Problem Collection for Introduction to Mathematical Reasoning

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Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs $2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Multiply together the numbers of fingers on each hand of all the human beings in the world—approximately 7 billion in all. What is the approximate answer?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. Suppose you have 6 toothpicks that are exactly the same length. Can you arrange the toothpicks so that 4 identical triangles are formed? Justify your answer.

Problem 5. I have 10 sticks in my bag. The length of each stick is an integer. No matter which 3 sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

Problem 6. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

(a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

Problem 7. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

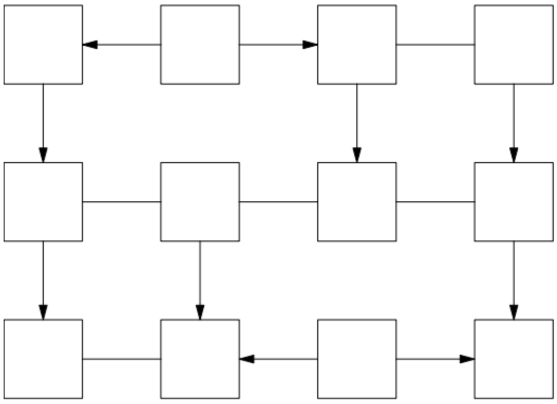
Problem 8. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 9. An ant is crawling along the edges of a unit cube. What is the maximum distance it can cover starting from a corner so that it does not cover any edge twice?

Problem 10. The grid below has 12 boxes and 15 edges connecting boxes. In each box, place one of the six integers from 1 to 6 such that the following conditions hold:

- For each possible pair of distinct numbers from 1 to 6, there is exactly one edge connecting two boxes with that pair of numbers.
- If an edge has an arrow, then it points from a box with a smaller number to a box with a larger number.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above.



Problem 11. In order to assess the reasoning skills of a newly developed android robot with artificial intelligence, the android's creator designs the following experiment. On Sunday, the creator describes the details of the experiment to the android and then turns the the android off. Once or twice, during the experiment, the android will be turned on, interviewed, and then turned back off. In addition, the creator will erase the awakening from the android's memory. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, the android will be awakened and interviewed on Monday only.
- If the coin comes up tails, the android will be awakened and interviewed on both Monday and Tuesday.

In either case, the android will be awakened on Wednesday without interview and the experiment ends. Any time the android is awakened and interviewed, it will not be able to tell which day it is or whether it has been awakened before. During the interview the android is asked: "What is your credence now for the proposition that the coin landed heads?". One way to interpret "credence" in this context is the android's determination of the probability that the coin landed on heads.

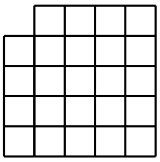
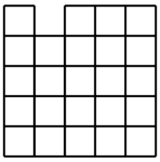
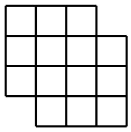
Problem 12. As a broke college student, you agree to take part in a recurring experiment. Each experiment begins on Sunday evening and ends on Wednesday morning. The experiment will be repeated 100 weeks in a row. You are told the details of the experiment in advance. Each Sunday evening, the experimenter describes the details of the experiment and then gives you a drug to put you to sleep. Once or twice, during the experiment, you will be awakened, interviewed, and then put back to sleep using a drug that includes an amnesia-inducing component that makes you forget the awakening. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, you will be awakened and interviewed on Monday only.
- If the coin comes up tails, you will be awakened and interviewed on both Monday and Tuesday.

In either case, you will be awakened on Wednesday without interview and the experiment ends. Any time you are awakened and interviewed, you will not be able to tell which day it is or whether you have been awakened before. During the interview you will be asked: "Is the coin heads or tails?". You are required to respond with either "heads" or "tails". The experimenter will record whether you were correct or not, but you will not be told whether you guessed correctly. At the end of the 100th run of the experiment, you will be given \$10 for each correct answer that you gave. What strategy should you employ in order to optimize your profit?

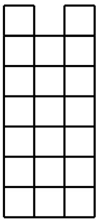
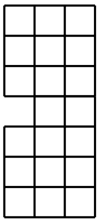
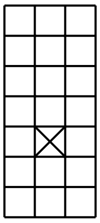
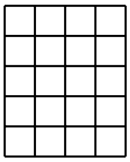
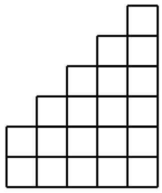
Problem 13. Four red ants and two black ants are walking along the edge of a one meter stick. The four red ants, called Albert, Bart, Debbie, and Edith, are all walking from left to right, and the two black ants, Cindy and Fred, are walking from right to left. The ants always walk at exactly one centimeter per second. Whenever they bump into another ant, they immediately turn around and walk in the other direction. And whenever they get to the end of a stick, they fall off. Albert starts at the left hand end of the stick, while Bart starts 20.2 cm from the left, Debbie is at 38.7cm, Edith is at 64.9cm and Fred is at 81.8cm. Cindy's position is not known—all we know is that he starts somewhere between Bart and Debbie. Which ant is the last to fall off the stick? And how long will it be before he or she does fall off?

Problem 14. Tile the following grids with dominoes. If a tiling is not possible, explain way.

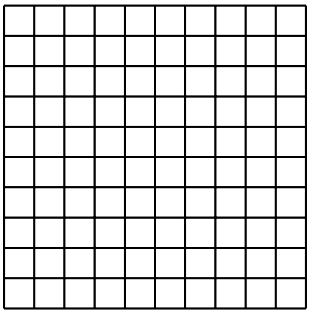
(a)  (b)  (c) 

Problem 15. Find all tetrominoes (polymonio with 4 cells).

