Cartesian Coordinates and Distance

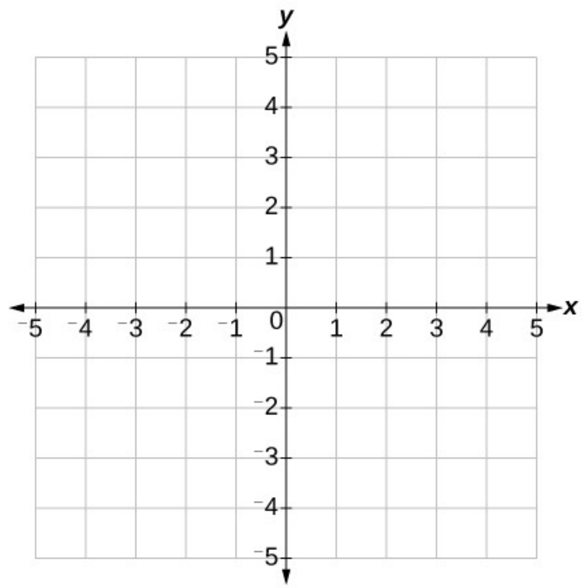
The Cartesian Coordinate System

A two-dimensional plane where the

x −axis is the horizontal axis

y −axis is the vertical axis

A point on the plane is defined as an ordered pair, (x, y), such that x is determined by its horizontal distance from the origin and y is determined by its vertical distance from the origin.



Example: Plotting Points in a Rectangular Coordinate System

Plot the following ordered pairs on the coordinate system and indicate what quadrant each is located.

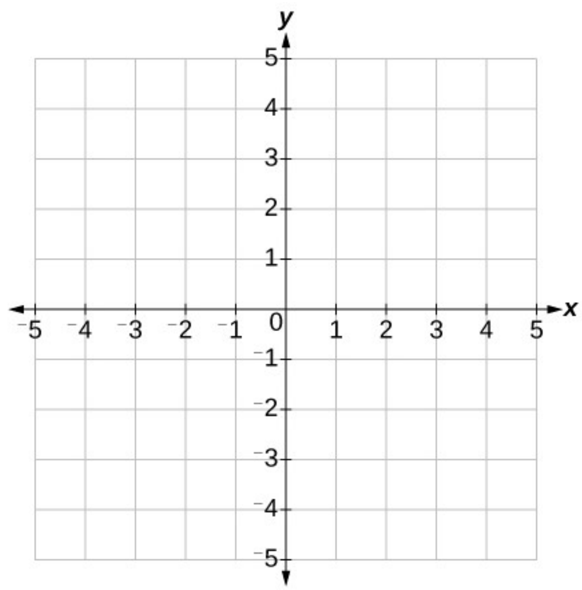
$A(3, 5)$

$B(-2, 5)$

$C(-4, 0)$

$D(1, -4)$

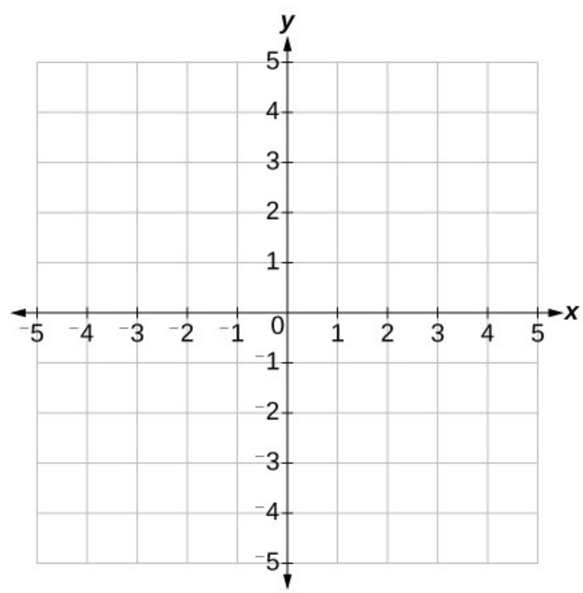
$E(0, -5)$



Graphing Equations by Plotting Points

Example: Graphing an Equation in Two Variables by Plotting Points

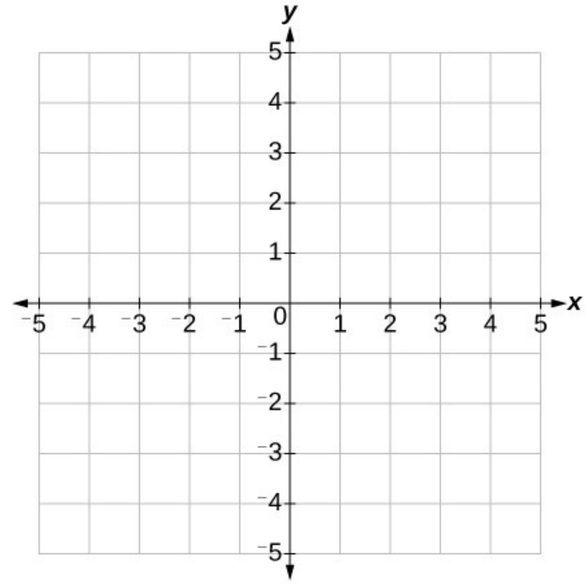
Construct a table and graph the equation by plotting points: $y = \frac{1}{2}x + 2$



