Chapter 2

Preliminaries and Review

In this course, we do assume certain background knowledge. This section is intended to review some of this, as well as to focus on specific items which may prove of use later.

2.1 Absolute Basics

There are some things I will not expand on at all.

We will take most normal mathematical and logical terminology from a proof transition course for granted.

We assume that sets exist. We will take the sets of counting ($\mathbb{N}$), integer ($\mathbb{Z}$), rational ($\mathbb{Q}$), and real ($\mathbb{R}$) numbers as known. In this course $\mathbb{N}$ starts with 1, not 0. We will denote the irrational numbers by $\mathbb{Q}^C$, a notation for the *complement* of the rationals with respect to the set $\mathbb{R}$.

In these sets we will assume the most basic properties of addition, subtraction, and multiplication, such as commutativity and distributivity. Likewise, we will later assume without proof basic facts about ordering numbers. At this time we explicitly assume that $0 < 1$ and that there are no integers between them. You will be able to use these facts to *prove* that is true for other consecutive pairs of integers. (The fact $0 < 1$ *does* require proof! Typically one uses a Well-Ordering type axiom.)

Although there is a lot of fun to be had with infinity, I do not foresee doing much with it other than using generic and specific infinite sets.

The rest of this preliminary chapter is intended to remind us of things for which I do think review could be useful.

2.2 Logic Review

In basic proof logic, we will usually be proving a statement of the nature

If P is true, then Q is also true.

Exercise 2.1. Prove that if x is an even integer, then x^2 is also an even integer.

However, there are more interesting examples of how to prove things. Recall the *converse* of the generic statement

If Q is true, then P is also true

and the *contrapositive* thereof

If Q is not true, then P is also not true.

Exercise 2.2. State the converse of the previous example, and then *its* contrapositive. Prove that final statement.

In (normal) mathematical logic, we say that the contrapositive and the original statement are *logically equivalent*. So you should use whichever one you find useful in proving theorems!

However, that isn't the case with the converse and the original statement.

Exercise 2.3. Take the statement "If n is a prime number, then n is a positive integer" and give a reason why it is true, but its converse is false.

In the first two examples, we have the highly unusual situation where both $P \implies Q$ and $Q \implies P$ are both true. We say that this is a *biconditional* statement and write $P \iff Q$, saying

P is true if and only if Q is true,

or "P is necessary and sufficient for Q", or some variation thereon.

There are a few other tricky logic issues that we should definitely practice. The logical connector *or* is a word that means many things. In most of math (and here), "A or B" means "A or B or *both*". This is important in statements like

If x or y is even, then xy is also even

since even if x and y were both even, xy would still be even.

Exercise 2.4. Come up with a theorem which is *not* true with this meaning of 'or' but *is* true with the so-called *exclusive or*.

The other remaining tricky issue is that of quantifiers—things like "for all" and "there exists". This is best tackled by trying it.

Exercise 2.5. Restate these as accurately as possible in terms of logic, and then do all of the following with as few double negatives as possible, using natural negations if possible:

- Negate the statement, "All professors at my college who live in my town are old and grey."
- Negate the statement, "There exists a student at my college who does not live in my town," in the most natural way possible (i.e., without double negatives).

- Give the contrapositive of the statement, "If there exists a professor at my college who is neither old nor grey, then all students at my college are young and bald."

Problem 2.6. Formally negate the statement, "For every positive number p, there is a positive integer N such that if an integer I is greater than N, then the reciprocal of I is less than p," with as few negatives and as naturally as possible. (But don't try to prove it!)

2.3 Sets and Functions

Recall that A is a *subset* of B (written $A \subseteq B$) if every element $x \in A$ is also an element of B. When a set is defined as a subset of another set with a given property, we often use "set builder" notation, such as

$$\{n \in \mathbb{Z} \mid 2 \text{ divides } n\}$$

for defining the even numbers. If a set has no elements, such as

$$\{P \text{ is a president of the USA before 2000} \mid P \text{ is female}\},$$

we call it the empty set ($\emptyset$). The familiar intersection ($\cap$) and union ($\cup$) operators are very useful. We distinguish between $\subseteq$ and $\subsetneq$, where in the latter the sets are not equal. Finally, we define $A \setminus B$ to be the set of all elements in A which are *not* in B.

Exercise 2.7. Prove that $\mathbb{Z} \subseteq \mathbb{Q}$.

Exercise 2.8. Decide whether $\mathbb{N} \cup \{-n \mid n \in \mathbb{N}\} = \mathbb{Z}$.

Exercise 2.9. Find three nonempty sets A, B, C such that $A \cap B = \emptyset$, $B \in C$, and $A \notin C$.

Exercise 2.10. Give an example where $A \neq B$ but $A \setminus B = \emptyset$.

More interestingly, we can define infinite combinations of these things, if we have a collection of sets X_α, one for each element $\alpha \in A$ an infinite set.

Problem 2.11. Find (nonempty) sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_{i+1} \subsetneq S_i$ and

$$\bigcap_{i=1}^{\infty} S_i = \emptyset.$$

Problem 2.12. Find (nonempty) sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_i \subsetneq S_{i+1}$ but

$$\bigcup_{i=1}^{\infty} S_i \neq \mathbb{N}.$$

We should also recall that, for us, a function is a set.

Definition 2.13. A *function* from a set A to a set B is a subset of the Cartesian product $A \times B$ such that there is precisely one $(a, b) \in f$ for each $a \in A$. We call A the *domain*, and say that $f(a) = b$ if $(a, b) \in f$. There is no standard name for B, but if you insist on having one, use *codomain*.

