# Advanced PDEs

Billy Sumners

April 22, 2019

# Contents

# Chapter 1

# Sobolev Spaces

## 1.1 Sobolev Spaces

Let $\Omega \subseteq \mathbb{R}^n$ be open, $u \in L^1_{\mathrm{loc}}(\Omega)$, and $\alpha \in \mathbb{N}_0^n$ a multiindex. A function $v \in L^1_{\mathrm{loc}}(\Omega)$ is a *weak derivative* of $u$ corresponding to $\alpha$ if

$$\int_\Omega u \partial^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\varphi \,\mathrm{d}x \tag{1.1}$$

for all $\varphi \in C_c^\infty(\Omega)$.

**Lemma 1.1.** *Suppose $v, w \in L^1_{\mathrm{loc}}(\Omega)$ are weak derivatives of $u \in L^1_{\mathrm{loc}}(\Omega)$ corresponding to $\alpha$. Then $v = w$ a.e.*

*Proof.* Given $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega (v - w)\varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega (u - u)\partial^\alpha \varphi \,\mathrm{d}x = 0. \tag{1.2}$$

The proof then follows from the following important lemma. $\square$

**Lemma 1.2** (Fundamental Lemma of the Calculus of Variations)**.** *Suppose $v \in L^1_{\mathrm{loc}}(\Omega)$ satisfies*

$$\int_\Omega v\varphi \,\mathrm{d}x = 0 \tag{1.3}$$

*for all $\varphi \in C_c^\infty(\Omega)$. Then $v = 0$ a.e.*

We will prove this later after introducing mollification. The idea is to approximate $\operatorname{sgn} v$ by a function in $C_c^\infty(\Omega)$.

If $u \in L^1_{\mathrm{loc}}(\Omega)$ has a weak derivative corresponding to $\alpha$, we will write $\partial^\alpha u$ for this weak derivative, interpreting it as a (necessarily unique by the preceding lemma) element of $L^1_{\mathrm{loc}}(\Omega)$. If all weak derivatives of order 1 for $u$ exist, we say $u$ is *weakly differentiable*, and we compile all its derivatives in the *weak gradient* $\nabla u := (\partial_1 u, \dots, \partial_n u)$.

Of course, integration by parts implies that if $u \in C^k(\Omega)$, then all its weak derivatives of order at most $k$ exist, and are equal to the corresponding classical derivatives. Furthermore, if $U \subseteq \Omega$ is open, then $C_c^\infty(U)$ embeds naturally in $C_c^\infty(\Omega)$ (extension by zero), so if $u \in L^1_{\text{loc}}(\Omega)$ has a weak derivative corresponding to $\alpha$, then its restriction to $U$ also has a weak derivative corresponding to $\alpha$, given by the restriction of $\partial^\alpha u$. From these facts, the following two examples follow naturally.

**Example 1.3.** On $\Omega = (-1,1) \subseteq \mathbb{R}$, define

$$u(x) := \begin{cases} 0 & x < 0, \\ x & x \geq 0. \end{cases} \tag{1.4}$$

Then on the open set $(-1,0)$, $u$ has classical derivative 0, and on $(0,1)$, $u$ has classical derivative 1. So if $u$ were weakly differentiable, its weak derivative would be

$$v(x) := \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases} \tag{1.5}$$

Let's check this. Fix $\varphi \in C_c^\infty(\Omega)$. Then

$$\int_{-1}^1 v\varphi \, \mathrm{d}x = \int_0^1 \varphi \, \mathrm{d}x = -\int_0^1 x\varphi'(x) \, \mathrm{d}x = -\int_{-1}^1 u\varphi' \, \mathrm{d}x. \tag{1.6}$$

It follows that $u'$ exists and equals $v$.

