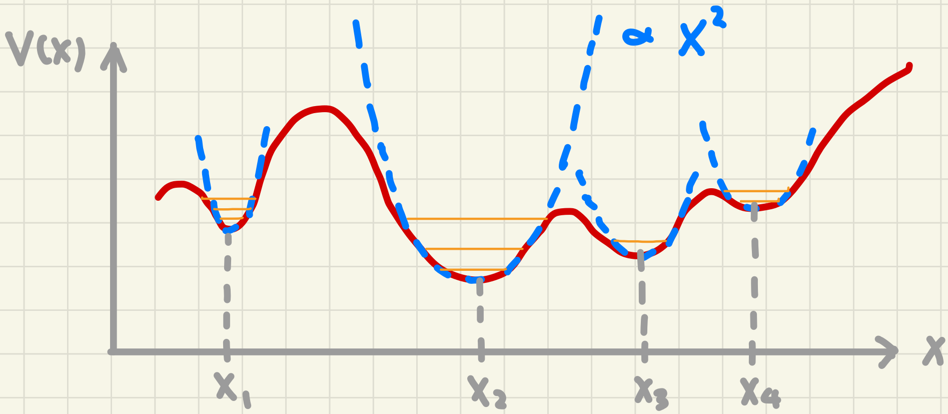
Chapter 1 Quantum Simple Harmonic Oscillator

In physics simple harmonic oscillator is so important because one can always use harmonic potential, i.e., the quadratic potential, to approximate arbitrary potential around its local minima through Taylor series

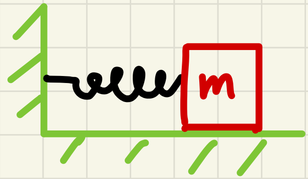


at local minima

$$V(x) = V(x_i) + \frac{1}{1!} V'(x_i)(x - x_i) + \underbrace{\frac{1}{2!} V''(x_i)(x - x_i)^2}_{\text{dominant term}} + \cdots\cdots$$

One can analyze the low excitations around each minimum without the precise knowledge of the whole potential. In this chapter we will show the importance of simple harmonic oscillator in quantum mechanics and give relevant examples,e.g., Landau level and quantization of electromagnetic waves and of LC circuit.

1-1 Analytic Method



The Schroedinger equation for harmonic oscillator reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi \qquad (1-1)$$

Let $x = ay \rightarrow dx = a\,dy$

$E = bK$

where y and K are dimensionless

$$\rightarrow -\frac{\hbar^2}{2m}\frac{1}{a^2}\frac{d^2\psi}{dy^2} + \frac{1}{2}m\omega^2a^2y^2\psi = bK\psi$$

$$\rightarrow \frac{d^2\psi}{dy^2} - \frac{m^2\omega^2}{\hbar^2}a^4y^2\psi = -\frac{2mba^2}{\hbar^2}K\psi$$

check the unit

$$\frac{m\omega}{\hbar} \sim \frac{\cancel{kg}\cdot\frac{1}{\cancel{s}}}{m\cdot\cancel{kg}\cdot\frac{m}{\cancel{s}}} \sim \frac{1}{m^2} \quad \text{unit of } \frac{1}{\text{length}^2}$$

let $\boxed{a = \sqrt{\frac{\hbar}{m\omega}}}$ (unit of length)

$$\rightarrow \frac{2mba^2}{\hbar^2} = \frac{2b}{\hbar\omega} \rightarrow \boxed{b = \frac{1}{2}\hbar\omega} \quad \text{(unit of energy)}$$

Eq. (1-1) becomes

$$\frac{d^2\psi}{dy^2} = (y^2 - k)\psi \qquad (1-2)$$

since k is the eigen value, a constant,

for $y \to \infty$

$$\frac{d^2\psi}{dy^2} = y^2\psi \rightarrow \psi \sim e^{-\frac{y^2}{2}} + \cancel{e^{\frac{y^2}{2}}} ,$$

nonphysical

$$\frac{d\psi}{dy} = -y e^{-\frac{y^2}{2}} ,$$

$$\frac{d^2\psi}{dy^2} = -e^{-y^2} + y^2 e^{-\frac{y^2}{2}} \sim y^2 e^{-\frac{y^2}{2}} .$$

Let $\psi(y) = e^{-\frac{y^2}{2}} H(y)$,

$$\rightarrow \frac{d\psi}{dy} = -y e^{-\frac{y^2}{2}} H + e^{-\frac{y^2}{2}} \frac{dH}{dy} ,$$

$$\frac{d^2\psi}{dy^2} = -e^{-\frac{y^2}{2}} H + y^2 e^{-\frac{y^2}{2}} H - 2y e^{-\frac{y^2}{2}} \frac{dH}{dy} + e^{-\frac{y^2}{2}} \frac{d^2H}{dy^2} .$$

Eq. (1-2) becomes the Hermit equation

$$\frac{d^2 H}{dy^2} - 2y\frac{dH}{dy} + (k-1)H = 0 \qquad (1-3)$$

The typical method to solve Eq. (1-3) is assuming

$$H(y) = \sum_{j=0}^{\infty} a_j y^j ,$$

$$\frac{dH}{dy} = \sum_{j=0}^{\infty} j a_j y^{j-1} ,$$

$$\frac{d^2H}{dy^2} = \sum_{j=0}^{\infty} j(j-1) a_j y^{j-2} .$$

Putting these into Eq. (1-3), we get

$$\sum_{j=0}^{\infty} \left[j(j-1) a_j y^{j-2} - 2j a_j y^j + (k-1) a_j y^j \right] = 0$$

$$\rightarrow \sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} - 2j a_j + (k-1) a_j \right] y^j = 0$$

$$\rightarrow \boxed{(j+2)(j+1) a_{j+2} + (k-1-2j) a_j = 0}$$

We then get the recursion relation

$$\boxed{a_{j+2} = \frac{2j+1-K}{(j+1)(j+2)} a_j} \quad (1-4)$$

In order to look for solutions which can be normalized, we truncate the solution at certain j by using

$$\boxed{K = 2n+1}, \quad \text{where } n \in N. \quad (1-5)$$

We therefore get Hermite polynomials H_n.

$$H_0(y) = a_0 \rightarrow H_0(y) = 1,$$

$$H_1(y) = a_1 y \rightarrow H_1(y) = 2y,$$

$$H_2(y) = a_0\left[\frac{0+1-5}{(0+1)(0+2)} y^2 + 1\right] = a_0(-2y^2+1)$$

$$\rightarrow H_2(y) = 4y^2 - 2,$$

$$H_3(y) = a_1\left[\frac{2+1-7}{(1+1)(1+2)} y^3 + y\right] = a_1\left(-\frac{2}{3}y^3 + y\right)$$

$$\rightarrow H_3(y) = 8y^3 - 12y,$$

$$\vdots$$

and the normalized stationary state

$$\boxed{\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-\frac{m\omega}{2\hbar}x^2}} \quad (1-6)$$

and the energy quantization

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega} \quad (1-7)$$

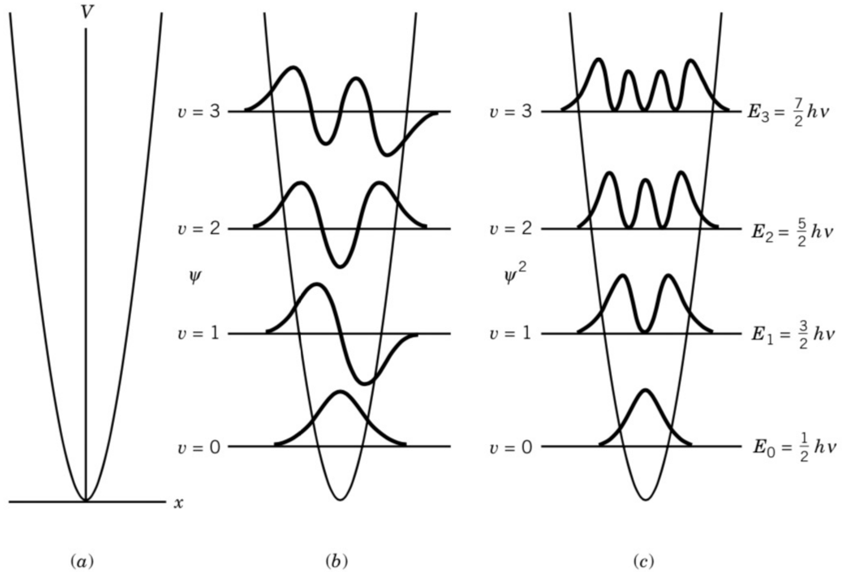


Figure 1-1 (*a*) Potential energy curve for a classical harmonic oscillator. (*b*) Allowed energy levels and wavefunctions for a quantum mechanical harmonic oscillator. (*c*) Probability density functions for a quantum mechanical harmonic oscillator. (See Computer Problems 9.B and 9.C.)

