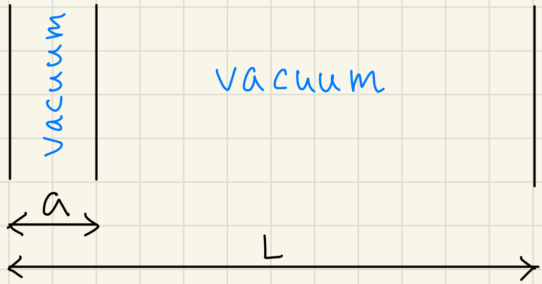
2-1 Casimir Effect (1948)

Hendrik Casimir
(1909-2000)



$$E_a(a) = \underset{\text{polarization}}{2} \cdot \frac{\hbar}{2} \sum_{lmn} W_{lmn} \quad \text{for only electromagnetic fields}$$

$$= \hbar \sum_{lmn} c \sqrt{\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{a}\right)^2}$$

$$= \hbar \sum_{n} \int_0^\infty \int_0^\infty c \; dl\, dm \sqrt{\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{a}\right)^2}$$

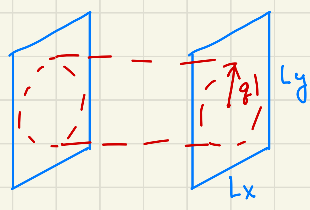
let $k_x = l\frac{\pi}{L_x}$, $k_y = m\frac{\pi}{L_y}$

$dk_x = \frac{\pi}{L_x} dl$, $dk_y = \frac{\pi}{L_y} dm$

$$\rightarrow E_a(a) = \frac{\hbar}{4} \frac{L_x L_y}{\pi^2} \sum_n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{a}\right)^2}\, dk_x\, dk_y$$

$$= A \frac{\hbar c}{2\pi} \sum_n \int_0^\infty \left[q^2 + \left(\frac{n\pi}{a}\right)^2\right]^{\frac{1}{2}} q\, dq \quad (\text{Cylindrical coordinate})$$

$$= A \frac{\hbar c}{6\pi} \sum_n \left[q^2 + \left(\frac{n\pi}{a}\right)^2\right]^{\frac{3}{2}} \Bigg|_0^\infty$$



$$\rightarrow E_b(L-a) = A \frac{\hbar c}{6\pi} \sum_n \left[q^2 + \left(\frac{n\pi}{L-a}\right)^2\right]^{\frac{3}{2}} \Bigg|_0^\infty$$

$k_x^2 + k_y^2 = q^2$

$$E_b(L-a) - E_a(a) = \Delta E(a)$$

$$= -A\frac{\hbar c}{6\pi}\sum_n \left[\left(\frac{n\pi}{L-a}\right)^3 - \left(\frac{n\pi}{a}\right)^3\right]$$

$$= A\frac{\hbar c}{6\pi}\sum_n \left(\frac{n\pi}{a}\right)^3\left[1 - \left(\frac{a}{L-a}\right)^3\right]$$

$$\lim_{\frac{a}{L}\to 0} \Delta E(a) = A\frac{\hbar c \pi^2}{6a^3}\sum_n n^3$$

$$= A\frac{\hbar c\pi^2}{6a^3}\zeta(-3)$$

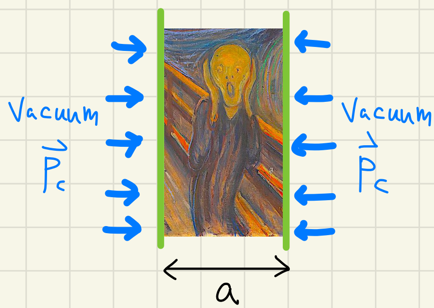
where Riemann Zeta function $\zeta(-3) = \frac{1}{120}$.

The Casimir force reads

$$\boxed{F_c(a) = \frac{d\Delta E}{da} = -A\frac{\hbar c\pi^2}{240a^4}.}$$

The pressure is

$$\boxed{P_c(a) = \frac{F_c}{A} = -\frac{\hbar c\pi^2}{240a^4}.}$$



$$|\vec{P}_c| \propto a^{-4}$$

2-2 Spontaneous Emission of an excited Atom in Vacuum

2-2-1 Dipole Approximation $\vec{r} \cdot \vec{E}$

When studying the interaction between an atom and electromagnetic waves, one has to solve the following equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{[-i\hbar \nabla - e\vec{A}]^2}{2m} \psi + V(\vec{r})\psi$$

For a plane wave the vector potential is

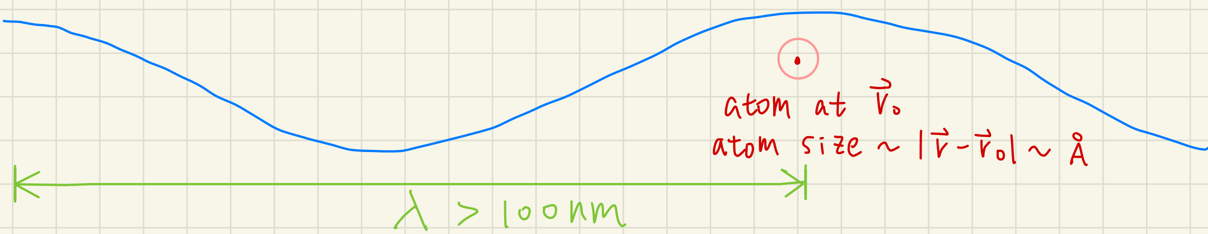
$$\vec{A}(\vec{r}, t) = \vec{A}(t)\, e^{i\vec{k}\cdot\vec{r}}$$

$$= \vec{A}(t)\, e^{i\vec{k}\cdot\vec{r}_0}\left[1 + i\vec{k}\cdot\underbrace{(\vec{r}-\vec{r}_0)}_{\text{size of an atom}} \cdots\right]$$

$2\pi \frac{|\vec{r}-\vec{r}_0|}{\lambda}$

When $\lambda \gg |\vec{r}-\vec{r}_0|$,

$$\vec{A}(\vec{r}, t) \cong \underbrace{\vec{A}(t)\, e^{i\vec{k}\cdot\vec{r}_0}}_{\text{atom only see the Time structure of an EM wave.}} = \vec{A}(\vec{r}_0, t)$$



The Hamiltonian becomes

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{[-i\hbar\nabla - e\vec{A}(\vec{r}_0, t)]^2}{2m}\Psi + V(\vec{r})\Psi \qquad (2.2.1)$$

Let $\Psi = f\,\Phi$ where $f(\vec{r})$ satisfy

$$[-i\hbar\nabla - e\vec{A}(\vec{r}_0, t)]\, f = 0 \rightarrow f(\vec{r}, t) = e^{i\frac{e}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}}$$

gauge transformation

$$\nabla f = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} e^{i\frac{e}{\hbar}(A_x x + A_y y + A_z z)} = i\frac{e}{\hbar} f \vec{A}$$

$$\rightarrow \quad \nabla f + \frac{e}{i\hbar} f\vec{A} = 0.$$

Substituting $\Psi = f\,\Phi$ into eq.(2.2.1)

$$\partial_t \Psi = i\frac{e}{\hbar}\left(\frac{\partial\vec{A}}{\partial t}\right)\cdot\vec{r}\, f\,\Phi + f\,\partial_t\Phi$$

$$-i\hbar\,\partial_j\Psi - e\vec{A}\Psi = e f\Phi\vec{A} - i\hbar f\partial_j\Phi - e f\Phi\vec{A} = -i\hbar f\nabla\Phi$$

$$|-i\hbar\partial_j - e\vec{A}|^2\Psi = (-i\hbar\partial_j - eA_j)(-i\hbar f\,\partial_j\Phi)$$

$$= -\hbar^2\left(i\frac{e}{\hbar} f A_j\partial_j\Phi + f\partial_j^2\Phi\right) + i\hbar e f A_j\partial_j\Phi$$

$$= -\hbar^2 f\nabla^2\Phi$$

Equation (2.2.1) becomes

$$i\hbar\,\partial_t\Phi = -\frac{\hbar^2}{2m}\nabla^2\Phi + V(\vec{r})\Phi + e\left[\vec{r}\cdot\partial_t\vec{A}(\vec{r}_0, t)\right]\Phi. \qquad (2.2.2)$$

By $\vec{E} = -\partial_t \vec{A}$, we get

$$\boxed{i\hbar \partial_t \Phi = \underbrace{-\frac{\hbar^2}{2m}\nabla^2 \Phi}_{\hat{\mathcal{H}}_A} + \underbrace{V(\vec{r})\Phi}_{\text{Coulomb potential}} \underbrace{- e\vec{r}}_{\text{dipole moment}} \cdot \vec{E}(\vec{r}_0, t)\Phi .} \quad (2.2.3)$$

Lamb pointed out how different eq. (2.2.3) from eq. (2.2.1) by what follows. When expanding eq. (2.2.1) the most important term is $-\frac{e}{m}\hat{p}\cdot\vec{A}$, one then has to compare $\hat{\mathcal{H}}_1 = -e\hat{r}\cdot\vec{E}$ and $\hat{\mathcal{H}}_2 = -\frac{e}{m}\hat{p}\cdot\vec{A}$.

