {}_AM_B)$, $f \mapsto f\psi$, of left A-modules.

We recall also that, given an A-B-bimodule ${}_AM_B$, the covariant **tensor product functors**

$$(-)\otimes_A M_B : \operatorname{Mod} A \longrightarrow \operatorname{Mod} B, \quad {}_AM \otimes_B (-) : \operatorname{Mod} B^{\mathrm{op}} \longrightarrow \operatorname{Mod} A^{\mathrm{op}}$$

are defined by associating to any right A-module X_A and to any left B-module ${}_BY$ the tensor products $X \otimes_A M_B$ and ${}_AM \otimes_B Y$ endowed with the natural right B-module and left A-module structure, respectively. It is well known that there exists an **adjunction isomorphism**

$$\operatorname{Hom}_B(X \otimes_A M_B, Z_B) \cong \operatorname{Hom}_A(X_A, \operatorname{Hom}_B({}_AM_B, Z_B)) \qquad \mathbf{(2.11)}$$

given by attaching to a B-module homomorphism $\varphi : X \otimes_A M_B \longrightarrow Z_B$ the A-module homomorphism

$$\overline{\varphi} : X_A \longrightarrow \operatorname{Hom}_B({}_AM_B, Z_B)$$

adjoint to φ defined by the formula $\overline{\varphi}(x)(m) = \varphi(x \otimes m)$, where $x \in X$ and $m \in M$. A straightforward calculation shows that the inverse to $\varphi \mapsto \overline{\varphi}$ is defined by $\psi \mapsto (x \otimes m \mapsto \psi(x)(m))$, where $x \in X$ and $m \in M$.

Formula (2.11) shows that the functor $(-) \otimes_A M_B$ is left adjoint to the functor $\operatorname{Hom}_B(-, {}_AM_B)$, and that $\operatorname{Hom}_B(-, {}_AM_B)$ is right adjoint to $(-) \otimes_A M_B$ (see (A.2.3) of the Appendix).

I.3 Semisimple modules and the radical of a module

Throughout, we assume that K is an algebraically closed field and that A is a finite dimensional K-algebra. A right A-module S is **simple** if S is nonzero and any submodule of S is either zero or S. A module M is **semisimple** if M is a direct sum of simple modules.

3.1. Schur's lemma. *Let S and S' be right A-modules, and $f : S \to S'$ be a nonzero A-homomorphism.*

(a) *If S is simple, then f is a monomorphism.*
(b) *If S' is simple, then f is an epimorphism.*
(c) *If S and S' are simple, then f is an isomorphism.*

Proof. Because $f : S \to S'$ is an A-module homomorphism, $\operatorname{Ker} h$ and $\operatorname{Im} h$ are A-submodules of S and S', respectively. Then $f \neq 0$ yields $\operatorname{Ker} h = 0$ if S is simple, and $\operatorname{Im} h = S'$ if S' is simple. The lemma follows. □

3.2. Corollary. *If S is a simple A-module, then there is a K-algebra isomorphism* $\operatorname{End} S \cong K$*.*

Proof. It follows from Schur's lemma that any nonzero element in $\operatorname{End} S$ is invertible and therefore $\operatorname{End} S$ is a skew field. Because S is simple, S is a cyclic A-module and therefore $\dim_K S$ is finite. It follows that $\dim_K \operatorname{End} S$ is finite and, for any nonzero element $\varphi \in \operatorname{End} S$, the elements $1_S, \varphi, \varphi^2, \ldots, \varphi^m, \ldots$ are linearly dependent over K. Consequently, there exists an irreducible nonzero polynomial $f(t) \in K[t]$ such that $f(\varphi) = 0$. Because the field K is algebraically closed, f is of degree 1 and therefore φ acts on S as the multiplication by a scalar $\lambda_\varphi \in K$. The correspondence $\varphi \mapsto \lambda_\varphi$ establishes a K-algebra isomorphism $\operatorname{End} S \cong K$. □

3.3. Lemma. (a) *A finite dimensional right A-module M is semisimple if and only if for any A-submodule N of M there exists a submodule L of M such that $L \oplus N = M$.*

(b) *A submodule of a semisimple module is semisimple.*

Proof. (a) Assume that $M = S_1 \oplus \cdots \oplus S_m$, where $S_1, \ldots . S_m$ are simple modules. Let N be a nonzero A-submodule of M and let $\{S_{j_1}, \ldots, S_{j_t}\}$ be a maximal family of modules in the set $\{S_1, \ldots, S_m\}$ such that the intersection of N with the module $L = S_{j_1} \oplus \cdots \oplus S_{j_t}$ is zero. It follows that $N \cap (L + S_t) \neq 0$, for all $t \notin \{j_1, \ldots, j_m\}$. This implies that $(L + N) \cap S_t \neq 0$ and hence we conclude that $S_t \subseteq L + N$, for all $t \notin \{j_1, \ldots, j_m\}$, because S_t is simple. Consequently, we get $M = L + N$ and therefore $M = L \oplus N$. The converse implication follows easily by induction on $\dim_K M$.

Because (b) is an immediate consequence of (a), the lemma is proved. □

For any right A-module M, the submodule $\operatorname{soc} M$ of M generated by all simple submodules of M is a semisimple module (see [2], [131]); it is called the **socle** of M. The main properties of the socle are listed in Exercise I.17.

Throughout, we frequently use the following well-known result.

3.4. Wedderburn–Artin theorem. *For any finite dimensional algebra A over an algebraically closed field K the following conditions are equivalent:*

(a) *The right A-module A_A is semisimple.*
(b) *Every right A-module is semisimple.*
(a′) *The left A-module ${}_AA$ is semisimple.*
(b′) *Every left A-module is semisimple.*
(c) $\operatorname{rad} A = 0$.
(d) *There exist positive integers $m_1, \ldots, m_s$ and a K-algebra isomorphism*

$$A \cong \mathbb{M}_{m_1}(K) \times \cdots \times \mathbb{M}_{m_s}(K).$$

Proof. See [2], [49], [61], [131], and [164]. □

A finite dimensional K-algebra A is called **semisimple** if one of the equivalent conditions in the Wedderburn–Artin theorem (3.4) is satisfied.

By (3.4), the commutative algebra $A = K[X_1, \ldots, X_n]/(X_1^{s_1}, \ldots, X_n^{s_n})$ of Example 1.5(a), where $s_1, \ldots, s_n$ are positive integers and $n \geq 1$, is semisimple if and only if $s_1 = \ldots = s_n = 1$.

In view of Example 1.5(b), the incidence K-algebra KI of a poset I is semisimple if and only if $a_i \not\preceq a_j$ for every pair of elements $a_i \neq a_j$ of I.

The semisimple group algebras KG are characterised as follows.

3.5. Maschke's theorem. *Let G be a finite group and let K be a field. Then the group algebra KG is semisimple if and only if the characteristic of K does not divide the order of G.*

Proof. See [61], [131], [164] and Section 5 of Chapter V. □

We now define the radical of a module.

3.6. Definition. Let M be a right A-module. The (Jacobson) **radical** $\operatorname{rad} M$ of M is the intersection of all the maximal submodules of M.

It follows from (1.2) that the radical $\operatorname{rad} A_A$ of the right A-module A_A is the radical $\operatorname{rad} A$ of the algebra A.

The main properties of the radical are collected in the following proposition.

3.7. Proposition. *Suppose that L, M, and N are modules in* $\operatorname{mod} A$.
(a) *An element $m \in M$ belongs to* $\operatorname{rad} M$ *if and only if $f(m) = 0$ for any $f \in \operatorname{Hom}_A(M, S)$ and any simple right A-module S.*
(b) $\operatorname{rad}(M \oplus N) = \operatorname{rad} M \oplus \operatorname{rad} N$.
(c) *If $f \in \operatorname{Hom}_A(M, N)$, then $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$.*
(d) $M \operatorname{rad} A = \operatorname{rad} M$.
(e) *Assume that L and M are A-submodules of N. If $L \subseteq \operatorname{rad} N$ and $L + M = N$, then $M = N$.*

Proof. The statement (a) follows immediately from the definition, because $L \subseteq M$ is a maximal submodule if and only if M/L is simple. The statements (b) and (c) follow immediately from (a). We leave them as an exercise.

(d) Take $m \in M$ and define a homomorphism $f_m : A \to M$ of right A-modules by the formula $f_m(a) = ma$ for $a \in A$. It follows from (c) that for $a \in \operatorname{rad} A$ we get $ma = f_m(a) \in f_m(\operatorname{rad} A) \subseteq \operatorname{rad} M$ and therefore $M\operatorname{rad} A \subseteq \operatorname{rad} M$. To prove the inclusion $\operatorname{rad} M \subseteq M\operatorname{rad} A$ we note that $(M/M\operatorname{rad} A)\operatorname{rad} A = 0$ and therefore the A-module $M/M\operatorname{rad} A$ is a module over the algebra $A/\operatorname{rad} A$ with respect to the action $(m + M\operatorname{rad} A) \cdot (a + \operatorname{rad} A) = ma + M\operatorname{rad} A$. By the Wedderburn–Artin theorem (3.4), the algebra $A/\operatorname{rad} A$ is semisimple and the finite dimensional $A/\operatorname{rad} A$-module $M/M\operatorname{rad} A$ is a direct sum of simple modules. Because the radical of any simple module is zero, (b) yields $\operatorname{rad}(M/M\operatorname{rad} A) = 0$. By (c), the canonical A-module epimorphism $\pi : M \to M/M\operatorname{rad} A$ carries $\operatorname{rad} M$ to zero, that is, $\operatorname{rad} M \subseteq \operatorname{Ker} \pi = M\operatorname{rad} A$ and we are done.

(e) Assume that $L \subseteq \operatorname{rad} N$ and $L + M = N$, and suppose to the contrary that $M \neq N$. Because N is finite dimensional, M is a submodule of a maximal submodule $X \neq N$ of N. It follows that $L \subseteq \operatorname{rad} N \subseteq X$ and we get $N = L + M \subseteq X + M = X$, contrary to our assumption. □

3.8. Corollary. *Suppose that M is a module in* $\operatorname{mod} A$.

(a) *The A-module $M/\operatorname{rad} M$ is semisimple and it is a module over the K-algebra $A/\operatorname{rad} A$.*

(b) *If L is a submodule of M such that M/L is semisimple, then $\operatorname{rad} M \subseteq L$.*

Proof. (a) We recall from (3.7)(d) that $\operatorname{rad} M = M\operatorname{rad} A$. It follows that $(M/\operatorname{rad} M)\operatorname{rad} A = 0$ and therefore the A-module $M/\operatorname{rad} M$ is a module over $A/\operatorname{rad} A$ with respect to the action $(m + M\operatorname{rad} A) \cdot (a + \operatorname{rad} A) = ma + M\operatorname{rad} A$. Now, by (3.4), the algebra $A/\operatorname{rad} A$ is semisimple, and the module $M/\operatorname{rad} M$ is semisimple.

