Lecture 9

Cubic spline.

$a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b$

$f_i = f(x_i)$, $i = 0, \ldots, N$.

$s(x)$ - spline:

1) on every segment $[x_{i-1}, x_i]$, $i = \overline{1,N}$

function $s(x)$ - is spline of the third order.

2) function $s(x)$ as its first and second derivatives are continious on $[a,b]$.

3) $s(x_i) = f(x_i)$, $i = \overline{0,N}$.

$s(x) = s_i(x)$ on every segment $[x_{i-1}, x_i]$,

$i = \overline{1,N}$:

$$s_i(x) = a_i + b_i(x - x_i) + \frac{c_i}{2}(x - x_i)^2 + \frac{d_i}{6}(x - x_i)^3$$

$$x_{i-1} \le x \le x_i, \quad i = \overline{1,N}. \qquad (1)$$

where a_i, b_i, c_i, d_i - coefficients of the spline.

$[\, s_i'(x) = b_i + c_i(x - x_i) + \frac{d_i}{2}(x - x_i)^2$

$$\begin{cases} \dots \\ s_i''(x) = c_i + d_i(x - x_i) \\ s_i'''(x) = d_i \end{cases}$$

$$a_i = s_i(x_i), \quad b_i = s_i'(x_i), \quad c_i = s_i''(x_i),$$

$$d_i = s_i'''(x_i).$$

$s(x_i) = f(x_i)$, $i = \overline{1, N}$, therefore

$$\boxed{a_i = f(x_i), \quad i = \overline{1, N}}$$

By the continuity condition, we have:

$$s_i(x_i) = s_{i+1}(x_i), \quad i = \overline{1, N-1}.$$

$$a_i = a_{i+1} + b_{i+1}(x_i - x_{i+1}) + \frac{c_{i+1}}{2}(x_i - x_{i+1})^2 + \frac{d_{i+1}}{6}(x_i - x_{i+1})^3;$$

$$h_i = x_i - x_{i-1}, \quad h_{i+1} = x_{i+1} - x_i;$$

$$f_i = f_{i+1} + b_{i+1}(-h_{i+1}) + \frac{c_{i+1}}{2}(-h_{i+1})^2 + \frac{d_{i+1}}{6}(-h_{i+1})^3;$$

$$f_i - f_{i+1} = -b_{i+1}h_{i+1} + \frac{c_{i+1}}{2}h_{i+1} - \frac{d_{i+1}}{6}h_{i+1}^3;$$

$$b_i h_i - \frac{c_i}{2}h_i + \frac{d_i}{6}h_i^3 = f_i - f_{i-1}, \quad i = \overline{1, N} \qquad (2)$$

Continuity condition for the first derivative:

$$s_i'(x_i) = s_{i+1}'(x_i), \quad i = \overline{1, N-1}.$$

$$b_i = b_{i+1} + c_{i+1}(x_i - x_{i+1}) + \frac{d_{i+1}}{2}(x_i - x_{i+1})^2;$$

$$b_i = b_{i+1} + c_{i+1}(-h_{i+1}) + \frac{d_{i+1}}{2}(-h_{i+1})^2;$$

$$b_i = b_{i+1} - c_{i+1}h_{i+1} + \frac{d_{i+1}}{2}h_{i+1};$$

$$c_i h_i - \frac{d_i}{2}h_i = b_i - b_{i-1}, \quad i = \overline{2, N} \qquad (3)$$

Using continuity condition for the second derivative:

$$s_i''(x_i) = s_{i+1}''(x_i)$$

$$c_i = c_{i+1} + d_{i+1}(x_i - x_{i+1})$$

$$c_i - c_{i+1} = d_{i+1}(-h_{i+1})$$

$$d_i h_i = c_i - c_{i-1}, \quad i = \overline{2, N}. \qquad (4)$$

Combining (2) - (4), we have 3N-2 equations system:

$$\begin{cases} b_i h_i - \frac{c_i}{2}h_i^2 + \frac{d_i}{6}h_i^3 = f_i - f_{i-1}, \\ c_i h_i - \frac{d_i}{2}h_i^2 = b_i - b_{i-1} \end{cases}$$

$$\left\{ d_i h_i = c_i - c_{i-1} \right.$$

if $f''(a) = f''(b) = 0 \Rightarrow s''(a) = s''(b) = 0$

$s_1''(x_0) = 0, \quad s_N''(x_N) = 0$, i.e.

