Examples and Counterexamples in the Theory of Tangent Measures

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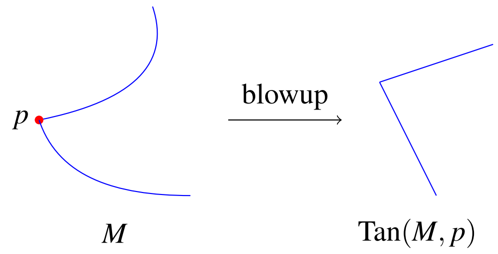

Figure 1: Blowing up a corner.

1 Introduction

Given a smooth submanifold $M \subseteq \mathbb{R}^d$, we know how its tangent space T_pM at a point $p \in M$ is defined, and what it represents. Namely, T_pM should represent the "best linear approximation" to M near p, much like the derivative of a differentiable function represents its best linear approximation near a point. Although T_pM is usually defined via quite abstract methods, the situation of being a subset of Euclidean space allows us to interpret T_pM in the very concrete sense of "zooming in" to p, or "blowing up" the manifold M around p. Such a procedure is applicable to far more general subsets of $\mathbb{R}^d$, and can often lead to more interesting results than a simple vector space. Typically, this "tangent cone" is denoted $\mathrm{Tan}(M, p)$ See figure 1 for an example of a set whose tangent cone at a point is not an affine subspace of $\mathbb{R}^d$.

It turns out that this blowup procedure applies to measures on $\mathbb{R}^d$, and is related to blowing up subsets of $\mathbb{R}^d$ in the sense that if $M \subseteq \mathbb{R}^d$ is an m-dimensional submanifold of $\mathbb{R}^d$, blowing up the m-dimensional Hausdorff measure on M (which we denote $\mathscr{H}^m \llcorner M$) at a point $p \in M$ gives us $\mathscr{H}^m \llcorner T_pM$. In section 3.1, we will see this in a more rigorous setting. Now, measures on $\mathbb{R}^d$ are far more numerous than subsets of $\mathbb{R}^d$, so we would like to ask how much worse can this blowup procedure for measures be compared to blowing up subsets of $\mathbb{R}^d$? It turns out that the answer is "a lot worse". In the following project, we will demonstrate the existence of a measure on $\mathbb{R}^d$ such that, at almost every point in its support, we can adjust the speed of blowing up to give us any measure imaginable on $\mathbb{R}^d$. As somewhat of a converse, we will also construct a measure on $\mathbb{R}$ which looks completely unlike Lebesgue measure on $\mathbb{R}$ (think a countable sum of δ measures), but blowing up at almost every point in its support returns Lebesgue measure.

2 Tangent Measures

2.1 Blowing up and Tangent Measures

To formalize the notion of "blowing up" a measure, fix $p \in \mathbb{R}^d$ and $r > 0$. Define the homothety $T^{p,r} \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$T^{p,r}(x) := \frac{x-p}{r}. \tag{2.1}$$

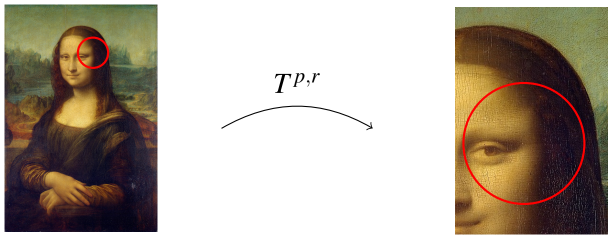

Figure 2: The homothety $T^{p,r}$.

Under this map, the ball $B(p,r)$ gets mapped to $B(0,1)$ (See figure 2). Given $\mu \in \mathscr{M}(\mathbb{R}^d)$, we write $\mu_{p,r} := T^{p,r}_* \mu$ for the pushforward of μ by $T^{p,r}$. More concretely, $\mu_{p,r} \in \mathscr{M}(\mathbb{R}^d)$ is the measure given by

$$\mu_{p,r}(A) = \mu((T^{p,r})^{-1}(A)) = \mu(p + rA) \quad \text{for } A \subseteq \mathbb{R}^d \text{ a Borel set.} \tag{2.2}$$

Suppose now there exists a sequence $r_j > 0$ with $r_j \downarrow 0$ as $j \to \infty$ and sequence $c_j > 0$ of normalizing constants such that $c_j \mu_{p,r_j}$ converges weakly* to some nonzero $\tau \in \mathscr{M}(\mathbb{R}^d)$. We then call τ a *tangent measure* to μ at p, and $c_j \mu_{p,r_j}$ a *blowup sequence* for τ. We write $\operatorname{Tan}(\mu, p)$ for the set of all tangent measures to μ at p. Some authors allow 0 to be a tangent measure, but we don't in order to avoid considering trivial cases.

We remark that the above definition does not require μ or τ to be positive measures at all, and works perfectly well for signed (or even vector-valued) measures. See [Rin18] for results on the structure of vector-valued tangent measures.

The structure of $\operatorname{Tan}(\mu, p)$ is of some interest, and we will be making use of the following lemma later.

Lemma 2.1. *Fix $\mu \in \mathscr{M}(\mathbb{R}^d)$ and $p \in \mathbb{R}^d$. Suppose $\tau \in \operatorname{Tan}(\mu, p)$. Then*

(a) *For all $c > 0$, the measure $c\tau$ is in $\operatorname{Tan}(\mu, p)$.*

(b) *For all $r > 0$, the measure $\tau_{0,r}$ is in $\operatorname{Tan}(\mu, p)$.*

(c) *The set $\operatorname{Tan}(\mu, p)$ is closed in $\mathscr{M}(\mathbb{R}^d)$ (with respect to the weak* topology).*

In the language of [Pre87], properties (a) and (b) of this lemma say $\operatorname{Tan}(\mu, p)$ is a *d-cone*.

Proof. Let $c_j \mu_{p,r_j}$ be a blowup sequence for τ. Then $cc_j \mu_{p,r_j}$ is easily seen to be a blowup sequence for $c\tau$, proving (a). Similarly, $c_j \mu_{p,rr_j}$ is a blowup sequence for $\tau_{0,r}$. Indeed, given $\varphi \in C_c(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} c_j \varphi \, \mathrm{d}\mu_{p,rr_j} = \int_{\mathbb{R}^d} c_j \varphi\left(\frac{1}{r}\frac{x-p}{r_j}\right) \mathrm{d}\mu(x) \to \int_{\mathbb{R}^d} \varphi\left(\frac{x}{r}\right) \mathrm{d}\tau(x) = \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\tau_{0,r}, \tag{2.3}$$

where we note that the function $x \mapsto \varphi(x/r)$ is in $C_c(\mathbb{R}^d)$. This proves (b).

To prove (c), let d be a metric on $\mathscr{M}(\mathbb{R}^d)$ as in (see lemma A.4). Let $\tau^{(n)}$ be a sequence in $\operatorname{Tan}(\mu,p)$ converging to $\tau \in \mathscr{M}(\mathbb{R}^d)$, and take a blowup sequence $c_j^{(n)}\mu_{p,r_j^{(n)}}$ for each $n \in \mathbb{N}$. We need to find a blowup sequence $\overline{c}_j\mu_{p,\overline{r}_j}$ converging to τ.

We do this construction inductively. Initially, take $n_1 = j_1 = 1$. Given $m \in \mathbb{N}$, assume that n_{m-1} and j_{m-1} have been found, and choose $n_m > n_{m-1}$ such that $d(\tau^{(n_m)},\tau) < \frac{1}{2m}$, and $j_m > j_{m-1}$ such that $d(c_{j_m}^{(n_m)}\mu_{p,r_{j_m}^{(n_m)}},\tau^{(n_m)}) < \frac{1}{2m}$ and $r_{j_m}^{(n_m)} < \min\left(r_{j_{m-1}}^{(n_{m-1})},\frac{1}{m}\right)$ (which is always possible since $r_j^{(n_m)} \downarrow 0$ as $j \to \infty$ by definition of a blowup sequence). Define $\overline{c}_m := c_{j_m}^{(n_m)}$ and $\overline{r}_m := r_{j_m}^{(n_m)}$. Then $\overline{r}_m \downarrow 0$ by construction, and

$$d(\overline{c}_m\mu_{p,\overline{r}_m},\tau) \leq d(\overline{c}_m\mu_{p,\overline{r}_m},\tau^{(n_m)}) + d(\tau^{(n_m)},\tau) < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}, \tag{2.4}$$

so $d(\overline{c}_m\mu_{p,\overline{r}_m},\tau) \to 0$ as $m \to \infty$. In order to apply lemma A.4, we just need to show the sequence $(\overline{c}_m\mu_{p,\overline{r}_m})_{m\in\mathbb{N}}$ is uniformly locally bounded. To see this, let $K \subseteq \mathbb{R}^d$ be compact. We use semicontinuity (lemma A.2) to estimate

$$\tau(K) \geq \limsup_{n\to\infty} \tau^{(n)}(K) \geq \limsup_{n\to\infty}\limsup_{j\to\infty} c_j^{(n)}\mu_{p,r_j^{(n)}}(K). \tag{2.5}$$

It follows immediately that we can bound $c_j^{(n)}\mu_{p,r_j^{(n)}}(K)$ uniformly in n and j, which is what we wanted to show. Lemma A.4 then shows that $\overline{c}_m\mu_{p,\overline{r}_m} \overset{*}{\rightharpoonup} \tau$ as $m \to \infty$. Therefore $\overline{c}_m\mu_{p,\overline{r}_m}$ is a blowup sequence for τ, and so τ lies in $\operatorname{Tan}(\mu,p)$, thereby proving (c). □

The normalizing constants c_j aren't particularly important. Indeed, suppose $c_j\mu_{p,r_j}$ is a blowup sequence for $\tau \in \operatorname{Tan}(\mu,p)$. Let $K > 0$ be such that $\tau(B(0,K)) > 0$. By semicontinuity (lemma A.2),

$$\begin{aligned}
0 < \tau(B(0,K)) &\leq \liminf_{j\to\infty} c_j\mu_{p,r_j}(B(0,K)) \\
&= \liminf_{j\to\infty} c_j\mu(B(p,Kr_j)) \\
&\leq \limsup_{j\to\infty} c_j\mu(\overline{B(p,Kr_j)}) \\
&= \limsup_{j\to\infty} c_j\mu_{p,r_j}(\overline{B(0,K)}) \\
&\leq \tau(\overline{B(0,K)}) < \infty.
\end{aligned} \tag{2.6}$$

Define $\widetilde{c}_j := \mu(B(p,Kr_j))^{-1}$. Since $\liminf_{j\to\infty} c_j\mu(B(p,Kr_j))$ is positive and finite, we may pass to a subsequence (not relabeled) such that $c_j\mu(B(p,Kr_j))$ converges to a positive value c. It follows that

$$\widetilde{c}_j\mu_{p,r_j} = \frac{\widetilde{c}_j}{c_j}c_j\mu_{p,r_j} \overset{*}{\rightharpoonup} c\tau. \tag{2.7}$$

So $\widetilde{c}_j\mu_{p,r_j}$ is a blowup sequence for $c\tau$. Dividing by c, we see $c^{-1}c_j\mu_{p,r_j}$ is a blowup sequence for τ.

2.2 Examples of Tangent Measures and their Local Properties

Fix $p \in \mathbb{R}^d$. Consider $\mu = \mathscr{L}^d$, and choose $\varphi \in C_c(\mathbb{R}^d)$. Then, for all $r > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi \, \mathrm{d}(r^{-d}\mu_{p,r}) &= \int_{\mathbb{R}^d} \varphi\left(\frac{x-p}{r}\right) r^{-d} \, \mathrm{d}\mathscr{L}^d(x) \\ &= \int_{\mathbb{R}^d} \varphi(y) r^{-d} r^d \, \mathrm{d}\mathscr{L}^d(y) \\ &= \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mathscr{L}^d, \end{aligned} \tag{2.8}$$

where we made the change of variables $y = r^{-1}(x-p)$. It follows that for any sequence $r_j \downarrow 0$, the sequence $r_j^{-d}\mu_{x_0,r_j}$ is a blowup sequence for $\mathscr{L}^d$, and therefore $\mathscr{L}^d$ is a tangent measure to itself at p.

More generally, let $f \in C(\mathbb{R}^d)$ be nonnegative, define $\mu = f\mathscr{L}^d$, and choose $\varphi \in C_c(\mathbb{R}^d)$. Similarly to the above, for all $r > 0$ we find

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi \, \mathrm{d}(r^{-d}\mu_{p,r}) &= \int_{\mathbb{R}^d} \varphi\left(\frac{x-p}{r}\right) f(x) r^{-d} \, \mathrm{d}\mathscr{L}^d(x) \\ &= \int_{\mathbb{R}^d} \varphi(y) f(p+ry) \, \mathrm{d}\mathscr{L}^d(y) \\ &\to \int_{\mathbb{R}^d} \varphi(y) f(p) \, \mathrm{d}\mathscr{L}^d(y). \end{aligned} \tag{2.9}$$

Where we again made the change of variables $y = r^{-1}(x-x_0)$, and used the dominated convergence theorem together with the fact that φ and f are continuous functions, along with the compact support property of φ. From this, we see that $f(p)\mathscr{L}^d$ is a tangent measure to $f\mathscr{L}^d$ at p.

