The National Council Of Teachers Of Mathematics Reasoning And Proof

Process Standard In High School Mathematics

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Author Note

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Abstract

I am interested in researching strategies to elicit logical reasoning and proof in high school students. My motivation for this research topic is that logical reasoning is severely underrepresented in mathematics classrooms, and that I believe high school students can benefit from knowing ‘how something works’ when studying mathematical topics. It is natural for learners to note patterns, structure, and regularities in both real-world and theoretical phenomena, and they naturally want to question whether these are coincidental or not. The process of identifying and questioning repetition is the drive for reasoning and proof (NCTM, 2000). I intend to uncover and explicitly define pedagogical strategies that will help students to think more robustly and syntactically in the math content areas.

Keywords: reasoning, proof, strategies, high school, mathematics, education

Background Material

Reasoning And Proof In Education

All of my sources that mention the Reasoning and Proof Standard of the NCTM agree that this standard is underrepresented in the mathematics classroom. Rigorous proof helps students better understand meaning and the concepts involved, and logical thinking is a natural process in cognition (Germain-McCarthy, 1999; Hanna, 1983; Ko & Knuth, 2009; O'Daffer & Thornquist 1993; Rips, 1994). The NCTM has stated that there should be less emphasis on formal proof in the geometry class specifically, and more emphasis on intuitive findings and ideas, 3D objects, and geometric modeling of problem situations (Germain-McCarthy, 1999; Hanna, 1983). However, logic and reasoning should be included in all classes regardless of content, and throughout all years of K-12 education (more relevantly, all four years of high school) (Ko & Knuth, 2009; NCTM, 2000). The NCTM recommends against teaching formal proof as a separate unit or course; rather, it should be integrated into all content areas of mathematics. Incidentally, all the Process Standards should be approached in this manner.

Ko and Knuth (2009) state that proficient knowledge in the math content areas (e.g. Algebra, Geometry, Trigonometry, Analysis, Calculus) and/or the Content Standards (e.g. Number Theory, Measurement, Probability) is required to verify a conjecture and avoid misconceptions. Verification of a conjecture must occur before attempting to prove or disprove it. Many students fail to correctly prove/refute a conjecture because they believe the validity of the conjecture to be the opposite (they believe the conjecture to be true when it is false, or *vice versa*). Since the approach to proving/refuting a conjecture depends on its validity, students who have misconceptions about the topic may seem lost when trying to complete the exercise.

Effective methods of teaching have been shown to increase conjecture-making and -testing in students: with a delivery approach, and by including students' input in lectures, teachers are more likely to get an idea of how their students are thinking. By asking questions, teachers can guide the students through the correct thought processes and help them grasp concepts and understand meaning more easily (Germain-McCarthy, 1999).

Measures Of Reasoning And Proof

Most sources agree that there are multiple approaches to mathematics, and more specifically, the field of logic. There are many schools of thought within the subject of math, and there are many methods of reasoning (Hanna, 1983; Weber, 2009). There seem to be varying levels of proof, and these levels can be organized into a 'spectrum of rigor' ranging from non-proof to informal proof to robust proof. Refutals can also fall on the spectrum, ranging from incorrect proof to adequate counterexample. Unfortunately, most students perform at the lower end of these spectrums, i.e. they take a more intuitive approach (Ko & Knuth, 2009; Weber, 2009).

Levels of cognitive demand. Proofs aside, there is a widely accepted scale on which to rate the demand required of a mathematical task and a student's performance on it. The four levels of cognitive demand were developed by Stein, et al. (2000) and correspond to memorization, procedures without connections, procedures with connections, and doing mathematics. The first two are lower-level demands and the last two are high-level demands. When constructing tasks for their students, teachers should aim as high as possible on this scale. There is some subjectivity involved when rating the tasks as well as the students' responses, but Stein, et al. (2000) make clear the differences between the four ratings.

Memorization tasks. These tasks involve producing previously learned information and do not elicit a problem to be solved. The students understand exactly what is expected, hence the tasks are non-ambiguous, and they do not require students to make connections to the meaning behind the tasks. Responses that earn this rating show the students engaged in memorizing facts and did not make connections between them and the underlying themes.

