COMBINATORIAL PROPERTIES OF HIGHER CLUSTER CATEGORIES

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Research Objectives

The main objective of this project is to understand and describe the combinatorics of higher-dimensional analogues of the path algebras (in the sense of higher representation theory developed by Iyama in [Iya10]) of affine Dynkin quivers and their associated generalized cluster categories (in the sense described in [OT11, Section 5], building upon [Ami09] and [Ami08], or [Guo12]), possibly and hopefully aiding in an eventual broader comprehension of the combinatorial structure of n-representation-infinite algebras (see [HIO14]).

The path towards the goal of this project can be divided into several steps, ordered chronologically and quite possibly also by level of difficulty.

1. **Combinatorial description of higher Auslander algebras of type D and E and the associated cluster categories.** The first step entails expanding upon the research done on the higher Auslander algebras of type A in [OT11] with the auxiliary goal of deepening my understanding of higher representation theory and the theory of cluster categories.

Dynkin quivers of type D arise (as well as those of type A) from triangulations of bordered surfaces with marked points – in this case the surface in question is a disk with finite number of marked points on its boundary and one marked point in its interior. The most natural higher-dimensional structure whose triangulations should describe basic tilting modules of their path algebras and also the basic cluster tilting objects of their associated cluster categories is the cyclic polytope with a distinguished point in its interior. As the higher Auslander algebras of such algebras are n-representation-finite, the necessary theory is already in place and it remains to successfully describe (in combinatorial terms) the tilting objects in question.

Quivers of Dynkin type E do not arise from triangulations of bordered surfaces and thus there are no obvious hints as to which combinatorial structure captures the nature of the tilting modules of the higher Auslander algebras of their path algebras. Nonetheless, as there are only three Dynkin diagrams of type E, at worst such structures can be found on a case-by-case basis.

2. **Generalization of the theory of higher Auslander algebras to path algebras of quivers of affine Dynkin type.** The iterative construction of higher Auslander algebras detailed in [Iya10] hinges on the property of the 'starting' algebra being representation-finite and hereditary and thus having a (up to multiplicity unique) cluster tilting object. In order to study representation-infinite path algebras from the same viewpoint, a generalization of this theory which compensates for the non-existence of such an object is paramount. An ideal generalization would be one that is also symmetric with respect to changes in orientation of the original quiver which now influences the structure of the associated cluster algebras and cluster categories.

The construction of n-represention-infinite algebras of type $\tilde{A}$ from [HIO14, Section 5] seems promising but still fully depends on the chosen orientation of the graph $\tilde{A}_n$ and in degree 0 yields path algebras of the resulting quiver only in cases when the chosen orientation is acyclic. I suspect, however, that the latter will be a prevailing issue and path algebras of the (exactly two) quivers of type $\tilde{A}$ with cyclic orientation will need to be either excluded or treated separately.

3. **Combinatorial description of the generalized higher Auslander algebras of quivers of affine Dynkin type.** Continuing in the same vein, I intend to find or construct a combinatorial structure which would encapsulate information about the (primarily tilting) modules of the generalized higher Auslander algebras from step 2. Such a structure will undoubtedly fail to be finite but I find it unreasonable to try to predict what such a structure could look like considering I have not yet conceptualized, much less studied, said generalization.

The division of the category of finitely generated modules over n-representation-infinite algebras into n-preprojective, n-regular and n-preinjective parts from [HIO14, Theorem 4.18] is bound to prove itself useful in this endeavour as it already allowed a deeper delve into the combinatorics of the AR quivers of these algebras ([GLL22]).

4. **Construction of a generalized cluster category associated to the generalized higher Auslander algebras of affine Dynkin type.** The final step is to construct a generalized cluster category in a manner similar to the one employed in [OT11, Section 5] for n-representation-infinite algebras or, at least, for the generalized higher Auslander algebras of affine Dynkin type from step 2. It is not yet clear which of the two approaches ([Ami09] or [Guo12]) is preferable, or if one shouldn't even seek generalization of a different kind entirely. In any case, the resulting cluster category must be n-Calabi-Yau, admit an m-cluster tilting object (n and m need not be equal but dependant), its cluster tilting objects should in the 2-dimensional case correspond to clusters of the connected cluster algebra and the combinatorial description of its cluster tilting objects must be compatible (in the vague sense of 'bearing a related structure') with the description from step 3.