Problem Collection for Introduction to Mathematical Reasoning

By Dana C. Ernst and Nándor Sieben
Northern Arizona University

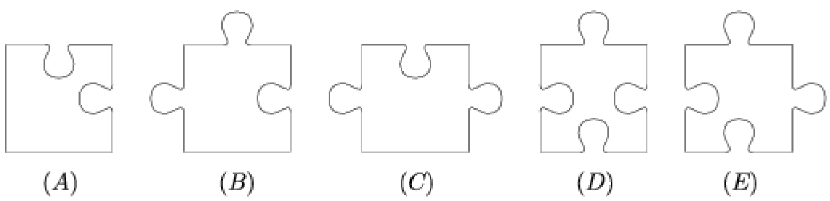
Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs \$2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Christine wants to take yoga classes to increase her strength and flexibility. In her neighborhood, there are two yoga studios: Namaste Yoga and Yoga Spirit. At Namaste Yoga, a student's first class costs \$12, and additional classes cost \$10 each. At Yoga Spirit, a student's first class costs \$24, and additional classes cost \$8 each. Because Christine wants to save money, she is interested in comparing the costs of the two studios. For what number of yoga classes do the two studios cost the same amount?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. The Sunny Day Juice Stand sells freshly squeezed lemonade and orange juice at the farmers' market. The juices are ladled out of large glass jars, each holding exactly the same amount of juice. Linda and Julie set up their stand early one Saturday morning. The first customer of the day ordered orange juice and Linda carefully ladled out 8 ounces into a paper cup. As she was about to hand the cup to the customer, he changed his mind and asked for lemonade instead. Accidentally, Linda dumped the cup of orange juice into the jar of lemonade. She quickly mixed up the juices, ladled out a cup of the mixture (mostly lemonade) and turned to hand it to the customer. "I've decided I don't want anything to drink right now," he said, and frazzled, Linda dumped the cupful of juice mixture into the orange juice jar. Linda's assistant, Julie, watched all of this with amusement. As the man walked away, she wondered aloud, "Now is there more orange juice in the lemonade or more lemonade in the orange juice?"

Problem 5. A rectangular puzzle that says "850 pieces" actually consists of 851 pieces. Each piece is identical to one of the 5 samples shown in the diagram. How many pieces of type (E) are there in the puzzle?



Problem 6. Describe where on Earth from which you can travel one mile south, then one mile east, and then one mile north and arrive at your original location. There is more than one such location. Find them all.

Problem 7. A soul swapping machine swaps the souls inside two bodies placed in the machine. Soon after the invention of the machine an unforeseen limitation is discovered: swapping only works on a pair of bodies once. Souls get more and more homesick as they spend time in another body and if a soul is not returned to its original body after a few days, it will kill its current host.

(a) Suppose Tom and Jerry swap souls and Garfield and Odie swap souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(b) Suppose Batman and Robin swap souls and then Robin's body and Flash utilize the machine. Argue that it is not possible to return the swapped souls to their original bodies using only Batman, Robin, and Flash.

(c) Consider the scenario of the previous problem. Suppose Wonder Woman and Superman are now available to sit in the machine after Batman, Robin, and Flash have already swapped souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(d) Now, suppose the soul swapping machine is used by the following pair of bodies (in the order listed): Adam and Alicia, Alicia and Gwen, Gwen and Blake. In addition, Pharrell and Miley are standing nearby. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

Problem 8. You are in a big city where all the streets go in one of two perpendicular directions. You take your car from its parking place and drive on a tour of the city such that you do not pass through the same intersection twice and return back to where you started. If you made 100 left turns, how many right turns did you make?

Problem 9. Find the rational number with smallest denominator between 1/3 and 3/8.

Problem 10. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

(a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

Problem 11. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

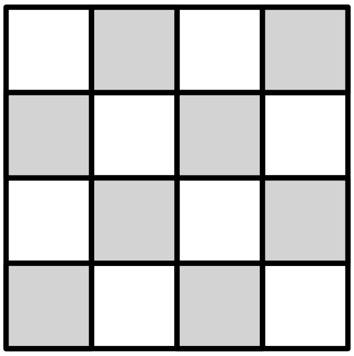
Problem 12. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 13. Suppose there are two bags of candy containing 8 pieces and 6 pieces, respectively. You and your friend are going to play a game and the winner gets to eat all of the candy. Here are the rules for the game:

1. You and your friend will alternate removing pieces of candy from the bags. Let's assume that you go first.
2. On each turn, the designated player selects a bag that still has candy in it and then removes at least one piece of candy. The designated player can only remove candy from a single bag and he/she must remove at least one piece.
3. The winner is the one that removes all the candy from the last remaining bag.

Does one of you have a guaranteed winning strategy? If so, describe that strategy. Can you generalize to handle any number of pieces of candy in either of the two bags?

Problem 14. Pennies and Paperclips is a two-player game played on a 4×4 checkerboard as shown below.



One player, "Penny", gets two pennies as her pieces. The other player, "Clip", gets a pile of paperclips as his pieces. Penny places her two pennies on any two different squares on the board. Once the pennies are placed, Clip attempts to cover the remainder of the board with paperclips - with each paperclip being required to cover two vertically or horizontally adjacent squares. Paperclips are not allowed to overlap. If the remainder of the board can be covered with paperclips then Clip is declared the winner. If the remainder of the board cannot be covered with paperclips then Penny is the winner.

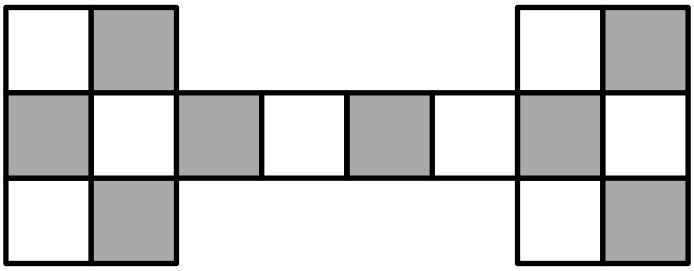
(a) Does either player have a winning strategy? If so, describe the winning strategies.

(b) State and prove a conjecture that determines precisely every situation in which Penny wins based on the placement of the pennies.

(c) State and prove a conjecture that determines precisely every situation in which Clip wins based on the placement of the pennies.

(d) Are there any situations in which neither player wins, or have you characterized all possible outcomes? Explain.

Problem 15. Consider the game Pennies and Paperclips described in the previous problem, but instead of playing on a 4×4 checkerboard, let's play on the following board.



State and prove a conjecture that determines precisely every situation in which Clip wins based on the placement of the pennies.

Problem 16. We call a game board for the Pennies and Paperclips game **fair**, if for each player there is at least one scenario in which they can win.

(a) Is the board from Problem 14 fair?

(b) Is the board from Problem 15 fair?

(c) Are there game boards that are not fair? That is, are there game boards on which one player can never win? If so, provide such a board and explain why it must be unfair. If not, explain why no such board exists.

(d) Can you create a fair board in which your conjecture from Problem 14(c) does not always hold?

Problem 17. Find all distinct pairs of numbers with largest gcd between and including 51 and 100. By distinct pair, we mean that you cannot choose the same number twice. Note that gcd starts for greatest common divisor. For example, $\gcd(14, 20) = 2$.