Differential Geometry

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April 19, 2019

Contents

Chapter 1

Connections

1.1 Connections on a Vector Bundle

Let $\pi\colon E \to M$ be a vector bundle. A *connection* on E is a linear map $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the *Leibniz rule*

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s \quad \text{for all } f \in C^\infty(M), s \in \Gamma(E). \tag{1.1.1}$$

Given a vector field $X \in \Gamma(TM)$, we write $\nabla_X s$ for the contraction of the section $X \otimes \nabla s \in \Gamma(TM \otimes T^*M \otimes E)$ over the first two indices. We call $\nabla_X s$ the *covariant derivative* of s in the direction X, and ∇s the *total covariant derivative* of s.

Much like standard differentiation, the covariant derivative is a local operator:

Lemma 1.1.1. *Fix $p \in M$, and let $U \subseteq M$ be an open neighborhood of M. Then $\nabla s|_p$ depends only on the values of s on U.*

Proof. Let $\psi \in C_c^\infty(M)$ be a bump function with $\psi = 1$ on U. Then

$$\nabla(\psi s)|_p = \mathrm{d}\psi_p \otimes s_p + \psi(p)\nabla s|_p = \nabla s|_p. \tag{1.1.2}$$

Since ψ was arbitrary, the proof is concluded. $\square$

Thanks to this lemma, given any open set $U \subseteq M$, ∇ restricts to an operator $\nabla^U : \Gamma(U;E) \to \Gamma(U;T^*M \otimes E)$ given by $\nabla^U s|p = \nabla(\psi s)|p$, where $\psi \in C_c^\infty(M)$ has support in U and is identially 1 on a neighborhood of p. Note that ∇^U satisfies the Leibniz rule, so is a connection on the restricted vector bundle $\pi_U : E_U \to U$. We will usually write ∇s in place of $\nabla^U s$.

Given a local frame field $s_1, \dots, s_k \in \Gamma(U;E)$ of E, we may write $\nabla s_i = \omega_i{}^j \otimes s_j$ for some collection $\omega_i{}^j \in \Omega^1(U)$ of 1-forms, called the *connection 1-forms* for ∇ over U with respect to s_i. Now choose another local frame field $\widetilde{s}_1, \dots, \widetilde{s}_k$ of E. Without loss of generality, the $\widetilde{s}_i$ are also defined over U. Write $\widetilde{s}_i = p_i{}^j s_j$ for some collection $p_i{}^j \in C^\infty(U)$, and let $P := (p_i{}^j)\colon U \to \mathrm{GL}(\mathbb{R}^k)$ be the corresponding matrix function. Also write $\widetilde{\omega}_i{}^j$ for the connection 1-forms of ∇ with respect to $\widetilde{s}_i$. We then have

$$\nabla\widetilde{s}_i = \widetilde{\omega}_i{}^k \otimes \widetilde{s}_k = p_k{}^j \widetilde{\omega}_i{}^k s_j, \tag{1.1.3}$$

and we also have

$$\nabla\widetilde{s}_i = \nabla(p_i{}^j s_j) = \mathrm{d}p_i{}^j \otimes s_j + p_i{}^k \omega_k{}^j \otimes s_j. \tag{1.1.4}$$

We therefore have the relation

$$p_k{}^j \widetilde{\omega}_i{}^k = \mathrm{d}p_i{}^j + \omega_k{}^j p_i{}^k. \tag{1.1.5}$$

Now, upon making the identification of linear maps $T_pU \to T_{P(p)}\mathrm{GL}(\mathbb{R}^k)$ with sections of $T_p^*U \otimes T_{P(p)}\mathrm{GL}(\mathbb{R}^k)$, the differential $\mathrm{d}P$ is of the form

$$\mathrm{d}P = \mathrm{d}p_i{}^j \otimes \frac{\partial}{\partial x_i{}^j}, \tag{1.1.6}$$

where $x_i{}^j$ are the standard coordinates on $\mathrm{GL}(\mathbb{R}^k)$. So, upon making the natural identification $T_p\mathrm{GL}(\mathbb{R}^k) \cong \mathbb{R}^{k\times k}$, we see $\mathrm{d}P$ is given by the matrix $(\mathrm{d}p_i{}^j)$ of 1-forms. If we write ω for the matrix $(\omega_i{}^j)$ of 1-forms, then (1.1.5) can be written as

$$\widetilde{\omega} = P^{-1}\mathrm{d}P + P^{-1}\omega P. \tag{1.1.7}$$

This equation will come in useful later.

A connection ∇ on E can be extended to a connection on E^* by requiring $\mathrm{d}(\theta(s)) = \nabla\theta(s) + \theta(\nabla s)$ for any $\theta \in \Gamma(E^*)$ and $s \in \Gamma(E)$. In particular, if s_i is a local frame for E and θ^i its dual frame for E^*, then we have

$$\begin{aligned} 0 = \mathrm{d}\delta_j{}^i &= \mathrm{d}(\theta^i(s_j)) \\ &= \nabla\theta^i(s_j) + \theta^i(\nabla s_j) \\ &= \nabla\theta^i(s_j) + \omega_j{}^k\theta^i(s_k) \\ &= \nabla\theta^i(s_j) + \omega_j{}^i, \end{aligned} \tag{1.1.8}$$

so $\nabla\theta^i = -\omega_j{}^i \otimes \theta^j$. Furthermore, it can be extended to any tensor product of E and E^* by requiring the Leibniz rule hold. For example, $\nabla(s\otimes t) = \nabla s \otimes t + s \otimes \nabla t$ for any $s,t \in \Gamma(E)$.

Let ∇ and $\widetilde{\nabla}$ be two connections on E. Choose a function $f \in C^\infty(M)$ and a section $s \in \Gamma(E)$. We then have

$$(\widetilde{\nabla} - \nabla)(fs) = (\mathrm{d}f \otimes s + f\widetilde{\nabla}s) - (\mathrm{d}f \otimes s + f\nabla s) = f(\widetilde{\nabla} - \nabla)s. \tag{1.1.9}$$

This means $\widetilde{\nabla} - \nabla$ is a section of the vector bundle $T^*M \otimes \mathrm{End}(E)$. Indeed, given $p \in M$ and $v \in E_p$, choose a local frame field s_i for E in a neighborhood of p, and extend by zero (using bump functions as before) to all of M. Choose a section $s = a^i s_i$ such that $s_p = v$. Define $(\widetilde{\nabla} - \nabla)_p \colon E_p \to T_p^*M \otimes E_p$ by

$$(\widetilde{\nabla} - \nabla)_p(v) := (\widetilde{\nabla} - \nabla)(s)|_p. \tag{1.1.10}$$

This is well-defined, since if $\widetilde{s} = b^i s_i$ is any other section with $\widetilde{s}_p = v$, then we have

$$(\widetilde{\nabla} - \nabla)(s)|_p - (\widetilde{\nabla} - \nabla)(\widetilde{s})|_p = (a^i(p) - b^i(p))(\widetilde{\nabla} - \nabla)(s_i)|_p = 0. \tag{1.1.11}$$

It follows that $(\widetilde{\nabla} - \nabla)_p$ is an element of $T_p^*M \otimes \mathrm{End}(E_p)$, as required. Actually, we may be more concrete: write ω for the matrix of connection 1-forms of ∇ with respect to s_i, and $\widetilde{\omega}$ for the corresponding matrix of connection 1-forms of $\widetilde{\nabla}$. Write θ^i for the frame of E^* dual to s_i. For $s = a^i s_i$, we then have

$$(\widetilde{\nabla} - \nabla)s = a^i(\widetilde{\omega}_i{}^j - \omega_i{}^j) \otimes s_j = (\widetilde{\omega}_i{}^j - \omega_i{}^j))\theta^i(s)s_j. \tag{1.1.12}$$

That is,

$$\widetilde{\nabla} - \nabla = (\widetilde{\omega}_i{}^j - \omega_i{}^j) \otimes \theta^i \otimes s_j. \tag{1.1.13}$$

We have therefore shown that the space of connections on a vector bundle E is an affine space over $\Gamma(T^*M \otimes \mathrm{End}(E))$.