**Example 1.4.** On the same $\Omega$, define

$$u(x) := \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases} \tag{1.7}$$

As before, $u$ is classically differentiable on $(-1,0)$ and on $(0,1)$ with classical derivative 0, so this would have to be the weak derivative of $u$ if it exists. However, we claim that $u$ is not weakly differentiable. To see this, fix $\varphi \in C_c^\infty(\Omega)$ with $\varphi(0) \neq 0$. Then

$$\int_{-1}^1 u\varphi' \, \mathrm{d}x = \int_0^1 \varphi' \, \mathrm{d}x = -\varphi(0) \neq 0 = -\int_{-1}^1 0\varphi(x) \, \mathrm{d}x, \tag{1.8}$$

as required.

Let's now define the spaces we will be using for the rest of the course. Let $\Omega \subseteq \mathbb{R}^n$ be open, $p \in [1,\infty]$, and $k \in \mathbb{N}_0$. The *Sobolev space* $W^{k,p}(\Omega)$ is defined to be the space of functions $u \in L^p(\Omega)$ such that for all multiindices $\alpha$ with $|\alpha| \leq k$, the weak derivative $\partial^\alpha u$ exists and lies in $L^p(\Omega)$. The norm on this space is given by

$$\|u\|_{W^{k,p}(\Omega)} := \left( \sum_{|\alpha| \leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p} \qquad p < \infty, \tag{1.9}$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)}. \tag{1.10}$$

Clearly, the Sobolev spaces are vector spaces. We will show their norms are actually norms by embedding them as particularly nice subspaces of a certain $L^p$ space. This embedding will automatically give us some other properties.

Define $\Omega_k$ to be the disjoint union $\coprod_{|\alpha|\leq k}\Omega = \prod_{|\alpha|\leq k}\{\alpha\}\times\Omega$, and equip it with the Lebesgue measure. Define $i\colon W^{k,p}(\Omega)\to L^p(\Omega_k)$ by

$$i(u)(\alpha,x) := \partial^\alpha u(x). \tag{1.11}$$

Then $i$ is a linear isometry as can be easily checked, from which it follows immediately that $\|\cdot\|_{W^{k,p}(\Omega)}$ is a norm. For more intricate properties, we will prove the following:

**Lemma 1.5.** *Under the embedding above, $W^{k,p}(\Omega)$ is a closed subspace of $L^p(\Omega_k)$.*

Indeed, this lemma tells us that $W^{k,p}(\Omega)$ is a Banach space for all $p\in[1,\infty]$, separable for $p\in[1,\infty)$, and reflexive for $p\in(1,\infty)$.

*Proof.* Let $u_i$ be a sequence in $W^{k,p}(\Omega)$ such that $i(u_i)$ is Cauchy in $L^p(\Omega_k)$. Then, for each multiindex $\alpha$, $\partial^\alpha u_i$ converges to some $u^{(\alpha)}$ in $L^p(\Omega)$. We claim that $u^{(0)}$ is in $W^{k,p}(\Omega)$, and $\partial^\alpha u^{(0)} = u^{(\alpha)}$ for all multiindices $\alpha$. Indeed, given $\varphi\in C_c^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega \varphi u^{(\alpha)}\,\mathrm{d}x &= \lim_{i\to\infty}\int_\Omega \varphi\partial^\alpha u_i\,\mathrm{d}x\\ &= \lim_{i\to\infty}(-1)^{|\alpha|}\int_\Omega \partial^\alpha\varphi u_i\,\mathrm{d}x\\ &= \int_\Omega \partial^\alpha\varphi u^{(0)}\,\mathrm{d}x,\end{aligned} \tag{1.12}$$

where passing to limits is possible by Hölder's inequality. Since each $u^{(\alpha)}$ lies in $L^p(\Omega)$, we have that $u^{(0)}$ is in $W^{k,p}(\Omega)$. Finally, since $i(u_i)\to i(u^{(0)})$ in $L^p(\Omega_k)$, it follows that $u_i\to u^{(0)}$ in $W^{k,p}(\Omega)$. □

Since finite-dimensional norms are all equivalent, there are many equivalent norms to put on Sobolev spaces. For example,

$$\|u\| := \sum_{|\alpha|\leq k}\|\partial^\alpha u\|_{L^p(\Omega)} \tag{1.13}$$

is a particularly nice one.