Okay, the quantum harmonic oscillator looks fancy! Then what?
Can we really realize this, namely, cooling down a mechanical oscillator down to its quantum ground state? This is related to one of Kip Thorne's famous speculations:

Will 21st century technology reveal quantum behavior in the realm of human-size objects?

This is a big quest, and we are on the way to it, check the following three great works:

Nature 464, 697 (2010),
Nature 478, 89 (2011) &
Science 367, 892 (2020).

1-2 Algebraic Method (very important !)

Important equality $\hat{x}^2 + \hat{y}^2 = (\hat{x} + i\hat{y})(\hat{x} - i\hat{y}) + i[\hat{x}, \hat{y}]$

The left hand side of Schroedinger equation reads

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\hat{x}^2\right)\psi$$

$$\rightarrow \hbar\omega\left(-\frac{\hbar}{2m\omega}\partial_x^2 + \frac{m\omega}{2\hbar}\hat{x}^2\right)\psi$$

$$\rightarrow \left(-\sqrt{\frac{\hbar}{2m\omega}}\partial_x + \sqrt{\frac{m\omega}{2\hbar}}\hat{x}\right)\left(\sqrt{\frac{\hbar}{2m\omega}}\partial_x + \sqrt{\frac{m\omega}{2\hbar}}\hat{x}\right)\psi$$

$$\rightarrow \left[-i(-i)\hbar\sqrt{\frac{1}{2m\hbar\omega}}\partial_x + \sqrt{\frac{m\omega}{2\hbar}}\hat{x}\right]\left(-i\cdot i\hbar\sqrt{\frac{1}{2m\hbar\omega}}\partial_x + \sqrt{\frac{m\omega}{2\hbar}}\hat{x}\right)\psi$$

$\hat{p} = -i\hbar\partial_x$, $[\hat{x}, \hat{p}] = i\hbar$

$$\rightarrow \underbrace{\left(-i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\hat{x}\right)}_{\hat{a}^+}\underbrace{\left(i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\hat{x}\right)}_{\hat{a}}\psi$$

let's define two new operators

$$\boxed{\begin{aligned} \hat{a}^+ &= -i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\hat{x} \quad \text{(creation operator)} \\ \hat{a} &= i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\hat{x} \quad \text{(annihilation operator)} \end{aligned}}$$

$$\rightarrow \quad \hat{a}^{+}\hat{a}\,\psi = \left[\frac{\hat{H}}{\hbar\omega} + \frac{i}{2\hbar}\left(\hat{x}\hat{p} - \hat{p}\hat{x}\right)\right]\psi$$

$$= \left(\frac{\hat{H}}{\hbar\omega} + \frac{i}{2\hbar}[\hat{x}, \hat{p}]\right)\psi$$

$$= \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right)\psi$$

$$\rightarrow \quad \hat{H}\psi = \left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)\hbar\omega\,\psi \qquad (1-8)$$

in view of Eq. (1-7)

$\hat{a}^{+}\hat{a}\,\psi = n\,\psi$, $n \in N \rightarrow \hat{a}^{+}\hat{a}$ is number operator

On the other hand we can investigate

$$\hat{a}\hat{a}^{+}\psi = \left[\frac{\hat{H}}{\hbar\omega} + \frac{i}{2\hbar}\left(\hat{p}\hat{x} - \hat{x}\hat{p}\right)\right]\psi$$

$$= \left(\frac{\hat{H}}{\hbar\omega} + \frac{1}{2}\right)\psi$$

$$\rightarrow \quad \hat{H}\psi = \left(\hat{a}\hat{a}^{+} - \frac{1}{2}\right)\hbar\omega\,\psi \qquad (1-9)$$

Given Eq. (1-8) and (1-9),

$$\hat{a}^{+}\hat{a} + \frac{1}{2} = \hat{a}\hat{a}^{+} - \frac{1}{2} \quad \rightarrow \quad \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} = 1$$

$$\rightarrow \quad \boxed{[\hat{a}, \hat{a}^{+}] = 1} \qquad (1-10)$$

Here we need to study more properties of $\hat{a}^+$ and $\hat{a}$ operator. Let's first look at

$$\hat{H}(\hat{a}^+\psi_n) = (\hat{a}^+\hat{a} + \tfrac{1}{2})\hat{a}^+\hbar\omega\psi_n$$

$$= (\hat{a}^+\hat{a}\hat{a}^+ + \tfrac{1}{2}\hat{a}^+)\hbar\omega\psi_n$$

$$= [\hat{a}^+(1+\hat{a}^+\hat{a}) + \tfrac{1}{2}\hat{a}^+]\hbar\omega\psi_n$$

$$= \hat{a}^+(\hat{a}^+\hat{a} + 1 + \tfrac{1}{2})\hbar\omega\psi_n$$

$$= (n + \tfrac{1}{2} + 1)\hbar\omega\,\hat{a}^+\psi_n$$

$$\rightarrow \quad \hat{H}(\hat{a}^+\psi_n) = (E_n + \hbar\omega)(\hat{a}^+\psi_n) = (E_n + \hbar\omega)\psi_{n+1}$$

and

$$\hat{H}(\hat{a}\psi) = (\hat{a}\hat{a}^+ - \tfrac{1}{2})\hat{a}\hbar\omega\psi_n$$

$$= (\hat{a}\hat{a}^+\hat{a} - \tfrac{1}{2}\hat{a})\hbar\omega\psi_n$$

$$= \hat{a}(\hat{a}^+\hat{a} - \tfrac{1}{2})\hbar\omega\psi_n$$

$$= (n + \tfrac{1}{2} - 1)\hbar\omega\,\hat{a}\psi_n .$$

$$\rightarrow \quad \hat{H}(\hat{a}\psi_n) = (E_n - \hbar\omega)(\hat{a}\psi_n) = (E_n - \hbar\omega)\psi_{n-1}$$

Therefore we find

$$\hat{a}^{+} \psi_n = C_{n+1} \psi_{n+1} ,$$

$$\hat{a} \psi_n = C_{n-1} \psi_{n-1} .$$

Here $\langle x | n \rangle = \psi_n (x)$. Instead of working in real space, i.e., x, we start to work in $|n\rangle$ basis

$$\hat{a}^{+} |n\rangle = C_{n+1} |n+1\rangle$$

$$\hat{a} |n\rangle = C_{n-1} |n-1\rangle .$$

Given eq. (1-8), $\hat{a}^{+} \hat{a} |n\rangle = n |n\rangle$

$$\rightarrow \quad \langle n | \hat{a}^{+} \hat{a} | n \rangle = | C_{n-1} |^2 = n$$

$$\rightarrow \quad C_{n-1} = \sqrt{n}$$

$$\rightarrow \quad \boxed{\hat{a} |n\rangle = \sqrt{n} \, |n-1\rangle .} \qquad (1\text{-}11)$$

And

$$\langle n | \hat{a} \hat{a}^{+} | n \rangle = | C_{n+1} |^2$$

$$= \langle n | (1 + \hat{a}^{+} \hat{a}) | n \rangle = n+1$$

$$\rightarrow \quad C_{n+1} = \sqrt{n+1}$$

$$\rightarrow \quad \boxed{\hat{a}^{+} |n\rangle = \sqrt{n+1} \, |n+1\rangle} \qquad (1\text{-}12)$$