(b) Assume that L is a submodule of M such that M/L is semisimple. Consider the canonical epimorphism $\varepsilon : M \to M/L$. Because (3.7)(c) yields $\varepsilon(\operatorname{rad} M) \subseteq \operatorname{rad}(M/L) = 0$, $\operatorname{rad} M \subseteq \operatorname{Ker} \varepsilon = L$, and (b) follows. □

It follows from (3.7)(d) that $(M/\operatorname{rad} M)\operatorname{rad} A = 0$ and therefore the module

$$\operatorname{top} M = M/\operatorname{rad} M,$$

called the **top** of M, is a right $A/\operatorname{rad} A$-module with respect to the action of $A/\operatorname{rad} A$ defined by the formula $(m + \operatorname{rad} M) \cdot (a + \operatorname{rad} A) = ma + \operatorname{rad} M$.

We remark that if $f : M \to N$ is an A-homomorphism, then $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$ and therefore f induces a homomorphism $\operatorname{top} f : \operatorname{top} M \longrightarrow \operatorname{top} N$

of $A/\operatorname{rad} A$-modules defined by the formula $(\operatorname{top} f)(m + \operatorname{rad} M) = f(m) + \operatorname{rad} N$.

3.9. Corollary. (a) *A homomorphism* $f : M \to N$ *in* $\operatorname{mod} A$ *is surjective if and only if the homomorphism* $\operatorname{top} f : \operatorname{top} M \longrightarrow \operatorname{top} N$ *is surjective.*

(b) *If* S *is a simple* A*-module, then* $S\operatorname{rad} A = 0$ *and* S *is a simple* $A/\operatorname{rad} A$*-module.*

(c) *An* A*-module* M *is semisimple if and only if* $\operatorname{rad} M = 0$.

Proof. (a) Assume that $\operatorname{top} f$ is surjective. Then $\operatorname{Im} f + \operatorname{rad} N = N$ and therefore f is surjective, because (3.7)(e) yields $\operatorname{Im} f = N$. Because the converse implication is easy, (a) follows.

(b) Because $S \neq 0$ and S is simple, S is cyclic and, by Nakayama's lemma (2.2), $S \neq S\operatorname{rad} A$. Hence $S\operatorname{rad} A = 0$ and (b) follows.

(c) If M is semisimple, then (b) yields $\operatorname{rad} M = 0$. The converse implication is a consequence of (3.7)(d) and (3.8)(a). □

Suppose that A is a finite dimensional K-algebra. If M is a module in $\operatorname{mod} A$, then there exists a chain $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M$ of submodules of M such that the module M_{j+1}/M_j is simple for $j = 0, 1, \ldots, m-1$ (see [2], [61], and [131]). This chain is called a **composition series** of M and the simple modules $M_1/M_0, \ldots, M_m/M_{m-1}$ are called the **composition factors** of M.

3.10. Jordan–Hölder theorem. *If* A *is a finite dimensional* K*-algebra and*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M,$$
$$0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_n = M$$

are two composition series of a module M *in* $\operatorname{mod} A$, *then* $m = n$, *and there exists a permutation* σ *of* $\{1, \ldots, m\}$ *such that, for any* $j \in \{0, 1, \ldots, m-1\}$, *there is an* A*-isomorphism* $M_{j+1}/M_j \cong N_{\sigma(j+1)}/N_{\sigma(j)}$.

Proof. See [2], [61], [131], and [164]. □

It follows from (3.10) that the number m of modules in a composition series $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M$ of M depends only on M; it is called the **length** of M and is denoted by $\ell(M)$.

As an immediate consequence of (3.10) we get the following.

3.11. Corollary. (a) *If* N *is an* A*-submodule of* M *in* $\operatorname{mod} A$, *then* $\ell(M) = \ell(N) + \ell(M/N)$.

(b) *If* L *and* N *are* A*-submodules of* M *in* $\operatorname{mod} A$, *then* $\ell(L + N) + \ell(L \cap N) = \ell(L) + \ell(N)$. □

I.4 Direct sum decompositions

In the study of indecomposable modules over a K-algebra A, an important rôle is played by idempotent elements of A. An element $e \in A$ is called an **idempotent** if $e^2 = e$. The idempotent e is said to be **central** if $ae = ea$ for all $a \in A$. The idempotents $e_1, e_2 \in A$ are called **orthogonal** if $e_1 e_2 = e_2 e_1 = 0$. The idempotent e is said to be **primitive** if e cannot be written as a sum $e = e_1 + e_2$, where e_1 and e_2 are nonzero orthogonal idempotents of A.

Every algebra A has two trivial idempotents 0 and 1. If the idempotent e of A is nontrivial, then $1 - e$ is also a nontrivial idempotent, the idempotents e and $1 - e$ are orthogonal, and there is a nontrivial right A-module decomposition $A_A = eA \oplus (1 - e)A$. Conversely, if $A_A = M_1 \oplus M_2$ is a nontrivial A-module decomposition and $1 = e_1 + e_2$, $e_i \in M_i$, then e_1, e_2 is a pair of orthogonal idempotents of A, and $M_i = e_i A$ is indecomposable if and only if e_i is primitive.

If e is a central idempotent, then so is $1-e$, and hence eA and $(1-e)A$ are two-sided ideals and they are easily shown to be K-algebras with identity elements $e \in eA$ and $1 - e \in (1 - e)A$, respectively. In this case the decomposition $A_A = eA \oplus (1-e)A$ is a direct product decomposition of the algebra A.

Because the algebra A is finite dimensional, the module A_A admits a direct sum decomposition $A_A = P_1 \oplus \cdots \oplus P_n$, where $P_1, \ldots, P_n$ are indecomposable right ideals of A. It follows from the preceding discussion that $P_1 = e_1 A, \ldots, P_n = e_n A$, where $e_1, \ldots, e_n$ are primitive pairwise orthogonal idempotents of A such that $1 = e_1 + \cdots + e_n$. Conversely, every set of idempotents with the preceding properties induces a decomposition $A_A = P_1 \oplus \cdots \oplus P_n$ with indecomposable right ideals $P_1 = e_1 A, \ldots, P_n = e_n A$.

Such a decomposition is called an **indecomposable decomposition** of A and such a set $\{e_1, \cdots, e_n\}$ is called a **complete set of primitive orthogonal idempotents** of A.

We say that an algebra A is **connected** (or indecomposable) if A is not a direct product of two algebras, or equivalently, if 0 and 1 are the only central idempotents of A.

4.1. Example. The K-subalgebra $A = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ K & K & K \end{bmatrix}$ of $\mathbb{M}_3(K)$ defined in (1.1)(c) is connected, $\dim_K A = 5$, and A_A has an indecomposable decomposition $A_A = e_1 A \oplus e_2 A \oplus e_3 A$, where $e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are primitive orthogonal idempotents of A such that $1_A =$

$e_1 + e_2 + e_3$. The right ideal e_jA consists of all matrices $\lambda = [\lambda_{st}]$ in A with $\lambda_{st} = 0$ for $s \neq j$, that is, $\lambda_{st} = 0$ outside the jth row. The right A-modules e_1A and e_2A are one-dimensional; hence they are simple. We also note that the right A-module $M = e_3A$ is of length 3. Indeed, the subspace M_1 of M consisting of the matrices $\lambda \in M$ such that $\lambda_{33} = \lambda_{32} = 0$ is a one-dimensional submodule of M (isomorphic to the simple ideal $e_{11}A$), the subspace M_2 consisting of the matrices $\lambda \in M$ such that $\lambda_{33} = 0$ is a two-dimensional submodule of M containing M_1, $\dim_K M_2/M_1 = 1$ and $\dim_K M/M_2 = 1$; hence $0 \subset M_1 \subset M_2 \subset M$ is a composition series of M and therefore $\ell(M) = 3$. □

Assume that $e \in A$ is an idempotent and that M is a right A-module. It is easy to check that the K-vector subspace eAe of A is a K-algebra and that e is the identity element of eAe. Note that eAe is a subalgebra of A if and only if $e = 1$. The K-vector subspace Me of M is a right eAe-module if we set $(me) \cdot (eae) = meae$ for all $m \in M$ and $a \in A$. In particular, Ae is a right eAe-module and eA is a left eAe-module. It follows that the K-vector space $\mathrm{Hom}_A(eA, M)$ is a right eAe-module with respect to the action $(\varphi \cdot eae)(x) = \varphi(eaex)$ for $x \in eA$, $a \in A$, $\varphi \in \mathrm{Hom}_A(eA, M)$.

The following useful fact is frequently used.

4.2. Lemma. *Let A be a K-algebra, $e \in A$ be an idempotent, and M be a right A-module.*

(a) *The K-linear map*

$$\theta_M : \mathrm{Hom}_A(eA, M) \longrightarrow Me, \tag{4.3}$$

defined by the formula $\varphi \mapsto \varphi(e) = \varphi(e)e$ for $\varphi \in \mathrm{Hom}_A(eA, M)$, is an isomorphism of right eAe-modules, and it is functorial in M.

(b) *The isomorphism $\theta_{eA} : \mathrm{End}\, eA \xrightarrow{\simeq} eAe$ of right eAe-modules induces an isomorphism of K-algebras.*

Proof. It is easy to see that the map θ_M is a homomorphism of right eAe-modules and it is functorial at the variable M. We define a K-linear map $\theta'_M : Me \to \mathrm{Hom}_A(eA, M)$ by the formula $\theta'_M(me)(ea) = mea$ for $a \in A$ and $m \in M$. A straightforward calculation shows that, given $m \in M$, the map $\theta'_M(me) : eA \to M$ is well-defined (does not depend of the choice of a in the presentation ea), it is a homomorphism of A-modules, moreover θ'_M is a homomorphism of eAe-modules and θ'_M is an inverse of θ_M. This proves (a). The statement (b) easily follows from (a). □

We also need the following technical but useful result.

4.4. Lemma (lifting idempotents). *For any K-algebra A the idempotents of the algebra $B = A/\mathrm{rad}\, A$ can be lifted modulo* $\mathrm{rad}\, A$*, that is, for* *any idempotent* $\eta = g + \operatorname{rad} A \in B$, $g \in A$, *there exists an idempotent* e *of* A *such that* $g - e \in \operatorname{rad} A$.

Proof. It follows from (2.3) that $(\operatorname{rad} A)^m = 0$ for some $m > 1$. Because $\eta^2 = \eta$, $g - g^2 \in \operatorname{rad} A$ and therefore $(g - g^2)^m = 0$. Hence, by Newton's binomial formula, we get $0 = (g - g^2)^m = g^m - g^{m+1}t$, where $t = \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j} g^{j-1}$. It follows that

(i) $g^m = g^{m+1}t$;

(ii) $gt = tg$.