The Sobolev space we will be using most often are $W^{k,2}(\Omega)$, also denoted $H^k(\Omega)$. These spaces gain an inner product, defined by

$$(u,v)_{H^k(\Omega)} := \int_\Omega uv\,\mathrm{d}x. \tag{1.14}$$

## 1.2 Mollification and Approximation

In this section, fix a nonnegative smooth test function $\eta\in C_c^\infty(B(0,1))$ such that $\|\eta\|_{L^1}=1$. Such an $\eta$ is called a *mollifier*. For $h>0$, we define its rescaling $\eta_h\in C_c^\infty(B(0,h))$ by

$$\eta_h(x) := \frac{1}{h^n}\eta\left(\frac{x}{h}\right). \tag{1.15}$$

Given $u \in L^1_{\mathrm{loc}}(\Omega)$, define its *mollification* at scale $h > 0$ by

$$u_h(x) := (\eta_h * u)(x) = \int_{B(x,h)} \eta_h(x-y)u(y)\,\mathrm{d}y, \tag{1.16}$$

whenever $x \in \Omega$ is such that $\overline{B(x,h)} \subseteq \Omega$. Strictly speaking, this condition is not absolutely necessary since we can extend any locally integrable function by zero to all of $\mathbb{R}^n$. It will be necessary shortly, however.

**Lemma 1.6.** *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and let $h > 0$. Then $u_h \in C^\infty(\mathbb{R}^n)$.*

*Proof.* Fix a multiindex $\alpha$. Then

$$\frac{\partial^\alpha}{\partial x^\alpha}\int_{\mathbb{R}^n} \eta_h(x-y)u(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \frac{\partial^\alpha}{\partial x^\alpha}\eta_h(x-y)u(y)\,\mathrm{d}y, \tag{1.17}$$

where we may move the derivative under the integral since the derivative

$$\frac{\partial^\alpha}{\partial x^\alpha}\eta_h(x-y)u(y) \tag{1.18}$$

is bounded by the integrable function $\|\partial^\alpha \eta_h\|_{L^\infty(\mathbb{R}^n)}\,|u|\,\mathbb{1}_{B(x,h)}$ independent of $x \in \mathbb{R}^n$. □

**Lemma 1.7.** *Fix a locally integrable function $u \in L^1_{\mathrm{loc}}(\Omega)$, and a multiindex $\alpha$ such that the weak derivative $\partial^\alpha u$ exists. Suppose $x \in \Omega$ and $h > 0$ is such that $\overline{B(x,h)} \subseteq \Omega$. Then the classical derivative $\partial^\alpha u_h$ exists, and $\partial^\alpha u_h(x) = (\partial^\alpha u)_h(x)$.*

*Proof.* This is a simple calculation:

$$\begin{aligned}\frac{\partial^\alpha}{\partial x^\alpha}\int_\Omega \eta_h(x-y)u(y)\,\mathrm{d}y &= \int_\Omega \frac{\partial^\alpha}{\partial x^\alpha}\eta_h(x-y)u(y)\,\mathrm{d}y \\ &= \int_\Omega (-1)^{|\alpha|}\frac{\partial^\alpha}{\partial x^\alpha}\eta_h(x-y)u(y)\,\mathrm{d}y \\ &= \int_\Omega \eta_h(x-y)\partial^\alpha(y)\,\mathrm{d}y.\end{aligned} \tag{1.19}$$

Again, we can move the derivative inside the integral by the proof of the previous lemma. In the last equality, we use the fact that $y \mapsto \eta_h(x-y)$ is smooth with compact support. □

We will now show how mollification can be used to approximate Sobolev functions by smooth functions.

