Convex Integration in Fluid Dynamics

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Abstract

1 Essence of Convex Integration

We consider the incompressible Euler equations in $n = 2, 3$ dimensions

$$\begin{cases} \partial_t v + \operatorname{div} v \otimes v + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases} \tag{1}$$

for some vector velocity field $v\colon \mathbb{T}^n \times [0,1] \to \mathbb{R}^n$, and some scalar pressure field $p\colon \mathbb{T}^n \times [0,1] \to \mathbb{R}$. Sufficiently smooth solutions of (1) conserve the total kinetic energy over time: taking the dot product of the first equation in (1) with v, we find the identity

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(v \left(\frac{|v|^2}{2} + p \right) \right) = 0. \tag{2}$$

Integrating over $x \in \mathbb{T}^n$, we see

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\mathbb{T}^n} |v(x,t)|^2 \,\mathrm{d}x = 0. \tag{3}$$

Such conservation of energy is not possible for solutions of lower regularity. In fact, Kolmogorov's physical theory of *turbulence* predicts anomalous dissipation of energy of solutions of the Navier-Stokes equations in the high Reynolds number regime. Lars Onsager proposed we investigate turbulence through the simpler Euler equations, and conjectured the following

Conjecture 1 (Onsager, 1949)**.**

1. Suppose $v \in L_t^\infty C_x^{0,\alpha}$ is a weak solution to (1) with $\alpha > \frac{1}{3}$. Then the total kinetic energy $\frac{1}{2}\int_{\mathbb{T}^n} |v(x,t)|^2 \,\mathrm{d}x$ is conserved over time.

2. There exist weak solutions $v \in L_t^\infty C_x^{0,\alpha}$ to (1) which dissipate energy whenever $\alpha < \frac{1}{3}$.

In this instance, $L^\infty_t C^{0,\alpha}_x$ denotes the set of vector fields v satisfying the Hölder condition

$$|v(x+h,t) - v(x,t)| \lesssim |h|^\alpha \text{ for all } x, h \in \mathbb{T}^n, t \in [0,1].$$

Furthermore, $(v,p)\colon \mathbb{T}^n \times [0,1] \to \mathbb{R}^n \times \mathbb{R}$ is a *weak solution* of (1) if

$$\int_{[0,1]} \int_{\mathbb{T}^n} v \cdot \partial_t \phi + \nabla \phi : v \otimes v + p \operatorname{div} v \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all test functions $\phi \in C^\infty_c(\mathbb{T}^n \times [0,1]; \mathbb{R}^n)$, and

$$\int_{[0,1]} \int_{\mathbb{T}^n} v \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all test functions $\psi \in C^\infty_c(\mathbb{T}^n \times [0,1]; \mathbb{R})$.

Part 1 of this conjecture was answered by [yo a cool dude], and 2 by Isett. The focus of this project will be on a 2013 paper by De Lellis and Székelyhidi, who proved the existence of merely continuous Euler flows which dissipate the total kinetic energy. This proved to be an important stepping stone for a number of results in this area, including Isett's proof.

2 Outline of DLS

Consider a (strictly) positive smooth function $e\colon [0,1] \to \mathbb{R}$. The aim is to find a *continuous* vector field $v\colon \mathbb{T}^d \times [0,1] \to \mathbb{R}^d$ ($d = 2,3$) solving the Euler equations, such that

$$\int_{\mathbb{T}^d} |v(x,t)|^2 \, \mathrm{d}x = e(t) \text{ for all } t \in [0,1]. \tag{4}$$

To prove this theorem, De Lellis and Székelyhidi considered smooth solutions of the perturbed system

$$\begin{cases} \partial_t v + \operatorname{div} v \otimes v + \nabla p = \operatorname{div} \mathring{R} \\ \operatorname{div} v = 0 \end{cases} \tag{5}$$

For $(v, p, \mathring{R})\colon \mathbb{T}^d \times [0,1] \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d\times d}_{\mathrm{sym},0}$, where $\mathbb{R}^{d\times d}_{\mathrm{sym},0}$ denotes the space of trace-free symmetric $d \times d$ matrices, denoted in DLS by $\mathscr{S}^{d\times d}_0$. System (5) is often called the *Euler-Reynolds system* and $\mathring{R}$ the *Reynolds stress tensor*.

The idea of the construction is that a continuous solution to the Euler system is a superposition of oscillations. Starting with the trivial solution $(0,0,0)$ to the Euler-Reynolds system, we reintroduce the oscillations in a controlled manner so that the total kinetic energy moves closer to $e(t)$, and we converge uniformly to a solution of the Euler system. The Reynolds stress $\mathring{R}$ measures, in some sense, how far we are from a solution of the Euler system.

