Chapter 3

Point Sets and Sequences

Definition 3.1. A *point* is simply an element of the real numbers $\mathbb{R}$.

Definition 3.2. A *point set* is a nonempty subset of the real numbers.

You might well wonder why we need extra terms for these concepts. The reasons are mainly historical. Not only do these ideas undergird much more abstract ideas of sets, but in addition, there is a tradition worth preserving of "Moore method" courses using this terminology since we are focusing more on the *geometric* properties of the real line in this chapter.

One of the interesting things about choosing to look at the whole real number line is that we have two additional useful axioms. They seem nearly trivial, but are *very* important.

Point Set Axiom 1. If p is a point, then there is a point less than p and a point greater than p.

Point Set Axiom 2. If $p \neq q$ are two points, there is a point between them.

Note that these are closely related to the last few problems – but we are starting from scratch here, and you should treat these are the primary axioms we need for now.

Example 3.3. If instead of $\mathbb{R}$ we chose as our 'background set' some subset C of $\mathbb{R}$, these axioms might not hold. We can see this with $C = [0,1] \cup [2,3]$, with $p = 0$ in the first case and $p = 1, q = 2$ in the second.

Definition 3.4. We say that a point set O is a (finite) *open interval* if there are points $a \neq b$ such that O is the point set consisting of all points between a and b. That is,

$$O = \{x \mid a < x < b\} = (a,b).$$

Definition 3.5. We say that a point set I is a (finite) *closed interval* if there are points $a \neq b$ such that I is the point set consisting of a, b, and all points between a and b. That is,

$$I = \{x \mid a \leq x \leq b\} = [a,b].$$

In either case, a and b are called the *endpoints* of the interval.

Exercise 3.6. Give some examples of point sets that are *not* intervals.

Definition 3.7. If M is a point set, we say that p is an *accumulation point of M* if **every** open interval containing p also contains a point of M different from p.

To word this in set notation, p is an accumulation point of M if, for every open interval O with $p \in O$, there is $x \in O$ such that $x \in M \setminus \{p\}$.

Problem 3.8. Show that if M is an open interval and $p \in M$, then p is an accumulation point of M.

Problem 3.9. Show that if M is a closed interval and $p \notin M$, then p is not an accumulation point of M.

Problem 3.10. Determine whether the endpoints of an open interval M are accumulation points of M.

It is worth exploring exactly how many points it is possible or impossible for M to have. The next two problems are just a start in investigating that.

Problem 3.11. Show that if M is a point set having an accumulation point, then M contains (at least) two points. Determine whether M must contain at least 3 points.

Problem 3.12. Consider $\mathbb{Z}$ as a point set; show that it has no accumulation points.

Problem 3.13. Given that H and K are point sets, determine whether each of the following is true or false. If the statement is true, prove it. Otherwise, provide a counterexample.

(a) If p is an accumulation point of $H \cap K$, then p is an accumulation point of both H and K.

(b) If p is an accumulation point of $H \cup K$, then p is an accumulation point of H or p is an accumulation point of K.

Problem 3.14. If M is the set of all reciprocals of elements of $\mathbb{N}$, then zero is an accumulation point of M.

We will now start to turn to connecting the concepts of point sets to more familiar ones from calculus, beginning with sequences.

Definition 3.15. A *sequence a* is a function from $\mathbb{N}$ to $\mathbb{R}$.

Sometimes these are called *point sequences* to distinguish them from other potential sequences.

Even though technically the sequence is a function a, for any given $i \in \mathbb{N}$ we will write $a_i = a(i)$. Then we may write a sequence $a(1), a(2), a(3), \cdots$ as $a_1, a_2, a_3, \cdots$—or, more formally, $\{a_i\}_{i=1}^{\infty}$. In order to keep the index visible (this is especially useful when we combine sequences), we will occasionally abuse notation slightly and write $\{a_i\}_{i=1}^{\infty}$ when referring to the function *or* the image set $\{a(i) \mid i \in \mathbb{N}\}$.

