Interesting Combinatorics In Higher Auslander Theory

Adam Klepáč

10th Day of Doctoral Students of the School of Mathematics

Charles University in Prague

Outline

Fundamentals

Algebras, Modules, Quivers

Auslander-Reiten Theory

Path Algebras, Representations, AR Quivers

Higher Auslander Algebras

Cluster Tilting Modules, Cyclic Polytopes

Fundamentals

Algebras, Modules, Quivers

Algebras

k-algebra

An algebra over a field k is a k-vector space equipped with a bilinear product.

Motivating examples

- Complex numbers as the vector space $\mathbb{R}^2$ with the typical product of complex numbers.
- Ring of polynomials (over k) with polynomial multiplication.
- Ring of square matrices with matrix multiplication.

k-algebra

An algebra over a field k is a k-vector space equipped with a bilinear product.

Motivating examples

- Complex numbers as the vector space $\mathbb{R}^2$ with the typical product of complex numbers.
- Ring of polynomials (over k) with polynomial multiplication.
- Ring of square matrices with matrix multiplication.

k-algebra

An algebra over a field k is a k-vector space equipped with a bilinear product.

Motivating examples

- Complex numbers as the vector space $\mathbb{R}^2$ with the typical product of complex numbers.
- Ring of polynomials (over k) with polynomial multiplication.
- Ring of square matrices with matrix multiplication.

k-algebra

An algebra over a field k is a k-vector space equipped with a bilinear product.

Motivating examples

- Complex numbers as the vector space $\mathbb{R}^2$ with the typical product of complex numbers.
- Ring of polynomials (over k) with polynomial multiplication.
- Ring of square matrices with matrix multiplication.

Modules

Λ-module

Let Λ be a k-algebra. A right Λ-module is a pair $(M, \cdot)$ where M is a k-vector space and $\cdot : M \times A \to M$ is a binary operation satisfying natural commutativity and associativity rules.

Examples

- Each algebra is a module (left or right) over itself.
- $k[x, y] = (k[x])[y]$ is a module (left or right) over $k[x]$.

Indecomposability ('prime' modules)

A (right) Λ-module M is indecomposable if $M \neq 0$ and $M = M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$.

Λ-module

Let Λ be a k-algebra. A right Λ-module is a pair $(M, \cdot)$ where M is a k-vector space and $\cdot : M \times A \to M$ is a binary operation satisfying natural commutativity and associativity rules.

Examples

- Each algebra is a module (left or right) over itself.
- $k[x, y] = (k[x])[y]$ is a module (left or right) over $k[x]$.

Indecomposability ('prime' modules)

A (right) Λ-module M is indecomposable if $M \neq 0$ and $M = M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$.

Λ-module

Let Λ be a k-algebra. A right Λ-module is a pair $(M, \cdot)$ where M is a k-vector space and $\cdot : M \times A \to M$ is a binary operation satisfying natural commutativity and associativity rules.

Examples

- Each algebra is a module (left or right) over itself.
- $k[x, y] = (k[x])[y]$ is a module (left or right) over $k[x]$.

Indecomposability ('prime' modules)

A (right) Λ-module M is indecomposable if $M \neq 0$ and $M = M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$.

Modules

Λ-module

Let Λ be a k-algebra. A right Λ-module is a pair $(M, \cdot)$ where M is a k-vector space and $\cdot : M \times A \to M$ is a binary operation satisfying natural commutativity and associativity rules.

Examples

- Each algebra is a module (left or right) over itself.
- $k[x, y] = (k[x])[y]$ is a module (left or right) over $k[x]$.

Indecomposability ('prime' modules)

A (right) Λ-module M is indecomposable if $M \neq 0$ and $M = M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$.

Module homomorphisms

Λ-module homomorphism

A map $f : M \to N$ between two (right) Λ-modules M and N is a Λ-module homomorphism if it's k-linear and respects $\cdot$, that is

$$f(m \cdot \lambda) = f(m) \cdot \lambda \text{ for } \lambda \in \Lambda, m \in M.$$

Section/retraction

A Λ-module homomorphism $f : M \to N$ is

- a section if $\exists g : N \to M$ such that $g \circ f = 1_N$.
- a retraction if $\exists h : N \to M$ such that $f \circ h = 1_M$.

