Estudo Orientado 1

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I. Expansion of Maxwell's Curl Equations in Cartesian Coordinates

The Maxwell's equations are:

$$\nabla \times \mathcal{E}(t) = -\partial_t \mathcal{B}(t), \quad (1)$$
$$\nabla \times \mathcal{H}(t) = \partial_t \mathcal{D}(t), \quad (2)$$
$$\nabla \cdot \mathcal{B}(t) = 0, \quad (3)$$
$$\nabla \cdot \mathcal{D}(t) = 0, \quad (4)$$

where $\partial_t \cdot = \frac{\partial \cdot}{\partial t}$.

The the constitutive relations are:

$$\mathcal{B}(t) = [\mu_0 \mu_r(t)] * \mathcal{H}(t), \quad (5)$$
$$\mathcal{D}(t) = [\epsilon_0 \epsilon_r(t)] * \mathcal{E}(t), \quad (6)$$

where $[\cdot]$ represents a tensor.

A. Normalizing the Electric Fields

It will be adopted the conventional approach in FDTD and the electric field will be normalized as:

$$\tilde{\mathcal{E}}(t) = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathcal{E}(t) = \frac{1}{\eta_0} \mathcal{E}(t). \quad (7)$$

Also, from now on, the time depency (t) will be ommited for cleaning notation reasons.

The other parameters related to the electric field must also be normalized:

$$\tilde{\mathcal{D}} = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \mathcal{D} = c_0 \mathcal{D}. \quad (8)$$

Therefore, the normalized Maxwell's equations become:

$$\nabla \times \tilde{\mathcal{E}} = -\partial_t \mathcal{B}, \quad (9)$$
$$\nabla \times \mathcal{H} = \partial_t \tilde{\mathcal{D}}, \quad (10)$$
$$\nabla \cdot \mathcal{B} = 0, \quad (11)$$
$$\nabla \cdot \tilde{\mathcal{D}} = 0. \quad (12)$$

B. Expanding Maxwell's Equations

To expand the equations, it will be assumed that $[\mu_r]$ and $[\epsilon_r]$ has only diagonal terms [1].

The equation $\nabla \times \tilde{\mathcal{E}} = -\frac{[\mu_r]}{c_0} \partial_t \mathcal{B}$ becomes:

$$\partial_z \tilde{\mathcal{E}}_y - \partial_y \tilde{\mathcal{E}}_z = \frac{\mu_{xx}}{c_0} \partial_t \mathcal{H}_x, \quad (13)$$
$$\partial_x \tilde{\mathcal{E}}_z - \partial_z \tilde{\mathcal{E}}_x = \frac{\mu_{yy}}{c_0} \partial_t \mathcal{H}_y, \quad (14)$$
$$\partial_y \tilde{\mathcal{E}}_x - \partial_x \tilde{\mathcal{E}}_y = \frac{\mu_{zz}}{c_0} \partial_t \mathcal{H}_z. \quad (15)$$

The equation $\nabla \times \mathcal{H} = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}$ becomes:

$$\partial_z \mathcal{H}_y - \partial_y \mathcal{H}_z = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_x, \quad (16)$$
$$\partial_x \mathcal{H}_z - \partial_z \mathcal{H}_x = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_y, \quad (17)$$
$$\partial_y \mathcal{H}_x - \partial_x \mathcal{H}_y = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_z. \quad (18)$$

Finally, the equation $\tilde{\mathcal{D}} = [\epsilon_r] \, \tilde{\mathcal{E}}$ becomes:

$$\tilde{\mathcal{D}}_x = \epsilon_{xx} \tilde{\mathcal{E}}_x, \quad (19)$$
$$\tilde{\mathcal{D}}_y = \epsilon_{yy} \tilde{\mathcal{E}}_y, \quad (20)$$
$$\tilde{\mathcal{D}}_z = \epsilon_{zz} \tilde{\mathcal{E}}_z. \quad (21)$$