Exercise 3.16. Write down several sequences a you are familiar with. If possible, give an algebraic formula for each a_i in terms of i.

Exercise 3.17. Give an example where the image set of a sequence $\{a_i\}_{i=1}^{\infty}$ is finite.

There is a deep connection between sequences and accumulation points, which the next few problems will elucidate. First, a definition—one you may have seen in calculus in a different form.

Definition 3.18. We say that the (point) sequence $p = \{p_i\}_{i=1}^{\infty}$ *converges to the point* x if, given an open interval S containing x, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ is also a positive integer then $p_n \in S$.

We simply say that p converges if there exists a point x to which the sequence converges. One could even write $p \to x$, informally. If a sequence does not converge to *any* point x, then we say it *diverges*.

The first problem about this should be used as a place to test ideas for how to prove convergence. Make sure you remember *all* the axioms you've learned and facts you've proved about real numbers—you may need them!

Problem 3.19. Consider the sequence given by $p_n = \frac{1}{n}$ (remember, $n \in \mathbb{N}$ is part of the definition of a sequence). Show that $p = \{p_i\}_{i=1}^{\infty}$ converges to 0.

As you tackle the next few problems, try actually writing down the first 10 or 12 elements of each sequence.

Problem 3.20. Consider the sequence given by $p_n = 1 - \frac{1}{n}$. Show that p converges to 1.

Problem 3.21. Consider the sequence with even terms $p_{2n} = \frac{1}{2n-1}$ and odd terms $p_{2n-1} = \frac{1}{2n}$. Show that p converges to 0.

Problem 3.22. Consider the sequence with odd terms $p_{2n-1} = \frac{1}{2n-1}$ and even terms $p_{2n} = 1 + \frac{1}{2n}$. Determine whether p converges to 0.

The following problem connects our two concepts. The most profound property of the real numbers is part of this connection, as we'll soon see.

Problem 3.23. Show that if p converges to the point x *and* for each $i \in \mathbb{N}$, $p_i \neq p_{i+1}$, then x is an accumulation point of the image set of $\{p_i\}_{i=1}^{\infty}$.

Exercise 3.24. Why do we need the restriction that $p_i \neq p_{i+1}$? Is this an absolutely necessary restriction for x to be an accumulation point of the image set?

Problem 3.25. Show that the sequence $\{\frac{1}{i}\}_{i=1}^{\infty}$ does not converge to a point other than zero.

Problem 3.26. Show that if p converges to the point x and y is a point different from x, then p does *not* converge to y.

Basic familiar facts about sequence convergence are recalled next. In these proofs, you may have to think a little more explicitly about what the intervals around x look like in order to combine sequences. Try doing some examples with explicit numbers first, to get a sense of how to approach them.

Problem 3.27. Show that if c is a number and $p = \{p_i\}_{i=1}^{\infty}$ converges to x, then cp (which means what you think it does) converges to cx.

Problem 3.28. Show that if $q = \{q_i\}_{i=1}^{\infty}$ converges to y and $p = \{p_i\}_{i=1}^{\infty}$ converges to x, then $\{q_i + p_i\}_{i=1}^{\infty}$ (which means what you think it does) converges to $y + x$.

Products and quotients of sequences behave like you think they will as well, and you can use these facts in the rest of the notes. We will include one special case soon.

Now we introduce a few more definitions that will lead us to the key axiom for the real numbers—*completeness*. We'll continue to see interplay between sequences and sets.

Definition 3.29. We say that a point set M is *bounded* if M is a subset of some closed interval.

Definition 3.30. We say that a point set M is *bounded above* if there is a point z such that if $x \in M$ then $x \leq z$; such a point is an *upper bound*.

Exercise 3.31. The property of a point set M being *bounded below* and the notion of a *lower bound* are defined similarly; try defining them.

Exercise 3.32. Show that a point set (in $\mathbb{R}$) being bounded is the same as it being bounded above and below.