We claim that the element $e = (gt)^m$ is the idempotent lifting η. First, we note that $e = g^m t^m = g^{m+1}t^{m+1} = \cdots = g^{2m}t^{2m} = ((gt)^m)^2 = e^2$ and therefore e is an idempotent. Next, we note that

(iii) $g - g^m \in \operatorname{rad} A$,

because the relation $g - g^2 \in \operatorname{rad} A$ yields the equalities $g - g^m = g(1 - g^{m-1}) = g(1 - g)(1 + g + \cdots + g^{m-2}) = (g - g^2)(1 + g + \cdots + g^{m-2}) \in \operatorname{rad} A$. Moreover, we have

(iv) $g - gt \in \operatorname{rad} A$,

because equalities (i)–(iii) yield

$$\begin{aligned} g + \operatorname{rad} A &= g^m + \operatorname{rad} A = g^{m+1}t + \operatorname{rad} A = (g^{m+1} + \operatorname{rad} A)(t + \operatorname{rad} A) = \\ &= (g^m + \operatorname{rad} A)(g + \operatorname{rad} A)(t + \operatorname{rad} A) = (g + \operatorname{rad} A)(g + \operatorname{rad} A)(t + \operatorname{rad} A) = \\ &= (g^2 + \operatorname{rad} A)(t + \operatorname{rad} A) = (g + \operatorname{rad} A)(t + \operatorname{rad} A) = gt + \operatorname{rad} A. \end{aligned}$$

Consequently, we get $e + \operatorname{rad} A = (gt)^m + \operatorname{rad} A = (gt + \operatorname{rad} A)^m = (g + \operatorname{rad} A)^m = g^m + \operatorname{rad} A = g + \operatorname{rad} A$ and our claim follows. $\square$

4.5. Proposition. *Let* $B = A/\operatorname{rad} A$. *The following statements hold.*

(a) *Every right ideal* I *of* B *is a direct sum of simple right ideals of the form* eB, *where* e *is a primitive idempotent of* B. *In particular, the right* B*-module* B_B *is semisimple.*

(b) *Any module* N *in* $\operatorname{mod} B$ *is isomorphic to a direct sum of simple right ideals of the form* eB, *where* e *is a primitive idempotent of* B.

(c) *If* $e \in A$ *is a primitive idempotent of* A, *then the* B*-module* $\operatorname{top} eA$ *is simple and* $\operatorname{rad} eA = e\operatorname{rad} A \subset eA$ *is the unique maximal proper submodule of* eA.

Proof. (a) Let S be a nonzero right ideal of B contained in I that is of minimal dimension. Then S is a simple B-module and $S^2 \neq 0$, because otherwise, in view of (1.4)(c), $0 \neq S \subseteq \operatorname{rad} B = 0$ and we get a contradiction. Hence $S^2 = S$ and there exists $x \in S$ such that $xS \neq 0$, $S = xS$ and $x = xe$ for some nonzero $e \in S$. Then, according to Schur's lemma, the B-homomorphism $\varphi : S \to S$ given by the formula $\varphi(y) = xy$ is bijective. Because $\varphi(e^2 - e) = x(e^2 - e) = xee - xe = xe - xe = 0$, $e^2 - e = 0$, the element $e \in S$ is a nonzero idempotent, and $S = eB$. It follows that $B = eB \oplus (1-e)B$ and $I = S \oplus (1-e)I$. Because $\dim_K (1-e)I < \dim_K I$, we can assume by induction that (a) is satisfied for $(1-e)I$ and therefore (a) follows.

(b) Let N be a B-module generated by the elements $n_1, \ldots, n_s$ and consider the B-module epimorphism $h : B^s \to N$ defined by the formula $h(\xi_i) = n_i$, where $\xi_1, \ldots, \xi_s$ is the standard basis of the B-module B^s. If N is simple, then $s = 1$ and (a) together with (3.3)(a) yields $N \cong eB$, where e is a primitive idempotent of B. Now suppose that N is arbitrary. Then, by (a), B^s is a direct sum of simple right ideals of the form eB, where e is a primitive idempotent of B, and it follows from (3.3)(a) that $B^s = \operatorname{Ker} h \oplus L$ for some B-submodule L of B^s. Then h induces an isomorphism $L \cong N$ and (b) follows from (3.3)(b).

(c) The element $\overline{e} = e + \operatorname{rad} A$ is an idempotent of B and $\operatorname{top} eA \cong \overline{e}B$. Assume to the contrary that $\overline{e}B$ is not simple. It follows from (a) that $\overline{e}B = \overline{e}_1 B \oplus \overline{e}_2 B$, where $\overline{e}_1, \overline{e}_2$ are nonzero idempotents of B such that $\overline{e} = \overline{e}_1 + \overline{e}_2$ and $\overline{e}_1\overline{e}_2 = \overline{e}_2\overline{e}_1 = 0$. Because $\overline{e}_1 = \overline{e}_1^2 = (\overline{e} - \overline{e}_2)\overline{e}_1 = \overline{e}\overline{e}_1$, $\overline{e}_1 = g_1 + \operatorname{rad} A$ for some $g_1 \in eA$. By (4.4), there exist $t \in A$ and $m \in \mathbb{N}$ such that the element $e_1 = (g_1 t)^m$ is an idempotent of A and $\overline{e}_1 = e_1 + \operatorname{rad} A$. It follows that $\operatorname{top} eA = \overline{e}B = \overline{e}_1 B \oplus \overline{e}_2 B$. Because $g_1 \in eA$, $e_1 \in eA$ and $e_1 A \subseteq eA$. Then the decomposition $A_A = e_1 A \oplus (1 - e_1)A$ induces the decomposition $eA = e_1 A \oplus \{(1 - e_1)A \cap eA\}$. It follows that $eA = e_1 A$, because the primitivity of e implies that eA is indecomposable. Hence $\overline{e}B = \operatorname{top} eA = \operatorname{top} e_1 A = \overline{e}_1 B$ and therefore $\overline{e}_2 B = 0$, contrary to our assumption. Consequently, the module $\operatorname{top} eA$ is simple and therefore $\operatorname{rad} eA = (eA)\operatorname{rad} A$ is a maximal proper A-submodule of eA. Now, if L is a proper A-submodule of eA that is not in $\operatorname{rad} eA$, then $L + \operatorname{rad} eA = eA$ and (3.7)(e) yields $L = eA$, a contradiction. This shows that $\operatorname{rad} eA$ contains all proper A-submodules of eA and finishes the proof. $\square$

An algebra A is said to be **local** if A has a unique maximal right ideal, or equivalently, if A has a unique maximal left ideal, see (4.6).

An example of a local algebra is the commutative algebra

$$A = K[X_1, \ldots, X_n]/(X_1^{s_1}, \ldots, X_n^{s_n}),$$

where $s_1, \ldots, s_n$ are nonzero natural numbers and $n \geq 1$. Indeed, it was shown in Example 1.5(a) that the radical rad A of A is a maximal ideal. It follows that rad A is the unique maximal ideal of A, that is, the algebra A is local.

Note that, in view of Example 1.5(b), the incidence K-algebra KI of a finite poset I is not local if $|I| \geq 2$.

Now we give a characterisation of algebras having only trivial idempotents.

4.6. Lemma. *Let A be a finite dimensional K-algebra. The following conditions are equivalent:*

(a) *A is a local algebra.*

(a$'$) *A has a unique maximal left ideal.*

(b) *The set of all noninvertible elements of A is a two-sided ideal.*

(c) *For any $a \in A$, one of the elements a or $1 - a$ is invertible.*

(d) *A has only two idempotents, 0 and 1.*

(e) *The K-algebra $A/\operatorname{rad} A$ is isomorphic to K.*

Proof. (a) implies (b). Because A is local, $\operatorname{rad} A$ is a unique proper maximal right ideal of A. It follows that $x \in \operatorname{rad} A$ if and only if x has no right inverse. Hence we conclude that any right invertible element $x \in A$ is invertible. Indeed, if $xy = 1$ then $(1 - yx)y = 0$. It follows that y has a right inverse and $1 - yx = 0$, because otherwise $y \in \operatorname{rad} A$, in view of (1.3), the element $1 - yx$ is invertible and we get $y = 0$, which is a contradiction.

This shows that $x \in \operatorname{rad} A$ if and only if x has no right inverse, or equivalently, if and only if x is not invertible. Then (b) follows.

That (a$'$) implies (b) follows in a similar way, and it is easy to see that (b) implies (c).

(c) implies (d). If $e \in A$ is an idempotent, then so is $1 - e$ and we have $e(1 - e) = 0$. It follows from (c) that $e = 0$ or $e = 1$.

(d) implies (e). Because, by (4.4), the idempotents of $A/\operatorname{rad} A$ can be lifted modulo $\operatorname{rad} A$, the semisimple algebra $B = A/\operatorname{rad} A$ has only two idempotents 0 and 1. By (4.5)(a), the right B-module B_B is simple and, in view of (3.2), there is a K-algebra isomorphism $\operatorname{End} B_B \cong K$. Hence we get K-algebra isomorphisms $B \cong \operatorname{Hom}_B(B_B, B_B) \cong K$ and (e) follows.

In view of (1.4), the statement (e) implies that $\operatorname{rad} A$ is the unique proper maximal right ideal and the unique proper maximal left ideal of A. Hence it follows that (e) implies (a) and that (e) implies (a$'$). The proof is complete. $\square$

We note that infinite dimensional algebras with only two idempotents 0 and 1 are not necessarily local. An example of such an algebra is the polynomial algebra $K[t]$, which is not local and has only two idempotents 0 and 1.

4.7. Corollary. *An idempotent $e \in A$ is primitive if and only if the algebra $eAe \cong \operatorname{End} eA$ has only two idempotents 0 and e, that is, the algebra eAe is local.* $\square$

4.8. Corollary. *Let A be an arbitrary K-algebra and M a right A-module.*

(a) *If the algebra $\operatorname{End} M$ is local, then M is indecomposable.*

(b) *If M is finite dimensional and indecomposable, then the algebra* $\operatorname{End} M$ *is local and any A-module endomorphism of M is nilpotent or is an isomorphism.*

Proof. (a) If M decomposes as $M = X_1 \oplus X_2$ with both X_1 and X_2 nonzero, then there exist projections $p_i : M \to X_i$ and injections $u_i : X_i \to M$ (for $i = 1, 2$) such that $u_1p_1 + u_2p_2 = 1_M$. Because u_1p_1 and u_2p_2 are nonzero idempotents in $\operatorname{End} M$, the algebra $\operatorname{End} M$ is not local, because otherwise 1_M belongs to the unique proper maximal ideal of $\operatorname{End} M$, a contradiction.

(b) Assume that M is finite dimensional and indecomposable. If $\operatorname{End} M$ is not local then, according to (4.6), the algebra $\operatorname{End} M$ has a pair of nonzero idempotents e_1, $e_2 = 1 - e_1$ and therefore $M \cong \operatorname{Im} e_1 \oplus \operatorname{Im} e_2$ is a nontrivial direct sum decomposition. Consequently, the algebra $\operatorname{End} M$ is local. By (4.6), every noninvertible A-module endomorphism $f : M \to M$ belongs to the radical of $\operatorname{End} M$ and therefore f is nilpotent, because $\operatorname{End} M$ is finite dimensional, and it follows from (2.3) that the radical of $\operatorname{End} M$ is nilpotent. $\square$

We note that infinite dimensional indecomposable modules over finite dimensional algebras do not necessarily have local endomorphism rings. An example of such a module over the Kronecker algebra (2.5) is presented in Exercise 4.15 of Chapter III.

