Cyclic Polytopes and Higher Auslander Algebras

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Background in Representation Theory

Fundamental notions

- A **k-algebra** (over a field k) is a k-vector space equipped with a bilinear product.
- A **right module** over a k-algebra Λ is a k-vector space with the additional operation of multiplication by elements of Λ from the right which satisfies natural compatibility conditions.
- A right Λ-module M is **indecomposable** if $M \neq 0$ and $M = M_1 \oplus M_2$ implies that either of M_1, M_2 is zero.
- A **morphism** $M \to N$ of right Λ-modules is a k-linear map which respects the multiplication by elements of Λ.
- A morphism f is **irreducible** if it's not invertible (from the left or right) and whenever it decomposes as $f = f_1 f_2$, then f_1 is invertible from the left or f_2 from the right.
- A **path algebra** of a quiver Q (an oriented graph with multiple edges and loops) is the k-algebra kQ with basis formed by all paths in Q and multiplication given by concatenation.

Auslander-Reiten quiver

We can represent the category $\operatorname{mod} \Lambda$ of right Λ-modules as a quiver by considering (isomorphisms classes of) indecomposable Λ-modules as vertices and, informally speaking, the irreducible morphisms (up to a linear combination) as arrows.

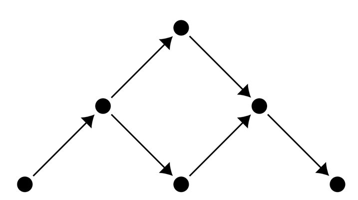

Figure 1. The Auslander-Reiten quiver of the path algebra $k(\bullet \leftarrow \bullet \leftarrow \bullet)$. This algebra has 6 indecomposable modules up to isomorphism.

Representation-finite Hereditary Algebras

Every (basic connected) k-algebra Λ is isomorphic to the quotient path algebra kQ/I for some quiver Q and an ideal I of kQ.

- The algebra Λ is **hereditary** if every submodule of a projective module is itself projective.
- The algebra Λ is **representation-finite** if it has only finitely many indecomposable modules up to isomorphism.

If Λ is hereditary, then it is isomorphic directly to some path algebra kQ. If it is in addition representation-finite, then Q has no loops or multiple edges and Q (as an undirected graph) is one of **Dynkin** graphs (depicted in Figure 2).

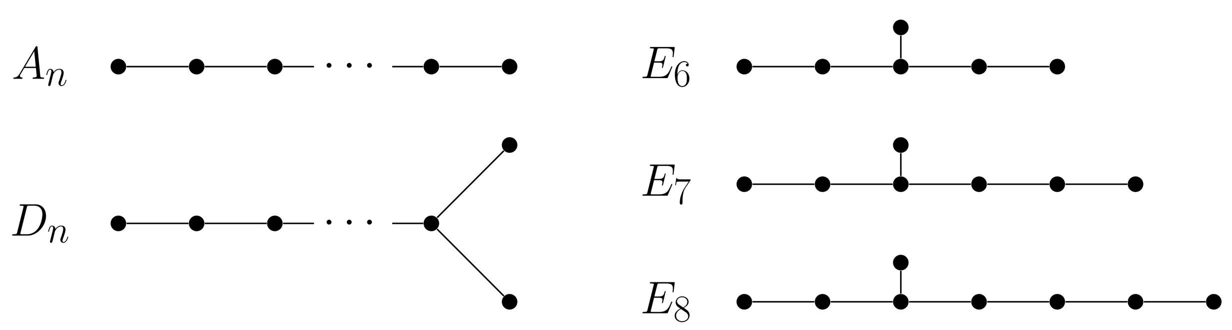

Figure 2. All the Dynkin graphs. The number n in A_n, D_n and E_n denotes the number of vertices.

Tilting Modules & Higher Auslander Algebras

A **d-fold extension** of a Λ-module M by N, is an exact sequence

$$0 \longrightarrow M \longrightarrow X_d \longrightarrow X_{d-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow N \longrightarrow 0,$$

where X_i are Λ-modules. The d-fold extensions of M by N form a group, which we denote $\operatorname{Ext}^d_\Lambda(M, N)$.

