Chapter IV

Auslander–Reiten theory

As we saw in the previous chapter, quiver-theoretical techniques provide a convenient way to visualise finite dimensional algebras and their modules. However, to actually compute the indecomposable modules and the homomorphisms between them, we need other tools. Particularly useful in this context are the notions of irreducible morphisms and almost split sequences. These were introduced by Auslander [13] and Auslander and Reiten [19], [20] while presenting a categorical proof of the first Brauer–Thrall conjecture (see Section 5 and [136] for a historical account). Their main theorem may be stated as follows.

Let A be a finite dimensional K-algebra and N_A be a finite dimensional indecomposable nonprojective A-module. Then there exists a nonsplit short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in $\operatorname{mod} A$ *such that*

(a) *L is indecomposable noninjective;*

(b) *if $u : L \to U$ is not a section, then there exists $u' : M \to U$ such that $u = u'f$; and*

(c) *if $v : V \to N$ is not a retraction, then there exists $v' : V \to M$ such that $v = gv'$.*

Further, the sequence is uniquely determined up to isomorphism. Dually, if L_A is indecomposable noninjective, a nonsplit short exact sequence as preceding exists, with N indecomposable nonprojective and satisfying the properties (b) *and* (c)*. It is again unique up to isomorphism.*

Such a sequence is called an almost split sequence ending with N (or starting with L). In this chapter, we introduce the notions of irreducible morphisms and almost split morphisms, then prove the preceding existence theorem for almost split sequences in module categories. This allows us to define a new quiver, called the Auslander–Reiten quiver, which can be considered as a first approximation for the module category. We then apply these results to prove the first Brauer–Thrall conjecture.

Throughout this chapter, we let A denote a finite dimensional K-algebra, K denote an algebraically closed field, and all A-modules are, unless otherwise specified, right finite dimensional A-modules.

IV.1. Irreducible morphisms and almost split sequences

This first section is devoted to introducing the notions of irreducible, minimal, and almost split morphisms in the category $\operatorname{mod} A$ of finite dimensional right A-modules. We recall that the ultimate aim of the representation theory of algebras is, given an algebra A, to describe the finite dimensional A-modules and the homomorphisms between them.

By the unique decomposition theorem (I.4.10), any module in $\operatorname{mod} A$ is a direct sum of indecomposable modules and such a decomposition is unique up to isomorphism and a permutation of its indecomposable summands. It thus suffices to describe the latter and the A-module homomorphisms between them.

Before stating the following definitions, we recall that an A-homomorphism is a section (or a retraction) whenever it admits a left inverse (or a right inverse, respectively).

1.1. Definition. Let L, M, N be modules in $\operatorname{mod} A$.

(a) An A-module homomorphism $f : L \to M$ is called **left minimal** if every $h \in \operatorname{End} M$ such that $hf = f$ is an automorphism.

(b) An A-module homomorphism $g : M \to N$ is called **right minimal** if every $k \in \operatorname{End} M$ such that $gk = g$ is an automorphism.

(c) An A-module homomorphism $f : L \to M$ is called **left almost split** if

(i) f is not a section and

(ii) for every A-homomorphism $u : L \to U$ that is not a section there exists $u' : M \to U$ such that $u'f = u$, that is, u' makes the following triangle commutative

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "u"'] & M \arrow[dl, "u'"] \\ U & \end{tikzcd}$$

(d) An A-homomorphism $g : M \to N$ is called **right almost split** if

(i) g is not a retraction and

(ii) for every A-homomorphism $v : V \to N$ that is not a retraction, there exists $v' : V \to M$ such that $gv' = v$, that is, v' makes the following triangle commutative

$$\begin{tikzcd} & V \arrow[dl, "v'"'] \arrow[d, "v"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

(e) An A-module homomorphism $f : L \to M$ is called **left minimal almost split** if it is both left minimal and left almost split.

(f) An A-module homomorphism $g : M \to N$ is called **right minimal almost split** if it is both right minimal and right almost split.

Clearly, each "right-hand" notion is the dual of the corresponding "left-hand" notion. As a first observation, we prove that left (or right) minimal almost split morphisms uniquely determine their targets (or sources, respectively).

1.2. Proposition. (a) *If the A-module homomorphisms $f : L \to M$ and $f' : L \to M'$ are left minimal almost split, then there exists an isomorphism $h : M \to M'$ such that $f' = hf$.*

(b) *If the A-module homomorphisms $g : M \to N$ and $g' : M' \to N$ are right minimal almost split, then there exists an isomorphism $k : M \to M'$ such that $g = g'k$.*

Proof. We only prove (a); the proof of (b) is similar. Because f and f' are almost split, there exist $h : M \to M'$ and $h' : M' \to M$ such that $f' = hf$ and $f = h'f'$. Hence $f = h'hf$ and $f' = hh'f'$. Because f and f' are minimal, hh' and $h'h$ are automorphisms. Consequently, h is an isomorphism. □

We now see that almost split morphisms are closely related to indecomposable modules.

1.3. Lemma. (a) *If $f : L \to M$ is a left almost split morphism in* $\operatorname{mod} A$*, then the module L is indecomposable.*

(b) *If $g : M \to N$ is a right almost split morphism in* $\operatorname{mod} A$*, then the module N is indecomposable.*

Proof. We only prove (a); the proof of (b) is similar. Assume that $L = L_1 \oplus L_2$, with both L_1 and L_2 nonzero and let $p_i : L \to L_i$ (with $i = 1, 2$) denote the corresponding projections. For any i (with $i = 1, 2$), the homomorphism p_i is not a section. Hence there exists a homomorphism $u_i : M \to L_i$ such that $u_i f = p_i$. But then $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : M \to L$ satisfies $uf = 1_L$, and this contradicts the fact that f is not a section. □

1.4. Definition. A homomorphism $f : X \to Y$ in $\operatorname{mod} A$ is said to be **irreducible** provided:

(a) f is neither a section nor a retraction and

(b) if $f = f_1 f_2$, either f_1 is a retraction or f_2 is a section

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f_2"'] & & Y \\ & Z \arrow[ur, "f_1"'] & \end{tikzcd}$$

Clearly, this notion is self-dual. An irreducible morphism in $\operatorname{mod} A$ is either a proper monomorphism or a proper epimorphism: indeed, if $f : X \to Y$ is irreducible but is not a proper epimorphism, and $f = jp$ is its canonical factorisation through $\operatorname{Im} f$, then j is not a retraction, and consequently p is a section, so that f is a proper monomorphism. The same argument shows that the irreducible morphisms are precisely those that admit no nontrivial factorisation.

1.5. Example. (a) Let $e \in A$ be a primitive idempotent. Then the right A-module eA is indecomposable and the inclusion $\operatorname{rad} eA \hookrightarrow eA$ is right almost split and is an irreducible morphism. Indeed, if $v \in \operatorname{Hom}_A(V, eA)$ and v is not a retraction, then $\operatorname{Im} v$ is a proper submodule of eA. It follows from (I.4.5)(c) that $\operatorname{Im} v \subseteq \operatorname{rad} eA$, that is, $v : V \to eA$ factors through $\operatorname{rad} eA$, and consequently, $\operatorname{rad} eA \hookrightarrow eA$ is right almost split. It follows from the maximality of $\operatorname{rad} eA$ in eA that $\operatorname{rad} eA \hookrightarrow eA$ is an irreducible morphism.

(b) Let S be a simple A-module, and let $E = E_A(S)$ be the injective envelope of S in $\operatorname{mod} A$. Then the canonical epimorphism $p : E \to E/S$ is left almost split and is an irreducible morphism. This follows from (a) by applying the duality functor $D : \operatorname{mod} A \longrightarrow \operatorname{mod} A^{\mathrm{op}}$ and (I.5.13).

We now reformulate the definition of irreducible morphisms using the notion of **radical** rad_A **of the category** $\operatorname{mod} A$ introduced in Section A.3 of the Appendix.

We recall that $\operatorname{rad}_A = \operatorname{rad}_{\operatorname{mod} A}$ denotes the radical $\operatorname{rad}_{\mathcal{C}}$ of the category $\mathcal{C} = \operatorname{mod} A$. If X and Y are indecomposable modules in $\operatorname{mod} A$, then $\operatorname{rad}_A(X, Y)$ is the K-vector space of all noninvertible homomorphisms from X to Y. Thus, if X is indecomposable, $\operatorname{rad}_A(X, X)$ is just the radical of the local algebra $\operatorname{End} X$. Further, if X and Y are arbitrary modules in $\operatorname{mod} A$, then $\operatorname{rad}_A(X, Y)$ is an $\operatorname{End} Y$–$\operatorname{End} X$-subbimodule of $\operatorname{Hom}_A(X, Y)$. This implies that $\operatorname{rad}_A(-, -)$ is a subfunctor of the bifunctor $\operatorname{Hom}_A(-, -)$.

Similarly, if X and Y are modules in $\operatorname{mod} A$, we define $\operatorname{rad}^2_A(X, Y)$ to consist of all A-module homomorphisms of the form gf, where $f \in \operatorname{rad}_A(X, Z)$ and $g \in \operatorname{rad}_A(Z, Y)$ for some (not necessarily indecomposable) object Z in $\operatorname{mod} A$. It is clear that $\operatorname{rad}^2_A(X, Y) \subseteq \operatorname{rad}_A(X, Y)$ and even that $\operatorname{rad}^2_A(X, Y)$ is an $\operatorname{End} Y$–$\operatorname{End} X$-subbimodule of $\operatorname{rad}_A(X, Y)$.

The next lemma shows that the quotient space $\operatorname{rad}_A(X, Y)/\operatorname{rad}^2_A(X, Y)$ measures the number of irreducible morphisms between indecomposable modules X and Y.

1.6. Lemma. *Let X, Y be indecomposable modules in* $\operatorname{mod} A$. *A morphism $f : X \to Y$ is irreducible if and only if $f \in \operatorname{rad}_A(X, Y) \setminus \operatorname{rad}^2_A(X, Y)$.*

Proof. Assume that f is irreducible. Then, clearly, $f \in \operatorname{rad}_A(X, Y)$. If $f \in \operatorname{rad}^2_A(X, Y)$, then f can be written as $f = gh$, where $h \in \operatorname{rad}_A(X, Z)$ and $g \in \operatorname{rad}_A(Z, Y)$ for some Z in $\operatorname{mod} A$. Decomposing Z into indecomposable summands as $Z = \bigoplus_{i=1}^{t} Z_i$, we can write $h = \begin{bmatrix} h_1 \\ \vdots \\ h_t \end{bmatrix} : X \longrightarrow \bigoplus_{i=1}^{t} Z_i$ and $g = [g_1 \dots g_t] : \bigoplus_{i=1}^{t} Z_i \longrightarrow Y$. Because f is irreducible, h is a section or g is a retraction. Assume the former, and let $h' = [h'_1 \dots h'_t] : \bigoplus_{i=1}^{t} Z_i \longrightarrow X$ be such that $1_X = h'h = \sum_{i=1}^{t} h'_i h_i$. Because h_i is not invertible (for any i), $h'_i h_i$ is not invertible either, and so $h'_i h_i \in \operatorname{rad}_A(X, X) = \operatorname{rad} \operatorname{End} X$. Because $\operatorname{End} X$ is local, we infer that $1_X \in \operatorname{rad} \operatorname{End} X$, a contradiction. Consequently, h is not a section. Similarly, g is not a retraction. This contradiction shows that $f \notin \operatorname{rad}^2_A(X, Y)$.

Conversely, assume that $f \in \operatorname{rad}_A(X, Y) \setminus \operatorname{rad}^2_A(X, Y)$. Because X, Y are indecomposable and f is not an isomorphism, it is clearly neither a section nor a retraction. Suppose that $f = gh$, where $h : X \to Z$, $g : Z \to Y$. Decompose Z into indecomposable summands as $Z = \bigoplus_{i=1}^{t} Z_i$ and write

$$h = \begin{bmatrix} h_1 \\ \vdots \\ h_t \end{bmatrix} : X \longrightarrow \bigoplus_{i=1}^{t} Z_i \text{ and } g = [g_1 \dots g_t] : \bigoplus_{i=1}^{t} Z_i \longrightarrow Y$$

so that $f = \sum_{i=1}^{t} g_i h_i$. Because $f \notin \operatorname{rad}^2_A(X, Y)$, there is either an index i such that h_i is invertible or an index j such that g_j is invertible. In the first case, h is a section; in the second, g is a retraction. □

In the following lemma, we characterise irreducible monomorphisms (or epimorphisms) in $\operatorname{mod} A$ by means of their cokernels (or kernels, respectively).

1.7 Lemma. *Let* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be a nonsplit short exact sequence in* $\operatorname{mod} A$.

(a) *The homomorphism* $f : L \to M$ *is irreducible if and only if, for every homomorphism* $v : V \to N$, *there exists* $v_1 : V \to M$ *such that* $v = gv_1$ *or* $v_2 : M \to V$ *such that* $g = vv_2$.

(b) *The homomorphism* $g : M \to N$ *is irreducible if and only if, for every homomorphism* $u : L \to U$, *there exists* $u_1 : M \to U$ *such that*

$u = u_1 f$ or $u_2 : U \to M$ such that $f = u_2 u$.

Proof. We only prove (a); the proof of (b) is similar. Assume first that $f : L \to M$ is irreducible, and let $v : V \to N$ be arbitrary. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f'} & E & \xrightarrow{g'} & V & \longrightarrow & 0 \\ & & \Big\downarrow 1_L & & \Big\downarrow u & & \Big\downarrow v & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

with exact rows, where E denotes the fibered product of V and M over N. Because $f = uf'$ is irreducible, f' is a section or u is a retraction. In the first case, g' is a retraction and there exists $v_1 : V \to M$ such that $gv_1 = v$. If $u' : V \to E$ is such that $g'u' = 1_V$, then $v_1 = uu'$ satisfies $gv_1 = v$. In the second case, there exists $v_2 : M \to V$ such that $g = vv_2$.

Conversely, assume that the stated condition is satisfied. Because the given sequence is not split, f is neither a section nor a retraction. Suppose that $f = f_1 f_2$, where $f_2 : L \to U$, $f_1 : U \to M$. Because f is a monomorphism, so is f_2 and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f_2} & U & \xrightarrow{u} & V & \longrightarrow & 0 \\ & & \Big\downarrow 1_L & & \Big\downarrow f_1 & & \Big\downarrow v & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

with exact rows, where $V = \operatorname{Coker} f_2$. In particular, by (A.5.3) of the Appendix, the module U is isomorphic to the fibered product of V and M over N. If there exists $v_1 : V \to M$ such that $v = gv_1$, then the universal property of the fibered product implies that u is a retraction and so f_2 is a section. If there exists $v_2 : M \to V$ such that $g = vv_2$, then, similarly, f_1 is a retraction. This shows that f is irreducible. □

As a first application of Lemma 1.7, we show that irreducible morphisms provide a useful method to construct indecomposable modules.

1.8. Corollary. (a) *If $f : L \to M$ is an irreducible monomorphism, then $N = \operatorname{Coker} f$ is indecomposable.*

(b) *If $g : M \to N$ is an irreducible epimorphism, then $L = \operatorname{Ker} g$ is indecomposable.*

Proof. We only prove (a); the proof of (b) is similar. Let $g : M \to N$ be the cokernel of f and assume that $N = N_1 \oplus N_2$ with N_1 and N_2 nonzero. Let $q_i : N_i \to N$ (with $i = 1, 2$) denote the corresponding inclusions. If there exists a morphism $u_i : M \to N_i$ such that $g = q_i u_i$, then, because g is an epimorphism, q_i is also an epimorphism and hence an isomorphism, contrary to the fact that $N_1 \neq 0$ and $N_2 \neq 0$. Then, by (1.7), there exists, for each $i = 1, 2$, a homomorphism $v_i : N_i \to M$ such that $gv_i = q_i$. Thus $v = [v_1\ v_2] : N_1 \oplus N_2 \to M$ satisfies $gv = 1_N$, so that g is a retraction. But then f is a section, and this contradicts the fact that f is irreducible. $\square$

The following easy lemma is needed in the proof of the next theorem.

1.9. Lemma. (a) *Let $f : L \to M$ be a nonzero A-module homomorphism, with L indecomposable. Then f is not a section if and only if* $\operatorname{Im} \operatorname{Hom}_A(f, L) \subseteq \operatorname{rad} \operatorname{End} L$.

(b) *Let $g : M \to N$ be a nonzero A-module homomorphism, with N indecomposable. Then g is not a retraction if and only if* $\operatorname{Im} \operatorname{Hom}_A(N, g) \subseteq \operatorname{rad} \operatorname{End} N$.

Proof. We prove (a); the proof of (b) is similar. Because L is indecomposable, $\operatorname{End} L$ is local. If $\operatorname{Im} \operatorname{Hom}_A(f, L) \not\subseteq \operatorname{rad} \operatorname{End} L$, there exists $h : M \to L$ such that $k = \operatorname{Hom}_A(f, L)(h) = hf$ is invertible. But then $k^{-1}hf = 1_L$ shows that f is a section. Conversely, if there exists h such that $hf = 1_L$, then $\operatorname{Hom}_A(f, L)(h) = 1_L$ shows that $\operatorname{Hom}_A(f, L)$ is an epimorphism. $\square$

We now relate the previous notions, showing that one may think of irreducible morphisms as components of minimal almost split morphisms.

1.10. Theorem. (a) *Let $f : L \to M$ be left minimal almost split in* $\operatorname{mod} A$. *Then f is irreducible. Further, a homomorphism $f' : L \to M'$ of A-modules is irreducible if and only if $M' \neq 0$ and there exists a direct sum decomposition $M \cong M' \oplus M''$ and a homomorphism $f'' : L \to M''$ such that $\begin{bmatrix} f' \\ f'' \end{bmatrix} : L \longrightarrow M' \oplus M''$ is left minimal almost split.*

(b) *Let $g : M \to N$ be right minimal almost split in* $\operatorname{mod} A$. *Then g is irreducible. Further, a homomorphism $g' : M' \to N$ of A-modules is irreducible if and only if $M' \neq 0$ and there exists a direct sum decomposition $M \cong M' \oplus M''$ and a homomorphism $g'' : M'' \to N$ such that $[g'\, g''] : M' \oplus M'' \longrightarrow N$ is right minimal almost split.*

Proof. We prove (a); the proof of (b) is similar. Let $f : L \to M$ be a left minimal almost split homomorphism in $\operatorname{mod} A$. By definition, f is not a section. Because, by (1.3), L is indecomposable and f is not an isomorphism, f is not a retraction either. Assume that $f = f_1 f_2$, where $f_2 : L \to X$ and $f_1 : X \to M$. We suppose that f_2 is not a section and prove that f_1 is a retraction. Because f is left almost split, there exists $f_2' : M \to X$ such that $f_2 = f_2' f$. Hence $f = f_1 f_2 = f_1 f_2' f$. Because f is left minimal, $f_1 f_2'$ is an automorphism and so f_1 is a retraction. This proves the first statement.

Let now $f' : L \to M'$ be an irreducible morphism in $\operatorname{mod} A$. Then clearly, $M' \neq 0$. Also, f' is not a section, hence there exists $h : M \to M'$ such that $f' = hf$. Because f' is irreducible and f is not a section, h is a retraction. Let $M'' = \operatorname{Ker} h$. Then there exists a homomorphism $q : M \to M''$ such that $\begin{bmatrix} h \\ q \end{bmatrix} : M \to M' \oplus M''$ is an isomorphism. It follows that $\begin{bmatrix} h \\ q \end{bmatrix} f = \begin{bmatrix} f' \\ qf \end{bmatrix} : L \to M' \oplus M''$ is left minimal almost split.

Conversely, assume that f' satisfies the stated condition; we must show that it is irreducible. Because L is indecomposable and f' is not an isomorphism, f' is not a retraction. On the other hand, if there exists h such that $hf' = 1_L$, then $[h \ 0] \begin{bmatrix} f' \\ f'' \end{bmatrix} = 1_L$ implies that $\begin{bmatrix} f' \\ f'' \end{bmatrix}$ is a section, a contradiction. Thus, f' is not a section. Assume that $f' = f_1 f_2$, where $f_2 : L \to X$ and $f_1 : X \to M'$. We suppose that f_2 is not a section and show that f_1 is a retraction. We have $\begin{bmatrix} f' \\ f'' \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_2 \\ f'' \end{bmatrix}$, where $\begin{bmatrix} f_2 \\ f'' \end{bmatrix} : L \to X \oplus M''$ and $\begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix} : X \oplus M'' \to M' \oplus M''$. Because f_2 is not a section, it follows from (1.9) that $\operatorname{Im} \operatorname{Hom}_A(f_2, L) \subseteq \operatorname{rad} \operatorname{End} L$. Similarly $\operatorname{Im} \operatorname{Hom}_A(f'', L) \subseteq \operatorname{rad} \operatorname{End} L$. Consequently, $\operatorname{Im} \operatorname{Hom}_A\left(\begin{bmatrix} f_2 \\ f'' \end{bmatrix}, L \right) \subseteq \operatorname{rad} \operatorname{End} L$, hence, again by (1.9), $\begin{bmatrix} f_2 \\ f'' \end{bmatrix}$ is not a section. Because $\begin{bmatrix} f' \\ f'' \end{bmatrix}$ is left minimal almost split and hence irreducible, $\begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix}$ is a retraction, and this implies that f_1 is a retraction. The proof is now complete. □

We now define a particular type of short exact sequence, which is particularly useful in the representation theory of algebras.

1.11. Definition. A short exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called an **almost split sequence** provided:

(a) f is left minimal almost split and
(b) g is right minimal almost split.

While the existence of almost split sequences is far from obvious, it follows from (1.3) that if such a sequence exists, then L and N are indecomposable modules. Also, an almost split sequence is never split (because f is not a section and g is not a retraction) so that L is not injective, and N is not projective. Finally, an almost split sequence is uniquely determined (up to isomorphism) by each of its end terms; indeed, if $0 \to L \to M \to N \to 0$ and $0 \to L' \to M' \to N' \to 0$ are two almost split sequences in $\operatorname{mod} A$, then (1.2) implies that the following assertions are equivalent:

(a) The two sequences are isomorphic.

(b) There is an isomorphism $L \cong L'$ of A-modules.

(c) There is an isomorphism $N \cong N'$ of A-modules.

1.12. Lemma. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

be a commutative diagram in $\operatorname{mod} A$*, where the rows are exact and not split.*

(a) *If L is indecomposable and w is an automorphism, then u and hence v are automorphisms.*

(b) *If N is indecomposable and u is an automorphism, then w and hence v are automorphisms.*

Proof. We only prove (a); the proof of (b) is similar. We may suppose that $w = 1_N$. If u is not an isomorphism, it must be nilpotent (because End L is local) and so there exists m such that $u^m = 0$. Then $v^m f = fu^m = 0$ and so v^m factors through the cokernel N of f, that is, there exists $h : N \to M$ such that $v^m = hg$. Because $gv^m = g$, we deduce that $ghg = g$ and consequently $gh = 1_N$ (because g is an epimorphism). This contradicts the fact that the given sequence is not split. $\square$

We end this section by giving several equivalent characterisations of almost split sequences.

1.13. Theorem. *Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a short exact sequence in* $\operatorname{mod} A$*. The following assertions are equivalent:*

(a) *The given sequence is almost split.*

(b) *L is indecomposable, and g is right almost split.*

(c) *N is indecomposable, and f is left almost split.*

(d) *The homomorphism f is left minimal almost split.*

(e) *The homomorphism g is right minimal almost split.*

(f) *L and N are indecomposable, and f and g are irreducible.*

Proof. By definition of almost split sequence, (a) implies (d) and (e). By (1.3), (a) implies (b) and (c). By (1.10) and (1.3), (a) implies (f) as well. To prove the equivalence of the first five conditions, we start by proving that (e) implies (b). Dually, (d) implies (c). Thus, the equivalence of the first three conditions implies that of the first five conditions. We show that (b) implies (c); the proof that (c) implies (b) is similar, and we prove that both conditions together imply (a). Finally, we show that (f) implies (b), which will complete the proof of the theorem.

Assume (e), that is, g is right minimal almost split. By (1.10), g is irreducible. Hence, by (1.8), $L = \operatorname{Ker} g$ is indecomposable. Thus, (e) implies (b).

To show that (b) implies (c), it suffices, by (1.3), to show that f is left almost split. Because g is not a retraction, f is not a section. Let $u : L \to U$ be such that $u'f \neq u$ for all $u' : M \to U$. We must prove that u is a section. It follows from (A.5.3) of the Appendix that there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle 1_N} & & \\ 0 & \longrightarrow & U & \xrightarrow{h} & V & \xrightarrow{k} & N & \longrightarrow & 0 \end{array}$$

with exact rows, where V is the amalgamed sum. The lower sequence is not split and hence k is not a retraction. Because g is right almost split, there exists $\overline{v} : V \to M$ such that $k = g\overline{v}$, and hence we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{u}u} & & \downarrow{\scriptstyle \overline{v}v} & & \downarrow{\scriptstyle 1_N} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

with exact rows, where $\overline{u}$ is derived from $\overline{v}$ and 1_N by passing to the kernels. By (1.12), $\overline{u}u$ is an automorphism. Hence u is a section.