Procedures without connections tasks. Tasks that are deemed procedures without connections are rated higher on the cognitive demand scale but are still low-level demands. These procedures are algorithmic, so they are explicitly requested or the students know that they are expected due to prior experience and drilling. There is little ambiguity involved with the processes required and there are no explanations to the procedures being used. Like memorization tasks, these responses show no connections between underlying themes and there is a focus on acquiring the result rather than the processes used to obtain it.

Procedures with connections tasks. Moving to higher-level cognitive demands, procedures with connections tasks focus more on the process of obtaining the answers. They aim to develop a deeper understanding of the underlying themes. Unlike low-level demands, these tasks are implicit and non-algorithmic regarding approach. They are usually represented in multiple ways. Responses show students have engaged with the underlying themes and have developed an understanding.

Doing mathematics. The highest rating on the scale of levels of cognitive demand, doing mathematics is earned by tasks that are non-algorithmic and ambiguous. There is explicit evidence of students' complex thinking, relevant knowledge, and an understanding of concepts, relationships, and connections. Students have shown self-monitoring and self-regulation throughout the task, and may have analyzed the task itself regarding methods, solutions, and overall solvability.

Van Heile model of reasoning. The van Heile model of development in reasoning in geometry is a scale of five levels of thinking. Contrary to the scale of cognitive demand, the levels in this model of reasoning are stages through which the reasoning mind progresses and develops, with each stage refining a reasoning process from the previous stage. Burger and Culpepper (1993) summarize the levels, indicating that each is distinguished by “particular language, symbols, and methods of inference” (p. 141). Although this model is geared toward geometric reasoning, I believe it can be used for reasoning in any context.

Holistic level. At the first stage, students use an intuitive language when comparing figures and relate new ideas to previous knowledge. They often use irrelevant information and omit significant information to describe shapes and the differences among them. Rather than focusing on specific attributes of shapes, students will view the shape as an entire object.

Analytic level. At the second stage, students zoom in and focus on details of a shape. Completely opposite of the holistic level, students see a shape as the sum of its parts, rather, its properties. They also have a tendency to classify objects into an incorrect category, or to not classify them into the correct category, because of its properties. (A memorable example of this phenomenon is one student’s response in my Letter-Writing Project in 2009. The student said integers were not rational because they were not expressed as ratios.)

Abstract level. The third stage of the van Hiele model is a turning point for the developing mind. Students begin to think in abstract terms and explicitly apply definitions and properties. They show understanding of logical statements such as implications and syllogism, and can form simple yet correct deductive arguments. Although logic is beginning to assume a larger role in the reasoning mind, students in this stage still only apply logical rules for a few steps and in a local context. It is stressed that students are able to understand the meaning of logical statements but may not be able to understand the distinction among the different types (axiom, definition, implication, theorem, biconditional, etc.).

Deductive level. Students in the fourth level fully understand the structure of the axiomatic system on which the discipline is built, and they view proof as the "final authority" (p. 142) in determining a statement's validity. They understand the difference among varying types of statements and the roles they play in the system, and they are able to reason logically given a system and its theorems. Students still have a sense of what is "correct" versus "incorrect." They have not yet grasped the notion that a system can change if its axioms were to change.

Rigorous level. At the final stage, students develop an appreciation for the discipline and the necessity for the robustness of a logical system. They begin to explore and compare different axiomatic systems and are able to reason effectively in them. They understand what is "correct" is only consistent with the given system, and that correctness can change if the system's axioms, and thus the system itself, changes.

Spectrums of proof and counterexample. Ko and Knuth (2009) developed a study in which they surveyed undergraduate calculus students' ability to decide the veracity of a theorem, and prove it if it was true or produce a counterexample if it was false. To better characterize the students' performance, seven categories in which to classify a proof and six categories in which to classify a counterexample were defined. Each category corresponds to a level of achievement and falls into a spectrum of low achievement to high achievement. Figure 1 was extracted directly from Ko and Knuth's study (2009, p. 71) and provides a summary of their ratings.

