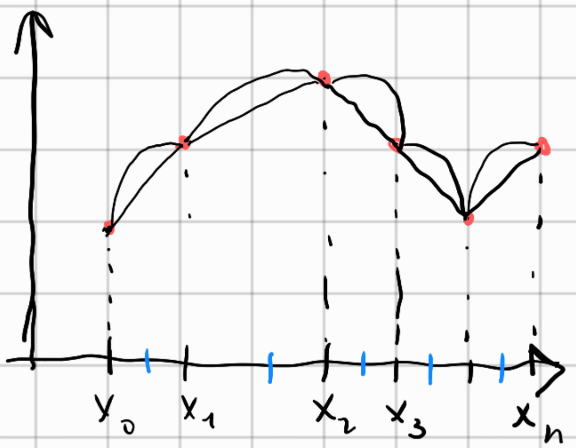
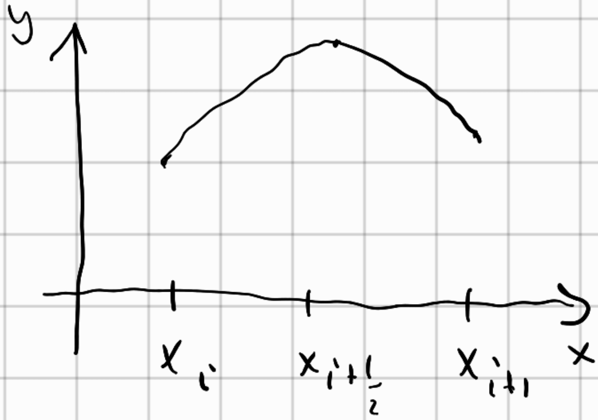
Lecture 8

Second order spline.



x	x_0	x_1	...	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	...	$f(x_n)$

$$S(x) = \bigcup_{i=0}^{n} S_i$$



$$x \in [a, b], \quad h = \frac{b-a}{N}$$

$$x_i = a + i \cdot h$$

$$x_{i+1} = x_i + h$$

$$x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$$

$$\begin{cases} s_i(x) = a_i + b_i\left(x - x_{i+\frac{1}{2}}\right) + \frac{c_i}{2}\left(x - x_{i+\frac{1}{2}}\right)^2 \\ x \in [x_i, x_{i+1}], \quad i = 0, \dots, n-1 \end{cases} \qquad (1)$$

$$(x_i, f(x_i)), \quad \left(x_{i+\frac{1}{2}}, f\left(x_{i+\frac{1}{2}}\right)\right), \quad (x_{i+1}, f(x_{i+1}))$$

$$\begin{cases} s_i(x_i) = a_i + b_i\left(-\frac{h}{2}\right) + \frac{c_i}{2}\left(-\frac{h}{2}\right)^2 \\ s_i\left(x_{i+\frac{1}{2}}\right) = a_i \\ s_i(x_{i+1}) = a_i + b_i\left(\frac{h}{2}\right) + \frac{c_i}{2}\left(\frac{h}{2}\right)^2 \end{cases} \qquad (2)$$

$$f(x_i) = s_i(x_i), \quad f\left(x_{i+\frac{1}{2}}\right) = s_i\left(x_{i+\frac{1}{2}}\right)$$

$$f(x_{i+1}) = S_i(x_{i+1})$$

$$\begin{cases} a_i = f(x_{i+\frac{1}{2}}) \\ a_i - b_i \cdot \frac{h}{2} + \frac{c_i}{8} \cdot h^2 = f(x_i) \\ a_i + b_i \cdot \frac{h}{2} + \frac{c_i}{8} \cdot h^2 = f(x_{i+1}) \end{cases} \qquad (3)$$

