Estudo Orientado 1

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I. Expansion of Maxwell's Curl Equations in Cartesian Coordinates

The Maxwell's equations are:

$$\nabla \times \mathcal{E}(t) = -\partial_t \mathcal{B}(t), \tag{1}$$
$$\nabla \times \mathcal{H}(t) = \partial_t \mathcal{D}(t), \tag{2}$$
$$\nabla \cdot \mathcal{B}(t) = 0, \tag{3}$$
$$\nabla \cdot \mathcal{D}(t) = 0, \tag{4}$$

where $\partial_t \cdot = \frac{\partial \cdot}{\partial t}$.

The the constitutive relations are:

$$\mathcal{B}(t) = [\mu_0 \mu_r(t)] * \mathcal{H}(t), \tag{5}$$
$$\mathcal{D}(t) = [\epsilon_0 \epsilon_r(t)] * \mathcal{E}(t), \tag{6}$$

where $[\cdot]$ represents a tensor.

A. Normalizing the Electric Fields

It will be adopted the conventional approach in FDTD and the electric field will be normalized as:

$$\tilde{\mathcal{E}}(t) = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathcal{E}(t) = \frac{1}{\eta_0} \mathcal{E}(t). \tag{7}$$

Also, from now on, the time depency (t) will be ommited for cleaning notation reasons.

The other parameters related to the electric field must also be normalized:

$$\tilde{\mathcal{D}} = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \mathcal{D} = c_0 \mathcal{D}. \tag{8}$$

Therefore, the normalized Maxwell's equations become:

$$\nabla \times \tilde{\mathcal{E}} = -\partial_t \mathcal{B}, \tag{9}$$
$$\nabla \times \mathcal{H} = \partial_t \tilde{\mathcal{D}}, \tag{10}$$
$$\nabla \cdot \mathcal{B} = 0, \tag{11}$$
$$\nabla \cdot \tilde{\mathcal{D}} = 0. \tag{12}$$

B. Expanding Maxwell's Equations

To expand the equations, it will be assumed that $[\mu_r]$ and $[\epsilon_r]$ has only diagonal terms [1].

The equation $\nabla \times \tilde{\mathcal{E}} = -\frac{[\mu_r]}{c_0} \partial_t \mathcal{B}$ becomes:

$$\partial_z \tilde{\mathcal{E}}_y - \partial_y \tilde{\mathcal{E}}_z = \frac{\mu_{xx}}{c_0} \partial_t \mathcal{H}_x, \tag{13}$$
$$\partial_x \tilde{\mathcal{E}}_z - \partial_z \tilde{\mathcal{E}}_x = \frac{\mu_{yy}}{c_0} \partial_t \mathcal{H}_y, \tag{14}$$
$$\partial_y \tilde{\mathcal{E}}_x - \partial_x \tilde{\mathcal{E}}_y = \frac{\mu_{zz}}{c_0} \partial_t \mathcal{H}_z. \tag{15}$$

The equation $\nabla \times \mathcal{H} = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}$ becomes:

$$\partial_z \mathcal{H}_y - \partial_y \mathcal{H}_z = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_x, \tag{16}$$
$$\partial_x \mathcal{H}_z - \partial_z \mathcal{H}_x = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_y, \tag{17}$$
$$\partial_y \mathcal{H}_x - \partial_x \mathcal{H}_y = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_z. \tag{18}$$

Finally, the equation $\tilde{\mathcal{D}} = [\epsilon_r] \tilde{\mathcal{E}}$ becomes:

$$\tilde{\mathcal{D}}_x = \epsilon_{xx} \tilde{\mathcal{E}}_x, \tag{19}$$
$$\tilde{\mathcal{D}}_y = \epsilon_{yy} \tilde{\mathcal{E}}_y, \tag{20}$$
$$\tilde{\mathcal{D}}_z = \epsilon_{zz} \tilde{\mathcal{E}}_z. \tag{21}$$

C. Notation for Curl Terms

$$C_x^E = \partial_z \tilde{\mathcal{E}}_y - \partial_y \tilde{\mathcal{E}}_z, \tag{22}$$
$$C_y^E = \partial_x \tilde{\mathcal{E}}_z - \partial_z \tilde{\mathcal{E}}_x, \tag{23}$$
$$C_z^E = \partial_y \tilde{\mathcal{E}}_x - \partial_x \tilde{\mathcal{E}}_y. \tag{24}$$

$$C_x^H = \partial_z \mathcal{H}_y - \partial_y \mathcal{H}_z, \tag{25}$$
$$C_y^H = \partial_x \mathcal{H}_z - \partial_z \mathcal{H}_x, \tag{26}$$
$$C_z^H = \partial_y \mathcal{H}_x - \partial_x \mathcal{H}_y. \tag{27}$$

