RESEARCH STATEMENT

ADAM KLEPÁČ

Abstract. Following the construction of d-representation-finite algebras in [2] and the description of the correspondence between certain types of cluster algebras and triangulations of bordered surfaces with marked points in [1], links have appeared connecting d-representation finite algebras to higher dimensional variants of said surface. One such link was discovered in [4] between higher Auslander algebras of the path algebra of linearly oriented Dynkin quiver A_n and cyclic polytopes. I wish to further study such kinds of connections, starting with the establishment of a similar type of link for path algebras of quivers of type D_n which, in the low-dimensional case, correspond to once punctured polygons; then, with a touch of expectation and naïvety, broadening it to include (special types of) cluster algebras of finite mutation type.

1. Introduction

This text serves primarily as an overview of relevant concepts regarding cluster algebras, bordered surfaces with marked points, higher dimensional cluster categories and d-representation-finite algebras interwoven with ideas of possible generalizations and caveats tied to such endeavour. So far, I have only scratched the surface of this topic, hence very few original results are present.

In Section 2, I give a summary of the theory of bordered surfaces with marked points. Section 3 is dedicated to (normalized skew-symmetrizable) cluster algebras and their connection to bordered surfaces with marked points is drawn. Sections 4 and 5 define d-representation-finite algebras and higher cluster categories, respectively. Section 6 summarizes relevant results from [4], regarding a higher-dimensional kind of connection described in Section 3. Finally, Section 7 is riddled with (splinters of) steps towards generalizations of the content of Section 6.

2. Bordered Surfaces with Marked Points

This section is a brief summary of [1], Section 2.

Definition 2.1 (Bordered surface with marked points)**.** Let $\mathbf{S}$ be a connected oriented 2-dimensional Riemann surface with boundary. We fix a finite set $\mathbf{M}$ of *marked points* in the closure of $\mathbf{S}$. Marked points lying in the interior of $\mathbf{S}$ are called *punctures*. The pair $(\mathbf{S}, \mathbf{M})$ is called a *bordered surface with marked points* if the following additional technical conditions are satisfied.

- The set $\mathbf{M}$ is non-empty.
- The pair $(\mathbf{S}, \mathbf{M})$ is not
  - a sphere with one or two punctures;
  - a monogon with zero or one puncture;
  - a digon without punctures;
  - a triangle without punctures.

Here, the term n-gon denotes a disk with n marked points on its boundary. Moreover, sphere with three punctures is also often excluded.

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Definition 2.2 (Arc)**.** An arc γ in a bordered surface with marked points $(\mathbf{S}, \mathbf{M})$ is a curve in $\mathbf{S}$ such that

- its endpoints are marked points;
- γ does not intersect itself, except that its endpoints may coincide;
- except for its endpoints, γ is disjoint from $\mathbf{M}$ and from the boundary of $\mathbf{S}$;
- γ is not contractible into $\mathbf{M}$ or into the boundary of $\mathbf{S}$.

We are interested in triangulations of $(\mathbf{S}, \mathbf{M})$. Vaguely speaking, triangulation is a division of $\mathbf{S}$ into 'triangles' by a series of 'cuts'. Here, 'triangles' are either disks with three marked points on their boundaries or, so-called *self-folded* triangles, once-punctured monogons with an arc connecting the unique marked point to the unique puncture. See figure 1.

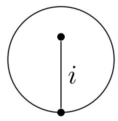

FIGURE 1. A self-folded triangle.

Definition 2.3 (Isotopy)**.** Let γ_1, γ_2 be two arcs in $(\mathbf{S}, \mathbf{M})$. An *isotopy* between γ_1 and γ_2 is a homotopy H between γ_1 and γ_2 such that $H(x, t)$ is an embedding for each fixed $t \in [0, 1]$. Isotopy is an equivalence relation on the set of all arcs in $(\mathbf{S}, \mathbf{M})$.

In the following text, each arc in $(\mathbf{S}, \mathbf{M})$ is considered up to isotopy.

Definition 2.4 (Compatibility of arcs)**.** Two arcs in $(\mathbf{S}, \mathbf{M})$ are called *compatible* if they (up to isotopy) do not intersect each other in the interior of $\mathbf{S}$.

Proposition 2.5. *Any collection of pairwise compatible arcs can be realized by curves in their respective isotopy classes which do not intersect in the interior of* $\mathbf{S}$.

Definition 2.6 (Ideal triangulation)**.** A maximal collection of pairwise compatible arcs is called an *ideal triangulation.* In fact, definition 2.1 excludes all cases where $(\mathbf{S}, \mathbf{M})$ cannot be triangulated. The arcs of an ideal triangulation cut $\mathbf{S}$ into *ideal triangles.* The three sides of an ideal triangle need not be distinct, leading to self-folded triangles, and two triangles can share more than one side.

The number of arcs in an ideal triangulation is an invariant of $(\mathbf{S}, \mathbf{M})$ and is called the *rank* of $(\mathbf{S}, \mathbf{M})$ to emphasize the connection between these surfaces and cluster algebras of the same rank, to be introduced in the next section.

The last concept we need to introduce is that of a *flip* of an ideal triangulation. These basically entail swapping one diagonal for another in some quadrilateral of an ideal triangulation.

Definition 2.7 (Flip)**.** A *flip* in an ideal triangulation T is a transformation which exchanges one arc $\gamma \in T$ for a different arc γ' which, together with the rest of arcs in T, forms a new ideal triangulation T'.

There is at most one way to flip and arc γ in an ideal triangulation. If γ is the 'folded' side of a self-folded triangle (the segment i in figure 1), then γ cannot be flipped. In all other cases, removing γ create a tetragonal face on $\mathbf{S}$, and the flipped arc γ' is define to be its other diagonal.

Of special import to the theory of cluster algebras is the following result.

Proposition 2.8. *Any two ideal triangulations are related by a series of flips.*

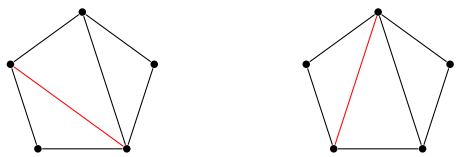

FIGURE 2. Two ideal triangulations of a pentagon, related by a flip.

3. Cluster Algebras

Before we reach the definition of a cluster algebras, we must discuss quivers and their mutations. This section is an altered version of [4], Section 4. The original text defines cluster algebras using signed adjacency matrices instead. This approach lessens the notational burden but also introduces a concept we do not use elsewhere. Naturally, quivers and adjacency matrices are tightly related, with the former being completely determined by the latter after a choice of orientation of a single arrow.

To each ideal triangulation T of a bordered surface with marked points $(\mathbf{S}, \mathbf{M})$ we associate a quiver $Q := Q(T)$ in the following manner. For a fixed ideal triangulation, we label its n arcs by natural numbers from 1 to n, keeping in mind that this labelling is arbitrary. The set of vertices of Q is then also $Q_0 := \{1, \dots, n\}$. Next, for each triangle Δ in T which is not self-folded, we draw an arrow $i \to j$ if

- i and j are sides of Δ with j following i in clockwise order;
- j is an arc 'folded' by l, i and l are sides of Δ and l follows i in clockwise order;
- i is an arc 'folded' by l, l and j are sides of Δ with j following l in clockwise order.

Finally, we remove all 2-cycles (meaning a configuration of arrows like this $\bullet \rightleftarrows \bullet$).

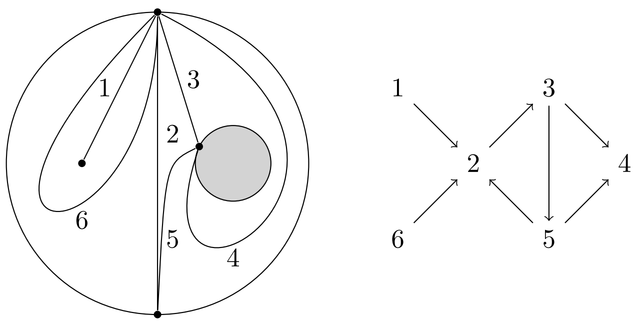

FIGURE 3. An ideal triangulation of a once-punctured annulus and its quiver.