4.9. Example. Let $A = \mathbb{T}_3(K) = \begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & K & K \end{bmatrix}$ be the K-subalgebra of $\mathbb{M}_3(K)$ defined in (1.1)(c), and let B be the subalgebra of A consisting of all matrices $\lambda = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$ in A such that $\lambda_{11} = \lambda_{22} = \lambda_{33}$. The algebra B is noncommutative and local; because $\operatorname{rad} B$ consists of all matrices $\begin{bmatrix} 0 & 0 & 0 \\ \lambda_{21} & 0 & 0 \\ \lambda_{31} & \lambda_{32} & 0 \end{bmatrix}$ in B, there is an algebra isomorphism $B/\operatorname{rad} B \cong K$ and (4.6) applies (compare with (1.5)(c)).

The following result is fundamental for the representation theory of finite dimensional algebras.

4.10. Unique decomposition theorem. *Let A be a finite dimensional K-algebra.*

(a) *Every module M in* $\operatorname{mod} A$ *has a decomposition $M \cong M_1 \oplus \cdots \oplus M_m$, where $M_1, \ldots, M_m$ are indecomposable modules and the endomorphism K-algebra* $\operatorname{End} M_j$ *is local for each $j = 1, \ldots, m$.*

(b) *If* $M \cong \bigoplus_{i=1}^{m} M_i \cong \bigoplus_{j=1}^{n} N_j$, *where M_i and N_j are indecomposable,* *then $m = n$ and there exists a permutation σ of $\{1, \dots, n\}$ such that $M_i \cong N_{\sigma(i)}$ for each $i = 1, \dots, n$.*

Proof. (a) Because $\dim_K M$ is finite, M has an indecomposable decomposition, that is, a decomposition into a direct sum of indecomposable modules. In view of (4.8), the endomorphism algebra of every indecomposable direct summand of M is local. Then M has a decomposition as required.

(b) Without loss of generality, we may suppose that $M = \bigoplus_{i=1}^{m} M_i = \bigoplus_{j=1}^{n} N_j$. We proceed by induction on m. If $m = 1$, then M is indecomposable and there is nothing to show. Assume that $m > 1$ and put $M_1' = \bigoplus_{i>1} M_i$. Denote the injections and projections associated to the direct sum decomposition $M = M_1 \oplus M_1'$ by u, u', p, p' and those associated to the direct sum decomposition $M = \bigoplus_{j=1}^{n} N_j$ by u_j, p_j (with $1 \leq j \leq n$). We have $1_{M_1} = pu = p\Big(\sum_{j=1}^n u_j p_j\Big)u = \sum_{j=1}^n p u_j p_j u$. Because $\operatorname{End} M_1$ is local, by (4.6)(c), there exists j with $1 \leq j \leq n$ such that $v = pu_j p_j u$ is invertible. Rearranging the indices if necessary, we may suppose that $j = 1$. Then $w = v^{-1} p u_1 : N_1 \to M_1$ satisfies $w p_1 u = 1_{M_1}$ so that $p_1 u w \in \operatorname{End} N_1$ is an idempotent. Because $\operatorname{End} N_1$ is local, it must equal 1_{N_1} or 0, because of (4.6)(d). If $p_1 u w = 0$, then $p_1 u = 0$ (because w is an epimorphism), a contradiction, because $v = p u_1 p_1 u$ is invertible. Thus $p_1 u w = 1_{N_1}$ and $f_{11} = p_1 u \in \operatorname{Hom}_A(M_1, N_1)$ is an isomorphism. Setting $N_1' = \bigoplus_{j>1} N_j$, we can put the identity homomorphism $1_M : M_1 \oplus M_1' \xrightarrow{\simeq} N_1 \oplus N_1'$ in the matrix form $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$. The wanted result would then follow from the induction hypothesis if we could show that $M_1' \cong N_1'$. Because the composite A-module homomorphism $g = \begin{bmatrix} 1 & 0 \\ -f_{21} f_{11}^{-1} & 1 \end{bmatrix} f = \begin{bmatrix} f_{11} & f_{12} \\ 0 & f_{22}' \end{bmatrix}$, where $f_{22}' = -f_{21} f_{11}^{-1} f_{12} + f_{22}$, is an isomorphism $M_1 \oplus M_1' \xrightarrow{\simeq} N_1 \oplus N_1'$, $f_{22}' : M_1' \xrightarrow{\simeq} N_1'$ is also an isomorphism and the proof is complete. $\square$

It follows that if $A_A = P_1 \oplus \cdots \oplus P_n$ is an indecomposable decomposition, then it is unique in the sense of the unique decomposition theorem.

We end this section by defining representation-finite algebras, a class we study in detail in the following chapters.

4.11. Definition. A finite dimensional K-algebra A is defined to be **representation–finite** (or an **algebra of finite representation type**) if the number of the isomorphism classes of indecomposable finite dimensional right A-modules is finite. A K-algebra A is called **representation–infinite** (or an **algebra of infinite representation type**) if A is not representation–finite.

It follows from the standard duality $D : \operatorname{mod} A \longrightarrow \operatorname{mod} A^{\mathrm{op}}$ that this definition is right-left symmetric. One can prove that if A is representation–finite then the number of the isomorphism classes of all indecomposable left A-modules is finite, or equivalently, that every indecomposable right (and left) A-module is finite dimensional (see [12], [13], [69], [147], and [151]).

I.5. Projective and injective modules

We start with some definitions. Let $h : M \to N$ and $u : L \to M$ be homomorphisms of right A-modules. We call an A-homomorphism $s : N \to M$ a **section** of h if $hs = 1_N$, and we call an A-homomorphism $r : M \to L$ a **retraction** of u if $ru = 1_L$. If s is a section of h, then h is surjective, s is injective, there are direct sum decompositions $M = \operatorname{Im} s \oplus \operatorname{Ker} h \cong N \oplus \operatorname{Ker} h$, and h is a retraction of s. Similarly, if r is a retraction of u, then r is surjective, u is injective, u is a section of r, and there exist direct sum decompositions $M = \operatorname{Im} u \oplus \operatorname{Ker} r \cong L \oplus \operatorname{Ker} r$.

An A-homomorphism $h : M \to N$ is called a **section** (or a **retraction**) if h admits a retraction (or a section, respectively).

A sequence $\cdots \longrightarrow X_{n-1} \xrightarrow{h_{n-1}} X_n \xrightarrow{h_n} X_{n+1} \xrightarrow{h_{n+1}} X_{n+2} \longrightarrow \cdots$ (infinite or finite) of right A-modules connected by A-homomorphisms is called **exact** if $\operatorname{Ker} h_n = \operatorname{Im} h_{n-1}$ for any n. In particular

$$0 \longrightarrow L \xrightarrow{u} M \xrightarrow{r} N \longrightarrow 0$$

is called a **short exact sequence** if u is a monomorphism, r is an epimorphism and $\operatorname{Ker} r = \operatorname{Im} u$. Note that the homomorphism u admits a retraction $p : M \to L$ if and only if r admits a section $v : N \to M$. In this case there are direct sum decompositions $M = \operatorname{Im} u \oplus \operatorname{Ker} p = \operatorname{Im} v \oplus \operatorname{Ker} r$ of M, and we say that the short exact sequence **splits**.

The following lemma is frequently used.

5.1. Snake lemma. *Assume that the following diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & L' & \xrightarrow{u'} & M' & \xrightarrow{v'} & N' & \longrightarrow & 0
\end{array}$$

in $\operatorname{mod} A$ *has exact rows and is commutative. Then there exists a connecting* A*-homomorphism* $\delta : \operatorname{Ker} h \to \operatorname{Coker} f$ *such that the induced sequence*

$$
\begin{array}{cccccccccc}
0 & \longrightarrow & \operatorname{Ker} f & \xrightarrow{u} & \operatorname{Ker} g & \xrightarrow{v} & \operatorname{Ker} h & & & \\
& & & \xrightarrow{\delta} & \operatorname{Coker} f & \xrightarrow{u'} & \operatorname{Coker} g & \xrightarrow{v'} & \operatorname{Coker} h & \longrightarrow & 0
\end{array}
$$

is exact.

Proof. See [49] , [112], [131], and [149]. □

5.2. Definition. (a) A right A-module F is **free** if F is isomorphic to a direct sum of copies of the module A_A.

(b) A right A-module P is **projective** if, for any epimorphism $h : M \to N$, the induced map $\operatorname{Hom}_A(P, h) : \operatorname{Hom}_A(P, M) \longrightarrow \operatorname{Hom}_A(P, N)$ is surjective, that is, for any epimorphism $h : M \to N$ and any $f \in \operatorname{Hom}_A(P, N)$, there is an $f' \in \operatorname{Hom}_A(P, M)$ such that the following diagram is commutative

$$
\begin{tikzcd}
 & P \arrow[dl, "f'"'] \arrow[d, "f"] & \\
M \arrow[r, "h"] & N \arrow[r] & 0
\end{tikzcd}
$$

(c) A right A-module E is **injective** if, for any monomorphism $u : L \to M$, the induced map $\operatorname{Hom}_A(u, E) : \operatorname{Hom}_A(M, E) \longrightarrow \operatorname{Hom}_A(L, E)$ is surjective, that is, for any monomorphism $u : L \to M$ and any $g \in \operatorname{Hom}_A(L, E)$, there is a $g' \in \operatorname{Hom}_A(M, E)$ such that the following diagram is commutative

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "u"] \arrow[d, "g"'] & M \arrow[dl, "g'"] \\
 & E &
\end{tikzcd}
$$

5.3. Lemma. (a) *A right A-module P is projective if and only if there exist a free A-module F and a right A-module P' such that $P \oplus P' \cong F$.*

(b) *Suppose that $A_A = e_1A \oplus \cdots \oplus e_nA$ is a decomposition of A_A into indecomposable submodules. If a right A-module P is projective, then $P = P_1 \oplus \cdots \oplus P_m$, where every summand P_j is indecomposable and isomorphic to some e_sA.*

(c) *Let M be an arbitrary right A-module. Then there exists an exact sequence*

$$
\cdots \to P_m \xrightarrow{h_m} P_{m-1} \to \cdots \to P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \to 0 \tag{5.4}
$$

in $\operatorname{Mod} A$*, where P_j is a projective right A-module for any $j \geq 0$. If, in addition, M is in* $\operatorname{mod} A$*, then there exists an exact sequence* (5.4)*, where P_j is a projective module in* $\operatorname{mod} A$ *for any $j \geq 0$.*

Proof. (a) It is easy to check that any free module is projective and that a direct summand of a free module is a projective module. Conversely, suppose that P is a projective module generated by elements $\{m_j;\, j \in J\}$.

If $F = \bigoplus_{j\in J} x_j A$ is a free module with the set $\{x_j,\, j \in J\}$ of free generators and $f : F \to P$ is the epimorphism defined by $f(x_j) = m_j$, then, by the projectivity of P, there exists a section $s : P \to F$ of f and therefore $F \cong P \oplus \operatorname{Ker} f$.