One can note that

$$\hat{p} = m\hat{v} = m\frac{d\hat{r}}{dt} = \frac{m}{i\hbar}[\hat{r}, \hat{\mathcal{H}}_0] = \frac{m}{i\hbar}(\hat{r}\hat{\mathcal{H}}_0 - \hat{\mathcal{H}}_0\hat{r}),$$

and for $\vec{E} = \vec{\mathcal{E}}\cos(wt)$, $\vec{A} = -\frac{1}{w}\vec{\mathcal{E}}\sin(wt)$. We compute

$$\left|\frac{\langle f|\hat{\mathcal{H}}_2|i\rangle}{\langle f|\hat{\mathcal{H}}_1|i\rangle}\right| = \left|\frac{\frac{e}{i\hbar}\langle f|(\hat{r}\hat{\mathcal{H}}_0 - \hat{\mathcal{H}}_0\hat{r})|i\rangle \cdot \frac{1}{w}\vec{\mathcal{E}}\sin(wt)}{-e\langle i|\hat{r}|f\rangle\cdot\vec{\mathcal{E}}\cos(wt)}\right|$$

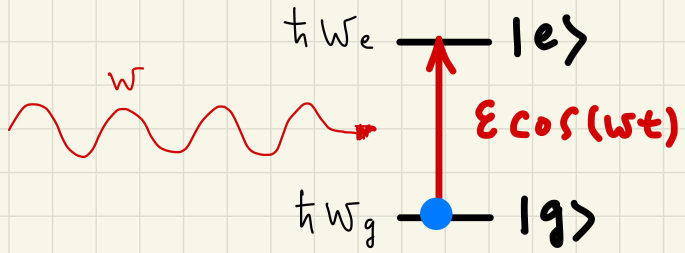
$\langle f|\hat{\mathcal{H}}_0 = \hbar w_f \langle f|$

$$= \left|\frac{\frac{e}{i}(w_i - w_f)\langle i|\hat{r}|f\rangle\cdot\frac{1}{w}\vec{\mathcal{E}}\sin(wt)}{-e\langle i|\hat{r}|f\rangle\cdot\vec{\mathcal{E}}\cos(wt)}\right| \approx \frac{w_i - w_f}{w}.$$

$\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ differ by the ratio of transition frequency over light frequency.

Equation (2.2.3) plays the important role for describing the light-matter interaction when the wavelength of light is much longer than the size of a single atom. This is the so-called dipole approximation or long wavelength approximation. Eq.(2.2.3) will be used in the following chapters.

2-2-2 Rotating Wave Approximation & Rabi Oscillation



In this section we are going to investigate the interaction between light and a two-level atom

$$|\psi\rangle = C_e(t)\, e^{-i\omega_e t}|e\rangle + C_g(t)\, e^{-i\omega_g t}|g\rangle = \begin{pmatrix} C_g(t)\, e^{-i\omega_g t} \\ C_e(t)\, e^{-i\omega_e t} \end{pmatrix}$$

by using Eq. (2.2.3) in perturbation region, namely,

$$\underbrace{\left|\langle\psi|\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)|\psi\rangle\right|}_{\langle\psi|\hat{H}_A|\psi\rangle} \gg \underbrace{\left|\langle\psi|\, e\hat{r}\cdot\vec{E}\, |\psi\rangle\right|}_{\langle\psi|\hat{H}_{AF}|\psi\rangle}$$

forming atomic structure   driving transition.

Moreover $\hat{H}_A|g\rangle = \hbar\omega_g|g\rangle$

$\hat{H}_A|e\rangle = \hbar\omega_e|e\rangle$, and $\underbrace{\vec{E} = \vec{\mathcal{E}}\cos(\omega t)}_{\text{classical field}}$

From Schrödinger equation to Matrix:

We first express $\hat{H}_A$ + $\hat{H}_{AF}$ in the basis of
① ②
$|e\rangle$ and $|g\rangle$ by

① $$(|g\rangle\langle g| + |e\rangle\langle e|)\hat{H}_A(|g\rangle\langle g| + |e\rangle\langle e|)$$

$$= |g\rangle\langle g|\hat{H}_A|g\rangle\langle g| + |e\rangle\langle e|\hat{H}_A|e\rangle\langle e|$$

$$= \hbar\omega_g|g\rangle\langle g| + \hbar\omega_e|e\rangle\langle e| .$$

Just like a spin 1/2 two-level system,

let
$$|g\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle g| = (1, 0)$$

$$|e\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle e| = (0, 1)$$

$$|g\rangle\langle g| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|e\rangle\langle e| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow \hat{H}_A = \hbar\begin{pmatrix} \omega_g & 0 \\ 0 & \omega_e \end{pmatrix}$$

② $$\left(|g\rangle\langle g| + |e\rangle\langle e|\right)\hat{H}_{AF}\left(|g\rangle\langle g| + |e\rangle\langle e|\right)$$

$$= -e\left(|g\rangle\langle g| + |e\rangle\langle e|\right)\hat{r}\left(|g\rangle\langle g| + |e\rangle\langle e|\right)\cdot\vec{E}$$

$$= -\left[e|g\rangle\langle g|\hat{r}|e\rangle\langle e| + e|e\rangle\langle e|\hat{r}|g\rangle\langle g|\right]\cdot\vec{E}$$

To calculate each matrix element we need state vector in space, e.g.,

$$e\langle g|\hat{r}|e\rangle = e\iint \langle g|\vec{r}\rangle\langle\vec{r}|\underbrace{\hat{r}|\vec{r}\,'\rangle}_{=\vec{r}\,'|\vec{r}\,'\rangle}\langle\vec{r}\,'|e\rangle\, d^3\vec{r}\, d^3\vec{r}\,'$$

$$= e\iint \psi_g^*(\vec{r})\,\vec{r}\,'\langle\vec{r}|\vec{r}\,'\rangle\,\psi_e(\vec{r}\,')\, d^3\vec{r}\, d^3\vec{r}\,'$$

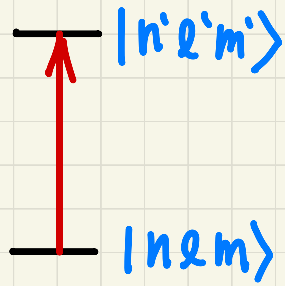
$$= e\iint \psi_g^*(\vec{r})\,\vec{r}\,'\,\delta(\vec{r}-\vec{r}\,')\,\psi_e(\vec{r}\,')\, d^3\vec{r}\, d^3\vec{r}\,'$$

$$= e\int \psi_g^*(\vec{r})\,\vec{r}\,\psi_e(\vec{r})\, d^3r = \vec{d}_{ge}$$

One can get each matrix element by either solving Schrödinger equation or experimental measurement.

For Hydrogen like particles, the dipole moment is proportional to $\langle n'\ell'm' | \hat{r} | n\ell m \rangle$ resulting in the so-called "SELECTION RULE".

$$\boxed{\begin{array}{l}\text{NO transitions occur unless} \\ \ell' - \ell = \Delta\ell = \pm 1, \text{ and} \\ m' - m = \Delta m = \pm 1 \text{ or } 0\end{array}}$$



This is due to the conservation of angular momentum.

Moreover
$$\langle g | \hat{r} | g \rangle = \int \psi_g^*(\vec{r})\, \vec{r}\, \psi_g(\vec{r})\, d^3r = 0,$$
$$\langle e | \hat{r} | e \rangle = \int \psi_e^*(\vec{r})\, \vec{r}\, \psi_e(\vec{r})\, d^3r = 0.$$
odd parity !!

$$\hat{H}_{AF} = -\left[\vec{d}_{ge} |g\rangle\langle e| + \vec{d}_{eg} |e\rangle\langle g| \right] \cdot \vec{E}$$
$$= \begin{pmatrix} 0 & -\vec{d}_{ge} \cdot \vec{E} \\ -\vec{d}_{eg} \cdot \vec{E} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\hbar\Omega_R \cos(\omega t) \\ -\hbar\Omega_R^* \cos(\omega t) & 0 \end{pmatrix},$$

where
$$\Omega_R = \frac{\vec{d}_{ge} \cdot \vec{\mathcal{E}}}{\hbar}, \quad \Omega_R^* = \frac{\vec{d}_{eg} \cdot \vec{\mathcal{E}}}{\hbar} \qquad (2,2,4)$$

is called Rabi frequency.

The Schrödinger equation $i\hbar\partial_t|\psi\rangle = (\hat{\mathcal{H}}_A + \hat{\mathcal{H}}_{AF})|\psi\rangle$ reads

$$i\hbar\partial_t\begin{pmatrix} C_g(t)\,e^{-i\omega_g t} \\ C_e(t)\,e^{-i\omega_e t}\end{pmatrix} = \hbar\begin{pmatrix} \omega_g & -\Omega_R\cos(\omega t) \\ -\Omega_R^*\cos(\omega t) & \omega_e\end{pmatrix}\begin{pmatrix} C_g(t)\,e^{-i\omega_g t} \\ C_e(t)\,e^{-i\omega_e t}\end{pmatrix}$$

$$\rightarrow\begin{pmatrix} i\dot{C}_g e^{-i\omega_g t} + \omega_g C_g e^{-i\omega_g t} \\ i\dot{C}_e e^{-i\omega_e t} + \omega_e C_e e^{-i\omega_e t}\end{pmatrix} = \begin{pmatrix} \omega_g C_g(t)\,e^{-i\omega_g t} - \Omega_R\cos(\omega t)C_e(t)\,e^{-i\omega_e t} \\ -\Omega_R^*\cos(\omega t)C_g(t)\,e^{-i\omega_g t} + \omega_e C_e(t)\,e^{-i\omega_e t}\end{pmatrix}$$