The above situation is true for far more measures than just Lebesgue measure. Fix a measure $\mu \in \mathscr{M}(\mathbb{R}^d)$. Let $f \in L^1(\mathbb{R}^d, \mu)$ be positive, and suppose it satisfies

$$\lim_{r \downarrow 0} \frac{1}{\mu(B(0,r))} \int_{B(p,r)} |f(x) - f(p)| \, \mathrm{d}\mu(x) = 0. \tag{2.10}$$

Consider the measure $f\mu$. Fix a tangent measure $\tau \in \mathrm{Tan}(\mu, p)$, and let $K > 0$ be such that $\tau(B(0,K)) > 0$. Choose $\varphi \in C_c(\mathbb{R}^d)$, and let $R > 0$ be such that $\operatorname{supp}\varphi \subseteq B(0,R)$. Define $C := \max\{K, R\}$. Choose a blowup sequence $c_j\mu_{p,r_j}$ for τ, where $c_j = c\mu(B(0,Cr_j))^{-1}$. Then

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi \, \mathrm{d}(c_j f\mu)_{p,r_j} &= \int_{\mathbb{R}^d} c_j \varphi\left(\frac{x-p}{r_j}\right) f(x) \, \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}^d} c_j \varphi\left(\frac{x-p}{r_j}\right) f(p) \, \mathrm{d}\mu(x) \\ &\quad + \int_{\mathbb{R}^d} c_j \varphi\left(\frac{x-p}{r_j}\right) (f(x) - f(p)) \, \mathrm{d}\mu(x). \end{aligned} \tag{2.11}$$

The first term converges to

$$\int_{\mathbb{R}^d} \varphi(x) f(p) \, \mathrm{d}\tau(x). \tag{2.12}$$

For the second term, note that $x \mapsto \varphi(r_j^{-1}(x-p))$ is supported in $B(p, Cr_j)$, so the magnitude of the second term is bounded, up to multiplication by a constant, by

$$\frac{1}{\mu(B(0, Cr_j))} \int_{B(p, Cr_j)} |f(x) - f(p)| \, \mathrm{d}\mu(x) \tag{2.13}$$

By assumption on f, this converges to zero as $j \to \infty$. It follows that $f(p)\tau$ is a tangent measure to $f\mu$ at p. In particular, $f(p)\operatorname{Tan}(\mu, p) \subseteq \operatorname{Tan}(f\mu, p)$.

We would now like to show the converse inclusion holds: namely, if τ is a tangent measure to $f\mu$ at p, then $\tau = f(p)\widetilde{\tau}$ for some $\widetilde{\tau} \in \operatorname{Tan}(\mu, p)$. Let $c_j(f\mu)_{p,r_j}$ be a blowup sequence for τ. Then, for all $\varphi \in C_c(\mathbb{R}^d)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi \, \mathrm{d}\tau &= \lim_{j\to\infty} \int_{\mathbb{R}^d} \varphi \, \mathrm{d}(c_j(f\mu)_{p,r_j}) \\
&= \lim_{j\to\infty} \int_{\mathbb{R}^d} c_j \varphi\left(\frac{x-p}{r_j}\right) f(x) \, \mathrm{d}\mu(x) \\
&= \lim_{j\to\infty} \int_{\mathbb{R}^d} c_j \varphi\left(\frac{x-p}{r_j}\right) f(p) \, \mathrm{d}\mu(x) \\
&\quad + \lim_{j\to\infty} \int_{\mathbb{R}^d} c_j \varphi\left(\frac{x-p}{r_j}\right) (f(x) - f(p)) \, \mathrm{d}\mu(x).
\end{aligned} \tag{2.14}$$

The same argument as above implies the second term is zero. Take $\widetilde{\tau} = f(p)^{-1}\tau$. We then see that $\widetilde{\tau}$ is a tangent measure to μ at p with blowup sequence $c_j \mu_{p,r_j}$. We have shown that $\operatorname{Tan}(f\mu, p) = f(p)\operatorname{Tan}(\mu, p)$.

Much like the Lebesgue Differentiation Theorem, it can be shown that for any $f \in L^1(\mathbb{R}^d, \mu)$ and μ-a.e. $p \in \mathbb{R}^d$, f satisfies

$$\lim_{r \downarrow 0} \frac{1}{\mu(B(p,r))} \int_{B(p,r)} |f(x) - f(p)| \, \mathrm{d}x = 0. \tag{2.15}$$

(See [Lel08] proposition 2.2.) In particular, if μ satisfies $\mu(B(p,r)) = \mu(B(0,r))$ for all $p \in \mathbb{R}^d$, then we must have $\operatorname{Tan}(f\mu, p) = f(p)\operatorname{Tan}(\mu, p)$ for μ-a.e. $p \in \mathbb{R}^d$. Thus, for any Borel measurable subset $A \subseteq \mathbb{R}^d$ containing p, we can take $f = \mathbb{1}_A$ to show that $\operatorname{Tan}(\mu \llcorner A, p) = \operatorname{Tan}(\mu, p)$ for μ-a.e. $p \in A$. This property captures the intuition that tangent measures should be a local property of measures.

3 Tangent Measures of Rectifiable Measures

3.1 Tangent Measures of Submanifolds

In this subsection, we will consider the tangent measures to $\mathscr{H}^k \llcorner M$, for M a k-dimensional submanifold of $\mathbb{R}^d$. The proof will follow the lines of theorem 4.8 from [Lel08]. One would reasonably expect that $\mathscr{H}^k \llcorner T_pM$ is a tangent measure, and indeed, this turns out to be the case. To start, let $E \subseteq \mathbb{R}^d$ be the graph of a C^1 map $f\colon U \subseteq \mathbb{R}^k \to \mathbb{R}^{d-k}$, where U is a bounded open set. That is, E is the image of the C^1 map $F\colon U \to \mathbb{R}^d$ defined by $F(x) := (x, f(x))$. Fix $p = (q, f(q)) \in E$. By translating f if necessary, we may assume $p = 0$. The *area formula* says that

$$\int_E \varphi \, \mathrm{d}\mathscr{H}^k = \int_U \varphi(x, f(x)) \mathrm{J}F(x) \, \mathrm{d}\mathscr{L}^k(x) \tag{3.1}$$

for any $\varphi \in C_c(\mathbb{R}^d)$. Here $\mathrm{J}F(x) := \sqrt{\det \nabla F(x)^{\mathrm{T}} \nabla F(x)}$ is the *Jacobian determinant* of F, where $\nabla F(x)$ is a $d \times k$ matrix with components $(\nabla F(x))_i^{\ j} = \partial_i F^j(x)$. If E is the graph of a real-valued function, i.e. $k = d-1$, then $\nabla F(x)$ is the vector $(1, \ldots, 1, \partial_1 f(x), \ldots, \partial_k f(x))$, and therefore

$$\mathrm{J}F(x) = \sqrt{1 + \sum_{i=1}^{k} \left| \frac{\partial f}{\partial x^i}(x) \right|^2}. \tag{3.2}$$

So the area formula agrees with the usual notion of the integral along a graph from calculus.

We may use the area formula to calculate the tangent measures to $\mathscr{H}^k \llcorner E$. Fix $\varphi \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$. For $r > 0$, we have

$$\frac{1}{r^k} \int_{\mathbb{R}^d} \varphi \, \mathrm{d}(\mathscr{H}^k \llcorner E)_{0,r} = \frac{1}{r^k} \int_U \varphi\left(\frac{F(x)}{r}\right) \mathrm{J}F(x) \, \mathrm{d}\mathscr{L}^k(x) \tag{3.3}$$

Let $R > 0$ be such that $\operatorname{supp} \varphi \subseteq B(0,R)$. Since U is bounded, F is uniformly Lipschitz on U. Let

$$\mathrm{Lip} F := \sup_{x,y \in U} \frac{|F(x) - F(y)|}{|x-y|} < \infty \tag{3.4}$$

be its Lipschitz constant. Note that $\mathrm{Lip} F$ is positive, since otherwise we would have that F is a constant function, implying U is the non-open set $\{0\}$. Suppose x lies outside the ball $B(0, rR(\mathrm{Lip} F)^{-1})$, i.e. $(\mathrm{Lip} F)\,|x| \geq rR$. By definition of the Lipschitz constant (in particular, taking $y = 0$ in the definition), it follows that $|F(x)| \geq rR$, which implies $\varphi(F(x)/r) = 0$. We may then write

$$\int_U \varphi\left(\frac{F(x)}{r}\right) \mathrm{J}F(x) \, \mathrm{d}x = \int_{U \cap B(0, rR(\mathrm{Lip} f)^{-1})} \varphi\left(\frac{F(x)}{r}\right) \mathrm{J}F(x) \, \mathrm{d}x. \tag{3.5}$$

Let $K = R/\mathrm{Lip} F$. Noting that for $r > 0$ sufficiently small such that $B(0, Kr) \subseteq U$, we have that $\mathrm{J}F$

is uniformly continuous on $B(0,Kr)$, and so we calculate

$$\begin{aligned}
&\lim_{r\downarrow 0}\left|\frac{1}{r^k}\int_{U\cap B(0,Kr)}\varphi\left(\frac{F(x)}{r}\right)\mathrm{J}F(x)\ \mathrm{d}x-\frac{1}{r^k}\int_{U\cap B(0,Kr)}\varphi\left(\frac{\mathrm{d}F(0)(x)}{r}\right)\mathrm{J}F(0)\ \mathrm{d}x\right| \\
&\quad\le\lim_{r\downarrow 0}\frac{1}{r^k}\int_{U\cap B(0,Kr)}|\mathrm{J}F(x)-\mathrm{J}F(0)|\ \mathrm{d}x\sup_{x\in U\cap B(0,Kr)}\left|\varphi\left(\frac{F(x)}{r}\right)-\varphi\left(\frac{\mathrm{d}F(0)(x)}{r}\right)\right| \\
&\quad\le\lim_{r\downarrow 0}\sup_{x\in U\cap B(0,Kr)}|\mathrm{J}F(x)-\mathrm{J}F(0)|\sup_{x\in U\cap B(0,Kr)}\frac{|F(x)-\mathrm{d}F(0)(x)|}{r} \\
&\quad=0.
\end{aligned}\tag{3.6}$$

Therefore

$$\begin{aligned}
\lim_{r\downarrow 0}\frac{1}{r^k}\int_U\varphi\left(\frac{F(x)}{r}\right)\mathrm{J}F(x)\ \mathrm{d}x&=\lim_{r\downarrow 0}\frac{1}{r^k}\int_{U\cap B(0,Kr)}\varphi\left(\frac{\mathrm{d}F(0)(x)}{r}\right)\mathrm{J}F(0)\ \mathrm{d}x \\
&=\int_{\mathbb{R}^k}\varphi(\mathrm{d}F(0)(y))\mathrm{J}F(0)\ \mathrm{d}y,
\end{aligned}\tag{3.7}$$

where we changed variables to $y=x/r$.

Now, the graph of $z\mapsto \mathrm{d}f(0)(z)$ is the tangent space T_0E, and $\mathrm{J}F(0)$ is the Jacobian of this map. Hence, by the area formula again, we conclude that (3.7) is equal to

$$\int_{T_0E}\varphi\ \mathrm{d}\mathscr{H}^k. \tag{3.8}$$

Since $\varphi\in C_c(\mathbb{R}^d)$ was arbitrary, we see $(\mathscr{H}^k\llcorner E)_{0,r}\overset{*}{\rightharpoonup}\mathscr{H}^k\llcorner T_0E$, so $\mathscr{H}^k\llcorner T_0E$ is a tangent measure to $\mathscr{H}^k\llcorner E$ at 0, thereby agreeing with our intuition. With a few minor modifications, the proof works to show $\mathscr{H}^k\llcorner T_pE$ is a tangent measure to $\mathscr{H}^k\llcorner E$ for any $p\in E$.

Recall that any C^1 submanifold of $\mathbb{R}^d$ can be considered locally as the graph of a C^1 function (see [Lee12] example 1.32 and proposition 5.16). More precisely, let $M\subseteq\mathbb{R}^d$ be a k-dimensional C^1 submanifold, and fix $p\in M$. Then there exists an open neighborhood E of p in M and an open set $U\subseteq\mathbb{R}^k$ such that E is the graph of a function $f\colon U\to\mathbb{R}^{d-k}$. Our calculations above show $\mathscr{H}^k\llcorner T_pE=\mathscr{H}^k\llcorner T_pM$ is a tangent measure to $\mathscr{H}^k\llcorner E$ at p. However, we also showed in section 2.2 that for $\mathscr{H}^k$-a.e. $p\in E$, the tangent measures to $\mathscr{H}^k\llcorner E=(\mathscr{H}^k\llcorner M)\llcorner E$ at p are precisely the tangent measures to $\mathscr{H}^k\llcorner M$ at p. Thus $\mathscr{H}^k\llcorner T_pM$ is a tangent measure to $\mathscr{H}^k\llcorner M$ at p.

The above theory holds more generally for Lipschitz submanifolds $M\subseteq\mathbb{R}^d$ by using Rademacher's theorem or its corollaries to approximate F with C^1 functions. See [EG15] theorem 6.11.

