Lecture 6

System of nonlinear algebraic equations (SNAE).

I. Newton's method.

$$\begin{cases} F(x,y) = 0 \\ G(x,y) = 0 \end{cases} \qquad (1)$$

$$x_{n+1} = x_n - \frac{1}{J(x_n,y_n)} \cdot \underbrace{\begin{vmatrix} F(x_n,y_n) & F'_y(x_n,y_n) \\ G(x_n,y_n) & G'_y(x_n,y_n) \end{vmatrix}}_{\Delta_x} =$$

$$= x_n - \frac{\Delta_x}{J(x_n,y_n)};$$

$$y_{n+1} = y_n - \frac{1}{J(x_n,y_n)} \cdot \underbrace{\begin{vmatrix} F'_x(x_n,y_n) & F(x_n,y_n) \\ G'_x(x_n,y_n) & G(x_n,y_n) \end{vmatrix}}_{\Delta_y} =$$

$$= y_n - \frac{\Delta_y}{J(x_n,y_n)};$$

$$J(x,y) = \begin{vmatrix} F'_x(x,y) & F'_y(x,y) \\ G'_x(x,y) & G'_y(x,y) \end{vmatrix} \neq 0.$$

(x^0, y^0) – graphical method.

Ex. 1.

$$\begin{cases} F(x,y) = 2x^3 - y^2 - 1 = 0 \\ G(x,y) = xy^3 - y - 4 = 0 \end{cases}$$

$x_0 = 1,2$
$y_0 = 1,7$

$$J(x,y) = \begin{vmatrix} 6x^2 & -2y \\ y^3 & 3xy^2 - 1 \end{vmatrix}.$$

$$J(1,2;\ 1,7) = \begin{vmatrix} 8,64 & -3,4 \\ 4,91 & 9,4 \end{vmatrix} = 97,91$$

$$x_1 = 1,2 - \frac{1}{97,91} \cdot \begin{vmatrix} -0,434 & -3,4 \\ 0,1956 & 9,4 \end{vmatrix} = \underline{1,2349},$$

$$y_1 = 1,7 - \frac{1}{97,91} \cdot \begin{vmatrix} 8,64 & -0,434 \\ 4,91 & 0,1956 \end{vmatrix} = \underline{1,6610}.$$

$$\Delta_x = |1,2349 - 1,2| = 0,0349$$

$$\Delta_y = |1,6610 - 17| = 0,039$$

$$\varepsilon = 10^{-4}$$

II. Method of simple iteration for SNAE.

$$\begin{cases} F_1(x,y) = 0, \\ F_2(x,y) = 0, \end{cases} \qquad (2)$$

$$\begin{cases} x = \varphi_1(x,y) \\ \ldots \end{cases} \qquad (3)$$

$$y = \varphi_2(x,y)$$

$$\begin{cases} x_{n+1} = \varphi_1(x_n, y_n) \\ y_{n+1} = \varphi_2(x_n, y_n) \end{cases} \quad n = 0, \dots \qquad (3)$$

(3) - method of simple iteration.

Theorem.

$R\,(a \le x \le A,\ b \le y \le B)$, $x = \xi$, $y = \eta$ - is one and only solution if:

1) functions φ_1 and φ_2 are definite and continiously differentiable on this domain.

2) x_0, y_0 and next iterations x_n, y_n $(n = 1, 2, \dots)$ belongs to R.

3) in R the inequalities are satisfied:

$$\begin{cases} \left|\dfrac{\partial \varphi_1}{\partial x}\right| + \left|\dfrac{\partial \varphi_2}{\partial x}\right| \le q_1 < 1, \\ \left|\dfrac{\partial \varphi_1}{\partial y}\right| + \left|\dfrac{\partial \varphi_2}{\partial y}\right| \le q_2 < 1 \end{cases} \qquad (4)$$

Therefore, the iterative process (3) is convergent to $x = \xi$, $y = \eta$, i.e.

$$\lim_{n \to \infty} x_n = \xi, \qquad \lim_{n \to \infty} y_n = \eta.$$

$$\begin{cases} \left|\frac{\partial \varphi_1}{\partial x}\right| + \left|\frac{\partial \varphi_1}{\partial y}\right| \leq q_1 < 1 \\ \left|\frac{\partial \varphi_2}{\partial x}\right| + \left|\frac{\partial \varphi_2}{\partial y}\right| \leq q_2 < 1 \end{cases} \qquad (4)$$

Ex 2.

$$\begin{cases} x^3+y^3-6x+3=0 \\ x^3-y^3-6y+2=0 \end{cases} \Rightarrow \begin{cases} x = \frac{x^3+y^3}{6} + \frac{1}{2} = \varphi_1(x,y) \\ y = \frac{x^3-y^3}{6} + \frac{1}{3} = \varphi_2(x,y) \end{cases}$$

$x_0 = \frac{1}{2}$, $y_0 = \frac{1}{2}$.

$$x_1 = \frac{1}{2} + \frac{\frac{1}{8}+\frac{1}{8}}{6} = 0,\underline{5}42$$

$$y_1 = \frac{1}{3} + \frac{\frac{1}{8}-\frac{1}{8}}{6} = 0,\underline{3}33$$

$$\Delta = |x^{n+1} - x^n| \text{ and } |y^{n+1} - y^n| < \varepsilon$$