The oscillations considered are so-called *Beltrami flows*, which are stationary solutions to the Euler equations having some important properties that allow for their use in this construction. Fix $\lambda_0 \geq 1$. In 2D, define

$$b_k(\xi) = i\frac{k^\perp}{|k|}e^{ik\cdot\xi}$$

for each $k \in \mathbb{Z}^2$, and choose $a_k \in \mathbb{C}$ such that $a_{-k} = \overline{a_k}$. The vector field

$$W(\xi) = \sum_{|k|=\lambda_0} a_k b_k(\xi) \tag{6}$$

is a Beltrami flow in 2D. In 3D, let $A_k \in \mathbb{R}^3$ be such that

$$A_k \cdot k = 0, |A_k| = \frac{1}{\sqrt{2}}, A_{-k} = A_k$$

for each $k \in \mathbb{Z}^3$ with $|k| = \lambda_0$. Set

$$B_k = A_k + i\frac{k}{|k|} \times A_k.$$

Let a_k be as above. Then

$$W(\xi) = \sum_{|k|=\lambda_0} a_k B_k e^{ik\cdot\xi} \tag{7}$$

is a Beltrami flow in 3D. DLS considers the B_k to be fixed throughout the paper. The possible choices of B_k give infinitely many continuous solutions to the Euler equations with the prescribed kinetic energy.

A key point of the construction is possible due to a so-called "geometric lemma", wherein for each $N \in \mathbb{N}$, we can find $r_0 > 0$, $\lambda_0 > 1$, pairwise disjoint and symmetric $\Lambda_j \subseteq \{k \in \mathbb{Z}^3 : |k| = \lambda_0\}$ $(j = 1, \ldots, N)$, and positive $\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{id}))$ $(j = 1, \ldots, N,\ k \in \Lambda_j)$, such that $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$, and for each $R \in B_{r_0}(\mathrm{id}) \subseteq \mathbb{R}^{d\times d}_{\mathrm{sym}}$, we have

$$R = \frac{1}{2}\sum_{k\in\Lambda_j}\left(\gamma_k^{(j)}\right)^2\left(\mathrm{id} - \frac{k}{|k|}\otimes\frac{k}{|k|}\right). \tag{8}$$

Though it is only stated for $d = 3$ in DLS, the proof can be translated *mutatis mutandis* into 2D.

Let $(v, p, \mathring{R})$ be a solution of the Euler-Reynolds system, and fix $\lambda_0 \geq 1$. With $B_k \in \mathbb{C}^3$ as above, write

$$W(y, s, \tau, \xi) = \sum_{|k|=\lambda_0} a_k(y, s, \tau)B_k e^{ik\cdot\xi} \tag{9}$$

for $y \in \mathbb{T}^d$, $s \in [0, 1]$, $\tau \in \mathbb{R}$, and $\xi \in \mathbb{T}^d$. We will choose coefficients $a_k(y, s, \tau)$ cleverly so that (modulo a certain correction term to ensure the resulting velocity field is divergence-free) $v(x,t) + W(x,t,\lambda t,\lambda x)$ is a new velocity field whose total kinetic energy is closer to $e(t)$, and solves the Euler-Reynolds system for some appropriate $(p_1, \mathring{R}_1)$, with $|\mathring{R}_1| \ll \operatorname{div}|\mathring{R}|$. This is achieved by defining an appropriate (time-dependent) partition of unity on the space of velocities, depending on a parameter μ. Our W involves the function

$$\frac{1}{d(2\pi)^2}\left(e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{T}^d} |v(x,t)|^2 \,\mathrm{d}x\right), \tag{10}$$

whose existence is to ensure our new velocity field has kinetic energy closer to $e(t)$. We also define $R(x,t) = \rho(t)\mathrm{id} - \mathring{R}(x,t)$. The tensor field R has trace ρd, and this is needed in the construction for estimating the energy. This is also why we choose to work with trace-free matrices in the Euler-Reynolds system. We use the geometric lemma and a property of Beltrami flows to find W such that

$$\frac{1}{4\pi^2}\int_{\mathbb{T}^d} (W\otimes W)(y,s,\tau,\xi)\,\mathrm{d}\xi = R.$$

Then we can write $W \otimes W$ as a Fourier series in ξ:

$$(W\otimes W)(y,s,\tau,\xi) = R(y,s) + \sum_{1\le |k|\le 2\lambda_0} U_k(y,s,\tau)e^{ik\cdot\xi}.$$

It seems as though we have some degrees of freedom to work with ρ and R. Can we do something with this?