$c_1 - d_1 h_1 = 0, \quad c_N = 0.$

$$\begin{cases} h_i d_i = c_i - c_{i-1}, \quad i = 1, \dots, N, \quad c_0 = c_N = 0 & (5) \\ h_i c_i - \frac{h_i^2}{2} d_i = b_i - b_{i-1}, \quad i = \overline{2, N} & (6) \\ h_i b_i - \frac{h_i^2}{2} c_i + \frac{h_i^3}{6} d_i = f_i - f_{i-1}, \quad i = \overline{1, N} & (7) \end{cases}$$

From (7):

$$b_i = \frac{h_i}{2} c_i - \frac{h_i^2}{6} d_i + \frac{f_i - f_{i-1}}{h_i},$$

$$b_{i-1} = \frac{h_{i-1}}{2} c_{i-1} - \frac{h_{i-1}^2}{6} d_{i-1} + \frac{f_i - f_{i-2}}{h_{i-1}},$$

Let's subtract the second equation from first:

$$b_i - b_{i-1} = \frac{h_i}{2} c_i - \frac{h_{i-1}}{2} c_{i-1} - \frac{h_i^2}{6} d_i + \frac{h_{i-1}^2}{6} d_{i-1} + \frac{f_i - f_{i-1}}{h_i} - \frac{f_{i-1} - f_{i-2}}{h_{i-1}};$$

$$b_i - b_{i-1} = \frac{1}{2}\left(h_i c_i - h_{i-1} c_{i-1}\right) - \frac{1}{6}\left(h_i^2 d_i - \right.$$

$$- h_{i-1}^2 d_{i-1}) + \frac{f_i - f_{i-1}}{h_i} - \frac{f_{i-1} - f_{i-2}}{h_{i-1}};$$

Put everything to rhs of (6):

$$h_i c_i - \frac{h_i^2}{2} d_i =$$

$$= \frac{1}{2}(h_i c_i - h_{i-1} c_{i-1}) - \frac{1}{6}(h_i^2 d_i -$$

$$- h_{i-1}^2 d_{i-1}) + \frac{f_i - f_{i-1}}{h_i} - \frac{f_{i-1} - f_{i-2}}{h_{i-1}};$$

$$\frac{1}{2} h_i c_i + \frac{1}{2} h_{i-1} c_{i-1} - \frac{1}{3} h_i^2 d_i - \frac{1}{6} h_{i-1}^2 d_{i-1} =$$

$$= \frac{f_i - f_{i-1}}{h_i} - \frac{f_{i-1} - f_{i-2}}{h_{i-1}}; \quad \Big| \cdot 2$$

$$h_i c_i + h_{i-1} c_{i-1} - \frac{2}{3} h_i^2 d_i - \frac{1}{3} h_{i-1}^2 d_{i-1} =$$

$$= 2\left(\frac{f_i - f_{i-1}}{h_i} - \frac{f_{i-1} - f_{i-2}}{h_{i-1}}\right); \qquad (8)$$

From (5), multiplying by h:

$$h_i^2 d_i = (c_i - c_{i-1}) h_i,$$

$$h_{i-1}^2 d_{i-1} = (c_{i-1} - c_{i-2}) h_{i-1},$$

Putting into (8):

$$h_i c_i + h_{i-1} c_{i-1} - \frac{2}{3}(c_i - c_{i-1}) h_i - \frac{1}{3}(c_{i-1} -$$

$$- c_{i-2}) h_{i-1} = 2\left(\frac{f_i - f_{i-1}}{h_i} - \frac{f_{i-1} - f_{i-2}}{h_{i-1}}\right)$$