Now, assume that both (b) and (c) hold; we must prove that f and g are minimal. To prove that f is left minimal, let $h \in \operatorname{End} M$ be such that $hf = f$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_L} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_N} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

with exact rows. By (1.12), h is an automorphism. Hence f is left minimal. Similarly, g is right minimal.

We now prove that (f) implies (b). By hypothesis, L is indecomposable and g is not a retraction. Assume that $v : V \to N$ is not a retraction. We may suppose that V is indecomposable (replacing it, if necessary, by one of its indecomposable summands). Because f is irreducible, (1.7) gives $v' : V \to M$ such that $v = gv'$ (and then we are done), or else $h : M \to V$ such that $g = vh$. But in this case, because g is irreducible and v is not a retraction, h must be a section. Because V is indecomposable, h is an isomorphism. But then $v' = h^{-1}$ satisfies $v = gv'$ and we have completed the proof of our theorem. $\square$

IV.2. The Auslander–Reiten translations

In this section and the next, we prove the existence of almost split sequences in the category $\operatorname{mod} A$ of finite dimensional A-modules, for A a finite dimensional K-algebra. We first consider the A-dual functor

$$(-)^t = \operatorname{Hom}_A(-, A) : \operatorname{mod} A \longrightarrow \operatorname{mod} A^{\mathrm{op}}.$$

We note that if P_A is a projective right A-module, then $P^t = \operatorname{Hom}_A(P, A)$ is a projective left A-module; indeed, if $P_A \cong eA$, with $e \in A$ a primitive idempotent, then $P^t = \operatorname{Hom}_A(eA, A) \cong Ae$, and our statement thus follows from the additivity of $(-)^t$. Moreover, one shows easily that the evaluation homomorphism $\epsilon_M : M \to M^{tt}$ defined by $\epsilon_M(z)(f) = f(z)$ (for $z \in M$ and $f \in M^t$) is functorial in M and is an isomorphism whenever M is projective. Thus, the functor $(-)^t$ induces a duality, also denoted by $(-)^t$, between the category $\operatorname{proj} A$ of projective right A-modules, and the category $\operatorname{proj} A^{\mathrm{op}}$ of projective left A-modules. We use this new duality to define a duality on an appropriate quotient of $\operatorname{mod} A$, and this duality is called the transposition.

We start by approximating each module M_A by projective modules. Let thus

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be a minimal projective presentation of M, that is, an exact sequence such that $p_0 : P_0 \to M$ and $p_1 : P_1 \to \operatorname{Ker} p_0$ are projective covers. Applying the (left exact, contravariant) functor $(-)^t$, we obtain an exact sequence of left A-modules

$$0 \longrightarrow M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow \operatorname{Coker} p_1^t \longrightarrow 0.$$

We denote $\operatorname{Coker} p_1^t$ by $\operatorname{Tr} M$ and call it the **transpose** of M.

We observe that the left A-module $\operatorname{Tr} M$ is uniquely determined up to isomorphism; this indeed follows from the fact that projective covers (and hence minimal projective presentations) are uniquely determined up to isomorphism.

We now summarise the main properties of the transpose Tr.

2.1. Proposition. *Let M be an indecomposable module in* $\operatorname{mod} A$.

(a) *The left A-module* $\operatorname{Tr} M$ *has no nonzero projective direct summands.*

(b) *If M is not projective, then the sequence*

$$P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow \operatorname{Tr} M \longrightarrow 0$$

induced from the minimal projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ of M is a minimal projective presentation of the left A-module $\operatorname{Tr} M$.

(c) *M is projective if and only if* $\operatorname{Tr} M = 0$. *If M is not projective, then* $\operatorname{Tr} M$ *is indecomposable and* $\operatorname{Tr}(\operatorname{Tr} M) \cong M$.

(d) *If M and N are indecomposable nonprojective, then $M \cong N$ if and only if* $\operatorname{Tr} M \cong \operatorname{Tr} N$.

Proof. If M is projective, then the term P_1 in the minimal projective presentation of M is zero, and therefore $\operatorname{Tr} M = 0$. Conversely, if $\operatorname{Tr} M = 0$, then p_1^t is an epimorphism, hence a retraction (because ${}_A(P_1^t)$ is projective). Thus, p_1 is a section, and M is projective. This shows the first part of (c).

Assume that M is not projective. Then $\operatorname{Tr} M \neq 0$. The sequence given in (b) is certainly a projective presentation of the left module $\operatorname{Tr} M$. We claim it is minimal. Indeed, if this is not the case, there exist nontrivial direct sum decompositions $P_0^t = E_0' \oplus E_0''$, $P_1^t = E_1' \oplus E_1''$ and an isomorphism $v : E_0'' \xrightarrow{\simeq} E_1''$ such that this sequence is isomorphic to the sequence

$$E_0' \oplus E_0'' \xrightarrow{\left[\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right]} E_1' \oplus E_1'' \longrightarrow \operatorname{Tr} M \longrightarrow 0,$$

where $u : E_0' \to E_1'$ is a homomorphism of left A-modules. But then applying $(-)^t$ yields a projective presentation of M of the form

$$E_1'^t \xrightarrow{u^t} E_0'^t \longrightarrow M \longrightarrow 0,$$

and this contradicts the minimality of the projective presentation of M. This shows our claim. Moreover, if $\operatorname{Tr} M$ has a nonzero projective direct summand, the homomorphism p_1^t has a direct summand of the form $(0 \to E)$, with ${}_AE$ projective. But, as earlier, this implies that p_1 has a direct summand of the form $(E^t \to 0)$, and we obtain another contradiction. We have thus shown (a) and (b).

Applying now $(-)^t$ to the exact sequence in (b), we get a commutative diagram

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] \arrow[d, "\varepsilon_{P_1}"', "\cong"] & P_0 \arrow[r, "p_0"] \arrow[d, "\varepsilon_{P_0}"', "\cong"] & M \arrow[r] & 0 \\
P_1^{tt} \arrow[r, "p_1^{tt}"] & P_0^{tt} \arrow[r, "p_0^{tt}"] & \operatorname{Tr}\operatorname{Tr} M \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Hence there is an isomorphism $M \cong \operatorname{Tr}\operatorname{Tr} M$ making the right square commutative. This proves (c), and (d) follows immediately. □

We have shown that the transpose Tr maps modules of $\operatorname{mod} A$ to modules of $\operatorname{mod} A^{\mathrm{op}}$ but does not define a duality $\operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$, because it annihilates the projectives. In order to make this correspondence a duality, we thus need to annihilate the projectives from $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$. This motivates the following construction.

For two A-modules M, N, let $\mathcal{P}(M, N)$ denote the subset of $\operatorname{Hom}_A(M, N)$ consisting of all homomorphisms that factor through a projective A-module.

We claim that this defines an ideal $\mathcal{P}$ in the category $\operatorname{mod} A$. First, for two modules M, N, the set $\mathcal{P}(M,N)$ is a subspace of the K-vector space $\operatorname{Hom}_A(M,N)$; indeed, if f, $f' \in \mathcal{P}(M,N)$, then f and f' can be respectively written as $f = hg$ and $f' = h'g'$, where the targets P of g and P' of g' are projective; consequently

$$f + f' = hg + h'g' = [h\ h'] \begin{bmatrix} g \\ g' \end{bmatrix}$$

factors through the projective module $P \oplus P'$. On the other hand, if $\lambda \in K$ and $f \in \mathcal{P}(M,N)$, then $\lambda f \in \mathcal{P}(M,N)$. Next, if $f \in \mathcal{P}(L,M)$ and $g \in \operatorname{Hom}_A(M,N)$, then $gf \in \mathcal{P}(L,N)$ and similarly, if $f \in \operatorname{Hom}_A(L,M)$ and $g \in \mathcal{P}(M,N)$, then $gf \in \mathcal{P}(L,N)$. This completes the proof that $\mathcal{P}$ is an ideal of $\operatorname{mod} A$.

We may thus consider the quotient category

$$\underline{\operatorname{mod}}\ A = \operatorname{mod} A/\mathcal{P}$$

called the **projectively stable category**. Its objects are the same as those of $\operatorname{mod} A$, but the K-vector space $\underline{\operatorname{Hom}}_A(M,N)$ of morphisms from M to N in $\underline{\operatorname{mod}}\ A$ is defined to be the quotient vector space

$$\underline{\operatorname{Hom}}_A(M,N) = \operatorname{Hom}_A(M,N)/\mathcal{P}(M,N)$$

of $\operatorname{Hom}_A(M,N)$ with the composition of morphisms induced from the composition in $\operatorname{mod} A$. There clearly exists a functor $\operatorname{mod} A \to \underline{\operatorname{mod}}\ A$ that is the identity on objects and associates to a homomorphism $f : M \to N$ in $\operatorname{mod} A$ its residual class modulo $\mathcal{P}(M,N)$ in $\underline{\operatorname{mod}}\ A$.

Dually, one may construct an ideal $\mathcal{I}$ in $\operatorname{mod} A$ by considering, for each pair (M,N) of A-modules, the K-subspace $\mathcal{I}(M,N)$ of $\operatorname{Hom}_A(M,N)$ consisting of all homomorphisms that factor through an injective A-module. The quotient category

$$\overline{\operatorname{mod}}\ A = \operatorname{mod} A/\mathcal{I}$$

is called the **injectively stable category**. Its objects are the same as those of $\operatorname{mod} A$, but the K-vector space $\overline{\operatorname{Hom}}_A(M,N)$ of morphisms from M to N in $\overline{\operatorname{mod}}\ A$ is given by the quotient vector space

$$\overline{\operatorname{Hom}}_A(M,N) = \operatorname{Hom}_A(M,N)/\mathcal{I}(M,N)$$

of $\operatorname{Hom}_A(M,N)$ with the composition of morphisms induced from the composition in $\operatorname{mod} A$. One again defines in the obvious way the residual class functor $\operatorname{mod} A \to \overline{\operatorname{mod}}\ A$.

We now see that, although the correspondence $M \mapsto \operatorname{Tr} M$ does not define a duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$, it does define one between the quotient categories $\underline{\mathrm{mod}}\, A$ and $\underline{\mathrm{mod}}\, A^{\mathrm{op}}$.

2.2. Proposition. *The correspondence* $M \mapsto \operatorname{Tr} M$ *induces a* K*-linear duality functor* $\operatorname{Tr} : \underline{\mathrm{mod}}\, A \longrightarrow \underline{\mathrm{mod}}\, A^{op}$.

Proof. To construct this duality, we start by giving an alternative construction of $\underline{\mathrm{mod}}\, A$ as a quotient category. Let $\overrightarrow{\mathrm{proj}}\, A$ denote the category whose objects are the triples (P_1, P_0, f), where P_1, P_0 are projective A-modules, and $f : P_1 \to P_0$ is a homomorphism in $\mathrm{mod}\, A$. (The notation $\overrightarrow{\mathrm{proj}}\, A$ is meant to suggest that we are dealing with homomorphisms between projective modules.) We define a morphism $(P_1, P_0, f) \longrightarrow (P_1', P_0', f')$ to be a pair (u_1, u_0) of homomorphisms in $\mathrm{mod}\, A$ such that $u_1 : P_1 \to P_1'$ and $u_0 : P_0 \to P_0'$ satisfy $f'u_1 = u_0 f$, that is, the following square is commutative

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_0 \\ \big\downarrow{\scriptstyle u_1} & & \big\downarrow{\scriptstyle u_0} \\ P_1' & \xrightarrow{f'} & P_0' \end{array}$$

The composition of the morphisms $(u_1, u_0) : (P_1, P_0, f) \longrightarrow (P_1', P_0', f')$ and $(u_1', u_0') : (P_1', P_0', f') \longrightarrow (P_1'', P_0'', f'')$ in the category $\overrightarrow{\mathrm{proj}}\, A$ is defined by the formula $(u_1', u_0')(u_1, u_0) = (u_1' u_1, u_0' u_0)$.

Let now $F : \overrightarrow{\mathrm{proj}}\, A \longrightarrow \underline{\mathrm{mod}}\, A$ denote the composition of the cokernel functor $\overrightarrow{\mathrm{proj}}\, A \longrightarrow \mathrm{mod}\, A$, given by $(P_1, P_0, f) \mapsto \operatorname{Coker} f$, with the residual class functor $\mathrm{mod}\, A \longrightarrow \underline{\mathrm{mod}}\, A$. Let $(u_1, u_0) : (P_1, P_0, f) \longrightarrow (P_1', P_0', f')$ be a morphism in $\overrightarrow{\mathrm{proj}}\, A$. We claim that $F(u_1, u_0) = 0$ if and only if there exists $w : P_0 \to P_1'$ such that $f'wf = u_0 f$. The situation can be visualised in the following diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_0 \\ \big\downarrow{\scriptstyle u_1} & w & \big\downarrow{\scriptstyle u_0} \\ P_1' & \xrightarrow{f'} & P_0' \end{array}$$

Indeed, assume that such a homomorphism w exists and consider the commutative diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle u_1} & w & \big\downarrow{\scriptstyle u_0} & v & \big\downarrow{\scriptstyle u} & & \\ P_1' & \xrightarrow{f'} & P_0' & \xrightarrow{g'} & M' & \longrightarrow & 0 \end{array}$$

with exact rows, where M and M' denote the cokernels of f and f', respectively, and u is induced from u_1 and u_0 by passing to the cokernels. Because $(u_0 - f'w)f = 0$, there exists $v : M \to P_0'$ such that $u_0 - f'w = vg$. But then $g'vg = g'u_0 = ug$ gives $g'v = u$ (because g is an epimorphism). Hence $u \in \mathcal{P}(M, M')$ and $F(u_1, u_0) = 0$. Conversely, assume that $F(u_1, u_0) = 0$. This means that the homomorphism u induced from u_1 and u_0 by passing to the respective cokernels of f and f' factors through a projective module. Because g' is an epimorphism, this implies the existence of $v : M \to P_0'$ such that $u = g'v$. But then $g'(u_0 - vg) = g'u_0 - g'vg = g'u_0 - ug = 0$ and there exists $w : P_0 \to P_1'$ such that $f'w = u_0 - vg$. Hence $f'wf = u_0 f$ and we have proved our claim.

This implies at once that the class $\overrightarrow{\mathrm{proj}}_1\, A$ of those morphisms (u_1, u_0) in $\overrightarrow{\mathrm{proj}}\, A$ such that $F(u_1, u_0) = 0$ forms an ideal in $\overrightarrow{\mathrm{proj}}\, A$. To see this, assume that $(u_1, u_0) : (P_1, P_0, f) \to (P_1', P_0', f')$ is a morphism in $\overrightarrow{\mathrm{proj}}_1 A$ and let $(v_1, v_0) : (P_1', P_0', f') \to (P_1'', P_0'', f'')$ be any morphism in $\overrightarrow{\mathrm{proj}}\, A$. It follows from the preceding claim that there exists $w : P_0 \to P_1'$ such that $f'wf = u_0 f$. But then $v_1 w : P_0 \to P_1''$ satisfies $f''(v_1 w)f = (f''v_1)wf = (v_0 f')wf = (v_0 u_0)f$ so that $(v_1 u_1, v_0 u_0)$ belongs to $\overrightarrow{\mathrm{proj}}_1\, A$. Similarly, if (u_1, u_0) is as earlier and $(w_1, w_0) : (Q_1, Q_0, g) \to (P_1, P_0, f)$ is any morphism in $\overrightarrow{\mathrm{proj}}\, A$, then $(u_1 w_1, u_0 w_0)$ belongs to $\overrightarrow{\mathrm{proj}}_1\, A$.

The foregoing considerations imply that the category $\underline{\mathrm{mod}}\, A$ is equivalent to the quotient of $\overrightarrow{\mathrm{proj}}\, A$ modulo $\overrightarrow{\mathrm{proj}}_1\, A$. Indeed, if M is an object in $\underline{\mathrm{mod}}\, A$, then we can write $M = F(P_1, P_0, f)$, where $P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$ is a minimal projective presentation of M and, given a morphism $u : M \to M'$ in $\underline{\mathrm{mod}}\, A$, where $M = F(P_1, P_0, f)$ and $M' = F(P_1', P_0', f')$, there exists a morphism $(u_1, u_0) : (P_1, P_0, f) \to (P_1', P_0', f')$ in $\overrightarrow{\mathrm{proj}}\, A$ making the following diagram commutative

$$\begin{array}{ccccccc}
P_1 & \xrightarrow{f} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle u_1} & & \big\downarrow{\scriptstyle u_0} & & \big\downarrow{\scriptstyle u} & & \\
P_1' & \xrightarrow{f'} & P_0' & \longrightarrow & M' & \longrightarrow & 0
\end{array}$$

(where the rows are minimal projective presentations), that is, $u = F(u_1, u_0)$. The morphism u equals zero in $\underline{\mathrm{mod}}\, A$ if and only if $F(u_1, u_0) = 0$, that is, if and only if (u_1, u_0) belongs to $\overrightarrow{\mathrm{proj}}_1\, A$. This shows that we have an "exact" sequence

$$0 \longrightarrow \overrightarrow{\mathrm{proj}}_1\, A \longrightarrow \overrightarrow{\mathrm{proj}}\, A \xrightarrow{F} \underline{\mathrm{mod}}\, A \longrightarrow 0.$$

We are now in a position to construct a duality $\underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A^{\mathrm{op}}$ induced by the correspondence $M \mapsto \mathrm{Tr}\, M$.

The duality $(-)^t : \mathrm{proj}\, A \xrightarrow{F} \mathrm{proj}\, A^{\mathrm{op}}$ induces obviously a duality $\overrightarrow{\mathrm{proj}}\, A \xrightarrow{F} \overrightarrow{\mathrm{proj}}\, A^{\mathrm{op}}$ given by the formula $(P_1, P_0, f) \mapsto (P_0^t, P_1^t, f^t)$. We also denote this duality by $(-)^t$. Now we claim that the restriction of $(-)^t$ to $\overrightarrow{\mathrm{proj}}_1\, A$ induces a duality $\overrightarrow{\mathrm{proj}}_1\, A \longrightarrow \overrightarrow{\mathrm{proj}}_1\, A^{\mathrm{op}}$. Indeed, let $(u_1, u_0) : (P_1, P_0, f) \to (P_1', P_0', f')$ belong to $\overrightarrow{\mathrm{proj}}_1\, A$; we must show that $(u_1^t, u_0^t) : (P_0'^t, P_1'^t, f'^t) \to (P_0^t, P_1^t, f^t)$ belongs to $\overrightarrow{\mathrm{proj}}_1\, A^{\mathrm{op}}$. But the hypothesis implies the existence of a homomorphism $w : P_0 \to P_1'$ such that $f'wf = u_0 f$. Hence $f^t w^t f'^t = f^t u_0^t = u_1^t f'^t$, and the conclusion follows.

We thus have a diagram with "exact rows" and commutative left square

$$\begin{tikzcd} 0 \arrow[r] & \overrightarrow{\mathrm{proj}}_1 A \arrow[r] \arrow[d, "(-)^t"] & \overrightarrow{\mathrm{proj}}\, A \arrow[r] \arrow[d, "(-)^t"] & \underline{\mathrm{mod}}\ A \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \overrightarrow{\mathrm{proj}}_1 A^{\mathrm{op}} \arrow[r] & \overrightarrow{\mathrm{proj}}\, A^{\mathrm{op}} \arrow[r] & \underline{\mathrm{mod}}\ A^{\mathrm{op}} \arrow[r] & 0 \end{tikzcd}$$

We define $\mathrm{Tr}\ : \underline{\mathrm{mod}}\ A \longrightarrow \underline{\mathrm{mod}}\ A^{\mathrm{op}}$ to be the unique functor that makes the right square commutative, namely, if $M = F(P_1, P_0, f)$, we set $\mathrm{Tr}\, M = F(P_0^t, P_1^t, f^t)$ and if $u : M \to M'$ is a morphism in $\underline{\mathrm{mod}}\ A$, where $M = F(P_1, P_0, f)$ and $M' = F(P_1', P_0', f')$, there exists a commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "u_1"] & P_0 \arrow[r] \arrow[d, "u_0"] & M \arrow[r] \arrow[d, "u"] & 0 \\ P_1' \arrow[r, "f'"] & P_0' \arrow[r] & M' \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Applying the functor $(-)^t$ yields a commutative diagram

$$\begin{tikzcd} P_0^t \arrow[r, "f^t"] & P_1^t \arrow[r] & \mathrm{Tr}\, M \arrow[r] & 0 \\ P_0'^t \arrow[r, "f'^t"] \arrow[u, "u_0^t"] & P_1'^t \arrow[r] \arrow[u, "u_1^t"] & \mathrm{Tr}\, M' \arrow[r] \arrow[u, dashed] & 0 \end{tikzcd}$$

with exact rows and a commutative left square. Let $\mathrm{Tr}\, u : \mathrm{Tr}\, M' \to \mathrm{Tr}\, M$ be the unique homomorphism that makes the right square commutative. It follows easily from these considerations that

$$\mathrm{Tr}\ : \underline{\mathrm{mod}}\ A \longrightarrow \underline{\mathrm{mod}}\ A^{\mathrm{op}}$$

is a well-defined functor and, in fact, a duality. □

The duality Tr defined in (2.2) is called the **transposition**. It transforms right A-modules into left A-modules and conversely. Thus, if we wish to define an endofunctor of $\mathrm{mod}\, A$, we need to compose it with another duality between right and left A-modules, namely the standard duality $D = \mathrm{Hom}_K(-, K)$.

2.3. Definition. The **Auslander–Reiten translations** are defined to be the compositions of D with Tr, namely, we set

$$\tau = D\mathrm{Tr} \qquad \text{and} \qquad \tau^{-1} = \mathrm{Tr}\, D.$$

In view of the importance of the translations in the sequel, we present in the following proposition a construction method for the Auslander–Reiten translate of a module.

We first recall that the **Nakayama functor** (see (III.2.8)),

$$\nu = D(-)^t = D\mathrm{Hom}_A(-, A) : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A,$$

induces two equivalences of categories $\mathrm{proj}\, A \underset{\nu^{-1}}{\overset{\nu}{\rightleftarrows}} \mathrm{inj}\, A$, where $\nu^{-1} = \mathrm{Hom}_A(DA, -)$ is quasi-inverse to ν.

2.4. Proposition. (a) *Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$ be a minimal projective presentation of an A-module M. Then there exists an exact sequence*

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \longrightarrow 0.$$

(b) *Let $0 \longrightarrow N \xrightarrow{i_0} E_0 \xrightarrow{i_1} E_1$ be a minimal injective presentation of an A-module N. Then there exists an exact sequence*

$$0 \longrightarrow \nu^{-1}N \xrightarrow{\nu^{-1}i_0} \nu^{-1}E_0 \xrightarrow{\nu^{-1}i_1} \nu^{-1}E_1 \longrightarrow \tau^{-1}N \longrightarrow 0.$$

Proof. (a) Applying successively the functors $(-)^t$ and D to the given minimal projective presentation of M, we obtain an exact sequence

$$0 \longrightarrow D\mathrm{Tr}\, M \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \longrightarrow 0$$

and (a) follows.

(b) Applying successively the functors D and $(-)^t$ to the given minimal injective presentation of N, we obtain an exact sequence

$$0 \longrightarrow (DN)^t \xrightarrow{(Di_0)^t} (DE_0)^t \xrightarrow{(Di_1)^t} (DE_1)^t \longrightarrow \mathrm{Tr}\, DN \longrightarrow 0.$$

For any A-module X we have a composed functorial isomorphism

$$(DX)^t \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DX, A) \cong \mathrm{Hom}_A(DA, DDX) \cong \mathrm{Hom}_A(DA, X) \cong \nu^{-1}X.$$

This isomorphism induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (DN)^t \arrow[r, "(Di_0)^t"] \arrow[d, "\cong"] & (DE_0)^t \arrow[r, "(Di_1)^t"] \arrow[d, "\cong"] & (DE_1)^t \arrow[r] \arrow[d, "\cong"] & \mathrm{Tr}\, DN \arrow[r] & 0 \\
0 \arrow[r] & \nu^{-1}N \arrow[r, "\nu^{-1}i_0"] & \nu^{-1}E_0 \arrow[r, "\nu^{-1}i_1"] & \nu^{-1}E_1 & &
\end{tikzcd}$$

with exact rows. Hence (b) follows. □

2.5. Example. Let A be given by the Kronecker quiver $1 \circ \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \circ 2$ and M_A be the representation $K \underset{0}{\overset{1}{\leftleftarrows}} K$, where 1 denotes, as usual, the identity homomorphism and 0 the zero homomorphism. Then M is indecomposable; indeed, an endomorphism f of M is given by a pair (a_1, a_2) of scalars such that $a_1 \cdot 1 = 1 \cdot a_2$ and $a_1 \cdot 0 = 0 \cdot a_2$. These two conditions yield $f = a \cdot 1_M$, where $a = a_1 = a_2 \in K$. Thus $\mathrm{End}\, M_A \cong K$ and so M is indecomposable. A minimal projective presentation of M_A is given by

$$0 \longrightarrow P(1) \xrightarrow{p_1} P(2) \xrightarrow{p_2} M_A \longrightarrow 0,$$

where $P(1) = S(1) = \big(K \overleftarrow{\qquad\qquad} 0\big)$ and $P(2) = (K^2 \overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overleftarrow{\qquad\qquad}}} K)$ are the indecomposable projective A-modules, p_1 is an isomorphism of $P(1)$ onto the direct summand of $\operatorname{rad} P(2)$ equal to $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] K \overleftarrow{\qquad\qquad} 0$, and p_2 is its cokernel homomorphism. Thus, in particular, M_A is not projective. By (2.4)(a), applying the Nakayama functor ν to this exact sequence, we get a short exact sequence

$$0 \longrightarrow \tau M \longrightarrow I(1) \xrightarrow{\nu p_1} I(2) \longrightarrow 0,$$

where $I(1) = (K \overset{[1\,0]}{\underset{[0\,1]}{\overleftarrow{\qquad\qquad}}} K^2)$ and $I(2) = S(2) = (0 \overleftarrow{\qquad\qquad} K)$ are the indecomposable injective A-modules. An obvious computation shows that the homomorphism $\nu\, p_1$ induces an isomorphism of the quotient module of $I(1)$ defined by $0 \overleftarrow{\qquad\qquad} \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] K)$ onto $I(2)$. Then $\tau M = \operatorname{Ker} \nu\, p_1$ is given by $K \overset{1}{\underset{0}{\overleftarrow{\qquad\qquad}}} K$, that is, $\tau M \cong M$.

