

Chapter 1

Introduction

1.1 What is This Course All About?

This course introduces basic concepts and methods of analysis. The course focuses on the theory of the real number system and calculus of functions of a real variable. The content will include:

1. The Real Number System: axioms; supremum and infimum.
2. Topology of the real number system including completeness, compactness.
3. Sequences and Convergence, including the algebra of limits.
4. Limits of Functions, including the algebra of limits.
5. Continuity, including the algebra of continuous functions, continuity of compositions, and uniform continuity.
6. Differentiation, including the algebra of derivatives, chain rule, Mean Value Theorem, Inverse Function Theorem, applications to behavior of functions, Taylor's Theorem and L'Hospital's Rule.
7. Riemann integration, including linearity and order properties, integrability of continuous functions, Riemann sums, the Fundamental Theorem of Calculus.

We will take an axiomatic approach (definition, theorem, and proof) to the subject, but along the way, you will develop intuition about the objects of real analysis and pick up more proof-writing skills. The emphasis of this course is on your ability to read, understand, and communicate mathematics in the context of real analysis.

The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.

Henri Poincaré

1.2 An Inquiry-Based Approach

This is not a lecture-oriented class or one in which mimicking prefabricated examples will lead you to success. You will be expected to work actively to construct your own understanding of the topics at hand with the readily available help of me and your classmates. Many of the concepts you learn and problems you work on will be new to you and ask you to stretch your thinking. You will experience *frustration* and *failure* before you experience *understanding*. This is part of the normal learning process. If you are doing things well, you should be confused at different points in the semester. The material is too rich for a human being to completely understand it immediately. Your viability as a professional in the modern workforce depends on your ability to embrace this learning process and make it work for you.

In order to promote a more active participation in your learning, we will incorporate ideas from an educational philosophy called inquiry-based learning (IBL). Loosely speaking, IBL is a student-centered method of teaching mathematics that engages students in sense-making activities. Students are given tasks requiring them to solve problems, conjecture, experiment, explore, create, and communicate. Rather than showing facts or a clear, smooth path to a solution, the instructor guides and mentors students via well-crafted problems through an adventure in mathematical discovery. According to Laursen and Rasmussen (2029), the Four Pillars of IBL are:

- Students engage deeply with coherent and meaningful mathematical tasks.
- Students collaboratively process mathematical ideas.
- Instructors inquire into student thinking.
- Instructors foster equity in their design and facilitation choices.

Don't fear failure. Not failure,
but low aim, is the crime. In
great attempts it is glorious
even to fail.

Bruce Lee

Much of the course will be devoted to students presenting their proposed solutions or proofs on the board and a significant portion of your grade will be determined by how much mathematics you produce. I use the word *produce* because I believe that the best way to learn mathematics is by doing mathematics. Someone cannot master a musical instrument or a martial art by simply watching, and in a similar fashion, you cannot master mathematics by simply watching; you must do mathematics!

In any act of creation, there must be room for experimentation, and thus allowance for mistakes, even failure. A key goal of our community is that we support each other—sharpening each other's thinking but also bolstering each other's confidence—so that we can make failure a productive experience. Mistakes are inevitable, and they should not

be an obstacle to further progress. It's normal to struggle and be confused as you work through new material. Accepting that means you can keep working even while feeling stuck, until you overcome and reach even greater accomplishments.

You will become clever through
your mistakes.

German Proverb

Furthermore, it is important to understand that solving genuine problems is difficult and takes time. You shouldn't expect to complete each problem in 10 minutes or less. Sometimes, you might have to stare at the problem for an hour before even understanding how to get started.

In this course, everyone will be required to

- read and interact with course notes and textbook on your own;
- write up quality solutions/proofs to assigned problems;
- present solutions/proofs on the board to the rest of the class;
- participate in discussions centered around a student's presented solution/proof;
- call upon your own prodigious mental faculties to respond in flexible, thoughtful, and creative ways to problems that may seem unfamiliar on first glance.

As the semester progresses, it should become clear to you what the expectations are.

Tell me and I forget, teach me
and I may remember, involve
me and I learn.

Benjamin Franklin

1.3 Rights of the Learner

As a student in this class, you have the right:

1. to be confused,
2. to make a mistake and to revise your thinking,
3. to speak, listen, and be heard, and
4. to enjoy doing mathematics.

You may encounter many
defeats, but you must not be
defeated.

Maya Angelou

1.4 Your Toolbox, Questions, and Observations

Throughout the semester, we will develop a list of *tools* that will help you understand and do mathematics. Your job is to keep a list of these tools, and it is suggested that you keep a running list someplace.