C. Notation for Curl Terms

$$C_x^E = \partial_z \tilde{\mathcal{E}}_y - \partial_y \tilde{\mathcal{E}}_z, \quad (22)$$
$$C_y^E = \partial_x \tilde{\mathcal{E}}_z - \partial_z \tilde{\mathcal{E}}_x, \quad (23)$$
$$C_z^E = \partial_y \tilde{\mathcal{E}}_x - \partial_x \tilde{\mathcal{E}}_y. \quad (24)$$

$$C_x^H = \partial_z \mathcal{H}_y - \partial_y \mathcal{H}_z, \quad (25)$$
$$C_y^H = \partial_x \mathcal{H}_z - \partial_z \mathcal{H}_x, \quad (26)$$
$$C_z^H = \partial_y \mathcal{H}_x - \partial_x \mathcal{H}_y. \quad (27)$$

D. Final Equations Form

$$C_x^E = \frac{\mu_{xx}}{c_0} \partial_t \mathcal{H}_x, \quad (28)$$
$$C_y^E = \frac{\mu_{yy}}{c_0} \partial_t \mathcal{H}_y, \quad (29)$$
$$C_z^E = \frac{\mu_{zz}}{c_0} \partial_t \mathcal{H}_z. \quad (30)$$

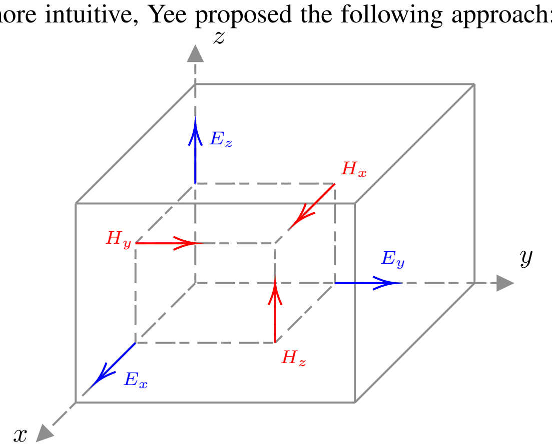
$$C_x^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_x, \tag{31}$$
$$C_y^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_y, \tag{32}$$
$$C_z^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_z. \tag{33}$$

$$\tilde{\mathcal{D}}_x = \epsilon_{xx}\tilde{\mathcal{E}}_x, \tag{34}$$
$$\tilde{\mathcal{D}}_y = \epsilon_{yy}\tilde{\mathcal{E}}_y, \tag{35}$$
$$\tilde{\mathcal{D}}_z = \epsilon_{zz}\tilde{\mathcal{E}}_z. \tag{36}$$

II. Finite-Difference Approximation to Maxwell's Equations

A. *Yee Grid*

A unit cell is constructed by dividing the 3 axis into discrete cells of size $(\Delta x, \Delta y, \Delta z)$. Inside this cell, it is necessary to put all the fields of the electromagnetic problem $(\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z, \mathcal{H}_x, \mathcal{H}_x, \mathcal{H}_z)$. Instead of putting all fields on the origin $(0, 0, 0)$, where is more intuitive, Yee proposed the following approach:



- $\mathcal{E}_x$ on $(\Delta x/2, 0, 0)$,
- $\mathcal{E}_y$ on $(0, \Delta y/2, 0)$,
- $\mathcal{E}_z$ on $(0, 0, \Delta z/2)$,
- $\mathcal{H}_x$ on $(0, \Delta y/2, \Delta z/2)$,
- $\mathcal{H}_y$ on $(\Delta x/2, 0, \Delta z/2)$,
- $\mathcal{H}_z$ on $(\Delta x/2, \Delta y/2, 0)$.

There are some reasons for using this scheme:

- The divergences are naturally zero.
- The physical boundary conditions are naturally satisfied.
- It is an elegant arrangement to approximate Maxwell's curl equations.