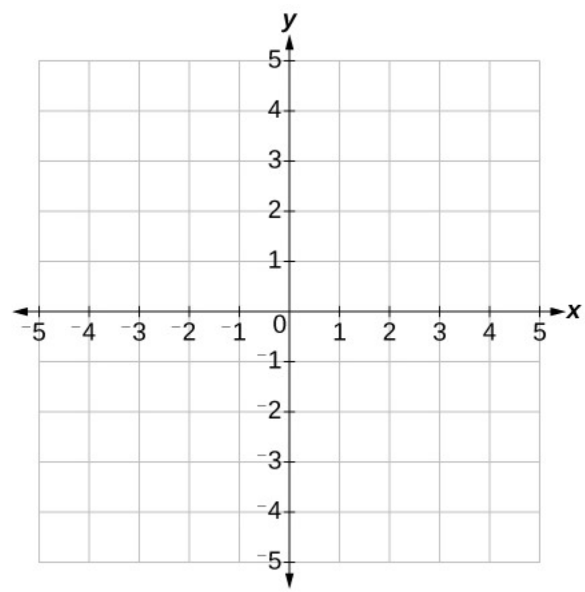
Graphing Equations with Technology

Example: Using a TI-83/84 and DESMOS to Graph

a. Use a TI-83/84 to graph the equation, then sketch the graph: $y = -\frac{2}{3}x + \frac{4}{3}$

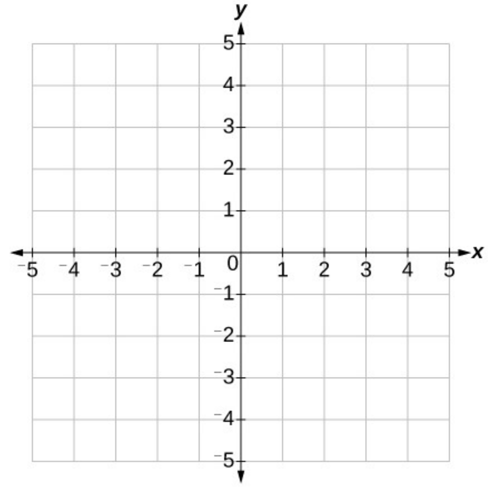


b. Use DESMOS.com to graph the equation, then sketch the graph: $y = \frac{3}{2}x - \frac{4}{7}$



Finding $x-$ and $y-$intercepts

$x-$intercepts:

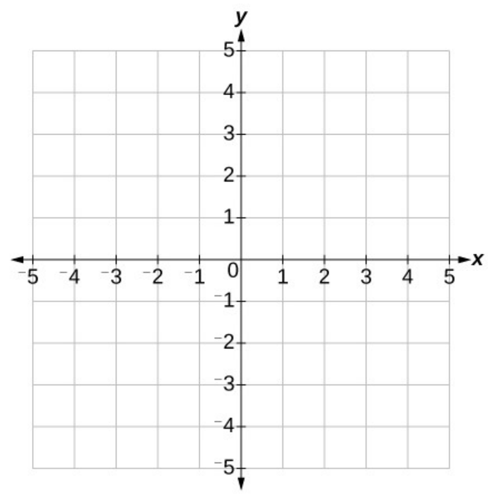


$y-$intercepts:

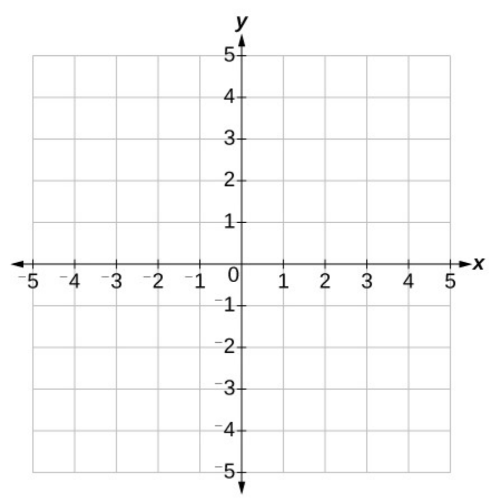
Example: Finding the Intercepts of the Given Equation

Find the $x-$ and $y-$intercepts for each equation, then graph using only the intercepts.

