Convex Integration in Fluid Dynamics

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Introduction

An *Euler flow* is a pair of functions $v\colon \mathbb{T}^3 \to \mathbb{R}^3$, $p\colon \mathbb{T}^3 \to \mathbb{R}$ solving the following *incompressible Euler equations*.

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0, \\ \operatorname{div} v = 0. \end{cases} \tag{1}$$

(Here, a *solution* is meant in the sense of [1].) For sufficently regular Euler flows, the kinetic energy $\int_{\mathbb{T}^3} |v(x)|^2 \,\mathrm{d}x$ is conserved. A natural question to ask is whether this is true for flows of lower regularity. Lars Onsager conjectured in 1949 that α-Hölder continuous Euler flows conserved energy for $\alpha > \frac{1}{3}$, but that there exist α-Hölder continuous Euler flows which dissipate the kinetic energy for $\alpha < \frac{1}{3}$.

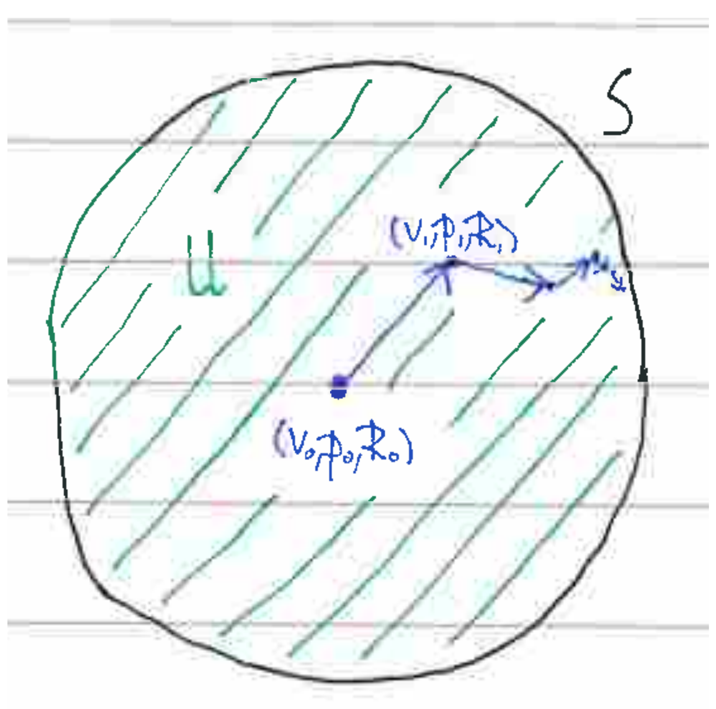
It was the work of Camillo De Lellis and László Székelyhidi in [2] to introduce a different approach to proving the Euler equations, mirroring the work of John Nash in differential geometry. This approach is called *convex integration*, and has proven to be a indispensible tool in proving Onsager's conjecture.

Current Research

This project is centered around the paper [1], whose existence comes from a recommendation by Peter Constantin to Camillo De Lellis and László Székelyhidi. In this paper, De Lellis and Székelyhidi (DLS) used convex integration to construct continuous solutions (v, p) of the Euler equations whose total kinetic energy (depending on time) could be any smooth function $e\colon [0, \infty) \to [0, \infty)$. In particular, they found a sequence of smooth solutions $(v_n, p_n, \mathring{R}_n)_{n\in\mathbb{N}}$ to a more general system, the *Euler-Reynolds system*:

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} \mathring{R}, \\ \operatorname{div} v = 0. \end{cases} \tag{2}$$

This sequence was obtained by adding highly oscillatory flows in a controlled fashion, in such a way that the total kinetic energy gets closer and closer to e, and then designing the matrix $\mathring{R}_n$ so that $(v_n, p_n, \mathring{R}_n)$ solves (2). This sequence of "subsolutions" converged in a strong manner to the desired continuous solution of the Euler system. Graphically, we can represent the set of solutions to (1) by a circle S, and the set of solutions to (2) as the interior of the circle U. Convex integration begins with an element of the interior, and coaxes it towards the boundary.



