Estudo Dirigido 1

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Outline

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Maxwell's Equations in Cartesian Coordinates

The Maxwell's Equations are:

$$\begin{aligned}\nabla \times \mathcal{E}(t) &= -\partial_t \mathcal{B}(t),\\ \nabla \times \mathcal{H}(t) &= \partial_t \mathcal{D}(t),\\ \nabla \cdot \mathcal{B}(t) &= 0,\\ \nabla \cdot \mathcal{D}(t) &= 0,\end{aligned}$$

where $\partial_t \cdot = \frac{\partial \cdot}{\partial t}$. The the constitutive relations are:

$$\mathcal{B}(t) = [\mu_0 \mu_r(t)] * \mathcal{H}(t), \tag{1}$$

$$\mathcal{D}(t) = [\epsilon_0 \epsilon_r(t)] * \mathcal{E}(t), \tag{2}$$

where $[\cdot]$ represents a tensor.

Normalizing the Electric Fields

It will be adopted the conventional approach in FDTD and the electric field will be normalized as:

$$\tilde{\mathcal{E}}(t) = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathcal{E}(t) = \frac{1}{\eta_0}\mathcal{E}(t). \tag{3}$$

The other parameters related to the electric field must also be normalized:

$$\tilde{\mathcal{D}} = \sqrt{\frac{1}{\epsilon_0\mu_0}}\mathcal{D} = c_0\mathcal{D}. \tag{4}$$

Therefore, the normalized Maxwell's equations become:

$$\nabla \times \tilde{\mathcal{E}} = -\partial_t \mathcal{B}, \tag{5}$$

$$\nabla \times \mathcal{H} = \partial_t \tilde{\mathcal{D}}, \tag{6}$$

$$\nabla \cdot \mathcal{B} = 0, \tag{7}$$

$$\nabla \cdot \tilde{\mathcal{D}} = 0. \tag{8}$$

To expand the equations, it will be assumed that $[\mu_r]$ and $[\epsilon_r]$ has only diagonal terms [1].

$$\partial_z \tilde{\mathcal{E}}_y - \partial_y \tilde{\mathcal{E}}_z = \frac{\mu_{xx}}{c_0} \partial_t \mathcal{H}_x \tag{9}$$

$$\partial_x \tilde{\mathcal{E}}_z - \partial_z \tilde{\mathcal{E}}_x = \frac{\mu_{yy}}{c_0} \partial_t \mathcal{H}_y \tag{10}$$

$$\partial_y \tilde{\mathcal{E}}_x - \partial_x \tilde{\mathcal{E}}_y = \frac{\mu_{zz}}{c_0} \partial_t \mathcal{H}_z \tag{11}$$

$$\partial_z \mathcal{H}_y - \partial_y \mathcal{H}_z = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_x \tag{12}$$

$$\partial_x \mathcal{H}_z - \partial_z \mathcal{H}_x = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_y \tag{13}$$

$$\partial_y \mathcal{H}_x - \partial_x \mathcal{H}_y = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_z \tag{14}$$

$$\tilde{\mathcal{D}}_x = \epsilon_{xx} \tilde{\mathcal{E}}_x \tag{15}$$

$$\tilde{\mathcal{D}}_y = \epsilon_{yy} \tilde{\mathcal{E}}_y \tag{16}$$

$$\tilde{\mathcal{D}}_z = \epsilon_{zz} \tilde{\mathcal{E}}_z \tag{17}$$

Notation for Curl Terms

$$C_x^E = \partial_z \tilde{\mathcal{E}}_y - \partial_y \tilde{\mathcal{E}}_z \tag{18}$$
$$C_y^E = \partial_x \tilde{\mathcal{E}}_z - \partial_z \tilde{\mathcal{E}}_x \tag{19}$$
$$C_z^E = \partial_y \tilde{\mathcal{E}}_x - \partial_x \tilde{\mathcal{E}}_y \tag{20}$$

$$C_x^H = \partial_z \mathcal{H}_y - \partial_y \mathcal{H}_z \tag{21}$$
$$C_y^H = \partial_x \mathcal{H}_z - \partial_z \mathcal{H}_x \tag{22}$$
$$C_z^H = \partial_y \mathcal{H}_x - \partial_x \mathcal{H}_y \tag{23}$$