Λ-module homomorphism

A map $f : M \to N$ between two (right) Λ-modules M and N is a Λ-module homomorphism if it's k-linear and respects $\cdot$, that is

$$f(m \cdot \lambda) = f(m) \cdot \lambda \text{ for } \lambda \in \Lambda, m \in M.$$

Section/retraction

A Λ-module homomorphism $f : M \to N$ is

- a section if $\exists g : N \to M$ such that $g \circ f = 1_N$.
- a retraction if $\exists h : N \to M$ such that $f \circ h = 1_M$.

Irreducibility (‘prime’ homomorphisms)

A Λ-module homomorphism $f : M \to N$ is irreducible if

- f is neither a **section** nor a **retraction**;
- whenever $f = f_2 \circ f_1$, then f_2 is a retraction or f_1 is a section.

We denote the k-vector space of irreducible homomorphisms $M \to N$ as $\mathrm{Irr}(M, N)$.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "f_1"'] & & N \\ & X \arrow[ur, "f_2"'] & \end{tikzcd}$$

Irreducibility (‘prime’ homomorphisms)

A Λ-module homomorphism $f : M \to N$ is irreducible if

- f is neither a **section** nor a **retraction**;
- whenever $f = f_2 \circ f_1$, then f_2 is a retraction or f_1 is a section.

We denote the k-vector space of irreducible homomorphisms $M \to N$ as $\mathrm{Irr}(M, N)$.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "f_1"'] & & N \\ & X \arrow[ur, "f_2"'] & \end{tikzcd}$$

Quivers

Quiver

A quiver is an oriented graph with multiple edges and loops.

Examples

Quiver

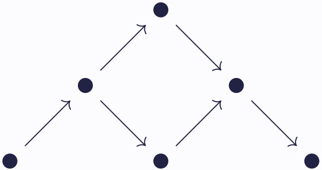
A quiver is an oriented graph with multiple edges and loops.

Examples

Quiver

A quiver is an oriented graph with multiple edges and loops.

Examples



Auslander-Reiten Theory

Path Algebras, Representations, AR Quivers

The path algebra of a quiver

Let Q be a quiver. The path algebra kQ of Q is the k-algebra whose k-vector space has as its basis all paths of length ≥ 0 in Q and the product of two basis elements is the concatenation of paths.

Path algebras – Example

Consider the quiver

$$\underset{1}{\bullet} \xleftarrow{\;a\;} \underset{2}{\bullet}$$

The basis of the path algebra kQ is the triple (e_1, e_2, a) (where e_i means 'stay at i') and its multiplication table is

	e_1	e_2	a
e_1	e_1	0	0
e_2	0	e_2	a
a	a	0	0

It's actually isomorphic to the k-algebra of lower triangular 2×2 matrices over k.

Consider the quiver

$$\underset{1}{\bullet} \xleftarrow{\ a\ } \underset{2}{\bullet}$$

The basis of the path algebra kQ is the triple (e_1, e_2, a) (where e_i means 'stay at i') and its multiplication table is

	e_1	e_2	a
e_1	e_1	0	0
e_2	0	e_2	a
a	a	0	0

It's actually isomorphic to the k-algebra of lower triangular 2×2 matrices over k.

Consider the quiver

$$\underset{1}{\bullet} \xleftarrow{\;a\;} \underset{2}{\bullet}$$

The basis of the path algebra kQ is the triple (e_1, e_2, a) (where e_i means 'stay at i') and its multiplication table is

	e_1	e_2	a
e_1	e_1	0	0
e_2	0	e_2	a
a	a	0	0

It's actually isomorphic to the k-algebra of lower triangular 2×2 matrices over k.

Every algebra is a path algebra

Theorem

Let k be an algebraically closed field and Λ a basic, connected and finite-dimensional algebra over k. Then there exists a finite connected quiver Q such that $\Lambda = kQ/I$ for some admissible ideal I of kQ.

Theorem

Let Λ be as above. Then, $\Lambda = kQ$ for a quiver Q if and only if Λ is hereditary (submodules of projective modules are projective).

Every algebra is a path algebra

Theorem

Let k be an algebraically closed field and Λ a basic, connected and finite-dimensional algebra over k. Then there exists a finite connected quiver Q such that $\Lambda = kQ/I$ for some admissible ideal I of kQ.

Theorem

Let Λ be as above. Then, $\Lambda = kQ$ for a quiver Q if and only if Λ is hereditary (submodules of projective modules are projective).

Theorem (Gabriel's)

*Let Λ be as above. Then, $\Lambda = kQ$ where Q (as an indirected graph) is **Dynkin** if and only if Λ is **representation-finite** (the number, up to isomorphism, of indecomposable finite-dimensional Λ-modules is finite).*

Dynkin quivers

A_n :

D_n :

E_6 :

E_7 :

E_8 :

The category $\operatorname{mod}\Lambda$

All the (right) Λ-modules and the homomorphisms between them form an abelian category, which we denote $\operatorname{mod}\Lambda$.

We wish to record the data of $\operatorname{mod}\Lambda$ in the form of a quiver.

The vertices are (isomorphism classes of) indecomposable Λ-modules and the number of arrows between two Λ-modules M, N is the dimension of $\operatorname{Irr}(M, N)$.

We call this quiver, the Auslander-Reiten quiver of Λ.

The category $\mathbf{mod}\,\Lambda$

All the (right) Λ-modules and the homomorphisms between them form an abelian category, which we denote $\operatorname{mod}\Lambda$.

We wish to record the data of $\operatorname{mod}\Lambda$ in the form of a quiver.

The vertices are (isomorphism classes of) indecomposable Λ-modules and the number of arrows between two Λ-modules M, N is the dimension of $\operatorname{Irr}(M, N)$.

We call this quiver, the Auslander-Reiten quiver of Λ.

The category $\operatorname{mod}\Lambda$

All the (right) Λ-modules and the homomorphisms between them form an abelian category, which we denote $\operatorname{mod}\Lambda$.

We wish to record the data of $\operatorname{mod}\Lambda$ in the form of a quiver.

The vertices are (isomorphism classes of) indecomposable Λ-modules and the number of arrows between two Λ-modules M, N is the dimension of $\operatorname{Irr}(M, N)$.

We call this quiver, the Auslander-Reiten quiver of Λ.

The category $\operatorname{mod}\Lambda$

All the (right) Λ-modules and the homomorphisms between them form an abelian category, which we denote $\operatorname{mod}\Lambda$.

We wish to record the data of $\operatorname{mod}\Lambda$ in the form of a quiver.

The vertices are (isomorphism classes of) indecomposable Λ-modules and the number of arrows between two Λ-modules M, N is the dimension of $\operatorname{Irr}(M, N)$.

We call this quiver, the Auslander-Reiten quiver of Λ.

The Auslander-Reiten quiver of the path algebra $k(\bullet \longleftarrow \bullet \longleftarrow \bullet)$ is

$$\begin{tikzcd}
 & & \bullet \arrow[dr] & & \\
 & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\
\bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet
\end{tikzcd}$$

Higher Auslander Algebras

Cluster Tilting Modules, Cyclic Polytopes

Module extensions

The group of extensions

An extension of a Λ-module M by a Λ-module N is a short exact sequence

$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0.$$

All the (equivalence classes of) extensions of M by N form a group which we denote $\mathrm{Ext}^1(M, N)$.

This construction can be extended to 'n-fold' extensions of M by N, that is, long exact sequences

$$0 \longrightarrow M \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow N \longrightarrow 0.$$

The group of n-fold extensions of M by N is denoted $\mathrm{Ext}^n(M, N)$.

Module extensions

The group of extensions

An extension of a Λ-module M by a Λ-module N is a short exact sequence

$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0.$$

All the (equivalence classes of) extensions of M by N form a group which we denote $\mathrm{Ext}^1(M, N)$.

This construction can be extended to 'n-fold' extensions of M by N, that is, long exact sequences

$$0 \longrightarrow M \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow N \longrightarrow 0.$$

The group of n-fold extensions of M by N is denoted $\mathrm{Ext}^n(M, N)$.

Cluster tilting modules

$\operatorname{add} M$

For a Λ-module M, we denote by $\operatorname{add} M$ the full subcategory of $\operatorname{mod} \Lambda$ whose objects are the direct sums of direct summands of M.

d-cluster tilting module

A Λ-module M is d-cluster tilting if

$$\operatorname{add} M = \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i(X, M) = 0 \ \forall i \in \{1, \ldots, d-1\}\}.$$

$\operatorname{add} M$

For a Λ-module M, we denote by $\operatorname{add} M$ the full subcategory of $\operatorname{mod} \Lambda$ whose objects are the direct sums of direct summands of M.

d-cluster tilting module

A Λ-module M is d-cluster tilting if

$$\operatorname{add} M = \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i(X, M) = 0 \ \forall i \in \{1, \ldots, d-1\}\}.$$

Construction (due to Iyama O.)

Let Λ be a representation-finite hereditary algebra. Then, Λ has a 1-cluster tilting module $M^{(1)}$. Put $\Lambda^{(1)} := \mathrm{End}(M^{(1)})$.

The k-algebra $\Lambda^{(1)}$ has a 2-cluster tilting module $M^{(2)}$. Put $\Lambda^{(2)} := \mathrm{End}(M^{(2)})$.

$$\vdots$$

The k-algebra $\Lambda^{(d)}$ has a d-cluster tilting module $M^{(d)}$. Put $\Lambda^{(d+1)} := \mathrm{End}(M^{(d)})$.

The k-algebra $\Lambda^{(d)}$ is called the d-Auslander algebra of Λ.

Construction (due to Iyama O.)

Let Λ be a representation-finite hereditary algebra. Then, Λ has a 1-cluster tilting module $M^{(1)}$. Put $\Lambda^{(1)} := \mathrm{End}(M^{(1)})$.

The k-algebra $\Lambda^{(1)}$ has a 2-cluster tilting module $M^{(2)}$. Put $\Lambda^{(2)} := \mathrm{End}(M^{(2)})$.

$$\vdots$$

The k-algebra $\Lambda^{(d)}$ has a d-cluster tilting module $M^{(d)}$. Put $\Lambda^{(d+1)} := \mathrm{End}(M^{(d)})$.

The k-algebra $\Lambda^{(d)}$ is called the d-Auslander algebra of Λ.

Construction (due to Iyama O.)

Let Λ be a representation-finite hereditary algebra. Then, Λ has a 1-cluster tilting module $M^{(1)}$. Put $\Lambda^{(1)} := \mathrm{End}(M^{(1)})$.

The k-algebra $\Lambda^{(1)}$ has a 2-cluster tilting module $M^{(2)}$. Put $\Lambda^{(2)} := \mathrm{End}(M^{(2)})$.

$$\vdots$$

The k-algebra $\Lambda^{(d)}$ has a d-cluster tilting module $M^{(d)}$. Put $\Lambda^{(d+1)} := \mathrm{End}(M^{(d)})$.

The k-algebra $\Lambda^{(d)}$ is called the d-Auslander algebra of Λ.

Construction (due to Iyama O.)

Let Λ be a representation-finite hereditary algebra. Then, Λ has a 1-cluster tilting module $M^{(1)}$. Put $\Lambda^{(1)} := \mathrm{End}(M^{(1)})$.

The k-algebra $\Lambda^{(1)}$ has a 2-cluster tilting module $M^{(2)}$. Put $\Lambda^{(2)} := \mathrm{End}(M^{(2)})$.

$$\vdots$$

The k-algebra $\Lambda^{(d)}$ has a d-cluster tilting module $M^{(d)}$. Put $\Lambda^{(d+1)} := \mathrm{End}(M^{(d)})$.

The k-algebra $\Lambda^{(d)}$ is called the d-Auslander algebra of Λ.

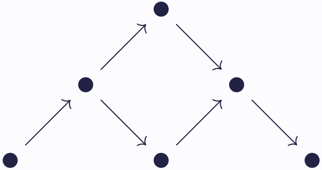
We apply Iyama's construction to $\Lambda := k(\bullet \longleftarrow \bullet \longleftarrow \bullet)$. The AR quivers of $\Lambda, \Lambda^{(1)}$ and $\Lambda^{(2)}$ are the following.

The AR quiver of Λ.

The AR quiver of $\Lambda^{(1)}$.

Higher Auslander algebras of type A

We apply Iyama's construction to $\Lambda := k(\bullet \longleftarrow \bullet \longleftarrow \bullet)$. The AR quivers of $\Lambda, \Lambda^{(1)}$ and $\Lambda^{(2)}$ are the following.

