Simplicial structures in higher Auslander–Reiten theory

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Dedicated to the memory of Thomas Poguntke

Abstract

We develop a novel combinatorial perspective on the higher Auslander algebras of type $\mathbb A$, a family of algebras arising in the context of Iyama's higher Auslander–Reiten theory. This approach reveals interesting simplicial structures hidden within the representation theory of these algebras and establishes direct connections to Eilenberg–MacLane spaces and higher-dimensional versions of Waldhausen's S_{\bullet} -construction in algebraic K-theory. As an application of our techniques we provide a generalisation of the higher reflection functors of Iyama and Oppermann to representations with values in stable ∞ -categories. The resulting combinatorial framework of slice mutation can be regarded as a higher-dimensional variant of the abstract representation theory of type $\mathbb A$ quivers developed by Groth and Šťovíček. Our simplicial point of view then naturally leads to an interplay between slice mutation, horn filling conditions, and the higher Segal conditions of Dyckerhoff and Kapranov. In this context, we provide a classification of higher Segal objects with values in any abelian category or stable ∞ -category.

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Introduction

In the context of higher Auslander–Reiten theory [Iya07b], Iyama introduced a remarkable family of finite-dimensional algebras $A_\ell^{(m)}$, parameterised by natural numbers ℓ and m, called higher Auslander algebras of type \mathbb{A} [Iya11]. For a fixed positive integer ℓ , these algebras are obtained by a recursive construction that starts with the hereditary algebra $A_\ell^{(1)}$ given by the path algebra of the quiver

$$1 \to 2 \to \cdots \to \ell$$

and proceeds inductively by setting

$$A_\ell^{(m+1)} \coloneqq \mathsf{End}_{A_\ell^{(m)}}(M)^{\mathsf{op}}$$

where M is a so-called m-cluster tilting $A_{\ell}^{(m)}$ -module. The module M is the direct sum of a skeleton of indecomposable objects in the m-cluster tilting subcategory

$$\mathcal{M}_{\ell}^{(m)} \subseteq \operatorname{mod} A_{\ell}^{(m)}$$

which is an m-abelian category in the sense of [Jas16].

The following results, proven in $\S 1$, are the starting point of this work:

(1) Let B be an abelian group. Then, for every $n \ge m \ge 1$, there is a canonical isomorphism

$$\operatorname{Hom}(K_0(\mathcal{M}_{n-m+1}^{(m)}), B) \cong \mathsf{K}(B, m)_n \tag{0.1}$$

where $K(B, m)_{\bullet}$ denotes an Eilenberg-MacLane space given in the form of its standard model as a simplicial abelian group. The relevance of the simplicial object $K(B, m)_{\bullet}$ in topology is, of course, its role as a classifying space for cohomology: For every CW complex Y, there is a bijection

$$[Y, |\mathsf{K}(B, m)_{\bullet}|] \cong H^m(Y, B).$$

between the set of homotopy classes of maps from Y to $|\mathsf{K}(B,m)_{\bullet}|$ and the mth cohomology group of Y with coefficients in B.

(2) Let \mathcal{B} be an abelian category. Then, for every $n \geq m \geq 1$, there is a canonical equivalence of categories

$$\operatorname{Fun}^{\operatorname{ex}}(\mathcal{M}_{n-m+1}^{(m)},\mathcal{B}) \simeq \mathsf{S}_n^{\langle m \rangle}(\mathcal{B}) \tag{0.2}$$

where $\mathsf{S}_{\bullet}^{\langle m \rangle}(\mathcal{B})$ denotes the *m*-dimensional Waldhausen S_{\bullet} -construction of \mathcal{B} as studied for m=1 in [Wal85], for m=2 in [HM15], and for all $m\geq 0$ in [Pog17]. The relevance of these simplicial objects in algebraic K-theory is their role in describing the deloopings of the K-theory spectrum of \mathcal{B} : We have homotopy equivalences

$$\Omega^m | \mathsf{S}^{\langle m \rangle}(\mathfrak{B})^{\cong} | \simeq K(\mathfrak{B})$$

exhibiting the K-theory space $K(\mathfrak{B})$ of \mathfrak{B} as an m-fold loop space, and, varying $m \geq 1$, as an Ω -spectrum (see [Gre07] for a survey on spectra).

The identifications (0.1) and (0.2) suggest that it is possible and natural to organise the various categories $\mathcal{M}_{\bullet-m+1}$ into a simplicial object. The goal of this article is to introduce a suitable combinatorial framework to achieve this and to study the resulting interplay between representation theoretic concepts from higher Auslander–Reiten theory and combinatorial concepts from the theory of simplicial structures.

We provide an overview of the contents of this work:

1 The higher Auslander algebras of type A

In this section, we develop a combinatorial approach to the higher Auslander algebras of type \mathbb{A} in terms of simplicial combinatorics: We provide a description of these algebras in terms of the posets $\Delta(m,n)$ of monotone functions between the standard ordinals [m] and [n]. This approach allows us to give a precise relationship between these algebras and fundamental objects in algebraic topology and algebraic K-theory: Eilenberg–MacLane spaces and higher versions of the Waldhausen S_{\bullet} -construction, respectively.

2 Higher reflection functors in type A

In this section, we develop a higher categorical approach to higher reflection functors based on the combinatorial description of the higher Auslander algebras of type \mathbb{A} from Proposition 1.10. This generalises the more traditional approach via tilting modules. A profitable side effect of our theory is that it allows us to study representations with values in arbitrary stable ∞ -categories, not necessarily linear over a field, much in the spirit of the abstract representation theory of Groth and Šťovíček. Indeed, these results can be seen as higher-dimensional generalisations of results in [GS16a] and [GS16b].

Our perspective on higher reflection functors brings a certain geometric appeal: Iyama's homological condition of "regularity in dimension m+1" is reflected geometrically by the requirement that representations of the poset $\Delta(m,n)$ map all rectilinear (m+1)-dimensional cubes to bi-Cartesian cubes. The reflection functors are then obtained by puncturing these cubes at their initial and final vertices, respectively.

We provide another instance of this phenomenon: We show that the derived category of coherent sheaves on projective m-space admits a combinatorial description in terms of representations of the Beilinson category $B_{\mathbb{Z}}^{(m)}$, with relations formulated again in terms of (m+1)-dimensional biCartesian cubes.

These results seem to be precursors of an axiomatic framework for investigating m-cluster tilting subcategories based on higher-dimensional biCartesian cubes in ∞ -categories.

In the related framework of stable derivators, further examples of tilting equivalences described in terms of posets of paracyclic simplices were recently obtained by Beckert in his doctoral thesis [Bec]. The abstract representation theory of cubical diagrams with various exactness conditions has been thoroughly investigated by Beckert and Groth in [BG18] also in the related framework of stable derivators.

3 Mutation, horn fillings, and higher Segal conditions

In this section, we explain how to interpret higher reflection functors of type \mathbb{A} in terms of horn filling conditions on simplicial objects. As a by-product of this observation we relate the notions of

- (i) slice mutation,
- (ii) outer horn filling conditions, and
- (iii) higher Segal conditions.

The main ingredient in the above comparison is an approach to objects of membranes in the sense of [DK12] by means of a Cartesian fibration

$$\bigcup : \operatorname{Cov}_{\Delta} \longrightarrow \Delta$$

whose fibres parameterise the simplicial subsets of the standard simplices.

4 Classification via the Dold-Kan correspondence

In this section, we provide a simple characterisation of so-called outer m-Kan complexes with values in either an abelian category or a stable ∞ -category. Our characterisation is given in terms of appropriate versions of the Dold–Kan correspondence. As a consequence, we show that the classes of outer m-Kan complexes and 2m-Segal objects coincide in this context.

A n-cubes in stable ∞ -categories

In this appendix we collect elementary results concerning n-cubes in stable ∞ -categories which are needed at various stages throughout the article.

Conventions

Throughout the article we use freely the language of ∞ -categories as developed in [Lur09, Lur17]. When there is no risk of confusion we identify an ordinary category with its nerve. In particular, if D is a small category and \mathbb{C} is an ∞ -category, we denote the ∞ -category of functors $\mathsf{N}(D) \to \mathbb{C}$ by $\mathsf{Fun}(D,\mathbb{C})$.

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1 The higher Auslander algebras of type \mathbb{A}

1.1 Higher Auslander–Reiten theory

Let $\mathbb K$ be a field. An Auslander algebra is a finite-dimensional $\mathbb K$ -algebra Γ that satisfies the homological constraints

$$\operatorname{gl.dim}(\Gamma) \leq 2 \leq \operatorname{dom.dim}(\Gamma), \tag{1.1}$$

where gl. $\dim(\Gamma)$ denotes the global dimension of Γ and dom. $\dim(\Gamma)$ denotes its dominant dimension: the largest natural number d such that the terms $I^0, I^1, \ldots, I^{d-1}$ in every minimal injective coresolution of Γ are projective.

Let A be a finite-dimensional \mathbb{K} -algebra of finite representation type and denote by M the direct sum of a complete set of representatives of the indecomposable A-modules. The endomorphism algebra $\Gamma = \operatorname{End}_A(M)^{\operatorname{op}}$ is an Auslander algebra and, up to Morita equivalence, every Auslander algebra is of this form [Aus71]. The functorial approach to Auslander–Reiten theory consists of relating the representation theory of A to that of Γ by means of the fully faithful left exact functor

$$G \colon \operatorname{mod} A \longrightarrow \operatorname{mod} \Gamma, \quad X \mapsto \operatorname{Hom}_A(M, X).$$
 (1.2)

Notably, under the functor G, the almost split exact sequences of A-modules correspond precisely to the minimal projective resolutions of simple Γ -modules of projective dimension 2. This is the seminal observation in Auslander–Reiten theory, see [ARS97] for further details.

More recently, Iyama proposed a higher-dimensional generalisation of Auslander–Reiten theory based on the notion of an (m + 1)-dimensional Auslander algebra, that is a finite-dimensional K-algebra Γ satisfying the homological constraint

$$\mathsf{gl.\,dim}(\Gamma) \le m+1 \le \mathsf{dom.\,dim}(\Gamma). \tag{1.3}$$

For m > 1, given a finite-dimensional K-algebra A, the existence of a corresponding (m + 1)-dimensional Auslander algebra hinges on the existence of an (m + 1)-cluster tilting module—an A-module M satisfying the following conditions:

- For every 0 < i < m we have $\operatorname{Ext}_A^i(M, M) = 0$.
- Every indecomposable A-module X such that, for every 0 < i < m, we have $\mathsf{Ext}_A^i(X, M) = 0$, is isomorphic to a direct summand of M. In particular each indecomposable projective A-module is a direct summand of M.
- Every indecomposable A-module Y such that, for every 0 < i < m, we have $\operatorname{Ext}_A^i(M,Y) = 0$, is isomorphic to a direct summand of M. In particular each indecomposable injective A-module is a direct summand of M.

The algebra $\Gamma = \operatorname{End}_A(M)^{\operatorname{op}}$, where M is an m-cluster tilting A-module, is an (m+1)-dimensional Auslander algebra and a central result of Iyama says that, up to Morita equivalence, every such algebra is of this form [Iya07a]. In analogy with the classical situation, a higher-dimensional variant of Auslander–Reiten theory, based on the obvious analogue of the functor (1.2), can be formulated in this context [Iya07b].

It is a nontrivial task to construct finite-dimensional K-algebras where higher Auslander–Reiten theory can be applied. In [Iya11] Iyama provides a remarkable recursive construction of a family $\{A^{(m)}\}_{m\geq 1}$ of (relative) higher Auslander algebras, where $A^{(1)}$ is an arbitrary finite-dimensional representation-finite K-algebra of global dimension 1. The recursion is defined by the formula

$$A^{(m+1)} := \operatorname{End}_{A^{(m)}}(M)^{\operatorname{op}}$$

where M is a canonical (relative) m-cluster tilting $A^{(m)}$ -module, the existence of which is guaranteed by one of the main results in [Iya11].

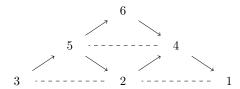
Of particular interest for this work is the case when $A^{(1)} = A_{\ell}^{(1)}$ is the K-linear path algebra of the quiver $1 \to 2 \to \cdots \to \ell$ of Dynkin type \mathbb{A}_{ℓ} . In this case the various higher Auslander algebras are related by fully faithful left exact functors

$$G \colon \operatorname{mod} A_{\ell}^{(m)} \longrightarrow \operatorname{mod} A_{\ell}^{(m+1)}, \quad X \mapsto \operatorname{Hom}_{A_{\ell}^{(m)}}(M,X). \tag{1.4}$$

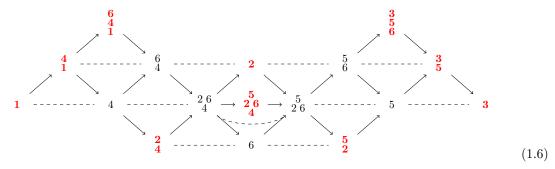
We refer to this family of algebras as the higher Auslander algebras of type \mathbb{A} .

We illustrate the first steps of Iyama's construction for the quiver $1 \to 2 \to 3$. The Auslander–Reiten quiver of the module category mod $A_3^{(1)}$ is given by

where, following standard procedure, we label the various indecomposable modules by the simple modules appearing in their composition series. The Auslander algebra $A_3^{(2)}$ is the quotient of the path algebra associated to this quiver modulo the relations given by the bottom dashed lines, which signify zero relations, and the middle dashed line, which expresses the commutativity of the square. As a matter of convenience, we relabel the vertices of this quiver as



We use this new labelling to describe the Auslander-Reiten quiver of $\operatorname{\mathsf{mod}} A_3^{(2)}$ which is given by



In order to define the 3-dimensional Auslander algebra $A_3^{(3)}$ we need to find a 2-cluster tilting $A_3^{(2)}$ -module M and set $A_3^{(3)} = \operatorname{End}_{A_3^{(2)}}(M)$. Iyama shows that, up to multiplicity of its direct summands, there is a unique such $A_3^{(2)}$ -module M and that it is given by the direct sum of the 10 indecomposable $A_3^{(2)}$ -modules highlighted in the Auslander–Reiten quiver of $\operatorname{mod} A_3^{(2)}$. The procedure continues $\operatorname{ad\ infinitum\ by\ finding\ a\ 3}$ -cluster tilting $A_3^{(3)}$ -module...

1.2 The simplicial combinatorics of higher Auslander algebras of type A

The starting point for the perspective we develop in this work is the observation that the structure of the higher Auslander algebras of type \mathbb{A} is governed by the combinatorics of simplices in a way we now explain.

Given a poset P, we may construct a category whose objects are the elements of P with a unique morphism from i to j whenever $i \leq j$. Below, we leave the distinction between a poset and its associated category implicit.

Definition 1.7. Let n be a natural number. Recall that the n-th standard ordinal is the poset $[n] := \{0 < 1 < \cdots < n\}$. Given two natural numbers m and n, we denote the set of monotone maps $f : [m] \to [n]$ by $\Delta(m,n)$. We endow $\Delta(m,n)$ with the structure of a poset by declaring $f \le f'$ if, for all $i \in [m]$, we have $f(i) \le f'(i)$. We further introduce the partition

$$\Delta(m,n) = \Delta(m,n)^{\sharp} \cup \Delta(m,n)^{\flat} \tag{1.8}$$

into the subset $\Delta(m,n)^{\sharp}$ of strictly monotone maps and its complement $\Delta(m,n)^{\flat}$.

Remark 1.9. The elements of the poset $\Delta(m,n)$ can be interpreted as m-simplices in the n-simplex Δ^n . The partition (1.8) then corresponds to the partition into non-degenerate and degenerate m-simplices. Further note that $\Delta(m,n)$ is a subposet of \mathbb{Z}^{m+1} with respect to the product order.

For natural numbers m and n, let $\mathbb{K}\Delta(m,n)$ denote the \mathbb{K} -category obtained from the category associated to the poset $\Delta(m,n)$ by passing to \mathbb{K} -linear envelopes of the morphism sets. We further form the \mathbb{K} -category $\underline{\mathbb{K}}\Delta(m,n)$ by replacing all morphism spaces in $\mathbb{K}\Delta(m,n)$ by the quotient modulo the subspace of morphisms that factor through an object in $\Delta(m,n)^{\flat}$. Finally, for each \mathbb{K} -category \mathcal{A} with finitely many objects, we introduce the \mathbb{K} -algebra

$$\operatorname{End}(\mathcal{A}) = \bigoplus_{(x,y) \in (\operatorname{ob}\mathcal{A})^2} \mathcal{A}(x,y)$$

with multiplication obtained from the composition law of A.

Let us illustrate how the simplicial combinatorics relate to the higher Auslander algebras of type \mathbb{A} in the special case $\ell=3$. The \mathbb{K} -category $\mathbb{K}\Delta(0,2)$ is the free \mathbb{K} -category on the quiver $0 \to 1 \to 2$. In other words there are algebra isomorphisms

$$A_3^{(1)} \cong \operatorname{End}(\mathbb{K}\Delta(0,2)) \cong \operatorname{End}(\underline{\mathbb{K}\Delta}(0,2)),$$

where the rightmost isomorphism is a consequence of the fact that there are no degenerate elements in $\Delta(0,2)$.

We now wish to describe the Auslander algebra $A_3^{(2)}$ in a similar way. For this, consider the poset $\Delta(1,3)$ whose Hasse quiver is given by

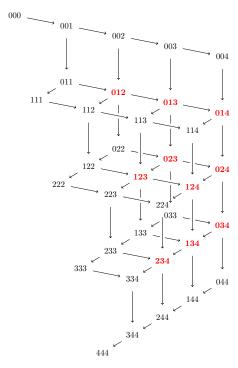
The elements of the set $\Delta(1,3)^{\sharp}$ of non-degenerate simplices are highlighted. Observe that the Hasse quiver of $\Delta(1,3)^{\sharp}$ agrees with the Auslander–Reiten quiver of mod $A_3^{(1)}$, see (1.5). Note, however, that neither of the algebras

$$\operatorname{End}(\mathbb{K}\Delta(1,3))$$
 and $\operatorname{End}(\mathbb{K}\Delta(1,3)^{\sharp})$

is isomorphic to $A_3^{(2)}$: the algebra $\operatorname{End}(\mathbb{K}\Delta(1,3))$ has too many simple modules while $\operatorname{End}(\mathbb{K}\Delta(1,3)^\sharp)$ is missing the zero relations. As it turns out, the appropriate way to account for the presence of non-degenerate simplices is to pass to the \mathbb{K} -category $\underline{\mathbb{K}\Delta}(1,3)$. Indeed, there is an isomorphism of algebras

$$A_3^{(2)} \cong \operatorname{End}(\underline{\mathbb{K}\Delta}(1,3)).$$

As a final example, let us describe the algebra $A_3^{(3)}$ in terms of posets of simplices. Consider the poset $\Delta(2,4)$ whose Hasse quiver is given by



Again, the elements of the set $\Delta(2,4)^{\sharp}$ of non-degenerate simplices are highlighted. Note that the Hasse quiver of $\Delta(2,4)^{\sharp}$ agrees with the full subquiver of the Auslander–Reiten quiver of $\operatorname{mod} A_3^{(2)}$ spanned by the summands of the unique 2-cluster tilting $A_3^{(2)}$ -module, see (1.6). In this case there is an isomorphism of algebras

$$A_3^{(3)} \cong \operatorname{End}(\underline{\mathbb{K}\Delta}(2,4)).$$

More generally, we have the following description of the higher Auslander algebras of type \mathbb{A} in terms of posets of simplices.

Proposition 1.10. Let $n \ge m \ge 1$ and set $\ell = n - m + 1$. There is an isomorphism of K-algebras

$$A_{\ell}^{(m+1)} \cong \operatorname{End}(\underline{\mathbb{K}\Delta}(m,n))$$

where $A_{\ell}^{(m+1)}$ denotes the (m+1)-dimensional Auslander algebra of type \mathbb{A}_{ℓ} .

Proof. Up to a minor modification of the object labels, a proof of this fact is explained in [JK16] based on the description given in [OT12].

Let ℓ be positive integer. Recall that $A_{\ell}^{(1)}$ denotes the \mathbb{K} -linear path algebra of the quiver $1 \to 2 \to \cdots \to \ell$. As explained in §1.1, the higher Auslander algebras of type \mathbb{A}_{ℓ} were originally constructed by Iyama by means of the recursion

$$A_{\ell}^{(m+1)} = \mathsf{End}_{A_{\ell}^{(m)}}(M) \tag{1.11}$$

where M is, up to multiplicity of its direct summands, the unique m-cluster-tilting $A_{\ell}^{(m)}$ -module. We now explain how this distinguished $A_{\ell}^{(m)}$ -module can be described in terms of the simplicial combinatorics.

Notation 1.12. For linearly ordered sets

$$I = \{i_0 < i_1 < \dots < i_m\}$$
 and $J = \{j_0 < j_1 < \dots < j_n\}$

let

$$I * J := \{i_0 < i_1 < \dots < i_m < j_0 < j_1 < \dots < j_n\}.$$

In particular for each $n \ge 0$ there is are (unique) isomorphisms of posets $[0] * [n] \cong [n+1]$ and $[n] * [0] \cong [n+1]$. These identifications allow us to consider the faithful endofunctors

$$([0]*-):\Delta\longrightarrow\Delta$$
 and $(-*[0]):\Delta\longrightarrow\Delta$.

Concretely, for a simplex $\sigma \colon [m] \to [n]$ we have

$$([0] * \sigma)_i = \begin{cases} 0 & i = 0\\ \sigma_{i-1} + 1 & 1 \le i \le m+1; \end{cases}$$

and

$$(\sigma * [0])_i = \begin{cases} \sigma_i & 0 \le i \le m \\ n+1 & i=m+1. \end{cases}$$

Remark 1.13. The endofunctor -*[0] is the basic ingredient in the definition of the *décalage* of a simplicial set.

The algebra isomorphism from Proposition 1.10 implies the existence of an equivalence of categories

$$\operatorname{mod} A_{\ell}^{(m)} \simeq \operatorname{Fun}_{\mathbb{K}}(\underline{\mathbb{K}\Delta}(m-1, n-1)^{\operatorname{op}}, \operatorname{mod} \mathbb{K}) \tag{1.14}$$

between the category of finitely generated right $A_\ell^{(m)}$ -modules and the category of contravariant \mathbb{K} -linear functors $\underline{\mathbb{K}}\underline{\Delta}(m-1,n-1)\to \mathsf{mod}\,\mathbb{K}$. The poset monomorphism

$$([0]*-): \Delta(m-1,n-1) \hookrightarrow \Delta(m,n)$$

preserves the partition of $\Delta(m-1, n-1)$ into non-degenerate and degenerate simplices whence it induces a fully faithful K-linear functor

$$\gamma : \underline{\mathbb{K}\Delta}(m-1, n-1)^{\mathsf{op}} \longrightarrow \underline{\mathbb{K}\Delta}(m, n)^{\mathsf{op}}.$$
 (1.15)

The pullback functor γ^* admits a fully faithful right adjoint

$$\gamma_* \colon \operatorname{Fun}_{\mathbb{K}}(\mathbb{K}\Delta(m-1,n-1)^{\operatorname{op}},\operatorname{mod}\mathbb{K}) \to \operatorname{Fun}_{\mathbb{K}}(\mathbb{K}\Delta(m,n)^{\operatorname{op}},\operatorname{mod}\mathbb{K})$$

given by $\operatorname{\mathsf{mod}} \mathbb{K}$ -enriched right Kan extension along γ . Under the equivalence (1.14), the functor γ_* is identified with the fully faithful left exact functor

$$G \colon \operatorname{mod} A_{\ell}^{(m)} \longrightarrow \operatorname{mod} A_{\ell}^{(m+1)}$$

of (1.4). Finally, consider the Yoneda functor

$$\iota \colon \Delta(m,n) \longrightarrow \operatorname{Fun}_{\mathbb{K}}(\underline{\mathbb{K}}\underline{\Delta}(m,n)^{\operatorname{op}},\operatorname{mod}\mathbb{K}), \quad \sigma \mapsto \operatorname{Hom}_{\mathbb{K}\underline{\Delta}(m,n)}(-,\sigma)$$

and form the composite functor

$$\varepsilon = \gamma^* \circ \iota : \Delta(m, n) \longrightarrow \operatorname{mod} A_{\ell}^{(m)}, \tag{1.16}$$

where the use of equivalence (1.14) is left implicit.

