Adam Klepáč

Assignment I

Dualities In Triangulated Categories

(1) Consider the arrow $X \xrightarrow{f} Y$. By the axiom (TR1), there exists a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{i} W \xrightarrow{j} \Sigma X.$$

Also, by the axiom (TR3), there exists an arrow $\alpha : W \to Z \oplus Z'$, such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\left(\begin{smallmatrix} 1_X \\ 0 \end{smallmatrix}\right)"] & Y \arrow[r, "i"] \arrow[d, "\left(\begin{smallmatrix} 1_Y \\ 0 \end{smallmatrix}\right)"] & W \arrow[r, "j"] \arrow[d, dashed, "\exists\alpha"] & \Sigma X \arrow[d, "\left(\begin{smallmatrix} 1_{\Sigma X} \\ 0 \end{smallmatrix}\right)"] \\
X \oplus X' \arrow[r, "f \oplus f'"] & Y \oplus Y' \arrow[r, "g \oplus g'"] & Z \oplus Z' \arrow[r, "h \oplus h'"] & \Sigma X \oplus \Sigma X'
\end{tikzcd}$$

commutes, where 1_A denotes the identity arrow of A and $\left(\begin{smallmatrix} 1_A \\ 0 \end{smallmatrix}\right)$ the (unique) inclusion arrow $A \hookrightarrow A \oplus A'$ for $A \in \{X, Y, \Sigma X\}$.

Finally, consider the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\left(\begin{smallmatrix} 1_X \\ 0 \end{smallmatrix}\right)"] & Y \arrow[r, "i"] \arrow[d, "\left(\begin{smallmatrix} 1_Y \\ 0 \end{smallmatrix}\right)"] & W \arrow[r, "j"] \arrow[d, dashed, "\exists\alpha"] & \Sigma X \arrow[d, "\left(\begin{smallmatrix} 1_{\Sigma X} \\ 0 \end{smallmatrix}\right)"] \\
X \oplus X' \arrow[r, "f \oplus f'"] \arrow[d, "(1_X\ 0)"] & Y \oplus Y' \arrow[r, "g \oplus g'"] \arrow[d, "(1_Y\ 0)"] & Z \oplus Z' \arrow[r, "h \oplus h'"] \arrow[d, "(1_Z\ 0)"] & \Sigma X \oplus \Sigma X' \arrow[d, "(1_{\Sigma X}\ 0)"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X,
\end{tikzcd}$$

where $(1_A\ 0)$ denote the projection arrows. The top rectangle is commutative by (TR3) and the commutativity of the bottom rectangle is obvious. Thus, the diagram is a morphism of triangles. Moreover, the maps $(1_A\ 0) \circ \left(\begin{smallmatrix} 1_A \\ 0 \end{smallmatrix}\right) = 1_A$ are all isomorphisms so **Lemma 2.14.** asserts that also $(1_Z\ 0) \circ \alpha$ is an isomorphism which in turn means that T is isomorphic to the distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{i} W \xrightarrow{j} \Sigma X$$

making it a distinguished triangle as well, by (TR0).

The argument for the distinction of T' is nearly identical.

(2)