Final Equations Form

$$C_x^E = \frac{\mu_{xx}}{c_0}\partial_t \mathcal{H}_x \tag{24}$$

$$C_y^E = \frac{\mu_{yy}}{c_0}\partial_t \mathcal{H}_y \tag{25}$$

$$C_z^E = \frac{\mu_{zz}}{c_0}\partial_t \mathcal{H}_z \tag{26}$$

$$C_x^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_x \tag{27}$$

$$C_y^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_y \tag{28}$$

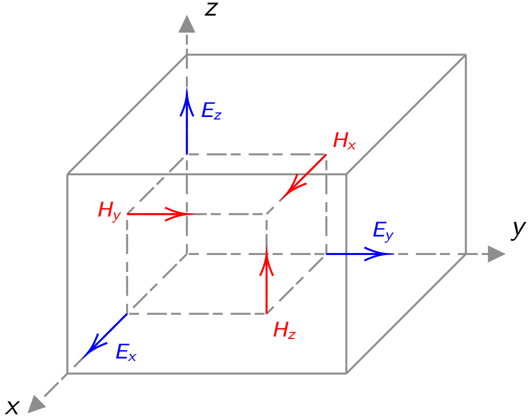
$$C_z^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_z \tag{29}$$

$$\tilde{\mathcal{D}}_x = \epsilon_{xx}\tilde{\mathcal{E}}_x \tag{30}$$

$$\tilde{\mathcal{D}}_y = \epsilon_{yy}\tilde{\mathcal{E}}_y \tag{31}$$

$$\tilde{\mathcal{D}}_z = \epsilon_{zz}\tilde{\mathcal{E}}_z \tag{32}$$

A unit cell is constructed by dividing the 3 axis into discrete cells of size $(\Delta x, \Delta y, \Delta z)$. Inside this cell, it is necessary to put all the fields of the electromagnetic problem $(\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z, \mathcal{H}_x, \mathcal{H}_x, \mathcal{H}_z)$. Instead of putting all fields on the origin $(0, 0, 0)$, where is more intuitive, Yee proposed the following approach:



Outline

Yee Grid

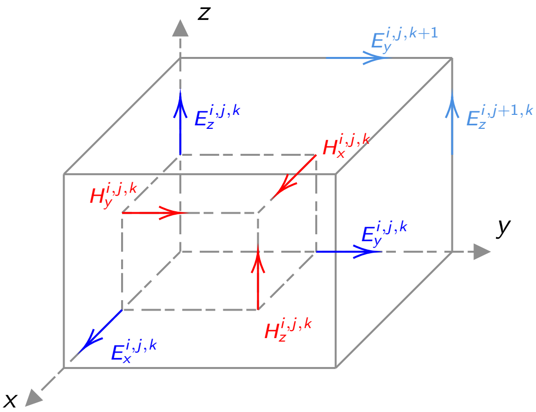
There are some reasons for using this scheme:

- The divergences are naturally zero.
- The physical boundary conditions are naturally satisfied.
- It is an elegant arrangement to approximate Maxwell's curl equations.

Additionaly, there are some consequences for using this scheme:

- Field components are in physically different locations.
- Field components may be in different materials even if they are in the same unit cell.
- Field components will be out of phase.

Finite-Difference Equations on Yee Grid



Based on this schematic, it is possible to write:

$$\frac{\partial \tilde{\mathcal{E}}_z\big|_t^{i,j,k}}{\partial y} = \frac{\tilde{\mathcal{E}}_z\big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_z\big|_t^{i,j,k}}{\Delta y} \tag{33}$$

$$\frac{\partial \tilde{\mathcal{E}}_y\big|_t^{i,j,k}}{\partial z} = \frac{\tilde{\mathcal{E}}_y\big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_y\big|_t^{i,j,k}}{\Delta y} \tag{34}$$

$$C_x^E = \frac{\tilde{\mathcal{E}}_z\big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_z\big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{\mathcal{E}}_y\big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_y\big|_t^{i,j,k}}{\Delta z} \tag{35}$$

Now, for the time derivative $\partial_t \mathcal{H}_x$ to exists at time t:

$$\partial_t \mathcal{H}_x \Big|_t^{i,j,k} = \frac{\mathcal{H}_x \Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x \Big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}. \tag{36}$$

So, the finite-difference equation for $\mathcal{H}_x$ becomes:

$$\frac{\tilde{\mathcal{E}}_z \Big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_z \Big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{\mathcal{E}}_y \Big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_y \Big|_t^{i,j,k}}{\Delta y} = \frac{\mu_{xx}\Big|^{i,j,k}}{c_0} \frac{\mathcal{H}_x \Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x \Big|_{t-\Delta t/2}^{i,j,k}}{\Delta t} \tag{37}$$

Finite-Difference Equations on Yee Grid

Similarly, it is possible to deduce the other components for the $\tilde{\mathcal{E}}$ field:

$$C_y^E = \frac{\tilde{\mathcal{E}}_x\Big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta z} - \frac{\tilde{\mathcal{E}}_z\Big|_t^{i+1,j,k} - \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\Delta x} \tag{38}$$

$$C_z^E = \frac{\tilde{\mathcal{E}}_y\Big|_t^{i+1,j,k} - \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\Delta x} - \frac{\tilde{\mathcal{E}}_x\Big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta y} \tag{39}$$