Proposition 1.17. Let $n \geq m \geq 1$ and set $\ell = n - m + 1$. Up to multiplicity of its direct summands, the unique m-cluster-tilting $A_{\ell}^{(m)}$ -module is given by

$$M = \bigoplus_{\sigma \in \Delta(m,n)} \varepsilon(\sigma) = \bigoplus_{\sigma \in \Delta(m,n)} \mathsf{Hom}_{\underline{\mathbb{K}\Delta}(m,n)}(\gamma(-),\sigma).$$

Proof. Indeed, the functor

$$G\colon \operatorname{mod} A_\ell^{(m)} \longrightarrow \operatorname{mod} A_\ell^{(m+1)}, \quad X \mapsto \operatorname{Hom}_{A_\ell^{(m)}}(M,X)$$

has a left adjoint

$$F\colon \operatorname{mod} A_{\ell}^{(m+1)} \longrightarrow \operatorname{mod} A_{\ell}^{(m)}, \quad Y \mapsto Y \otimes_{A_{\ell}^{(m+1)}} M,$$

where we use the recursion (1.11) to view M as a (left) module over $A_{\ell}^{(m+1)}$. Clearly, the functor F identifies the regular representation $A_{\ell}^{(m+1)}$ with the $A_{\ell}^{(m)}$ -module M. Finally, under the equivalence (1.14) the adjoint pair $F \dashv G$ is identified with the adjoint pair $\gamma^* \dashv \gamma_*$. The claim follows.

As we explain below, the functor ε reveals interesting simplicial structures hidden within the representation theory of the higher Auslander algebras of type \mathbb{A} .

1.3 Relation to Eilenberg-MacLane spaces

Let B be an abelian group and m a positive integer. An Eilenberg-MacLane space of type K(B, m) is a (path connected) topological space characterised up to weak homotopy equivalence by the property

$$\pi_k(\mathsf{K}(B,m)) \simeq \begin{cases} B & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases}$$
(1.18)

One important aspect of Eilenberg–MacLane spaces is that they are classifying spaces for singular cohomology with coefficients in B: for every CW-complex X, there is a canonical bijection between $H^n(X, B)$ and the set of homotopy classes of maps $X \to \mathsf{K}(B, m)$, see Corollary III.2.7 in [GJ99].

An explicit combinatorial model for an Eilenberg-MacLane space of type K(B, m) can be obtained via the Dold–Kan correspondence [Dol58, Kan58]. Under this correspondence, the aforementioned model is the simplicial abelian group that corresponds to the chain complex B[m] consisting only of the abelian group B placed in homological degree m. With some abuse of notation, we denote this simplicial abelian group by K(B, m). The characterising property (1.18) is an immediate consequence of the fact that, under the Dold–Kan correspondence, the cohomology groups of a given chain complex are identified with the homotopy groups of the corresponding simplicial abelian group, see Corollary III.2.5 in [GJ99]. We provide an explicit description of the abelian group of n-simplices of K(B, m).

Definition 1.19. Let $m, n \geq 0$. Recall that $\Delta(m, n)$ denotes the set of m-simplices in Δ^n . Then

$$\mathsf{K}(B,m)_n \subset \mathsf{Map}(\Delta(m,n),B)$$

is the subgroup of those (set-theoretic) functions $f : \Delta(m,n) \to B$ which have the following properties:

- (1) The function f maps all degenerate m-simplices to 0.
- (2) For every non-degenerate (m+1)-simplex $\sigma: [m+1] \to [n]$ the equation

$$\sum_{i=0}^{m+1} f(\sigma \circ d^i) = 0$$

is satisfied, where $d^i : [m] \to [m+1]$ is the unique strictly monotone map whose image does not contain i.

The simplicial structure of K(B, m) is obtained via the functoriality of $\Delta(m, n)$ in $[n] \in \Delta$.

Theorem 1.20. Let $n \ge m \ge 1$ and set $\ell = n - m + 1$. We denote the Grothendieck group of the abelian category mod $A_{\ell}^{(m)}$ by $K_0(A_{\ell}^{(m)})$. The following statements hold:

(1) The map

$$e: \Delta(m,n) \longrightarrow K_0(A_{\ell}^{(m)})$$

induced by the functor $\varepsilon \colon \Delta(m,n) \to \operatorname{mod} A_{\ell}^{(m)}$ described in (1.16) defines an n-simplex in $\mathsf{K}(K_0(A_{\ell}^{(m)}),m)$.

(2) The n-simplex e is universal in the following sense: For every abelian group B the map

$$\operatorname{\mathsf{Hom}}_{\mathsf{Ab}}(K_0(A_\ell^{(m)}),B) \longrightarrow \mathsf{K}(B,m)_n$$

induced by pullback along ε is an isomorphism.

(3) There is a co-simplicial abelian group

$$\mathsf{K}_0(A^{(m)}_{\bullet-m+1}) \colon \Delta \longrightarrow \mathsf{Ab}$$

where, by convention, $K_0(A_{n-m+1}^{(m)}) = 0$ for n < m. Moreover, the n-simplices e from statement (1) exhibit $K_0(A_{\bullet - m + 1}^{(m)})$ as a quotient

$$e_{\bullet} \colon \mathbb{Z}\Delta(m,-) \twoheadrightarrow K_0(A_{\bullet-m+1}^{(m)})$$

of the free co-simplicial abelian group $\mathbb{Z}\Delta(m,-)$.

In particular, for every abelian group B the map

$$e_{ullet}^* \colon \operatorname{Hom}_{\operatorname{Ab}}(K_0(A_{ullet-m+1}^{(m)}),B) \longrightarrow \operatorname{K}(B,m)$$

 $is\ an\ isomorphism\ of\ simplicial\ abelian\ groups.$

Proof. In view of the explicit model for K(B, m) given in Definition 1.19, the statements are equivalent to saying that the map of abelian groups

$$\mathbb{Z}\Delta(m,n) \longrightarrow K_0(A_\ell^{(m)})$$
 (1.21)

induced by e is surjective with kernel R(m, n) generated by:

• the elements of $\Delta(m,n)$ corresponding to degenerate m-simplices and

• the elements of $\Delta(m,n)$ of the form

$$\sum_{i=0}^{m+1} \tau \circ d^i$$

where $\tau \colon [m+1] \to [n]$ is a non-degenerate (m+1)-simplex.

It is straightforward to verify that R(m, -) is a co-simplicial subgroup of $\mathbb{Z}\Delta(m, -)$. The map (1.21) is surjective since all indecomposable projectives in

$$\operatorname{mod} A_{\ell}^{(m)} \simeq \operatorname{Fun}_{\mathbb{K}}(\underline{\mathbb{K}\Delta}(m-1,n-1)^{\operatorname{op}},\operatorname{mod} \mathbb{K})$$

are in the image of ε and, since the algebra $A_{\ell}^{(m)}$ has finite global dimension, the indecomposable projectives freely generate $K_0(A_{\ell}^{(m)})$. Indeed, an indecomposable projective module represented by an (m-1)-simplex σ is the image of $\gamma(\sigma)$:

$$\operatorname{Hom}_{\mathbb{K}\Delta(m-1,n-1)}(-,\sigma) \cong \operatorname{Hom}_{\mathbb{K}\Delta(m,n)}(\gamma(-),\gamma(\sigma)) = \varepsilon(\gamma(\sigma)).$$

It is obvious that all degenerate m-simplices map to 0. We now show that the kernel of the map (1.21) is given by R(m,n). Each non-degenerate simplex $\tau \colon [m+1] \to [n]$ induces a sequence

$$\tau \circ d^{m+1} \longrightarrow \tau \circ d^m \longrightarrow \ldots \longrightarrow \tau \circ d^1 \longrightarrow \tau \circ d^0$$

in the poset $\Delta(m, n)$. It is straightforward to verify that this sequence is mapped, under the functor ε of (1.16), to an exact sequence

$$0 \longrightarrow \varepsilon(\tau \circ d^{m+1}) \longrightarrow \varepsilon(\tau \circ d^m) \longrightarrow \ldots \longrightarrow \varepsilon(\tau \circ d^1) \longrightarrow \varepsilon(\tau \circ d^0) \longrightarrow 0$$

in $\operatorname{\mathsf{mod}} A_\ell^{(m)}$. Indeed, this exact sequence is described in Proposition 3.19 in [OT12], albeit in a different combinatorial framework. This implies the claimed relations.

Remark 1.22. The short exact sequence

$$0 \longrightarrow R(m,n) \longrightarrow \mathbb{Z}\Delta(m,n) \longrightarrow K_0(A_{\ell}^{(m)}) \longrightarrow 0$$

constructed in the proof of Theorem 1.20 corresponds to the presentation of the Grothendieck group associated to the m-cluster tilting $A_{\ell}^{(m)}$ -module M, which can be defined using the ideas of [BT14].

1.4 Relation to the higher Waldhausen S_•-constructions

Segal's Γ -space delooping framework [Seg74] provides a beautiful means to understand the algebraic K-theory spectrum of a ring R as the group completion of the direct sum operation on the classifying space of finitely generated projective R-modules. In [Wal85] Waldhausen introduced the S $_{\bullet}$ -construction as a variant of Segal's construction that also accounts for the presence of possibly non-split short exact sequences. This construction works in the much larger generality of what are now called Waldhausen categories, where Waldhausen himself used it to define his algebraic K-theory of spaces.

We recall a higher-dimensional generalisation of the S_{\bullet} -construction introduced by Hesselholt and Madsen in dimension 2 [HM15] and by Poguntke in arbitrary positive dimensions [Pog17].

Definition 1.23. For $m, n \geq 0$, let $\Delta(m, n)$ be the poset of m-simplices in Δ^n . For an abelian category \mathcal{B} , we define

$$\mathsf{S}_n^{\langle m \rangle}(\mathcal{B}) \subset \mathsf{Fun}(\Delta(m,n),\mathcal{B})$$

to be the full subcategory of functors $X: \Delta(m,n) \to \mathcal{B}$ satisfying the following conditions:

(1) The functor X maps degenerate simplices to zero objects in \mathcal{B} .

(2) For every non-degenerate (m+1)-simplex σ , the sequence

$$0 \longrightarrow X(\sigma \circ d^{m+1}) \longrightarrow X(\sigma \circ d^m) \longrightarrow \ldots \longrightarrow X(\sigma \circ d^0) \longrightarrow 0$$

is exact.

Via the functoriality of $\Delta(m,n)$ in $[n] \in \Delta$ we obtain a simplicial category $S_{\bullet}^{\langle m \rangle}(\mathcal{B})$ that we call the m-dimensional Waldhausen S_{\bullet} -construction of \mathcal{B} .

Remark 1.24. Note that Definition 1.23 is a categorical version of Definition 1.19. In fact, there is a direct relation between the two: Treating $S_{\bullet}^{\langle m \rangle}(\mathcal{B})$ as a simplicial exact category, it follows from additivity that we have an isomorphism of simplicial abelian groups

$$K_0(\mathsf{S}^{\langle m \rangle}_{\bullet}(\mathcal{B})) \cong K(K_0(\mathcal{B}), m).$$

Remark 1.25. The simplicial category $S_{\bullet}^{\langle 1 \rangle}(\mathcal{B})$ is Waldhausen's original S_{\bullet} -construction. The construction $S_{\bullet}^{\langle 2 \rangle}(\mathcal{B})$ was introduced by Hesselholt and Madsen in [HM15] as a model for real algebraic K-theory. Finally, the higher versions $S_{\bullet}^{\langle m \rangle}(\mathcal{B})$ were introduced and studied by Poguntke in [Pog17] from the point of view of higher Segal spaces.

Remark 1.26. The relevance of these constructions for algebraic K-theory is the following: Given a category \mathcal{C} , we denote by \mathcal{C}^{\simeq} the groupoid obtained by discarding all noninvertible morphisms in \mathcal{C} . Then, for an abelian category \mathcal{B} , the higher algebraic K-groups of \mathcal{B} are given by the formula

$$K_i(\mathfrak{B}) = \pi_{i+m}(|S_{\bullet}^{\langle m \rangle}(\mathfrak{B})^{\simeq}|)$$

where |-| denotes geometric realization. Further, the sequence of spaces

$$|S_{\bullet}^{\langle 1 \rangle}(\mathcal{B})^{\simeq}|, |S_{\bullet}^{\langle 2 \rangle}(\mathcal{B})^{\simeq}|, |S_{\bullet}^{\langle 3 \rangle}(\mathcal{B})^{\simeq}|, \dots$$

forms an Ω -spectrum which is a model for the connective algebraic K-theory spectrum of \mathcal{B} . This can be seen, for example, by using straightforward higher-dimensional generalisations of the additivity results of Hesselholt and Madsen, or a totalization construction introduced by Poguntke.

We now come to the relation between the higher Waldhausen S_{\bullet} -constructions and the higher Auslander algebras of type \mathbb{A} . This is captured in the following result which can be considered a categorified version of Theorem 1.20, see also Remark 1.22.

Theorem 1.27. Let $n \ge m \ge 1$ and set $\ell = n - m + 1$. Define the subcategory

$$\mathfrak{M}_{\ell}^{(m)} \coloneqq \left\{ \varepsilon(\sigma) \in \operatorname{mod} A_{\ell}^{(m)} \ \middle| \ \sigma \in \Delta(m,n) \right\}.$$

The following statements hold:

(1) The functor

$$\varepsilon \colon \Delta(m,n) \longrightarrow \mathfrak{M}_{\ell}^{(m)} \hookrightarrow \operatorname{mod} A_{\ell}^{(m)}$$

from (1.16) defines an n-simplex in $S_{\bullet}^{\langle m \rangle}(\operatorname{mod} A_{\ell}^{(m)})$.

(2) The n-simplex ε is universal in the following sense: For every K-linear abelian category B, the functor

$$\operatorname{Fun}_{\mathbb{K}}^{\operatorname{ex}}(\mathbb{M}_{\ell}^{(m)},\mathbb{B}) \longrightarrow \mathsf{S}_{n}^{\langle m \rangle}(\mathbb{B})$$

induced by pullback along ε is an equivalence, where the leftmost category is the category of exact \mathbb{K} -linear functors $\mathfrak{M}_{\ell}^{(m)} \to \mathfrak{B}$, that is functors which send sequences

$$0 \longrightarrow M_{m+1} \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0 \tag{1.28}$$

in $\mathcal{M}_{\ell}^{(m)}$ which are exact in mod $A_{\ell}^{(m)}$ to exact sequences in \mathcal{B} .

Proof. The first statement is established as part of the proof of Theorem 1.20. The second claim follows immediately from Lemma 1.30 below and Remark 5.4 in [Pog17] which shows that the exactness conditions imposed on the *n*-simplices $F \colon \Delta(m,n) \to \mathcal{B}$ in $\mathsf{S}_n^{\langle m \rangle}(\mathcal{B})$ are equivalent to the induced functor $\mathcal{M}_\ell^{(m)} \cong \underline{\mathbb{K}\Delta}(m,n) \to \mathcal{B}$ being exact.

Remark 1.29. After passing to the additive envelope, the resulting subcategory $\mathfrak{M}_{\ell}^{(m)}$ of Theorem 1.27 is an m-abelian category in the sense of [Jas16].

For the convenience of the reader, we translate Proposition 3.19 in [OT12] to our combinatorial framework.

Lemma 1.30. In the setting of Theorem 1.27, the exact sequences in $\mathfrak{M}_{\ell}^{(m)}$ of the form (1.28) are precisely the sequences of the form

$$0 \longrightarrow \varepsilon(q_{(0,\dots,0)}) \longrightarrow \bigoplus_{|v|=1} \varepsilon(q_v) \longrightarrow \cdots \longrightarrow \bigoplus_{|v|=m} \varepsilon(q_v) \longrightarrow \varepsilon(q_{(1,\dots,1)}) \longrightarrow 0$$

where $q: \{0,1\}^{m+1} \to \Delta(m,n)$ is a rectilinear (m+1)-cube in the sense of Definition 2.8, and $|v| = \sum v_i$.

2 Higher reflection functors in type \mathbb{A}

2.1 The derived representation theory of higher Auslander algebras of type \mathbb{A}

Let \mathbb{K} be a field. We begin this section with a description of the derived representation theory of the higher Auslander algebras of type \mathbb{A} in terms of the simplicial combinatorics. Given a differential graded (=dg) \mathbb{K} -algebra A or, more generally, a small dg \mathbb{K} -category, we denote its (unbounded) derived ∞ -category by $\mathcal{D}A$, see Section 1.3 in [Lur17] for details.

2.1.1 Derived representation theory of the higher Auslander algebras of type $\mathbb A$ via the simplicial combinatorics

Let \mathbb{K} be a field and let m and ℓ be positive integers. Consider the right derived functor

$$\mathsf{R}G \colon \mathcal{D}A_{\ell}^{(m)} \longrightarrow \mathcal{D}A_{\ell}^{(m+1)}, \quad X \mapsto \mathsf{RHom}_{A_{\ell}^{(m)}}(M,X) \tag{2.1}$$

associated to the left exact functor from (1.4). The m-cluster tilting $A_\ell^{(m)}$ -module M is a compact generator of the derived category $\mathcal{D}A_\ell^{(m)}$. Indeed, since $A_\ell^{(m)}$ has finite global dimension, M is certainly compact; the fact that M is a generator is also clear since each indecomposable projective $A_\ell^{(m)}$ -module is a direct summand of M. In particular, we could have considered the endomorphism dg algebra

$$\mathsf{R}A_\ell^{(m+1)} = \mathsf{RHom}_{A_\ell^{(m)}}(M,M)^{\mathsf{op}},$$

instead of its underived variant

$$A_{\ell}^{(m+1)} = H^0(\mathsf{R} A_{\ell}^{(m+1)}) = \mathsf{Hom}_{A_{\ell}^{(m)}}(M,M),$$

to obtain an exact equivalence

$$\mathcal{D}A_{\ell}^{(m)} \stackrel{\simeq}{\longrightarrow} \mathcal{D}\mathsf{R}A_{\ell}^{(m+1)}, \quad X \mapsto \mathsf{RHom}_{A_{\ell}^{(m)}}(M,X). \tag{2.2}$$

The functor $\mathsf{R} G$ is then obtained as the composite

$$\mathcal{D}A_{\ell}^{(m)} \xrightarrow{\simeq} \mathcal{D}\mathsf{R}A_{\ell}^{(m+1)} \xrightarrow{\iota^*} \mathcal{D}A_{\ell}^{(m+1)} \tag{2.3}$$

of the equivalence (2.2) and the restriction along the canonical morphism $\iota \colon A_\ell^{(m+1)} \to \mathsf{R} A_\ell^{(m+1)}$. The morphism ι^* should be regarded as a forgetful functor that forgets the action of the higher structure present in the dg algebra $\mathsf{R} A_\ell^{(m+1)}$.

Using the combinatorial description

$$A_{\ell}^{(m)} \cong \operatorname{End}(\mathbb{K}\Delta(m-1, n-1)) \tag{2.4}$$

of the higher Auslander algebra from Proposition 1.10, our first goal will be to describe the functor (2.1) as well as the factorization (2.3) in terms of representations of the posets of simplices $\Delta(m,n)$. To this end we work more generally with representations in an arbitrary stable ∞ -category \mathcal{C} , recovering the classical situation as the case $\mathcal{C} = \mathcal{DK}$.

Definition 2.5. Let $m, n \geq 0$, let $S \subset \Delta(m, n)$, and let \mathcal{C} be a stable ∞ -category. A functor $X \colon S \to \mathcal{C}$ is called *reduced* if it sends all objects in the subset $S \cap \Delta(m, n)^{\flat}$ of degenerate simplices to zero objects in \mathcal{C} . We further denote by

$$\operatorname{\mathsf{Fun}}_*(S,\mathfrak{C}) \subset \operatorname{\mathsf{Fun}}(S,\mathfrak{C})$$

the full subcategory spanned by the reduced functors.

Using this terminology, the following result provides a combinatorial description of the derived category $\mathcal{D}A_{\ell}^{(m)}$.

Proposition 2.6. Let $n \ge m \ge 1$ and set $\ell = n - m + 1$. There is a canonical equivalence

$$\mathcal{D}A_{\ell}^{(m)} \simeq \operatorname{Fun}_*(\Delta(m-1,n-1)^{\operatorname{op}},\mathcal{D}\mathbb{K})$$

of stable ∞ -categories.

Proof. The existence of the required equivalence is a consequence of Proposition 1.10 and the strictification result Proposition 4.2.4.4 from [Lur09] applied to the category of (unbounded) chain complexes of \mathbb{K} -modules endowed with the projective model structure.

Notation 2.7. Let m and n be natural numbers. Note that there is a poset isomorphism $(-)^*: \Delta(m,n) \to \Delta(m,n)^{op}$ given by sending an m-simplex σ to the simplex

$$\sigma_i^* = n - \sigma_{m-i}, \quad i \in [m].$$

To simplify the exposition, in what follows we work with functors on $\Delta(m, n)$, rather on $\Delta(m, n)^{op}$. In view of the above identification, this is immaterial as it amounts to a relabelling of the objects.

In what follows we use freely the terminology and results about cubical diagrams in stable ∞ -categories from Appendix A. Further, given vectors v and w in \mathbb{Z}^d , we define the *Hadamard product*

$$v \circ w = (v_1 w_1, \dots, v_d w_d)$$

given by their coordinate-wise product.

Definition 2.8. Let $d \ge 1$ be an integer. We say that a d-cube $q: I^d \to \mathbb{Z}^d$ is rectilinear if all of its edges are parallel to the standard coordinate vectors in \mathbb{Z}^d . This condition can be expressed in terms of the Hadamard product by saying that, for each $v \in I^d$, the equality

$$q_v = q_{0\cdots 0} + v \circ (q_{1\cdots 1} - q_{0\cdots 0})$$

is satisfied.

Definition 2.9. Let $m, n \geq 0$, let $S \subset \Delta(m, n)$, and let \mathcal{C} be a stable ∞ -category. A reduced functor $X: S \to \mathcal{C}$ is called *exact* if the restriction of X along every rectilinear (m+1)-cube in S is biCartesian. We further denote by

$$\operatorname{Fun}^{\operatorname{ex}}_*(S,\mathfrak{C}) \subset \operatorname{Fun}_*(S,\mathfrak{C})$$

the full subcategory spanned by the exact functors.

The following result should be regarded as a combinatorial version of (2.2), see also Corollary 2.13. It can also be regarded as a higher-dimensional version of Theorem 4.6 in [GS16a], which deals with the case m = 1.

Proposition 2.10. Let m and n be positive integers and

$$\gamma = ([0] * -) : \Delta(m-1, n-1) \hookrightarrow \Delta(m, n).$$

For a stable ∞ -category \mathfrak{C} , the restriction functor

$$\gamma^* \colon \operatorname{Fun}_*^{\operatorname{ex}}(\Delta(m,n),\mathcal{C}) \longrightarrow \operatorname{Fun}_*(\Delta(m-1,n-1),\mathcal{C})$$

is an equivalence of ∞ -categories.

Proof. We factor the morphism γ as the composite

$$\Delta(m-1,n-1) \stackrel{h}{\longleftrightarrow} \Delta(m-1,n-1)' \stackrel{g}{\longleftrightarrow} \Delta(m-1,n-1) \cup \Delta(m,n)^{\flat} \stackrel{f}{\longleftrightarrow} \Delta(m,n)$$

where $\Delta(m-1,n-1)'$ is the union of $\Delta(m-1,n-1)$ with the degenerate m-simplices of the form 00... Let \mathcal{C} be a stable ∞ -category and consider the functor

$$f_!g_*h_! \colon \operatorname{\mathsf{Fun}}_*(\Delta(m-1,n-1),\mathfrak{C}) \longrightarrow \operatorname{\mathsf{Fun}}_*(\Delta(m,n),\mathfrak{C})$$

given by

- 1. left Kan extension along h, followed by
- 2. right Kan extension along g, followed by
- 3. left Kan extension along f.

Since Kan extensions along fully faithful functors are fully faithful (Corollary 4.3.2.16 in [Lur09]), this composite functor is fully faithful. We now claim that a diagram

$$X: \Delta(m,n) \longrightarrow \mathcal{C}$$

is in the essential image of the functor $f_!g_*h_!$ if and only if X is exact. This claim is verified by means of the pointwise criterion for the Kan extensions: the Kan extensions along h and g have the effect of extending a given diagram $Y \colon \Delta(m-1,n-1) \to \mathcal{C}$ by assigning zero objects to all degenerate edges. The remaining left Kan extension along f completes the various punctured rectilinear cubes in $\Delta(m-1,n-1) \cup \Delta(m,n)^{\flat}$ to biCartesian cubes in $\Delta(m,n)$ so that the required exactness conditions are satisfied. We leave the straightforward combinatorial details to the reader.

Remark 2.11. The auxiliary rectilinear cubes introduced in the proof of Proposition 2.10 should be compared the minimal projective resolutions described in Proposition 3.17 in [OT12].

Remark 2.12. The proof of Theorem 5.27 in [IO11] can be used to give a constructive proof of Proposition 2.10 along the lines of the proof of Theorem 4.6 in [GS16a], see also Proposition 2.25 below.

Corollary 2.13. Let $m \ge n \ge 1$ and set $\ell = n - m + 1$. There is a canonical equivalence

$$\mathcal{D}A_{\ell}^{(m)} \simeq \operatorname{Fun}^{\operatorname{ex}}_*(\Delta(m,n)^{\operatorname{op}}, \mathcal{D}\mathbb{K}) \tag{2.14}$$

of stable ∞ -categories.

Remark 2.15. Consider equivalence (2.14). The condition on M to be an m-cluster tilting $A_{\ell}^{(m)}$ -module includes that the only self-extensions of M lie in degree m. This condition is nicely reflected by the fact that the exactness conditions for a functor $X: \Delta(m,n)^{\text{op}} \to \mathcal{DK}$ only involve cubes of dimension m+1.

In the situation of Proposition 2.10, we may choose a quasi-inverse of the functor γ^* thus obtaining the sequence of functors

$$\operatorname{\mathsf{Fun}}_*(\Delta(m-1,n-1),\mathfrak{C}) \xrightarrow{\simeq} \operatorname{\mathsf{Fun}}_*^{\mathsf{ex}}(\Delta(m,n),\mathfrak{C}) \hookrightarrow \operatorname{\mathsf{Fun}}_*(\Delta(m,n),\mathfrak{C}).$$

For $\mathcal{C} = \mathcal{D}\mathbb{K}$, this yields the promised combinatorial description of the sequence (2.3).

2.1.2 Cluster tilting in the derived category via paracyclic combinatorics

In §2.1.1 we discussed a combinatorial counterpart of the description of the derived category $\mathcal{D}A_{\ell}^{(m)}$ in terms of a compact generator given by the m-cluster tilting $A_{\ell}^{(m)}$ -module M. In this section we develop an analogous combinatorial viewpoint on the description of $\mathcal{D}A_{\ell}^{(m)}$ in terms of a natural (infinite) subcategory associated to M.

Let \mathcal{U} be the subcategory of $\mathcal{D}A_{\ell}^{(m)}$ spanned by the complexes of the form $\varepsilon(\sigma)[mi]$ where σ is an m-simplex in Δ^n and $i \in \mathbb{Z}$. We view \mathcal{U} as a dg \mathbb{K} -category and set

$$\mathsf{R} A^{(m+1)}_{\mathbb{Z}\ell} = \mathfrak{U}^\mathsf{op} \qquad \text{and} \qquad A^{(m+1)}_{\mathbb{Z}\ell} = H^0(\mathsf{R} A^{(m+1)}_\ell).$$

Note that there is an equivalence of stable ∞ -categories

$$\mathcal{D}RA_{\mathbb{Z}\ell}^{(m)} \xrightarrow{\simeq} \mathcal{D}RA_{\ell}^{(m)} \tag{2.16}$$

given by restricting along the morphism $RA_{\ell}^{(m)} \to RA_{\mathbb{Z}\ell}^{(m)}$ induced by the inclusion of M into \mathcal{U} as the complexes concentrated in degree 0.

Remark 2.17. The subcategory \mathcal{U} is an m-cluster tilting subcategory of the perfect derived category perf $A_{\ell}^{(m)}$, see Theorem 1.21 in [Iya11]. Moreover, \mathcal{U} is clearly invariant under the action of the m-fold suspension functor of perf $A_{\ell}^{(m)}$. This makes $H^0(\mathcal{U})$ into an (m+2)-angulated category in the sense of [GKO13], see also Remark 1.29.

Arguing as in (2.3), we obtin a sequence of functors

$$\mathcal{D}A_{\ell}^{(m)} \xrightarrow{\simeq} \mathcal{D}\mathsf{R}A_{\mathbb{Z}\ell}^{(m+1)} \longrightarrow \mathcal{D}A_{\mathbb{Z}\ell}^{(m+1)} \longrightarrow \mathcal{D}A_{\ell}^{(m+1)}, \tag{2.18}$$

where the rightmost functor given by restriction along the morphism $A_{\ell}^{(m+1)} \to A_{\mathbb{Z}\ell}^{(m+1)}$ As we will demonstrate in this section, the combinatorial counterpart of the passage from $A_{\ell}^{(m)}$ to $\mathsf{R}A_{\mathbb{Z}\ell}^{(m)}$ is the passage from simplices to paracyclic simplices. In particular, using this paracyclic perspective we will obtain a combinatorial description of the sequence (2.18).

Definition 2.19. We denote by $\Lambda_{\infty}(m,n)$ the set of those monotone maps $f: \mathbb{Z} \to \mathbb{Z}$ that satisfy, for all $i \in \mathbb{Z}$, the equivariance condition

$$f(i+m+1) = f(i) + n + 1.$$

Again, we interpret $\Lambda_{\infty}(m,n)$ as a poset by declaring $f \leq f'$ if, for all $i \in \mathbb{Z}$, we have $f(i) \leq f(i')$. We also introduce the partition

$$\Lambda_{\infty}(m,n) = \Lambda_{\infty}(m,n)^{\sharp} \cup \Lambda_{\infty}(m,n)^{\flat} \tag{2.20}$$

into the subset $\Lambda_{\infty}(m,n)^{\sharp}$ of strictly monotone maps and its complement $\Lambda_{\infty}(m,n)^{\flat}$.

Remark 2.21. The underlying set of the poset $\Lambda_{\infty}(m,n)$ is the set of morphisms between objects \widetilde{m} and \widetilde{n} in the *paracyclic category* Λ_{∞} , an enlargement of the simplex category introduced independently in [Nis90], [FL91] and [GJ93]. The poset structure on $\Lambda_{\infty}(m,n)$ that we consider makes Λ_{∞} a 2-category.

Remark 2.22. The paracyclic category admits a presentation by generators and relations extending that of the simplex category. More precisely, Λ is generated by coface and codegeneracy morphisms satisfying the usual relations in Δ together with an additional automorphism $t = t_{n+1} : \tilde{n} \to \tilde{n}$ of infinite order subject to the relations

$$t \circ d^{i} = d^{i-1} \circ t \qquad 1 \le i \le n, \qquad \text{and} \qquad t \circ d^{0} = d^{n},$$

$$t \circ s^{i} = s^{i-1} \circ t \qquad 1 \le i \le n, \qquad \text{and} \qquad t \circ s^{0} = s^{n} \circ t^{2}.$$
(2.23)

Remark 2.24. Let $m, n \geq 0$. A paracyclic simplex $\sigma \in \Lambda_{\infty}(m, n)$ is determined by its restriction to $[m] \subset \mathbb{Z}$ so that we obtain an embedding $\Lambda_{\infty}(m, n) \subset \mathbb{Z}^{m+1}$. In particular, we may talk about rectilinear (m+1)-cubes in $\Lambda_{\infty}(m, n)$ in the sense of Definition 2.8.

The following result is a combinatorial version of the equivalence (2.16).

Proposition 2.25. Let $m, n \geq 0$ and let \mathfrak{C} be a stable ∞ -category. We identify $\Delta(m, n)$ with the subset of $\Lambda_{\infty}(m, n)$ consisting of those maps $f \colon \mathbb{Z} \to \mathbb{Z}$ that map $[m] \subset \mathbb{Z}$ into $[n] \subset \mathbb{Z}$. Then, the induced restriction functor

$$\operatorname{\mathsf{Fun}}^{\mathsf{ex}}_{\star}(\Lambda_{\infty}(m,n),\mathfrak{C}) \longrightarrow \operatorname{\mathsf{Fun}}^{\mathsf{ex}}_{\star}(\Delta(m,n),\mathfrak{C})$$

is an equivalence of stable ∞ -categories.

Proof. This follows by exhibiting a quasi-inverse of the above functor by means of successive Kan extensions similar to the proof of Proposition 2.10. We leave the details to the reader.

The poset isomorphism $\Delta(m,n) \to \Delta(m,n)^{op}$ described in Notation 2.7 extends to a poset isomorphism $\Lambda_{\infty}(m,n) \to \Lambda_{\infty}(m,n)^{op}$. In particular, we have the following corollary of Corollary 2.13 and Proposition 2.25.

Corollary 2.26. Let $m \ge n \ge 1$ and set $\ell = n - m + 1$. There is a canonical equivalence

$$\mathcal{D}A_{\ell}^{(m)} \simeq \operatorname{Fun}^{\operatorname{ex}}_*(\Lambda_{\infty}(m,n)^{\operatorname{op}}, \mathfrak{D}\mathbb{K})$$

of stable ∞ -categories.

Combining the equivalences from Proposition 2.10 and Proposition 2.25 yields the sequence of functors

$$\operatorname{\mathsf{Fun}}_*(\Delta(m-1,n-1),\operatorname{\mathcal{C}}) \xrightarrow{\simeq} \operatorname{\mathsf{Fun}}^{\operatorname{ex}}_*(\Lambda_\infty(m,n),\operatorname{\mathcal{C}}) \hookrightarrow \operatorname{\mathsf{Fun}}_*(\Lambda_\infty(m,n),\operatorname{\mathcal{C}}) \longrightarrow \operatorname{\mathsf{Fun}}_*(\Delta(m,n),\operatorname{\mathcal{C}})$$

which, for $\mathcal{C} = \mathcal{D}\mathbb{K}$, provides the promised combinatorial construction of (2.18).

2.2 Higher reflection functors in type \mathbb{A} and slice mutation

Let m and n be natural numbers and consider the composite inclusion of posets

$$j \colon \Delta(m-1,n-1) \xrightarrow{[0]*-} \Delta(m,n) \hookrightarrow \Lambda_{\infty}(m,n).$$

We denote the image of this inclusion by \underline{S}_P and let $S_P = \underline{S}_P \cap \Lambda(m,n)^{\sharp}$. From Proposition 2.25 and Proposition 2.10, we deduce that, for a stable ∞ -category \mathcal{C} , the restriction functor

$$j^* : \operatorname{\mathsf{Fun}}^{\mathsf{ex}}_*(\Lambda_{\infty}(m,n), \mathfrak{C}) \longrightarrow \operatorname{\mathsf{Fun}}_*(\underline{S}_P, \mathfrak{C})$$
 (2.27)

is an equivalence of stable ∞ -categories. We wish to explain the following:

- The subset $S_P \subset \Lambda_\infty(m,n)^{\sharp}$ is an example of a *slice* in $\Lambda_\infty(m,n)^{\sharp}$.
- An equivalence analogous to (2.27) can be obtained more generally for each slice $S \subset \Lambda_{\infty}(m,n)^{\sharp}$.
- We can mutate between different slices via higher-dimensional analogues of reflection functors.

To this end, we begin with the description of certain symmetries of the poset paracyclic simplices.

Notation 2.28. For a natural number n, we denote by $t_n : \mathbb{Z} \to \mathbb{Z}$ the monotone map given by $i \mapsto i+1$ and view it as an element of $\Lambda_{\infty}(n,n)$.

Definition 2.29. Let $n \ge m \ge 1$ be integers.

(1) The Heller automorphism of $\Lambda_{\infty}(m,n)$ is the poset automorphism

$$\Sigma_m := - \circ t_m : \Lambda_{\infty}(m, n) \longrightarrow \Lambda_{\infty}(m, n).$$

Thus, given $\sigma \in \Lambda(m,n)$ we have $\Sigma_m(\sigma)_i = \sigma_{i+1}$ for each $i \in \mathbb{Z}$.

(2) The Coxeter automorphism of $\Lambda_{\infty}(m,n)$ is the poset automorphism

$$\Phi_m := t_n^{-1} \circ -: \Lambda_{\infty}(m, n) \longrightarrow \Lambda_{\infty}(m, n).$$

Thus, given $\sigma \in \Lambda(m,n)$ we have $\Phi_m(\sigma)_i = \sigma_i - 1$ for each $i \in \mathbb{Z}$.

(3) The Nakayama automorphism is the poset automorphism

$$\mathbf{S} := \Phi_m \circ \Sigma_m : \Lambda_{\infty}(m,n) \longrightarrow \Lambda_{\infty}(m,n).$$

Thus, given $\sigma \in \Lambda(m,n)$ we have $\mathbf{S}(\sigma)_i = \sigma_{i+1} - 1$ for each $i \in \mathbb{Z}$.

Remark 2.30. Let \mathbb{K} be a field, let $m, n \geq 0$, and set $\ell = n - m + 1$. Recall from Corollary 2.26 that there is an equivalence

$$\mathcal{D}A_{\ell}(m) \simeq \operatorname{Fun}^{\operatorname{ex}}_{*}(\Lambda_{\infty}(m,n), \mathcal{D}\mathbb{K}).$$

Utilising methods similar to those employed in Section 5 in [GS16a], one can show that the Heller, Coxeter, and Nakayama automorphisms of the poset $\Lambda_{\infty}(m,n)$ induce, via pullback, familiar autoequivalences on the derived category of $A_{\ell}^{(m)}$. More precisely

$$\Sigma_m^* \cong [m]$$
 and $\mathbf{S}^* \cong - \otimes_{A_\ell^{(m)}}^{\mathbb{L}} \operatorname{\mathsf{Hom}}_{\mathbb{K}}(A_\ell^{(m)}, \mathbb{K}).$

In particular \mathbf{S}^* is a Serre functor on perf $A_{\ell}^{(m)}$. Moreover, $\Phi_m^* = (\mathbf{S} \circ \Sigma_m^-)^*$ corresponds to the derived m-Auslander–Reiten translation functor introduced by Iyama in [Iya11]. Moreover, by the very definition of the poset $\Lambda_{\infty}(m,n)$, there is an equality

$$\mathbf{S}_m^{n+1} = \Sigma_m^{n-m}$$

which reflects the fractional Calabi–Yau dimension of $A_{\ell}^{(m)}$, see [HI11] for details.

What follows is a translation of Definition 5.20 in [IO11] to our combinatorial framework. Recall that, for positive integers m and n, the subset $\Lambda_{\infty}(m,n)^{\sharp}$ consists of the strictly monotone maps in $\Lambda_{\infty}(m,n)$.

Definition 2.31. Let $n \ge m \ge 1$ and let $S \subset \Lambda_{\infty}(m,n)^{\sharp}$ a finite subset. The subset S is a *slice* if the following conditions are satisfied:

- (1) The subset S consists of a complete system of representatives of the Φ_m -orbits of non-degenerate paracyclic m-simplices.
- (2) For every pair of paracyclic m-simplices σ and τ in S, the interval $[\sigma, \tau] \subset \Lambda_{\infty}(m, n)^{\sharp}$ is contained in S.

In this context, slices should be thought of higher-dimensional analogues of Dynkin quivers of type \mathbb{A} with an arbitrary orientation. More precisely, we have the following observation.

Remark 2.32. Let $n \geq 1$ and let $S \subset \Lambda_{\infty}(1,n)^{\sharp}$ be a slice. In this case the underlying graph of the Hasse quiver of S is the Dynkin diagram \mathbb{A}_n and the Hasse quivers for the various slices exhaust all possible choices of orientation.

Example 2.33. Let $n \ge m \ge 1$ be integers. Recall the construction of the subset $S_P \subset \Lambda_\infty(m, n)$ as the image of the composite inclusion

$$\Delta(m-1, n-1) \stackrel{[0]*-}{\longleftrightarrow} \Delta(m, n) \hookrightarrow \Lambda_{\infty}(m, n).$$

Then S_P is a slice, called the *projective slice*. Dually, denote by $S_I \subset \Lambda_{\infty}(m,n)$ the image of the composite inclusion

$$\Delta(m-1, n-1) \stackrel{-*[0]}{\longleftrightarrow} \Delta(m, n) \hookrightarrow \Lambda_{\infty}(m, n).$$

Then S_I is a slice, called the *injective slice*. This terminology stems from the fact that the image of S_P (resp. S_I) under the functor

$$\varepsilon \colon \Delta(m,n) \longrightarrow \operatorname{mod} A_{\ell}^{(m)}$$

from (1.16) is a complete set of representatives of the indecomposable projective (resp. injective) $A_{\ell}^{(m)}$ -modules.

Our interest in the notion of a slice arises from the existence of the following mutation operations which can be thought of as higher-dimensional versions of sink-source reflections, *c.f.* Definition 5.25 in [IO11].

Proposition 2.34. Let $n \geq m \geq 1$ be integers and let $S \subset \Lambda_{\infty}(m,n)^{\sharp}$ be a slice.

(1) Let $\sigma \in S$ be a minimal element. The subset

$$\mu_{\sigma}^{R}(\sigma) = (S \setminus \{\sigma\}) \cup \{\Phi_{m}^{-1}(\sigma)\} \subset \Lambda_{\infty}(m,n)^{\sharp}$$

is a slice, called the right mutation of S at σ , and $\Phi_m^{-1}(\sigma)$ is a maximal element therein.

(2) Let $\sigma \in S$ be a maximal element. The subset

$$\mu_{\sigma}^{L}(\sigma) = (S \setminus \{\sigma\}) \cup \{\Phi_{m}(\sigma)\} \subset \Lambda_{\infty}(m, n)^{\sharp}$$

is a slice, called the left mutation of S at σ , and $\Phi_m(\sigma)$ is a minimal element therein.

(3) If $\sigma \in S$ is a minimal element, then $\mu_{\sigma}^{L}(\mu_{\sigma}^{R}(S)) = S$ and, dually, if $\sigma \in S$ is a maximal element, then $\mu_{\sigma}^{R}(\mu_{\sigma}^{L}(S)) = S$.

Proof. See statements (3) and (4) in Proposition 5.26 in [IO11].

Remark 2.35. Let $n \ge m \ge 1$ and let $\ell = n - m + 1$. Slices in $\Lambda_{\infty}(m,n)^{\sharp}$ classify a distinguished class of tilting complexes in the perfect derived category $\operatorname{perf}(A_{\ell}^{(m)})$ which are higher-dimensional analogues of the APR tilting complexes of [APR79] (cf. Section 5 in [IO11]). For m = 1, these tilting complexes and their mutations are closely related to the classical reflection functors of [BGP73].

The following is the main theorem in the context of slice mutation, c.f. Theorem 5.27 in [IO11].

Theorem 2.36 (Iyama and Oppermann). Let $n \ge m \ge 1$ be integers. Iterated slice mutation acts transitively on the set of slices in $\Lambda_{\infty}(m,n)^{\sharp}$.

To apply Theorem 2.36 in our higher categorical context, we prove a minor refinement of statements (1) and (2) in Proposition 5.26 in [IO11].

Lemma 2.37. Let $n \geq m \geq 1$ be integers. Let $S \subset \Lambda_{\infty}(m,n)^{\sharp}$ be a slice and let $\sigma \in S$ a minimal element. Then all the non-degenerate paracyclic simplices of the (m+1)-cube $\sigma + I^{m+1}$ are contained in $S \cup \{\Phi_m^{-1}(\sigma)\}$.

Proof. Let $v \in I^{m+1}$ be such that $\sigma + v$ is non-degenerate. We need to prove that $\sigma + v$ belongs to $S \cup \{\Phi_m^{-1}(\sigma)\}$. The claim is clear if |v| = 0 or |v| = m+1 since, by definition, $\Phi_m^{-1}(\sigma) = \sigma + (1, \dots, 1)$. Suppose that 0 < |v| < m+1. Given that S is a slice in $\Lambda_\infty(m,n)$ there exists an integer a such that $\Phi_m^a(\sigma + v) \in S$. The minimality of σ in S readily implies that $a \le 0$, as otherwise we would have $\Phi_m^a(\sigma + v) < \sigma$. Suppose that a < 0. If this is the case, then the inequalities

$$\sigma < \Phi_m^{-1}(\sigma) \leq \Phi_m^a(\sigma + v)$$

are satisfied. But the convexity of S implies that $\Phi_m^{-1}(\sigma) \in S$, which contradicts the fact that S is a slice in $\Lambda_{\infty}(m,n)$ for S cannot contain both σ and $\Phi_m^{-1}(\sigma)$. Whence a=0 and $\sigma+v$ belongs to S as required. This finishes the proof.

Definition 2.38. Let $n \ge m \ge 1$ be integers. Let $S \subset \Lambda_{\infty}(m,n)$ be a slice, let $\sigma \in S$ be a minimal element, and let $S' = \mu_{\sigma}^{R}(\sigma)$ be the right mutation of S at σ . We define the poset

$$S \diamond S' := S \cup \{\Phi_m(\sigma)\} = \{\sigma\} \cup S'.$$

Remark 2.39. Let $n \ge m \ge 1$ be integers. Let $S \subset \Lambda_{\infty}(m,n)$ be a slice, let $\sigma \in S$ be a minimal element, and let $S' = \mu_{\sigma}^{R}(\sigma)$ be the right mutation of S at σ . The canonical inclusions

$$S \subset S \diamond S' \supset S'$$

allow us to describe the passage from S to S' by means of the rectilinear (m+1)-cube

$$\sigma + I^{m+1} = \Phi_m^{-1}(\sigma) - I^{m+1},$$

see Lemma 2.37.

We introduce the following auxiliary definition which allows us to meaningfully consider slices in the higher categorical context.

Definition 2.40. Let $n \geq m \geq 1$ and $J \subset \Lambda_{\infty}(m,n)^{\sharp}$. We enlarge J to the poset

$$J := \{ \rho \in \Lambda_{\infty}(m, n) \mid \exists \sigma, \tau \in J \colon \sigma \le \rho \le \tau \}.$$

Note that, if S is a slice, then $\underline{S} \setminus S$ consists only of degenerate paracyclic simplices.

We will now prove the main result of this section, which can be regarded as a higher-dimensional generalisation of Theorem 4.15 in [GS16a].

Theorem 2.41. Let $n \ge m \ge 1$, let \mathfrak{C} be a stable ∞ -category, and let $S \subset \Lambda_{\infty}(m,n)^{\sharp}$ be a slice. The functor

$$\operatorname{Fun}^{\operatorname{ex}}_*(\Lambda_\infty(m,n),\mathfrak{C}) \xrightarrow{\simeq} \operatorname{Fun}_*(\underline{S},\mathfrak{C}),$$

induced by restriction to \underline{S} is an equivalence of stable ∞ -categories.

Theorem 2.41 will be deduced from Theorem 2.36 and the following result.

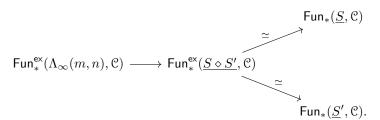
Proposition 2.42. Let $n \geq m \geq 1$, let C be a stable ∞ -category. Let S be a slice in $\Lambda_{\infty}(m,n)$ and let $S' = \mu_{\sigma}^{R}(S)$ the right mutation of S at a minimal element $\sigma \in S$. Then, the restriction functors comprising the zig-zag

$$\operatorname{Fun}_*(S,\mathfrak{C}) \xrightarrow{\simeq} \operatorname{Fun}_*^{\operatorname{ex}}(S \diamond S',\mathfrak{C}) \xrightarrow{\simeq} \operatorname{Fun}_*(S',\mathfrak{C}).$$

are equivalences of ∞ -categories.