We set $w_o(x,t) = W(x,t,\lambda t,\lambda x)$. To ensure our new vector field is divergence-free, we use the classical Leray projector $\mathbb{P}$ and set $v_1 = v + \mathbb{P}w_o$. The pressure p_1 is defined by subtracting $\frac{1}{2}|w_o|^2$, and the Reynolds stress tensor $\mathring{R}_1$ is effectively designed to automatically solve the Euler-Reynolds system, by introducing a right inverse (modulo some number) to the divergence operator.

The required estimates on $(v_1, p_1, \mathring{R}_1)$ come from our choice of w_o, and are derived using Schauder estimates for the Laplace operator, and the important Fourier estimate

$$\left|\int_{\mathbb{T}^d} a(x)e^{i\lambda k\cdot x}\,\mathrm{d}x\right| \le \frac{[a]_{C^m(\mathbb{T}^d)}}{\lambda^m} \tag{11}$$

for any $a \in C^\infty(\mathbb{T}^d)$ and $m \in \mathbb{N}$, where

$$[a]_{C^m(\mathbb{T}^d)} = \sup_{|\beta|=m} \|\partial^\beta a\|_{C(\mathbb{T}^d)}.$$

3 Questions Considered in this Project

Question 1. Let $F\colon [0,\infty) \to [0,1]$ be a distribution function (i.e. nondecreasing, right-continuous, $F(0) = 0$, and $\lim_{x\to\infty} F(x) = 1$). Can we use convex integration to construct a stationary weak solution (v,p) to the Euler equation, such that $\frac{1}{2}|v|^2$ has the probability distribution given by F?

This means if we begin with a solution to the Euler-Reynolds system with a simple distribution function for its kinetic energy, is it possible to continually add highly oscillatory flows to perturb the distribution function of the velocity fields towards F?

It may be helpful to send this problem to a more abstract plane. In particular, take a probability measure μ on $[0,\infty)$, corresponding to a distribution function F. Can we use convex integration to find a sequence $(v_n)_{n\in\mathbb{N}}$ of velocity fields converging in some sense to v, such that the distribution of $\frac{1}{2}|v|^2$ is μ? There are a couple of simplifying assumptions we can make. Firstly, we can assume μ has a density with respect to (normalized - i.e. $|\mathbb{T}^d| = 1$) Lebesgue measure $\mathrm{d}x$ on $\mathbb{T}^d$. That is, $\mathrm{d}\mu = f\,\mathrm{d}x$ for some measurable $f\colon \mathbb{T}^d \to \mathbb{R}$. The second is to assume that μ is the distribution of some measurable function $e\colon \mathbb{T}^d \to [0,\infty)$. That is, $\mathrm{d}\mu = e_*\,\mathrm{d}x$.

The second assumption gives us a lot of tools at our disposal (in fact, I have this suspicion that every probability measure μ on $[0,\infty)$ is the distribution of some measurable function e on $\mathbb{T}^d$). For if we can show that $\frac{1}{2}|v_n|^2$ converges almost everywhere, in L^p $(1 \le p \le \infty)$, in probability, or merely in distribution to e, then the distribution functions F_n of $\frac{1}{2}|v_n|^2$ will converge to the distribution function F of μ at every point of continuity of F. Given the structure of proposition 2.2 in DLS, it seems that aiming for L^1 convergence may be our best shot. Indeed, proposition 2.2 says that given a velocity field v satisfying

$$\frac{3\delta}{4}e(t) \le e(t) - \frac{1}{2}\int_{\mathbb{T}^n} |v(x,t)|^2\,\mathrm{d}x \le \frac{5\delta}{4}e(t),$$

among other conditions, then we can find another velocity field v_1 with the same bounds divided by 2. Perhaps if we now require

$$\frac{3\delta}{4}K \le \int_{\mathbb{T}^3} \left|\frac{1}{2}|v(x)|^2 - e(x)\right|\,\mathrm{d}x \le \frac{5\delta}{4}K$$

for some K, then we can alter the construction find v_1 satisfying the same bounds divided by 2. This would imply L^1 convergence of $\frac{1}{2}|v_n|^2$ to e. Perhaps a possible route is to also ask that $\frac{1}{2}|v|^2 \le e$, and then alter the construction somewhat to find v_1 with $\frac{1}{2}|v|^2 \le \frac{1}{2}|v_1|^2 \le e$, and

$$\frac{3\delta}{8}\hat{e} \le \hat{e} - \frac{1}{2}\int_{\mathbb{T}^d} |v_1|^2\,\mathrm{d}x \le \frac{5\delta}{8}\hat{e},$$

where $\hat{e} = \int_{\mathbb{T}^d} e\,\mathrm{d}x$. This is much, much closer to the original formulation in DLS. Enforcing the requirement that $\frac{1}{2}|v_n|^2$ is nondecreasing in n and less than e ensures convergence in L^1. However, this may be challenging to accomplish - there hasn't yet been a construction so that $\frac{1}{2}|v_n|^2$ converges pointwise to $\hat{e}$, so asking for