(b) Let P be a projective module. By (a), there exist a free A-module F and a right A-module P' such that $P \oplus P' \cong F$. By our assumption, F is a direct sum of copies of the indecomposable modules $e_1A, \dots, e_nA$. Because by (4.8) the algebra $\operatorname{End} e_jA$ is local for each $j = 1, \dots, n$, (b) is a consequence of the unique decomposition theorem (4.10).

(c) It was shown in (a) that, for any module M (or M in $\operatorname{mod} A$), there is an epimorphism $f : F \to M$, where F is a free module in $\operatorname{Mod} A$ (or in $\operatorname{mod} A$, respectively). We set $P_0 = F$ and $h_0 = f$. Let $f_1 : F_1 \to \operatorname{Ker} h_0$ be an epimorphism with a free module F_1 in $\operatorname{Mod} A$. We set $P_1 = F_1$ and we take for h_1 the composition of f_1 with the embedding $\operatorname{Ker} h_0 \subseteq P_0$. If M is in $\operatorname{mod} A$, then the free module F_1 can be chosen in $\operatorname{mod} A$, because A is finite dimensional, hence $\dim_K M$ and $\dim_K F_0$ are finite, and therefore $\operatorname{Ker} h_0$ is in $\operatorname{mod} A$. Continuing this procedure, we construct by induction the required exact sequence (5.4). $\square$

We define a **projective resolution** of a right A-module M to be a complex

$$P_\bullet : \quad \cdots \to P_m \xrightarrow{h_m} P_{m-1} \to \cdots \to P_1 \xrightarrow{h_1} P_0 \to 0$$

of projective A-modules together with an epimorphism $h_0 : P_0 \xrightarrow{h_0} M$ of right A-modules such that the sequence (5.4) is exact. For the sake of simplicity, we call the sequence (5.4) a projective resolution of the A-module M. By (5.3), any module M in $\operatorname{mod} A$ has a projective resolution in $\operatorname{mod} A$.

We define an **injective resolution** of M to be a complex

$$I^\bullet : \quad 0 \to I^0 \xrightarrow{d^1} I^1 \to \cdots \to I^m \xrightarrow{d^{m+1}} I^{m+1} \to \cdots$$

of injective A-modules together with a monomorphism $d^0 : M \to I^0$ of right A-modules such that the sequence

$$0 \to M \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \to \cdots \to I^m \xrightarrow{d^{m+1}} I^{m+1} \to \cdots$$

is exact. For the sake of simplicity, we call this sequence an injective resolution of the A-module M. We show later that any module M in $\operatorname{mod} A$ has an injective resolution in $\operatorname{mod} A$.

First, we show that if A is a finite dimensional K-algebra, then any module M in $\operatorname{mod} A$ admits an exact sequence (5.4) in $\operatorname{mod} A$, where the epimorphisms $h_j : P_j \to \operatorname{Im} h_j$ are minimal for all $j \geq 0$ in the following sense.

5.5. Definition. (a) An A-submodule L of M is **superfluous** if for every submodule X of M the equality $L + X = M$ implies $X = M$.

(b) An A-epimorphism $h : M \to N$ in $\mathrm{mod}\, A$ is **minimal** if $\mathrm{Ker}\, h$ is superfluous in M. An epimorphism $h : P \to M$ in $\mathrm{mod}\, A$ is called a **projective cover** of M if P is a projective module and h is a minimal epimorphism.

It follows from (3.7)(e) that the submodule $\mathrm{rad}\, M$ of M is superfluous if M is a finitely generated module over a finite dimensional algebra.

Now we give a useful characterisation of projective covers.

5.6. Lemma. *An epimorphism $h : P \to M$ is a projective cover of an A-module M if and only if P is projective and for any A-homomorphism $g : N \to P$ the surjectivity of hg implies the surjectivity of g.*

Proof. Assume that $h : P \to M$ is a projective cover of M and let $g : N \to P$ be a homomorphism such that hg is surjective. It follows that $\mathrm{Im}\, g + \mathrm{Ker}\, h = P$ and therefore g is surjective, because by assumption $\mathrm{Ker}\, h$ is superfluous in P. This shows the sufficiency.

Conversely, assume that $h : P \to M$ has the stated property. Let N be a submodule of P such that $N + \mathrm{Ker}\, h = P$. If $g : N \hookrightarrow P$ is the natural inclusion, then $hg : N \to M$ is surjective. Hence, by hypothesis, g is surjective. This shows that $\mathrm{Ker}\, h$ is superfluous and finishes the proof. □

5.7. Definition. (a) An exact sequence

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

in $\mathrm{mod}\, A$ is called a **minimal projective presentation** of an A-module M if the A-module homomorphisms $P_0 \xrightarrow{p_0} M$ and $P_1 \xrightarrow{p_1} \mathrm{Ker}\, p_0$ are projective covers.

(b) An exact sequence (5.4) in $\mathrm{mod}\, A$ is called a **minimal projective resolution** of M if $h_j : P_j \to \mathrm{Im}\, h_j$ is a projective cover for all $j \geq 1$ and $P_0 \xrightarrow{h_0} M$ is a projective cover.

It follows from the next result that any module M in $\mathrm{mod}\, A$ admits a minimal projective presentation and a minimal projective resolution in $\mathrm{mod}\, A$.

5.8. Theorem. *Let A be a finite dimensional K-algebra and let $A_A = e_1 A \oplus \cdots \oplus e_n A$, where $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A.*

(a) *For any A-module M in* $\mathrm{mod}\, A$ *there exists a projective cover*

$$P(M) \xrightarrow{h} M \longrightarrow 0$$

where $P(M) \cong (e_1 A)^{s_1} \oplus \cdots \oplus (e_n A)^{s_n}$ and $s_1 \geq 0, \ldots, s_n \geq 0$. The homomorphism h induces an isomorphism $P(M)/\mathrm{rad}\, P(M) \cong M/\mathrm{rad}\, M$.

(b) *The projective cover* $P(M)$ *of a module* M *in* $\operatorname{mod} A$ *is unique in the sense that if* $h' : P' \to M$ *is another projective cover of* M*, then there exists a commutative diagram*

$$\begin{tikzcd} & & 0 \\ P(M) \arrow[rr, "h"] & & M \arrow[u] \arrow[r] & 0 \\ & & P' \arrow[u, "h'"] \arrow[ull, "g"] \end{tikzcd}$$

where g *is an isomorphism.*

Proof. We set $B = A/\operatorname{rad} A$, $\overline{e}_j = e_j + \operatorname{rad} A \in B$ and let $p : A \to B$ be the residual class K-algebra epimorphism. Because $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents of A, $\{\overline{e}_1, \dots, \overline{e}_n\}$ is a complete set of primitive orthogonal idempotents of B and $B_B = \overline{e}_1 B \oplus \cdots \oplus \overline{e}_n B$ is an indecomposable decomposition. It follows from (4.5)(c) that $\operatorname{rad} e_j A \subset e_j A$ is the unique maximal A-submodule of $e_j A$, then $\operatorname{top} e_j A \cong \overline{e}_j B$ is a simple B-module and the epimorphism $p_j : e_j A \to \operatorname{top} e_j A$ induced by p is a projective cover of $\operatorname{top} e_j A$.

Let M be a module in $\operatorname{mod} A$. Then $\operatorname{top} M = M/\operatorname{rad} M$ is a module in $\operatorname{mod} B$ and, according to (3.8) and (4.5), there exist B-module isomorphisms

$$\operatorname{top} M \cong (\overline{e}_1 B)^{s_1} \oplus \cdots \oplus (\overline{e}_n B)^{s_n} \cong (\operatorname{top} e_1 A)^{s_1} \oplus \cdots \oplus (\operatorname{top} e_n A)^{s_n},$$

for some $s_1 \geq 0, \dots, s_n \geq 0$. We set $P(M) = (e_1 A)^{s_1} \oplus \cdots \oplus (e_n A)^{s_n}$. By the projectivity of the module $P(M)$, there exists an A-module homomorphism $h : P(M) \to M$ making the diagram

$$\begin{tikzcd} P(M) \arrow[r, "h"] \arrow[d, "t"] & M \arrow[d, "t'"] \\ \operatorname{top} P(M) \arrow[r, "\operatorname{top} h"] & \operatorname{top} M \end{tikzcd}$$

commutative, where t and t' are the canonical epimorphisms. It follows that $\operatorname{top} h$ is an isomorphism and, from (3.9)(a), we infer that h is an epimorphism. Moreover, the commutativity of the diagram yields

$$\operatorname{Ker} h \subseteq \operatorname{Ker} t = (\operatorname{rad} e_1 A)^{s_1} \oplus \cdots \oplus (\operatorname{rad} e_n A)^{s_n} = \operatorname{rad} P(M).$$

Because, according to (3.7)(e), the module $\operatorname{rad} P(M)$ is superfluous in $P(M)$, $\operatorname{Ker} h$ is also superfluous in $P(M)$. Therefore the epimorphism h is a projective cover of M.

(b) The existence of a homomorphism $g : P' \to P(M)$ making the diagram shown in (b) commutative follows from the projectivity of P'. Because

$hg = h'$ is surjective, $\operatorname{Im} g + \operatorname{Ker} h = P(M)$ and therefore g is surjective, because $\operatorname{Ker} h$ is superfluous in $P(M)$. It follows that $\ell(P') \geq \ell(P(M))$. The preceding argument with $P(M)$ and P' interchanged shows that $\ell(P(M)) \geq \ell(P')$. Hence g is an isomorphism and the proof is complete. $\square$

Remark. The proof of (5.8) gives us a recipe for constructing the projective cover $P(M) \to M$ of any module in $\operatorname{mod} A$. We also refer simply to the module $P(M)$ as being a projective cover of M.

5.9. Corollary. *If P is a projective module in* $\operatorname{mod} A$*, then the canonical epimorphism $t : P \to \operatorname{top} P$ is a projective cover of* $\operatorname{top} P$ *and there exists an A-isomorphism $P \cong (e_1A)^{s_1} \oplus \cdots \oplus (e_nA)^{s_n}$ for some $s_1 \geq 0, \ldots,\ s_n \geq 0$.* $\square$

5.10. Corollary. *Let A be a K-algebra. Any module M in* $\operatorname{mod} A$ *admits a minimal projective presentation and a minimal projective resolution in* $\operatorname{mod} A$*.*

Proof. Let M be a module in $\operatorname{mod} A$. By (5.8), there is a projective cover $p_0 : P_0 \to M$ in $\operatorname{mod} A$. Then $\operatorname{Ker} p_0$ is finite dimensional and, according to (5.8), there is a projective cover $p_1 : P_0 \to \operatorname{Ker} p_0$. This yields a minimal projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$ of M. Continuing this procedure, we get by induction a minimal projective resolution of M in $\operatorname{mod} A$. $\square$

Now we shift our attention from projective to injective modules. For this purpose we recall from (2.9) that the functor $D(-) = \operatorname{Hom}_K(-, K)$ defines two dualities

$$\operatorname{mod} A \xrightarrow{D} \operatorname{mod} A^{\mathrm{op}} \xrightarrow{D} \operatorname{mod} A$$

such that there are natural equivalences of functors $D \circ D \cong 1_{\operatorname{mod} A}$ and $D \circ D \cong 1_{\operatorname{mod} A^{\mathrm{op}}}$. This allows us to study the injective modules in $\operatorname{mod} A$ by means of the projective modules in $\operatorname{mod} A^{\mathrm{op}}$.