Table 2
Seven types of proof productions

Production	Description
No response	Left blank, no relevant knowledge, presented as a guess
Restatement	Restated the problem with students' own language but no basis for constructing a proof
Counterexample	Gave an incorrect counterexample attempts to refute a true proposition
Empirical	Used examples as demonstrations
Non-referential symbolic	Manipulated symbols behind the meanings involved in problem situations with logical errors but did not produce a proof
Structural	Presented mathematical definitions, relevant axioms or theorems that could construct a valid proof but making logical errors
Completeness	Provided a complete proof

Table 3
Six types of counterexample productions

Production	Description
No response	Left blank or no relevant knowledge presented as a guess
Proof	Gave an incorrect proof attempts to prove a false proposition
Inadequate	Provided a counterexample that failed to refute a false proposition or did not exist
Justification	Narrated a proposition that was false instead of providing a counterexample to refute it
Incomplete	Provided a counterexample that succeeded by refuting a false proposition but making logical errors
Adequate	Provided a complete counterexample

Figure 1.

Exploration Of Processes

General teaching recommendations are suggested by Germain-McCarthy (1999) and the NCTM and can apply to teachers of all content areas, not just mathematics teachers. These suggestions include practicing teaching based on reform-based research and maintaining contact with other colleagues, students, and parents. Similar suggestions are provided for administrators and teacher educators.

Aside from general teaching methods, there are more specific strategies that can be used both by teachers and by students that will help develop better reasoning abilities and proof-writing in students. The more strategies teachers have to offer their students, the more they will develop logical thinking and reasoning abilities. I believe that logical thinking is a skill useful in other areas outside of the mathematics classroom, and that these abilities should be fostered in children so they can use them as tools later on in life.

Mathematical Reasoning

O'Daffer and Thornquist (1993) classify inductive and deductive reasoning under a superordinate category, mathematical reasoning, which, in turn, can be categorized under mathematical thinking. Mathematical reasoning involves forming generalizations (inductive reasoning) and drawing conclusions (deductive reasoning) about ideas and their relationships.

Inductive reasoning. Inductive reasoning involves making general statements about specific examples. O'Daffer and Thornquist (1993) emphasize a clear distinction between inductive reasoning and proof by induction: "inductive reasoning might be used to discover a generalization about natural numbers, while mathematical induction would be required to prove it" (p. 50). Nonetheless, there is a relationship between making generalizations from observations and showing the validity of those generalizations. Proof by induction is not part of the high school curriculum, but students think inductively often by making generalizations and stereotypes. As a result, many misconceptions arise from false inductive reasoning. For example, a student familiar with the commutative property of multiplication of real numbers may incorrectly assume that multiplication of matrices is also commutative. Another related mistake is to assume that if the product of two matrices is the zero matrix, at least one of the multiplicands must be the zero matrix.

Making observations and devising conjectures based on those observations helps students to see patterns and make generalizations. Whether these generalizations are correct or not, the process of "forming a conjecture is a major step in mathematical investigations" (Reynolds, 2006, p. 21). I will use an example in the domain of computer science to illustrate this point. A student is asked to write a program that will take a list '*A*' of five values and create a list '*B*' with those values in reverse order. The program runs by setting 'B_1' equal to 'A_5,' 'B_2'equal to 'A_4,' etc. The instructor now asks the student to use change the program so that it will reverse a list with five hundred values. The student reluctantly adds many more lines of code to expand the program. Now the teacher asks the student to make the program work for a list of any number of values. The student will have to generalize the program into abstract terms.

Another example of a student using inductive reasoning is this: after discovering the formula for calculating the probability of the union of two mutually exclusive sets, the student develops the formula for calculating the probability of the union of any two sets, whether disjoint or not. Taking a rule and expanding it to form a more general form is the essence of inductive reasoning.

Deductive reasoning and formal proof. Deductive reasoning involves making specific statements from general examples. O'Daffer and Thornquist (1993) and Reynolds (2006) give three valid patterns of deductive reasoning, each of which correspond to a particular method of proof. The first two patterns, *modus ponens* and *modus tollens*, are used in direct and indirect proofs, which are described in the following few subsections of this document. An example of the third pattern, called syllogism (more colloquially, the "chain rule") is illustrated by the following scenario: given that any square is a rectangle, and that any rectangle is a parallelogram, a student can deduce that all squares are parallelograms. In abstract language, "if both 'if *p* then *q*' and 'if *q* then *r*' are true, then it follows that 'if *p* then *r*' must be true." O'Daffer and Thornquist (1993) also give examples of two patterns that are not necessarily valid in all cases: assuming the converse, and assuming the inverse, of a given conditional statement. Although these patterns of thinking may work in some situations, they are not always true.