Question 2. In which parts of De Lellis and Székelyhidi's construction is it impossible to prescribe $\frac{1}{2}|v(x,t)|^2$ for all x and t? (Note that the construction prescribes $\frac{1}{2}\int_{\mathbb{T}^n} |v(x,t)|^2\,\mathrm{d}x$ for all t.)

This question means that if, for every instance in which they appear, we naïvely replace $\frac{1}{2}\int_{\mathbb{T}^n}|v(x,t)|^2\,\mathrm{d}x$ by $\frac{1}{2}|v|^2$, and the goal function $e(t)$ by some $e(x,t)$ now depending on space and time, where in the construction do we find we can't proceed?

From what I can gather, the construction seems relatively stable up until lemma 6.3. In particular, we can take

$$\rho(x,t) = \frac{1}{3(2\pi)^2}\left(e(x,t)\left(1-\frac{\delta}{2}\right) - \frac{1}{2}|v(x,t)|^2\right).$$

The construction of the constants η and M still makes sense, since $\mathbb{T}^3 \times [0,1]$ is compact and we require e to be positive. The proof of lemma 6.3 relies crucially on the key proposition 5.2, whose statement is the integral estimate

$$\left|\int_{\mathbb{T}^3} a(x)e^{i\lambda k\cdot x}\,\mathrm{d}x\right| \leq \frac{[a]_{C^m(\mathbb{T}^3)}}{\lambda^m}$$

for any $k \in \mathbb{Z}^3 \setminus \{0\}, \lambda \geq 1, m \in \mathbb{N}$, and $a \in C^\infty(\mathbb{T}^3)$. (In fact, this result is fundamentally why highly oscillatory flows are chosen to be the key tools in convex integration.) We obtain the desired estimate for $\frac{1}{2}\int_{\mathbb{T}^n}|v(x,t)|^2\,\mathrm{d}x$ by using proposition 5.2 to estimate the integrals

$$\int_{\mathbb{T}^3}|w_o|^2 - \operatorname{tr} R\,\mathrm{d}x \text{ and } \int_{\mathbb{T}^3} v\cdot w_o\,\mathrm{d}x,$$

and then recalling that $\operatorname{tr} R = 3\rho$.

The second fault comes from lemma 7.2. In fact, it is explicitly stated in the proof - ρ depends only on t, so we can put it inside the divergence operator defining the oscillation part. If we add additional x dependence, this property goes away. Can we find a way to control $\operatorname{div}\rho(x,t)$? As a matter of interest, note that the proof of lemma 7.2 also relies on a corollary of proposition 5.2.

The first fault seems much more difficult to manage, since it relies fundamentally on the property of integrals of oscillatory functions.

Question 3. The existence of energy dissipative continuous Euler flows has been shown in $\mathbb{T}^n$ for $n = 2,3$. The case for $n = 1$ is probably easy (but maybe still worth considering). What about for $n > 3$? What about for $\mathbb{R}^n$?

For $n = 1$, it quickly becomes evident that the construction in De Lellis-Székelyhidi becomes trivial. Suppose $(v_n)_{n\in\mathbb{N}}$ is a sequence of smooth 1-dimensional solution to the Euler-Reynolds system converging to a weak solution v of the Euler system, constructed as in the paper. The only trace-free 1×1 matrix is the zero matrix, and the divergence operator becomes differentiation in the only space variable. As such, the right hand side of the Euler-Reynolds system disappears, and by the divergence-free condition on the velocity field, so too does the nonlinear term disappear. We are left with the system

$$\begin{cases}\partial_t v_n + \partial_x p_n = 0\\ \partial_x v_n = 0\end{cases}$$

The second condition forces v_n to be constant in space for any $n \in \mathbb{N}$, and so the weak solution v of the Euler system must also be constant, and therefore a classical solution of the Euler system. It follows that any 1-dimensional weak solution constructed according to De Lellis-Székelyhidi is conserved in one dimension. Actually, if a function has weak derivative zero, is it true that the function is constant (almost everywhere)? If so, this would mean any 1-dimensional weak solution (not just those constructed according to DLS) would conserve energy.