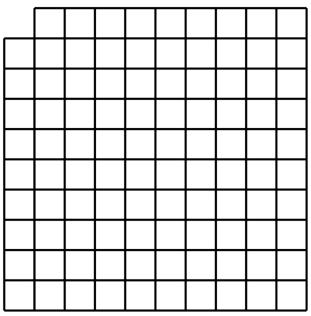
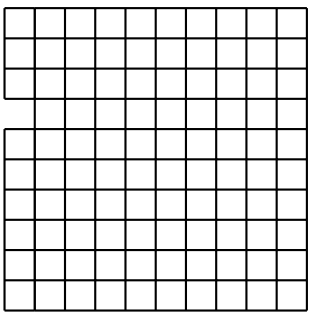
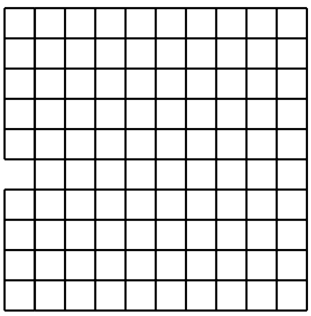
Problem 16. Tile the following grids using every tetromino exactly once. The X in (c) denotes an absence of an available square in the grid. If a tiling is not possible, explain way.

(a)  (b)  (c)  (d)  (e) 

Problem 17. Consider the 10×10 grid of squares below. Show that you can color the squares of the grid with 3 colors so that every consecutive row of 3 squares and every consecutive column of 3 squares uses all 3 colors.



Problem 18. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain way.

(a)  (b)  (c) 

Problem 19. Take 15 poker chips or coins, divide into any number of piles with any number of chips in each pile. Arrange piles in adjacent columns. Take the top chip off every column and make a new column to the left. Repeat forever. What happens? Make conjectures about what happens when we change the number of chips.

Problem 20. The nth *triangular number* is defined via $t_n := 1+2+\cdots+n$. For example, $t_4 = 1+2+3+4 = 10$. Find a visual proof of the following fact. By "visual proof" we mean a sufficiently general picture that is convincing enough to justify the claim.

For all $n \in \mathbb{N}$, $t_n = \dfrac{n(n+1)}{2}$.

Problem 21. Let t_n denote the nth triangular number. Find both an algebraic proof and a visual proof of the following fact.

For all $n \in \mathbb{N}$, $t_n + t_{n+1} = (n+1)^2$.

Problem 22. Find a visual proof of the following fact. *Warning:* This problem is not about triangular numbers.

For $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-1) = n^2$.

Problem 23. Suppose you randomly cut a stick into 3 pieces. What is the probability that you can form a triangle out of these 3 pieces?

Problem 24. Suppose you randomly pick 3 distinct points on a circle. What is the probability that the center of the circle lies in the interior of the triangle formed by these 3 points?

Problem 25. There is a plate of 40 cookies. You and your friend are going to take turns taking either 1 or 2 cookies from the plate. However, it is a faux pas to take the last cookie, so you want to make sure that you do not take the last cookie. How can you guarantee that you will never be the one taking the last cookie? What about n cookies?

Problem 26. The Sylver Coinage Game is a game in which 2 players alternately name positive integers that are not the sum of nonnegative multiples of previously named integers. The person who names 1 is the loser! Here is a sample game between A and B:

1. A opens with 5. Now neither player can name $5, 10, 15, \ldots$
2. B names 4. Now neither player can name 4, 5, 8, 9, 10, or any number greater than 11.
3. A names 11. Now the only remaining numbers are 1, 2, 3, 6, and 7.
4. B names 6. Now the only remaining numbers are 1, 2, 3, and 7.
5. A names 7. Now the only remaining numbers are 1, 2, and 3.
6. B names 2. Now the only remaining numbers are 1 and 3.
7. A names 3, leaving only 1.
8. B is forced to name 1 and loses.

If player A names 3, can you find a strategy that guarantees that the second player wins? If so, describe the strategy? If such a strategy is not possible, then explain why?

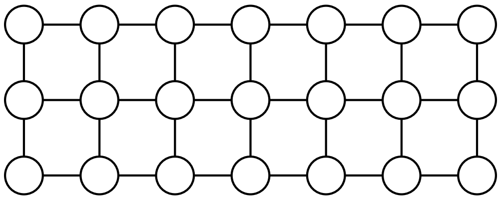
Problem 27. Suppose someone draws 20 distinct random lines in the plane. What is the maximum number of intersections of these lines?

Problem 28. A mouse eats her way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 $1 \times 1 \times 1$ subcubes. If she starts at one corner and always moves to an uneaten subcube by passing through a face of a subcube, can she finish at the center of the cube?

Problem 29. An overfull prison has decided to terminate some prisoners. The jailer comes up with a game for selecting who gets terminated. Here is his scheme. 10 prisoners are to be lined up all facing the same direction. On the back of each prisoner's head, the jailer places either a black or a red dot. Each prisoner can only see the color of the dot for all of the prisoners in front of them and the prisoners do not know how many of each color there are. The jailer may use all black dots, or perhaps he uses 3 red and 7 black, but the prisoners do not know. The jailer tells the prisoners that if a prisoner can guess the color of the dot on the back of their head, they will live, but if they guess incorrectly, they will be terminated. The jailer will call on them in order starting at the back of the line. Before lining up the prisoners and placing the dots, the jailer allows the prisoners 5 minutes to come up with a plan that will maximize their survival. What plan can the prisoners devise that will maximize the number of prisoners that survive? Some more info: each prisoner can hear the answer of the prisoner behind them and they will know whether the prisoner behind them has lived or died. Also, each prisoner can only respond with the word "black" or "red." What if there are n prisoners?