We start by recalling the following important result.

5.11. Baer's criterion. *A right A-module E is injective if for any right ideal I of A and any A-homomorphism $f : I \to E$ there exists an A-homomorphism $f' : A_A \to E$ such that $f = f'u$, where u is the inclusion $u : I \hookrightarrow A$.*

Proof. See [2], [48], and [149]. $\square$

The notions dual to minimal epimorphism and to projective cover are defined as follows.

5.12. Definition. An A-module monomorphism $u : L \to M$ in $\operatorname{mod} A$ is **minimal** if every nonzero submodule X of M has a nonzero intersection with $\operatorname{Im} u$. A monomorphism $u : L \to E$ in $\operatorname{mod} A$ is called an **injective envelope** of L if E is an injective module and u is a minimal monomorphism.

Now we are able to state the main transfer theorem via the standard duality.

5.13. Theorem. *Let A be a finite dimensional K-algebra and let $D : \operatorname{mod} A \longrightarrow \operatorname{mod} A^{\mathrm{op}}$ be the standard duality $D(-) = \operatorname{Hom}_K(-, K)$ (2.9). Then the following hold.*

(a) *A sequence $0 \longrightarrow L \xrightarrow{u} N \xrightarrow{h} M \longrightarrow 0$ in $\operatorname{mod} A$ is exact if and only if the induced sequence $0 \longrightarrow D(M) \xrightarrow{D(h)} D(N) \xrightarrow{D(u)} D(L) \longrightarrow 0$ is exact in $\operatorname{mod} A^{\mathrm{op}}$.*

(b) *A module E in $\operatorname{mod} A$ is injective if and only if the module $D(E)$ is projective in $\operatorname{mod} A^{\mathrm{op}}$. A module P in $\operatorname{mod} A$ is projective if and only if the module $D(P)$ is injective in $\operatorname{mod} A^{\mathrm{op}}$.*

(c) *A module S in $\operatorname{mod} A$ is simple if and only if the module $D(S)$ is simple in $\operatorname{mod} A^{\mathrm{op}}$.*

(d) *A monomorphism $u : M \to E$ in $\operatorname{mod} A$ is an injective envelope if and only if the epimorphism $D(u) : D(E) \to D(M)$ is a projective cover in $\operatorname{mod} A^{\mathrm{op}}$. An epimorphism $h : P \to M$ in $\operatorname{mod} A$ is a projective cover if and only if the $D(h) : D(M) \to D(P)$ is an injective envelope in $\operatorname{mod} A^{\mathrm{op}}$.*

Proof. This is straightforward and left to the reader (see [61]). □

5.14. Corollary. *Every module M in $\operatorname{mod} A$ has an injective envelope $u : M \to E(M)$ and the module $E(M)$ is uniquely determined by M, up to isomorphism.*

Proof. Let M be a module in $\operatorname{mod} A$. By (5.8), the left A-module $D(M)$ has a projective cover $h : P \to D(M)$. It follows from (5.13)(d) that the monomorphism $M \cong DD(M) \xrightarrow{D(h)} D(P)$ is an injective envelope of M in $\operatorname{mod} A$. We set $E(M) = D(P)$. By (5.8), the left A-module P is uniquely determined by $D(M)$, up to isomorphism. It follows that the right module $E(M) = D(P)$ is uniquely determined by M, up to isomorphism. □

We refer simply to the module $E(M)$ as being an injective envelope of M.

5.15. Definition. (a) An exact sequence $0 \longrightarrow N \xrightarrow{u^0} I^0 \xrightarrow{u^1} I^1$ is a **minimal injective presentation** of an A-module N if the monomorphisms $u^0 : N \to I^0$ and $\operatorname{Im} u^1 \hookrightarrow I^1$ are injective envelopes.

(b) An injective resolution $0 \to M \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \to \cdots \to I^m \xrightarrow{d^{m+1}} I^{m+1} \to \cdots$ of a module M in $\operatorname{mod} A$ is said to be **minimal** if $\operatorname{Im} d^m \to I^m$ is an injective envelope for all $m \geq 1$ and $d^0 : M \to I^0$ is an injective envelope.

5.16. Corollary. *Every module M in* $\operatorname{mod} A$ *has a minimal injective presentation and a minimal injective resolution in* $\operatorname{mod} A$.

Proof. Let M be a module in $\operatorname{mod} A$. By (5.8), the left A-module $D(M)$ has a minimal projective presentation and a minimal projective resolution in $\operatorname{mod} A^{\mathrm{op}}$. It follows from (5.13) that the standard duality $D : \operatorname{mod} A^{\mathrm{op}} \longrightarrow \operatorname{mod} A$ carries a minimal projective presentation and a minimal projective resolution of $D(M)$ to a minimal injective presentation and a minimal injective resolution of the module $M \cong DD(M)$, respectively. $\square$

5.17. Corollary. *Suppose that $A_A = e_1 A \oplus \cdots \oplus e_n A$ is a decomposition of A into indecomposable submodules.*

(a) *Every simple right A-module is isomorphic to one of the modules*

$$S(1) = \operatorname{top} e_1 A, \ldots, S(n) = \operatorname{top} e_n A.$$

(b) *Every indecomposable projective right A-module is isomorphic to one of the modules*

$$P(1) = e_1 A,\ P(2) = e_2 A, \ldots, P(n) = e_n A.$$

Moreover, $e_i A \cong e_j A$ if and only if $S(i) \cong S(j)$.

(c) *Every indecomposable injective right A-module is isomorphic to one of the modules*

$$I(1) = D(Ae_1) \cong E(S(1)), \ldots, I(n) = D(Ae_n) \cong E(S(n)),$$

where $E(S(j))$ is an injective envelope of the simple module $S(j)$.

Proof. Apply (4.5), (4.7), (4.10), (5.9), and (5.13). $\square$

5.18. Example. Let $A = \mathbb{M}_2(K)$ and let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then e_1, e_2 are primitive orthogonal idempotents of A such that $1_A = e_1 + e_2$ and $A_A = e_1 A \oplus e_2 A$. The algebra A is semisimple, $S(1) = P(1) = I(1) \cong S(2) = P(2) = I(2)$ and $\dim_K S(1) = \dim_K S(2) = 2$. $\square$

I.6 Basic algebras and embeddings of module categories

Throughout, we need essentially the following class of algebras (see [73], [125], and [131] for historical notes).

6.1. Definition. Assume that A is a K-algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. The algebra A is called **basic** if $e_i A \not\cong e_j A$, for all $i \neq j$.

It is clear that every local finite dimensional algebra is basic. It follows from the following proposition that the algebras of Examples (1.1)(c) and (1.1)(d) are basic.

6.2. Proposition. (a) *A finite dimensional K-algebra A is basic if and only if the algebra $B = A/\operatorname{rad} A$ is isomorphic to a product $K \times K \times \cdots \times K$ of copies of K.*

(b) *Every simple module over a basic K-algebra is one-dimensional.*

Proof. (a) Let $A_A = e_1 A \oplus \cdots \oplus e_n A$ be an indecomposable decomposition of A. Then $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A, the element $\overline{e}_j = e_j + \operatorname{rad} A$ is an idempotent of $B = A/\operatorname{rad} A$, and in view of (4.5)(c) $\overline{e}_j B = \operatorname{top} e_j A$ is a simple B-module. Hence $B_B = \overline{e}_1 B \oplus \cdots \oplus \overline{e}_n B$ is an indecomposable decomposition of B_B. By (5.9), $e_j A \cong P(\overline{e}_j B)$ and therefore $e_j A \cong e_i A$ if and only if $\overline{e}_j B \cong \overline{e}_i B$.

It follows that if A is basic, then B is basic. Moreover, Schur's lemma (3.1) yields $\operatorname{Hom}_B(\overline{e}_i B, \overline{e}_j B) = 0$ for $i \neq j$, and (3.2) yields $\operatorname{End} \overline{e}_j B \cong K$ for $j = 1, \ldots, n$. Hence, given an element $b \in B$ and $j \leq n$, the multiplication map $b_j : \overline{e}_j B \to B_B$ defined by the formula $b_j(y) = \overline{e}_j b y$, for $y \in \overline{e}_j B$, induces a homomorphism $b'_j : \overline{e}_j B \to \overline{e}_j B$ of right B-modules and the K-algebra homomorphism $\sigma_j : B \to \operatorname{End} \overline{e}_j B \cong K$ defined by the formula $\sigma_j(b) = b'_j$. Hence we get the K-algebra homomorphism

$$\sigma : B \longrightarrow \operatorname{End}(\overline{e}_1 B) \times \cdots \times \operatorname{End}(\overline{e}_n B) \cong K \times \cdots \times K$$

defined by $\sigma(b) = (\sigma_1(b), \ldots, \sigma_n(b))$, for $b \in B$. Because σ is obviously injective, by comparing the dimensions, we see that it is bijective. The sufficiency part of (a) follows.

Assume now that B is a product $K \times \cdots \times K$. Then B is commutative and $\overline{e}_1, \ldots, \overline{e}_n$ are central primitive pairwise orthogonal idempotents of B. It follows that $\overline{e}_i B \not\cong \overline{e}_j B$ for $i \neq j$ and (5.8) yields $e_i A \cong P(\overline{e}_i B) \not\cong P(\overline{e}_j B) \cong e_j A$. Consequently A is basic and (a) follows.

The statement (b) follows from (a) because, by (3.9)(b), any simple A-module S is a module over the quotient algebra $B = A/\operatorname{rad} A$ and, by (a), B is isomorphic to a product $K \times \cdots \times K$ if A is basic. Hence $\dim_K S = 1$ and the proof is complete. $\square$

6.3. Definition. Assume that A is a K-algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. A **basic algebra** associated to A is the algebra

$$A^b = e_A A e_A,$$

where $e_A = e_{j_1} + \cdots + e_{j_a}$, and $e_{j_1}, \ldots, e_{j_a}$ are chosen such that $e_{j_i} A \not\cong e_{j_t} A$ for $i \neq t$ and each module $e_s A$ is isomorphic to one of the modules $e_{j_1} A, \ldots, e_{j_a} A$.

Example 6.4. Let $A = \mathbb{M}_n(K)$ and $\{e_1, \dots, e_n\}$ be the standard set of matrix orthogonal idempotents of A. Then $e_iA \cong e_jA$ for all i, j, $e_A = e_1$ and $A^b \cong K$.