Finite-Difference Equations on Yee Grid

And also for $\mathcal{H}$:

$$C_x^H = \frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y} - \frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k-1}}{\Delta z} \tag{40}$$

$$C_y^H = \frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k-1}}{\Delta z} - \frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x} \tag{41}$$

$$C_z^H = \frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x} - \frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y} \tag{42}$$

Finite-Difference Equations on Yee Grid

Finally, the finite-difference equations are, for $\mathcal{H}_y$:

$$\frac{\tilde{\mathcal{E}}_x\Big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta z} - \frac{\tilde{\mathcal{E}}_z\Big|_t^{i+1,j,k} - \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\Delta x} = \frac{\mu_{yy}\Big|^{i,j,k}}{c_0} \frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta t} \tag{43}$$

for $\mathcal{H}_z$:

$$\frac{\tilde{\mathcal{E}}_y\Big|_t^{i+1,j,k} - \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\Delta x} - \frac{\tilde{\mathcal{E}}_x\Big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta y} = \frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta t} \tag{44}$$

Finite-Difference Equations on Yee Grid

for $\tilde{\mathcal{E}}$:

$$\frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y} - \frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k-1}}{\Delta z} = \frac{\epsilon_{xx}\Big|^{i,j,k}}{c_0}\frac{\tilde{\mathcal{E}}_x\Big|_{t+\Delta t}^{i,j,k} - \tilde{\mathcal{E}}_x\Big|_{t}^{i,j,k}}{\Delta t} \tag{45}$$

$$\frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k-1}}{\Delta z} - \frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x} = \frac{\epsilon_{yy}\Big|^{i,j,k}}{c_0}\frac{\tilde{\mathcal{E}}_y\Big|_{t+\Delta t}^{i,j,k} - \tilde{\mathcal{E}}_y\Big|_{t}^{i,j,k}}{\Delta t} \tag{46}$$

$$\frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x} - \frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y} = \frac{\epsilon_{zz}\Big|^{i,j,k}}{c_0}\frac{\tilde{\mathcal{E}}_z\Big|_{t+\Delta t}^{i,j,k} - \tilde{\mathcal{E}}_z\Big|_{t}^{i,j,k}}{\Delta t} \tag{47}$$

Outline

The Perfect Matching Layer

The Maxwell's Equations on the frequency domain are:

$$\nabla \times \mathbf{E}(\omega) = -j\omega\mu_0 \left[\mu_r\right] \mathbf{H}(\omega) \tag{48}$$
$$\nabla \times \mathbf{H}(\omega) = \sigma\mathbf{E}(\omega) + j\omega \left[S\right] \mathbf{D}(\omega) \tag{49}$$
$$\mathbf{D}(\omega) = \epsilon_0 \left[\epsilon_r\right] \mathbf{E}(\omega) \tag{50}$$

According to [1], the PML $[S]$ can be incorporated as:

$$\nabla \times \mathbf{E}(\omega) = -j\omega\mu_0 \left[\mu_r\right] \left[S\right] \mathbf{H}(\omega) \tag{51}$$
$$\nabla \times \mathbf{H}(\omega) = \sigma\mathbf{E}(\omega) + j\omega\mathbf{D}(\omega) \tag{52}$$
$$\mathbf{D}(\omega) = \epsilon_0 \left[\epsilon_r\right] \mathbf{E}(\omega) \tag{53}$$

Hence, the normalized equations become:

$$\nabla \times \tilde{\mathbf{E}}(\omega) = -j\omega \frac{[\mu_r]}{c_0} [S]\, \mathbf{H}(\omega) \tag{54}$$

$$\nabla \times \mathbf{H}(\omega) = \eta_0 \sigma \tilde{\mathbf{E}}(\omega) + \frac{j\omega}{c_0} [S]\, \tilde{\mathbf{D}}(\omega) \tag{55}$$

$$\mathbf{D}(\omega) = [\epsilon_r]\, \tilde{\mathbf{E}}(\omega) \tag{56}$$

The tensor $[S]$

The tensor $[S]$ is used to incorporate loss on all directions using a fake conductivity σ'_i on every propagation direction i. Also, to avoid reflection, there is a impedance matching.

$$[S] = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix} \tag{57}$$

$$s_i = 1 + \frac{\sigma'_i}{j\omega\epsilon_0}, i \in (x, y, z) \tag{58}$$

$$\sigma'_i(i) = \frac{\epsilon_0}{2\Delta t}\left(\frac{i}{L_i}\right)^3 \tag{59}$$

$$i \in (x, y, z)$$

Incorporating PML into Maxwell's Equations

Considering only the diagonal terms in $[\mu_r]$, $[\epsilon_r]$ and $[\sigma] = 0$, the final form of the Maxwell's Equations with UPML are [2]:

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)\mathbf{H}_x = -\frac{c_0}{\mu_{xx}}\mathbf{C}^E_x \tag{60}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1}\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)\mathbf{H}_y = -\frac{c_0}{\mu_{yy}}\mathbf{C}^E_y \tag{61}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1}\mathbf{H}_z = -\frac{c_0}{\mu_{zz}}\mathbf{C}^E_z \tag{62}$$

Incorporating PML into Maxwell's Equations

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)\tilde{\mathbf{D}}_x = c_0\mathbf{C}^H_x - \frac{\sigma_{xx}}{\epsilon_0}\tilde{\mathbf{E}}_x \tag{63}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1}\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)\tilde{\mathbf{D}}_y = c_0\mathbf{C}^H_y - \frac{\sigma_{yy}}{\epsilon_0}\tilde{\mathbf{E}}_y \tag{64}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1}\tilde{\mathbf{D}}_z = c_0\mathbf{C}^H_z - \frac{\sigma_{zz}}{\epsilon_0}\tilde{\mathbf{E}}_z \tag{65}$$

$$\tilde{\mathbf{D}}_x = \epsilon_{xx} \tilde{\mathbf{E}}_x, \tag{66}$$

$$\tilde{\mathbf{D}}_y = \epsilon_{yy} \tilde{\mathbf{E}}_y, \tag{67}$$

$$\tilde{\mathbf{D}}_z = \epsilon_{zz} \tilde{\mathbf{E}}_z. \tag{68}$$

Note that keeping $[S]$ separaed from $[\mu_r]$ and $[\mu_r]$ allows the PML to be handled independently from the materials and devices being simulated.

Conversion to the Time-Domain

Starting from (60):

$$j\omega \mathbf{H}_x + \frac{\sigma'_y + \sigma'_z}{\epsilon_0}\mathbf{H}_x + \frac{1}{j\omega}\frac{\sigma'_y\sigma'_z}{\epsilon_0^2}\mathbf{H}_x = -\frac{c_0}{\mu_{xx}}\mathbf{C}_x^E - \frac{1}{j\omega}\frac{c_0\sigma'_x}{\epsilon_0\mu_{xx}}\mathbf{C}_x^E \tag{69}$$

In the time-domain becomes:

$$\partial_t \mathcal{H}_x + \frac{\sigma'_y + \sigma'_z}{\epsilon_0}\mathcal{H}_x + \int_{-\infty}^{t} \frac{\sigma'_y\sigma'_z}{\epsilon_0^2}\mathcal{H}_x(\tau)d\tau = -\frac{c_0}{\mu_{xx}}C_x^E - \int_{\infty}^{t} \frac{c_0\sigma'_x}{\epsilon_0\mu_{xx}}C_x^E(\tau)d\tau \tag{70}$$

For the term 1, the time approximation will be the same as used before:

$$\partial_t \mathcal{H}_x(t) \approx \frac{\left.\mathcal{H}_x\right|_{t+\Delta t/2}^{i,j,k} - \left.\mathcal{H}_x\right|_{t-\Delta t/2}^{i,j,k}}{\Delta t} \tag{71}$$

For the term 2, it is necessary to approximate $\mathcal{H}_x(t)$, that will be done by averaging the values at $t + {}^{\Delta t}/_2$ and $t - {}^{\Delta t}/_2$:

$$\frac{\sigma'_y + \sigma'_z}{\epsilon_0}\mathcal{H}_x(t) \approx \frac{\sigma'_y + \sigma'_z}{\epsilon_0}\frac{\left.\mathcal{H}_x\right|_{t+\Delta t/2}^{i,j,k} + \left.\mathcal{H}_x\right|_{t-\Delta t/2}^{i,j,k}}{2} \tag{72}$$

Numerical Approximations

For the term 3, it is necessary to approximate the integral with a summation:

$$\int_{-\infty}^{t} \frac{\sigma_y'\sigma_z'}{\epsilon_0^2}\mathcal{H}_x(\tau)d\tau \approx \frac{\sigma_y'\sigma_z'}{\epsilon_0^2}\sum_{T=\Delta t/2}^{t+\Delta t/2} \mathcal{H}_x\Big|_T^{i,j,k}\Delta t \tag{73}$$

However, in this way the summation is going future on time. The fix is simple: just pull out the last term from summation and do the integration over half a time step:

$$\int_{-\infty}^{t} \frac{\sigma_y'\sigma_z'}{\epsilon_0^2}\mathcal{H}_x(\tau)d\tau \approx \frac{\sigma_y'\sigma_z'}{\epsilon_0^2}\left(\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k}\frac{\Delta t}{2} + \sum_{T=\Delta t/2}^{t-\Delta t/2} \mathcal{H}_x\Big|_T^{i,j,k}\Delta t\right)$$

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} \mathcal{H}_x(\tau) d\tau \approx \frac{\sigma'_y \sigma'_z \Delta t}{\epsilon_0^2} \left(\frac{\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k} - \mathcal{H}_x\Big|_{(t-\Delta t/2)}^{i,j,k}}{4} + \sum_{T=\Delta t/2}^{t-\Delta t/2} \mathcal{H}_x\Big|_{T}^{i,j,k} \right) \tag{74}$$