Problem 30. Four prisoners are making plans to escape from jail. Their current plan requires them to cross a narrow bridge in the dark that has no handrail. In order to successfully cross the bridge, they must use a flashlight. However, they only have a single flashlight. To complicate matters, at most two people can be on the bridge at the same time. So, they will need to make multiple trips across the bridge, returning the flashlight back to the first side of the bridge by having someone walk it back. Unfortunately, they can't throw the flashlight. It takes 1, 2, 5, and 10 minutes for prisoner A, prisoner B, prisoner C, and prisoner D to cross the bridge and when two prisoners are walking together with the flashlight, it takes the time of the slower prisoner. What is the minimum total amount of time it takes all four prisoners to get across the bridge?

Problem 31. In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black vertices (and vertical and horizontal sides).



Problem 32. Each point of the plane is colored red or blue. Show that there is a rectangle whose vertices are all the same color.

Problem 33. A certain fast-food chain sells a product called "nuggets" in boxes of $6, 9$, and 20. A number n is called *nuggetable* if one can buy exactly n nuggets by buying some number of boxes. For example, 21 is nuggetable since you can buy two boxes of six and one box of nine to get 21. Here are the first few nuggetable numbers:

$$6, 9, 12, 15, 18, 20, 21, 24, 26, 27, \ldots$$

and here are the first few non-nuggetable numbers:

$$1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots$$

What is the largest non-nuggetable number?

Problem 34. Our space ship is at a Star Base with coordinates $(1, 2)$. Our hyper drive allows us to jump from coordinates (a, b) to either coordinates $(a, a + b)$ or to coordinates $(a + b, b)$. How can we reach the impending enemy attack at coordinates $(8, 13)$?

Problem 35. Consider our Star Base from the previous problem. Recall that our hyper drive allows us to jump from coordinates (a, b) to either coordinates $(a, a + b)$ or to coordinates $(a + b, b)$. If we start at $(1, 0)$, which points in the plane can we get to by using our hyper drive? Justify your answer.

Problem 36. You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

Your objective is to identify a single real coin that Zoltar has not taken as tribute.

Problem 37. Welcome to Circle-Dot[1]. We'll approach Circle-Dot as a game, where the object of the game is to construct a word made entirely of $\circ$'s and $\bullet$'s. Circle-Dot begins with two words; called axioms. Using the two axioms and three rules of inference, we can create new Circle-Dot words, which are theorems in the Circle-Dot System. The process of creating Circle-Dot words using the axioms and rules of inference are proofs in the system.

On each of your "turns" in the game you can apply one of the 5 available axioms or rules to your current list of constructed Circle-Dot words to produce a new word. Also, once you have produced a new word, you can use this theorem in future "games."

Below are the axioms for Circle-Dot. Note that $\circ$ and $\bullet$ are valid symbols in the system while w and v are variables that stand for any sequence of $\circ$'s and $\bullet$'s.

Axiom A. $\circ\bullet$

Axiom B. $\bullet\circ$

At any time in your proof, you may quote an axiom. Below are the rules for generating new statements from known statements.

Rule 1. Given wv and vw, conclude w

Rule 2. Given w and v, conclude $w \bullet v$

Rule 3. Given $wv\bullet$, conclude $w\circ$

As an example, let's try to prove the following theorem.

Theorem C. $\bullet$ (just a single dot)

At the moment, the only tools we have for getting started are the axioms. As we prove theorems, we'll be able to incorporate them into our proofs, as well. To get started, let's apply Axiom A and see what that gets us. Applying Axiom A, we get $\circ\bullet$. Looking at Rules 2 and 3, it should be moderately clear that they won't help us get a single dot. So, perhaps Rule 1 will be useful, but to use it, we see that we need to have wv and vw. Applying Axiom B, we get $\bullet\circ$. Now, if we let $w = \bullet$ and $v = \circ$, then $wv = \bullet\circ$ and $vw = \circ\bullet$. Applying Rule 1, we can conclude that $\bullet$ holds. Putting this altogether, we can write something like the following.

Proof of Theorem C.

1. $\circ\bullet$ by Axiom A
2. $\bullet\circ$ by Axiom B
3. $\bullet$ by Rule 1 (using lines 2 and 1)

Now, try proving the following theorems.

Theorem D. $\circ$

Theorem E. $\bullet\bullet\bullet$

Theorem F. $\bullet\bullet\circ$

Theorem G. $\bullet\circ\circ$

Theorem H. $\circ\bullet\bullet\circ$

Theorem I. $\circ\circ\circ\circ$

Theorem J. $\bullet\circ\bullet$

Theorem K. $\bullet\circ\circ\circ$

Make a conjecture about which sequences of $\circ$'s and $\bullet$'s are theorems in the Circle-Dot system.

[1] The Circle-Dot System was developed by Ken Monks from the University of Scranton.

Problem 38. How many ways can 110 be written as the sum of 14 different positive integers? *Hint:* First, figure out what the largest possible integer could be in the sum. Note that the largest integer in the sum will be maximized when the other 13 numbers are as small as possible. Finish off the problem by doing an analysis of cases.

Problem 39. Let t_n denote the nth triangular number. Find an algebraic and a visual proof of the following fact.

For all $a, b \in \mathbb{N}$, $t_{ab} = t_a t_b + t_{a-1} t_{b-1}$.

Problem 40. We have two strings of pyrotechnic fuse. The strings do not look homogeneous in thickness but both of them have a label saying 4 minutes. So we can assume that it takes 4 minutes to burn through either of these fuses. How can we measure a one minute interval?

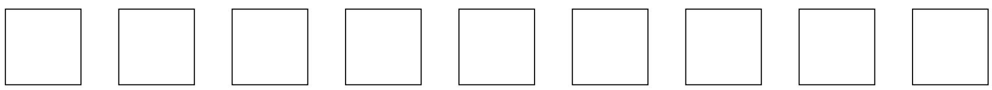
Problem 41. In the game Turnaround, you are given a permutation of the numbers from 1 to n. Your goal is to get them in the natural order $12\cdots n$. At each stage, your only option is to reverse the order of the first k places (you get to pick k at each stage). For instance, given 6375142, you could reverse the first four to get 5736142 and then reverse the first six to get 4163752. Solve the following sequence in as few moves as possible: 352614.