Additionaly, there are some consequences for using this scheme:

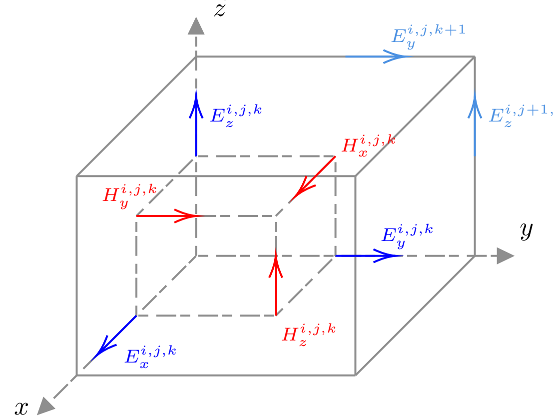
- Field components are in physically different locations.
- Field components may be in different materials even if they are in the same unit cell.
- Field components will be out of phase.

B. *Finite-Difference Equations on Yee Grid*

Each cell on the grid is identified by the coordines $(i\Delta x, j\Delta y, k\Delta z)$, where (i, j, k) are the index of the cell.

Note that on each face of the Yee cell there is the fields of the adjacent cell.

Consider, first, the grid for $\mathcal{H}_x$:



Based on this schematic, it is possible to write [1]:

$$\frac{\partial \tilde{\mathcal{E}}_z^{i,j,k}}{\partial y}(t) = \frac{\tilde{\mathcal{E}}_z^{i,j+1,k}(t) - \tilde{\mathcal{E}}_z^{i,j,k}(t)}{\Delta y}, \tag{37}$$
$$\frac{\partial \tilde{\mathcal{E}}_y^{i,j,k}}{\partial z}(t) = \frac{\tilde{\mathcal{E}}_y^{i,j,k+1}(t) - \tilde{\mathcal{E}}_y^{i,j,k}(t)}{\Delta y}. \tag{38}$$

Note that this space derivatives exists at time instant t and they exist at the same point as $\mathcal{H}_x^{i,j,k}$.

We need to explicitly write the time on the Yee grid equations, since it is essencial to write all the members of equations on the same time instant.

Hence, the C_x^E final equation is:

$$C_x^E = \frac{\tilde{\mathcal{E}}_z^{i,j+1,k}(t) - \tilde{\mathcal{E}}_z^{i,j,k}(t)}{\Delta y} - \frac{\tilde{\mathcal{E}}_y^{i,j,k+1}(t) - \tilde{\mathcal{E}}_y^{i,j,k}(t)}{\Delta z} \tag{39}$$

Now, for the time derivative $\partial_t \mathcal{H}_x$ to exists at time t:

$$\partial_t \mathcal{H}_x^{i,j,k}(t) = \frac{\mathcal{H}_x^{i,j,k}(t + {}^{\Delta t}/_2) - \mathcal{H}_x^{i,j,k}(t - {}^{\Delta t}/_2)}{\Delta t}. \tag{40}$$

So, the finite-difference equation for $\mathcal{H}_x$ becomes:

$$\frac{\tilde{\mathcal{E}}_z^{i,j+1,k}(t) - \tilde{\mathcal{E}}_z^{i,j,k}(t)}{\Delta y} - \frac{\tilde{\mathcal{E}}_y^{i,j,k+1}(t) - \tilde{\mathcal{E}}_y^{i,j,k}(t)}{\Delta y} = \frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\mathcal{H}_x^{i,j,k}(t + {}^{\Delta t}/_2) - \mathcal{H}_x^{i,j,k}(t - {}^{\Delta t}/_2)}{\Delta t}. \tag{41}$$