D. Final Equations Form

$$C_x^E = \frac{\mu_{xx}}{c_0} \partial_t \mathcal{H}_x, \tag{28}$$
$$C_y^E = \frac{\mu_{yy}}{c_0} \partial_t \mathcal{H}_y, \tag{29}$$
$$C_z^E = \frac{\mu_{zz}}{c_0} \partial_t \mathcal{H}_z. \tag{30}$$

$$C_x^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_x, \tag{31}$$

$$C_y^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_y, \tag{32}$$

$$C_z^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_z. \tag{33}$$

$$\tilde{\mathcal{D}}_x = \epsilon_{xx}\tilde{\mathcal{E}}_x, \tag{34}$$

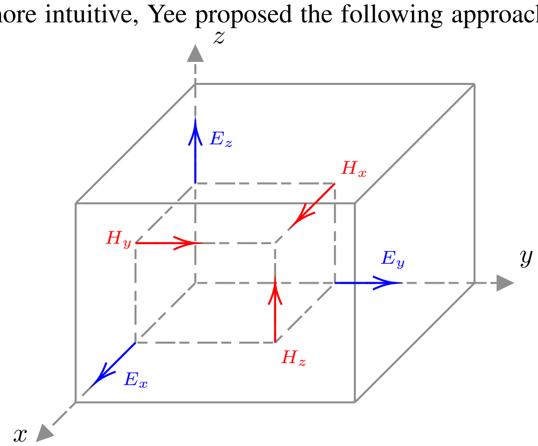
$$\tilde{\mathcal{D}}_y = \epsilon_{yy}\tilde{\mathcal{E}}_y, \tag{35}$$

$$\tilde{\mathcal{D}}_z = \epsilon_{zz}\tilde{\mathcal{E}}_z. \tag{36}$$

II. Finite-Difference Approximation to Maxwell's Equations

A. Yee Grid

A unit cell is constructed by dividing the 3 axis into discrete cells of size $(\Delta x, \Delta y, \Delta z)$. Inside this cell, it is necessary to put all the fields of the electromagnetic problem $(\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z, \mathcal{H}_x, \mathcal{H}_x, \mathcal{H}_z)$. Instead of putting all fields on the origin $(0,0,0)$, where is more intuitive, Yee proposed the following approach:



- $\mathcal{E}_x$ on $(\Delta x/2, 0, 0)$,
- $\mathcal{E}_y$ on $(0, \Delta y/2, 0)$,
- $\mathcal{E}_z$ on $(0, 0, \Delta z/2)$,
- $\mathcal{H}_x$ on $(0, \Delta y/2, \Delta z/2)$,
- $\mathcal{H}_y$ on $(\Delta x/2, 0, \Delta z/2)$,
- $\mathcal{H}_z$ on $(\Delta x/2, \Delta y/2, 0)$.

There are some reasons for using this scheme:

- The divergences are naturally zero.
- The physical boundary conditions are naturally satisfied.
- It is an elegant arrangement to approximate Maxwell's curl equations.

Additionaly, there are some consequences for using this scheme:

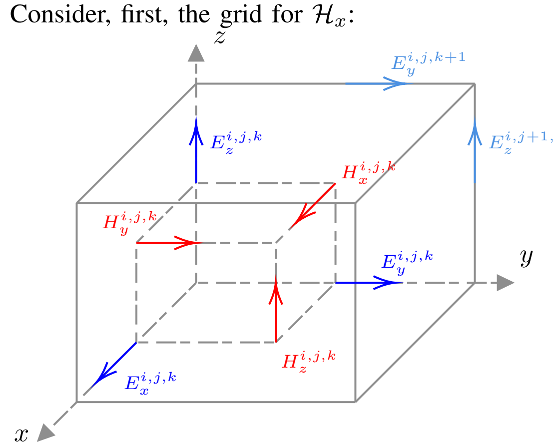
- Field components are in physically different locations.
- Field components may be in different materials even if they are in the same unit cell.
- Field components will be out of phase.

B. Finite-Difference Equations on Yee Grid

Each cell on the grid is identified by the coordines $(i\Delta x, j\Delta y, k\Delta z)$, where (i,j,k) are the index of the cell.

Note that on each face of the Yee cell there is the fields of the adjacent cell.

Consider, first, the grid for $\mathcal{H}_x$:



Based on this schematic, it is possible to write [1]:

$$\frac{\partial \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\partial y} = \frac{\tilde{\mathcal{E}}_z\Big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\Delta y}, \tag{37}$$

$$\frac{\partial \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\partial z} = \frac{\tilde{\mathcal{E}}_y\Big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\Delta y}. \tag{38}$$

Note that this space derivatives exists at time instant t and they exist at the same point as $\mathcal{H}_x^{i,j,k}$.

We need to explicitly write the time on the Yee grid equations, since it is essencial to write all the members of equations on the same time instant.

Hence, the C_x^E final equation is:

$$C_x^E = \frac{\tilde{\mathcal{E}}_z\Big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{\mathcal{E}}_y\Big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\Delta z} \tag{39}$$

Now, for the time derivative $\partial_t \mathcal{H}_x$ to exists at time t:

$$\partial_t \mathcal{H}_x\Big|_t^{i,j,k} = \frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}. \tag{40}$$

So, the finite-difference equation for $\mathcal{H}_x$ becomes:

$$\frac{\tilde{\mathcal{E}}_z\Big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{\mathcal{E}}_y\Big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\Delta y} = \frac{\mu_{xx}\Big|^{i,j,k}}{c_0} \frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}. \tag{41}$$