Problem 42. A signed permutation of the numbers 1 through n is a fixed arrangement of the numbers 1 through n, where each number can be either be positive or negative. For example, $(-2, 1, -4, 5, 3)$ is a signed permutation of the numbers 1 through 5. In this case, think of positive numbers as being right-side-up and negative numbers as being upside-down. A *reversal* of a signed permutation is the act of performing a 180-degree rotation to some consecutive subsequence of the permutation. That is, a reversal swaps the order of a subsequence of numbers while changing the sign of each number in the subsequence. Performing a reversal to a signed permutation results in a new signed permutation. For example, if we perform a reversal on the second, third, and fourth entries in $(-2, 1, -4, 5, 3)$, we obtain $(-2, -5, 4, -1, 3)$. The *reversal distance* of a signed permutation of 1 through n is the minimum number of reversals required to transform the given signed permutation into $(1, 2, \ldots, n)$. It turns out that the reversal distance of $(3, 1, 6, 5, -2, 4)$ is 5. Find a sequence of 5 reversals that transforms $(3, 1, 6, 5, -2, 4)$ into $(1, 2, 3, 4, 5, 6)$.

Problem 43. Consider a tournament with 15 teams. If every team plays every other team, how many games were played?

Problem 44. There are 8 frogs and 9 rocks on a field. The 9 rocks are laid out in a horizontal line. The 8 frogs are evenly divided into 4 green frogs and 4 brown frogs. The green frogs sit on the first 4 rocks facing right and the brown frogs sit on the last 4 rocks facing left. The fifth rock is vacant for now. Switch the places of the green and brown frogs by using the following rules:

- A frog can only jump forward
- A frog can hop to an vacant rock one place ahead
- A frog can leap over its neighbor frog to a vacant rock two places ahead



Can we generalize this problem and find how many jumps are necessary to switch n green and n brown frogs?

Problem 45. Consider a 4×4 grid with light-up squares. In the starting configuration, some subset of the squares are lit up. At each stage, a square lights up if at least two of its immediate neighbors (horizontal or vertical) were "on" during the previous stage. It's easy to see that for the starting configuration in which four squares along a diagonal of the board are lit up, the entire board is eventually covered by "on" squares. Several other simple starting configurations with four "on" squares also result in the entire board being covered. Is it possible for a starting configuration with fewer than four squares to cover the entire board? If yes, find it; if no, give a proof.

Problem 46. Consider the scenario of the previous problem, except this time suppose we have an 8×8 grid. Is it possible for a starting configuration with fewer than eight squares to cover the entire board? If yes, find it; if no, give a proof. Can you generalize to the $n \times n$ case?

Problem 47. In the game Light Up, two players alternately choose unlit squares from an $m \times n$ grid of light-up squares. The objective of the game is to be the first to light up the entire grid. At the beginning of the game, all squares are turned off. On each player's turn, the player selects any square that is currently off and then the selected square gets lit up. Moreover, additional squares get lit up if at least two of its immediate neighbors (horizontal or vertical) are lit up. This process continues until no new squares are lit up and then it is the next player's turn. The loser of the game is the player that no longer has an available square to light up. Determine which player has a winning strategy for the following grid sizes: 1×3, 1×4, 1×5, 2×2, 2×3, 3×3.

Problem 48. Two different positive numbers a and b each differ from their reciprocal by 1. What is $a + b$?

Problem 49. My Uncle Robert owns a stable with 25 race horses. He wants to know which three are the fastest. He owns a race track that can accommodate five horses at a time. What is the minimum number of races required to determine the fastest three horses?

Problem 50. A father has 20 one dollar bills to distribute among his five sons. He declares that the oldest son will propose a scheme for dividing up the money and all five sons will vote on the plan. If a majority agree to the plan, then it will be implemented, otherwise dad will simply split the money evenly among his sons. Assume that all the sons act in a manner to maximize their monetary gain but will opt for evenly splitting the money, all else being equal. What proposal will the oldest son put forth, and why?

Problem 51. Imagine that in the scenario of the previous problem the father decides that after the oldest son's plan is unveiled, the second son will have the opportunity to propose a different division of funds. The sons will then vote on which plan they prefer. Assume that the sons still act to maximize their monetary gain, but will vote for the older son's plan if they stand to receive the same amount of money either way. What will transpire in this case, and why?

Problem 52 (Two Deep)**.** Consider the equation below. If a is a number, what number is it?

$$a = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cdots}}}}}$$

Problem 53. We have the following information about three integers:

(a) Their product is an integer;

(b) Their product is a prime;

(c) One of them is the average of the other two.

What are these numbers? *Hint*: You need to find all such triples and show that there are no others.

Problem 54. Three boxes, one with black, one with white, and one with black and white balls. Each of the boxes is labeled B, W, and BW, but unfortunately, *all* the boxes are labeled incorrectly. Moreover, you cannot see inside each of the boxes, but you can reach in and pull a ball out. What is the minimum number of balls that need to be pulled before you can relabel all the boxes correctly?

Problem 55. Given enough space, the population of a certain type of bacteria doubles every minute. Suppose one bacterium is placed in a bottle at 11:00AM and an hour later, the bottle is full.

(a) At what time is the bottle half full?

(b) Suppose that at 11:15AM an intelligent bacterium recognizes the space limitations her fellow bacteria are going to have in 45 minutes. The bacteria look around the room and notice 3 empty bottles nearby. Shortly thereafter, the bacteria start emigrating to the empty bottles in an attempt to prolong their existence. At what time will all 4 bottles be full?