For the term 4, the curl approximation for will be the same as in (35):

$$-\frac{c_0}{\mu_{xx}} C_x^E \approx -\frac{c_0}{\mu_{xx}\Big|^{i,j,k}} C_x^E\Big|_{t}^{i,j,k} \tag{75}$$

Finally, for the term 5, as for the term 3, the integral will be approximated with a summation:

$$
\begin{aligned}
-\int_{\infty}^{t} \frac{c_0\sigma'_x}{\epsilon_0\mu_{xx}} C_x^E(\tau)d\tau &= -\frac{c_0\sigma'_x}{\epsilon_0\mu_{xx}} \int_{\infty}^{t} C_x^E(\tau)d\tau \\
&\approx -\frac{c_0\sigma_x^H\Big|^{i,j,k}}{\epsilon_0\mu_{xx}\Big|^{i,j,k}} \sum_{T=0}^{t} C_x^E\Big|_T^{i,j,k} \Delta t \\
&\approx -\frac{c_0\Delta t\sigma_x^H\Big|^{i,j,k}}{\epsilon_0\mu_{xx}\Big|^{i,j,k}} \sum_{T=0}^{t} C_x^E\Big|_T^{i,j,k}
\end{aligned} \tag{76}
$$

Update equations

Starting from the numerical approximation of (70):

$$\frac{\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k} - \mathcal{H}_x\Big|_{(t-\Delta t/2)}^{i,j,k}}{\Delta t} + \frac{\sigma'_y + \sigma'_z}{\epsilon_0}\frac{\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k} + \mathcal{H}_x\Big|_{(t-\Delta t/2)}^{i,j,k}}{2} = -\frac{c_0}{\mu_{xx}\Big|^{i,j,k}} C_x^E\Big|_t^{i,j,k} - \frac{c_0 \Delta t \sigma_x^H\Big|^{i,j,k}}{\epsilon_0 \mu_{xx}\Big|^{i,j,k}} \sum_{T=0}^{t} C_x^E\Big|_T^{i,j,k}$$

it is possible to isolate $\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k}$ as:

$$\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} = m_{x_1}\Big|^{i,j,k} \mathcal{H}_x\Big|_{t-\Delta t/2}^{i,j,k} + m_{x_2}\Big|^{i,j,k} C_x^E\Big|_t^{i,j,k} + m_{x_3} I_{C_x^E}\Big|_t^{i,j,k} + m_{x_4} I_{H_x}\Big|_t^{i,j,k} \tag{77}$$

Update equation for $\mathcal{H}_x$

$$\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} = m_{x_1}\Big|^{i,j,k}\mathcal{H}_x\Big|_{t-\Delta t/2}^{i,j,k} + m_{x_2}\Big|^{i,j,k} C_x^E\Big|_t^{i,j,k} + m_{x_3} I_{C_x^E}\Big|_t^{i,j,k} + m_{x_4} I_{H_x}\Big|_t^{i,j,k} \tag{78}$$

$$m_{x_0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma'_y\Big|^{i,j,k} + \sigma'_z\Big|^{i,j,k}}{2\epsilon_0} + \frac{\sigma'_y\Big|^{i,j,k}\sigma'_z\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}$$

$$m_{x_1}\Big|^{i,j,k} = \frac{1}{m_{x_0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma'_y\Big|^{i,j,k} + \sigma'_z\Big|^{i,j,k}}{2\epsilon_0} - \frac{\sigma'_y\Big|^{i,j,k}\sigma'_z\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}\right]$$

$$m_{x_2}\Big|^{i,j,k} = -\frac{1}{m_{x_0}\Big|^{i,j,k}}\frac{c_0}{\mu_{xx}\Big|^{i,j,k}} \qquad m_{x_3}\Big|^{i,j,k} = -\frac{1}{m_{x_0}\Big|^{i,j,k}}\frac{c_0\Delta t}{\epsilon_0}\frac{\sigma'_x\Big|^{i,j,k}}{\mu_{xx}\Big|^{i,j,k}}$$

$$m_{x_4}\Big|^{i,j,k} = -\frac{1}{m_{x_0}\Big|^{i,j,k}}\frac{\Delta t}{\epsilon_0^2}\sigma'_y\Big|^{i,j,k}\sigma'_z\Big|^{i,j,k}$$

$$I_{C_x^E}\Big|_t^{i,j,k} = \sum_{T=0}^{t} C_x^E\Big|_T^{i,j,k}$$

$$I_{H_x}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} \mathcal{H}_x\Big|_T^{i,j,k}$$