$$\rightarrow\begin{pmatrix} \dot{C}_g e^{-i\omega_g t} \\ \dot{C}_e e^{-i\omega_e t}\end{pmatrix} = \begin{pmatrix} i\Omega_R\cos(\omega t)C_e(t)\,e^{-i\omega_e t} \\ i\Omega_R^*\cos(\omega t)C_g(t)\,e^{-i\omega_g t}\end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix} \Omega_R C_e(t)\left(e^{i\omega t} + e^{-i\omega t}\right)e^{-i\omega_e t} \\ \Omega_R^* C_g(t)\left(e^{i\omega t} + e^{-i\omega t}\right)e^{-i\omega_g t}\end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix} \Omega_R C_e(t)\left[e^{i(\omega-\omega_e)t} + e^{-i(\omega+\omega_e)t}\right] \\ \Omega_R^* C_g(t)\left[e^{i(\omega-\omega_g)t} + e^{-i(\omega+\omega_g)t}\right]\end{pmatrix}$$

$$\rightarrow\begin{pmatrix} \dot{C}_g \\ \dot{C}_e\end{pmatrix} = \frac{i}{2}\begin{pmatrix} \Omega_R C_e\left\{e^{i[\omega-(\omega_e-\omega_g)]t} + e^{-i[\omega+(\omega_e-\omega_g)]t}\right\} \\ \Omega_R^* C_g\left\{e^{i[\omega+(\omega_e-\omega_g)]t} + e^{-i[\omega-(\omega_e-\omega_g)]t}\right\}\end{pmatrix}.$$

(fast oscillation
counter-rotating)

(slow oscillation
rotating)

The counter-rotating term is fast oscillation and averaged out in the time scale of interest, i.e.,

$$\begin{pmatrix} \dot{C}_g \\ \dot{C}_e \end{pmatrix} \cong \frac{i}{2} \begin{pmatrix} \Omega_R C_e e^{i\Delta t} \\ \Omega_R^* C_g e^{-i\Delta t} \end{pmatrix}, \qquad (2.2.5)$$

where $\Delta = \omega - (\omega_e - \omega_g)$ is called detuning.

In the last step we keep only rotating terms, which is so-called ROTATING WAVE approximation.

We now solve Eq.(2.2.5) with the initial condition

$$\begin{pmatrix} C_g(0) \\ C_e(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

namely, the atom is initially in the ground state $|g\rangle$.

First, let $\Omega_R^* = \Omega_R$ a constant, and then take the second derivative of C_e.

$$\ddot{C}_e = \frac{i}{2} \Omega_R \dot{C}_g e^{-i\Delta t} + \frac{1}{2} \Delta \Omega_R C_g e^{-i\Delta t}$$

Substitute $\begin{cases} \dot{C}_g = \frac{i}{2} \Omega_R C_e e^{i\Delta t} \\ C_g = \frac{2}{i\Omega_R} \dot{C}_e e^{i\Delta t} \end{cases}$ into above equation one gets $\ddot{C}_e + i\Delta \dot{C}_e + \frac{1}{4}\Omega_R^2 C_e = 0$. (2.2.6)

The trial solution $C_e = e^{iat}$ can be used to solve Eq. (2.2.6), and results in

$$a^2 + \Delta a - \frac{1}{4}\Omega_R^2 = 0$$

$$\rightarrow \quad a = \frac{1}{2}\left(-\Delta \pm \sqrt{\Delta^2 + \Omega_R^2}\right).$$

Let's define $\boxed{\Omega = \sqrt{\Delta^2 + \Omega_R^2}}$, and we get the solution

$$\begin{cases} C_e(t) = \left(a_1 e^{i\frac{\Omega}{2}t} + a_2 e^{-i\frac{\Omega}{2}t}\right) e^{-i\frac{\Delta}{2}t} \\ C_g(t) = \left(\frac{\Omega - \Delta}{\Omega_R} a_1 e^{i\frac{\Omega}{2}t} - \frac{\Omega + \Delta}{\Omega_R} a_2 e^{-i\frac{\Omega}{2}t}\right) e^{i\frac{\Delta}{2}t}. \end{cases}$$

Now, we need the initial condition to get a_1 and a_2, namely,

$$\begin{cases} C_e(0) = a_1 + a_2 = 0 \\ C_g(0) = \frac{\Omega - \Delta}{\Omega_R} a_1 - \frac{\Omega + \Delta}{\Omega_R} a_2 = 1 \end{cases}$$

$$\rightarrow \quad a_1 = \frac{\Omega_R}{2\Omega}, \quad a_2 = -\frac{\Omega_R}{2\Omega}$$

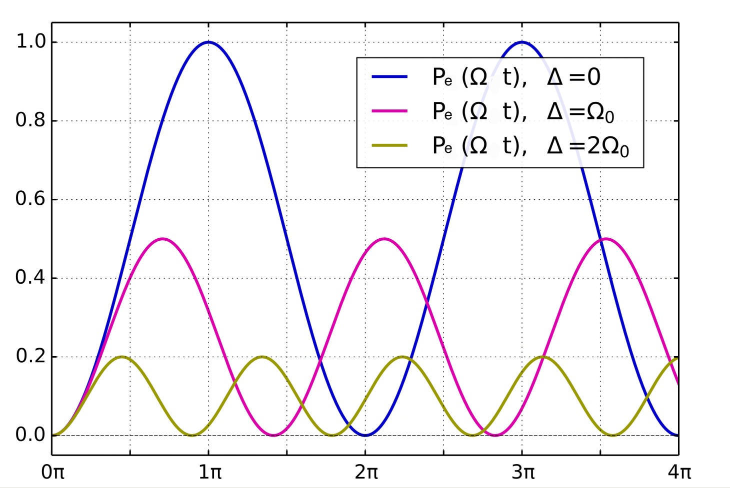
The solution reads

$$\begin{cases} C_e(t) = i\dfrac{\Omega_R}{\Omega}\sin\left(\dfrac{\Omega}{2}t\right)e^{-i\frac{\Delta}{2}t} \\[2ex] C_g(t) = \left[\cos\left(\dfrac{\Omega}{2}t\right) - i\dfrac{\Delta}{\Omega}\sin\left(\dfrac{\Omega}{2}t\right)\right]e^{i\frac{\Delta}{2}t}. \end{cases} \tag{2.2.7}$$

The populations are

$$\begin{cases} |C_e(t)|^2 = P_e(t) = \dfrac{\Omega_R^2}{\Omega^2}\sin^2\left(\dfrac{\Omega}{2}t\right) \\[2ex] |C_g(t)|^2 = P_g(t) = \dfrac{\Delta^2}{\Omega^2}\sin^2\left(\dfrac{\Omega}{2}t\right) + \cos^2\left(\dfrac{\Omega}{2}t\right). \end{cases} \tag{2.2.8}$$

$P_e(t)$

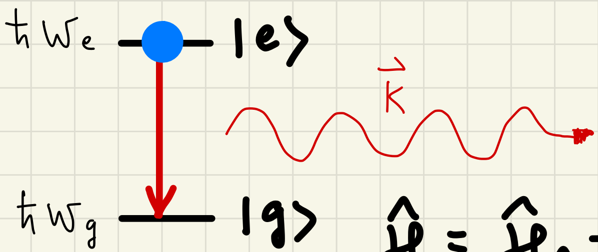


Ωt

Isidor Isaac Rabi
(1898-1988)

As a result, when a two-level atom interacts with a classical laser, its population oscillates between two states with the angular frequency Ω. This is the so-called Rabi oscillation. CHECK the AMPLITUDE!

2-2-3 Interaction Picture & Rotating Wave Approximation



$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_F + \hat{\mathcal{H}}_{AF}$$

In the previous section, we show how a classical laser interacts with a two-level atom and causes Rabi oscillation. In that case, one does NOT care about the annihilation and creation of a photon which is negligible, comparing to the classical laser energy. However, when considering an initially excited atom interacts with vacuum fields, the atomic eigenstate is NOT stationary anymore. Instead, an atom in an excited state will decay and emit photons such that the annihilation and creation of a photon make difference!

In previous sections, we have replaced

$\hat{\mathcal{H}}_A = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$ with its matrix form

$\hat{\mathcal{H}}_A = \begin{pmatrix} \hbar\omega_g & 0 \\ 0 & \hbar\omega_e \end{pmatrix}$. Here we need to add

$\hat{\mathcal{H}}_F = \sum_{\vec{k}} \hbar\omega_k (\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2})$ for the energy of the free fields, and use the electric field operator

$$\hat{E}(z,t) = i\sum_{\vec{k}} \hat{e}_{\vec{k}} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \left(\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_0} - \hat{a}^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_0}\right)$$

rather than the classical electric field $\mathcal{E}\cos(\omega_k t)$.