2.6. Example. Let A be given by the quiver

$$\begin{tikzcd}
\overset{1}{\circ} & & \overset{4}{\circ} \arrow[dl, "\beta"'] & \\
& \underset{3}{\circ} \arrow[ul, "\lambda"'] \arrow[dl, "\mu"] & & \circ\, 6 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
\underset{2}{\circ} & & \underset{5}{\circ} \arrow[ul, "\delta"] &
\end{tikzcd}$$

bound by $\alpha\beta = \gamma\delta$, $\delta\mu = 0$, and $\beta\lambda = 0$. Take the simple injective module

$$S(6):\quad \begin{tikzcd}
0 & & 0 \arrow[dl] & \\
& 0 \arrow[ul] \arrow[dl] & & K \arrow[ul] \arrow[dl] \\
0 & & 0 \arrow[ul] &
\end{tikzcd}$$

The projective cover of $S(6)$ is $P(6)$ and the kernel L of the canonical epimorphism $P(6) \to S(6)$ is the indecomposable module

$$L:\quad \begin{tikzcd}
0 & & K \arrow[dl, "1"'] & \\
& K \arrow[ul] \arrow[dl] & & 0 \arrow[ul] \arrow[dl] \\
0 & & K \arrow[ul, "1"] &
\end{tikzcd}$$

Because the top of L is isomorphic to $S(4)\oplus S(5)$, then the projective cover of L is isomorphic to $P(4)\oplus P(5)$ and therefore the module $S(6)$ has a minimal projective presentation of the form $P(4) \oplus P(5) \xrightarrow{p_1} P(6) \xrightarrow{p_2} S(6) \longrightarrow 0$ (see (I.5.8)). By (2.4)(a), applying the functor ν to the exact sequence, we get an exact sequence $0 \longrightarrow \tau S(6) \longrightarrow I(4) \oplus I(5) \xrightarrow{\nu\, p_1} I(6) \longrightarrow 0$, because $\nu\, p_1 \neq 0$ and $I(6) = S(6)$ is simple. Hence we get

$$\tau S(6): \quad \begin{tikzcd}[column sep=small, row sep=small] 0 & & K \arrow[dl] & \\ & 0 \arrow[ul] \arrow[dl] & & K \arrow[ul, "1"'] \arrow[dl, "1"] \\ 0 & & K \arrow[ul] & \end{tikzcd}$$

and obviously $\tau S(6) \ncong S(6)$.

This proposition yields at once an easy and useful criterion for a module to have projective, or injective, dimension at most one.

2.7. Lemma. *Let M be a module in* $\operatorname{mod} A$.
(a) $\operatorname{pd}_A M \leq 1$ *if and only if* $\operatorname{Hom}_A(DA, \tau M) = 0$.
(b) $\operatorname{id}_A M \leq 1$ *if and only if* $\operatorname{Hom}_A(\tau^{-1} M, A) = 0$.

Proof. We only prove (a); the proof of (b) is similar. Applying the left exact functor $\nu^{-1} = \operatorname{Hom}_A(DA, -)$ to the exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \longrightarrow 0$$

given in (2.4) we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \nu^{-1}\tau M \arrow[r] & \nu^{-1}\nu P_1 \arrow[r] \arrow[d, "\cong"] & \nu^{-1}\nu P_0 \arrow[d, "\cong"] & & \\ 0 \arrow[r] & \operatorname{Ker} p_1 \arrow[r] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & M \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Thus $\operatorname{Hom}_A(DA, \tau M) = \nu^{-1}\tau M \cong \operatorname{Ker} p_1$ vanishes if and only if $\operatorname{pd} M \leq 1$. $\square$

The previous results yield formulas for the dimension vector of the Auslander–Reiten translate in terms of the Coxeter transformation $\mathbf{\Phi}_A : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ of any algebra A of finite global dimension (see (III.3.14)).

2.8. Lemma. (a) *Let M be an indecomposable nonprojective module in* $\operatorname{mod} A$ *and* $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$ *be a minimal projective presentation of M. Then*

$$\mathbf{dim}\, \tau M = \mathbf{\Phi}_A(\mathbf{dim}\, M) - \mathbf{\Phi}_A(\mathbf{dim}\, \operatorname{Ker} p_1) + \mathbf{dim}\, \nu M.$$

(b) *Let N be an indecomposable noninjective module in* $\operatorname{mod} A$ *and let* $0 \longrightarrow N \xrightarrow{i_0} E_0 \xrightarrow{i_1} E_1$ *be a minimal injective presentation of N. Then*

$$\mathbf{dim}\, \tau^{-1} N = \mathbf{\Phi}_A^{-1}(\mathbf{dim}\, N) - \mathbf{\Phi}_A^{-1}(\mathbf{dim}\, \operatorname{Coker} i_1) + \mathbf{dim}\, \nu^{-1} N.$$

Proof. We only prove (a); the proof of (b) is similar. The exact sequence $0 \longrightarrow \operatorname{Ker} p_1 \longrightarrow P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$ yields

$$\mathbf{dim}\, M - \mathbf{dim}\, \operatorname{Ker} p_1 = -\mathbf{dim}\, P_1 + \mathbf{dim}\, P_0.$$

Applying the Coxeter transformation $\mathbf{\Phi}_A$ and using (III.3.16)(a), we get

$$\mathbf{\Phi}_A(\mathbf{dim}\, M) - \mathbf{\Phi}_A(\mathbf{dim}\, \mathrm{Ker}\, p_1) = \mathbf{dim}\, \nu\, P_1 - \mathbf{dim}\, \nu\, P_0.$$

Then the injective presentation $0 \longrightarrow \tau\, M \longrightarrow \nu\, P_1 \longrightarrow \nu\, P_0 \longrightarrow \nu\, M \longrightarrow 0$ of $\tau\, M$ yields $\mathbf{dim}\, \tau\, M = \mathbf{dim}\, \nu\, P_1 - \mathbf{dim}\, \nu\, P_0 + \mathbf{dim}\, \nu\, M = \mathbf{\Phi}_A(\mathbf{dim}\, M) - \mathbf{\Phi}_A(\mathbf{dim}\, \mathrm{Ker}\, p_1) + \mathbf{dim}\, \nu\, M$. □

2.9. Corollary. (a) *If M is an indecomposable module in* $\mathrm{mod}\, A$ *such that* $\mathrm{pd}_A M \leq 1$ *and* $\mathrm{Hom}_A(M, A) = 0$*, then* $\mathbf{dim}\, \tau\, M = \mathbf{\Phi}_A(\mathbf{dim}\, M)$.

(b) *If N is an indecomposable module in* $\mathrm{mod}\, A$ *such that* $\mathrm{id}_A N \leq 1$ *and* $\mathrm{Hom}_A(DA, N) = 0$*, then* $\mathbf{dim}\, \tau^{-1}\, N = \mathbf{\Phi}_A^{-1}(\mathbf{dim}\, N)$.

Proof. We only prove (a); the proof of (b) is similar. By our assumption, M is not projective and $\nu\, M = D\mathrm{Hom}_A(M, A) = 0$. Then (a) is a consequence of (2.8), because $\mathrm{pd}_A M \leq 1$ implies $\mathrm{Ker}\, p_1 = 0$, in the notation of (2.8). □

The following proposition records some of the most elementary properties of Auslander–Reiten translations.

2.10. Proposition. *Let M and N be indecomposable modules in* $\mathrm{mod}\, A$.

(a) *The module τM is zero if and only if M is projective.*

(a′) *The module $\tau^{-1} N$ is zero if and only if N is injective.*

(b) *If M is a nonprojective module, then τM is indecomposable noninjective and $\tau^{-1}\tau M \cong M$.*

(b′) *If N is a noninjective module, then $\tau^{-1} N$ is indecomposable nonprojective and $\tau\tau^{-1} N \cong N$.*

(c) *If M and N are nonprojective, then $M \cong N$ if and only if there is an isomorphism $\tau M \cong \tau N$.*

(c′) *If M and N are noninjective, then $M \cong N$ if and only if there is an isomorphism $\tau^{-1} M \cong \tau^{-1} N$.*

Proof. Because the translations τ and τ^{-1} are compositions of the transposition Tr and the duality D, the proposition follows directly from (2.1), (I.5.13), and the definitions. A detailed proof is left as an exercise (see (IV.7.25)). □

2.11. Corollary. *The Auslander–Reiten translations τ and τ^{-1} induce mutually inverse equivalences* $\underline{\mathrm{mod}}\, A \underset{\tau^{-1}}{\overset{\tau}{\rightleftarrows}} \overline{\mathrm{mod}}\, A$.

Proof. This follows directly from (2.2) and (2.10). □

For an A-module X, we consider the functorial homomorphism

$$\varphi^X : (-) \otimes_A X^t \longrightarrow \mathrm{Hom}_A(X, -)$$

defined on a module Y_A by

$$\begin{array}{rcccl}\varphi_Y^X & : & Y \otimes_A X^t & \longrightarrow & \operatorname{Hom}_A(X,Y) \\ & & y \otimes f & \mapsto & (x \mapsto yf(x)),\end{array}$$

where $x \in X$, $y \in Y$ and $f \in X^t$. It is easily seen that if X is projective, then φ^X is a functorial isomorphism and that if Y is projective, then φ_Y^X is an isomorphism. We prove that the cokernel of φ_Y^X coincides with $\underline{\operatorname{Hom}}_A(X,Y)$.

2.12. Lemma. *For any A-modules X and Y, there is an exact sequence*

$$Y \otimes_A X^t \xrightarrow{\varphi_Y^X} \operatorname{Hom}_A(X,Y) \longrightarrow \underline{\operatorname{Hom}}_A(X,Y) \longrightarrow 0$$

with all homomorphisms functorial in both variables.

Proof. For an A-module Y, let $f : P \to Y$ be an epimorphism with P projective. We claim that for any A-module X, there is an exact sequence

$$\operatorname{Hom}_A(X,P) \xrightarrow{\operatorname{Hom}_A(X,f)} \operatorname{Hom}_A(X,Y) \longrightarrow \underline{\operatorname{Hom}}_A(X,Y) \longrightarrow 0.$$

Indeed, it is sufficient to show that $\operatorname{Im}\operatorname{Hom}_A(X,f) = \mathcal{P}(X,Y)$. Because, clearly, $\operatorname{Im}\operatorname{Hom}_A(X,f) \subseteq \mathcal{P}(X,Y)$, we take $g \in \mathcal{P}(X,Y)$. By definition, there exist a projective module P'_A and homomorphisms $g_2 : X \to P'$, $g_1 : P' \to Y$ such that $g = g_1 g_2$. Because $f : P \to Y$ is an epimorphism and P' is projective, there exists $h : P' \to P$ such that $g_1 = fh$. Then $g = g_1 g_2 = fhg_2 = \operatorname{Hom}_A(X,f)(hg_2) \in \operatorname{Im}\operatorname{Hom}_A(X,f)$ and we have proved our claim.

Because $\varphi_P^X : P \otimes_A X^t \to \operatorname{Hom}_A(X,P)$ is an isomorphism and φ^X is functorial, we have a commutative diagram

$$\begin{array}{ccccccc} P \otimes_A X^t & \xrightarrow{f \otimes X^t} & Y \otimes_A X^t & \longrightarrow & 0 & & \\ {\scriptstyle \varphi_P^X}\big\downarrow \cong & & {\scriptstyle \varphi_Y^X}\big\downarrow & & & & \\ \operatorname{Hom}_A(X,P) & \xrightarrow{\operatorname{Hom}_A(X,f)} & \operatorname{Hom}_A(X,Y) & \longrightarrow & \underline{\operatorname{Hom}}_A(X,Y) & \longrightarrow & 0 \end{array}$$

with exact rows. Consequently

$$\begin{aligned}\operatorname{Im}\varphi_Y^X &= \varphi_Y^X(f \otimes X^t)(P \otimes X^t) \\ &= \operatorname{Hom}_A(X,f)\varphi_P^X(P \otimes X^t) \\ &\cong \operatorname{Im}\operatorname{Hom}_A(X,f) = \mathcal{P}(X,Y)\end{aligned}$$

and therefore $\operatorname{Coker}\varphi_Y^X \cong \underline{\operatorname{Hom}}_A(X,Y)$. $\square$

2.13. Theorem (the Auslander–Reiten formulas). *Let A be a K-algebra and M, N be two A-modules in* $\operatorname{mod} A$. *Then there exist isomorphisms*

$$\mathrm{Ext}_A^1(M,N) \;\cong\; D\underline{\mathrm{Hom}}_A(\tau^{-1}N,M) \;\cong\; D\overline{\mathrm{Hom}}_A(N,\tau M)$$

that are functorial in both variables.

Proof. We only prove the first isomorphism; the proof of the second is similar. Clearly, it suffices to prove our claim for modules N having no injective direct summand. In view of (2.10), we can suppose that $N = \tau L$, where $L = \tau^{-1}N$. Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \longrightarrow 0$ be a minimal projective presentation of L. Applying the functor $\nu = D(-)^t$, we obtain the exact sequence (see (2.4)(a))

$$0 \longrightarrow \tau L \longrightarrow DP_1^t \xrightarrow{Dp_1^t} DP_0^t \xrightarrow{Dp_0^t} DL^t \longrightarrow 0,$$

where both DP_1^t and DP_0^t are injective. The functor $\mathrm{Hom}_A(M,-)$ yields the complex

$$0\to\mathrm{Hom}_A(M,\tau L)\to\mathrm{Hom}_A(M,DP_1^t)\xrightarrow{\overline{p}_1}\mathrm{Hom}_A(M,DP_0^t)\xrightarrow{\overline{p}_0}\mathrm{Hom}_A(M,DL^t),$$

where, for brevity, we write $\overline{p}_1$ for $\mathrm{Hom}_A(M,Dp_1^t)$ and $\overline{p}_0$ for $\mathrm{Hom}_A(M,Dp_0^t)$. Thus we have

$$\mathrm{Ext}_A^1(M,N) = \mathrm{Ext}_A^1(M,\tau L) = \mathrm{Ker}\,\overline{p}_0/\mathrm{Im}\,\overline{p}_1.$$

On the other hand, applying the right exact functor $D\mathrm{Hom}_A(-,M)$ to the minimal projective presentation of L yields an exact sequence

$$D\mathrm{Hom}_A(P_1,M)\xrightarrow{\widetilde{p}_1}D\mathrm{Hom}_A(P_0,M)\xrightarrow{\widetilde{p}_0}D\mathrm{Hom}_A(L,M)\longrightarrow 0,$$

where, for brevity, we write $\widetilde{p}_1$ for $D\mathrm{Hom}_A(p_1,M)$ and $\widetilde{p}_0$ for $D\mathrm{Hom}_A(p_0,M)$. Now associated to an A-module X there exists a functorial morphism $\varphi^X : (-)\otimes_A X^t \longrightarrow \mathrm{Hom}_A(X,-)$ introduced earlier. The composition of the dual homomorphism $D\varphi^X : D\mathrm{Hom}_A(X,-)\longrightarrow D((-)\otimes_A X^t)$ with the adjunction isomorphism $\eta^X : D((-)\otimes_A X^t)\xrightarrow{\simeq}\mathrm{Hom}_A(-,DX^t)$ yields a functorial morphism

$$\omega^X = \eta^X D\varphi^X : D\mathrm{Hom}_A(X,-)\longrightarrow\mathrm{Hom}_A(-,DX^t),$$

which is an isomorphism whenever X is projective. We thus have a commutative diagram with exact lower row

$$\begin{tikzcd}
\mathrm{Hom}_A(M,DP_1^t) \arrow[r, "\overline{p}_1"] & \mathrm{Hom}_A(M,DP_0^t) \arrow[r, "\overline{p}_0"] & \mathrm{Hom}_A(M,DL^t) & \\
D\mathrm{Hom}_A(P_1,M) \arrow[r, "\widetilde{p}_1"] \arrow[u, "\omega_M^{P_1}", "\cong"'] & D\mathrm{Hom}_A(P_0,M) \arrow[r, "\widetilde{p}_0"] \arrow[u, "\omega_M^{P_0}", "\cong"'] & D\mathrm{Hom}_A(L,M) \arrow[r] \arrow[u, "\omega_M^{L}"] & 0
\end{tikzcd}$$

The homomorphism $\widetilde{p}_0(\omega_M^{P_0})^{-1}$ of A-modules induces a homomorphism $\psi : \mathrm{Ker}\,\overline{p}_0 \to \mathrm{Ker}\,\omega_M^L$. Because $\widetilde{p}_0$ is an epimorphism and $\omega_M^{P_0}$ an isomorphism,

ψ must be an epimorphism. Because $\operatorname{Ker}\widetilde{p}_0 = \operatorname{Im}\widetilde{p}_1$ and the maps $\omega_M^{P_0}$, $\omega_M^{P_1}$ are isomorphisms, we deduce that $\operatorname{Ker}\psi \cong \operatorname{Im}\overline{p}_1$. Consequently, we have

$$\begin{array}{rcccc} \operatorname{Ker}\overline{p}_0/\operatorname{Im}\overline{p}_1 & \cong & \operatorname{Ker}\overline{p}_0/\operatorname{Ker}\psi & \cong & \operatorname{Ker}\omega_M^L \\ & = & \operatorname{Ker} D\varphi_M^L & \cong & D\operatorname{Coker}\varphi_M^L. \end{array}$$

Thus there exist an isomorphism $\operatorname{Ext}_A^1(M,N) \cong D\operatorname{Coker}\varphi_M^L$ and, by (2.12), $\operatorname{Coker}\varphi_M^L \cong \underline{\operatorname{Hom}}_A(L,M) = \underline{\operatorname{Hom}}_A(\tau^{-1}N,M)$. The proof is complete. $\square$

2.14. Corollary. *Let A be a K-algebra and M, N be two modules in* $\operatorname{mod} A$.

(a) *If* $\operatorname{pd} M \leq 1$ *and N is arbitrary, then there exists a K-linear isomorphism*

$$\operatorname{Ext}_A^1(M,N) \cong D\operatorname{Hom}_A(N,\tau M).$$

(b) *If* $\operatorname{id} N \leq 1$ *and M is arbitrary, then there exists a K-linear isomorphism*

$$\operatorname{Ext}_A^1(M,N) \cong D\operatorname{Hom}_A(\tau^{-1}N,M).$$

Proof. The Auslander–Reiten formulas (2.13) give an isomorphism $\operatorname{Ext}_A^1(M,N) \cong D\overline{\operatorname{Hom}}_A(N,\tau M)$. Now $\operatorname{pd} M \leq 1$ gives $\operatorname{Hom}_A(DA,\tau M) = 0$ (by (2.7)). Hence $\mathcal{I}(N,\tau M) = 0$, because every injective module in $\operatorname{mod} A$ is a direct summand of $(DA)^s$, for some $s \geq 1$. Consequently, $\overline{\operatorname{Hom}}_A(N,\tau M) = \operatorname{Hom}_A(N,\tau M)$ and (a) follows. The proof of (b) is similar to that of (a). $\square$

2.15. Corollary. *Let A be a K-algebra and M, N be two modules in* $\operatorname{mod} A$.

(a) *If* $\operatorname{pd} M \leq 1$ *and* $\operatorname{id} N \leq 1$, *then there exists a K-linear isomorphism*

$$\operatorname{Hom}_A(N,\tau M) \cong \operatorname{Hom}_A(\tau^{-1}N,M).$$

(b) *If* $\operatorname{pd} M \leq 1$, $\operatorname{id}\tau N \leq 1$ *and N is indecomposable nonprojective, then there is a K-linear isomorphism*

$$\operatorname{Hom}_A(\tau N,\tau M) \cong \operatorname{Hom}_A(N,M).$$

(c) *If* $\operatorname{pd}\tau^{-1}M \leq 1$, $\operatorname{id} N \leq 1$ *and M is indecomposable noninjective, then there is a K-linear isomorphism*

$$\operatorname{Hom}_A(\tau^{-1}N,\tau^{-1}M) \cong \operatorname{Hom}_A(N,M).$$

Proof. The statement (a) is an immediate consequence of (2.14). Finally, (b) and (c) follow from (a) and (2.10). $\square$

IV.3. The existence of almost split sequences

We are now able, using the results of Section 2, to prove the main existence theorem for almost split sequences, due to Auslander and Reiten. In this section, as in the previous one, we let A denote a fixed finite dimensional K-algebra, and we denote by rad_A the radical of the category $\operatorname{mod} A$.

3.1. Theorem. (a) *For any indecomposable nonprojective A-module M_A, there exists an almost split sequence* $0 \to \tau M \to E \to M \to 0$ *in* $\operatorname{mod} A$.

(b) *For any indecomposable noninjective A-module N_A, there exists an almost split sequence* $0 \to N \to F \to \tau^{-1}N \to 0$ *in* $\operatorname{mod} A$.

Proof. We only prove (a); the proof of (b) is similar. Let M be an indecomposable nonprojective A-module. By the Auslander–Reiten formulas (2.13), there exists an isomorphism

$$D\underline{\operatorname{Hom}}_A(L, M) \cong \operatorname{Ext}^1_A(M, \tau L)$$

for any indecomposable module L, which is functorial in both variables. Let $S(L, M) = \operatorname{Hom}_A(L, M)/\operatorname{rad}_A(L, M)$. Because $\mathcal{P}(L, M) \subseteq \operatorname{rad}_A(L, M)$, we have a canonical K-linear epimorphism $p_{L,M} : \underline{\operatorname{Hom}}_A(L, M) \to S(L, M)$ and hence a canonical monomorphism $Dp_{L,M} : DS(L, M) \to D\underline{\operatorname{Hom}}_A(L, M)$.

Now, M being indecomposable, $\operatorname{End} M$ and hence $\underline{\operatorname{End}}\ M$ are local. Because we have an epimorphism

$$p_{M,M} : \underline{\operatorname{End}}\ M \to S(M, M) = \operatorname{End} M/\operatorname{rad}\operatorname{End} M,$$

$S(M, M)$ is isomorphic to the simple top of $\underline{\operatorname{End}}\, M$ considered as a left or right $\operatorname{End} M$-module, and its image under $Dp_{M,M}$ is the simple socle of the $\operatorname{End} M$-module $D\underline{\operatorname{Hom}}_A(M, M)$. Let ξ' be a nonzero element in $DS(M, M)$ and ξ be its image in $\operatorname{Ext}^1_A(M, \tau M) \cong D\underline{\operatorname{Hom}}_A(M, M)$. We claim that if ξ is represented by the short exact sequence

$$0 \longrightarrow \tau M \xrightarrow{\ f\ } E \xrightarrow{\ g\ } M \longrightarrow 0,$$

then this sequence is almost split.

First, this sequence is not split, and by (2.10), the module τM is indecomposable. It suffices thus, by (1.13), to show that g is right almost split. Because ξ is a nonzero element in $\operatorname{Ext}^1_A(M, \tau M)$, g is not a retraction. Let $v : V \to M$ be a homomorphism that is not a retraction. We may assume that V is indecomposable. Then v is not an isomorphism. It follows from the functoriality that we have a commutative diagram

$$\begin{array}{ccccc} DS(M,M) & \xrightarrow{Dp_{M,M}} & D\underline{\mathrm{Hom}}_A(M,M) & \xrightarrow{\cong} & \mathrm{Ext}^1_A(M,\tau M) \\ \Big\downarrow{\scriptstyle DS(M,v)} & & \Big\downarrow{\scriptstyle D\underline{\mathrm{Hom}}_A(M,v)} & & \Big\downarrow{\scriptstyle \mathrm{Ext}^1_A(v,\tau M)} \\ DS(M,V) & \xrightarrow{Dp_{M,V}} & D\underline{\mathrm{Hom}}_A(M,V) & \xrightarrow{\cong} & \mathrm{Ext}^1_A(V,\tau M) \end{array}$$

where the vertical maps are induced by v. By hypothesis, $v \in \mathrm{rad}_A(V,M)$ and therefore $DS(M,v)(\xi') = 0$. Consequently, the image $\mathrm{Ext}^1_A(v,\tau M)(\xi)$ of ξ in $\mathrm{Ext}^1_A(V,\tau M)$ is zero, that is, there exists a commutative diagram

$$\begin{array}{lccccccccc} \mathrm{Ext}^1_A(v,\tau M)(\xi): & 0 & \longrightarrow & \tau M & \xrightarrow{f'} & E' & \xrightarrow{g'} & V & \longrightarrow & 0 \\ & & & \Big\downarrow{\scriptstyle 1_{\tau M}} & & \Big\downarrow{\scriptstyle w} & & \Big\downarrow{\scriptstyle v} & & \\ \xi: & 0 & \longrightarrow & \tau M & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

with exact rows, where the upper sequence splits. Let thus $g'' : V \to E'$ be such that $g'g'' = 1_V$. Then $v' = wg''$ satisfies $gv' = gwg'' = vg'g'' = v$. This completes the proof that g is right almost split and hence the proof of the theorem. $\square$

The next corollary provides examples of almost split sequences.