**Theorem 1.8.** (a) *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $u \in L^1_{\mathrm{loc}}(\Omega)$. Then $u_h \to u$ a.e. as $h \downarrow 0$. If $u$ is continuous, then $u_h \to u$ locally uniformly.*

*Choose a smaller open set $\Omega' \Subset \Omega$, let $p \in [1,\infty]$, and let $h > 0$ be sufficiently small such that*

$$\left\{x \in \mathbb{R}^n : d(x,\Omega') < h\right\} \subseteq \Omega. \tag{1.20}$$

(b) *If $u \in L^p(\Omega)$, then $\|u_h\|_{L^p(\Omega')} \leq \|u\|_{L^p(\Omega)}$. Furthermore, if $p \in [1,\infty)$, then $u_h \to u$ in $L^p(\Omega')$.*

(c) *If* $u \in W^{k,p}(\Omega)$*, then* $\|u_h\|_{W^{k,p}(\Omega')} \leq \|u\|_{W^{k,p}(\Omega)}$*. Furthermore, if* $p \in [1,\infty)$*, then* $u_h \to u$ *in* $W^{k,p}(\Omega')$*.*

*Proof.* (a) By the Lebesgue differentiation theorem, we have

$$\lim_{h\downarrow 0} \fint_{B(x,h)} |u(x)-u(y)| \, \mathrm{d}y \tag{1.21}$$

for a.e. $x \in \Omega$. Choose such an $x$. Then

$$\begin{aligned}\lim_{h\downarrow 0}\left|u(x) - \int_{B(x,h)} \eta_h(x-y)u(y)\,\mathrm{d}y\right| &\leq \lim_{h\downarrow 0}\int_{B(x,h)} \eta_h(x-y)\,|u(x)-u(y)|\,\mathrm{d}y \\ &\leq \lim_{h\downarrow 0} C \fint_{B(x,h)} |u(x)-u(y)|\,\mathrm{d}y \\ &= 0.\end{aligned} \tag{1.22}$$

This shows $u_h \to u$ a.e.

For local uniform convergence, note that $u$ is uniformly continuous on any ball in $\Omega$. Let $\varepsilon > 0$, and choose $\delta > 0$ such that $h < \delta$ implies $|u(x)-u(y)| < \varepsilon$ for all $x, y \in \Omega$ with $|x-y| < h$. The rest follows by following the above argument for convergence a.e.

(b) Suppose first that $p = \infty$. Then, for all $x \in \Omega'$, we have

$$|u_h(x)| \leq \int_{B(x,h)} \eta_h(x-y)\,|u(y)|\,\mathrm{d}y \leq \|u\|_{L^\infty(\Omega)}. \tag{1.23}$$

Suppose on the other hand that $p \in [1,\infty)$. We use Hölder's inequality with respect to the measure $\eta_h(x-y)\,\mathrm{d}y$ to find

$$\begin{aligned}|u_h(x)| &\leq \int_{B(x,h)} u(y)\eta_h(x-y)\,\mathrm{d}y \\ &\leq \left(\int_{B(x,h)} |u(y)|^p\,\eta_h(x-y)\,\mathrm{d}y\right)^{\frac{1}{p}} \left(\int_{B(x,h)} \eta_h(x-y)\,\mathrm{d}y\right)^{\frac{1}{p'}} \\ &\leq \left(\int_{B(x,h)} |u(y)|^p\,\eta_h(x-y)\,\mathrm{d}y\right)^{\frac{1}{p}}.\end{aligned} \tag{1.24}$$

Taking to the power $p$ and integrating over $x \in \Omega'$, we have

$$\begin{aligned}\int_{\Omega'} |u_h(x)|^p\,\mathrm{d}x &\leq \int_{\Omega'}\int_{B(x,h)} |u(y)|^p\,\eta_h(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{B(x,h)} |u(y)|^p \int_{\Omega'} \eta_h(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \int_{\Omega} |u(y)|^p\,\mathrm{d}y.\end{aligned} \tag{1.25}$$

This shows the required inequality.