Question 4. On $\mathbb{T}^2$, Let v be the constant vector field $(1,0)$. Let $k = (1,0)$, and define the oscillatory flow $w_k(x) = \frac{k^\perp}{|k|}\sin(k \cdot x)$. Calculate the distribution functions

$$F_v(s) = |\{x \in \mathbb{T}^2 \colon |v(x)|^2 \leq s\}| \text{ and } F_{v+w_k}(s) = |\{x \in \mathbb{T}^2 \colon |(v+w_k)(x)|^2 \leq s\}|$$

for $s \in \mathbb{R}$.

This question is intended as a manual labour question to investigate the effects of oscillations on the distribution of the kinetic energy $\frac{1}{2}|v|^2$, as a companion to question 1. For the context of this question, we will identify $\mathbb{T}^1$ with $[-\pi, \pi]$.

For v, we know easily that $|v|^2 = 1$, so

$$F_v(s) = \begin{cases} 0 & s < 1; \\ 2\pi & s \geq 1. \end{cases}$$

The vector field $v + w_k$ is more involved. We readily calculate

$$|(v+w_k)(x)|^2 = |(1, \sin x^1)|^2 = 1 + (\sin x^1)^2.$$

So all of the mass of $|v + b_k|^2$ is contained in the interval $[1,2]$. Fix $s \in [1,2]$. Then $|v + b_k|^2 \leq s$ if and only if $\sin x^1 \in [-\sqrt{s-1}, \sqrt{s-1}]$, which holds for $x^1 \in \mathbb{T}^1$ if and only if

$$\begin{aligned} x^1 \in & [-\pi, -\pi + \arcsin\sqrt{s-1}] \\ & \cup [-\arcsin\sqrt{s-1}, \arcsin\sqrt{s-1}] \\ & \cup [\pi - \arcsin\sqrt{s-1}, \pi]. \end{aligned}$$

Noting that $\{x \in \mathbb{T}^2 : |(v+w_k)(x)|^2 \leq s\} = \{x^1 \in \mathbb{T}^1 : 1 + (\sin x^1)^2 \leq s\} \times \mathbb{T}^1$, we find, for $s \in [1,2]$,

$$F_{v+w_k}(s) = |\mathbb{T}^1||\{x^1 \in \mathbb{T}^1 : 1 + (\sin x^1)^2 \leq s\}| = (2\pi)4\arcsin\sqrt{s-1}.$$

In conclusion,

$$F_{v+w_k}(s) = \begin{cases} 0 & s < 1, \\ 8\pi \arcsin\sqrt{s-1} & 1 \leq s < 2, \\ 4\pi^2 & s \geq 2. \end{cases}$$

We can also work with higher frequency oscillations which are still perpendicular to v - if $k = (k^1, 0)$ for some $k^1 \in \mathbb{Z}$, then one can show that the distribution function of $|v + w_k|^2$ is the same as the one calculated above.

More generally, let's consider $v(x) + w_{(1,0)}(\lambda x)$ for some $\lambda \geq 1$. Write the corresponding distribution function as $F_{(1,0),\lambda}$. We have that

$$\left|v(x) + w_{(1,0)}(\lambda x)\right|^2 = 1 + (\sin \lambda x^1)^2 = 1 + \frac{1}{2}(1 - \cos 2\lambda x^1).$$

So $\left|v(x) + w_{(1,0)}(\lambda x)\right|^2 \leq s$ if and only if $\cos 2\lambda x^1 \geq 3-2s$. For $s \geq 2$, this is true for all $x \in \mathbb{T}^2$. For $s < 1$, this is not true for any $x \in \mathbb{T}^2$. For $1 \geq s < 2$, we have that for $x^1 \in \mathbb{T}^1 = [0, 2\pi)$, $\cos 2\lambda x^1 \geq 3 - 2s$ if and only if $2\lambda x^1 \in [2\pi m, 2\pi m + \arccos(3-2s)] \cup [2\pi m - \arccos(3-2s), 2\pi m]$ for some $m \in \mathbb{N} \cup \{0\}$. For all $m \leq 2\lambda$, this set is entirely contained in $2\mathbb{T}^1$. So $\left|\{x^1 \in \mathbb{T}^1 : \cos 2\lambda x^1 \geq 3 - 2s\}\right| = \frac{\lfloor 2\lambda \rfloor}{\lambda} \arccos(3 - 2s) + \epsilon_\lambda$ for some ε_λ which depends only on the fractional part of λ, is zero if $\lambda \in \mathbb{N}$, and becomes small for $\lambda \gg 1$.