The atom-field operator becomes

$$\begin{aligned} \hat{\mathcal{H}}_{AF} &= -e\left(|g\rangle\langle g| + |e\rangle\langle e|\right)\hat{r}\left(|g\rangle\langle g| + |e\rangle\langle e|\right)\cdot\hat{E} \\ &= -\left(e|g\rangle\langle g|\hat{r}|e\rangle\langle e| + e|e\rangle\langle e|\hat{r}|g\rangle\langle g|\right)\cdot\hat{E} \\ &= -\hbar\sum_{\vec{k}}\left(g^{\vec{k}}_{ge}|g\rangle\langle e| + g^{\vec{k}}_{eg}|e\rangle\langle g|\right)\left(\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_0} - \hat{a}^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_0}\right) \\ &= -\hbar\sum_{\vec{k}}\begin{pmatrix} 0 & i g^{\vec{k}}_{ge} \\ i g^{\vec{k}}_{eg} & 0 \end{pmatrix}\left(\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_0} - \hat{a}^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_0}\right), \end{aligned}$$

where $g^{\vec{k}}_{ge} = \vec{d}_{ge}\cdot\hat{e}_{\vec{k}}\sqrt{\frac{\omega_k}{2\hbar\epsilon_0 V}}$, $g^{\vec{k}}_{eg} = \vec{d}_{eg}\cdot\hat{e}_{\vec{k}}\sqrt{\frac{\omega_k}{2\hbar\epsilon_0 V}}$, The Hamiltonian reads

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_F + \hat{\mathcal{H}}_{AF} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{AF}$$

$$= \underbrace{\begin{pmatrix} \hbar\omega_g & 0 \\ 0 & \hbar\omega_e \end{pmatrix} + \sum_{\vec{k}} \hbar\omega_k \left(\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2} \right)}_{\hat{\mathcal{H}}_0}$$

$$- \hbar \sum_{\vec{k}} \begin{pmatrix} 0 & i g^{\vec{k}}_{ge} \\ i g^{\vec{k}}_{eg} & 0 \end{pmatrix} \left(\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_0} - \hat{a}^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_0} \right). \quad (2.2.9)$$

Because all interesting dynamics result from $\hat{\mathcal{H}}_{AF}$, we use unitary transformation (see Appendix 2-1) $\hat{U} = e^{\frac{i}{\hbar}\hat{\mathcal{H}}_0 t}$ to simplify $\hat{\mathcal{H}}$, i.e.,

$$\hat{\mathcal{H}}_i = \hat{U} \hat{\mathcal{H}} \hat{U}^\dagger + i\hbar (\partial_t \hat{U}) \hat{U}^\dagger$$

$$= e^{\frac{i}{\hbar}\hat{\mathcal{H}}_0 t} (\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{AF}) e^{-\frac{i}{\hbar}\hat{\mathcal{H}}_0 t} + i\hbar \frac{i}{\hbar} \hat{\mathcal{H}}_0 e^{\frac{i}{\hbar}\hat{\mathcal{H}}_0 t} e^{-\frac{i}{\hbar}\hat{\mathcal{H}}_0 t}$$

$$= e^{\frac{i}{\hbar}\hat{\mathcal{H}}_0 t} \hat{\mathcal{H}}_{AF} e^{-\frac{i}{\hbar}\hat{\mathcal{H}}_0 t} \quad \text{since } [\hat{\mathcal{H}}_0, e^{\frac{i}{\hbar}\hat{\mathcal{H}}_0 t}] = 0.$$

This is so-called INTERACTION PICTURE where one can investigate dynamics in more convenient way than the Schrödinger's picture. However, one has to figure out how $\hat{\mathcal{H}}_i$ looks like.

$$\hat{\mathcal{H}}_i = \sum_{\vec{k}} \begin{pmatrix} e^{i\omega_g t} & 0 \\ 0 & e^{i\omega_e t} \end{pmatrix} (-i\hbar) \begin{pmatrix} 0 & g^{\vec{k}}_{ge} \\ g^{\vec{k}}_{eg} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_g t} & 0 \\ 0 & e^{-i\omega_e t} \end{pmatrix}$$

$$\cdot\, e^{i\omega_k \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} t} \left(\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_0} - \hat{a}^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_0} \right) e^{-i\omega_k \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} t}$$

$$= -i\hbar \sum_{\vec{k}} \begin{pmatrix} 0 & g^{\vec{k}}_{ge} e^{-i(\omega_e - \omega_g)t} \\ g^{\vec{k}}_{eg} e^{i(\omega_e - \omega_g)t} & 0 \end{pmatrix}$$

$$\cdot\, e^{i\omega_k \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} t} \left(\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_0} - \hat{a}^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_0} \right) e^{-i\omega_k \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} t}$$

$$\boxed{e^{i\omega_k \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} t} = \sum_m \frac{(i\omega_k t)^m}{m!} (\underbrace{\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}}}_{\hat{n}_{\vec{k}}})^m = \sum_m \frac{(i\omega_k t)^m}{m!} \hat{n}^m_{\vec{k}}}$$

$$e^{i\omega_k \hat{n}_{\vec{k}} t} \hat{a}_{\vec{k}} e^{-i\omega_k \hat{n}_{\vec{k}} t} |n\rangle_{\vec{k}} = \sum_m \frac{(i\omega_k t)^m}{m!} \hat{n}^m_{\vec{k}} \hat{a}_{\vec{k}} \sum_j \frac{(-i\omega_k t)^j}{j!} \hat{n}^j_{\vec{k}} |n\rangle_{\vec{k}}$$

$$= \sum_m \frac{(i\omega_k t)^m}{m!} (n-1)^m \sqrt{n} \sum_j \frac{(-i\omega_k t)^j}{j!} n^j |n-1\rangle_{\vec{k}}$$

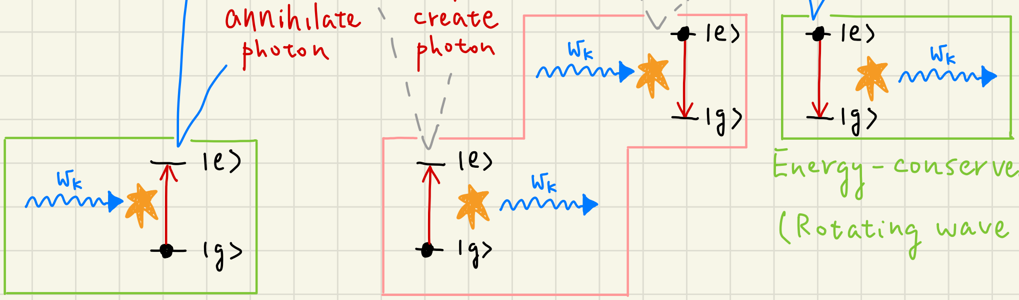
$$= e^{i(n-1)\omega_k t} e^{-in\omega_k t} \sqrt{n}\, |n-1\rangle$$

$$\rightarrow \boxed{\begin{aligned} e^{i\omega_k \hat{n}_{\vec{k}} t} \hat{a}_{\vec{k}} e^{-i\omega_k \hat{n}_{\vec{k}} t} &= e^{-i\omega_k t} \hat{a}_{\vec{k}}, \\ e^{i\omega_k \hat{n}_{\vec{k}} t} \hat{a}^\dagger_{\vec{k}} e^{-i\omega_k \hat{n}_{\vec{k}} t} &= e^{i\omega_k t} \hat{a}^\dagger_{\vec{k}}. \end{aligned}} \qquad (2.2.10)$$

We arrive at the Hamiltonian in the interaction picture with $\vec{r}_0 = 0$ for simplificity :

$$\hat{\mathcal{H}}_i = -i\hbar \sum_{\vec{k}} \begin{pmatrix} 0 & g_{ge}^{\vec{k}} e^{-i\omega t} \\ g_{eg}^{\vec{k}} e^{i\omega t} & 0 \end{pmatrix} \left(\hat{a}_{\vec{k}} e^{-i\omega_k t} - \hat{a}_{\vec{k}}^{\dagger} e^{i\omega_k t} \right)$$

$$= -i\hbar \sum_{\vec{k}} \begin{pmatrix} 0 & \underbrace{g_{ge}^{\vec{k}}}_{\text{deexcite atom}} \big(\underbrace{\hat{a}_{\vec{k}}}_{\text{annihilate photon}} e^{-i\sigma t} - \underbrace{\hat{a}_{\vec{k}}^{\dagger}}_{\text{create photon}} e^{i\delta t} \big) \\ \underbrace{g_{eg}^{\vec{k}}}_{\text{excite atom}} \big(\underbrace{\hat{a}_{\vec{k}}}_{\text{annihilate photon}} e^{-i\delta t} - \underbrace{\hat{a}_{\vec{k}}^{\dagger}}_{\text{create photon}} e^{i\sigma t} \big) & 0 \end{pmatrix}.$$



Energy-conserved. (Rotating wave)

Each process is NOT energy-conserved, and so must occur together (counter-Rotating wave, virtual process).

Energy-conserved. (Rotating wave)

CHECK :

$$|g\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |e\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

this term de-excite atom

$$\begin{pmatrix} 0 & 0 \\ \underbrace{\Omega_{eg}}_{\text{this term excites an atom!}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Omega_{eg} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \Omega_{ge} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \Omega_{ge} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Neglect two energy-non-conservation terms, namely, rotating wave approximation, one gets

$$\boxed{\hat{\mathcal{H}}_i = \hbar \sum_{\vec{k}} \begin{pmatrix} 0 & \Omega_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}} e^{i\delta t} \\ \Omega^{*}_{\vec{k}} \hat{a}_{\vec{k}} e^{-i\delta t} & 0 \end{pmatrix}} \quad (2.2.11)$$

where $\Omega_{\vec{k}} = i e^{-i\vec{k}\cdot\vec{r}_0} \vec{d}_{ge} \cdot \hat{e}_{\vec{k}} \sqrt{\frac{\omega_k}{2\hbar\epsilon_0 V}}$,

and $\Omega^{*}_{\vec{k}} = -i e^{i\vec{k}\cdot\vec{r}_0} \vec{d}_{eg} \cdot \hat{e}_{\vec{k}} \sqrt{\frac{\omega_k}{2\hbar\epsilon_0 V}}$,

are single-photon Rabi frequency.