Choose local coordinates x^α for M. We may then write $\omega_i{}^j = \Gamma^j_{\alpha i} \mathrm{d}x^\alpha$ for some smooth functions $\Gamma^j_{\alpha i}$, called *Christoffel symbols*. The covariant derivative of a section $s = a^i s_i$ in the direction X is then given by

$$\nabla_X s = \left(X^\alpha \frac{\partial a^i}{\partial x^\alpha} + a^j X^\alpha \Gamma^i_{\alpha j} \right) s_i =: X^\alpha a^i{}_{;\alpha} s_i. \tag{1.1.14}$$

Let $u\colon N \to M$ be a smooth map. Given a connection ∇ on a vector bundle $E \to M$, we define the *pullback connection* $u^*\nabla$ on the pullback bundle $u^*E \to N$ locally: given a local frame s_i of E with connection 1-forms $\omega_i{}^j$, note that $s_i \circ u$ is a local frame of u^*E. We define $u^*\nabla$ by

$$u^*\nabla(s_i \circ u) := u^*\omega_i{}^j \otimes (s_j \circ u). \tag{1.1.15}$$

A special case is when $N = [0,1]$, and we have a smooth curve $\gamma\colon [0,1] \to M$. In this case, we usually write $D_t := \gamma^*\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}$. For a section $s(t) = a^i(t)s_i(\gamma(t))$ along γ, we calculate

$$\begin{aligned} D_t s &= \left(\frac{\mathrm{d}a^i}{\mathrm{d}t} + a^j \gamma^* \omega_j{}^i \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \right) s_i \\ &= \left(\frac{\mathrm{d}a^i}{\mathrm{d}t} + a^j \Gamma^i_{\alpha j} \frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}t} \right) s_i. \end{aligned} \tag{1.1.16}$$

So s is parallel along γ in the domain of the s_i if and only if it satisfies the system

$$\frac{\mathrm{d}a^i}{\mathrm{d}t} + a^j \Gamma^i_{\alpha j} \frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}t} = 0 \tag{1.1.17}$$

for all t. Given initial values $a^i(0)$, some ODE theory and a patchwork job along all the domains of local frames guarantees the existence and uniqueness of a parallel section s along γ with initial value $s(0)$. We call s the *parallel transport* of $s(0)$ along γ. In a sense, parallel transport is a way of connecting vectors in $E_{\gamma(0)}$ and $E_{\gamma(1)}$. Hence the name "connection".

We will now define the curvature of ∇. First, we extend ∇ by defining the *covariant exterior derivative* $\mathrm{d}^\nabla\colon \Gamma(\Lambda^k T^*M \otimes E) \to \Gamma(\Lambda^{k+1} T^*M \otimes E)$ to be

$$\mathrm{d}^\nabla(\eta \otimes s) := \mathrm{d}\eta \otimes s + (-1)^k \eta \wedge \nabla s, \tag{1.1.18}$$

and extending by linearity. We note that d^∇ satisfies the specialized Leibniz rule

$$\mathrm{d}^\nabla(fs) = \mathrm{d}f \wedge s + f\mathrm{d}^\nabla s \tag{1.1.19}$$

for any $f \in C^\infty(M)$ and $s \in \Gamma(\Lambda^k T^*M \otimes E)$. Now, although the standard exterior derivative satisfies $\mathrm{d}^2 = 0$, this is not true for the covariant exterior derivative. We define the *Riemann curvature* *tensor* of ∇ to be $R^\nabla := (\mathrm{d}^\nabla)^2 \colon \Gamma(E) \to \Gamma(\Lambda^2 T^*M \otimes E)$. Concretely, given a local frame s_i of E, with connection 1-forms ω, we calculate

$$\begin{aligned} R^\nabla s_i &= \mathrm{d}^\nabla(\nabla s_i) \\ &= \mathrm{d}^\nabla(\omega_i{}^j s_j) \\ &= (\mathrm{d}\omega_i{}^j - \omega_i{}^k \wedge \omega_k{}^j) \otimes s_j. \end{aligned} \tag{1.1.20}$$

The matrix $\Omega := \mathrm{d}\omega + \omega \wedge \omega$ of 2-forms is called the matrix of *curvature 2-forms* for ∇ with respect to s_i. Choosing local coordinates x^α for M, we will write $\Omega_i{}^j = \frac{1}{2} R_{\alpha\beta i}{}^j \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta$. It turns out that R^∇ is also a tensor: fix $f \in C^\infty(M)$ and $s \in \Gamma(E)$. We then calculate

$$\begin{aligned} R^\nabla(fs) &= \mathrm{d}^\nabla(\mathrm{d}f \otimes s + f\nabla s) \\ &= \mathrm{d}^2 f \otimes s - \mathrm{d}f \wedge \nabla s + \mathrm{d}f \wedge \nabla s + fR^\nabla s \\ &= fR^\nabla s. \end{aligned} \tag{1.1.21}$$

So R^∇ is a section of the bundle $\Lambda^2 T^*M \otimes \mathrm{End}(E)$, and we can locally write it as $R^\nabla = \Omega_i{}^j \otimes \theta^i \otimes s_j$. It turns out that 2 derivatives is the most we can take:

$$\begin{aligned} \mathrm{d}^\nabla R^\nabla &= \mathrm{d}\Omega_i{}^j \otimes \theta^i \otimes s_j + \Omega_i{}^j \wedge (\nabla\theta^i \otimes s_j + \theta^i \otimes \nabla s_j) \\ &= \mathrm{d}\Omega_i{}^j \otimes \theta^i \otimes s_j + \Omega_i{}^j \wedge (-\omega_k{}^i \otimes \theta^k \otimes s_j + \omega_j{}^k \otimes \theta^i \otimes s_k) \\ &= (\mathrm{d}\Omega_i{}^j - \Omega_k{}^j \wedge \omega_i{}^k + \Omega_i{}^k \wedge \omega_k{}^j) \otimes \theta^i \otimes s_j. \end{aligned} \tag{1.1.22}$$

Now, since $\Omega_i{}^j = \mathrm{d}\omega_i{}^j + \omega_k{}^j \wedge \omega_i{}^k$, we have

$$\begin{aligned} \mathrm{d}\Omega_i{}^j &= \mathrm{d}\omega_k{}^j \wedge \omega_i{}^k - \omega_k{}^j \wedge \mathrm{d}\omega_i{}^k \\ &= (\Omega_k{}^j - \omega_l{}^j \wedge \omega_k{}^l) \wedge \omega_i{}^k - \omega_k{}^j \wedge (\Omega_i{}^k - \omega_l{}^k \wedge \omega_i{}^l) \\ &= \Omega_k{}^j \wedge \omega_i{}^k - \omega_k{}^j \wedge \Omega_i{}^k. \end{aligned} \tag{1.1.23}$$

Plugging this into the above equation, we see $\mathrm{d}^\nabla R^\nabla = 0$. This is called the *second Bianchi identity*.

1.2 The Levi-Civita Connection

Given a vector bundle $\pi \colon E \to M$, a *bundle metric* is a section g of the bundle $E^* \otimes E^*$ such that at each point $p \in M$, g_p is an inner product on E_p. A connection ∇ on E is *compatible with* g if $\nabla g = 0$. In other words,

$$\mathrm{d}(g(s,t)) = g(\nabla s, t) + g(s, \nabla t) \tag{1.2.1}$$

for all sections $s, t \in \Gamma(E)$. Choosing a local frame $s_1, \ldots, s_k$ for E and writing $\nabla s_i = \omega_i{}^j \otimes s_j$, we see

$$\mathrm{d}g_{ij} = g(\nabla s_i, s_j) + g(s_i, \nabla s_j) = \omega_i{}^k g_{kj} + \omega_j{}^k g_{ik} = \omega_{ij} + \omega_{ji}. \tag{1.2.2}$$

In particular, if the s_i are orthonormal, then the matrix ω is skew-symmetric. Of course, this is sufficient to show ∇ is compatible with g, since if θ^i is the orthonormal coframe for E^* dual to s_i, then

$$\begin{aligned}
\nabla g &= \nabla(\delta_{ij}\theta^i \otimes \theta^j) \\
&= \delta_{ij}\nabla\theta^i \otimes \theta^j + \delta_{ij}\theta^i \otimes \nabla\theta^j \\
&= \delta_{ij}\omega_k{}^i \otimes \theta^k \otimes \theta^j + \delta_{ij}\omega_k{}^j \otimes \theta^i \otimes \theta^k \\
&= \omega_{kj}\theta^k \otimes \theta^j + \omega_{ki} \otimes \theta^i \otimes \theta^k \\
&= 0.
\end{aligned} \tag{1.2.3}$$

In fact, this also implies the curvature matrix Ω is skew symmetric, since

$$\begin{aligned}
\Omega_{ij} &= \delta_{jk}\Omega_i^{\ k} \\
&= \delta_{jk}(\mathrm{d}\omega_i{}^k + \omega_l{}^k \wedge \omega_i{}^l) \\
&= \mathrm{d}\omega_{ij} + \delta^{lr}\omega_{lj} \wedge \omega_{ir} \\
&= -\mathrm{d}\omega_{ji} - \delta^{lr}\omega_{ri} \wedge \omega_{jl} \\
&= -\delta_{ik}(\mathrm{d}\omega_j{}^k + \omega_l{}^k \wedge \omega_j{}^l) \\
&= -\Omega_{ji}.
\end{aligned} \tag{1.2.4}$$

With $\Omega_{ij} = \frac{1}{2}R_{\alpha\beta ij}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta$ as before, we then have the following two symmetries of the curvature tensor:

$$R_{\alpha\beta ij} + R_{\beta\alpha ij} = 0, \tag{1.2.5}$$
$$R_{\alpha\beta ij} + R_{\alpha\beta ji} = 0. \tag{1.2.6}$$

Later, we will see some more symmetries of R^∇.

