Estudo Orientado 1

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I. Expansion of Maxwell's Curl Equations in Cartesian Coordinates

The Maxwell's equations are:

$$\nabla \times \mathcal{E}(t) = -\partial_t \mathcal{B}(t), \quad (1)$$
$$\nabla \times \mathcal{H}(t) = \partial_t \mathcal{D}(t), \quad (2)$$
$$\nabla \cdot \mathcal{B}(t) = 0, \quad (3)$$
$$\nabla \cdot \mathcal{D}(t) = 0, \quad (4)$$

where $\partial_t \cdot = \frac{\partial \cdot}{\partial t}$.

The the constitutive relations are:

$$\mathcal{B}(t) = [\mu_0 \mu_r(t)] * \mathcal{H}(t), \quad (5)$$
$$\mathcal{D}(t) = [\epsilon_0 \epsilon_r(t)] * \mathcal{E}(t), \quad (6)$$

where $[\cdot]$ represents a tensor.

A. Normalizing the Electric Fields

It will be adopted the conventional approach in FDTD and the electric field will be normalized as:

$$\tilde{\mathcal{E}}(t) = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathcal{E}(t) = \frac{1}{\eta_0} \mathcal{E}(t). \quad (7)$$

Also, from now on, the time depency (t) will be ommited for cleaning notation reasons.

The other parameters related to the electric field must also be normalized:

$$\tilde{\mathcal{D}} = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \mathcal{D} = c_0 \mathcal{D}. \quad (8)$$

Therefore, the normalized Maxwell's equations become:

$$\nabla \times \tilde{\mathcal{E}} = -\partial_t \mathcal{B}, \quad (9)$$
$$\nabla \times \mathcal{H} = \partial_t \tilde{\mathcal{D}}, \quad (10)$$
$$\nabla \cdot \mathcal{B} = 0, \quad (11)$$
$$\nabla \cdot \tilde{\mathcal{D}} = 0. \quad (12)$$

B. Expanding Maxwell's Equations

To expand the equations, it will be assumed that $[\mu_r]$ and $[\epsilon_r]$ has only diagonal terms.

The equation $\nabla \times \tilde{\mathcal{E}} = -\frac{[\mu_r]}{c_0} \partial_t \mathcal{B}$ becomes:

$$\partial_z \tilde{\mathcal{E}}_y - \partial_y \tilde{\mathcal{E}}_z = \frac{\mu_{xx}}{c_0} \partial_t \mathcal{H}_x, \quad (13)$$
$$\partial_x \tilde{\mathcal{E}}_z - \partial_z \tilde{\mathcal{E}}_x = \frac{\mu_{yy}}{c_0} \partial_t \mathcal{H}_y, \quad (14)$$
$$\partial_y \tilde{\mathcal{E}}_x - \partial_x \tilde{\mathcal{E}}_y = \frac{\mu_{zz}}{c_0} \partial_t \mathcal{H}_z. \quad (15)$$

The equation $\nabla \times \mathcal{H} = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}$ becomes:

$$\partial_z \mathcal{H}_y - \partial_y \mathcal{H}_z = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_x, \quad (16)$$
$$\partial_x \mathcal{H}_z - \partial_z \mathcal{H}_x = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_y, \quad (17)$$
$$\partial_y \mathcal{H}_x - \partial_x \mathcal{H}_y = \frac{1}{c_0} \partial_t \tilde{\mathcal{D}}_z. \quad (18)$$

Finally, the equation $\tilde{\mathcal{D}} = [\epsilon_r] \tilde{\mathcal{E}}$ becomes:

$$\tilde{\mathcal{D}}_x = \epsilon_{xx} \tilde{\mathcal{E}}_x, \quad (19)$$
$$\tilde{\mathcal{D}}_y = \epsilon_{yy} \tilde{\mathcal{E}}_y, \quad (20)$$
$$\tilde{\mathcal{D}}_z = \epsilon_{zz} \tilde{\mathcal{E}}_z. \quad (21)$$