6.5. Lemma. *Let $A^b = e_AAe_A$ be a basic algebra associated to A.*
(a) *The idempotent $e_A \in A^b$ is the identity element of A^b and there is a K-algebra isomorphism $A^b \cong \mathrm{End}(e_{j_1}A \oplus \cdots \oplus e_{j_a}A)$.*
(b) *The algebra A^b does not depend on the choice of the sets $e_1, \dots, e_n$ and $e_{j_1}, \dots, e_{j_a}$, up to a K-algebra isomorphism.*

Proof. (a) By (4.2) applied to the A-module $M = e_AA$, there is a K-algebra isomorphism $\mathrm{End}\, e_AA \cong e_AAe_A$. Because there exists an A-module isomorphism $e_AA \cong e_{j_1}A \oplus \cdots \oplus e_{j_a}A$, we derive K-algebra isomorphisms

$$A^b = e_AAe_A \cong \mathrm{Hom}_A(e_AA, e_AA) \cong \mathrm{End}(e_{j_1}A \oplus \cdots \oplus e_{j_a}A).$$

(b) It follows from the unique decomposition theorem (4.10) that the A-module e_AA depends only on A and not on the choice of the sets $\{e_1, \dots, e_n\}$ and $\{e_{j_1}, \dots, e_{j_a}\}$, up to isomorphism of A-modules. Then the statement (b) is a consequence of the K-algebra isomorphisms $A^b \cong \mathrm{End}\, e_AA \cong \mathrm{End}(e_{j_1}A \oplus \cdots \oplus e_{j_a}A)$. $\square$

We will show in (6.10) that the algebra A^b is basic and that there is an equivalence of categories $\mathrm{mod}\, A \cong \mathrm{mod}\, A^b$.

In the study of $\mathrm{mod}\, A$ we frequently use two embeddings of module categories induced by an algebra idempotent defined as follows.

Suppose that $e \in A$ is an idempotent in a finite dimensional K-algebra A and consider the algebra $B = eAe \cong \mathrm{End}\, eA$ with the identity element $e \in B$. We define three additive K-linear covariant functors

$$\mathrm{mod}\, B \underset{\mathrm{res}_e}{\overset{T_e,\, L_e}{\rightleftarrows}} \mathrm{mod}\, A \qquad (\mathbf{6.6})$$

by the formulas

$$\mathrm{res}_e(-) = (-)e, \quad T_e(-) = - \otimes_B eA, \quad L_e(-) = \mathrm{Hom}_B(Ae, -).$$

If $f : X \to X'$ is a homomorphism of A-modules, we define a homomorphism of B-modules $\mathrm{res}_e(f) : \mathrm{res}_e(X) \to \mathrm{res}_e(X')$ by the formula $xe \mapsto f(x)e$, that is, $\mathrm{res}_e(f)$ is the restriction of f to the subspace Xe of X. We call res_e the **restriction functor**. The K-linear functors T_e, L_e are called **idempotent embedding functors**.

Example 6.7. Suppose that $A = KI \subseteq \mathbb{M}_n(K)$ is the incidence algebra of a poset $(I, \preceq)$, where $I = \{1, \dots, n\}$ (see (1.1)(d)). Let J be a subposet of I and take for e the idempotent $e_J = \sum_{j \in J} e_j \in KI$, where $e_1, \dots, e_n \in KI$

are the standard matrix idempotents. A simple calculation shows that if $\lambda' = [\lambda'_{pq}] \in KI$ and $\lambda = e_J \lambda' e_J$, then λ has an $n \times n$ matrix form $\lambda = [\lambda_{pq}] \in KI$, where $\lambda_{pq} = 0$ whenever $p \in I \setminus J$ or $q \in I \setminus J$. This shows that $e_J(KI)e_J$ is the K-vector subspace of KI consisting of all matrices $\lambda = [\lambda_{pq}] \in KI$ with $\lambda_{pq} = 0$ whenever $p \in I \setminus J$ or $q \in I \setminus J$. Therefore there is a K-algebra isomorphism $e_J(KI)e_J \cong KJ$.

The following result is very useful in applications.

Theorem 6.8. *Suppose that A is a finite dimensional K-algebra and that $e \in A$ is an idempotent, and let $B = eAe$. The functors T_e, L_e* (6.6) *associated to $e \in A$ satisfy the following conditions.*

(a) *T_e and L_e are full and faithful K-linear functors such that* $\mathrm{res}_e T_e \cong 1_{\mathrm{mod}\, B} \cong \mathrm{res}_e L_e$*, the functor L_e is right adjoint to* res_e *and T_e is left adjoint to* res_e*, that is, there are functorial isomorphisms*

$$\begin{array}{rcl} \mathrm{Hom}_A(X_A, L_e(Y_B)) & \cong & \mathrm{Hom}_B(\mathrm{res}_e(X_A), Y_B) \\ \mathrm{Hom}_A(T_e(Y_B), X_A) & \cong & \mathrm{Hom}_B(Y_B, \mathrm{res}_e(X_A)) \end{array}$$

for every A-module X_A and every B-module Y_B.

(b) *The restriction functor* res_e *is exact, T_e is right exact, and L_e is left exact.*

(c) *The functors T_e and L_e preserve indecomposability, T_e carries projectives to projectives, and L_e carries injectives to injectives.*

(d) *A module X_A is in the category* $\mathrm{Im}\, T_e$ *if and only if there is an exact sequence $P_1 \xrightarrow{h} P_0 \longrightarrow X_A \longrightarrow 0$, where P_1 and P_0 are direct sums of summands of eA.*

Proof. (a) By (4.2), the map θ_X, $f \mapsto f(e) = f(e)e$, is a functorial B-module isomorphism $\mathrm{Hom}_A(eA, X_A) \xrightarrow{\simeq} Xe$. Hence, in view of the adjoint formula (2.11), we get

$$\begin{array}{rcl} \mathrm{Hom}_A(T_e(Y_B), X_A) & = & \mathrm{Hom}_A(Y \otimes_B eA, X_A) \\ & \cong & \mathrm{Hom}_B(Y, \mathrm{Hom}_A(eA, X_A)) \\ & \cong & \mathrm{Hom}_B(Y, Xe) \cong \mathrm{Hom}_B(Y_B, \mathrm{res}_e(X_A)), \end{array}$$

and similarly we get the first isomorphism required in (a). Moreover, there are isomorphisms $\mathrm{res}_e T_e(Y_B) = (Y \otimes_B eA)e \cong Y \otimes_B (eAe) = Y \otimes_B B \cong Y_B$ and $\mathrm{res}_e L_e(Y_B) \cong Y_B$. As a consequence, we get functorial isomorphisms

$$\begin{array}{rcl} \mathrm{Hom}_B(Y_B, Y'_B) & \cong & \mathrm{Hom}_B(Y_B, \mathrm{res}_e T_e(Y'_B)) \\ & \cong & \mathrm{Hom}_A(T_e(Y_B), T_e(Y'_B)) \end{array}$$

and $\mathrm{Hom}_B(Y_B, Y'_B) \cong \mathrm{Hom}_A(L_e(Y_B), L_e(Y'_B))$ such that $f \mapsto T_e(f)$ and $f \mapsto L_e(f)$, respectively. This proves that T_e and L_e are full and faithful and (a) follows.

(b) The exactness of the functor res_e is obvious. The functor T_e is right exact, because the tensor product functor is right exact. Because the functor $\mathrm{Hom}_A(M, -)$ is left exact, the functor L_e is left exact and (b) follows.

(c) It follows from (a) that L_e and T_e induce the algebra isomorphisms $\mathrm{End}\, X \cong \mathrm{End}\, L_e X$ and $\mathrm{End}\, X \cong \mathrm{End}\, T_e X$. Hence they preserve indecomposability, because of (4.8).

Now assume that P is a projective module in $\mathrm{mod}\, B$ and let $h : M \to N$ be an epimorphism in $\mathrm{mod}\, A$. In view of the natural isomorphism in (6.8)(a) for the functor T_e, there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_A(T_e(P), M) & \xrightarrow{\mathrm{Hom}_A(T_e(P),h)} & \mathrm{Hom}_A(T_e(P), N) \\
\cong \downarrow & & \cong \downarrow \\
\mathrm{Hom}_B(P, \mathrm{res}_e(M)) & \xrightarrow{\mathrm{Hom}_B(P,\mathrm{res}_e(h))} & \mathrm{Hom}_B(P, \mathrm{res}_e(N)).
\end{array}$$

Because P is projective, the homomorphism $\mathrm{Hom}_B(P, \mathrm{res}_e(h))$ is surjective. It follows that $\mathrm{Hom}_A(T_e(P), h)$ is also surjective and therefore the A-module $T_e(P)$ is projective. If E is injective, then we show that $L_e(E)$ is injective.

(d) Assume that $e = e_{j_1} + \ldots + e_{j_s}$ and $e_{j_1}, \ldots, e_{j_s}$ are primitive orthogonal idempotents. It follows that $B = e_{j_1}B \oplus \ldots \oplus e_{j_s}B$ and the modules $e_{j_1}B, \ldots, e_{j_s}B$ are indecomposable.

First, we show that the multiplication map

$$m_{j_i} : e_{j_i}B \otimes_B eA \to e_{j_i}A, \qquad \textbf{(6.9)}$$

$e_{j_i}x \otimes ea \mapsto e_{j_i}xea$, is an A-module isomorphism for $i = 1, \ldots, s$. It is clear that m_{j_i} is well-defined and an A-module epimorphism. Because m_{j_i} is the restriction of the A-module isomorphism $m : B \otimes_B eA \to eA$, $x \otimes ea \mapsto xea$, to the direct summand $e_{j_i}B \otimes_B eA$ of $B \otimes_B eA \cong eA$, m_{j_i} is injective and we are done.

To prove (d), assume that $\overline{P}_1 \to \overline{P}_0 \to Y_B \to 0$ is an exact sequence in $\mathrm{mod}\, B$, where $\overline{P}_0$, $\overline{P}_1$ are projective. Then the induced sequence

$$\overline{P}_1 \otimes_B eA \;\to\; \overline{P}_0 \otimes_B eA \;\to\; Y \otimes_B eA \;\to\; 0$$

in $\mathrm{mod}\, A$ is exact and the modules $P_1 = \overline{P}_1 \otimes_B eA$, $P_0 = \overline{P}_0 \otimes_B eA$ satisfy the conditions required in (d) because, according to (5.3), the modules $\overline{P}_1$ and $\overline{P}_0$ are direct sums of indecomposable modules isomorphic to some of the modules $e_{j_1}B, \ldots, e_{j_s}B$, and the preceding observation applies.