$$\frac{1}{3} h_i c_i + \frac{2}{3} h_{i-1} c_{i-1} + \frac{1}{3} c_{i-2} h_{i-1} + \frac{2}{3} c_{i-1} h_i =$$

$$= 2\left(\frac{f_i - f_{i-1}}{h_i} - \frac{f_{i-1} - f_{i-2}}{h_{i-1}}\right); \quad \Big| \cdot 3$$

$$h_{i-1} c_{i-2} + 2(h_{i-1} + h_i) c_{i-1} + h_i c_i =$$

$$= 6 \cdot \left(\frac{f_i - f_{i-1}}{h_i} - \frac{f_{i-1} - f_{i-2}}{h_{i-1}}\right);$$

Finally:

$$\underbrace{h_i}_{=S} c_{i-1} + \underbrace{2(h_i + h_{i+1})}_{B} c_i + \underbrace{h_{i+1}}_{=A} c_{i+1} =$$

$$= \underbrace{6 \cdot \left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right)}_{D}; \qquad (9)$$

$i = 1, \dots, N-1$

$c_0 = 0, \quad c_N = 0.$

Thomas method:

$$A c_{i+1} + B c_i + S c_{i-1} = D$$

i: $c_i = \alpha_{i+1} c_{i+1} + \beta_{i+1}$

$i-1$: $c_{i-1} = \alpha_i c_i + \beta_i$

$$A c_{i+1} + B c_i + S(\alpha_i c_i + \beta_i) = D$$

i: $$c_i = -\frac{A}{B + S\alpha_i} c_{i+1} + \frac{D - S\beta_i}{B + S\alpha_i}$$

α_{i+1} $\qquad\qquad$ β_{i+1}

$$\alpha_{i+1} = -\frac{A}{B+S\alpha_i}, \quad \beta_{i+1} = \frac{D - S\beta_i}{B + S\alpha_i};$$

Known: $\alpha_1, \beta_1, \underline{c_N}$

I. $\alpha_i, \beta_i, i = 2, \dots, N$ $\qquad$ $\alpha, \beta \longrightarrow$

II. $c_i, i = N-1, \dots, 1.$ $\qquad$ $c \longleftarrow$

$c_0 = \alpha_1 c_1 + \beta_1 = 0$ $\qquad$ $A = h_{i+1}$

$\alpha_1, \beta_1 = 0$ $\qquad$ $B = 2(h_i + h_{i+1})$

$C = h_i$

$$D = 6 \cdot \left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i} \right)$$

$$d_i = \frac{c_i - c_{i-1}}{h_i}, \quad b_i = \frac{h_i}{2} c_i - \frac{h_i^2}{6} d_i + \frac{f_i - f_{i-1}}{h_i} \quad (10)$$

$i = \overline{1, N}.$

$$\Delta = |S_i(x) - f(x)| \le M_4 h^4$$

$$M_4 = \sup_{x \in [a,b]} f^{(IV)}(x)$$

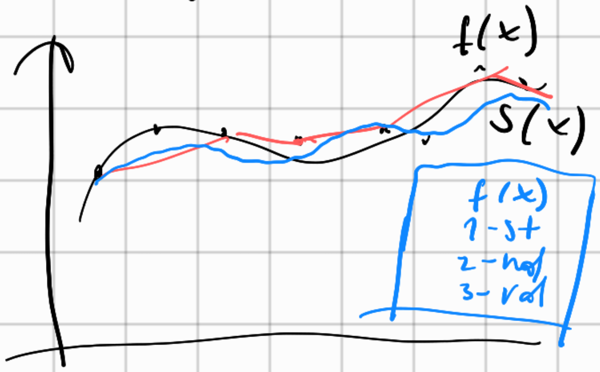
Lab 8. Submit a report (1 file).

1) Code (w/o any existing spline methods)

2) output $x \in [a, b]$, N

x	f(x)	S(x)	Δ

3) Graphs



1) graphs of 1-st, 2-nd, 3-rd order splines (together).

2) Mean average error distribution (its dependence by N).