Proof. The claim follows immediately from Lemma 2.37, Remark 2.39 and the fact that the space of colimits of a given finite shape in \mathcal{C} is contractible, see Proposition 1.2.12.9 and Remark 1.2.13.5 in [Lur17].

Proof of Theorem 2.41. By Theorem 2.36 and Proposition 2.42, for each slice S and its right mutation $S' \subset \Lambda_{\infty}(m,n)$ at a minimal element, we have a diagram



By two-out-of-three, we deduce that the slice restriction $\operatorname{Fun}^{\operatorname{ex}}_*(\Lambda_\infty(m,n),{\mathbb C}) \to \operatorname{Fun}_*(\underline{S},{\mathbb C})$ is an equivalence if and only if the slice restriction $\operatorname{Fun}^{\operatorname{ex}}_*(\Lambda_\infty(m,n),{\mathbb C}) \to \operatorname{Fun}_*(\underline{S}',{\mathbb C})$ is an equivalence. Therefore, to prove the claim, by the transitivity of slice mutation, it suffices to check that a particular slice restriction is an equivalence. To this end, we note that we have established, in (2.27), such an equivalence

$$\operatorname{Fun}^{\operatorname{ex}}_*(\Lambda_\infty(m,n),\operatorname{\mathcal{C}}) \xrightarrow{\ \ \, \simeq \ \ } \operatorname{Fun}_*(\underline{S}_P,\operatorname{\mathcal{C}})$$

given by restriction to the projective slice.

Remark 2.43. A proof of Theorem 2.41 in the case m=1 which utilises a version of the 'knitting algorithm' is given in Theorem 4.15 in [GS16a], see also Section 6 in [GS16b]. In contrast, an important feature of our proof of Theorem 2.41 is the use of slices in the poset of paracyclic simplices and the description of their mutations by means of biCartesian cubes.

2.3 Relation to the higher Waldhausen S_•-constructions

Let K be a field and $n \ge m \ge 1$. In Corollary 2.13 and Corollary 2.26 we established equivalences

$$\mathcal{D}A_{\ell}^{(m)} \simeq \operatorname{Fun}_{*}^{\operatorname{ex}}(\Delta(m,n), \mathcal{D}\mathbb{K}) \tag{2.44}$$

and

$$\mathcal{D}A_{\ell}^{(m)} \simeq \operatorname{Fun}_{*}^{\operatorname{ex}}(\Lambda_{\infty}(m, n), \mathcal{D}\mathbb{K}), \tag{2.45}$$

respectively. In §2.2 we have seen that these equivalences allow us to develop a combinatorial approach to higher reflection functors in type \mathbb{A} . However, there is more to be learnt: It follows from equivalence (2.44) that the various derived categories $\mathcal{D}A^{(m)}_{\bullet-m+1}$ can be organised into a simplicial object and equivalence (2.45) implies that this simplicial object carries a canonical paracyclic structure. We formulate this statement more generally as follows.

Notation 2.46. Following Section 1.1.4 in [Lur17], we denote the ∞ -category of (small) stable ∞ -categories and exact functors between them by $\mathcal{C}at_{\infty}^{\mathsf{ex}}$.

Proposition 2.47. Let C be a (small) stable ∞ -category and let $m \geq 0$.

(1) The association

$$[n] \mapsto \operatorname{Fun}^{\operatorname{ex}}_*(\Delta(m,n), \mathfrak{C})$$

extends to define a simplicial object

$$\mathsf{S}^{\langle m \rangle}_{\bullet}(\mathfrak{C}) \colon \Delta^{\mathsf{op}} \longrightarrow \mathfrak{C}at^{\mathsf{ex}}_{\infty},$$

called the m-dimensional S_{\bullet} -construction of \mathcal{C} .

(2) The association

$$\widetilde{n} \mapsto \operatorname{\mathsf{Fun}}^{\mathsf{ex}}_*(\Lambda_\infty(m,n), \mathfrak{C})$$

extends to define a paracyclic object

$$\widetilde{\mathsf{S}}_{ullet}^{(m)}(\mathfrak{C}) \colon \Lambda_{\infty}^{\mathsf{op}} \longrightarrow \mathfrak{C}at_{\infty}^{\mathsf{ex}},$$

called the m-dimensional paracyclic S_{\bullet} -construction of C.

(3) Restriction along the inclusion $\Delta \hookrightarrow \Lambda_{\infty}$ induces an equivalence

$$\widetilde{\mathsf{S}}_{ullet}^{\langle m \rangle}(\mathfrak{C})|_{\Delta} \simeq \mathsf{S}_{ullet}^{\langle m \rangle}(\mathfrak{C}).$$

Proof. In view of Proposition 2.25, we only need to verify that the values of the simplicial (resp. paracyclic) objects are really stable ∞ -categories and that the functoriality is provided by exact functors. This can be verified using the results in Appendix A.

Remark 2.48. Let \mathcal{C} be a stable ∞ -category and $n \geq m \geq 1$ integers. The paracyclic autoequivalence t_n on $\mathsf{S}_n^{\langle m \rangle}(\mathcal{C})$ is induced by the Coxeter automorphism Φ_m on $\Lambda_\infty(m,n)$.

2.4 Ladders of recollements

We now wish to express the inductive nature of the higher-dimensional versions of the Waldhausen S_{\bullet} -construction in terms of certain ladders of recollements in the sense of [BlGS88] and [AHKLY17].

Proposition 2.49. Let C be a stable ∞ -category and m a natural number. For each $n \geq 0$ the functors

$$\mathsf{S}_{n}^{\langle m \rangle}(\mathfrak{C}) \overset{\longleftarrow}{\underset{\leftarrow}{\overset{d_{0}}{\longrightarrow}}} \mathsf{S}_{n+1}^{\langle m+1 \rangle}(\mathfrak{C})$$

$$\vdots$$

$$\underset{\leftarrow}{\overset{s_{n}}{\longrightarrow}} \mathsf{S}_{n+1}^{\langle m+1 \rangle}(\mathfrak{C})$$

are part of a string of adjunctions $d_0 \dashv s_0 \dashv d_1 \dashv s_1 \dashv \cdots \dashv d_n \dashv s_n \dashv d_{n+1}$.

Proof. This is a straightforward consequence of the following observations:

- The poset structure on the sets of morphisms in Δ make it into a 2-category Δ . Moreover, there is a chain of adjunctions $d_0 \dashv s_0 \dashv d_1 \dashv s_1 \dashv \cdots \dashv d_n \dashv s_n \dashv d_{n+1}$ in the 1-cell dual 2-category $\Delta^{(\mathsf{op},-)}$.
- The higher S_{\bullet} -constructions are defined in terms of the representable 2-functor $Fun(-,\mathcal{C})$.

We leave the details to the reader.

Lemma 2.50. Let C be a stable ∞ -category and $m \ge 1$ an integer. For each $n \ge 0$ and for each $i \in [n+1]$ there is an equivalence of ∞ -categories

$$\mathsf{S}_n^{\langle m \rangle}(\mathfrak{C}) \simeq \ker d_i \subseteq \mathsf{S}_{n+1}^{\langle m+1 \rangle}(\mathfrak{C}).$$

In particular the stable ∞ -category $\ker d_i$ is independent of $i \in [n]$.

Proof. The paracyclic identities (2.23) imply the existence of equivalences of ∞ -categories ker $d_i \simeq \ker d_{i+1}$ for each $i \in [n]$. Hence it is enough to prove the claim for i = 0. Firstly, recall from (2.27) the equivalence

$$\mathsf{S}_{n+1}^{\langle m+1\rangle}(\mathfrak{C}) = \mathsf{Fun}_*^{\mathsf{ex}}(\Lambda_\infty(m+1,n+1),\mathfrak{C}) \xrightarrow{\simeq} \mathsf{Fun}_*(\underline{S}_P,\mathfrak{X}),$$

where $S_P \subset \Lambda_{\infty}(m+1, n+1)$. Secondly, Lemma A.22 and Proposition A.24 imply that

$$\ker d_0 \simeq \operatorname{Fun}_*^{\operatorname{ex}}(\underline{S}_P, \mathcal{C}) \subseteq \operatorname{Fun}_*(\underline{S}_P, \mathcal{C}).$$

Finally, the inclusion ([0] * -): $\Delta(m,n) \to \Delta(m+1,n+1)$ identifies $\Delta(m,n)$ with S_P yielding equivalences

$$\ker d_0 \simeq \operatorname{Fun}^{\operatorname{ex}}_*(\underline{S}_P, \mathfrak{C}) \simeq \operatorname{Fun}^{\operatorname{ex}}_*(\Delta(m,n), \mathfrak{C}) = \operatorname{S}_n^{\langle m \rangle}(\mathfrak{C})$$

of stable ∞ -categories, which is what we needed to prove.

As an easy consequence of Proposition 2.49 and Lemma 2.50 we obtain the following result.

Proposition 2.51. Let C be a stable ∞ -category and $m \geq 1$ an integer. Then, for each $n \geq 0$ there is a ladder of recollements of stable ∞ -categories

$$\mathsf{S}_{n}^{\langle m+1\rangle}(\mathfrak{C}) \overset{\longleftarrow}{\underset{\leftarrow}{\overset{s_{0}}{\longrightarrow}}} \mathsf{S}_{n+1}^{\langle m+1\rangle}(\mathfrak{C}) \overset{\longleftarrow}{\underset{\leftarrow}{\overset{\cdots}{\longrightarrow}}} \mathsf{S}_{n}^{\langle m\rangle}(\mathfrak{C}).$$

in the sense of [AHKLY17].

Proof. Let $i \in [n]$. According to Proposition A.8.20 in [Lur17], the sequence of adjunctions $d_i \dashv s_i \dashv d_{i+1}$ is part of a recollement

$$\mathsf{S}_{n}^{\langle m+1\rangle}(\mathfrak{C}) \xrightarrow{\longleftarrow s_{i}} \overset{\leftarrow}{\longrightarrow} \mathsf{S}_{n+1}^{\langle m+1\rangle}(\mathfrak{C}) \xrightarrow{\longleftarrow j_{*}} \overset{\rightarrow}{\longrightarrow} \mathfrak{D}$$

where the stable ∞ -category \mathcal{D} can be identified with the full subcategory of $\mathsf{S}_{n+1}^{\langle m+1 \rangle}$ spanned by the objects X such that for every object Y of $\mathsf{S}_n^{\langle m+1 \rangle}(\mathcal{C})$ the mapping space $\mathsf{Map}_{\mathsf{S}_{n+1}^{\langle m+1 \rangle}}(s_i(Y),X)$ is contractible. We claim that \mathcal{D} can be equivalently identified with the kernel of the right adjoint of s_i , that is the kernel of d_{i+1} . Indeed, the unit transformation of the adjunction $s_i \dashv d_{i+1}$ induces an isomorphism

$$\mathsf{Map}_{\mathsf{S}_{n+1}^{\langle m+1 \rangle}(\mathfrak{C})}(s_i(Y), X) \xrightarrow{\simeq} \mathsf{Map}_{\mathsf{S}_n^{\langle m+1 \rangle}(\mathfrak{C})}(Y, d_{i+1}(X))$$

in the homotopy category of spaces, see Definition 5.2.2.7 and Proposition 5.2.2.8 in [Lur17], which readily implies that \mathcal{D} contains the kernel of d_{i+1} . Conversely, suppose that X is an object of \mathcal{D} . Then, the unit transformation of the adjunction $s_i \dashv d_{i+1}$ induces an isomorphism

$$\mathsf{Map}_{\mathsf{S}_{n+1}^{\langle m+1\rangle}(\mathfrak{C})}(s_id_{i+1}(X),X) \xrightarrow{\simeq} \mathsf{Map}_{\mathsf{S}_{n}^{\langle m+1\rangle}(\mathfrak{C})}(d_{i+1}(X),d_{i+1}(X)).$$

in the homotopy category of spaces. Since the leftmost space is contractible by assumption, we conclude that $d_{i+1}(X)$ is a zero object of the stable ∞ -category $S_n^{\langle m+1\rangle}(\mathcal{C})$. This shows that \mathcal{D} is contained in the kernel of d_{i+1} . The statement of the proposition now follows from Lemma 2.50. \square

2.5 Tilting and cluster tilting for projective space

In this section, we discuss another instance of the phenomenon that the presence of an m-cluster tilting subcategory goes hand in hand with a diagrammatic description of the derived category in terms of (m+1)-dimensional biCartesian cubes.

Let \mathbb{K} be a field and m a natural number. Let $\mathbb{P}^m = \mathbb{P}^m_{\mathbb{K}}$ be the projective m-space over \mathbb{K} and \mathfrak{DP}^m the derived ∞ -category of complexes of quasi-coherent sheaves on \mathbb{P}^m , see Section 1.3 in [Lur17] for details. A famous theorem of Beĭlinson [Beĭ78] shows that the object

$$T = \bigoplus_{0 \leq i \leq m} \mathcal{O}(i)$$

is a compact generator of \mathcal{DP}^m ; in fact, it is even a tilting bundle. As a consequence, we obtain an equivalence of stable ∞ -categories

$$\mathcal{DP}^m \simeq \mathcal{D} \operatorname{End}(T)^{\operatorname{op}}. \tag{2.52}$$

Choosing homogeneous coordinates $x_0, x_1, \ldots, x_m \in \Gamma(\mathcal{O}(1))$, the algebra $\mathsf{End}(T)$, commonly referred to as the *Beilinson algebra*, admits the following combinatorial description. Denote by $B^{(m)}$ the category with objects $0, 1, \ldots, m$, and morphisms from i to j given by monomials in $\{x_0, x_1, \ldots, x_m\}$ of degree j-i. The composition law in $B^{(m)}$ is given by multiplication of monomials. It is then immediate that we have an isomorphism

$$\operatorname{End}(T) \cong \operatorname{End} \mathbb{K}B^{(m)}$$
.

In particular, we may reformulate the tilting equivalence (2.52) as an equivalence

$$\mathfrak{DP}^m \simeq \operatorname{Fun}(B^{(m)}, \mathfrak{DK}).$$

Remark 2.53. The algebra $\operatorname{End}(\mathbb{K}B^{(m)})$ is the prototypical example of an m-representation infinite algebra in the sense of [HIO14], see Example 2.15 therein. From the point of view of higher Auslander–Reiten theory, these algebras are higher-dimensional analogues of hereditary algebras of infinite representation type.

Now, instead of the tilting bundle T, consider the subcategory $\mathsf{Line}\,\mathbb{P}^m$ of $\mathfrak{D}\mathbb{P}^m$ spanned by the line bundles $\mathfrak{O}(i),\,i\in\mathbb{Z}$. The subcategory $\mathsf{Line}\,\mathbb{P}^m$ admits a combinatorial description in terms of the category $B^{(m)}_{\mathbb{Z}}$ with set of objects \mathbb{Z} and morphisms from i to j given by the set of monomials in $\{x_0,x_1,\ldots,x_m\}$ of degree j-i.

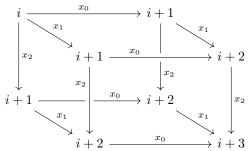
For every $i \in \mathbb{Z}$ and every sequence $a_0, \ldots, a_m \geq 1$ of positive integers, we exhibit a cube

$$q\colon I^{m+1}\longrightarrow B^{(m)}_{\mathbb{Z}}$$

uniquely determined by the following properties:

- (1) We have $q_{(0,...,0)} = i$.
- (2) For $0 \le j \le m$, let e_j denote the jth coordinate vector of \mathbb{Z}^{m+1} . Then every edge of q that is parallel to e_j gets mapped to multiplication by $x_j^{a_j}$ in $B_{\mathbb{Z}}^{(m)}$.

We refer to the cubes of this form as the *rectilinear cubes* in $B_{\mathbb{Z}}^{(m)}$. We call those cubes with $a_0 = a_1 = \cdots = a_m = 1$ minimal. For example, for m = 2 a typical minimal rectilinear cube in $B^{(m)}$ has the form



Definition 2.54. Given a stable ∞ -category \mathcal{C} , we call a functor $B_{\mathbb{Z}}^{(m)} \to \mathcal{C}$ exact, if it maps all rectilinear cubes in $B_{\mathbb{Z}}^{(m)}$ to biCartesian cubes in \mathcal{C} . We denote by

$$\operatorname{Fun}^{\operatorname{ex}}(B^{(m)}_{\mathbb{Z}}, \mathfrak{C}) \subset \operatorname{Fun}(B^{(m)}_{\mathbb{Z}}, \mathfrak{C})$$

the full subcategory of exact functors.

Remark 2.55. Since every rectilinear cube can be decomposed into minimal rectilinear cubes, it is enough to require minimal rectilinear cubes to be sent to biCartesian cubes.

In this context, we have the following analogue of Proposition 2.10 and Proposition 2.25.

Proposition 2.56. Let $m \geq 1$, and let \mathfrak{C} be a stable ∞ -category. The restriction functor

$$\operatorname{\mathsf{Fun}}^{\operatorname{\mathsf{ex}}}(B^{(m)}_{\mathbb{Z}},\mathfrak{C}) \stackrel{\simeq}{\longrightarrow} \operatorname{\mathsf{Fun}}(B^{(m)},\mathfrak{C})$$

is an equivalence of ∞ -categories.

Proof. We shall exhibit a quasi-inverse to the restriction functor

$$\operatorname{Fun}^{\operatorname{ex}}(B^{(m)}_{\mathbb{Z}}, \mathfrak{C}) \longrightarrow \operatorname{Fun}(B^{(m)}, \mathfrak{C}).$$

For a subset $J \subseteq \mathbb{Z}$, we define $B_J^{(m)}$ to be the full subcategory of $B_{\mathbb{Z}}^{(m)}$ spanned by the objects in J. Factorise the inclusion ι into the composite

$$B^{(m)} \xrightarrow{f} B^{(m)}_{[0,\infty)} \xrightarrow{g} B^{(m)}_{\mathbb{Z}}.$$

We claim that the essential image of the fully faithful functor

$$g_*f_! \colon \operatorname{Fun}(B^{(m)}, \mathfrak{C}) \longrightarrow \operatorname{Fun}(B^{(m)}_{\mathbb{Z}}, \mathfrak{C})$$

given by

- 1. left Kan extension along f, followed by
- 2. right Kan extension along g.

is precisely $\mathsf{Fun}^\mathsf{ex}(B^{(m)}_{\mathbb{Z}}, \mathfrak{C})$. We make the following observations:

- A functor $F: B_{[0,\infty)}^{(m)} \to \mathcal{C}$ is a left Kan extension of its restriction $F|_{B^{(m)}}: B^{(m)} \to \mathcal{C}$ if and only if for each i > 0 the restriction of F to $B_{[0,i+m+1]}^{(m)}$ is a left Kan extension of $F|_{B_{[0,i+m]}^{(m)}}$.
- Let i > 0. We identify the punctured cube $I^{m+1}_{|v| < m+1}$ with the poset of non-empty subsets of [m]. There is a cofinal functor $I^{m+1}_{|v| < m+1} \to \left(B^{(m)}_{[0,i+m]}\right)_{/i+m+1}$ given by mapping a subset $v \subsetneq [m]$ to the morphism

$$i + |v| \xrightarrow{\prod_{k \notin v} x_k} i + m + 1$$

and an inclusion $v \subseteq w$ to the morphism

$$i + |v| \xrightarrow{\prod_{k \in w \setminus v} x_k} i + |w|.$$

The composite

$$I^{m+1}_{|v| < m+1} \longrightarrow \left(B^{(m)}_{[0,i+m]}\right)_{/i+m+1} \longrightarrow B^{(m)}_{\mathbb{Z}}$$

is a punctured minimal rectilinear cube in $B_{[0,\infty)}^{(m)}$. All punctured minimal rectilinear cubes in $B_{[0,\infty)}^{(m)}$ are of this form.

It follows that the essential image of $f_!$ consists of those functors $B^{(m)}_{[0,\infty)} \to \mathbb{C}$ whose restriction to each minimal rectilinear cube in $B^{(m)}_{[0,\infty)}$ is biCartesian. A similar argument shows that the essential image of g_* consists precisely of those functors whose restriction to each minimal rectilinear cube in $B_{(-\infty,m]}$ is biCartesian. This finishes the proof because every minimal rectilinear cube in $B_{\mathbb{Z}}$ lies either in $B_{[0,\infty)}$ or in $B_{(-\infty,m]}$.

The connection to higher Auslander–Reiten theory is as follows: As shown in [HIMO14] (see also Example 6.7 in [Jas16]), the subcategory Line \mathbb{P}^m is an m-cluster tilting subcategory of the exact category Vect \mathbb{P}^m of vector bundles on \mathbb{P}^m . Again, in our combinatorial context the condition of homological purity in degree m is reflected geometrically by the fact that only (m+1)-dimensional cubes appear in the exactness constraints of Proposition 2.56.

Various features of the discussion from $\S 2.1$ reappear in the current context of projective m-space, be it in a somewhat elementary fashion:

- In analogy to the combinatorial description of the Coxeter functor for higher Auslander algebras of type \mathbb{A} , the autoequivalence of $\mathcal{D}(\mathbb{P}^m)$ given by the tensor action of the line bundle $\mathcal{O}(1)$ is described by pulling back along the shift autoequivalence $i \mapsto i+1$ of $B_{\mathbb{Z}}^{(m)}$. In other words, the action of the Picard group $\operatorname{Pic}(\mathbb{P}^m) \cong \mathbb{Z}$ admits a purely combinatorial description in terms of $B_{\mathbb{Z}}^{(m)}$.
- The analogues of the slices defined in §2.1 are the full subcategories of $B_{\mathbb{Z}}^{(m)}$ spanned by the intervals [j,j+m] of length m. The restriction functors from $\mathsf{Fun}^\mathsf{ex}(B_{\mathbb{Z}}^{(m)},\mathfrak{C})$ to each of these subcategories are equivalences and the procedure of mediating between these different slices can be regarded as a version of slice mutation. Note, however, that all slices are isomorphic to $B^{(m)}$.

3 Mutation, horn fillings, and higher Segal conditions

Let $n \ge m \ge 1$ and \mathcal{C} be a stable ∞ -category. In §2.2 we established a combinatorial approach to slice mutation, based on the equivalences

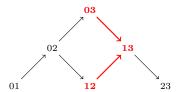
$$\operatorname{Fun}^{\operatorname{ex}}_*(\Delta(m,n),\operatorname{\mathfrak{C}}) \xrightarrow{\ \ \, \simeq \ \ } \operatorname{Fun}(\underline{S},\operatorname{\mathfrak{C}})$$

for the various slices $S \subset \Delta(m,n)$. In §2.3 we then observed that, for a fixed natural number m, the various stable ∞ -categories $\mathsf{Fun}^{\mathsf{ex}}_*(\Delta(m,n),\mathfrak{C})$ organise into a simplicial object $\mathsf{S}^{\langle m \rangle}_{\bullet}(\mathfrak{C})$, which we called the m-dimensional S_{\bullet} -construction of the stable ∞ -category \mathfrak{C} .

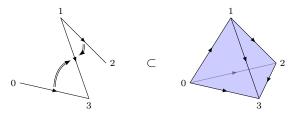
The goal of this section is to analyse the interplay between our combinatorial framework for slice mutation and the simplicial object $S^{\langle m \rangle}_{\bullet}(\mathfrak{C})$. Our main insight is that slice mutation is intimately related to

- certain 2-categorical outer horn filling conditions and
- the higher Segal conditions introduced in [DK12, Pog17].