We can then construct eigen states of harmonic oscillator by using eq. (1-11) and (1-12)

$$|1\rangle = \hat{a}^{+}|0\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}}\hat{a}^{+}|1\rangle = \frac{1}{\sqrt{2!}}(\hat{a}^{+})^{2}|0\rangle$$

$$|3\rangle = \frac{1}{\sqrt{3}}\hat{a}^{+}|2\rangle = \frac{1}{\sqrt{3!}}(\hat{a}^{+})^{3}|0\rangle$$

$$\vdots$$

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}|0\rangle .} \qquad (1\text{-}13)$$

Summary of $\hat{a}$, $\hat{a}^{+}$ property

definition	$\hat{a}^{+} = -i\sqrt{\frac{1}{2m\hbar\omega}}\,\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}$	
	$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega}}\,\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}$	
commutator	$[\hat{a}, \hat{a}^{+}] = 1$	(1-10)
action on $\vert n\rangle$	$\hat{a}\vert n\rangle = \sqrt{n}\,\vert n-1\rangle$	(1-11)
	$\hat{a}^{+}\vert n\rangle = \sqrt{n+1}\,\vert n+1\rangle$	(1-12)
number operator	$\hat{a}^{+}\hat{a}\vert n\rangle = n\vert n\rangle$	
eigen states	$\vert n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}\vert 0\rangle$	(1-13)

What can we learn from

$$\hat{a}|0\rangle = 0 \ ?$$

Let's go back to the definition of

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega}}\,\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}$$

$\hat{p} = -i\hbar\partial_x$

$$= \hbar\sqrt{\frac{1}{2m\hbar\omega}}\,\partial_x + \sqrt{\frac{m\omega}{2\hbar}}\,x$$

$$\rightarrow \quad \hat{a}|0\rangle = 0$$

$$\rightarrow \quad \hbar\sqrt{\frac{1}{2m\hbar\omega}}\,\partial_x \psi_0 = -\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0$$

$$\rightarrow \quad \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}x\,dx$$

$$\rightarrow \quad \psi_0 = A\,e^{-\frac{m\omega}{2\hbar}x^2}$$

By normalization $\int_{-\infty}^{\infty} A^2 e^{-\frac{m\omega}{\hbar}x^2}dx = 1$

$$\rightarrow \quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}.$$

We get the ground wave function!!

1-3 The Uncertainty Principle of Quantum Harmonic Oscillator

In this section, we study the property of harmonic oscillator by using $\hat{a}$, $\hat{a}^{+}$. From the definition of $\hat{a}$ and $\hat{a}^{+}$, we know

$$\boxed{\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}^{+} + \hat{a}) \qquad (1\text{-}14) \\ \hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^{+} - \hat{a}) \qquad (1\text{-}15) \end{aligned}}$$

We will use Eq. (1-14) and (1-15) to calculate the uncertainty principle $\Delta x \Delta p$, where

$$\begin{aligned} \Delta x^2 &= \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \\ &= \langle n | \left(\frac{\hbar}{2m\omega}\right) (\hat{a}^{+} + \hat{a})(\hat{a}^{+} + \hat{a}) | n \rangle \\ &\quad - \langle n | \sqrt{\frac{\hbar}{2m\omega}} (\cancel{\hat{a}^{+}} + \cancel{\hat{a}}) | n \rangle^2 \\ &= \frac{\hbar}{2m\omega} \langle n | (\cancel{\hat{a}^{+}\hat{a}^{+}} + \hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+} + \cancel{\hat{a}\hat{a}}) | n \rangle \\ &= \frac{\hbar}{2m\omega} (n + n + 1) \end{aligned}$$

$$\rightarrow \quad \Delta x = \sqrt{\frac{(2n+1)\hbar}{2m\omega}}$$

$$\Delta p^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$$

$$= -\frac{m\hbar\omega}{2} \langle n | (\cancel{\hat{a}^\dagger \hat{a}^\dagger} - \hat{a}^\dagger \hat{a} - \hat{a} \hat{a}^\dagger + \cancel{\hat{a} \hat{a}}) | n \rangle$$

$$= (2n+1) \frac{m\hbar\omega}{2}$$

$$\rightarrow \quad \Delta p = \sqrt{\frac{2n+1}{2} m\hbar\omega}$$

$$\rightarrow \quad \boxed{\Delta x \, \Delta p = (2n+1) \frac{\hbar}{2} \geq \frac{\hbar}{2}} \qquad (1\text{-}16)$$

This example shows that one can use eq. (1-14) and (1-15) to calculate many quantities of quantum harmonic oscillator.

In what follows, we will give three examples to show the power of quantum harmonic oscillator:

Landau level

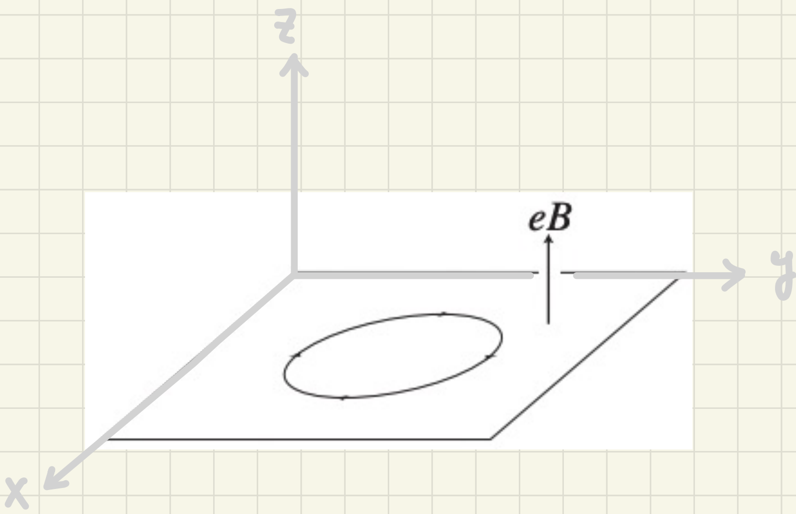
Quantization of L-C circuit,

Quantization of electromagnetic wave.

1-4 Landau Level

Understanding the famous Landau level is the first and crucial step to learn, e.g, the Abrikosov flux lattice in type-II superconductor, quantum Hall effect and topological material so on. In this section we are going to show the first powerful example of quantum harmonic oscillator.

Imagine there are bunches of electron moving in a two dimensional x-y plane applied with a uniform magnetic field along z-axis. Let's solve this system with quantum mechanics.



We need the correct form of Hamiltonian. Let's start with classical Lorentz force

$$\frac{d\vec{P}}{dt} = e(\vec{E} + \vec{v} \times \vec{B}) = e\begin{pmatrix} E_x + V_y B \\ E_y - V_x B \\ 0 \end{pmatrix}$$

$$= e\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \underbrace{\vec{v} \times \nabla \times \vec{A}}\right)$$

We need to convert this term by vector analysis.