3.2. Corollary. (a) *If* $0 \to \tau M \to E \to M \to 0$ *is an almost split sequence in* $\mathrm{mod}\, A$ *then it represents a nonzero element* ξ *of the simple socle of the* $\mathrm{End}\, M$-$\mathrm{End}\, M$*-bimodule* $\mathrm{Ext}^1_A(M,\tau M) \cong D\underline{\mathrm{Hom}}_A(M,M)$.

(b) *Let* M *be an indecomposable nonprojective module in* $\mathrm{mod}\, A$*. Then* $\underline{\mathrm{End}}\, M$ *is a skew field if and only if* $\overline{\mathrm{End}}\, \tau M$ *is a skew field, and in this case, any nonsplit exact sequence* $0 \to \tau M \to E \to M \to 0$ *is almost split and* $\underline{\mathrm{End}}\, M \cong K$.

(c) *Let* N *be an indecomposable noninjective module in* $\mathrm{mod}\, A$*. Then* $\overline{\mathrm{End}}\, N$ *is a skew field if and only if* $\underline{\mathrm{End}}\, \tau^{-1}N$ *is a skew field, and in this case, any nonsplit exact sequence* $0 \to N \to F \to \tau^{-1}N \to 0$ *is almost split and* $\overline{\mathrm{End}}\, N \cong K$.

Proof. The statement (a) follows from the proof of (3.1). We only prove (b); the proof of (c) is similar. The first statement of (a) follows from (2.11). Assume that $\underline{\mathrm{End}}\, M$ is a skew field. Because $\dim_K \underline{\mathrm{End}}\, M$ is finite and the field K is algebraically closed, $\underline{\mathrm{End}}\, M \cong K$ and $\mathrm{Ext}^1_A(M,\tau M)$ is a one-dimensional K-vector space (because it has simple socle, by (a)). Hence, by the proof of (3.1), any nonsplit extension represents an element in the socle of $\mathrm{Ext}^1_A(M,\tau M)$ and thus is almost split. $\square$

3.3. Example. Let A be the K-algebra given by the Kronecker quiver $1 \circ \underset{\beta}{\overset{\alpha}{\longleftarrow\!\!\longleftarrow}} \circ 2$ and M be the representation $K \underset{0}{\overset{1}{\longleftarrow\!\!\longleftarrow}} K$. As we have seen before, $\mathrm{End}\, M \cong K$ and $\tau M \cong M$. It follows from (3.2) that any nonsplit extension $0 \to M \to E \to M \to 0$ is an almost split sequence. Let E be the representation

$$\begin{tikzcd} K^2 & K^2 \arrow[l, shift right, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"'] \arrow[l, shift left, "{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]}"] \end{tikzcd}$$

The subrepresentation E' of E given by $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] K \overset{1}{\underset{0}{\leftleftarrows}} \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] K$ is clearly isomorphic to M, and moreover $E/E' \cong M$. We thus have a short exact sequence as required. To prove that it is almost split, we show it is not split, and it suffices to show that E is indecomposable. To do this, we observe that any endomorphism f of E is given by a pair of matrices $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right)$ such that

$$\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right), \text{ and } \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right).$$

These two conditions yield $a = a' = d = d'$, $b = b' = 0$, and $c = c'$. Thus $f = a \cdot 1_E + g$, where $a \in K$ and $g \in \operatorname{End} E$ is nilpotent. Let now $I = \{f \in \operatorname{End} E \mid a = 0\}$. Then I is a nilpotent ideal of $\operatorname{End} E$. Because moreover $(\operatorname{End} E)/I \cong K$, I is a maximal ideal of $\operatorname{End} E$. Therefore $I = \operatorname{rad} \operatorname{End} E$ and $\operatorname{End} E$ is local. Thus, E is indecomposable.

3.4. Example. Let A be the K-algebra given by the quiver

$$\begin{tikzcd} \overset{1}{\circ} & & \overset{4}{\circ} \arrow[dl, "\beta"'] & \\ & \underset{3}{\circ} \arrow[ul, "\lambda"'] \arrow[dl, "\mu"] & & \circ\ 6 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ \underset{2}{\circ} & & \underset{5}{\circ} \arrow[ul, "\delta"] & \end{tikzcd}$$

bound by $\alpha\beta = \gamma\delta$, $\delta\mu = 0$, $\beta\lambda = 0$. It was shown in Example 2.6 that there is an exact sequence $0 \longrightarrow \tau S(6) \longrightarrow I(4) \oplus I(5) \xrightarrow{\nu p_1} I(6) \longrightarrow 0$. It is clear that $\operatorname{End} \tau S(6) \cong K$, hence $\overline{\operatorname{End}}\, \tau S(6) \cong K$. In view of the unique decomposition theorem (I.4.10), this sequence does not split. It then follows from (3.2)(b) that the sequence is almost split.

It also follows from (3.1) that there exists a right (or left) minimal almost split morphism ending (or starting, respectively) at any indecomposable nonprojective (or noninjective, respectively) module. We now want to show the existence of such a homomorphism ending (or starting) at an indecomposable projective (or injective, respectively) module.

3.5. Proposition. (a) *Let P be an indecomposable projective module in* $\operatorname{mod} A$. *An A-module homomorphism $g : M \to P$ is right minimal almost split if and only if g is a monomorphism with image equal to* $\operatorname{rad} P$.

(b) *Let I be an indecomposable injective module. An A-module homomorphism $f : I \to M$ is left minimal almost split if and only if f is an* *epimorphism with kernel equal to* $\operatorname{soc} I$*.*

Proof. We only prove (a); the proof of (b) is similar. It suffices, by (1.2), to show that the inclusion homomorphism $g : \operatorname{rad} P \to P$ is right minimal almost split. Because g is a monomorphism, g is right minimal. Clearly, g is not a retraction. Let thus $v : V \to P$ be a homomorphism that is not a retraction. Because P is projective, by (I.4.5), the module $\operatorname{rad} P$ is the unique maximal submodule of P. Because v is not an epimorphism, $v(V) \subseteq \operatorname{rad} P$, that is, v factors through g. □

3.6. Corollary. *Let X be an indecomposable module in* $\operatorname{mod} A$*.*

(a) *There exists a right minimal almost split morphism $g : M \to X$. Moreover $M = 0$ if and only if X is simple projective.*

(b) *There exists a left minimal almost split morphism $f : X \to M$. Moreover, $M = 0$ if and only if X is simple injective.*

Proof. The proof follows directly from (3.1) and (3.5). □

3.7. Example. Let A be the K-algebra given by the quiver $1 \circ \longleftarrow \circ 2$. Consider the short exact sequence $0 \longrightarrow S(1) \xrightarrow{f} P(2) \xrightarrow{g} S(2) \longrightarrow 0$ in $\operatorname{mod} A$, where f is the embedding of $S(1)$ as the radical of $P(2)$ and g is the canonical homomorphism of $P(2)$ onto its top. Because $P(2) = I(1)$, it follows from (3.5) that f is right minimal almost split and g is left minimal almost split. On the other hand, it will be shown in (3.11) that, because the middle term is projective-injective, the sequence is almost split (thus, f is also left minimal almost split and g is right minimal almost split).

3.8. Proposition. (a) *Let M be an indecomposable nonprojective module in* $\operatorname{mod} A$*. There exists an irreducible morphism $f : X \to M$ if and only if there exists an irreducible morphism $f' : \tau M \to X$.*

(b) *Let N be an indecomposable noninjective module in* $\operatorname{mod} A$*. There exists an irreducible morphism $g : N \to Y$ if and only if there exists an irreducible morphism $g' : Y \to \tau^{-1} N$.*

Proof. We only prove (a); the proof of (b) is similar. Assume that $f : X \to M$ is irreducible. By (1.10), there exists $h : Y \to M$ such that $[f\ h] : X \oplus Y \to M$ is right minimal almost split. But then $[f\ h]$ is an epimorphism, because M is not projective. Therefore, by (1.8), $L = \operatorname{Ker}[f\ h]$ is indecomposable, and thus, by (1.13), the short exact sequence

$$0 \longrightarrow L \xrightarrow{\left[\begin{smallmatrix} f' \\ h' \end{smallmatrix}\right]} X \oplus Y \xrightarrow{[f\ h]} M \longrightarrow 0$$

is almost split. Consequently, there exists an isomorphism $g : \tau M \xrightarrow{\simeq} L$ and the homomorphism $f'g : \tau M \to X$ is irreducible. The proof of the converse is similar. $\square$

3.9. Corollary. (a) *Let S be a simple projective noninjective module in* $\operatorname{mod} A$. *If $f : S \to M$ is irreducible, then M is projective.*

(b) *Let S be a simple injective nonprojective module in* $\operatorname{mod} A$. *If $g : M \to S$ is irreducible, then M is injective.*

Proof. We only prove (a); the proof of (b) is similar. We may clearly assume M to be indecomposable. If M is not projective, there exists, by (3.8), an irreducible morphism $\tau M \to S$, and this contradicts (3.6). $\square$

This corollary allows us to construct examples of almost split sequences. Indeed, let S be simple projective noninjective and $f : S \to P$ be left minimal almost split. By (3.9), P is projective and by (3.5), for each indecomposable summand P' of P, the corresponding component $f' : S \to P'$ of f is a monomorphism with image a summand of $\operatorname{rad} P'$. It follows that, if P is the direct sum of all such indecomposable projectives P', then the sequence $0 \longrightarrow S \xrightarrow{f} P \longrightarrow \operatorname{Coker} f \longrightarrow 0$ is almost split.

3.10. Example. Assume that A is a K-algebra given by the quiver $\overset{1}{\circ} \longleftarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \longleftarrow \overset{4}{\circ}$. Then $S(3)$ is a simple projective noninjective summand of $\operatorname{rad} P(2)$ and is equal to $\operatorname{rad} P(4)$. Thus we have an almost split sequence

$$0 \longrightarrow S(3) \longrightarrow P(2) \oplus P(4) \longrightarrow (P(2) \oplus P(4))/S(3) \longrightarrow 0.$$

The preceding remark is essentially used in the next section. We conclude this section with a further example of an almost split sequence.

3.11. Proposition. *Let P be a nonsimple indecomposable projective-injective module, $S = \operatorname{soc} P$, and $R = \operatorname{rad} R$. Then the sequence*

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} q \\ i \end{smallmatrix}\right]} R/S \oplus P \xrightarrow{[-j\, p]} P/S \longrightarrow 0$$

is almost split, where i, j are the inclusions and p, q the projections.

Proof. Because R has simple socle S, it is indecomposable. Hence $i : R \to P$ is, up to isomorphism, the unique nontrivial irreducible morphism ending in P (by (3.5)). Dually, the module P/S is indecomposable and $p : P \to P/S$ is, up to isomorphism, the unique nontrivial irreducible morphism starting with P. It follows from (3.8) that $R \cong \tau(P/S)$. Because the given exact sequence is not split, it remains to show (by (1.13)) that the monomorphism $\left[\begin{smallmatrix} q \\ i \end{smallmatrix}\right] : R \to R/S \oplus P$ is left almost split. Assume that $u : R \to U$ is not a section. If u is a monomorphism, then, because P is injective, u factors through P and we are done. If not, there exists a factorisation $u = u'u''$, with $u'' : R \to U'$ a proper epimorphism and $u' : U' \to U$ a monomorphism. Because $\operatorname{Ker} u \neq 0$, the simple socle S of R is contained in $\operatorname{Ker} u = \operatorname{Ker} u''$. Thus the epimorphism u'' factors through R/S, that is, there exists $u_1 : R/S \to U'$ such that $u'' = u_1 q$. Hence $\overline{u} = [u'u_1, 0]$ satisfies $\overline{u}\left[\begin{smallmatrix} q \\ i \end{smallmatrix}\right] = [u'u_1,\ 0]\left[\begin{smallmatrix} q \\ i \end{smallmatrix}\right] = u'u_1 q = u'u'' = u$. $\square$

3.12. Example. Let A be the K-algebra given by the quiver

$$\begin{tikzcd}
 & & \overset{2}{\circ} \arrow[dll, "\beta"'] & & \\
1\circ & & \overset{3}{\circ} \arrow[ll, "\delta"'] & & \circ 6 \arrow[ull, "\alpha"'] \arrow[ll, "\gamma"'] \arrow[dl, "\lambda"] \\
 & \underset{4}{\circ} \arrow[ul, "\nu"] & & \underset{5}{\circ} \arrow[ll, "\mu"] &
\end{tikzcd}$$

bound by the commutativity relations: $\alpha\beta = \gamma\delta$ and $\gamma\delta = \lambda\mu\nu$. The A-module $P(6) = I(1)$ is projective-injective and the almost split sequence described in (3.11) with $P = P(6)$ is of the form

$$0 \longrightarrow \operatorname{rad} P(6) \longrightarrow S(2) \oplus S(3) \oplus \frac{P(5)}{S(1)} \oplus P(6) \longrightarrow \frac{P(6)}{S(1)} \longrightarrow 0.$$

IV.4. The Auslander–Reiten quiver of an algebra

Let A be a finite dimensional K-algebra. We may wish to record the information we have on the category $\operatorname{mod} A$ in the form of a quiver. Then it seems clear that points should represent modules and arrows should represent homomorphisms. Because any module in $\operatorname{mod} A$ decomposes as the direct sum of indecomposable modules uniquely determined up to isomorphism, we should let the points represent isomorphism classes of indecomposable modules. Similarly, the homomorphisms that admit no nontrivial factorisation are the irreducible morphisms; thus our arrows should correspond to the irreducible morphisms. But to be more precise, we need some additional considerations on irreducible morphisms.

Let M and N be indecomposable modules in $\operatorname{mod} A$. We have seen in (1.6) that an A-homomorphism $f : M \to N$ is an irreducible morphism if and only if $f \in \operatorname{rad}_A(M,N) \setminus \operatorname{rad}_A^2(M,N)$. Thus the quotient

$$\operatorname{Irr}(M,N) = \operatorname{rad}_A(M,N)/\operatorname{rad}_A^2(M,N) \qquad \textbf{(4.1)}$$

of the K-vector spaces $\operatorname{rad}_A(M,N)$ and $\operatorname{rad}_A^2(M,N)$ measures the number of irreducible morphisms from M to N. It is called the **space of irreducible morphisms**. It is easily seen (see (1.6)) that $\operatorname{Irr}(M,N)$ is in fact an $\operatorname{End} N$–$\operatorname{End} M$-bimodule, annihilated on the left by $\operatorname{rad}_A(N,N) = \operatorname{rad} \operatorname{End} N$ and on the right by $\operatorname{rad}_A(M,M) = \operatorname{rad} \operatorname{End} M$.

We now give the relation between the space of irreducible morphisms and minimal almost split morphisms.

4.2. Proposition. *Let* $M = \bigoplus_{i=1}^{t} M_i^{n_i}$ *be a module in* $\operatorname{mod} A$*, with the* M_i *indecomposable and pairwise nonisomorphic.*

(a) *Let* $f : L \to M$ *be a homomorphism in* $\operatorname{mod} A$ *with* L *indecomposable,* $f = \begin{bmatrix} f_1 \\ \vdots \\ f_t \end{bmatrix}$*, where* $f_i = \begin{bmatrix} f_{i1} \\ \vdots \\ f_{in_i} \end{bmatrix} : L \longrightarrow M_i^{n_i}$*. Then* f *is left minimal almost split if and only if the* f_{ij} *belong to* $\operatorname{rad}_A(L, M_i)$ *and their residual classes* $\overline{f}_{i1}, \ldots, \overline{f}_{in_i}$ *modulo* $\operatorname{rad}_A^2(L, M_i)$ *form a* K*-basis of* $\operatorname{Irr}(L, M_i)$ *for all* i*, and if there is an indecomposable module* M' *in* $\operatorname{mod} A$ *such that* $\operatorname{Irr}(L, M') \neq 0$*, then* $M' \cong M_i$ *for some* i*.*

(b) *Let* $g : M \to N$ *be a homomorphism in* $\operatorname{mod} A$ *with* N *indecomposable,* $g = [\, g_1 \quad \ldots \quad g_t \,]$*, where* $g_i = [\, g_{i1} \quad \ldots \quad g_{in_i} \,] : M_i^{n_i} \longrightarrow N$*. Then* g *is right minimal almost split if and only if the* g_{ij} *belong to* $\operatorname{rad}_A(M_i, N)$ *and their residual classes* $\overline{g}_{i1}, \ldots, \overline{g}_{in_i}$ *modulo* $\operatorname{rad}_A^2(M_i, N)$ *form a* K*-basis of* $\operatorname{Irr}(M_i, N)$ *for all* i*, and, if there is an indecomposable module* M' *in* $\operatorname{mod} A$ *such that* $\operatorname{Irr}(M', N) \neq 0$*, then* $M' \cong M_i$ *for some* i*.*

Proof. We only prove (a); the proof of (b) is similar. Assume thus that f is left minimal almost split. Note that, by the statement (a) of (1.10), if $u : U \to V$ is irreducible and $v : V \to W$ is a retraction, then $vu : U \to W$ is irreducible. Because, again by (1.10), $f : L \to M$ is irreducible, this remark implies that each $f_{ij} : L \to M_i$ is irreducible and thus belongs to $\operatorname{rad}_A(L, M_i)$ (by (1.6)).

On the other hand, (1.10) also shows that if there is an indecomposable module M' such that $\operatorname{Irr}(L, M') \neq 0$, so that there is an irreducible morphism $L \to M'$, then $M' \cong M_i$ for some i. We now want to show that for each i, $\{\overline{f}_{i1}, \ldots \overline{f}_{in_i}\}$ is a K-basis of $\operatorname{Irr}(L, M_i)$.

Let $\overline{h} \in \operatorname{Irr}(L, M_i)$ be the residual class of $h \in \operatorname{rad}_A(L, M_i)$. Because h is not a section, it factors through f, that is, there exists a homomorphism $u = [u_1, \ldots, u_t] : \bigoplus_{k=1}^{t} M_k^{n_k} \to M_i$, with $u_k = [u_{k1}, \ldots, u_{kn_k}] : M_k^{n_k} \to M_i$ such that

$$h = uf = \sum_{k=1}^{t} \sum_{j=1}^{n_k} u_{kj} f_{kj}.$$

Any u_{ij} is an endomorphism of M_i. Because $\operatorname{End} M_i$ is local and the base field K is algebraically closed, we have that $\operatorname{End} M_i / \operatorname{rad} \operatorname{End} M_i \cong K$, so that $u_{ij} = \lambda_j \cdot 1_{M_i} + u'_{ij}$ with $\lambda_j \in K$ and $u'_{ij} \in \operatorname{rad}_A(M_i, M_i) = \operatorname{rad} \operatorname{End} M_i$. On the other hand, if $k \neq i$, then $u_{kj} \in \operatorname{rad}_A(M_k, M_i)$. Because $f_{kj} \in$

$\mathrm{rad}_A(L, M_k)$, we have $u_{kj} f_{kj} \in \mathrm{rad}_A^2(L, M_i)$ for $k \neq i$. Thus

$$\overline{h} = \sum_k \sum_j \overline{u}_{kj} \overline{f}_{kj} = \sum_j \lambda_j \cdot \overline{f}_{ij}.$$

This shows that $\{\overline{f}_{i1}, \ldots, \overline{f}_{in_i}\}$ generates $\mathrm{Irr}(L, M_i)$ as a K-vector space. To prove the linear independence of this set, assume that $\sum_j \lambda_j \overline{f}_{ij} = 0$ in $\mathrm{Irr}(L, M_i)$, where $\lambda_j \in K$. Thus the homomorphism $v = \sum_j \lambda_j f_{ij}$ belongs to $\mathrm{rad}_A^2(L, M_i)$. Assume that $\lambda_j \neq 0$ for some j; then the homomorphism $l = [\lambda_1, \ldots, \lambda_{n_i}] : M_i^{n_i} \to M_i$ is a retraction, and, by the first remark, $v = l f_i$ is irreducible, a contradiction, because $v \in \mathrm{rad}_A^2(L, M_i)$. Consequently, $\lambda_j = 0$. We have completed the proof that $\{\overline{f}_{i1}, \ldots, \overline{f}_{in_i}\}$ is a K-basis of $\mathrm{Irr}(L, M_i)$ and thus of the necessity.

For the sufficiency, assume that for each j, $\{\overline{f}_{j1}, \ldots, \overline{f}_{jn_j}\}$ is a basis of the K-vector space $\mathrm{Irr}(L, M_j)$ and consider a left minimal almost split morphism $f' : L \to U$ (see (3.6)). It follows that $f : L \to M$ is not a section and applying the necessity part to U yields that $U \cong M$. Indeed, let $U = \bigoplus_{k=1}^{s} U_k^{m_k}$ be a decomposition of U, where $U_1, \ldots, U_s$ are pairwise nonisomorphic indecomposable modules. For each k, $\mathrm{Irr}(L, U_k) \neq 0$ yields $U_k \cong M_j$ for some j and $m_k = \dim_K \mathrm{Irr}(L, U_k) = \dim_K \mathrm{Irr}(L, M_j) = n_j$. Analogously, for each j, $\mathrm{Irr}(L, M_j) \neq 0$ yields $M_j \cong U_k$ for some k. Hence we deduce that $U = \bigoplus_{k=1}^{s} U_k^{m_k} \cong \bigoplus_{j=1}^{t} M_j^{n_j} = M$.

Without loss of generality we may assume that $U = M$ and $f' : L \to M$ is left minimal almost split. Applying the necessity part to f' yields that $f' = [f'_{js}] : L \to \bigoplus_{j=1}^{t} M_j^{n_j}$ and, for each j, the set $\{\overline{f}'_{j1}, \ldots, \overline{f}'_{jn_j}\}$ is a basis of the K-vector space $\mathrm{Irr}(L, M_j)$. Because f is not a section, there exists $h : M \to M$ such that $f = hf'$. Hence we conclude that h is an isomorphism. Consequently, f is a left minimal almost split morphism. $\square$

4.3. Remark. Let $P(a) = e_a A$ be an indecomposable projective A-module and $I(a) = D(Ae_a)$ be an indecomposable injective A-module.

(a) The embedding $\mathrm{rad}\, P(a) \hookrightarrow P(a)$ is an irreducible morphism and is right minimal almost split. If $X_1, \ldots X_t$ are indecomposable and pairwise nonisomorphic A-modules such that $\mathrm{rad}\, P(a) \cong X_1^{n_1} \oplus \cdots \oplus X_t^{n_t}$, then $n_j = \dim_K \mathrm{Irr}(X_j, P(a))$ and every indecomposable A-module X with $\mathrm{Irr}(X, P(a)) \neq 0$ is isomorphic to X_j for some j.

(b) The natural epimorphism $I(a) \to I(a)/\mathrm{soc}\, I(a)$ is an irreducible morphism and is left minimal almost split. If $Y_1, \ldots Y_s$ are indecomposable and pairwise nonisomorphic such that $I(a)/\mathrm{soc}\, I(a) \cong Y_1^{m_1} \oplus \cdots \oplus Y_t^{m_t}$, then $m_j = \dim_K \mathrm{Irr}(I(a), Y_j)$ and every indecomposable A-module Y with $\mathrm{Irr}(I(a), Y) \neq 0$ is isomorphic to Y_j for some j.

The first statement of (a) follows from (3.5)(a). The remaining part of (a) is a consequence of (4.2) and the unique decomposition theorem (I.4.10).

The first statement of (b) follows from (3.5)(b). The remaining part of (b) follows easily by applying the duality $D : \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod} A$.

We collect some of the previous results in the following useful corollary.