Similarly, it is possible to write the curl equations for the other components of $\tilde{\mathcal{E}}$ and for $\mathcal{H}$:

$$C_y^E = \frac{\tilde{\mathcal{E}}_x^{i,j,k+1}(t) - \tilde{\mathcal{E}}_x^{i,j,k}(t)}{\Delta z} - \frac{\tilde{\mathcal{E}}_z^{i+1,j,k}(t) - \tilde{\mathcal{E}}_z^{i,j,k}(t)}{\Delta x} \tag{42}$$

$$C_z^E = \frac{\tilde{\mathcal{E}}_y^{i+1,j,k}(t) - \tilde{\mathcal{E}}_y^{i,j,k}(t)}{\Delta x} - \frac{\tilde{\mathcal{E}}_x^{i,j+1,k}(t) - \tilde{\mathcal{E}}_x^{i,j+1,k}(t)}{\Delta y} \tag{43}$$

$$C_x^H = \frac{\mathcal{H}_z^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_z^{i,j-1,k}(t + {}^{\Delta t}/2)}{\Delta y} - \frac{\mathcal{H}_y^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_y^{i,j,k-1}(t + {}^{\Delta t}/2)}{\Delta z} \tag{44}$$

$$C_y^H = \frac{\mathcal{H}_x^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_x^{i,j,k-1}(t + {}^{\Delta t}/2)}{\Delta z} - \frac{\mathcal{H}_z^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_z^{i-1,j,k}(t + {}^{\Delta t}/2)}{\Delta x} \tag{45}$$

$$C_z^H = \frac{\mathcal{H}_y^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_y^{i-1,j,k}(t + {}^{\Delta t}/2)}{\Delta x} - \frac{\mathcal{H}_x^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_x^{i,j-1,k}(t + {}^{\Delta t}/2)}{\Delta y} \tag{46}$$

Finally, the finite-difference equations are, for $\mathcal{H}_y$:

$$\frac{\tilde{\mathcal{E}}_x^{i,j,k+1}(t) - \tilde{\mathcal{E}}_x^{i,j,k}(t)}{\Delta z} - \frac{\tilde{\mathcal{E}}_z^{i+1,j,k}(t) - \tilde{\mathcal{E}}_z^{i,j,k}(t)}{\Delta x} = \frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\mathcal{H}_y^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_y^{i,j,k}(t - {}^{\Delta t}/2)}{\Delta t}, \tag{47}$$

for $\mathcal{H}_z$:

$$\frac{\tilde{\mathcal{E}}_y^{i+1,j,k}(t) - \tilde{\mathcal{E}}_y^{i,j,k}(t)}{\Delta x} - \frac{\tilde{\mathcal{E}}_x^{i,j+1,k}(t) - \tilde{\mathcal{E}}_x^{i,j+1,k}(t)}{\Delta y} = \frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\mathcal{H}_z^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_z^{i,j,k}(t - {}^{\Delta t}/2)}{\Delta t}, \tag{48}$$

for $\tilde{\mathcal{E}}_x$:

$$\frac{\mathcal{H}_z^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_z^{i,j-1,k}(t + {}^{\Delta t}/2)}{\Delta y} - \frac{\mathcal{H}_y^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_y^{i,j,k-1}(t + {}^{\Delta t}/2)}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{\tilde{\mathcal{E}}_x^{i,j,k}(t + {}^{\Delta t}/2) - \tilde{\mathcal{E}}_x^{i,j,k}(t)}{\Delta t}, \tag{49}$$

for $\tilde{\mathcal{E}}_y$:

$$\frac{\mathcal{H}_x^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_x^{i,j,k-1}(t + {}^{\Delta t}/2)}{\Delta z} - \frac{\mathcal{H}_z^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_z^{i-1,j,k}(t + {}^{\Delta t}/2)}{\Delta x} = \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{\tilde{\mathcal{E}}_y^{i,j,k}(t + {}^{\Delta t}/2) - \tilde{\mathcal{E}}_y^{i,j,k}(t)}{\Delta t}, \tag{50}$$

and for $\tilde{\mathcal{E}}_z$:

$$\frac{\mathcal{H}_y^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_y^{i-1,j,k}(t + {}^{\Delta t}/2)}{\Delta x} - \frac{\mathcal{H}_x^{i,j,k}(t + {}^{\Delta t}/2) - \mathcal{H}_x^{i,j-1,k}(t + {}^{\Delta t}/2)}{\Delta y} = \frac{\epsilon_{zz}^{i,j,k}}{c_0} \frac{\tilde{\mathcal{E}}_z^{i,j,k}(t + {}^{\Delta t}/2) - \tilde{\mathcal{E}}_z^{i,j,k}(t)}{\Delta t}. \tag{51}$$

To ease the implementation, the vector $\tilde{\mathcal{E}}$ will exists at integer step times $(0, \Delta t, 2\Delta t, \dots)$ meanwhile the vector $\mathcal{H}$ will exists at half time steps $({}^{\Delta t}/2, 3{}^{\Delta t}/2, 5{}^{\Delta t}/2, \dots)$.

