Chapter VI

Tilting theory

Tilting theory is one of the main tools in the representation theory of algebras. It originated with the study of reflection functors [32], [18]. The first set of axioms for a tilting module is due to Brenner and Butler [46]; the one generally accepted now is due to Happel and Ringel [89]. The main idea of tilting theory is that when the representation theory of an algebra A is difficult to study directly, it may be convenient to replace A with another simpler algebra B and to reduce the problem on A to a problem on B. We then construct an A-module T, called a tilting module, which can be thought of as being close to the Morita progenerators such that, if $B = \operatorname{End} T_A$, then the categories $\operatorname{mod} A$ and $\operatorname{mod} B$ are reasonably close to each other (but generally not equivalent). As will be seen, the knowledge of one of these module categories implies the knowledge of two distinguished full subcategories of the other, which form a torsion pair and thus determine up to extensions the whole module category. Because this procedure can be seen as generalising Morita theory, it is reasonable to give special attention to the full subcategory $\operatorname{Gen} T_A$ of all A-modules generated by T and to use the adjoint functors $\operatorname{Hom}_A(T, -)$ and $-\otimes_B T$ to compare $\operatorname{mod} A$ and $\operatorname{mod} B$.

Some notation is useful. Throughout this chapter, we let A denote an algebra, by which is meant, as usual, a finite dimensional, basic, and connected algebra over a fixed algebraically closed field K. For an A-module M, we denote by $\operatorname{add} M$ the smallest additive full subcategory of $\operatorname{mod} A$ containing M, that is, the full subcategory of $\operatorname{mod} A$ whose objects are the direct sums of direct summands of the module M. In many places, we consider the restriction to a subcategory $\mathcal{C}$ of a functor F defined originally on a module category, and we denote it by $F|_{\mathcal{C}}$.

VI.1. Torsion pairs

It is a well-known fact from elementary abelian group theory that there exists no nonzero homomorphism from a torsion group to a torsion-free one and that these two classes of abelian groups are maximal for this property. Generalising this situation, we obtain the concept of a torsion pair, valid in any abelian category, but which we need only for module categories. The following definition is due to Dickson [53].

1.1. Definition. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\operatorname{mod} A$ is called a **torsion pair** (or a **torsion theory**) if the following conditions are satisfied:

(a) $\operatorname{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.
(b) $\operatorname{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.
(c) $\operatorname{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

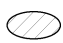
The first condition of the definition says that there is no nonzero homomorphism from an object in $\mathcal{T}$ to one in $\mathcal{F}$, and the other two conditions say that these two subcategories are maximal for this property. In analogy with the situation for abelian groups, the subcategory $\mathcal{T}$ is called the **torsion class**, and its objects are called **torsion objects**, while the subcategory $\mathcal{F}$ is called the **torsion-free class**, and its objects are called **torsion-free objects**. It follows directly from the definition that the torsion class and the torsion-free class determine uniquely each other.

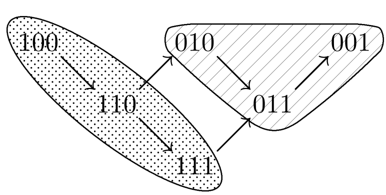
1.2. Examples. (a) An arbitrary class $\mathcal{C}$ of A-modules induces a torsion pair as follows: let $\mathcal{F} = \{N \mid \operatorname{Hom}_A(-, N)|_{\mathcal{C}} = 0\}$ and $\mathcal{T} = \{M \mid \operatorname{Hom}_A(M, -)|_{\mathcal{F}} = 0\}$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair, and $\mathcal{T}$ is in fact the smallest torsion class containing $\mathcal{C}$. The dual construction yields the smallest torsion-free class containing $\mathcal{C}$.

(b) If $(\mathcal{T}, \mathcal{F})$ is a torsion pair in the category $\operatorname{mod} A$ of all finite dimensional right A-modules, and $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ denotes the standard duality, then $(D\mathcal{F}, D\mathcal{T})$ is a torsion pair in $\operatorname{mod} A^{\mathrm{op}}$.

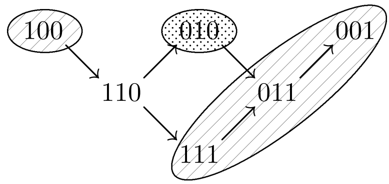
(c) Let A be the path algebra of the quiver

$$\overset{1}{\circ} \longleftarrow \overset{2}{\circ} \longleftarrow \overset{3}{\circ}$$

and let $\mathcal{T} = \operatorname{add}\{010 \oplus 011 \oplus 001\}$, $\mathcal{F} = \operatorname{add}\{100 \oplus 110 \oplus 111\}$ (where the indecomposable A-modules are represented by their dimension vectors). Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair. We may illustrate $(\mathcal{T}, \mathcal{F})$ in the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A, adopting the convention (which we keep throughout this chapter and the next) to shade the class $\mathcal{T}$ as  and the class $\mathcal{F}$ as  :



(d) Let A be as in (c). Then we have another torsion pair $(\mathcal{T}, \mathcal{F})$, illustrated as follows in $\Gamma(\operatorname{mod} A)$:



Our first objective is to give an intrinsic characterisation of torsion (or torsion-free) classes. For this purpose, we need one further definition.

1.3. Definition. A subfunctor t of the identity functor on $\operatorname{mod} A$ is called an **idempotent radical** if, for every module M_A, we have $t(tM) = tM$ and $t(M/tM) = 0$.

We recall that a subfunctor of the identity functor on $\operatorname{mod} A$ is a functor $t : \operatorname{mod} A \longrightarrow \operatorname{mod} A$ that assigns to each module M a submodule $tM \subseteq M$ such that each homomorphism $M \longrightarrow N$ restricts to a homomorphism $tM \longrightarrow tN$. As we now show, each torsion pair induces an idempotent radical and conversely.

1.4. Proposition. (a) *Let $\mathcal{T}$ be a full subcategory of* $\operatorname{mod} A$. *The following conditions are equivalent:*

(i) *$\mathcal{T}$ is the torsion class of some torsion pair $(\mathcal{T}, \mathcal{F})$ in* $\operatorname{mod} A$.
(ii) *$\mathcal{T}$ is closed under images, direct sums, and extensions.*
(iii) *There exists an idempotent radical t such that $\mathcal{T} = \{M \mid tM = M\}$.*

(b) *Let $\mathcal{F}$ be a full subcategory of* $\operatorname{mod} A$. *The following conditions are equivalent:*

(i) *$\mathcal{F}$ is the torsion-free class of some torsion pair $(\mathcal{T}, \mathcal{F})$ in* $\operatorname{mod} A$.
(ii) *$\mathcal{F}$ is closed under submodules, direct products, and extensions.*
(iii) *There exists an idempotent radical t such that $\mathcal{F} = \{N \mid tN = 0\}$.*

Proof. We only prove (a); the proof of (b) is similar.

(i) implies (ii). A short exact sequence $0 \to M' \to M \to M'' \to 0$ of A-modules induces a left exact sequence of functors

$$0 \longrightarrow \operatorname{Hom}_A(M'', -)|_{\mathcal{F}} \longrightarrow \operatorname{Hom}_A(M, -)|_{\mathcal{F}} \longrightarrow \operatorname{Hom}_A(M', -)|_{\mathcal{F}}.$$

Hence $M \in \mathcal{T}$ implies $M'' \in \mathcal{T}$ and, similarly, $M', M'' \in \mathcal{T}$ imply $M \in \mathcal{T}$. The statement follows.

(ii) implies (iii). Let M be any A-module and tM denote the **trace** of $\mathcal{T}$ in M, that is, the sum of the images of all A–homomorphisms from modules in $\mathcal{T}$ to M. Because $\mathcal{T}$ is closed under images and direct (hence arbitrary) sums, tM is the largest submodule of M that lies in $\mathcal{T}$. The trace defines a subfunctor of the identity: if $f : M \to N$ is a homomorphism, then $f(tM) \subseteq tN$ for, if $g : X \to M$ is a homomorphism with $X \in \mathcal{T}$, then $fg : X \to N$ has its image lying in tN. Moreover, we clearly have $t(tM) = tM$ and $M \in \mathcal{T}$ if and only if $tM = M$. Finally, let M be arbitrary and assume that $t(M/tM) = M'/tM$ with $tM \subseteq M' \subseteq M$. Because $\mathcal{T}$ is closed under extensions, tM, $M'/tM \in \mathcal{T}$ yield $M' \in \mathcal{T}$. Hence $M' \subseteq tM$ and $t(M/tM) = 0$.

(iii) implies (i). Let $\mathcal{F} = \{N \mid tN = 0\}$. Clearly, $\mathrm{Hom}_A(M, -)|_{\mathcal{F}} = 0$ for all $M \in \mathcal{T}$. We claim that, conversely, $\mathrm{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$. Indeed, $t(M/tM) = 0$ gives $M/tM \in \mathcal{F}$. The canonical surjection $M \to M/tM$ being zero, we have $M/tM = 0$ so that $M = tM \in \mathcal{T}$. Similarly, $\mathrm{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies that $N \in \mathcal{F}$. □

An immediate consequence is that a torsion (or a torsion-free) class is an additive, hence K-linear, subcategory of $\operatorname{mod} A$, closed under isomorphic images, extensions, and direct summands.

The idempotent radical t attached to a given torsion pair is called the **torsion radical**. It follows from its definition that, for any module M_A, we have $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. The uniqueness follows from the next proposition, which also says that any module can be written in a unique way as the extension of a torsion-free module by a torsion module.

1.5. Proposition. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\operatorname{mod} A$ *and M be an A-module. There exists a short exact sequence*

$$0 \longrightarrow tM \longrightarrow M \longrightarrow M/tM \longrightarrow 0$$

with $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. This sequence is unique in the sense that, if $0 \to M' \to M \to M'' \to 0$ is exact with $M' \in \mathcal{T}$, $M'' \in \mathcal{F}$, then the two sequences are isomorphic.

Proof. Only the second statement needs a proof. Because $M' \in \mathcal{T}$ and tM is the largest torsion submodule of M, there exists a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "j"'] & M \arrow[r] \arrow[d, "1_M"'] & M'' \arrow[r] \arrow[d, dashed, "f"'] & 0 \\
0 \arrow[r] & tM \arrow[r] & M \arrow[r] & M/tM \arrow[r] & 0
\end{tikzcd}$$

where j denotes the inclusion and f is obtained by passing to the cokernels. The Snake lemma (I.5.1) yields $tM/M' \cong \operatorname{Ker} f$. Because $tM/M' \in \mathcal{T}$ and $\operatorname{Ker} f \in \mathcal{F}$, we get $M'' \cong M/tM$ and $tM/M' = 0$. □

A short exact sequence as in the proposition is called the **canonical sequence** for M. For instance, in Example 1.2 (d), the canonical sequence for the indecomposable module $M = 110$ (which is neither torsion nor torsion-free) is $0 \longrightarrow 100 \longrightarrow 110 \longrightarrow 010 \longrightarrow 0$. The following obvious corollary is sometimes useful.

1.6. Corollary. *Every simple module is either torsion or torsion-free.* □

A torsion pair $(\mathcal{T}, \mathcal{F})$ such that each indecomposable A-module lies either in $\mathcal{T}$ or in $\mathcal{F}$ is called **splitting**. This is the case in example (1.2)(c) (but not in (1.2)(d)). Splitting torsion pairs are characterised as follows.

1.7. Proposition. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\operatorname{mod} A$*. The following conditions are equivalent:*

(a) *$(\mathcal{T}, \mathcal{F})$ is splitting.*
(b) *For each A-module M, the canonical sequence for M splits.*
(c) $\operatorname{Ext}^1_A(N, M) = 0$ *for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.*
(d) *If $M \in \mathcal{T}$, then $\tau^{-1}M \in \mathcal{T}$.*
(e) *If $N \in \mathcal{F}$, then $\tau N \in \mathcal{F}$.*

Proof. (a) implies (b). Let M_A be any module and M' (or M'') denote the direct sum of all the indecomposable summands of M that belong to $\mathcal{T}$ (or $\mathcal{F}$, respectively). We have a split short exact sequence $0 \to M' \to M \to M'' \to 0$ with $M' \in \mathcal{T}$, $M'' \in \mathcal{F}$, which is, by (1.5), isomorphic to the canonical sequence.

(b) implies (c). Any short exact sequence $0 \to M \to E \to N \to 0$ with $M \in \mathcal{T}$ and $N \in \mathcal{F}$ is a canonical sequence, by (1.5).

(c) implies (a). Let M be indecomposable. The hypothesis implies that the canonical sequence for M splits. Hence $M \cong tM \oplus (M/tM)$ so that either $M \cong tM$ or $M \cong M/tM$.

(a) implies (d). Let $0 \to M \to \bigoplus_{i=1}^n E_i \to \tau^{-1}M \to 0$ be the almost split sequence starting with M, where the modules $E_1, \ldots, E_n$ are indecomposable. Because $\operatorname{Hom}_A(M, E_i) \neq 0$ for all i, the hypothesis implies that $E_i \in \mathcal{T}$ for all i. Hence $\bigoplus_{i=1}^n E_i \in \mathcal{T}$ so that $\tau^{-1}M \in \mathcal{T}$. We prove similarly that (a) implies (e).

(d) implies (c). Let $M \in \mathcal{T}$ and $N \in \mathcal{F}$. By the Auslander–Reiten formulas (IV.2.13), $\operatorname{Ext}^1_A(N, M) \cong D\underline{\operatorname{Hom}}_A(\tau^{-1}M, N)$. Because $\tau^{-1}M \in \mathcal{T}$ and $N \in \mathcal{F}$, we have $\operatorname{Hom}_A(\tau^{-1}M, N) = 0$. Hence $\operatorname{Ext}^1_A(N, M) = 0$. We prove similarly that (e) implies (c). □

Let T be an arbitrary A-module. We define $\operatorname{Gen} T$ to be the class of all modules M in $\operatorname{mod} A$ generated by T, that is, the modules M such that there exist an integer $d \geq 0$ and an epimorphism $T^d \to M$ of A-modules. Dually, we define $\operatorname{Cogen} T$ to be the class of all modules N in $\operatorname{mod} A$ cogenerated by T, that is, the modules N such that there exist an integer $d \geq 0$ and a monomorphism $N \to T^d$ of A-modules.

We ask when the class $\operatorname{Gen} T$ is a torsion class and when the class $\operatorname{Cogen} T$ is a torsion-free class. It is clear that $\operatorname{Gen} T$ is closed under images, $\operatorname{Cogen} T$ is closed under submodules, and both classes are closed under direct sums. There remains thus, by (1.4), to see when they are closed under extensions. This is generally not the case: let A be an algebra having two nonisomorphic simple modules S, S' such that $\operatorname{Ext}_A^1(S, S') \neq 0$; then neither $\operatorname{Gen}(S \oplus S')$ nor $\operatorname{Cogen}(S \oplus S')$ is closed under extensions.

Before answering these questions, we derive a necessary and sufficient condition for an A-module to belong to $\operatorname{Gen} T$ (or to $\operatorname{Cogen} T$). We write $B = \operatorname{End} T_A$ so that T is endowed with a natural left B-module structure, compatible with the action of A, making it a B–A-bimodule.

1.8. Lemma. *Let M be an A-module.*

(a) $M \in \operatorname{Gen} T$ *if and only if the canonical homomorphism*

$$\varepsilon_M : \operatorname{Hom}_A(T, M) \otimes_B T \longrightarrow M$$

defined by $f \otimes t \mapsto f(t)$ is surjective, where $B = \operatorname{End} T_A$.

(b) $M \in \operatorname{Cogen} T$ *if and only if the canonical homomorphism*

$$\eta_M : M \longrightarrow \operatorname{Hom}_B(\operatorname{Hom}_A(M, T), T)$$

defined by $x \mapsto (g \mapsto g(x))$ is injective.

Proof. We only prove (a); the proof of (b) is similar. Assume $M \in \operatorname{Gen} T$ and let $f_1, \dots, f_d$ be a basis of the K-vector space $\operatorname{Hom}_A(T, M)$. Then $f = [f_1 \dots f_d] : T^d \to M$ is an epimorphism. Indeed, there exist $m > 0$ and an epimorphism $g : T^m \to M$. It follows from the definition of f that there exists $h : T^m \to T^d$ such that $g = fh$, so that f is surjective. Let $L = \operatorname{Ker} f$, and apply $\operatorname{Hom}_A(T, -)$ to the short exact sequence

$$0 \longrightarrow L \longrightarrow T^d \xrightarrow{f} M \longrightarrow 0.$$

Because $\operatorname{Hom}_A(T, f)$ is an epimorphism by the definition of f, this yields a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, L) \longrightarrow \operatorname{Hom}_A(T, T^d) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, M) \longrightarrow 0.$$

Applying $- \otimes_B T$, we obtain the upper row in the commutative diagram with exact rows

$$\begin{tikzcd}
 & \operatorname{Hom}_A(T,L) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_L"'] & \operatorname{Hom}_A(T,T^d) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_{T^d}"'] & \operatorname{Hom}_A(T,M) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_M"'] & 0 \\
0 \arrow[r] & L \arrow[r] & T^d \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The composite homomorphism

$$\varepsilon_{T^d} : \mathrm{Hom}_A(T, T^d) \otimes_B T \cong B^d \otimes_B T \cong T^d$$

is an isomorphism. By the commutativity of the right square, the homomorphism ε_M is surjective.

Conversely, because $\mathrm{Hom}_A(T, M)$ is a finitely generated B-module, there exist $m > 0$ and an epimorphism $g : B^m \to \mathrm{Hom}_A(T, M)$, hence an epimorphism

$$T^m \cong B^m \otimes_B T \xrightarrow{g \otimes T} \mathrm{Hom}_A(T, M) \otimes_B T \xrightarrow{\varepsilon_M} M,$$

so $M \in \mathrm{Gen}\, T$. □

The following lemma answers our questions.

1.9. Lemma. (a) *Assume that* $\mathrm{Ext}^1_A(T, -)|_{\mathrm{Gen}\, T} = 0$*; then* $\mathrm{Gen}\, T$ *is a torsion class. If this is the case, then the corresponding torsion-free class is the class* $\{M \mid \mathrm{Hom}_A(T, M) = 0\}$.

(b) *Assume that* $\mathrm{Ext}^1_A(-, T)|_{\mathrm{Cogen}\, T} = 0$*; then* $\mathrm{Cogen}\, T$ *is a torsion-free class. If this is the case, then the corresponding torsion class is the class* $\{M \mid \mathrm{Hom}_A(M, T) = 0\}$.

Proof. We only prove (a); the proof of (b) is similar. Assume that

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence with $M', M'' \in \mathrm{Gen}\, T$. Because $\mathrm{Ext}^1_A(T, M') = 0$, we have a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(T, M') \longrightarrow \mathrm{Hom}_A(T, M) \longrightarrow \mathrm{Hom}_A(T, M'') \longrightarrow 0,$$

which yields, after applying $- \otimes_B T$, the upper row in the commutative diagram with exact rows

$$\begin{tikzcd}
 & \mathrm{Hom}_A(T, M') \otimes_B T \arrow[r] \arrow[d, "\varepsilon_{M'}"'] & \mathrm{Hom}_A(T, M) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_M"'] & \mathrm{Hom}_A(T, M'') \otimes_B T \arrow[r] \arrow[d, "\varepsilon_{M''}"'] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

Because, by (1.8), $\varepsilon_{M'}$ and $\varepsilon_{M''}$ are epimorphisms, so is ε_M. A further application of (1.8) yields that $M \in \mathrm{Gen}\, T$ so that $\mathrm{Gen}\, T$ is indeed closed under extensions.

For the second statement, we notice that every torsion-free module M satisfies $\mathrm{Hom}_A(T, M) = 0$. Conversely, if $\mathrm{Hom}_A(T, M) = 0$ and $X \in \mathrm{Gen}\, T$, there exist $m > 0$ and an epimorphism $T^m \to X$. But this implies that $\mathrm{Hom}_A(X, M) = 0$. □

1.10. Definition. Let $\mathcal{C}$ be a full K-subcategory of $\operatorname{mod} A$. An A-module $M \in \mathcal{C}$ is called **Ext-projective** in $\mathcal{C}$ if $\operatorname{Ext}^1_A(M,-)|_{\mathcal{C}} = 0$. Dually, it is called **Ext-injective** in $\mathcal{C}$ if $\operatorname{Ext}^1_A(-,M)|_{\mathcal{C}} = 0$.

This definition, due to Auslander and Smalø [22], is clearly motivated by Lemma 1.9. Thus $\operatorname{Gen} T$ is a torsion class if T is Ext-projective in $\operatorname{Gen} T$ and, dually, $\operatorname{Cogen} T$ is a torsion-free class if T is Ext-injective in $\operatorname{Cogen} T$. The following proposition characterises completely Ext-projectives and Ext-injectives in torsion or torsion-free classes.

1.11. Proposition. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\operatorname{mod} A$ *and M be an indecomposable A-module.*

(a) *Assume that M lies in $\mathcal{T}$.*
   (i) *M is* Ext-*projective in $\mathcal{T}$ if and only if $\tau M \in \mathcal{F}$.*
   (ii) *M is* Ext-*injective in $\mathcal{T}$ if and only if there exist an injective module $E \notin \mathcal{F}$ and an isomorphism $M \cong tE$.*

(b) *Assume that M lies in $\mathcal{F}$.*
   (i) *M is* Ext-*injective in $\mathcal{F}$ if and only if $\tau^{-1} M \in \mathcal{T}$.*
   (ii) *M is* Ext-*projective in $\mathcal{F}$ if and only if there exist a projective module $P \notin \mathcal{T}$ and an isomorphism $M \cong P/tP$.*

Proof. We only prove (a); the proof of (b) is similar. Suppose $\tau M \in \mathcal{F}$. Then, for any $X \in \mathcal{T}$, we have

$$\operatorname{Ext}^1_A(M, X) \cong D\overline{\operatorname{Hom}}_A(X, \tau M) \subseteq D\operatorname{Hom}_A(X, \tau M) = 0.$$

Thus, M is Ext-projective in $\mathcal{T}$. Conversely, if $\tau M \notin \mathcal{F}$, then, in the canonical sequence

$$0 \longrightarrow t(\tau M) \xrightarrow{u} \tau M \xrightarrow{v} \tau M/t(\tau M) \longrightarrow 0,$$

the epimorphism v is not an isomorphism and, in particular, is not a retraction. Considering the almost split sequence

$$0 \longrightarrow \tau M \xrightarrow{f} N \xrightarrow{g} M \longrightarrow 0,$$

we deduce the existence of a homomorphism $h : N \to \tau M/t(\tau M)$ such that $hf = v$. Because v is surjective, so is h, and we have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & t(\tau M) & \xrightarrow{f'} & \operatorname{Ker} h & \xrightarrow{g'} & M & \longrightarrow & 0 \\
 & & {\scriptstyle u}\downarrow & & \downarrow & & {\scriptstyle 1}\downarrow & & \\
0 & \longrightarrow & \tau M & \xrightarrow{f} & N & \xrightarrow{g} & M & \longrightarrow & 0 \\
 & & {\scriptstyle v}\downarrow & & {\scriptstyle h}\downarrow & & & & \\
 & & \tau M/t(\tau M) & \xrightarrow{1} & \tau M/t(\tau M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The first row is not split (for if g' were a retraction, so would be g) and consequently $\operatorname{Ext}_A^1(M, t(\tau M)) \neq 0$. Thus, M is not Ext-projective in $\mathcal{T}$.

