Cholesky decomposition

$$A x = b$$

1. A is symmetric (if $A = A^T$)
2. A is positive definite (if $z \neq 0$, $z^T A z > 0$)

E.g.

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$$

- $A = A^T$
- $z \neq 0$, $z^T A z > 0$

Decomposition

$$A = L \cdot L^T$$

- L - lower triangular matrix

$$A x = b \longrightarrow L L^T \cdot x = b$$

Solving by 2 steps:

1) $L y = b$ $\rightarrow$ forward substitution

2) $L^T x = y \rightarrow$ back substitution

Process of filling L.

A is $n \times n$ matrix $\rightarrow$ L is $n \times n$ with elements l_{ij}.

- diag. elements:

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}$$

- off diag. elements:

$$l_{ij} = \frac{1}{l_{jj}}\left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} \cdot l_{jk}\right)$$

Example:

$Ax = b$ $\qquad A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix}$ $\quad b = \begin{pmatrix} 6 \\ 25 \\ -35 \end{pmatrix}$

$L - ?$ $\quad x - ?$

1) $L = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$

$$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{12}{2} = 6$$

$$l_{31} = \frac{a_{31}}{l_{11}} = \frac{16}{2} = -8$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{37 - 6^2} = 1$$

$$l_{32} = \frac{a_{32} - l_{31} \cdot l_{21}}{l_{22}} = \frac{-43 - (-8)6}{1} = 5$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{98 - (-8)^2 - 5^2} =$$

$$= \sqrt{98 - 64 - 25} = \sqrt{9} = 3$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{pmatrix}$$

2) $Ly = b$ $$\begin{pmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 25 \\ -35 \end{pmatrix}$$

$$y_1 = 3$$

$$y_2 = 25 - 6 \cdot 3 = 7$$

$$3 y_3 = -35 + 24 - 35$$

$$y_3 = -\frac{46}{3}$$

3) $L^T x = y$ $$\begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -\frac{46}{3} \end{pmatrix}$$

$$x_3 = -\frac{46}{3 \cdot 3} = -\frac{46}{9}$$

$$x_2 = 7 + \frac{3\,[illegible]}{9};$$

$$x_2 = \frac{3 - 6x_2}{2}$$