In a representation-finite hereditary algebra Λ, a module M

- is **tilting** if $\operatorname{Ext}^1_\Lambda(M, M) = 0$ and the number direct summands of M equals the number of simple Λ-modules.
- is **d-cluster tilting** if

$$\operatorname{add} M = \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(M, X) = 0 \; \forall i \in \{1, \ldots, d-1\}\},$$

  where $\operatorname{add} M$ denotes the category of direct summands of direct copies of M.

The **tilting theorem** [1, Theorem 3.8, Chapter VI] establishes a strong connection between the categories $\operatorname{mod} \Lambda$ and $\operatorname{mod} \operatorname{End}_\Lambda(M)$ where M is a **tilting** Λ-module. In case Λ is hereditary, $\operatorname{End}_\Lambda(M)$ is a so-called **tilted algebra**, an important object of study in contemporary representation theory.

The **global dimension** of Λ is the minimal length of a projective resolution across all Λ-modules. Iyama [2] describes an iterative construction of d-representation-finite algebras (algebras of global dimension $\leq d$ admitting a d-cluster tilting module). It goes like this.

1. Start with a representation-finite hereditary (also called 1-representation-finite) algebra $\Lambda = \Lambda^{(1)}$. This algebra has a 1-cluster tilting module $M = M^{(1)}$. Set $\Lambda^{(2)} = \operatorname{End}_\Lambda(M)$.
2. The algebra $\Lambda^{(2)}$ is 2-representation-finite with a 2-cluster tilting module $M^{(2)}$. Set $\Lambda^{(3)} = \operatorname{End}_{\Lambda^{(2)}}(M^{(2)})$.

k. The algebra $\Lambda^{(k)}$ is k-representation-finite with a k-cluster tilting module $M^{(k)}$. Set $\Lambda^{(k+1)} = \operatorname{End}_{\Lambda^{(k)}}(M^{(k)})$. We call $\Lambda^{(k)}$ the **k-Auslander algebra** of Λ.

Higher Auslander Algebras of Type A & Cyclic Polytopes

A **cyclic polytope** $C(m, d)$ is the convex hull of a set of m points on the moment curve $t \mapsto (t, t^2, \ldots, t^d)$ in $\mathbb{R}^d$. A **triangulation** of $C(m, d)$ is its division into d-dimensional simplices which share vertices with it.

In [3], Oppermann and Thomas uncover an intriguing **correspondence**

$$\{\text{triangulations of } C(n+2d, 2d)\} \longleftrightarrow \{\text{tilting modules over } \Lambda^{(d)}\},$$

where $\Lambda = kA_n$ (the path algebra of a linearly ordered quiver of Dynkin type A, see Figure 2).

Figure 3. The direct sum of the coloured three modules in the Auslander-Reiten quiver of kA_3 is a tilting module corresponding to the depicted triangulation of (an illustrative version of) $C(5, 2)$.

Higher Auslander Algebras of Type D

We currently work on expanding this correspondence to higher Auslander algebras of kD_n – the path algebras of **Dynkin quivers of type D** (see Figure 2).

- There's an isomorphism $kD_n \cong kA_{2(n-1)-1}/I$ for an adequate orientation of A_n and D_n and an adequate ideal I (see Figure 4).
- Using this isomorphism we can partially translate the problem of describing tilting modules of higher Auslander algebras of type D back to the already understood type A.
- It is beneficial to view $kA_{2(n-1)-1}$ as two linearly oriented kA_{n-1}'s. This way, we can find a correspondence between triangulations of two cyclic polytopes (while identifying certain pairs of simplices) and tilting modules of higher Auslander algebras of type D.

Figure 4. Illustration of the isomorphism $kD_4 \cong kA_5/I$. We basically orient A_5 toward the central vertex and 'glue' the paths that start at the same distance from the central vertex but not at the farthest vertices.

Further Research Goals

- Expand the described correspondence to **higher cluster categories** of type D (as is done for type A in [3, Section 6]).
- Focus on path algebras of **affine Dynkin quivers**, which are no longer representation-finite, but are close enough so that a generalized notion of a higher Auslander algebra is plausible.
- Generalize the construction of a cluster category from [3, Section 5] to also include path algebras not necessarily representation-finite.
- Devise a **combinatorial framework** which would describe tilting modules over such 'generalized' higher Auslander algebras and tilting objects of the corresponding cluster categories.

References

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