For the convergence, note that $C(\Omega)$ is dense in $L^p(\Omega)$, so let $\varepsilon > 0$, and choose $v \in C(\Omega)$ such that $\|u-v\|_{L^p(\Omega)} < \frac{\varepsilon}{3}$. By part (a), we can choose $h > 0$ sufficiently small so that $\|v_h - v\|_{L^p(\Omega')} < \frac{\varepsilon}{3}$. Then

$$\begin{aligned} \|u_h - u\|_{L^p(\Omega')} &\leq \|u_h - v_h\|_{L^p(\Omega')} + \|v_h - v\|_{L^p(\Omega')} + \|v-u\|_{L^p(\Omega')} \\ &\leq \|u-v\|_{L^p(\Omega)} + \|v_h - v\|_{L^p(\Omega')} + \|v-u\|_{L^p(\Omega)} \\ &< \varepsilon, \end{aligned} \tag{1.26}$$

as required.

(c) This follows immediately from part (b) and lemma 1.7.

□

Having approximated Sobolev functions locally by smooth functions, we would now like to do it globally.

**Lemma 1.9.** *For $k \in \mathbb{N}$ and $p \in [1,\infty]$, let $u \in W^{k,p}(\Omega)$ and $\psi \in C^\infty(\Omega)$. Then $\psi u \in W^{k,p}(\Omega)$.*

*Proof.* We claim $\psi u$ has weak derivative

$$v_\alpha := \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \psi \partial^\beta u \in L^p(\Omega) \tag{1.27}$$

corresponding to the multiindex $\alpha$ with $|\alpha| \leq k$. By an induction argument, it suffices to prove this for $|\alpha| = 1$. Fix $i \in \{1, \ldots, n\}$, and a test function $\varphi \in C_c^\infty(\Omega)$. Then

$$\begin{aligned} \int_\Omega v_i \varphi \, \mathrm{d}x &= \int_\Omega (u \partial_i \psi + \psi \partial_i u) \varphi \, \mathrm{d}x \\ &= \int_\Omega u(\varphi \partial_i \psi - \partial_i(\psi\varphi)) \, \mathrm{d}x \\ &= -\int_\Omega u \psi \partial_i \varphi \, \mathrm{d}x, \end{aligned} \tag{1.28}$$

therefore proving the claim, and hence the lemma. □

**Theorem 1.10.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}_0$ and $p \in [1,\infty)$. Then, for all $u \in W^{k,p}(\Omega)$ and $\varepsilon > 0$, there exists $v \in (C^\infty \cap W^{k,p})(\Omega)$ such that $\|u-v\|_{W^{k,p}(\Omega)} < \varepsilon$.*

*Proof.* partition of unity wrt an increasing sequence $\emptyset = \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_i \subseteq \cdots \subseteq \Omega$. □

# Chapter 2

# Embeddings of Sobolev Spaces

## 2.1 Integrability of Sobolev Functions

**Theorem 2.1** (Sobolev Embedding)**.** *For $p \in [1,n)$, there exists $C = C_{n,p} > 0$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{2.1}$$

*for all $u \in W_0^{1,p}(\mathbb{R}^n)$, where $p^* := \frac{np}{n-p}$ is the Sobolev conjugate of $p$. In other words, $W_0^{1,p}(\mathbb{R}^n)$ embeds continuously in $L^{p^*}(\mathbb{R}^n)$.*

A similar result holds for other $W_0^{k,p}(\mathbb{R}^n)$ by a little bootstrapping.