III. The Perfect Matching Layer

The tensors for the permittivity and permeability will be [1]:

$$[\epsilon_{r,x}] = [\mu_{r,x}] = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix}, \tag{52}$$

for a wave travelling at x direction,

$$[\epsilon_{r,y}] = [\mu_{r,y}] = \begin{bmatrix} s_y & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & s_y \end{bmatrix}, \tag{53}$$

for a wave travelling at y direction, and

$$[\epsilon_{r,z}] = [\mu_{r,z}] = \begin{bmatrix} s_z & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & \frac{1}{s_z} \end{bmatrix}, \tag{54}$$

for a wave travelling at z direction.

And, for absorb all waves in all boundaries,

$$[\epsilon_{r,\text{UPML}}] = [\mu_{r,\text{UPML}}] = [S] = [\epsilon_{r,x}]\,[\epsilon_{r,y}]\,[\epsilon_{r,z}]$$

$$[S] = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}. \tag{55}$$

The loss is incorporated into the permittivity through the electrical conductivity σ as $\tilde{\epsilon} = \epsilon_r + \dfrac{\sigma}{j\omega\epsilon_0}$.

A. Incorporating PML into Maxwell's Equations

The Maxwell's Equations in the frequency domain are:

$$\nabla \times \mathbf{E}(\omega) = -j\omega\mu_0 [\mu_r] \mathbf{H}(\omega) \tag{56}$$
$$\nabla \times \mathbf{H}(\omega) = \sigma \mathbf{E}(\omega) + j\omega [S] \mathbf{D}(\omega) \tag{57}$$
$$\mathbf{D}(\omega) = \epsilon_0 [\epsilon_r] \mathbf{E}(\omega) \tag{58}$$

The PML $[S]$ can be incorporated as:

$$\nabla \times \mathbf{E}(\omega) = -j\omega\mu_0 [\mu_r] [S] \mathbf{H}(\omega) \tag{59}$$
$$\nabla \times \mathbf{H}(\omega) = \sigma \mathbf{E}(\omega) + j\omega \mathbf{D}(\omega) \tag{60}$$
$$\mathbf{D}(\omega) = \epsilon_0 [\epsilon_r] \mathbf{E}(\omega) \tag{61}$$

Normalizing the electric field:

$$\tilde{\mathbf{E}}(\omega) = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{E}(\omega) = \frac{1}{\eta_0} \mathbf{E}(\omega) \tag{62}$$

$$\tilde{\mathbf{D}}(\omega) = \sqrt{\frac{1}{\epsilon_0\mu_0}} \mathbf{D}(\omega) = c_0 \mathbf{D}(\omega) \tag{63}$$

Hence, the equations become:

$$\nabla \times \tilde{\mathbf{E}}(\omega) = -j\omega \frac{[\mu_r]}{c_0} [S] \mathbf{H}(\omega) \tag{64}$$
$$\nabla \times \mathbf{H}(\omega) = \eta_0 \sigma \tilde{\mathbf{E}}(\omega) + \frac{j\omega}{c_0} [S] \tilde{\mathbf{D}}(\omega) \tag{65}$$
$$\mathbf{D}(\omega) = [\epsilon_r] \tilde{\mathbf{E}}(\omega) \tag{66}$$

Keeping $[S]$ separate from $[\mu_r]$ and $[\epsilon_r]$ allows the PML to be handled independently from the materials being simulated.