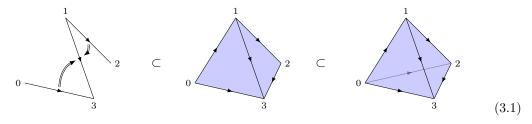
We illustrate these relations by means of an explicit example. Consider the slice in $\Delta(1,3)$ given by the subset $S = \{12, 13, 03\}$. We depict S as a subquiver of the Auslander–Reiten quiver of mod $A_3^{(1)}$ as follows:



Alternatively, we may emphasise the role of the elements of $\Delta(1,3)$ as edges in the 3-simplex and visualise the slice geometrically as



where the notation for the arrows is chosen to signify their interpretation as 2-morphisms in the 2-category \triangle . In this presentation, it becomes apparent that we may interpret S as a 1-dimensional subcomplex of the 2-dimensional subcomplex of Δ^3 formed by the union of triangles contained in the following triangulation of the square:



Let \mathcal{C} be a stable ∞ -category and denote the Waldhausen S_{\bullet} -construction of \mathcal{C} by \mathcal{X}_{\bullet} . The 2-Segal property of the S_{\bullet} -construction, as established in [DK12], provides an equivalence of ∞ -categories

$$\chi_3 \xrightarrow{\simeq} \chi_{\{1,2,3\}} \times_{\chi_{\{1,3\}}} \chi_{\{0,1,3\}}$$

$$\tag{3.2}$$

while restriction to the slice S yields an equivalence

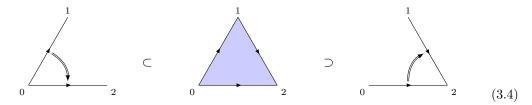
$$\chi_3 \xrightarrow{\simeq} \operatorname{Fun}(12 \to 13 \leftarrow 03, \mathcal{C}).$$
(3.3)

In comparing the equivalences (3.2) and (3.3), we make the following observations:

(1) The description of the stable ∞ -category \mathcal{X}_3 provided by the 2-Segal equivalence (3.2) contains, in comparison with (3.3), a certain amount of redundancy: the datum corresponding to each 2-simplex amounts to biCartesian squares

in the stable ∞ -category \mathcal{C} , while the data retained in the slice equivalence (3.3) are only the morphisms f and g.

(2) According to the pictorial presentation of (3.1), the discrepency between (3.3) and (3.2) can be accounted for by unique fillings of *outer* horns:



(3) However, and this is a crucial point, the outer horns in (3.4) need to be considered in a suitable 2-categorical framework: The data they capture is not just comprised of the two separate edges, but also includes a representation of the 2-morphism that relates the edges.

The starting point that enables us to realise the 2-categorical horn filling conditions of (3) is the observation that, for every $m \geq 0$ and every stable ∞ -category $\mathsf{S}^{\langle m \rangle}_{\bullet}(\mathfrak{C})$ admits a natural extension to a 2-simplicial object in the ∞ -bicategory of stable ∞ -categories (cf. [Dyc17]).

However, in order to formulate the 2-categorical horn filling conditions and their relation to the Segal conditions one needs a systematic framework of limits and Kan extensions in ∞ -bicategories. Since, to our knowledge, this theory has not been developed in the literature to the extent needed for our purposes, we will postpone the realization of this program to a sequel [DJW] where we establish the necessary foundational results and establish 2-categorical counterparts of the results established in this section.

In this article, we allow ourselves to work in the ∞ -categorical context by means of the following modification of the above challenge: Instead of studying 2-simplicial stable ∞ -categories \mathfrak{X}_{\bullet} themselves, we focus on understanding the shadow of the interplay of slice mutation, horn filling, and the higher Segal conditions after passing to

- Grothendieck groups $K_0(-)$ so that we obtain simplicial abelian groups,
- K-theory spectra K(-) so that we obtain simplicial objects in spectra.

To formulate our results in this context we introduce the following terminology, c.f. [DK12, Pog17].

Definition 3.5. Let K be a (finite) simplicial set. Recall that the category of simplices of K is the slice category $\Delta_{/K}$, where we view Δ as a full subcategory of the category of simplicial sets via the Yoneda embedding $[n] \mapsto \Delta^n$. Let \mathcal{C} be an ∞ -category and X a simplicial object in \mathcal{C} . We obtain a diagram

$$X|_{(\Delta_{/K})^{\operatorname{op}}}\colon (\Delta_{/K})^{\operatorname{op}} \longrightarrow \Delta^{\operatorname{op}} \xrightarrow{X} \mathfrak{C}$$

where $\Delta_{/K} \to \Delta$ is the forgetful functor. Assuming that \mathcal{C} has (finite) limits, we further refer to the object

$$X_K\coloneqq \lim_{(\Delta_{/K})^{\operatorname{op}}} X|_{(\Delta_{/K})^{\operatorname{op}}}.$$

as the *object of K-membranes in X*.

We are interested in the following descent conditions on simplicial sets, which are made more precise in §3.1 below.

Definition 3.6 (Sketch). Let \mathcal{C} be an ∞ -category and let X be a simplicial object in \mathcal{C} .

(1) We say that X satisfies m-dimensional slice descent if, for every n > m, and every m-dimensional slice $S \subset \Delta^n$, the restriction map

$$X_n \longrightarrow X_S$$

is an equivalence in C.

(2) We say that X is an outer m-Kan complex, or that it has unique outer horn fillers above dimension m if, for every n > m, the restriction maps

$$X_n \longrightarrow X_{\Lambda_0^n}$$
 and $X_n \longrightarrow X_{\Lambda_n^n}$

are equivalences in \mathcal{C} .

(3) We say that X is a 2m-Segal object if, for every n > 2m and every triangulation $\mathcal{T} \subset \Delta^n$ of the 2m-dimensional cyclic polytope C([n], 2m) on the vertices [n], the restriction map

$$X_n \longrightarrow X_{\mathfrak{T}}$$

is an equivalence in \mathcal{C} .

The interrelations between these notions are captured by the following summary of results of this section:

Theorem 3.7. Let C be an ∞ -category with finite limits, and let X a simplicial object in C. For a fixed m > 0, consider the following conditions on X:

- (I) The simplicial object X satisfies m-dimensional slice descent.
- (II) The simplicial object X satisfies outer horn descent above dimension m.
- (III) The simplicial object X is a 2m-Segal object.

Then the following hold:

- (1) We always have the implications $(I) \Longrightarrow (II) \Longrightarrow (III)$.
- (2) Assume m = 1. Then we have $(I) \iff (II)$.

In [DJW], we will show that Theorem 3.7 remains true, mutatis mutandis, for 2-simplicial objects with values in ∞ -bicategories. In particular, via the categorified Dold-Kan correspondence of [Dyc17], we will obtain a classification of 2m-Segal objects in the ∞ -bicategory of stable ∞ -categories.

3.1 Membranes, covers and refinements

We introduce the following framework which allows us to work efficiently with objects of membranes associated to simplicial subsets of the standard simplices.

Definition 3.8. A cover of $[n] \in \Delta$ is a collection \mathcal{F} of subsets of [n] such that $\bigcup \mathcal{F} = [n]$. We write $\mathcal{F} \models [n]$ to indicate that \mathcal{F} is a cover of [n]. Given two covers \mathcal{F}' and \mathcal{F} of n, we say that \mathcal{F}' is a refinement of \mathcal{F} if for every $f' \in \mathcal{F}'$ there is an $f \in \mathcal{F}$ with $f' \subseteq f$. In this case we write $\mathcal{F}' \preceq \mathcal{F}$.

Every cover $\mathcal{F} \models [n]$ refines the trivial cover $\{[n]\} \models [n]$; if the converse is true (i.e. $[n] \in \mathcal{F}$) then we call $\mathcal{F} \models [n]$ a degenerate cover. More generally, we call a refinement $\mathcal{F}' \preceq \mathcal{F}$ degenerate, if \mathcal{F} also refines \mathcal{F}' . For each $[n] \in \Delta$, covers of [n] and refinements between them are naturally assembled into a category $\mathsf{Cov}_{[n]}$ which is equivalent to a poset; the isomorphisms are the degenerate refinements. The categories $\mathsf{Cov}_{[n]}$ can be assembled into a Cartesian fibration

$$\bigcup: \operatorname{Cov}_{\Delta} \longrightarrow \Delta, \tag{3.9}$$

where there is a unique morphism from $\mathcal{F}' \models [n]$ to $\mathcal{F} \models [m]$ over $\alpha \colon [m] \to [n]$ provided that for each $f' \in \mathcal{F}'$ there is an $f \in \mathcal{F}$ such that $\alpha(f') \subseteq f$. A cover $\mathcal{F} \models [n]$ gets mapped to $[n] = \bigcup \mathcal{F} \in \Delta$. The assignment which sends [n] to the trivial cover $\{[n]\} \models [n]$ gives rise to a fully faithful right adjoint

$$\{\}: \Delta \hookrightarrow Cov_{\Delta}$$

to the functor (3.9). From now on we will identify Δ with its image in Cov_{Δ} .

Given an ∞ -category $\mathcal C$ with finite limits, right Kan extension along $\{\}^{\mathsf{op}}$ yields a fully faithful embedding

$$\operatorname{\mathsf{Fun}}(\Delta^{\operatorname{op}},{\mathfrak C}) \hookrightarrow \operatorname{\mathsf{Fun}}(\operatorname{Cov}^{\operatorname{op}}_\Delta,{\mathfrak C})$$

given by evaluating a functor X on a cover $X \models [n]$ via the formula

$$X_{\mathcal{F}} \simeq \lim \left(\left(\Delta_{/\mathcal{F}} \right)^{\mathsf{op}} o \Delta^{\mathsf{op}} \overset{X}{\longrightarrow} \mathfrak{C} \right).$$

Every cover $\mathcal{F} \models [n]$ gives rise to a simplicial subset

$$K_{\mathfrak{F}} \colon [n] \mapsto \mathsf{Hom}_{\mathsf{Cov}_{\Delta}}([n], \mathfrak{F})$$

of Δ^n . Two covers of [n] give rise to the same simplicial set if and only if they are isomorphic in Cov_{Δ} . It is immediate from the definition that $X_{\mathcal{F}}$ agrees with the membrane space $X_{K_{\mathcal{F}}}$ defined above

Given a cover $\mathcal{F} \models [n]$, we denote by $\mathcal{F}_{\cap} \subset \mathcal{P}([n])$ the subposet of those non-empty subsets of [n] which are of the form $\bigcap F$ for some $\emptyset \neq F \subseteq \mathcal{F}$.

Remark 3.10. The fully faithful inclusion $\mathcal{F}_{\cap} \hookrightarrow \Delta_{/\mathcal{F}}$ has a left adjoint given on objects by

$$(\alpha \colon [m] \to \mathcal{F}) \mapsto \bigcap \{ f \in \mathcal{F} \mid \alpha([m]) \subseteq f \}.$$

In particular, by cofinality, we can compute $X_{\mathcal{F}}$ as

$$X_{\mathcal{F}} \simeq \lim \left(\mathcal{F}^{\mathsf{op}}_{\cap} \hookrightarrow \Delta^{\mathsf{op}} \overset{X}{\longrightarrow} \mathfrak{C} \right).$$

3.1.1 Descent

Let \mathcal{C} be an ∞ -category with all finite limits and $X: \Delta^{\mathsf{op}} \to \mathcal{C}$ a simplicial object.

Definition 3.11. We say that a cover $\mathcal{F} \models [n]$ is X-local, or that X satisfies descent with respect to \mathcal{F} , if X, viewed as a functor $\operatorname{Cov}_{\Delta}^{\mathsf{op}} \to \mathcal{C}$ sends the structure map $\mathcal{F} \to [n]$ to an equivalence

$$X_n \longrightarrow X_{\mathcal{F}}.$$

More generally, we say that a refinement of covers $\mathcal{F}' \preceq \mathcal{F}$ is X-local if it is sent by X to an equivalence

$$X_{\mathfrak{T}} \longrightarrow X_{\mathfrak{T}'}$$

in C.

All of the conditions we impose on simplicial objects in this article are descent-conditions with respect to certain specific covers. We will now develop some basic techniques for comparing different such descent conditions to each other.

Remark 3.12. Degenerate refinements $\mathcal{F}' \cong \mathcal{F}$ are precisely the isomorphisms in the category $\operatorname{Cov}_{\Delta}$, hence they are X-local for every X. In particular, every degenerate cover is X-local for every X.

Lemma 3.13. Let $\mathfrak{F}' \preceq \mathfrak{F} \models [n]$ be a refinement of covers. Assume that for every $F \subseteq \mathfrak{F}$ with $\emptyset \neq I := \bigcap F$, the functor $X : \Delta^{\mathsf{op}} \to \mathfrak{C}$ satisfies descent with respect to the cover

$$I^*(F') := \{ f' \cap I \mid f' \in \mathcal{F}' \} \models I.$$

Then $\mathfrak{F}' \preceq \mathfrak{F}$ is X-local.

Proof. Consider the composite functor

$$g\colon \left(\Delta_{/\mathfrak{F}'}\right)^{\mathrm{op}} \longrightarrow \left(\Delta_{/\mathfrak{F}}\right)^{\mathrm{op}} \longrightarrow \mathfrak{F}_{\cap}^{\mathrm{op}}$$

where the first map is induced by the refinement $\mathcal{F}' \leq \mathcal{F}$ and the second one is (the opposite of) the left adjoint to $\mathcal{F}_{\cap} \hookrightarrow \Delta_{/\mathcal{F}}$.

We claim that $X|_{\mathcal{F}^{op}_{\cap}}$ is a right Kan extension of $X|_{(\Delta_{/\mathcal{F}'})^{op}}$ along g. For each $I = \bigcap F \in \mathcal{F}_{\cap}$, we have to prove that the natural map

$$X|_{\mathcal{F}^{\mathrm{op}}_{\cap}}(I) \longrightarrow g_*\left(X|_{\left(\Delta_{/\mathcal{F}'}\right)^{\mathrm{op}}}\right)(I)$$

(induced by unit of the adjunction $\Delta_{/\mathcal{F}} \leftrightarrow \mathcal{F}_{\cap}$) is an equivalence. Observe that for each $I \in \mathcal{F}_{\cap}$ we have

$$\left(\Delta_{/\mathcal{F}'}\right)^{\mathsf{op}}_{I/} = \left(\Delta_{/I^*(F')}\right)^{\mathsf{op}}$$

hence using the pointwise formula for Kan extensions we reduce to showing that the natural map

$$X(I) \longrightarrow \lim \left(\left(\Delta_{/I^*(F')} \right)^{\operatorname{op}} \hookrightarrow \Delta^{\operatorname{op}} \overset{X}{\longrightarrow} \mathfrak{C} \right) \simeq X(I^*(F'))$$

is an equivalence; this is precisely the assumption on X. The result follows because we have

$$X_{\mathcal{F}'} \simeq \lim (X|_{\left(\Delta_{/\mathcal{F}'}\right)^{\mathrm{op}}}) \simeq \lim (X|_{\mathcal{F}_{\cap}}) \simeq X_{\mathcal{F}}. \qquad \qquad \Box$$

Remark 3.14. If there is an $f \in F$ which is contained in some $f' \in \mathcal{F}'$ then $(\bigcap F)^*(F') \models \bigcap F$ is a degenerate cover, hence the assumption of Lemma 3.13 is automatic in this case.

Notation 3.15. Given a cover $\mathcal{F}' \models [n]$ and a subset $f \subset [n]$, we introduce a new cover

$$\mathfrak{F}' \uplus f := \{f\} \cup \{f' \in \mathfrak{F}' \mid f' \not\subset f\} \models [n]$$

by adding the element f and then removing the redundant elements of \mathcal{F}' . We then have a refinement $\mathcal{F}' \leq \mathcal{F}' \uplus f$.

Corollary 3.16. Let $\mathfrak{F}' \models [n]$ be a cover and $f \subset [n]$ a subset. If $f^*(\mathfrak{F}') \models f$ is an X-local cover then $\mathfrak{F}' \preceq \mathfrak{F}' \uplus f$ is an X-local refinement.

Proof. Follows by direct application of Lemma 3.13 since, by Remark 3.14, we only need to check the case $F = \{f\}$, which is true by assumption.

Let P be a set of covers, which we identify with the full subcategory $P \subset \text{Cov}_{\Delta}$ which they span. We denote by $P_{[n]}$ the set/category of covers $\mathcal{F} \models [n]$ in P, i.e. the fibre over [n] of the composition $P \subset \text{Cov}_{\Delta} \xrightarrow{\bigcup} \Delta$.

Definition 3.17. We say a cover $\mathcal{F} \models [n]$ is P-local, if for every $X \colon \Delta^{\mathsf{op}} \to \mathcal{C}$ we have the implication

every
$$\mathfrak{F}' \in P$$
 is X-local $\Longrightarrow \mathfrak{F}$ is X-local

We denote the set of all P-local covers by \overline{P} .

Definition 3.18. Let P be a set of non-degenerate covers. We say that P is *saturated* if it has the following properties:

- (1) Given a non-degenerate refinement $\mathcal{F}' \leq \mathcal{F}$ of covers in P_n , there is an element $f \in \mathcal{F}$ such that
 - The cover $\mathcal{F}' \uplus f$ is isomorphic to a cover in P_n .
 - The cover $f^*(\mathcal{F}') \models f$ is isomorphic to a cover in P_f .
- (2) For each $[n] \in \Delta$, the category $P_{[n]}$ is connected.

Lemma 3.19 (Refinement Principle). Let P be a saturated cover. For each $n \in \mathbb{N}$ where $P_{[n]}$ is non-empty, choose an element $p_n \in P_{[n]}$. Then we have

$$\overline{P} = \overline{\{p_n \mid n \in \mathbb{N}, P_{[n]} \neq \emptyset\}}.$$

Proof. The inclusion " \supseteq " is trivial. To prove that the converse inclusion holds, assume that X satisfies descent with respect to the covers $p_n \models [n]$; we prove by induction that for each n all covers in P_n are X-local. Let $\mathcal{F}' \preceq \mathcal{F}$ be a refinement in P_n . By repeated application of assumption (1), we can write this refinement as a composition

$$\mathfrak{F}' = \mathfrak{F}'_0 \preceq (\mathfrak{F}'_0 \uplus f_1) \cong \mathfrak{F}'_1 \preceq (\mathfrak{F}'_1 \uplus f_2) \cong \mathfrak{F}'_2 \ldots \preceq (\mathfrak{F}'_{l-1} \uplus f_l) \cong \mathfrak{F}'$$

where each \mathcal{F}'_i lies in P and where all the covers $f_i^*(\mathcal{F}'_{i-1}) \models f_i$ lie in P. Since f_i is a proper subset of [n], we know by induction that the covers $f_i^*(\mathcal{F}'_{i-1}) \models f_i$ are X-local. It follows from Corollary 3.16 that all the refinements $\mathcal{F}'_{i-1} \preceq \mathcal{F}'_{i-1} \uplus f_i$ are X-local; hence $\mathcal{F}' \preceq \mathcal{F}$ is X-local. Since the category $P_{[n]}$ is connected and every refinement in $P_{[n]}$ is X-local, we conclude that all covers in $P_{[n]}$ are X-local if any one of them $(e.g.\ p_n)$ is.

3.2 Outer horn descent and slice mutations

We introduce the following classes of covers.

Definition 3.20. Let $m \ge 0$ and consider the following classes of covers:

 \bullet The class of left horns above dimension m is

$$\Lambda^{>m}_{-} \coloneqq \{\Lambda^n_0 \models [n] \mid n > m\}.$$

 \bullet The class of right horns above dimension m is

$$\Lambda_+^{>m} := \{\Lambda_n^n \models [n] \mid n > m\}.$$

• The class of *projective m-slices* is

$$P(m) := \{ P(m, n) := \{ f \subset [n] \mid 0 \in f, |f| = m + 1 \} \models [n] \mid n > m \}.$$

• The class of *injective* m-slices is

$$I(m) \coloneqq \{I(m,n) \coloneqq \{f \subset [n] \mid n \in f, |f| = m+1\} \models [n] \mid n > m\}.$$

• The class of all m-dimensional slices is

$$S(m) := \{S \models [n]\}$$

where S ranges over all those slices $S \subset \Lambda_{\infty}(m,n)^{\sharp}$ (as in Definition 2.31) which are contained in $\Delta(m,n)^{\sharp}$. Here we identify elements of $\Delta(m,n)^{\sharp}$ with their image (a subset of [n] of cardinality m+1), so that we can interpret $S \subset \Delta(m,n)$ as a cover $S \models [n]$.

Notation 3.21. Consider the class \mathbf{P}^m of those non-degenerate covers $\mathcal{F} \models [n]$ satisfying:

- (1) $0 \in \bigcap \mathcal{F}$
- (2) For every $I \subset [n]$ with |I| = m there is a $f \in \mathcal{F}$ with $I \subset f$.

We define the class \mathbf{I}^m analogously with $n \in \bigcap \mathcal{F}$ instead of $0 \in \bigcap \mathcal{F}$.

Proposition 3.22. Let \mathcal{C} be an ∞ -category with finite limits and X a simplicial object in \mathcal{C} .

- (1) The following statements are equivalent:
 - The simplicial object X satisfies left horn descent above dimension m.
 - The simplicial object X satisfies descent with respect to all covers in \mathbf{P}^m .
 - The simplicial object X satisfies m-dimensional projective slice descent.
- (2) The following statements are equivalent

- The simplicial object X satisfies right horn descent above dimension m.
- The simplicial object X satisfies descent with respect to all covers in \mathbf{I}^m .
- The simplicial object X satisfies m-dimensional injective slice descent.

Proof. For $n \leq m$, the category $\mathbf{P}_{[n]}^m$ is empty. Observe that for n > m, the covers Λ_0^n and P(m,n) are terminal and initial in $\mathbf{P}_{[n]}^m$, respectively. It is straightforward to check condition (1) of Definition 3.18 so that \mathbf{P}^m is a saturated class. Two applications of the refinement principle (Lemma 3.19) yield $\overline{\Lambda_{-}^{>m}} = \overline{\mathbf{P}^m} = \overline{P(m)}$, hence the first claim. The second claim is analogous using the saturated class \mathbf{I}^m .

Lemma 3.23. A cover $\mathcal{F} \models [n]$ is a equal to a 1-dimensional slice $S \subset \Delta(1,n)$ if and only if \mathcal{F} is the trivial cover $\{[1]\} \models [1]$ or it is of the form $\mathcal{F} = \mathcal{F}' \cup \{f\}$ where f = [n-1] or $f = \{1, \ldots, n\}$.

Proof. The "if" direction is clear. To prove the converse, let $S \subset \Delta(m,n)^{\sharp}$ be a slice. Since $\{0,n\}$ is the unique element in its Φ_m^{-1} -orbit, we must have $\{0,n\} \in S$. If n=1, then we are done; otherwise exactly one of $\{0,n-1\}$ and $\{1,n\}$ must be in S, without loss of generality the first case. No element of the form $\{i,n\}$ with i>0 can then be in S because otherwise by convexity also $\{1,n\} \in S$ which is false. We conclude that $S \setminus \{\{0,n\}\}$ is a subset of $\Delta(m,n-1)^{\sharp}$ where it still satisfies the axioms to be a slice S'. The result follows by induction.

Proposition 3.24. If a simplicial object X satisfies m-dimensional slice descent then it also satisfies outer horn descent above dimension m. The converse "only if" is true for m = 1.

Proof. The first statement follows by combining the two parts of Proposition 3.22. To prove the converse, assume that X satisfies outer horn descent above dimension 1 and let $S \models [n]$ be a 1-dimensional slice. We write $S = S' \cup \{f\}$ as in Lemma 3.23 and observe that by induction the cover $\mathcal{F} \cap f \cong \mathcal{F}' \preceq f$ is X-local; hence by Corollary 3.16 we conclude that the refinement $\mathcal{F} \preceq \mathcal{F} \cup \{f\} \cong \{\{0,n\},f\}$ is X-local. The result follows because the covers $\{\{0,n\},[n-1]\} \models [n]$ and $\{\{0,n\},\{1,\ldots,n\}\} \models [n]$ are $\Lambda_+^{\geq 1}$ -local and $\Lambda_-^{\geq 1}$ -local, respectively, by Proposition 3.22. \square

3.3 Higher Segal objects

We recall from [DK12] and [Pog17] the notion of a higher Segal object in an ∞ -category with finite limits. For the sake of brevity we keep our exposition rather terse.