Check

$$\vec{v}\times\nabla\times\vec{A} = \varepsilon_{ijk} v_j (\nabla\times\vec{A})_k = \varepsilon_{ijk} v_j \varepsilon_{klm} \partial_l A_m$$

$$= \varepsilon_{kij}\varepsilon_{klm} v_j \partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) v_j \partial_l A_m$$

$$= v_m \partial_i A_m - v_l \partial_l A_i = \partial_i (v_m A_m) - A_m \partial_i \cancel{v_m}^{\,0} - v_l \partial_l A_i$$

$$= \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}$$

only function of TIME !!

$$\frac{d\vec{p}}{dt} = e\left[-\nabla\phi - \frac{\partial\vec{A}}{\partial t} + \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}\right]$$

$$= -e\left[\frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A} + \nabla(\phi - \vec{v}\cdot\vec{A})\right]$$

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \frac{\partial A_j}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}$$

$$\rightarrow \quad \frac{d}{dt}(\vec{p} + e\vec{A}) = -e\nabla(\phi - \vec{v}\cdot\vec{A})$$

$$\rightarrow \quad \frac{d}{dt}\underbrace{(\vec{p} + e\vec{A})}_{\text{canonical momentum } \vec{\mathbb{P}}} = -\nabla\underbrace{\left[e(\phi - \vec{v}\cdot\vec{A})\right]}_{\text{genelized potential } U}$$

$$\rightarrow \quad \frac{d\vec{\mathbb{P}}}{dt} = -\nabla U$$

☆☆ One should always use $\vec{\mathbb{P}}$ and U to construct Lagrangian or Hamiltonian. ☆☆

Here we wish to demonstrate how one should use $\vec{\mathbb{P}}$ and U to construct Lagrangian $\mathcal{L}$ and Hamiltonian $\mathcal{H}$.

$$\boxed{\begin{aligned} &\mathcal{L} = T - U && (1\text{-}17) \\ &\frac{\partial \mathcal{L}}{\partial X_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial V_i}\right) = 0 && (1\text{-}18) \end{aligned}}$$

$(1\text{-}18) \rightarrow$ gauge invariant

For $\vec{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$, vector potential has at least three choices of gauge

$$\underbrace{\vec{A}_1 = \begin{pmatrix} 0 \\ XB \\ 0 \end{pmatrix}, \quad \vec{A}_2 = \begin{pmatrix} -yB \\ 0 \\ 0 \end{pmatrix}}_{\text{Landau gauge}}, \quad \underbrace{\vec{A}_3 = \frac{B}{2}\begin{pmatrix} -y \\ X \\ 0 \end{pmatrix}}_{\text{symmetric gauge}}$$

For scalar potential $V=0$, you may write down

$$\mathcal{L} = \frac{1}{2}m|\vec{v}|^2 + e\vec{v}\cdot\vec{A} \qquad (1\text{-}19)$$

(i) Landau gauge $\vec{A}_1 = \begin{pmatrix} 0 \\ xB \\ 0 \end{pmatrix}$

$$\mathcal{L} = \frac{1}{2}m(v_x^2+v_y^2) + e v_y x B$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_x}\right) = e v_y B - m\dot{v}_x = 0 \qquad \text{ok !!}$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_y}\right) = -e v_x B - m\dot{v}_y = 0 \qquad \text{ok !!}$$

(ii) Symmetric gauge $\vec{A} = \frac{B}{2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$

$$\mathcal{L} = \frac{1}{2}m(v_x^2+v_y^2) + \frac{eB}{2}(v_y x - v_x y)$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_x}\right) = e v_y B - m\dot{v}_x = 0 \qquad \text{ok !!}$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_y}\right) = -e v_x B - m\dot{v}_y = 0 \qquad \text{ok !!}$$

So the form of Eq. (1-19) leads to gauge invariant equation of motion, it's ok.

Next question is how to construct Hamiltonian

which momentum this should be ?

$$\mathcal{H} = \sum_j p_j \dot{x}_j - \mathcal{L} \qquad (1\text{-}20)$$

$$\frac{dx_j}{dt} = \frac{\partial \mathcal{H}}{\partial p_j} \qquad (1\text{-}21)$$

$$\frac{dp_j}{dt} = -\frac{\partial \mathcal{H}}{\partial q_j} \qquad (1\text{-}22)$$

$$p_j = \frac{\partial \mathcal{L}}{\partial V_i}$$

(I) Landau gauge

$$\mathcal{H} = m(V_x^2 + V_y^2) - \frac{1}{2}m(V_x^2 + V_y^2) - e V_y X B$$
$$= \frac{1}{2}m(V_x^2 + V_y^2) - e V_y X B$$

$$m\dot{V}_x = -\frac{\partial \mathcal{H}}{\partial X} = e V_y B \qquad \text{ok !!}$$

$$m\dot{V}_y = -\frac{\partial \mathcal{H}}{\partial Y} = 0 \qquad \text{NOT ok}$$

(II) Symmetric gauge

$$\mathcal{H} = \frac{1}{2}m(v_x^2 + v_y^2) - \frac{eB}{2}(V_y x - V_x y)$$

$$m\dot{V}_x = -\frac{\partial \mathcal{H}}{\partial x} = \frac{1}{2} e V_y B$$ NOT ok :(

$$m\dot{V}_y = -\frac{\partial \mathcal{H}}{\partial y} = -\frac{1}{2} e V_x B$$ NOT ok :(

This shows that Eq. (1-20) needs a modification, otherwise the equation of motion is gauge dependent !!

1-4-1 Hamiltonian for Charged Particle interacting with EM Fields

Let's try to use canonical momentum

$$\boxed{\mathbb{P}_i = \frac{\partial \mathcal{L}}{\partial V_i}} \qquad (1\text{-}23)$$

i.e., $\vec{\mathbb{P}} = \vec{p} + e\vec{A}$ in eq. (1-20)

$$\mathcal{H} = \vec{\mathbb{P}} \cdot \vec{V} - \mathcal{L} \qquad (1\text{-}24)$$

(I) Landau gauge

$$\vec{\mathbb{P}} = \vec{p} + e\vec{A} = \begin{pmatrix} mV_x \\ mV_y + exB \\ 0 \end{pmatrix}$$

$$\mathcal{H} = \vec{\mathbb{P}} \cdot \vec{V} - \mathcal{L}$$

$$= \mathbb{P}_x V_x + \mathbb{P}_y V_y - \frac{1}{2}m(V_x^2 + V_y^2) - eV_y xB$$

$$= \frac{1}{2}m(V_x^2 + V_y^2)$$

$$= \frac{|\vec{p}|^2}{2m}$$

$$= \frac{|\vec{\mathbb{P}} - e\vec{A}|^2}{2m} = \frac{1}{2m}\sum_i (\mathbb{P}_i - eA_i)^2$$

the equation of motion reads

$$\frac{d\mathbb{P}_j}{dt} = \frac{dP_j}{dt} + e\partial_t A_j + eV_i \partial_i A_j$$

$$-\frac{\partial \mathcal{H}}{\partial q_j} = \frac{-1}{m}\sum_i (\mathbb{P}_i - eA_i)(-e)\frac{\partial A_i}{\partial q_j}$$

j-th component of Hamiltonian equation of motion

$$\boxed{\dot{p}_j + e\partial_t A_j + e v_i \partial_i A_j = \frac{e}{m}(\mathbb{P}_i - eA_i)\frac{\partial A_i}{\partial q_j} \quad (1\text{-}25)}$$

where $\vec{p} = m\begin{pmatrix} V_x \\ V_y \end{pmatrix}$, $\vec{A} = \begin{pmatrix} 0 \\ xB \end{pmatrix}$,

and $\vec{\mathbb{P}} = \begin{pmatrix} mV_x \\ mV_y + exB \end{pmatrix}$

$$\frac{\partial A_i}{\partial q_j} = \begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} \end{pmatrix}$$

$$\rightarrow m\begin{pmatrix} \dot{V}_x \\ \dot{V}_y \end{pmatrix} + e\left[V_x \partial_x \begin{pmatrix} 0 \\ xB \end{pmatrix} + V_y \partial_y \begin{pmatrix} 0 \\ xB \end{pmatrix}\right]$$

$$= \frac{e}{m} m \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

$$\rightarrow m\begin{pmatrix} \dot{V}_x \\ \dot{V}_y \end{pmatrix} + \begin{pmatrix} 0 \\ eV_xB \end{pmatrix} = \begin{pmatrix} eV_yB \\ 0 \end{pmatrix}$$

OK!!