a. $y=-\frac{3}{4}x+3$



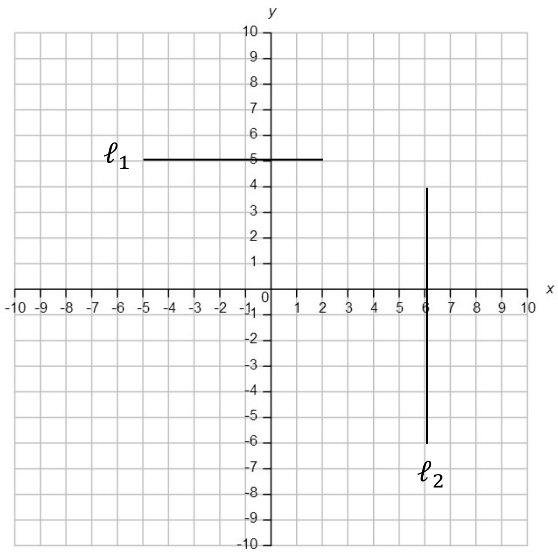
b. $x+2y=4$

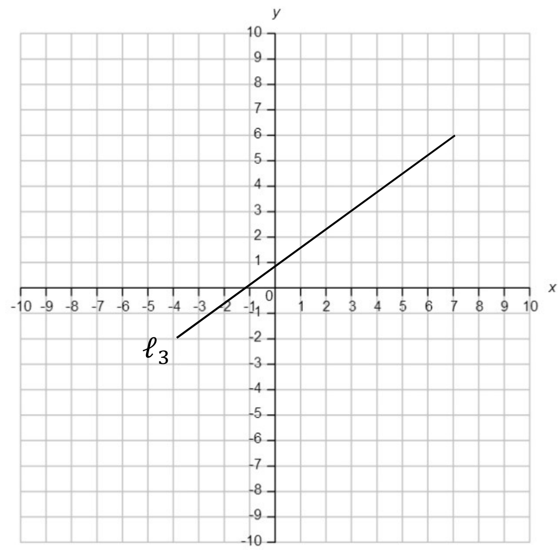


The Distance Formula

Let's see if we can determine the distance formula.

1. Let's consider the graphs below. Label the ends of the line segments in the form (x, y).





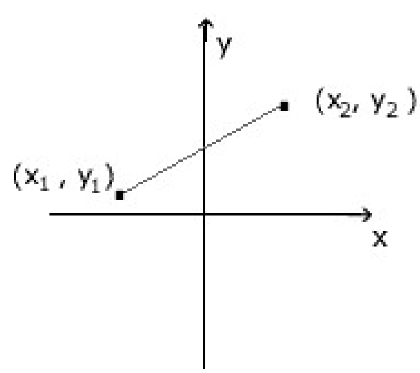
2. What are the lengths of each of the line segments above? [Hint: we can use the Pythagorean Theorem]

ℓ_1:

ℓ_2:

ℓ_3:

3. Using a method similar to what we used to find the length of ℓ_3 , what is the length of the generic line pictured?



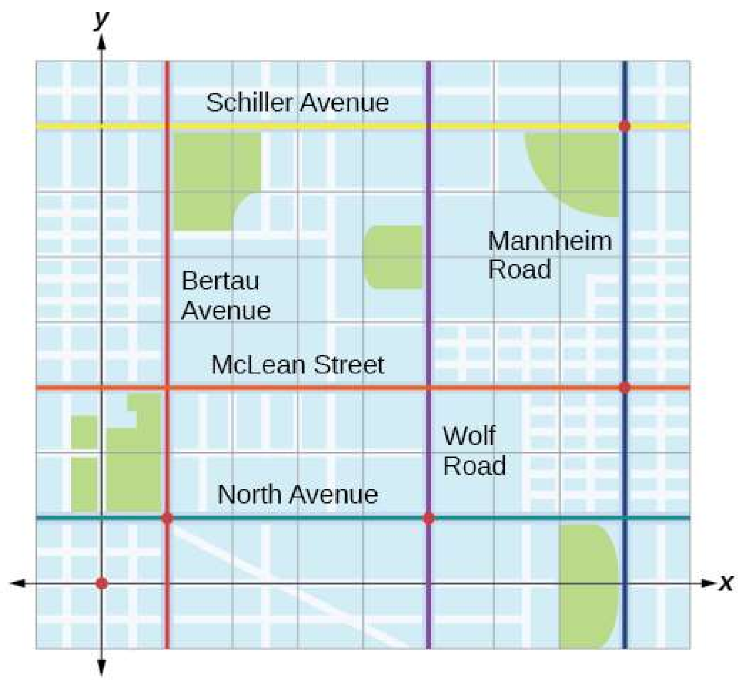
Example: Finding the Distance Between Two Points

Find the distance between the points: $(-1,4)$ and $(11,9)$.

Example: Finding the Distance Between Two Locations

Let us return to Tracie's trip. Recall that Tracie set out from Elmhurst, IL to go to Franklin Park, and that on the way she made some stops (indicated by the red dots).

a. Find the total distance Tracie traveled (i.e. walked).



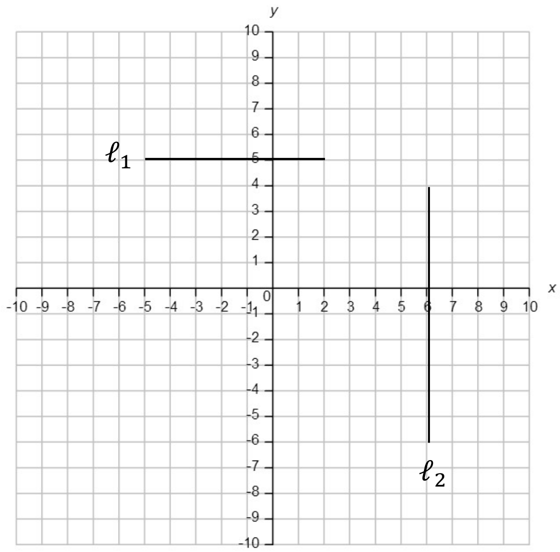
b. Find the distance between Tracie's starting point and end point "as the crow flies."