Conversely, assume there is an exact sequence $P_1 \xrightarrow{h} P_0 \to X_A \to 0$, in $\operatorname{mod} A$ with P_0, P_1 direct sums of summands of eA. Then $P_0 e$ and $P_1 e$ are obviously finite dimensional projective B-modules and by the observation, there are A-module isomorphisms $T_e(P_0 e) = P_0 e \otimes_B eA \cong P_0$, $T_e(P_1 e) = P_1 e \otimes_B eA \cong P_1$. If Y_B denotes the cokernel of the restriction $he : P_1 e \to P_0 e$ of h to $\operatorname{res}_e(P_1) = P_1 e$, then we derive a commutative diagram

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & X_A & \longrightarrow & 0 \\ {\scriptstyle f_1}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle f_0}\Big\downarrow{\scriptstyle \cong} & & & & \\ T_e(P_1 e) & \longrightarrow & T_e(P_0 e) & \longrightarrow & T_e(Y_B) & \longrightarrow & 0 \end{array}$$

with exact rows and bijective vertical maps f_1, f_0. Hence we get an isomorphism $X_A \cong T_e(Y_B)$ induced by f_0 and the proof is complete. $\square$

6.10. Corollary. *Let $A^b = e_A A e_A$ be a basic K-algebra associated with A (see (6.3)). The algebra A^b is basic and the functors*

$$\operatorname{mod} A^b \underset{\operatorname{res}_{e_A}}{\overset{T_{e_A}}{\Longleftrightarrow}} \operatorname{mod} A$$

are K-linear equivalences of categories quasi-inverse to each other.

Proof. Assume that $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents of A, $e_A = e_{j_1} + \cdots + e_{j_a}$ and $e_{j_1}, \dots, e_{j_a}$ are chosen as in (6.3). Then $e_{j_1}, \dots, e_{j_a}$ are orthogonal idempotents of A^b,

$$A^b = e_A A^b = e_{j_1} A^b \oplus \ldots \oplus e_{j_a} A^b,$$

and $e_{j_t} A^b e_{j_t} = e_{j_t} e_A A e_A e_{j_t} = e_{j_t} A e_{j_t}$ for all t. It follows from (4.7) that the algebra $\operatorname{End} e_{j_t} A^b \cong e_{j_t} A^b e_{j_t}$ is local, because $e_{j_t} A$ is indecomposable in $\operatorname{mod} A$. Hence e_{j_t} is a primitive idempotent of A^b. To show that the algebra A^b is basic, assume that $e_{j_t} A^b \cong e_{j_r} A^b$. Because we have shown in (6.9) that the multiplication map $m_{j_i} : e_{j_i} A^b \otimes_{A^b} e_A A \to e_{j_i} A$, $e_{j_i} x \otimes e_A a \mapsto e_{j_i} x e_A a$, is an A-module isomorphism for $i = 1, \dots, a$, we get A-module isomorphisms

$$e_{j_t} A \cong e_{j_t} A^b \otimes_{A^b} e_A A \cong e_{j_r} A^b \otimes_{A^b} e_A A \cong e_{j_r} A$$

and therefore $t = r$ by the choice of $e_{j_1}, \dots, e_{j_a}$ in (6.3).

By (6.8), the functor T_{e_A} is full and faithful. Because

$$e_A A \cong e_{j_1} A \oplus \cdots \oplus e_{j_a} A,$$

each $e_{j_t} A$ is isomorphic to a summand of $e_A A$. This, together with (6.3) and (6.8), shows that every module X in $\operatorname{mod} A$ admits an exact sequence

$P' \to P \to X \to 0$, where P and P' are direct sums of summands of $e_A A$. It then follows from (6.8)(d) that any module X_A belongs to the image of the functor T_{e_A}. Consequently, T_{e_A} is dense, and according to (A.2.5) of the Appendix, the full and faithful K-linear functor T_{e_A} is an equivalence of categories. Therefore res_{e_A} is a quasi-inverse of T_{e_A}. □

6.11. Corollary. *Let A be a K-algebra. For each $n \geq 1$, there exists a K-linear equivalence of categories* $\mathrm{mod}\, A \cong \mathrm{mod}\, \mathbb{M}_n(A)$.

Proof. Let $B = \mathbb{M}_n(A)$ and let $\xi_1, \dots, \xi_n \in B$ be the standard set of matrix idempotents in B, that is, ξ_j is the matrix with 1 on the position (j, j) and zeros elsewhere. Because $B = \xi_1 B \oplus \cdots \oplus \xi_n B$, $\xi_1 B \cong \xi_2 B \cong \cdots \cong \xi_n B$ and $\xi_1 B \xi_1 \cong A$, applying (6.8) to $e = \xi_1 \in B$, we conclude as in the proof of (6.10) that the composite functor $\mathrm{mod}\, A \cong \mathrm{mod}\, \xi_1 B \xi_1 \xrightarrow{T_{\xi_1}} \mathrm{mod}\, \mathbb{M}_n(A)$ is an equivalence of categories. □

I.7. Exercises

1. Let $f : A \to B$ be a homomorphism of K-algebras. Prove that $f(\mathrm{rad}\, A) \subseteq \mathrm{rad}\, B$.

2. Let A be the polynomial K-algebra $K[t_1, t_2]$. Prove that
(a) the algebra A is not local,
(b) the elements 0 and 1 are the only idempotents of A, and
(c) the radical of A is zero.

3. Prove that a homomorphism $u : L \to M$ of right A-modules admits a retraction $p : M \to L$ if and only if u is injective and $M = \mathrm{Im}\, u \oplus N$, where N is a submodule of M.

4. Prove that a homomorphism $r : M \to N$ of right A-modules admits a section $v : N \to M$ if and only if r is surjective and $M = L \oplus \mathrm{Ker}\, r$, where L is a submodule of M.

5. Suppose that the sequence $0 \longrightarrow L \xrightarrow{u} M \xrightarrow{r} N \longrightarrow 0$ of right A-modules is exact. Prove that the homomorphism u admits a retraction $p : M \to L$ if and only if r admits a section $v : N \to M$.

6. Let N be a submodule of a right A-module M. Prove that
(a) $\mathrm{rad}(M/N) \supseteq (N + \mathrm{rad}\, M)/N$, and
(b) if $N \subseteq \mathrm{rad}\, M$, then $\mathrm{rad}(M/N) = (\mathrm{rad}\, M)/N$.

7. Let $A = K[t]$. Prove that the cyclic A-module $M = K[t]/(t^3)$ has no projective cover in $\mathrm{Mod}\, A$.

8. Let A be a K-algebra and let $Z(A)$ be the **centre** of A, that is, the subalgebra of A consisting of all elements $a \in A$ such that $ay = ya$ for all $y \in A$. Show that the following three conditions are equivalent:

(a) The algebra A is connected.

(b) The algebra $Z(A)$ is connected.

(c) The elements 0 and 1 are the only central idempotents of A.

9. Assume that A is a K-algebra, $e \in A$ is an idempotent of A, and M is a right A-module. Prove the following statements:

(a) The K-subspace eAe of A is a K-algebra with respect to the multiplication of A, and e is the identity element of eAe.

(b) The K-vector space Me is a right eAe-module, and the K-vector space $\mathrm{Hom}_A(eA, M)$ is a right eAe-module with respect to the multiplication $(f, a) \mapsto fa$ for $f \in \mathrm{Hom}_A(eA, M)$ and $a \in A$, where we set $(fa)(x) = f(xa)$ for all $x \in eA$.

(c) The K-linear map $\theta_M : \mathrm{Hom}_A(eA, M) \longrightarrow Me$, $f \mapsto f(e)$, is an isomorphism of right eAe-modules, and it is functorial in M.

(d) The map $\theta_{eA} : \mathrm{Hom}_A(eA, eA) \longrightarrow eAe$ is a K-algebra isomorphism.

(e) The map $M \otimes_A Ae \longrightarrow Me$, $m \otimes x \mapsto mx$, is an isomorphism of right eAe-modules, and it is functorial in M.

10. Assume that A is a finite dimensional K-algebra. Prove that A is local if and only if every element of A is invertible or nilpotent.

11. Let KI be the incidence K-algebra of a poset $(I, \preceq)$ (see (1.5)(d)) and let B be the K-subalgebra of KI consisting of the matrices $\lambda = [\lambda_{ij}] \in KI$ such that $\lambda_{ii} = \lambda_{jj}$ for all $i, j \in I$. Prove the following statements:

(a) The algebra KI is basic, and KI is semisimple if and only if $a_i \not\preceq a_j$ for every pair of elements $a_i \neq a_j$ of I.

(b) The algebra KI is local if and only if $|I| = 1$.

(c) The subalgebra B of KI is local.

(d) The algebra B is noncommutative if and only if there is a triple a_i, a_j, a_s of pairwise different elements of I such that $a_i \prec a_j \prec a_s$.

12. Let M be a module in $\mathrm{mod}\, A$. Prove that there is a functorial isomorphism $\mathrm{soc}\, DM \xrightarrow{\simeq} D(M/\mathrm{rad}\, M)$, where D is the standard duality.

13. Let $A = \mathbb{M}_n(K)$, where $n \geq 1$, and let M be an indecomposable A-module. Show that $\ell(M) = 1$ and $\dim_K M = n$.

14. Let A be a basic finite dimensional algebra over an algebraically closed field K, and let M be a finite dimensional right A-module. Show that $\ell(M) = \dim_K M$.

15. Let A be a finite dimensional K-algebra over an algebraically closed field K. Prove that the following three conditions are equivalent:

(a) The algebra A is basic.
(b) Every simple right A-module is one-dimensional.
(c) $\dim_K M = \ell(M)$, for any module M in $\operatorname{mod} A$.
Hint: Apply (6.2).

16. Let A be any of the two subalgebras

$$\begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & K & K & K \end{bmatrix} \subset \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & K & K & 0 \\ K & K & K & K \end{bmatrix}$$

of the full matrix algebra $\mathbb{M}_4(K)$ defined in Examples 1.1(c) and 1.1(d). Let $e_1 = e_{11}, e_2 = e_{22}, e_3 = e_{33}, e_4 = e_{44}$ be the standard complete set of primitive orthogonal idempotents in A. Show that

(a) the algebra A is basic,

(b) there is an isomorphism $Ae_1 \cong D(e_4A)$ of left A-modules, where D is the standard duality,

(c) the right ideal $S(1) = e_1A$ of A is simple and $\operatorname{soc} A_A = \begin{bmatrix} K & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ K & 0 & 0 & 0 \end{bmatrix}$,

and

(d) the indecomposable projective right ideal $P(4) = e_4A$ is an injective envelope of $S(1)$, and the indecomposable projective right ideals $P(1) = e_1A$, $P(2) = e_2A$ and $P(3) = e_3A$ are not injective.

17. Assume that A is a finite dimensional K-algebra, $f : M \to N$ is a homomorphism in $\operatorname{mod} A$, and $M \neq 0$. Prove the following statements:

(a) The socle $\operatorname{soc} M$ of M is a nonzero semisimple submodule of M and $f(\operatorname{soc} M) \subseteq \operatorname{soc} N$.

(b) If $f(\operatorname{soc} M) \neq 0$, then $f \neq 0$.

(c) The inclusion homomorphism $\operatorname{soc} M \subseteq M$ induces an A-module isomorphism $E(\operatorname{soc} M) \xrightarrow{\simeq} E(M)$ of the injective envelopes $E(\operatorname{soc} M)$ and $E(M)$ of $\operatorname{soc} M$ and M, respectively.

(d) The module M is indecomposable if and only if the injective envelope $E(M)$ of M is indecomposable.