Similarly, it is possible to write the curl equations for the other components of $\tilde{\mathcal{E}}$ and for $\mathcal{H}$:

$$C_y^E = \frac{\tilde{\mathcal{E}}_x\Big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta z} - \frac{\tilde{\mathcal{E}}_z\Big|_t^{i+1,j,k} - \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\Delta x} \quad (42)$$

$$C_z^E = \frac{\tilde{\mathcal{E}}_y\Big|_t^{i+1,j,k} - \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\Delta x} - \frac{\tilde{\mathcal{E}}_x\Big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta y} \quad (43)$$

$$C_x^H = \frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta y} - \frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta z} \quad (44)$$

$$C_y^H = \frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta z} - \frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta x} \quad (45)$$

$$C_z^H = \frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta x} - \frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta y} \quad (46)$$

Finally, the finite-difference equations are, for $\mathcal{H}_y$:

$$\frac{\tilde{\mathcal{E}}_x\Big|_t^{i,j,k+1} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta z} - \frac{\tilde{\mathcal{E}}_z\Big|_t^{i+1,j,k} - \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\Delta x} = \frac{\mu_{yy}\Big|^{i,j,k}}{c_0}\frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}, \quad (47)$$

for $\mathcal{H}_z$:

$$\frac{\tilde{\mathcal{E}}_y\Big|_t^{i+1,j,k} - \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\Delta x} - \frac{\tilde{\mathcal{E}}_x\Big|_t^{i,j+1,k} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta y} = \frac{\mu_{zz}^{i,j,k}}{c_0}\frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}, \quad (48)$$

for $\tilde{\mathcal{E}}_x$:

$$\frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y} - \frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k-1}}{\Delta z} = \frac{\epsilon_{xx}\Big|^{i,j,k}}{c_0}\frac{\tilde{\mathcal{E}}_x\Big|_{t+\Delta t}^{i,j,k} - \tilde{\mathcal{E}}_x\Big|_t^{i,j,k}}{\Delta t}, \quad (49)$$

for $\tilde{\mathcal{E}}_y$:

$$\frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k-1}}{\Delta z} - \frac{\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_z\Big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x} = \frac{\epsilon_{yy}\Big|^{i,j,k}}{c_0}\frac{\tilde{\mathcal{E}}_y\Big|_{t+\Delta t}^{i,j,k} - \tilde{\mathcal{E}}_y\Big|_t^{i,j,k}}{\Delta t}, \quad (50)$$

and for $\tilde{\mathcal{E}}_z$:

$$\frac{\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_y\Big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x} - \frac{\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y} = \frac{\epsilon_{zz}^{i,j,k}}{c_0}\frac{\tilde{\mathcal{E}}_z\Big|_{t+\Delta t}^{i,j,k} - \tilde{\mathcal{E}}_z\Big|_t^{i,j,k}}{\Delta t}. \quad (51)$$

To ease the implementation, the vector $\tilde{\mathcal{E}}$ will exists at integer step times $(0, \Delta t, 2\Delta t, \dots)$ meanwhile the vector $\mathcal{H}$ will exists at half time steps $(\Delta t/2, 3\Delta t/2, 5\Delta t/2, \dots)$.

III. The Perfect Matching Layer

The tensors for the permittivity and permeability will be [1]:

$$[\epsilon_{r,x}] = [\mu_{r,x}] = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix}, \quad (52)$$

for a wave travelling at x direction,

$$[\epsilon_{r,y}] = [\mu_{r,y}] = \begin{bmatrix} s_y & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & s_y \end{bmatrix}, \quad (53)$$

for a wave travelling at y direction, and

$$[\epsilon_{r,z}] = [\mu_{r,z}] = \begin{bmatrix} s_z & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & \frac{1}{s_z} \end{bmatrix}, \quad (54)$$

for a wave travelling at z direction.

And, for absorb all waves in all boundaries,

$$[\epsilon_{r,\text{UPML}}] = [\mu_{r,\text{UPML}}] = [S] = [\epsilon_{r,x}]\,[\epsilon_{r,y}]\,[\epsilon_{r,z}]$$

$$[S] = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}. \quad (55)$$

The loss is incorporated into the permittivity through the electrical conductivity σ as $\tilde{\epsilon} = \epsilon_r + \frac{\sigma}{j\omega\epsilon_0}$.

A. The PML Parameters

From [1]:

$$\sigma'_x(x) = \frac{\epsilon_0}{2\Delta t}\left(\frac{x}{L_x}\right)^3 \tag{56}$$

$$\sigma'_y(y) = \frac{\epsilon_0}{2\Delta t}\left(\frac{y}{L_y}\right)^3 \tag{57}$$

$$\sigma'_z(z) = \frac{\epsilon_0}{2\Delta t}\left(\frac{z}{L_z}\right)^3, \tag{58}$$

where L_i is the length of the PML extending in the i direction.