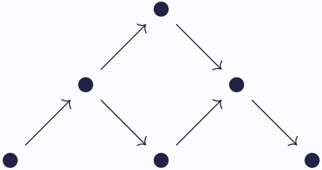


The AR quiver of Λ.

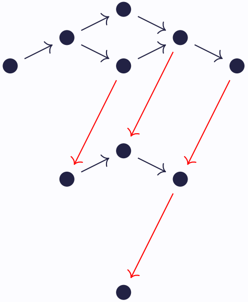
The AR quiver of $\Lambda^{(1)}$.

Higher Auslander algebras of type A

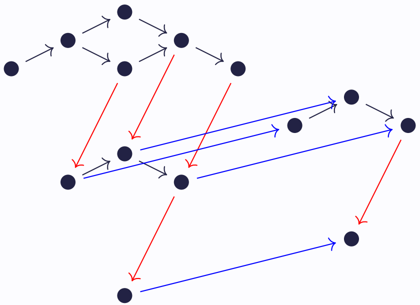
We apply Iyama's construction to $\Lambda := k(\bullet \longleftarrow \bullet \longleftarrow \bullet)$. The AR quivers of $\Lambda, \Lambda^{(1)}$ and $\Lambda^{(2)}$ are the following.



The AR quiver of Λ.



The AR quiver of $\Lambda^{(1)}$.



The AR quiver of $\Lambda^{(2)}$.

Cyclic polytopes

Cyclic polytope

The moment curve is the map $p(t) = (t, t^2, t^3, \ldots, t^n) \subseteq \mathbb{R}^n$ for $t \in \mathbb{R}$. A cyclic polytope $C(m, n)$ in $\mathbb{R}^n$ is the convex hull of the set $\{p(t_1), \ldots, p(t_m)\}$ where $t_1 < t_2 < \ldots < t_m$.

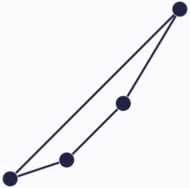
Example. The moment curve in $\mathbb{R}^2$ is just a parabola and $C(4, 2)$ is the following shape (for $t_i = i - 1$).

Cyclic polytopes

Cyclic polytope

The moment curve is the map $p(t) = (t, t^2, t^3, \ldots, t^n) \subseteq \mathbb{R}^n$ for $t \in \mathbb{R}$. A cyclic polytope $C(m, n)$ in $\mathbb{R}^n$ is the convex hull of the set $\{p(t_1), \ldots, p(t_m)\}$ where $t_1 < t_2 < \ldots < t_m$.

Example. The moment curve in $\mathbb{R}^2$ is just a parabola and $C(4, 2)$ is the following shape (for $t_i = i - 1$).



A triangulation of a cyclic polytope

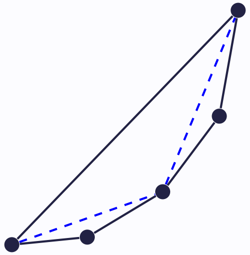
By a triangulation of $C(m, n)$ we mean its division into n-dimensional simplices that share vertices with $C(m, n)$.

Example. One possible triangulation of $C(5, 2)$.

A triangulation of a cyclic polytope

By a triangulation of $C(m, n)$ we mean its division into n-dimensional simplices that share vertices with $C(m, n)$.

Example. One possible triangulation of $C(5, 2)$.



Theorem (Thomas H., Oppermann S.)

There are bijections

$$\left\{\begin{array}{c}\text{Some } 2d\text{-simplices}\\ \text{of } C(n+2d,2d)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Indecomposable}\\ \text{modules over } \Lambda^{(d)}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{Triangulations}\\ \text{of } C(n+2d,2d)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Tilting modules over } \Lambda^{(d)}\\ \text{contained in } \operatorname{add} M^{(d)}\end{array}\right\}$$

Triangulations and tilting modules

Theorem (Thomas H., Oppermann S.)

There are bijections

$$\left\{\begin{array}{c}\text{Some } 2d\text{-simplices}\\ \text{of } C(n+2d,2d)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Indecomposable}\\ \text{modules over } \Lambda^{(d)}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{Triangulations}\\ \text{of } C(n+2d,2d)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Tilting modules over } \Lambda^{(d)}\\ \text{contained in } \operatorname{add} M^{(d)}\end{array}\right\}$$