Problem 56 (All Different)**.** In the figure below, each row of four is to have one each of the letters A, B, C, D. No two rows are the same. No two vertically adjacent cells have the same letter. Fill in the rest of the missing letters.

			A
			D
		B	
	D		
		C	
			B
	A	C	
D	B		
		B	A
A			
	C	B	
B			
		C	
	B		
			D
	B	C	
A			
		C	
B			A
	B		

Problem 57. Let P be a point inside the triangle ABC. Show that $PA + PB < CA + CB$.

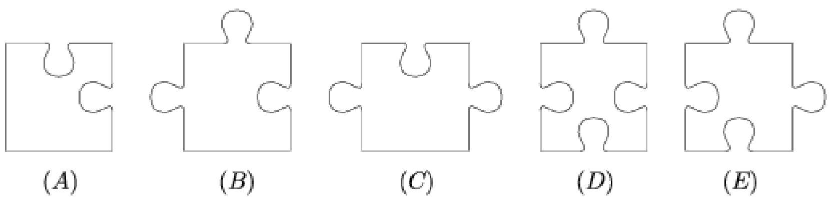
Problem 58 (Magic!)**.** In each of the three grids below, place a number in each square so that every row, every column, and each of the two long diagonals add up to the same number.

20	3	
13		

22	3	
13		

24	3	
13		

Problem 59. A rectangular puzzle that says "850 pieces" actually consists of 851 pieces. Each piece is identical to one of the 5 samples shown in the diagram. How many pieces of type (E) are there in the puzzle?



Problem 60. The first vote counts of the papal conclave resulted in 33 votes each for candidates A and B and 34 votes for candidate C. The cardinals then discussed the candidates in pairs. In the second round each pair of cardinals with differing first votes changed their votes to the third candidate they did not vote for in the first round. The new vote counts were 16, 17 and 67. They were about to start the smoke signal when Cardinal Ordinal shouted "wait". What was his reason?

Problem 61. There are five students at a party. We ask how many friends they have in the group. Here are the answers:

Alex: I have 4 friends.

Bob: I have fewer friends than Alex has.

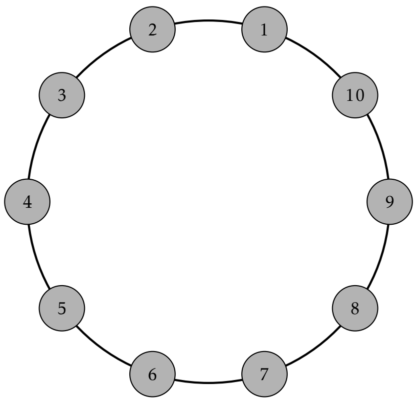
Camille: I have as many friends as Doug.

Doug: Edit has one more friend than I have.

Edit: I have an odd number of friends.

Are Camille and Doug friends?

Problem 62. Ten people form a circle. Each picks a number and tells it to the two neighbors adjacent to him in the circle. Then each person computes and announces the average of the numbers of his two neighbors. The figure shows the average announced by each person. What is number picked by the person who announced 6?



Problem 63. Annie, Bob, and Cristy are sitting by a campfire when Cristy announces that she is thinking of a 3-digit number. She then tells Annie and Bob that the number she is thinking of is one of the following:

$$515, 516, 519, 617, 618, 714, 716, 814, 815, 817.$$

Next, Cristy whispers the leftmost digit in Annie's ear and then whispers the remaining two digits in Bob's ear. The following conversation then takes place:

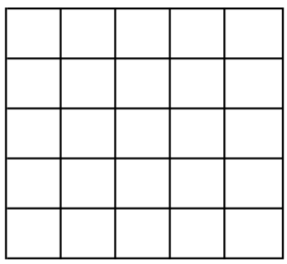
Annie: I don't know what the number is, but I know Bob doesn't know too.

Bob: At first I didn't know what the number was, but now I know.

Annie: Ah, then I know the number, too.

From that information, determine Cristy's number.

Problem 64 (Quilt)**.** How many ways are there to place the letters A, B, C, D, E into the grid below, one per box, so that each row, each column and each of the two long diagonals contain one of each letter?



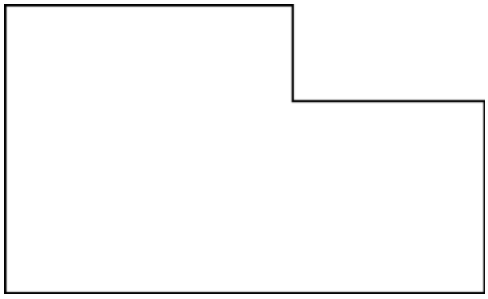
Problem 65. Show that in any group of 6 students there are 3 students who know each other or 3 students who do not know each other.

Problem 66. Place five stones on an 8×8 grid in such a way that every square consisting of 9 cells has only one stone in it.

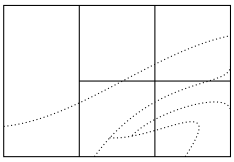
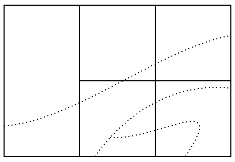
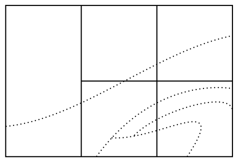
Problem 67. Consider an 8×8 grid of squares. What numbers of stones can you place on the grid such a way that every square consisting of 9 cells has only one stone in it?

Problem 68. Find a solution to the equation $28x + 30y + 31z = 365$, where x, y, and z are positive whole numbers.

Problem 69. This shape below is made by joining two squares, one 3×3, one 2×2. Divide it into a few pieces which can be re-assembled to make a square.



Problem 70. There are five countries on an island. The island also has several species of frogs. The frog territories do not overlap. During an international conference of frog experts, each country wants to create a frog exhibit featuring one of the frog species that live in the country. No two countries want to pick the same kind of frog. How should they choose between the frog species? The map of the island is shown below. Solid lines represent the border between the countries, while dotted lines are the boundaries between the frog territories.