Let's show that $p^*$ is the only possible index. Indeed, suppose the inequality $\|u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$ holds for some $C = C_{n,p} > 0$, $q \in [1,\infty]$, and all $u \in W^{1,p}(\mathbb{R}^n)$. Note first that $q \neq \infty$, since we can take $u(x) = |x|^{-s} - 1$ on $B(0,1)$ for some $s \in (0, \frac{n-1}{p})$, $u(x) = 0$ elsewhere. Then $u$ is unbounded, yet lies in $W_0^{1,p}(\mathbb{R}^n)$. Thus we now suppose $q \in [1,\infty)$. For $\lambda > 0$, define $u_\lambda(x) := u(\lambda x)$. Then

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(\lambda x)|^q \, \mathrm{d}x\right)^{\frac{1}{q}} = \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}, \tag{2.2}$$

and similarly,

$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\lambda \nabla u(\lambda x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} = \lambda^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}. \tag{2.3}$$

So in order for the estimate $\|u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$ to hold independent of $u$, we need $-\frac{n}{q} = 1 - \frac{n}{p}$. Indeed, the given estimate implies

$$\lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{2.4}$$

for all $u \in W_0^{1,p}(\mathbb{R}^n)$ and $\lambda > 0$. So if $-\frac{n}{q} < 1 - \frac{n}{p}$, then we can take $\lambda \to 0$ to obtain a contradiction, and in the case $-\frac{n}{q} > 1 - \frac{n}{p}$, we take $\lambda \to \infty$. Solving $-\frac{n}{q} = 1 - \frac{n}{p}$ gives us $q = \frac{np}{n-p} = p^*$

## 2.2 Hölder Continuity of Sobolev Functions

Choose a set $A \subseteq \mathbb{R}^n$ (not necessarily open), and let $\alpha \in (0,1]$. A function $u\colon A \to \mathbb{R}$ is *uniformly $\alpha$-Hölder continuous* if there exists $C > 0$ such that

$$|u(x) - u(y)| \leq C\,|x-y|^{\alpha} \tag{2.5}$$

for all $x, y \in A$. The *$\alpha$-Hölder seminorm* is defined by

$$[u]_{C^{0,\alpha}(A)} := \sup_{x,y\in A} \frac{|u(x)-u(y)|}{x-y}^{\alpha}. \tag{2.6}$$

More generally, $u$ is *locally $\alpha$-Hölder continuous* if it is uniformly $\alpha$-Hölder continuous on any compact subset of $A$. Now take an open set $\Omega \subseteq \mathbb{R}^n$. We let $C^{k,\alpha}(\Omega)$ denote the set of all $u \in C^k(\Omega)$ whose derivatives up to order $k$ are all locally $\alpha$-Hölder continuous. If $\Omega$ has the property $(\overline{\Omega})^\circ = \Omega$, we define $C^{k,\alpha}(\overline{\Omega})$ to be the set of functions $u \in C^{k,\alpha}(\Omega)$ such that the $\alpha$-Hölder norm

$$\|u\|_{C^0(\Omega)} + \sum_{|\beta|\leq k} [\partial^\beta u]_{C^{0,\alpha}(\Omega)} \tag{2.7}$$

is finite. The definition is ambigious when $\Omega = \mathbb{R}^n$, so we take $C^{k,\alpha}(\mathbb{R}^n)$ to be the $C^{k,\alpha}(\overline{\Omega})$ definition.

**Lemma 2.2** (Morrey's Inequality)**.** *For $p \in (n,\infty]$ and $r > 0$, there exists $C = C_{n,p} > 0$ such that*

$$|u(x) - u(y)| \leq C r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B(0,r))} \tag{2.8}$$

*for a.e. $x, y \in B(0,r)$ and all $u \in W^{1,p}(B(0,r))$.*

The following theorem follows nicely:

**Theorem 2.3** (Morrey Embedding)**.** *For $p \in (n,\infty]$, there exists $C = C_{n,p} > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^n)$, there exists a version $\widetilde{u}$ of $u$ which is uniformly $\alpha$-Hölder continuous, and*

$$\|\widetilde{u}\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{2.9}$$

*In other words, $W^{1,p}(\mathbb{R}^n)$ embeds continuously in $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.*