4.4. Corollary. *Let* $0 \longrightarrow L \xrightarrow{f} \bigoplus_{i=1}^{t} M_i^{n_i} \xrightarrow{g} N \longrightarrow 0$ *be a short exact sequence in* $\operatorname{mod} A$ *with* L, N *indecomposable and the* M_i *indecomposable and pairwise nonisomorphic. Write* $f = \begin{bmatrix} f_1 \\ \vdots \\ f_t \end{bmatrix}$ *and* $g = [g_1 \dots g_t]$, *where* $f_i = \begin{bmatrix} f_{i1} \\ \vdots \\ f_{in_i} \end{bmatrix} : L \longrightarrow M_i^{n_i}$ *and* $g = [g_{i1} \dots g_{in_i}] : M_i^{n_i} \longrightarrow N$. *The following conditions are equivalent:*

(a) *The given sequence is almost split.*

(b) *For each* i, *the homomorphisms* f_{ij} *belong to* $\operatorname{rad}_A(L, M_i)$, *their residual classes* $\overline{f}_{ij}$ *modulo* $\operatorname{rad}^2_A(L, M_i)$ *form a* K*-basis of* $\operatorname{Irr}(L, M_i)$, *and if there exists an indecomposable module* M' *with* $\operatorname{Irr}(L, M') \neq 0$, *then* $M' \cong M_i$ *for some* i.

(c) *For each* i, *the homomorphisms* g_{ij} *belong to* $\operatorname{rad}_A(M_i, N)$, *their residual classes* $\overline{g}_{ij}$ *modulo* $\operatorname{rad}^2_A(M_i, N)$ *form a* K*-basis of* $\operatorname{Irr}(M_i, N)$, *and if there exists an indecomposable module* M' *with* $\operatorname{Irr}(M', N) \neq 0$, *then* $M' \cong M_i$ *for some* i.

Further, if these equivalent conditions hold, then for each i,

$$\dim_K \operatorname{Irr}(L, M_i) = \dim_K \operatorname{Irr}(M_i, N).$$

Proof. The equivalence of these conditions follows from (4.2), and the last statement from (b) and (c). $\square$

4.5. Corollary. *Let* X *and* Y *be indecomposable modules in* $\operatorname{mod} A$.

(a) *If* $\tau X \neq 0$ *and* $\tau Y \neq 0$, *then there exists a* K*-linear isomorphism* $\operatorname{Irr}(\tau X, \tau Y) \cong \operatorname{Irr}(X, Y)$.

(b) *If* $\tau^- X \neq 0$ *and* $\tau^- Y \neq 0$, *then there exists a* K*-linear isomorphism* $\operatorname{Irr}(\tau^- X, \tau^- Y) \cong \operatorname{Irr}(X, Y)$.

Proof. We only prove (a); the proof of (b) is dual. Because $\tau X \neq 0$ and $\tau Y \neq 0$, X is not projective, Y is not projective, and there exist almost split sequences $0 \longrightarrow \tau X \longrightarrow U \xrightarrow{u} X \longrightarrow 0$ and $0 \longrightarrow \tau Y \longrightarrow V \xrightarrow{v} Y \longrightarrow 0$ in $\operatorname{mod} A$. First, we prove that $\operatorname{Irr}(X, Y) \neq 0$ implies $\operatorname{Irr}(\tau X, \tau Y) \cong \operatorname{Irr}(X, Y)$. Assume that $\operatorname{Irr}(X, Y) \neq 0$. Because v is a right minimal almost split morphism, according to (4.2)(b), the module X is isomorphic to a direct summand of V, and by (3.8) there is an irreducible morphism $\tau Y \to X$.

Then, by (4.4), there is a K-linear isomorphism $\operatorname{Irr}(\tau Y, X) \cong \operatorname{Irr}(X, Y)$. Because u is a right minimal almost split morphism and $\operatorname{Irr}(\tau Y, X) \neq 0$, then, according to (4.2)(b), the module τY is isomorphic to a direct summand of U and, according to (4.4), there is a K-linear isomorphism $\operatorname{Irr}(\tau Y, X) \cong \operatorname{Irr}(\tau X, \tau Y)$. Consequently, we get a K-linear isomorphism $\operatorname{Irr}(\tau X, \tau Y) \cong \operatorname{Irr}(X, Y)$.

Using these arguments, we also prove that $\operatorname{Irr}(\tau X, \tau Y) \cong \operatorname{Irr}(X, Y)$ if $\operatorname{Irr}(\tau X, \tau Y) \neq 0$. This finishes the proof. $\square$

We are now able to define the quiver of the category $\operatorname{mod} A$.

4.6. Definition. Let A be a basic and connected finite dimensional K-algebra The **quiver** $\Gamma(\operatorname{mod} A)$ of $\operatorname{mod} A$ is defined as follows:

(a) The points of $\Gamma(\operatorname{mod} A)$ are the isomorphism classes $[X]$ of indecomposable modules X in $\operatorname{mod} A$.

(b) Let $[M]$, $[N]$ be the points in $\Gamma(\operatorname{mod} A)$ corresponding to the indecomposable modules M, N in $\operatorname{mod} A$. The arrows $[M] \to [N]$ are in bijective correspondence with the vectors of a basis of the K-vector space $\operatorname{Irr}(M, N)$.

The quiver $\Gamma(\operatorname{mod} A)$ of the module category $\operatorname{mod} A$ is called the **Auslander–Reiten quiver** of A.

We may define in exactly the same way the quiver $\Gamma(\mathcal{C})$ of an arbitrary additive subcategory $\mathcal{C}$ of $\operatorname{mod} A$ that is closed under direct sums and summands. We leave to the reader the verification that if $\mathcal{C} = \operatorname{proj} A$, the quiver $\Gamma(\operatorname{proj} A)$ is the opposite of the ordinary quiver of A. In the rest of this section, we examine the combinatorial structure of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A.

It follows from the definition that the points of $\Gamma(\operatorname{mod} A)$ are the isomorphism classes of indecomposable A-modules, and that there exists an arrow $[L] \to [M]$ if and only if $\operatorname{Irr}(L, M) \neq 0$, that is, if and only if there exists an irreducible morphism $L \to M$. By (4.2), (3.1), and (3.5), the set $[M]^-$ of the immediate predecessors of $[M]$ coincides with the set of those points $[L]$ such that L is either an indecomposable direct summand of $\operatorname{rad} M$, if M is projective, or an indecomposable direct summand of the middle term of the almost split sequence ending with M, if M is not projective. Similarly, the set $[M]^+$ of the immediate successors of M coincides with the set of those points $[N]$ such that N is either an indecomposable summand of $M/\operatorname{soc} M$, if M is injective, or an indecomposable direct summand of the middle term of the almost split sequence starting with M, if M is not injective. In particular, for every M, the sets $[M]^+$ and $[M]^-$ are finite. This shows that each point of $\Gamma(\operatorname{mod} A)$ has only finitely many neighbours.

A quiver having this property, that is, such that each point has only finitely may neighbours, is called **locally finite**.

An obvious consequence is that each connected component of an Auslander–Reiten quiver has at most countably many points. Indeed, let x be an arbitrary fixed point of a locally finite quiver Γ. Denote by N_1 the set of neighbours of x, and for each $i \geq 2$ define N_i to be the set of neighbours of points from N_{i-1}. Because Γ is locally finite, each N_i is finite. Because Γ is connected, the set $\Gamma_0 = \bigcup_{i \geq 1} N_i$ is a connected component consisting of at most countably many points.

It is clear that $\Gamma(\operatorname{mod} A)$ is finite (or, equivalently, has finitely many points) if and only if A is representation–finite, that is, the number of the isomorphism classes of indecomposable finite dimensional right A-modules is finite (see (I.4.11)). In fact, we show in the next section that if $\Gamma(\operatorname{mod} A)$ has a finite connected component Γ, then $\Gamma(\operatorname{mod} A) = \Gamma$ and, consequently, A is representation–finite.

We recall that A is called representation–infinite if A is not representation–finite.

A second observation is that every irreducible morphism $f : M \to N$ is either a proper monomorphism or a proper epimorphism; see (1.4). Moreover, if $M = N$, then, because M is finite dimensional as a K-vector space, f should be an isomorphism. This shows that the source and the target of this homomorphism must be distinct and therefore an Auslander–Reiten quiver has no loops.

The Auslander–Reiten quiver is actually endowed with an additional structure. Let Γ_0' (or Γ_0'') denote the set of those points in $\Gamma(\operatorname{mod} A)$ that correspond to a projective (or an injective, respectively) indecomposable module. For each $[N] \in \Gamma(\operatorname{mod} A)_0 \setminus \Gamma_0'$, the Auslander–Reiten translate τN of N exists, and, by (2.10), we have $[\tau N] \in \Gamma(\operatorname{mod} A)_0 \setminus \Gamma_0''$. This defines a bijection

$$\tau : \Gamma(\operatorname{mod} A)_0 \setminus \Gamma_0' \longrightarrow \Gamma(\operatorname{mod} A)_0 \setminus \Gamma_0'',$$

also denoted by τ. Thus, for each indecomposable nonprojective module N, we have $\tau[N] = [\tau N]$. The inverse bijection is denoted by

$$\tau^{-1} : \Gamma(\operatorname{mod} A)_0 \setminus \Gamma_0'' \longrightarrow \Gamma(\operatorname{mod} A)_0 \setminus \Gamma_0'$$

and, for each indecomposable noninjective module L, we have $\tau^{-1}[L] = [\tau^{-1} L]$. We say that τ is the **translation** of the quiver $\Gamma(\operatorname{mod} A)$. Let thus N be an indecomposable nonprojective A-module, and let

$$0 \longrightarrow \tau N \longrightarrow \bigoplus_{i=1}^{t} M_i^{n_i} \longrightarrow N \longrightarrow 0$$

be an almost split sequence ending with N, with the M_i indecomposable and pairwise nonisomorphic. By (4.4), for each i, we have

$$n_i = \dim_K \operatorname{Irr}(M_i, N) = \dim_K \operatorname{Irr}(\tau N, M_i).$$

Hence, corresponding to this almost split sequence is the following "mesh" in $\Gamma(\operatorname{mod} A)$:

$$\begin{tikzcd}
 & {[M_1]} \arrow[dr, "\beta_{1n_1} \cdots \beta_{11}"] & \\
{[\tau N]} \arrow[ur, "\alpha_{11} \cdots \alpha_{1n_1}"] \arrow[dr, "\alpha_{t1} \cdots \alpha_{1n_t}"'] \arrow[rr, dashed, no head] & \vdots & {[N]} \\
 & {[M_t]} \arrow[ur, "\beta_{tn_t} \cdots \beta_{t1}"'] &
\end{tikzcd}$$

In particular, we see that $[\tau N]^+ = [N]^-$ and that for each $[M_i]$ in this set, there exists a bijection between the set $\{\alpha_{i1}, \ldots, \alpha_{in_i}\}$ of arrows from $[\tau N]$ to $[M_i]$ and the set $\{\beta_{i1}, \ldots, \beta_{in_i}\}$ of arrows from $[M_i]$ to $[N]$.

We may thus define a new combinatorial structure.

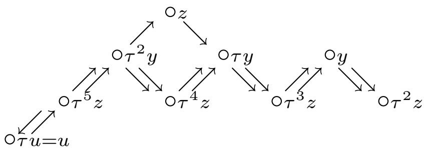
4.7. Definition. Let Γ be a locally finite quiver without loops and τ be a bijection whose domain and codomain are both subsets of Γ_0. The pair (Γ, τ) (or more briefly, Γ) is said to be a **translation quiver** if for every $x \in \Gamma_0$ such that τx exists, and every $y \in x^-$, the number of arrows from y to x is equal to the number of arrows from τx to y.

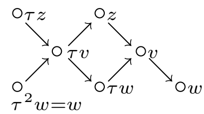
A **full translation subquiver** of a translation quiver (Γ, τ) is a translation quiver (Γ', τ') such that Γ' is a full subquiver of Γ and $\tau' x = \tau x$, whenever x is a vertex of Γ' such that τx belongs to Γ'.

It follows directly from the definition that, if $x \in \Gamma_0$ is such that τx exists, then $(\tau x)^+ = x^-$. The bijection τ is called the **translation** of Γ. The points of Γ, where τ (or τ^{-1}) is not defined are called **projective points** (or **injective points**, respectively). The full subquiver of Γ consisting of a nonprojective point $x \in \Gamma_0$, its translate τx, and the points of $(\tau x)^+ = x^-$ is called the **mesh** ending with x and starting with τx. Let Γ_1' denote the subset of Γ_1 consisting of the arrows with nonprojective target. Because, for $x \in \Gamma_0$ nonprojective there exists a bijection between the arrows having x as target and those having τx as source, we can define an injective mapping $\sigma : \Gamma_1' \to \Gamma_1$ such that if $\alpha \in \Gamma_1'$ has target x, then $\sigma\alpha$ has source τx. Such a mapping is called a **polarisation** of Γ. Clearly, if Γ has no multiple arrows, there exists a unique polarisation on Γ. Otherwise, there usually exist many polarisations. We have already proven the following lemma.

4.8. Lemma. *The Auslander–Reiten quiver* $\Gamma(\operatorname{mod} A)$ *of an algebra* A *is a translation quiver, the translation* τ *being defined for all points* $[M]$ *such that* M *is not a projective module, by* $\tau[M] = [\tau M]$. □

It is, of course, easy to construct examples of translation quivers that are not necessarily Auslander–Reiten quivers, for instance





In most cases we consider the Auslander–Reiten quiver has no multiple arrows. This is the case for representation–finite algebras.

4.9. Proposition. *Let A be a representation–finite algebra. Then* $\Gamma(\operatorname{mod} A)$ *has no multiple arrows.*

Proof. We must show that, for each pair M, N of indecomposable A-modules, we have $\dim_K \operatorname{Irr}(M,N) \leq 1$. We assume that this is not the case, that is, that there exists a pair M, N such that $\dim_K \operatorname{Irr}(M,N) \geq 2$. In particular, $\operatorname{Irr}(M,N) \neq 0$. Because every irreducible morphism $M \to N$ is an epimorphism or a monomorphism, we must have $\dim_K M \neq \dim_K N$. Suppose $\dim_K M > \dim_K N$ (the other case is dual). In particular, N cannot be projective, and there exists an almost split sequence of the form $0 \longrightarrow \tau N \longrightarrow M^2 \oplus E \longrightarrow N \longrightarrow 0$. Hence we get

$$\begin{aligned} \dim_K \tau N \quad &= 2\dim_K M + \dim_K E - \dim_K N \\ &> \dim_K M > \dim_K N. \end{aligned}$$

Furthermore, $\dim_K \operatorname{Irr}(\tau N, M) \geq 2$. An obvious induction shows that, for any two natural numbers i, j such that $i > j$, we have

$$\dim_K \tau^i M > \dim_K \tau^i N > \dim_K \tau^j M > \dim_K \tau^j N.$$

This implies that the mapping $\mathbb{N} \to \Gamma(\operatorname{mod} A)_0$ given by $i \mapsto \tau^i[N]$ is injective, and the connected component of $\Gamma(\operatorname{mod} A)$ containing $[N]$ is infinite, which contradicts the hypothesis that A is representation–finite. $\square$

We now turn to the construction of the Auslander–Reiten quiver of an algebra A. In many simple cases, it is possible to construct $\Gamma(\operatorname{mod} A)$ without constructing explicitly all the almost split sequences in $\operatorname{mod} A$. We illustrate the procedure with examples. In these examples, we agree to identify isomorphic modules and homomorphisms.

4.10. Example. Let A be the path K-algebra of the linear quiver $\underset{1}{\circ} \xleftarrow{\ \beta\ } \underset{2}{\circ} \xleftarrow{\ \alpha\ } \underset{3}{\circ}$. We have a complete list of the indecomposable projective or injective A-modules, given as representations (see (III.2)):

$$\begin{array}{ll} P(1) = (K \longleftarrow 0 \longleftarrow 0) & = S(1) \\ P(2) = (K \xleftarrow{1} K \longleftarrow 0) & \\ P(3) = (K \xleftarrow{1} K \xleftarrow{1} K) = I(1) & \\ I(2) = (0 \longleftarrow K \xleftarrow{1} K) & \\ I(3) = (0 \longleftarrow 0 \longleftarrow K), & \end{array}$$

and we also have a simple module $S(2)$, which is neither projective nor injective. Further, we have

$$\begin{array}{ll} P(1) = \operatorname{rad} P(2) & P(2) = \operatorname{rad} P(3) \\ I(3) = I(2)/S(2) & I(2) = I(1)/S(1) = P(3)/S(1). \end{array}$$

Because the A-module $P(1)$ is simple projective and noninjective, by (3.9), the target of each irreducible morphism starting with $P(1)$ is projective. Because $P(1) = \operatorname{rad} P(2)$, and $P(1)$ is not a summand of $\operatorname{rad} P(3)$, the inclusion $i : P(1) \to P(2)$ is the only such irreducible morphism and is actually the only right minimal almost split morphism ending with $P(2)$. Thus we have an almost split sequence $0 \longrightarrow P(1) \xrightarrow{i} P(2) \longrightarrow \operatorname{Coker} i \longrightarrow 0$. It is easily seen that $\operatorname{Coker} i = P(2)/P(1) = S(2)$.

Now consider $P(2)$. We have just seen that there exists an irreducible morphism $P(2) \to S(2)$. On the other hand $\operatorname{rad} P(3) = P(2)$, hence there exists an irreducible (inclusion) morphism $P(2) \to P(3)$. Now $P(3) = I(1)$ is projective-injective, hence, by (3.11), we have an almost split sequence of the form $0 \longrightarrow P(2) \longrightarrow P(3) \oplus S(2) \longrightarrow I(2) \longrightarrow 0$. On the other hand, the homomorphism $I(2) \to I(2)/S(2) = I(3) = S(3)$ is left minimal almost split, with kernel $S(2)$, so that we have an almost split sequence $0 \longrightarrow S(2) \longrightarrow I(2) \longrightarrow S(3) \longrightarrow 0$. Putting together the information we obtained, $\Gamma(\operatorname{mod} A)$ is the quiver

$$\begin{tikzcd} {[P(1)]} \arrow[dr] & & {[S(2)]} \arrow[dr] & & {[S(3)]} \\ & {[P(2)]} \arrow[ur] \arrow[dr] & & {[I(2)]} \arrow[ur] & \\ & & {[P(3)]} \arrow[ur] & & \end{tikzcd}$$

It is customary, when drawing $\Gamma(\operatorname{mod} A)$, to put the translate τx of a nonprojective point x on the same horizontal line as x. We always follow this convention.

4.11. Example. Let A be given by the quiver $\underset{1}{\circ} \xleftarrow{\gamma} \underset{2}{\circ} \xleftarrow{\beta} \underset{3}{\circ} \xleftarrow{\alpha} \underset{4}{\circ}$ bound by $\alpha\beta\gamma = 0$. We have the following list of indecomposable projective or injective A-modules (see (III.2)):

$$\begin{aligned}
P(1) &= S(1);\\
P(2) &= (K \overset{1}{\longleftarrow} K \longleftarrow 0 \longleftarrow 0);\\
P(3) &= (K \overset{1}{\longleftarrow} K \overset{1}{\longleftarrow} K \longleftarrow 0) = I(1);\\
P(4) &= (0 \longleftarrow K \overset{1}{\longleftarrow} K \overset{1}{\longleftarrow} K) = I(2);\\
I(3) &= (0 \longleftarrow 0 \longleftarrow K \overset{1}{\longleftarrow} K);\\
I(4) &= S(4).
\end{aligned}$$

We thus have two right minimal almost split morphisms $P(1) \to P(2)$, $P(2) \to P(3)$ and two left minimal almost split morphisms $I(2) \to I(3)$, $I(3) \to I(4)$. Because $P(3)$ and $P(4)$ are projective-injective, we have almost split sequences (by (3.11))

$$0 \longrightarrow P(2) \longrightarrow P(3) \oplus \frac{P(2)}{S(1)} \longrightarrow \frac{P(3)}{S(1)} \longrightarrow 0\ ;$$

$$0 \longrightarrow \operatorname{rad} P(4) \longrightarrow P(4) \oplus \frac{\operatorname{rad} P(4)}{S(2)} \longrightarrow \frac{P(4)}{S(2)} \longrightarrow 0.$$

Here we observe that $P(2)/S(1) = S(2)$, $P(4)/S(2) = I(3)$, and $\operatorname{rad} P(4) = P(3)/S(1)$ is the indecomposable module M in $\operatorname{mod} A$ given by the diagram $(0 \longleftarrow K \overset{1}{\longleftarrow} K \longleftarrow 0)$, and $(\operatorname{rad} P(4))/S(2) = S(3)$. Computing successively kernels and cokernels, we obtain $\Gamma(\operatorname{mod} A)$ of the form

$$\begin{tikzcd}
{[P(1)]} \arrow[dr] & & {[S(2)]} \arrow[dr] & & {[S(3)]} \arrow[dr] & & {[S(4)]} \\
& {[P(2)]} \arrow[ur] \arrow[dr] & & {[M]} \arrow[ur] \arrow[r] & {[P(4)]} \arrow[r] & {[I(3)]} \arrow[ur] & \\
& & {[P(3)]} \arrow[ur] & & & &
\end{tikzcd}$$

We remark that, if we replace each indecomposable module by its dimension vector, we obtain

$$\begin{tikzcd}
1\,0\,0\,0 \arrow[dr] & & 0\,1\,0\,0 \arrow[dr] & & 0\,0\,1\,0 \arrow[dr] & & 0\,0\,0\,1 \\
& 1\,1\,0\,0 \arrow[ur] \arrow[dr] & & 0\,1\,1\,0 \arrow[ur] \arrow[r] & 0\,1\,1\,1 \arrow[r] & 0\,0\,1\,1 \arrow[ur] & \\
& & 1\,1\,1\,0 \arrow[ur] & & & &
\end{tikzcd}$$

Thus, for each mesh of $\Gamma(\operatorname{mod} A)$ of the form

$$\begin{tikzcd}
& {[M_1]} \arrow[dr] & \\
& \vdots & \\
{[\tau N]} \arrow[uur] \arrow[ddr] \arrow[rr, dashed, no head] & & {[N]} \\
& \vdots & \\
& {[M_t]} \arrow[uur] &
\end{tikzcd}$$

one has $\mathbf{dim}\, N + \mathbf{dim}\, \tau N = \sum\limits_{i=1}^{t} \mathbf{dim}\, M_i$; this follows from the fact that the corresponding almost split sequence is exact. This seemingly innocent (and trivial) remark gives a method of construction we illustrate in the next example.