Another form of deductive reasoning is the process of elimination. Using this process, students literally deduce the correct answer by eliminating nonsensical possibilities according to a given set of axioms or rules. The popular mind game "Sudoku" is an excellent tool to enhance this type of thinking.

Deductive reasoning and formal proofs go hand in hand. Formal proofs use deductive reasoning strategies and require a certain language. "At the most general level, a formal proof is a finite sequence of sentences … in which each sentence is either a premise, an axiom of the logical system, or a sentence that follows from preceding sentences by one of the system's rules" (Rips, 1994, p. 34). Proofs are essays that show a given proposition as true or false. Writing a formal proof requires a process of demonstrating that statements logically follow from one another, and that the logical conclusions lead the reader completely certain of the veracity of the proposition in question (Abbott, 2001; Reynolds, 2006). Here, "logically follows" indicates that statements must be justified from either an accepted axiom, a premise, or a result, so that every conclusion is drawn from previously known facts. No new information is fabricated into the system; the only information we use (directly or indirectly) is the axioms on which the system is founded.

It is important to emphasize at this point that the previously known premises in an axiomatic system are derived from a set of agreed-upon rules, called axioms. Axioms are elements on which a system is founded, hence axioms cannot be proven using smaller elements. They are self-evident (Abbott, 2001). We must blindly accept them as true, or we should be willing to accept the logical consequences that follow. (Take the Saccheri Quadrilateral as an example of a phenomenon in which we are willing to abandon an axiom, namely, Euclid's Fifth Postulate.) The axiomatic system will provide theorems that are consistent with the axioms presented, and our proofs to these theorems must also be consistent. Thus, it is not a question of whether a statement is "true" or "false," rather, whether it is "consistent" or "inconsistent" with the universally accepted axioms and thus the logical system.

There are four suggested strategies for proving theorems: disproof by counterexample (proof by example), direct proof, indirect proof, and proof by induction. Each strategy is most useful for a particular premise (Durbin, 2005; O'Daffer & Thornquist 1993; Reynolds, 2006).

Disproof by counterexample. The easiest proof is actually a disproof. A disproof by counterexample is executed by finding at least one example that does not fit into a proposed generalization, and then showing how that example disproves the generalization (Arnold, 2002; O'Daffer & Thornquist 1993). The best way for students to use this method is to be able to recognize patterns and anomalies. Patterns help students to form conjectures, and teachers should encourage their students to test the conjectures they make. Anomalies (counterexamples) disprove false conjectures and force students to reconsider their thinking. An example of using an anomaly to disprove a conjecture is a student considering the number 2 after supposing that all prime numbers are odd. Students that make false conjectures, discover they are false, and correct them are more likely to remember and understand the topics they are working with (Germain-McCarthy, 1999).

Proof by example. A disproof of a conjecture is considered a proof of the negation of the conjecture. Thus, sometimes this method is called "proof by counterexample" (O'Daffer &

Thornquist 1993). Despite this label, we use examples for proof and counterexamples for disproof. Not all statements can be proven by example. Only negations of conjectures can be proven by example. Consider the statement, “There exists at least one function ‘*f*’ such that ‘*f*’ has pointwise continuity across the real numbers, but does not have uniform continuity.” This is really a negation of the conjecture, “All functions that are pointwise continuous are necessarily uniformly continuous.” To prove the original statement by example, we disprove its negation by counterexample. An adequate counterexample could be any arbitrary quadratic function and a demonstration of how it satisfies the statement. (After the proof by example is complete, one could supply another proof that all quadratic functions satisfy the given theorem, although it is not required to prove the original theorem.)

Direct proof. A direct proof is the one with which we are most familiar. It involves using the format suggested by Rips (1994) and Abbott (2001) described above, showing that a given statement is true by using a pattern of previously defined axioms, definitions, and theorems. *Modus ponens* dictates, “if ‘if *p* then *q*’ is true and if ‘*p*’ is true, then ‘*q*’ must be true.” In an implication, the ‘*p*’ statement is called the hypothesis and the ‘*q*’ statement is called the conclusion (Durbin, 2005). *Modus ponens* is the most common pattern used in direct proofs.