Update equation for $\mathcal{H}_y$

$$\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} = m_{y_1}\Big|^{i,j,k}\mathcal{H}_y\Big|_{t-\Delta t/2}^{i,j,k} + m_{y_2}\Big|^{i,j,k} C_y^E\Big|_t^{i,j,k} + m_{y_3} I_{C_y^E}\Big|_t^{i,j,k} + m_{y_4} I_{H_y}\Big|_t^{i,j,k} \tag{79}$$

$$m_{y_0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_x'\Big|^{i,j,k} + \sigma_z'\Big|^{i,j,k}}{2\epsilon_0} + \frac{\sigma_x'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}$$

$$m_{y_4}\Big|^{i,j,k} = -\frac{1}{m_{y_0}\Big|^{i,j,k}}\frac{\Delta t}{\epsilon_0^2}\sigma_x'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}$$

$$m_{y_1}\Big|^{i,j,k} = \frac{1}{m_{y_0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma_x'\Big|^{i,j,k} + \sigma_z'\Big|^{i,j,k}}{2\epsilon_0} - \frac{\sigma_x'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}\right]$$

$$I_{C_y^E}\Big|_t^{i,j,k} = \sum_{T=0}^{t} C_y^E\Big|_T^{i,j,k}$$

$$m_{y_2}\Big|^{i,j,k} = -\frac{1}{m_{y_0}\Big|^{i,j,k}}\frac{c_0}{\mu_{yy}\Big|^{i,j,k}}$$

$$m_{y_3}\Big|^{i,j,k} = -\frac{1}{m_{y_0}\Big|^{i,j,k}}\frac{c_0\Delta t}{\epsilon_0}\frac{\sigma_y'\Big|^{i,j,k}}{\mu_{yy}\Big|^{i,j,k}}$$

$$I_{H_y}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2}\mathcal{H}_y\Big|_T^{i,j,k}$$

Update equation for $\mathcal{H}_z$

$$\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} = m_{z_1}\Big|^{i,j,k}\mathcal{H}_z\Big|_{t-\Delta t/2}^{i,j,k} + m_{z_2}\Big|^{i,j,k} C_z^E\Big|_t^{i,j,k} + m_{z_3} I_{C_z^E}\Big|_t^{i,j,k} + m_{z_4} I_{H_z}\Big|_t^{i,j,k} \tag{80}$$

$$m_{z_0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma'_x\Big|^{i,j,k} + \sigma'_y\Big|^{i,j,k}}{2\epsilon_0} + \frac{\sigma'_x\Big|^{i,j,k}\sigma'_y\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}$$

$$m_{z_4}\Big|^{i,j,k} = -\frac{1}{m_{z_0}\Big|^{i,j,k}}\frac{\Delta t}{\epsilon_0^2}\sigma'_x\Big|^{i,j,k}\sigma'_y\Big|^{i,j,k}$$

$$m_{z_1}\Big|^{i,j,k} = \frac{1}{m_{z_0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma'_x\Big|^{i,j,k} + \sigma'_y\Big|^{i,j,k}}{2\epsilon_0} - \frac{\sigma'_x\Big|^{i,j,k}\sigma'_y\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}\right]$$

$$I_{H_z}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2}\mathcal{H}_z\Big|_T^{i,j,k}$$

$$m_{z_2}\Big|^{i,j,k} = -\frac{1}{m_{z_0}\Big|^{i,j,k}}\frac{c_0}{\mu_{zz}\Big|^{i,j,k}} \qquad m_{z_3}\Big|^{i,j,k} = -\frac{1}{m_{z_0}\Big|^{i,j,k}}\frac{c_0\Delta t}{\epsilon_0}\frac{\sigma'_z\Big|^{i,j,k}}{\mu_{zz}\Big|^{i,j,k}}$$

$$I_{C_z^E}\Big|_t^{i,j,k} = \sum_{T=0}^{t} C_z^E\Big|_t^{i,j,k}$$

Update equation for $\tilde{\mathcal{D}}_x$

$$\tilde{\mathcal{D}}_x\Big|_{t+\Delta t/2}^{i,j,k} = m_{x_1}\Big|^{i,j,k}\tilde{\mathcal{D}}_x\Big|_{t}^{i,j,k} + m_{x_2}\Big|^{i,j,k} C_x^H\Big|_{t+\Delta t/2}^{i,j,k} + m_{x_3} I_{C_x^H}\Big|_{t-\Delta t/2}^{i,j,k} + m_{x_4} I_{D_x}\Big|_{t}^{i,j,k} \tag{81}$$

$$I_{C_x^H}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} C_x^H\Big|_{T}^{i,j,k} \tag{82}$$

$$I_{D_x}\Big|_{t}^{i,j,k} = \sum_{T=0}^{t} \tilde{\mathcal{D}}_x\Big|_{T}^{i,j,k} \tag{83}$$