Definition 3.1 (Quiver mutation)**.** For a quiver Q, we construct the quiver $\mu_k(Q)$, called its *mutation* at k-th vertex by

- reversing all arrows with k as source or as target;
- adding an arrow $j \to i$ for each path $i \to k \to j$;
- deleting all 2-cycles.

The mutation at k-th vertex is an involution of Q, that is, $\mu_k^2(Q) = Q$ for every vertex $k \in Q_0$. Now, we proceed to define initial seeds and cluster algebras.

We fix a free abelian group $(\mathbb{P}, \cdot)$ on variables $y_1, \ldots, y_n$ and define an operation $\oplus$ on $\mathbb{P}$ by the formula

$$\prod_i y_i^{a_i} \oplus \prod_i y_i^{b_i} = \prod_i y_i^{\min(a_i, b_i)}.$$

Let $\mathbb{ZP}$ denote the group ring of $\mathbb{P}$ and $\mathbb{QP}$ the field of fractions of $\mathbb{ZP}$. Finally, we let $\mathcal{F} := \mathbb{QP}(x_1, \ldots, x_n)$ be the field of rational functions in variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{QP}$.

Definition 3.2 (Initial seed)**.** An *initial seed* is a triple $(\mathbf{x}, \mathbf{y}, Q)$ consisting of the following data:

(1) an n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ of variables of $\mathcal{F}$, the so-called *initial cluster*;
(2) an n-tuple $\mathbf{y} = (y_1, \ldots, y_n)$ of generators of $\mathbb{P}$, the so-called *initial coefficients tuple*;
(3) a quiver Q without loops and 2-cycles.

Definition 3.3 (Seed mutation)**.** A *seed mutation $\mu(k)$ in direction k* is a transformation of an initial seed $(\mathbf{x}, \mathbf{y}, Q)$ into a new seed $(\mathbf{x}', \mathbf{y}', Q')$ defined as follows:

- $\mathbf{x}'$ is the n-tuple of variables constructed by replacing the cluster variable x_k in $\mathbf{x}$ by a new cluster variable x'_k defined by the *exchange relation*

$$x_k x'_k = \frac{1}{y_k \oplus 1}\left(y_k \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i \right);$$

- $\mathbf{y}' = (y'_1, \ldots, y'_n)$ is a new n-tuple of coefficients, where

$$y'_i := \begin{cases} y_k^{-1} & \text{if } i = k; \\ y_i \prod_{k \to i} y_k (y_k \oplus 1)^{-1} \prod_{k \leftarrow i} (y_k \oplus 1) & \text{if } i \neq k. \end{cases}$$

- Q' is the mutation of Q at the k-th vertex.

Seed mutations are involutions, that is, $\mu_k^2(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}, \mathbf{y}, Q)$.

Definition 3.4 (Cluster algebra)**.** Let $(\mathbf{x}, \mathbf{y}, Q)$ be an initial seed and $\mathcal{X}$ be the set of all cluster variables generated by repeated mutation of $(\mathbf{x}, \mathbf{y}, Q)$. The *cluster algebra* $\mathcal{A} := \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$.

We say that a cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is

- of *finite type*, if the set $\mathcal{X}$ of cluster variables if finite;
- of *finite mutation type*, if the number of quivers mutation equivalent (those Q can mutate into) to Q is finite;
- of *acyclic type*, if Q is mutation equivalent to a quiver without oriented cycles;
- of *surface type*, if Q is a quiver coming from a triangulation of a bordered surface with marked points.

We are particularly interested in quivers whose underlying graph is one of (simply-laced) Dynkin diagrams of type $A_n (n \geq 1)$, $D_n (n \geq 4)$ or $E_n (6 \leq n \leq 8)$ (see figure 4). By [1], Theorem 6.5, a cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is of finite type if and only if the underlying graph of Q is a disjoint union of the aforementioned Dynkin diagrams. In this particular case, it is also true (by [1], Lemma 6.4) that quivers given by two different orientations of a Dynkin diagram are mutation equivalent, hence the structure of $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is independent of the choice of orientation.