B. Incorporating PML into Maxwell's Equations

The Maxwell's Equations in the frequency domain are:

$$\nabla \times \mathbf{E}(\omega) = -j\omega\mu_0 [\mu_r] \mathbf{H}(\omega) \tag{59}$$

$$\nabla \times \mathbf{H}(\omega) = \sigma\mathbf{E}(\omega) + j\omega [S] \mathbf{D}(\omega) \tag{60}$$

$$\mathbf{D}(\omega) = \epsilon_0 [\epsilon_r] \mathbf{E}(\omega) \tag{61}$$

The PML $[S]$ can be incorporated as:

$$\nabla \times \mathbf{E}(\omega) = -j\omega\mu_0 [\mu_r] [S] \mathbf{H}(\omega) \tag{62}$$

$$\nabla \times \mathbf{H}(\omega) = \sigma\mathbf{E}(\omega) + j\omega\mathbf{D}(\omega) \tag{63}$$

$$\mathbf{D}(\omega) = \epsilon_0 [\epsilon_r] \mathbf{E}(\omega) \tag{64}$$

Normalizing the electric field:

$$\tilde{\mathbf{E}}(\omega) = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{E}(\omega) = \frac{1}{\eta_0}\mathbf{E}(\omega) \tag{65}$$

$$\tilde{\mathbf{D}}(\omega) = \sqrt{\frac{1}{\epsilon_0\mu_0}}\mathbf{D}(\omega) = c_0\mathbf{D}(\omega) \tag{66}$$

Hence, the equations become:

$$\nabla \times \tilde{\mathbf{E}}(\omega) = -j\omega\frac{[\mu_r]}{c_0}[S]\mathbf{H}(\omega) \tag{67}$$

$$\nabla \times \mathbf{H}(\omega) = \eta_0\sigma\tilde{\mathbf{E}}(\omega) + \frac{j\omega}{c_0}[S]\tilde{\mathbf{D}}(\omega) \tag{68}$$

$$\mathbf{D}(\omega) = [\epsilon_r]\tilde{\mathbf{E}}(\omega) \tag{69}$$

Keeping $[S]$ separate from $[\mu_r]$ and $[\epsilon_r]$ allows the PML to be handled independently from the materials being simulated.

The ω will be ommitted from the equations.

Considering only the diagonal terms in $[\mu_r]$, $[\epsilon_r]$ and $[\sigma]$, the final form of the Maxwell's Equations with UPML are [2]:

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)\mathbf{H}_x = -\frac{c_0}{\mu_{xx}}\mathbf{C}^E_x \tag{70}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1}\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)\mathbf{H}_y = -\frac{c_0}{\mu_{yy}}\mathbf{C}^E_y \tag{71}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1}\mathbf{H}_z = -\frac{c_0}{\mu_{zz}}\mathbf{C}^E_z \tag{72}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)\tilde{\mathbf{D}}_x = c_0\mathbf{C}^H_x - \frac{\sigma_{xx}}{\epsilon_0}\tilde{\mathbf{E}}_x \tag{73}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1}\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)\tilde{\mathbf{D}}_y = c_0\mathbf{C}^H_y - \frac{\sigma_{yy}}{\epsilon_0}\tilde{\mathbf{E}}_y \tag{74}$$

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1}\tilde{\mathbf{D}}_z = c_0\mathbf{C}^H_z - \frac{\sigma_{zz}}{\epsilon_0}\tilde{\mathbf{E}}_z \tag{75}$$

$$\tilde{\mathbf{D}}_x = \epsilon_{xx}\tilde{\mathbf{E}}_x, \tag{76}$$

$$\tilde{\mathbf{D}}_y = \epsilon_{yy}\tilde{\mathbf{E}}_y, \tag{77}$$

$$\tilde{\mathbf{D}}_z = \epsilon_{zz}\tilde{\mathbf{E}}_z. \tag{78}$$

C. Conversion to the Time-Domain

First, assume no conductivity: $[\sigma] = 0$.

Starting from (70):

$$j\omega\mathbf{H}_x + \frac{\sigma'_y+\sigma'_z}{\epsilon_0}\mathbf{H}_x + \frac{1}{j\omega}\frac{\sigma'_y\sigma'_z}{\epsilon_0^2}\mathbf{H}_x = -\frac{c_0}{\mu_{xx}}\mathbf{C}^E_x - \frac{1}{j\omega}\frac{c_0\sigma'_x}{\epsilon_0\mu_{xx}}\mathbf{C}^E_x \tag{79}$$

In the time-domain becomes:

$$\partial_t\mathcal{H}_x + \frac{\sigma'_y+\sigma'_z}{\epsilon_0}\mathcal{H}_x + \int_{-\infty}^{t}\frac{\sigma'_y\sigma'_z}{\epsilon_0^2}\mathcal{H}_x(\tau)d\tau = -\frac{c_0}{\mu_{xx}}C^E_x - \int_{\infty}^{t}\frac{c_0\sigma'_x}{\epsilon_0\mu_{xx}}C^E_x(\tau)d\tau \tag{80}$$

Similarly, for the other components:

$$\partial_t \mathcal{H}_y + \frac{\sigma'_x + \sigma'_z}{\epsilon_0}\mathcal{H}_y + \int_{-\infty}^{t} \frac{\sigma'_x \sigma'_z}{\epsilon_0^2}\mathcal{H}_y(\tau)d\tau$$
$$= -\frac{c_0}{\mu_{yy}}C_y^E - \int_{\infty}^{t} \frac{c_0\sigma'_y}{\epsilon_0\mu_{yy}}C_y^E(\tau)d\tau \tag{81}$$

$$\partial_t \mathcal{H}_z + \frac{\sigma'_x + \sigma'_y}{\epsilon_0}\mathcal{H}_z + \int_{-\infty}^{t} \frac{\sigma'_x \sigma'_y}{\epsilon_0^2}\mathcal{H}_z(\tau)d\tau$$
$$= -\frac{c_0}{\mu_{zz}}C_z^E - \int_{\infty}^{t} \frac{c_0\sigma'_z}{\epsilon_0\mu_{zz}}C_z^E(\tau)d\tau \tag{82}$$