(II) symmetric gauge

Let's substitute

$$\vec{p} = m\begin{pmatrix} V_x \\ V_y \end{pmatrix}, \quad \vec{A} = \begin{pmatrix} -\frac{1}{2}By \\ \frac{1}{2}Bx \end{pmatrix}$$

and $\vec{\mathbb{P}} = \begin{pmatrix} mV_x - \frac{1}{2}eBy \\ mV_y + \frac{1}{2}eBx \end{pmatrix}$ into eq. (1.25),

we get

$$m\begin{pmatrix} \dot{V}_x \\ \dot{V}_y \end{pmatrix} + \frac{1}{2}eB\left[V_x \partial_x \begin{pmatrix} -y \\ x \end{pmatrix} + V_y \partial_y \begin{pmatrix} -y \\ x \end{pmatrix}\right]$$

$$= \frac{1}{2}eB \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

$$\Rightarrow \quad m\begin{pmatrix} \dot{V}_x \\ \dot{V}_y \end{pmatrix} + \frac{1}{2}eB\begin{pmatrix} -V_y \\ V_x \end{pmatrix} = \frac{1}{2}eB\begin{pmatrix} V_y \\ -V_x \end{pmatrix}$$ ok!! #

Conclusion :

$$\mathcal{L} = \frac{1}{2}m|\vec{v}|^2 - e(\phi - \vec{v}\cdot\vec{A}) \qquad (1.26)$$

$$\mathbb{P}_i = \frac{\partial \mathcal{L}}{\partial V_i} \qquad (1.23)$$

$$\mathcal{H} = \frac{|\vec{\mathbb{P}} - e\vec{A}|^2}{2m} + e\phi \qquad (1.27)$$

1-4-2 Schroedinger Equation & Gauge (Phase) Transform

In quantum mechanics, we replace the classical canonical momentum with an operator $\vec{P} \to \hat{P}$, where $\hat{P} = -i\hbar\nabla$.

The Schrödinger equation of a charged particle in electromagnetic fields is

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = \frac{\left|-i\hbar\nabla - e\vec{A}\right|^2}{2m}\psi + e\phi\psi . \qquad (1.28)}$$

Now, we'll demonstrate the effect of gauge transformation on quantum wavefunction ψ.

Consider gauge transformation

$$\phi' = \phi - \frac{\partial f}{\partial t}$$

$$\vec{A}' = \vec{A} + \nabla f ,$$

where $f(\vec{r}, t)$ is an arbitrary real function.

The Schrödinger equation under gauge transformation reads

$$i\hbar \partial_t \psi' = \frac{1}{2m}\left[\frac{\hbar}{i}\nabla - e(\vec{A}+\nabla f)\right]^2 \psi' + e(\phi - \partial_t f)\psi' \tag{1.29}$$

Let's assume $\psi' = \psi \cdot F$, where ψ satisfies Eq. (1.28), i.e.,

$$i\hbar \partial_t \psi = \frac{1}{2m}\Big[-\hbar^2 \nabla^2 \psi + i\hbar e(\nabla \cdot \vec{A})\psi + 2i\hbar e \vec{A}\cdot\nabla\psi + e^2 |\vec{A}|^2 \psi\Big] + e\phi\psi \tag{1.30}$$

(Appendix 4)

Substitute ψ' into eq. (1.29), we get

$$\begin{aligned} & i\hbar(\underline{F\partial_t\psi} + \psi\partial_t F) \\ &= -\hbar^2(\underline{F\nabla^2\psi} + 2\nabla F\cdot\nabla\psi + \psi\nabla^2 F) \\ &+ i\hbar e\left[2(\underline{F\nabla\psi} + \psi\nabla F)\cdot(\vec{A}+\nabla f) + (\underline{\nabla\cdot\vec{A}} + \nabla^2 f)\psi F\right] \\ &+ e^2|\vec{A}+\nabla f|^2 \psi F \\ &+ e(\underline{\phi} - \partial_t f)\psi F \end{aligned} \tag{1.31}$$

Eq. (1.30) − Eq. (1.31) results in

$$
\begin{aligned}
& i\hbar \psi \partial_t F = \\
&= -\hbar^2 \left(2 \nabla F \cdot \nabla \psi + \psi \nabla^2 F \right) \\
&+ i\hbar e \left[2 \left(F \nabla \psi \cdot \nabla f + \psi \vec{A} \cdot \nabla F + \psi \nabla F \cdot \nabla f \right) + \psi F \nabla^2 f \right] \\
&+ e^2 \left(2 \vec{A} \cdot \nabla f + |\nabla f|^2 \right) \psi F \\
&- e \psi F \partial_t f
\end{aligned}
$$

Let

$$
\begin{cases}
i\hbar \psi \partial_t F = - e \psi F \partial_t f & \rightarrow \quad \partial_t F = i \frac{e}{\hbar} F \partial_t f , \\
\left(-2\hbar^2 \nabla F + i 2 \hbar e F \nabla f \right) \cdot \nabla \psi = 0 & \rightarrow \quad \nabla F = i \frac{e}{\hbar} F \nabla f , \\
\left(2 i \hbar e \nabla F + 2 e^2 F \nabla f \right) \cdot \vec{A} = 0 & \rightarrow \quad \nabla F = i \frac{e}{\hbar} F \nabla f , \\
-\hbar^2 \nabla^2 F + 2 i \hbar e \nabla F \cdot \nabla f + i \hbar e F \nabla^2 f + e^2 |\nabla f|^2 F = 0 &
\end{cases}
$$

$$
\rightarrow \left(\frac{\hbar}{i} \nabla - e \nabla f \right)^2 F = 0 .
$$

All four equations indicate $F = e^{i \frac{ef}{\hbar}}$. The wavefunction under gauge transformation then reads

$$
\boxed{\psi'(\vec{r}, t) = \psi(\vec{r}, t) \, e^{i \frac{e}{\hbar} f(\vec{r}, t)} .} \qquad (1.32)
$$

The gauge transformation leads to a PHASE transformation on wavefunction in quantum mechanics !!

1-4-3 Landau Levels in Landau Gauge

We start investigating Landau level by first using Landau gauge $\vec{A} = \begin{pmatrix} 0 \\ xB \end{pmatrix}$ and $\phi = 0$.

The Schrödinger equation reads

$$i\hbar \partial_t \psi = \frac{1}{2m}\left[\hat{p}_x^2 + (\hat{p}_y - eBx)^2\right]\psi, \qquad (1.33)$$

where $\hat{p}_x = -i\hbar\partial_x$ and $\hat{p}_y = -i\hbar\partial_y$.

In Landau gauge, the Hamiltonian manifests translational symmetry along y-direction but NOT in x-direction. We therefore use following ansatz

$$\psi(x,y,t) = e^{iky} f_k(x)\, e^{-i\frac{E}{\hbar}t}$$

Eq. (1.33) then becomes

$$E f_k = \frac{1}{2m}\left[-\hbar^2\partial_x^2 + (\hbar k - eBx)^2\right] f_k$$

$$= -\frac{\hbar^2}{2m}\partial_x^2 f_k + \frac{e^2B^2}{2m}\left(x - \frac{\hbar k}{eB}\right)^2 f_k$$

$$= -\frac{\hbar^2}{2m}\partial_x^2 f_k + \frac{1}{2}m\omega_B^2(x - k l_B^2)^2 f_k \,, \quad (1.34)$$

where $\omega_B = \frac{eB}{m}$ (cyclotron frequency)

$l_B = \sqrt{\frac{\hbar}{eB}}$ (magnetic length).

Compare eq. (1.34) and eq. (1-1), the present system is a shifted harmonic oscillator from origin $x=0$ to $x = k l_B^2$. One can therefore write down its solution

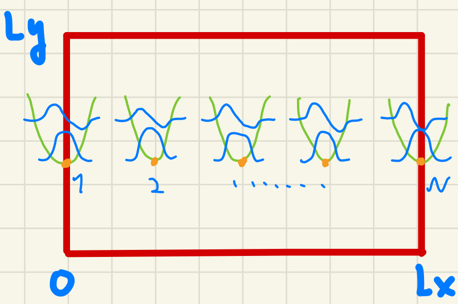
$$\psi_n(x,y,t) = f_n(x)\, e^{iky - i\frac{E_n}{\hbar}t}$$

$$f_n(x) = \left(\frac{m\omega_B}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\left[\sqrt{\frac{m\omega_B}{\hbar}}(x - k l_B^2)\right] e^{-\frac{m\omega_B}{2\hbar}(x - k l_B^2)^2}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_B$$

$\Psi_n(x, y, t)$ is the wavefunction of Landau levels lebeled by quantum number n. Remarkably, Landau levels are independent on y-directional wave number k, thus have infinite degeneracy!! (Particle of various k has the same E_n)