Let $E \notin \mathcal{F}$ be injective and $X \in \mathcal{T}$. The functor $\operatorname{Hom}_A(X, -)$ applied to the short exact sequence $0 \to tE \to E \to E/tE \to 0$ yields

$$0 = \operatorname{Hom}_A(X, E/tE) \longrightarrow \operatorname{Ext}_A^1(X, tE) \longrightarrow \operatorname{Ext}_A^1(X, E) = 0.$$

Thus tE is Ext-injective in $\mathcal{T}$. Conversely, let $M \in \mathcal{T}$ be Ext-injective and E be its injective envelope. Because $M \subseteq E$, we have $M \subseteq tE$. Consider the short exact sequence $0 \to M \to tE \to tE/M \to 0$. Because $tE \in \mathcal{T}$, we have $tE/M \in \mathcal{T}$. The Ext-injectivity of M in $\mathcal{T}$ implies that the sequence splits. Hence M is a direct summand of tE. The statement follows. $\square$

In example (1.2)(c), $\mathcal{T} = \operatorname{Gen}(010 \oplus 011)$, the indecomposable Ext-projectives in $\mathcal{T}$ are 010 and 011, and the indecomposable Ext-injectives are 001 and 011, whereas $\mathcal{F} = \operatorname{Cogen}(111)$ and every indecomposable in $\mathcal{F}$ is both Ext-injective and Ext-projective.

VI.2. Partial tilting modules and tilting modules

We now introduce a class of modules that induce torsion pairs in a natural way.

2.1. Definition. Let A be an algebra. An A-module T is called a **partial tilting module** if the following two conditions are satisfied:

(T1) $\operatorname{pd} T_A \leq 1$,
(T2) $\operatorname{Ext}_A^1(T, T) = 0$.

A partial tilting module T is called a **tilting module** if it also satisfies the following additional condition:

(T3) There exists a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T', T'' in $\operatorname{add} T$.

Thus, any projective A-module is trivially a partial tilting module, and any Morita progenerator is a tilting module. In fact, the axioms can be understood to mean that a partial tilting module is a module that is "close enough" to a projective module, and a tilting module is a module that is "close enough" to a Morita progenerator. The third condition (T3) may be reformulated to say that a partial tilting module T_A is a tilting module if and only if, for any indecomposable projective A-module P, there exists a short exact sequence

$$0 \longrightarrow P_A \longrightarrow T'_A \longrightarrow T''_A \longrightarrow 0$$

with T', T'' in $\operatorname{add} T$.

One easy consequence of (T3) is that every tilting module is faithful. We recall that an A-module is **faithful** if its **right annihilator**

$$\operatorname{Ann} M = \{a \in A \mid Ma = 0\}$$

vanishes. We need the following characterisation of faithful modules.

2.2. Lemma. *Let A be an algebra and M be an A-module. The following conditions are equivalent:*

(a) *M_A is faithful.*
(b) *For any basis $\{f_1, \dots, f_d\}$ of the K-vector space* $\operatorname{Hom}_A(A, M)$*, the K-linear map $f = [f_1 \dots f_d]^t : A_A \longrightarrow M^d$ is injective.*
(c) *A_A is cogenerated by M_A.*
(d) *DA_A is generated by M_A.*

Proof. Let $\{f_1, \dots, f_d\}$ be a basis of the K-vector space $\operatorname{Hom}_A(A, M)$. Then M is faithful if and only if

$$f = [f_1 \dots f_d]^t : A_A \longrightarrow M^d$$

is a monomorphism; indeed, $f(a) = 0$ for some $a \in A$ if and only if $g(a) = 0$ for some $a \in A$ and any $g \in \operatorname{Hom}_A(A, M)$. Using the canonical isomorphism $M_A \cong \operatorname{Hom}_A(A, M)$, this is equivalent to saying that $Ma = 0$ for some $a \in A$. This implies the equivalence of (a), (b), and (c).

The right annihilator $\{a \in A \mid Ma = 0\}$ of M_A coincides with the left annihilator $\{a \in A \mid aDM = 0\}$ of ${}_ADM$. Therefore, M_A is faithful if and only if ${}_AA$ is cogenerated by ${}_ADM$ or, equivalently, DA_A is generated by $D(DM) \cong M$. □

Applying the equivalence of (a) and (c), the monomorphism $A_A \to T'_A$ of (T3) shows that every tilting module is faithful.

Given a partial tilting module T_A, we ask whether the class $\operatorname{Gen} T$ is a torsion class. We also consider the full subcategory $\mathcal{T}(T)$ of $\operatorname{mod} A$ defined by $\mathcal{T}(T) = \{M_A \mid \operatorname{Ext}_A^1(T, M) = 0\}$.

2.3. Lemma. *Let T be a partial tilting module. Then*

(a) $\operatorname{Gen} T$ *is a torsion class in which T is* Ext*-projective, and the corresponding torsion-free class is* $\mathcal{F}(T) = \{M_A \mid \operatorname{Hom}_A(T, M) = 0\}$*;*

(b) $\mathcal{T}(T)$ *is a torsion class in which T is* Ext*-projective; and and the corresponding torsion-free class is* $\operatorname{Cogen} \tau T$*; and*

(c) $\operatorname{Gen} T \subseteq \mathcal{T}(T)$.

Proof. Assume that $M \in \operatorname{Gen} T$. There exist $m > 0$ and an epimorphism $T^m \to M$. Because $\operatorname{pd} T \leq 1$, this epimorphism induces an epimorphism $0 = \operatorname{Ext}_A^1(T, T^m) \to \operatorname{Ext}_A^1(T, M)$. Hence $\operatorname{Ext}_A^1(T, M) = 0$. Thus the functor $\operatorname{Ext}_A^1(T, -)|_{\operatorname{Gen} T}$ equals zero and, by (1.9)(a), $\operatorname{Gen} T$ is a torsion class in which T is Ext-projective. Moreover, we have shown that $\operatorname{Gen} T \subseteq \mathcal{T}(T)$ and (1.9)(a) implies that the torsion-free class corresponding to $\operatorname{Gen} T$ is $\mathcal{F}(T)$. This shows (a) and (c).

To prove (b), let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Applying $\operatorname{Hom}_A(T, -)$ yields a right exact sequence

$$\operatorname{Ext}_A^1(T, M') \longrightarrow \operatorname{Ext}_A^1(T, M) \longrightarrow \operatorname{Ext}_A^1(T, M'') \longrightarrow 0;$$

hence $M', M'' \in \mathcal{T}(T)$ imply $M \in \mathcal{T}(T)$ and $M \in \mathcal{T}(T)$ implies $M'' \in \mathcal{T}(T)$. Because $\mathcal{T}(T)$ is closed under direct sums, it is a torsion class, in which T is clearly Ext-projective. For the corresponding torsion-free class, we observe that, because $\operatorname{pd} T \leq 1$, we have, by (IV.2.14), that $\operatorname{Ext}_A^1(T, M) \cong D\operatorname{Hom}_A(M, \tau T)$ and thus $M \in \mathcal{T}(T)$ if and only if $\operatorname{Hom}_A(M, \tau T) = 0$. Moreover, for each X in $\operatorname{Cogen} \tau T$, we have

$$\operatorname{Ext}_A^1(X, \tau T) \cong D\underline{\operatorname{Hom}}_A(T, X) \subseteq D\operatorname{Hom}_A(T, X) = 0,$$

because $\operatorname{Hom}_A(T, \tau T) = 0$. It follows that the restriction of $\operatorname{Ext}_A^1(-, \tau T)$ to $\operatorname{Cogen} \tau T$ is zero. Hence, by (1.9)(b), $\operatorname{Cogen} \tau T$ is a torsion-free class whose corresponding torsion class is $\{M \mid \operatorname{Hom}_A(M, \tau T) = 0\} = \mathcal{T}(T)$. □

It is easy to see that every injective A-module is torsion in the torsion pair $(\mathcal{T}(T), \operatorname{Cogen} \tau T)$. Also, if a projective module P lies in $\operatorname{Gen} T$, then $P \in \operatorname{add} T$. Indeed, if $P \in \operatorname{Gen} T$, there exist $m > 0$ and an epimorphism $T^m \to P$ that must split, because P is projective.

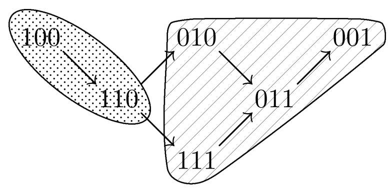
In Example 1.2 (c), the module $T = 010 \oplus 011$ is a partial tilting module. Indeed, $\operatorname{pd} T \leq 1$, as seen from the projective resolutions

$$0 \longrightarrow P(1) \longrightarrow P(2) \longrightarrow 010 \longrightarrow 0,$$
$$0 \longrightarrow P(1) \longrightarrow P(3) \longrightarrow 011 \longrightarrow 0.$$

In fact, it is easy to see that in this example, we have $\operatorname{gl.dim} A = 1$. Algebras with global dimension one are called hereditary and are studied in detail in the following chapters. Because 011 is injective,

$$\begin{aligned}\operatorname{Ext}_A^1(T,T) &\cong \operatorname{Ext}_A^1(010 \oplus 011, 010) \cong D\operatorname{Hom}_A(010, \tau(010 \oplus 011)) \\ &\cong D\operatorname{Hom}_A(010, 100 \oplus 110) = 0.\end{aligned}$$

The torsion pair illustrated in Example 1.2 (c) is the pair $(\operatorname{Gen} T, \mathcal{F}(T))$; the pair $(\mathcal{T}(T), \operatorname{Cogen} \tau T)$ is illustrated as follows:



In this case, the inclusion of (2.3)(c) is proper.

In Example 1.2 (d), the module $T = 100 \oplus 111 \oplus 001$ is a partial tilting module. Indeed, $\operatorname{pd} T \leq 1$ because $\operatorname{gl.dim} A = 1$. Because $100 \oplus 111$ is projective, whereas $001 \oplus 111$ is injective, we have

$$\begin{aligned}\operatorname{Ext}_A^1(T,T) &\cong \operatorname{Ext}_A^1(001, 100) \cong D\operatorname{Hom}_A(100, \tau(001)) \\ &\cong D\operatorname{Hom}_A(100, 010) = 0.\end{aligned}$$

In fact, T is even a tilting module: because $P(1), P(3) \in \operatorname{add} T$, the short exact sequence

$$0 \longrightarrow P(2) \longrightarrow 111 \longrightarrow 001 \longrightarrow 0$$

shows that (T3) is satisfied. In this case, the classes $(\operatorname{Gen} T, \mathcal{F}(T))$ and $(\mathcal{T}(T), \operatorname{Cogen} \tau T)$ coincide and are illustrated in Example 1.2 (d).

As the reader may have noticed, the formula of (IV.2.14), asserting that $\operatorname{Ext}_A^1(T, M) \cong D\operatorname{Hom}_A(M, \tau T)$ whenever $\operatorname{pd} T \leq 1$, is extremely useful in these computations.

The following lemma, known as **Bongartz's lemma** [33], justifies the name of partial tilting module; it asserts that a partial tilting module may always be completed to a tilting module.

2.4. Lemma. *Let T_A be a partial tilting module. There exists an A-module E such that $T \oplus E$ is a tilting module.*

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be a basis of the K-vector space $\operatorname{Ext}^1_A(T, A)$. Represent each $\mathbf{e}_i$ by a short exact sequence $0 \longrightarrow A \xrightarrow{f_i} E_i \xrightarrow{g_i} T \longrightarrow 0$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & 0 & \longrightarrow & A^d & \xrightarrow{f} & \bigoplus_{i=1}^{d} E_i & \xrightarrow{g} & T^d & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 1} & & \\
(*) & 0 & \longrightarrow & A & \xrightarrow{v} & E & \xrightarrow{w} & T^d & \longrightarrow & 0
\end{array}$$

where $f = \begin{bmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_d \end{bmatrix}$, $g = \begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_d \end{bmatrix}$ and $k = [1, \dots, 1]$ is the codiagonal homomorphism. We denote by $\mathbf{e}$ the element of $\operatorname{Ext}^1_A(T^d, A)$ represented by the lower sequence $(*)$. Let $u_i : T \to T^d$ be the inclusion homomorphism in the ith coordinate. We claim that $\mathbf{e}_i = \operatorname{Ext}^1_A(u_i, A)\mathbf{e}$ for each i with $1 \leq i \leq d$. Indeed, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & 0 & \longrightarrow & A & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & T & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle u_i''} & & \downarrow{\scriptstyle u_i'} & & \downarrow{\scriptstyle u_i} & & \\
 & 0 & \longrightarrow & A^d & \xrightarrow{f} & \bigoplus_{i=1}^{d} E_i & \xrightarrow{g} & T^d & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 1} & & \\
(*) & 0 & \longrightarrow & A & \xrightarrow{v} & E & \xrightarrow{w} & T^d & \longrightarrow & 0
\end{array}$$

where u_i', u_i'' denote the respective inclusion homomorphisms in the ith coordinate. Because $ku_i'' = 1_A$, we deduce a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & 0 & \longrightarrow & A & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & T & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle uu_i'} & & \downarrow{\scriptstyle u_i} & & \\
(*) & 0 & \longrightarrow & A & \xrightarrow{v} & E & \xrightarrow{w} & T^d & \longrightarrow & 0
\end{array}$$

hence our claim. Applying $\operatorname{Hom}_A(T, -)$ to $(*)$ yields an exact sequence

$$\cdots \to \operatorname{Hom}_A(T, T^d) \xrightarrow{\delta} \operatorname{Ext}^1_A(T, A) \longrightarrow \operatorname{Ext}^1_A(T, E) \longrightarrow \operatorname{Ext}^1_A(T, T^d) = 0.$$

Because $\mathbf{e}_i = \mathrm{Ext}^1_A(u_i, A)\mathbf{e} = \delta(u_i)$, each basis element of $\mathrm{Ext}^1_A(T, A)$ lies in the image of the connecting homomorphism δ, which is therefore surjective. Hence $\mathrm{Ext}^1_A(T, E) = 0$. Applying now $\mathrm{Hom}_A(-, T)$ and $\mathrm{Hom}_A(-, E)$ to $(*)$ yields respectively

$$0 = \mathrm{Ext}^1_A(T^d, T) \longrightarrow \mathrm{Ext}^1_A(E, T) \longrightarrow \mathrm{Ext}^1(A, T) = 0,$$
$$0 = \mathrm{Ext}^1_A(T^d, E) \longrightarrow \mathrm{Ext}^1_A(E, E) \longrightarrow \mathrm{Ext}^1(A, E) = 0;$$

hence $\mathrm{Ext}^1_A(E \oplus T, E \oplus T) = 0$. It follows from the short exact sequence $(*)$ that $\mathrm{pd}\, E \le 1$, hence that $\mathrm{pd}\,(T \oplus E) \le 1$ and the module $T \oplus E$ satisfies the axiom (T3). $\square$

The short exact sequence $(*)$ constructed in the proof of the lemma is referred to as **Bongartz's exact sequence**. As a first consequence, we obtain the following characterisation of tilting modules.

2.5. Theorem. *Let T_A be a partial tilting module. The following conditions are equivalent:*

(a) *T_A is a tilting module.*
(b) $\mathrm{Gen}\, T = \mathcal{T}(T)$.
(c) *For every module $M \in \mathcal{T}(T)$, there exists a short exact sequence $0 \to L \to T_0 \to M \to 0$ with $T_0 \in \mathrm{add}\, T$ and $L \in \mathcal{T}(T)$.*
(d) *Let X be an A-module. Then $X \in \mathrm{add}\, T$ if and only if X is* Ext-*projective in $\mathcal{T}(T)$.*
(e) $\mathcal{F}(T) = \mathrm{Cogen}\, \tau T$.

Proof. Because (b) and (e) are clearly equivalent (by (2.3)), it suffices to establish the equivalence of the first four conditions.

(a) implies (b). Assume that T is a tilting module and let $M \in \mathcal{T}(T)$. We must show that $M \in \mathrm{Gen}\, T$ or, equivalently, that $M \cong tM$, where t is the torsion radical associated to the torsion pair $(\mathrm{Gen}\, T, \mathcal{F}(T))$. Applying $\mathrm{Hom}_A(T, -)$ to the canonical sequence $0 \to tM \to M \to M/tM \to 0$ yields an epimorphism $\mathrm{Ext}^1_A(T, M) \to \mathrm{Ext}^1_A(T, M/tM)$. Because $\mathrm{Ext}^1_A(T, M) = 0$, we have $\mathrm{Ext}^1_A(T, M/tM) = 0$. Further, because $M/tM \in \mathcal{F}(T)$, we have $\mathrm{Hom}_A(T, M/tM) = 0$. On the other hand, because T is a tilting module, there exists a short exact sequence $0 \to A \to T' \to T'' \to 0$ with $T', T'' \in \mathrm{add}\, T$. Applying the functor $\mathrm{Hom}_A(-, M/tM)$ to this sequence yields an exact sequence $0 = \mathrm{Hom}_A(T', M/tM) \to \mathrm{Hom}_A(A, M/tM) \to \mathrm{Ext}^1_A(T'', M/tM) = 0$ so that $M/tM \cong \mathrm{Hom}_A(A, M/tM) = 0$ and $M = tM \in \mathrm{Gen}\, T$.

(b) implies (c). Let $M \in \mathcal{T}(T)$ and $f_1, \ldots, f_d$ be a basis of the K-vector space $\mathrm{Hom}_A(T, M)$. Because $M \in \mathrm{Gen}\, T$, the homomorphism $f = [f_1 \ldots f_d] : T^d \to M$ is surjective (see the proof of (1.8)). Letting $L = \mathrm{Ker}\, f$

and applying $\operatorname{Hom}_A(T, -)$ to the short exact sequence $0 \to L \to T^d \xrightarrow{f} M \to 0$ yields an exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_A(T, T^d) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, M) \longrightarrow \operatorname{Ext}_A^1(T, L) \longrightarrow 0.$$

By construction, $\operatorname{Hom}_A(T, f)$ is an epimorphism. Hence $\operatorname{Ext}_A^1(T, L) = 0$ and $L \in \mathcal{T}(T)$.

(c) implies (d). Let $X \in \operatorname{add} T$; then X is clearly Ext-projective in $\mathcal{T}(T) = \{M \mid \operatorname{Ext}_A^1(T, M) = 0\}$. Conversely, let X be Ext-projective in $\mathcal{T}(T)$, and consider the exact sequence $0 \to L \to T_0 \to X \to 0$ with $T_0 \in \operatorname{add} T$ and $L \in \mathcal{T}(T)$. Because X is Ext-projective in $\mathcal{T}(T)$, this sequence splits and $X \in \operatorname{add} T$.

(d) implies (a). Let $0 \to A \to E \to T^d \to 0$ be Bongartz's exact sequence corresponding to the partial tilting module T. To show that T is a tilting module, it suffices to show that $E \in \operatorname{add} T$ or, equivalently, that E is Ext-projective in $\mathcal{T}(T)$. First, we observe that, because $T \oplus E$ is a tilting module by (2.4), we have $\operatorname{Ext}_A^1(T, E) = 0$ so that $E \in \mathcal{T}(T)$. Letting $M \in \mathcal{T}(T)$ and applying $\operatorname{Hom}_A(-, M)$ to the previous Bongartz sequence yields an exact sequence

$$0 = \operatorname{Ext}_A^1(T^d, M) \longrightarrow \operatorname{Ext}_A^1(E, M) \longrightarrow \operatorname{Ext}_A^1(A, M) = 0.$$

Hence $\operatorname{Ext}_A^1(E, M) = 0$. □

2.6. Corollary. *Let T_A be a tilting module and $M \in \mathcal{T}(T)$. Then there exists an exact sequence*

$$\cdots \to T_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with all T_i in $\operatorname{add} T$.

Proof. This follows from (2.5)(c) and an obvious induction. □

In the sequel, if T_A is a tilting module, we refer to the torsion pair $(\operatorname{Gen} T, \mathcal{F}(T_A)) = (\mathcal{T}(T_A), \operatorname{Cogen} \tau T)$ as the **torsion pair induced by** T in $\operatorname{mod} A$, and we usually denote it by $(\mathcal{T}(T_A), \mathcal{F}(T_A))$.

As another consequence of (2.5), we can refine the result of (1.8)(a), in the case where T is a tilting module.

2.7. Corollary. *Let T_A be a tilting module, and $B = \operatorname{End} T_A$. Then $M \in \mathcal{T}(T)$ if and only if the canonical A-module homomorphism ε_M : $\operatorname{Hom}_A(T, M) \otimes_B T \to M$ is bijective.*

Proof. The sufficiency follows from (1.8) and (2.5). For the necessity, we apply twice (2.5)(c) and find short exact sequences

$$0 \longrightarrow L_0 \longrightarrow T_0 \longrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow L_1 \longrightarrow T_1 \longrightarrow L_0 \longrightarrow 0,$$

with $T_0, T_1 \in \operatorname{add} T$ and $L_0, L_1 \in \mathcal{T}(T)$. Applying $\operatorname{Hom}_A(T,-)$ yields short exact sequences

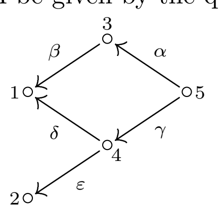
$$\begin{gathered} 0 \longrightarrow \operatorname{Hom}_A(T, L_0) \longrightarrow \operatorname{Hom}_A(T, T_0) \longrightarrow \operatorname{Hom}_A(T, M) \longrightarrow 0, \\ 0 \longrightarrow \operatorname{Hom}_A(T, L_1) \longrightarrow \operatorname{Hom}_A(T, T_1) \longrightarrow \operatorname{Hom}_A(T, L_0) \longrightarrow 0, \end{gathered}$$

because $\operatorname{Ext}^1_A(T, L_0) = 0$ and $\operatorname{Ext}^1_A(T, L_1) = 0$. Applying the right exact functor $\operatorname{Hom}_A(T,-) \otimes_B T$ to the exact sequence $T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$ we get the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_A(T,T_1) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_{T_1}"'] & \operatorname{Hom}_A(T,T_0) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_{T_0}"'] & \operatorname{Hom}_A(T,M) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_M"'] & 0 \\ T_1 \arrow[r] & T_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Because ε_T is just the canonical A-module isomorphism $\operatorname{Hom}_A(T,T) \otimes_B T \cong B \otimes_B T_A \cong T_A$, it follows that ε_{T_0}, ε_{T_1} are isomorphisms. Hence so is ε_M. $\square$

2.8. Examples. (a) Let A be given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\gamma\varepsilon = 0$. Representing the indecomposable A-modules by their dimension vectors, we consider the module

$$T_A = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix}.$$

Then T_A is a tilting module. Indeed, we have the following

(T1) $\operatorname{pd} T_A \le 1$, because the modules $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ 0 & & \end{smallmatrix} = P(1)$, $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ 1 & & \end{smallmatrix} = P(4)$, $\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix} = P(5)$ are projective, and we have projective resolutions for the other two summands of T

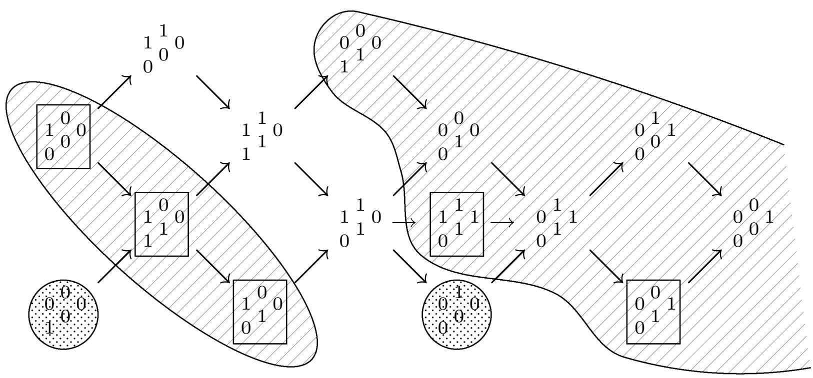
$$\begin{gathered} 0 \longrightarrow P(2) \longrightarrow P(4) \longrightarrow \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ 0 & & \end{smallmatrix} \longrightarrow 0, \\ 0 \longrightarrow P(3) \longrightarrow P(5) \longrightarrow \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix} \longrightarrow 0. \end{gathered}$$

(T2) $\mathrm{Ext}^1_A(T,T)=0$. Because $\begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ 0 & & \end{smallmatrix}$ is projective and $\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ 0 & & \end{smallmatrix}$ is injective, this follows from

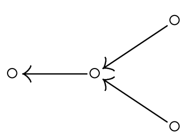
$$\begin{aligned}
\mathrm{Ext}^1_A(T,T) &\cong \mathrm{Ext}^1_A\left(\begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ 0 & & \end{smallmatrix} ,\ \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 0 & & \end{smallmatrix} \right) \\
&\cong D\mathrm{Hom}_A\left(\begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 0 & & \end{smallmatrix} ,\ \tau\left(\begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ 0 & & \end{smallmatrix} \right)\right) \\
&\cong D\mathrm{Hom}_A\left(\begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ 0 & & \end{smallmatrix} ,\ \begin{smallmatrix} & 0 & \\ 0 & 0 & 0 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \\ 0 & & \end{smallmatrix} \right) = 0.
\end{aligned}$$

(T3) There exists, for each point a in the quiver of A, a short exact sequence $0 \to P(a) \to T' \to T'' \to 0$ with $T', T'' \in \operatorname{add} T$. Because $P(1), P(4), P(5) \in \operatorname{add} T$, it suffices to consider the two short exact sequences presented in (T1).

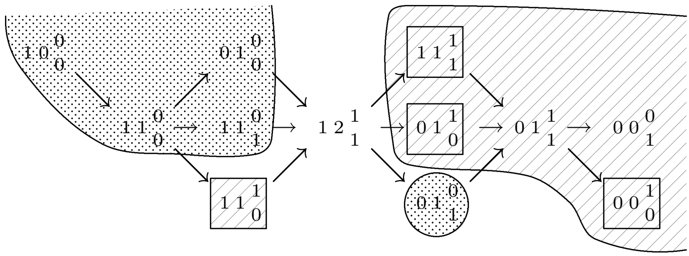
The torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ induced by T in $\operatorname{mod} A$ is illustrated as follows in $\Gamma(\operatorname{mod} A)$, where we represent the indecomposable summands of T by squares:



(b) Let A be given by the quiver



and consider the module $T_A = 1\,1\,{}^{1}_{0} \oplus 1\,1\,{}^{1}_{1} \oplus 0\,1\,{}^{1}_{0} \oplus 0\,0\,{}^{1}_{0}$. We leave it to the reader to verify that T is a tilting module and that the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ induced by T in $\operatorname{mod} A$ is as illustrated here:



(c) The following class of tilting modules, whose construction is due to Auslander, Platzeck, and Reiten [18] (and, accordingly, are called APR-tilting modules), were at the origins of the theory. Let A be an algebra and $S(a)_A$ be a simple projective that is not injective (thus, the corresponding point a is a sink in the quiver of A and there exists at least one arrow having a as a target). We claim that

$$T_A = T[a] = \tau^{-1}S(a) \oplus (\bigoplus_{b\neq a} P(b))$$

is a tilting module.

First, we note that, according to (IV.3.9) and (IV.4.4), the almost split sequence in $\operatorname{mod} A$ starting from the simple projective module $S(a) = P(a)$ has the form

$$0 \longrightarrow S(a) \longrightarrow \bigoplus_{c\neq a} P(c)^{m_c} \longrightarrow \tau^{-1}S(a) \longrightarrow 0,$$

where $m_c = \dim_K \operatorname{Irr}(S(a), P(c))$. This immediately yields (T1) and (T3). The statement (T2) is a consequence of $\operatorname{Ext}^1_A(T,T) \cong D\operatorname{Hom}_A(T,\tau T) = 0$, because $\tau T = S(a)$ is simple projective. In this case, the only indecomposable A-module lying in $\mathcal{F}(T_A)$ is $S(a)$, whereas $\mathcal{T}(T_A)$ is the additive subcategory generated by all remaining indecomposables. Indeed, if M_A is indecomposable, then $M \in \mathcal{T}(T)$ if and only if $0 = \operatorname{Ext}^1_A(T,M) \cong D\operatorname{Hom}_A(M,S(a))$ if and only if $M \ncong S(a)$. In particular, $(\mathcal{T}(T),\mathcal{F}(T))$ is splitting.

For instance, if A is as in (a), then there exist two APR-tilting modules $T[1]$ and $T[2]$, whereas, if A is as in (b), then there exists a unique APR-tilting module corresponding to the only sink in the quiver of A.

The reader may have observed that in all of the examples, the number of indecomposable nonisomorphic summands of a tilting A-module is equal to the number of nonisomorphic simple A-modules (that is, to the rank of the Grothendieck group $K_0(A)$ of A). This is no accident, as will be shown in (4.4).

VI.3. The tilting theorem of Brenner and Butler

Tilting theory aims at comparing the module categories of two finite dimensional algebras. Namely, let A be an algebra, T_A be a tilting module, and $B = \operatorname{End} T_A$. Because T_A is, by definition, a module "close to" a Morita progenerator, thus "close to" A_A, it turns out that $B = \operatorname{End} T_A$ is "close to" $\operatorname{End} A_A \cong A$. An obvious functor allowing to pass from $\operatorname{mod} A$ to $\operatorname{mod} B$ is the functor $\operatorname{Hom}_A(T, -)$. The following easy lemma shows that this functor maps the objects in $\operatorname{add} T$ onto the projective B-modules. For this reason, the procedure of passing from an algebra to the endomorphism algebra of one of its modules is sometimes called **projectivisation**; see [21].

3.1. Lemma. *Let A be an algebra, T be any A-module, and $B = \operatorname{End} T_A$.*

(a) *For each module $T_0 \in \operatorname{add} T$ and each A-module M, the K-linear map $f \mapsto \operatorname{Hom}_A(T, f)$ induces a functorial isomorphism*

$$\operatorname{Hom}_A(T_0, M) \cong \operatorname{Hom}_B(\operatorname{Hom}_A(T, T_0), \operatorname{Hom}_A(T, M)).$$

(b) *The functor $\operatorname{Hom}_A(T, -)$ induces an equivalence of categories between $\operatorname{add} T$ and the subcategory $\operatorname{proj} B$ of $\operatorname{mod} B$ consisting of the projective modules.*

Proof. (a) This follows from the additivity of the functors and from the fact that the defined map is an isomorphism when $T_0 = T$.

(b) Clearly, P_B is an indecomposable projective B-module if and only if P is an indecomposable summand of

$$B_B = (\operatorname{End} T_A)_B = \operatorname{Hom}_A({}_B T_A, T_A),$$

if and only if $P_B \cong \operatorname{Hom}_A({}_B T_A, T_0)$ for some indecomposable summand T_0 of T. Thus the functor $\operatorname{Hom}_A(T, -)|_{\operatorname{add} T}$ maps into $\operatorname{proj} B$ and is dense. Also, (a) shows that it is full and faithful. □

As an obvious consequence of (3.1)(b) we get that B is a basic algebra if and only if two distinct indecomposable summands of T are not isomorphic (we then say that T is **multiplicity-free**).

In (3.1), no assumption on T was necessary. Until the end of this section, we assume that T is a tilting A-module and

$$B = \operatorname{End} T_A.$$

We consider the functor

$$\operatorname{Hom}_A(T, -) : \mathcal{T}(T_A) \longrightarrow \operatorname{mod} B.$$

The following lemma ensures that this functor embeds $\mathcal{T}(T)$ as a full subcategory of $\operatorname{mod} B$, closed under extensions.

3.2. Lemma. *Let $M, N \in \mathcal{T}(T)$; then we have functorial isomorphisms:*
(a) $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_B(\mathrm{Hom}_A(T,M), \mathrm{Hom}_A(T,N))$.
(b) $\mathrm{Ext}^1_A(M,N) \cong \mathrm{Ext}^1_B(\mathrm{Hom}_A(T,M), \mathrm{Hom}_A(T,N))$.

Proof. By (2.6), there exists an exact sequence

$$T_*: \quad \cdots \to T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} M \longrightarrow 0$$

with $T_i \in \operatorname{add} T$ for all i. Applying $\mathrm{Hom}_A(-,N)$ to the right exact sequence $T_1 \to T_0 \to M \to 0$ yields a left exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(T_0,N) \longrightarrow \mathrm{Hom}\,(T_1,N).$$

By (3.1)(a), we have a commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}_A(M,N) \arrow[r, dashed] \arrow[d] & \mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,N)) \arrow[d] \\
\mathrm{Hom}_A(T_0,N) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_B(\mathrm{Hom}_A(T,T_0),\mathrm{Hom}_A(T,N)) \arrow[d] \\
\mathrm{Hom}_A(T_1,N) \arrow[r, "\cong"] & \mathrm{Hom}_B(\mathrm{Hom}_A(T,T_1),\mathrm{Hom}_A(T,N))
\end{tikzcd}$$

where the dotted arrow is induced by the others. This shows (a) by passing to the kernels. For (b), let $L = \operatorname{Im} d_1$; we have a short exact sequence

$$0 \longrightarrow L \xrightarrow{j} T_0 \xrightarrow{d_0} M \longrightarrow 0,$$

to which we apply $\mathrm{Hom}_A(-,N)$, thus obtaining an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(T_0,N) \xrightarrow{\mathrm{Hom}_A(j,N)} \mathrm{Hom}_A(L,N) \longrightarrow \mathrm{Ext}^1_A(M,N) \longrightarrow 0$$

so that $\mathrm{Ext}^1_A(M,N) \cong \operatorname{Coker}\mathrm{Hom}_A(j,N)$ is isomorphic to the first cohomology group of the complex $\mathrm{Hom}_A(T_*,N)$. On the other hand, if we apply $\mathrm{Hom}_A(T,-)$ to the complex T_*, we obtain, by (3.1)(b), a projective resolution $\mathrm{Hom}_A(T,T_*)$ of $\mathrm{Hom}_A(T,M)$ in $\operatorname{mod} B$, because $\operatorname{Ker} d_i \in \mathcal{T}(T)$ and hence $\mathrm{Ext}^1_A(T, \operatorname{Ker} d_i) = 0$ for any $i \geq 1$. Therefore $\mathrm{Ext}^1_B(\mathrm{Hom}_A(T,M), \mathrm{Hom}_A(T,N))$ is isomorphic to the first cohomology group of the complex $\mathrm{Hom}_B(\mathrm{Hom}_A(T,T_*), \mathrm{Hom}_A(T,N))$, which is, by (3.1)(a), isomorphic (as a complex) to $\mathrm{Hom}_A(T_*,N)$. This completes the proof of (b). □

The key observation of tilting theory is that the tilting module T_A induces a tilting B-module, which is the left B-module ${}_BT$. Moreover, the algebra A can be recovered from B and ${}_BT$.

3.3. Lemma. *Let T_A be a tilting A-module and $B = \operatorname{End} T_A$.*

(a) $D({}_BT) \cong \operatorname{Hom}_A(T, DA)$.

(b) ${}_BT$ *is a tilting left B-module.*

(c) *The canonical K-algebra homomorphism $A \to \operatorname{End}({}_BT)^{\mathrm{op}}$, given by $a \mapsto (t \mapsto ta)$, is an isomorphism.*

Proof. (a) $D({}_BT) \cong D({}_BT_A \otimes_A A) \cong \operatorname{Hom}_A(T, DA)$.

(b) We verify the axioms of tilting module:

(T1) $\operatorname{pd}{}_BT \leq 1$. Indeed, because T_A is a tilting module, there exists a short exact sequence $0 \to A_A \to T' \to T'' \to 0$ with $T', T'' \in \operatorname{add} T$. Applying $\operatorname{Hom}_A(-, {}_BT_A)$, we get a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T'', {}_BT_A) \longrightarrow \operatorname{Hom}_A(T', {}_BT_A) \longrightarrow \operatorname{Hom}_A(A, {}_BT_A) \longrightarrow 0.$$

Because

$\operatorname{Hom}_A(A, {}_BT_A) \cong {}_BT$ and $\operatorname{Hom}_A(T', {}_BT_A), \operatorname{Hom}_A(T'', {}_BT_A) \in \operatorname{add}({}_BB)$, we are done.

(T2) $\operatorname{Ext}^1_B(T, T) = 0$. Indeed, using (a) and the fact that $DA \in \mathcal{T}(T)$, we get, by (3.2)(b),

$$\begin{aligned}\operatorname{Ext}^1_B(DT, DT) &\cong \operatorname{Ext}^1_B(\operatorname{Hom}_A(T, DA), \operatorname{Hom}_A(T, DA)) \\ &\cong \operatorname{Ext}^1_A(DA, DA) = 0,\end{aligned}$$

hence the result.

(T3) Let $0 \to P_1 \to P_0 \to T_A \to 0$ be a projective resolution. Applying $\operatorname{Hom}_A(-, {}_BT_A)$, we get a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, {}_BT_A) \longrightarrow \operatorname{Hom}_A(P_0, {}_BT_A) \longrightarrow \operatorname{Hom}_A(P_1, {}_BT_A) \longrightarrow 0.$$

Because

$\operatorname{Hom}_A(T, {}_BT_A) \cong {}_BB$ and $\operatorname{Hom}_A(P_0, {}_BT_A), \operatorname{Hom}_A(P_1, {}_BT_A) \in \operatorname{add}({}_BT)$, we are done.

(c) Let $a \in A$ belong to the kernel of this homomorphism. Then $Ta = 0$. But every tilting module is faithful, hence $a = 0$. Thus the given homomorphism is injective. By (a) and the fact that $DA \in \mathcal{T}(T)$, (3.2)(a) yields vector space isomorphisms

$$A \cong \operatorname{End} DA \cong \operatorname{End} \operatorname{Hom}_A(T, DA) \cong \operatorname{End} DT,$$

so that $\dim_K A = \dim_K \operatorname{End}({}_BT)$ and the canonical homomorphism is an isomorphism. □

A first consequence of (3.3) is that B is a connected algebra. In fact, we show more, namely that the centre is preserved under the tilting process.

3.4. Lemma. *Let A be an algebra and T_A be a tilting A-module. Then the centre $Z(A)$ of A is isomorphic to the centre $Z(B)$ of $B = \operatorname{End} T_A$.*

Proof. We define $\varphi : Z(A) \to Z(B)$ by $a \mapsto (\rho_a : t \mapsto ta)$. Indeed, let $a \in Z(A)$; then ρ_a is an endomorphism of T_A for, if $t_1, t_2 \in T$ and $a_1, a_2 \in A$, then we have

$$\rho_a(t_1a_1 + t_2a_2) = t_1a_1a + t_2a_2a = t_1aa_1 + t_2aa_2 = \rho_a(t_1)a_1 + \rho_a(t_2)a_2.$$

Also, ρ_a is central for, if $f \in \operatorname{End} T_A = B$ and $t \in T$, we have $(\rho_a f)(t) = f(t)a = f(ta) = (f\rho_a)(t)$. Finally, φ is an algebra homomorphism for, if $a_1, a_2 \in Z(A)$ then $\varphi(a_1a_2) = \rho_{a_1a_2} = \rho_{a_2a_1} = \varphi(a_1)\varphi(a_2)$ and, clearly, $\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$ and $\varphi(1) = 1$.

To show that φ is an isomorphism, we construct its inverse. Following (3.3)(c), we identify the algebra A with $\operatorname{End}({}_BT)^{\mathrm{op}}$ via $a \mapsto \rho_a$, then we define $\psi : Z(B) \to Z(A)$ by $b \mapsto (\lambda_b : t \mapsto bt)$. By (3.3)(b) and the first part, ψ is an algebra homomorphism. Let $a \in Z(A)$ and consider $\psi\varphi(a) = \lambda_{\rho_a}$; it is given by $\lambda_{\rho_a} : t \mapsto \rho_a(t) = ta$, that is, by the element $a \in A$ as identified to the endomorphism $\rho_a \in \operatorname{End}({}_BT)$. Thus $\psi\varphi(a) = a$ for every $a \in Z(A)$ and $\psi\varphi = 1_{Z(A)}$. By symmetry, we have $\varphi\psi = 1_{Z(B)}$. □

3.5. Corollary. *Let A be an algebra. If T_A is a tilting A-module, then the algebra $B = \operatorname{End} T_A$ is connected.*

Proof. Note that an algebra is connected if and only if its centre is (see Exercise 8.8 in Chapter I), and then apply (3.4). □

Another consequence of (3.3) and the considerations in Section 2 is that ${}_BT$ induces a torsion pair $(\mathcal{T}({}_BT), \mathcal{F}({}_BT))$ in the category of left B-modules, where, as before,

$$\begin{aligned} \mathcal{T}({}_BT) &= \operatorname{Gen}({}_BT) = \{{}_BU \mid \operatorname{Ext}^1_B(T, U) = 0\}, \\ \mathcal{F}({}_BT) &= \operatorname{Cogen} \tau({}_BT) = \{{}_BV \mid \operatorname{Hom}_B(T, V) = 0\}. \end{aligned}$$

Because we are interested in the category $\operatorname{mod} B$ of right B-modules, we must rather consider the torsion pair (see Example 1.2 (b))

$$(\mathcal{X}(T_A), \mathcal{Y}(T_A)) = (D\mathcal{F}({}_BT), D\mathcal{T}({}_BT)).$$

3.6. Corollary. *Let A be an algebra. Any tilting A-module T_A induces a torsion pair $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ in the category $\operatorname{mod} B$, where $B = \operatorname{End} T_A$ and*

$$\begin{aligned} \mathcal{X}(T_A) &= \{X_B \mid \operatorname{Hom}_B(X, DT) = 0\} = \{X_B \mid X \otimes_B T = 0\}, \\ \mathcal{Y}(T_A) &= \{Y_B \mid \operatorname{Ext}^1_B(Y, DT) = 0\} = \{Y_B \mid \operatorname{Tor}^B_1(Y, T) = 0\}. \end{aligned}$$

Proof. This follows from the remark and the functorial isomorphisms $\mathrm{Hom}_B(X, DT) \cong D(X \otimes_B T)$ and $\mathrm{Ext}^1_B(Y, DT) \cong D\mathrm{Tor}^B_1(Y, T)$. The first is the adjoint isomorphism. The second is a consequence of (A.4.11) in the Appendix. $\square$

Note that $\mathcal{Y}(T_A)$ contains all the projective B-modules. This subcategory of $\operatorname{mod} B$ plays a rô le fairly similar to that of $\mathcal{T}(T_A)$ in $\operatorname{mod} A$. In fact, we have the following analogue of (2.5)(c) and (2.7).

3.7. Lemma. *Let A be an algebra, T_A be a tilting A-module, $B = \operatorname{End} T_A$, and $Y_B \in \mathcal{Y}(T_A)$.*

(a) *There exists a short exact sequence $0 \to Y \to T^* \to Z \to 0$ with T^* in $\operatorname{add} DT$ and Z in $\mathcal{Y}(T_A)$.*

(b) *The canonical homomorphism $\delta_Y : Y_B \to \mathrm{Hom}_A(T, Y \otimes_B T)$ defined by $y \mapsto (t \mapsto y \otimes t)$ is an isomorphism.*

Proof. (a) Because ${}_BT$ is a tilting module and $D(Y_B) \in \mathcal{T}({}_BT)$, there exists a short exact sequence $0 \to {}_BY' \to {}_BT' \to {}_B(DY) \to 0$ with $T' \in \operatorname{add}({}_BT)$, $Y' \in \mathcal{T}({}_BT)$. Taking $T^* = DT'$ and $Z = DY'$ completes the proof.

(b) The duality isomorphism $\mathrm{Hom}_B(X, DT) \cong D(X \otimes_B T)$ yields $DA \cong D\mathrm{Hom}_B(T, T) \cong DT \otimes_B T$, so that $\delta_{DT} : D({}_BT) \to \mathrm{Hom}_A(T, DA) \cong \mathrm{Hom}_A(T, DT \otimes_B T)$ is an isomorphism. Therefore, so is δ_{T^*}, for any $T^* \in \operatorname{add} DT$. Applying (a) twice to $Y \in \mathcal{Y}(T_A)$, we obtain short exact sequences $0 \to Y \to T^*_0 \to Y_0 \to 0$ and $0 \to Y_0 \to T^*_1 \to Y_1 \to 0$ with $T^*_0, T^*_1 \in \operatorname{add} DT$ and $Y_0, Y_1 \in \mathcal{Y}(T_A)$, and so $\mathrm{Tor}^B_1(Y_0, T) = 0$ and $\mathrm{Tor}^B_1(Y_1, T) = 0$. Applying $- \otimes_B T$ yields short exact sequences

$$0 \to Y \otimes_B T \to T^*_0 \otimes_B T \to Y_0 \otimes_B T \to 0 \text{ and}$$
$$0 \to Y_0 \otimes_B T \to T^*_1 \otimes_B T \to Y_1 \otimes_B T \to 0.$$

These combine to a left exact sequence

$$0 \longrightarrow Y \otimes_B T \longrightarrow T^*_0 \otimes_B T \longrightarrow T^*_1 \otimes_B T$$

to which we apply $\mathrm{Hom}_A(T, -)$, thus obtaining the lower row of the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\delta_Y"'] & T^*_0 \arrow[r] \arrow[d, "\delta_{T^*_0}"', "\cong"] & T^*_1 \arrow[d, "\delta_{T^*_1}"', "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_A(T, Y \otimes_B T) \arrow[r] & \mathrm{Hom}_A(T, T^*_0 \otimes_B T) \arrow[r] & \mathrm{Hom}_A(T, T^*_1 \otimes_B T)
\end{tikzcd}$$

Because $\delta_{T^*_0}$ and $\delta_{T^*_1}$ are isomorphisms, so is δ_Y. $\square$

We are now able to prove the main result of this section, which is known as the **Brenner–Butler theorem** or the **tilting theorem**.

3.8. Theorem. *Let A be an algebra, T_A be a tilting module, $B = \operatorname{End} T_A$, and $(\mathcal{T}(T_A), \mathcal{F}(T_A))$, $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ be the induced torsion pairs in* $\operatorname{mod} A$ *and* $\operatorname{mod} B$*, respectively. Then T has the following properties:*

(a) *${}_BT$ is a tilting module, and the canonical K-algebra homomorphism $A \to \operatorname{End}({}_BT)^{\mathrm{op}}$ defined by $a \mapsto (t \mapsto ta)$ is an isomorphism.*
(b) *The functors $\operatorname{Hom}_A(T, -)$ and $- \otimes_B T$ induce quasi-inverse equivalences between $\mathcal{T}(T_A)$ and $\mathcal{Y}(T_A)$.*
(c) *The functors $\operatorname{Ext}^1_A(T, -)$ and $\operatorname{Tor}^B_1(-, T)$ induce quasi-inverse equivalences between $\mathcal{F}(T_A)$ and $\mathcal{X}(T_A)$.*

Proof. Because (a) is (3.3)(b) and (3.3)(c), we prove (b). Let $M \in \mathcal{T}(T_A)$. The duality isomorphism established in (3.6) yields

$$D\operatorname{Hom}_A(T, M) \cong {}_BT_A \otimes DM \in \operatorname{Gen}({}_BT),$$

and therefore $\operatorname{Hom}_A(T, M) \in \operatorname{Cogen} DT = \mathcal{Y}(T)$. By (2.7), we have $M \cong \operatorname{Hom}_A(T, M) \otimes_B T$. Conversely, if $Y \in \mathcal{Y}(T_A)$, then $Y \otimes_B T_A \in \operatorname{Gen} T_A = \mathcal{T}(T_A)$ and, by (3.7), we have $Y \cong \operatorname{Hom}_A(T, Y \otimes_B T)$.

To show (c), we take $N \in \mathcal{F}(T_A)$. There is a short exact sequence $0 \longrightarrow N \longrightarrow E \longrightarrow L \longrightarrow 0$ with E injective. In particular, $E \in \mathcal{T}(T_A)$ and hence $L \in \mathcal{T}(T_A)$. Applying $\operatorname{Hom}_A(T, -)$, we get a short exact sequence $0 \longrightarrow \operatorname{Hom}_A(T, E) \longrightarrow \operatorname{Hom}_A(T, L) \longrightarrow \operatorname{Ext}^1_A(T, N) \longrightarrow 0$. Applying $- \otimes_B T$, we get the left column in the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\operatorname{Tor}^B_1(\operatorname{Ext}^1_A(T, N), T) \arrow[d] \arrow[r, dashed] & N \arrow[d] \\
\operatorname{Hom}_A(T, E) \otimes_B T \arrow[d] \arrow[r, "\varepsilon_E", "\cong"'] & E \arrow[d] \\
\operatorname{Hom}_A(T, L) \otimes_B T \arrow[d] \arrow[r, "\varepsilon_L", "\cong"'] & L \arrow[d] \\
\operatorname{Ext}^1_A(T, N) \otimes_B T \arrow[d] & 0 \\
0 &
\end{tikzcd}$$

with exact columns, because $L \in \mathcal{T}(T)$ implies $\operatorname{Tor}^B_1(\operatorname{Hom}_A(T, L), T) = 0$, by (b). Therefore we get $\operatorname{Ext}^1_A(T, N) \otimes_B T = 0$ (hence $\operatorname{Ext}^1_A(T, N) \in \mathcal{X}(T_A)$)

and $N \cong \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, N), T)$. Dually, let $X_B \in \mathcal{X}(T)$ and consider the short exact sequence

$$0 \to Y \to P \to X \to 0$$

with P projective. Then $P \in \mathcal{Y}(T)$ and $Y \in \mathcal{Y}(T)$. Applying $- \otimes_B T$, we get a short exact sequence

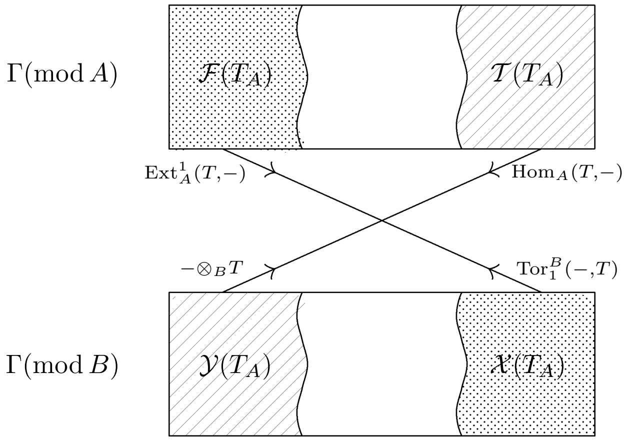
$$0 \longrightarrow \mathrm{Tor}_1^B(X, T) \longrightarrow Y \otimes_B T \longrightarrow P \otimes_B T \longrightarrow 0.$$

Applying $\mathrm{Hom}_A(T, -)$, we get the right column in the commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & \mathrm{Hom}_A(T, \mathrm{Tor}_1^B(X, T)) \arrow[d] \\
Y \arrow[r, "\delta_Y", "\cong"'] \arrow[d] & \mathrm{Hom}_A(T, Y \otimes_B T) \arrow[d] \\
P \arrow[r, "\delta_P", "\cong"'] \arrow[d] & \mathrm{Hom}_A(T, P \otimes_B T) \arrow[d] \\
X \arrow[r, dashed] \arrow[d] & \mathrm{Ext}_A^1(T, \mathrm{Tor}_1^B(X, T)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

because $\mathrm{Ext}_A^1(T, Y \otimes_B T) = 0$ by (b). Therefore $\mathrm{Hom}_A(T, \mathrm{Tor}_1^B(X, T)) = 0$ (hence $\mathrm{Tor}_1^B(X, T) \in \mathcal{F}(T_A)$) and $X \cong \mathrm{Ext}_A^1(T, \mathrm{Tor}_1^B(X, T))$. □

It is possible to visualise the equivalence of (3.8) in the Auslander–Reiten quivers of the algebras A and B. If one keeps in mind that $\mathcal{T}(T_A)$ contains the injective A-modules and thus lies (roughly speaking) "at the right" of $\Gamma(\mathrm{mod}\, A)$, while $\mathcal{F}(T_A)$ lies "on the left" of $\mathcal{T}(T_A)$ (because there is no homomorphism from a torsion module to a torsion-free one) and, similarly, $\mathcal{Y}(T_A)$ contains the projective B-modules and thus lies "at the left" of $\Gamma(\mathrm{mod}\, B)$, while $\mathcal{X}(T_A)$ lies "on its right," one obtains the following picture, which also shows the quasi-inverse equivalences:



Here, and in the sequel, the equivalent subcategories $\mathcal{T}(T_A)$ and $\mathcal{Y}(T_A)$ are shaded as and the equivalent subcategories $\mathcal{F}(T_A)$ and $\mathcal{X}(T_A)$ are shaded as .

The following corollary asserts that the composition of any two of the four functors $\operatorname{Hom}_A(T,-)$, $\operatorname{Ext}^1_A(T,-)$, $-\otimes_B T$, and $\operatorname{Tor}^B_A(-,T)$, which are not quasi-inverse to each other on one of the shaded subcategories, vanishes.

3.9. Corollary. (a) *Let M be an arbitrary A-module. Then*

(i) $\operatorname{Tor}^B_1(\operatorname{Hom}_A(T,M),T)=0$*;*
(ii) $\operatorname{Ext}^1_A(T,M)\otimes_B T=0$*; and*
(iii) *the canonical sequence of M in $(\mathcal{T}(T_A),\mathcal{F}(T_A))$ is*

$$0\longrightarrow \operatorname{Hom}_A(T,M)\otimes_B T\xrightarrow{\varepsilon_M} M\longrightarrow \operatorname{Tor}^B_1(\operatorname{Ext}^1_A(T,M),T))\longrightarrow 0.$$

(b) *Let X be an arbitrary B-module. Then*

(i) $\operatorname{Hom}_A(T,\operatorname{Tor}^B_1(X,T))=0$*;*
(ii) $\operatorname{Ext}^1_A(T,X\otimes_B T)=0$*; and*
(iii) *the canonical sequence of X in $(\mathcal{X}(T_A),\mathcal{Y}(T_A))$ is*

$$0\longrightarrow \operatorname{Ext}^1_A(T,\operatorname{Tor}^B_1(X,T))\longrightarrow X\xrightarrow{\delta_X}\operatorname{Hom}_A(T,X\otimes_B T)\longrightarrow 0.$$

Proof. We only prove (a); the proof of (b) is similar. Indeed, let

$$0\to tM\to M\to M/tM\to 0$$

be the canonical sequence of M in $(\mathcal{T}(T),\mathcal{F}(M))$. Applying the functor $\operatorname{Hom}_A(T,-)$, we obtain isomorphisms $\operatorname{Hom}_A(T,M)\cong\operatorname{Hom}_A(T,tM)$ and $\operatorname{Ext}^1_A(T,M)\cong\operatorname{Ext}^1_A(T,M/tM)$. Therefore $tM\in\mathcal{T}(T)$ implies that

$$\operatorname{Tor}^B_1(\operatorname{Hom}_A(T,M),T)\cong\operatorname{Tor}^B_1(\operatorname{Hom}_A(T,tM),T)=0$$

and

$$tM \cong \operatorname{Hom}_A(T, tM) \otimes_B T \cong \operatorname{Hom}_A(T, M) \otimes_B T.$$

Similarly, $M/tM \in \mathcal{F}(T)$ implies that

$$\operatorname{Ext}^1_A(T, M) \otimes_B T \cong \operatorname{Ext}^1_A(T, M/tM) \otimes_B T = 0$$

and

$$M/tM \cong \operatorname{Tor}_1^B((\operatorname{Ext}^1_A(T, M/tM), T) \cong \operatorname{Tor}_1^B(\operatorname{Ext}^1_A(T, M), T). \qquad \square$$

To illustrate these statements on examples it is useful to have formulas for the dimension vectors of modules in $\mathcal{X}(T_A)$ and $\mathcal{Y}(T_A)$.

3.10. Lemma. *Assume that T_A is a multiplicity-free tilting A-module, $T_A = T_1 \oplus \ldots \oplus T_n$ is its decomposition into a direct sum of indecomposable modules, and $B = \operatorname{End} T_A$. Let $e_i \in \operatorname{End} T_A$ be the composition of the canonical projection $p_i : T \to T_i$ with the canonical injection $u_i : T_i \to T$.*

(a) *The elements $e_1, \ldots, e_n$ are primitive orthogonal idempotents of B such that $1 = e_1 + \ldots + e_n$; there is a B-module isomorphism $e_a B \cong \operatorname{Hom}_A(T, T_a)$, for all a; and there exist K-linear isomorphisms*

$$e_a B e_b \cong \operatorname{Hom}_A(T_b, T_a) \text{ and } \operatorname{Ext}^1_A(e_a T, N) \cong \operatorname{Ext}^1_A(T, N) e_a$$

for all a, b and for any A-module N.

(b) *For any pair of A-modules $M \in \mathcal{T}(T_A)$ and $N \in \mathcal{F}(T_A)$, we have*

$$\begin{aligned} \mathbf{dim}\operatorname{Hom}_A(T, M) &= [\dim_K \operatorname{Hom}_A(T_1, M) \ \ldots \ \dim_K \operatorname{Hom}_A(T_n, M)]^t \text{ and} \\ \mathbf{dim}\operatorname{Ext}^1_A(T, N) &= [\dim_K \operatorname{Hom}_A(N, \tau T_1) \ \ldots \ \dim_K \operatorname{Hom}_A(N, \tau T_n)]^t. \end{aligned}$$

Proof. We recall that, for any L in $\operatorname{mod} A$, the vector space $\operatorname{Hom}_A(T, L)$ has a right B-module structure defined by $fb = f \circ b$ for $f \in \operatorname{Hom}_A(T, L)$ and $b \in B$, where $f \circ b$ means the composition of $b : T \to T$ with $f : T \to L$. It follows from (3.1)(b) and from the assumption that T_A is multiplicity-free that the B-modules $\operatorname{Hom}_A(T, T_1), \ldots, \operatorname{Hom}_A(T, T_n)$ form a complete set of pairwise nonisomorphic indecomposable projective B-modules and, obviously, there is a B-module isomorphism

$$B \cong \operatorname{Hom}_A(T, T_1) \oplus \ldots \oplus \operatorname{Hom}_A(T, T_n).$$

It is easy to see that for any j the B-module homomorphism $\operatorname{Hom}_A(T, T_j) \to e_j B$, defined by $f \mapsto u_j f = e_j u_j f$, is an isomorphism, and the first part of

(a) follows. The isomorphism $\mathrm{Hom}_B(e_bB, e_aB) \cong e_aBe_b$, defined by $h \mapsto h(e_b)$ (see (I.4.2)), together with (3.8)(b) yields $e_aBe_b \cong \mathrm{Hom}_B(e_bB, e_aB) \cong \mathrm{Hom}_B(\mathrm{Hom}_A(T, T_b), \mathrm{Hom}_A(T, T_a)) \cong \mathrm{Hom}_A(T_b, T_a)$.

Because $p_i = p_i \circ e_i$, for each A-module L, the K-linear map

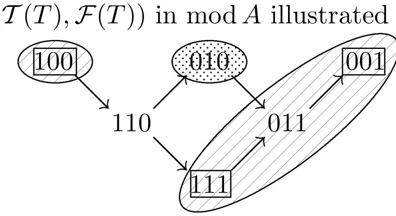
$$\mathrm{Hom}_A(e_iT, L) \longrightarrow \mathrm{Hom}_A(T, L)e_i$$

$g \mapsto g \circ p_i = (g \circ p_i)e_i$ is a K-linear isomorphism, which is functorial in L. Hence, if $I^\bullet$ is an injective resolution of an A-module N, there is an isomorphism $\mathrm{Hom}_A(e_iT, I^\bullet) \cong \mathrm{Hom}_A(T, I^\bullet)e_i$ of complexes and it induces K-linear isomorphisms of the cohomology spaces. In view of (A.4.1) in the Appendix, this yields the isomorphisms $\mathrm{Ext}^1_A(e_iT, N) \cong H^1(\mathrm{Hom}_A(e_iT, I^\bullet)) \cong H^1(\mathrm{Hom}_A(T, I^\bullet)e_i) \cong H^1(\mathrm{Hom}_A(T, I^\bullet))e_i \cong \mathrm{Ext}^1_A(T, N)e_i$. It follows that the ith coordinates of the vectors $\mathbf{dim}\,\mathrm{Hom}_A(T, M)$ and $\mathbf{dim}\,\mathrm{Ext}^1_A(T, N)$ are as follows:

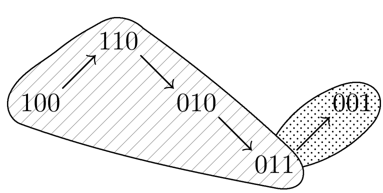
$$\begin{aligned}
(\mathbf{dim}\,\mathrm{Hom}_A(T, M))_i &= \dim_K \mathrm{Hom}_A(T, M)e_i = \dim_K \mathrm{Hom}_A(e_iT, M)\\
&= \dim_K \mathrm{Hom}_A(e_i(T), M) = \dim_K \mathrm{Hom}_A(T_i, M),\\
(\mathbf{dim}\,\mathrm{Ext}^1_A(T, N))_i &= \dim_K \mathrm{Ext}^1_A(T, N)e_i = \dim_K \mathrm{Ext}^1_A(e_iT, N)\\
&= \dim_K \mathrm{Ext}^1_A(e_i(T), N) = \dim_K \mathrm{Ext}^1_A(T_i, N)\\
&= \dim_K D\mathrm{Hom}_A(N, \tau T_i) = \dim_K \mathrm{Hom}_A(N, \tau T_i),
\end{aligned}$$

because $\mathrm{pd}\, T_i \leq 1$ yields $\mathrm{Ext}^1_A(T_i, N) \cong D\mathrm{Hom}_A(N, \tau T_i)$, by (IV.2.14). $\square$

3.11. Examples. (a) Consider, as in Example 1.2 (d), the algebra A given by the quiver $\circ \longleftarrow \circ \longleftarrow \circ$. The tilting module $T_A = 100 \oplus 111 \oplus 001$ induces a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\mathrm{mod}\, A$ illustrated as follows:



Hence, $B = \mathrm{End}\, T_A$ is given by the quiver $\circ \xleftarrow{\mu} \circ \xleftarrow{\lambda} \circ$ bound by $\lambda\mu = 0$. The induced torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\mathrm{mod}\, B$ is illustrated in $\Gamma(\mathrm{mod}\, B)$ as follows:



The effect of the functors $\operatorname{Hom}_A(T,-)$ and $\operatorname{Ext}^1_A(T,-)$ can easily be computed. We have

$$\begin{aligned} \operatorname{Hom}_A(T,100) &\cong 100, & \qquad \operatorname{Hom}_A(T,111) &\cong 110,\\ \operatorname{Hom}_A(T,011) &\cong 010, & \qquad \operatorname{Hom}_A(T,001) &\cong 011, \end{aligned}$$

and finally $\operatorname{Ext}^1_A(T,010) \cong 001$.

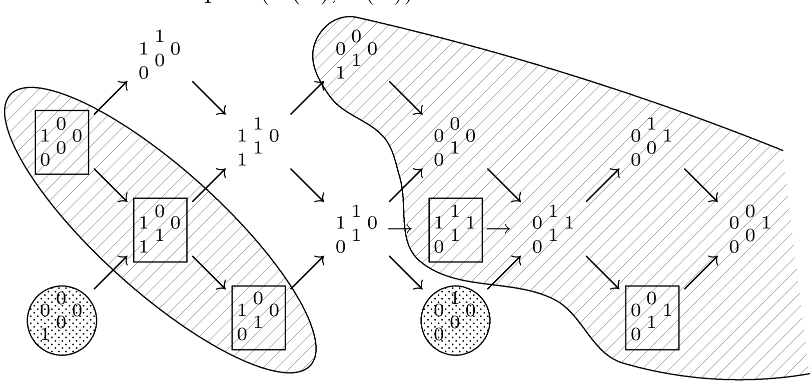
(b) Consider, as in Example 2.8 (a), the algebra A given by the quiver

$$\begin{tikzcd} & \overset{3}{\circ} \arrow[dl, "\beta"'] & \\ 1\,\circ & & \circ\,5 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & \underset{4}{\circ} \arrow[ul, "\delta"] \arrow[dl, "\varepsilon"] & \\ 2\,\circ & & \end{tikzcd}$$

bound by $\alpha\beta = \gamma\delta$ and $\gamma\varepsilon = 0$. The tilting module

$$T_A = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ 0 & & \end{smallmatrix} \;\oplus\; \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ 1 & & \end{smallmatrix} \;\oplus\; \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ 0 & & \end{smallmatrix} \;\oplus\; \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix} \;\oplus\; \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix}$$

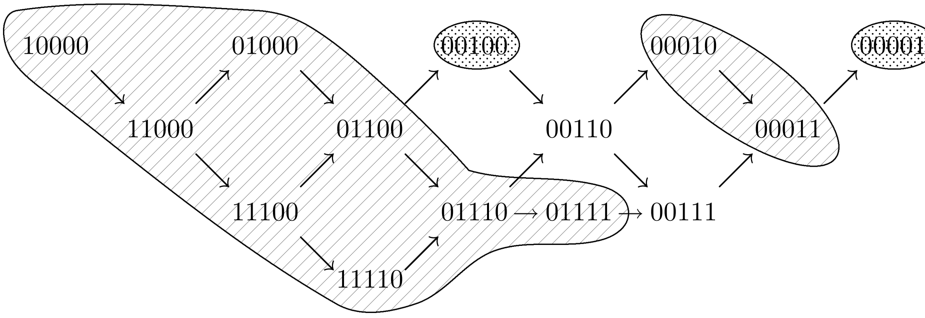
induces a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ illustrated as follows:



Hence, $B = \operatorname{End} T_A$ is given by the quiver

$$\circ \xleftarrow{\ \eta\ } \circ \xleftarrow{\ \nu\ } \circ \xleftarrow{\ \mu\ } \circ \xleftarrow{\ \lambda\ } \circ$$

bound by $\lambda\mu\nu\eta = 0$. The induced torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ is illustrated as follows:



Here, we have

$$
\begin{aligned}
&\operatorname{Hom}_A\left(T, \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{smallmatrix}\right) = 10000, \ \operatorname{Hom}_A\left(T, \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{smallmatrix}\right) = 11000, \operatorname{Hom}_A\left(T, \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{smallmatrix}\right) = 11100,\\
&\operatorname{Hom}_A\left(T, \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{smallmatrix}\right) = 11110, \ \operatorname{Hom}_A\left(T, \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{smallmatrix}\right) = 01000, \operatorname{Hom}_A\left(T, \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{smallmatrix}\right) = 01100,\\
&\operatorname{Hom}_A\left(T, \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{smallmatrix}\right) = 01110, \ \operatorname{Hom}_A\left(T, \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{smallmatrix}\right) = 01111, \operatorname{Hom}_A\left(T, \begin{smallmatrix} & 1 & \\ 0 & 0 & 1 \\ & 0 & \end{smallmatrix}\right) = 00010,\\
&\operatorname{Hom}_A\left(T, \begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{smallmatrix}\right) = 00011, \quad \operatorname{Ext}^1_A\left(T, \begin{smallmatrix} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{smallmatrix}\right) = 00100, \quad \operatorname{Ext}^1_A\left(T, \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{smallmatrix}\right) = 00001.
\end{aligned}
$$

Observe that

$$(DT)_B = \operatorname{Hom}_A(T, DA) = 11110 \oplus 01000 \oplus 01111 \oplus 00010 \oplus 00011.$$

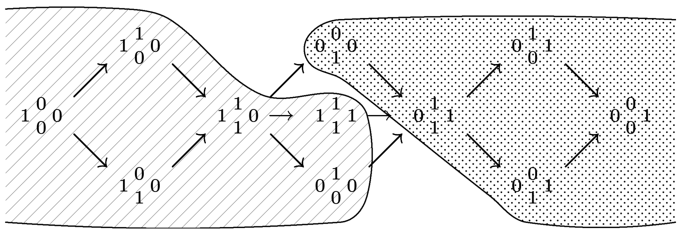
(c) Consider, as in Example 2.8 (b), the algebra A given by the quiver

$$
\begin{tikzcd}
 & & \circ \arrow[dl] \\
\circ & \circ \arrow[l] & \\
 & & \circ \arrow[ul]
\end{tikzcd}
$$

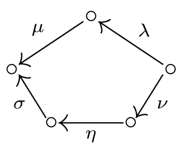
and the tilting module $T_A = \begin{smallmatrix} & & 1 \\ 1 & 1 & \\ & & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 \\ 1 & 1 & \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 \\ 0 & 1 & \\ & & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 \\ 0 & 0 & \\ & & 0 \end{smallmatrix}$. Here, $B = \operatorname{End} T_A$ is given by the quiver

$$
\begin{tikzcd}
 & \circ \arrow[dl, "\beta"'] & \\
\circ & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
 & \circ \arrow[ul, "\delta"] &
\end{tikzcd}
$$

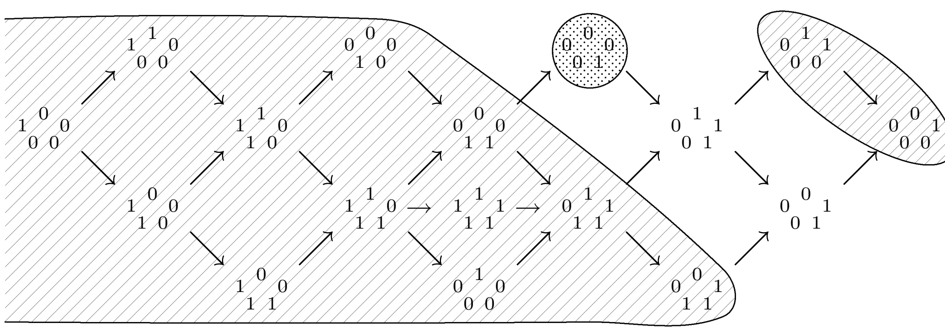
bound by $\alpha\beta = \gamma\delta$. The induced torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ is illustrated as:



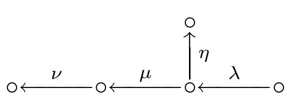
(d) Consider the algebra A of Example (b), with the APR-tilting module $T[2]$. Here, $B = \operatorname{End} T[2]_A$ is given by the quiver



bound by $\lambda\mu = \nu\eta\sigma$. The induced torsion pair $(\mathcal{X}(T[2]), \mathcal{Y}(T[2]))$ in $\operatorname{mod} B$ is illustrated as:



If, on the other hand, one considers the APR-tilting module $T[1]$, one obtains the algebra $\operatorname{End} T[1]_A$ given by the quiver



bound by the relation $\lambda\mu\nu = 0$. We leave to the reader the calculation of $(\mathcal{X}(T[1]), \mathcal{Y}(T[1]))$.

