Reasoning and Proof Problem Set

Mathematical Reasoning

1. Inductive Reasoning

Base Case $n=1$. The determinant of a 1×1 matrix is the value of the entry of that matrix.

$$\left\|\begin{bmatrix} a_{1,1} \end{bmatrix}\right\| = a_{1,1}$$

Case $n=2$. The determinant of a 2×2 matrix is the computed using the following formula:

$$\begin{aligned}
\left\|\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\right\| &= \left(a_{1,1}\cdot\left\|\begin{bmatrix} a_{2,2} \end{bmatrix}\right\|\cdot(-1)^{1-1}\right) + \left(a_{1,2}\cdot\left\|\begin{bmatrix} a_{2,1} \end{bmatrix}\right\|\cdot(-1)^{2-1}\right) \\
&= \left(a_{1,1}\cdot a_{2,2}\cdot(1)\right) + \left(a_{1,2}\cdot a_{2,1}\cdot(-1)\right) \\
&= a_{1,1}a_{2,2} - a_{1,2}a_{2,1}
\end{aligned}$$

Case $n=3$. Find the determinant of this 3×3 matrix. It is sufficient to reduce this case to the case $n=2$. (You need not find the determinants of any 2×2 matrices.)

$$\left\|\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}\right\| =$$

Case $n=n$. Write down the steps to find the determinant of an $n\times n$ matrix, for any natural n. (Hint: you can use recursion, i.e. using a word in its definition.)
It is not necessary to prove this conjecture.

2. Syllogism

Assume that both of the following statements are true: A. There are as many integers as there are natural numbers. B. There are as many rational numbers as there are integers. What can we conclude about how many rational numbers there are?
We can conclude that there are as many rational numbers as there are natural numbers.

3. Process of Elimination

Fill in the chart with the letters A through H. You can only use each letter once, and you must use all letters. The chart must follow the rules below. After you finish, please describe the general actions you took to solve this problem.

1	2	3	4	5	6	7	8
G	A	F	D	E	B	H	C

Rules:

I. B, C, D, G cannot go in prime boxes. (1 is not considered prime.)

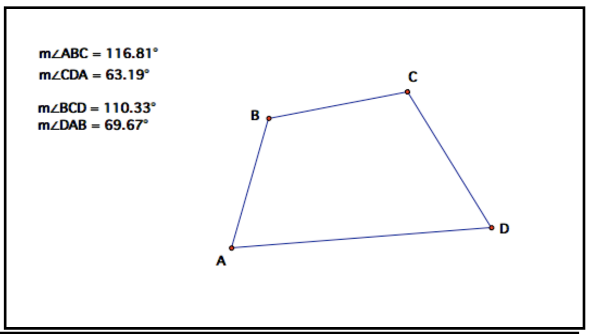
II. E, F, G, H cannot go in even boxes.

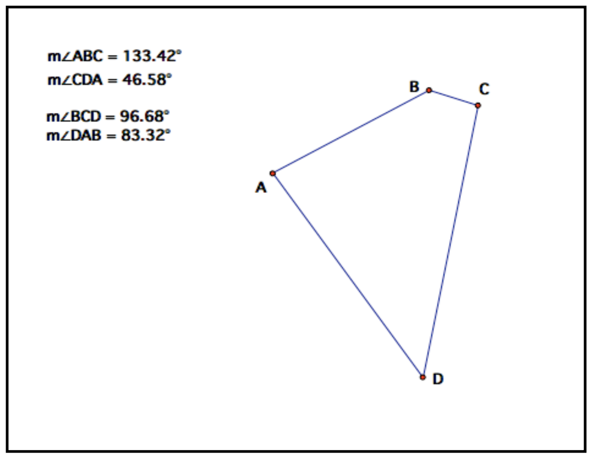
III. B, E, F, H cannot go in boxes with a power of 2. (1 is a power of 2.)

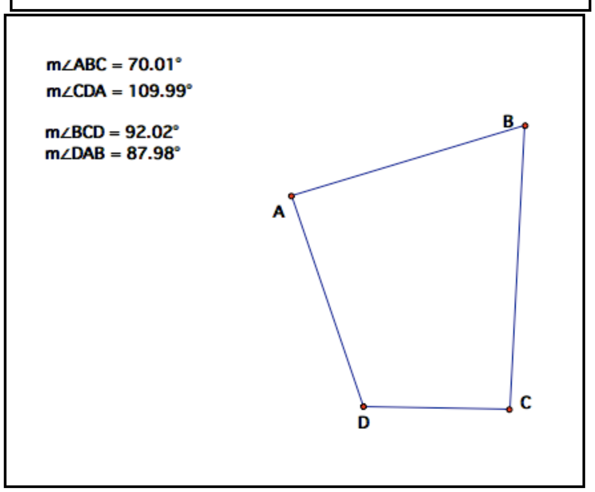
IV. The right border of a certain box must contain the following letter of that box. For example, [X][Y].

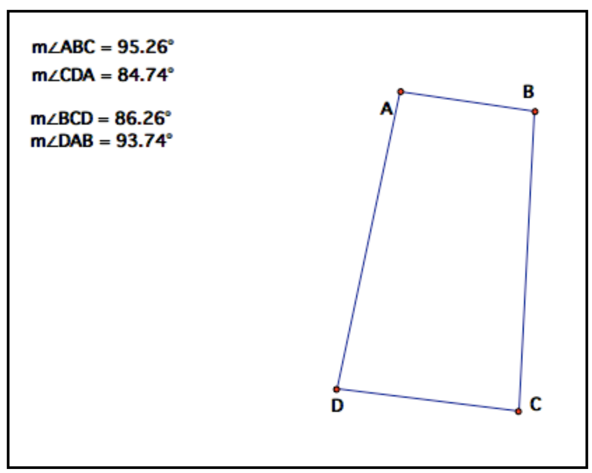
V. F must go to the left of H.

4. Disproof by Counterexample



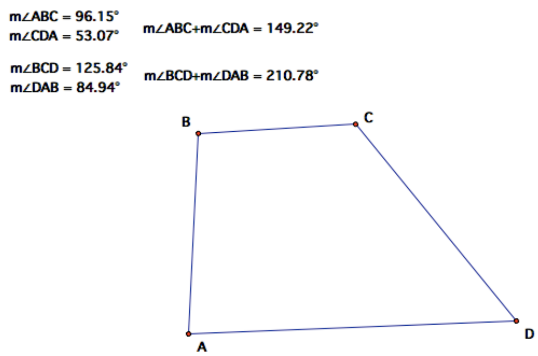






Pretend you are a much younger student. Make a false conjecture about the quadrilaterals you see above. Then provide a counterexample to show your younger self that the conjecture is incorrect.

Conjecture: A quadrilateral's opposite angles are supplementary.



Disproof:

5. Direct Proof

Prove that the following conditional statement is true by using a direct approach.
Theorem: "Every first-degree polynomial function has at least one solution."
Proof: A first-degree polynomial function can be represented as $f(x) = ax + b$. To find a solution, we set $f(x) = ax + b = 0$ and solve for x. So $x = \frac{-b}{a}$. A solution exists. Q.e.d.
We have shown that a solution exists, and for extra credit, we will show that there is only one solution. To prove that this solution is unique, assume to the contrary (see problem 7) that there is another solution, $x_0 \neq \frac{-b}{a}$. If x_0 is indeed a solution, then $f(x_0) = ax_0 + b = 0$. But solving for x_0 gives us $x_0 = \frac{-b}{a}$. Thus, $x_0 \neq x_0$. A contradiction! There must not be more than one solution. Therefore, every first-degree polynomial function has one and only one solution. Q.e.d.

6. Indirect Proof

Prove that the following implication is true by showing that its contrapositive is true.
Theorem: "If n^2 is even, then n is even."
Proof: Suppose n is odd. We will show that n^2 is odd, thus proving the contrapositive. If n is odd, it can be represented as $2k+1$ for any integer k. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Let $m = 2k^2 + 2k$. Note m is an integer. Then $n^2 = 2m + 1$, which is odd. Hence if n is odd then n^2 must be odd. So it's impossible to have n^2 even with n odd. Therefore, n^2 is even implies n is even. Q.e.d.

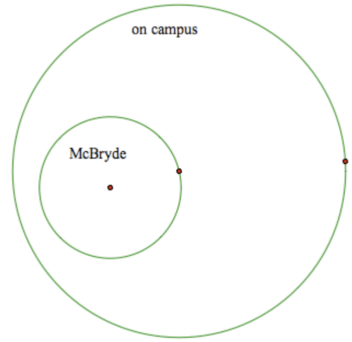
7. Proof by Contradiction

Prove that the following statement is true by showing that it cannot be false.
Theorem: "There are an infinity of natural numbers."
Proof: Suppose to the contrary that there are finitely many natural numbers. Then there must exist some greatest natural number, n_0. Consider $n_0 + 1 > n_0$. But $n_0 + 1$ is natural, so n_0 is not the greatest natural number, $n_0 + 1$ is. A contradiction! There cannot be a greatest natural number. Therefore, they must continue on forever. Q.e.d.

English-Logic Inconsistencies

8. Euler Circles

Euler circles are pictures that represent logical statements. Each circle represents one individual statement. For example, the following picture represents, "If I'm in McBryde, then I'm on campus." Draw Euler circles to represent the following scenario: "All timpani are percussion instruments. Some pitched instruments are timpani."



9. Venn Diagrams

A. Draw a Venn diagram and shade in the area that represents the following scenario: "I never use complete sentences OR proper punctuation."

B. Describe this set: "Let *S* be the set of all even AND odd numbers." Now, draw a Venn diagram to the corresponding events and shade in the appropriate section that represents set *S*.