Next, it is of utmost importance that you work to understand every proof. (Every!) Questions are often your best tool for determining whether you understand a proof. Therefore, here are some sample questions that apply to any proof that you should be prepared to ask of yourself or the presenter:

- What method(s) of proof are you using?
- What form will the conclusion take?
- How did you know to set up that [equation, set, whatever]?
- How did you figure out what the problem was asking?
- Was this the first thing you tried?
- Can you explain how you went from this line to the next one?
- What were you thinking when you introduced this?
- Could we have ... instead?
- Would it be possible to ...?
- What if ...?

Another way to help you process and understand proofs is to try and make observations and connections between different ideas, proof statements and methods, and to compare approaches used by different people. Observations might sound like some of the following:

- When I tried this proof, I thought I needed to ... But I didn't need that, because ...
- I think that ... is important to this proof, because ...
- When I read the statement of this theorem, it seemed similar to this earlier theorem. Now I see that it [is/isn't] because ...

1.5 Rules of the Game

You should *not* look to resources outside the context of this course for help. That is, you should not be consulting the Internet, other texts, other faculty, or students outside of our course. On the other hand, you may use each other, the course notes, me, and your own intuition. In this class, earnest failure outweighs counterfeit success; you need not feel pressure to hunt for solutions outside your own creative and intellectual reserves. For more details, check out the Syllabus.

1.6 Structure of the Notes

As you read the notes, you will be required to digest the material in a meaningful way. It is your responsibility to read and understand new definitions and their related concepts. However, you will be supported in this sometimes difficult endeavor. In addition, you will be asked to complete exercises aimed at solidifying your understanding of the material. Most importantly, you will be asked to make conjectures, produce counterexamples, and prove theorems.

Most items in the notes are labelled with a number. A **Definition** is just that. However, in this type of course, it is extremely important to use definitions very accurately, and we try to use as few as possible to avoid confusion and over-saturation. A **Problem** is something to be proved. Some will be very easy, a few extremely hard. Most will be somewhere in between, to enhance your senses of confidence and accomplishment. Most class time will be hearing and responding to your solutions to these items; they are the core of the course. An **Exercise** is not at the same level of importance as a problem, but it should usually be very helpful in understanding the content (and hopefully proofs) of the problems. It may be a worked-out example provided in class, or it may be some instructions for trying your own examples. It is important to point out that there are very few examples in the notes. This is intentional. One of the goals of the items labelled as **Exercise** is for you to produce the examples. Lastly, there are many situations where you will want to refer to an earlier definition or problem. In this case, you should reference the statement by number. For example, you might write something like, “By Problem 2.20, we see that...”

1.7 Some Minimal Guidance

Especially in the opening sections, it won't be clear what facts from your prior experience in mathematics we are “allowed” to use. Unfortunately, addressing this issue is difficult and is something we will sort out along the way. However, in general, here are some minimal and vague guidelines to keep in mind.

First, there are times when we will need to do some basic algebraic manipulations. You should feel free to do this whenever the need arises. But you should show sufficient work along the way. You do not need to write down justifications for basic algebraic manipulations (e.g., adding 1 to both sides of an equation, adding and subtracting the same amount on the same side of an equation, adding like terms, factoring, basic simplification, etc.).

On the other hand, you do need to make explicit justification of the logical steps in a proof. When necessary, you should cite a previous definition, theorem, etc. by number.

Unlike the experience many of you had writing proofs in geometry, our proofs will be written in complete sentences. You should break sections of a proof into paragraphs and use proper grammar. There are some pedantic conventions for doing this that I will point out along the way. Initially, this will be an issue that most students will struggle with, but after a few weeks everyone will get the hang of it.

Ideally, you should rewrite the statements of theorems before you start the proof.

Moreover, for your sake and mine, you should label the statement with the appropriate number. I will expect you to indicate where the proof begins by writing “*Proof.*” at the beginning. Also, we will conclude our proofs with the standard “proof box” (i.e., \square or \blacksquare), which is typically right-justified.

Lastly, every time you write a proof, you need to make sure that you are making your assumptions crystal clear. Sometimes there will be some implicit assumptions that we can omit, but at least in the beginning, you should get in the habit of stating your assumptions up front. Typically, these statements will start off “Assume...” or “Let...”.

This should get you started. We will discuss more as the semester progresses. Now, go have fun and kick some butt!

If you want to sharpen a sword,
you have to remove a little
metal.

Unknown