VI.4. Consequences of the tilting theorem

In this section, we investigate the connection between an algebra A and the endomorphism algebras of its tilting modules, using the tilting theorem of Brenner and Butler. Throughout, we keep the notation used in Section 3.

Our first result says that, under tilting, the global dimension of an algebra changes by at most one. As a consequence, this entails that the class of algebras of finite global dimension is closed under the tilting process. We need one lemma.

4.1. Lemma. *Let A be an algebra, T_A be a tilting module, and $B = \operatorname{End} T_A$. If $M \in \mathcal{T}(T)$, then $\operatorname{pd} \operatorname{Hom}_A(T, M) \leq \operatorname{pd} M$.*

Proof. We use induction on $n = \operatorname{pd} M$. If $n = 0$, then M is projective. Because $M \in \mathcal{T}(T) = \operatorname{Gen} T$, this implies that $M \in \operatorname{add} T$. Therefore $\operatorname{Hom}_A(T, M)$ is projective (by (3.1)(b)), and we are done.

Now, assume $n \geq 1$. By (2.5)(c), there exists a short exact sequence

$$0 \longrightarrow L \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with $T_0 \in \operatorname{add} T$ and $L \in \mathcal{T}(T)$. Therefore we have a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, L) \longrightarrow \operatorname{Hom}_A(T, T_0) \longrightarrow \operatorname{Hom}_A(T, M) \longrightarrow 0.$$

Assume $n = 1$. Then the first short exact sequence yields an exact sequence of functors

$$0 = \operatorname{Ext}^1_A(T_0, -)|_{\mathcal{T}(T)} \longrightarrow \operatorname{Ext}^1_A(L, -)|_{\mathcal{T}(T)} \longrightarrow \operatorname{Ext}^2_A(M, -)|_{\mathcal{T}(T)} = 0;$$

therefore $\operatorname{Ext}^1_A(L, -)|_{\mathcal{T}(T)} = 0$, that is, L is Ext-projective in $\mathcal{T}(T)$. By (2.5)(d), $L \in \operatorname{add} T$, so that $\operatorname{Hom}_A(T, L)$ is projective and the second exact sequence implies that $\operatorname{pd} \operatorname{Hom}_A(T, M) \leq 1$. Finally, assume $n \geq 2$. Then, according to (A.4.7) of the Appendix, the first short exact sequence yields $\operatorname{pd} L \leq n - 1$, because $\operatorname{pd} T_0 \leq 1$. By the induction hypothesis, this implies that $\operatorname{pd} \operatorname{Hom}_A(T, L) \leq n - 1$. Hence the second short exact sequence gives

$$\operatorname{pd} \operatorname{Hom}_A(T, M) \leq 1 + \operatorname{pd} \operatorname{Hom}_A(T, L) \leq 1 + (n-1) = n. \qquad \square$$

4.2. Theorem. *Let A be an algebra, T_A be a tilting module, and $B = \operatorname{End} T_A$. Then $|\operatorname{gl.dim} A - \operatorname{gl.dim} B| \leq 1$.*

Proof. Let X be any B-module. There exists a short exact sequence

$$0 \longrightarrow Y \longrightarrow P \longrightarrow X \longrightarrow 0$$

with P projective. Because $P \in \mathcal{Y}(T)$, we have $Y \in \mathcal{Y}(T)$ as well. By the tilting theorem (3.8), there exists $M \in \mathcal{T}(T)$ such that $Y = \operatorname{Hom}_A(T, M)$. By (4.1), we have $\operatorname{pd} Y \leq \operatorname{pd} M$. Hence $\operatorname{pd} X \leq 1 + \operatorname{pd} Y \leq 1 + \operatorname{pd} M \leq 1 + \operatorname{gl.dim} A$, and consequently $\operatorname{gl.dim} B \leq 1 + \operatorname{gl.dim} A$. Because, again by the tilting theorem, ${}_BT$ is also a tilting module, we have $\operatorname{gl.dim} A \leq 1 + \operatorname{gl.dim} B$. □

In Example 3.11 (a), we have $\operatorname{gl.dim} B = 2$, whereas $\operatorname{gl.dim} A = 1$ (hence the bound of (4.2) is sharp). In Example 3.11 (b), we have $\operatorname{gl.dim} A = \operatorname{gl.dim} B = 2$.

There are the following other relations between the homological dimensions in $\operatorname{mod} A$ and $\operatorname{mod} B$ (see Exercise 20):

(a) If $N \in \mathcal{F}(T)$, then $\operatorname{pd} \operatorname{Ext}^1_A(T, N) \leq 1 + \max(1, \operatorname{pd} N)$.
(b) If $M \in \mathcal{T}(T)$, then $\operatorname{id} \operatorname{Hom}_A(T, M) \leq 1 + \operatorname{id} M$.
(c) If $N \in \mathcal{F}(T)$, then $\operatorname{id} \operatorname{Ext}^1_A(T, N) \leq \operatorname{id} N$.

In our next application, we show that the number of simple modules is preserved under the tilting process. For this purpose we recall from (III.3.5) that the Grothendieck group $K_0(A)$ of A is free abelian and that the elements $[S]$, where S ranges over a complete set of representatives of the isomorphism classes of simple A-modules, constitute a basis of $K_0(A)$. The map $[X] \mapsto \mathbf{dim}\, X$ defines a group isomorphism

$$\mathbf{dim} : K_0(A) \xrightarrow{\cong} \mathbb{Z}^n,$$

where n is the number of the isomorphism classes of simple A-modules. Throughout, we identify the group $K_0(A)$ with $\mathbb{Z}^n$ and the element $[X]$ of $K_0(A)$ with the dimension vector $\mathbf{dim}\, X$ in $\mathbb{Z}^n$, for any module X in $\operatorname{mod} A$.

4.3. Theorem. *Let A be an algebra, T_A be a tilting module, and $B = \operatorname{End} T_A$. Then the correspondence*

$$\mathbf{dim}\, M \mapsto \mathbf{dim} \operatorname{Hom}_A(T, M) - \mathbf{dim} \operatorname{Ext}^1_A(T, M),$$

where M is an A-module, induces an isomorphism $f : K_0(A) \to K_0(B)$ of the Grothendieck groups of A and B.

Proof. Because $\operatorname{pd} T_A \leq 1$, any short exact sequence $0 \to L_A \to M_A \to N_A \to 0$ in $\operatorname{mod} A$ induces an exact cohomology sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A(T, L) \longrightarrow \operatorname{Hom}_A(T, M) \longrightarrow \operatorname{Hom}_A(T, N) \\ \longrightarrow \operatorname{Ext}^1_A(T, L) \longrightarrow \operatorname{Ext}^1_A(T, M) \longrightarrow \operatorname{Ext}^1_A(T, N) \longrightarrow 0 \end{aligned}$$

in $\operatorname{mod} B$, from which we deduce the equality

$$\begin{aligned} \mathbf{dim} \operatorname{Hom}_A(T, M) - \mathbf{dim} \operatorname{Ext}^1_A(T, M) = \\ = [\mathbf{dim} \operatorname{Hom}_A(T, L) - \mathbf{dim} \operatorname{Ext}^1(T, L)] + \\ + [\mathbf{dim} \operatorname{Hom}_A(T, N) - \mathbf{dim} \operatorname{Ext}^1(T, N)] \end{aligned}$$

in $K_0(B)$ (see (III.3.3) and (III.3.5)). Hence the given correspondence defines indeed a group homomorphism $f : K_0(A) \to K_0(B)$.

Let S be a simple B-module. Because $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair, we have $S \in \mathcal{X}(T)$ or $S \in \mathcal{Y}(T)$ (by (1.6)). In the latter case, we have $S \cong \operatorname{Hom}_A(T, S \otimes_B T)$ while $\operatorname{Ext}^1_A(T, S \otimes_B T) = 0$, so that $\mathbf{dim}\, S = f(\mathbf{dim}\, S \otimes_B T)$. In the former case, we have $S \cong \operatorname{Ext}^1_A(T, \operatorname{Tor}^B_1(S, T))$ while $\operatorname{Hom}_A(T, \operatorname{Tor}^B_1(S, T)) = 0$, so that $\mathbf{dim}\, S = f(-\mathbf{dim}\operatorname{Tor}^B_1(S, T))$. In either case, $\mathbf{dim}\, S$ lies in the image of f. Because, according to (III.3.5), the vectors of the form $\mathbf{dim}\, S$, where S ranges over a complete set of representatives of the isomorphism classes of simple B-modules, constitute a basis of $K_0(B)$, this shows that f is surjective. Consequently, the rank of $K_0(A)$ is greater than or equal to that of $K_0(B)$. Because ${}_BT$ is also a tilting module and $A \cong \operatorname{End}({}_BT)^{\mathrm{op}}$, we have, by symmetry, that the rank of $K_0(B)$ is greater than or equal to that of $K_0(A)$. Therefore these ranks are equal, and the group epimorphism f is an isomorphism. $\square$

For instance, in Example 3.11 (a), it is easily seen that $f(100) = (100)$, $f(010) = -(001)$, and $f(001) = (011)$. Hence the matrix $\mathbf{F}$ of f in the canonical bases of $K_0(A)$ and $K_0(B)$ is of the form

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

(where the elements of $K_0(A)$ and $K_0(B)$ are considered as column vectors). Thus, the image of the dimension vector of the torsion module $I(2) = 011$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

that is, is the dimension vector of the B-module 010.

We deduce from (4.3) and Bongartz's lemma (2.4) a very useful criterion for deciding whether a partial tilting module is a tilting module or not.

4.4. Corollary. *Let T_A be a partial tilting module. Then T_A is a tilting module if and only if the number of pairwise nonisomorphic indecomposable summands of T equals the number of pairwise nonisomorphic simple modules (that is, the rank of $K_0(A)$).*

Proof. If T_A is a tilting module, and $B = \operatorname{End} T_A$, then by (3.1)(b), the number t of pairwise nonisomorphic indecomposable summands of T equals the rank of $K_0(B)$. Hence, by (4.3), t equals the rank of $K_0(A)$.

Conversely, assume that T_A is a partial tilting module satisfying the stated condition. By Bongartz's lemma (2.3), there exists an A-module E such that $T \oplus E$ is a tilting module. The necessity part says that the number of pairwise nonisomorphic indecomposable summands of $T \oplus E$ equals the rank of $K_0(A)$, hence, by hypothesis, equals the number of pairwise nonisomorphic indecomposable summands of T. Therefore $E \in \operatorname{add} T$ and T is indeed a tilting module. $\square$

Assume now that A is an algebra of finite global dimension. We recall from (III.3.11) and (III.3.13) that the Euler characteristic of A is the bilinear form on $K_0(A)$ defined by

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_A = \sum_{s=0}^{\infty} (-1)^s \dim_K \operatorname{Ext}_A^s(M, N),$$

where M, N are modules in $\operatorname{mod} A$. The preceding sum is finite due to our hypothesis on A. We next show that the Euler characteristic of A is preserved under tilting; namely, that the isomorphism between the Grothendieck groups of A and B defined in (4.3) is an isometry of the Euler characteristics of A and B.

4.5. Proposition. *Let A be an algebra of finite global dimension, T_A be a tilting module, $B = \operatorname{End} T_A$, and $f : K_0(A) \to K_0(B)$ be the isomorphism of* (4.3). *Then for any A-modules M and N we have*

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_A = \langle f(\mathbf{dim}\, M), f(\mathbf{dim}\, N) \rangle_B.$$

Proof. Let $T_1, \ldots, T_n$ denote the pairwise nonisomorphic indecomposable summands of T. We claim that the vectors $\mathbf{dim}\, T_i$, where $1 \le i \le n$, constitute a basis of $K_0(A)$. Indeed, by (3.1)(b), the B-modules

$$\operatorname{Hom}_A(T, T_1), \ldots, \operatorname{Hom}_A(T, T_n)$$

form a complete set of representatives of the isomorphism classes of indecomposable projective modules. Because, by (4.2), B also has finite global dimension, the vectors $f(\mathbf{dim}\, T_i) = \mathbf{dim} \operatorname{Hom}_A(T, T_i)$, where $1 \le i \le n$, constitute a basis of $K_0(B)$. Because, by (4.3), f is an isomorphism, this implies our claim.

Also, the projectivity of the B-modules $\operatorname{Hom}_A(T, T_i)$ and the tilting theorem imply that, for any i, j such that $1 \le i, j \le n$,

$$\begin{aligned}
\langle f(\mathbf{dim}\, T_i), f(\mathbf{dim}\, T_j) \rangle_B &= \langle \mathbf{dim} \operatorname{Hom}_A(T, T_i), \mathbf{dim} \operatorname{Hom}_A(T, T_j) \rangle_B \\
&= \dim_K \operatorname{Hom}_B(\operatorname{Hom}_A(T, T_i), \operatorname{Hom}_A(T, T_j)) \\
&= \dim_K \operatorname{Hom}_A(T_i, T_j) = \langle \mathbf{dim}\, T_i, \mathbf{dim}\, T_j \rangle_A,
\end{aligned}$$

because $\operatorname{Ext}_A^1(T_i, T_j) = 0$. The conclusion follows from our claim. □

Let $\mathbf{A}$ and $\mathbf{B}$ be the matrices defining the Euler characteristics of the algebras A and B, respectively, and let $\mathbf{F}$ denote the matrix defining the isomorphism f of (4.3). It follows from (4.3) that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{F}$ are all square matrices of the same size, and from the explicit expression of f that the matrix $\mathbf{F}$ has integral coefficients. Because for $\mathbf{x}, \mathbf{y} \in K_0(A)$, we have

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \mathbf{x}^t \mathbf{A} \mathbf{y} \text{ and } \langle f(\mathbf{x}), f(\mathbf{y}) \rangle_B = (\mathbf{F}\mathbf{x})^t \mathbf{B} (\mathbf{F}\mathbf{y}) = \mathbf{x}^t (\mathbf{F}^t \mathbf{B} \mathbf{F}) \mathbf{y},$$

we infer from (4.5) that $\mathbf{x}^t \mathbf{A} \mathbf{y} = \mathbf{x}^t (\mathbf{F}^t \mathbf{B} \mathbf{F}) \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in K_0(A)$. That is, $\mathbf{A} = \mathbf{F}^t \mathbf{B} \mathbf{F}$; the matrices $\mathbf{A}$ and $\mathbf{B}$ are $\mathbb{Z}$-congruent.

We deduce the following corollary.

4.6. Corollary. *Let A be an algebra of finite global dimension, T_A be a tilting module, and $B = \operatorname{End} T_A$. Then the Cartan matrices $\mathbf{C}_A$ of A and $\mathbf{C}_B$ of B are $\mathbb{Z}$-congruent.*

Proof. By (III.3.11) and the preceding discussion, we have $\mathbf{A} = (\mathbf{C}_A^{-1})^t$ and $\mathbf{B} = (\mathbf{C}_B^{-1})^t$. Thus, the equality $\mathbf{A} = \mathbf{F}^t \mathbf{B} \mathbf{F}$ can be written as $(\mathbf{C}_A^{-1})^t = \mathbf{F}^t (\mathbf{C}_B^{-1})^t \mathbf{F}$, or, equivalently, as $\mathbf{C}_B = \mathbf{F} \mathbf{C}_A \mathbf{F}^t$. □

These considerations also apply to the integral Euler quadratic form $q_A : K_0(A) \to \mathbb{Z}$ attached to the Euler characteristic of A by the formula

$$q_A(\mathbf{dim}\, M) = \langle \mathbf{dim}\, M, \mathbf{dim}\, M \rangle_A,$$

where M is an A-module; see (III.3.11). The equality $\mathbf{A} = \mathbf{F}^t \mathbf{B} \mathbf{F}$ yields the following corollary.

4.7. Corollary. *Let A be an algebra of finite global dimension, T_A be a tilting module, and $B = \operatorname{End} T_A$. Then the Euler quadratic forms q_A and q_B are $\mathbb{Z}$-congruent.* □

Let, for instance, A be as in Example 3.11 (a), that is, A is given by the quiver

$$\overset{1}{\circ} \longleftarrow \overset{2}{\circ} \longleftarrow \overset{3}{\circ}$$

Then

$$\mathbf{C}_A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and consequently

$$\mathbf{A} = (\mathbf{C}_A^{-1})^t = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

so that

$$q_A(\mathbf{x}) = \mathbf{x}^t\mathbf{A}\mathbf{x} = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3, \text{ for } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in K_0(A).$$

We tilt A to B, where B is given by the quiver

$$\underset{1}{\circ} \xleftarrow{\mu} \underset{2}{\circ} \xleftarrow{\lambda} \underset{3}{\circ}$$

bound by $\lambda\mu = 0$. Then

$$\mathbf{C}_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and consequently

$$\mathbf{B} = (\mathbf{C}_B^{-1})^t = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix},$$

so that

$$q_B(\mathbf{x}) = \mathbf{x}^t\mathbf{B}\mathbf{x} = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 + x_1x_3, \text{ for } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in K_0(B).$$

We have already observed that the matrix $\mathbf{F}$ defining the group isomorphism $f \ : \ K_0(A) \xrightarrow{\cong} K_0(B)$ is of the form

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Finally, it is easily verified that

$$\begin{aligned} \mathbf{F}^t\mathbf{B}\mathbf{F} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \mathbf{A}. \end{aligned}$$

As a third and final application of the tilting theorem, we consider those almost split sequences in $\operatorname{mod} B$ whose left term lies in $\mathcal{Y}(T)$ and whose right term lies in $\mathcal{X}(T)$; such sequences are called **connecting sequences**. The following easy lemma shows that there are only finitely many connecting sequences.

4.8. Lemma. *If* $0 \to Y_B \to E_B \to X_B \to 0$ *is a connecting sequence, then there exists an indecomposable injective* A*-module* $I(a)$ *such that* $Y \cong \mathrm{Hom}_A(T, I(a))$.

Proof. Because $Y \in \mathcal{Y}(T)$, according to (3.8), there exists $M \in \mathcal{T}(T)$ such that $Y \cong \mathrm{Hom}_A(T, M)$. Let $f : M \to N$ be an injective envelope in $\mathrm{mod}\, A$ and consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{f} N \longrightarrow N/M \longrightarrow 0.$$

Because $N \in \mathcal{T}(T)$, this sequence lies entirely in $\mathcal{T}(T)$. Applying the functor $\mathrm{Hom}_A(T, -)$ yields a short exact sequence in $\mathcal{Y}(T)$

$$0 \longrightarrow Y \xrightarrow{\mathrm{Hom}_A(T,f)} \mathrm{Hom}_A(T, N) \xrightarrow{\qquad\qquad} \mathrm{Hom}_A(T, N/M) \longrightarrow 0.$$

Since $\tau^{-1} Y = X \in \mathcal{X}(T)$, we deduce from (1.11)(b) that Y is Ext-injective in $\mathcal{Y}(T)$. Therefore the preceding short exact sequence splits, that is, $\mathrm{Hom}_A(T, f)$ is a section. Applying $- \otimes_B T$ shows that f is a section. We have thus shown that M is injective. Its indecomposability follows from the indecomposability of Y. Hence M is isomorphic to an indecomposable injective module $I(a)$. □

Of course, not all indecomposable injective A-modules correspond to connecting sequences. The next lemma, known as the **connecting lemma**, characterises those that do and gives the right term of such a sequence. More precisely, one can show, exactly as in (4.8), that the right term X of a connecting sequence $0 \to Y \to E \to X \to 0$ satisfies $X \cong \mathrm{Ext}^1_A(T, P)$ for some indecomposable projective A-module P. The connecting lemma says that the top of P is isomorphic to the socle of I, and that $P \notin \mathrm{add}\, T$.

4.9. Connecting lemma. *Let* A *be an algebra,* T_A *be a tilting module, and* $B = \mathrm{End}\, T_A$*. Let* $P(a)$ *be the projective cover of a simple module* $S(a)_A$ *and* $I(a)$ *be its injective envelope. Then*

$$\tau^{-1} \mathrm{Hom}_A(T, I(a)) \cong \mathrm{Ext}^1_A(T, P(a)).$$

In particular, $P(a) \in \mathrm{add}\, T$ *if and only if* $\mathrm{Hom}_A(T, I(a))$ *is an injective* B*-module.*

Proof. Let $P = P(a) = e_a A$ and $I = I(a) = D(Ae_a)$, where $e_a \in A$ is a primitive idempotent. By (III.2.11), there is a functorial isomorphism $D\mathrm{Hom}_A(T, I) \cong \mathrm{Hom}_A(P, T)$. We need to show that the transpose Tr of $\mathrm{Hom}_A(P, T)$ is isomorphic to $\mathrm{Ext}^1_A(T, P)$. For this purpose, we use the definition of the transpose (IV.2).