Consider also $v + \varepsilon w_k$ for some $\varepsilon > 0$. Following the derivation above carefully, we find that

$$F_{v+\varepsilon w_k}(s) = \begin{cases} 0 & s < 1, \\ 2\pi \arccos(1 - \frac{s-1}{\varepsilon/2}) & 1 \leq s < 1 + \varepsilon, \\ (2\pi)^2 & s \geq 1 + \varepsilon. \end{cases}$$

Let's consider k now perpendicular to v. In particular, set $k = (0, 1)$. Then $(v + b_k)(x) = (1 - \sin x^2, 0)$, so $|(v + b_k)(x)|^2 = (1 - \sin x^2)^2$. For $s \geq 0$, we see that $(1 - \sin x^2)^2 \leq s$ if and only if $1 - \sqrt{s} \leq \sin x^2 \leq \sqrt{s} + 1$. The upper inequality is true for all $s \geq 0$. For the lower inequality, consider first $s \in [0, 1]$, corresponding to $1 - \sqrt{s} \in [0, 1]$. Then $\sin x^2 \geq 1 - \sqrt{s}$ if and only if $x^2 \in [\arcsin(1-\sqrt{s}), \pi - \arcsin(1-\sqrt{s})]$, and this set has measure $\pi + 2\arcsin(\sqrt{s}-1)$. Consider next $s \in [1, 4]$, corresponding to $1-\sqrt{s} \in [-1, 0]$. Then $\sin x^2 \geq 1-\sqrt{s}$ if and only if $x^2 \in [-\pi, -\pi + \arcsin(\sqrt{s}-1)] \cup [-\arcsin(\sqrt{s}-1), \pi]$, and this set also has measure $\pi + 2\arcsin(\sqrt{s} - 1)$. We conclude that

$$\begin{aligned} F_{v+b_k}(s) &= |\mathbb{T}^1||\{x^1 \in \mathbb{T}^1 : (1 - \sin x^2)^2 \leq s\}| \\ &= \begin{cases} 0 & s < 0, \\ 2\pi^2 + 4\pi \arcsin(\sqrt{s} - 1) & 0 \leq s < 4, \\ 4\pi^2 & s \geq 4. \end{cases} \end{aligned}$$

As for the case of $k = (k^1, 0)$, if $k = (0, k^2)$ for some $k^2 \in \mathbb{Z}$, then the distribution function of $|v + b_k|^2$ is the same as that of $|v + b_{(0,1)}|^2$.

Consider a general $k = (k^1, k^2) \in \mathbb{Z}^2$. Then

$$w_k(x) = \frac{(-k^2, k^1)}{|k|} \sin k \cdot x.$$

So

$$
\begin{aligned}
|(v+w_k)(x)|^2 &= \left(1-\frac{k^2}{|k|}\sin k\cdot x\right)^2+\left(\frac{k^1}{|k|}\sin k\cdot x\right)^2 \\
&= 1-\frac{2k^2}{|k|}\sin k\cdot x+(\sin k\cdot x)^2 .
\end{aligned}
$$

Thus $|(v+w_k)(x)|\leq s$ if and only if

$$
(\sin k\cdot x)^2-\frac{2k^2}{|k|}\sin k\cdot x+(1-s)=\left(\sin k\cdot x-\frac{k^2}{|k|}\right)^2+\left(1-s-\frac{(k^2)^2}{|k|^2}\right)\leq 0.
$$

For $s<1-\frac{(k^2)^2}{|k|}$, this does not hold for any $x\in\mathbb{T}^2$. Otherwise, we need

$$
-\sqrt{\frac{(k^2)^2}{|k|^2}+s-1}+\frac{k^2}{|k|}\leq \sin k\cdot x\leq \sqrt{\frac{(k^2)^2}{|k|^2}+s-1}+\frac{k^2}{|k|}.
$$

Thus $k\cdot x\in[A_m,B_m]\cup[A'_m,B'_m]$ for some $m\in\mathbb{Z}$, where

$$
\begin{aligned}
A_m &= 2\pi m+\arcsin\left(-\sqrt{\frac{(k^2)^2}{|k|^2}+s-1}+\frac{k^2}{|k|}\right), \\
B_m &= 2\pi m+\arcsin\left(\sqrt{\frac{(k^2)^2}{|k|^2}+s-1}+\frac{k^2}{|k|}\right), \\
A'_m &= (2m+1)\pi-\arcsin\left(-\sqrt{\frac{(k^2)^2}{|k|^2}+s-1}+\frac{k^2}{|k|}\right), \\
B'_m &= (2m+1)\pi+\arcsin\left(\sqrt{\frac{(k^2)^2}{|k|^2}+s-1}+\frac{k^2}{|k|}\right),
\end{aligned}
$$

whenever this makes sense.

