

Interesting Combinatorics In Higher Auslander Theory

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Fundamentals

Algebras, Modules, Quivers

Auslander-Reiten Theory

Path Algebras, Representations, AR Quivers

Higher Auslander Algebras

Cluster Tilting Modules, Cyclic Polytopes

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Algebras, Modules, Quivers

k -algebra

An algebra over a field k is a k -vector space equipped with a **bilinear product**.

Motivating examples

- Complex numbers as the vector space \mathbb{R}^2 with the typical product of complex numbers.
- Ring of polynomials (over k) with polynomial multiplication.
- Ring of square matrices with matrix multiplication.

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Λ -module

Let Λ be a k -algebra. A right Λ -module is a pair (M, \cdot) where M is a k -vector space and $\cdot : M \times \Lambda \rightarrow M$ is a binary operation satisfying natural commutativity and associativity rules.

Examples

- Each algebra is a module (left or right) over itself.
- $k[x, y] = (k[x])[y]$ is a module (left or right) over $k[x]$.

Indecomposability ('prime' modules)

A (right) Λ -module M is **indecomposable** if $M \neq 0$ and $M = M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$.

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Module homomorphisms

Λ -module homomorphism

A map $f : M \rightarrow N$ between two (right) Λ -modules M and N is a Λ -module homomorphism if it's k -linear and respects \cdot , that is

$$f(m \cdot \lambda) = f(m) \cdot \lambda \text{ for } \lambda \in \Lambda, m \in M.$$

Section/retraction

A Λ -module homomorphism $f : M \rightarrow N$ is

- a **section** if $\exists g : N \rightarrow M$ such that $g \circ f = 1_M$.
- a **retraction** if $\exists h : N \rightarrow M$ such that $f \circ h = 1_N$.

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Module homomorphisms

Irreducibility ('prime' homomorphisms)

A Λ -module homomorphism $f : M \rightarrow N$ is **irreducible** if

- f is neither a **section** nor a **retraction**;
- whenever $f = f_2 \circ f_1$, then f_2 is a retraction or f_1 is a section.

We denote the k -vector space of irreducible homomorphisms $M \rightarrow N$ as $\text{Irr}(M, N)$.



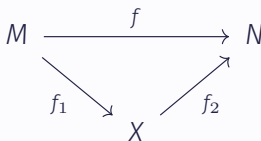
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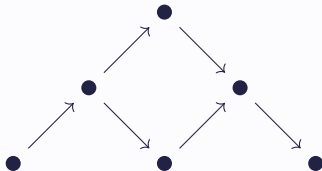


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Auslander-Reiten Theory

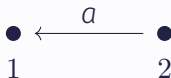
Path Algebras, Representations, AR Quivers

The path algebra of a quiver

Let Q be a quiver. The **path algebra** kQ of Q is the k -algebra whose k -vector space has as its basis all paths of length ≥ 0 in Q and the product of two basis elements is the concatenation of paths.

Path algebras – Example

Consider the quiver



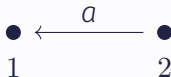
The basis of the path algebra kQ is the triple (e_1, e_2, a) (where e_i means ‘stay at i ’) and its multiplication table is

	e_1	e_2	a
e_1	e_1	0	0
e_2	0	e_2	a
a	a	0	0

It's actually isomorphic to the k -algebra of lower triangular 2×2 matrices over k .

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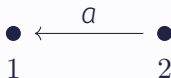
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Every algebra is a path algebra

Theorem

Let k be an algebraically closed field and Λ a basic, connected and finite-dimensional algebra over k . Then there exists a finite connected quiver Q such that $\Lambda = kQ/I$ for some admissible ideal I of kQ .

Theorem

*Let Λ be as above. Then, $\Lambda = kQ$ for a quiver Q if and only if Λ is **hereditary** (submodules of projective modules are projective).*

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Theorem (Gabriel's)

Let Λ be as above. Then, $\Lambda = kQ$ where Q (as an undirected graph) is **Dynkin** if and only if Λ is **representation-finite** (the number, up to isomorphism, of indecomposable finite-dimensional Λ -modules is finite).

Dynkin quivers

$$A_n : \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet$$

$$D_n : \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \begin{array}{l} \nearrow \bullet \\ \searrow \bullet \end{array}$$

$$E_6 : \begin{array}{c} \bullet \\ | \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array}$$

$$E_7 : \begin{array}{c} \bullet \\ | \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array}$$

$$E_8 : \begin{array}{c} \bullet \\ | \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array}$$

The category $\text{mod } \Lambda$

All the (right) Λ -modules and the homomorphisms between them form an **abelian category**, which we denote $\text{mod } \Lambda$.

We wish to record the data of $\text{mod } \Lambda$ in the form of a quiver.

The vertices are (isomorphism classes of) indecomposable Λ -modules and the number of arrows between two Λ -modules M, N is the dimension of $\text{Irr}(M, N)$.

We call this quiver, the **Auslander-Reiten quiver** of Λ .