$$\partial_t \tilde{\mathcal{D}}_x + \frac{\sigma'_y + \sigma'_z}{\epsilon_0}\tilde{\mathcal{D}}_x + \frac{\sigma'_y \sigma'_z}{\epsilon_0^2}\int_{-\infty}^{t} \tilde{\mathcal{D}}_x(\tau)d\tau$$
$$= c_0 C_x^H + \frac{c_0\sigma'_x}{\epsilon_0}\int_{-\infty}^{t} C_x^H(\tau)d\tau \tag{83}$$

$$\partial_t \tilde{\mathcal{D}}_y + \frac{\sigma'_x + \sigma'_z}{\epsilon_0}\tilde{\mathcal{D}}_y + \frac{\sigma'_x \sigma'_z}{\epsilon_0^2}\int_{-\infty}^{t} \tilde{\mathcal{D}}_y(\tau)d\tau$$
$$= c_0 C_y^H + \frac{c_0\sigma'_y}{\epsilon_0}\int_{-\infty}^{t} C_y^H(\tau)d\tau \tag{84}$$

$$\partial_t \tilde{\mathcal{D}}_z + \frac{\sigma'_x + \sigma'_y}{\epsilon_0}\tilde{\mathcal{D}}_z + \frac{\sigma'_x \sigma'_y}{\epsilon_0^2}\int_{-\infty}^{t} \tilde{\mathcal{D}}_z(\tau)d\tau$$
$$= c_0 C_z^H + \frac{c_0\sigma'_z}{\epsilon_0}\int_{-\infty}^{t} C_z^H(\tau)d\tau \tag{85}$$

The update equations for $\tilde{\mathcal{D}}$ are the same as (34), (35) and (36).

D. Numerical Approximations

Remark: Yee cell will be considered for numerical implementation.

Starting from (80), each term of equation will be approximated as shown.

For the term 1, the time approximation will be the same as used before:

$$\partial_t \mathcal{H}_x(t) \approx \frac{\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k} - \mathcal{H}_x\Big|_{(t-\Delta t/2)}^{i,j,k}}{\Delta t} \tag{86}$$

For the term 2, it is necessary to approximate $\mathcal{H}_x(t)$, that will be done by averaging the values at $t + \Delta t/2$ and $t - \Delta t/2$:

$$\frac{\sigma'_y + \sigma'_z}{\epsilon_0}\mathcal{H}_x(t) \approx \frac{\sigma'_y + \sigma'_z}{\epsilon_0}\frac{\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k} + \mathcal{H}_x\Big|_{(t-\Delta t/2)}^{i,j,k}}{2} \tag{87}$$

For the term 3, it is necessary to approximate the integral with a summation:

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2}\mathcal{H}_x(\tau)d\tau \approx \frac{\sigma'_y \sigma'_z}{\epsilon_0^2}\sum_{T=\Delta t/2}^{t+\Delta t/2} \mathcal{H}_x\Big|_T^{i,j,k}\Delta t \tag{88}$$

However, in this way the summation is going future on time. The fix is simple: just pull out the last term from summation and do the integration over half a time step:

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2}\mathcal{H}_x(\tau)d\tau$$
$$\approx \frac{\sigma'_y \sigma'_z}{\epsilon_0^2}\left(\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k}\frac{\Delta t}{2} + \sum_{T=\Delta t/2}^{t-\Delta t/2} \mathcal{H}_x\Big|_T^{i,j,k}\Delta t\right)$$
$$\approx \frac{\sigma'_y \sigma'_z \Delta t}{\epsilon_0^2}\left(\frac{\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k} - \mathcal{H}_x\Big|_{(t-\Delta t/2)}^{i,j,k}}{4}\right.$$
$$\left. + \sum_{T=\Delta t/2}^{t-\Delta t/2} \mathcal{H}_x\Big|_T^{i,j,k}\right) \tag{89}$$

For the term 4, the curl approximation for will be the same as in (39):

$$-\frac{c_0}{\mu_{xx}}C_x^E \approx -\frac{c_0}{\mu_{xx}\Big|^{i,j,k}}C_x^E\Big|_t^{i,j,k} \tag{90}$$

For the term 5, as for the term 3, the integral will be approximated with a summation:

$$-\int_{\infty}^{t} \frac{c_0\sigma'_x}{\epsilon_0\mu_{xx}}C_x^E(\tau)d\tau = -\frac{c_0\sigma'_x}{\epsilon_0\mu_{xx}}\int_{\infty}^{t} C_x^E(\tau)d\tau$$
$$\approx -\frac{c_0\sigma_x^H\Big|^{i,j,k}}{\epsilon_0\mu_{xx}\Big|^{i,j,k}}\sum_{T=0}^{t} C_x^E\Big|_T^{i,j,k}\Delta t$$
$$\approx -\frac{c_0\Delta t\sigma_x^H\Big|^{i,j,k}}{\epsilon_0\mu_{xx}\Big|^{i,j,k}}\sum_{T=0}^{t} C_x^E\Big|_T^{i,j,k} \tag{91}$$

