Chapter 5

Continuity

Definition 5.1. We say that a function f is *continuous at a point* x in its domain (or at the point $(x, f(x))$) if, for any open interval S containing $f(x)$, there is an open interval T containing x such that if $t \in T$ is in the domain of f, then $f(t) \in S$.

Definition 5.2. A function f is *continuous* if it is continuous at every point in its domain.

Let's show that this definition of continuity behaves the way we expect from calculus.

Exercise 5.3. Show that each of the following functions is continuous.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x$.

(b) $g : \mathbb{R} \to \mathbb{R}$ defined via $g(x) = 2x$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined via $h(x) = x + 3$.

Problem 5.4. Show that any linear function given by $f(x) = mx + b$ is continuous for all $x \in \mathbb{R}$.

The next problem tells us that we can reframe continuity in terms of distance.

Problem 5.5. Let f be a function. Prove that f is continuous at x if and only if for every $\epsilon > 0$, then there exists $\delta > 0$ so that if t is in the domain of f and $|t - x| < \delta$, then $|f(t) - f(x)| < \epsilon$.

The previous characterization is typically referred to as the "$\epsilon - \delta$ definition of continuity", although for us it is a theorem instead of a definition. This characterization is used as the definition of continuity in metric spaces.

Problem 5.6. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Find all points x where f is continuous and justify your answer.

Problem 5.7. Define $g : \{0\} \to \mathbb{R}$ via $g(0) = 0$. Show that g is continuous at $x = 0$.

Problem 5.8. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Find all points x where f is continuous and justify your answer.

Problem 5.9. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. Prove that f is continuous.

Exercise 5.10. Find a continuous function f and an open interval U such that the preimage $f^{-1}(U)$ is not an open interval.

Problem 5.11. Let f be a function. Prove that f is continuous if and only if the preimage $f^{-1}(U)$ of every open set U is an open set intersected with the domain of f.

The previous characterization of continuity is often referred to as the "open set definition of continuity" and is the definition used in topology.

It turns out that there is a deep connection between continuity and sequences!

Definition 5.12. We say that a function f is *sequentially continuous at a point* x if, for every sequence $(x_i)_{i=1}^{\infty}$ (in the domain of f) converging to x, it is also true that $(f(x_i))_{i=1}^{\infty}$ converges to $f(x)$.

Problem 5.13. Let f be a function. Prove that f is continuous at x if and only if f is sequentially continuous at x.

The upshot of the previous problem is that the notions of being *continuous at a point* and *sequentially continuous at a point* are equivalent on the real numbers. However, there are contexts in mathematics where the two are not equivalent. This is topic in a branch of mathematics called *topology*. If you want to know more, check out the following YouTube video:

https://www.youtube.com/watch?v=sZ5fBHGYurg

The sequential way of thinking of continuity often makes proving some basic facts concerning continuity easier.

At this point, we have four different ways of thinking about continuity.

- Definition 5.1 using open intervals.
- Problem 5.5 using ϵ and δ.
- Problem 5.11 using inverse images of open sets.
- Problem 5.13 using sequential continuity.

You should take the time to review each one. Moreover, it is worth pointing out that three of the four characterizations involve continuity at a point. Which one does not? For the remainder of the book, feel free to use which ever characterization you'd like.

Problem 5.14. Suppose f and g are functions that are continuous at x and let $c \in \mathbb{R}$. Prove that each of the following functions are also continuous at x.

(a) cf

(b) $f + g$

(c) $f - g$

(d) fg

Exercise 5.15. Prove that every polynomial is continuous on all of $\mathbb{R}$.

To be continued...