4.12. Example. Let A be the K-algebra given by the quiver

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \overset{2}{\circ} \arrow[dl, "\beta"'] & \\
1\ \circ & & \circ\ 4 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
 & \underset{3}{\circ} \arrow[ul, "\delta"] & \\
 & & \circ\ 5 \arrow[ul, "\varepsilon"']
\end{tikzcd}$$

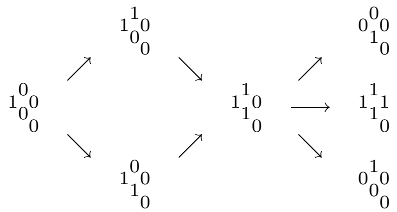
bound by $\alpha\beta = \gamma\delta$, $\varepsilon\delta = 0$. Any algebra A whose ordinary quiver Q_A is acyclic admits at least one simple projective module. In our case, there exists only one, namely $P(1)$, whose dimension vector is $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & & 0 \end{smallmatrix}$. We know that no arrow of $\Gamma(\operatorname{mod} A)$ ends in $P(1)$ and that the target of each arrow starting at $P(1)$ is projective. In our case, we find two such arrows, namely $[P(1)] \to [P(2)]$ and $[P(1)] \to [P(3)]$ (indeed, $P(1) = \operatorname{rad} P(2) = \operatorname{rad} P(3)$), which are our first two arrows. Moreover, these are the only arrows of targets $P(2)$ and $P(3)$, respectively. Because $P(1)$ is not injective, we have an almost split sequence

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(3) \longrightarrow \tau^{-1}P(1) \longrightarrow 0.$$

Moreover, $\mathbf{dim}\, \tau^{-1}P(1) = \mathbf{dim}\, P(2) + \mathbf{dim}\, P(3) - \mathbf{dim}\, P(1) = \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \\ & & 0 \end{smallmatrix} + \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ & & 0 \end{smallmatrix} - \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & & 0 \end{smallmatrix} = \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \\ & & 0 \end{smallmatrix}$. We see at once that $\tau^{-1}P(1) = \operatorname{rad} P(4)$, and hence there is a unique arrow of target $P(4)$, namely $[\tau^{-1}P(1)] \to [P(4)]$. This gives us the beginning of $\Gamma(\operatorname{mod} A)$ (where the isomorphism classes of indecomposable A-modules are replaced by their dimension vectors):

$$\begin{tikzcd}
 & \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \\ & & 0 \end{smallmatrix} \arrow[dr] & & \\
\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & & 0 \end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \\ & & 0 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & & 0 \end{smallmatrix} \\
 & \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ & & 0 \end{smallmatrix} \arrow[ur] & &
\end{tikzcd}$$

The calculation of the almost split sequences starting at $P(2)$ and $P(3)$, respectively, gives



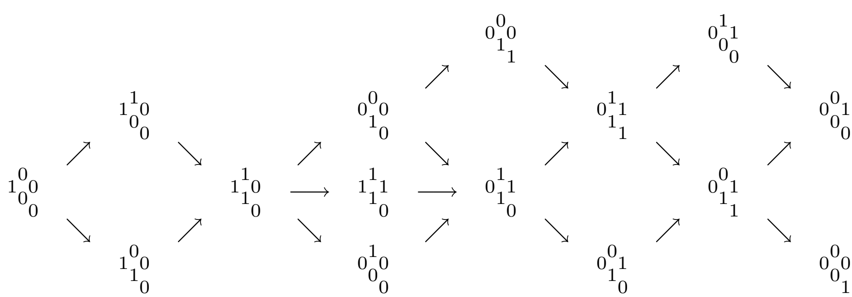
Because $S(3) = \operatorname{rad} P(5)$, there exists a unique arrow of target $P(5)$, namely $[S(3)] \to [P(5)]$. In this way, all the projectives have been obtained. All other indecomposable modules are thus of the form $\tau^{-1}L$, with L indecomposable: to obtain the dimension vector of such a module, we consider the almost split sequence

$$0 \longrightarrow L \longrightarrow M_1 \oplus \ldots \oplus M_t \longrightarrow \tau^{-1}L \longrightarrow 0.$$

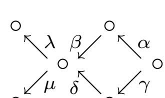
Because we can assume by induction that $\mathbf{dim}\, L$ and $\mathbf{dim}\, M_i$ (for all i with $1 \leq i \leq t$) are known, we deduce $\mathbf{dim}\, \tau^{-1}L = \sum_{i=1}^{t} \mathbf{dim}\, M_i - \mathbf{dim}\, L$. This allows us to construct the rest of $\Gamma(\operatorname{mod} A)$. The construction stops when we reach the injectives; indeed, the left minimal almost split morphism starting at an indecomposable injective $I(a)$ is the projection onto its socle factor $I(a)/S(a)$, and

$$\dim_K I(a) = 1 + \dim_K I(a)/S(a) > \dim_K I(a)/S(a).$$

Thus the previous method would give a dimension vector with negative coordinates, a contradiction. Continuing the construction yields the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$



4.13. Example. Let A be the K-algebra given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\delta\mu = 0$, and $\beta\lambda = 0$. Then $\Gamma(\operatorname{mod} A)$ can be constructed as earlier and is of the form

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & & \begin{smallmatrix}0&&1&\\&1&&0\\1&&0&\end{smallmatrix} \arrow[dr] & & & & & & \\
\begin{smallmatrix}1&&0&\\&0&&0\\0&&0&\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0&&0&\\&1&&0\\1&&0&\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}0&&1&\\&1&&0\\0&&0&\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0&&0&\\&0&&0\\0&&1&\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0&&1&\\&0&&1\\0&&0&\end{smallmatrix} \arrow[dr] & \\
 & \begin{smallmatrix}1&&0&\\&1&&0\\1&&0&\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}0&&0&\\&1&&0\\0&&0&\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}0&&1&\\&1&&0\\0&&1&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0&&1&\\&1&&1\\0&&1&\end{smallmatrix} \arrow[r] & \begin{smallmatrix}0&&1&\\&0&&1\\0&&1&\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}0&&0&\\&0&&1\\0&&0&\end{smallmatrix} \\
\begin{smallmatrix}0&&0&\\&0&&0\\1&&0&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1&&0&\\&1&&0\\0&&0&\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}0&&0&\\&1&&0\\0&&1&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0&&1&\\&0&&0\\0&&0&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0&&0&\\&0&&1\\0&&1&\end{smallmatrix} \arrow[ur] & \\
 & & & \begin{smallmatrix}1&&0&\\&1&&0\\0&&1&\end{smallmatrix} \arrow[ur] & & & & & &
\end{tikzcd}
$$

Let M, N, and L be the simple A-modules such that $\mathbf{dim}\, M = \begin{smallmatrix}0&&0&\\&1&&0\\0&&0&\end{smallmatrix}$, $\mathbf{dim}\, N = \begin{smallmatrix}0&&0&\\&0&&1\\0&&0&\end{smallmatrix}$, and $\mathbf{dim}\, L = \begin{smallmatrix}0&&0&\\&1&&0\\0&&0&\end{smallmatrix}$. Because $\mathbf{dim}\, \tau M = \begin{smallmatrix}1&&0&\\&1&&0\\1&&0&\end{smallmatrix}$, we get $\operatorname{Hom}_A(DA, \tau M) = 0$, and (2.7)(a) yields $\operatorname{pd}_A M = 1$.

On the other hand, $\operatorname{pd}_A N \geq 2$, because $\mathbf{dim}\, \tau N = \begin{smallmatrix}0&&1&\\&0&&1\\0&&1&\end{smallmatrix}$ and therefore there is a nonzero homomorphism from the indecomposable injective A-module E of dimension vector $\begin{smallmatrix}0&&1&\\&1&&1\\0&&1&\end{smallmatrix}$ to the module τN. Then we get $\operatorname{Hom}_A(DA, \tau N) \neq 0$ and (2.7)(a) yields $\operatorname{pd}_A N \geq 2$. Actually, $\operatorname{pd}_A N = 2$, because the minimal projective resolution of N has the form

$$
0 \longrightarrow \begin{smallmatrix}1&&0&\\&1&&0\\1&&0&\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&&0&\\&1&&0\\0&&1&\end{smallmatrix} \bigoplus \begin{smallmatrix}0&&1&\\&1&&0\\1&&0&\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&&1&\\&1&&1\\0&&1&\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&&0&\\&0&&1\\0&&0&\end{smallmatrix} \longrightarrow 0
$$

Similarly, $\operatorname{id}_A L \geq 2$, because $\mathbf{dim}\, \tau^{-1} L = \begin{smallmatrix}0&&1&\\&1&&0\\0&&1&\end{smallmatrix}$ and there is a nonzero homomorphism from $\tau^{-1}L$ to the indecomposable projective module P of dimension vector $\begin{smallmatrix}0&&1&\\&1&&1\\0&&1&\end{smallmatrix}$. It follows that $\operatorname{Hom}_A(\tau^{-1}L, A) \neq 0$ and (2.7)(b) yields $\operatorname{id}_A L \geq 2$.

The method presented in these examples works perfectly well for all finite and acyclic Auslander–Reiten quivers. An interesting remark in this case is that, as suggested by the examples, every indecomposable module is (up to isomorphism) uniquely determined by its dimension vector. This is shown later.

4.14. Example. Let A be the K-algebra given by the quiver

$$
\begin{tikzcd}[column sep=small]
 & \overset{2}{\circ} \arrow[dl, "\beta"'] & \\
1\ \circ & & \circ\ 3 \arrow[ul, "\alpha"'] \arrow[ll, "\gamma"]
\end{tikzcd}
$$

bound by $\alpha\beta = 0$. Then $\Gamma(\operatorname{mod} A)$ is given by

$$\begin{tikzcd}
& {}_{1}{}^{1}{}_{0} \arrow[dr] & & {}_{0}{}^{1}{}_{1} \arrow[dr] & \\
{}_{1}{}^{0}{}_{0} \arrow[ur] \arrow[dr] & & {}_{1}{}^{2}{}_{1} \arrow[ur] \arrow[dr] & & {}_{0}{}^{0}{}_{1} \\
& {}_{1}{/}^{1}{}_{1} \arrow[ur] \arrow[dr] & & {}_{1}{}^{1}{\backslash}_{1} \arrow[ur] \arrow[dr] & \\
{}_{0}{}^{1}{}_{0} \arrow[ur] & & {}_{1}{}^{0}{}_{1} \arrow[ur] & & {}_{0}{}^{1}{}_{0}
\end{tikzcd}$$

where modules are replaced by their dimension vectors and one must identify the two copies of $S(2) = {}_{0}{}^{1}{}_{0}$, thus forming a cycle. Here, ${}_{1}{/}^{1}{}_{1}$ represents the indecomposable projective module $P(3) = \begin{tikzcd}[sep=small] & K \arrow[dl, "0"'] & \\ K & & K \arrow[ul, "1"'] \arrow[ll, "1"] \end{tikzcd}$, while ${}_{1}{}^{1}{\backslash}_{1}$ represents the indecomposable injective module $I(1) = \begin{tikzcd}[sep=small] & K \arrow[dl, "1"'] & \\ K & & K \arrow[ul, "0"'] \arrow[ll, "1"] \end{tikzcd}$. It follows that indecomposable modules are not uniquely determined by their dimension vectors, because $P(3) \ncong I(1)$ and $\mathbf{dim}\, P(3) = \mathbf{dim}\, I(1)$.

IV.5. The first Brauer–Thrall conjecture

At the origin of many recent developments of representation theory are the following two conjectures attributed to Brauer and Thrall.

Conjecture 1. *A finite dimensional K-algebra is either representation–finite or there exist indecomposable modules with arbitrarily large dimension.*

Conjecture 2. *A finite dimensional algebra over an infinite field K is either representation–finite or there exists an infinite sequence of numbers $d_i \in \mathbb{N}$ such that, for each i, there exists an infinite number of nonisomorphic indecomposable modules with K-dimension d_i.*

The first statement has now been shown to hold true, whenever the field K is arbitrary (see [13], [14], [140], [147], [148], [151], [154], [170]), and the second one when K is algebraically closed (see [26], [27], [124], [140], [162], and for historical notes see [83]). Our objective in this section is to give a simple proof of the first conjecture.

Let A be a finite dimensional K-algebra. A sequence of irreducible morphisms in $\operatorname{mod} A$ of the form

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t$$

with all the M_i indecomposables is called a **chain of irreducible morphisms** from M_0 to M_t of length t.

5.1. Lemma. *Let $t \in \mathbb{N}$ and let M and N be indecomposable right A-modules with $\operatorname{Hom}_A(M,N) \neq 0$. Assume that there exists no chain of irreducible morphisms from M to N of length $< t$.*

(a) *There exists a chain of irreducible morphisms*

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_t} M_t$$

and a homomorphism $g : M_t \to N$ with $gf_t \dots f_2 f_1 \neq 0$.

(b) *There exists a chain of irreducible morphisms*

$$N_t \xrightarrow{g_t} N_{t-1} \xrightarrow{g_{t-1}} \cdots \longrightarrow N_1 \xrightarrow{g_1} N_0 = N$$

and a homomorphism $f : M \to N_t$ with $g_1 \dots g_t f \neq 0$.

Proof. We only prove (a); the proof of (b) is similar. We proceed by induction on t. For $t = 0$, there is nothing to show. Assume thus that M and N are given with $\operatorname{Hom}_A(M,N) \neq 0$ and that there is no chain of irreducible morphisms from M to N of length $< t+1$. By the induction hypothesis, there exists a chain of irreducible morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t$$

and a homomorphism $g : M_t \to N$ with $gf_t \dots f_1 \neq 0$. The induction hypothesis implies that g cannot be an isomorphism. Because M_t and N are indecomposable, g is not a section. We consider the left minimal almost split morphism starting with M_t

$$h = \begin{bmatrix} h_1 \\ \vdots \\ h_1 \end{bmatrix} : \ M_t \xrightarrow{\qquad\qquad} \bigoplus_{j=1}^{s} L_j,$$

where the modules $L_1, \dots, L_s$ are indecomposable. Then g factors through h, that is, there exists $u = [u_1, \dots, u_s] : \bigoplus_{j=1}^{s} L_j \longrightarrow N$ such that $g = uh = \sum_{j=1}^{s} u_j h_j$. Thus, because $0 \neq gf_t \dots f_1 = \sum_{j=1}^{s} u_j h_j f_t \dots f_1$, there exists j such that $1 \leq j \leq s$ and $u_j h_j f_t \dots f_1 \neq 0$. Setting $M_{t+1} = L_j$, $f_{t+1} = h_j$ and $g' = u_j$, our claim follows from the fact that h_j is irreducible. $\square$

5.2. Lemma (Harada and Sai). *For a natural number b, let*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \ \to \ \cdots \ \to \ M_{2^b-1} \xrightarrow{f_{2^b-1}} M_{2^b}$$

be a chain of nonzero nonisomorphisms in $\operatorname{mod} A$*, with all* M_i *indecomposables of length* $\leq b$*. Then* $f_{2^b-1} \dots f_2 f_1 = 0$*.*

Proof. We show by induction on n that if

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to \cdots \to M_{2^n-1} \xrightarrow{f_{2^n-1}} M_{2^n}$$

is a sequence of nonzero nonisomorphisms between indecomposable modules of length $\leq b$, then the length of the image of the composite homomorphism $f_{2^n-1} \dots f_2 f_1$ is $\leq b-n$. This will imply the statement upon setting $b = n$.

Let $n = 1$. If the length $\ell(\operatorname{Im} f_1)$ of $\operatorname{Im} f_1$ is equal to b, then f_1 is an isomorphism, a contradiction that shows that $\ell(\operatorname{Im} f_1) \leq b - 1$. Assume that the statement holds for n, and let

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \to M_{2^n-1} \xrightarrow{f_{2^n-1}} M_{2^n} \xrightarrow{f_{2^n}} M_{2^n+1} \xrightarrow{f_{2^n+1}} \cdots \xrightarrow{f_{2^{n+1}-1}} M_{2^{n+1}}$$

be a sequence of nonzero nonisomorphisms between indecomposable modules of length $\leq b$. We consider the two homomorphisms $f = f_{2^n-1} \dots f_2 f_1$ and $h = f_{2^{n+1}-1} \dots f_{2^n+1}$. By the induction hypothesis, $\ell(\operatorname{Im} f) \leq b - n$ and $\ell(\operatorname{Im} h) \leq b - n$. If at least one of these two inequalities is strict, we are done. We may thus suppose that $\ell(\operatorname{Im} f) = \ell(\operatorname{Im} h) = b - n > 0$. Let $g = f_{2^n}$. We must show that $\ell(\operatorname{Im} hgf) \leq b - n - 1$.

We claim that if this is not the case, then g is an isomorphism, a contradiction that completes the proof. Assume thus that $\ell(\operatorname{Im} hgf) > b - n - 1$. Because $\ell(\operatorname{Im} hgf) \leq \ell(\operatorname{Im} f) = b - n$, this implies that $\ell(\operatorname{Im} hgf) = b - n$. Now

$$\ell(\operatorname{Im} hgf) = \ell\left(\frac{\operatorname{Im} f}{\operatorname{Im} f \cap \operatorname{Ker} hg}\right) = \ell(\operatorname{Im} f) - \ell(\operatorname{Im} f \cap \operatorname{Ker} hg).$$

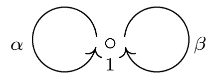
This implies that $\ell(\operatorname{Im} f \cap \operatorname{Ker} hg) = 0$, hence $\operatorname{Im} f \cap \operatorname{Ker} hg = 0$. On the other hand, $\operatorname{Im} hgf \subseteq \operatorname{Im} hg \subseteq \operatorname{Im} h$ and $\ell(\operatorname{Im} hgf) = \ell(\operatorname{Im} h) = b - n$ give $\ell(\operatorname{Im} hg) = b - n$. Consequently,

$$\ell(\operatorname{Ker} hg) = \ell(M_{2^n}) - \ell(\operatorname{Im} hg) = \ell(M_{2^n}) - (b - n) = \ell(M_{2^n}) - \ell(\operatorname{Im} f).$$

This shows that $M_{2^n} = \operatorname{Im} f \oplus \operatorname{Ker} hg$. Because M_{2^n} is indecomposable and $f \neq 0$, we have $\operatorname{Ker} hg = 0$. Therefore hg is a monomorphism. Hence g itself is a monomorphism. Similarly, one shows that $\operatorname{Im} gf \cap \operatorname{Ker} h = 0$, hence that $M_{2^n+1} = \operatorname{Im} gf \oplus \operatorname{Ker} h$. Because $gf \neq 0$ and the module M_{2^n+1} is indecomposable then we get $M_{2^n+1} = \operatorname{Im} gf$, so that gf and therefore g are epimorphisms. This completes the proof that g is an isomorphism, and hence of the lemma. □

The following example shows that the bounds given in the Harada–Sai lemma are the best bounds possible.

5.3. Example. Let A be given by the quiver



consisting of two loops α and β, bound by $\alpha^2 = 0$, $\beta^2 = 0$, $\alpha\beta = 0$, and $\beta\alpha = 0$.

We construct 7 indecomposable A-modules of length ≤ 3 and 6 nonisomorphisms between them with nonzero composition.

The algebra A admits a unique simple module S_A and any A-module can be written in a form of a triple $(V, \varphi_\alpha, \varphi_\beta)$, where V is a finite dimensional K-vector space and $\varphi_\alpha, \varphi_\beta : V \to V$ are K-linear endomorphisms satisfying the conditions $\varphi_\alpha^2 = 0$, $\varphi_\beta^2 = 0$, $\varphi_\alpha\varphi_\beta = \varphi_\beta\varphi_\alpha = 0$, and a morphism $(V, \varphi_\alpha, \varphi_\beta) \to (V', \varphi'_\alpha, \varphi'_\beta)$ is a K-linear map $f : V \to V'$ such that $\varphi'_\alpha f = f\varphi_\alpha$ and $\varphi'_\beta f = f\varphi_\beta$. Let thus

$$\begin{array}{lllllll}
M_1 & = & M_5 & = & A_A & = & \left(K^3, \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\1&0&0 \end{bmatrix}\right),\\
M_2 & = & M_6 & = & A_A/S & = & \left(K^2, \begin{bmatrix} 0&0\\1&0 \end{bmatrix}, 0\right),\\
M_3 & = & M_7 & = & (DA)_A & = & \left(K^3, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}\right),\\
M_4 & = & S_A & = & (K, 0, 0). & &
\end{array}$$

Each of these modules has a simple top or a simple socle and hence is indecomposable. Let now

$$\begin{array}{lllllllllllll}
f_1 & = & \begin{bmatrix} 1&0&0\\0&0&1 \end{bmatrix} & : & M_1 & \longrightarrow & M_2, & f_2 & = & \begin{bmatrix} 0&1\\0&0\\1&0 \end{bmatrix} & : & M_2 \longrightarrow & M_3,\\
f_3 & = & \begin{bmatrix} 1&0&0 \end{bmatrix} & : & M_3 & \longrightarrow & M_4, & f_4 & = & \begin{bmatrix} 0\\0\\1 \end{bmatrix} & : & M_4 \longrightarrow & M_5,\\
f_5 & = & \begin{bmatrix} 1&0&0\\0&0&1 \end{bmatrix} & : & M_5 & \longrightarrow & M_6, & f_6 & = & \begin{bmatrix} 1&0\\0&0\\0&1 \end{bmatrix} & : & M_6 \longrightarrow & M_7.
\end{array}$$

It is easily checked that each of these matrices defines an A-module homomorphism, and $f_6f_5f_4f_3f_2f_1 = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix} \neq 0$.

We are now able to prove our criterion of representation–finiteness, which was announced in the previous section and implicitly used in the construction of Auslander–Reiten quivers.

5.4. Theorem. *Assume that A is a basic and connected finite dimensional K-algebra. If $\Gamma(\operatorname{mod} A)$ admits a connected component $\mathcal{C}$ whose* *modules are of bounded length, then $\mathcal{C}$ is finite and $\mathcal{C} = \Gamma(\operatorname{mod} A)$. In particular, A is representation–finite.*

Proof. Let b be a bound for the length of the indecomposable modules X with $[X]$ in $\mathcal{C}$. Let M, N be two indecomposable A-modules such that $\operatorname{Hom}_A(M, N) \neq 0$. If $[M] \in \mathcal{C}_0$, there exists a chain of irreducible morphisms from M to N of length smaller than $2^b - 1 = t$, and in particular $[N] \in \mathcal{C}_0$. Indeed, if this is not the case, there exists, by (5.1), a chain of irreducible morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{t-1} \xrightarrow{f_t} M_t$$

and a homomorphism $g : M_t \to N$ with $g f_t \dots f_1 \neq 0$. However, (5.2) yields $f_t \dots f_1 = 0$, a contradiction that shows our claim. Similarly, if $[N] \in \mathcal{C}_0$, we have $[M] \in \mathcal{C}_0$.

Let now $[M] \in \mathcal{C}_0$ be arbitrary. There exists an indecomposable projective module P_A such that $\operatorname{Hom}_A(P, M) \neq 0$; hence we also have $[P] \in \mathcal{C}_0$. It follows from (II.3.4) and (I.5.17) that, for any other indecomposable projective P', there exists a sequence of indecomposable projective modules $P = P_0, P_1, \dots, P_s = P'$ such that $\operatorname{Hom}_A(P_{i-1}, P_i) \neq 0$ or $\operatorname{Hom}_A(P_i, P_{i-1}) \neq 0$ for each $1 \leq i \leq s$, because the algebra A is connected, $P \cong e_a A$ and $P' \cong e_b A$ for some primitive orthogonal idempotents e_a, e_b of A, and (I.4.2) yields $\operatorname{Hom}_A(e_a A, e_b A) \cong e_b A e_a$. Hence $[P'] \in \mathcal{C}_0$. We deduce that any indecomposable A-module X corresponds to a point $[X]$ in $\mathcal{C}$, because there exists an indecomposable projective A-module P' such that $\operatorname{Hom}_A(P', X) \neq 0$. This shows that $\mathcal{C} = \Gamma(\operatorname{mod} A)$.

On the other hand, for each indecomposable projective A-module P and each indecomposable A-module M such that $\operatorname{Hom}_A(P, M) \neq 0$, we know that there exists a chain of irreducible morphisms from P to M of length smaller than $t = 2^b - 1$. Because there are only finitely many nonisomorphic indecomposable projectives, there are only finitely many nonisomorphic indecomposable modules corresponding to points in $\mathcal{C}$. Hence A is representation–finite. □

As a consequence of (5.4) we get the validity of the first Brauer–Thrall conjecture.

5.5. Corollary. *Any algebra is either representation–finite or admits indecomposable modules of arbitrary length.* □

We end this section with the following corollary, which underlines the importance of the irreducible morphisms and hence of the Auslander–Reiten quiver, for the description of the module category of a representation–finite algebra.

5.6. Corollary. *Let A be a representation–finite algebra. Any nonzero nonisomorphism between indecomposable modules in* $\operatorname{mod} A$ *is a sum of compositions of irreducible morphisms.*

Proof. Let M, N be indecomposable A-modules and $t \geq 1$. Denote by $\operatorname{rad}_A^t(M, N)$ the K-subspace of $\operatorname{rad}_A(M, N)$ consisting of the K-linear combinations of compositions $f_1 f_2 \dots f_t$, where $f_1, f_2, \dots, f_t$ are nonisomorphisms between indecomposable A-modules. Because A is representation–finite, the lengths of the indecomposable A-modules are bounded; hence, by the Harada–Sai lemma (5.2), there exists $m \geq 1$ such that $\operatorname{rad}_A^{m+1}(M, N) = 0$ for all M and N.

Let $g \in \operatorname{rad}_A(M, N)$ be nonzero. If $g \notin \operatorname{rad}_A^2(M, N)$, then g is irreducible and there is nothing to prove. If $g \in \operatorname{rad}_A^2(M, N)$, there exists s such that $2 \leq s \leq m$ and $g \in \operatorname{rad}_A^s(M, N) \setminus \operatorname{rad}_A^{s+1}(M, N)$.

We prove our statement by descending induction on s. If $s = m$, then g is a sum of nonzero compositions $g_1 \cdot g_2 \cdot \ldots \cdot g_m$ of nonisomorphisms $g_1, g_2, \dots, g_m$ between indecomposable modules. Because $\operatorname{rad}_A^{m+1}(M, N) = 0$, the homomorphisms $g_1, \dots, g_m$ do not belong to the square of the radical and therefore are irreducible. This proves the statement for $s = m$. Suppose that $s \leq m - 1$. Then g is a sum of nonzero compositions $g_1 g_2 \dots g_s$ of nonisomorphisms between indecomposable modules. Let g' denote the sum of all the summands $g_1 g_2 \dots g_s$ of g in which all the homomorphisms $g_1, g_2, \dots, g_s$ are irreducible. Then $g'' = g - g' \in \operatorname{rad}_A^{s+1}(M, N)$. If $g'' = 0$, the statement is trivial. If $g'' \neq 0$, then, by the induction hypothesis, g'' is a sum of compositions of irreducible morphisms and therefore so is $g = g' + g''$. The proof is now complete. $\square$

IV.6. Functorial approach to almost split sequences

Let A be a finite dimensional K-algebra. We present in this section an interpretation of the almost split sequences in $\operatorname{mod} A$ in terms of the projective resolutions of the simple objects in the categories $\mathcal{F}un^{\mathrm{op}} A$ and $\mathcal{F}un\, A$ of the contravariant, and covariant, respectively, K-linear functors from the category $\operatorname{mod} A$ of finitely generated right A-modules into the category $\operatorname{mod} K$ of finite dimensional K-vector spaces. These categories are defined in Section A.2 of the Appendix and are both seen to be abelian. We recall that, given a pair of functors F and G in the category $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$), we denote by $\operatorname{Hom}(F, G)$ the set of functorial morphisms

$\varphi : F \to G$.

Of particular interest in our study is the following classical result.