Exercise 2.14. Write some function from $\mathbb{Z}$ to itself this way.

Recall a few properties functions might have.

- A function from A to B is *onto* if, for each $b \in B$, there is an $a \in A$ such that $f(a) = b$.
- A function from A to B is *one-to-one* if when $a \neq a' \in A$, then $f(a) \neq f(a') \in B$.
- A function from A to B is a *bijection* if it is one-to-one and onto.

Exercise 2.15. Suppose that $f(x) = x^2$, where $A = B = \mathbb{Z}$. Change A or B so that it is one-to-one. Find a way to change it to be onto.

Exercise 2.16. Find a bijection from $\mathbb{Z}$ to itself that is *not* the identity. If you remember what an inverse function is, find that, too.

There are two more important sets relevant to a given function.

Definition 2.17. Let $f : A \to B$ be a function.

- If $S \subseteq A$, we call the *image* of S under f, denoted $f(S)$, the set

$$\{f(x) \mid x \in S\}.$$

- If $T \subseteq B$, we call the *preimage* of T under f, denoted $f^{-1}(T)$, the set

$$\{x \in A \mid f(x) \in T\}.$$

Notice that the preimage exists whether or not the inverse function exists (and the notation f^{-1} refers only to the preimage in this course). We call the image of the entire domain the *image of the function*, $f(A)$.

Exercise 2.18. If $f(x) = x^2$ from $\mathbb{Z}$ to itself, find $f(\{-2,-1,0,1,2\})$ and $f^{-1}(\{0,1,4\})$.

Problem 2.19. Find functions f and g and sets S and T such that $f(f^{-1}(T)) \neq T$ and $g^{-1}(g(S)) \neq S$. (Note that this is not composition!)

Problem 2.20. With all sets appropriately defined, only one of two following statements is necessarily true. Prove one and disprove the other.

$$f(X \cap Y) = f(X) \cap f(Y)$$

$$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$$

2.4 Order Properties of the Real Numbers

We are almost ready to start some actual analysis, where the functions and sets will be defined on the real numbers. (As mentioned earlier, we will tacitly assume that numbers in this course are real numbers, unless we state otherwise.) In order to do this, we need some facts about real numbers, including the following **Order Axioms** previously promised.

Given $a, b, c \in \mathbb{R}$:

Order Axiom 1 If $a < b$ and $b < c$ then $a < c$.

Order Axiom 2 If $a \neq b$ then either $a < b$ or $b < a$ (but not both!).

Order Axiom 3 If $a < b$, then $a + c < b + c$.

Order Axiom 4 If $a < b$ and $c > 0$, then $ac < bc$; if $c < 0$, then $ac > bc$.

Order Axiom 5 Given $a \in \mathbb{R}$, then there is an integer n such that $n \leq a < n + 1$.

Exercise 2.21. Why are n and $n + 1$ are the smallest possible such integers?

Definition 2.22. Given these axioms, we say that the real numbers $\mathbb{R}$ are *linearly ordered* because "less than" and "greater than" have meaning.

The last axiom is sometimes called 'Archimedean'. We call numbers greater than zero *positive*, and those greater than zero or equal to zero *nonnegative*; there is a similar definition for *negative* and *nonpositive*.

Adding the order axioms to the most basic facts about addition, subtraction, and multiplication already can allow us to prove slightly more advanced facts.

Exercise 2.23. Under these axioms, the various familiar relationships between positive, negative, addition, and multiplication are true – such as 'a positive times a negative is negative' and 'a positive plus a positive is positive'.

Here is another useful fact related to the axioms. It may seem just as expected, but is quite different in style.

Problem 2.24. Prove that for any positive $a \in \mathbb{R}$, there is an integer N such that $0 < \frac{1}{N} < a$.

Finally, there is a special function which should be introduced now.

Definition 2.25. Given $a \in \mathbb{R}$, we define the *absolute value of a*, denoted $|a|$, to be a if $a \geq 0$ and $-a$ otherwise.

Exercise 2.26. Show that $|-4|$ is what you expect, using this definition.

Exercise 2.27. Show that $|a| \geq 0$, with equality only if $a = 0$.

Exercise 2.28. Show that $|a|^2 = a^2$.

The next two statements are pretty useful, and best proved (for now) by cases, though they have more clever proofs too.

Problem 2.29. Show that $|ab| = |a||b|$.

Problem 2.30. Show that $|a+b| \leq |a| + |b|$.

The latter is usually called the *triangle inequality*, and we have seen it before. A related statement is sometimes called the *reverse triangle inequality*.

Problem 2.31. Show that $|a-b| \geq \big||a| - |b|\big|$.

To prove the following, be sure to review properties of the integers we assume without proof.

Problem 2.32. Assume that there is a positive element of the preimage of $\{2\}$ under the function $f(x) = x^2$ from the reals to the reals; that is, assume $\sqrt{2}$ exists. Show $1 < \sqrt{2} < 2$.

We do assume here that in a proof transition or other experience you have seen that $\sqrt{2}$ is not a rational number, so now we have an irrational between two rationals, by a previous example.

The following problems are a good challenge to generalize this. You may need some other typical facts from an introductory proof course.

Problem 2.33. Prove that between any two distinct real numbers there is a rational number.

Problem 2.34. Prove that between any two distinct real numbers there is an irrational number.

We now turn to the main matters under consideration in this course.