Chapter 6

Differentiation

It's time for calculus!

Definition 6.1. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. For real number D, we say that f has *derivative* D at the point a if the following two conditions hold:

1. The point a is an accumulation point of the domain of f.

2. If S is an open interval containing D, then there is an open interval T containing a such that if $t \in T$, $t \neq a$, and t is in the domain of f, then

$$\frac{f(t) - f(a)}{t - a} \in S.$$

In this case, we say that f is *differentiable* at a. If f does indeed have a derivative at some points in its domain, then the derivative of f is the function denoted by f', such that for each number x at which f is differentiable, $f'(x)$ is the derivative of f at x.

Note that the definition of derivative automatically excludes the kind of behavior we saw with continuous functions, where a function defined only at a single point was continuous.

Exercise 6.2. Explain why any function defined only on $\mathbb{Z}$ cannot have a derivative.

Exercise 6.3. Find and prove a formula for the derivative of $f(x) = 3$.

Problem 6.4. Find and prove a formula for the derivative of $g(x) = 2x - 5$.

The following problem provides an alternative definition for the derivative.

Problem 6.5. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Prove that f has derivative D at the point a if and only if the following two conditions hold:

1. The point a is an accumulation point of the domain of f.

2. If $\epsilon > 0$, then there exists $\delta > 0$ such that if t is in the domain of f and $|t - a| < \delta$, then

$$\left| \frac{f(t) - f(a)}{t - a} - D \right| < \epsilon.$$

Problem 6.6. Find the derivative of $h(x) = x^2 - x + 1$ at $x = 2$.

Problem 6.7. Find the derivative of $h(x) = x^2 + ax + b$ for any $a, b \in \mathbb{R}$.

Problem 6.8. If f is differentiable at x and $c \in \mathbb{R}$, show that the function cf also has a derivative at x and $(cf)'(x) = cf'(x)$.

Problem 6.9. If f and g are differentiable at x, show that the function $f + g$ also has a derivative at x and $(f + g)'(x) = f'(x) + g'(x)$.

More coming soon...