$$\boxed{a_i = f(x_{i+\frac{1}{2}})}$$

$$2a_i + \frac{c_i}{4} h^2 = f_i + f_{i+1}$$

$$\boxed{c_i = 4 \cdot \frac{f_i - 2f_{i+\frac{1}{2}} + f_{i+1}}{h^2}}$$

$$\boxed{b_i = \frac{f_{i+1} - f_i}{h}}$$

$$\bar{x}_i = x_i + \frac{h}{k}, \quad k \geq 1, \quad k \neq 2$$

$$\bar{x}_i - x_{i+\frac{1}{2}} = x_i + \frac{h}{k} - x_i - \frac{h}{2} = h\left(\frac{1}{k} - \frac{1}{2}\right)$$

$$\boxed{S_i(\bar{x}_i) = a_i + b_i\, h\left(\frac{1}{k} - \frac{1}{2}\right) + \frac{c_i}{2} \cdot h^2 \left(\frac{1}{k} - \frac{1}{2}\right)^2}$$

$$\Delta_i = \left| f\left(x_i + \frac{h}{k}\right) - S_i\left(x_i + \frac{h}{k}\right) \right|$$

$$f\left(x_i + \frac{h}{k}\right) = f(x_i) + f'(x_i) \cdot \frac{h}{k} + f''(x_i) \frac{h^2}{2k^2} +$$

$$+ f'''(x_i) \frac{h^3}{6k^3} + O(h^4)$$

$$s_i\left(x_i + \frac{h}{k}\right) = f\left(x_i + \frac{h}{2}\right) + \frac{f(x_i+h) - f(x_i)}{\cancel{h}} \cdot \cancel{h} \cdot \left(\frac{1}{k} - \frac{1}{2}\right) + \frac{\cancel{4}\,\cancel{h^2}}{\cancel{2}} \, \frac{f(x_i) - 2f\left(x_i + \frac{h}{2}\right) + f(x_i+h)}{\cancel{2h^2}} \cdot \left(\frac{1}{k} - \frac{1}{2}\right)^2;$$

$$f\left(x_i + \frac{h}{2}\right) = f(x_i) + f'(x_i)\cdot\frac{h}{2} + f''(x_i)\frac{h^2}{8} + f'''(x_i)\frac{h^3}{48} + O(h^4)$$

$$f(x_i + h) = f(x_i) + f'(x_i)h + f''(x_i)\frac{h^2}{2} + f'''(x_i)\cdot\frac{h^3}{6} + O(h^4)$$

$$f(x_i+h) - f(x_i) = f'(x_i)h + f''(x_i)\frac{h^2}{2} + f'''(x_i)\cdot\frac{h^3}{6} + O(h^4)$$

$$f(x_i) - 2f\left(x_i + \frac{h}{2}\right) + f(x_i+h) = \cancel{f(x_i)} - 2\cancel{f(x_i)} - \cancel{f'(x_i)h} - f''(x_i)\frac{h^2}{4} - f'''(x_i)\frac{h^3}{24} + \cancel{f(x_i)} + \cancel{f'(x_i)}h + f''(x_i)\frac{h^2}{2} + f'''(x_i)\frac{h^3}{6} + O(h^4) = f''(x_i)\frac{h^2}{4} + f'''(x_i)\frac{h^3}{8} + O(h^4)$$

$$s_i(\bar{x}) = f(\cancel{x_i}) + f'(x_i)\cdot h \quad + f''(x_i) h^2$$

$$S_i(x_i) = f(x_i) + f'(x_i)\frac{h}{2} + f''(x_i)\frac{h^2}{8} +$$

$$+ f'''(x_i)\frac{h^3}{48} + \left(f'(x_i)\cdot h + f''(x_i)\cdot \frac{h^2}{2} +\right.$$

$$\left. + f'''(x_i)\cdot\frac{h^3}{6}\right)\left(\frac{1}{k} - \frac{1}{2}\right) + \left(f''(x_i)\frac{h^2}{4} +\right.$$

$$\left. f'''(x_i)\cdot\frac{h^3}{8}\right)\left(\frac{1}{k} - \frac{1}{2}\right)^2 + O(h^4)$$