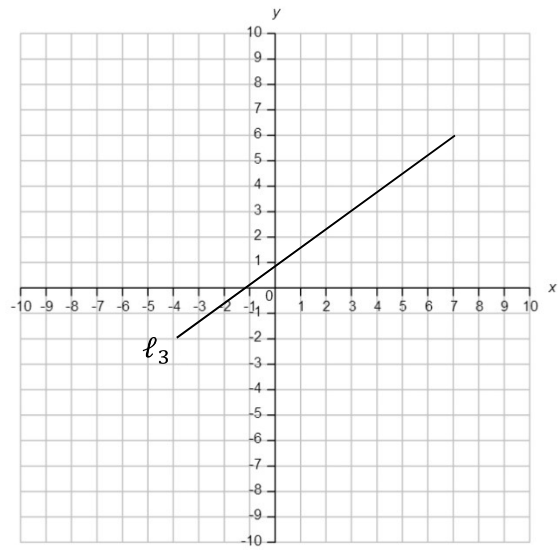
Using The Midpoint Formula

Let’s see if we can determine the midpoint formula.

Earlier we looked at the graphs below and labeled the ends of the line segments in the form of (x, y).

Let’s do that again.





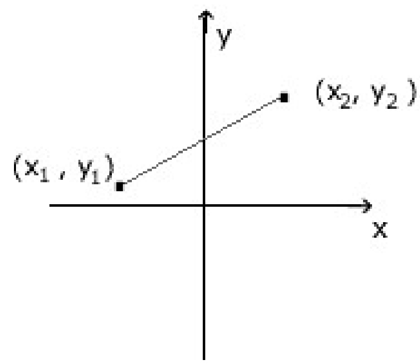
How do we find them middle of each of the line segments above?

ℓ_1:

ℓ_2:

ℓ_3:

Using a method similar to what we used to find the length of ℓ_3 , what is the length of the generic line pictured?

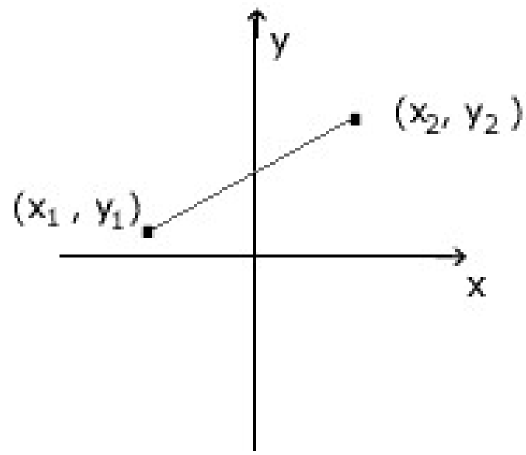


Example: Finding the Midpoint of a Line Segment

Find the midpoint of the line segment with endpoints $(-2, -1)$ and $(-8, 6)$.

Graphing and Equations of Circles

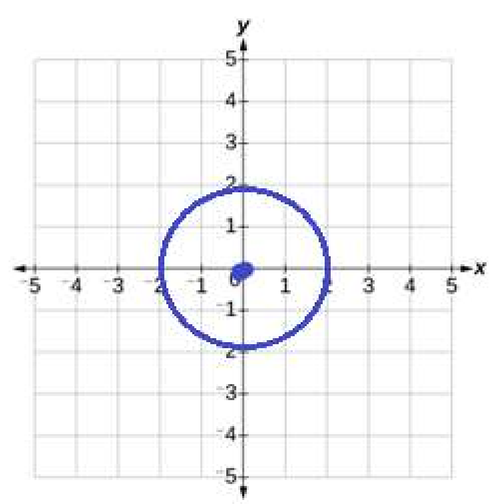
Let us return to when we discovered the distance formula and find the equation of a circle.



Example: Finding the Equation of a Circle Centered at the Origin

a. Find the equation of a circle whose center is the origin and whose radius is 4.

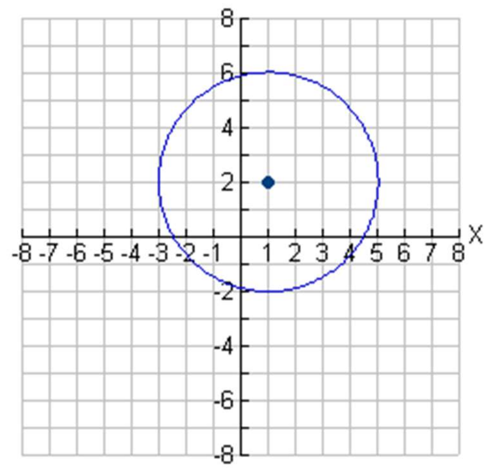
b. Find the equation of the circle in the graph.



Example: Finding the Equation of a Circle Centered at a Point Other Than the Origin

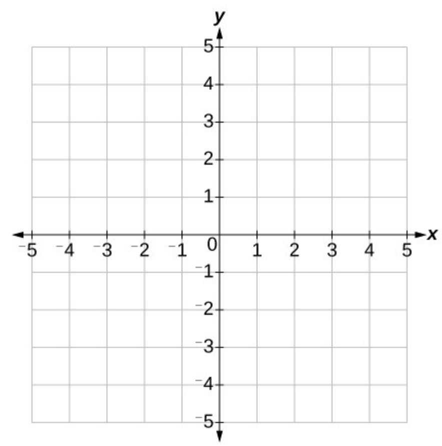
a. Find the equation of a circle whose center is (-2, 1) and whose radius is 3.