We now restrict attention to the cotangent bundle T^*M. Given a connection ∇ on T^*M (or, equivalently, on TM), define the *torsion* of ∇ to be the map

$$\tau := d - 2\,\mathrm{Alt}_2 \circ \nabla \colon\, \Omega^1(M) \to \Omega^2(M). \tag{1.2.7}$$

Given $f \in C^\infty(M)$ and $\theta \in \Omega^1(M)$, we have

$$\begin{aligned}
\tau(f\theta) &= \mathrm{d}(f\theta) - 2\,\mathrm{Alt}_2(\nabla(f\theta)) \\
&= \mathrm{d}f \wedge \theta + f\mathrm{d}\theta - 2\,\mathrm{Alt}_2(\mathrm{d}f \otimes \theta + f\nabla\theta) \\
&= f(\mathrm{d}\theta - 2\,\mathrm{Alt}_2(\nabla\theta)) \\
&= f\tau(\theta).
\end{aligned} \tag{1.2.8}$$

It follows that τ is a section of $\Lambda^2T^*M \otimes (T^*M)^* \cong \Lambda^2T^*M \otimes TM$. Explicitly, if e_i is a frame for TM and θ^i its dual coframe, then, upon writing $\nabla e_i = \omega_i{}^j \otimes e_j$, we have

$$\tau(\theta^i) = \mathrm{d}\theta^i + \omega_j{}^i \wedge \theta^j, \tag{1.2.9}$$

so $\tau = (\mathrm{d}\theta^i + \omega_j{}^i \wedge \theta^j) \otimes e_i$. Choosing local coordinates x^i, we have

$$\tau = \left(\omega_j{}^i \wedge \mathrm{d}x^j\right) \otimes \frac{\partial}{\partial x^i} = \left(\Gamma^i_{kj}\mathrm{d}x^k \wedge \mathrm{d}x^j\right) \otimes \frac{\partial}{\partial x^i}. \tag{1.2.10}$$

So if ∇ is torsion-free, then $\Gamma^i_{kj} = \Gamma^i_{jk}$ for all i, j, k. Also, given vector fields $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned}\tau(X,Y) &= (\mathrm{d}\theta^i(X,Y) + \omega_j{}^i(X)\theta^j(Y) - \omega_j{}^i(Y)\theta^j(X)) \otimes e_i \\ &= (XY^i - YX^i - \theta^i([X,Y]) + \omega_j{}^i(X)Y^j - \omega_j{}^i(Y)X^j) \otimes e_i \\ &= \nabla_X Y - \nabla_Y X - [X,Y].\end{aligned} \tag{1.2.11}$$

Finally, if ∇ is torsion-free, we differentiate both sides of $\mathrm{d}\theta^i + \omega_j{}^i \wedge \theta^j = 0$ to find

$$\begin{aligned}0 &= \mathrm{d}\omega_j{}^i \wedge \theta^j - \omega_j{}^i \wedge \mathrm{d}\theta^j \\ &= \mathrm{d}\omega_j{}^i \wedge \theta^j + \omega_j{}^i \wedge \omega_k{}^j \wedge \theta^k \\ &= \Omega_j{}^i \wedge \theta^j.\end{aligned} \tag{1.2.12}$$

Write $\Omega_j{}^i = \frac{1}{2} R_{\alpha\beta j}{}^i \theta^\alpha \wedge \theta^\beta$. Plugging this into the above equation, we find

$$0 = \Omega_j{}^i \wedge \theta^j = \frac{1}{2} R_{\alpha\beta j}{}^i \theta^\alpha \wedge \theta^\beta \wedge \theta^j. \tag{1.2.13}$$

We therefore obtain the *first Bianchi identity*:

$$R_{\alpha\beta j}{}^i + R_{j\alpha\beta}{}^i + R_{\beta j\alpha}{}^i = 0. \tag{1.2.14}$$

Theorem 1.2.1 (Fundamental Theorem of Riemannian Geometry)**.** *Let (M,g) be a Riemannian manifold (i.e. M is a manifold and g a bundle metric on TM). Then there exists a unique connection on TM which is torsion-free and compatible with g, called the Levi-Civita connection.*

Proof. Let e_i be a local frame for TM and θ^i its dual coframe. Suppose ∇ is a Levi-Civita connection, and write $\nabla e_i = \omega_i{}^j \otimes e_j$, $\omega_i{}^j = c^j_{ki}\theta^k$. We will derive conditions on the coefficients c^j_{ki} which determine ∇ uniquely. We also write $\mathrm{d}\theta^i = b^i_{jk}\theta^j \otimes \theta^k$ for some b^i_{jk} satisfying $b^i_{jk} + b^i_{kj} = 0$. Since ∇ is torsion-free, we have

$$\begin{aligned}b^i_{jk}\theta^j \otimes \theta^k &= \mathrm{d}\theta^i \\ &= \theta^j \wedge \omega_j{}^i \\ &= c^i_{kj}\theta^j \wedge \theta^k \\ &= c^i_{kj}(\theta^j \otimes \theta^k - \theta^k \otimes \theta^j) \\ &= (c^i_{kj} - c^i_{jk})\theta^j \otimes \theta^k.\end{aligned} \tag{1.2.15}$$

We therefore have the relation $b^i_{jk} = c^i_{kj} - c^i_{jk}$. We will need the following two additional equations obtained by permuting indices:

$$\begin{aligned}b^j_{ki} &= c^j_{ik} - c^j_{ki}, \\ b^k_{ij} &= c^k_{ji} - c^k_{ij}.\end{aligned} \tag{1.2.16}$$

By skew-symmetry of ω, we have $c^i_{kj} = -c^j_{ki}$ for all i, j, k. We then compute

$$\begin{aligned} c^i_{kj} &= b^i_{jk} + c^i_{jk} \\ &= b^i_{jk} - c^k_{ji} \\ &= b^i_{jk} - b^k_{ij} - c^k_{ij} \\ &= b^i_{jk} - b^k_{ij} + c^j_{ik} \\ &= b^i_{jk} - b^k_{ij} + b^j_{ki} + c^j_{ki} \\ &= b^i_{jk} - b^k_{ij} + b^j_{ki} - c^i_{kj}. \end{aligned} \tag{1.2.17}$$

So $c^i_{kj} = \frac{1}{2}(b^i_{jk} - b^k_{ij} + b^j_{ki})$. □

There are a number of other ways to compute the Levi-Civita connection, one of the most common being the *Koszul formula* given by

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X), \tag{1.2.18}$$

or the formula for the Christoffel symbols in local coordinates:

$$\Gamma^i_{jk} = \frac{1}{2} g^{li} \left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right). \tag{1.2.19}$$

The Levi-Civita connection on a Riemannian manifold produces some useful operations on smooth functions. Given $f \in C^\infty(M)$, define its *gradient* to be

$$\operatorname{grad} f := (\mathrm{d}f)^\# \in \Gamma(TM). \tag{1.2.20}$$

So if e_i is a local frame for TM, then $\operatorname{grad} f = g^{ij} f_i e_j$, where $f_i = e_i(f)$. In particular, for local coordinates x^i, we have

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}. \tag{1.2.21}$$

The *Hessian* of f is defined by

$$\operatorname{Hess} f := \nabla(\mathrm{d}f) \in \Gamma(T^*M \otimes T^*M). \tag{1.2.22}$$

