Exam 1 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 1 is worth a total of 24 points and is worth 30% of your overall score on Exam 1. This take-home exam is due at the beginning of class on **Friday, March 1**. Your overall score on Exam 1 is worth 20% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems or problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem X or Problem Y, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

We need to introduce a few definitions.

Recall that a binary relation $\leq$ is a *total order* on a set X if the following conditions hold for all $a, b, c \in X$:

(a) If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry);

(b) If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity);

(c) Either $a \leq b$ or $b \leq a$ (connex).

For each (non-strict) total order $\leq$ there is an associated asymmetric relation $<$, called a *strict total order* given by $a < b$ if $a \leq b$ and $a \neq b$.

Now, let X be a set totally ordered by $<$. Define $\mathcal{B}_<$ to be the collection of all subsets of X that are any of the following forms:

$$\{x \in X \mid x < a\} \quad \text{or} \quad \{x \in X \mid a < x\} \quad \text{or} \quad \{x \in X \mid a < x < b\}$$

for $a, b \in X$. It is easy to verify that $\mathcal{B}_<$ is a basis for a topology, called the *order topology* on X.

Given sets A and B, recall that their Cartesian product is given by

$$A \times B := \{(a, b) \mid a \in A, b \in B\}.$$

If A and B are totally ordered by $<_A$ and $<_B$, respectively, then the *lexicographic order* (or *dictionary order*) $<$ on $A \times B$ is defined via $(a_1, b_1) < (a_2, b_2)$ if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$.

The square $[0,1] \times [0,1]$ with the lexicographic order and its associated order topology is called the *lexicographically ordered square*.

Let X and Y be two sets. Recall that the projection functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are defined via

$$\pi_X(x, y) = x \quad \text{and} \quad \pi_Y(x, y) = y.$$

If sets X and Y have topologies, there is a natural topology on $X \times Y$. If particular, if X and Y are topological spaces, then the *product topology* on $X \times Y$ is the topology whose basis is all sets of the form $U \times V$ where U is an open set in X and V is an open set in Y. It is easy to verify that this collection of basic open sets satisfies the conditions of Theorem 4.3, thus confirming that this collection is the basis for a topology.

1. (2 points each) In the lexicographically ordered square find the closures of the following subsets and briefly justify your answers.
   (a) $C := \{(x, 0) \mid 0 < x < 1\}$
   (b) $D := \left\{\left(x, \frac{1}{2}\right) \mid 0 < x < 1\right\}$
   (c) $E := \left\{\left(\frac{1}{2}, y\right) \mid 0 < y < 1\right\}$
2. (4 points) Let $(X, \mathcal{T})$ be a topological space with subspace $(Y, \mathcal{T}_Y)$. Prove that if $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_Y := \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.*

*This is Theorem 4.30.

3. (2 points each) For each of the following subspaces in the given topological space, describe the associated subspace topology and briefly justify your answer.
   (a) In $\mathbb{H}_{bub}$, let X be the set of points on the x-axis.
   (b) In the lexicographically ordered square, consider the set D from Problem 1.
   (c) In the lexicographically ordered square, consider the set E from Problem 1.
4. (2 points each) Consider the product topology on $\mathbb{R} \times \mathbb{R}$, where we endow each factor of $\mathbb{R}$ with the standard topology.
   (a) Find an open set in $\mathbb{R} \times \mathbb{R}$ that is not the product of open sets.
   (b) Find a closed set in $\mathbb{R} \times \mathbb{R}$ that is not the product of closed sets.
5. (4 points) Let X_1 and X_2 be topological spaces. Let C_1 and C_2 be closed sets in X_1 and X_2, respectively. Determine whether $C_1 \times C_2$ is a closed set in the product space $X_1 \times X_2$. Justify your answer with a proof or counterexample.