6.1. Theorem (Yoneda's lemma). *Let $\mathcal{C}$ be an additive K-category and X be an object in $\mathcal{C}$.*

(a) *For any contravariant functor $F : \mathcal{C} \longrightarrow \operatorname{mod} K$, the correspondence $\pi : \varphi \mapsto \varphi_X(1_X)$ defines a bijection between the set $\operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F)$ of functorial morphisms $\varphi : \operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$ and the set $F(X)$.*

(b) *For any covariant functor $F : \mathcal{C} \longrightarrow \operatorname{mod} K$, the correspondence $\pi : \varphi \mapsto \varphi_X(1_X)$ defines a bijection between the set $\operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(X,-), F)$ of functorial morphisms $\varphi : \operatorname{Hom}_{\mathcal{C}}(X,-) \longrightarrow F$ and the set $F(X)$.*

Proof. We only prove (a); the proof of (b) is similar. For a functorial morphism $\varphi : \operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$, we have $\varphi_X(1_X) \in F(X)$, so π defines a map $\operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F) \longrightarrow F(X)$. We now construct its inverse

$$\sigma : F(X) \longrightarrow \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F).$$

Let $a \in F(X)$ and Y be an arbitrary object in $\mathcal{C}$. We define the map $\sigma(a)_Y : \operatorname{Hom}_{\mathcal{C}}(Y,X) \longrightarrow F(Y)$ to be given by $\sigma(a)_Y(f) = F(f)(a)$, for $f \in \operatorname{Hom}_{\mathcal{C}}(Y,X)$.

To show that $\sigma(a) : \operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$ is a functorial morphism, we must show that, for any morphism $g : Y \to Z$, the following diagram is commutative

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{C}}(Y,X) & \xrightarrow{\sigma(a)_Y} & F(Y) \\
{\scriptstyle \operatorname{Hom}_{\mathcal{C}}(g,X)}\uparrow & & \uparrow{\scriptstyle F(g)} \\
\operatorname{Hom}_{\mathcal{C}}(Z,X) & \xrightarrow{\sigma(a)_Z} & F(Z)
\end{array}$$

Let thus $f \in \operatorname{Hom}_{\mathcal{C}}(Z,X)$; then $F(g)\sigma(a)_Z(f) = F(g)F(f)(a) = F(f \circ g)(a)$, while $\sigma(a)_Y \operatorname{Hom}_{\mathcal{C}}(g,X)(f) = \sigma(a)_Y(f \circ g) = F(f \circ g)(a)$.

It remains to show that π and σ are mutually inverse.

(i) Let $a \in F(X)$. To prove that $\pi\sigma(a) = a$, we note that

$$\pi\sigma(a) = \sigma(a)_X(1_X) = F(1_X)(a) = 1_{F(X)}(a) = a.$$

(ii) Let $\varphi \in \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F)$. To prove that $\sigma\pi(\varphi) = \varphi$, we show that, for any object Y in $\mathcal{C}$, we have $\sigma\pi(\varphi)_Y = \varphi_Y$. By definition, for any $f \in \operatorname{Hom}_{\mathcal{C}}(Y,X)$, we have

$$\sigma\pi(\varphi)_Y(f) = F(f)(\pi(\varphi)) = F(f)\varphi_X(1_X).$$

Because φ is a functorial morphism, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,X) & \xrightarrow{\varphi_X} & F(X) \\ {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f,X)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}_{\mathcal{C}}(Y,X) & \xrightarrow{\varphi_Y} & F(Y) \end{array}$$

That is, $F(f)\varphi_X = \varphi_Y \mathrm{Hom}_{\mathcal{C}}(f,X)$. Thus we have

$$\sigma\pi(\varphi)_Y(f) = \varphi_Y \mathrm{Hom}_{\mathcal{C}}(f,X)(1_X) = \varphi_Y(f)$$

and the proof is complete. $\square$

6.2. Corollary. *Let $\mathcal{C}$ be an additive K-category and let X be an object in $\mathcal{C}$.*

(a) *Let F be a subfunctor of* $\mathrm{Hom}_{\mathcal{C}}(-,X)$. *The map $f \mapsto \mathrm{Hom}_{\mathcal{C}}(-,f)$ is a bijection $F(X) \cong \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(-,X),F)$. In particular, for any object Y in $\mathcal{C}$, the map* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(-,X),\mathrm{Hom}_{\mathcal{C}}(-,Y))$ *given by $f \mapsto \mathrm{Hom}_{\mathcal{C}}(-,f)$ is a bijection.*

(b) *Let F be a subfunctor of* $\mathrm{Hom}_{\mathcal{C}}(X,-)$. *The map $f \mapsto \mathrm{Hom}_{\mathcal{C}}(f,-)$ is a bijection $F(X) \cong \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(X,-),F)$. In particular, for any object Y in $\mathcal{C}$, the map* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(Y,-),\mathrm{Hom}_{\mathcal{C}}(X,-))$ *given by $f \mapsto \mathrm{Hom}_{\mathcal{C}}(f,-)$ is a bijection.*

Proof. We only prove (a); the proof of (b) is similar. Let $f \in F(X) \subseteq \mathrm{Hom}_{\mathcal{C}}(X,X)$. It was shown that the inverse of the bijection π in Yoneda's lemma 6.1 is given by $\sigma(f) : \mathrm{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$. We show that $\sigma(f) = \mathrm{Hom}_{\mathcal{C}}(-,f)$. Indeed, let Y be an object in $\mathcal{C}$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$; then $\sigma(f)_Y(g) = F(g)(f) = f \circ g = \mathrm{Hom}_{\mathcal{C}}(Y,f)(g)$ because, by definition, $F(g) \in F(Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(Y,X)$. This shows the first assertion. The second follows from the first applied to the functor $F = \mathrm{Hom}_{\mathcal{C}}(-,Y)$. $\square$

In particular, it follows from (6.2) that the categories $\mathcal{F}un^{\mathrm{op}} A$ and $\mathcal{F}un\, A$ are not only abelian, they are also additive K-categories. As a second corollary, we now show that a Hom functor uniquely determines the representing object.

6.3. Corollary. *Let $\mathcal{C}$ be an additive K-category and let X, Y be two objects in $\mathcal{C}$.*

(a) *$X \cong Y$ if and only if* $\mathrm{Hom}_{\mathcal{C}}(-,X) \cong \mathrm{Hom}_{\mathcal{C}}(-,Y)$.

(b) *$X \cong Y$ if and only if* $\mathrm{Hom}_{\mathcal{C}}(X,-) \cong \mathrm{Hom}_{\mathcal{C}}(Y,-)$.

Proof. We only prove (a); the proof of (b) is similar. Clearly, $X \cong Y$ implies $\mathrm{Hom}_{\mathcal{C}}(-,X) \cong \mathrm{Hom}_{\mathcal{C}}(-,Y)$. Conversely, assume that there is an isomorphism $\mathrm{Hom}_{\mathcal{C}}(-,X) \cong \mathrm{Hom}_{\mathcal{C}}(-,Y)$ of functors. By (6.2), there exist morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(-,f) : \mathrm{Hom}_{\mathcal{C}}(-,X) \to \mathrm{Hom}_{\mathcal{C}}(-,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(-,g) : \mathrm{Hom}_{\mathcal{C}}(-,Y) \to \mathrm{Hom}_{\mathcal{C}}(-,X)$

are mutually inverse functorial isomorphisms. Thus the equalities
$\mathrm{Hom}_{\mathcal{C}}(-,1_X) = 1_{\mathrm{Hom}_{\mathcal{C}}(-,X)} = \mathrm{Hom}_{\mathcal{C}}(-,g)\circ\mathrm{Hom}_{\mathcal{C}}(-,f) = \mathrm{Hom}_{\mathcal{C}}(-,g\circ f)$
give $g\circ f = 1_X$, by (6.2) again. Similarly, $f\circ g = 1_Y$. □

An object P in $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$) is said to be **projective** if for any functorial epimorphism $\varphi : F \to G$, the induced map of K-vector spaces $\mathrm{Hom}(P,\varphi) : \mathrm{Hom}(P,F) \longrightarrow \mathrm{Hom}(P,G)$, given by $\psi \mapsto \varphi\psi$, is surjective.

We now observe that Yoneda's lemma also gives projective objects in the categories $\mathcal{F}un^{\mathrm{op}} A$ and $\mathcal{F}un\, A$.

6.4. Corollary. *Let A be a K-algebra and M be a module in* $\mathrm{mod}\, A$.
(a) *The functor* $\mathrm{Hom}_A(-,M)$ *is a projective object in $\mathcal{F}un^{op} A$.*
(b) *The functor* $\mathrm{Hom}_A(M,-)$ *is a projective object in $\mathcal{F}unA$.*

Proof. We only prove (a); the proof of (b) is similar. We must prove that, for any functorial epimorphism $\varphi : F \to G$, the induced map

$$\mathrm{Hom}(\mathrm{Hom}_A(-,M),\varphi) : \mathrm{Hom}(\mathrm{Hom}_A(-,M),F) \longrightarrow \mathrm{Hom}(\mathrm{Hom}_A(-,M),G)$$

given by $\psi \mapsto \varphi\psi$, is surjective. We claim that the following diagram

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{Hom}_A(-,M),F) \arrow[rr, "\mathrm{Hom}(\mathrm{Hom}_A(-,M),\varphi)"] \arrow[d, "\pi^F"', "\cong"] & & \mathrm{Hom}(\mathrm{Hom}_A(-,M),G) \arrow[d, "\cong"', "\pi^G"] \\ F(M) \arrow[rr, "\varphi_M"] & & G(M) \end{tikzcd}$$

is commutative, where π^F and π^G denote the bijection π in Yoneda's lemma 6.1 applied to F and G, respectively. Indeed, let $\psi \in \mathrm{Hom}(\mathrm{Hom}_A(-,M),F)$, then

$$\begin{aligned} \varphi_M\pi^F(\psi) &= \varphi_M\psi_M(1_M) = (\varphi\psi)_M(1_M) = \pi^G(\varphi\psi) \\ &= \pi^G\mathrm{Hom}(\mathrm{Hom}_A(-,M),\varphi)(\psi). \end{aligned}$$

On the other hand, φ_M is surjective, because φ is a functorial epimorphism. Hence so is $\mathrm{Hom}(\mathrm{Hom}_A(-,M),\varphi)$. □

A functor F in $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$) is called **finitely generated** if F is isomorphic to a quotient of a functor of the form $\mathrm{Hom}_A(-,M)$ (or $\mathrm{Hom}_A(M,-)$, respectively) for some A-module M, that is, there exists a functorial epimorphism $\mathrm{Hom}_A(-,M) \longrightarrow F \longrightarrow 0$, (or a functorial epimorphism $\mathrm{Hom}_A(M,-) \longrightarrow F \longrightarrow 0$, respectively).

We now characterise the finitely generated projective objects in our functor categories $\mathcal{F}un^{\mathrm{op}} A$ and $\mathcal{F}un\, A$.

6.5. Lemma. (a) *An object in $\mathcal{F}un^{op} A$ is finitely generated projective if and only if it is isomorphic to a functor of the form* $\mathrm{Hom}_A(-,M)$, *for*

M an A-module. Such a functor is indecomposable if and only if M is indecomposable.

(b) *An object in $\mathcal{F}un\,A$ is finitely generated projective if and only if it is isomorphic to a functor of the form $\mathrm{Hom}_A(M,-)$, for M an A-module. Such a functor is indecomposable if and only if M is indecomposable.*

Proof. We only prove (a); the proof of (b) is similar. The projectivity of the finitely generated functor $\mathrm{Hom}_A(-,M)$ follows from (6.4). Conversely, let F be a finitely generated projective object in $\mathcal{F}un^{\mathrm{op}}A$, then there exists a functorial epimorphism $\varphi : \mathrm{Hom}_A(-,X) \longrightarrow F$, for some A-module X. Because F is projective, φ is a retraction and so there exists a functorial monomorphism $\psi : F \longrightarrow \mathrm{Hom}_A(-,X)$ such that $\varphi\psi = 1_F$. Let $\pi = \psi\varphi : \mathrm{Hom}_A(-,X) \longrightarrow F \longrightarrow \mathrm{Hom}_A(-,X)$ (thus, $F = \mathrm{Im}\,\pi$). By (6.2), there exists an endomorphism f of X such that $\pi = \mathrm{Hom}_A(-,f)$. Because π is an idempotent, we have $\mathrm{Hom}_A(-,f^2) = \mathrm{Hom}_A(-,f)^2 = \pi^2 = \pi = \mathrm{Hom}_A(-,f)$ thus $f^2 = f$, again by (6.2), that is, f is an idempotent. Consequently, $M = \mathrm{Im}\,f$ is a direct summand of X. Because $\mathrm{Hom}_A(-,M)$ is the image of $\mathrm{Hom}_A(-,f)$, we deduce that $F \cong \mathrm{Hom}_A(-,M)$. The same argument shows the last assertion. $\square$

We now show that if M is an indecomposable module, the Hom functors $\mathrm{Hom}_A(-,M)$ and $\mathrm{Hom}_A(M,-)$ behave, in their respective categories, in a similar way to the finitely generated indecomposable projective modules over a finite dimensional algebra, in the sense that they have simple tops.

6.6. Lemma. *Let M be an indecomposable A-module.*

(a) *The functor $\mathrm{rad}_A(-,M)$ is the unique maximal subfunctor of the functor $\mathrm{Hom}_A(-,M)$.*

(b) *The functor $\mathrm{rad}_A(M,-)$ is the unique maximal subfunctor of the functor $\mathrm{Hom}_A(M,-)$.*

Proof. We only prove (a); the proof of (b) is similar. It suffices to show that any proper subfunctor F of $\mathrm{Hom}_A(-,M)$ is contained in $\mathrm{rad}_A(-,M)$, that is, for any indecomposable A-module N, we have $F(N) \subseteq \mathrm{rad}_A(N,M)$. If $N \not\cong M$, this follows from the fact that, by (A.3.5) of the Appendix, $\mathrm{rad}_A(N,M) = \mathrm{Hom}_A(N,M)$. Assume thus $N \cong M$ and let $f : M \to M$ belong to $F(M)$. By (6.2), $\mathrm{Hom}_A(-,f)$ maps $\mathrm{Hom}_A(-,M)$ to F, which is a proper subfunctor of $\mathrm{Hom}_A(-,M)$. Consequently, the functorial morphism $\mathrm{Hom}_A(-,f) : \mathrm{Hom}_A(-,M) \longrightarrow F \longrightarrow \mathrm{Hom}_A(-,M)$ is not an isomorphism. Hence neither is f and thus $f \in \mathrm{rad}_A(M,M)$. $\square$

A nonzero functor is called **simple** if it has no nontrivial subfunctor.

Lemma 6.6 thus implies the following corollary.

6.7. Corollary. *Let M be an indecomposable A-module.*
(a) *The functor $S^M = \mathrm{Hom}_A(-,M)/\mathrm{rad}_A(-,M)$ is simple in $\mathcal{F}un^{op}A$.*
(b) *The functor $S_M = \mathrm{Hom}_A(M,-)/\mathrm{rad}_A(M,-)$ is simple in $\mathcal{F}un\,A$.*
□

In particular, $S^M(M) \cong S_M(M) \cong \mathrm{End}\,M/\mathrm{rad}\,\mathrm{End}\,M$ is a one-dimensional K-vector space (because the module M is indecomposable). By (6.2), this implies that $\mathrm{Hom}(\mathrm{Hom}_A(-,M), S^M)$ and $\mathrm{Hom}(\mathrm{Hom}_A(M,-), S_M)$ are also one-dimensional K-vector spaces and hence there exist nonzero functorial morphisms

$$\pi^M : \mathrm{Hom}_A(-,M) \longrightarrow S^M \quad \text{and} \quad \pi_M : \mathrm{Hom}_A(M,-) \longrightarrow S_M$$

that are uniquely determined up to a scalar multiple. Moreover, π^M and π_M are necessarily epimorphisms, because their targets are simple.

On the other hand, Corollary 6.7 also implies that if X is an indecomposable A-module not isomorphic to M, we have $S^M(X) = 0$ and $S_M(X) = 0$. Therefore the explicit expression of the functorial morphisms π^M and π_M follows from the proof of Yoneda's lemma, that is, if X is an indecomposable A-module, the morphisms $\pi^M(X) : \mathrm{Hom}_A(X,M) \longrightarrow S^M(X)$ and $\pi_M(X) : \mathrm{Hom}_A(M,X) \longrightarrow S_M(X)$ are both isomorphic to the canonical surjection $\mathrm{End}\,M \longrightarrow \mathrm{End}\,M/\mathrm{rad}\,\mathrm{End}\,M$ if $X \cong M$ and are zero otherwise.

Following (I.5.6), a functorial epimorphism $\varphi : F \to G$ in $\mathcal{F}un^{\mathrm{op}}A$ (or in $\mathcal{F}un\,A$) is called **minimal** if, for each functorial morphism $\psi : H \to F$, the composite morphism $\varphi\psi$ is an epimorphism if and only if ψ is an epimorphism. A minimal functorial epimorphism $\varphi : F \to G$, with F projective, is called a **projective cover** of G.

An exact sequence $F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} G \longrightarrow 0$ in $\mathcal{F}un^{\mathrm{op}}A$ (or in $\mathcal{F}unA$) is called a **projective presentation** of G. If, in addition, $\varphi_0 : F_0 \longrightarrow G$ is a projective cover and $\varphi_1 : F_1 \xrightarrow{\varphi_1} \mathrm{Im}\,\varphi_1$ is a projective cover, the sequence is called a **minimal projective presentation** of G.

We now prove the converse of Corollary 6.7, namely, we show that any simple contravariant (or covariant) functor is of the form described in (a) (or in (b), respectively) of the corollary.

6.8. Lemma. (a) *Let S be a simple object in $\mathcal{F}un^{op}A$. There exists, up to isomorphism, a unique indecomposable A-module M such that $S(M) \neq 0$. Further, $S \cong S^M$, the functorial morphism $\pi^M : \mathrm{Hom}_A(-,M) \longrightarrow S^M$ is a projective cover and $S(X) \neq 0$ if and only if M is isomorphic to a direct summand of X.*

(b) *Let S be a simple object in $\mathcal{F}un\,A$. There exists, up to isomorphism, a unique indecomposable A-module M such that $S(M) \neq 0$. Further, $S \cong$*

S_M, the functorial morphism $\pi_M : \mathrm{Hom}_A(M,-) \longrightarrow S_M$ is a projective cover, and $S(X) \neq 0$ if and only if M is isomorphic to a direct summand of X.

Proof. We only prove (a); the proof of (b) is similar. Let S be a simple functor. We first note that, by Yoneda's lemma (6.1), $S(X) \neq 0$ for some A-module X if and only if there exists a nonzero functorial morphism $\pi^X : \mathrm{Hom}_A(-,X) \longrightarrow S$ that is necessarily an epimorphism, because S is simple. Because $S \neq 0$, there exists an indecomposable A-module M such that $S(M) \neq 0$. Let X be an arbitrary module such that $S(X) \neq 0$. We thus have functorial epimorphisms $\pi^M : \mathrm{Hom}_A(-,M) \longrightarrow S$ and $\pi^X : \mathrm{Hom}_A(-,X) \longrightarrow S$. By the projectivity of the functors $\mathrm{Hom}_A(-,M)$ and $\mathrm{Hom}_A(-,X)$ (see(6.4)), we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_A(-,M) \arrow[r, "\pi^M"] \arrow[d, "\mathrm{Hom}_A(-,f)"'] & S \arrow[r] \arrow[d, "1_S"] & 0 \\
\mathrm{Hom}_A(-,X) \arrow[r, "\pi^X"] \arrow[d, "\mathrm{Hom}_A(-,g)"'] & S \arrow[r] \arrow[d, "1_S"] & 0 \\
\mathrm{Hom}_A(-,M) \arrow[r, "\pi^M"] & S \arrow[r] & 0
\end{tikzcd}$$

where the existence of the morphisms $f : M \to X$ and $g : X \to M$ follows from (6.2). Because M is indecomposable, $\mathrm{End}\, M$ is local, hence $gf \in \mathrm{End}\, M$ must be nilpotent or invertible, by (I.4.6). However, if $(gf)^m = 0$ for some $m \geq 1$, we obtain $\pi^M = \pi^M \mathrm{Hom}_A(-,(gf)^m) = 0$, a contradiction. Hence gf is invertible so that f is a section and g is a retraction. Consequently, the functorial morphism $\mathrm{Hom}_A(-,g)$ is a retraction. This shows that $\pi^M : \mathrm{Hom}_A(-,M) \longrightarrow S$ is a projective cover. The uniqueness up to isomorphism of the indecomposable module M follows from the uniqueness up to isomorphism of the projective cover and (6.4). Finally, because, by (6.6), $\mathrm{Hom}_A(-,M)$ has $\mathrm{rad}(-,M)$ as unique maximal subfunctor, we infer the existence of a functorial isomorphism $S \cong \mathrm{Hom}_A(-,M)/\mathrm{rad}_A(-,M) = S^M$. □

We have thus exhibited a bijective correspondence $M \mapsto S^M$ (or $M \mapsto S_M$) between the isomorphism classes of indecomposable A-modules and of simple objects in $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$, respectively). We now show that almost split morphisms in $\mathrm{mod}\, A$ correspond to projective presentations of these simple objects.

6.9. Lemma. (a) *Let N be an indecomposable A-module. A homomorphism $g : M \to N$ of A-modules is a right almost split morphism if and only if the induced sequence of functors*

$$\mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a projective presentation of S^N in $\mathcal{F}un^{op}A$.

(b) *Let L be an indecomposable A-module. A homomorphism $f : L \to M$ of A-modules is a left almost split morphism if and only if the induced sequence of functors*

$$\mathrm{Hom}_A(M,-) \xrightarrow{\mathrm{Hom}_A(f,-)} \mathrm{Hom}_A(L,-) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

is a projective presentation of S_L in $\mathcal{F}un\,A$.

Proof. We only prove (a); the proof of (b) is similar. Assume that g is right almost split. To prove that the induced sequence of functors is a projective presentation of S^N in $\mathcal{F}un^{\mathrm{op}}A$, it suffices, by (6.4), to prove it is exact, or equivalently, by (6.7), to prove that $\mathrm{Im}\,\mathrm{Hom}_A(-,g) = \mathrm{rad}_A(-,N)$. Thus, we must show that, for every indecomposable A-module X, $\mathrm{Im}\,\mathrm{Hom}_A(X,g) = \mathrm{rad}_A(X,N)$.

Let $h \in \mathrm{rad}_A(X,N)$. Then $h : X \to N$ is not an isomorphism. Because g is a right almost split morphism, there exists $k : X \to M$ such that $h = gk = \mathrm{Hom}_A(X,g)(k)$. Thus $\mathrm{rad}_A(X,N) \subseteq \mathrm{Im}\,\mathrm{Hom}_A(X,g)$. For the reverse inclusion, assume first $X \ncong N$, then $\mathrm{rad}_A(X,N) = \mathrm{Hom}_A(X,N)$ and clearly $\mathrm{Im}\,\mathrm{Hom}_A(X,g) \subseteq \mathrm{Hom}_A(X,N)$; on the other hand, if $X \cong N$, this follows from the fact that g is not a retraction and (1.9). We have thus shown the necessity.

For the sufficiency, assume that the given sequence of functors is exact. We must show that g is right almost split. Suppose first that g is a retraction and $g' : N \to M$ is such that $gg' = 1_N$. Then, for any $h \in \mathrm{End}\,N$, we have $h = gg'h = \mathrm{Hom}_A(N,g)(g'h) \in \mathrm{Im}\,\mathrm{Hom}_A(N,g) = \mathrm{Ker}\,\pi^N_N$. This implies that $S^N(N) = 0$, a contradiction. Hence g is not a retraction. Let X be indecomposable, and $h : X \to N$ be a nonisomorphism, that is, $h \in \mathrm{rad}_A(X,N)$. Because the given sequence of functors is exact, evaluating these functors at X yields $\mathrm{rad}_A(X,N) = \mathrm{Ker}\,\pi^N_X = \mathrm{Im}\,\mathrm{Hom}_A(X,g)$. Hence there exists $k : X \to M$ such that $h = \mathrm{Hom}_A(X,g)(k) = gk$. Thus g is right almost split. □

Furthermore, minimal almost split morphisms in $\mathrm{mod}\,A$ correspond to minimal projective presentations of simple functors, as we show in the following lemma.

6.10. Lemma. (a) *Let N be an indecomposable A-module. A homomorphism $g : M \to N$ of A-modules is a right minimal almost split morphism if and only if the induced sequence of functors*

$$\mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective presentation of S^N in $\mathcal{F}un^{op}A$.

(b) *Let L be an indecomposable A-module. A homomorphism $f : L \to M$ of A-modules is a left minimal almost split morphism if and only if the induced sequence of functors*

$$\mathrm{Hom}_A(M,-) \xrightarrow{\mathrm{Hom}_A(f,-)} \mathrm{Hom}_A(L,-) \xrightarrow{\ \pi_L\ } S_L \longrightarrow 0$$

is a minimal projective presentation of S_L in $\mathcal{F}un\, A$.