C. Notation for Curl Terms

$$C_x^E = \partial_z \tilde{\mathcal{E}}_y - \partial_y \tilde{\mathcal{E}}_z, \quad (22)$$
$$C_y^E = \partial_x \tilde{\mathcal{E}}_z - \partial_z \tilde{\mathcal{E}}_x, \quad (23)$$
$$C_z^E = \partial_y \tilde{\mathcal{E}}_x - \partial_x \tilde{\mathcal{E}}_y. \quad (24)$$

$$C_x^H = \partial_z \mathcal{H}_y - \partial_y \mathcal{H}_z, \quad (25)$$
$$C_y^H = \partial_x \mathcal{H}_z - \partial_z \mathcal{H}_x, \quad (26)$$
$$C_z^H = \partial_y \mathcal{H}_x - \partial_x \mathcal{H}_y. \quad (27)$$

D. Final Equations Form

$$C_x^E = \frac{\mu_{xx}}{c_0} \partial_t \mathcal{H}_x, \quad (28)$$
$$C_y^E = \frac{\mu_{yy}}{c_0} \partial_t \mathcal{H}_y, \quad (29)$$
$$C_z^E = \frac{\mu_{zz}}{c_0} \partial_t \mathcal{H}_z. \quad (30)$$

$$C_x^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_x, \quad (31)$$

$$C_y^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_y, \quad (32)$$

$$C_z^H = \frac{1}{c_0}\partial_t \tilde{\mathcal{D}}_z. \quad (33)$$

$$\tilde{\mathcal{D}}_x = \epsilon_{xx}\tilde{\mathcal{E}}_x, \quad (34)$$

$$\tilde{\mathcal{D}}_y = \epsilon_{yy}\tilde{\mathcal{E}}_y, \quad (35)$$

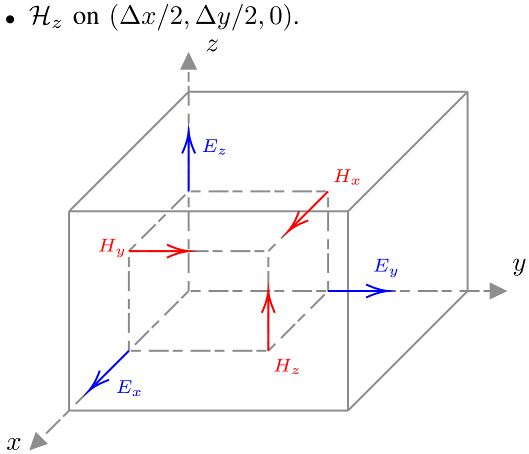
$$\tilde{\mathcal{D}}_z = \epsilon_{zz}\tilde{\mathcal{E}}_z. \quad (36)$$

II. Finite-Difference Approximation to Maxwell's Equations

A. Yee Grid

A unit cell is constructed by dividing the 3 axis into discrete cells of size $(\Delta x, \Delta y, \Delta z)$. Inside this cell, it is necessary to put all the fields of the electromagnetic problem $(\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z, \mathcal{H}_x, \mathcal{H}_x, \mathcal{H}_z)$. Instead of putting all fields on the origin $(0,0,0)$, where is more intuitive, Yee proposed the following approach:

- $\mathcal{E}_x$ on $(\Delta x/2, 0, 0)$,
- $\mathcal{E}_y$ on $(0, \Delta y/2, 0)$,
- $\mathcal{E}_z$ on $(0, 0, \Delta z/2)$,
- $\mathcal{H}_x$ on $(0, \Delta y/2, \Delta z/2)$,
- $\mathcal{H}_y$ on $(\Delta x/2, 0, \Delta z/2)$,
- $\mathcal{H}_z$ on $(\Delta x/2, \Delta y/2, 0)$.



There are some reasons for using this scheme:

- The divergences are naturally zero.
- The physical boundary conditions are naturally satisfied.
- It is an elegant arrangement to approximate Maxwell's curl equations.

Additionaly, there are some consequences for using this scheme:

- Field components are in physically different locations.
- Field components may be in different materials even if they are in the same unit cell.
- Field components will be out of phase.

B. Finite-Difference Equations on Yee Grid

Each cell on the grid is identified by the coordines $(i\Delta x, j\Delta y, k\Delta z)$, where (i,j,k) are the index of the cell.

Note that on each face of the Yee cell there is the fields of the adjacent cell. Consider, first, the equation for $\mathcal{H}_x$: