Problem Collection for Introduction to Mathematical Reasoning

By Dana C. Ernst and Nándor Sieben
Northern Arizona University

Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs $2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Christine wants to take yoga classes to increase her strength and flexibility. In her neighborhood, there are two yoga studios: Namaste Yoga and Yoga Spirit. At Namaste Yoga, a student's first class costs $12, and additional classes cost $10 each. At Yoga Spirit, a student's first class costs $24, and additional classes cost $8 each. Because Christine wants to save money, she is interested in comparing the costs of the two studios. For what number of yoga classes do the two studios cost the same amount?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. The Sunny Day Juice Stand sells freshly squeezed lemonade and orange juice at the farmers' market. The juices are ladled out of large glass jars, each holding exactly the same amount of juice. Linda and Julie set up their stand early one Saturday morning. The first customer of the day ordered orange juice and Linda carefully ladled out 8 ounces into a paper cup. As she was about to hand the cup to the customer, he changed his mind and asked for lemonade instead. Accidentally, Linda dumped the cup of orange juice into the jar of lemonade. She quickly mixed up the juices, ladled out a cup of the mixture (mostly lemonade) and turned to hand it to the customer. "I've decided I don't want anything to drink right now," he said, and frazzled, Linda dumped the cupful of juice mixture into the orange juice jar. Linda's assistant, Julie, watched all of this with amusement. As the man walked away, she wondered aloud, "Now is there more orange juice in the lemonade or more lemonade in the orange juice?"

Problem 5. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

(a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

Problem 6. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 7. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 8. Describe where on Earth from which you can travel one mile south, then one mile east, and then one mile north and arrive at your original location. There is more than one such location. Find them all.

Problem 9. I have 10 sticks in my bag. The length of each stick is an integer. No matter which 3 sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

Problem 10. Suppose there are two bags of candy containing 8 pieces and 6 pieces, respectively. You and your friend are going to play a game and the winner gets to eat all of the candy. Here are the rules for the game:

1. You and your friend will alternate removing pieces of candy from the bags. Let's assume that you go first.
2. On each turn, the designated player selects a bag that still has candy in it and then removes at least one piece of candy. The designated player can only remove candy from a single bag and he/she must remove at least one piece.
3. The winner is the one that removes all the candy from the last remaining bag.

Does one of you have a guaranteed winning strategy? If so, describe that strategy. Can you generalize to handle any number of pieces of candy in either of the two bags?

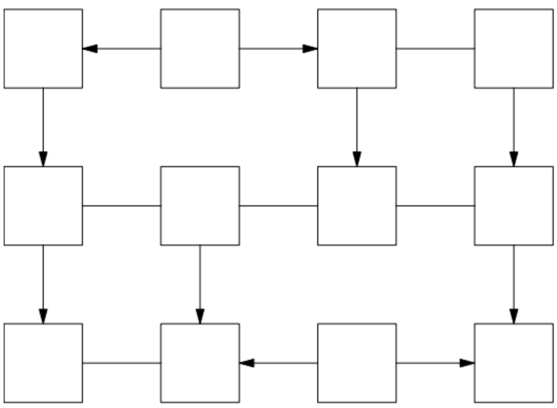
Problem 11. Suppose you have 6 toothpicks that are exactly the same length. Can you arrange the toothpicks so that exactly 4 identical triangles are formed? You cannot cut, break, or bend the toothpicks. Moreover, each vertex of a triangle must be formed when the tips of two toothpicks meet.

Problem 12. An ant is crawling along the edges of a unit cube. What is the maximum distance it can cover starting from a corner so that it does not cover any edge twice?

Problem 13. The grid below has 12 boxes and 15 edges connecting boxes. In each box, place one of the six integers from 1 to 6 such that the following conditions hold:

- For each possible pair of distinct numbers from 1 to 6, there is exactly one edge connecting two boxes with that pair of numbers.
- If an edge has an arrow, then it points from a box with a smaller number to a box with a larger number.