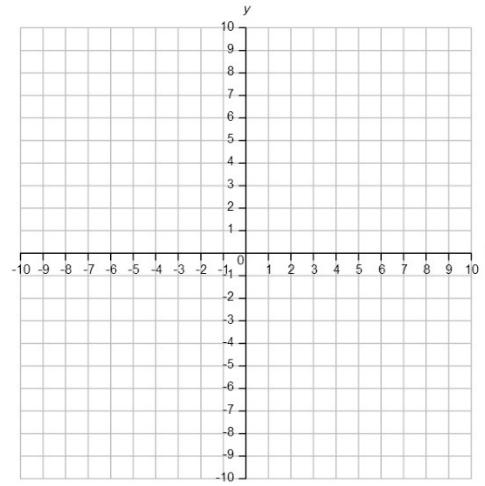
b. Find the equation of the circle in the graph.



Example: Graph the Equation of a Circle Centered at the Origin

Graph $(x)^2 + (y)^2 = 9$.





Example: Graph the Equation of a Circle Centered at a Point Other Than the Origin

Graph $(x-2)^2+(y)^2=25$.

Example: Finding the Center of a Circle

The diameter of a circle has endpoints $(-1,1)$ and $(7,-4)$, what is the center of the circle?

Solving Linear Equations in One Variable

- **Linear Equation:**

- **Solution Set:**

- **Identity Equation:**

- **Conditional Equation:**

- **Inconsistent equation:**

Example: Solving an Equation in One Variable

Solve the equation: $2x + 1 = -9$

Example: Solving an Equation Algebraically When the Variable Appears on Both Sides

Solve the equation: $-2(3x - 1) + x = 14 - x$

Solving Rational Equations

- **Rational Expression:**

- **Rational Equation:**

- **Excluded Values:**

Example: Solving a Rational Equation

Solve the equation: $\frac{3}{x} - \frac{1}{3} = \frac{1}{6}$

Example: Solving a Rational Equations without Factoring

Solve the equation: $\frac{2}{3x} = \frac{1}{4} - \frac{1}{6x}$

Example: Solving a Rational Equation by Factoring the Denominator

Solve the equation: $-\frac{5}{2x} + \frac{3}{4x} = -\frac{7}{4}$

Example: Solving Rational Equations with a Binomial Denominator

Solve the equation: $\frac{-3}{2x+1} = \frac{4}{3x+1}$

Example: Solving a Rational Equations with Factored Denominators

Solve the equation: $\frac{2}{x-2} + \frac{1}{x+1} = \frac{1}{x^2-x-2}$

A Few More Rational Equation Examples

$$\frac{1}{x-6} - \frac{1}{x} = \frac{6}{x^2-6x}$$

$$\frac{x^2}{x-3}=\frac{9}{x-3}$$

$$\frac{2}{3x+6}+\frac{1}{x^2-4}=\frac{4}{x-2}$$

$$\frac{x}{x-4} - \frac{4}{x+4} = \frac{32}{x^2-16}$$

Finding a Linear Equation

- **Slope-intercept form:**

- **Slope:**

Example: Finding the Slope of a Line Given Two Points

Find the slope of the line that passes through the points $(-2, 6)$ and $(1, 4)$.

Exampl: Identifying the Slope and y −intercept of a Line Given an Equation

Identify the slope and the y −intercept in the equation: $y = -\frac{4}{5}x + 1$

- **Point-Slope Formula:**

Example: Finding the Equation of a Line Given the Slope and One Point

Find the equation of the line in slope-intercept form that has a slope of 4 and passes through the point $(2, 5)$.

Example: Finding the Equation of a Line Passing Through Two Given Points

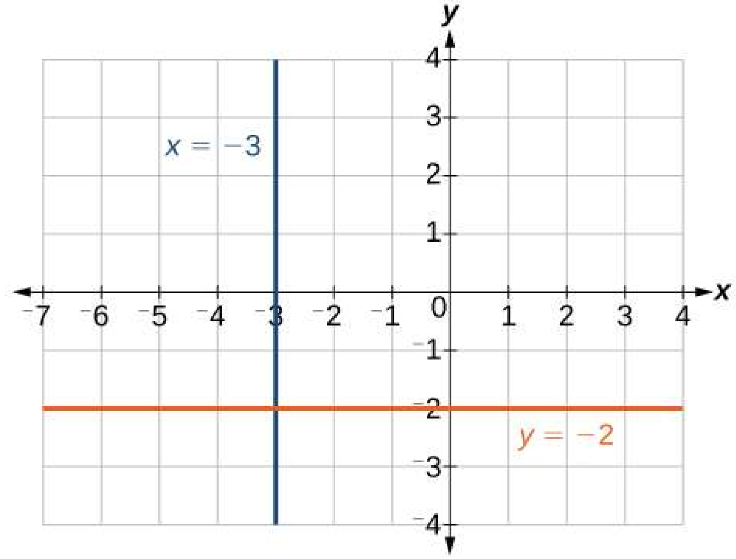
Find the equation of the line in slope-intercept form that passes through the points $(-3\ 10)$ and $(5, -6)$.

- **Standard Form of a Line:**

Example: Finding the Equation of a Line and Writing it in Standard Form

Find the equation of the line in standard form with a slope of $-\frac{1}{3}$ and passes through the point $\left(1, \frac{1}{3}\right)$.

