Reasoning and Proof

The National Council of Teachers of Mathematics reasoning and proof process standard in high school mathematics

Christopher H. Harvey

Introduction

- Which is more important?
  - knowing how to do something
  - knowing how something works
- The drive for proof:
  - Recognition of patterns, structure, and regularities
  - Question if they are coincidental or not
- Eliciting a better understanding

R&P in Education

- R&P is severely underrepresented in the mathematics classroom
- Why? Logical thinking is a natural cognitive process
- R&P and logical thinking should be developed throughout students' entire education

Content and Process

- To prove a conjecture:
  - determine its validity
  - avoid misconceptions
  - have proficient knowledge in content areas and content standards

Measures of R&P

- Levels of Cognitive Demand
- Van Heile Model of Reasoning
- spectrums of proof and counterexample

Levels of Cognitive Demand

- Memorization
- Procedures without Connections
- Procedures with Connections
- Doing Mathematics

Van Heile Model of Reasoning

- Holistic Level
- Analytic Level
- Abstract Level
- Deductive Level
- Rigorous Level

Proofs and Counterexamples

- Proofs:
  - no response
  - restatement
  - counterexample
  - empirical
  - symbolic
  - structural
  - completeness
- Counterexample:
  - no response
  - proof
  - inadequate
  - justification
  - incomplete
  - adequate

Table 2
Seven types of proof productions

Production	Description
No response	Left blank, no relevant knowledge, presented as a guess
Restatement	Restated the problem with students' own language but no basis for constructing a proof
Counterexample	Gave an incorrect counterexample attempts to refute a true proposition
Empirical	Used examples as demonstrations
Non-referential symbolic	Manipulated symbols behind the meanings involved in problem situations with logical errors but did not produce a proof
Structural	Presented mathematical definitions, relevant axioms or theorems that could construct a valid proof but making logical errors
Completeness	Provided a complete proof

Table 3
Six types of counterexample productions

Production	Description
No response	Left blank or no relevant knowledge presented as a guess
Proof	Gave an incorrect proof attempts to prove a false proposition
Inadequate	Provided a counterexample that failed to refute a false proposition or did not exist
Justification	Narrated a proposition that was false instead of providing a counterexample to refute it
Incomplete	Provided a counterexample that succeeded by refuting a false proposition but making logical errors
Adequate	Provided a complete counterexample

(Ko & Knuth, 2009, p. 71)

Mathematical Reasoning

Inductive Reasoning

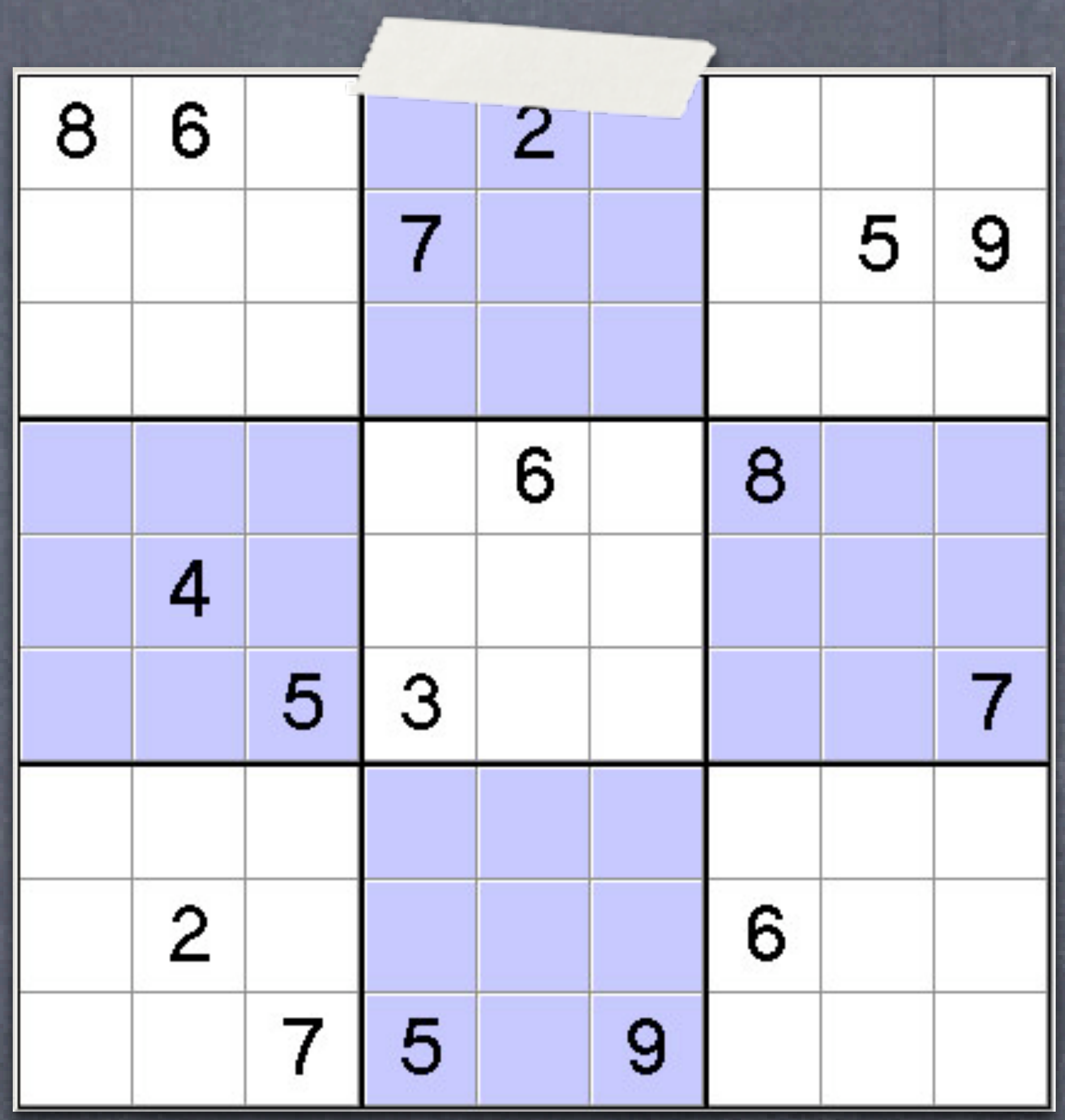
- Make general statements about specific examples
- Not proof by induction
- commonly associated with false generalizations, e.g. multiplication of matrices is commutative
- inductive reasoning is a useful skill, e.g. expanding a specific rule to a general form