$$\Delta_i = \left| f\left(x_i + \frac{h}{k}\right) - S_i\left(x_i + \frac{h}{k}\right)\right| =$$

$$= \left| f'(x_i)\cdot\frac{h}{k} + f''(x_i)\frac{h^2}{2k^2} + f'''(x_i)\frac{h^3}{6k^3} - \right.$$

$$- \left[f'(x_i)\cdot\frac{h}{2} + f''(x_i)\frac{h^2}{8} + \right.$$

$$f'''(x_i)\frac{h^3}{48} + \left(f'(x_i)\cdot h + f''(x_i)\cdot\frac{h^2}{2} + \right.$$

$$\left. + f'''(x_i)\cdot\frac{h^3}{6}\right)\left(\frac{1}{k} - \frac{1}{2}\right) + \left(f''(x_i)\frac{h^2}{4} + \right.$$

$$\left.\left. f'''(x_i)\cdot\frac{h^3}{8}\right)\left(\frac{1}{k} - \frac{1}{2}\right)^2 + O(h^4)\right]\Bigg| =$$

$$= \left| f'(x_i)\cdot\left(\cancel{\frac{h}{k}} - \cancel{\frac{h}{2}} - \cancel{\frac{h}{k}} + \cancel{\frac{h}{2}}\right) + \right.$$

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$$+ f''(x_i)\cdot\left(\frac{h^2}{2k^2} - \frac{h^2}{8} - \frac{h^2}{2k} + \frac{h^2}{4} - \frac{h^2}{4}\left(\frac{1}{k} - \frac{1}{2}\right)^2\right)$$

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$$+ f'''(x_i)\left(\frac{h^3}{6k^3} - \frac{h^3}{48} - \frac{h^3}{6k} + \frac{h^3}{\ldots} - \frac{h^3}{\ldots}\left(\frac{1}{k} - \frac{1}{2}\right)^2\right.$$

$$+ O(h^4) \Big| =$$

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① $$\frac{h^2}{2k^2} - \frac{h^2}{8} - \frac{h^2}{2k} + \frac{h^2}{4} - \frac{h^2}{4}\left(\frac{1}{k^2} - \frac{1}{k} + \frac{1}{4}\right) =$$

$$= h^2\left(\frac{1}{2k^2} + \frac{1}{4} - \frac{1}{2k} - \frac{1}{4k^2} + \frac{1}{4k} - \frac{1}{16}\right) =$$

$$= h^2\left(\frac{1}{4k^2} - \frac{1}{4k} - \frac{3}{16}\right)$$

② $$\frac{h^3}{6k^3} - \frac{h^3}{48} - \frac{h^3}{6k} + \frac{h^3}{12} - \frac{h^3}{8}\left(\frac{1}{k^2} - \frac{1}{k} + \frac{1}{4}\right)$$

$$= h^3\left(\frac{1}{6k^3} - \frac{1}{48} - \frac{1}{6k} + \frac{1}{12} - \frac{1}{4k^2} + \frac{1}{4k} - \frac{1}{16}\right) = h^3\left(\frac{1}{6k^3} + \frac{1}{12k} - \frac{1}{4k^2} + \frac{1}{16} - \frac{1}{16}\right) =$$

$$= h^3\left(\frac{1}{6k^3} + \frac{1}{12k} - \frac{1}{4k^2}\right) = \frac{h^3}{2k}\left(\frac{1}{3k^2} + \frac{1}{6} - \frac{1}{2k}\right)$$

$$= \left| f''(x_i)\cdot\frac{h^2}{2}\left(\frac{1}{k^2} + \frac{1}{4}\right) + f'''(x_i)\frac{h^3}{2k}\left(\frac{1}{3k^2} + \frac{1}{6} - \frac{1}{2k}\right) + O(h^4)\right| = \left| f''\frac{h^2}{2}\left(\frac{1}{k^2} + \frac{1}{4}\right) + \right.$$

$$+ O(h^3)\Big| = O(h^3)$$