3.2 Rectifiable Sets and Measures

Having shown that $\mathscr{H}^k\llcorner T_pM$ is a tangent measure to $\mathscr{H}^k\llcorner M$ for any k-dimensional Lipschitz submanifold $M\subseteq\mathbb{R}^d$ and $\mathscr{H}^k$-a.e. $p\in M$, we now ask when the opposite is true. Namely, can

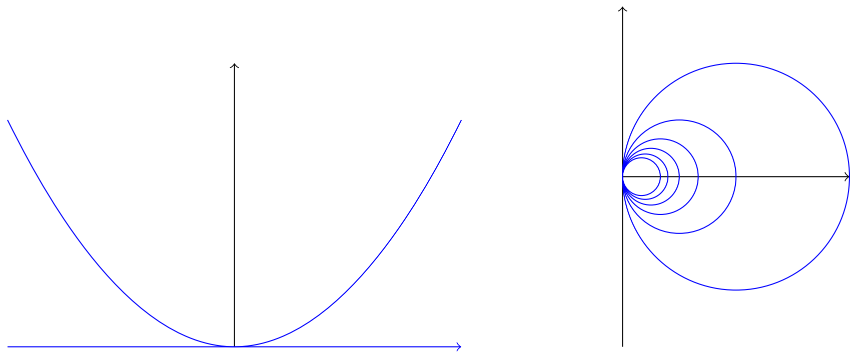

Figure 3: Two examples of sets which are rectifiable, but not manifolds.

we judge that a set is a submanifold by showing it has approximate tangent spaces a.e., where an *approximate tangent space* to a set $S \subseteq \mathbb{R}^d$ at $p \in S$ is a k-dimensional vector space $V \subseteq \mathbb{R}^d$ such that $\mathscr{H}^k \llcorner V \in \operatorname{Tan}(\mathscr{H}^k \llcorner S, p)$. Immediately, we see this is not true at all, as illustrated by figure 3.2. First, the union of the x-axis and the parabola $y = x^2$ in $\mathbb{R}^2$ has an approximate tangent space at every point, but it fails to be a manifold near the origin.

More absurdly, take H to be the union of circles centered at $(\frac{1}{n}, 0)$ with radius $\frac{1}{n}$ in $\mathbb{R}^2$ ($n \in \mathbb{N}$), the so-called *Hawaiian earring*. The fundamental group $\pi_1(H)$ is famously complicated. In particular, it is uncountable, and therefore cannot be a topological manifold (see [Lee11] theorem 7.21). Despite this, it has an approximate tangent spaces at a.e. point. As a matter of interest, we also see that that $\mathscr{H}^k \llcorner H$ is a tangent measure at the origin.

The thread linking our two counterexamples is that they are countable unions of submanifolds of $\mathbb{R}^d$, and this will turn out to be the correct generalization. We say a $\mathscr{H}^k$-measurable subset $E \subseteq \mathbb{R}^d$ is called *k-rectifiable* if there exist Lipschitz maps $f_i \colon \mathbb{R}^k \to \mathbb{R}^d$ for $i \in \mathbb{N}$, such that

$$\mathscr{H}^k \left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k) \right) = 0 \tag{3.9}$$

Rectifiable sets are in abundance. For example, given any Borel set in $\mathbb{R}^d$ with locally finite perimeter, its measure-theoretic boundary turns out to be rectifiable. See [KP99] section 3.7.

As a technical aside to state the next theorem, we need to quickly introduce Hausdorff densities. Fix a Radon measure $\mu \in \mathscr{M}(\mathbb{R}^d)$, a point $p \in \mathbb{R}^d$, and $s > 0$. Define the *upper Hausdorff s-density* of μ at p to be

$$\Theta^{s*}(\mu, p) := \limsup_{r \downarrow 0} \frac{\mu(B(p,r))}{r^s}, \tag{3.10}$$

and the *lower Hausdorff s-density* to be

$$\Theta^{s}_{*}(\mu, p) := \liminf_{r \downarrow 0} \frac{\mu(B(p,r))}{r^s}. \tag{3.11}$$

If both densities agree, we write $\Theta^s(\mu, p)$ for their common value, and call it the *Hausdorff s-*

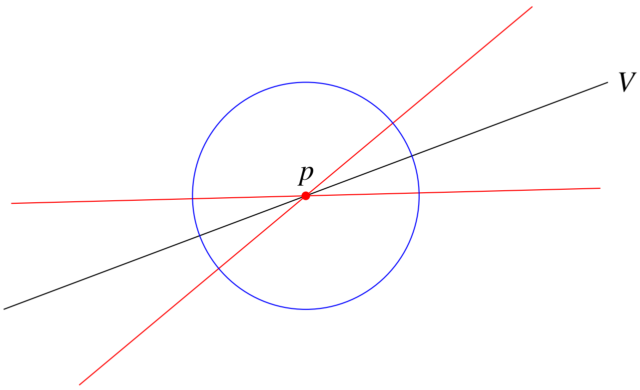

Figure 4: Truncated "light cone" around a point p.

density.

The following theorem characterizes rectifiable sets.

> **Theorem 3.1.** *Fix $k \in \mathbb{N}$, and let $E \subseteq \mathbb{R}^d$ be $\mathscr{H}^k$-measurable with finite $\mathscr{H}^k$-measure, and such that $\Theta^k_*(E,p) > 0$ for $\mathscr{H}^k$-a.e. $p \in E$. Then E is k-rectifiable if and only if for $\mathscr{H}^k$-a.e. $p \in E$, there exists an k-dimensional vector subspace $V_p \subseteq \mathbb{R}^d$ such that $\operatorname{Tan}(\mathscr{H}^k \llcorner E, p) = \{c\mathscr{H}^k \llcorner V_p : c > 0\}$.*

As with most other things in geometric measure theory, the proof of theorem 3.1 is highly technical, involving study of the truncated "light cone"-shaped set

$$E \cap B(p,r) \cap \{x \in \mathbb{R}^d : d(x-p,V) < s|x-p|\}, \tag{3.12}$$

where $r,s > 0$ and V is a k-dimensional subspace of $\mathbb{R}^d$ (figure 4). See [Mat95] chapters 15-16 for more details on the proof.

More generally, a Radon measure $\mu \in \mathscr{M}(\mathbb{R}^d)$ is said to be *k-rectifiable* if μ is absolutely continuous with respect to $\mathscr{H}^k$, and there exists a k-rectifiable Borel set $E \subseteq \mathbb{R}^d$ such that $\mu(\mathbb{R}^d \setminus E) = 0$. We can generalize theorem 3.1 to measures in the following way:

> **Theorem 3.2.** *Fix $k \in \mathbb{N}$, and let $\mu \in \mathscr{M}(\mathbb{R}^d)$ be a Radon measure such that $0 < \Theta^k_*(\mu,p) \leq \Theta^{k*}(\mu,p) < \infty$ for μ-a.e. $p \in \mathbb{R}^d$. Then μ is k-rectifiable if and only if for $\mathscr{H}^k$-a.e. $p \in \mathbb{R}^d$, there exists a k-dimensional vector subspace $V_p \subseteq \mathbb{R}^d$ such that $\operatorname{Tan}(\mu,p) = \{c\mathscr{H}^k \llcorner V_p : c > 0\}$.*

The additional condition on the upper Hausdorff density of μ is vital, as we will see in section 5.3.

Notes and Remarks

The notion of rectifiability is hardly a new one, and is present in any good textbook on geometric measure theory, for example [Fed96], [Mat95], or [Kra]. Other tools such as Hausdorff densities, the area formula and its aptly named dual, the coarea formula, are similarly found in such textbooks.

4 Measures with Multiple Tangent Measures

All measures we have constructed so far have had unique tangent measures up to multiplication. Do there exist measures μ such that $\mathrm{Tan}(\mu, p)$ contains distinct tangent measures for some point $p \in \mathbb{R}^d$? There certainly are - example 14.2(3) in [Mat95] is one. In this section, we will construct a measure μ such that for μ-a.e. $p \in \mathbb{R}^d$, $\mathrm{Tan}(\mu, p)$ is all of $\mathscr{M}(\mathbb{R}^d)$. Naturally, this is the worst it can possibly be, although we will see later on that this is not actually uncommon for Radon measures.

The following theorem was introduced by Toby O'Neil in [O'N95], and the proof will follow his method.

> **Theorem 4.1** (O'Neil)**.** *Fix $d \in \mathbb{N}$. There exists a Radon measure $\mu \in \mathscr{M}(\mathbb{R}^d)$ such that for μ-a.e. $p \in \mathbb{R}^d$, we have* $\mathrm{Tan}(\mu, p)$.

We will spend the rest of this section proving this result. Define the set $\mathscr{S} \subseteq \mathscr{M}(\mathbb{R}^d)$ to be the set of convex linear combinations of Dirac measures δ_x for $x \in \mathbb{Q}^d \cap B(0,1)$, with δ_0 always included in the convex linear combination. That is,

$$\mathscr{S} := \left\{ \alpha_0 \delta_0 + \sum_{i=1}^{N} \alpha_i \delta_{x_i} \;\middle|\; \begin{array}{l} N \in \mathbb{N}, \alpha_i \in \mathbb{Q} \cap (0,1), \sum_{i=0}^{N} \alpha_i = 1, \\ x_i \in \mathbb{Q}^d \cap B(0,1), x_i \neq x_j \text{ for } i \neq j \end{array} \right\}. \tag{4.1}$$

We claim the set $\{p\nu_{0,q} : p, q \in \mathbb{Q}^+, \nu \in \mathscr{S}\}$ is weakly* dense in $\mathscr{M}(\mathbb{R}^d)$. To prove this, first note that the set of measures with compact support in $\mathscr{M}(\mathbb{R}^d)$ is dense in $\mathscr{M}(\mathbb{R}^d)$. This can be shown, for example, by considering the compactly supported measures $\mu \llcorner B(0,N)$ for $N \in \mathbb{N}$. Fix, therefore, a compactly supported $\mu \in \mathscr{M}(\mathbb{R}^d)$. Given $m \in \mathbb{N}$, let $\mathscr{Q}_m$ be the family of half-open cubes in $\mathbb{R}^d$ with side length $\frac{1}{m}$ congruent to the set

$$Q_0 := \prod_{i=1}^{d} \left[-\frac{1}{2m}, \frac{1}{2m} \right). \tag{4.2}$$

Let x_Q be the midpoint of $Q \in \mathscr{Q}_m$. Note that x_Q lies in $\mathbb{Q}^d$. For $\varphi \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu - \sum_{Q\in\mathscr{Q}_m} \varphi(x_Q)\mu(Q)\right| &= \left|\int_{\mathbb{R}^d} \varphi - \sum_{Q\in\mathscr{Q}_m} \varphi(x_Q)\mathbb{1}_Q \, \mathrm{d}\mu\right| \\ &\leq \int_{\mathbb{R}^d} \sum_{Q\in\mathscr{Q}_m} |\varphi(x) - \varphi(x_Q)| \, \mathbb{1}_Q(x) \, \mathrm{d}\mu(x) \\ &\leq \int_{\mathbb{R}^d} \sum_{Q\in\mathscr{Q}_m} |x - x_Q| \, \mathbb{1}_Q(x) \, \mathrm{d}\mu(x) \\ &\to 0 \quad \text{as } m \to \infty,\end{aligned} \tag{4.3}$$

noting that the convergence is justified by the dominated convergence theorem since μ is finite and the integrands are bounded by 1 uniformly in m. In particular, we see that

$$\lim_{m\to\infty} \sum_{Q\in\mathscr{Q}_m} \varphi(x_Q)\mu(Q) = \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu \tag{4.4}$$

independently of $\varphi \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$. Fix $\varepsilon > 0$, and choose $m_0 \in \mathbb{N}$ such that $m \geq m_0$ implies

$$\left|\int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu - \sum_{Q\in\mathscr{Q}_m} \varphi(x_Q)\mu(Q)\right| < \varepsilon \tag{4.5}$$

for all $f \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$. Index the elements Q_i of $\mathscr{Q}_m$ by $i \in \mathbb{N} \cup \{0\}$, with Q_0 being the cube centered at 0 as above. For each $i \in \mathbb{N} \cup \{0\}$, let $p_{Q_i} \in \mathbb{Q}^+$ be such that $|\mu(Q_i) - p_{Q_i}| < \frac{\varepsilon}{2^{i+1}}$. For $\varphi \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$, we then have

$$\begin{aligned}\left|\sum_{Q\in\mathscr{Q}_m} \varphi(x_Q)\mu(Q) - \sum_{Q\in\mathscr{Q}_m} \varphi(x_Q)p_Q\right| &\leq \sum_{Q\in\mathscr{Q}_m} \varphi(x_Q) \, |\mu(Q) - p_Q| \\ &\leq \|\varphi\|_{L^\infty} \sum_{i=0}^{\infty} \frac{\varepsilon}{2^{i+1}} \\ &\leq \|\varphi\|_{L^\infty} \varepsilon.\end{aligned} \tag{4.6}$$

Since μ has compact support, there exists $N \in \mathbb{N}$ such that $\mu(Q) = 0$ whenever $Q \cap B(0, N-1) = \emptyset$, so that

$$\sum_{Q\in\mathscr{Q}_m} \varphi(x_Q)\mu(Q) = \sum_{\substack{Q\in\mathscr{Q}_m \\ Q\cap B(0,N-1)\neq\emptyset}} \varphi(x_Q)\mu(Q) \tag{4.7}$$

Write $\widetilde{\mathscr{Q}}_m = \{Q \in \mathscr{Q}_m : Q \cap B(0, N-1) \neq \emptyset\}$. For each $Q \in \widetilde{\mathscr{Q}}_m$, set $y_Q := N^{-1}x_Q \in \mathbb{Q}^n \cap B(0,1)$.