Deductive Reasoning

- Make specific statements from general examples
- Patterns of thinking involve applying known facts to specific situations to arrive at a logical consequence
- Axiomatic systems
- Formal proofs

some examples

- process of elimination
- syllogism



http://www.sharewarecentral.com/

Axiomatic System

- This is an important topic but I will cover it briefly
- systems founded on axioms, or self-evident truths
- van Heile's Rigorous Level: comparison of different systems
- truth is relative

Formal Proofs

- Disproof by Counterexample
  - Proof by Example
- Direct Proof (*modus ponens*)
- Indirect Proof (*modus tollens*)
  - Proof by Contradiction (*reductio ad absurdum*)
- Proof by Induction
  - Weak Induction
  - Strong Induction

Disproof by Counterexample

- counterexamples disprove general conjectures, examples prove specific ones
- "All primes are odd." Counterexample: 2
- "There exists an even prime." Example: 2
- disproving a conjecture by counterexample is proving its negation by example

Direct Proof

$$\big((p \to q) \wedge p\big) \to q$$

- *Modus ponens* (*modus ponendo ponens*) "the way that affirms by affirming"
- if the implication and hypothesis are true, then the conclusion must be true

p	q	p→q	... ∧ p	... → q
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

Indirect Proof

$$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$$

- *Modus tollens* (*modus tollendo tollens*) "the way that denies by denying"
- if the implication is true and the conclusion is false, then the hypothesis must be false

p	q	p→q	… ∧ ¬q	… → ¬p
F	F	T	T	T
F	T	T	F	T
T	F	F	F	T
T	T	T	F	T

Proof by Contradiction

- *Reductio ad absurdum* "reduction to the absurd"
- type of indirect proof
- used if there is no contrapositive
  - assume the negation is true and arrive at a contradiction with a previously known fact
  - this contradiction dictates that the negation cannot be true, a.k.a. the original statement cannot be false
  - then the original statement must be true

Proof by Induction

- used to show a property is true for all natural numbers
- base case(s): show it is true for one or some cases
- inductive hypothesis: assume it is true for any given case
- inductive step: show that if it is true for a given case, it is true for the case after that
- finally, deduce that it must be true for all cases

English versus Logic

Logical AND

- "This party is full of math AND music majors."
- Are there any non-music math majors at the party? Are there any non-math music majors at the party?
- No. Everyone at the party is a math and music double-major.

Logical OR

- "This party is full of math OR music majors."
- What type of party is it, a math party or a music party?
- Both. There are math majors, music majors, and possibly double-majors at the party.