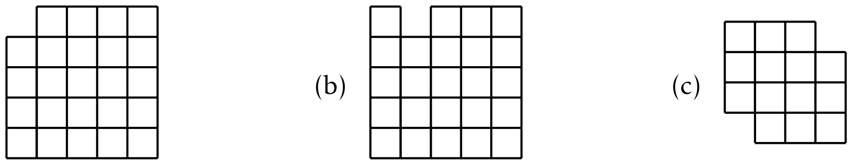
You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above.



Problem 14. Four red ants and two black ants are walking along the edge of a one meter stick. The four red ants, called Albert, Bart, Debbie, and Edith, are all walking from left to right, and the two black ants, Cindy and Fred, are walking from right to left. The ants always walk at exactly one centimeter per second. Whenever they bump into another ant, they immediately turn around and walk in the other direction. And whenever they get to the end of a stick, they fall off. Albert starts at the left hand end of the stick, while Bart starts 20.2 cm from the left, Debbie is at 38.7cm, Edith is at 64.9cm and Fred is at 81.8cm. Cindy's position is not known—all we know is that he starts somewhere between Bart and Debbie. Which ant is the last to fall off the stick? And how long will it be before he or she does fall off?

Problem 15. Tile the following grids with dominoes. If a tiling is not possible, explain why.

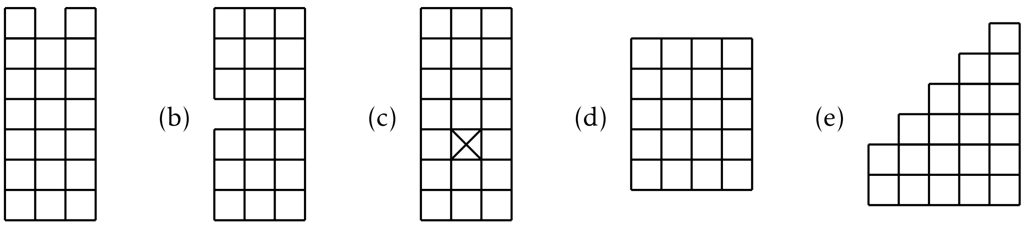
(a) (b) (c)



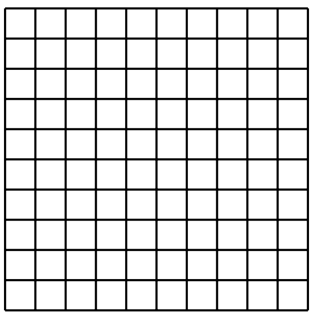
Problem 16. Find all tetrominoes (polymonio with 4 cells).

Problem 17. Tile the following grids using every tetromino exactly once. The X in (c) denotes an absence of an available square in the grid. If a tiling is not possible, explain why.

(a) (b) (c) (d) (e)

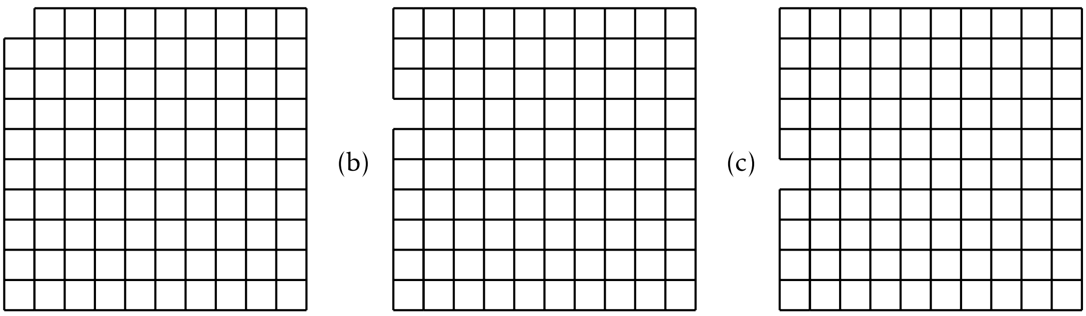


Problem 18. Consider the 10×10 grid of squares below. Show that you can color the squares of the grid with 3 colors so that every consecutive row of 3 squares and every consecutive column of 3 squares uses all 3 colors.



Problem 19. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain why.

(a) (b) (c)



Problem 20. A mouse eats her way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 $1 \times 1 \times 1$ subcubes. If she starts at one corner and always moves to an uneaten subcube by passing through a face of a subcube, can she finish at the center of the cube?