In seeking higher-dimensional geometric counterparts to cluster algebras of finite type, it is of course beneficial to – at least at first – focus on those that are also of surface type. Here, one has a starting idea as to what the higher-dimensional object in question should be. The only such cluster algebras are of type A_n and

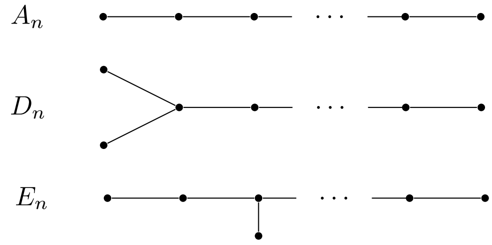

FIGURE 4. Simply-laced Dynkin diagrams of types A_n, D_n and E_n.

D_n. Quivers given by orientations of E_n do not arise from any bordered surface with marked points.

Moreover, based on the results in [3], for purposes of categorisation, one need not consider the entire class of cluster algebras with a chosen Dynkin quiver. The combinatorial properties of $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ are in fact governed entirely by so-called *decorated representations* of the path algebra of Q (see [3], Section 2).

Hence, studying the higher Auslander algebras (to be introduced promptly) of path algebras of Dynkin quivers appears to be a sensible endeavour in this direction.

By a considerable extension, one might also consider studying higher-dimensional variants of cluster algebras of 'affine' type, whose quiver is one of so-called *affine* Dynkin diagrams. For these, however, Lemma 6.4 from [1] does not apply and thus the choice of orientation matters. Furthermore, the path algebras of such diagrams are in general representation-infinite leading to caveats in applying the higher Auslander theory developed in [2].

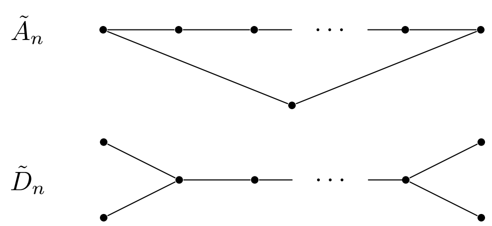

FIGURE 5. Affine Dynkin diagrams of types $\tilde{A}_n$ and $\tilde{D}_n$.

4. HIGHER AUSLANDER THEORY

In this section, we summarize results from [2], Section 1. We fix a finite-dimensional algebra Λ over a field k.

Definition 4.1 (d-cluster tilting module)**.** A module $M \in \operatorname{mod}\Lambda$ is called *d-cluster tilting* if

$$\operatorname{add} M = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, M) = 0 \;\forall i \in \{1, \dots, d-1\}\}.$$

We note that a 1-cluster tilting module is just an additive generator of the category $\operatorname{mod}\Lambda$.

Definition 4.2 (d-Auslander algebra)**.**

(1) An algebra Λ is called *d-representation-finite* if $\operatorname{gl.dim}\Lambda < \infty$ and Λ has a d-cluster tilting module.

(2) Let Λ be a d-representation-finite algebra and M its d-cluster tilting module. We call $\mathrm{End}_\Lambda(M)$ the *d-Auslander algebra* of Λ. We denote it $\Lambda^{(d)}$.

By [2], Theorem 1.6, if Λ is d-representation-finite then its d-cluster tilting module is unique up to multiplicity. A 1-representation-finite algebra is simply called representation-finite.

One of the main results in [2] concerns an iterative construction of d-Auslander algebras of a representation-finite hereditary algebra Λ. In this particular case, $\Lambda^{(d)}$ is a $(d+1)$-representation-finite algebra and has a $(d+1)$-cluster tilting module for every $d \geq 1$. See [2], 1.13 - 1.16. Starting with Λ, we denote its 1-cluster tilting module by ${}_\Lambda M$. Then, $\Lambda^{(1)} := \mathrm{End}_\Lambda(M)$ is a 2-representation-finite algebra with a 2-cluster tilting module, which we denote ${}_{\Lambda^{(1)}}M$. And so forth.

References

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