Indirect proof. An indirect proof is used to prove an implication by proving its contrapositive. *Modus tollens* is a specific pattern that dictates, “if ‘if *p* then *q*’ is true and if ‘*q*’ is false, then ‘*p*’ must be false.” This method means that the contrapositive ‘if *not q* then *not p*’ of an implication ‘if *p* then *q*’ is logically equivalent to the original implication. This can be easily verified by the truth table. We take an indirect approach to the given conditional statement by using a direct proof on its contrapositive. Proving that the contrapositive holds proves that the original conjecture holds (Durbin, 2005). Therefore, this approach can is identified as "proof by contraposition" (Reynolds, 2006) or "proof by transposition" (Turner, 2007).

Proof by contradiction. Another indirect approach is proof by contradiction, also known as *reductio ad absurdum*. Because not all given statements are conditional, we cannot use proof by contraposition every time. In the case that a plain statement '*s*,' such as, "the square root of 2 is irrational," does not have a contrapositive, we must approach the proof in this manner: we assume that the negation '*not s*' of the given statement is true and use a process of *modus ponens* to arrive at a conclusion '*r*' that contradicts a previously known fact '*not r*.' (This fact may or may not be the given proposition, that is, '*not r*' may or may not be the same as '*s*'). Our arrival at a contradiction '*r* and *not r*' forces us to abandon our supposition, that the negation of the given statement was true. In other words, we prove that the proposition cannot be false, thus it must be true (Abbott, 2001).

The latter approach is also applicable to conditional statements. In the case that we not use proof by contraposition, we let an implication 'if *p* then *q*' be the statement '*s*' to be proven and start by assuming '*s*' to be false. To do so, we must assume the negation '*p* and *not q*' of the original implication to be true. Then by *modus ponens*, the proof is completed normally (Durbin, 2005; Reynolds, 2006).

Proof by induction. The last and most complex type of proof is proof by induction. Proof by induction is used to prove a general statement about the natural numbers. The basic idea is to show that it is true under certain circumstances, and show that if it is true under those circumstances, it must be true under all circumstances. Durbin (2005) and Turner (2007) present two types of induction: weak induction and strong induction.

Weak induction. This proof involves three steps: (1) the base case: given a property about natural numbers, show it is true for $n = 1$, (2) the inductive hypothesis: assume this property holds for any given natural number n, with $n > 1$, and (3) the inductive step: show this property is true for $n + 1$. We complete the proof by deducing that the property must be true for all natural numbers n. The domino effect illustrates an analogy for this type of proof (O'Daffer & Thornquist 1993; Turner, 2007).

Strong induction. Similar to weak induction, strong induction is executed in the same manner, except that there are multiple base cases (step 1), and that the strong inductive hypothesis (step 2) involves assuming the property holds for integers k with $1 < k \leq n$. We proceed to step 3 by showing the property is true for $n + 1$ and complete the proof normally (Turner, 2007).

Both forms of induction are equivalent, but strong induction may be required in the case that weak induction is insufficient (Durbin, 2005). Strong induction is required for theorems with multiple base cases and recursive formulas (Arnold, 2002).

English-Logic Inconsistencies

O'Daffer and Thornquist (1993) have shown that students reasoning abilities have improved with a teacher's implicit, yet consistent, "if-then" language in the classroom. They state that high school students have trouble with the language of mathematical logic, including conditional statements and their derivatives (inverse, converse, and contrapositive). This may be because the English language and the language of logic are inconsistent with each other. For example, the words "if… then…," "and," "or," and "not" have different meanings in math than they do in English. It is possible that students apply their previous knowledge about these words to the theorems they learn in math, and thus develop misconceptions.

Implications and their derivatives. The implication 'if *p* then *q*' logically dictates that if '*p*' is true, then '*q*' must necessarily be true. It says nothing about what happens if '*p*' is false. In fact, the truth table will show us that when '*p*' is false, the implication 'if *p* then *q*' is true independently of '*q*.' We have already discovered that given an implication, its contrapositive 'if *not q* then *not p*' is logically equivalent. Even so, students will develop misconceptions about the validity of the implication's converse 'if *q* then *p*' and the implication's inverse 'if *not p* then *not q*' (O'Daffer & Thornquist 1993).