Explicitly,

$$\operatorname{Hess} f = \nabla(f_i \theta^i) = (\mathrm{d}f_j - f_i {\omega_j}^i) \otimes \theta^j, \tag{1.2.23}$$

where θ^i is the coframe for T^*M dual to e_i. Given two vector fields $X, Y \in \Gamma(TM)$, we may compute

$$\begin{aligned} \operatorname{Hess} f(X,Y) &= (\mathrm{d}f_j(X) - f_i {\omega_j}^i(X)) Y^j \\ &= X(Yf) - (\nabla_X Y) f. \end{aligned} \tag{1.2.24}$$

Note that since ∇ is torsion-free, we have $\mathrm{Alt}_2(\nabla \mathrm{d}f) = 0$, so $\operatorname{Hess} f = \operatorname{Sym}_2(\nabla \mathrm{d}f) + \mathrm{Alt}_2(\nabla \mathrm{d}f) = \operatorname{Sym}_2(\nabla \mathrm{d}f)$. That is, $\operatorname{Hess} f$ is a symmetric section of $T^*M \otimes T^*M$. Finally, the *Laplacian* of f is defined by

$$\Delta f := g^{ij} \operatorname{Hess} f(e_i, e_j). \tag{1.2.25}$$

In coordinates, we can calculate this to be

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right). \tag{1.2.26}$$

Chapter 2

Riemannian Submanifolds

2.1 Decomposition of the Levi-Civita Connection

Let M be an n-manifold, (N,g) an $(n+p)$-dimensional Riemannian manifold, and $u\colon M \to N$ an immersion. Equip M with the pullback metric u^*g, traditionally called the *first fundamental form*. Since the bundle map $\mathrm{d}u\colon TM \to u^*TN$ is injective by assumption, we may take the orthonormal decomposition $u^*TN \cong TM \oplus NM$, and we call NM the *normal bundle*. Explicitly, $Y_p \in T_{u(p)}N$ is in N_pM if and only if $g_{u(p)}(Y_p, \mathrm{d}u_p(X_p)) = 0$ for all $X_p \in T_pM$. Similarly, we decompose $u^*T^*N \cong T^*M \oplus N^*M$, and call N^*M the *conormal bundle*. From now on, we will suppress the $\mathrm{d}u$, identifying $\mathrm{d}u_p(X_p)$ and X_p. Similarly, we will suppress "$\circ u$" whenever it appears. This means if $X \in \Gamma(TN)$ is a vector field on N and $p \in M$ is a point, then X_p means $X_{u(p)}$.

We will denote indices by $A,B,C,\dots \in \{1,\dots,n+p\}$, $i,j,k,\dots \in \{1,\dots,n\}$, and $\alpha,\beta,\gamma,\dots \in \{n+1,\dots,n+p\}$. Suppose e_A is a local g-orthonormal frame for N with dual coframe θ^A, and write $\nabla^N e_A = \omega_A{}^B \otimes e_B$, where ∇^N is the Levi-Civita connection on N. Assume e_A is an *adapted frame*, which means e_i (more precisely, $(\mathrm{d}u)^{-1}(e_i \circ u)$) is a u^*g-orthonormal frame for TM, and e_α (more precisely, $e_\alpha \circ u$) is a $g\circ u$-orthonormal frame for NM. Now, the pullback $u^*\theta^\alpha$ is evidently zero. Also, since ∇^N is torsion-free, we have

$$\mathrm{d}\theta^i + \omega_j{}^i \wedge \theta^j + \omega_\beta{}^i \wedge \theta^\beta = 0. \tag{2.1.1}$$

We pull this back to find

$$\mathrm{d}u^*\theta^i + u^*\omega_j{}^i \wedge u^*\theta^j = 0. \tag{2.1.2}$$

Since the matrix ω is skew-symmetric, the matrix $u^*\omega$ of pulled-back forms must also be skew-symmetric, implying (along with the above torsion-free property) that $u^*\omega_i{}^j$ is the matrix of connection 1-forms for the Levi-Civita connection ∇^M on M with respect to the frame e_i, or equivalently, with respect to the coframe $u^*\theta^i$. Of course, $u^*\omega_A{}^B$ is the matrix of connection 1-forms for the pullback connection $u^*\nabla^N$. We now decompose

$$\begin{aligned} u^*\nabla^N e_i &= (u^*\nabla^N e_i)^\top + (u^*\nabla^N e_i)^\perp \\ &= u^*\omega_i{}^j \otimes e_j + u^*\omega_i{}^\beta \otimes e_\beta \\ &= \nabla^M e_i + A e_i, \end{aligned} \tag{2.1.3}$$

where $A\colon \Gamma(TM) \to \Gamma(T^*M \otimes NM)$ is given by $Ae_i := (u^*\nabla^N e_i)^\perp$. Immediately from the definition, we see that A is $C^\infty(M)$-linear, and therefore a section of $T^*M \otimes NM \otimes T^*M$. Define the *second fundamental form* $\mathrm{II}\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(NM)$ by

$$\mathrm{II}(X,Y) := AX(Y) = (u^*\nabla^N_Y X)^\perp. \tag{2.1.4}$$

More precisely, this should be $(u^*\nabla^N_Y \mathrm{d}u(X))^\perp$. The second fundamental form is symmetric. Indeed, given the torsion-free condition $\mathrm{d}\theta^\alpha + \omega_j{}^\alpha \wedge \theta^j + \omega_\beta{}^\alpha \wedge \theta^\beta = 0$, we may pullback to find $u^*\omega_j{}^\alpha \wedge u^*\theta^j = 0$. It follows that $u^*\omega_j{}^\alpha = h^\alpha_{ji} u^*\theta^i$ for some smooth functions satisfying $h^\alpha_{ji} = h^\alpha_{ij}$. With this, we have

$$\begin{aligned} \mathrm{II}(e_i, e_j) &= u^*\omega_i{}^\beta(e_j)e_\beta \\ &= h^\beta_{ij} e_\beta \\ &= h^\beta_{ji} e_\beta \\ &= \mathrm{II}(e_j, e_i). \end{aligned} \tag{2.1.5}$$

Having decomposed $u^*\nabla^N$ on $\Gamma(TM)$, we now decompose it on $\Gamma(NM)$. We have

$$\begin{aligned} u^*\nabla^N e_\alpha &= (u^*\nabla^N e_\alpha)^\top + (u^*\nabla^N e_\alpha)^\perp \\ &= u^*\omega_\alpha{}^j \otimes e_j + u^*\omega_\alpha{}^\beta \otimes e_\beta \\ &= Se_\alpha + \nabla^\perp e_\alpha, \end{aligned} \tag{2.1.6}$$

where $S\colon \Gamma(NM) \to \Gamma(T^*M \otimes TM)$ is the *shape operator* or *Weingarten map*, and $\nabla^\perp\colon \Gamma(NM) \to \Gamma(T^*M \otimes NM)$ is the induced connection on NM. Similarly to A, the shape operator is $C^\infty(M)$-linear, and hence a section of $TM \otimes T^*M \otimes N^*M$. The shape operator and second fundamental form are related:

$$g(SX(Z), Y) = -g(X, \mathrm{II}(Y,Z)) \quad \text{for all } X \in \Gamma(NM) \text{ and } Y, Z \in \Gamma(TM). \tag{2.1.7}$$

To see this, we calculate

$$\begin{aligned} g(Se_\alpha(Z), e_i) &= g(u^*\omega_\alpha{}^j(Z)e_j, e_i) \\ &= u^*\omega_{\alpha i}(Z) \\ &= -u^*\omega_{i\alpha}(Z) \\ &= -g(e_\alpha, u^*\omega_i{}^\beta(Z)e_\beta) \\ &= -g(e_\alpha, \mathrm{II}(e_i, Z)). \end{aligned} \tag{2.1.8}$$

Linearity implies the result holds for general vector fields.