Finding the $x\in\mathbb{T}^2=[0,2\pi)^2$ for which this is true is the next challenge.

We introduce the following notation. For $v\in\mathbb{R}^2$ and $k\in\mathbb{Z}^2\setminus\{0\}$, we write v^k and $v^{k^\perp}$ for the unique real numbers satisfying

$$
v=v^k\frac{k}{|k|}+v^{k^\perp}\frac{k^\perp}{|k|}.
$$

Given $v\in\mathbb{R}^2$, $A>0$, $k\in\mathbb{Z}^2\setminus\{0\}$, and $\lambda\geq 1$, we write $F^\lambda_{v,A,k}$ for the distribution function

$$
F^\lambda_{v,A,k}(s):=\left|\{x\in\mathbb{T}^2:|v+Aw_k(\lambda(x))|^2\leq s\}\right|.
$$

We also write $F_{v,A,k}$ for the pointwise limit $\lim_{\lambda\to\infty}F^\lambda_{v,A,k}$.

Lemma 1. Let $v \in \mathbb{R}^2$, $A > 0$, $k \in \mathbb{Z}^2 \setminus \{0\}$, and $\lambda \in \mathbb{N}$. Assume $\left|v^{k^\perp}\right| > A$. If $v^{k^\perp} > A$, then

$$F_{v,A,k}(s) = \begin{cases} 0 & s < s_{-A} \\ 2\pi(\pi + 2\arcsin(A^{-1}(\sqrt{s-(v^k)^2} - v^{k^\perp}))) & s_{-A} \leq s < s_A \\ 4\pi^2 & s \geq s_2, \end{cases} \tag{12}$$

where $s_{-A} := (v^{k^\perp} - A)^2 + (v^k)^2$, and $s_A := (v^{k^\perp} + A)^2 + (v^k)^2$. Otherwise, if $-v^{k^\perp} > A$, then

$$F_{v,A,k}(s) = \begin{cases} 0 & s < s_A \\ 2\pi(\pi + 2\arcsin(A^{-1}(\sqrt{s-(v^k)^2} + v^{k^\perp}))) & s_A \leq s < s_{-A} \\ 4\pi^2 & s \geq s_{-A}, \end{cases} \tag{13}$$

Proof. For $y \in \mathbb{T}^2$, we have

$$v + Aw_k(y) = v^k \frac{k}{|k|} + \left(v^{k^\perp} + A\sin(k \cdot y)\right) \frac{k^\perp}{|k|}.$$

So

$$|v + Aw_k(y)|^2 = (v^k)^2 + (v^{k^\perp} + A\sin(k \cdot y))^2.$$

For $s \geq (v^k)^2$, $|v + Aw_k(y)|^2 \leq s$ if and only if

$$-\sqrt{s-(v^k)^2} - v^{k^\perp} \leq A\sin(k \cdot y) \leq \sqrt{s-(v^k)^2} - v^{k^\perp}. \tag{14}$$

In a sense, $\left|v^k\right|^2$ represents the absolute minimum that the kinetic energy of $v + Aw_k$ could possibly be, since w_k is oscillating in the $k^\perp$ direction. If s is lower than this value, then $F_{v,A,k}(s) = 0$. From now on, we assume $s \geq (v^k)^2$. Here, we make the assumption that $A < \left|v^{k^\perp}\right|$. Then either $A < v^{k^\perp}$, or $A < -v^{k^\perp}$. For now, suppose the former. Then $-v^{k^\perp} < -A$, so in (14), the left-hand bound (which is decreasing) remains less than $-A$ for all $s \geq (v^k)^2$. So we just need to find all $y \in \mathbb{T}^2$ such that

$$A\sin(k \cdot y) \leq \sqrt{s-(v^k)^2} - v^{k^\perp}. \tag{15}$$

Write $r(s)$ for the right hand side. We see that $r(s) = -A$ if and only if $s = s_{-A} := (v^{k^\perp} - A)^2 + (v^k)^2$, and $r(s) = A$ if and only if $s = s_A := (v^{k^\perp} + A)^2 + (v^k)^2$. For $s < s_{-A}$, there is no $y \in \mathbb{R}^2$ satisfying (15), so $F_{v,A,k}(s) = 0$. For $s \geq s_A$, every $y \in \mathbb{R}^2$ satisfies (15), so $F_{v,A,k}(s) = \left|\mathbb{T}^2\right| = 4\pi^2$. From now until we say otherwise, assume $s_1 \leq s < s_2$.