For a sample of finite size, one can calculate the number of states occupy E_n states



L_x = Maximum $k l_B^2$

Number of state $\rightarrow k_{max} = \dfrac{L_x}{l_B^2}$

$$\mathcal{N} = \frac{1}{\left(\frac{2\pi}{L_y}\right)} \int_0^{\frac{L_x}{l_B^2}} dk = \frac{L_x L_y}{2\pi l_B^2} = \frac{L_x L_y B}{\left(\frac{2\pi\hbar}{e}\right)} = \frac{AB}{\Phi_0},$$

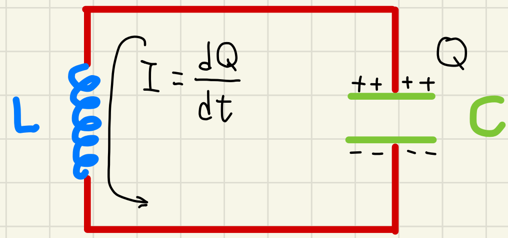
where AB is the magnetic flux,

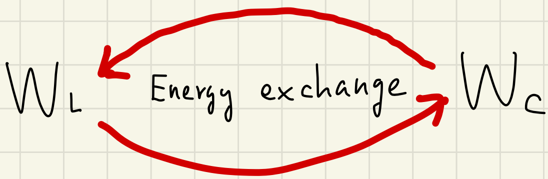
and $\Phi_0 = \dfrac{2\pi\hbar}{e}$ is the quantum of flux.

1-5 Quantization of L-C Circuit

$$V = \dot{\Phi} = \frac{Q}{C}$$

$$\Phi = LI$$





The energy stored in a L-C circuit is

1. $W_C = \frac{1}{2} C V^2 = \frac{1}{2} C \left(\frac{d\Phi}{dt}\right)^2 = \frac{1}{2} C \dot{\Phi}^2$, (capacitor)

2. $W_L = \frac{1}{2} L I^2 = \frac{1}{2L} \Phi^2$, (inductance)

where magnetic flux $\Phi = LI$.

Compare above forms of energy with harmonic oscillator:

$\Phi \sim x$ $\qquad$ $W_L \propto \Phi^2 \sim x^2$ (potential)

$\frac{d\Phi}{dt} \sim \frac{dx}{dt} \sim v$ $\qquad$ $W_C \propto \dot{\Phi}^2 \sim v^2$ (kinetic)

The Lagrangian is

$$\mathcal{L} = T - U = W_c - W_L$$

$$\rightarrow \quad \boxed{\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2 .} \qquad (1.35)$$

The canonical momentum is

$$\mathbb{P} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C \dot{\Phi} = Q$$

The Hamiltonian then reads

$$\mathcal{H} = \mathbb{P} V - \mathcal{L}$$

$$\rightarrow \quad \boxed{\mathcal{H} = \frac{1}{2} C \dot{\Phi}^2 + \frac{1}{2L} \Phi^2} \qquad (1.36)$$

The classical equation of motion is

$$\dot{\mathbb{P}} = C \ddot{\Phi} = - \frac{\partial \mathcal{H}}{\partial \Phi} = - \frac{\Phi}{L}$$

$$\rightarrow \quad \ddot{I} + \frac{1}{LC} I = 0 \quad \rightarrow \quad \omega = \frac{1}{\sqrt{LC}}$$

To get the quantum version, we use

$$\left.\begin{aligned} C\dot{\Phi} = \hat{Q} &\rightarrow \hat{p} \\ \hat{\Phi} &\rightarrow \hat{x} \end{aligned}\right\} \quad \boxed{\hat{Q} = -i\hbar \frac{\partial}{\partial \Phi}}$$

$$\hat{\mathcal{H}} = \frac{\hat{Q}^2}{2C} + \frac{1}{2}\frac{1}{L}\hat{\Phi}^2 \qquad (1.37)$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$\rightarrow \quad m = C, \qquad \omega = \frac{1}{\sqrt{LC}}$$

Also,

$$\hat{\mathcal{H}} = \left(i\frac{\hat{Q}}{\sqrt{2C}} + \frac{\hat{\Phi}}{\sqrt{2L}}\right)\left(-i\frac{\hat{Q}}{\sqrt{2C}} + \frac{\hat{\Phi}}{\sqrt{2L}}\right) + \frac{i}{2\sqrt{LC}}\underbrace{[\hat{\Phi}, \hat{Q}]}_{=[\hat{x},\hat{p}] = i\hbar} \qquad (1.38)$$

$$\rightarrow \quad \hat{\mathcal{H}} = \hat{a}\hat{a}^{\dagger} - \frac{1}{2}\hbar\omega = \left(\hat{n} + \frac{1}{2}\right)\hbar\omega$$

$$\hat{a}^{\dagger} = -i\sqrt{\frac{1}{2m\hbar\omega}}\,\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} = -i\sqrt{\frac{\sqrt{LC}}{2C\hbar}}\,\hat{Q} + \sqrt{\frac{C}{2\hbar\sqrt{LC}}}\,\hat{\Phi},$$

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega}}\,\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} = i\sqrt{\frac{\sqrt{LC}}{2C\hbar}}\,\hat{Q} + \sqrt{\frac{C}{2\hbar\sqrt{LC}}}\,\hat{\Phi}.$$

One can express $\hat{Q}$ and $\hat{\Phi}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, i.e.,

$$\hat{\Phi} = \sqrt{\frac{\hbar Z}{2}} \left(\hat{a}^\dagger + \hat{a} \right),$$

$$\hat{Q} = i\sqrt{\frac{\hbar}{2Z}} \left(\hat{a}^\dagger - \hat{a} \right),$$

where $Z = \sqrt{\frac{L}{C}}$ is the impedance of the circuit. One can then solve $\hat{a}|0\rangle = 0$ to get the ground state wavefunction, namely,

$$\left(\hbar \sqrt{\frac{\sqrt{LC}}{2C\hbar}} \frac{\partial}{\partial \Phi} + \sqrt{\frac{C}{2\hbar\sqrt{LC}}}\, \Phi \right) \psi_0 = 0 .$$

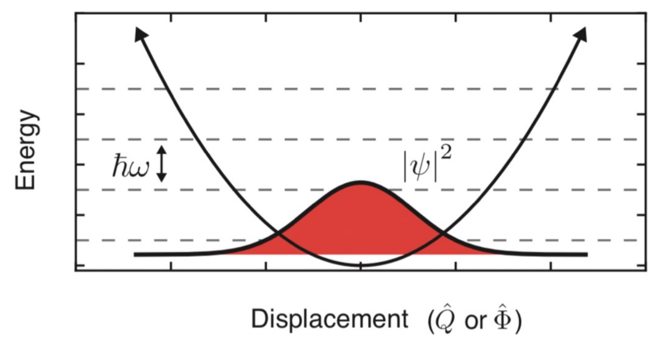


Figure 1.2: Energy levels of the quantum harmonic oscillator. The quadratic potential yields evenly spaced energy eigenstates ($\Delta E = \hbar\omega$). The ground state of the system is Gaussian distributed in the conjugate variables of motion, e.g. charge $\hat{Q}$ and flux $\hat{\Phi}$. Note that the circuit has finite probability $|\psi|^2$ of being detected at a nonzero value of $\hat{Q}$ or $\hat{\Phi}$ for the ground state. This phenomenon is known as zero-point fluctuations of the circuit and leads to a number of important consequences as we see in this chapter.

Check out this nice paper: Science 209, 547 (1980).

1-6 Quantization of Electromagnetic Fields

Let's start with electromagnetic Hamiltonian

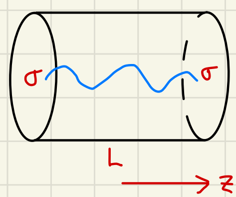
$$\mathcal{H} = \int \left(\frac{1}{2} \epsilon_0 |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \right) dv$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

$$\rightarrow \mathcal{H} = \frac{1}{2} \int \Big(\underbrace{\epsilon_0 \left| -\frac{\partial \vec{A}}{\partial t} \right|^2}_{①} + \underbrace{\frac{1}{\mu_0} |\nabla \times \vec{A}|^2}_{②} \Big) dv \qquad (1,6,1)$$

$V = \sigma L$

We replace $\vec{A}(z,t)$ with its inverse Fourier transformation



$$\vec{A}(z,t) = \frac{1}{\sqrt{V}} \int_{-\infty}^{\infty} \vec{a}(k)\, e^{i(kz - wt)}\, dk$$

① $$\left| -\frac{\partial \vec{A}}{\partial t} \right|^2 = \frac{1}{V} \int w\, \vec{a}(k)\, e^{i(kz-wt)}\, dk \cdot \int w'\, \vec{a}^*(k')\, e^{-i(k'z - w't)}\, dk'$$

② $$|\nabla \times \vec{A}|^2 = \mathcal{E}_{ijk}\, \partial_j A_k\, \mathcal{E}_{ilm}\, \partial_l A_m^*$$

$$= \mathcal{E}_{ijk}\, \mathcal{E}_{ilm}\, (\partial_j A_k)(\partial_l A_m^*)$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(\partial_j A_k)(\partial_l A_m^*)$$

$$= (\partial_l A_m)(\partial_l A_m^*) - (\partial_m A_l)(\partial_l A_m^*)$$

$$= |\partial_z A_x|^2 + |\partial_z A_y|^2$$

Coulomb gauge
$(\nabla \cdot \vec{A} = 0,\ \vec{k} = k\hat{z})$

$$= \frac{1}{V} \int k\, \vec{a}(k)\, e^{i(kz-wt)}\, dk \cdot \int k'\, \vec{a}^*(k')\, e^{-i(k'z-w't)}\, dk'$$

$$\mathcal{H} = \frac{1}{2}\iint\left[\epsilon_0\left|-\frac{\partial\vec{A}}{\partial t}\right|^2 + \frac{1}{\mu_0}\left|\nabla\times\vec{A}\right|^2\right]dV$$

Parseval's theorem, $w = ck$, $\mu_0\epsilon_0 = \frac{1}{c^2}$.

$$= \epsilon_0\int_{-\infty}^{\infty} w^2\,\vec{a}^*(k)\cdot\vec{a}(k)\,dk. \qquad (1,6,2)$$

(Appendix 2)

$$= \sum_{j=1}^{2}\int_0^{\infty}\left\{\frac{1}{2\epsilon_0}\left|\underbrace{\frac{i}{\sqrt{2}}\epsilon_0 w\left[a_j(k) - a_j^*(k)\right]}_{\mathbb{P}_j(k)}\right|^2 + \frac{k^2}{2\mu_0}\left|\underbrace{\frac{1}{\sqrt{2}}\left[a_j(k) + a_j^*(k)\right]}_{\mathbb{X}_j(k)}\right|^2\right\}dk \qquad (1,6,3)$$

polarizations A_x, A_y

(Appendix 3)

Equation (1,6,2) shows that

all EM modes of $\vec{k}$ and polarization are decoupled!!

We define generalized position and momentum

for j-th polarization and k mode

$$\mathbb{X}_j(k) = \frac{1}{\sqrt{2}}\left[a_j(k) + a_j^*(k)\right],$$

$$\mathbb{P}_j(k) = \frac{i}{\sqrt{2}}\epsilon_0 w\left[a_j(k) - a_j^*(k)\right],$$

$$\mathcal{H} = \sum_{j=1}^{2}\int_0^{\infty}\left(\frac{1}{2\epsilon_0}\left|\mathbb{P}_j(k)\right|^2 + \frac{k^2}{2\mu_0}\left|\mathbb{X}_j(k)\right|^2\right)dk$$

$$= \sum_{j=1}^{2}\int_0^{\infty}\left(\frac{1}{2\epsilon_0}\left|\mathbb{P}_j(k)\right|^2 + \frac{1}{2}\epsilon_0 w^2\left|\mathbb{X}_j(k)\right|^2\right)dk \qquad (1,6,4)$$

To quantize Hamiltonian (1.6.4), one can use

$$\boxed{\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}^{+} + \hat{a}) \qquad (1\text{-}14) \\ \hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^{+} - \hat{a}) \qquad (1\text{-}15) \end{aligned}}$$

We replace $m \to \epsilon_0$, and get

$$\boxed{\begin{aligned} \hat{x}_j(k) &= \sqrt{\frac{\hbar}{2\epsilon_0\omega}}\left[\hat{a}_j^{+}(k) + \hat{a}_j(k)\right], \qquad (1.6.5) \\ \hat{p}_j(k) &= i\sqrt{\frac{\epsilon_0\hbar\omega}{2}}\left[\hat{a}_j^{+}(k) - \hat{a}_j(k)\right], \qquad (1.6.6) \\ E_n(\omega) &= \left[\hat{a}_j^{+}(k)\hat{a}_j(k) + \frac{1}{2}\right]\hbar\omega . \qquad (1.6.7) \end{aligned}}$$

Here $[\hat{a}_j(k), \hat{a}_l^{+}(q)] = \delta_{jl}\delta_{kq}$,

$$[\hat{a}_j^{+}(k), \hat{a}_l^{+}(q)] = 0,$$

$$[\hat{a}_j(k), \hat{a}_l(q)] = 0.$$

The electric field operator is $\hat{E} = -\frac{1}{\epsilon_0}\hat{P}$

$$\hat{E}(z,t) = i\sum_{j=1}^{2}\int_0^{\infty}\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left[\hat{a}_j(k)e^{i(kz-\omega t)} - \hat{a}_j^{+}(k)e^{-i(kz-\omega t)}\right]dk \qquad (1.6.8)$$

Great! In summary we arrive at two Hamiltonians:

① Charged particle in electromagnetic fields

$$\hat{\mathcal{H}} = \frac{[\frac{\hbar}{i}\nabla - e\overset{\text{vector potential}}{\vec{A}}]^2}{2m} + e\underset{\text{scalar potential}}{\phi} \qquad (1.28)$$

② Electromagnetic field in vacuum

$$\hat{\mathcal{H}}_F = \sum_{\underset{\text{polarization}}{j},\,\underset{\text{wavenumber}}{k}} \hbar\omega_k \left(\hat{a}^\dagger_{jk}\hat{a}_{jk} + \underbrace{\frac{1}{2}}_{\text{zero-point energy}}\right) \qquad (1.6.7)$$

which construct the framework of light-matter interaction in quantum optics.

Appendix 1 Check the EM generalized momentum from Lagrangian

Let's start with electromagnetic Lagrangian

$$\mathcal{L} = \int \left(\frac{1}{2} \epsilon_0 |\vec{E}|^2 - \frac{1}{2\mu_0} |\vec{B}|^2 \right) dV$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

$$\rightarrow \mathcal{L} = \frac{1}{2} \int \left(\epsilon_0 \left| -\frac{\partial \vec{A}}{\partial t} \right|^2 - \frac{1}{\mu_0} |\nabla \times \vec{A}|^2 \right) dv$$

We can then calculate generalized momentum

$$\mathbb{P}_j = \frac{\partial \mathcal{L}}{\partial \dot{A}_j(\vec{r}_0)} = \epsilon_0 \int \dot{A}_i(\vec{r}) \frac{\partial \dot{A}_i(\vec{r})}{\partial \dot{A}_j(\vec{r}_0)} \, dv$$

$$= \epsilon_0 \int \dot{A}_i(\vec{r}) \, \delta_{ij} \, \delta(\vec{r} - \vec{r}_0) \, dv$$

$$= \epsilon_0 \dot{A}_j(\vec{r}_0)$$

$$= -\epsilon_0 E_j(\vec{r}_0)$$

only at $\vec{r} = \vec{r}_0$ the defferential $\neq 0$

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \int \left(\epsilon_0 \left| -\frac{\partial \vec{A}}{\partial t} \right|^2 - \frac{1}{\mu_0} |\nabla \times \vec{A}|^2 \right) dv$$

$$= \frac{1}{2} \int \left(\frac{1}{\epsilon_0} |\vec{\mathbb{P}}|^2 - \frac{k^2}{\mu_0} |\vec{A}(k)|^2 \right) dV$$

Appendix 2 Derivation of equation (1.6.2)

In what follows, we show the detailed derivation.

$$\mathcal{H} = \frac{1}{2}\int \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right) dv \quad \text{(Hamiltonian)}$$

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{V}}\int_{-\infty}^{\infty} \vec{a}(\vec{k})\, e^{i(\vec{k}\cdot\vec{r}-wt)}\, d^3k \quad \left(\begin{array}{c}\text{vector}\\ \text{potential}\end{array}\right)$$

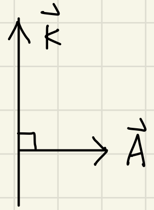
$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}}{\partial t} = \frac{-i}{\sqrt{V}}\int_{-\infty}^{\infty} w\, \vec{a}(\vec{k})\, e^{i(\vec{k}\cdot\vec{r}-wt)}\, d^3k$$

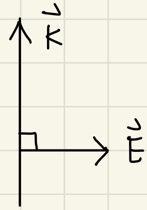
$$\vec{B}(\vec{r},t) = \nabla\times\vec{A} = \frac{i}{\sqrt{V}}\int_{-\infty}^{\infty} \vec{k}\times\vec{a}(\vec{k})\, e^{i(\vec{k}\cdot\vec{r}-wt)}\, d^3k$$

For simplicity we will work with Coulomb gauge $\nabla\cdot\vec{A}=0$ and $\nabla\cdot\vec{E}=0$ and use $\vec{k}=(0,0,k)$

$$\nabla\cdot\vec{A} \propto \vec{k}\cdot\vec{A}=0 \rightarrow \vec{A}=(A_x, A_y, 0)$$

$$\nabla\cdot\vec{E} \propto \vec{k}\cdot\vec{E}=0 \rightarrow \vec{E}=(E_x, E_y, 0)$$





$$\vec{A}(z,t) = \frac{1}{\sqrt{V}} \int_{-\infty}^{\infty} \vec{a}(k)\, e^{i(kz-wt)}\, dk$$

$$\vec{E}(z,t) = -\frac{\partial \vec{A}}{\partial t} = \frac{-iw}{\sqrt{V}} \int_{-\infty}^{\infty} \vec{a}(k)\, e^{i(kz-wt)}\, dk$$

$$\vec{B}(z,t) = \nabla \times \vec{A} = \frac{i}{\sqrt{V}} \int_{-\infty} k\, \hat{e}_z \times \vec{a}(k)\, e^{i(kz-wt)}\, dk$$

$$\mathcal{H} = \frac{\sigma}{2} \int_0^L \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2\right) dz$$

$L \int_0^L |\vec{E}|^2\, dz$ (the electric integration)

$$= \int_0^L \int_{-\infty}^{\infty} w\, \vec{a}(k)\, e^{i(kz-wt)}\, dk \cdot \int_{-\infty}^{\infty} w'\, \vec{a}^*(k')\, e^{-i(k'z-w't)}\, dk'\, dz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^L w w'\, \vec{a}(k) \cdot \vec{a}^*(k')\, e^{i(k-k')z - i(w-w')t}\, dz\, dk'\, dk$$

$$= L \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w w'\, \vec{a}(k) \cdot \vec{a}^*(k')\, \delta_{kk'}\, dk'\, dk$$

$$= L \int_{-\infty}^{\infty} w^2\, \vec{a}(k) \cdot \vec{a}^*(k)\, dk$$

$$L \int_0^L |\vec{B}|^2 \, dz \quad (\text{the magnetic integration})$$

$$= \int_0^L \int_{-\infty}^{\infty} k \hat{e}_z \times \vec{a}(k) e^{i(kz-wt)} dk \cdot \int_{-\infty}^{\infty} k' \hat{e}_z \times \vec{a}^*(k') e^{-i(k'z-wt)} dk'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^L k k' \varepsilon_{j3m} a_m(k) \varepsilon_{j3n} a_n^*(k') e^{i(k-k')z - i(w-w')t} dz\, dk'\, dk$$

due to $\nabla \cdot \vec{A} = 0$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^L k k' (\delta_{33} \delta_{mn} - \underbrace{\cancel{\delta_{3n}\delta_{3m}}}_{0}) a_m(k) a_n^*(k') \cdot e^{i(k-k')z - i(w-w')t} dz\, dk'\, dk$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^L k k' a_m(k) a_m^*(k') e^{i(k-k')z - i(w-w')t} dz\, dk'\, dk$$

$$= L \int_{-\infty}^{\infty} k^2 \vec{a}(k) \cdot \vec{a}^*(k) \, dk$$

The total energy (the Hamiltonian) reads

$$\mathcal{H} = \frac{1}{2}\int_0^L \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right) dz$$

$$= \frac{1}{2}\epsilon_0 \int_{-\infty}^{\infty} \omega^2\, \vec{a}^*(k)\cdot\vec{a}(k)\, dk + \frac{1}{2\mu_0}\int_{-\infty}^{\infty} k^2\, \vec{a}^*(k)\cdot\vec{a}(k)\, dk$$

$$= \epsilon_0 \int_{-\infty}^{\infty} \omega^2\, \vec{a}^*(k)\cdot\vec{a}(k)\, dk$$

$$\boxed{\mathcal{H} = \epsilon_0 \int_{-\infty}^{\infty} \omega^2\, \vec{a}^*(k)\cdot\vec{a}(k)\, dk} \qquad (1.6.2)$$

Appendix 3 Check equation (1.6.3)

$$\frac{1}{2\epsilon_0}|\mathbb{P}_j|^2 + \frac{k^2}{2\mu_0}|\mathbb{X}_j|^2 = -\frac{\epsilon_0}{4}\omega^2\left[a_j(k) - a_j^*(k)\right]\cdot\left[a_j(k) - a_j^*(k)\right]$$

$$+ \frac{\epsilon_0}{4}c^2k^2\left[a_j(k) + a_j^*(k)\right]\cdot\left[a_j(k) + a_j^*(k)\right]$$

$$= \epsilon_0\, \omega^2\, a_j^*(k)\cdot a_j(k) \qquad (1.6.3)$$

Appendix 4 Check equation (1.30)

$$i\hbar\partial_t\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)\psi + e\phi\psi$$

$$= \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)\cdot\left(\frac{\hbar}{i}\nabla\psi - e\psi\vec{A}\right) + e\phi\psi$$

$$= \frac{1}{2m}\Big[-\hbar^2\nabla^2\psi - \frac{\hbar e}{i}\left(\nabla\psi\cdot\vec{A} + \psi\,\nabla\cdot\vec{A}\right)$$

$$- \frac{\hbar e}{i}\vec{A}\cdot\nabla\psi + e^2\psi|\vec{A}|^2\Big]\psi + e\phi\psi$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2\psi + i\hbar e(\nabla\cdot\vec{A})\psi + 2i\hbar e\vec{A}\cdot\nabla\psi + e^2|\vec{A}|^2\psi\right]$$

$$+ e\phi\psi \qquad (1.30)$$

Appendix 5 Classical Poisson bracket

$$\mathcal{H} = \sum_j p_j\dot{x}_j - \mathcal{L}$$

$$\frac{dx_j}{dt} = \frac{\partial\mathcal{H}}{\partial p_j} = \sum_i\left(\frac{\partial x_j}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial x_j}{\partial p_i}\frac{dp_i}{dt}\right)$$

$$= \sum_i\left(\frac{\partial x_j}{\partial x_i}\frac{\partial\mathcal{H}}{\partial p_i} - \frac{\partial x_j}{\partial p_i}\frac{\partial\mathcal{H}}{\partial x_i}\right)$$

$$= \frac{\partial x_j}{\partial x_j}\frac{\partial\mathcal{H}}{\partial p_j} - \frac{\partial x_j}{\partial p_j}\frac{\partial\mathcal{H}}{\partial x_j}$$

$$= \{x_j, \mathcal{H}\}$$

$$\frac{dp_j}{dt} = -\frac{\partial\mathcal{H}}{\partial q_j} = \{p_j, \mathcal{H}\}$$