(a)  (b)  (c) 

Problem 71. During a class period students used their cell phones once. In fact, for any two students there was a time when both of the students used their phones. Show that there was a time when nobody listened to the instructor.

Problem 72. Rodd and Deven are pi-ous ninth century monks. It is the summer of 888 AD, and they have agreed they will share the job of writing the town records every day. Rodd does every day that contains an ODD digit in the date. Deven does all other days. They begin

- 20.08.888 Deven
- 21.08.888 Rodd
- 22.08.888 Deven
- 23.08.888 Rodd
- 24.08.888 Deven
- 25.08.888 Rodd
- 26.08.888 Deven
- 27.08.888 Rodd
- 28.08.888 Deven

When is the next day when Deven has to work?

Problem 73. A rectangle that is not a square is folded along a diagonal. Prove that the perimeter of the resulting pentagon is smaller than the perimeter of the original rectangle.

Problem 74. After Thor is captured by Loki, Loki sets Thor the following challenge in order to gain his freedom. Thor is presented three closed doors, numbered 1–3. Thor's hammer (which he is unable to summon due to a spell Loki cast on the hammer) is behind one of the doors and there are wolves behind the other two doors. If Thor can guess which door his hammer is behind, Loki will return the hammer and let Thor go. Otherwise, Loki will cast a spell that turns Thor into a goat. Thor picks door number 1. Because Loki is mischievous and knows what is behind each door, he decides to show Thor what is behind door number 3, which happens to be a wolf. Loki says, "Do you want to pick door number 2 or stick with your original choice of door 1?" Is it to Thor's advantage to switch his choice?

Problem 75. Consider the scenario of the previous problem, except now assume that there are $n \geq 4$ doors, behind one of which is Thor's hammer and there are wolves behind the remaining $n-1$ doors. Moreover, assume that Loki shows Thor $k \geq 2$ incorrect choices after Thor's initial guess. Loki says, "Do you want to pick a different door or stick with your original choice?" Should Thor modify his initial guess or not? In particular, explain what Thor should do in the extreme case when Loki opens $k = n-2$ incorrect doors.

Problem 76. You and your two friends Thor and Valkyrie are captured by Loki. In order to gain your freedom, Loki sets you the following challenge. The three of you are put in adjacent cells. In each cell is a quantity of stones. Each of you can count the number of stones in your own cell, but not in anyone else's. You are told that each cell has at least one stone but at most nine stones, and no two cells have the same number of stones. The rules of the challenge are as follows: The three of you will ask Loki a single question each, which he will answer truthfully "Yes" or "No". Every one hears the questions and the answers. Loki will set all of you free only if one of you can correctly determine the total number of stones in all the cells. Here is the initial conversation.

> Thor: Is the total an even number?
>
> Loki: No.
>
> Valkyrie: Is the total a prime number?
>
> Loki: No

You have five stones in your cell. What question will you ask? You should assume that Thor and Valkyrie are just as good at logic as you are.

Problem 77. A box contains two red hats and three green hats. Azalea, Barnaby, and Caleb close their eyes, take a hat from the box and put it on. When they open their eyes they can see each other's hats but not their own. They do not know which hats are left in the box. We can assume that all the protagonists are perfect logicians who tell the truth. They know all the information in the above paragraph. In addition, one of them is colorblind. They all know who the colorblind person is.

> Azalea says: "I don't know the color of my hat."
>
> Barnaby says: "I don't know the color of my hat."
>
> Caleb says: "I don't know the color of my hat."
>
> Azalea says: "I don't know the color of my hat."

Who is the colorblind person, and what color is their hat?

Problem 78. Show that in any set of seven different positive integers there are three numbers such that the greatest common divisor of any two of them leaves the same remainder when divided by three.

Problem 79. In a PE class, everyone has 5 friends. Friendships are mutual. Two students in the class are appointed captains. The captains take turns selecting members for their teams, until everyone is selected. Prove that at the end of the selection process there are the same number of friendships within each team.

Problem 80. In the senate of the Klingon home world no senator has more than three enemies. Show that the senate can be separated into two houses so that nobody has more than one enemy in the same house.

Problem 81 (One Overs)**.** Find positive odd integers $A < B < C$ such that

$$\frac{1}{3} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C}.$$

Problem 82. There are $2n$ Federation ambassadors invited to a Ferengi banquet. Every ambassador has at most $n-1$ enemies. Show that the ambassadors can be seated around a round table avoiding enemies sitting next to each other.

Problem 83. A colony of chameleons on an island currently comprises 13 green, 15 blue, and 17 red individuals. When two chameleons of different colors meet, they both change their colors to the third color. Is it possible that all chameleons in the colony eventually have the same color?

Problem 84. 100 prisoners are isolated in individual jail cells with no way to communicate. They are currently serving life sentences. Due to an overcrowded prison, the jailer decides to offer the prisoners the following deal. There is a a room with nothing in it except a light switch (that starts in the off position). At random, the jailer will escort a single prisoner into the room with the light switch. After 5 seconds, the jailer will escort the prisoner back to his/her jail cell. The jailer will repeat this over and over again. He tells each of the prisoners that if one of the prisoners can indicate when every prisoner has been in the room with the light switch at least once, he will let all the prisoners go. However, if a prisoner erroneously states that each prisoner has been in the room with the light switch, then all the prisoners will be executed. Before beginning, the jailer gets all 100 prisoners together and gives them 5 minutes to come up with a plan. What should their plan be? It's important to note that the jailer is choosing prisoners at random to take in the room. That is, by chance, the same prisoner may be escorted to the room several times in a row. Also, your task is to devise a scheme for the prisoners to communicate with the light switch. You shouldn't bother searching for other ways for the prisoners to communicate.