Then

$$
\begin{aligned}
\sum_{Q\in\widetilde{\mathscr{Q}}_m} p_Q\delta_{x_Q} &= \sum_{Q\in\widetilde{\mathscr{Q}}_m} p_Q\delta_{Ny_Q} \\
&= \left(\sum_{Q\in\widetilde{\mathscr{Q}}_m} p_Q\delta_{y_Q}\right)_{0,N^{-1}} \\
&= \left(\sum_{Q\in\widetilde{\mathscr{Q}}_m} p_Q\right)\left(\sum_{Q\in\widetilde{\mathscr{Q}}_m} \frac{p_Q}{\sum_{Q\in\widetilde{\mathscr{Q}}_m} p_Q}\delta_{y_Q}\right)_{0,N^{-1}} \\
&=: p\nu_{0,N^{-1}}.
\end{aligned}
\tag{4.8}
$$

Let $f \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$. By our calculations above, we have that for $m \geq m_0$,

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d}\varphi\,\mathrm{d}\mu - \int_{\mathbb{R}^d}\varphi\,\mathrm{d}(p\nu_{0,N^{-1}})\right| &= \left|\int_{\mathbb{R}^d}\varphi\,\mathrm{d}\mu - \sum_{Q\in\widetilde{\mathscr{Q}}_m}\varphi(x_Q)p_Q\right| \\
&= \left|\int_{\mathbb{R}^d}\varphi\,\mathrm{d}\mu - \sum_{Q\in\mathscr{Q}_m}\varphi(x_Q^*)p_Q\right| \\
&\leq \left|\int_{\mathbb{R}^d}\varphi\,\mathrm{d}\mu - \sum_{Q\in\mathscr{Q}_m}\varphi(x_Q)\mu(Q)\right| \\
&\quad + \left|\sum_{Q\in\mathscr{Q}_m}\varphi(x_Q)\mu(Q) - \sum_{Q\in\mathscr{Q}_m}\varphi(x_Q)p_Q\right| \\
&\leq \varepsilon + \varepsilon\,\|\varphi\|_{L^\infty} \\
&= \varepsilon(1+\|\varphi\|_{L^\infty}).
\end{aligned}
\tag{4.9}
$$

By construction, ν is an element of $\mathscr{S}$. Since $\varepsilon > 0$ was arbitrary, we are done.

Because of this and lemma 2.1, it suffices to find a measure μ such that $\mathscr{S} \subseteq \mathrm{Tan}(\mu, p)$ for μ-a.e. $p \in \mathbb{R}^d$. Indeed, lemma 2.1 parts (a) and (b) mean the set $\{c\nu_{0,r} : c, r > 0\}$ is contained in $\mathrm{Tan}(\mu, p)$, and by the above analysis, this means $\mathrm{Tan}(\mu, p)$ is dense in $\mathscr{M}(\mathbb{R}^d)$. By part (c) of lemma 2.1, we infer that $\mathrm{Tan}(\mu, p)$ is all of $\mathscr{M}(\mathbb{R}^d)$ for μ-a.e. $p \in \mathbb{R}^d$.

Note that $\mathscr{S}$ is a countable set. Index the elements $\widetilde{\nu}_i$ of $\mathscr{S}$ by $i \in \mathbb{N}$. Define the sequence $(i_k)_{k\in\mathbb{N}\cup\{0\}}$ by

$$
i_k = k + 1 - \frac{1}{2}n(n+1) \text{ for } k \in \left[\frac{1}{2}n(n+1), \frac{1}{2}(n+1)(n+2)\right), \quad n \in \mathbb{N}\cup\{0\}. \tag{4.10}
$$

That is, i_k is the sequence $(1,1,2,1,2,3,1,2,3,4,\dots)$. If $k_n = \frac{1}{2}n(n+1)$ for $n \in \mathbb{N}\cup\{0\}$, then $i_{k_n} = 1$. More generally, for $m \in \mathbb{N}$, define

$$
k_m(n) = \frac{1}{2}(n+m-1)(n+m) + (m-1) \text{ for all } n \in \mathbb{N}\cup\{0\}. \tag{4.11}
$$

Then $i_{k_m(n)} = m$ for $n \in \mathbb{N}\cup\{0\}$. We set $\nu_k := \widetilde{\nu}_{i_k}$. It follows that every element of $\mathscr{S}$ occurs

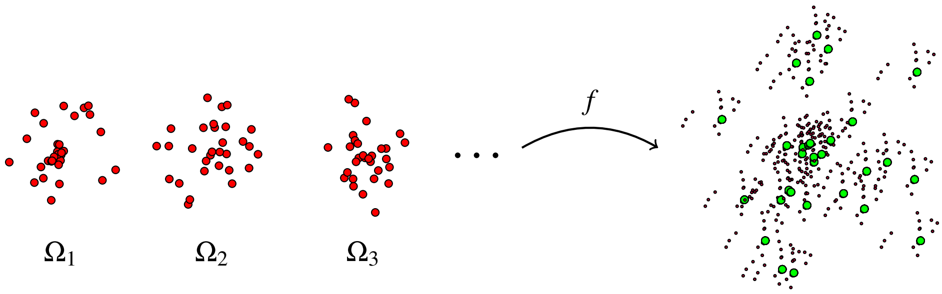

Figure 5: Mapping the product space $(\Omega, \mathscr{A}, \alpha)$ into $B(0,1)$.

infinitely many times in the sequence (ν_k). In particular, $\nu_{k_m(n)} = \widetilde{\nu}_m$ for all $n \in \mathbb{N} \cup \{0\}$.

For each $k \in \mathbb{N}$, write

$$\nu_k = \alpha_k(0)\delta_0 + \sum_{j=1}^{N_k} \alpha_k(x_{k,j})\delta_{x_{k,j}}, \tag{4.12}$$

and define $\Omega_k := \{0, x_{k,1}, \ldots, x_{k,N_k}\}$. Each α_k can be interpreted as the probability mass function for the measure ν_k on $(\Omega_k, 2^{\Omega_k})$. We then set $\Omega := \prod_{k=0}^{\infty} \Omega_k$. Consider the σ-algebra $\mathscr{A}$ on Ω generated by the *cylinder sets* of the form $\eta|_j = \{(\eta_0, \ldots, \eta_j)\} \times \prod_{k=j+1}^{\infty} \Omega_k$ for $\eta \in \Omega$, and define a probability measure α on $(\Omega, \mathscr{A})$ by

$$\alpha(\eta|_j) := \prod_{k=0}^{j} \alpha_k(\eta_k). \tag{4.13}$$

More abstractly, $(\Omega, \mathscr{A}, \alpha)$ is the product space $\prod_{k=0}^{\infty} (\Omega_k, 2^{\Omega_k}, \nu_k)$.

For each $k \in \mathbb{N}$, define

$$\sigma_k := \min\{|x-y| : x, y \in \Omega_k, x \neq y\}. \tag{4.14}$$

Inductively define the sequence r_k by choosing $r_0 > 1$, and defining $r_k := \frac{r_0^{k+1}}{\sigma_{k-1}} r_{k-1}$. Define $f \colon \Omega \to B(0,1)$ by

$$f(\eta) := \sum_{k=0}^{\infty} \frac{\eta_k}{r_k} \tag{4.15}$$

See figure 5. Define $\mu := f_*\alpha$ to be the pushforward measure of α by f. We will now show μ is the measure we want.

Given $m \in \mathbb{N}$, define

$$V_m := \{\eta \in \Omega : \eta_{k_m(n)} = 0 \text{ i.o. in } n\}. \tag{4.16}$$

That is, V_m is the event of choosing the point 0 infinitely often along the subsequence where $\nu_k = \widetilde{\nu}_m$. Now, since $\alpha(\eta_{k_m(n)} = 0) = \alpha_{k_m(n)}(0)$ is positive and independent of n, we have that

$$\sum_{n=0}^{\infty} \alpha(\eta_{k_m(n)} = 0) = \infty. \tag{4.17}$$

Furthermore, the events $\{\eta_{k_m(n)} = 0\}$ (over $n \in \mathbb{N} \cup \{0\}$) are independent, so we may conclude by the Borel-Cantelli lemma that $\alpha(V_m) = 1$. Define $V = \bigcap_{m=1}^{\infty} V_m$. Immediately, we have $\alpha(V) = 1$, and therefore $\mu(f(V)) = 1$.

We will show that $\mathscr{S} \subseteq \mathrm{Tan}(\mu, p)$ for all $p \in f(V)$. Fix $p \in f(V)$, and write $p = f(\pi) = \sum_{k=0}^{\infty} \frac{\pi_k}{r_k}$ for some $\pi \in V$. Fix $m \in \mathbb{N}$, and let $(n_j)_{j\in\mathbb{N}}$ be a sequence with values in $\mathbb{N} \cup \{0\}$, with $n_j \to \infty$, and such that $\pi_{k_m(n_j)} = 0$ for all j. This is possible by definition of V. Once we show $\int_{\mathbb{R}^d} \varphi \, \mathrm{d}(c_j \mu_{p,s_j}) \to \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\nu_m$ for some appropriate choice of c_j and s_j and all $\varphi \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$, the proof will be concluded by lemma A.3.

For $j \in \mathbb{N}$, define $\Omega^{(j)} := \left\{\eta \in \Omega : \eta_k = \pi_k \text{ for } k = 0, \ldots, k_m(n_j) - 1\right\}$, let $c_j = \alpha(\Omega^{(j)})^{-1}$, and let $s_j = 1/r_{k_m(n_j)}$. We have

$$\int_{\mathbb{R}^d} \varphi \, \mathrm{d}(c_j \mu_{p,s_j}) = c_j \int_{\mathbb{R}^d} \varphi\left(\frac{x-p}{s_j}\right) \, \mathrm{d}\mu(x) = c_j \int_{\Omega} \varphi\left(r_{k_m(n_j)} \sum_{k=0}^{\infty} \frac{\eta_k - \pi_k}{r_k}\right) \, \mathrm{d}\alpha(\eta). \tag{4.18}$$

We also have

$$\int_{\mathbb{R}^d} \varphi \, \mathrm{d}\widetilde{\nu}_m = \int_{\Omega_{k_m(n_j)}} \varphi \, \mathrm{d}\nu_{k_m(n_j)} = \int_{\Omega} \varphi(\eta_{k_m(n_j)}) \, \mathrm{d}\alpha(\eta) = c_j \int_{\Omega^{(j)}} \varphi(\eta_{k_m(n_j)}) \, \mathrm{d}\alpha(\eta), \tag{4.19}$$

where the last equality comes from the fact that the functions (random variables in this context) $\mathbb{1}_{\Omega^{(j)}}$ and $\eta \mapsto \varphi(\eta_{k_m(n_j)})$ are independent.

We would like to show, at least for j large enough, that

$$\int_{\Omega} \varphi\left(r_{k_m(n_j)} \sum_{k=0}^{\infty} \frac{\eta_k - \pi_k}{r_k}\right) \, \mathrm{d}\alpha(\eta) = \int_{\Omega^{(j)}} \varphi\left(r_{k_m(n_j)} \sum_{k=0}^{\infty} \frac{\eta_k - \pi_k}{r_k}\right) \, \mathrm{d}\alpha(\eta). \tag{4.20}$$

Indeed, suppose this were true, and let η be in this set. Then

$$\left|\eta_{k_m(n_j)} - r_{k_m(n_j)} \sum_{k=0}^{\infty} \frac{\eta_k - \pi_k}{r_k}\right| \leq \sum_{k=k_m(n_j)+1}^{\infty} \frac{|\eta_k - \pi_k|}{r_k} \leq \sum_{k=k_m(n_j)}^{\infty} \frac{2}{r_k}, \tag{4.21}$$

since all η_k and π_k are contained in $B(0,1)$, and $\pi_{k_m(n_j)} = 0$ by assumption. Since φ is Lipschitz with constant at most 1, this estimate implies

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \varphi \, \mathrm{d}(c_j \mu_{p,s_j}) - \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\widetilde{\nu}_m\right| \\
&= \left|c_j \int_{\Omega^{(j)}} \varphi\left(r_{k_m(n_j)} \sum_{k=0}^{\infty} \frac{\eta_k - \pi_k}{r_k}\right) \, \mathrm{d}\alpha(\eta) - c_j \int_{\Omega^{(j)}} \varphi(\eta_{k_m(n_j)}) \, \mathrm{d}\alpha(\eta)\right| \\
&\leq c_j \int_{\Omega^{(j)}} \sum_{k=k_m(n_j)}^{\infty} \frac{2}{r_k} \, \mathrm{d}\alpha \\
&= \sum_{k=k_m(n_j)}^{\infty} \frac{2}{r_k} \\
&\to 0 \quad \text{as } j \to \infty,
\end{aligned} \tag{4.22}$$

noting that $\sum_{k=0}^{\infty} \frac{1}{r_k} < \infty$.