Problem 21. There is a plate of 40 cookies. You and your friend are going to take turns taking either 1 or 2 cookies from the plate. However, it is a faux pas to take the last cookie, so you want to make sure that you do not take the last cookie. How can you guarantee that you will never be the one taking the last cookie? What about n cookies?

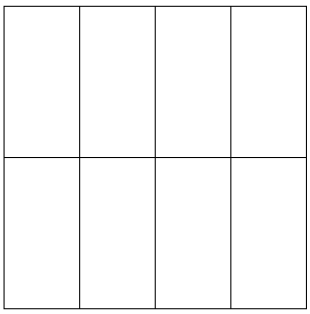
Problem 22. The Sylver Coinage Game is a game in which 2 players alternately name positive integers that are not the sum of nonnegative multiples of previously named integers. The person who names 1 is the loser! Here is a sample game between A and B:

1. A opens with 5. Now neither player can name $5, 10, 15, \ldots$
2. B names 4. Now neither player can name 4, 5, 8, 9, 10, or any number greater than 11.
3. A names 11. Now the only remaining numbers are 1, 2, 3, 6, and 7.
4. B names 6. Now the only remaining numbers are 1, 2, 3, and 7.
5. A names 7. Now the only remaining numbers are 1, 2, and 3.
6. B names 2. Now the only remaining numbers are 1 and 3.
7. A names 3, leaving only 1.
8. B is forced to name 1 and loses.

If player A names 3, can you find a strategy that guarantees that the second player wins? If so, describe the strategy? If such a strategy is not possible, then explain why?

Problem 23. How many factors of 10 are there in 50! (i.e., 50 factorial)?

Problem 24. How many rectangles are in the figure below?



Problem 25. Four prisoners are making plans to escape from jail. Their current plan requires them to cross a narrow bridge in the dark that has no handrail. In order to successfully cross the bridge, they must use a flashlight. However, they only have a single flashlight. To complicate matters, at most two people can be on the bridge at the same time. So, they will need to make multiple trips across the bridge, returning the flashlight back to the first side of the bridge by having someone walk it back. Unfortunately, they can't throw the flashlight. It takes 1, 2, 5, and 10 minutes for prisoner A, prisoner B, prisoner C, and prisoner D to cross the bridge and when two prisoners are walking together with the flashlight, it takes the time of the slower prisoner. What is the minimum total amount of time it takes all four prisoners to get across the bridge?

Problem 26. In order to assess the reasoning skills of a newly developed android robot with artificial intelligence, the android's creator designs the following experiment. On Sunday, the creator describes the details of the experiment to the android and then turns the the android off. Once or twice, during the experiment, the android will be turned on, interviewed, and then turned back off. In addition, the creator will erase the awakening from the android's memory. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, the android will be awakened and interviewed on Monday only.
- If the coin comes up tails, the android will be awakened and interviewed on both Monday and Tuesday.

In either case, the android will be awakened on Wednesday without interview and the experiment ends. Any time the android is awakened and interviewed, it will not be able to tell which day it is or whether it has been awakened before. During the interview the android is asked: "What is your credence now for the proposition that the coin landed heads?". One way to interpret "credence" in this context is the android's determination of the probability that the coin landed on heads. How should/would the android answer the interviewer's question?

Problem 27. As a broke college student, you agree to take part in a recurring experiment. Each experiment begins on Sunday evening and ends on Wednesday morning. The experiment will be repeated 100 weeks in a row. You are told the details of the experiment in advance. Each Sunday evening, the experimenter describes the details of the experiment and then gives you a drug to put you to sleep. Once or twice, during the experiment, you will be awakened, interviewed, and then put back to sleep using a drug that includes an amnesia-inducing component that makes you forget the awakening. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, you will be awakened and interviewed on Monday only.
- If the coin comes up tails, you will be awakened and interviewed on both Monday and Tuesday.

In either case, you will be awakened on Wednesday without interview and the experiment ends. Any time you are awakened and interviewed, you will not be able to tell which day it is or whether you have been awakened before. During the interview you will be asked: "Is the coin heads or tails?". You are required to respond with either "heads" or "tails". The experimenter will record whether you were correct or not, but you will not be told whether you guessed correctly. At the end of the 100th run of the experiment, you will be given \$10 for each correct answer that you gave. What strategy should you employ in order to optimize your profit?