E. Update equations

Starting from the numerical approximation of (80):

$$\frac{\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k}-\mathcal{H}_x\Big|_{(t-\Delta t/2)}^{i,j,k}}{\Delta t}+\frac{\sigma_y'+\sigma_z'}{\epsilon_0}\frac{\mathcal{H}_x\Big|_{(t+\Delta t/2)}^{i,j,k}+\mathcal{H}_x\Big|_{(t-\Delta t/2)}^{i,j,k}}{2}=-\frac{c_0}{\mu_{xx}\Big|^{i,j,k}}C_x^E\Big|_t^{i,j,k}-\frac{c_0\Delta t\sigma_x^H\Big|^{i,j,k}}{\epsilon_0\mu_{xx}\Big|^{i,j,k}}\sum_{T=0}^{t}C_x^E\Big|_T^{i,j,k} \tag{92}$$

it is possible to isolate $\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k}$ as:

$$\begin{aligned}\mathcal{H}_x\Big|_{t+\Delta t/2}^{i,j,k}&=m_{x_1}\Big|^{i,j,k}\mathcal{H}_x\Big|_{t-\Delta t/2}^{i,j,k}\\&+m_{x_2}\Big|^{i,j,k}C_x^E\Big|_t^{i,j,k}+m_{x_3}I_{C_x^E}\Big|_t^{i,j,k}\\&+m_{x_4}I_{H_x}\Big|_t^{i,j,k},\end{aligned}\tag{93}$$

where:

$$m_{x_0}\Big|^{i,j,k}=\frac{1}{\Delta t}+\frac{\sigma_y'\Big|^{i,j,k}+\sigma_z'\Big|^{i,j,k}}{2\epsilon_0}+\frac{\sigma_y'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}\tag{94}$$

$$m_{x_1}\Big|^{i,j,k}=\frac{1}{m_{x_0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t}-\frac{\sigma_y'\Big|^{i,j,k}+\sigma_z'\Big|^{i,j,k}}{2\epsilon_0}-\frac{\sigma_y'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}\right]\tag{95}$$

$$m_{x_2}\Big|^{i,j,k}=-\frac{1}{m_{x_0}\Big|^{i,j,k}}\frac{c_0}{\mu_{xx}\Big|^{i,j,k}}\tag{96}$$

$$m_{x_3}\Big|^{i,j,k}=-\frac{1}{m_{x_0}\Big|^{i,j,k}}\frac{c_0\Delta t}{\epsilon_0}\frac{\sigma_x'\Big|^{i,j,k}}{\mu_{xx}\Big|^{i,j,k}}\tag{97}$$

$$m_{x_4}\Big|^{i,j,k}=-\frac{1}{m_{x_0}\Big|^{i,j,k}}\frac{\Delta t}{\epsilon_0^2}\sigma_y'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}\tag{98}$$

$$I_{C_x^E}\Big|_t^{i,j,k}=\sum_{T=0}^{t}C_x^E\Big|_T^{i,j,k}\tag{99}$$

$$I_{H_x}\Big|_{t-\Delta t/2}^{i,j,k}=\sum_{T=\Delta t/2}^{t-\Delta t/2}\mathcal{H}_x\Big|_T^{i,j,k}\tag{100}$$

Similarly, is possible to deduce the update equations for the other components starting from (81), (82), (83), (84) and (85). The results are as following.

For $\mathcal{H}_y$:

$$\begin{aligned}\mathcal{H}_y\Big|_{t+\Delta t/2}^{i,j,k}&=m_{y_1}\Big|^{i,j,k}\mathcal{H}_y\Big|_{t-\Delta t/2}^{i,j,k}\\&+m_{y_2}\Big|^{i,j,k}C_y^E\Big|_t^{i,j,k}+m_{y_3}I_{C_y^E}\Big|_t^{i,j,k}\\&+m_{y_4}I_{H_y}\Big|_t^{i,j,k},\end{aligned}\tag{101}$$

where:

$$m_{y_0}\Big|^{i,j,k}=\frac{1}{\Delta t}+\frac{\sigma_x'\Big|^{i,j,k}+\sigma_z'\Big|^{i,j,k}}{2\epsilon_0}+\frac{\sigma_x'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}\tag{102}$$

$$m_{y_1}\Big|^{i,j,k}=\frac{1}{m_{y_0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t}-\frac{\sigma_x'\Big|^{i,j,k}+\sigma_z'\Big|^{i,j,k}}{2\epsilon_0}-\frac{\sigma_x'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}\right]\tag{103}$$

$$m_{y_2}\Big|^{i,j,k}=-\frac{1}{m_{y_0}\Big|^{i,j,k}}\frac{c_0}{\mu_{yy}\Big|^{i,j,k}}\tag{104}$$

$$m_{y_3}\Big|^{i,j,k}=-\frac{1}{m_{y_0}\Big|^{i,j,k}}\frac{c_0\Delta t}{\epsilon_0}\frac{\sigma_y'\Big|^{i,j,k}}{\mu_{yy}\Big|^{i,j,k}}\tag{105}$$

$$m_{y_4}\Big|^{i,j,k}=-\frac{1}{m_{y_0}\Big|^{i,j,k}}\frac{\Delta t}{\epsilon_0^2}\sigma_x'\Big|^{i,j,k}\sigma_z'\Big|^{i,j,k}\tag{106}$$

$$I_{C_y^E}\Big|_t^{i,j,k}=\sum_{T=0}^{t}C_y^E\Big|_T^{i,j,k}\tag{107}$$

$$I_{H_y}\Big|_{t-\Delta t/2}^{i,j,k}=\sum_{T=\Delta t/2}^{t-\Delta t/2}\mathcal{H}_y\Big|_T^{i,j,k}\tag{108}$$