It remains to check (4.20). Suppose $\eta \in \Omega \setminus \Omega^{(j)}$, and let $K < k_m(n_j)$ be the smallest integer with $\eta_K \neq \pi_K$. Let $R > 0$ be such that $\operatorname{supp} \varphi \subseteq B(0,R)$. Then

$$\sum_{k=0}^{\infty} \frac{\eta_k - \pi_k}{r_k} = \frac{\eta_K - \pi_K}{r_K} + \sum_{k=K+1}^{\infty} \frac{\eta_k - \pi_k}{r_k}. \tag{4.23}$$

By definition, we have

$$\left| \frac{\eta_K - \pi_K}{r_K} \right| \geq \frac{\sigma_K}{r_K}, \tag{4.24}$$

and we also have the estimate

$$\begin{aligned}
\left| \sum_{k=K+1}^{\infty} \frac{\eta_k - \pi_k}{r_k} \right| &\leq \sum_{k=K+1}^{\infty} \frac{2}{r_k} \\
&= 2 \sum_{k=K+1}^{\infty} \frac{\sigma_{k-1}}{r_0^{k+1}} \frac{1}{r_{k-1}} \\
&= \cdots \\
&= 2 \sum_{k=K+1}^{\infty} \frac{\sigma_{k-1} \cdots \sigma_K}{r_0^{k+1} \cdots r_0^{K+2}} \frac{1}{r_K} \\
&\leq 2 \frac{\sigma_K}{r_K} \sum_{k=K+1}^{\infty} \frac{1}{r_0^{k+1}} \\
&= 2 \frac{\sigma_K}{r_K} \frac{r_0^{-(K+2)}}{1 - r_0^{-1}} \\
&= 2 \frac{\sigma_K}{r_K} \frac{r_0^{-(K+1)}}{r_0 - 1},
\end{aligned} \tag{4.25}$$

where we note that $\sigma_k < 1$ for all $k \in \mathbb{N} \cup \{0\}$, since $0 \in \Omega_k$ by assumption. We therefore have

$$\begin{aligned}
\left| r_{k_m(n_j)} \sum_{k=1}^{\infty} \frac{\eta_k - \pi_k}{r_k} \right| &\geq r_{k_m(n_j)} \left(\left| \frac{\eta_K - \pi_K}{r_K} \right| - \left| \sum_{k=K+1}^{\infty} \frac{\eta_k - \pi_k}{r_k} \right| \right) \\
&\geq r_{k_m(n_j)} \frac{\sigma_K}{r_K} \left(1 - \frac{r_0^{-(K+1)}}{1 - r_0} \right) \\
&\geq r_{k_m(n_j)} \frac{\sigma_K}{r_K} \frac{r_0(1 - r_0) - 1}{r_0(1 - r_0)} \\
&= \frac{r_0(1 - r_0) - 1}{r_0(1 - r_0)} \frac{r_{k_m(n_j)-1}}{r_K} \frac{\sigma_K}{\sigma_{k_m(n_j)-1}} r_0^{k_m(n_j)+1} \\
&\geq r_0 \sigma_K \frac{r_0(1 - r_0) - 1}{r_0(1 - r_0)} r_0^{k_m(n_j)}.
\end{aligned} \tag{4.26}$$

For j sufficiently large, this expression is greater than R, so that

$$\varphi\left(r_{k_m(n_j)}\sum_{k=1}^{\infty}\frac{\eta_k-\pi_k}{r_k}\right)=0, \tag{4.27}$$

thus proving (4.20). $\square$

Notes and Remarks

Note that definition (4.1) of the set $\mathscr{S}$ grants us some leeway in choosing μ. Indeed, rather than taking $x_i \in \mathbb{Q}^d \cap B(0,1)$, we could ask that they come from a dense subset of another open ball, and the same proof would carry on over verbatim. One might wonder how restrictive the condition $\operatorname{Tan}(\mu,x_0)=\mathscr{M}(\mathbb{R}^d)$ actually is. All our examples in the last two chapters certainly didn't satisfy this, and it even took us a lot of work to show the existence of even one of these measures. As it turns out, this is not so restrictive at all. Tuomas Sahlsten in [Sah12] and Toby O'Neil in [One] independently managed to prove the following:

‖ **Theorem 4.2.** *For a typical* $\mu \in \mathscr{M}(\mathbb{R}^d)$*, we have* $\operatorname{Tan}(\mu,x_0)=\mathscr{M}(\mathbb{R}^d)$.

Here, *typical* means this result holds for all μ in a residual subset (a countable union of dense open sets) of $\mathscr{M}(\mathbb{R}^d)$. The proof employed by Sahlsten for theorem 4.2 is effectively a much more technical version of the construction to prove theorem 4.1, making use of a set $\mathscr{S}$ similar to the above, and also explicitly defining the residual subset $\mathscr{R} \subseteq \mathscr{M}(\mathbb{R}^d)$.

5 Tangent Measures of Singular Measures

5.1 Pushforward Measures

Let $\mu \in \mathscr{M}(\mathbb{R}^m)$, and let $f\colon \mathbb{R}^m \to \mathbb{R}^d$ be a C^1 diffeomorphism. We would like to investigate the tangent measures of the pushforward measure $f_*\mu$. This investigation will follow many of the same lines as in section 3.1 Choose a tangent measure $\tau \in \operatorname{Tan}(\mu,q)$, and let $c_j\mu_{q,r_j}$ be a blowup sequence for τ. By translating if necessary, we may assume $f(q)=0$. Fix $\varphi \in \operatorname{Lip}_{\le 1}(\mathbb{R}^d)$, and let $R>0$ be such that $\operatorname{supp}\varphi \subseteq B(0,R)$. Now consider

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi\,\mathrm{d}(c_jf_*\mu)_{0,r_j} &= c_j\int_{\mathbb{R}^d}\varphi\left(\frac{x}{r_j}\right)\,\mathrm{d}(f_*\mu)(x)\\ &= c_j\int_{\mathbb{R}^m}\varphi\left(\frac{f(y)}{r_j}\right)\,\mathrm{d}\mu(y).\end{aligned} \tag{5.1}$$

Note that since f is a diffeomorphism, the set $f^{-1}(B(0,R))$ is bounded, so f is uniformly Lipschitz on this set. Let $\operatorname{Lip} f$ be its Lipschitz constant on this set. If $y \in \mathbb{R}^m$ lies outside the set $B(0,rR/\operatorname{Lip} f)$, then $f(y)/r$ lies outside the set $B(0,R)$ (again, see section 3.1). So for $K :=$

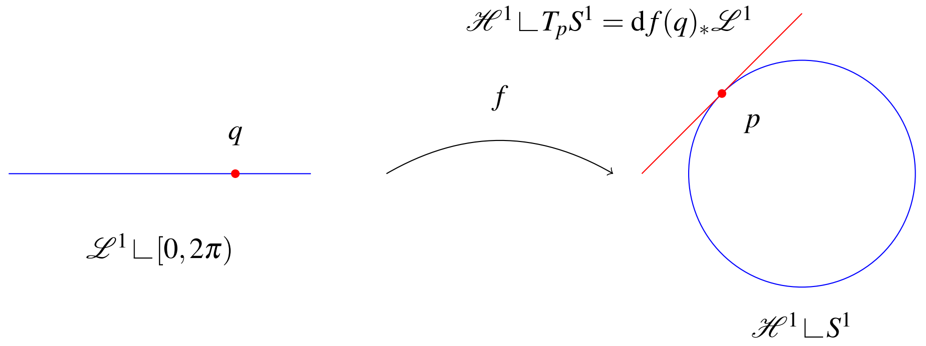

Figure 6: Pushforward of Lebesgue measure.

$R/\operatorname{Lip} f$, the above integral is

$$c_j \int_{B(0,Kr_j)} \varphi\left(\frac{f(y)}{r_j}\right) \, \mathrm{d}\mu(y). \tag{5.2}$$

Now, we also have

$$\begin{aligned} &\lim_{j\to\infty} \left| c_j \int_{B(0,Kr_j)} \varphi\left(\frac{f(y)}{r_j}\right) \, \mathrm{d}\mu(y) - c_j \int_{B(0,Kr_j)} \varphi\left(\frac{\mathrm{d}f(0)(y)}{r_j}\right) \, \mathrm{d}\mu(y) \right| \\ &\quad \le \lim_{j\to\infty} c_j \mu(0, Kr_j) \sup_{y\in B(0,Kr_j)} \frac{|f(y) - \mathrm{d}f(0)(y)|}{r_j} \\ &= 0, \end{aligned} \tag{5.3}$$

Here, we remark that $c_j \mu(0, Kr_j)$ is finite by our estimates in section 2.1. Finally, we then have

$$\begin{aligned} \lim_{j\to\infty} c_j \int_{B(0,Kr_j)} \varphi\left(\frac{\mathrm{d}f(0)(y)}{r_j}\right) \, \mathrm{d}\mu(y) &= \lim_{j\to\infty} c_j \int_{\mathbb{R}^m} \varphi\left(\frac{\mathrm{d}f(0)(y)}{r_j}\right) \, \mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}^m} \varphi\left(\mathrm{d}f(0)(y)\right) \, \mathrm{d}\tau(y) \\ &= \int_{\mathbb{R}^d} \varphi \, \mathrm{d}(\mathrm{d}f(0)_* \tau). \end{aligned} \tag{5.4}$$

It follows that $\mathrm{d}f(0)_* \tau$ is a tangent measure to $f_* \mu$ at 0.

Let's consider the example suggested by figure 6. Define the map $f \colon [0, 2\pi) \subseteq \mathbb{R} \to \mathbb{R}^2$ by $f(s) := (\cos s, \sin s)$, whose image is a circle. Recall the area formula (3.1). Since the derivative of f at s is just the (column) vector $f'(s) = (-\sin s, \cos s)$, we may calculate its Jacobian determinant to be

$$\mathrm{J}f(s) = \sqrt{f'(s)^{\mathrm{T}} f'(s)} = 1. \tag{5.5}$$

Given $\varphi \in C_c(\mathbb{R}^2)$, we use the area formula to calculate

$$\begin{aligned}
\int_{\mathbb{R}^2} \varphi \, \mathrm{d}(\mathscr{H}^1 \llcorner S^1) &= \int_{S^1} \varphi \, \mathscr{H}^1 \\
&= \int_0^{2\pi} \varphi(f(s)) \mathrm{J} f(s) \, \mathrm{d}\mathscr{L}^1(s) \\
&= \int_0^{2\pi} \varphi(f(s)) \, \mathrm{d}\mathscr{L}^1(s) \\
&= \int_{\mathbb{R}} \varphi \, \mathrm{d}(f_*(\mathscr{L}^1 \llcorner [0, 2\pi])).
\end{aligned} \tag{5.6}$$

It follows that $\mathscr{H}^1 \llcorner S^1$ is the pushforward measure $f_* \mathscr{L}^1$. Now fix a point $p = f(q) \in S^1$. Our above calculations for tangent measures of pushforward measure shows that $\mathrm{d}f(q)_* \mathscr{L}^1$ is a tangent measure to $f_*(\mathscr{L}^1 \llcorner [0, 2\pi])$ at p. However, we know that $\mathrm{d}f(q)$ is given by $f'(q) = (-\sin q, \cos q)$, so the image of $\mathrm{d}f(q)$ is precisely the tangent space $T_p S^1$. This agrees with our calculations in section 3.1.

5.2 Hausdorff Density and Singularities

Consider the measure $\mu = \delta_0 + \mathscr{L}^d$ on $\mathbb{R}^d$. Certainly, this measure is not singular with respect to $\mathscr{L}^d$ in the sense that its null sets are completely indistinct from those of Lebesgue measure, but there is certainly a problem at the origin. This invites us to consider a notion of "singularity" at a point in $\mathbb{R}^d$.

Recall the definition of Hausdorff density from section 3.2. Immediately from the defition, we see $\Theta^d(\mathscr{L}^d, p) = \omega_d$ for all $p \in \mathbb{R}^d$, where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Some authors put a constant in the denominator of the definition of the upper and lower Hausdorff densities to ensure $\Theta^d(\mathscr{L}^d, p) = 1$, but the exact value is not important, at least for our purposes. More generally, we have $\Theta^s(\mathscr{H}^s, p) = \omega_s$ for some $\omega_s > 0$. For our measure $\mu = \delta_0 + \mathscr{L}^d$ as above, it is easy to see that $\Theta^d(\mu, p) = \omega_d$ for $p \neq 0$, and $\Theta^d(\mu, 0) = \infty$.

Actually, David Preiss in [Pre87] introduced tangent measures to prove the following conjecture in geometric measure theory:

> **Theorem 5.1** (Marstrand). *Let $\mu \in \mathscr{M}(\mathbb{R}^d)$ be a Radon measure. Suppose there exists $s > 0$ such that the Hausdorff density $\Theta^s(\mu, p)$ exists and is positive and finite for all p in a set of positive μ-measure. Then s is an integer.*

In fact, Preiss proved the following stronger theorem:

> **Theorem 5.2** (Preiss). *Let $\mu \in \mathscr{M}(\mathbb{R}^d)$ be a Radon measure, and $k \in \mathbb{N}$ a positive integer. If the Hausdorff density $\Theta^k(\mu, p)$ exists and is positive and finite for all p in the support of μ, then μ is k-rectifiable.*

Theorem 3.2 says that it suffices to show this condition on the Hausdorff density implies we have flat tangent measures at μ-a.e. point. See [Lel08] for a more thorough outline of the proof, chapter

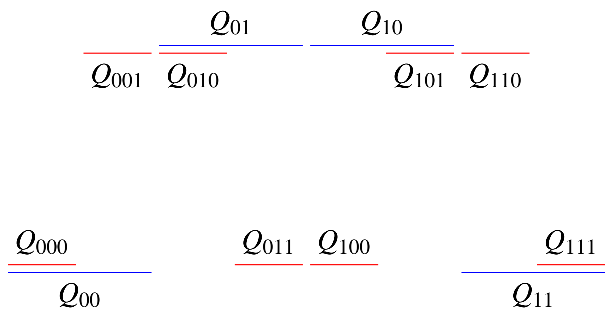

Figure 7: Construction intervals and corresponding parity.

14 of [Mat95] for a direct proof of Marstrand's theorem, or section 4.2 [Fal85] for an easier proof of Marstrand's theorem in the case $0 < s < 1$. In the latter instance, Marstrand's theorem says $\Theta^s(\mu, p)$ does not exist μ-a.e. for $0 < s < 1$.