Implications

- an implication is logically equivalent to its contrapositive
- an implication's converse is logically equivalent to the inverse (they are contrapositives)
- the converse and inverse are equivalent to each other, but are not always equivalent to the implication
- only biconditionals hold this property

Conditional Statement

p	q	s $p \to q$	CP $\neg q \to \neg p$	CV $q \to p$	IV $\neg p \to \neg q$
F	F	T	T	T	T
F	T	T	T	F	F
T	F	F	F	T	T
T	T	T	T	T	T

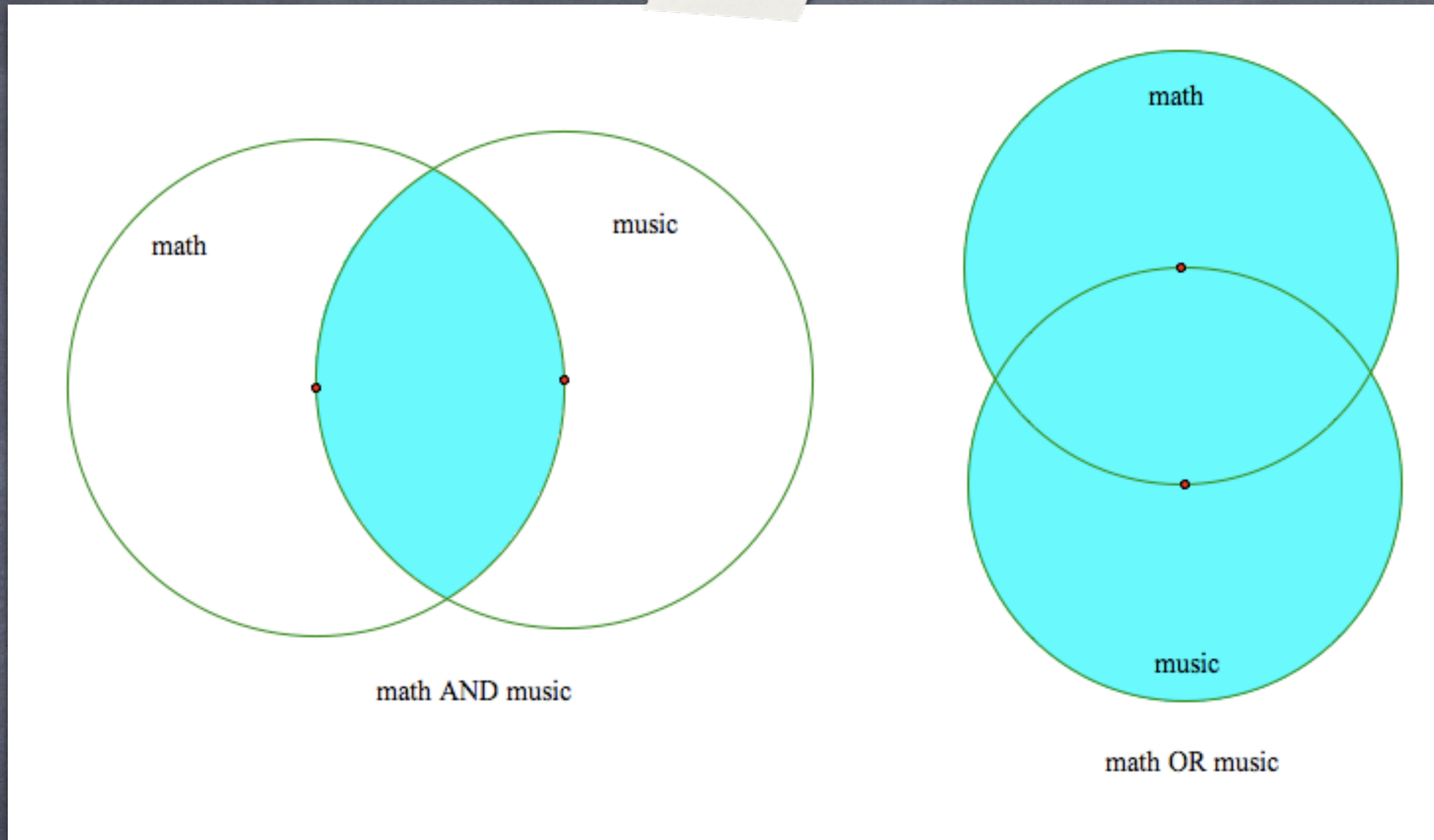
Biconditional Statement

p	q	BC $p \leftrightarrow q$	$s_1 = CV_2$ $p \to q$	$s_2 = CV_1$ $q \to p$	$CP_1 = IV_2$ $\neg q \to \neg p$	$CP_2 = IV_1$ $\neg p \to \neg q$
F	F	T	T	T	T	T
F	T	F	T	F	T	T
T	F	F	F	T	F	F
T	T	T	T	T	T	T

Pictorial Depictions

- useful tools in developing reasoning skills and set theory
  - truth tables
  - Euler circles
  - Venn diagrams

Euler Circles



Venn Diagrams

Wrap-Up