One limitation of the construction in DLS is that we would like to be able to control $|u|$ in more specific ways than just the total kinetic energy $\int_{\mathbb{T}^3} |u|^2 \,\mathrm{d}x$. As it turns out, we can partially control $|u(x, t)|$, but so far only for u bounded, and not necessarily continuous (cf. [2]).

Energy Distributions

In light of current research, we asked ourselves (now working in two dimensions) whether it was possible to use convex integration to find a continuous flow u for which the quantity

$$F(s) := |\{x \in \mathbb{T}^2 : |u|^2 \le s\}| \tag{3}$$

is equal to a prescribed function. We call F the *distribution* of u (or more precisely, of its kinetic energy). In order to do this, we must first find out how distributions change upon adding oscillations, especially high-frequency ones. So we start with a given divergence-free field v, choose a 'prototype oscillation' ϕ which is continuously differentiable and 2π-periodic (usually $\phi = \sin$ or $\cos$), choose a frequency $k \in \mathbb{Z}^2$, and consider the vector field $u(x) = v(x) + \frac{k^\perp}{|k|}\phi(k \cdot x)$. We write $e = |u|^2$ for the kinetic energy of u, and F the distribution of e as defined above. The goal was to investigate how F acts as we take the frequency k very large.

We made use of the *coarea formula*

$$F'(s) = \int_{\{e=s\}} |\nabla e|^{-1} \,\mathrm{d}\mathscr{H}^1. \tag{4}$$

Thus it was necessary to find conditions simplifying the level set $\{e = s\} = \{x \in \mathbb{T}^2 : e(x) = s\}$, and the gradient norm $|\nabla e| = ((\partial_1 e)^2 + (\partial_2 e)^2)^{1/2}$. We considered the very simple case $v(x_1, x_2) = (0, c)$ for some $c \in \mathbb{R}$. We also required $k = (\lambda, 0)$ for some scalar frequency $\lambda \in \mathbb{Z}$. This reduced the problem to essentially 1 dimension. If we define $A_{s,c} = \{\theta \in [0, 2\pi] : \phi(\theta) = \sqrt{s} - c\}$, we found

$$F'(s) = \sum_{\theta \in A_{s,c}} \frac{\pi}{|\phi'(\theta)|\sqrt{s}} \tag{5}$$

whenever $\phi'(\theta) \neq 0$ for all $\theta \in A_{s,c}$. Note the independence of F on λ.

Forthcoming Research

With this project, we have managed to find a simple expression for the distribution of a simple divergence-free vector field to which a single oscillation is added. There is clearly much more work to be done towards our goal of finding a bounded/continuous/integrable/etc Euler flow with prescribed distribution. Using a loose argument about dividing $\mathbb{T}^2$ into an $N \times N$ grid for N large, we hypothesized that if $v\colon \mathbb{T}^2 \to \mathbb{R}^2$ is a suitably integrable vector field, then

$$F(s) = \int_{\mathbb{T}^2} F_{u_x}(s) \,\mathrm{d}x, \tag{6}$$

where u_x is the vector field

$$u_x(y) = v(x) + \frac{k^\perp}{|k|}\phi(k \cdot y), \tag{7}$$

and F_{u_x} its distribution. A concrete proof of this result may be possible using Riemann sums.

References

[1] Camillo De Lellis and László Székelyhidi. "Dissipative continuous Euler flows". In: *Inventiones mathematicae* 193 (2 2013), pp. 377–407.

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[3] Philip Isett. "A Proof of Onsager's Conjecture". preprint available at https://arxiv.org/abs/1608.08301, to be published in Annals of Mathematics.

[4] Lars Onsager. "Statistical hydrodynamics". In: *Il Nuovo Cimento* 6 (Mar. 1949), pp. 279–287.