The ω will be ommitted from the equations.

Considering only the diagonal terms in $[\mu_r]$, $[\epsilon_r]$ and $[\sigma]$, the final form of the Maxwell's Equations with UPML are [2]:

$$j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) \mathbf{H}_x = -\frac{c_0}{\mu_{xx}} \mathbf{C}^E_x \tag{67}$$

$$j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) \mathbf{H}_y = -\frac{c_0}{\mu_{yy}} \mathbf{C}^E_y \tag{68}$$

$$j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1} \mathbf{H}_z = -\frac{c_0}{\mu_{zz}} \mathbf{C}^E_z \tag{69}$$

$$j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) \tilde{\mathbf{D}}_x = c_0 \mathbf{C}^H_x - \frac{\sigma_{xx}}{\epsilon_0} \tilde{\mathbf{E}}_x \tag{70}$$

$$j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) \tilde{\mathbf{D}}_y = c_0 \mathbf{C}^H_y - \frac{\sigma_{yy}}{\epsilon_0} \tilde{\mathbf{E}}_y \tag{71}$$

$$j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1} \tilde{\mathbf{D}}_z = c_0 \mathbf{C}^H_z - \frac{\sigma_{zz}}{\epsilon_0} \tilde{\mathbf{E}}_z \tag{72}$$

$$\tilde{\mathbf{D}}_x = \epsilon_{xx} \tilde{\mathbf{E}}_x, \tag{73}$$
$$\tilde{\mathbf{D}}_y = \epsilon_{yy} \tilde{\mathbf{E}}_y, \tag{74}$$
$$\tilde{\mathbf{D}}_z = \epsilon_{zz} \tilde{\mathbf{E}}_z. \tag{75}$$

B. Conversion to the Time-Domain

First, assume no conductivity: $[\sigma] = 0$.
Starting from (67):

$$j\omega \mathbf{H}_x + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} \mathbf{H}_x + \frac{1}{j\omega} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} \mathbf{H}_x = -\frac{c_0}{\mu_{xx}} \mathbf{C}^E_x - \frac{1}{j\omega} \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} \mathbf{C}^E_x \tag{76}$$

In the time-domain becomes:

$$\partial_t \mathcal{H}_x + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} \mathcal{H}_x + \int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} \mathcal{H}_x(\tau) d\tau = -\frac{c_0}{\mu_{xx}} C^E_x - \int_{\infty}^{t} \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C^E_x(\tau) d\tau \tag{77}$$

Similarly, for the other components:

$$\partial_t \mathcal{H}_y + \frac{\sigma'_x + \sigma'_z}{\epsilon_0} \mathcal{H}_y + \int_{-\infty}^{t} \frac{\sigma'_x \sigma'_z}{\epsilon_0^2} \mathcal{H}_y(\tau) d\tau = -\frac{c_0}{\mu_{yy}} C^E_y - \int_{\infty}^{t} \frac{c_0 \sigma'_y}{\epsilon_0 \mu_{yy}} C^E_y(\tau) d\tau \tag{78}$$

$$\partial_t \mathcal{H}_z + \frac{\sigma'_x + \sigma'_y}{\epsilon_0} \mathcal{H}_z + \int_{-\infty}^{t} \frac{\sigma'_x \sigma'_y}{\epsilon_0^2} \mathcal{H}_z(\tau) d\tau = -\frac{c_0}{\mu_{zz}} C^E_z - \int_{\infty}^{t} \frac{c_0 \sigma'_z}{\epsilon_0 \mu_{zz}} C^E_z(\tau) d\tau \tag{79}$$

C. The PML Parameters

References

[1] R. Rumpf, *Electromagnetic and Photonic Simulation for the Beginner: Finite-Difference Frequency-Domain in MATLAB*, 01 2022.

[2] Rumpf, Raymond. Derivation of 3D update equations with a UPML. [Online]. Available: https://empossible.net/wp-content/uploads/2020/01/Lecture-Derivation-of-3D-Update-Equations-w-PML.pdf