Definition 3.25. Let $I \subset [n]$ be a subset of [n]. An index $i \in [n] \setminus I$ is called an *odd gap of I* if the cardinality of the set $\{j \in I \mid j > i\}$ is odd and an *even gap of I* if this cardinality is even. We define the following covers:

- The upper k-Segal cover $\mathfrak{I}_{n,k}^+ \models [n]$ consists of all even subsets $I \subset [n]$ of cardinality k+1.
- The lower k-Segal cover $\mathfrak{T}_{n,k}^- \models [n]$ consists of all even subsets $I \subset [n]$ of cardinality k+1.

Remark 3.26. Let $n \ge k \ge 1$ be integers. The sets $\mathfrak{T}_{n,k}^+$ and $\mathfrak{T}_{n,k}^-$ can be identified, respectively, with the minimal and the maximal element of the set of triangulations of a k-dimensional cyclic polytope on the vertex set [n] with respect to an appropriate partial order, see Section 6.1 in [DLRS10] for details.

Definition 3.27. Let \mathcal{C} be an ∞ -category with finite limits, X a simplicial object in \mathcal{C} and $k \geq 1$ an integer.

- (1) The simplicial object X is called lower k-Segal if it satisfies descent with respect to all lower k-Segal covers $\mathcal{T}_{n,k}^+ \models [n]$.
- (2) The simplicial object X is called upper k-Segal if it satisfies descent with respect to all upper k-Segal covers $\mathfrak{T}_{n,k}^+ \models [n]$.
- (3) The simplicial object X is called k-Segal if is both lower and upper k-Segal.

Before moving on, we recall from [Pog17] the following explicit characterization of the lower and upper 2m-Segal covers.

Remark 3.28. Let $n \ge 2m - 1$.

(1) The lower (2m-1)-Segal cover $\mathfrak{T}_{n,2m-1}^- \models [n]$ consists precisely of subsets $I \subseteq [n]$ of the form

$$I = \bigcup_{j=1}^{m} \{i_j, i_j + 1\}$$

where the i_j are such that this union is disjoint (i.e. $0 < i_j < i_j + 1 < i_{j+1}$ for all j). Let $n \ge 2m$.

(1) The lower 2m-Segal cover $\mathfrak{T}_{n,2m}^- \models [n]$ consists precisely of subsets $I \subseteq [n]$ of the form

$$I = \{0\} \cup I'$$

where I' is an element of the lower (2m-1)-Segal cover $\mathfrak{T}_{n-1,2m-1}^- \models \{1,\ldots,n\}$.

(2) The upper 2m-Segal cover $\mathfrak{T}_{n,2m}^+ \models [n]$ consists precisely of subsets $I \subseteq [n]$ of the form

$$I = I' \cup \{n\}$$

where I' is an element of the lower (2m-1)-Segal cover $\mathfrak{T}_{n-1,2m-1}^-\models [n-1].$

3.3.1 Outer horn descent and even-dimensional Segal objects

The following result relates outer horn descent conditions to even Segal conditions in ∞ -categories with finite limits.

Theorem 3.29. Let \mathcal{C} be an ∞ -category with finite limits, let $m \geq 1$ be an integer, and X a simplicial object in \mathcal{C} .

- (1) Suppose that X satisfies left horn descent above dimension m. Then X is a lower 2m-Segal object in \mathfrak{C} .
- (2) Suppose that X satisfies right horn descent above dimension m. Then X is an upper 2m-Segal object in \mathbb{C} .

In particular, if X satisfies outer horn descent above dimension m, then X is a 2m-Segal object in \mathfrak{C} .

Proof. Using the explicit description of the Segal covers (Remark 3.28) it is clear that $\mathfrak{T}_{n,2m}^-$ belongs to the class \mathbf{P}^m and $\mathfrak{T}_{n,2m}^+$ belongs to the class \mathbf{I}^m . Hence claims (1) and (2) follow from Proposition 3.22. The last statement is an immediate consequence of the first two.

3.3.2 Inner horn descent and odd-dimensional Segal objects

Notation 3.30. Let $m \geq 0$. Consider the following classes of covers:

 \bullet The class of almost right horns above dimension m is

$$\Lambda_{\mathbf{a}+}^{>m} \coloneqq \left\{ \Lambda_{n-1}^n \models [n] \mid n > m \right\}.$$

 \bullet The class of almost left horns above dimension m is

$$\Lambda_{\mathtt{a}-}^{>m} \coloneqq \{\Lambda_1^n \models [n] \mid n > m\}.$$

 \bullet The class of inner horns above dimension m is

$$\Lambda_{\text{inn}}^{>m} := \{ \Lambda_i^n \models [n] \mid 0 < i < n > m \}.$$

Theorem 3.31. Let C be an ∞ -category with finite limits, let $m \geq 1$ be an integer and X a simplicial object in C. Suppose that X satisfies almost right horn descent above dimension m or almost left horn descent above dimension m. Then X is lower (2m-1)-Segal.

In particular, if X satisfies inner horn descent above dimension m then X is lower (2m-1)Segal.

Proof. We treat the case of almost right horns; the case of almost left horns is analogous. We fix the following notation:

- Let P be the class consisting of those non-degenerate covers $\mathcal{F} \models [n]$ such that
 - $[n-1] = \{0, \dots, n-1\} \in \mathcal{F}.$
 - for all $[n-1] \neq f \in \mathcal{F}$ we have $n-1, n \in f$.
 - for all $I \subset [n]$ with |I| = m there is an $f \in \mathcal{F}$ with $I \subseteq f$.
- For each n > 2m-1, put $U^n := \mathfrak{T}_{n,2m-1}^+ \uplus [n-1] \models [n]$ and $E_n := \{\mathfrak{T}_{n,2m-1}^+ \preceq U^n\}$

Observe that for each n > m the almost right horn Λ_{n-1}^n is terminal in $P_{[n]}$; it is straightforward to check that P is saturated. The class $E := \bigcup_{n > 2m-1} E_n$ is also saturated because the cover $[n-1]^*(\mathcal{T}_{n,2m-1}^+) \models [n-1]$ can be identified with $\mathcal{T}_{n-1,2m-1}^+ \models [n-1]$ (which is the trivial cover for n=2m). Observe, finally, that for all n > 2m-1, the cover $U^n \models [n]$ lies in P_n . We conclude

$$\overline{\Lambda_{\mathbf{a}+}^{>m}} = \overline{P} \supset \overline{\{U^n \mid n > 2m-1\}} = \overline{E} = \overline{\left\{\Im_{n,2m-1}^+ \mid n > 2m-1\right\}}$$

by repeated application of the refinement principle.

4 Classification via the Dold–Kan correspondence

We introduce the following terminology.

Definition 4.1. Let \mathcal{C} be an ∞ -category with finite limits and $m \geq 0$. An outer m-Kan complex in \mathcal{C} is a simplicial object X which satisfies descent with respect to all covers in $\Lambda^{\geq m} \cup \Lambda^{\geq m}_+$, see Definition 3.20. In other words, X satisfies both left and right horn descent above dimension m.

Let m be a positive integer. In this section we show that, if \mathcal{C} is an abelian category or a stable ∞ -category, then the classes of outer m-Kan complexes and of 2m-Segal objects in \mathcal{C} coincide, see Theorems 4.12 and 4.27. Our proofs rely on a characterisation of the outer m-Kan complexes expressed in terms of appropriate versions of the Dold-Kan correspondence, see Theorems 4.5 and 4.22.

4.1 Simplicial objects in abelian categories

4.1.1 The Dold-Kan correspondence

Let \mathcal{A} be an abelian category and X a simplicial object in \mathcal{A} . Recall that the Moore chain complex of X is the connective chain complex (X, ∂) where $\partial \colon X_n \to X_{n-1}$ is given by the formula

$$\partial \coloneqq \sum_{i=0}^{n} (-1)^i d_i.$$

The normalised Moore chain complex of X is the subcomplex (\overline{X}, ∂) of (X, ∂) given by

$$\overline{X}_n := \bigcap_{i=1}^n \ker d_i.$$

In particular $\partial = d_0$ on \overline{X}_n . The passage from a simplicial object to its normalised Moore chain complex yields a functor

$$C: \mathcal{A}_{\Delta} \longrightarrow \mathsf{Ch}_{\geq 0}(\mathcal{A})$$

from the category \mathcal{A}_{Δ} of simplicial objects in \mathcal{A} to the category $\mathsf{Ch}_{\geq 0}(\mathcal{A})$ of connective chain complexes in \mathcal{A} . The functor C admits a right adjoint

$$N \colon \operatorname{Ch}_{\geq 0}(\mathcal{A}) \longrightarrow \mathcal{A}_{\Delta}$$

which associates to a connective chain complex in \mathcal{A} its Dold-Kan nerve. The Dold-Kan correspondence [Dol58, Kan58] is the following fundamental equivalence of categories.

Theorem 4.2 (Dold-Kan correspondence). The functors

$$C: \mathcal{A}_{\Delta} \longrightarrow \mathsf{Ch}_{\geq 0}(\mathcal{A}): N$$

form a pair of adjoint equivalences of categories.

Remark 4.3. Let \mathcal{A} be an abelian category and X a simplicial object in \mathcal{A} . Up to natural isomorphism, the normalised Moore chain complex of X can be alternatively defined by setting

$$\overline{X}_n \coloneqq \bigcap_{i=0}^{n-1} \ker d_i$$

with the differential given by $d_n : \overline{X}_n \to \overline{X}_{n-1}$.

4.1.2 Outer m-Kan complexes in abelian categories

Definition 4.4. Let \mathcal{A} be an abelian category and m a natural number. A connective chain complex X in \mathcal{A} is (strictly) m-truncated if for each n > m the object X_m is a zero object of \mathcal{A} .

Outer m-Kan complexes in \mathcal{A} admit the following simple characterisation.

Theorem 4.5. Let A be an abelian category and m natural number. The Dold-Kan correspondence

$$C \colon \mathcal{A}_{\Delta} \stackrel{}{\longleftarrow} \mathsf{Ch}_{\geq 0}(\mathcal{A}) \colon N$$

restricts to an equivalence of categories between the full subcategory of A_{Δ} spanned by the outer m-Kan complexes and the full subcategory of $\mathsf{Ch}_{>0}(\mathcal{A})$ spanned by the m-truncated chain complexes.

In fact, Theorem 4.5 is an immediate consequence of the following general statement.

Proposition 4.6. Let A be an abelian category and X a simplicial object in A.

(1) For each $n \ge 1$ there are split short exact sequences

$$0 \to \overline{X}_n \to X_n \to X_{\Lambda_n^n} \to 0$$
 and $0 \to \overline{X}_n \to X_n \to X_{\Lambda_n^n} \to 0$,

where (\overline{X}, ∂) is the normalised Moore chain complex of X and $X_n \to X_{\Lambda_0^n}$ (resp. $X_n \to X_{\Lambda_n^n}$) is the Segal map.

- (2) For a non-empty subset $J \subseteq [n]$ we write $\overline{X}_J := \overline{X}_{|J|-1}$.
 - (a) There is a direct sum decomposition

$$X_{\Lambda_0^n} \cong \bigoplus_{0 \in J \subsetneq [n]} \overline{X}_J, \tag{4.7}$$

with the following property: For each $i \neq 0$, the action of the i-th face map on $X_n \cong \overline{X}_n \oplus X_{\Lambda_0}^n$ corresponds to the projection onto those direct summands \overline{X}_J such that J does not contain i.

(b) There is a direct sum decomposition

$$X_{\Lambda_n^n} \cong \bigoplus_{n \in J \subsetneq [n]} \overline{X}_J, \tag{4.8}$$

with the following property: For each $i \neq n$, the action of the i-th face map on $X_n \cong \overline{X}_n \oplus X_{\Lambda_n}^n$ corresponds to the projection onto those direct summands \overline{X}_J such that J does not contain i.

Remark 4.9. Let \mathcal{A} be an abelian category, X a simplicial object in \mathcal{A} and (\overline{X}, ∂) its normalised Moore chain complex. Consider the commutative diagram

$$\bigoplus_{\substack{[n] \twoheadrightarrow [k]}} \overline{X}_k \xleftarrow{\cong} X_n \xrightarrow{\cong} \bigoplus_{0 \in J \subseteq [n]} \overline{X}_J$$

$$\downarrow^{d_i} \qquad \qquad \downarrow^{d_i} \qquad \qquad \downarrow^{d_i}$$

$$\bigoplus_{\substack{[n-1] \twoheadrightarrow [k]}} \overline{X}_k \xleftarrow{\cong} X_{n-1} \xrightarrow{\cong} \bigoplus_{0 \in J \subseteq [n] \backslash \{i\}} \overline{X}_J$$

where the right hand square is given in Proposition 4.6 and the left hand square corresponds to the standard description of the Dold–Kan nerve, see for example Section III.2 in [GJ99]. Although there is a canonical bijection between surjective monotone maps $[n] \rightarrow [k]$ and subsets of [n] which contain 0 of cardinality k+1 given by

$$(f \colon [n] \twoheadrightarrow [k]) \mapsto \left\{ \min f^{-1}(i) \mid i \in [k] \right\},\,$$

the horizontal composites in the above diagram are complicated to describe; in particular, even after identifying the respective labelling sets via the above bijection these composites do not simply correspond to permuting summands. The advantage of the direct sum decomposition (4.7) (resp. (4.8)) is that all but the 0-th (resp. n-th) face maps correspond to canonical projections onto some direct summands. This is well suited for our purposes, since we are mostly interested on the relation between the simplicial object and various membrane objects defined by simplicial subsets of Λ_0^n (resp. Λ_n^n). Of course, the complexity of the simplicial object has not disappeared: the 0-th (resp. n-th) face map and all of the degeneracy maps have non-trivial descriptions with respect to this alternative direct sum decomposition.

Proof of Proposition 4.6. In view of Remark 4.3 it is enough to consider the case of 0-th horns. By definition, the Segal map $X_n \to X_{\Lambda_0^n}$ fits into a commutative diagram

$$0 \longrightarrow \overline{X}_{n} \longrightarrow X_{n} \xrightarrow{[d_{1} \cdots d_{n}]^{\top}} \bigoplus_{i=1}^{n} X_{n-1}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$0 \longrightarrow X_{\Lambda_{0}^{n}} \xrightarrow{[p_{1} \cdots p_{n}]^{\top}} \bigoplus_{i=1}^{n} X_{n-1}$$

with exact rows. Following the proof of Lemma I.3.4 in [GJ99], we define a map

$$f = f^{(1)} \colon X_{\Lambda_0^n} \longrightarrow X_n$$

recursively by setting $f^{(n)} := s_{n-1}p_n$ and

$$f^{(i)} = f^{(i+1)} - s_{i-1}d_i f^{(i+1)} + s_{i-1}p_i$$

for $1 \le i \le n$. It is straightforward to verify that $d_i f = p_i$ for $i \ne 0$ (using the simplicial identities and the fact that for i < j we have $d_i p_j = d_{j-1} p_i$). It follows that f is a right inverse to the Segal map $X_n \to X_{\Lambda_0^n}$.

The desired direct sum decomposition of $X_{\Lambda_0^n}$ follows by induction on n. Indeed, for n=1 there is a split short exact sequence

$$0 \longrightarrow \overline{X}_1 \longrightarrow X_1 \xrightarrow{\stackrel{g_0}{\xrightarrow{d_1}}} X_0 \longrightarrow 0$$

and $X_0 = X_{\Lambda_0^1}$; the claims are clear in this case. Let $n \ge 1$ and suppose that the desired direct sum decomposition has been established for all $1 \le k \le n$. It follows that

$$X_{\Lambda_0^{n+1}} = \lim_{0 \in J \subsetneq [n+1]} X_J \cong \lim_{0 \in J \subsetneq [n+1]} \bigoplus_{0 \in I \subseteq J} \overline{X}_I \cong \bigoplus_{0 \in K \subsetneq [n+1]} \overline{X}_K$$

since, by induction, the limit is taken over the projections onto the appropriate direct summands. In particular the maps $p_i \colon X_{\Lambda_0^{n+1}} \to X_n, i \neq 0$ can be taken as the projections onto the direct sum of those \overline{X}_J such that J does not contain i. This finishes the proof.

Remark 4.10. Following the general spirit of this article, we give a description of the splitting map $f: X_{\Lambda_n^n} \to X_n$ in terms of cubical diagrams.¹

Given i < j, we define two embeddings $z_i^j, u_i^j \colon [1]^{i-1} \to [1]^{j-1}$ via

$$z_i^j(w_1, \dots, w_{i-1}) := (w_1, \dots, w_{i-1}, 0, \dots, 0)$$

 $u_i^j(w_1, \dots, w_{i-1}) := (w_1, \dots, w_{i-1}, 0, 1, \dots, 1)$

Fix $n \ge 1$ and define a cube $q^n : [1]^{n-1} \to \Delta(n, n-1)$ by

$$v \mapsto q_v^n \colon \begin{cases} 0 \mapsto 0 \\ i \mapsto i - 1 + v_i, & \text{for } 1 \le i \le n - 1 \\ n \mapsto n - 1 \end{cases}$$

For every $1 \le i \le n$ we can then define a map $\iota_i \colon X_{n-1} \to X_n$ by totalising the action of the (i-1)-cube $q^n \circ z_i^n$; explicitly:

$$\iota_i := \sum_{w \in [1]^{i-1}} (-1)^{|w|} (q_{z_i^n(w)}^n)^*. \tag{4.11}$$

Then we have

$$f = \sum_{i=1}^{n} (-1)^{i-1} \iota_i p_i.$$

The essential relation $d_j f = p_j$ is explained by the following observations:

- For all i > j, composing the cube $q: [1]^{i-1} \to \Delta(n, n-1)$ with the coface map $d^j: [n-1] \to [n]$ yields a cube $q: [1]^{i-1} \to \Delta(n-1, n-1)$ with two identical faces along the j-th direction. Therefore $d_j \iota_i$ (and hence $d_j \iota_i p_i$) vanishes.
- For all i < j we have a commutative diagram

$$\begin{aligned} [1]^{j-1} & \xrightarrow{z_j^n} & [1]^{n-1} & \xrightarrow{q^n} & \Delta(n,n-1) & \xrightarrow{-\circ d^j} & \Delta(n-1,n-1) \\ u_i^j \uparrow & & & \\ [1]^{i-1} & \xrightarrow{z_i^{n-1}} & [1]^{n-2} & \xrightarrow{q^{n-1}} & \Delta(n-1,n-2) & \xrightarrow{d^j - 1} & \\ & & & \\ z_i^n & & & \\ \end{bmatrix} \\ [1]^{n-1} & \xrightarrow{q^n} & \Delta(n,n-1) & \xrightarrow{-\circ d^j} & \Delta(n-1,n-1) \end{aligned}$$

so that the relation $d_i p_j = d_{j-1} p_i$ implies that in the sum for $d_j f$ the summands of $d_j \iota_i p_i$ cancel with those summands of $d_j \iota_j p_j$ which are indexed by an element of the form $u_i^j(w) \in [1]^{j-1}$ for some $w \in [1]^{i-1}$.

The only summand in the sum for $d_j f$ which does not cancel is the one corresponding to $(1, \ldots, 1) \in [1]^{j-1}$; this summand is equal to $d_j s_{j-1} p_j = p_j$.

Proof of Theorem 4.5. Let \mathcal{A} be an abelian category and m a natural number. By definition, a simplicial object X in \mathcal{A} is an outer m-Kan complex if, for each n > m, the Segal maps

$$X_n \longrightarrow X_{\Lambda_n^n}$$
 and $X_n \longrightarrow X_{\Lambda_n^n}$

are isomorphisms in \mathcal{A} . By Proposition 4.6 the latter condition is equivalent to the statement that for each n > m the object \overline{X}_n is a zero object of \mathcal{A} , where (\overline{X}, ∂) is the normalised Moore chain complex of X. In other words, the simplicial object X in \mathcal{A} is an outer m-Kan complex if and only if the normalised Moore chain complex of X is m-truncated, which is what we needed to prove. \square

This description is related to the projector $\pi\colon X_n\to X_n$ onto the complement of $X_{\Lambda_0^n}$ defined in Section 2 of [Dyc17].

4.1.3 Classification of 2m-Segal objects in abelian categories

The goal of this section is to prove the following characterization of 2m-Segal objects in abelian categories.

Theorem 4.12. Let X be a simplicial object in an abelian category A and $m \ge 1$. The following are equivalent:

- (1) The simplicial object X is 2m-Segal.
- (2) The simplicial object X is an outer m-Kan complex.
- (3) The connective chain complex $\overline{X} := C(X)$ is m-truncated.

Proof. The implications $(3) \Longrightarrow (2) \Longrightarrow (1)$ are established in Theorem 4.5 and Theorem 3.29. It remains to prove the implication $(1) \Longrightarrow (3)$. Recall from Proposition 2.10 in [Pog17] that every 2m-Segal object is l-Segal for every $l \ge 2m$. In particular, if X is 2m-Segal then it is also 2k-Segal for all k > m.

Suppose that there exists k > m such that \overline{X}_k is non-zero. Let $I = \{0, 3, ..., 3k\}$; we view I as a subset of [3k]. The smallest even subset of [3k] that contains I has cardinality 2k + 1. Therefore I is not contained in the lower 2k-Segal cover $\mathcal{T}_{3k,2k}$. Now, the object \overline{X}_I is a direct summand of

$$X_{3k} \cong \bigoplus_{0 \in J \subseteq [3k]} \overline{X}_J.$$

From the explicit description of the face maps of $X \cong N(C(X))$ given in Proposition 4.6, we conclude immediately that \overline{X}_I is sent to zero by the Segal maps $X_{3k} \to X_{\mathcal{T}_{3k,2k}^-}$. A similar argument shows that I is not contained in the upper 2k-Segal cover $\mathcal{T}_{3k,2k}^+$ and therefore \overline{X}_I is sent to zero by the Segal map $X_{3k} \to X_{\mathcal{T}_{3k,2k}^+}$. In particular these Segal maps cannot be equivalences, contradicting the fact that X is 2k-Segal.

4.2 Simplicial objects in stable ∞ -categories

4.2.1 Lurie's ∞-categorical Dold-Kan correspondence

Let \mathcal{C} be a stable ∞ -category. To extend the classical Dold–Kan correspondence to the ∞ -categorical context, one first needs to adapt the notions of simplicial objects and connective chain complexes. The first notion is straightforward, as functors $\Delta^{op} \to \mathcal{C}$ (or more precisely $N(\Delta)^{op} \to \mathcal{C}$) already capture the correct notion; see Definition 6.1.2.2 in [Lur09]. In contrast, the naive notion of a connective chain complex in \mathcal{C} , namely a connective chain complex

$$\cdots \xrightarrow{d} X_n \xrightarrow{d} \cdots \xrightarrow{d} X_1 \xrightarrow{d} X_0 \to 0 \to \cdots$$

in the homotopy category $Ho(\mathcal{C})$, requires further refinement, see Section 1.2.2 in [Lur17]. This is due to the fact that the space of identifications $d^2 \cong 0$ need not be contractible.

Definition 4.13. Let \mathcal{C} be a stable ∞ -category. A *filtered object in* \mathcal{C} is a functor $\mathsf{N}(\mathbb{N}) \to \mathcal{C}$, where we view \mathbb{N} as a poset with respect to the usual order.

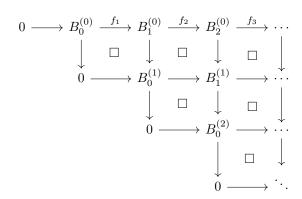
Remark 4.14. Let \mathcal{C} be a stable ∞ -category. Recall that the homotopy category $\mathsf{Ho}(\mathcal{C})$ has the structure of a triangulated category whence, in particular, $\mathsf{Ho}(\mathcal{C})$ is an additive category, see Section 1.1.2 in [Lur17]. A filtered object B in \mathcal{C} can be visualised as a diagram

$$B: B_0 \xrightarrow{f_1} B_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} B_n \to \cdots$$

Note that B induces a connective chain complex \overline{B} in $Ho(\mathcal{C})$ where $\overline{B_0} := B_0$ and

$$\overline{B}_n := \Sigma^{-n} \operatorname{cofib}(f_n)$$

for $n \ge 1$, see Remark 1.2.2.3 in [Lur17]. The data encoded by the filtered object $B = B^{(0)}$ can be visualised as a diagram



in which all marked squares are biCartesian.