Problem 28. Take 15 poker chips or coins, divide into any number of piles with any number of chips in each pile. Arrange piles in adjacent columns. Take the top chip off every column and make a new column to the left. Repeat forever. What happens? Make conjectures about what happens when we change the number of chips.

Problem 29. The nth *triangular number* is defined via $t_n := 1+2+\cdots+n$. For example, $t_4 = 1+2+3+4 = 10$. Find a visual proof of the following fact. By "visual proof" we mean a sufficiently general picture that is convincing enough to justify the claim.

For all $n \in \mathbb{N}$, $t_n = \dfrac{n(n+1)}{2}$.

Problem 30. Let t_n denote the nth triangular number. Find both an algebraic proof and a visual proof of the following fact.

For all $n \in \mathbb{N}$, $t_n + t_{n+1} = (n+1)^2$.

Problem 31. Find a visual proof of the following fact. *Warning:* This problem is not about triangular numbers.

For $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-1) = n^2$.

Problem 32. Suppose someone draws 20 distinct random lines in the plane. What is the maximum number of intersections of these lines?

Problem 33. A certain fast-food chain sells a product called "nuggets" in boxes of 6, 9, and 20. A number n is called *nuggetable* if one can buy exactly n nuggets by buying some number of boxes. For example, 21 is nuggetable since you can buy two boxes of six and one box of nine to get 21. Here are the first few nuggetable numbers:

$$6, 9, 12, 15, 18, 20, 21, 24, 26, 27, \ldots$$

and here are the first few non-nuggetable numbers:

$$1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots$$

What is the largest non-nuggetable number?

Problem 34. Let t_n denote the nth triangular number. Find an algebraic and a visual proof of the following fact.

For all $a, b \in \mathbb{N}$, $t_{ab} = t_a t_b + t_{a-1} t_{b-1}$.

Problem 35. How many ways can 110 be written as the sum of 14 different positive integers? *Hint:* First, figure out what the largest possible integer could be in the sum. Note that the largest integer in the sum will be maximized when the other 13 numbers are as small as possible. Finish off the problem by doing an analysis of cases.

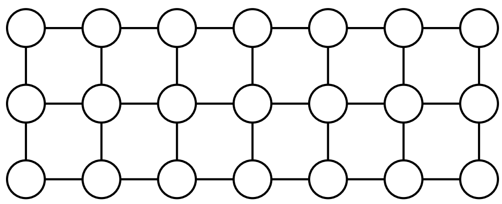
Problem 36. Suppose you randomly cut a stick into 3 pieces. What is the probability that you can form a triangle out of these 3 pieces?

Problem 37. Suppose you randomly pick 3 distinct points on a circle. What is the probability that the center of the circle lies in the interior of the triangle formed by these 3 points?

Problem 38. Consider a gambler who tosses a coin at most 6 times, and if it comes out heads (H), wins a dollar, and if it comes out tails (T), loses a dollar. He is kicked out as soon as he is in the red, i.e., has negative capital. In how many ways can he survive to 6 rounds, but at the end break even?

Problem 39. An overfull prison has decided to terminate some prisoners. The jailer comes up with a game for selecting who gets terminated. Here is his scheme. 10 prisoners are to be lined up all facing the same direction. On the back of each prisoner's head, the jailer places either a black or a red dot. Each prisoner can only see the color of the dot for all of the prisoners in front of them and the prisoners do not know how many of each color there are. The jailer may use all black dots, or perhaps he uses 3 red and 7 black, but the prisoners do not know. The jailer tells the prisoners that if a prisoner can guess the color of the dot on the back of their head, they will live, but if they guess incorrectly, they will be terminated. The jailer will call on them in order starting at the back of the line. Before lining up the prisoners and placing the dots, the jailer allows the prisoners 5 minutes to come up with a plan that will maximize their survival. What plan can the prisoners devise that will maximize the number of prisoners that survive? Some more info: each prisoner can hear the answer of the prisoner behind them and they will know whether the prisoner behind them has lived or died. Also, each prisoner can only respond with the word "black" or "red." What if there are n prisoners?