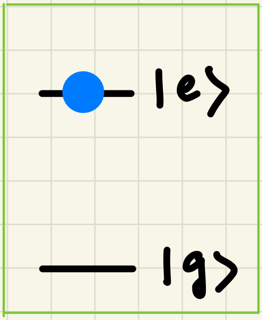
CHECK: A simple way to derive the electric field amplitude of a single photon.

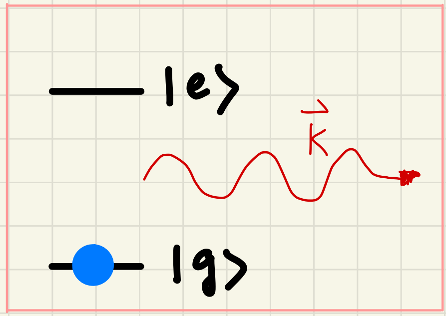
$$2\epsilon_0 |\vec{E}|^2 V = \hbar\omega_k \rightarrow |\vec{E}| = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}$$

$$\Omega_{\vec{k}} = \frac{\vec{d}\cdot\vec{E}}{\hbar} = \vec{d}\cdot\hat{e}_{\vec{k}} \sqrt{\frac{\omega_k}{2\hbar\epsilon_0 V}}$$

2-2-4 Weisskopf-Wigner Theory of spontaneous emission

We are ready to investigate spontaneous emission by using Eq.(2.2.11). Let's first write down the state $|\psi\rangle = C_e(t)|e,0\rangle + \sum_{\vec{k}} C_{g\vec{k}}(t)|g,1\rangle_{\vec{k}}$. (2.2.12)

$|\psi\rangle = C_e(t)$ 

$+\sum_{\vec{k}} C_{g\vec{k}}(t)$ 

with $C_e(0) = 1$, $C_{g\vec{k}}(0) = 0$. Note that the initial electric dipolement $\vec{d} \propto \langle e|\hat{r}|e\rangle = 0 \rightarrow$ No emission!

The spontaneous emisson is due to vacuum quantum fluctuation.

From the Schrödinger equation with $\delta = \omega_k - \omega$

$$i\hbar\partial_t \begin{pmatrix} \sum_{\vec{k}} C_{g\vec{k}}|1\rangle_{\vec{k}} \\ C_e|0\rangle \end{pmatrix} = \hbar \sum_{\vec{k}} \begin{pmatrix} 0 & \Omega_{\vec{k}}\, \hat{a}^\dagger_{\vec{k}}\, e^{i\delta t} \\ \Omega^*_{\vec{k}}\, \hat{a}_{\vec{k}}\, e^{-i\delta t} & 0 \end{pmatrix} \begin{pmatrix} \sum_{\vec{k}} C_{g\vec{k}}|1\rangle_{\vec{k}} \\ C_e|0\rangle \end{pmatrix}$$

$$\rightarrow \begin{cases} i\sum_{\vec{k}} \dot{C}_{g\vec{k}}|1\rangle_{\vec{k}} = \sum_{\vec{k}} \Omega_{\vec{k}}\, e^{i\delta t} C_e|1\rangle_{\vec{k}} \\ i\dot{C}_e|0\rangle = \sum_{\vec{k}} \Omega^*_{\vec{k}}\, e^{-i\delta t} C_{g\vec{k}}|0\rangle \end{cases}$$

$$\rightarrow \begin{cases} \dot{C}_{g\vec{k}} = -i\Omega_{\vec{k}} e^{i\delta t} C_e & (2.2.12) \\ \dot{C}_e = -i\sum_{\vec{k}} \Omega^*_{\vec{k}} e^{-i\delta t} C_{g\vec{k}} & (2.2.13) \end{cases}$$

To solve above equations, we first integrate eq.(2.2.12)

$$C_{g\vec{k}}(t) = -i\Omega_{\vec{k}} \int_0^t e^{i\delta\tau} C_e(\tau)\, d\tau , \qquad (2.2.14)$$

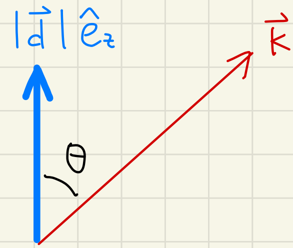
and substitute it into eq.(2.2.13)

$$\dot{C}_e(t) = -\sum_{\vec{k}} |\Omega_{\vec{k}}|^2 e^{-i\delta t} \int_0^t e^{i\delta\tau} C_e(\tau)\, d\tau . \qquad (2.2.15)$$

Let's take the continious limit of $\vec{k}$ in the spherical coordinate

$$\sum_{\vec{k}} \rightarrow \underbrace{2}_{\text{two polarizations}} \frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} k^2\, dk , \qquad (2.2.16)$$

$$|\Omega_{\vec{k}}|^2 = \frac{\omega_k}{2\hbar\epsilon_0 V} |\vec{d}|^2 \cos^2\theta .$$



With $\omega_k = ck$, eq. (2.2.15) becomes

$$\dot{C}_e(t) = -\frac{c|\vec{d}|^2}{4\pi^2\hbar\epsilon_0}\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta \int_0^{\infty} k^3 e^{-i\delta t}\int_0^t e^{i\delta\tau} C_e(\tau)\, d\tau\, dk$$

$$= -\frac{|\vec{d}|^2}{6\pi^2\hbar\epsilon_0 c^3}\int_0^{\infty}\omega_k^3\int_0^t e^{i(\omega_k-\omega)(\tau-t)} C_e(\tau)\, d\tau\, d\omega_k .$$

Here is the trick! Weisskopf and Wigner argue that in the emission spectra ω_k is centered about the atomic transition angular frequency ω and varies a little. They therefore approximated $\omega_k \sim \omega$ and get

$$\dot{C}_e(t) \cong -\frac{|\vec{d}|^2\omega^3}{6\pi^2\hbar\epsilon_0 c^3}\int_0^t C_e(\tau)\, \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} e^{i(\omega_k-\omega)(\tau-t)}\, d\omega_k}_{2\pi\delta(\tau-t)}\, d\tau$$

$$= -\frac{1}{2}\frac{|\vec{d}|^2\omega^3}{3\pi\hbar\epsilon_0 c^3}\int_0^t C_e(\tau)\,\delta(\tau-t)\, d\tau$$

$$= -\frac{\Gamma}{2}C_e(t), \qquad (2.2.17)$$

where

$$\boxed{\Gamma = \frac{|\vec{d}|^2\omega^3}{3\pi\hbar\epsilon_0 c^3}} \qquad (2.2.18)$$

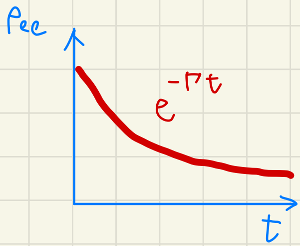
very important!!

is the spontaneous decay rate of a single atom.

The solution of eq. (2.2.17) reads

$$C_e(t) = e^{-\frac{\Gamma}{2}t}, \qquad (2.2.19)$$

$$|C_e(t)|^2 = e^{-\Gamma t}.$$



This shows that an excited atom in VACUUM decays exponentially in time with the lifetime

$$\boxed{\tau_e = \frac{1}{\Gamma}}. \qquad (2.2.20)$$

Eq. (2.2.14) reads

$$C_{g\vec{k}}(t) = -i\Omega_{\vec{k}} \int_0^t e^{i\delta\tau} e^{-\frac{\Gamma}{2}\tau} d\tau,$$

$$= \Omega_{\vec{k}} \frac{1 - e^{[i(\omega_k - \omega) - \frac{\Gamma}{2}]t}}{(\omega_k - \omega) + i\frac{\Gamma}{2}}.$$

The solution is

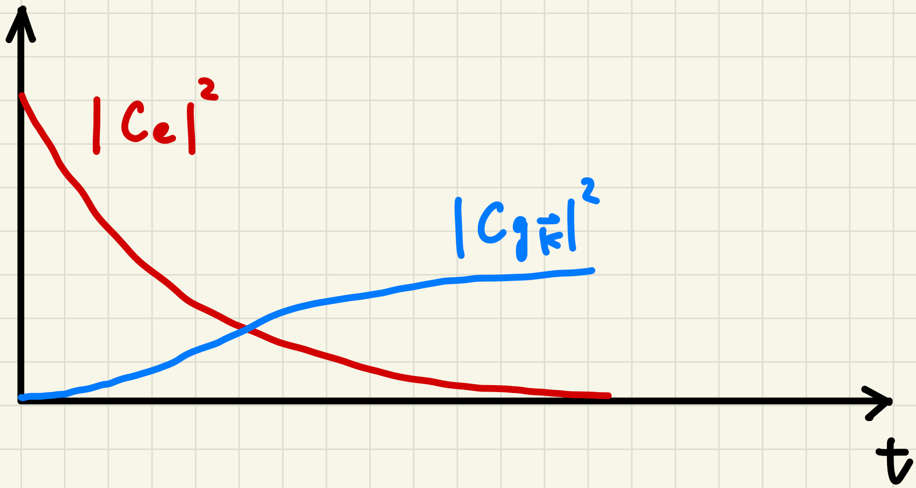
$$|\psi(t)\rangle = e^{-\frac{\Gamma}{2}t}|e,0\rangle + |g\rangle \sum_{\vec{k}} \Omega_{\vec{k}} \frac{1 - e^{[i(\omega_k - \omega) - \frac{\Gamma}{2}]t}}{(\omega_k - \omega) + i\frac{\Gamma}{2}} |1\rangle_{\vec{k}}.$$