- **Vertical Lines:**



- **Horizontal Lines:**

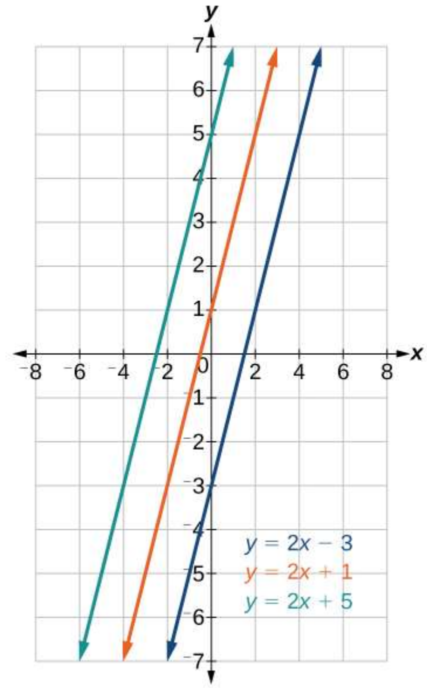
Example: Finding the Equation of a Line Passing Through the Given Points

a. Find the equation of the line that passes through the points: $(1, -3)$ and $(1, 4)$.

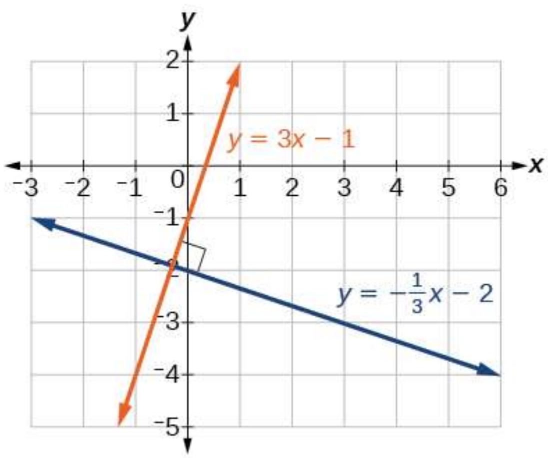
b. Find the equation of the line that passes through the points: $(-5, 2)$ and $(2, 2)$.

Determining Whether Lines are Parallel or Perpendicular

- **Parallel Lines:**



- **Perpendicular Lines:**



Example: Determining Whether the Lines are Parallel, Perpendicular, or Neither

a. Determine whether the lines are parallel, perpendicular, or neither:
$$2y - x = 10 \text{ and } 2y = x + 4$$

b. Determine whether the lines are parallel, perpendicular, or neither:
$$3x - 2y = 5 \text{ and } 2y - 9x = 6$$

c. Determine whether the lines are parallel, perpendicular, or neither:
$$x = 4 \text{ and } y = -3$$

Writing Equations of Parallel and Perpendicular Lines

Example: Writing the Equation of a Line Parallel to a Given Line and Passing Through a Given Point

Find the equation of the line parallel to $5x = y + 7$ and passing through the point $(-1, -2)$.

Example: Finding the Equation of a Line Perpendicular Line and Passing Through a Given Point.

Find the equation of the line perpendicular to $3y = x - 4$ and passing through the point $(-2, 1)$.

A Few More Parallel and Perpendicular Lines Examples

Write the equation of the line, in slope-intercept form, that is

a. Parallel to the given line and passes through the given point.
b. Perpendicular to the given line and passes through the given point.

$$(-1, 6); y = 2x + 9$$

Write the equation of the line, in slope-intercept form, that is

a. Parallel to the given line and passes through the given point.
b. Perpendicular to the given line and passes through the given point.

$$(3, -2); 3x + 4y = 5$$

Real-World Applications

Example: Real-World Applications

a. To be ADA compliant, the maximum slope for a wheelchair ramp is $\frac{1}{12}$. If the vertical distance from the group to the door bottom is 2.5 feet, find the distance the ramp must extend from the building in order to comply with the needed slope.

b. The cost of renting a car is \$57 per week plus \$0.25 per mile traveled during the week.
   i. Determine the equation for the total cost you would pay if you rented this car for a week and traveled x miles.

   ii. Suppose you have to travel 50 miles (one way) to get to the resort you are staying at. What is your total cost for the rental if you do not travel anywhere else during the week?

   iii. If your total cost was \$64.75, how many miles were you charged for traveling?

   iv. Suppose you only had a maximum of \$100 to spend on your car rental. What is the maximum number of miles you could travel that week?

Using a Linear Equation to Solve a Real-World Problem

Given a real-world problem, model a linear equation to fit it.

1. Identify known quantities
2. Assign a variable to represent the unknown quantity
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure the solution can be explained in words, including the units of measure.

Example: Modeling a Linear Equation to Solve an Unknown Number Problem

Find a linear equation to solve for the following unknown quantitates:

One number is three more than twice another number. If the sum of the two numbers is 36, find the two numbers.

Example: Setting up a Linear Equation to Solve a Real-World Application

a. There are two cell phone companies that offer different packages. Company A charges a monthly fee of $34 plus $0.05 per minute of talk-talk. Company B charges a monthly fee of $40 plus $0.04 per minute of talk-time.

   i. Write a linear equation that models the packages offered by both companies.

   ii. If the average number of minutes used each month is 1,160, which company offers the better plan?

   iii. If the average number of minutes used each month is 420, which company offers the better plan?

   iv. How many minutes of talk-time would yield equal monthly statements from both companies?

b. Find a linear equation to model the situation: It costs ABC electronics company \$2.50 per unit to produce a part used in a popular brand of desktop computers. The company has monthly operating expenses of \$350 for utilities and \$3,300 for salaries. What are the company's monthly expenses?