Problem 85. Suppose you have 12 coins, all identical in appearance and weight except for one that is either heavier or lighter than the other 11 coins. What is the minimum number of weighings one must do with a two-pan scale in order to identify the counterfeit?

Problem 86. Consider the situation in the previous problem, but suppose you have n coins. For which n is it possible to devise a procedure for identifying the counterfeit coin in only 3 weighings with a two-pan scale?

Problem 87. Let's revisit the counterfeit coin problem presented in Problems 85 and 86. In Problem 85, we discovered that we could detect the counterfeit coin in at most 3 weighings regardless of whether we knew in advance whether the counterfeit was heavier or lighter than the non-counterfeit coins. One feature of our algorithm was that after our 3 weighings, we could not only tell which coin was the counterfeit but also whether it was in fact heavier or lighter. It's certainly believable that 3 weighings is the best we can guarantee with 12 coins, but we did not prove this.

In Problem 86, we were asked to determine which number of coins we could start with and guarantee that we could identify which coin is counterfeit in at most 3 weighings. Certainly, $n = 12$ works. What about $n < 12$ and $n > 12$? It certainly seems believable that if we could handle 12 coins in 3 weighings, we could handle less. But is this true? It's not obvious at all what happens with more than 12 coins.

Let's do some exploring. Let n be the number of coins. Assume that exactly one coin is counterfeit so that the remaining $n-1$ coins are not counterfeit. Further suppose that we do not know whether the counterfeit coin is heavier or lighter than the others but we do know that the counterfeit coin is one of these. Let k denote the number of weighing used to detect the counterfeit coin. We will attempt to find a relationship between n and k. It is clear that if $n = 1$, we only have a counterfeit coin we are done without having to do any weighings. So, let's assume that $n \geq 2$. Even if you can't do one of the earlier parts, you should still try to use the results to do the later parts.

(a) Argue that n cannot be 2 so that $n \geq 3$.

(b) Suppose that on the first weighing, you take two piles of m coins where $2m < n$ and weigh them. There are two possibilities. Either the two sets of m coins balance on the scale or they don't. Let's first consider the case where the scales balance on the first weighing. In this case, the counterfeit must be one of the remaining $n - 2m$ coins. We must be able to detect the counterfeit in the remaining $k-1$ weighings. Argue that

$$2(n-2m) - 1 \leq 3^{k-1}. \tag{1}$$

(c) Next, let's assume that the scale was unbalanced on the first weighing when we weighed the two piles of m coins. Argue that

$$2 \cdot 2m \leq 2 \cdot 3^{k-1}. \tag{2}$$

Hint: The 2 on the righthand side comes from the fact that the heavy side of the scale may be on the left or the right.

(d) Starting with inequality (2), show that

$$2m \leq 3^{k-1} - 1. \tag{3}$$

Hint: First, simplify (2) in the obvious way and then observe that the righthand side is odd.

(e) Next, start with $2(n-2m)-1+4m$ and then use (1) and (3) to show that

$$n \leq \frac{3^k - 1}{2}. \tag{4}$$

(f) Use inequality (4) to show that the number of coins must be less than or equal to 13 if we are only allowed 3 weighings.

(g) We've already seen that we could handle $n = 12$ coins in $k = 3$ weighings. However, just because we found out that the number of coins must be less than or equal to 13 if we are only allowed 3 weighings does not guarantee that we can actually pull this off. Can you adapt the strategy for 12 coins to handle 13 coins? If we can actually handle 13 coins in 3 weighings, we will show that our bound given in inequality (4) is optimal when $k = 3$. In this case, we say that the bound is "sharp."

(h) Use one of the facts above to prove that we cannot handle 12 coins in only 2 weighings.

(i) Sort out which numbers of coins we can handle when $k = 2$. Verify that your answer is correct.

Problem 88 (The Martian Artifacts)**.** Recent archaeological work on Mars discovered a site containing a pile of white spheres, each about the size of a tennis ball. A plaque near the mound states that each sphere contains a jewel that come in many different colors while strictly more than half of the spheres contain jewels of the same color. When two spheres are brought together, they both glow white if their internal jewels are the same color; otherwise, no glow. In how few tests can you find a sphere that you are certain holds a jewel of the majority color if the number of spheres in the pile is $2, 3, 4, 5, 6, 7, 8$, or 9? You should provide an answer with justification for each of the different values.

Problem 89. A soul swapping machine swaps the souls inside two bodies placed in the machine. Soon after the invention of the machine an unforeseen limitation is discovered: swapping only works on a pair of bodies once. Souls get more and more homesick as they spend time in another body and if a soul is not returned to its original body after a few days, it will kill its current host.

(a) Suppose Tom and Jerry swap souls and Garfield and Odie swap souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(b) Suppose Batman and Robin swap souls and then Robin's body and Flash utilize the machine. Argue that it is not possible to return the swapped souls to their original bodies using only Batman, Robin, and Flash.

(c) Consider the scenario of the previous problem. Suppose Wonder Woman and Superman are now available to sit in the machine after Batman, Robin, and Flash have already swapped souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(d) Now, suppose the soul swapping machine is used by the following pair of bodies (in the order listed): Adam and Alicia, Alicia and Gwen, Gwen and Blake. In addition, Pharrell and Miley are standing nearby. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(e) Suppose n different people have been involved in a finite sequence of soul swaps. Note that it's possible that an individual body may use the machine more than once during this soul swapping bonanza. Is it possible to return all swapped souls back to their original bodies? You may assume some innocent bystanders are nearby.

Problem 90. Adam, Bob, Chloe, and Dolores are friends and they want to decide who is the coolest person in the group. They ask each other what they think about the others. Adam thinks only Dolores is cool because he has a crush on her. Bob thinks all of his friends are cool. Chloe thinks Adam is cool while Dolores thinks Bob is cool. Who is the coolest in the group?