Given a biconditional statement '*p* if and only if *q*' as true, the implication 'if *p* then *q*' and its converse 'if *q* then *p*' must both be true, by definition. Because the converse and inverse are contrapositives of each other, they are logically equivalent regardless of their validity. Since the converse is true, the inverse 'if *not p* then *not q*' must also be true. Hence the inverse and converse of an implication are true in the case of true biconditional statements. I will use a concrete example to illustrate these abstract concepts. Consider the biconditional statement, "The working lamp is on if and only if it is giving off light." The implication dictates, "If the working lamp is on, then it is giving off light." The converse dictates, "If the working lamp is giving off light, then it is on." We are forced to accept that both the implication and converse are true. Further, the inverse dictates, "If the working lamp is off, then it is not giving off light," which is true by contrapositive of the converse. It is easy to see why students are quick to assume that the converse and inverse are necessarily true.

The inverse and converse do not hold true for non-biconditional statements, however, because ‘*p*’ and ‘*q*’ are not equivalent. An example of a common mistake children make by assuming the inverse to be true is illustrated by the following situation: parents will tell their children, “If you don’t eat your vegetables, you won’t get dessert.” Children will interpret this to be logically equivalent to its inverse, “If you eat your vegetables, you will get dessert,” which may not always be the case. If students are confused between conditional statements and biconditional statements, they might become confused on the validity of an implication’s converse and inverse. As O’Daffer & Thornquist (1993) state, “[High school students] often interpret an if-then statement as if *and only* if… Most do not recognize invalid converse and inverse reasoning patterns… [Teachers should] create opportunities to illustrate the idea that an if-then statement is not the same as an if-and-only-if statement” (pp. 46, 48).

The conjunction and disjunction operators. The word “and” has a different meaning in logic than it does in English. If the statement ‘*p* and *q*’ is true, then both ‘*p*’ and ‘*q*’ must be true. If at least one is false, then the conjunction operator will return false (Reynolds, 2006). On the contrary, we sometimes use the word “and” when we mean to use the logical connective “or.” For instance, a logical program will interpret the statement, “This party is full of math and music majors” to mean that each person at the party is majoring in both math and music, which is a correct interpretation of the statement but an inaccurate depiction of the situation. We intuitively interpret the statement instead, as an accurate depiction, that all the people at the party are majoring in either math or music, or possibly both. To logically represent the situation we ought to say, “This party is full of math or music majors.” Although we then might intuitively interpret this to mean that the party is either one of math or one of music. This example serves as a segue to my next topic.

Another misconception involves the “or” connective. Mathematically, it compliments “and.” If the statement ‘*p* or *q*’ is false, then both ‘*p*’ and ‘*q*’ must be false. If at least one is true, then the disjunction operator will return true (Reynolds, 2006). We intuitively think of “or” as meaning “either one or the other but not both.” For another example, “Do you want the red shirt or the blue shirt?” Our intuitive minds wouldn’t make sense of the option to have both, but it is a logically acceptable answer.

Reynolds (2006) states, “Truth tables and Venn diagrams can also be helpful tools for logical reasoning” (p. 53), although I would take it a step further and say that not only are they helpful, but they are essential for helping students understand logical relationships. “Euler circles” (Rips, 1994, p. 24) are diagrams that use circles to represent statements. The visual relationships among these circles represent the logical dependancies of the statements. Venn diagrams and Euler circles can be used to explain logical operators, implications, and their derivatives in a way that truth tables cannot, and they could also be used for teaching set theory, to which logic is a parallel discipline. Incidentally, they are helpful for students that are visually and spatially inclined. The NCTM Representations process standard would benefit from these visual representations of abstract concepts (NCTM, 2000).

Preparation For Data Collection

In preparation for a pilot study, I have constructed a problem set that elicits the different strategies of reasoning discussed throughout this document. Of nine total problems, each one corresponds to one strategy or method. The problem set has a high school target audience, though it is at a college undergraduate difficulty level and can be rated on the higher ends of the scales described in this document. Although data collection has not been gathered at this point, my hopes are for teachers to use this study to gear their tasks more around reasoning and proof. I have attached the problem set, accompanied with selected solutions, to this paper.