Exercise 3.33. Find all upper bounds for $(0,1)$, $[0,1]$, and $(0,1) \cap \mathbb{Q}^C$ (irrationals between 0 and 1).

In the next problem, remember that we 'abuse notation' by using $\{p_i\}_{i=1}^{\infty}$ to mean more than one mathematical object.

Problem 3.34. If the sequence p converges to the point x, then the image set $\{p_i\}_{i=1}^{\infty}$ is bounded.

You can use this concept to prove some of the more difficult sequence convergence properties.

Problem 3.35. Show that if q converges to 0 and p converges to x, then $\{q_i \cdot p_i\}_{i=1}^{\infty}$ (which means what you think it does) converges to 0.

Now we start to approach the heart of why calculus works.

Definition 3.36. We say p is a *least upper bound* (or *supremum*) of a point set M if p is an upper bound of M and $p \leq q$ for any other upper bound q of M.

Exercise 3.37. Define the *greatest lower bound/infimum* by analogy.

Problem 3.38. Find the suprema of $(0,1)$, and $(0,1)\cap\mathbb{Q}^C$. If we could apply the definition of supremum to $\emptyset$, what would its supremum be?

Problem 3.39. Prove that the supremum of a point set is unique, if it exists.

Problem 3.40. If M and N are point sets with suprema, characterize the supremum of $M \cup N$.

If M and N are point sets, define $cM := \{cx \mid x \in M\}$ and $M+N := \{x+y \mid x \in M, y \in N\}$.

Problem 3.41. Assuming M and N have suprema, prove *either* that $\sup(cM)$ is $c\sup(M)$ (given $c > 0$) *or* that $\sup(M+N) = \sup(M) + \sup(N)$.

Exercise 3.42. Show that $c\inf(M) = \sup(cM)$ if $c < 0$. What other properties are there relating inf, sup, and c?

The reason the supremum is so important is the following fundamental axiom.

Axiom of Completeness of the Real Numbers. If M is a point set and is bounded above, then M has a supremum.

Exercise 3.43. Come up with a sequence p such that the image set is unbounded $\{p_i\}_{i=1}^{\infty}$ and hence does not have a supremum.

Exercise 3.44. Show that the axiom is *not* true if one requires that the supremum be a rational number.

It will be quite useful in the future to show that there is an equivalent way to formulate completeness in terms of sequences.

Definition 3.45. We say that a sequence p is *nondecreasing* if $p_i \le p_{i+1}$ for all $i \in \mathbb{N}$. (The concept of nonincreasing is defined similarly.)

Exercise 3.46. Replace the $\le$ above with $<$ to define the notion of (strictly) *increasing*. Find examples of nondecreasing sequences which are not increasing (and similarly for nonincreasing/decreasing).

Problem 3.47. If p is a nondecreasing sequence such that the image set $\{p_i\}_{i=1}^{\infty}$ is bounded above, then p converges to some point x.

Problem 3.48. Assuming the previous problem is true, prove the completeness axiom for point sets.

Why is all this so important? One reason is that we can use the completeness of the reals to *prove* one of the axioms we snuck in earlier. It may be thought of as the 'real' reason why the following is true, since open intervals can be as small as we need them to be.

Problem 3.49. Using the previous problem (but not its proof!), show that for any point x, there is an $n \in \mathbb{Z}$ such that $n > x$.[1]

[1] *Hint:* Use a proof by contradiction.

A use of the *alternate* definition of completeness is proving properties of bounded sequences. The most famous of these is the following theorem

Theorem 3.50 (The Bolzano-Weierstrass Theorem)**.** Let us call a sequence $\{b_k\}_{k=1}^{\infty}$ a *subsequence* of another sequence $\{a_n\}_{n=1}^{\infty}$ if there is a sequence of natural numbers $\{n_i\}_{i=1}^{\infty}$ with $n_i < n_{i+1}$ such that $b_k = a_{n_k}$. Then every sequence with bounded image set has a convergent subsequence.