Update equation for $\tilde{\mathcal{D}}_y$

$$\tilde{\mathcal{D}}_y\Big|_{t+\Delta t/2}^{i,j,k} = m_{y_1}\Big|^{i,j,k}\tilde{\mathcal{D}}_y\Big|_{t}^{i,j,k} \quad + m_{y_2}\Big|^{i,j,k} C_y^H\Big|_{t+\Delta t/2}^{i,j,k} + m_{y_3} I_{C_y^H}\Big|_{t-\Delta t/2}^{i,j,k} \quad + m_{y_4} I_{D_y}\Big|_{t}^{i,j,k} \tag{84}$$

$$I_{C_y^H}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} C_y^H\Big|_{T}^{i,j,k} \tag{85}$$

$$I_{D_y}\Big|_{t}^{i,j,k} = \sum_{T=0}^{t} \tilde{\mathcal{D}}_y\Big|_{T}^{i,j,k} \tag{86}$$

Update equation for $\tilde{\mathcal{D}}_z$

$$\tilde{\mathcal{D}}_z\Big|_{t+\Delta t/2}^{i,j,k} = m_{z_1}\Big|^{i,j,k}\tilde{\mathcal{D}}_z\Big|_{t}^{i,j,k} \quad + m_{z_2}\Big|^{i,j,k} C_z^H\Big|_{t+\Delta t/2}^{i,j,k} + m_{z_3} I_{C_z^H}\Big|_{t-\Delta t/2}^{i,j,k} \quad + m_{z_4} I_{D_z}\Big|_{t}^{i,j,k} \tag{87}$$

$$I_{C_z^H}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} C_z^H\Big|_{T}^{i,j,k} \tag{88}$$

$$I_{D_z}\Big|_{t}^{i,j,k} = \sum_{T=0}^{t} \tilde{\mathcal{D}}_z\Big|_{T}^{i,j,k} \tag{89}$$

Update equation for $\tilde{\mathcal{E}}$

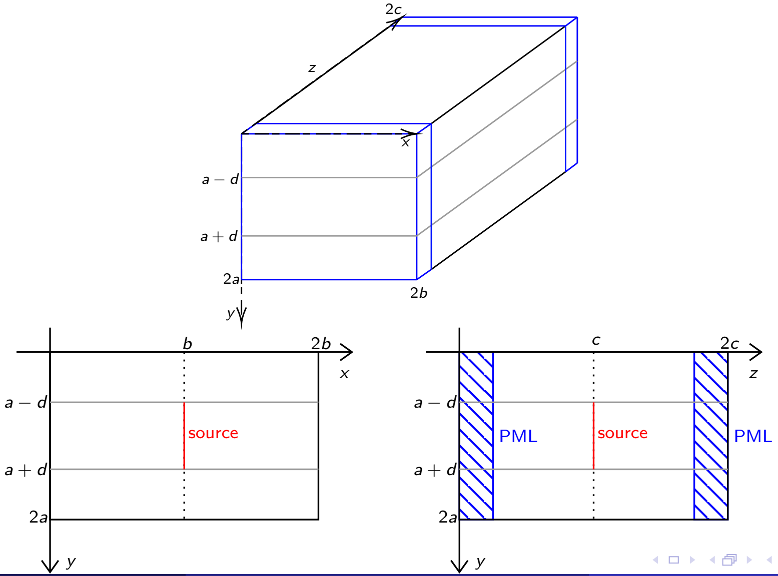
$$\tilde{\mathcal{E}}_x\Big|_{t+\Delta t}^{i,j,k} = m_{E_{x_1}}\tilde{\mathcal{D}}_x\Big|_{t+\Delta t}^{i,j,k} \tag{90}$$

$$\tilde{\mathcal{E}}_y\Big|_{t+\Delta t}^{i,j,k} = m_{E_{y_1}}\tilde{\mathcal{D}}_y\Big|_{t+\Delta t}^{i,j,k} \tag{91}$$

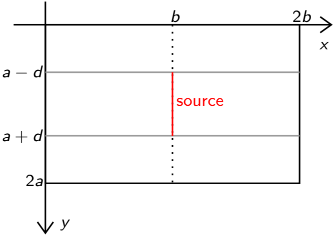
$$\tilde{\mathcal{E}}_z\Big|_{t+\Delta t}^{i,j,k} = m_{E_{z_1}}\tilde{\mathcal{D}}_z\Big|_{t+\Delta t}^{i,j,k} \tag{92}$$

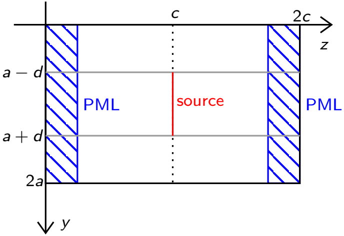
Outline

Simulation Scenary



Simulation Scenary





Parameter	Description	Value
a	Half-length on y-axis	5 cm
b	Half-length on x-axis	5 cm
c	Half-length on z-axis	10 cm
d	Half-distance between strips	2 cm
N_x	Number of cells on x-axis	40
N_y	Number of cells on y-axis	40
N_z	Number of cells on z-axis	70
N_{PML}	Number of cells with PML on z-axis	10
ϵ_{r_2}	Relative permittivity between strips	5