Using a Formula to Solve Real-World Applications

Example: Solving an Application Using a Formula

a. It takes Andrew 30 minutes to drive to work in the morning. He drives home using the same route, but it takes him 10 minutes longer, and he averages 10 miles per hour less than in the morning. How far does Andrew drive to work?

b. On Saturday morning, it took Jennifer 3.6 hours to drive to her mother's house for the weekend. On Sunday, due to heavy traffic, it took Jennifer 4 hours to return home. Her speed is 5 miles per hour slower on Sunday than on Saturday. What was her speed on Sunday?

Perimeter:

Example: Solving a Perimeter Problem

a. The perimeter of a rectangular outdoor patio is 54 feet. The length is 3 feet longer than the width. What are the dimensions of the patio?

b. Find the dimensions of a rectangle give that the perimeter is 110 cm, and the length is 1 cm more than twice the width.

Area:

Example: Solving an Area Problem

a. The perimeter of a tablet of graph paper is 48 inches. The length is 6 inches more than the width. Find the area of the graph paper.

b. A game room has a perimeter of 70 feet. The length is five more than twice the width. How many square feet of carpeting should be ordered?

Volume:

Example: Solving a Volume Problem

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 inches, and the volume is 1,600 cubic inches.

Example: Solving a Formula for a Given Variable

a. Solve $P = 2L + 2W$ for W.

b. Solve $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ for f.

Example: Solving a Problem with Percentages

a. A retired woman has \$50,000 to invest but needs to make \$6,000 a year from the interest to meet certain living expenses. One bond investment pays 15% annual interest. The rest of it she wants to put into a CD that pays 7% annual interest. How much should she invest in each option to sustain a \$6,000 annual return?

b. A chemistry teacher needs 20 quarts of a 40% salt solution for an experiment. She has a 30% salt solution and a 70% salt solution in her supplies. How many quarts of each solution does the teacher need to mix to get her desired result?

Basics of Complex Numbers

Expressing Square Roots of Negative Numbers as Multiples of i

What is the number i?

- **Complex Number:**

Example: Expressing an Imaginary Number in Standard Form

Express each of the following in terms of i:

a. $\sqrt{-7}$

b. $\sqrt{-16}$

c. $-\sqrt{-13}$

d. $-\sqrt{-64}$

e. $\sqrt{-48}$

Simplifying Powers of i

Example: Simplifying Powers of i

Simplify each of the following:

a. i^{29}

b. i^{58}

c. i^{2012}

d. i^{77}

Operations on Complex Numbers

Adding and Subtracting Complex Numbers

Example: Adding and Subtracting Complex Numbers

Compute the following:

a. $(8 + 6i) + (3 + 2i)$

b. $(4 + 5i) - (6 - 3i)$

Multiplying Complex Numbers

Example: Multiplying a Complex Number by a Real Number

Multiply: $\frac{1}{2}(5 - 2i)$

Example: Multiplying a Complex Number by a Complex Number

Multiply each of the following:

a. $(8i)(-8i)$

b. $(3-4i)(2+3i)$

c. $(5+7i)(5-7i)$

d. $(5-2i)(3i)$

Dividing Complex Numbers

- **Complex Conjugate:**

Example: Finding Complex Conjugates

Find the complex conjugate of each of the following:

a. $2 + i\sqrt{5}$

b. $-\frac{1}{2}i$

c. $-3 + 4i$

Example: Dividing with Complex Numbers

Divide and simplify each of the following:

a. $\frac{3+4i}{2-i}$

b. Divide $2-5i$ by $1-6i$

c. $\frac{2+\sqrt{-12}}{2}$

d. $\frac{-5+3i}{2i}$

- **Quadratic Equation:**

Solving Quadratic Equations by Factoring

- **Zero-Product Property:**

Example: Factoring and Solving a Quadratic Equation

Solve each of the equations by factoring:

a. $x^2 - 5x - 6 = 0$

b. $x^2 + 8x + 15 = 0$

c. $12x^2 + 11x + 2 = 0$

d. $5x^2 = 5x + 30$

e. $-3x^3 - 5x^2 - 2x = 0$

Using the Square Root Property

- **Square Root Property:**

Example: Solving a Simple Quadratic Equation Using the Square Root Property

Solve each of the equations using the square root property:

a. $x^2 - 9 = 0$

b. $x^2 = 8$

Example: Solving a Quadratic Equation Using the Square Root Property.

Solve each of the equations using the square root property:

a. $4x^2 + 1 = 7$

b. $3(x-4)^2 = 15$

Solving Equations using Completing the Square

- **Completing the Square:**

Example: Solving a Quadratic by Completing the Square with an Even "b"

Solve by completing the square: $x^2 - 6x = 13$

Example: Solving a Quadratic by Completing the Square with an Odd "b"

Solve by completing the square: $x^2 - 3x - 5 = 0$

Using the Quadratic Formula

- **The Quadratic Formula:**

Example: Solve the Quadratic Equation Using the Quadratic Formula (Real Solutions)

Use the quadratic formula to solve each of the following equations:

a. $x^2 + 5x = -1$

b. $9x^2 + 3x - 2 = 0$

Example: Solving a Quadratic Equation Using the Quadratic Formula (Imaginary Solutions)

Use the quadratic formula to solve the equation: $x^2 + x + 2 = 0$

The Discriminant

What is the discriminant?