The first seven tasks are the most important, falling under the category of mathematical reasoning. Problem 1 addresses inductive reasoning. Although there are many drawbacks to students making incorrect conjectures based on faulty observations, inductive reasoning is a process that helps students to open their minds. In any problem involving a recursive formula, such as the formula for finding the determinant of a matrix, an initial case is required (Arnold 2002). For example, the formula for finding the factorial of a whole number is calculated by multiplying that number by the factorial of the previous one. The point at which this recursion stops is the base case zero, which must unavoidably be defined.

In the problem involving finding determinants of matrices, I have included a base case, namely, a one-by-one matrix. In addition to the base case, I have demonstrated the formula for the determinant of a two-by-two matrix. The student's task is to use this formula to calculate the determinant a three-by-three matrix, and then to expand in a general form to find the determinant of any sized matrix. The third case is an introduction to inductive reasoning. The student does not actually generalize any formulas, and is only asked to reduce the problem to the second case. The last step of the problem is where the inductive reasoning occurs, expanding the problem into an abstract form. The student is not required to prove this reasoning. Incidentally, I did not include a proof by induction in this problem set because I consider it too difficult a task for high school students.

The second and third problems are short, simple tasks involving syllogism and process of elimination, respectively. The second task provides two implications that are to be assumed true (regardless of their validity) and the student is supposed to deduce, through a double process of *modus ponens*, a conclusion. Problem 3 aims for the student to place letters in a chart, in the correct order, based on a set of given rules. Moreover, the rules describe where letters *cannot* go, rather than where they *should* go. Thus the student must eliminate illogical possibilities to complete the chart. After they complete the task, I ask them to briefly describe their thought processes and strategies.

Problem 4 addresses proof by counterexample. I have presented four figures, each one of a quadrilateral and its angle measures. All the quadrilaterals are cyclic, although the student is not given this information. The student is supposed to make an observation and a false conjecture, likely that the opposite angles of a quadrilateral are supplementary, and then provide an adequate counterexample (any non-cyclic quadrilateral will do).

The fifth task asks the student to prove a conjecture with a direct approach. The student should assume the hypothesis, that a function is a polynomial of first degree, and arrive at the conclusion, that there is one and only one solution. The proof involves simply finding the general solution of a first-degree polynomial. Converse of the direct proof, the next problem poses an indirect proof of the theorem, “if a natural number’s square is even, it must be even.” The approach to this indirect proof is to directly prove the theorem’s contrapositive: that if a natural number is odd, its square must be odd. The student will assume the negation of the conclusion and arrive at the negation of the hypothesis. Furthermore, other than ending the proof at that, the student should show that it is impossible to have the hypothesis and original assumption hold together. That is to say, “an odd number’s square is even” is a false statement. This method is somewhat similar to proving by contradiction, posed in problem 7.

In the seventh and last problem involving mathematical reasoning, the student is supposed to show that the given theorem must be true because it cannot be false. With a pattern of *reductio ad absurdum*, the student assumes the negation and arrives at a contradiction, thus forcing the assumption to be false. If the student shows that the assumption is false, and the assumption is that the theorem is false, the student has shown that the theorem is true.

The last two problems deal with inconsistencies between the English and logic languages. They ask the student to draw Euler circles and Venn diagrams representing given scenarios. Part B of problem 9 is particularly interesting. When teachers say, “this and that,” they must be wary that they are implying both “this” and “that.” In the set of “even and odd” numbers, students will commonly refer to the integers. However, the “and” operator implies the intersection, not union, of those sets. The intersection of the set of even numbers and the set of odd numbers is null.

Summary

Through rigorous study and research, I have accomplished this project’s goal of explicitly defining and describing different aspects of the Reasoning and Proof process of the NCTM. Not only have I separated distinct types of logical thinking, mathematical reasoning, and proof-writing strategies, but I have discussed the importance of such processes, expanded on methods of measuring them in students, and summarized important recommendations, provided and backed by scholarly research, for teachers in the mathematics classroom. Additionally, I have supplemented a diverse arsenal of tasks that elicit reasoning and proof processes in students, to be used for conducting a study some time in the future.

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