The introduction of filtered objects in stable ∞ -categories is justified by the following ∞ -categorical version of the Dold–Kan correspondence, see Theorem 1.2.4.1 in [Lur17].

Theorem 4.15 (Lurie's ∞ -categorical Dold–Kan correspondence). Let \mathfrak{C} be a stable ∞ -category. There is a canonical equivalence

$$\operatorname{Fun}(\Delta^{\operatorname{op}}, \operatorname{\mathcal{C}}) \xrightarrow{\cong} \operatorname{Fun}(\mathbb{N}, \operatorname{\mathcal{C}})$$

of stable ∞ -categories.

For our purposes it is necessary to recall a particular description of the equivalence between $\operatorname{\mathsf{Fun}}(\Delta^{\operatorname{\mathsf{op}}},\mathbb{C})$ and $\operatorname{\mathsf{Fun}}(\mathbb{N},\mathbb{C})$ which is implicit in Lurie's proof of Theorem 4.15. Recall that, for a given integer $n\geq 0$, the full subcategory of Δ spanned by the finite ordinals $\{[m]\mid 0\leq m\leq n\}$ is denoted by $\Delta_{\leq n}$.

Notation 4.16. We denote by \mathcal{J}_n the subcategory of the overcategory $(\Delta_{\leq n})_{/[n]}$ spanned by the *injective* monotone maps $[m] \to [n]$. The composite of canonical functors

$$N(\mathcal{J}_n) \longrightarrow N(\Delta_{\leq n})_{/[n]} \longrightarrow N(\Delta_{\leq n}) \longrightarrow N(\Delta)$$

yields a functor $j \colon \mathsf{N}(\mathcal{J}_n) \to \mathsf{N}(\Delta)$. Note that there is a fully faithful functor $i \colon \mathcal{J}_n \hookrightarrow \mathcal{J}_{n+1}$ induced by the canonical inclusion $[n] \subset [n+1]$. In the sequel we identify \mathcal{J}_n with the poset of non-empty subsets of [n]. In particular, \mathcal{J}_n^{\dashv} can be identified with the poset of all subsets of [n] and is therefore isomorphic to the (n+1)-cube I^{n+1} .

Construction 4.17. Let \mathcal{C} be a stable ∞ -category and X a simplicial object in \mathcal{C} . We associate to X a filtered object B in \mathcal{C} as follows. Let $n \geq 0$ be an integer. We define B_n to be a colimit of the punctured (n+1)-cube $X|_{\mathcal{J}_n^{op}} \colon \mathsf{N}(\mathcal{J}_n)^{op} \to \mathcal{C}$ obtained from X by restriction along the composite of canonical functors

$$N(\mathcal{J}_n)^{op} \longrightarrow N(\Delta_{\leq n})^{op} \longrightarrow N(\Delta)^{op}.$$

The required maps $f_{n+1} \colon B_n \to B_{n+1}$ are induced by the canonical fully faithful functors $i \colon \mathcal{J}_n \hookrightarrow \mathcal{J}_{n+1}$. The fact that B is equivalent to the filtered object corresponding to X under the equivalence in Theorem 4.15 is a consequence of Lemma 1.2.4.17 in [Lur17] which states that the functor $j \colon \mathsf{N}(\mathcal{J}_n) \to \mathsf{N}(\Delta_{\leq n})$ is right cofinal.

Example 4.18. Let \mathcal{C} be a stable ∞ -category and X a simplicial object in \mathcal{C} . According to Construction 4.17, the terms B_0 , B_1 , and B_2 of the filtered object B in \mathcal{C} associated to X are determined by the biCartesian cubes

$$X_{\{0\}} \xrightarrow{\sim} B_0 \qquad X_{\{0,1\}} \xrightarrow{X_{\{0,1,2\}}} X_{\{1,2\}} \xrightarrow{X_{\{0,1,2\}}} X_{\{0,2\}} \xrightarrow{X_{\{0,2\}}} X_{\{2\}} \xrightarrow{X_{\{0\}}} X_{\{0\}} \xrightarrow{f_1} B_1 \qquad X_{\{0,1\}} \xrightarrow{X_{\{0,1\}}} X_{\{0\}} \xrightarrow{X_{\{0,1\}}} B_2$$

The required maps $f_{n+1}: B_n \to B_{n+1}$ are induced by the universal property of these diagrams in the obvious way. For example, the map $f_1: B_0 \to B_1$ is indicated above while the map $f_2: B_1 \to B_2$ is induced by the 2-cube

$$\begin{array}{ccc} X_{\{0,1\}} & \longrightarrow & X_{\{1\}} \\ \downarrow & & \downarrow \\ X_{\{0\}} & \longrightarrow & B_2 \end{array}$$

Remark 4.19. Let \mathcal{C} be a stable ∞ -category and X a simplicial object in \mathcal{C} . Replacing the canonical inclusions $[n] \subset [n+1]$, which are given by the co-face maps $d^n \colon [n] \hookrightarrow [n+1]$, by the co-face maps $d^0 \colon [n] \hookrightarrow [n+1]$ in Construction 4.17 gives an equivalent way to construct the filtered object B in \mathcal{C} associated to X by the equivalence in Theorem 4.15, see Remark 4.19 for comparison.

4.2.2 Outer m-Kan complexes in stable ∞ -categories

Definition 4.20. Let \mathcal{C} be a stable ∞ -category and m a natural number. A filtered object B in \mathcal{C} is (strictly) m-truncated if for each n > m the map $f_n \colon B_{n-1} \to B_n$ is an equivalence.

Remark 4.21. Let \mathcal{C} be a stable ∞ -category and m a natural number. A filtered object B in \mathcal{C} is m-truncated if and only if the associated connective chain complex \overline{B} in the homotopy category $\mathsf{Ho}(\mathcal{C})$ is m-truncated in the sense of Definition 4.4, see Remark 4.14.

Our aim is to prove the following theorem, which is an ∞ -categorical version of Theorem 4.5.

Theorem 4.22. Let C be a stable ∞ -category and m a natural number. Lurie's ∞ -categorical Dold-Kan correspondence

$$\operatorname{Fun}(\Delta^{\operatorname{op}}, \mathcal{C}) \longrightarrow \operatorname{Fun}(\mathbb{N}, \mathcal{C})$$

restricts to an equivalence of ∞ -categories between the full subcategory of $\operatorname{Fun}(\Delta^{\operatorname{op}}, \mathfrak{C})$ spanned by the outer m-Kan complexes and the full subcategory of $\operatorname{Fun}(\mathbb{N}, \mathfrak{C})$ spanned by the m-truncated filtered objects.

In fact, Theorem 4.22 is a consequence of the following general statement which is an analogue of Proposition 4.6 for stable ∞ -categories.

Proposition 4.23. Let C be a stable ∞ -category, X a simplicial object in C, and $B = B_X$ the filtered object in C associated to X under the equivalence in Theorem 4.15.

(1) For each $n \geq 1$ the Segal maps $X_n \to X_{\Lambda_0^n}$ and $X_n \to X_{\Lambda_n^n}$ fit into fibre sequences

$$\begin{array}{cccc} \overline{X}_n & \longrightarrow & X_n & & \overline{X}_n & \longrightarrow & X_n \\ \downarrow & \Box & \downarrow & & \text{and} & & \downarrow & \Box & \downarrow \\ 0 & \longrightarrow & X_{\Lambda_n^n} & & & 0 & \longrightarrow & X_{\Lambda_n^n} \end{array}$$

in \mathbb{C} , where $\overline{X}_n := \Sigma^{-n}(\mathsf{cofib}(B_{n-1} \xrightarrow{f_n} B_n))$ is the n-th term of the connective chain complex \overline{X} in $\mathsf{Ho}(\mathbb{C})$ induced by B.

- (2) For a non-empty subset $J \subseteq [n]$ we write $\overline{X}_J := \overline{X}_{|J|-1}$.
 - (a) There is a direct sum decomposition

$$X_{\Lambda_0^n} \simeq \bigoplus_{0 \in J \subsetneq [n]} \overline{X}_J, \tag{4.24}$$

with the following property: For each $i \neq 0$, the action of the i-th face map on $X_n \simeq \overline{X}_n \oplus X_{\Lambda_0}^n$ corresponds to the projection onto those direct summands \overline{X}_J such that J does not contain i.

(b) There is a direct sum decomposition

$$X_{\Lambda_n^n} \simeq \bigoplus_{n \in J \subsetneq [n]} \overline{X}_J, \tag{4.25}$$

with the following property: For each $i \neq n$, the action of the i-th face map on $X_n \simeq \overline{X}_n \oplus X_{\Lambda_n}^n$ corresponds to the projection onto those direct summands \overline{X}_J such that J does not contain i.

Before giving a proof of Proposition 4.23 we use it to prove Theorem 4.22.

Proof of Theorem 4.22. Let \mathcal{C} be a stable ∞ -category and m a natural number. A simplicial object X of \mathcal{C} is an outer m-Kan complex if, by definition, for each n > m the Segal maps

$$X_n \longrightarrow X_{\Lambda_n^n}$$
 and $X_n \longrightarrow X_{\Lambda_n^n}$

are equivalences in \mathcal{C} . By Proposition 4.23 the latter condition is equivalent to the statement that for each n > m the object $\overline{X}_n = \Sigma^{-n} \operatorname{cofib}(B_{n-1} \to B_n)$ is a zero object in \mathcal{C} , where B is the filtered object of \mathcal{C} associated to X under the equivalence in Theorem 4.15. In other words, the simplicial object X is an outer m-Kan complex if and only if for each n > m the map $B_{n-1} \to B_n$ is an equivalence in \mathcal{C} if and only if the filtered object B of \mathcal{C} is m-truncated, see Remark 4.21. \square

We now give a proof of statement (1) in Proposition 4.23.

Proof of statement (1) in Proposition 4.23. Let X be a simplicial object in \mathcal{C} and B the filtered object in \mathcal{C} corresponding to X under the equivalence in Theorem 4.15. Fix an integer $n \geq 1$. Consider the following auxiliary diagrams.

- Let \widetilde{X} be an (n+1)-cube obtained from $X|_{\mathfrak{J}_n^{\mathsf{op}}}$ by left Kan extension to $(\mathfrak{J}_n^{\triangleleft})^{\mathsf{op}}$. Thus \widetilde{X} is a coCartesian (n+1)-cube in \mathfrak{C} and $\widetilde{X}_{\emptyset} \simeq B_n$.
- Let $\widetilde{X}|_{0\in J}$ be the *n*-cube obtained from \widetilde{X} by restricting to the subsets of [n] which contain 0. Note that $\widetilde{X}|_{0\in J}=X|_{0\in J}$.
- Similarly, let $\widetilde{X}|_{0\notin J}$ be the *n*-cube obtained from \widetilde{X} by restricting to the subsets of [n] which do not contain 0.

The *n*-cubes $\widetilde{X}|_{0 \in J}$ and $\widetilde{X}|_{0 \notin J}$ are opposite facets of the (n+1)-cube \widetilde{X} . Since \widetilde{X} is coCartesian, Corollary A.32 implies the existence of an equivalence

$$\mathsf{tot\text{-}fib}(\widetilde{X}|_{0\in J})\simeq \Sigma^{-n}\,\mathsf{tot\text{-}cofib}(\widetilde{X}|_{0\not\in J}).$$

Moreover, by definition

$$X_{\Lambda_0^n} = \lim_{0 \in J \subsetneq [n]} X_J.$$

Therefore

$$\mathsf{tot\text{-}fib}(\widetilde{X}|_{0\in J}) = \mathsf{fib}(X_n \to X_{\Lambda_0^n}).$$

Furthermore, in view of Remark 4.19 we can identify the map $f_n: B_{n-1} \to B_n$ with the canonical map

$$\underset{0 \notin J \neq \emptyset}{\mathsf{colim}} \, \widetilde{X}_J \to \widetilde{X}_{\emptyset}.$$

In particular

$$\mathsf{tot\text{-}cofib}(\widetilde{X}|_{0\not\in J}) = \mathsf{cofib}(B_{n-1} \xrightarrow{f_n} B_n).$$

Piecing together the above equivalences, we conclude that there is an equivalence

$$\overline{X}_n \coloneqq \Sigma^{-n} \operatorname{cofib}(B_{n-1} \xrightarrow{f_n} B_n) \simeq \operatorname{fib}(X_n \to X_{\Lambda_0^n}),$$

which is what we needed to show.

We delay the proof of (2) in Proposition 4.23 to establish some notation.

Notation 4.26. Let \mathcal{C} be a stable ∞ -category and X a simplicial object in \mathcal{C} . Let $[n] \in \Delta$. We introduce the following auxiliary posets.

- We denote by Ω_n^0 the poset of all proper subsets $0 \in I \subsetneq [n]$ that contain 0.
- Let Q_n^{\triangleright} be the right cone over Q_n^0 ; we denote the cone point by Λ_0^n so that we can write $I < \Lambda_0^n$ for all $I \in Q_n^0$.
- We denote by $\overline{\mathbb{Q}_n^{\triangleright}}$ the poset obtained from $\mathbb{Q}_n^{\triangleright}$ by adjoining new elements $\overline{I'}$ for every element $I' \in \mathbb{Q}_n^0$; we declare $\overline{I'} < I'$ (hence also $\overline{I'} < \Lambda_0^n$ and $\overline{I'} < I$ for every $I' \subseteq I \in \mathbb{Q}_n^0$).
- Let $x \in \mathcal{Q}_n^{\triangleright}$ (this includes the case $x = \Lambda_0^n$ and x = I for $0 \in I \subsetneq [n]$). We denote by $\mathcal{P}_n(x) \subset \overline{\mathcal{Q}_n^{\triangleright}}$ the subposet defined by

$$\mathcal{P}_n(x) := \{x\} \cup \{\overline{I'} \mid I' < x\}.$$

Note that $\mathcal{P}_n(x)$ is a cone with vertex x on the discrete (po)set $\{\overline{I'} \mid 0 \in I' \subsetneq [n]\}$ (for $x = \Lambda_0^n$) or $\{\overline{I'} \mid 0 \in I' \subseteq I\}$ (for $0 \in x = I \subsetneq [n]$).

Proof of statement (2) in Proposition 4.23. Let X be a simplicial object in a stable ∞ -category \mathcal{C} . We argue by induction on n. In the base case n=0 there is nothing to show. For the rest of the proof, fix a value for $n \geq 1$ and assume the result is proven for all smaller values.

The simplicial object X induces a diagram $X: (\mathfrak{Q}_n)^{\mathsf{op}} \to \mathfrak{C}$ via the obvious functor $\mathfrak{Q}_n^0 \to \Delta$. By right Kan extension we extend this to a diagram $\widetilde{X}: (\mathfrak{Q}_n^{\mathsf{p}})^{\mathsf{op}} \to \mathfrak{C}$ so that

(i) the diagram $\widetilde{X}: (\mathfrak{Q}_n^{\triangleright})^{\mathsf{op}} \to \mathfrak{C}$ is a limit cone;

the value of \widetilde{X} at the cone point of $\Omega_n^{\triangleright}$ is by definition the membrane space $X_{\Lambda_0^n}$. We can extend \widetilde{X} further to a diagram $\widetilde{X} : (\overline{\Omega_n^{\triangleright}})^{\sf op} \to \mathfrak{C}$ by putting $\widetilde{X}(\overline{I}) := \overline{X}_I$ for each $0 \in I \subsetneq [n]$ and by declaring the map

$$X_I \simeq igoplus_{0 \in I' \subset I} \overline{X}_{I'} \longrightarrow \overline{X}_I$$

corresponding to $\overline{I} < I$ to be the projection onto the "top summand" of the direct sum decomposition which exists by induction. By construction,

(ii) for each $0 \in I \subsetneq [n]$ the restriction of \widetilde{X} to the cone $\mathcal{P}_n(I)$ is a product cone in \mathcal{C} .

By the pointwise formula (and using the fact that $\mathcal{Q}_n^{\triangleright}$ is cofinal in $\overline{\mathcal{Q}_n^{\triangleright}}$), (i) and (ii) imply that

- (iii) $\widetilde{X}: (\overline{\mathbb{Q}_n^{\triangleright}})^{\mathsf{op}} \to \mathcal{C}$ is a right Kan extension of its restriction to the discrete poset $\{\overline{I} \mid 0 \in I \subsetneq [n]\}$. which implies that
- (iv) the restriction of \widetilde{X} to $\mathcal{P}_n(\Lambda_0^n)$ is a product cone in \mathcal{C} .

We conclude that $\widetilde{X}|_{\mathcal{P}_n(\Lambda_0^n)}$ exhibits a direct sum decomposition

$$X_{\Lambda_0^n} \stackrel{\cong}{\longrightarrow} \bigoplus_{0 \in I \subsetneq [n]} \overline{X}_I.$$

Furthermore, we can identify the structure maps $p_i \colon X_{\Lambda_0^n} \to X_{n-1}$ (which are induced by the relation $([n] \setminus \{i\}) < \Lambda_0^n$ in $\mathcal{Q}_n^{\triangleright}$) with the projection onto those summands \overline{X}_I where $i \notin I$. Since products agree with homotopy products (see Example 1.2.13.1 in [Lur09]), statements (ii) and (iv) remain true in the homotopy category. We have the implication (iv)&(ii) \iff (iii) \implies (i) also in the homotopy category; hence we conclude that $\widetilde{X}|_{\mathcal{Q}_n^{\triangleright}}$ is still a limit cone in the homotopy category \mathcal{C} .

We define a map $f: X_{\Lambda_0^n} \to X_n$ using the formulas from the proof of Proposition 4.6. The same calculation (which uses only the simplicial identities and the defining properties of the projections $p_i: X_{\Lambda_0^n} \to X_{n-1}$) shows that for each $i = 1, \ldots, n$, the composition

$$X_{\Lambda_0^n} \xrightarrow{f} X_n \longrightarrow X_{\Lambda_0^n} \xrightarrow{p_i} X_{n-1}$$

is equal in the homotopy category to the structure map $p_i: X_{\Lambda_0^n} \to X_{n-1}$ of the cone $\widetilde{X}|_{\mathbb{Q}_n^{\triangleright}}$. Since we have established this cone to be a limit cone in the homotopy category, it follows that the composition

$$X_{\Lambda_0^n} \xrightarrow{f} X_n \longrightarrow X_{\Lambda_0^n}$$

is equal to the identity in Ho C. We conclude that the fibre sequences of statement (1) in Proposition 4.23 is split and exhibits X_n as the direct sum of \overline{X}_n and $X_{\Lambda_0^n}$. The result follows.

4.2.3 Classification of 2m-Segal objects in stable ∞ -categories

We are ready to establish the following characterisation of 2m-Segal objects in stable ∞ -categories, in analogy to Theorem 4.12 which deals with simplicial objects in abelian categories.

Theorem 4.27. Let X be a simplicial object in a stable ∞ -category \mathfrak{C} and $m \geq 1$. The following statements are equivalent:

- (1) The simplicial object X is 2m-Segal.
- (2) The simplicial object X is an outer m-Kan complex.
- (3) Thee filtered object $B = B_X$ in $\mathfrak C$ associated to X under the equivalence in Theorem 4.15 is m-truncated.

Proof. The implications $(3) \Longrightarrow (2) \Longrightarrow (1)$ follow from Theorem 4.22 and Theorem 3.29. In view of the explicit description of the simplicial object X given in statement (2) in Proposition 4.23, the implication $(1) \Longrightarrow (3)$ can be shown to hold exactly as in Theorem 4.12. Indeed, the proof in Theorem 4.12 does not use the 0-th face map of the simplicial object.

A n-cubes in stable ∞ -categories

The idea of studying (co)Cartesian cubes and the interplay between total and iterated fibres goes back at least to Goodwillie [Goo92]. A systematic study of such cubes in the stable context was carried out by Beckert and Groth in [BG18] in the related framework of stable derivators.

A.1 coCartesian *n*-cubes in stable ∞ -categories

Let $n \ge 0$ be an integer. The *n*-cube is the poset

$$I^n := \underbrace{[1] \times \dots \times [1]}_{n \text{ times}}.$$
 (A.1)

For $v \in I^n$ we define $|v| := v_1 + \cdots + v_n$. In particular I^0 consists of a single vertex \emptyset with $|\emptyset| = 0$.

Definition A.2. Let \mathcal{C} be an ∞ -category and $n \geq 0$ an integer. An n-cube in \mathcal{C} is an object of the ∞ -category $\mathsf{Fun}(I^n,\mathcal{C})$, that is a functor $X \colon \mathsf{N}(I^n) \to \mathcal{C}$.

Remark A.3. Let $n \geq 0$ be an integer and J a finite set of cardinality n. Each bijection $j \colon \{1, \ldots, n\} \to J$ yields an isomorphism between I^n and the poset 2^J of subsets of J given by associating to $v \in I^n$ the subset j_v of J whose characteristic function is v. In specific contexts it is often more convenient to consider J-cubes, that is functors $\mathsf{N}(2^J) \to \mathfrak{C}$, instead of n-cubes in the sense of Definition A.2.

We are interested in the following exactness conditions on n-cubes, see for example Definition 6.1.1.2 in [Lur17].

Definition A.4. Let \mathcal{C} be an ∞ -category and $n \geq 0$ an integer.

(1) An n-cube X in $\mathfrak C$ is Cartesian if the canonical map

$$X_{0\cdots 0} \longrightarrow \lim_{0 < |v|} X_v \tag{A.5}$$

is an equivalence in \mathcal{C} .

(2) An *n*-cube X in \mathcal{C} is *coCartesian* if the canonical map

$$\operatorname{colim}_{|v| < n} X_v \longrightarrow X_{1 \dots 1} \tag{A.6}$$

is an equivalence in \mathcal{C} .

(3) An n-cube X in \mathcal{C} is biCartesian if it is both Cartesian and coCartesian.

We are chiefly interested in the properties of n-cubes in $stable \infty$ -categories. The following result, Proposition 1.2.4.13 in [Lur17], justifies our focus on coCartesian n-cubes in what follows.

Proposition A.7. Let C be a stable ∞ -category and $n \ge 0$ an integer. An n-cube X in C is Cartesian if and only if it is coCartesian if and only if it is biCartesian.

Let us illustrate Proposition A.7 in low dimensions.

Example A.8. Let \mathcal{C} be a stable ∞ -category, $n \geq 0$ an integer, and X an n-cube in \mathcal{C} .

- (1) If n = 0, then X can be identified with an object X_{\emptyset} of \mathfrak{C} . In this case X is coCartesian if and only if X_{\emptyset} a zero object of \mathfrak{C} .
- (2) If n = 1, then X can be identified with a map $X_0 \to X_1$ in \mathcal{C} . In this case X is coCartesian if and only if the map $X_0 \to X_1$ is an equivalence in \mathcal{C} .
- (3) If n=2, then X classifies a (coherent) commutative square

$$\begin{array}{ccc}
X_{00} & \longrightarrow & X_{01} \\
\downarrow & & \downarrow \\
X_{10} & \longrightarrow & X_{11}
\end{array}$$

in \mathcal{C} . In this case X is coCartesian if and only if the above diagram is a biCartesian square.

Notation A.9. Let \mathcal{C} be a stable ∞ -category and suppose given a 2-cube in \mathcal{C} of the form

$$\begin{array}{ccc} X_{00} & \xrightarrow{f} & X_{01} \\ \downarrow & & \downarrow \\ X_{10} & \xrightarrow{g} & X_{11} \end{array}$$

Functoriality of the cofibre induces a map $\operatorname{cofib}(f) \to \operatorname{cofib}(g)$ which can be subsequently identified with a 1-cube in $\mathbb C$. More generally, the isomorphism $I^{n+1} \cong [1] \times I^n$ allows us to identify an (n+1)-cube X in $\mathbb C$ with a morphism $X|_{\{0\}\times I^n} \to X|_{\{1\}\times I^n}$ in the stable ∞ -category $\operatorname{Fun}(I^n,\mathbb C)$. In particular X has an associated cofibre

$$\mathsf{cofib}(X) := \mathsf{cofib}\left(X|_{\{0\}\times I^n} \to X|_{\{1\}\times I^n}\right) \tag{A.10}$$

which itself is an object of $\operatorname{Fun}(I^n, \mathcal{C})$, that is an n-cube in \mathcal{C} .