Parameter	Description	Value
N_x^c	Index centered on x	20
N_y^c	Index centered on y	20
N_z^c	Index centered on z	35
$N_{\text{plate}}^{\text{high}}$	Y-index of top strip	32
$N_{\text{plate}}^{\text{low}}$	Y-index of low strip	8
Δx	$2b/N_x$	0.25 cm
Δy	$2a/N_y$	0.25 cm
Δz	$2c/N_z$	0.286 cm

Boundary Conditions

On Source

The source is located at $\left(N_x^c, N_{\text{plate}}^{\text{low}} : N_{\text{plate}}^{\text{high}}, N_z^c\right)$. It will be considered that the source is on $\tilde{\mathcal{E}}_y$ as shown below:

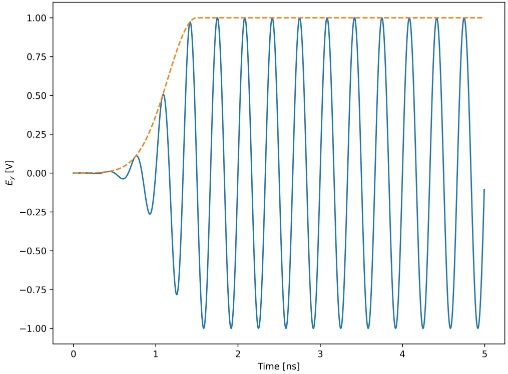


Figure: Input electric field with frequency $f = 3.0\,\mathrm{GHz}$, peak time $t = 1.5\,\mathrm{ns}$ and duration $t_f = 5.0\,\mathrm{ns}$

On Conductor Strips

There is one strip located at $\left(:, N^{\text{low}}_{\text{plate}}, :\right)$ and another at $\left(:, N^{\text{high}}_{\text{plate}}, :\right)$. At these positions is valid:

$$\tilde{\mathcal{E}}_x = \tilde{\mathcal{E}}_z = \mathcal{H}_y = 0 \tag{93}$$

Conductor Walls

The condition for conductor walls is the same as for conductor strips. Their positions are at: $(0, :, :)$, $(N_x - 1, :, :)$, $(:, 0, :)$,$(:, N_y - 1, :)$, $(:, :, 0)$ and $(:, :, N_z - 1)$.

Time Step

The time step was chosen to obey the Courant Stability condition:

$$\Delta t \leq \frac{1}{c_0 \sqrt{\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2 + \left(\frac{1}{\Delta z}\right)^2}} \tag{94}$$

Simulation Results

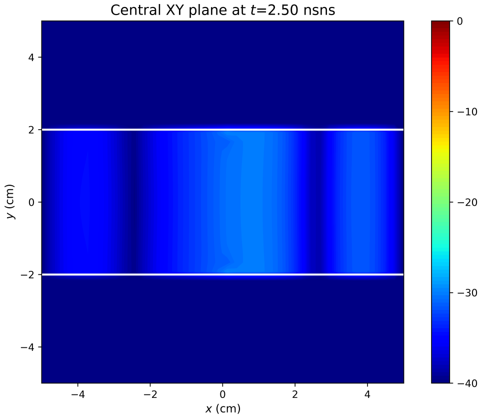


Figure: Central XY Plane.

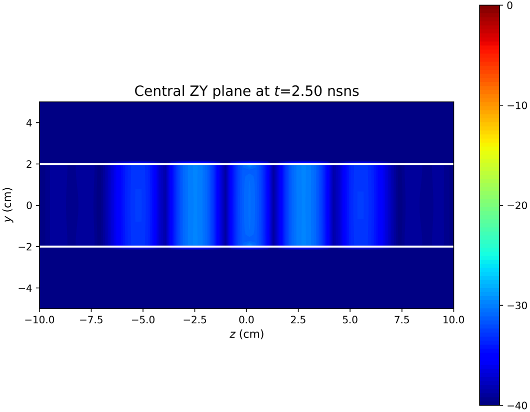


Figure: Central ZY Plane.

Outline

Next Steps

- Choose indicators to measure performance of the developed codes,
- Deduce the equations for the proposed problem using Geometric Algebra,
- Implement the solution using Geometric Algebra in Python/Cython,
- Study the possibility to solve this type of problem with Quantum Computing.

References I

R. Rumpf, *Electromagnetic and Photonic Simulation for the Beginner: Finite-Difference Frequency-Domain in MATLAB*, 01 2022.

Rumpf, Raymond. Derivation of 3D update equations with a UPML. [Online]. Available: https://empossible.net/wp-content/uploads/2020/01/Lecture-Derivation-of-3D-Update-Equations-w-PML.pdf