Exam 2 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 2 is worth a total of 32 points and is worth 30% of your overall score on Exam 2. This take-home exam is due at the beginning of class on **Monday, October 29**. Your overall score on Exam 2 is worth 18% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 5.35, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

1. (4 points) Let G be a cyclic group of order 6 with generator g. Define $\phi : \mathbb{R}_6 \to G$ via
$$\phi(e) = e, \phi(r) = g, \phi(r^2) = g^3, \phi(r^3) = g^2, \phi(r^4) = g^4, \phi(r^5) = g^5.$$
Determine whether ϕ is an isomorphism. Justify your answer.

2. Let G be a group and let $H \leq G$. Define the relation $\sim$ on G via
$$a \sim b \text{ if and only if } a^{-1}b \in H.$$
   (a) (4 points) Prove that $\sim$ is an equivalence relation on G. *Note:* It turns out that there is a nice description of the corresponding equivalence classes. If $a \in G$, then let $[a]$ denote the equivalence class containing a. It isn't too hard to prove that $[a] = aH$, where $aH := \{ah \mid h \in H\}$. In the interest of time, let's take this for granted.*

   (b) (4 points) Define $\phi : H \to aH$ via $\phi(h) = ah$. Prove that ϕ is one-to-one and onto.†

   (c) (4 points) Prove that if G is a finite group, then $|H|$ divides $|G|$.

   (d) (2 points) Prove that if G is a finite group and $a \in G$, then $|a|$ divides $|G|$.

   You may use the results above (even if you were not able to prove them) on the rest of the exam.

3. Suppose $(G, *)$ and $(H, \circ)$ are groups. Define $\star$ on $G \times H$ via $(g_1, h_1) \star (g_2, h_2) = (g_1 * g_2, h_1 \circ h_2)$.‡ It turns out that $(G \times H, \star)$ is a group. We will prove this later (it's not too difficult), but for now, let's take it for granted. If e_G and e_H are the identity elements of G and H, respectively, then (e_G, e_H) is the identity element in $G \times H$. Moreover, if $(g, h) \in G \times H$, then $(g, h)^{-1} = (g^{-1}, h^{-1})$.

   (a) (3 points) Consider $S_2 \times R_4$ (using the operation of S_2 in the first component and the operation of R_4 in the second component). Find a generating set for $S_2 \times R_4$ and then create a Cayley diagram for this group.

   (b) (3 points) Argue that $S_2 \times R_4$ cannot be isomorphic to any of D_4, R_8, Q_8, and L_3.

4. (4 points each) Prove any **two** of the following theorems.

   Theorem 1. If G is a group, then $|gh| = |hg|$ for all $g, h \in G$.

   Theorem 2. Suppose $(G_1, *)$ and $(G_2, \circ)$ are groups and the function $\phi : G_1 \to G_2$ satisfies the homomorphic property. If $g \in G_1$ such that g has finite order, then $|\phi(g)|$ divides $|g|$.

   Theorem 3. Suppose G is a finite nontrivial cyclic group such that $|G| = n$. Then G has no proper nontrivial subgroups if and only if n is prime.

   Theorem 4. Suppose p and q are distinct primes. If G is any group of order pq, then G has either an element of order p or an element of order q.§

*In case you are interested, the proof involves showing two set containments: $[a] \subseteq aH$ and $aH \subseteq [a]$. Both arguments are straightforward.

†The function ϕ is not intended to satisfy the homomorphic property. In fact, it doesn't.

‡This looks fancier than it is. We're just doing the operation of each group in the appropriate component.

§Recall that in mathematics, "or" is inclusive unless specified otherwise. So, this statement allows for both an element of order p and an element of order q. It turns out that G must have both an element of order p and an element of order q, but you don't need to prove this. One approach to tackling this theorem is to first consider the case when G is cyclic.