2.2 Decomposition of the Curvature

We write $\Omega^N = \mathrm{d}\omega + \omega \wedge \omega$ for the matrix of curvature 2-forms of ∇^N with respect to the frame e_A. Pulling back to M, we have

$$\begin{aligned} u^*(\Omega^N)_i{}^j &= (\mathrm{d}(u^*\omega_i{}^j) + u^*\omega_k{}^j \wedge u^*\omega_i{}^k) + u^*\omega_\alpha{}^j \wedge u^*\omega_i{}^\alpha \\ &= u^*(\Omega^M)_i{}^j + u^*\omega_\alpha{}^j \wedge u^*\omega_i{}^\alpha \\ &= u^*(\Omega^M)_i{}^j - \delta_{\alpha\beta} h^\alpha_{kj} h^\beta_{li} \theta^k \wedge \theta^l. \end{aligned} \tag{2.2.1}$$

These is called the *Gauss equations*. We call $u^*(\Omega^N)_i{}^j$ the *ambient curvature* of M, $(\Omega^M)_i{}^j$ the *intrinsic curvature* of M, and $u^*\omega_\alpha{}^j \wedge u^*\omega_i{}^\alpha$ the *extrinsic curvature* of M. In the special case where $n = 2$ and $N = \mathbb{R}^3$, i.e. M is a surface embedded in three-dimensional Euclidean space, the ambient curvature is zero, and skew-symmetry of implies the only nonzero elements of Ω^M are $\Omega^M_{12} = -\Omega^M_{21}$. By the Gauss equation,

$$\Omega^M_{12} = -(h_{22}h_{11} - (h_{12})^2)\theta^1 \wedge \theta^2 = -\det \mathrm{II}\, \theta^1 \wedge \theta^2. \tag{2.2.2}$$

We call $K := \det \mathrm{II}$ the *Gaussian curvature* of M. More generally, consider the immersion $M^n \hookrightarrow \mathbb{R}^{n+1}$ of a hypersurface in Euclidean space. The Gauss equations read

$$\Omega^M_{ij} = h_{kj}h_{li}\theta^k \wedge \theta^l. \tag{2.2.3}$$

Diagonalizing the matrix (h_{ij}), we obtain an orthonormal coframe θ^i such that $h_{ij} = \kappa_i \delta_{ij}$ for some smooth functions κ_i, called the *principal curvatures* of M. We therefore have

$$\Omega^M_{ij} = \kappa_j \kappa_i \theta^j \wedge \theta^i. \tag{2.2.4}$$

Define the *curvature operator* $\mathscr{R}\colon \Omega^2(M) \to \Omega^2(M)$ by $\mathscr{R}(\theta^i \wedge \theta^j) = -\Omega^M_{ij}$. The above equation shows that $\theta_i \wedge \theta_j$ are eigenvectors of $\mathscr{R}$. In general, eigenvectors of the curvature operator can have high rank, where the *rank* of $\theta \in \Omega^2(M)$ is the least $r \geq 0$ such that $\theta = \sum_{k=1}^r \alpha_k \theta_{i_k} \wedge \theta_{j_k}$. Our calculations show that if the eigenvectors of the curvature operator have rank strictly greater than 1, then M cannot be isometrically immersed as a hypersurface in Euclidean space.

Next, we consider the curvature $\Omega^\perp$ of the normal bundle (namely, of the connection $\nabla^\perp$). As before, we decompose

$$\begin{aligned} u^*(\Omega^N)_\alpha{}^\beta &= (\Omega^\perp)_\alpha{}^\beta + u^*\omega_i{}^\beta \wedge u^*\omega_\alpha{}^i \\ &= (\Omega^\perp)_\alpha{}^\beta - \delta^{ij} h^\beta_{ki} h^\alpha_{lj} \theta^k \wedge \theta^l. \end{aligned} \tag{2.2.5}$$

These is called the *Ricci equations*.

Finally, the *Codazzi-Mainardi equations* are effectively a tautology:

$$u^*\Omega_i{}^\alpha = \mathrm{d}(u^*\omega_i{}^\alpha) + u^*\omega_j{}^\alpha \wedge u^*\omega_i{}^j + u^*\omega_\beta{}^\alpha \wedge u^*\omega_i{}^\beta. \tag{2.2.6}$$

Chapter 3

Curvature

We will use this chapter to study the curvature operator $\mathscr{R}\colon \Omega^2(M) \to \Omega^2(M)$ further. One thing to note is that $\mathscr{R}$ is symmetric with respect to the induced metric on $\Lambda^2 T^*M$. Indeed, given an orthonormal coframe θ^i for M, we have

$$\begin{aligned} g(\mathscr{R}(\theta^i \wedge \theta^j), \theta^k \wedge \theta^l) &= g(-\frac{1}{2}R_{rsij}\theta^r \wedge \theta^s, \theta^k \wedge \theta^l) \\ &= -R_{klij} \\ &= -R_{ijkl} \\ &= g(\mathscr{R}(\theta^k \wedge \theta^l), \theta^i \wedge \theta^j). \end{aligned} \tag{3.0.1}$$

where the interchange symmetry $R_{klij} = R_{ijkl}$ comes from the following calculation:

$$\begin{aligned} R_{klij} &= -R_{iklj} - R_{likj} \\ &= +R_{ikjl} + R_{lijk} \\ &= -R_{jikl} - R_{kjil} - R_{jlik} - R_{ijlk} \\ &= R_{jilk} + R_{kjli} + R_{jlki} + R_{ijkl} \\ &= 2R_{ijkl} - R_{lkji} \\ &= 2R_{ijkl} - R_{klij}. \end{aligned} \tag{3.0.2}$$

Adding R_{klij} to both sides and dividing by 2 gives the desired symmetry. From this, we see that $\mathscr{R}$ is a section of the peculiar bundle $\mathrm{Sym}^2(\Lambda^2 T^*M)$. The idea is to decompose this space into "irreducible" parts.

Let V be a vector space, and choose operators $S, T \in \mathrm{Sym}^2(V)$. Define the *Kulkarni-Nomizu product* $S \owedge T\colon \Lambda^2 V \to \Lambda^2 V$ of S and T by

$$(S \owedge T)(v \wedge w) := Sv \wedge Tw - Sw \wedge Tv. \tag{3.0.3}$$

Then $S \owedge T$ lies in $\mathrm{Sym}^2(\Lambda^2 V)$. In the other direction, define the *Ricci contraction* $\mathrm{ric}\colon \mathrm{Sym}^2(\Lambda^2 V) \to \mathrm{Sym}^2(V)$ to be a certain trace given by

$$\mathrm{ric}\,\mathscr{R}(v,w) = \sum_{i=1}^{n} g(\mathscr{R}(v \wedge e_i), w \wedge e_i), \tag{3.0.4}$$

where e_i is an orthonormal basis for V. In components,

$$\begin{aligned}
R_{ij} &= \operatorname{ric}\mathscr{R}(e_i, e_j) \\
&= \sum_{k=1}^{n} g(\mathscr{R}(e_i \wedge e_k), e_j \wedge e_k) \\
&= -\sum_{k=1}^{n} g\left(\frac{1}{2} R^{rs}{}_{ik} e_r \wedge e_s, e_j \wedge e_k\right) \\
&= -\sum_{k=1}^{n} R_{jkik} \\
&= -R_{jki}{}^{k} \\
&= R_{kji}{}^{k}.
\end{aligned} \tag{3.0.5}$$

We define the *scalar curvature* S of $\mathscr{R}$ to be the trace of $\operatorname{ric}\mathscr{R}$. Thus

$$S = \delta^{ij} R_{ij} = \delta^{ij} R_{kji}{}^{k}. \tag{3.0.6}$$

Chapter 4

Geodesics

4.1 The Exponential Map

Let M be a manifold, and let ∇ be a connection on TM. Fix a smooth path $\gamma\colon [0,1] \to M$, and let $D_t = \gamma^* \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}$. We say γ is a *geodesic* with respect to ∇ if $D_t \gamma' = 0$. That is, if γ' is parallel along γ. In local coordinates, $D_t \gamma'$ is written

$$\left(\frac{\mathrm{d}^2 \gamma^i}{\mathrm{d}t^2} + \frac{\mathrm{d}\gamma^k}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \Gamma^i_{kj} \right) \frac{\partial}{\partial x^i}. \tag{4.1.1}$$

Therefore γ is (locally) a geodesic if and only if it satisfies the *geodesic equation*

$$\frac{\mathrm{d}^2 \gamma^i}{\mathrm{d}t^2} + \frac{\mathrm{d}\gamma^k}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \Gamma^i_{kj} = 0 \tag{4.1.2}$$

Given initial conditions $\gamma(0) = p \in M$ and $\gamma'(0) = v \in T_p M$, ODE theory guarantees the existence of a maximal interval $I_{p,v} \subseteq \mathbb{R}$ and a unique maximal solution $\gamma_{p,v} \colon I_{p,v} \to M$ depending smoothly on the initial conditions. We say M is *geodesically complete* if $I_{p,v} = \mathbb{R}$ for all $(p,v) \in TM$.