For $\mathcal{H}_z$:

$$\begin{aligned}\mathcal{H}_z\Big|_{t+\Delta t/2}^{i,j,k}&=m_{z_1}\Big|^{i,j,k}\mathcal{H}_z\Big|_{t-\Delta t/2}^{i,j,k}\\&+m_{z_2}\Big|^{i,j,k}C_z^E\Big|_t^{i,j,k}+m_{z_3}I_{C_z^E}\Big|_t^{i,j,k}\\&+m_{z_4}I_{H_z}\Big|_t^{i,j,k},\end{aligned}\tag{109}$$

where:

$$m_{z_0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma'_x\Big|^{i,j,k} + \sigma'_y\Big|^{i,j,k}}{2\epsilon_0} + \frac{\sigma'_x\Big|^{i,j,k}\sigma'_y\Big|^{i,j,k}\Delta t}{4\epsilon_0^2} \tag{110}$$

$$m_{z_1}\Big|^{i,j,k} = \frac{1}{m_{z_0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma'_x\Big|^{i,j,k} + \sigma'_y\Big|^{i,j,k}}{2\epsilon_0} - \frac{\sigma'_x\Big|^{i,j,k}\sigma'_y\Big|^{i,j,k}\Delta t}{4\epsilon_0^2}\right] \tag{111}$$

$$m_{z_2}\Big|^{i,j,k} = -\frac{1}{m_{z_0}\Big|^{i,j,k}}\frac{c_0}{\mu_{zz}\Big|^{i,j,k}} \tag{112}$$

$$m_{z_3}\Big|^{i,j,k} = -\frac{1}{m_{z_0}\Big|^{i,j,k}}\frac{c_0\Delta t}{\epsilon_0}\frac{\sigma'_z\Big|^{i,j,k}}{\mu_{zz}\Big|^{i,j,k}} \tag{113}$$

$$m_{z_4}\Big|^{i,j,k} = -\frac{1}{m_{z_0}\Big|^{i,j,k}}\frac{\Delta t}{\epsilon_0^2}\sigma'_x\Big|^{i,j,k}\sigma'_y\Big|^{i,j,k} \tag{114}$$

$$I_{C_z^E}\Big|_t^{i,j,k} = \sum_{T=0}^{t} C_z^E\Big|_t^{i,j,k} \tag{115}$$

$$I_{H_z}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} \mathcal{H}_z\Big|_T^{i,j,k} \tag{116}$$

For $\tilde{\mathcal{D}}_x$:

$$\tilde{\mathcal{D}}_x\Big|_{t+\Delta t/2}^{i,j,k} = m_{x_1}\Big|^{i,j,k}\tilde{\mathcal{D}}_x\Big|_t^{i,j,k} + m_{x_2}\Big|^{i,j,k}C_x^H\Big|_{t+\Delta t/2}^{i,j,k} + m_{x_3}I_{C_x^H}\Big|_{t-\Delta t/2}^{i,j,k} + m_{x_4}I_{D_x}\Big|_t^{i,j,k}, \tag{117}$$

where:

$$I_{C_x^H}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} C_x^H\Big|_T^{i,j,k} \tag{118}$$

$$I_{D_x}\Big|_t^{i,j,k} = \sum_{T=0}^{t} \tilde{\mathcal{D}}_x\Big|_T^{i,j,k} \tag{119}$$

For $\tilde{\mathcal{D}}_y$:

$$\tilde{\mathcal{D}}_y\Big|_{t+\Delta t/2}^{i,j,k} = m_{y_1}\Big|^{i,j,k}\tilde{\mathcal{D}}_y\Big|_t^{i,j,k} + m_{y_2}\Big|^{i,j,k}C_y^H\Big|_{t+\Delta t/2}^{i,j,k} + m_{y_3}I_{C_y^H}\Big|_{t-\Delta t/2}^{i,j,k} + m_{y_4}I_{D_y}\Big|_t^{i,j,k}, \tag{120}$$

where:

$$I_{C_y^H}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} C_y^H\Big|_T^{i,j,k} \tag{121}$$

$$I_{D_y}\Big|_t^{i,j,k} = \sum_{T=0}^{t} \tilde{\mathcal{D}}_y\Big|_T^{i,j,k} \tag{122}$$

For $\tilde{\mathcal{D}}_z$:

$$\tilde{\mathcal{D}}_z\Big|_{t+\Delta t/2}^{i,j,k} = m_{z_1}\Big|^{i,j,k}\tilde{\mathcal{D}}_z\Big|_t^{i,j,k} + m_{z_2}\Big|^{i,j,k}C_z^H\Big|_{t+\Delta t/2}^{i,j,k} + m_{z_3}I_{C_z^H}\Big|_{t-\Delta t/2}^{i,j,k} + m_{z_4}I_{D_z}\Big|_t^{i,j,k}, \tag{123}$$

where:

$$I_{C_z^H}\Big|_{t-\Delta t/2}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} C_z^H\Big|_T^{i,j,k} \tag{124}$$

$$I_{D_z}\Big|_t^{i,j,k} = \sum_{T=0}^{t} \tilde{\mathcal{D}}_z\Big|_T^{i,j,k} \tag{125}$$