5.3 The Preiss Measure

Our interest is showing that a partial converse to theorem 5.2 for $d = 1$ is not true: namely, we can find a measure μ which has flat tangent measures at μ-a.e. point, but whose Hausdorff 1-density at μ-a.e. point is infinite. Stated as a theorem:

Theorem 5.3 (Preiss)**.** *There exists a Radon measure $\mu \in \mathscr{M}(\mathbb{R})$ such that for μ-a.e. $p \in \mathbb{R}$, we have $\Theta^{1*}(\mu, p) = \infty$, and* $\operatorname{Tan}(\mu, p) = \{c\mathscr{L}^1 : c > 0\}$.

We will follow the details of Preiss's proof in [Pre87] section 5.8. See [FP90] for an alternative probabilistic proof.

The Preiss measure will be a pushforward measure of the form $\mu = f_*\mathscr{L} \llcorner [0,1)$ for some function $f\colon [0,1) \to \mathbb{R}$. By our calculations in section 5.1, we would like any "approximate gradient" of f to be approximately constant.

The construction employed is dyadic. We will construct a sequence g_k of "approximate gradients" of f, and show their antiderivatives converge to the thing we're looking for. This construction will be based on a collection of dyadic subintervals of $[0,1)$ of the form $[i2^{-k}, (i+1)2^{-k})$. We will define them rigorously by induction. To start let $Q := [0,1)$. Next, given $Q_{i_1,\dots,i_{k-1}} \subseteq Q$ define $Q_{i_1,\dots,i_{k-1},0}$ to be the first half-open subinterval of $Q_{i_1,\dots,i_{k-1}}$ of size 2^{-k}, and $Q_{i_1,\dots,i_{k-1},1}$ to be the second. The construction intervals Q_i are therefore enumerated by binary sequences $i = (i_1, \dots, i_k) \in \{0,1\}^k$. The integer k is called the *generation* of the interval Q_i. The *children* of Q_i are Q_{i0} and Q_{i1}, and Q_i is the children's *parent*. We write $\mathscr{Q}_k$ for the set of all construction intervals of generation k. Given a binary sequence $i = (i_1, \dots, i_k)$, its *parity* $|i|$ is defined to be the sum $i_{k-1} + i_k$, with addition taken mod 2. For example, the parity of the sequence 011010 is $1 + 0 = 1$, and the parity of 1010111 is $1 + 1 = 0$. See figure 7.

Having defined the construction intervals, we define the functions $g_k\colon Q \to \mathbb{R}$ inductively. Each

g_k is defined from the previous by adding an additional oscillation in a controlled manner so that $g_k \to 0$ in a sense we will make precise shortly. Define $g_0 \colon Q \to \mathbb{R}$ by $g_0(t) = 1$ for all $t \in Q$, and fix $\varepsilon_0 \in (0, \sqrt{5} - 2))$ (the strange upper bound will be important later). Suppose g_{k-1} and ε_{k-1} have been defined, and set

$$g_k(t) := \left(1 + \sum_{i \in \{0,1\}^k} s_i \mathbb{1}_{Q_i}(t)\right) g_{k-1}(t), \tag{5.7}$$

where $s_i = 0$ if $|g_{k-1}(s)| \leq \varepsilon_{k-1}$ for some (hence all) s in the parent of Q_i, and $s_i = (-1)^{|i|}\varepsilon_{k-1}$ otherwise. Note that if $i \in \{0,1\}^{k-1}$, then $s_{i,0} = -s_{i,1}$. We also let $\varepsilon_k = \varepsilon_{k-1}$ if

$$\begin{aligned} &\left|\left\{t \in Q : \left|g_j(t)\right| > \varepsilon_{k-1} \text{ for } j = 0, \ldots, k\right\}\right| \geq \varepsilon_{k-1} \\ \text{or } &\left|\left\{t \in Q : \left|g_j(t)\right| < 1 \text{ for } j = 0, \ldots, k-1\right\}\right| \geq \varepsilon_{k-1}, \end{aligned} \tag{5.8}$$

and $\varepsilon_k = \varepsilon_{k-1}/2$ otherwise. Thus if too much of g_k is too large or too small, we decrease the size of the oscillation. Immediately from the construction, we see

$$\begin{aligned} g_k(t) &= \left(1 + \sum_{i \in \{0,1\}^k} s_i \mathbb{1}_{Q_i}(t)\right) g_{k-1}(t) \\ &\leq (1 + \varepsilon_k) \left(1 + \sum_{i \in \{0,1\}^{k-1}} s_i \mathbb{1}_{Q_i}(t)\right) g_{k-2}(t) \\ &\leq \cdots \\ &\leq (1 + \varepsilon_0)^k \end{aligned} \tag{5.9}$$

for all $t \in Q$ and $k \in \mathbb{N}$, and we similarly have $g_k(t) \geq (1 - \varepsilon_0)^k$. See figure 8 for some iterations of g.

We now define

$$f_k(t) := \int_0^t g_k(s)\, \mathrm{d}s. \tag{5.10}$$

For all construction intervals Q_i of generation k, we claim $f_{k+1}(a) = f_k(a)$ whenever a is one of the endpoints of Q_i. Indeed, given such an a, there exists a subset $A \subseteq \{0,1\}^k$ enumerating all the generation k construction intervals $Q_{j'}$ which come before a. Writing $g_k|_Q$ for the value of g_k on

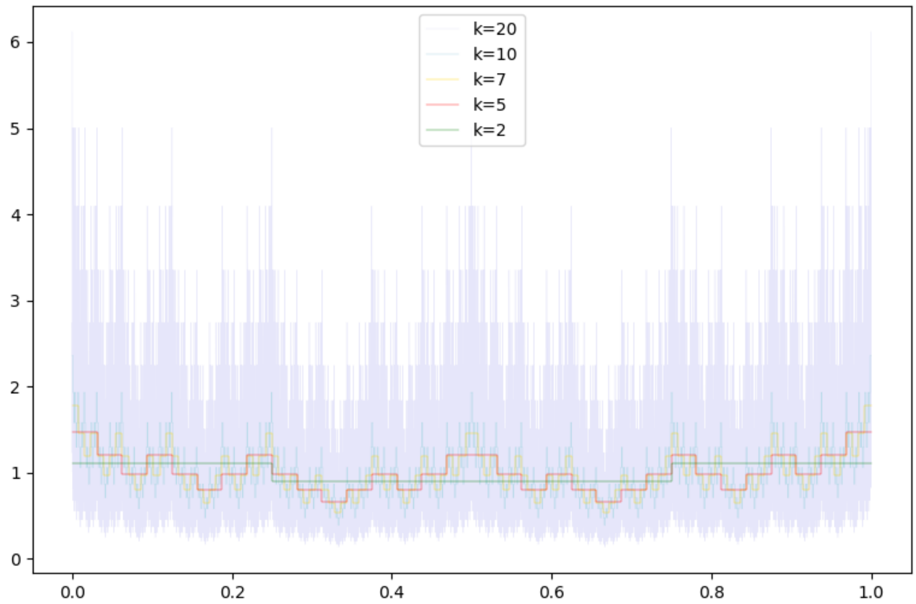

Figure 8: Some iterations of g_k for $\varepsilon_0 = 0.1$.

$Q \in \mathscr{Q}_k$, we have

$$\begin{aligned} f_{k+1}(a) - f_k(a) &= \int_0^a g_{k+1}(t) - g_k(t) \, \mathrm{d}t \\ &= \int_0^a \sum_{j \in \{0,1\}^{k+1}} s_j \mathbb{1}_{Q_j}(t) g_k(t) \, \mathrm{d}t \\ &= \sum_{j' \in A} \int_{Q_{j'}} s_{j',0} \mathbb{1}_{Q_{j',0}}(t) g_k|_{Q_{j'}} + s_{j',1} \mathbb{1}_{Q_{j',1}}(t) g_k|_{Q_{j'}} \, \mathrm{d}t \\ &= \sum_{j' \in A} (s_{j',0} - s_{j',0}) 2^{-(k+1)} g_k|_{Q_{j'}} \\ &= 0. \end{aligned} \tag{5.11}$$

which proves the claim.

Fix $t \in Q$, and let $Q_i = [a,b)$ be the generation k construction interval in which t lies. Then

$$\begin{aligned} f_{k+1}(t) - f_k(t) &= (f_{k+1}(t) - f_{k+1}(a)) - (f_k(t) - f_k(a)) \\ &= \int_a^t g_{k+1}(s) - g_k(s) \, \mathrm{d}s \\ &\leq \int_a^t \varepsilon_k g_k(s) \, \mathrm{d}s \\ &= (t-a)\varepsilon_k g_k(a) \\ &\leq 2^{-k} \varepsilon_k (1+\varepsilon_0)^k, \end{aligned} \tag{5.12}$$

since g_k is constant on $[a,t)$. Taking suprema over $t \in [0,1]$, it follows that $\|f_{k+1} - f_k\|_{L^\infty} \le 2^{-k}\varepsilon_0(1+\varepsilon_0)^k$. Since $\sum_{k=1}^\infty \left(\frac{1+\varepsilon_0}{2}\right)^k < \infty$, the sequence f_k is Cauchy in $C([0,1])$, so converges uniformly to some $f \in C([0,1])$. For any $t \in Q$, we also have the estimate

$$\begin{aligned}
f(t) - f_k(t) &= \sum_{j=k}^\infty f_{j+1}(t) - f_j(t) \\
&\le \sum_{j=k}^\infty 2^{-j}\varepsilon_j g_j(t) \\
&\le \sum_{j=k}^\infty 2^{-j}\varepsilon_k (1+\varepsilon_0)^{j-k} g_k(t) \\
&= \frac{\varepsilon_k g_k(t)}{(1+\varepsilon_0)^k} \sum_{j=k}^\infty \left(\frac{1+\varepsilon_0}{2}\right)^j \\
&= \frac{\varepsilon_k g_k(t)}{(1+\varepsilon_0)^k} \frac{(1+\varepsilon_0)^k}{2^k} \frac{1}{1-\frac{1+\varepsilon_0}{2}} \\
&= \frac{2\varepsilon_k}{1-\varepsilon_0} 2^{-k} g_k(t).
\end{aligned} \tag{5.13}$$

The aim now is to show $\mu := f_*\mathscr{L}^1 \llcorner Q$ is the measure we want. This will follow by finding a function $h\colon Q \times (0,\infty) \to (0,\infty)$ such that for $\mathscr{L}^1$-a.e. $t \in Q$, the following three conditions hold:

$$\lim_{r \downarrow 0} \frac{\mu(B(f(t),r))}{h(t,r)} = 1, \tag{5.14}$$

$$\limsup_{r \downarrow 0} \frac{h(t,r)}{r} = \infty, \text{ and} \tag{5.15}$$

$$\lim_{r \downarrow 0} \frac{h(t,sr)}{h(t,r)} = s \text{ for all } s > 0. \tag{5.16}$$

Indeed, (5.14) and (5.15) immediately imply $\Theta^{1*}(\mu, f(t)) = \infty$. Meanwhile for $\tau \in \operatorname{Tan}(\mu, f(t))$ with $c\mu(B(f(t),Kr_j)^{-1}\mu_{f(t),r_j} \xrightarrow{*} \tau$, we use lemma A.2, (5.14), and (5.16) to estimate

$$\begin{aligned}
\tau(B(0,r)) &\le \liminf_{j\to\infty} \frac{c\mu_{f(t),r_j}(B(0,r))}{\mu(B(f(t),Kr_j))} \\
&= \liminf_{j\to\infty} c \frac{h(t,Kr_j)}{\mu(B(f(t),Kr_j))} \frac{\mu(B(f(t),r_j r))}{h(t,r_j r)} \frac{h(t,r_j r)}{h(t,Kr_j)} \\
&= cK^{-1}r.
\end{aligned} \tag{5.17}$$

On the other hand, the same lemma gives us $\tau(\overline{B(0,r)}) \ge cK^{-1}r$. This immediately implies $\tau(B(0,r)) = cK^{-1}r$, since

$$cK^{-1}r \ge \tau(B(0,r)) = \sup\{\tau(K) : K \subseteq B(0,r) \text{ compact}\} \ge \sup\left\{\tau(\overline{B(0,r')}) : r' < r\right\} = cK^{-1}r. \tag{5.18}$$

To show $\tau(B(p,r)) = cK^{-1}r$ for all other $p \in \mathbb{R}$, we apply the following trick (see [Pre87] theorem

2.12): For μ-a.e. $y \in \mathbb{R}$ and all $\tau \in \operatorname{Tan}(\mu, y)$, $x \in \mathbb{R}$, the measure $\tau_{x,1}$ lies in $\operatorname{Tan}(\mu, y)$. Thus, whenever $f(t)$ is such that this property holds, we see that $\tau_{p,1}$ is in $\operatorname{Tan}(\mu, f(t))$, and so our above calculations imply

$$\tau(B(p,r)) = \tau_{p,1}(B(0,r)) = cK^{-1}r, \tag{5.19}$$

as required.