Given a smooth path $\gamma\colon I \to M$ and $\lambda \in \mathbb{R} \setminus \{0\}$, define $\gamma_\lambda \colon \lambda^{-1} I \to M$ by $\gamma_\lambda(t) = \gamma(\lambda t)$. Then $\gamma_\lambda'(t) = \lambda \gamma'(t)$, and so

$$\frac{\mathrm{d}^2 \gamma_\lambda^i}{\mathrm{d}t^2} + \frac{\mathrm{d}\gamma_\lambda^k}{\mathrm{d}t} \frac{\mathrm{d}\gamma_\lambda^j}{\mathrm{d}t} \Gamma^i_{kj} = \lambda^2 \left(\frac{\mathrm{d}^2 \gamma^i \mathrm{d}t^2}{+} \frac{\mathrm{d}\gamma^k}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \right). \tag{4.1.3}$$

Therefore, if γ is the geodesic with initial conditions $\gamma(0) = p$, $\gamma'(0) = v$, and maximal interval $I_{p,v} \subseteq \mathbb{R}$, then γ_λ is a geodesic with initial conditions $\gamma_\lambda(0) = p$, $\gamma'(0) = \lambda v$, and maximal interval $\lambda^{-1} I_{p,v}$. In particular, $\gamma_{p,\lambda v}(t) = \gamma_{p,v}(\lambda t)$. Thus, for $\lambda > 0$ sufficiently small, the maximal interval $I_{p,\lambda v} = \lambda^{-1} I_{p,v}$ contains 1. It follows that there exists an open set $U_p \subseteq T_p M$ (which we may choose to be star-shaped) containing 0 such that $\gamma_{p,v}(1)$ exists for all $v \in U_p$. Define the *exponential map* $\exp_p^\nabla \colon U_p \to M$ by $\exp_p^\nabla(v) := \gamma_{p,v}(1)$. We also define $U := \coprod_{p \in M} U_p \subseteq TM$, which is a neighborhood of (the image of) the zero section. Define $\mathrm{Exp}^\nabla \colon U \to M \times M$ by $\mathrm{Exp}^\nabla(p,v) := (p, \exp_p^\nabla(v))$. Since $\gamma_{p,v}$ depends smoothly on initial conditions, the maps $\exp_p^\nabla$ and Exp^∇ are smooth.

Let's calculate the differentials of the exponential maps. We identify T_0T_pM and T_pM in the natrual way. Given $w \in T_pM$, define the curve $\xi(t) = tw \in T_pM$. Then $\xi(0) = 0$ and $\xi'(t) = w$, so we then calculate

$$\begin{aligned} \mathrm{d}\exp_p^\nabla(0)(w) &= \frac{\mathrm{d}}{\mathrm{d}t}\exp_p^\nabla(\xi(t))|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma_{p,tw}(1)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma_{p,w}(t)|_{t=0} \\ &= w. \end{aligned} \tag{4.1.4}$$

It follows that $\mathrm{d}\exp_p^\nabla(0) = \mathrm{id}_{T_pM}$. In particular, $\mathrm{d}\operatorname{Exp}^\nabla(p,0)(0,w) = (0,w)$. On the other hand, choose a curve $\xi(t)$ in M with $\xi(0) = p$ and $\xi'(0) = v$. Then

$$\begin{aligned} \mathrm{d}\operatorname{Exp}^\nabla(p,0)(v,0) &= \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Exp}^\nabla(\xi(t),0)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\xi(t),\xi(t))|_{t=0} \\ &= (v,v). \end{aligned} \tag{4.1.5}$$

It follows that $\mathrm{d}\operatorname{Exp}^\nabla(p,0)$ is given by the matrix

$$\begin{pmatrix} \mathrm{id}_{T_pM} & 0 \\ \mathrm{id}_{T_pM} & \mathrm{id}_{T_pM} \end{pmatrix} \tag{4.1.6}$$

Since this is invertible, the Inverse Function Theorem ensures there is a (potentially smaller) neighborhood U of the zero section on which $\operatorname{Exp}^\nabla$ is a diffeomorphism onto its image. In particular, $\exp_p^\nabla$ is a diffeomorphism from U_p onto its image.

4.2 Variations of Length and Energy

Fix a Riemannian manifold (M,g), and let ∇ be the Levi-Civita connection. A *path* in M is a smooth map $\gamma\colon I \to M$, and its image is a *curve*. A path is *regular* if it is an immersion, and the image of a regular path is a *smooth* curve. The *length* of a path $\gamma\colon [0,1] \to M$ (and its corresponding curve) is defined by

$$\ell(\gamma) := \int_0^1 \left|\gamma'(t)\right| \,\mathrm{d}t, \tag{4.2.1}$$

and its *energy* is

$$E(\gamma) := \frac{1}{2}\int_0^1 \left|\gamma'(t)\right|^2 \,\mathrm{d}t. \tag{4.2.2}$$

A *variation* of a path $\gamma\colon [0,1] \to M$ is a smooth map $F\colon [0,1] \times (-\varepsilon,\varepsilon) \to M$ such that $F(\cdot,0) = \gamma$. We will usually write $\gamma_s = F(\cdot,s)$. The interval $[0,1]$ has coordinate t and $(-\varepsilon,\varepsilon)$ has coordinate s. We define $V := \mathrm{d}F\left(\frac{\partial}{\partial s}\right) \in \Gamma(F^*(TM))$ to be the *variation vector field*, and $T := \mathrm{d}F\left(\frac{\partial}{\partial t}\right) \in \Gamma(F^*(TM))$ to be the *tangent vector field*. We also write $D_s := F^*\nabla_{\frac{\partial}{\partial s}}$ and

$D_t := F^*\nabla_{\frac{\partial}{\partial t}}$. Since ∇ is compatible with g, the pullback connection $F^*\nabla$ is compatible with $g \circ F$. We then calculate

$$\begin{aligned}\frac{\partial}{\partial s}E(\gamma_s) &= \frac{1}{2}\int_0^1 \frac{\partial}{\partial s}\left|\gamma_s'\right|^2\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^1 \mathrm{d}(g(T,T))\left(\frac{\partial}{\partial s}\right)\,\mathrm{d}t \\ &= \int_0^1 g(D_sT,T)\,\mathrm{d}t.\end{aligned} \tag{4.2.3}$$

To proceed, we will use the fact that $D_sT = D_tV$. To see this, pick local coordinates x^i on M. Let ω be the corresponding matrix of Levi-Civita connection 1-forms. By the torsion-free property of ∇, we have $\omega_j{}^i \wedge \mathrm{d}x^j = 0$. Pulling this back by F and applying it to $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$, we have

$$F^*\omega_j{}^i\left(\frac{\partial}{\partial s}\right)\mathrm{d}x^j(T) = F^*\omega_j{}^i\left(\frac{\partial}{\partial t}\right)\mathrm{d}x^j(V). \tag{4.2.4}$$

By definition,

$$D_tV = \left(\frac{\partial}{\partial t}\mathrm{d}x^i(V) + F^*\omega_j{}^i\left(\frac{\partial}{\partial t}\right)\mathrm{d}x^j(V)\right)\frac{\partial}{\partial x^i}. \tag{4.2.5}$$

It then suffices to notice that

$$\mathrm{d}F\left(\left[\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right]\right) = \left(\frac{\partial}{\partial s}\mathrm{d}x^i(T) - \frac{\partial}{\partial t}\mathrm{d}x^i(V)\right)\frac{\partial}{\partial x^i}. \tag{4.2.6}$$

But since $[\frac{\partial}{\partial s},\frac{\partial}{\partial t}] = 0$, we are done. With this "symmetry lemma" in hand, we return to equation (4.2.3) and calculate

$$\begin{aligned}\int_0^1 g(D_sT,T)\,\mathrm{d}t &= \int_0^1 g(D_tV,T)\,\mathrm{d}t \\ &= \int_0^1 \frac{\partial}{\partial t}g(V,T) - g(V,D_tT)\,\mathrm{d}t \\ &= g(V,T)|_{t=0}^1 - \int_0^1 g(V,D_tT)\,\mathrm{d}t.\end{aligned} \tag{4.2.7}$$

Noting that this expression depends only on V and γ, and not the particular variation F used to generate V, we define the *first variation of energy* to be

$$\delta E(\gamma)(V) := \left.\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s)\right|_{s=0}. \tag{4.2.8}$$

We can do a similar calculation for the length functional ℓ. In this case, suppose $F\colon [0,1]\times$

$(-\varepsilon,\varepsilon) \to M$ is a variation of γ such that γ_s is a regular path for each $s \in (-\varepsilon,\varepsilon)$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\ell(\gamma_s) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} g(T,T)^{1/2}\,\mathrm{d}t \\ &= \int_0^1 g\left(D_sT, \frac{T}{|T|}\right)\,\mathrm{d}t \\ &= \int_0^1 g\left(D_tV, \frac{T}{|T|}\right)\,\mathrm{d}t \\ &= g\left(V, \frac{T}{|T|}\right)\Bigg|_{t=0}^1 - \int_0^1 g\left(V, D_t\frac{T}{|T|}\right)\,\mathrm{d}t.\end{aligned} \tag{4.2.9}$$

Note that geodesics have constant speed. Indeed, if $\gamma\colon [0,1] \to M$ is a geodesic, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\gamma'\right|^2 = 2g(D_t\gamma',\gamma') = 0. \tag{4.2.10}$$

From this, we see that critical points for E are precisely the critical points of ℓ parameterized at constant speed.