Value of Discriminant	Results

Example: Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Determine the discriminant, and then state how many solutions there are and the nature of the solutions.

a. $2x^2 - 6x + 7 = 0$

b. $6x^2 - x - 2 = 0$

c. $x^2 + 4x + 7 = 0$

d. $9x^2 - 30x + 25 = 0$

Example: Real-World Applications

An epidemiological study of the spread of a certain influenza strain that hit a small school population found that the total number of students, P, who contracted the flu t days after it broke out is given by the model $P = -t^2 + 13t + 130$, where $1 \leq t \leq 6$. Find the day that 160 students had the flu. Recall that the restriction on t is at most 6.

Using the Pythagorean Theorem

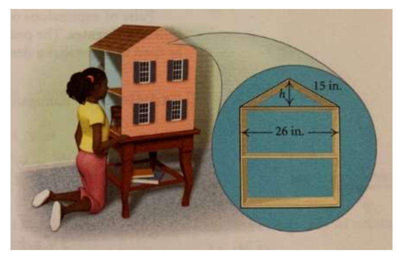
- **The Pythagorean Theorem:**

Example: Finding the Length of the Missing Side of a Right Triangle.

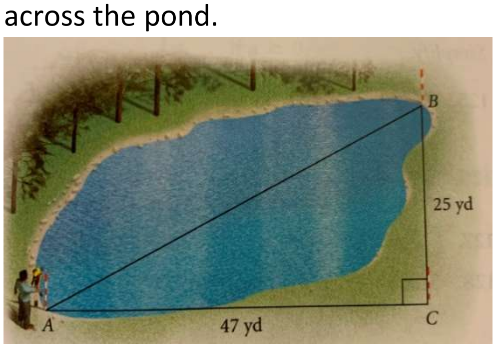
A right triangle has a leg that measures 4 units, and a hypotenuse of 5 units. What is the length of the other leg?

Example: Real-World Applications

a. Roy is building a doll house for his granddaughter. The doll house measures 26 inches across, and the slanted side of the roof measures 15 inches. Find the height of the roof.



b. A surveyor places poles at points A, B, and C to measure the distance across the pond. The distances AC and BC are measured to be 47 yards and 25 yards, respectively. Find the distance AB across the pond.



Solving Equations Involving Rational Exponents

- **Rational Exponents:**

Example: Evaluating a Number Raised to a Rational Exponent

Evaluate each expression:

a. $8^{\frac{2}{3}}$

b. $64^{-\frac{1}{3}}$

Example: Solve the Equation Including a Variable Raised to a Rational Exponent

Solve each equation:

a. $x^{\frac{5}{4}} = 32$

b. $x^{\frac{3}{2}} = 125$

Example: Solving an Equation Involving Rational Exponents and Factoring

Solve each equation:

a. $3x^{\frac{3}{4}} = x^{\frac{1}{2}}$

b. $(x+5)^{\frac{3}{2}} = 8$

Solving Equations Using Factoring

- **Polynomial Equations:**

Example: Solving a Polynomial by Factoring

Solve each equation:

a. $5x^4 = 80x^2$

b. $12x^4 = 3x^2$

Example: Solve a Polynomial by Grouping

Solve each equation:

a. $x^3 + x^2 - 9x - 9 = 0$

b. $5x^3 + 45x = 2x^2 + 18$

Solving Radical Equations

- **Radical Equation:**

- **Extraneous Solutions:**

Example: Solving an Equation with One Radical

Solve each equation:

a. $\sqrt{15-2x}=x$

b. $\sqrt{x+3}=3x-1$

Example: Solving a Radical Equation Containing Two Radicals

Solve each equation:

a. $\sqrt{2x+3}+\sqrt{x-2}=4$

b. $\sqrt{3x+7}+\sqrt{x+2}=1$

Solving Absolute Value Equations

- **Absolute Value Equations:**

Example: Solving Absolute Value Equations

Solve each equation:

a. $|6x + 4| = 8$

b. $|3x + 4| = -9$

c. $|1 - 4x| + 8 = 13$

d. $|-5x + 10| = 0$

Solving Other Types of Equations

Example: Solving a Fourth-Degree Equation in Quadratic Form

Solve each equation:

a. $3x^4 - 2x^2 - 1 = 0$

b. $x^4 - 8x^2 - 9 = 0$

Example: Solving an Equation in Quadratic Form Containing a Binomial

Solve each equation:

a. $(x+2)^2 + 11(x+2) - 12 = 0$

b. $(x-5)^2 - 4(x-5) - 21 = 0$

Solving Rational Equations Which Leads to a Quadratic

Example: Solving a Rational Equation Leading to a Quadratic

Solve each equation:

a. $$\frac{-4x}{x-1}+\frac{4}{x+1}=\frac{-8}{x^2-1}$$

b. $$\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$$

Use Radicals in Applications

Example: Real-World Application

One model for body surface area, BSA, is that $BSA = \sqrt{\frac{wh}{3600}}$ where w = weight in kg and h = height in cm.

a. Find the height of a 72-kg female to the nearest cm whose BSA = 1.8.

b. Find the weight of a 177 cm male to the nearest kg whose BSA = 2.1.