*Proof.* Let $x, y \in \mathbb{R}^n$, and set $r := 2\,|x-y|$. Then $u \in W^{1,p}(B(x,r))$, so Morrey's inequality implies there exists $C = C_{n,p} > 0$ such that

$$|u(x) - u(y)| \leq C_{n,p}\,|x-y|^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B(x,r))}. \tag{2.10}$$

□

The Sobolev and Morrey embedding theorems give us our first set of Poincaré-like inequalities:

**Theorem 2.4** (Friedrichs-Poincaré). *For $p \in [1,\infty]$ and $\Omega \subseteq \mathbb{R}^n$ open and with finite measure, let $q$ lie in one of the following intervals*

- $[1, p^*]$ *if* $p \in [1,n)$,
- $[1,\infty)$ *if* $p = n$,
- $[1,\infty]$ *if* $p \in (n,\infty]$.

*Then there exists $C = C_{n,p,q} > 0$ such that*

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \tag{2.11}$$

*for all $u \in W_0^{1,p}(\Omega)$.*

*Proof.* Suppose first that $p \in [1,n)$. By the Sobolev embedding theorem, there exists $C = C_{n,p} > 0$ such that

$$\|u\|_{L^{p^*}(\Omega)} = \|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{n,p} \|\nabla u\|_{L^p(\mathbb{R}^n)} = C_{n,p} \|\nabla u\|_{L^p(\Omega)}, \tag{2.12}$$

for all $u \in W_0^{1,p}(\Omega)$, which embeds in $W_0^{1,p}(\mathbb{R}^n)$ by extension by zero. Since $\Omega$ has finite measure, we have $L^{p^*}(\Omega) \subseteq L^q(\Omega)$, and $\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)}$ for all $u \in L^{p^*}(\Omega)$. The desired inequality follows.

Now suppose $p = n$. First, take $n > 1$. Choose $q \in [\frac{n}{n-1}, \infty)$, and set $p' = \frac{nq}{n+q}$. Then $p' \in [1,n)$, and $(p')^* = q$. By the previous estimates, we have

$$\|u\|_{L^q(\Omega)} \leq C_{n,q} \|\nabla u\|_{L^{p'}(\Omega)} \leq C_{n,q} \|\nabla u\|_{L^n(\Omega)}. \tag{2.13}$$

The case $n = 1$ is what.

Finally, suppose $p \in (n,\infty]$. Take $u \in W_0^{1,p}(\Omega)$, and let $\widetilde{u} \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ be its continuous version.Since $\Omega$ has finite measure, we can choose $r > 0$ such that $B(y,r) \setminus \Omega$ is nonempty. Choose $x$ in this set. Then by Morrey's inequality, noting $\widetilde{u} \in W^{1,p}(B(y,r))$, we have

$$\begin{aligned} |\widetilde{u}(y)| &= |\widetilde{u}(y) - \widetilde{u}(x)| \\ &\leq C_{n,p} r^{1-\frac{n}{p}} \|\nabla \widetilde{u}\|_{L^p(B(y,r))} \\ &= C_{n,p,\Omega} \|\nabla \widetilde{u}\|_{L^p(\Omega)}. \end{aligned} \tag{2.14}$$

The inequality follows. □

## 2.3 Compact Embeddings

**Theorem 2.5** (Rellich-Kondrachov). *For $p \in [1,n)$ and $\Omega \subseteq \mathbb{R}^n$ open and bounded, let $q \in [1,p^*)$. Then $W_0^{1,p}(\mathbb{R}^n)$ embeds compactly in $L^q(\mathbb{R}^n)$.*

The proof comes from this absolutely fat theorem:

**Theorem 2.6** (Arzelà-Ascoli)**.** *Let $A \subseteq \mathbb{R}^n$ be some set, and let $u_i \in C(A)$ be a bounded and uniformly equicontinuous sequence. Then $u_i$ has a locally uniformly convergent subsequence.*

## 2.4 Extension and Approximation

# Chapter 3

# Weak Solutions to PDEs