(2.2.21)

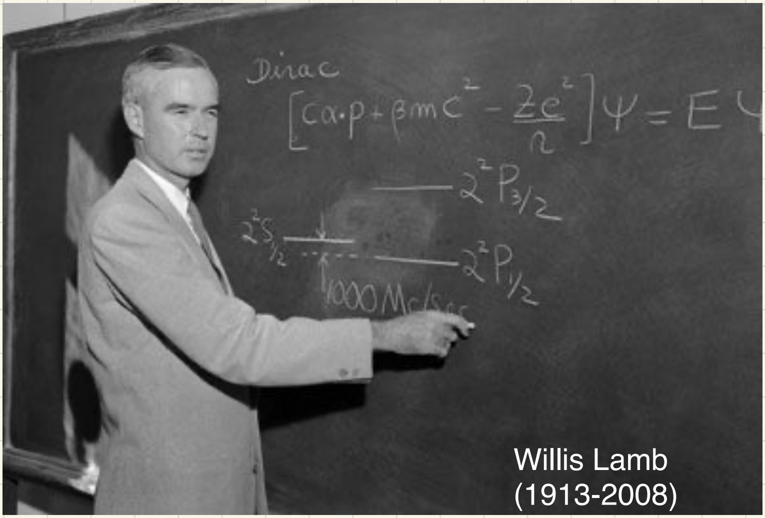
Let's calculate the probability

$$|C_e(t)|^2 = e^{-\Gamma t}$$

$$|C_{g,\vec{k}}(t)|^2 = |\Omega_{\vec{k}}|^2 \frac{1 - 2e^{-\frac{\Gamma}{2}t}\cos(\omega_k - \omega)t + e^{-\Gamma t}}{(\omega_k - \omega)^2 + \frac{\Gamma^2}{4}}$$



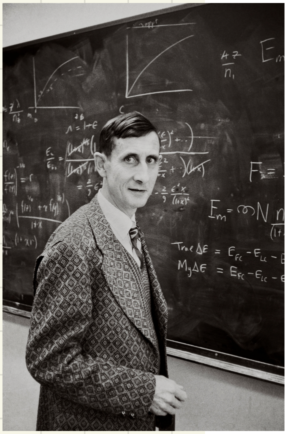
2-3 Lamb Shift (1955 Nobel Prize in Physics)



Willis Lamb
(1913-2008)

NEW
Spectral Line
↓
NEW
PHYSICS!

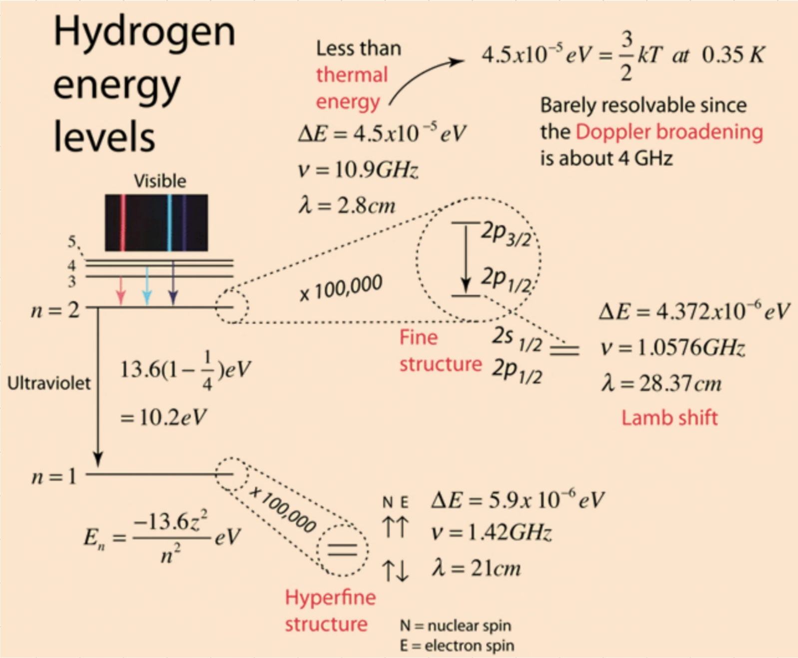
On the occasion of Lamb's sixty-fifth birthday, Freeman Dyson* wrote:


Freeman Dyson
(1923-2020)

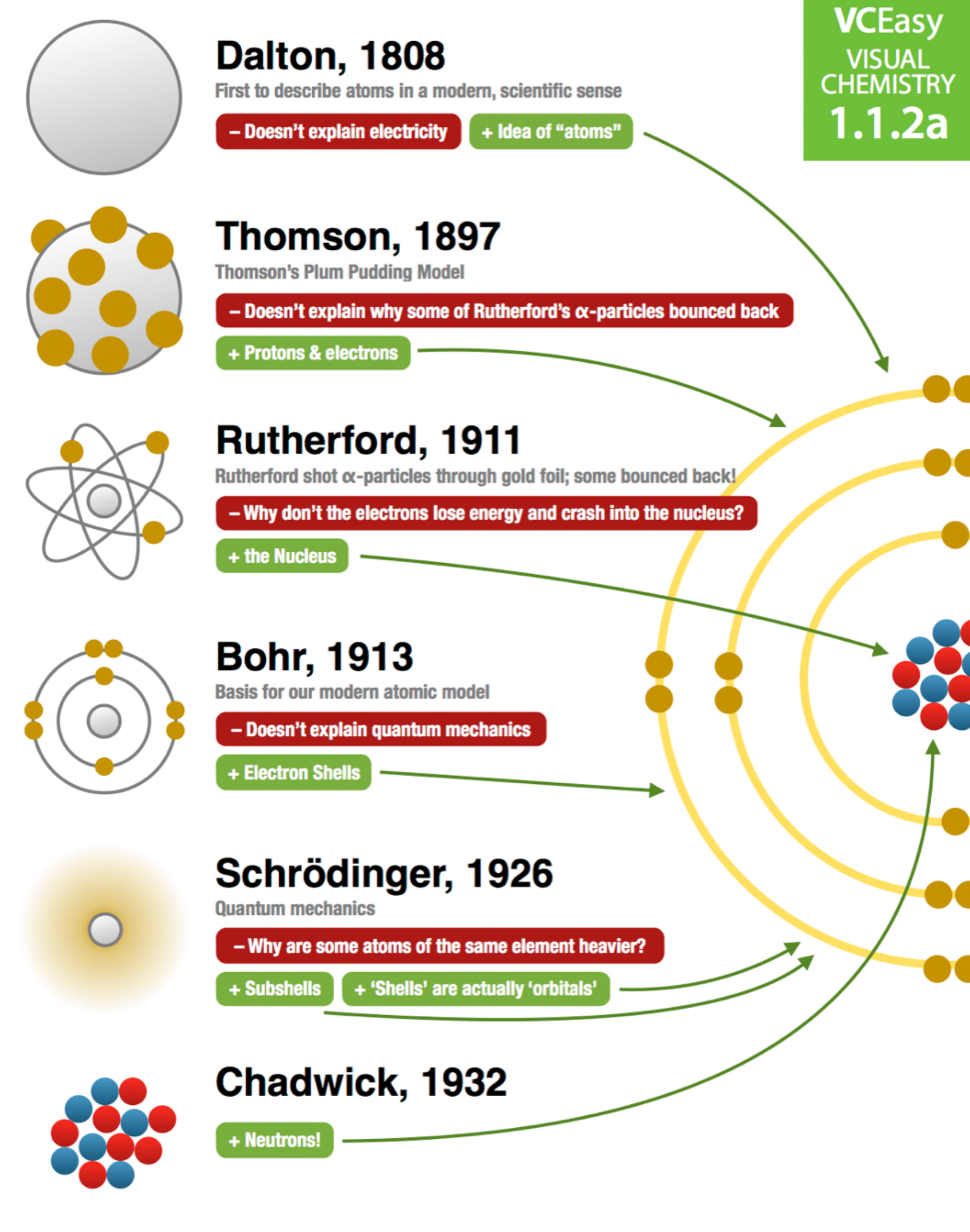
> Your work on the hydrogen fine structure led directly to the wave of progress in quantum electrodynamics on which I took a ride to fame and fortune. You did the hard, tedious, exploratory work. Once you had started the wave rolling, the ride for us theorists was easy. And after we had zoomed ashore with our fine, fancy formalisms, you still stayed with your stubborn experiment. For many years thereafter you were at work, carefully coaxing the hydrogen atom to give us the accurate numbers which provided the solid foundations for all our speculations...
>
> Those years, when the Lamb shift was the central theme of physics, were golden years for all the physicists of my generation. You were the first to see that that tiny shift, so elusive and hard to measure, would clarify in a fundamental way our thinking about particles and fields.

* Dyson [1978].