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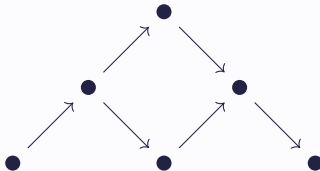
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The Auslander-Reiten quiver of the path algebra $k(\bullet \leftarrow \bullet \leftarrow \bullet)$ is



Higher Auslander Algebras

Cluster Tilting Modules, Cyclic Polytopes

Module extensions

The group of extensions

An **extension** of a Λ -module M by a Λ -module N is a **short exact sequence**

$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0.$$

All the (equivalence classes of) extensions of M by N form a group which we denote $\text{Ext}^1(M, N)$.

This construction can be extended to ' n -fold' extensions of M by N , that is, **long exact sequences**

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Cluster tilting modules

$\text{add } M$

For a Λ -module M , we denote by $\text{add } M$ the full subcategory of $\text{mod } \Lambda$ whose objects are the direct sums of direct summands of M .

d -cluster tilting module

A Λ -module M is d -cluster tilting if

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Construction (due to Iyama O.)

Let Λ be a representation-finite hereditary algebra. Then, Λ has a 1-cluster tilting module $M^{(1)}$. Put $\Lambda^{(1)} := \text{End}(M^{(1)})$.

The k -algebra $\Lambda^{(1)}$ has a 2-cluster tilting module $M^{(2)}$. Put $\Lambda^{(2)} := \text{End}(M^{(2)})$.

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The k -algebra $\Lambda^{(d)}$ has a d -cluster tilting module $M^{(d)}$. Put $\Lambda^{(d+1)} := \text{End}(M^{(d)})$.

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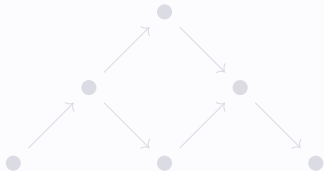
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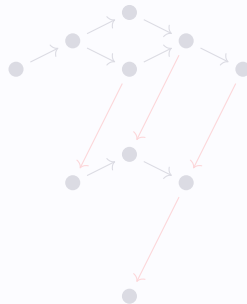
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Higher Auslander algebras of type A

We apply Iyama's construction to $\Lambda := k(\bullet \leftarrow \bullet \leftarrow \bullet)$. The AR quivers of Λ , $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are the following.



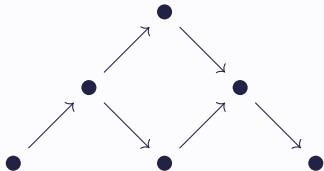
The AR quiver of Λ .



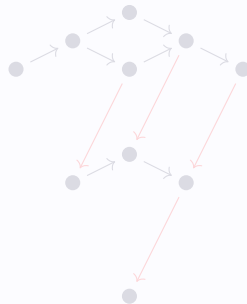
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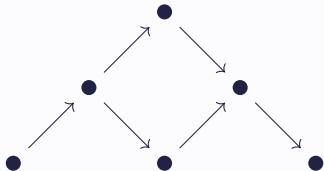
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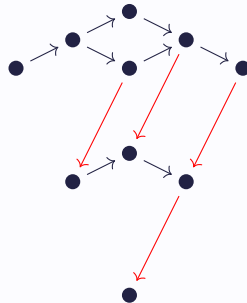
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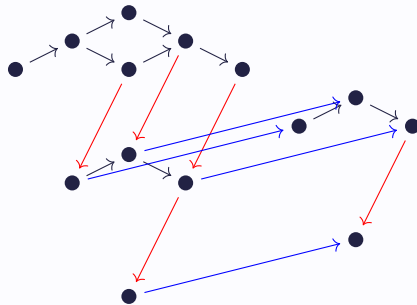


The AR quiver of Λ .



The AR quiver of $\Lambda^{(1)}$.

Higher Auslander algebras of type A



The AR quiver of $\Lambda^{(2)}$.

Cyclic polytopes

Cyclic polytope

The **moment curve** is the map $p(t) = (t, t^2, t^3, \dots, t^n) \subseteq \mathbb{R}^n$ for $t \in \mathbb{R}$. A **cyclic polytope** $C(m, n)$ in \mathbb{R}^n is the convex hull of the set $\{p(t_1), \dots, p(t_m)\}$ where $t_1 < t_2 < \dots < t_m$.

Example. The moment curve in \mathbb{R}^2 is just a parabola and $C(4, 2)$ is the following shape (for $t_i = i - 1$).

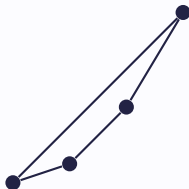


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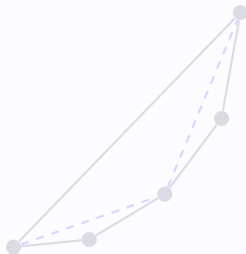


Triangulations and tilting modules

A triangulation of a cyclic polytope

By a **triangulation** of $C(m, n)$ we mean its division into n -dimensional simplices that share vertices with $C(m, n)$.

Example. One possible triangulation of $C(5, 2)$.

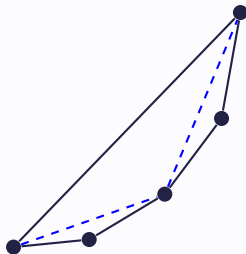


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Theorem (Thomas H., Oppermann S.)

There is a bijection

$$\left\{ \begin{array}{l} \text{Triangulations} \\ \text{of } C(n + 2d, 2d) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Tilting modules of } \Lambda^{(d)} \\ \text{contained in } \text{add } M^{(d)} \end{array} \right\}$$