Problem 40. In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black corners.



Problem 41. Each point of the plane is colored red or blue. Show that there is a rectangle whose corners are all the same color.

Problem 42. You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

Your objective is to identify a single real coin that Zoltar has not taken as tribute.

Problem 43. Our space ship is at a Star Base with coordinates $(1,2)$. Our hyper drive allows us to jump from coordinates (a,b) to either coordinates $(a, a+b)$ or to coordinates $(a+b, b)$. How can we reach the impending enemy attack at coordinates $(8,13)$?

Problem 44. Consider our Star Base from the previous problem. Recall that our hyper drive allows us to jump from coordinates (a,b) to either coordinates $(a,a+b)$ or to coordinates $(a+b,b)$. If we start at $(1,0)$, which points in the plane can we get to by using our hyper drive? Justify your answer.

Problem 45. Find all integers a,b,c,d, and e, such that

$$\begin{aligned}
a^2 &= a+b-2c+2d+e-8\\
b^2 &= -a-2b-c+2d+2e-6\\
c^2 &= 3a+2b+c+2d+2e-31\\
d^2 &= 2a+b+c+2d+2e-2\\
e^2 &= a+2b+3c+2d+e-8.
\end{aligned}$$

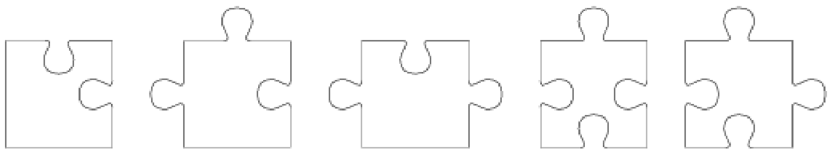
Problem 46. There are 30 red, 40 yellow, 50 blue, and 60 green balls in a box. We take out balls from the box with closed eyes. On the first turn we take out 1 ball, on the second turn we take out 2, and so on. On the nth turn we take out n balls. What is the minimum number of balls we need to take out to guarantee the following:

(a) We have a blue ball;

(b) We have a red and a green ball;

(c) We have all four colors.

Problem 47. A rectangular puzzle that says "850 pieces" actually consists of 851 pieces. Each piece is identical to one of the 5 samples shown in the diagram. How many pieces of type (E) are there in the puzzle?



(A) (B) (C) (D) (E)

Problem 48. In the game Turnaround, you are given a permutation of the numbers from 1 to n. Your goal is to get them in the natural order $12\cdots n$. At each stage, your only option is to reverse the order of the first k places (you get to pick k at each stage). For instance, given 6375142, you could reverse the first four to get 5736142 and then reverse the first six to get 4163752. Solve the following sequence in as few moves as possible: 352614.

Problem 49. A signed permutation of the numbers 1 through n is a fixed arrangement of the numbers 1 through n, where each number can be either be positive or negative. For example, $(-2,1,-4,5,3)$ is a signed permutation of the numbers 1 through 5. In this case, think of positive numbers as being right-side-up and negative numbers as being upside-down. A *reversal* of a signed permutation is the act of performing a 180-degree rotation to some consecutive subsequence of the permutation. That is, a reversal swaps the order of a subsequence of numbers while changing the sign of each number in the subsequence. Performing a reversal to a signed permutation results in a new signed permutation. For example, if we perform a reversal on the second, third, and fourth entries in $(-2,1,-4,5,3)$, we obtain $(-2,-5,4,-1,3)$. The *reversal distance* of a signed permutation of 1 through n is the minimum number of reversals required to transform the given signed permutation into $(1,2,\ldots,n)$. It turns out that the reversal distance of $(3,1,6,5,-2,4)$ is 5. Find a sequence of 5 reversals that transforms $(3,1,6,5,-2,4)$ into $(1,2,3,4,5,6)$.

Problem 50. Consider a tournament with 15 teams. If every team plays every other team, how many games were played?

Problem 51. Two different positive numbers a and b each differ from their reciprocal by 1. What is $a+b$?

Problem 52. Let X be the intersection of the diagonals of the trapezoid $ABCD$ with parallel sides AB and CD. Show that the areas of triangles AXD and BXC are the same.