2-3-1 Hydrogen Energy Levels

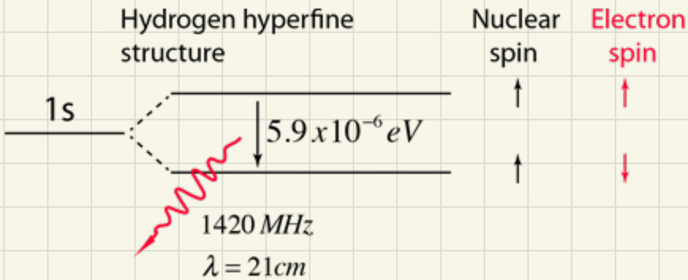


	Year	Special Relativity	Electron spin	Proton spin	Vacuum fluctuation
Bohr model	1913	No	No	No	No
Sommerfeld correction	1916	Yes	No	No	No
Schroedinger Equation	1926	No	No	No	No
Dirac Equation by W. Gordon	1928	Yes	Yes	No	No
21 cm line	1930s	Yes	Yes	Yes	No
Lamb shift	1947	Yes	Yes	No	Yes

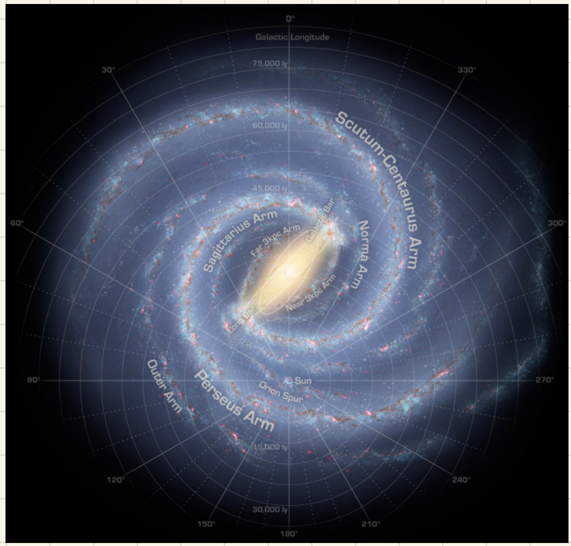


atomic theory - historical development of the model of atomic theory with contributions from Dalton to Chadwick

1) Discovery of 21cm (1420.4 MHz) Line



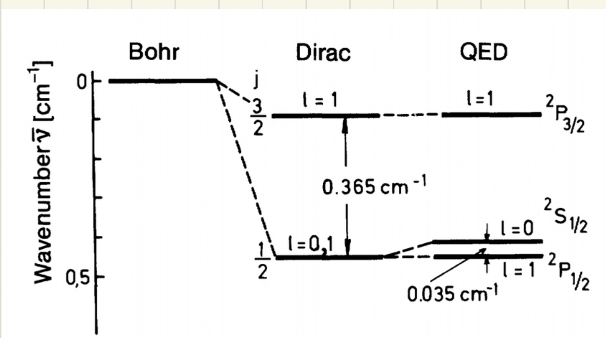
1930s-1940, Jan Hendrik Oort observed that the radio waves seemed to propagate from the centre of the Galaxy.
1944, Hendrik van de Hulst predicted the existence of the 21 cm hyperfine line of neutral interstellar hydrogen.
1951, H. I. Ewan and E. M. Purcell first detected the 21 cm line.
1950s, van Woerden et al. discovered the Near 3 kpc Arm through 21-centimeter radio measurements of HI.

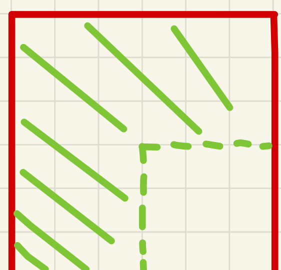


http://galaxymap.org/drupal/node/202

2) Discovery of Lamb Shift

A very long story! Read what follows.





75% baryonic mass is consituted by H atom!!
H is important for observations in astronomy.

WILLIS E. LAMB, JR.

Fine structure of the hydrogen atom

Nobel Lecture, December 12, 1955

When the Nobel Prizes were first awarded in 1901, physicists knew something of just two objects which are now called « elementary particles »: the electron and the proton. A deluge of other « elementary » particles appeared after 1930; neutron, neutrino, μ meson, π meson, heavier mesons, and various hyperons. I have heard it said that « the finder of a new elementary particle used to be rewarded by a Nobel Prize, but such a discovery now ought to be punished by a $10,000 fine ».

In order to determine the properties of elementary particles experimentally it is necessary to subject them to external forces or to allow them to interact with each other. The hydrogen atom which is the union of the first known elementary particles: electron and proton, has been studied for many years and its spectrum has taught us much about the electron.

In 1885, Balmer found that the wavelengths of fourteen lines of the hydrogen spectrum were given by a simple equation. In 1887, Michelson and Morley discovered a fine structure of some of these lines. The quantum theory was founded by Planck in 1900, and in 1913 Bohr gave rules of quantization which permitted a derivation of Balmer's formula. Sommerfeld showed in 1916 that the fine structure of Bohr's energy levels was caused by relativistic corrections. In 1924, De Broglie attributed wave properties to the electron and soon a quantum mechanics of the hydrogen atom emerged from the hands of Heisenberg, Born, and Schroedinger. Spin and magnetic moment of the electron were suggested by Uhlenbeck and Goudsmit in1925, and their dynamical equations were worked out by Thomas a year later. In 1928, Dirac discovered an equation which described an electron with wave properties, charge, spin, magnetic moment and a mass depending on velocity as required by relativity theory. The energy levels of hydrogen were given by Dirac's theory with high precision.

Of special interest to us are his predictions, as shown in Fig. 1, of the $n = 2$ group of energy levels which are 10.2 electron volts above the $n = 1$ ground state. The fine structure splitting $2^2P_{3/2} - 2^2P_{1/2}$, which according to Dirac's theory arises from spin-orbit interaction, agrees exactly with the sep-

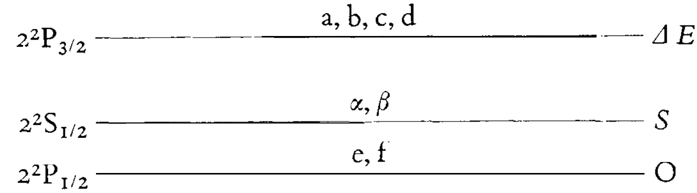


Fig. 1. Fine structure of $n = 2$ levels of hydrogen. According to the Dirac theory the $2^2S_{1/2}$ and $2^2P_{1/2}$ levels coincide exactly. The letters a, b, c, d, e, f, α and β denote the magnetic sub-levels which are split in a magnetic field.

aration of the two levels of Sommerfeld's 1916 theory. The exact coincidence in energy of the $2^2S_{1/2}$ and $2^2P_{1/2}$ states is a consequence of the assumed Coulomb law of attraction between electron and proton. Any departure from this law would cause a separation of these levels.

Many spectroscopic studies of the hydrogen fine structure were made to test the Dirac theory, but by 1940 had failed to establish clearly a decision, although there was evidence strongly supporting the theory. (We now know that the work of Houston[1] and Williams[2] indicated a discrepancy which should have been taken seriously.)

For the subsequent developments, some chapters from my own peculiar history may be of interest. After undergraduate training as a chemist, I studied theoretical physics under Professor J. R. Oppenheimer at the University of California from 1934 to 1938. My thesis[3] dealt with field theories of nucleons which predicted a very small discrepancy from Coulomb's law about a proton. At Columbia University after 1938, I came into close relation with Professor I. I. Rabi and members of the molecular beam laboratory. My attention was drawn briefly to metastable atoms[4] in connection with a proposed atomic beam experiment. During the war, at the Columbia Radiation Laboratory, I received some first-hand acquaintance with microwave radar and vacuum-tube construction techniques. One of the wartime projects in the Laboratory was the determinations of the absorption coefficient of centimeter waves in atmospheric water vapor, and my interest was started in what was to become the very active postwar field of microwave spectroscopy.

In teaching a summer session class in atomic physics in 1945 using a textbook[6] by Herzberg, I found references to some attempts[7] made in 1932-1935 to detect absorption of short-wavelength radio waves in a gas discharge of atomic hydrogen. At first it seemed to me that these experiments had failed because of inadequate microwave techniques. I thought of repeating them

2-3-2 Hydrogen atom's fine Structure from the Dirac Equation

$$E_{j\,n} = -m\,c^2\left[1-\left(1+\left[\frac{\alpha}{n-j-\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\alpha^2}}\right]^2\right)^{-1/2}\right]$$

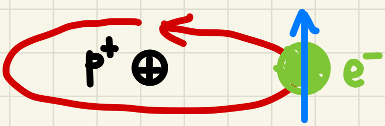
m: electron rest mass
e : elementary charge
c : speed of light in vacuum
n : principal quantum number
j: l+s (orbital + electron spin)

n = 1,2,3...
l = 0,1,2,...,n-1 (s,p,d,f,...)
m = -l,...,l (l projection)
s = 1/2 (electron spin)

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \simeq \frac{1}{137.036}$$ fine structure constant

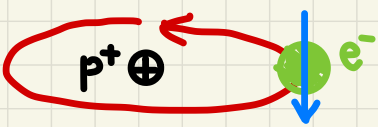
The fine structure of $2P_{3/2}$, $2P_{1/2}$ and $2S_{1/2}$ levels