We can now prove the following lemma, whose overall message is "there exist nice coordinates on a Riemannian manifold".

Lemma 4.2.1 (Gauss)**.** *Choose $p \in M$, and let $\exp_p\colon U_p \to M$ be the exponential map with respect to the Levi-Civita connection. Suppose $v,w \in U_p$ are orthogonal. Then $\mathrm{d}\exp_p(v)(v)$ and $\mathrm{d}\exp_p(v)(w)$ are orthogonal.*

Proof. Define $\gamma\colon [0,1] \to T_pM$ by $\gamma(t) = (1+t)v$. Then $\gamma(0) = \gamma'(0) = v$, so

$$\begin{aligned}\mathrm{d}\exp_p(v)(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\exp_p(\gamma(t))\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma_{p,(1+t)v}(1)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma_{p,v}(1+t)\Big|_{t=0} \\ &= \gamma'_{p,v}(1).\end{aligned} \tag{4.2.11}$$

On the other hand, define $\sigma\colon (-\varepsilon,\varepsilon) \to T_pM$ by $\sigma(s) := (\cos s)v + (\sin s)w$. Then $\sigma(0) = v$ and $\sigma'(0) = w$, so

$$\mathrm{d}\exp_p(v)(w) = \frac{\mathrm{d}}{\mathrm{d}s}\exp_p(\sigma(s))\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_{p,(\cos s)v+(\sin s)w}(1)\Big|_{s=0}. \tag{4.2.12}$$

Consider the variation $F\colon [0,1]\times(-\varepsilon,\varepsilon) \to M$ of $\gamma_{p,v}$ defined by $F(t,s) = \gamma_{p,(\cos s)v+(\sin s)w}(t)$. Since $\gamma_{p,v}$ is a geodesic, the first variation formula tells us

$$g(V(0,0),\gamma'_{p,v}(0)) = g(V(0,1),\gamma'_{p,v}(1)), \tag{4.2.13}$$

where V is the variation vector field of F. By our above calculation, we see $V(0,1) = \mathrm{d}\exp_p(v)(w)$. On the other hand, we have $\gamma'_{p,v}(1) = \mathrm{d}\exp_p(v)(v)$ by the previous calculation, and $V(0,0) = 0$ clearly. It follows that $g(\mathrm{d}\exp_p(v)(w),\mathrm{d}\exp_p(v)(v)) = 0$, as required. $\square$

Fix $p \in M$. Let $\varepsilon_p > 0$ be such that $\exp_p$ is a diffeomorphism from the open ball $B(0,\varepsilon_p)$ to its image, denoted $B(p,\varepsilon_p)$. ...

We also have that geodesics are locally minimizing in the following sense: pick $q \in B(p,\varepsilon_p)$, and let $v \in B(0,\varepsilon_p)$ be such that $q = \exp_p(v)$. The length of the geodesic $\gamma_{p,v}$ joining p and q is given by

$$\ell(\gamma_{p,v}) = \int_0^1 \left|\gamma_{p,v}'\right| \,\mathrm{d}t = \int_0^1 |v| \,\mathrm{d}t = |v|, \tag{4.2.14}$$

using the fact that geodesics have constant speed, and $\gamma_{p,v}'(0) = v$. Is threre a curve joining p and q of length less than $|v|$? Such a curve must leave $B(p,\varepsilon_p)$ since if it minimizes length, then it is parameterized by a geodesic. But of course, by uniqueness, the only geodesic joining p and q in $B(p,\varepsilon_p)$ is $\gamma_{p,v}$. Now, let $\gamma\colon [0,1] \to M$ be a path joining p and q which leaves $B(p,\varepsilon_p)$. Then we can find $t^* \in (0,1)$ such that $\gamma(t) \in B(p,\varepsilon_p)$ for all $t \in [0,t^*)$, but $\gamma(t^*) \notin B(p,\varepsilon_p)$. Write $\gamma(t) = \exp_p(r(t)\omega(t))$ for all $t \in [0,t^*)$, where $r(t) \in [0,\varepsilon_p)$ and $\omega(t) \in \partial B(0,1) \subseteq T_pM$. By the Gauss lemma,

$$\begin{aligned}
\ell(\gamma) &= \int_0^1 \left|\gamma'\right| \,\mathrm{d}t \\
&> \int_0^{t^*} \left|\gamma'\right| \,\mathrm{d}t \\
&\geq \int_0^{t^*} r' \,\mathrm{d}t \\
&= r(t^*) \\
&= \varepsilon_p \\
&> |v| \\
&= \ell(\gamma_{p,v}),
\end{aligned} \tag{4.2.15}$$

therefore showing that any curve which leaves $B(p,\varepsilon_p)$ cannot minimize the length between p and q.

4.3 Metric Space Structure

Given a Riemannian manifold (M,g), we define a metric on M via

$$d(p,q) := \inf\{\ell(\gamma) : \gamma \text{ is a curve joining } p \text{ and } q\}. \tag{4.3.1}$$

The only metric space axiom we need to check is $d(p,q) = 0$ implies $p = q$. To see this, suppose $d(p,q) = 0$, and let $\varepsilon_p > 0$ be such that $\exp_p\colon B(0,\varepsilon_p) \to B(p,\varepsilon_p)$ is a diffeomorphism. Since $d(p,q) = 0$, we can find a curve γ of length less than ε_p joining p and q. By our observations in the previous section, this means γ lies in $B(p,\varepsilon_p)$, and so we can write $q = \exp_p(v)$ for some $v \in B(0,\varepsilon_p)$. Again by the previous section, the unique curve of shortest length joining p and q in $B(p,\varepsilon_p)$ is given by $\gamma_{p,v}$. Of course, since $d(p,q) = 0$, this means $|v| = \ell(\gamma_{p,v}) = 0$, and so $q = \exp_p(v) = p$.

This metric turns out to generate the topology on M, so a Riemannian manifold is metrizable. The following theorem concerns the global topology of a Riemannian manifold:

Theorem 4.3.1 (Hopf-Rinow)**.** *Let (M,g) be a Riemannian manifold, and d the above metric on M. Then d is a complete metric if and only if M is geodesically complete. Furthermore, if M is geodesically complete, then for all $p,q \in M$, there exists a geodesic joining p and q.*

Completeness is important - even $\mathbb{R}^2 \setminus \{0\}$ doesn't have a geodesic joining $(-1,1)$ and $(1,-1)$.

