Exam 2 (Option 2)

Your Name:

Names of Any Collaborators:

Instructions

For Exam 3, you have two options:

1. **Option 1:** Complete a 30-minute oral exam with me in my office. The questions will come directly from homework and previous exams. The oral exams will take place during Monday, November 25 to Thursday, December 12. Available time slots are first come, first served.
2. **Option 2:** Complete this take-home exam.

Regardless of which option you choose, Exam 3 will be worth 10% of your overall grade. This take-home exam is worth a total of 18 points is due at the beginning of class on **Friday, December 6**.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 5.35, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

1. (4 points) Let G be a group. Since the elements of the center $Z(G)$ commute with all the elements of G, the left and right cosets of $Z(G)$ will be equal, and hence $Z(G)$ is normal in G. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

2. (4 points each) Complete **two** of the following. Problem 1 is useful on at least one of the problems below.

   (a) Prove that $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)\rangle \cong V_4$.*

   (b) Prove that if $|G| = pq$, where p and q are primes (not necessarily distinct), then either $Z(G) = \{e\}$ or G is abelian.

   (c) Let G be a group and define the *automorphism group of G* via

   $$\operatorname{Aut}(G) := \{\phi : G \to G \mid \phi \text{ is an isomorphism}\}.$$

   Prove that $\operatorname{Aut}(G)$ is a group under function composition.

   (d) Suppose G is an infinite abelian group and define $F = \{g \in G \mid g \text{ has finite order}\}$. Prove that F is a normal subgroup of G.

   (e) If $N \trianglelefteq G$ and $H \leq G$, prove that $NH := \{nh \mid n \in N, h \in H\}$ is a subgroup of G.

3. (2 points each) Define $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ (i.e, the set of complex numbers without zero). It turns out that $\mathbb{C}^*$ is a group under multiplication, where 1 is the identity of the group. You can take this for granted. Define $\phi : \mathbb{R} \to \mathbb{C}^*$ via $\phi(t) = \cos(2\pi t) + i\sin(2\pi t)$, where $i^2 = -1$. It isn't too difficult to see that the image of ϕ is a circle of radius 1 centered at the origin in the complex plane. If ϕ is a group homomorphism, then Theorem 7.5 (a weakened version of Theorem 3.63) implies that this circle is a subgroup of $\mathbb{C}^*$.

   (a) Prove that ϕ is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{C}^*, \cdot)$.†

   (b) Prove that $\ker(\phi) = \mathbb{Z}$.

   (c) In light of the First Isomorphism Theorem, what conclusion can you make? Be as specific as possible.

*Note that $\langle(0,2)\rangle$ is a normal subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$ since $\mathbb{Z}_2 \times \mathbb{Z}_4$ is abelian, and hence $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)\rangle$ is a sensible quotient group.

†The following trigonometric identities might come in handy:

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