2P3/2



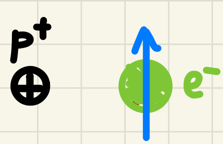
$$j = 1 + \frac{1}{2} = \frac{3}{2}$$

2P1/2



=

2S1/2

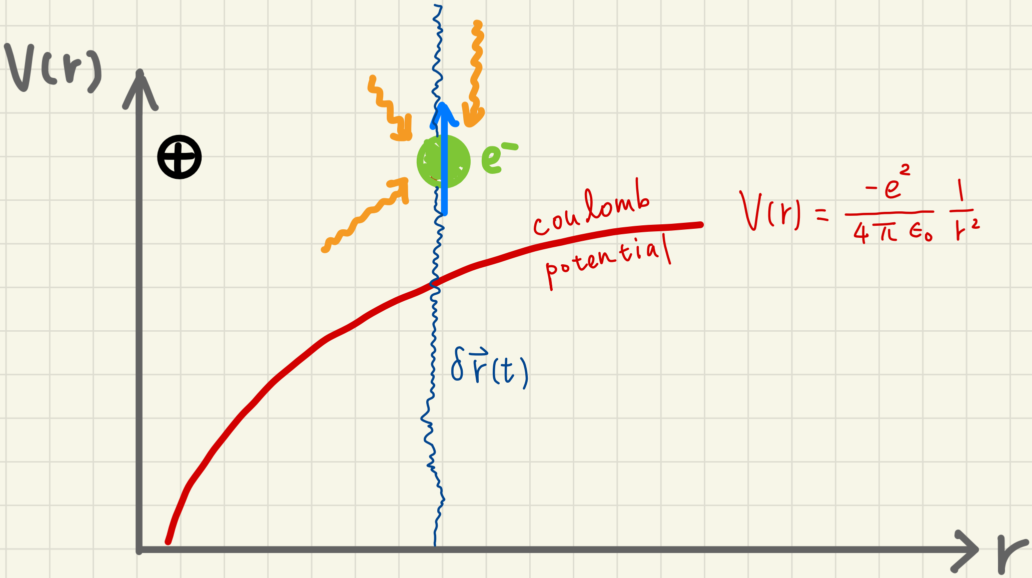


$$j = 1 - \frac{1}{2} = \frac{1}{2}$$

$$j = 0 + \frac{1}{2} = \frac{1}{2}$$

Degenerate in Dirac equation !!

2-3-3 Welton's Model of Lamb Shift



Imagine the position of the bound electron is perturbed by vacuum fluctuation's electric field, the Newton's law reads

$$m\frac{d^2\delta\hat{r}}{dt^2} = -e\hat{E}_k$$

$$\rightarrow \quad \delta\hat{r}_k \cong \frac{e}{mc^2k^2}\hat{E}_k$$

$$= \frac{ie}{mc^2k^2}\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left[\hat{a}_{\vec{k}}e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \hat{a}^\dagger_{\vec{k}}e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

Take $\hat{\delta r}_k$ into account by Taylor's expansion

$$
\begin{aligned}
\Delta \hat{V} &= V(\vec{r} + \hat{\delta r}) - V(\vec{r}) \\
&= \hat{\delta r} \cdot \nabla V + \frac{1}{2} (\hat{\delta r} \cdot \nabla)^2 V + \cdots\cdots .
\end{aligned}
$$

When putting an atom in state $|n\ell m\rangle$ in vacuum, the total state reads $|\underbrace{n\ell m}_{\text{atom}}, \underbrace{0}_{\text{vacuum}}\rangle$.

The energy shift is

$$
\begin{aligned}
\langle \Delta \hat{V} \rangle &= \langle n\ell m\, 0 | \Delta \hat{V} | n\ell m\, 0 \rangle \\
&= \langle 0 | \cancelto{0}{\hat{\delta r}} | 0 \rangle \cdot \langle n\ell m | \Delta V | n\ell m \rangle \\
&\quad + \frac{1}{2} \langle n\ell m\, 0 | (\hat{\delta r} \cdot \nabla)^2 V \, | n\ell m\, 0 \rangle ,
\end{aligned}
$$

where

$$
\begin{aligned}
(\hat{\delta r} \cdot \nabla)^2 &= (\delta x\, \partial x + \delta y\, \partial y + \delta z\, \partial z)^2 \\
&= \delta x^2 \partial x^2 + \cancelto{0}{\delta x \delta y}\, \partial x \partial y + \cancelto{0}{\delta x \delta z}\, \partial x \partial z \\
&\quad + \cancelto{0}{\delta y \delta x}\, \partial y \partial x + \delta y^2 \partial y^2 + \cancelto{0}{\delta y \delta z}\, \partial y \partial z \\
&\quad + \cancelto{0}{\delta z \delta x}\, \partial y \partial z + \cancelto{0}{\delta z \delta y}\, \partial z \partial y + \delta z^2 \partial z^2 \\
&= \frac{1}{3} (\delta x^2 + \delta y^2 + \delta z^2)(\partial x^2 + \partial y^2 + \partial z^2) \\
&= \frac{1}{3} \hat{\delta r}^2 \nabla^2 .
\end{aligned}
$$

The energy shift becomes

$$\langle \Delta \hat{V} \rangle \cong \frac{1}{6} \langle 0 | \hat{\delta r}^2 | 0 \rangle \langle n l m | \frac{-e^2}{4\pi\epsilon_0} \nabla^2 \frac{1}{r} | n l m \rangle$$

$$= -\frac{e^2}{24\pi\epsilon_0} \langle 0 | \hat{\delta r}^2 | 0 \rangle \langle n l m | (-4\pi) \delta(\vec{r}) | n l m \rangle$$

$$= \frac{e^2}{6\epsilon_0} \langle 0 | \hat{\delta r}^2 | 0 \rangle \langle n l m | \delta(\vec{r}) | n l m \rangle$$

$$= \frac{e^2}{6\epsilon_0} \underbrace{\langle 0 | \hat{\delta r}^2 | 0 \rangle}_{\substack{\text{vacuum} \\ \text{fluctuation} \\ ②}} \underbrace{\left| \psi_{nlm}(0) \right|^2}_{\substack{\text{electronic} \\ \text{probability at center} \\ ①}} .$$

We see that the energy shift is caused by

① electronic probability at the center and

② vacuum fluctuation, and

we calculate both in what follows:

① electronic probability at the center

$$\left| \psi_{200}(0) \right|^2 = \frac{1}{\underbrace{8\pi a_0^3}}$$

$$\left| \psi_{21m}(0) \right|^2 = 0 \quad \rightarrow \text{2P has no shift !!}$$

② vacuum fluctuation $\hat{\delta r}_k^2 \propto \hat{a}_k\hat{a}_k - \hat{a}_k\hat{a}_k^\dagger - \hat{a}_k^\dagger\hat{a}_k + \hat{a}_k^\dagger\hat{a}_k^\dagger$

$$\langle 0|\hat{\delta r}^2|0\rangle = \sum_{\vec{k}} \frac{\hbar e^2}{2\epsilon_0 V m^2 c^3 k^3}\langle 0|\hat{a}_{\vec{k}}\hat{a}_{\vec{k}}^\dagger|0\rangle$$

$$= \sum_{\vec{k}} \frac{\hbar e^2}{2\epsilon_0 V m^2 c^3 k^3}$$

$$= \frac{\hbar e^2}{2\epsilon_0 V m^2 c^3}\cdot 2\cdot\frac{V}{(2\pi)^3}4\pi\int_{\frac{\pi}{a_0}}^{\frac{mc}{\hbar}}\frac{dk}{k}$$

$$= \frac{\hbar e^2}{2\epsilon_0 m^2 c^3 \pi^2}\ln\left(\frac{m a_0 c}{\hbar\pi}\right),$$

where Bohr radius $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$.

The Lamb shift reads

$$\langle\Delta\hat{V}\rangle_L = \frac{\hbar e^4}{96\,\epsilon_0^2 m^2 c^3 \pi^3 a_0^3}\ln\left(\frac{4\epsilon_0\hbar c}{e^2}\right)$$

$$\boxed{\langle\Delta\hat{V}\rangle_L = \frac{m e^{10}}{6144\,\epsilon_0^5 c^3 \pi^6 \hbar^5}\ln\left(\frac{4\epsilon_0\hbar c}{e^2}\right).}$$

~ 3.78 rad · GHz

This is very close to the measurement of 6.64 rad GHz !!

Appendix 2-1 Unitary Transformation

$$i\hbar \partial_t \psi = \hat{\mathcal{H}} \psi$$

$$\rightarrow i\hbar \hat{U} \partial_t \psi = \hat{U} \hat{\mathcal{H}} \psi$$

$$= \hat{U} \hat{\mathcal{H}} \hat{U}^{\dagger} \hat{U} \psi \quad \text{where} \quad \hat{U}^{\dagger} \hat{U} = \hat{I}$$

Let $\psi_u = \hat{U} \psi$ and $\psi = \hat{U}^{\dagger} \psi_u$

and then $\partial_t (\hat{U} \psi) = (\partial_t \hat{U}) \psi + \hat{U} \partial_t \psi$.

$$\rightarrow \hat{U} \partial_t \psi = \partial_t \psi_u - (\partial_t \hat{U}) \psi$$

$$= \partial_t \psi_u - (\partial_t \hat{U}) \hat{U}^{\dagger} \psi_u$$

$$\rightarrow i\hbar \partial_t \psi_u - i\hbar (\partial_t \hat{U}) \hat{U}^{\dagger} \psi_u = \hat{U} \hat{\mathcal{H}} \hat{U}^{\dagger} \psi_u$$

$$\rightarrow i\hbar \partial_t \psi_u = \left[\hat{U} \hat{\mathcal{H}} \hat{U}^{\dagger} + i\hbar (\partial_t \hat{U}) \hat{U}^{\dagger} \right] \psi_u$$

$$= \hat{\mathcal{H}}_u \psi_u .$$

We get the so-called UNITARY TRANSFORMATION.

$$\boxed{\begin{aligned} \hat{\mathcal{H}}_u &= \hat{U} \hat{\mathcal{H}} \hat{U}^{\dagger} + i\hbar (\partial_t \hat{U}) \hat{U}^{\dagger} \\ \psi_u &= \hat{U} \psi \end{aligned}}$$