Proof. We only prove (a); the proof of (b) is similar. Assume that g is right minimal almost split. It follows from (6.9) that the induced sequence of functors is a projective presentation. We claim it is minimal, that is, by (6.6), $\mathrm{Hom}_A(-,g) : \mathrm{Hom}_A(-,M) \longrightarrow \mathrm{rad}_A(-,N)$ is a projective cover. Let thus $\varphi : \mathrm{Hom}_A(-,X) \longrightarrow \mathrm{rad}_A(-,N)$ be a functorial epimorphism. It follows from (6.4) and (6.2) that there exist morphisms $u : M \to X$ and $v : X \to M$ such that we have a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_A(-,M) \arrow[r, "\mathrm{Hom}_A(-,g)"] \arrow[d, "\mathrm{Hom}_A(-,u)"'] & \mathrm{rad}_A(-,N) \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{Hom}_A(-,X) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Hom}_A(-,v)"'] & \mathrm{rad}_A(-,N) \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{Hom}_A(-,M) \arrow[r, "\mathrm{Hom}_A(-,g)"] & \mathrm{rad}(-,N) \arrow[r] & 0
\end{tikzcd}$$

that is, $\mathrm{Hom}_A(-,g) \circ \mathrm{Hom}_A(-,v) \circ \mathrm{Hom}_A(-,u) = \mathrm{Hom}_A(-,g)$. By (6.2) again, $g(vu) = g$. Because g is right minimal, vu is an automorphism. Consequently, v is a retraction and therefore $\mathrm{Hom}_A(-,v)$ is a retraction. This shows that $\mathrm{Hom}_A(-,g) : \mathrm{Hom}_A(-,M) \longrightarrow \mathrm{rad}_A(-,N)$ is a projective cover.

Conversely, if the shown sequence of functors is a minimal projective presentation, it follows from (6.9) that g is right almost split. We must show that it is right minimal. Assume $h : M \to M$ is such that $gh = g$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_A(-,M) \arrow[r, "\mathrm{Hom}_A(-,g)"] \arrow[d, "\mathrm{Hom}_A(-,h)"'] & \mathrm{rad}_A(-,N) \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{Hom}_A(-,M) \arrow[r, "\mathrm{Hom}_A(-,g)"] & \mathrm{rad}(-,N) \arrow[r] & 0
\end{tikzcd}$$

Because $\mathrm{Hom}_A(-,g)$ is a projective cover, $\mathrm{Hom}_A(-,h)$ is an isomorphism and hence so is h. $\square$

We are now able to prove the main theorem of this section, which shows that almost split sequences in $\mathrm{mod}\, A$ correspond to minimal projective resolutions of simple functors in $\mathcal{F}un^{\mathrm{op}} A$ and in $\mathcal{F}un\, A$ defined in a usual way.

6.11. Theorem. (a) *Let N be an indecomposable A-module.*

(i) *N is projective, and $g : M \to N$ is right minimal almost split if and only if the induced sequence of functors*

$$0 \longrightarrow \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\ \pi^N\ } S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}un^{op} A$.

(ii) *N is not projective, and the sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact and almost split if and only if the induced sequence of functors*

$$0 \longrightarrow \mathrm{Hom}_A(-,L) \xrightarrow{\mathrm{Hom}_A(-,f)} \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\ \pi^N\ } S^N \longrightarrow 0$$

(*where $L \neq 0$*) *is a minimal projective resolution of S^N in $\mathcal{F}un^{op} A$.*

(b) *Let L be an indecomposable A-module.*

(i) *L is injective, and $f : L \to M$ is left minimal almost split if and only if the induced sequence of functors*

$$0 \longrightarrow \mathrm{Hom}_A(M,-) \xrightarrow{\mathrm{Hom}_A(f,-)} \mathrm{Hom}_A(L,-) \xrightarrow{\ \pi_L\ } S_L \longrightarrow 0$$

is a minimal projective resolution of S_L in $\mathcal{F}un\, A$.

(ii) *L is not injective, and the sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is exact and almost split if and only if the induced sequence of functors*

$$0 \longrightarrow \mathrm{Hom}_A(N,-) \xrightarrow{\mathrm{Hom}_A(g,-)} \mathrm{Hom}_A(M,-) \xrightarrow{\mathrm{Hom}_A(f,-)} \mathrm{Hom}_A(L,-) \xrightarrow{\ \pi_L\ } S_L \longrightarrow 0$$

(*where $N \neq 0$*) *is a minimal projective resolution of S_L in $\mathcal{F}un\, A$.*

Proof. We only prove (a); the proof of (b) is similar.

(i) Assume that N is projective, and $g : M \to N$ is right minimal almost split. By (3.5), g is a monomorphism with image equal to $\operatorname{rad} N$. By the left exactness of the Hom functor, $\mathrm{Hom}_A(-,g) : \mathrm{Hom}_A(-,M) \longrightarrow \mathrm{Hom}_A(-,N)$ is a monomorphism. Thus, it follows from (6.10) that the induced sequence of functors

$$0 \longrightarrow \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\ \pi^N\ } S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}un^{\mathrm{op}} A$. Conversely, if the sequence of functors is a minimal projective resolution of S^N in $\mathcal{F}un^{\mathrm{op}} A$, it follows from (6.10) that g is right minimal almost split. Evaluating the sequence of functors at A_A yields that g is a monomorphism. But, by the description of right minimal almost split morphisms in (3.1) and (3.2), this implies that N is projective.

(ii) Assume that N is not projective, and let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an almost split sequence. By the left exactness of the Hom functor, we derive an exact sequence of projective functors

$$0 \longrightarrow \mathrm{Hom}_A(-,L) \xrightarrow{\mathrm{Hom}_A(-,f)} \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N).$$

Because $g : M \to N$ is right minimal almost split, (6.10) yields that the induced sequence of functors

$$0 \longrightarrow \mathrm{Hom}_A(-,L) \xrightarrow{\mathrm{Hom}_A(-,f)} \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}un^{\mathrm{op}} A$. Conversely, assume that the sequence of functors (where $L \neq 0$) is a minimal projective resolution of S^N in $\mathcal{F}un^{\mathrm{op}} A$. First, we claim that N is not projective. Indeed, if this were the case, then S^N has, by (a), a minimal projective resolution of the form

$$0 \longrightarrow \mathrm{Hom}_A(-,\mathrm{rad}\, N) \longrightarrow \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0,$$

where the first morphism is induced from the canonical inclusion of $\mathrm{rad}\, N$ into N. We thus have a short exact sequence of functors

$$0 \longrightarrow \mathrm{Hom}_A(-,L) \xrightarrow{\mathrm{Hom}_A(-,f)} \mathrm{Hom}_A(-,M) \longrightarrow \mathrm{Hom}_A(-,\mathrm{rad}\, N) \longrightarrow 0$$

that splits, because $\mathrm{Hom}_A(-,\mathrm{rad}\, N)$ is projective. In particular, the morphism $\mathrm{Hom}_A(-,f)$ is a section, a contradiction to the minimality of the given projective resolution. This shows our claim that N is not projective. In particular, N is not isomorphic to a direct summand of A_A hence, by (6.8), $S^N(A_A) = 0$. Evaluating the given projective resolution at A_A yields a short exact sequence of A-modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0,$$

where, by (6.10), g is right minimal almost split. But this implies, by (1.13), that the sequence is almost split. $\square$

It is useful to observe that it follows from (6.11)(a) that, for any projective A-module P, there exists a functorial isomorphism $\mathrm{rad}_A(-,P) \cong \mathrm{Hom}_A(-,\mathrm{rad}\, P)$. Dually, for any injective A-module I, there exists a functorial isomorphism $\mathrm{rad}_A(I,-) \cong \mathrm{Hom}_A(I/\mathrm{soc}\, I,-)$.

IV.7. Exercises

1. Let $f : M \longrightarrow N$ be a homomorphism in $\operatorname{mod} A$. Show that the following conditions are equivalent:

(a) For every epimorphism $h : L \longrightarrow N$, there exists $g : M \longrightarrow L$ such that $f = hg$.

(b) For every epimorphism $h : L \longrightarrow N$ with L projective there exists $g : M \longrightarrow L$ such that $f = hg$.

(c) $f \in \mathcal{P}(M, N)$, that is, f factors through a projective A-module.

2. State and prove the dual of Exercise 1.

3. Let M be a left A-module without projective direct summand. Show that there is a functorial isomorphism $\underline{\operatorname{Hom}}_{A^{\mathrm{op}}}(M, -) \cong \operatorname{Tor}_1^{A^{\mathrm{op}}}(M, -)$.

4. Let p be a prime, $n > 0$, and $\mathbb{Z}_{p^j} = \mathbb{Z}/(p^j)$. Show that the exact sequence in $\operatorname{mod} \mathbb{Z}$

$$0 \longrightarrow \mathbb{Z}_{p^n} \xrightarrow{\left[\begin{smallmatrix} u_n \\ \pi_n \end{smallmatrix}\right]} \mathbb{Z}_{p^{n+1}} \oplus \mathbb{Z}_{p^{n-1}} \xrightarrow{[\pi_{n+1}\ u_{n-1}]} \mathbb{Z}_{p^n} \longrightarrow 0$$

is almost split, where $u_j : \mathbb{Z}_{p^j} \to \mathbb{Z}_{p^{j+1}}$ is the monomorphism given by $\overline{x} \mapsto \overline{px}$ and $\pi_j : \mathbb{Z}_{p^j} \to \mathbb{Z}_{p^{j-1}}$ is the canonical epimorphism.

5. Let M be an indecomposable nonprojective right A-module and let $\xi : 0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$ be a nonsplit exact sequence. Show that the following conditions are equivalent:

(a) ξ is almost split.

(b) For every homomorphism $u : \tau M \longrightarrow U$ that is not a section, we have $\operatorname{Ext}_A^1(M, u)(\xi) = 0$.

(c) For every homomorphism $v : V \longrightarrow M$ that is not a retraction, we have $\operatorname{Ext}_A^1(v, \tau M)(\xi) = 0$.

6. Let M be an indecomposable nonprojective right A-module and let $\xi : 0 \xrightarrow{f} \tau M \xrightarrow{g} E \longrightarrow M \longrightarrow 0$ be a nonsplit exact sequence. Show that the following conditions are equivalent:

(a) The sequence ξ is almost split.

(b) For every indecomposable A-module U and every nonisomorphism $u : \tau M \longrightarrow U$, there exists $\overline{u} : E \longrightarrow U$ such that $\overline{u} f = u$.

(c) For every indecomposable A-module V and every nonisomorphism $v : V \longrightarrow M$, there exists $\overline{v} : V \longrightarrow E$ such that $g\overline{v} = v$.

7. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an almost split sequence in $\operatorname{mod} A$. Prove the following statements:

(a) If N' is a nonzero proper submodule of N, then the short exact sequence $0 \longrightarrow L \longrightarrow g^{-1}(N') \longrightarrow N' \longrightarrow 0$ is split.

(b) If L' is a nonzero submodule of L, then the short exact sequence $0 \longrightarrow L/L' \longrightarrow M/f(L') \longrightarrow N' \longrightarrow 0$ is split.

8. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an almost split sequence in $\operatorname{mod} A$. Prove the following statements:

(a) For every nonsplit exact sequence $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} N \longrightarrow 0$ and every commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & N & \longrightarrow & 0 \end{array}$$

there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle k'} & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

such that $h'h = 1_L$ and $k'k = 1_M$. In particular, h and k are sections.

(b) For every nonsplit exact sequence $0 \longrightarrow L \xrightarrow{u} X \xrightarrow{v} Y \longrightarrow 0$ and every commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{u} & X & \xrightarrow{v} & Y & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle k'} & & \\ 0 & \longrightarrow & L & \xrightarrow{u} & X & \xrightarrow{v} & Y & \longrightarrow & 0 \end{array}$$

such that $hh' = 1_M$ and $kk' = 1_N$. In particular, h and k are retractions.

9. Let $\xi : 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a nonsplit short exact sequence in $\operatorname{mod} A$. Prove the following statements:

(a) The homomorphism f is irreducible if and only if

(i) $\operatorname{Im} f$ is a direct summand of every proper submodule M' of M such that $\operatorname{Im} f \subseteq M'$, and

(ii) if X is an A-module and $\eta \in \mathrm{Ext}_A^1(N, X)$, then either there exists an A-module homomorphism $u : X \longrightarrow L$ such that $\mathrm{Ext}_A^1(N, u)(\eta) = \xi$ or an A-module homomorphism $v : L \longrightarrow X$ such that $\mathrm{Ext}_A^1(N, v)(\xi) = \eta$.

(b) The homomorphism g is irreducible if and only if

(i) $g : M/L' \longrightarrow N$ is a retraction if L' is a nonzero submodule of $L = \mathrm{Ker}\, g$, and

(ii) if X is a module and $\eta \in \mathrm{Ext}_A^1(X, L)$, then either there exists a homomorphism $u : N \to X$ such that $\mathrm{Ext}_A^1(u, L)(\eta) = \xi$ or a homomorphism $v : X \to N$ such that $\mathrm{Ext}_A^1(v, L)(\xi) = \eta$.

10. (a) Let $f : L \longrightarrow M$ be an irreducible monomorphism in $\mathrm{mod}\, A$, with M indecomposable. Let $h : X \longrightarrow N$ be an irreducible morphism, where $N = \mathrm{Coker}\, f$. Show that h is an epimorphism.

(b) Let $g : M \longrightarrow N$ be an irreducible epimorphism in $\mathrm{mod}\, A$, with M indecomposable. Let $h : L \longrightarrow X$ be an irreducible morphism, where $L = \mathrm{Ker}\, g$. Show that h is a monomorphism.

11. Let $f : L \longrightarrow M$ be an irreducible morphism in $\mathrm{mod}\, A$, and X be a right A-module.

(a) Show that $\mathrm{Ext}_A^1(X, f) : \mathrm{Ext}_A^1(X, L) \to \mathrm{Ext}_A^1(X, M)$ is a monomorphism, if $\mathrm{Hom}_A(M, X) = 0$.

(b) Show that $\mathrm{Ext}_A^1(f, X) : \mathrm{Ext}_A^1(M, X) \to \mathrm{Ext}_A^1(L, X)$ is a monomorphism, if $\mathrm{Hom}_A(X, L) = 0$.

12. Let $g : M \longrightarrow N$ be a right almost split epimorphism. If $\mathrm{Ker}\, g$ is not indecomposable, show that there exists a right almost split morphism $g_1 : M_1 \longrightarrow N$ such that $\ell(M_1) < \ell(M)$. Deduce that if M is of minimal length such that there exists a right almost split epimorphism $g : M \longrightarrow N$, then the short exact sequence $0 \longrightarrow \mathrm{Ker}\, g \longrightarrow M \xrightarrow{g} N \longrightarrow 0$ is almost split.

13. State and prove the dual of Exercise 12.

14. Let $0 \longrightarrow \tau M \longrightarrow \bigoplus_{i=1}^n E_i \longrightarrow M \longrightarrow 0$ be an almost split sequence, with the E_i indecomposable. Show that, for every i, we have $\ell(E_i) \neq \ell(M)$ and $\ell(E_i) \neq \ell(\tau M)$ so that no E_i is isomorphic to M or τM.

15. Let X be a nonzero module in $\mathrm{mod}\, A$. Show that there exists at most finitely many nonisomorphic almost split sequences

$$0 \longrightarrow L_i \longrightarrow M_i \longrightarrow N_i \longrightarrow 0$$

with X isomorphic to a direct summand of M_i.

16. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an almost split sequence in the category $\mathrm{mod}\, A$ and suppose that M is not indecomposable. Show that $\mathrm{Hom}_A(L, N) \neq 0$.

17. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an almost split sequence in the category $\operatorname{mod} A$. Show that if P is a nonzero projective module, the following conditions are equivalent:

(a) P is isomorphic to a direct summand of M.

(b) There exists an irreducible morphism $P \longrightarrow N$.

(c) There exists an irreducible morphism $L \longrightarrow P$.

(d) L is isomorphic to a direct summand of $\operatorname{rad} P$.

(e) There is an indecomposable direct summand R of $\operatorname{rad} P$ such that $N \cong \tau^{-1} R$.

(f) If $f : X \longrightarrow N$ is an epimorphism in $\operatorname{mod} A$ that is not a retraction, then P is isomorphic to a direct summand of X.

18. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an almost split sequence in $\operatorname{mod} A$. Prove the following statements:

(a) If there exists an irreducible epimorphism $h : P \longrightarrow N$ with P indecomposable projective, then $N \cong P/S$, where S is a simple submodule of P.

(b) If $N/\operatorname{rad} N$ is simple and M has a nonzero projective direct summand, there exists an irreducible epimorphism $h : P \longrightarrow N$, with P indecomposable projective.

19. Let A be the K-algebra of Example 4.13. Let M and N be the simple A-modules such that $\mathbf{dim}\, M = \begin{smallmatrix} & & 0 & \\ 0 & 1 & & 0 \\ 0 & & 0 & \end{smallmatrix}$ and $\mathbf{dim}\, N = \begin{smallmatrix} & & 0 & \\ 0 & 0 & & 1 \\ 0 & & 0 & \end{smallmatrix}$. Show that $\mathbf{dim}\, \tau M = \begin{smallmatrix} & & 0 & \\ 1 & 1 & & 0 \\ 1 & & 0 & \end{smallmatrix}$, and that $\operatorname{Hom}_A(DA, \tau M) = 0$.

20. Let A be given by the quiver
$$\begin{tikzcd} \underset{2'}{\circ} \arrow[r, shift left, "\alpha'"] & \underset{1'}{\circ} \arrow[l, shift left, "\beta'"] & & \underset{1}{\circ} \arrow[ll, "\gamma"'] \arrow[r, shift left, "\alpha"] & \underset{2}{\circ} \arrow[l, shift left, "\beta"] \end{tikzcd}$$
bound by the relations $\beta\alpha = 0$, $\beta'\alpha' = 0$, and $\alpha\beta\gamma = \gamma\beta'\alpha'$. Show that $P(1) \cong I(1')$, $P(2) \cong I(2')$ and deduce the almost split sequences having as middle terms $P(1)$ and $P(2)$, respectively.

21. Construct the Auslander–Reiten quiver of the algebra defined by each of the following bound quivers:

(a)
$$\begin{tikzcd} & \circ \arrow[dl, "\beta"'] & \\ \circ & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & \circ \arrow[ul, "\delta"] & \end{tikzcd} \qquad \alpha\beta = \gamma\delta;$$

(b)
$$\begin{tikzcd} & \circ \arrow[dl, "\beta"'] & \\ \circ & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & \circ \arrow[ul, "\delta"] & \end{tikzcd} \qquad \alpha\beta = 0,\ \gamma\delta = 0;$$

(c)
$$\begin{tikzcd} & \circ \arrow[dl, "\beta"'] & \\ \circ & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & \circ \arrow[ul, "\delta"] & \end{tikzcd} \qquad \alpha\beta = 0.$$

In each case describe the structure of each indecomposable module.

22. Construct the Auslander–Reiten quiver of the algebra defined by each of the following bound quivers:

(a)
$$\begin{tikzcd}
& \circ \arrow[dl, "\beta"'] & \\
\circ & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
& \circ \arrow[ul, "\delta"] \arrow[dl, "\varepsilon"] & \\
\circ & &
\end{tikzcd}$$
$\alpha\beta = 0,\ \gamma\delta = 0;$

(b)
$$\begin{tikzcd}
& & \circ \arrow[dl, "\gamma"'] & & \\
\circ & \circ \arrow[l, "\lambda"'] & & \circ \arrow[ul, "\beta"'] \arrow[dl, "\delta"] & \circ \arrow[l, "\alpha"'] \\
& & \circ \arrow[ul, "\varepsilon"] & &
\end{tikzcd}$$
$\alpha\beta = 0,\ \gamma\lambda = 0,\ \beta\gamma = \delta\varepsilon;$

(c)
$$\begin{tikzcd}
\circ & & & \\
& \circ \arrow[ul, "\lambda"'] \arrow[dl, "\xi"] & \circ \arrow[l, "\alpha"'] & \circ \arrow[l, "\mu"'] \\
\circ & & &
\end{tikzcd}$$
$\mu\alpha = 0;$

(d)
$$\begin{tikzcd}
& & \circ \arrow[dll, "\beta"'] & & \\
\circ & & & & \circ \arrow[ull, "\alpha"'] \arrow[dl, "\gamma"] \\
& \circ \arrow[ul, "\varepsilon"] & & \circ \arrow[ll, "\delta"] &
\end{tikzcd}$$
$\alpha\beta = 0,\ \gamma\delta = 0,\ \delta\varepsilon = 0;$

(e)
$$\begin{tikzcd}
\circ & & \circ \arrow[dl, "\mu"'] & \\
\circ & \circ \arrow[l, "\eta"] \arrow[ul, "\lambda"'] & & \circ \arrow[ul, "\nu"'] \arrow[ll, "\xi"]
\end{tikzcd}$$
$\xi\eta = 0,\ \mu\lambda = 0,\ \nu\mu = 0;$

(f)
$$\begin{tikzcd}
\circ & \circ \arrow[l, "\delta"'] \arrow[d, "\varepsilon"'] & \circ \arrow[l, "\gamma"'] & \\
& \circ & \circ \arrow[l, "\beta"] & \circ \arrow[l, "\alpha"]
\end{tikzcd}$$
$\gamma\delta = 0,\ \alpha\beta = 0;$

(g)
$$\begin{tikzcd}
& & \circ \arrow[dl, "\beta"'] & & \\
& \circ \arrow[dl, "\varepsilon"'] & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] & \\
\circ & & \circ \arrow[ul, "\delta"] & & \circ \arrow[ul, "\lambda"'] \arrow[dll, "\mu"] \\
& & \circ \arrow[ull, "\nu"] & &
\end{tikzcd}$$
$\alpha\beta = \gamma\delta,\ \lambda\alpha\beta\varepsilon = \mu\nu;$

(h)
$$\begin{tikzcd}
& & \circ \arrow[dl, "\gamma"'] & & \\
\circ & \circ \arrow[l, "\delta"'] & & \circ \arrow[ul, "\beta"'] \arrow[dl, "\eta"] & \\
& & \circ \arrow[ul, "\nu"] & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\lambda"] \\
& & & \circ \arrow[ul, "\mu"] &
\end{tikzcd}$$
$\alpha\eta = \lambda\mu,\ \beta\gamma = \eta\nu;$

(i)
$$\begin{tikzcd}
& \circ \arrow[dl, "\beta"'] \arrow[dd, "\mu"'] & \\
\circ & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
& \circ \arrow[ul, "\delta"] &
\end{tikzcd}$$
$\alpha\beta = \gamma\delta$, $\alpha\mu = 0$, $\mu\delta = 0$;

(j)
$$\begin{tikzcd}
\circ & & & \\
& \circ \arrow[ul, "\mu"'] & & \\
& & \circ \arrow[ul, "\lambda"'] \arrow[dl, "\beta"'] & \\
\circ & \circ \arrow[l, "\varepsilon"'] & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
& & \circ \arrow[ul, "\delta"] &
\end{tikzcd}$$
$\alpha\beta = \gamma\delta$, $\alpha\beta\varepsilon = 0$;

(k)
$$\begin{tikzcd}
& \circ \arrow[dl, "\gamma"'] & & \\
\circ & & \circ \arrow[ul, "\beta"'] \arrow[dl, "\eta"] & \\
& \circ \arrow[ul, "\nu"] & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\lambda"] \\
& & \circ \arrow[ul, "\mu"] &
\end{tikzcd}$$
$\alpha\eta = \lambda\mu$, $\beta\gamma = \eta\nu$, $\alpha\beta = 0$, $\mu\nu = 0$;

(l)
$$\begin{tikzcd}
\circ \arrow[r, shift left, "\alpha"] & \circ \arrow[l, shift left, "\beta"]
\end{tikzcd}$$
$\alpha\beta = 0$;

(m)
$$\begin{tikzcd}
\circ \arrow[r, shift left, "\alpha"] & \circ \arrow[l, shift left, "\beta"]
\end{tikzcd}$$
$\alpha\beta = 0$, $\beta\alpha = 0$.

23. Let Q be either of the following quivers:

(a)
$$\begin{tikzcd}
\circ & \circ \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

(b)
$$\begin{tikzcd}
\circ \arrow[dr] & & \circ \arrow[dl] \\
& \circ & \\
\circ \arrow[ur] & & \circ \arrow[ul]
\end{tikzcd}$$

Construct the component of the Auslander–Reiten quiver of the path K-algebra $A = KQ$ containing the indecomposable projective modules, and show that it contains no injective modules.

24. Let A be a K-algebra such that $\operatorname{rad}_A^m = \operatorname{rad}_{\operatorname{mod} A}^m = 0$ for some $m \geq 1$. Prove that any nonzero nonisomorphism between indecomposable modules in $\operatorname{mod} A$ is a sum of compositions of irreducible morphisms.

Hint: Follow the proof of (5.6).

25. Complete the proof of Proposition 2.10.

26. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a nonsplit short exact sequence in $\operatorname{mod} A$. Prove the following statements:

(a) f is irreducible if and only if, for every subfunctor F of the functor $\operatorname{Hom}_A(-, N)$, F either contains or is contained in the image of the functorial morphism $\operatorname{Hom}_A(-, g) : \operatorname{Hom}_A(-, M) \longrightarrow \operatorname{Hom}_A(-, N)$.

(b) g is irreducible if and only if, for every subfunctor F of $\operatorname{Hom}_A(L, -)$, F either contains or is contained in the image of the functorial morphism $\operatorname{Hom}_A(f, -) : \operatorname{Hom}_A(M, -) \longrightarrow \operatorname{Hom}_A(L, -)$.