The following statement is a special case of Lemma 1.2.4.15 in [Lur17]. It provides an inductive characterisation of coCartesian n-cubes in stable ∞ -categories.

Proposition A.11. Let \mathcal{C} be a stable ∞ -category and $n \geq 0$ an integer. Let X be an (n+1)-cube in \mathcal{C} which we identify with a morphism $X|_{\{0\}\times I^n} \to X|_{\{1\}\times I^n}$ in the stable ∞ -category $\operatorname{Fun}(I^n,\mathcal{C})$. Then, X is a coCartesian (n+1)-cube if and only if its cofibre $\operatorname{cofib}(X)$ is a coCartesian n-cube.

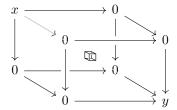
Proof. Since stable ∞ -categories admit all finite colimits (see Proposition 1.1.3.4 in [Lur17]), the claim follows by applying Lemma 1.2.4.15 in [Lur17] in the case $K = I^n$.

We illustrate Proposition A.11 with a simple but important example.

Example A.12. Let \mathcal{C} be a stable ∞ -category and x an object of \mathcal{C} . By definition, the suspension of x is characterised by the existence of a coCartesian square

$$\begin{array}{ccc}
x & \longrightarrow & 0 \\
\downarrow & \Box & \downarrow \\
0 & \longrightarrow & \Sigma(x)
\end{array}$$

Consider now a coCartesian 3-cube in $\mathcal C$ of the form



According to Proposition A.11, taking point-wise cofibres of the above 3-cube yields a coCartesian square

$$\begin{array}{ccc} \Sigma(x) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & y \end{array}$$

We conclude that there is an equivalence $y \simeq \Sigma^2(x)$ in \mathcal{C} . More generally, if $n \geq 1$ and X is a coCartesian (n+1)-cube in \mathcal{C} such that for each $v \in I^{n+1}$ with 0 < |v| < n+1 the object X_v is a zero object of \mathcal{C} , then there is an equivalence $X_{1\cdots 1} \simeq \Sigma^n(X_{0\cdots 0})$ in \mathcal{C} .

As an immediate consequence of Proposition A.11 we obtain an inductive criterion to verify whether an n-cube in a stable ∞ -category is coCartesian.

Corollary A.13. Let C be a stable ∞ -category and $n \geq 1$ an integer. An n-cube X in C is coCartesian if and only if

$$\operatorname{cofib}^{n}(X) := \underbrace{\operatorname{cofib}(\operatorname{cofib}(\cdots \operatorname{cofib}(X)))}_{n \ times} \tag{A.14}$$

is a zero object of C.

Proof. The claim follows by iterated application of Proposition A.11 since a 0-cube in \mathbb{C} , that is an object of \mathbb{C} , is coCartesian if and only if it is a zero object of \mathbb{C} .

Remark A.15. Let \mathcal{C} be a stable ∞ -category, $n \geq 1$ an integer, and X an n-cube in \mathcal{C} . The object $\mathsf{cofib}^n(X)$ appearing in Corollary A.13 is sometimes called the 'iterated cofibre of X', see for example [BG18].

We illustrate Corollary A.13 with a simple observation.

Corollary A.16. Let C be a stable ∞ -category and $n \ge 1$ an integer. Let X be an (n+1)-cube in C. If two out of the three cubical diagrams X, $X|_{\{0\}\times I^n}$, and $X|_{\{1\}\times I^n}$ are coCartesian, then so is the third.

Proof. The claim follows by applying Corollary A.13 to the cofibre sequence

$$\operatorname{cofib}^n(X|_{\{0\}\times I^n}) \longrightarrow \operatorname{cofib}^n(X|_{\{1\}\times I^n}) \longrightarrow \operatorname{cofib}^{n+1}(X),$$

of objects in \mathcal{C} , keeping in mind the elementary fact that if two out of the three objects in a cofibre sequence in a stable ∞ -category are zero objects, then so is the third.

Corollary A.17. Let \mathcal{C} be a stable ∞ -category and $n \geq 1$ an integer. The subcategory $\mathsf{Fun}^\mathsf{ex}(I^n,\mathcal{C})$ of the stable ∞ -category $\mathsf{Fun}(I^n,\mathcal{C})$ spanned by the coCartesian n-cubes in \mathcal{C} is closed under fibres and cofibres. In particular $\mathsf{Fun}^\mathsf{ex}(I^n,\mathcal{C})$ is a stable ∞ -category.

Proof. Recall that an n-cube in \mathcal{C} is coCartesian if and only if it is biCartesian, see Proposition A.7. Let $X \colon [1] \times I^n \to \mathcal{C}$ be an (n+1)-cube which we identify with a morphism in $\mathsf{Fun}(I^n,\mathcal{C})$. By Corollary A.16, if the n-cubes $X|_{\{0\}\times I^n}$ and $X|_{\{1\}\times I^n}$ are coCartesian, then X is also coCartesian. Finally, Proposition A.11 and its dual show that both the cofibre (resp. the fibre) of X, taken in the stable ∞ -category $\mathsf{Fun}(I^n,\mathcal{C})$, is coCartesian (resp. Cartesian). The claim follows. \square

As a further application of Proposition A.11 we sketch a proof of the 'pasting lemma' for coCartesian n-cubes in stable ∞ -categories.

Corollary A.18. Let \mathcal{C} be a stable ∞ -category and $n \geq 1$ an integer. Let $X \colon I^n \times [2] \to \mathcal{C}$ be a diagram in \mathcal{C} . Suppose that the (n+1)-cube $X|_{I^n \times \{0,1\}}$ is coCartesian. Then, the (n+1)-cube $X|_{I^n \times \{1,2\}}$ is coCartesian if and only if the (n+1)-cube $X|_{I^n \times \{0,2\}}$ is coCartesian.

Proof. The case n=1 is an ∞ -categorical version of the classical 'pasting lemma' for coCartesian squares and is proven in Lemma 4.4.2.1 in [Lur09]. The general case can be reduced to the case n=1 by iterated application of Proposition A.11. We leave the details to the reader.

Example A.19. Let \mathcal{C} be a stable ∞ -category. A diagram $X: I^2 \times [2] \to \mathcal{C}$ can be visualised as two 3-cubes

$$X_{000} \xrightarrow{\hspace{1cm}} X_{001} \xrightarrow{\hspace{1cm}} X_{002}$$

$$\downarrow X_{010} \xrightarrow{\hspace{1cm}} X_{011} \xrightarrow{\hspace{1cm}} X_{011} \xrightarrow{\hspace{1cm}} X_{012}$$

$$\downarrow X_{100} \xrightarrow{\hspace{1cm}} X_{101} \xrightarrow{\hspace{1cm}} X_{102} \xrightarrow{\hspace{1cm}} X_{112}$$

$$\downarrow X_{110} \xrightarrow{\hspace{1cm}} X_{111} \xrightarrow{\hspace{1cm}} X_{112}$$

glued at a common 2-cube. Provided that the leftmost 3-cube is coCartesian, Corollary A.18 states that the rightmost 3-cube is coCartesian if and only if the composite 3-cube

$$\begin{array}{c} X_{000} \xrightarrow{\longrightarrow} X_{002} \\ \downarrow & X_{010} \xrightarrow{\downarrow} X_{012} \\ X_{100} \xrightarrow{\searrow} \downarrow & X_{102} \downarrow \\ & X_{110} \xrightarrow{\longrightarrow} X_{112} \end{array}$$

is coCartesian.

A.2 The total cofibre of an *n*-cube in a stable ∞ -category

Recall the inductive criterion to verify whether an n-cube in a stable ∞ -category is coCartesian provided by Corollary A.13. In view of Definition A.4 and the aforementioned criterion, the following definition is rather natural.

Definition A.20. Let \mathcal{C} be a stable ∞ -category, $n \geq 0$ an integer, and X an n-cube in \mathcal{C} .

(1) The total cofibre of X, denoted by $\mathsf{tot\text{-}cofib}(X)$, is defined as the cofibre of the canonical map $\mathsf{colim}_{|v| < n} \, X_v \to X_{1\cdots 1}$.

(2) Dually, the total fibre of X, denoted by $\mathsf{tot}\text{-fib}(X)$, is defined as the fibre of the canonical map $X_{0\cdots 0} \to \mathsf{lim}_{0<|v|} X_v$.

Note that if n=0, then there are equivalences tot-cofib $(X)\simeq X_{\emptyset}$ and tot-fib $(X)\simeq X_{\emptyset}$.

Remark A.21. The total cofibre of an *n*-cube is investigated in [BG18] in the related framework of (stable) derivators, albeit with a slightly different focus.

The total cofibre of an n-cube in a stable ∞ -category behaves much like the cofibre of a morphism. We begin to construct this analogy with an elementary observation.

Lemma A.22. Let \mathbb{C} be a stable ∞ -category and $n \geq 0$ an integer. An n-cube X in \mathbb{C} is coCartesian if and only if tot-cofib(X) is a zero object of \mathbb{C} .

Proof. The claim follows immediately from Definition A.4 and the definition of the total cofibre of X, keeping in mind that a morphism in a stable ∞ -category is an equivalence if and only if its cofibre is a zero object of \mathcal{C} .

Let \mathcal{C} be a stable ∞ -category. By definition, the cofibre of a morphism $f \colon x \to y$ in \mathcal{C} is characterised by the existence of a coCartesian square of the form

$$\begin{array}{ccc} x & \stackrel{f}{\longrightarrow} y \\ \downarrow & \Box & \downarrow \\ 0 & \longrightarrow \mathsf{cofib}(f) \end{array}$$

The total cofibre of an n-cube in a stable ∞ -category is characterised by an analogous universal property. To prove this we need the following auxiliary lemma which is a special case of Corollary 4.2.3.10 in [Lur09].

Lemma A.23. Let \mathcal{P} be a poset and $\{\mathcal{P}_I \subset \mathcal{P} \mid I \in \mathcal{J}\}$ a collection of subposets of \mathcal{P} such that $\mathcal{P} = \bigcup_{I \in \mathcal{J}} \mathcal{P}_I$. We view \mathcal{J} as a poset with respect to the partial order induced by \mathcal{P} , that is $I \leq J$ if $\mathcal{P}_I \subseteq \mathcal{P}_J$. Suppose that for each finite chain $\sigma = \{\sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_m\}$ in \mathcal{P} the poset

$$\mathcal{J}_{\sigma} := \{ I \in \mathcal{J} \mid \sigma \subseteq \mathcal{P}_I \}$$

has a contractible nerve. Let C be an ∞ -category which admits colimits of shape P_I for each $I \in \mathcal{J}$. Then, for each functor $p \colon \mathsf{N}(\mathcal{P}) \to \mathcal{C}$, colimits of p can be identified with colimits of a diagram $q \colon \mathsf{N}(\mathcal{J}) \to \mathcal{C}$ characterised by the following properties:

- (1) The object q_I of \mathbb{C} is a colimit of the diagram $p|_{\mathbb{P}_I} \colon \mathbb{P}_I \to \mathbb{C}$.
- (2) For each $I, J \in \mathcal{J}$ such that $I \leq J$ the map $q_I \to q_J$ is induced by the inclusion $\mathfrak{P}_I \subseteq \mathfrak{P}_J$ via the universal property of q_I .

Proof. The claim follows by applying Corollary 4.2.3.10 in [Lur09] in the case $K := \mathsf{N}(\mathcal{P})$ to the functor $F : \mathcal{J} \to 2^{\mathcal{J}}$ given by $I \mapsto \mathcal{P}_I$, see also Remark 4.2.3.9 in [Lur09].

Proposition A.24. Let \mathcal{C} be a stable ∞ -category, n a positive integer, and X an n-cube in \mathcal{C} . Then, there exists an (n+1)-cube $\widetilde{X}: [1] \times I^n \to \mathcal{C}$ with the following properties:

- (1) The n-cube $\widetilde{X}|_{\{0\}\times I^n}$ agrees with X.
- (2) For each $v \in \{1\} \times I^n$ with |v| < n+1 the object \widetilde{X}_v is a zero object of \mathfrak{C} .
- (3) The (n+1)-cube \widetilde{X} is coCartesian.

Moreover, for every (n+1)-cube Y in $\mathbb C$ satisfying properties (1)-(3) there is an equivalence $Y_{1,1\cdots 1} \simeq \mathsf{tot}\text{-}\mathsf{cofib}(X)$ in $\mathbb C$.

Proof. The existence of an (n+1)-cube \widetilde{X} in \mathcal{C} satisfying properties (1)-(3) above is immediate from the existence of a zero object and of finite colimits in \mathcal{C} (taking into account the obvious fact that there are no morphisms in the poset I^{n+1} from objects in $\{1\} \times I^n$ to objects in $\{0\} \times I^n$).

We prove the second claim. Let Y be an (n+1)-cube in \mathbb{C} satisfying properties (1)-(3). Observe that, since the (n+1)-cube Y is coCartesian, the object $Y_{1,1...1}$ is a colimit of the diagram $Y|_{|v|< n+1}$. We shall use Lemma A.23 to exhibit $Y_{1,1...1}$ as the total cofibre of X.

Let \mathcal{J} be the poset of proper subsets of [1] and $\mathcal{P} := I^{n+1} \setminus \{(1, \dots, 1)\}$. We define an order preserving map from \mathcal{J} to the poset of subsets of \mathcal{P} by

$$\begin{split} \mathcal{P}_{\emptyset} &:= \left\{ v \in \left\{ 0 \right\} \times I^n \mid |v| < n \right\}, \\ \mathcal{P}_{\left\{ 0 \right\}} &:= \left\{ 0 \right\} \times I^n, \text{ and} \\ \mathcal{P}_{\left\{ 1 \right\}} &:= \mathcal{P}_{\emptyset} \cup \left\{ v \in \left\{ 1 \right\} \times I^n \mid |v| < n+1 \right\}. \end{split}$$

and note that $\mathcal{P} = \mathcal{P}_{\emptyset} \cup \mathcal{P}_{\{0\}} \cup \mathcal{P}_{\{1\}}$. We claim that the collection $\{\mathcal{P}_{\emptyset}, \mathcal{P}_{\{0\}}, \mathcal{P}_{\{1\}}\}$ satisfies the hypothesis of Lemma A.23. Indeed, the only subposet of \mathcal{J} whose nerve is not contractible is $\mathcal{J}' \coloneqq \{\{0\}, \{1\}\}$. But it is easy to verify that if a chain σ in \mathcal{P} is contained both in $\mathcal{P}_{\{0\}}$ and $\mathcal{P}_{\{1\}}$, then σ must be contained in \mathcal{P}_{\emptyset} . We conclude that if $\mathcal{J}' \subseteq \mathcal{J}_{\sigma}$, then $\mathcal{J}_{\sigma} = \mathcal{J}$ in this case.

The conclusion of Lemma A.23 implies that $Y_{1,1...1}$ is a colimit of the diagram

$$\begin{array}{ccc} \operatornamewithlimits{colim}_{v\in\mathcal{P}_\emptyset} Y_v & \longrightarrow & \operatornamewithlimits{colim}_{v\in\mathcal{P}_{\{0\}}} Y_v \\ & & \downarrow \\ \operatornamewithlimits{colim}_{v\in\mathcal{P}_{\{1\}}} Y_v \end{array}$$

where the maps are induced by the inclusions $\mathcal{P}_{\emptyset} \subset \mathcal{P}_{\{0\}}$ and $\mathcal{P}_{\emptyset} \subset \mathcal{P}_{\{0\}}$ via the universal property of $\operatorname{colim}_{v \in \mathcal{P}_{\emptyset}} Y_v$. Moreover, by construction $\operatorname{colim}_{v \in \mathcal{P}_{\emptyset}} Y_v = \operatorname{colim}_{|v| < n} X_v$ while

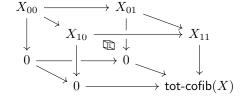
$$\operatorname*{colim}_{v\in\mathcal{P}_{\{0\}}}Y_v=Y_{(0,1\cdots 1)}=X_{1\cdots 1}$$

given that $(0,1,\ldots,1)$ is the maximal element in $\mathcal{P}_{\{0\}}$. Finally, utilising the fact that Y satisfies property (1) it is straightforward to verify that the colimit of the diagram $Y|_{\mathcal{P}_{\{1\}}}$ is a zero object in \mathcal{C} . In other words, $Y_{1,1\cdots 1}$ is part of a coCartesian square

$$\begin{array}{ccc} \operatorname{colim}_{|v| < n} X_v & \longrightarrow X_{1 \cdots 1} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow Y_{1,1 \cdots 1} \end{array}$$

in \mathcal{C} . This shows that $Y_{1,1\cdots 1}$ can be identified with the total cofibre of X, which is what we needed to prove.

Example A.25. Let \mathcal{C} be a stable ∞ -category and X a 2-cube in \mathcal{C} . Proposition A.24 implies that there exists a coCartesian 3-cube of the form



The total cofibre of an n-cube in a stable ∞ -category admits the following alternative characterisation.

Proposition A.26. Let C be a stable ∞ -category, $n \ge 1$ an integer, and X an n-cube in C. Then, there exists an equivalence tot-cofib $(X) \simeq \operatorname{cofib}^n(X)$. In particular, X induces a cofibre sequence

$$\mathsf{tot\text{-}cofib}(X|_{\{0\}\times I^n}) \longrightarrow \mathsf{tot\text{-}cofib}(X|_{\{1\}\times I^n}) \longrightarrow \mathsf{tot\text{-}cofib}(X) \tag{A.27}$$

in \mathcal{C} .

Proof. By Proposition A.24 there exists a coCartesian (n+1)-cube \widetilde{X} such that $\widetilde{X}|_{\{0\}\times I^n}$ and for each $v \in \{1\} \times I^n$ with |v| < n+1 the object \widetilde{X}_v is a zero object of \mathfrak{C} . Moreover, there is an equivalence $\widetilde{X}_{1,1\cdots 1} \simeq \mathsf{tot\text{-}cofib}(X)$ in \mathfrak{C} . Also, by definition there is a cofibre sequence

$$\operatorname{cofib}^{n}(\widetilde{X}|_{\{0\}\times I^{n}}) \longrightarrow \operatorname{cofib}^{n}(\widetilde{X}|_{\{1\}\times I^{n}}) \longrightarrow \operatorname{cofib}^{n+1}(\widetilde{X}). \tag{A.28}$$

Since the (n+1)-cube \widetilde{X} is coCartesian, the object $\mathsf{cofib}^{n+1}(\widetilde{X})$ is a zero object of \mathcal{C} , see Corollary A.13. Consequently, given that \mathcal{C} is stable, the map

$$\mathsf{cofib}^n(\widetilde{X}|_{\{0\}\times I^n}) \longrightarrow \mathsf{cofib}^n(\widetilde{X}|_{\{1\}\times I^n})$$

is an equivalence in \mathcal{C} . The claim follows since $\widetilde{X}|_{\{0\}\times I^n}=X$ by construction and since there are equivalences

$$\operatorname{cofib}^n(\widetilde{X}|_{\{1\}\times I^n})\longrightarrow \widetilde{X}|_{1\cdots 1}\simeq \operatorname{tot-cofib}(X),$$

where the leftmost equivalence can be easily established using the fact for each $v \in \{1\} \times I^n$ with |v| < n+1 the object \widetilde{X}_v is a zero object of \mathcal{C} . Finally, the existence of the required cofibre sequence A.27 follows immediately from the above equivalence and the cofibre sequence A.28. \square

Remark A.29. A version of Proposition A.26 in the related framework of pointed derivators is proven in Theorem 8.25 in [BG18].

Let \mathcal{C} be a stable ∞ -category and suppose given a coCartesian square

$$\begin{array}{ccc} X_{00} & \stackrel{f}{\longrightarrow} X_{01} \\ \downarrow & \Box & \downarrow \\ X_{10} & \stackrel{g}{\longrightarrow} X_{11} \end{array}$$

in C. Proposition A.7 implies that the induced map $\mathsf{cofib}(f) \to \mathsf{cofib}(g)$ is an equivalence in C. The following result shows that there is an analogous relationship between the total cofibres of parallel facets of a coCartesian (n+1)-cube in a stable ∞ -category.

Corollary A.30. Let $\mathfrak C$ be a stable ∞ -category, $n \geq 0$ an integer, and X a coCartesian (n+1)-cube in $\mathfrak C$. Then, the induced map $\mathsf{tot\text{-}cofib}(X|_{\{0\}\times I^n}) \to \mathsf{tot\text{-}cofib}(X|_{\{1\}\times I^n})$ is an equivalence in $\mathfrak C$

Proof. The claim follows immediately from Lemma A.22 and the cofibre sequence A.27.

Let \mathcal{C} be a stable ∞ -category and $f: x \to y$ a morphism in \mathcal{C} . Recall that there is an equivalence $\mathsf{cofib}(f) \simeq \Sigma(\mathsf{fib}(x))$, witnessed by the existence of a diagram

in which each square is biCartesian. The following result shows that the total fibre and the total cofibre of an n-cube in \mathcal{C} satisfy an analogous relationship.

Proposition A.31. Let \mathfrak{C} be a stable ∞ -category, $n \geq 1$ an integer and X an n-cube in \mathfrak{C} . Then, there is an equivalence $\mathsf{tot\text{-}cofib}(X) \simeq \Sigma^n \, \mathsf{tot\text{-}fib}(X)$.

Proof. Utilising Proposition A.24 an its dual we construct an auxiliary diagram

$$F: I^n \times \{-1, 0, 1\} \longrightarrow \mathcal{C}$$

with the following properties:

- The *n*-cube $F|_{I^n \times \{0\}}$ agrees with X.
- The (n+1)-cube $F|_{I^n \times \{0,1\}}$ is obtained from X as in Proposition A.24.
- The (n+1)-cube $F|_{I^n \times \{-1,0\}}$ is obtained from X as in the dual of Proposition A.24.

By construction, there are equivalences

$$F_{(1\cdots 1,1)} \simeq \mathsf{tot\text{-}cofib}(X)$$
 and $F_{(0\cdots 0,-1)} \simeq \mathsf{tot\text{-}fib}(X)$

in \mathcal{C} . Finally, the composite (n+1)-cube $F|_{I^n \times \{-1,1\}}$ exhibits $F_{(1\cdots 1,1)}$ as the n-fold suspension of $F_{(0\cdots 0,-1)}$, see Example A.12. We deduce the existence of an equivalence $\mathsf{tot}\mathsf{-cofib}(X) \simeq \Sigma^n \mathsf{tot}\mathsf{-fib}(X)$ in \mathcal{C} , which is what we needed to prove. \square

We obtain the following consequence of Corollary A.30 and Proposition A.31.

Corollary A.32. Let \mathcal{C} be a stable ∞ -category, $n \geq 1$ an integer, and X a coCartesian (n+1)-cube in \mathcal{C} . Then, there exists an equivalence tot-cofib $(X|_{\{1\}\times I^n})\simeq \Sigma^n(\mathsf{tot}\mathsf{-fib}(X|_{\{0\}\times I^n}))$ in \mathcal{C} .

Proof. By Proposition A.31 applied to the *n*-cube $X|_{\{1\}\times I^n}$ and the dual of Corollary A.30 applied to the *n*-cube $X|_{\{0\}\times I^n}$ there are equivalences

$$\mathsf{tot\text{-}cofib}(X|_{\{1\}\times I^n}) \simeq \Sigma^n(\mathsf{tot\text{-}fib}(X|_{\{1\}\times I^n})) \simeq \Sigma^n(\mathsf{tot\text{-}fib}(X|_{\{0\}\times I^n})).$$

The claim follows. \Box

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