Because T_A is a tilting module, there exists a short exact sequence

$$0 \longrightarrow P_A \longrightarrow T'_A \xrightarrow{f} T''_A \longrightarrow 0$$

with $T', T'' \in \operatorname{add} T$. Applying $\operatorname{Hom}_A(-, {}_BT_A)$ yields a short exact sequence

$$0 \to \operatorname{Hom}_A(T'', {}_BT_A) \xrightarrow{\operatorname{Hom}_A(f,T)} \operatorname{Hom}_A(T', {}_BT_A) \to \operatorname{Hom}_A(P(a), {}_BT_A) \to 0,$$

which is a projective resolution for the left B-module $\operatorname{Hom}_A(P,T)$. The transpose (in $\operatorname{mod} B$) of $\operatorname{Hom}_A(P,T)$ is obtained by applying to the previous sequence the functor $(-)^t = \operatorname{Hom}_B(-, B) = \operatorname{Hom}_B(-, \operatorname{Hom}_A(T,T))$. If $T_0 \in \operatorname{add} T$, we have a functorial isomorphism in $\operatorname{add} T$ given by

$$\operatorname{Hom}_A(T, T_0) \cong \operatorname{Hom}_B(\operatorname{Hom}_A(T_0, T), \operatorname{Hom}_A(T,T)).$$

Indeed, such an isomorphism exists when $T_0 = T$ and the functors are additive. Hence the commutative square

$$\begin{tikzcd}
\operatorname{Hom}_B(\operatorname{Hom}_A(T',T),\operatorname{Hom}_A(T,T)) \arrow[r, "\cong"] \arrow[d, "\operatorname{Hom}_B(\operatorname{Hom}_A(f,T),\operatorname{Hom}_A(T,T))"'] & \operatorname{Hom}_A(T,T') \arrow[d, "\operatorname{Hom}_A(T,f)"'] \\
\operatorname{Hom}_B(\operatorname{Hom}_A(T'',T),\operatorname{Hom}_A(T,T)) \arrow[r, "\cong"'] & \operatorname{Hom}_A(T,T'')
\end{tikzcd}$$

shows that $\operatorname{Hom}_A(f,T)^t \cong \operatorname{Hom}_A(T,f)$. On the other hand, applying $\operatorname{Hom}_A(T,-)$ to the first short exact sequence yields an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T,P) \longrightarrow \operatorname{Hom}_A(T,T') \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T,T'') \longrightarrow \operatorname{Ext}^1_A(T,P) \longrightarrow 0.$$

By definition of the transpose, we deduce, as required

$$\operatorname{Ext}^1_A(T,P) \cong \operatorname{Tr}\operatorname{Hom}_A(P,T) \cong \operatorname{Tr} D\operatorname{Hom}_A(T,I) = \tau^{-1}\operatorname{Hom}_A(T,I).$$

The second statement follows from the fact that a projective module P lies in $\operatorname{add} T$ if and only if it lies in $\mathcal{T}(T) = \operatorname{Gen} T$, that is, if and only if $\operatorname{Ext}^1_A(T,P) = 0$. □

The middle term of a connecting sequence, on the other hand, can only be approximated by means of its canonical sequence.

4.10. Corollary. *Let $P(a)$, $I(a)$, and $S(a)$ be as in* (4.9)*, with $P(a) \notin \operatorname{add} T$. Consider the connecting sequence*

$$0 \longrightarrow \operatorname{Hom}_A(T, I(a)) \xrightarrow{u} E_B \xrightarrow{v} \operatorname{Ext}_A^1(T, P(a)) \longrightarrow 0.$$

The canonical sequence of E_B in the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is

$$0 \longrightarrow \operatorname{Ext}_A^1(T, \operatorname{rad} P(a)) \longrightarrow E_B \longrightarrow \operatorname{Hom}_A(T, I(a)/S(a)) \longrightarrow 0.$$

Proof. Because $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion pair, the simple module $S(a)$ lies in either $\mathcal{T}(T)$ or $\mathcal{F}(T)$ (by (1.6)).

(a) Assume that $S(a) \in \mathcal{T}(T)$; then $\operatorname{Ext}_A^1(T, S(a)) = 0$. Hence the short exact sequence

$$0 \to S(a) \to I(a) \to I(a)/S(a) \to 0$$

induces a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, S(a)) \xrightarrow{f} \operatorname{Hom}_A(T, I(a)) \xrightarrow{g} \operatorname{Hom}_A(T, I(a)/S(a)) \longrightarrow 0.$$

On the other hand, $P(a) \notin \operatorname{add} T$ implies $P(a) \notin \mathcal{T}(T)$ so that $P(a) \neq tP(a)$ and hence $tP(a) \subseteq \operatorname{rad} P(a)$, which yields a K-linear isomorphism $\operatorname{Hom}_A(T, P(a)) \cong \operatorname{Hom}_A(T, \operatorname{rad} P(a))$ and the exact sequence in $\operatorname{mod} A$

$$0 \to \operatorname{rad} P(a) \to P(a) \to S(a) \to 0$$

induces a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, S(a)) \xrightarrow{h} \operatorname{Ext}_A^1(T, \operatorname{rad} P(a)) \xrightarrow{k} \operatorname{Ext}_A^1(T, P(a)) \longrightarrow 0.$$

This sequence does not split; otherwise, there would exist a nonzero homomorphism from the torsion B-module

$$\operatorname{Ext}_A^1(T, \operatorname{rad} P(a)) \cong \operatorname{Ext}_A^1(T, \operatorname{rad} P(a)/t\operatorname{rad} P(a))$$

to the torsion-free module $\operatorname{Hom}_A(T, S(a))$ (see (3.9)), a contradiction. In particular, k is not a retraction. Because the given connecting sequence is almost split, there exists a homomorphism $f' : \operatorname{Ext}_A^1(T, \operatorname{rad} P(a)) \to E$ such that $k = vf'$. By passing to the kernels, there exists a homomorphism $\operatorname{Hom}_A(T, S(a)) \to \operatorname{Hom}_A(T, I(a))$ whose composition with u equals $f'h$. But the K-vector space $\operatorname{Hom}_B(\operatorname{Hom}_A(T, S(a)), \operatorname{Hom}_A(T, I(a))) \cong \operatorname{Hom}_A(S, I(a))$ is one-dimensional. Hence this homomorphism can be taken equal to f, after replacing h, if necessary, by one of its scalar multiples, so that we have a commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}_A(T, S(a)) \arrow[r, "h"] \arrow[d, "f"'] & \mathrm{Ext}_A^1(T, \mathrm{rad}\, P(a)) \arrow[r, "k"] \arrow[d, "f'"'] & \mathrm{Ext}_A^1(T, P(a)) \arrow[r] \arrow[d, "1"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(T, I(a)) \arrow[r, "u"] \arrow[d, "g"'] & E \arrow[r, "v"] \arrow[d, "g'"'] & \mathrm{Ext}_A^1(T, P(a)) \arrow[r] & 0 \\
 & \mathrm{Hom}_A(T, I(a)/S(a)) \arrow[r, "1"] \arrow[d] & \mathrm{Hom}_A(T, I(a)/S(a)) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

The middle column yields the result.

(b) Assume that $S(a) \in \mathcal{F}(T)$; then $\mathrm{Hom}_A(T, S(a)) = 0$ and hence we have short exact sequences

$$
0 \longrightarrow \mathrm{Ext}^1(T, \mathrm{rad}\, P(a)) \longrightarrow \mathrm{Ext}_A^1(T, P(a)) \longrightarrow \mathrm{Ext}_A^1(T, S(a)) \longrightarrow 0,
$$
$$
0 \longrightarrow \mathrm{Hom}_A(T, I(a)) \longrightarrow \mathrm{Hom}_A(T, I(a)/S(a)) \longrightarrow \mathrm{Ext}_A^1(T, S(a)) \longrightarrow 0.
$$

The second sequence does not split and we deduce, exactly as in (a), a commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{Ext}_A^1(T, \mathrm{rad}\, P(a)) \arrow[r, "1"] \arrow[d] & \mathrm{Ext}_A^1(T, \mathrm{rad}\, P(a)) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_A(T, I(a)) \arrow[r, "u"] \arrow[d, "1"'] & E \arrow[r, "v"] \arrow[d] & \mathrm{Ext}_A^1(T, P(a)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(T, I(a)) \arrow[r] & \mathrm{Hom}_A(T, I(a)/S(a)) \arrow[r] \arrow[d] & \mathrm{Ext}_A^1(T, S(a)) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

Again the middle column yields the result. $\square$

For instance, in Example 3.11 (a), the only connecting sequence is the sequence

$$0 \longrightarrow 010 \longrightarrow 011 \longrightarrow 001 \longrightarrow 0.$$

Here, $S_A = 010$, $I_A = 011$, $P_A = 110$ and we have $\operatorname{Hom}_A(T, I) = 010$ and $\operatorname{Ext}^1_A(T, P) = 001$. The middle term E lies entirely in $\mathcal{Y}(T)$, hence

$$E \cong \operatorname{Hom}_A(T, I/S) = \operatorname{Hom}_A(T, 001) = 011.$$

In Example 3.11 (c), the connecting sequence

$$0 \longrightarrow \; 1\,{}^{1}_{1}\,0 \; \longrightarrow \; 0\,{}^{0}_{1}\,0 \,\oplus\, 1\,{}^{1}_{1}\,1 \,\oplus\, 0\,{}^{1}_{0}\,0 \; \longrightarrow \; 0\,{}^{1}_{1}\,1 \; \longrightarrow 0$$

corresponds to the simple A-module $S = 0\,1\,{}^{0}_{0}$. Here, $I_A = 0\,1\,{}^{1}_{1}$, $P_A = 1\,1\,{}^{0}_{0}$, $\operatorname{Hom}_A(T, I) = 1\,{}^{1}_{1}\,0$, $\operatorname{Ext}^1_A(T, P) = 0\,{}^{1}_{1}\,1$. The middle term E is a direct sum of three indecomposable modules. Indeed, $I/S = 0\,0\,{}^{1}_{0} \oplus 0\,0\,{}^{0}_{1}$ so that

$$\operatorname{Hom}_A(T, I/S) = \operatorname{Hom}_A\left(T,\ 0\,0\,{}^{1}_{0}\right) \oplus \operatorname{Hom}_A\left(T,\ 0\,0\,{}^{0}_{1}\right) = \ 1\,{}^{1}_{1}\,1 \,\oplus\, 0\,{}^{1}_{0}\,0\,,$$

whereas $\operatorname{rad} P = 1\,0\,{}^{0}_{0}$, so that $\operatorname{Ext}^1_A(T, \operatorname{rad} P) = 0\,{}^{0}_{1}\,0$.

The reader may have noticed that in Examples (3.11), it turns out that the indecomposable summands of E are either torsion or torsion-free (that is, the corresponding canonical sequence splits). This is generally not the case, as will be shown in Exercise 14.

VI.5. Separating and splitting tilting modules

It is reasonable to consider those tilting modules that induce splitting torsion pairs, one in $\operatorname{mod} A$ and the other in $\operatorname{mod} B$, where $B = \operatorname{End} T_A$. This leads to the following definition.

5.1. Definition. Let A be an algebra, T_A be a tilting module, and $B = \operatorname{End} T_A$. Then

(a) T_A is said to be **separating** if the induced torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ is splitting, and
(b) T_A is said to be **splitting** if the induced torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ is splitting.

For instance, let, as in Example 3.11 (a), A be given by the quiver

$$\overset{1}{\circ} \longleftarrow \overset{2}{\circ} \longleftarrow \overset{3}{\circ}$$

Then the shown tilting module $T_A = 100 \oplus 111 \oplus 001$ is splitting but not separating. On the other hand, it is easily seen that, over the same algebra A, the APR-tilting module $T[1]_A$ is both splitting and separating. In general, however, an APR-tilting module is necessarily separating, as we showed in Example 2.8 (c), but it is not always splitting, as was seen in (3.11)(d). Finally, Example 3.11 (b) showed a tilting module that is neither separating nor splitting.

Clearly, if T_A is a splitting tilting module, then every indecomposable B-module is the image of an indecomposable A-module via one of the functors $\operatorname{Hom}_A(T,-)$ or $\operatorname{Ext}^1_A(T,-)$, so that B has fewer indecomposable modules than A (in particular, if A is representation–finite, then so is B). Moreover, the almost split sequences in $\operatorname{mod} B$ are easily characterised.

5.2. Proposition. *Let A be an algebra, T_A be a splitting tilting module, and $B = \operatorname{End} T_A$. Then any almost split sequence in* $\operatorname{mod} B$ *lies entirely in either $\mathcal{X}(T)$ or $\mathcal{Y}(T)$, or else it is of the form*

$$0 \to \operatorname{Hom}_A(T, I) \to \operatorname{Hom}_A(T, I/\operatorname{soc} I) \oplus \operatorname{Ext}^1_A(T, \operatorname{rad} P) \to \operatorname{Ext}^1_A(T, P) \to 0,$$

where P is an indecomposable projective module not lying in $\operatorname{add} T$ *and I is the indecomposable injective module such that $P/\operatorname{rad} P \cong \operatorname{soc} I$.*

Proof. Let $0 \to E' \to E \to E'' \to 0$ be an almost split sequence in $\operatorname{mod} B$. Because $(\mathcal{X}(T), \mathcal{Y}(T))$ is a splitting torsion pair, either this sequence lies entirely in one of the subcategories $\mathcal{X}(T)$ and $\mathcal{Y}(T)$ or we have $E' \in \mathcal{Y}(T)$ and $E'' \in \mathcal{X}(T)$; that is, it is a connecting sequence. In this last case, it follows from (4.8) and (4.9) that it is of the form

$$0 \longrightarrow \operatorname{Hom}_A(T, I) \longrightarrow E_B \longrightarrow \operatorname{Ext}^1_A(T, P) \longrightarrow 0,$$

where P and I are as required. Further, it follows from (4.10) that the canonical sequence for E in $(\mathcal{X}(T), \mathcal{Y}(T))$ is of the form

$$0 \longrightarrow \operatorname{Ext}^1_A(T, \operatorname{rad} P) \longrightarrow E_B \longrightarrow \operatorname{Hom}_A(T, I/\operatorname{soc} I) \longrightarrow 0.$$

Because $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting, this canonical sequence splits (1.7) so that $E \cong \operatorname{Ext}^1_A(T, \operatorname{rad} P) \oplus \operatorname{Hom}_A(T, I/\operatorname{soc} I)$. $\square$

The following lemma shows that the almost split sequences in $\operatorname{mod} A$ lying entirely inside one of the classes $\mathcal{T}(T)$ and $\mathcal{F}(T)$ give rise to almost split sequences in $\operatorname{mod} B$.

5.3. Lemma. *Let A be an algebra, T_A be a splitting tilting module, and $B = \operatorname{End} T_A$. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an almost split sequence in* $\operatorname{mod} A$.

(a) *If the modules L, M, and N lie in $\mathcal{T}(T)$, then*

$$0 \to \operatorname{Hom}_A(T, L) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, M) \xrightarrow{\operatorname{Hom}_A(T,g)} \operatorname{Hom}_A(T, N) \to 0$$

is an almost split sequence in $\operatorname{mod} B$, *all of whose terms lie in $\mathcal{Y}(T)$.*

(b) *If the modules L, M, and N lie in $\mathcal{F}(T)$, then*

$$0 \ \to \ \operatorname{Ext}^1_A(T, L) \xrightarrow{\operatorname{Ext}^1_A(T,f)} \operatorname{Ext}^1_A(T, M) \xrightarrow{\operatorname{Ext}^1_A(T,g)} \operatorname{Ext}^1_A(T, N) \ \to \ 0$$

is an almost split sequence in $\operatorname{mod} B$, *all of whose terms lie in $\mathcal{X}(T)$.*

Proof. We only prove (a); the proof of (b) is similar. Because the modules L, M, and N lie in $\mathcal{T}(T) = \operatorname{Gen} T_A$, $\operatorname{Ext}^1_A(T, L) = 0$ and the sequence of B-modules

$$0 \to \operatorname{Hom}_A(T, L) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, M) \xrightarrow{\operatorname{Hom}_A(T,g)} \operatorname{Hom}_A(T, N) \to 0$$

is exact. Moreover, the B-modules $\operatorname{Hom}_A(T, L)$ and $\operatorname{Hom}_A(T, N)$ are indecomposable, because N and L are. By (IV.1.13), it suffices to show that $\operatorname{Hom}_A(T, f)$ and $\operatorname{Hom}_A(T, g)$ are irreducible. By (3.8), the functor $\operatorname{Hom}_A(T, -)$ induces an equivalence of categories $\mathcal{Y}(T) \xrightarrow{\cong} \mathcal{T}(T)$, and therefore the homomorphism $\operatorname{Hom}_A(T, f)$ is neither a section nor a retraction. Assume that there exist $u : \operatorname{Hom}_A(T, L) \to Y$ and $v : Y \to \operatorname{Hom}_A(T, M)$ in $\operatorname{mod} B$ such that $\operatorname{Hom}_A(T, f) = vu$. Because $u \neq 0$ (because $f \neq 0$), $Y \in \mathcal{Y}(T)$ and there exists $E \in \mathcal{T}(T)$ such that $Y \cong \operatorname{Hom}_A(T, E)$. Moreover, there exist homomorphisms of A-modules $u' : L \to E$ and $v' : E \to M$ such that $u = \operatorname{Hom}_A(T, u')$ and $v = \operatorname{Hom}_A(T, v')$. It follows that $f = v'u'$, and therefore u' is a retraction or v' is a section. Hence u is a retraction, or v is a section. This shows that $\operatorname{Hom}_A(T, f)$ is an irreducible morphism. The proof that $\operatorname{Hom}_A(T, g)$ is an irreducible morphism is similar. $\square$

The following technical property will be needed in Chapter VIII.

5.4. Lemma. *Let A be an algebra, I be an indecomposable injective A-module, T_A be a splitting tilting module, and $B = \operatorname{End} T_A$.*

(a) *If $Y_B \in \mathcal{Y}(T)$ is indecomposable, then there exists an irreducible morphism $\operatorname{Hom}_A(T, I) \to Y$ in $\operatorname{mod} B$ if and only if there exists an indecomposable A-module J such that $Y \cong \operatorname{Hom}_A(T, J)$ and J is isomorphic to a direct summand of $I/\operatorname{soc} I$.*

(b) *If $X_B \in \mathcal{X}(T)$ is indecomposable, then there exists an irreducible morphism $\operatorname{Hom}_A(T, I) \to X$ in $\operatorname{mod} B$ if and only if there exists an indecomposable injective A-module J such that $\tau X \cong \operatorname{Hom}_A(T, J)$ and I is a direct summand of $J/\operatorname{soc} J$. Further, in this case, $X \cong \operatorname{Ext}^1_A(T, P)$, where P is the projective cover of* $\operatorname{soc} J$.

Proof. Let $p : I \to I/\operatorname{soc} I$ be the canonical surjection. We claim that the homomorphism $f = \operatorname{Hom}_A(T, p)$ is irreducible in $\operatorname{mod} B$. By (3.8), the functor $\operatorname{Hom}_A(T, -)$ induces an equivalence of categories $\mathcal{Y}(T) \xrightarrow{\cong} \mathcal{T}(T)$, and therefore f is neither a section nor a retraction. Assume that $f = hg$, where $g : \operatorname{Hom}_A(T, I) \to Z$ and $h : Z \to \operatorname{Hom}_A(T, I/socI)$ are in $\operatorname{mod} B$. Because $h \neq 0$ (because $f \neq 0$), $Z \notin \mathcal{X}(T)$ and therefore $Z \in \mathcal{Y}(T)$, because T_A is a splitting tilting module. By (3.8)(b), there exists $M \in \mathcal{T}(T)$ such that $Z \cong \operatorname{Hom}_A(T, M)$. Moreover, there exist homomorphisms of A-modules $g' : I \to M$ and $h' : M \to I/\operatorname{soc} I$ such that $g = \operatorname{Hom}_A(T, g')$ and $h = \operatorname{Hom}_A(T, h')$. It follows that $p = h'g'$, and therefore h' is a retraction or g' is a section. Hence h is a retraction or g is a section. This shows that $\operatorname{Hom}_A(T, p)$ is an irreducible morphism. The sufficiency follows from (IV.1.10) and (IV.4.2).

For the necessity, let $Y_B \in \mathcal{Y}(T)$ be an indecomposable module and $f : \operatorname{Hom}_A(T, I) \to Y$ be an irreducible morphism in mod B. Then there exists an indecomposable A-module J such that $Y \cong \operatorname{Hom}_A(T, J)$ and a homomorphism of B-modules $f' : I \to J$ such that $f = \operatorname{Hom}_A(T, f')$. Because, according to (IV.3.5)(b), $p : I \longrightarrow I/\operatorname{soc} I$ is left minimal almost split, there exists $g' : I/\operatorname{soc} I \longrightarrow J$ such that $f' = g'p$. Moreover, because f is irreducible, so is f' (by the equivalence $\mathcal{Y}(T) \xrightarrow{\cong} \mathcal{T}(T)$). Therefore g' is a retraction and so J is isomorphic to a direct summand of $I/\operatorname{soc} I$.

(b) Let $f : \operatorname{Hom}_A(T, I) \to X_B$ be irreducible with $X_B \in \mathcal{X}(T)$ indecomposable. Because all the projective B-modules lie in $\mathcal{Y}(T)$, the module X is not projective, hence there exists an irreducible morphism $\tau X \to \operatorname{Hom}_A(T, I)$. Because $\operatorname{Hom}_A(T, I) \in \mathcal{Y}(T)$, we deduce that $\tau X \in \mathcal{Y}(T)$. By (5.2), the almost split sequence ending with X is a connecting sequence, so that there exists an indecomposable injective A-module J such that $\tau X \cong \operatorname{Hom}_A(T, J)$. If P denotes the projective cover of $\operatorname{soc} J$, then $X \cong \operatorname{Ext}^1_A(T, P)$. By (a), the existence of an irreducible morphism $g : \operatorname{Hom}_A(T, J) \to \operatorname{Hom}_A(T, I)$ implies that I is isomorphic to a direct summand of $J/\operatorname{soc} J$. This shows the necessity.

Conversely, assume that J_A is an indecomposable injective module such that $\tau X \cong \operatorname{Hom}_A(T, J)$ and I a direct summand of $J/\operatorname{soc} J$. Then (a) yields an irreducible morphism $\tau X \to \operatorname{Hom}_A(T, I)$. Hence, in view of (IV.3.8), there exists an irreducible morphism $\operatorname{Hom}_A(T, I) \to X$. $\square$

There exists a characterisation of separating and splitting tilting modules, due to Hoshino [94]. To prove it, we need the following lemma.