Let's now show such a function h does exist. We would like to find a subsequence g_{k_j} along which $\varepsilon_{k_j+1} = \varepsilon_{k_j}/2$. Suppose, for a contradiction, that there does not exist such a subsequence. Then the sequence ε_k eventually stabilizes, so we can choose $k_0 \in \mathbb{N}$ such that $\varepsilon_k = \varepsilon_{k_0}$ for all $k \geq k_0$. Then $E := \{t \in Q : |g_k(t)| > \varepsilon_{k_0} \text{ for all } k\}$ has positive measure, or $E' := \{t \in Q : |g_k(t)| < 1 \text{ for all } k\}$ has positive measure. Indeed, this is since

$$|\{t \in Q : |g_k(t)| > \varepsilon_{k_0} \text{ for all } k\}| = \lim_{k\to\infty} \left|\left\{t \in Q : \left|g_j(t)\right| > \varepsilon_{k_0} \text{ for } j = 1,\dots,k\right\}\right| \tag{5.20}$$

and similarly for E'. The set E' certainly cannot have positive measure, since $g_0(t) = 1$ for all $t \in Q$. We will therefore assume E has positive measure. By definition of the g_k, we have

$$g_k(t) = \left(1 + \sum_{i\in\{0,1\}^k} (-1)^{|i|} \varepsilon_{k-1} \mathbb{1}_{Q_i}(t)\right) g_{k-1}(t) \tag{5.21}$$

for all $t \in E$ and $k \geq k_0 + 1$. Thus

$$\begin{aligned} \liminf_{k\to\infty} g_k(t) &= \liminf_{k\to\infty} \prod_{j=1}^{k} \left(1 + \sum_{i\in\{0,1\}^j} (-1)^{|i|} \varepsilon_{j-1} \mathbb{1}_{Q_i}(t)\right) \\ &\leq \liminf_{k\to\infty} \exp\left(\sum_{j=1}^{k} \sum_{i\in\{0,1\}^j} (-1)^{|i|} \varepsilon_{j-1} \mathbb{1}_{Q_i}(t)\right) \\ &\leq \liminf_{k\to\infty} \exp\left(\sum_{j=k_0}^{k} \varepsilon_{k_0} \sum_{i\in\{0,1\}^j} (-1)^{|i|} \mathbb{1}_{Q_i}(t)\right) \\ &= 0 \end{aligned} \tag{5.22}$$

for $\mathscr{L}^1$-a.e. $t \in E$. This contradicts $|E| > 0$. We conclude there exists a subsequence g_{k_j} along which $\varepsilon_{k_j+1} = \varepsilon_{k_j}/2$.

Having found g_{k_j}, by definition we see

$$\left|\left\{t \in Q : |g_k(t)| > \varepsilon_{k_j} \text{ for } k = 0,\dots,k_j\right\}\right| < \varepsilon_{k_j} \tag{5.23}$$

for all $j \in \mathbb{N}$. Fix $\delta > 0$. Noting that $\lim_{j\to\infty} \varepsilon_{k_j} = 0$, we have that for j large enough,

$$\left|\left\{t \in Q : \left|g_{k_j}(t)\right| > \delta\right\}\right| \leq \left|\left\{t \in Q : \left|g_{k_j}(t)\right| > \varepsilon_{k_j}\right\}\right| \leq \varepsilon_{k_j}. \tag{5.24}$$

Letting $j \to \infty$ on both sides and recalling $\delta > 0$ was arbitrary, we conclude $g_{k_j} \to 0$ in measure. In particular, there exists a further subsequence (not relabeled) with $g_k \to 0$ a.e. Let $\widetilde{Q} \subseteq Q$ be the set of all t with $g_k(t) \to 0$. From here on out, we will pass to this subsequence.

Given $t \in \widetilde{Q}$ and $r > 0$, define $k(t,r)$ to be the largest integer such that $|g_k(t)| \geq 2^k r$. Note that $\lim_{r\downarrow 0} k(t,r) = \infty$ since, for all $K \in \mathbb{N}$, we can find $r > 0$ such that $2^{-K}(1+\varepsilon_0)^K < r$. Define

$$h(t,r) := \frac{r}{g_{k(t,r)}(t)}. \tag{5.25}$$

Immediately, we see

$$\limsup_{r\downarrow 0} \frac{h(t,r)}{r} = \limsup_{r\downarrow 0} \frac{1}{g_{k(t,r)}(t)} = \infty, \tag{5.26}$$

therefore proving (5.15).

We next prove (5.16). By definition of h, it suffices to show

$$\lim_{r\downarrow 0} \frac{g_{k(t,r)}(t)}{g_{k(t,sr)}(t)} = 1 \text{ for all } s > 0. \tag{5.27}$$

Suppose $s \in [\frac{1+\varepsilon_0}{2}, 1]$. Then $k(t,r) \leq k(t,sr)$. Furthermore, if $j \geq k(t,r)+2$, then

$$\begin{aligned} g_j(t) &\leq (1+\varepsilon_0)^{j-k(t,r)-1} g_{k(t,r)+1}(t) \\ &< (1+\varepsilon_0)^{j-k(t,r)-1} 2^{k(t,r)+1} r \\ &\leq \left(\frac{1+\varepsilon_0}{2}\right)^{j-k(t,r)-1} 2^j r \\ &\leq 2^j sr. \end{aligned} \tag{5.28}$$

This implies $k(t,sr) \leq k(t,r)+1$. Thanks to this, if $s \in [\frac{1+\varepsilon_0}{2}, 1]$, then we have

$$(1-\varepsilon_{k(t,r)+1}) g_{k(t,r)}(t) \leq g_{k(t,sr)}(t) \leq (1+\varepsilon_{k(t,r)+1}) g_{k(t,r)}(t). \tag{5.29}$$

Dividing through by $g_{k(t,r)}(t)$ and taking $r \downarrow 0$, noting $\varepsilon_{k(t,r)} \to 0$, we conclude (5.27) for $s \in [\frac{1+\varepsilon_0}{2}, 1]$. For $s < \frac{1+\varepsilon_0}{2}$, choose a sequence $s_0, ..., s_k$ with $s_0, \frac{s_i}{s_{i-1}}, \frac{s}{s_k} \in [\frac{1+\varepsilon_0}{2}]$. We then find

$$\lim_{r\downarrow 0} \frac{g_{k(t,r)}(t)}{g_{k(t,sr)}(t)} = \lim_{r\downarrow 0} \frac{g_{k(t,s_k r)}(t)}{g_{k(t,sr)}(t)} \frac{g_{k(t,s_{k-1} r)}(t)}{g_{k(t,s_k r)}(t)} \cdots \frac{g_{k(t,s_0 r)}(t)}{g_{k(t,s_1 r)}(t)} \frac{g_{k(t,r)}(t)}{g_{k(t,sr)}(t)} = 1. \tag{5.30}$$

For $s > 1$, use the fact that $s^{-1} < 1$, and

$$\lim_{r\downarrow 0} \frac{g_{k(t,r)}(t)}{g_{k(t,sr)}(t)} = \lim_{r\downarrow 0} \frac{g_{k(t,s^{-1}r)}(t)}{g_{k(t,r)}(t)}. \tag{5.31}$$

Therefore proving (5.27), and hence (5.16).

Property (5.14) turns out to be the trickiest to prove. We will prove there exists $c_{t,r}$ such that

$$\begin{aligned}(t-(1+c_{t,r})^{-1}h(t,r),t+(1+c_{t,r})^{-1}h(t,r)) &\subseteq f^{-1}(B(f(t),r)) \\ &\subseteq (t-(1-c_{t,r})^{-1}h(t,r),t+(1-c_{t,r})^{-1}h(t,r))\end{aligned} \tag{5.32}$$

for all $t \in \widetilde{Q}$ and all sufficiently small $r > 0$, and with the property $c_{t,r}^{-1} \to 0$ as $r \downarrow 0$. Property (5.14) is then immediate. Proving these two inclusions will follow from quantifying what we mean when we say g_k is an "approximate gradient" for f.

Let Q_i be a construction interval of generation k, and pick $s,t \in Q_i \cap \widetilde{Q}$ with $t > s$ and $2^{-(k+1)} \leq t-s$. In particular, t and s do not lie in the same construction interval of generation $k+1$. Then

$$|f_k(t)-f_k(s)-(t-s)g_k(t)| = 0 \tag{5.33}$$

by construction. Using (5.13), we estimate

$$\begin{aligned}|f(t)-f(s)-(t-s)g_k(t)| &\leq |f(t)-f_k(t)|+|f_k(t)-f_k(s)-(t-s)g_k(t)|+|f_k(s)-f(s)| \\ &\leq \frac{4\varepsilon_k}{1-\varepsilon_0}2^{-k}g_k(t).\end{aligned} \tag{5.34}$$

The following two-sided estimate then follows immediately:

$$\begin{aligned}\left(1-\frac{8\varepsilon_k}{1-\varepsilon_0}\right)g_k(t)(t-s) &\leq \left((t-s)-\frac{4\varepsilon_k}{1-\varepsilon_0}2^{-k}\right)g_k(v) \\ &\leq |f(t)-f(s)| \\ &\leq \left((t-s)+\frac{4\varepsilon_k}{1-\varepsilon_0}2^{-k}\right)g_k(t) \\ &\leq \left(1+\frac{8\varepsilon_k}{1-\varepsilon_0}\right)g_k(t)(t-s),\end{aligned} \tag{5.35}$$

using the fact that $2^{-1}2^{-k} = 2^{-(k+1)} \leq t-s$.

Having shown this, fix $r > 0$. Let $t,s \in \widetilde{Q}$ be such that $|f(t)-f(s)| < r$. Let $j \in \mathbb{N}$ be such that t,s lie in a construction interval of generation j, and $2^{-(j+1)} \leq t-s$. Suppose $i \geq j$. Using (5.35), we estimate

$$\begin{aligned}g_i(t) &\leq 2^{i-j}g_j(t) \\ &\leq 2^{i-j}\left(1-\frac{8\varepsilon_j}{1-\varepsilon_0}\right)^{-1}\frac{r}{t-s} \\ &\leq 2^i\frac{2}{1+\varepsilon_0}r.\end{aligned} \tag{5.36}$$

It follows that $k(t,\frac{2}{1+\varepsilon_0}r) \leq j-1$. By (5.28), we know $k(t,\frac{2}{1+\varepsilon_0}r) \geq k(t,r)-1$. Hence $j \geq k(t,r)$.

If $j = k(t,r)$, we may again use (5.35) to estimate

$$\begin{aligned} t-s &\le \left(1-\frac{8\varepsilon_{k(t,r)}}{1-\varepsilon_0}\right)^{-1} \frac{|f(t)-f(s)|}{g_{k(t,r)}(t)} \\ &< \left(1-\frac{8\varepsilon_{k(t,r)}}{1-\varepsilon_0}\right)^{-1} h(t,r). \end{aligned} \tag{5.37}$$

On the other hand, if $j > k(t,r)$, we estimate

$$\begin{aligned} t-s \le 2^{-j} &= 2^{-(k(t,r)+1)} 2^{k(t,r)+1-j} \\ &< \frac{r}{g_{k(t,r)+1}(t)} \\ &\le (1-\varepsilon_{k(t,r)})^{-1} h(t,r) \\ &\le \left(1-\frac{8\varepsilon_{k(t,r)}}{1-\varepsilon_0}\right)^{-1} h(t,r). \end{aligned} \tag{5.38}$$

The right-hand inclusion in (5.32) has therefore been shown, noting that by our choice of subsequence above, $\varepsilon_{k(t,r)} \to 0$ as $r \downarrow 0$.

For the left-hand inclusion, we see that if

$$|t-s| < \left(1+\frac{\varepsilon_{k(t,r)}}{1-\varepsilon_0}\right)^{-1} h(t,r), \tag{5.39}$$

then $|t-s| \le 2^{-k(t,r)}$. Therefore t and s lie in a construction interval of generation $j \ge k(t,r)$, and $2^{-(j+1)} \le |t-s|$. Without loss of generality, $t > s$. If $j = k(t,r)$, then (5.35) immediately implies $|f(t)-f(s)| \le r$. If $j = k(t,r)+1$, then we have

$$\begin{aligned} |f(t)-f(s)| &\le \left(1+\frac{8\varepsilon_{k(t,r)+1}}{1-\varepsilon_0}\right)\left(1+\frac{8\varepsilon_{k(t,r)}}{1-\varepsilon_0}\right)^{-1} \frac{g_{k(t,r)+1}(t)}{g_{k(t,r)}(t)} r \\ &\le \frac{1-\varepsilon_0+4\varepsilon_{k(t,r)}}{1-\varepsilon_0+8\varepsilon_{k(t,r)}}(1+\varepsilon_{k(t,r)}) r \\ &\le r. \end{aligned} \tag{5.40}$$

Suppose now that $j \geq k(t,r)+2$. We estimate

$$\begin{aligned}
|f(t)-f(s)| &\leq \left(1+\frac{8\varepsilon_j}{1-\varepsilon_0}\right) g_j(t)(t-s) \\
&\leq \left(1+\frac{8\varepsilon_j}{1-\varepsilon_0}\right)(1+\varepsilon_0)^{j-k(t,r)-1} g_{k(t,r)+1}(t) 2^{-j} \\
&\leq \left(1+\frac{8\varepsilon_j}{1-\varepsilon_0}\right)(1+\varepsilon_0)^{j-k(t,r)-1} 2^{k(t,r)+1} r 2^{-j} \\
&\leq \left(1+\frac{2\varepsilon_0}{1-\varepsilon_0}\right)\left(\frac{1+\varepsilon_0}{2}\right)^{j-k(t,r)-1} r \\
&\leq \left(1+\frac{2\varepsilon_0}{1-\varepsilon_0}\right)\left(\frac{1+\varepsilon_0}{2}\right) r
\end{aligned} \tag{5.41}$$

where we note that we must have $j \geq 2$, so $8\varepsilon_j \leq 2\varepsilon_0$ by our choice of subsequence previously. Now, our upper bound on ε_0 is a root of the polynomial x^2+4x-1. However, the bound

$$\left(1+\frac{2\varepsilon_0}{1-\varepsilon_0}\right)\left(\frac{1+\varepsilon_0}{2}\right) < 1 \tag{5.42}$$

holds if and only if

$$1+2\varepsilon_0+\varepsilon_0^2 < 2-2\varepsilon_0, \tag{5.43}$$

which is precisely true if and only if $-2-\sqrt{5} < \varepsilon_0 < -2+\sqrt{5}$. The estimate $|f(t)-f(s)| < r$ then follows, therefore finishing the proof of (5.32), and hence the proof of theorem 5.3. □

