Homotopy Theory

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Chapter 1

Homotopy Groups

1.1 Definitions

Let I^n be the n-dimensional unit cube. For a pointed topological space (X, x_0), we define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f\colon (I^n, \partial I^n) \to (X, x_0)$. Giving this set a group structure is similar to the case of the fundamental group: given maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$, we define

$$(f+g)(s_1, \ldots, s_n) := \begin{cases} f(2s_1, s_2, \ldots, s_n) & s_1 \in [0, \frac{1}{2}], \\ g(2s_1 - 1, s_2, \ldots, s_n) & s_1 \in [\frac{1}{2}, 1]. \end{cases} \tag{1.1.1}$$

We then define $[f] + [g] := [f+g]$, where $[f]$ denotes the homotopy class of f as usual. To check this is well-defined, we need to check that if $f_0 \simeq f_1$, then $f_0 + g \simeq f_1 + g$. Indeed, if f_t is a homotopy from f_0 to f_1 (through maps $(I^n, \partial I^n) \to (X, x_0)$), then $f_t + g$ is a homotopy from $f_0 + g$ to $f_1 + g$.

To check that $+$ is a group operation,