5.5. Lemma. *Let A be an algebra, T_A be a tilting module, and $B = \operatorname{End} T_A$. If $M \in \mathcal{T}(T)$ and $N \in \mathcal{F}(T)$, then, for any $j \geq 1$, there is an* *isomorphism*

$$\mathrm{Ext}_A^j(M,N) \cong \mathrm{Ext}_B^{j-1}(\mathrm{Hom}_A(T,M), \mathrm{Ext}_A^1(T,N)).$$

Proof. Let $0 \to N \to I \to N' \to 0$ be a short exact sequence, with I injective. Thus I and N' belong to $\mathcal{T}(T)$. Applying $\mathrm{Hom}_A(T,-)$ yields a short exact sequence in $\mathrm{mod}\, B$

$$0 \longrightarrow \mathrm{Hom}_A(T,I) \longrightarrow \mathrm{Hom}_A(T,N') \longrightarrow \mathrm{Ext}_A^1(T,N) \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_B(\mathrm{Hom}_A(T,M),-)$, we obtain the long exact cohomology sequence

$$\begin{aligned}&0\to\mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,I))\to\mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,N'))\\ &\to \mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Ext}_A^1(T,N))\to\mathrm{Ext}_B^1(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,I))\\ &\to \ldots\end{aligned}$$

$$\begin{aligned}&\ldots\to\mathrm{Ext}_B^j(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,I))\to\mathrm{Ext}_B^j(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,N'))\\ &\to \mathrm{Ext}_B^j(\mathrm{Hom}_A(T,M),\mathrm{Ext}_A^1(T,N)) \to \mathrm{Ext}_B^{j+1}(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,I))\\ &\to \ldots .\end{aligned}$$

By the tilting theorem (3.8), we have

$$\mathrm{Ext}_B^j(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,I)) \cong \mathrm{Ext}_A^j(M,I) = 0,$$

for all $j \geq 1$, because I is injective. Then the sequence

$$\begin{aligned}0\to\mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,I))\to&\mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,N'))\\ &\to \mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Ext}_A^1(T,N)) \to 0\end{aligned}$$

is exact, and there is an isomorphism

$$\mathrm{Ext}_B^j(\mathrm{Hom}_A(T,M),\mathrm{Ext}_A^1(T,N)) \;\cong\; \mathrm{Ext}_B^j(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,N'))$$

for all $j \geq 1$. Compare this exact sequence with the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M,I) \longrightarrow \mathrm{Hom}_A(M,N') \longrightarrow \mathrm{Ext}_A^1(M,N) \longrightarrow 0$$

obtained by applying the functor $\mathrm{Hom}_A(M,-)$ to the short exact sequence $0 \to N \to I \to N' \to 0$, using the injectivity of I and the fact that $N \in \mathcal{F}(T)$. Because, by the tilting theorem (3.8), there are isomorphisms

$$\mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,E)) \cong \mathrm{Hom}_A(M,E),$$
$$\mathrm{Hom}_B(\mathrm{Hom}_A(T,M),\mathrm{Hom}_A(T,N')) \cong \mathrm{Hom}_A(M,N'),$$

by passing to the cokernels, we obtain an isomorphism

$$\operatorname{Hom}_B(\operatorname{Hom}_A(T,M),\operatorname{Ext}_A^1(T,N)) \cong \operatorname{Ext}_A^1(M,N),$$

which is the required statement whenever $j=1$. Assume now $j \geq 1$. Then the tilting theorem (3.8) again gives

$$\begin{aligned}\operatorname{Ext}_B^j(\operatorname{Hom}_A(T,M),\operatorname{Ext}_A^1(T,N)) &\cong \operatorname{Ext}_B^j(\operatorname{Hom}_A(T,M),\operatorname{Hom}_A(T,N'))\\ &\cong \operatorname{Ext}_A^j(M,N') \cong \operatorname{Ext}_A^{j+1}(M,N). \quad \square\end{aligned}$$

5.6. Theorem. *Let A be an algebra, T_A be a tilting A-module, and $B = \operatorname{End} T_A$.*

(a) *T_A is separating if and only if* $\operatorname{pd} X = 1$ *for every* $X_B \in \mathcal{X}(T)$.

(b) *T_A is splitting if and only if* $\operatorname{id} N = 1$ *for every* $N_A \in \mathcal{F}(T)$.

Proof. We only prove (b); (a) follows using that ${}_BT$ is a tilting module. We first show the sufficiency of the condition. Assume that, for every $N \in \mathcal{F}(T)$, we have $\operatorname{id} N = 1$. Let $X \in \mathcal{X}(T)$ and $Y \in \mathcal{Y}(T)$. Then there exist $M \in \mathcal{T}(T)$ and $N \in \mathcal{F}(T)$ such that $X \cong \operatorname{Ext}_A^1(T,N)$ and $Y \cong \operatorname{Hom}_A(T,M)$. Hence, by (5.5),

$$\operatorname{Ext}_B^1(Y,X) \cong \operatorname{Ext}_B^1(\operatorname{Hom}_A(T,M),\operatorname{Ext}_A^1(T,N)) \cong \operatorname{Ext}_A^2(M,N) = 0,$$

because $\operatorname{id} N = 1$. Therefore, by (1.7), the pair $(\mathcal{X}(T),\mathcal{Y}(T))$ is splitting.

Conversely, assume that $(\mathcal{X}(T),\mathcal{Y}(T))$ is splitting and let $N \in \mathcal{F}(T)$. Take an injective resolution of N

$$0 \longrightarrow N \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \xrightarrow{d^2} I^2 \longrightarrow \cdots.$$

Let $L^0 = \operatorname{Im} d^1$ and $L^1 = \operatorname{Im} d^2$. Then, by (5.5), because $L^1 \in \mathcal{T}(T)$ and $N \in \mathcal{F}(T)$, we have

$$\operatorname{Ext}_A^1(L^1,L^0) \cong \operatorname{Ext}_A^2(L^1,N) \cong \operatorname{Ext}_B^1(\operatorname{Hom}_A(T,L^1),\operatorname{Ext}_A^1(T,N)) = 0,$$

because $\operatorname{Hom}_A(T,L^1) \in \mathcal{Y}(T)$ and $\operatorname{Ext}_A^1(T,N) \in \mathcal{X}(T)$, and $(\mathcal{X}(T),\mathcal{Y}(T))$ is splitting (see (1.7)). This implies that the short exact sequence $0 \to L^0 \to I^1 \to L^1 \to 0$ splits. Therefore, L^0 is injective and consequently $\operatorname{id} N \leq 1$. Finally, because $N \in \mathcal{F}(T)$, N cannot be injective so that $\operatorname{id} N = 1$. $\square$

If A is an algebra and $P(a)$ is simple projective noninjective, then the APR-tilting module $T[a]$ (which is always separating, by (2.8)(c)) is splitting if and only if $\operatorname{id} P(a) = 1$. Moreover, we have the following corollary.

5.7. Corollary. *If* gl.dim $A \leq 1$, *then every tilting A-module is splitting.*

This is the case for the algebras of Examples 3.11 (a) and (c). These algebras are studied in detail in future chapters.

Let T_A be a tilting A-module and let $T_1, \dots, T_n$ denote the pairwise nonisomorphic indecomposable summands of T. By (3.1), the modules $\mathrm{Hom}_A(T, T_1), \dots, \mathrm{Hom}_A(T, T_n)$ form a complete set of pairwise nonisomorphic indecomposable projective modules over the algebra $B = \mathrm{End}\, T_A$. It is less easy in general to describe the indecomposable injective B-modules. In the splitting case, however, we have the following result.

5.8. Proposition. *Let A be an algebra, T_A be a splitting tilting module, $B = \mathrm{End}\, T_A$, and $T_1, \dots, T_n$ be a complete set of pairwise nonisomorphic indecomposable direct summands of T. Assume that the modules $T_1, \dots, T_m$ are projective, the remaining modules $T_{m+1}, \dots, T_n$ are not projective and $I_1, \dots, I_m$ are indecomposable injective A-modules with* $\mathrm{soc}\ I_j \cong T_j/\mathrm{rad}\ T_j$, *for $j = 1, \dots, m$. Then the right B-modules*

$$\mathrm{Hom}_A(T, I_1), \dots, \mathrm{Hom}_A(T, I_m), \mathrm{Ext}_A^1(T, \tau T_{m+1}), \dots, \mathrm{Ext}_A^1(T, \tau T_n)$$

form a complete set of pairwise nonisomorphic indecomposable injective modules.

Proof. It follows from (4.9) that $\mathrm{Hom}_A(T, I_1), \dots, \mathrm{Hom}_A(T, I_m)$ are paiwise non-isomorphic indecomposable injective B-modules, and belong to $\mathcal{Y}(T)$. If $m = n$, they form a complete set of pairwise nonisomorphic indecomposable injective B-modules.

Assume that $m < n$. Clearly, $\mathrm{Ext}_A^1(T, \tau T_{m+1}), \dots, \mathrm{Ext}_A^1(T, \tau T_n)$ are pairwise nonisomorphic objects of the torsion class $\mathcal{X}(T_A)$ of $\mathrm{mod}\, B$. It then suffices to show that, for each i such that $m+1 \leq i \leq n$, the B-module $\mathrm{Ext}_A^1(T, \tau T_i)$ is injective. Indeed, if this is not the case, then there exists an almost split sequence $0 \longrightarrow \mathrm{Ext}_A^1(T, \tau T_i) \longrightarrow F_B \longrightarrow X_B \longrightarrow 0$ in $\mathrm{mod}\, B$. Because, by our assumption, the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\mathrm{mod}\, B$ is splitting and $\mathrm{Ext}_A^1(T, \tau T_i)$ maps to no module from $\mathcal{Y}(T)$, we deduce that $F_B \in \mathcal{X}(T)$, and similarly $X_B \in \mathcal{X}(T)$. Thus, there exist an A-module E and an indecomposable A-module N in $\mathcal{F}(T)$ such that $F_B \cong \mathrm{Ext}_A^1(T, E)$ and $X_B \cong \mathrm{Ext}_A^1(T, N)$, and the almost split exact sequence becomes

$$0 \longrightarrow \mathrm{Ext}_A^1(T, \tau T_i) \longrightarrow \mathrm{Ext}_A^1(T, E) \longrightarrow \mathrm{Ext}_A^1(T, N) \longrightarrow 0.$$

The equivalence $\mathcal{X}(T) \cong \mathcal{F}(T)$ yields a short exact sequence in $\mathcal{F}(T)$

$$0 \longrightarrow \tau T_i \longrightarrow E \longrightarrow N \longrightarrow 0.$$

Because $T_i = \tau^{-1}(\tau T_i) \in \mathcal{T}(T)$, by (1.11)(b), the A-module τT_i is Ext-injective in $\mathcal{F}(T)$. Therefore, the short exact sequence splits, and applying $\mathrm{Ext}_A^1(T, -)$ to it yields a split–almost split sequence, a contradiction. $\square$

VI.6. Torsion pairs induced by tilting modules

It is natural to ask which torsion pairs $(\mathcal{T}, \mathcal{F})$ in a module category $\operatorname{mod} A$ are in fact induced by tilting modules, that is, are such that there exists a tilting module T_A such that $\mathcal{T} = \mathcal{T}(T_A)$ and $\mathcal{F} = \mathcal{F}(T_A)$. This is useful in practice, because in many applications it is easier to start by constructing the torsion pairs and then finding the corresponding tilting module. Clearly, because a torsion class induced by a tilting module T is of the form $\operatorname{Gen} T$, we may start our investigation by asking what the properties of a module U are so that the class $\operatorname{Gen} U$ is a torsion class. We need one definition.

An A-module U will be called Gen-**minimal** if, whenever $U = U' \oplus U''$, $U' \notin \operatorname{Gen} U''$. We define dually Cogen-**minimal** modules.

Our first lemma is a partial converse of (1.9).

6.1. Lemma. *Let A be an algebra.*

(a) *Let U be a* Gen-*minimal A-module such that* $\operatorname{Gen} U$ *is a torsion class. Then U is* Ext-*projective in* $\operatorname{Gen} U$.

(b) *Let V be a* Cogen-*minimal A-module such that* $\operatorname{Cogen} V$ *is a torsion-free class. Then V is* Ext-*injective in* $\operatorname{Cogen} V$.

Proof. We only prove (a); the proof of (b) is similar. Under the stated assumptions, let $M \in \operatorname{Gen} U$ be such that $\operatorname{Ext}_A^1(U, M) \neq 0$. Then there exists an indecomposable summand U_0 of U such that $\operatorname{Ext}_A^1(U_0, M) \neq 0$, and hence a nonsplit extension

$$0 \longrightarrow M \xrightarrow{u} E \xrightarrow{v} U_0 \longrightarrow 0.$$

Because $M, U_0 \in \operatorname{Gen} U$, and $\operatorname{Gen} U$ is a torsion class, we have $E \in \operatorname{Gen} U$, and thus there exists an epimorphism $p : U^m \to E$ for some $m > 0$. Let $U^m = R \oplus U_0^m$; then the composition $f = vp : U^m \to U_0$ can be written as $f = [g, f_1, \dots, f_m]$ with $g \in \operatorname{Hom}_A(R, U_0)$ and $f_i \in \operatorname{End} U_0$ for each i.

The surjectivity of f means that $U_0 = g(R) + \sum_{i=1}^m f_i(U_0)$. Because v is not a retraction, no f_i is an isomorphism, and consequently, $f_i(U_0) \subseteq (\operatorname{rad} \operatorname{End} U_0) \cdot U_0$ (because the indecomposability of U_0 implies that $\operatorname{End} U_0$ is local) for any i such that $1 \leq i \leq m$. So $U_0 = g(R) + (\operatorname{rad} \operatorname{End} U_0) \cdot U_0$. Applying Nakayama's lemma (I.2.2) to the left $\operatorname{End} U_0$-module U_0, we get that $U_0 = g(R)$ so that g is an epimorphism. This, however, contradicts the Gen-minimality of U. Thus $\operatorname{Ext}_A^1(U, M) = 0$ for all M in $\operatorname{Gen} U$. $\square$

6.2. Corollary. *Let A be an algebra.*

(a) *Let U be a* Gen-*minimal A-module. Then* $\operatorname{Gen} U$ *is a torsion class if and only if U is* Ext-*projective in* $\operatorname{Gen} U$.

(b) *Let V be a* Cogen-*minimal A-module. Then* $\operatorname{Cogen} V$ *is a torsion-free class if and only if V is* Ext-*injective in* $\operatorname{Cogen} V$.

Proof. This follows from (1.9) and (6.1). $\square$

6.3. Corollary. *Let A be an algebra and let U be a* Gen*-minimal faithful A-module such that* $\operatorname{Gen} U$ *is a torsion class. Then U is a partial tilting module.*

Proof. Because $U \in \operatorname{Gen} U$, (6.1) yields $\operatorname{Ext}^1_A(U,U) = 0$. On the other hand, because U is faithful, by (2.2), we have $DA \in \operatorname{Gen} U$, whereas the Ext-projectivity of U in the torsion class $\operatorname{Gen} U$ implies, by (1.11), that τU lies in the corresponding torsion-free class. Thus, we have $\operatorname{Hom}_A(DA, \tau U) = 0$. Therefore, by (IV.2.7), we have $\operatorname{pd} U \leq 1$. $\square$

6.4. Lemma. *Let A be an algebra.*

(a) *If $\mathcal{T} = \operatorname{Gen} U$ is a torsion class, then the numbers of isomorphism classes of indecomposable* Ext*-projectives in $\mathcal{T}$ and of indecomposable* Ext*-injectives in $\mathcal{T}$ are finite and equal.*
(b) *If $\mathcal{F} = \operatorname{Cogen} V$ is a torsion-free class, then the number of isomorphism classes of indecomposable* Ext*-projectives in $\mathcal{F}$ and of indecomposable* Ext*-injectives in $\mathcal{F}$ are finite and equal.*

Proof. We only prove (a); the proof of (b) is similar. Because there clearly exists a direct summand U_0 of U that is Gen-minimal and such that $\operatorname{Gen} U = \operatorname{Gen} U_0$, we may assume from the start that U is Gen-minimal. Because, on the other hand, U is clearly faithful as an $A/\operatorname{Ann} U$-module and we have embeddings

$$\mathcal{T} \hookrightarrow \operatorname{mod}(A/\operatorname{Ann} U) \hookrightarrow \operatorname{mod} A,$$

we may also assume that U is faithful.

By (6.3) and (6.1), U is a partial tilting module and is Ext-projective in $\mathcal{T}$. Because $DA \in \operatorname{Gen} U$ (by (2.2)), all the indecomposable injective A-modules are torsion and so, by (1.11), they coincide with the indecomposable Ext-injectives in $\mathcal{T}$.

Let $u_1, \ldots, u_d$ be a basis of the K-vector space $\operatorname{Hom}_A(A, U)$ and consider the homomorphism $u = \begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix} : A_A \longrightarrow U^d_A$. Because U is faithful, according to (2.2), the map u is injective. We thus have a short exact sequence

$$0 \longrightarrow A \xrightarrow{u} U^d \longrightarrow U' \longrightarrow 0,$$

where $U' = \operatorname{Coker} u$. Notice that $U' \in \mathcal{T}$. Also, because $\operatorname{pd} U \leq 1$, we have $\operatorname{pd} U' \leq 1$. We now show that U' is Ext-projective in $\mathcal{T}$. Let $M \in \mathcal{T}$ and apply $\operatorname{Hom}_A(-, M)$ to the preceding sequence. This yields an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_A(U', M) \longrightarrow \mathrm{Hom}_A(U^d, M) &\xrightarrow{\mathrm{Hom}_A(u,M)} \mathrm{Hom}_A(A, M) \\ &\longrightarrow \mathrm{Ext}^1_A(U', M) \longrightarrow 0, \end{aligned}$$

because $\mathrm{Ext}^1_A(U^d, M) = 0$ due to the Ext-projectivity of U in $\mathcal{T}$. We claim that $\mathrm{Hom}_A(u, M)$ is surjective. Because $M \in \mathcal{T}$, there exists an epimorphism $p : U^m \to M$ for some $m > 0$. Because A_A is a projective module, the homomorphism $\mathrm{Hom}_A(A, p) : \mathrm{Hom}_A(A, U^m) \to \mathrm{Hom}_A(A, M)$ is surjective. On the other hand, it follows from the definition of u that $\mathrm{Hom}_A(u, U^m) : \mathrm{Hom}_A(U^d, U^m) \to \mathrm{Hom}_A(A, U^m)$ is surjective. Therefore the composition $\mathrm{Hom}_A(u, p) : \mathrm{Hom}_A(U^d, U^m) \to \mathrm{Hom}_A(A, M)$ is surjective. Because $\mathrm{Hom}_A(u, p) = \mathrm{Hom}_A(u, M) \circ \mathrm{Hom}_A(U^d, p)$, this shows that $\mathrm{Hom}_A(u, M)$ is surjective. Therefore $\mathrm{Ext}^1_A(U, M) = 0$, and hence U' is Ext-projective in $\mathcal{T}$.

We deduce that $T_A = U \oplus U'$ is a tilting module. Indeed, $\mathrm{pd}\, T \leq 1$ and the Ext-projectivity of both U and U' implies that $\mathrm{Ext}^1_A(T, T) = 0$. Finally, the short exact sequence $0 \longrightarrow A \xrightarrow{u} U^d \longrightarrow U' \longrightarrow 0$ shows that T is indeed a tilting module. It follows from (2.5) that $\mathcal{T}(T) = \mathrm{Gen}\, T = \mathrm{Gen}\, U = \mathcal{T}$. By (2.5)(d), the pairwise nonisomorphic indecomposable Ext-projectives in $\mathcal{T}$ coincide with the pairwise nonisomorphic indecomposable direct summands of T. Therefore, by (4.4), their number equals the rank of $K_0(A)$ and thus equals the number of pairwise nonisomorphic indecomposable Ext-injectives in $\mathcal{T} = \mathcal{T}(T)$. $\square$

6.5. Theorem. *Let A be an algebra and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\mathrm{mod}\, A$. *Then there exists a tilting module T_A such that $\mathcal{T} = \mathcal{T}(T_A)$ if and only if $\mathcal{T} = \mathrm{Gen}\, M$ for some A-module M, and $\mathcal{T}$ contains the injectives.*

Proof. Because the necessity is obvious, we only show the sufficiency. Let $\mathcal{T}$ be a torsion class containing all the injectives such that $\mathcal{T} = \mathrm{Gen}\, M$ for some A-module M. Let $T_1, \ldots, T_t$ be a complete set of pairwise nonisomorphic indecomposable Ext-projectives in $\mathcal{T}$, and let $T_A = \bigoplus_{i=1}^t T_i$. We claim that T_A is a tilting module. Indeed, the Ext-projectivity of T_A in $\mathcal{T}$ implies that $\mathrm{Ext}^1_A(T, T) = 0$. On the other hand,

$$\mathrm{Hom}_A(DA, \tau T) = \bigoplus_{i=1}^{t} \mathrm{Hom}_A(DA, \tau T_i) = 0$$

(because τT_i is zero or torsion-free, by (1.11)(a), whereas $DA \in \mathcal{T}$ by hypothesis). Hence, by (IV.2.7), $\mathrm{pd}\, T \leq 1$. Also, by (6.4), t equals the number of pairwise nonisomorphic indecomposable injective A-modules. Therefore t equals the rank of $K_0(A)$ and so T is a tilting module, by (4.4).

Because M is itself Ext-projective in $\mathcal{T}$, its indecomposable direct summands are also summands of T. Therefore $\mathcal{T} \subseteq \mathcal{T}(T)$. Because $T \in \operatorname{Gen} M$, we also have $\mathcal{T}(T) \subseteq \mathcal{T}$ so that $\mathcal{T}(T) = \mathcal{T}$. □

We give an application of this theorem, but first we prove two important corollaries. The first is obvious.

6.6. Corollary. *Let A be a representation–finite algebra and $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\operatorname{mod} A$. *Then there exists a tilting module T_A such that $\mathcal{T} = \mathcal{T}(T_A)$ and $\mathcal{F} = \mathcal{F}(T_A)$ if and only if $\mathcal{T}$ contains the injectives.*

Proof. Let $\{M_1, \ldots, M_r\}$ be a complete set of pairwise nonisomorphic indecomposable modules in $\mathcal{T}$ (such a set is finite, because A is representation–finite), and let $M = M_1 \oplus \ldots \oplus M_r$. Then $\mathcal{T} = \operatorname{Gen} M$, and the required equivalence is a direct consequence of (2.5) and (6.5). □

6.7. Corollary. *Let B be an algebra and $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in* $\operatorname{mod} B$. *Then there exists an algebra A and a tilting module T_A such that $B = \operatorname{End} T_A$, $\mathcal{X} = \mathcal{X}(T_A)$ and $\mathcal{Y} = \mathcal{Y}(T_A)$ if and only if $\mathcal{Y} = \operatorname{Cogen} Y$ for some B-module Y, and $\mathcal{Y}$ contains the projectives.*

Proof. We first show the necessity. Let A be an algebra and T_A be a tilting module such that $B = \operatorname{End} T_A$. It follows from (3.1)(b) that $\mathcal{Y}(T_A)$ contains the projective B-modules. We claim that $\mathcal{Y}(T_A)$ is the class cogenerated by the B-module $D({}_BT) = \operatorname{Hom}_A(T, DA) \in \mathcal{Y}(T_A)$. Let $Y \in \mathcal{Y}(T_A)$; there exists an A-module $M \in \mathcal{T}(T)$ such that $Y = \operatorname{Hom}_A(T, M)$. There exists an injective A-module U and a monomorphism $M \to U$ and hence a monomorphism $Y = \operatorname{Hom}_A(T, M) \to \operatorname{Hom}_A(T, U)$. Because $\operatorname{Hom}_A(T, U) \in \operatorname{add} D({}_BT)$, we deduce from (3.3)(a) that $\mathcal{Y}(T) \subseteq \operatorname{Cogen} D({}_BT)$. Because, on the other hand, $D({}_BT) \in \mathcal{Y}(T)$, we have established our claim.