Problem 91. The Infinite Hotel has rooms numbered $1, 2, 3, 4, \ldots$. Every room in the Infinite Hotel is currently occupied. Is it possible to make room for one more guest (assuming they want a room all to themselves)? An infinite number of new guests, say $g_1, g_2, g_3, \ldots$, show up in the lobby and each demands a room. Is it possible to make room for all the new guests even in the hotel is already full?

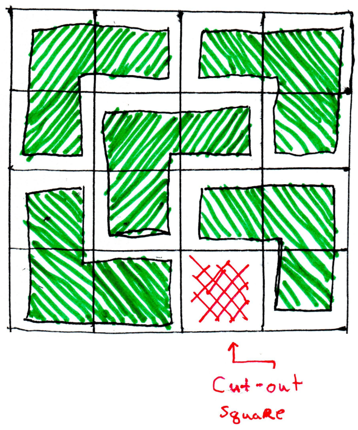
Problem 92 (The Good Teacher)**.** You are teaching Calculus I, and you wish to give the students a cubic polynomial and have them find its three x-intercepts, its two critical points and its one inflection point. Because you are a kind person, you want these 6 points to all have integer coordinates and you want the cubic to have integer coefficients that are not too horribly large. Find one.

Problem 93. Consider the regular hexagon $ABCDEF$. Let X be the midpoint of CD and let Y be the midpoint of DE. Let Z be the common point of AX and BY. Which polygon has larger area, ABZ or $DXZY$?

Problem 94. Suppose we draw n lines in the plane that have the maximum number of unique intersections. This partitions the plane into disjoint regions (some of which are polygons with finite area and some are not). Suppose we color each of the regions so that no two adjacent regions (i.e., share a common edge) have the same color. What is the fewest colors we could use to accomplish this? Justify your answer.

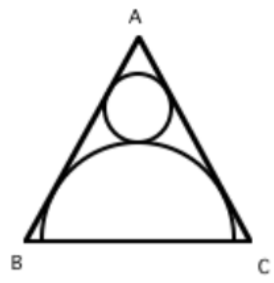
Problem 95. Prove that every natural number can be written as the sum of distinct powers of two.

Problem 96. Consider a grid of squares that is 2^n squares wide by 2^n squares tall such that one of the squares has been cut out, but you don't know which one! You have a bunch of L-shaped trominoes made up of 3 squares. Prove that you can perfectly cover this grid with trominoes (with no overlap) for any $n \in \mathbb{N}$. The figure below depicts one possible covering for the case involving $n = 2$. *Hint:* Use induction.

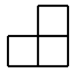


Problem 97. In a certain kind of tournament, every player plays every other player exactly once and either wins or loses (there are no ties). Define a *top player* to be a player who, for every other player x, either beats x or beats a player y who beats x. (There may be more than one top player.) Then every n-player tournament has a top player.

Problem 98. The figure below shows an equilateral triangle ABC with an inscribed semicircle of radius R that is tangent to sides AB and AC, and inscribed circle of radius r that is tangent to the triangle and the semicircle. Find the value of r/R.



Problem 99. What size rectangles can be tiled with the following tromino?



Problem 100. A line of soldiers is receiving matching commands from a drill instructor. Turn left, says the instructor. Of course there are mistakes. Some of them turn left, and some right. Then the instructor gives the correction command several times. At each command, if a soldier faces a neighbor, (s)he assumes a mistake and turns around. Does the line become stable after a while?

Problem 101. There are 30 red, 40 yellow, 50 blue, and 60 green balls in a box. We take out balls from the box with closed eyes. On the first turn we take out 1 ball, on the second turn we take out 2, and so on. On the nth turn we take out n balls. What is the minimum number of balls we need to take out to guarantee the following:

(a) We have a blue ball;

(b) We have a red and a green ball;

(c) We have all four colors.

Problem 102. A frog jumps along the number line. It starts at 0 and every second it jumps n units to the right (the same positive integer n each time). We want to catch the frog. It's dark, we can't see the frog, and we do not know what n is. (For all we know, it might be a super-frog, so n could be arbitrarily large.) However, at any given second, we are allowed to choose an integer and search there. If the frog is on that integer, we catch it; if not, we have to try again.

(a) How can we catch the frog? We need to know which integer to check at each second.

(b) Now suppose the frog is allowed to start by going either to the left or to the right; once it chooses a direction, it always jumps n units in that direction.

(c) What if the conditions in part (b) hold, but we don't know which integer point the frog started at?

Problem 103. Each integer on the number line is colored with exactly one of three possible colors—red, green, or blue—according to the following two rules:

- The negative of a red number must be colored blue;
- The sum of two blue numbers (not necessarily distinct) must be colored red.

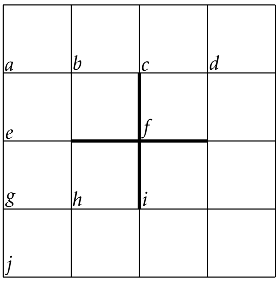
Using this information, answer the following questions.

(a) Show that the negative of a blue number must be colored red and the sum of two red numbers must be colored blue.

(b) Determine all possible colorings of the integers that satisfy these rules.

Problem 104. Three actors and their three agents want to cross a river in a boat that is capable of holding only two people at a time, under the constraint that no actor can be in the presence of another agent unless their own agent is also present, because each agent is worried their rivals will poach their client. How should they cross the river with the least amount of rowing?

Problem 105. Write a single digit into each square of the cross-number puzzle using the clues.



Across

g. The sum of across-f and across-i.
j. The cube of down-i.

Down

a. The cube of down-b.
b. The reverse of down-i.
c. Divisible by down-i.
d. All digits are the same.