For $\tilde{\mathcal{E}}_x$:

$$\tilde{\mathcal{E}}_x\Big|_{t+\Delta t}^{i,j,k} = m_{E_{x_1}}\tilde{\mathcal{D}}_x\Big|_{t+\Delta t}^{i,j,k} \tag{126}$$

$$\tilde{\mathcal{E}}_y\Big|_{t+\Delta t}^{i,j,k} = m_{E_{y_1}}\tilde{\mathcal{D}}_y\Big|_{t+\Delta t}^{i,j,k} \tag{127}$$

$$\tilde{\mathcal{E}}_z\Big|_{t+\Delta t}^{i,j,k} = m_{E_{z_1}}\tilde{\mathcal{D}}_z\Big|_{t+\Delta t}^{i,j,k} \tag{128}$$

IV. FDTD FOR A TRANSMISSION LINE

The following schematic will be considered for simulation purposes:

Consider the parameters:

Parameter	Description	Value
a	Half-length on y-axis	5 cm
b	Half-length on x-axis	5 cm
c	Half-length on z-axis	10 cm
d	Half-distance between strips	2 cm
N_x	Number of cells on x-axis	40
N_y	Number of cells on y-axis	40
N_z	Number of cells on z-axis	70
N_{PML}	Number of cells with PML on z-axis	10

From this table it is possible to derive other parameters:

Parameter	Description	Value
N_x^c	Index centered on x	20
N_y^c	Index centered on y	20
N_z^c	Index centered on z	35
$N_{\text{plate}}^{\text{high}}$	Y-index of top strip	32
$N_{\text{plate}}^{\text{low}}$	Y-index of low strip	8
Δx	$2b/N_x$	40
Δy	$2a/N_y$	70
Δz	$2c/N_z$	10

A. Boundary Conditions

1) On Source:

The source is located at $\left(N_x^c, N_{\text{plate}}^{\text{low}} : N_{\text{plate}}^{\text{high}}, N_z^c\right)$. It will be considered that the source is on $\tilde{\mathcal{E}}_y$ as shown below:

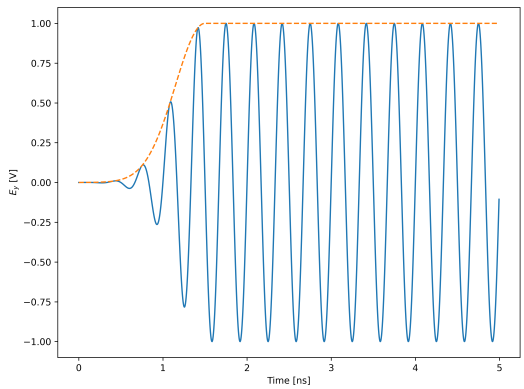


Figure 1: Input electric field with frequency $f_c = 3.0\,\text{GHz}$, peak time $t_p = 1.5\,\text{ns}$ and duration $t_f = 5.0\,\text{ns}$. The Gaussian that limits the sine function has $\tau = 0.5\,\text{ns}$.

2) On Conductor Strips:

There is one strip located at $\left(:, N_{\text{plate}}^{\text{low}}, :\right)$ and another at $\left(:, N_{\text{plate}}^{\text{high}}, :\right)$. At these positions is valid:

$$\tilde{\mathcal{E}}_x = \tilde{\mathcal{E}}_z = \mathcal{H}_y = 0 \tag{129}$$

3) Conductor Walls:

The condition for conductor walls is the same as for conductor strips. Their positions are at: $(0, :, :)$, $(N_x - 1, :, :)$, $(:, 0, :)$,$(:, N_y - 1, :)$, $(:, :, 0)$ and $(:, :, N_z - 1)$.

B. Step Time

The step time was chosen to obey the Courant Stability condition:

$$\Delta t \leq \frac{1}{c_0\sqrt{\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2 + \left(\frac{1}{\Delta z}\right)^2}} \tag{130}$$

C. Simulation Results

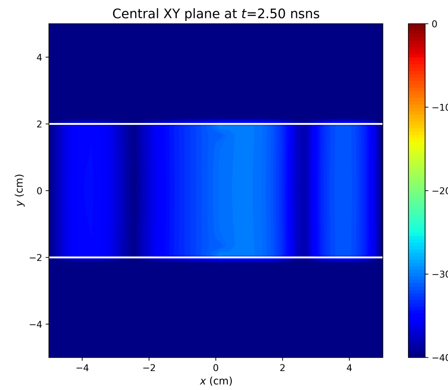


Figure 2: Central XY Plane.

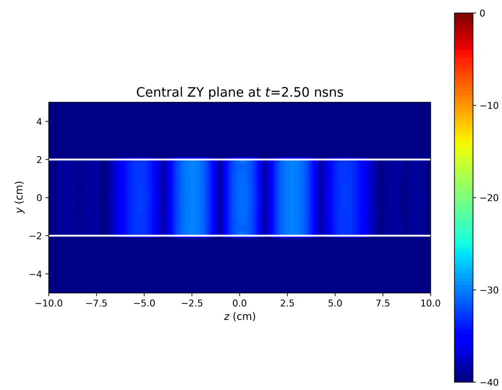


Figure 3: Central ZY Plane.

References

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