4.4 Second Variation Formula

Recall our earlier calculation of the first variation $\delta E(\gamma)$ of a path $\gamma\colon [0,1] \to M$. In particular, we calculated

$$\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s) = \int_0^1 g(D_t V, T)\ \mathrm{d}t. \tag{4.4.1}$$

We differentiate this further to find

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(\gamma_s) &= \int_0^1 g(D_s D_t V, T) + g(D_t V, D_s T)\ \mathrm{d}t \\
&= \int_0^1 g(R(V,T)V - D_t D_s V, T) + g(D_t V, D_s T)\ \mathrm{d}t \\
&= \int_0^1 g(R(V,T)V, T)\ \mathrm{d}t - g(D_s V, T)\big|_{t=0}^1 + \int_0^1 g(D_s V, D_t T)\ \mathrm{d}t + \int_0^1 g(D_t V, D_t V)\ \mathrm{d}t \\
&= -\int_0^1 g(R(V,T)T + D_t D_t V, V)\ \mathrm{d}t + \int_0^1 g(D_s V, D_t T)\ \mathrm{d}t + \left(-g(D_s V, T) + g(D_t V, V)\right)\big|_{t=0}^1 .
\end{aligned} \tag{4.4.2}$$

Here, R is the curvature of M. To justify the above expression involving the curvature, given a frame e_i for M, we have the expression

$$\begin{aligned}
D_s D_t (e_i \circ F) - D_t D_s (e_i \circ F) &= R^{F^*\nabla}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)(e_i \circ F) \\
&= \Omega_i{}^j (V,T)(e_j \circ F) \\
&= R(V,T)(e_i \circ F).
\end{aligned} \tag{4.4.3}$$

Since R is tensorial, it follows that $R(V,T)V = D_s D_t V - D_t D_s V$. Suppose now that two things are true: γ is a geodesic, in which case $D_t T(t,0) = 0$ for all $t \in [0,1]$, and γ_s is a so-called *proper variation* of γ, meaning endpoints are preserved. Then $V(0,s) = V(1,s) = D_s V(0,s) = D_s V(1,s) = 0$ for all $s \in (-\varepsilon, \varepsilon)$. We then evaluate the above expression at $s = 0$ to find the *second variation formula*

$$\delta^2 E(\gamma_s)(V) = \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(\gamma_s)\right|_{s=0} = -\int_0^1 g(R(V,T)T + D_t D_t V, V)\ \mathrm{d}t \tag{4.4.4}$$

A vector field J along a geodesic γ is called a *Jacobi field* if it satisfies

$$R(J,\gamma')\gamma' + D_t D_t J = 0. \tag{4.4.5}$$

Let's now calculate some Jacobi fields in a certain special case. Take $e_n = \gamma'/|\gamma'|$, and extend this to a parallel orthonormal frame e_i along γ. Write

$$R(e_i,\gamma')\gamma' = |\gamma'|^2 R(e_i,e_n)e_n = |\gamma'|^2 R_{inn}{}^j e_j. \tag{4.4.6}$$

Then for a vector field $J = J^i e_i$ along γ, the Jacobi field equation reads

$$\frac{\mathrm{d}^2 J^i}{\mathrm{d}t^2} + J^j R_{jnn}{}^i |\gamma'|^2 = 0. \tag{4.4.7}$$

Suppose M has constant sectional curvature, meaning we may write $R_{jnn}{}^i = c\delta_j{}^i$ for some $c \in \mathbb{R}$ and all $i, j = 1,\dots,n-1$. Then the Jacobi field equation reads

$$\begin{aligned} \frac{\mathrm{d}^2 J^i}{\mathrm{d}t^2} + J^i |\gamma'|^2 = 0, &\quad \text{for } i = 1,\dots,n-1, \\ \frac{\mathrm{d}^2 J^n}{\mathrm{d}t^2} = 0. & \end{aligned} \tag{4.4.8}$$

We can solve this explicitly as follows: we immediately have $J^n(t) = a^n t + b^n$. For $i = 1,\dots,n-1$, we have solutions for three cases of c. Namely,

$$\begin{aligned} J^i(t) &= a^i \sin(\sqrt{c}t) + b^i \cos(\sqrt{c}t) \quad \text{for } c > 0, \\ J^i(t) &= a^i t + b^i \quad \text{for } c = 0, \\ J^i(t) &= a^i \sinh(\sqrt{-c}t) + b^i \cosh(\sqrt{-c}t) \quad \text{for } c < 0. \end{aligned} \tag{4.4.9}$$

Lemma 4.4.1. *Let γ_s be a one-parameter family of geodesics with associated variation vector field V. Then $V(t,0)$ is a Jacobi field.*

Proof. Since each γ is a geodesic, we have $D_t T = 0$, and therefore $D_s D_t T = 0$. It follows that

$$0 = D_s D_t T = R(V,T)T + D_t D_s T = R(V,T)T + D_t D_t V. \tag{4.4.10}$$

Restrict to $s = 0$ to conclude. ☐

In fact, the opposite is true. Namely, every Jacobi field is the variation vector field of a one-parameter family of geodesics.

The theory of ODEs guarantees the existence of a unique Jacobi field J along a geodesic γ with initial conditions $J(0) = v$ and $D_t J(0) = w$ for $v, w \in T_{\gamma(0)}M$. With this in mind, we have the following proposition:

Proposition 4.4.2. *Given $(p,v) \in TM$ such that $\exp_p(v)$ is defined, define the geodesic $\gamma(t) := \exp_p(tv)$ for $t \in [0,1]$. That is, $\gamma = \gamma_{p,v}$. Fix another $w \in T_pM$, and let J be the unique Jacobi field along γ with $J(0) = 0$, $D_t J(0) = w$. Then $J(1) = \mathrm{d}\exp_p(v)(w)$.*

Proof. Define the following one-parameter family of geodesics:

$$\gamma_s(t) := \exp_p(t(v+sw)). \tag{4.4.11}$$

The associated variation vector field V satisfies

$$V(t,s) = \mathrm{d}\exp_p(t(v+sw))(w). \tag{4.4.12}$$

Thus $V(0,0) = 0$ and

$$D_t V(0,0) = D_s T(0,0) = D_s \mathrm{d}\exp_p(0)(v+sw)|_{s=0} = w. \tag{4.4.13}$$

By the previous lemma, we know that $V(t,0)$ is a Jacobi field along γ. It follows by uniqueness that $V(t,0) = J(t)$, and so $J(1) = V(1,0) = \mathrm{d}\exp_p(v)(w)$. $\square$

Two points $p,q \in M$ on a geodesic γ are *conjugate* if there exists a nontrivial Jacobi field along γ vanishing at both p and q.

Corollary 4.4.3. *If p and $\exp_p(v)$ are not conjugate along $\exp_p(tv)$, then $\mathrm{d}\exp_p(v)$ is injective.*

One can show that if a geodesic extends past its first conjugate point, then it cannot be a minimizing geodesic.

We can use Jacobi fields to relate the curvature of a manifold with its topology. Our first step will be the following observation: let J be a Jacobi field along a geodesic γ. Then

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}|J|^2 &= \frac{\mathrm{d}}{\mathrm{d}t} 2g(D_t J, J) \\ &= 2g(D_t D_t J, J) + 2|D_t J|^2 \\ &= 2|D_t J|^2 - 2g(R(J,\gamma')\gamma', J). \end{aligned} \tag{4.4.14}$$

Note that $g(R(J,\gamma')\gamma',J)$ is the sectional curvature of the (parameterized family of) two-planes spanned by J and γ'. Using this, we deduce the following:

Lemma 4.4.4. *Let γ be a geodesic such that the sectional curvature of M along γ is nonpositive. Then γ carries no conjugate points. Thus if M is complete with everywhere nonpositive sectional curvature, then $\mathrm{d}\exp_p(v)$ is invertible for all $(p,v) \in TM$.*

Proof. Let J be a Jacobi field along γ vanishing at 0. Then by (4.4.14), we must have $J = 0$ everywhere, or $|J(t)| > 0$ for all sufficiently large t. For the second statement, recall by the Hopf-Rinow theorem that completeness of M implies $\exp_p(v)$ exists for all $(p,v) \in TM$. Corollary 4.4.3 then implies the statement. $\square$

Theorem 4.4.5 (Cartan-Hadamard)**.** *Let M be a complete and connected Riemannian manifold with nonpositive sectional curvature everywhere. Then, for all $p \in M$, the exponential map $\exp_p$ is a covering map. In particular, the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.*

Theorem 4.4.6 (Bonnet-Myers)**.** *Let M be a complete and connected Riemannian manifold with $\mathrm{Ric}(v,v) \geq (n-1)/R^2$ for all $v \in TM$. Then $\mathrm{diam}(M) \leq \pi R$. In particular, M is compact, and its fundamental group is finite.*

Theorem 4.4.7 (Synge)**.** *Let M be a compact, even dimensional, and oriented Riemannian manifold with strictly positive sectional curvature. Then M is simply connected.*

Theorem 4.4.8 (Killing-Hopf)**.** *Let M be a complete Riemannian manifold with constant sectional curvature c, equal to $-1, 0, 1$ by scaling without loss of generality. Then its universal cover is isometric to*

(i) S^n *if* $c = 1$,

(ii) $\mathbb{R}^n$ *if* $c = 0$,

(iii) $\mathbb{H}^n$ *if* $c = -1$.