To prove the sufficiency, we notice that, by (6.4), the torsion class of left B-modules $D\mathcal{Y}$ is induced by a tilting module, that is, there exists a left B-module ${}_BT$ such that $D\mathcal{Y} = \mathcal{T}({}_BT)$ and $D\mathcal{X} = \mathcal{F}({}_BT)$. Letting $A = \operatorname{End}({}_BT)^{\mathrm{op}}$, we deduce from (3.3) that T_A is a tilting A-module and $B = \operatorname{End} T_A$. Moreover, by (3.6), $\mathcal{Y}(T_A) = D\mathcal{T}({}_BT) = \mathcal{Y}$ and $\mathcal{X}(T_A) = D\mathcal{F}({}_BT) = \mathcal{X}$. □

To apply Corollary 6.7 in examples, we need the following easy computational lemma.

6.8. Lemma. *Assume that the torsion pair $(\mathcal{X}, \mathcal{Y})$ in* $\operatorname{mod} B$ *satisfies the equivalent conditions of* (6.7). *Then $D({}_BT)$ equals the direct sum of a complete set of pairwise nonisomorphic indecomposable* Ext-*injectives in $\mathcal{Y}$.*

Proof. We recall that $D({}_BT) = \operatorname{Hom}_A(T, DA)$ equals the direct sum of modules of the form $\operatorname{Hom}_A(T, I(a))$, where $I(a)$ runs over a complete set of indecomposable injective A-modules. Let $I(a)_A$ be indecomposable injective. By the connecting lemma (4.9), either $\operatorname{Hom}_A(T, I(a))$ is injective in $\operatorname{mod} B$ (if the corresponding indecomposable projective lies in $\operatorname{add} T_A$) or $\tau^{-1}\operatorname{Hom}_A(T, I(a)) \in \mathcal{X}$. By (1.11), $\operatorname{Hom}_A(T, I(a))$ is Ext-projective in $\mathcal{Y}$.

Conversely, let Y be indecomposable Ext-injective in $\mathcal{Y}$; then $\tau^{-1}Y \in \mathcal{X}$. If $\tau^{-1}Y \neq 0$; then, by (4.8), there exists an indecomposable injective A-module $I(a)$ such that $Y \cong \operatorname{Hom}_A(T, I(a))$. Assume now that $\tau^{-1}Y = 0$, that is, Y is injective. Because $Y \in \mathcal{Y}$, there exists an indecomposable A-module $M \in \mathcal{T}(T_A)$ such that $Y \cong \operatorname{Hom}_A(T, M)$. Let $M \to E$ be an injective envelope of M in $\operatorname{mod} A$. Applying $\operatorname{Hom}_A(T, -)$ to the short exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0$$

yields an exact sequence in $\operatorname{mod} B$

$$0 \longrightarrow Y \longrightarrow \operatorname{Hom}_A(T, E) \longrightarrow \operatorname{Hom}_A(T, E/M) \longrightarrow 0,$$

because $\operatorname{Ext}^1_A(T, M) = 0$. Because, by hypothesis, Y is Ext-injective in $\mathcal{Y}$ and the previous sequence lies in $\mathcal{Y}$, it splits. Hence Y is isomorphic to a direct summand of $\operatorname{Hom}_A(T, E)$, that is, there exists an indecomposable summand $I(a)$ of E such that $Y \cong \operatorname{Hom}_A(T, I(a))$. $\square$

Assume thus that $(\mathcal{X}, \mathcal{Y})$ satisfies the conditions of (6.7). We indicate how to find an algebra A and a tilting module T_A from which $(\mathcal{X}, \mathcal{Y})$ arises. We first compute $D({}_BT)$ using (6.8): Let $Y_1, \ldots, Y_n$ be a complete set of pairwise nonisomorphic indecomposable Ext-injectives in $\mathcal{Y}$, then $D({}_BT) = \bigoplus_{i=1}^n Y_i$. We next find

$$A = \operatorname{End}_{B^{\mathrm{op}}}({}_BT) = \operatorname{End}_B(D({}_BT)) = \operatorname{End}_B(\bigoplus_{i=1}^n Y_i).$$

In doing the last calculation, we associate each of the Y_i to a point in the quiver of A. Thus, without loss of generality, we may assume that $Y_i = \operatorname{Hom}_A(T, I(i))$ for each i such that $1 \leq i \leq n$. Letting $T = \bigoplus_{j=1}^n T_j$, we have

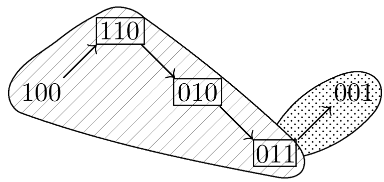
$$\begin{aligned}
(T_j)_i &= \operatorname{Hom}_A(P(i)_A, T_j) \\
&\cong D\operatorname{Hom}_A(T_j, I(i)) \\
&\cong D\operatorname{Hom}_B(\operatorname{Hom}_A(T, T_j), \operatorname{Hom}_A(T, I(i))) \\
&\cong D\operatorname{Hom}_B(P(j)_B, Y_i).
\end{aligned}$$

Thus, in particular, $\dim_K(T_j)_i$ is the jth coordinate of Y_i. This gives $\mathbf{dim}\, T_j$. The method is explained in the following example.

6.9. Examples. (a) Let B be given by the quiver

$$\overset{1}{\circ} \xleftarrow{\ \mu\ } \overset{2}{\circ} \xleftarrow{\ \lambda\ } \overset{3}{\circ}$$

bound by $\lambda\mu = 0$ and $(\mathcal{X}, \mathcal{Y})$ be the shown torsion pair in mod B (compare with (3.11)(a))



where $\mathcal{Y}$ is shaded as and $\mathcal{X}$ as . Clearly, $(\mathcal{X}, \mathcal{Y})$ satisfies the conditions of (6.7). To find an algebra A and a tilting module T_A from which $(\mathcal{X}, \mathcal{Y})$ arises, we consider the indecomposable Ext-injectives in $\mathcal{Y}$; these are $Y_1 = 110$, $Y_2 = 010$, $Y_3 = 011$. Thus $D({}_BT) = 110 \oplus 010 \oplus 011$. Hence $A = \operatorname{End}_{B^{\mathrm{op}}}({}_BT) = \operatorname{End}_B(D({}_BT)) = \operatorname{End}_B(\bigoplus_{i=1}^3 Y_i)$ is given by the quiver

$$\overset{1}{\circ} \longleftarrow \overset{2}{\circ} \longleftarrow \overset{3}{\circ}$$

where the point i corresponds to Y_i (for each i with $1 \le i \le 3$). To recover T_A, we notice that, in the preceding notation,

$$\operatorname{Hom}_A(T, I(1)) = 110, \quad \operatorname{Hom}_A(T, I(2)) = 010, \quad \operatorname{Hom}_A(T, I(3)) = 011.$$

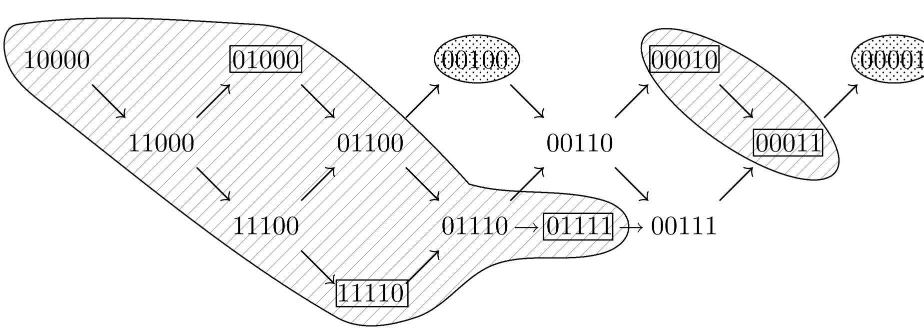
Thus, if one writes $T = T_1 \oplus T_2 \oplus T_3$, with T_1, T_2, T_3 indecomposable, one gets

$$T_1 = 100, \qquad T_2 = 111, \qquad T_3 = 001.$$

(b) Let B be given by the quiver

$$\underset{1}{\circ} \xleftarrow{\ \eta\ } \underset{2}{\circ} \xleftarrow{\ \nu\ } \underset{3}{\circ} \xleftarrow{\ \mu\ } \underset{4}{\circ} \xleftarrow{\ \lambda\ } \underset{5}{\circ}$$

bound by $\lambda\mu\nu\eta = 0$ and $(\mathcal{X}, \mathcal{Y})$ be the shown torsion pair in mod B (compare with (3.11)(b))



where $\mathcal{Y}$ is shaded as and $\mathcal{X}$ as . Clearly, $(\mathcal{X}, \mathcal{Y})$ satisfies the conditions of (6.7). The indecomposable Ext-injective modules in $\mathcal{Y}$ are $Y_1 = 11110$, $Y_2 = 01000$, $Y_3 = 00010$, $Y_4 = 01111$, and $Y_5 = 00011$. Thus, $A = \operatorname{End}\left(\bigoplus_{i=1}^{5} Y_i\right)$ is given by the quiver

$$\begin{tikzcd}
 & \overset{3}{\circ} \arrow[dl, "\beta"'] & \\
1\circ & & \circ 5 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
 & \underset{4}{\circ} \arrow[ul, "\delta"] \arrow[dl, "\varepsilon"] & \\
2\circ & &
\end{tikzcd}$$

bound by $\alpha\beta = \gamma\delta$ and $\gamma\varepsilon = 0$, where the point i corresponds to Y_i (for each i with $1 \leq i \leq 5$). To recover T_A, we notice that

$$\operatorname{Hom}_A(T, I(1)) = 11110, \operatorname{Hom}_A(T, I(2)) = 01000, \operatorname{Hom}_A(T, I(3)) = 00010,$$
$$\operatorname{Hom}_A(T, I(4)) = 01111, \operatorname{Hom}_A(T, I(5)) = 00011.$$

Thus if one writes $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$, with T_1, T_2, T_3, T_4, T_5 indecomposable, one gets

$$T_1 = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ 0 & & \end{smallmatrix}, \quad T_2 = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ 1 & & \end{smallmatrix}, \quad T_3 = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ 0 & & \end{smallmatrix}, \quad T_4 = \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix}, \quad T_5 = \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix}$$

VI.7. Exercises

1. Show that a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\operatorname{mod} A$ is a torsion pair if and only if it satisfies the following four conditions:

(a) $\mathcal{T} \cap \mathcal{F} = \{0\}$;
(b) $\mathcal{T}$ is closed under images;
(c) $\mathcal{F}$ is closed under submodules; and
(d) for every module M, there exists a short exact sequence $0 \to M' \to M \to M'' \to 0$ with $M' \in \mathcal{T}$ and $M'' \in \mathcal{F}$.

2. Verify the assertions in Example 1.2 (a).

3. A torsion pair $(\mathcal{T}, \mathcal{F})$ is called **hereditary** if $\mathcal{T}$ is closed under submodules. Give an example of a hereditary torsion pair. Show that a torsion pair $(\mathcal{T}, \mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is closed under injective envelopes.

4. Let T_A be an A-module. Show that:

(a) $\operatorname{Gen} T$ is a torsion class if and only if $\operatorname{Ext}_A^1(T, T'') = 0$ for every quotient T'' of T.
(b) $\operatorname{Cogen} T$ is a torsion-free class if and only if $\operatorname{Ext}_A^1(T', T) = 0$ for every submodule T' of T.

5. Assume that $\operatorname{Gen} T$ is a torsion class for some module T_A. Show that τT belongs to the corresponding torsion-free class.

6. Assume that $\operatorname{Gen} T$ is a torsion class for some module T_A.

(a) Show that if T_A is faithful, then T_A is a partial tilting module.
(b) Give an example showing that if T_A is not faithful, then T_A is generally not a partial tilting module.

7. Let T_A be a partial tilting module. Show that:

(a) If $\mathcal{T}$ is a torsion class such that T_A is Ext-projective in $\mathcal{T}$, then $\operatorname{Gen} T \subseteq \mathcal{T} \subseteq \mathcal{T}(T)$.
(b) $\mathcal{T}(T)$ is induced by a tilting module having T as a summand.

8. Let T_A be a partial tilting module and E be the middle term of Bongartz's exact sequence. Show that any indecomposable direct summand E' of E is projective or satisfies $\operatorname{Hom}_A(E', T) \neq 0$.

9. An A-module M is called **sincere** if $\operatorname{Hom}_A(P, M) \neq 0$ for any projective A-module P. Show that any faithful module is sincere (consequently, any tilting module is sincere).

10. Let T_A be a tilting module. Show that any indecomposable projective-injective A-module is a direct summand of T.

11. Let T_A be a tilting module and $(\mathcal{T}(T), \mathcal{F}(T))$ be the induced torsion pair in $\operatorname{mod} A$. Show that if $M_3 \to M_2 \to M_1 \to M_0$ is exact with $M_i \in \mathcal{T}(T)$ for all i, then the induced sequence

$$\operatorname{Hom}_A(T, M_2) \longrightarrow \operatorname{Hom}_A(T, M_1) \longrightarrow \operatorname{Hom}_A(T, M_0)$$

is exact.

12. Let T_A be a tilting module and $\mathcal{X}(T)$ be the induced torsion class in $\operatorname{mod} B$. Show that $\mathcal{X}(T) = \operatorname{Gen} \operatorname{Ext}^1_A(T, A)$.

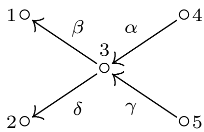
13. Let T_A be a tilting module and E_A be injective. Show that if $N \in \mathcal{F}(T)$, then we have a functorial isomorphism

$$\operatorname{Hom}_A(N, E) \cong \operatorname{Ext}^1_B(\operatorname{Ext}^1_A(T, N), \operatorname{Hom}_A(T, E)).$$

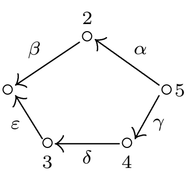
14. Let A be a K-algebra given by each of the bound quivers (i)–(iv).

(a) Verify that the given module T_A is a tilting module.
(b) Compute the bound quiver of $B = \operatorname{End} T_A$.
(c) Illustrate in $\Gamma(\operatorname{mod} A)$ and $\Gamma(\operatorname{mod} B)$ the classes $\mathcal{T}(T)$, $\mathcal{F}(T)$, $\mathcal{X}(T)$, and $\mathcal{Y}(T)$.

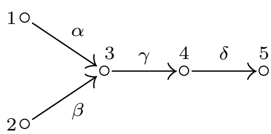
(d) Describe explicitly the equivalences $\mathcal{T}(T) \cong \mathcal{Y}(T)$, $\mathcal{X}(T) \cong \mathcal{F}(T)$.
(e) Compute the global dimensions of A and B.
(f) Describe all connecting sequences in $\operatorname{mod} A$ and $\operatorname{mod} B$. For which ones is the canonical sequence of the middle term not split?
(g) Find the matrix $\mathbf{F}$ of the isomorphism $K_0(A) \to K_0(B)$, the matrices $\mathbf{A}$ and $\mathbf{B}$ of the Euler characteristics for A and B, respectively, and verify the relation $\mathbf{A} = \mathbf{F}^t\mathbf{B}\mathbf{F}$.

(i) 
bound by $\alpha\beta = 0$, $\gamma\delta = 0$,

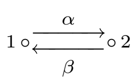
$$T_A = \begin{smallmatrix}1&&0\\&0&\\0&&0\end{smallmatrix} \oplus \begin{smallmatrix}1&&0\\&1&\\0&&1\end{smallmatrix} \oplus \begin{smallmatrix}0&&1\\&1&\\0&&1\end{smallmatrix} \oplus \begin{smallmatrix}0&&1\\&1&\\0&&0\end{smallmatrix} \oplus \begin{smallmatrix}0&&1\\&1&\\1&&0\end{smallmatrix}$$

(ii) 
bound by $\alpha\beta = \gamma\delta\varepsilon$,

$$T_A = \begin{smallmatrix}&&0&&\\1&&&&0\\&1&&1&\end{smallmatrix} \oplus \begin{smallmatrix}&&1&&\\1&&&&1\\&1&&1&\end{smallmatrix} \oplus \begin{smallmatrix}&&0&&\\0&&&&0\\&1&&0&\end{smallmatrix} \oplus \begin{smallmatrix}&&0&&\\0&&&&0\\&1&&1&\end{smallmatrix} \oplus \begin{smallmatrix}&&0&&\\0&&&&1\\&1&&1&\end{smallmatrix}$$

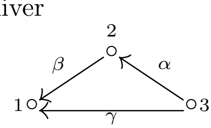
(iii) 
bound by $\gamma\delta = 0$,

$$T_A = \begin{smallmatrix}0&&&\\&0&0&1\\0&&&\end{smallmatrix} \oplus \begin{smallmatrix}0&&&\\&0&1&1\\0&&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&&\\&1&1&0\\0&&&\end{smallmatrix} \oplus \begin{smallmatrix}0&&&\\&1&1&0\\1&&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&&\\&1&1&0\\1&&&\end{smallmatrix}$$

(iv) 
bound by $\beta\alpha = 0$,

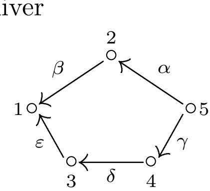
$T_A = \begin{pmatrix}1\\2\\1\end{pmatrix} \oplus (1)$ (in the notation of (V.2.7))

15. Let A be given by the quiver



bound by $\alpha\beta = 0$. Find all (nontrivial, multiplicity-free) tilting A-modules and compute the bound quiver of the endomorphism algebra of each.

16. Let A be given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$, and $\delta\varepsilon = 0$. Compute the bound quiver of the endomorphism algebra B of the unique APR-tilting module and the Auslander–Reiten quivers of each of A and B and then describe the equivalences $\mathcal{T}(T) \cong \mathcal{Y}(T)$, $\mathcal{F}(T) \cong \mathcal{X}(T)$.

17. Repeat Exercise 16 with A given by the quiver

$$\begin{tikzcd}
 & \overset{2}{\circ} \arrow[dl, "\beta"'] & & \\
1\,\circ & & \overset{4}{\circ} \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] & \circ\,5 \arrow[l, "\varepsilon"'] \\
 & \underset{3}{\circ} \arrow[ul, "\delta"] & &
\end{tikzcd}$$

bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, and $\varepsilon\gamma = 0$.

18. Let T_A be a tilting module and $B = \operatorname{End} T_A$. Show that if $J_B \in \mathcal{Y}(T)$ is an indecomposable injective B-module, then there exists an indecomposable injective A-module E_A such that $J \cong \operatorname{Hom}_A(T, E)$ and the indecomposable projective P_A such that $P/\operatorname{rad} P \cong \operatorname{soc} I$ and P_A are not in $\operatorname{add} T$.

19. Let T_A be a tilting module and $B = \operatorname{End} T_A$. If, for a point a of Q_A, both $P(a)$ and $I(a)$ are in $\operatorname{add} T$, then show that $\operatorname{Hom}_A(T, I(a))$ is a projective-injective B-module and, conversely, show that every indecomposable projective-injective B-module is of this form.

20. Let T_A be a tilting module. Prove the following implications:

(a) If $N \in \mathcal{F}(T)$, then $\operatorname{pd} \operatorname{Ext}^1_A(T, N) \le 1 + \max(1, \operatorname{pd} N)$.
(b) If $M \in \mathcal{T}(T)$, then $\operatorname{id} \operatorname{Hom}_A(T, M) \le 1 + \operatorname{id} M$.
(c) If $N \in \mathcal{F}(T)$, then $\operatorname{id} \operatorname{Ext}^1_A(T, N) \le \operatorname{id} N$.

Hint: See the remark following (4.2).

21. The following construction, due to Brenner and Butler, generalises that of the APR-tilting modules. Let A be an algebra and $S(a)$ be a simple A-module such that: (i) $\operatorname{pd} \tau^{-1} S(a) \le 1$ and (ii) $\operatorname{Ext}^1_A(S(a), S(a)) = 0$. Show that

(a) $T = \tau^{-1} S(a) \oplus (\bigoplus_{b \ne a} P(b))$ is a tilting module,
(b) $\mathcal{F}(T) = \operatorname{add} S(a)$.

Let A be as in Exercise 14 (ii). Find a simple A-module $S(a)$ satisfying (i) and (ii), construct the corresponding tilting module T as in (a), compute the bound quiver and the Auslander–Reiten quiver of $B = \operatorname{End} T$, and describe the equivalences $\mathcal{T}(T) \cong \mathcal{Y}(T)$, $\mathcal{F}(T) \cong \mathcal{X}(T)$.

22. An A-module T_A is called a **partial cotilting module** if T satisfies

(CT1) $\operatorname{id} T \le 1$ and
(CT2) $\operatorname{Ext}^1_A(T, T) = 0$

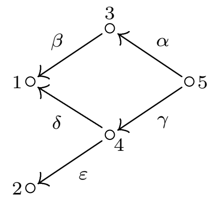
and a **cotilting module** if it also satisfies

(CT3) the number of pairwise nonisomorphic indecomposable summands of T equals the rank of $K_0(A)$.

Show that T_A is a (partial) cotilting module if and only if ${}_ADT$ is a (partial) tilting module. Then state and prove the analogues for (partial) cotilting modules of the results of Sections 2 and 3.

23. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod} A$. Show that there exists a tilting module T_A such that $\mathcal{T} = \mathcal{T}(T_A)$, $\mathcal{F} = \mathcal{F}(T_A)$ if and only if $\mathcal{F}$ is cogenerated by a module N such that $\operatorname{pd}(\tau^{-1}N) \le 1$.

24. Let A be given by the quiver



bound by $\alpha\beta = \gamma\delta$.

(a) Show that $\mathcal{X} = \operatorname{add}\left\{ \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \\ 0 & & \end{smallmatrix} \right\}$ is a torsion-free class in $\operatorname{mod} A$.

(b) Find a class $\mathcal{Y}$ such that $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\operatorname{mod} A$.

(c) Show that there exists an algebra C and a tilting module T_C such that $A = \operatorname{End} T_C$, $\mathcal{X} = \mathcal{X}(T_C)$, and $\mathcal{Y} = \mathcal{Y}(T_C)$. Compute the algebra C and the module T_C.