Notes and Remarks

The Preiss measure was constructed to be a measure on $\mathbb{R}$. Indeed, the application of the integral on $[0,t]$ prevents our proof from extending to $\mathbb{R}^d$ for general d. One might then wonder if there exists an analog of the Preiss measure on $\mathbb{R}^d$ for $d \geq 2$. In fact, this is an open problem:

Question 5.4. *Let $d \geq 2$ be a positive integer. Does there exist $\mu \in \mathscr{M}(\mathbb{R}^d)$ such that* $\operatorname{Tan}(\mu,p) = \left\{c\mathscr{L}^d : c > 0\right\}$ *and $\Theta^{d*}(\mu,p) = \infty$ for μ-a.e. $p \in \mathbb{R}^d$?*

For a real number $s \leq d$, a weaker problem is to find $\mu \in \mathscr{M}(\mathbb{R}^d)$, for μ-a.e. $p \in \mathbb{R}^d$, we have $\Theta^{s*}(\mu,p) = \infty$ and all tangent measures to μ at p are m-uniform, meaning

$$\tau(B(x,r)) = cr^s \tag{5.44}$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Of course, if $s = d$, this is just question 5.4. The most natural examples of m-uniform measures are the m-flat measures, defined in section 3.2. However, there are less obvious m-uniform measures, and their classification is of interest. For example, in [KP87], Kowalski and Preiss proved that the m-uniform measures in dimension $d = m+1$ are either m-flat, or equal to

$$\mathscr{H}^m \llcorner \left\{x \in \mathbb{R}^d : (x^4)^2 = (x_1)^2+(x_2)^2+(x_3)^2\right\}. \tag{5.45}$$

Furthermore, in [KP02], it was shown by Kirchheim and Preiss that the support of an m-uniform measure lies in an analytic variety in $\mathbb{R}^d$. See [Nim16] for more examples of m-uniform measures.

Another direction to go from theorem 5.3 is to worsen the type of singularity. Recall the type of singularity we have constructed is $\Theta^{1*}(\mu, p) = \infty$ for μ-a.e. $p \in \mathbb{R}$. In [OS11], Orponen and Sahlsten proved that there exists $\mu \in \mathscr{M}(\mathbb{R})$ such that for μ-a.e. $p \in \mathbb{R}$, every tangent measure of μ at p is equivalent to Lebesgue measure, but the following *doubling constant*

$$D(\mu, p) := \limsup_{r \downarrow 0} \frac{\mu(B(p, 2r))}{B(p, r)} \tag{5.46}$$

is infinite. The condition $D(\mu, p) = \infty$ is equivalent to

$$\sup_{\tau \in \operatorname{Tan}(\mu, p)} \frac{\tau(B(0, R))}{\tau(B(0, 1))} = \infty \quad \text{for all } R > 1, \tag{5.47}$$

so there are two conditions on elements of $\operatorname{Tan}(\mu, p)$ which are somehow "fighting" against one another. Orponen and Sahlsten again applied a dyadic construction (albeit in a slightly different way) to prove this result. In particular, the proof relies extensively on a function especially suited to $\mathbb{R}$ and constructing a highly oscillatory function from it, suggesting perhaps that to prove similar results for dimensions other than $d = 1$, we need to find an appropriate notion of oscillation which generalizes to $\mathbb{R}^d$.

A Appendix

Fix the dimension $d \in \mathbb{N}$. We denote by $\mathscr{M}(\mathbb{R}^d)$ the set of all locally finite Radon measures on $\mathbb{R}^d$. That is, if $\mu \in \mathscr{M}(\mathbb{R}^d)$, then μ is defined on Borel subets of $\mathbb{R}^d$, μ is finite on compact subsets of $\mathbb{R}^d$, and for all Borel $A \subseteq \mathbb{R}^d$, we have

$$\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ compact}\} = \inf\{\mu(U) : U \supseteq A \text{ open}\}. \tag{A.1}$$

Given $\mu \in \mathscr{M}(\mathbb{R}^d)$ and a Borel set $A \subseteq \mathbb{R}^d$, we write $\mu \llcorner A$ for the restriction of μ to A. That is, $(\mu \llcorner A)(B) := \mu(A \cap B)$ for any Borel $B \subseteq \mathbb{R}^d$.

Write $C_c(\mathbb{R}^d)$ for the space of all continuous functions with compact support on $\mathbb{R}^d$. The *Riesz representation theorem* says that given a positive linear functional f on $C_c(\mathbb{R}^d)$, there exists a unique $\mu \in \mathscr{M}(\mathbb{R}^d)$ such that

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu \tag{A.2}$$

for all $\varphi \in C_c(\mathbb{R}^d)$. As a corollary, we can define a measure simply by specifying its action on elements of $C_c(\mathbb{R}^d)$. A good reference for the Riesz representation theorem can be found in any decent textbook on measure theory. For example, [AFP00], [Coh13], or [Els11].

Identifying $\mathscr{M}(\mathbb{R}^d)$ with the dual of $C_c(\mathbb{R}^d)$, we see $\mathscr{M}(\mathbb{R}^d)$ inherits a natural notion of convergence. Namely, we say that a sequence $\mu_j \in \mathscr{M}(\mathbb{R}^d)$ *converges weakly** to μ in $\mathscr{M}(\mathbb{R}^d)$, and we write $\mu_j \overset{*}{\rightharpoonup} \mu$, if

$$\int_{\mathbb{R}^d} \varphi \ \mathrm{d}\mu_k \to \int_{\mathbb{R}^d} \varphi \ \mathrm{d}\mu \text{ for all } \varphi \in C_c(\mathbb{R}^d). \tag{A.3}$$

Example A.1. Here are two examples of weak* convergence in $\mathscr{M}(\mathbb{R}^d)$.

(1) Define $\mu_j := j^{-1}\mathscr{L}^d$ for $j \in \mathbb{N}$. Then, for $\varphi \in C_c(\mathbb{R}^d)$, we have

$$\left| \int_{\mathbb{R}^d} \varphi \ \mathrm{d}\mu_j \right| \leq \frac{1}{j} \int_{\mathbb{R}^d} |\varphi| \ \mathrm{d}\mathscr{L}^d = \frac{1}{j} \|\varphi\|_{L^1(\mathbb{R}^d)} \to 0. \tag{A.4}$$

Therefore $\mu_j \overset{*}{\rightharpoonup} 0$ in $\mathscr{M}(\mathbb{R}^d)$.

(2) Define the sequence $\mu_j := \sin(2\pi jx) \ \mathrm{d}\mathscr{L}^1(x)$ for $j \in \mathbb{N}$. Then, for $\varphi \in C_c(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}} \varphi \ \mathrm{d}\mu_j &= \int_{\mathbb{R}} \varphi(x) \sin(2\pi jx) \ \mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbb{R}} \varphi(x) \left[\sin(2\pi jx) - \sin\left(2\pi j \left(x - \frac{1}{2j}\right)\right)\right] \ \mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbb{R}} \left[\varphi(y) - \varphi\left(y + \frac{1}{2j}\right)\right] \sin(2\pi jy) \ \mathrm{d}x \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{A.5}$$

So $\mu_j \overset{*}{\rightharpoonup} 0$. This example is a little more abstract than the previous one, since there is no clear notion of "size" for this sequence of measures (whereas those of the previous example had a factor of j^{-1}). However, the sequence $\varphi(x)\sin(2\pi jx)$ can be thought of as versions of φ which oscillate faster and faster, so that as $j \to \infty$, the oscillations should cancel out upon integrating.

When estimating the measure of sets after applying a blowup procedure, the following lemma will be very useful:

Lemma A.2. *Let $\mu_j \in \mathscr{M}(\mathbb{R}^d)$ be a sequence of measures converging weakly* to $\mu \in \mathscr{M}(\mathbb{R}^d)$. Then for all open sets $U \subseteq \mathbb{R}^d$, we have*

$$\mu(U) \leq \liminf_{j\to\infty} \mu_j(U), \tag{A.6}$$

and for all compact sets $K \subseteq \mathbb{R}^d$, we have

$$\mu(K) \geq \limsup_{j\to\infty} \mu(K). \tag{A.7}$$

See proposition 1.62 from [AFP00] for a proof.

We will write $\mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$ for the set of all nonnegative Lipschitz functions φ on $\mathbb{R}^d$ with compact support and with Lipschitz constant at most 1, and $\mathrm{Lip}\, f$ for the Lipschitz constant of a Lipschitz continuous function. The following easy lemma gives an alternative characterization of weak* convergence:

Lemma A.3. *Let $(\mu_j)_{j\in\mathbb{N}}$ be a sequence in $\mathscr{M}(\mathbb{R}^d)$. Then μ_j converges weakly* to some $\mu \in \mathscr{M}(\mathbb{R}^d)$ if and only if $\int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu_j \to \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu$ for all $\varphi \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$.*

Proof. Temporarily write $\mu_j \rightsquigarrow \mu$ for the latter notion of convergence. Since $\mathrm{Lip}_{\leq 1}(\mathbb{R}^d) \subseteq C_c(\mathbb{R}^d)$, we easily see that $\mu_j \xrightarrow{*} \mu$ implies $\mu_j \rightsquigarrow \mu$. Conversely, suppose $\mu_j \rightsquigarrow \mu$, and fix a nonnegative $\varphi \in C_c(\mathbb{R}^d)$ with support in $K \subseteq \mathbb{R}^d$. The measures $\mu_j(K)$ can be bounded uniformly in j: indeed, choose $\psi \in \mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$ such that $\psi = 1$ on K. Then

$$\mu_j(K) \leq \int_{\mathbb{R}^d} \psi \, \mathrm{d}\mu_j \to \int_{\mathbb{R}^d} \psi \, \mathrm{d}\mu. \tag{A.8}$$

Thus we can find $C_K > 0$ with $\mu_j(K) \leq C$ for all $j \in \mathbb{N}$. Define $C > 0$ to be the maximum of C_K and $\mu(K)$ Let $\varepsilon > 0$. By the Stone-Weierstrass theorem, we can find a nonnegative Lipschitz continuous ψ with $\|\varphi - \psi\|_{L^\infty} < \frac{\varepsilon}{3C}$. Then $(\mathrm{Lip}\,\psi)^{-1}\psi$ is in $\mathrm{Lip}_{\leq 1}(\mathbb{R}^d)$. Choose $j_0 \in \mathbb{N}$ such that $j \geq j_0$ implies

$$\frac{1}{\mathrm{Lip}\,\psi}\left|\int_{\mathbb{R}^d} \frac{\psi}{\mathrm{Lip}\,\psi} \, \mathrm{d}\mu_j - \int_{\mathbb{R}^d} \frac{\psi}{\mathrm{Lip}\,\psi} \, \mathrm{d}\mu\right| < \frac{\varepsilon}{3} \tag{A.9}$$

For $j \geq j_0$, we therefore have the estimate

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu_j - \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu\right| &\leq \left|\int_{\mathbb{R}^d} \varphi - \psi \, \mathrm{d}\mu_j\right| \\
&\quad + \frac{1}{\mathrm{Lip}\,\psi}\left|\int_{\mathbb{R}^d} \frac{\psi}{\mathrm{Lip}\,\psi} \, \mathrm{d}\mu_j - \int_{\mathbb{R}^d} \frac{\psi}{\mathrm{Lip}\,\psi} \, \mathrm{d}\mu\right| \\
&\quad + \left|\int_{\mathbb{R}^d} \psi - \varphi \, \mathrm{d}\mu\right| \\
&< \mu_j(K)\,\|\varphi - \psi\|_{L^\infty} + \frac{\varepsilon}{3} + \mu(K)\,\|\varphi - \psi\|_{L^\infty} \\
&< C_K \frac{\varepsilon}{3C} + \frac{\varepsilon}{3} + \mu(K)\frac{\varepsilon}{3} \\
&\leq \varepsilon,
\end{aligned} \tag{A.10}$$

as required. ☐

Since $C_c(\mathbb{R}^d)$ is separable, the weak* topology on $\mathscr{M}(\mathbb{R}^d)$ gains a nice metric:

Lemma A.4. *There exists a separable and complete metric d on $\mathscr{M}(\mathbb{R}^d)$ such that given any sequence $(\mu_j)_{j\in\mathbb{N}}$ and measure μ in $\mathscr{M}(\mathbb{R}^d)$, we have $\mu_j \xrightarrow{*} \mu$ as $j \to \infty$ if and only if $(\mu_j)_{j\in\mathbb{N}}$ is uniformly locally bounded, and $d(\mu_j, \mu) \to 0$ as $j \to \infty$.*

Here, *uniformly locally bounded* means for all compact $K \subseteq \mathbb{R}^d$, there exists $C_K > 0$ such that $\mu_j(K) \leq C_K$ for all $j \in \mathbb{N}$. A reference for the proof of lemma A.4 is proposition 2.6 from [Lel08].

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