Dividing by A, we see that $y \in \mathbb{R}^2$ satisfies (15) if and only if

$$k \cdot y \in A_m := \left[(2m-1)\pi - \arcsin\left(A^{-1}r(s)\right), 2m\pi + \arcsin\left(A^{-1}r(s)\right)\right]$$

for some $m \in \mathbb{Z}$. Define

$$E = \left\{ x^k \frac{k}{|k|} + x^{k^\perp} \frac{k^\perp}{|k|} : 0 \leq x^k, x^{k^\perp} \leq 2\pi \right\}.$$

There is a natural isometry $E \to \mathbb{T}^2$. We will calculate the measure of the set $\{y \in E : |v + Aw_k(y)|^2 \leq s\}$. Writing $y = y^k \frac{k}{|k|} + y^{k^\perp} \frac{k^\perp}{|k|}$ as usual, we see that $k \cdot y = y^k |k|$. So

$$\begin{aligned} \left| \left\{ y \in E : |v + Aw_k(y)|^2 \leq s \right\} \right| &= 2\pi \left| \left\{ y^k \in [0, 2\pi] : y^k |k| \in \bigcup_{m \in \mathbb{Z}} A_m \right\} \right| \\ &= 2\pi \left| [0, 2\pi] \cap \bigcup_{m \in \mathbb{Z}} \frac{1}{|k|} A_m \right| \end{aligned}$$

where, as usual, $\frac{1}{|k|} A_m = \left\{ \frac{x}{|k|} : x \in A_m \right\}$. The sets $\frac{1}{|k|} A_m$ are disjoint, and $\frac{1}{|k|} A_m \subseteq [0, 2\pi]$ if and only if

$$\begin{cases} (2m-1)\pi - \arcsin(A^{-1} r(s)) \geq 0, \\ 2m\pi + \arcsin(A^{-1} r(s)) \leq 2\pi |k|. \end{cases}$$

The former condition holds if and only if $m \geq 1$ (no matter whether $\arcsin(A^{-1} r(s))$ is greater than or at most 0). The latter condition is slightly more complicated. For $\arcsin(A^{-1} r(s)) \leq 0$, it holds if and only if $m \leq |k|$, whereas for $\arcsin(A^{-1} r(s)) > 0$, it holds if and only if $m \leq |k| - 1$. So for $\arcsin(A^{-1} r(s)) \leq 0$, this means

$$\left| [0, 2\pi] \cap \bigcup_{m \in \mathbb{Z}} \frac{1}{|k|} A_m \right| = \frac{1}{|k|} (|k| + 1) |A_m| + \left| [0, 2\pi] \cap \frac{1}{|k|} A_0 \right| + \left| [0, 2\pi] \cap \frac{1}{|k|} A_{|k|} \right|.$$

The two terms involving A_0 and $A_{|k|}$ correspond to the borderline cases which intersect $[0, 2\pi]$ but are not entirely contained in it. Of course, as $|k| \to \infty$, these terms vanish. We then have that

$$\begin{aligned} F^\lambda_{v,A,k}(s) &= \left| \left\{ x \in E : |v + Aw_k(\lambda x)|^2 \leq s \right\} \right| \\ &= \left| \left\{ y \in E : |v + Aw_{\lambda k}(y)|^2 \leq s \right\} \right| \\ &\to 2\pi \left(\pi + 2 \arcsin \left(A^{-1} \sqrt{s - (v^k)^2} - v^{k^\perp} \right) \right), \end{aligned}$$

as $\lambda \to \infty$, as required. For $\arcsin(A^{-1} r(s)) > 0$, we have

$$\left| [0, 2\pi] \cap \bigcup_{m \in \mathbb{Z}} \frac{1}{|k|} A_m \right| = \frac{1}{|k|} |k| |A_m| + \left| [0, 2\pi] \cap \frac{1}{|k|} A_0 \right| + \left| [0, 2\pi] \cap \frac{1}{|k|} A_{|k|} \right|,$$

so we obtain the same result.

Having proved the result for $A < v^{k^\perp}$, we now assume $A < -v^{k^\perp}$. In this case, the right-hand bound in (14) remains greater than A for all $s \geq (v^k)^2$, so we just need to find all $y \in \mathbb{T}^2$ such that

$$-\sqrt{s-(v^k)^2} - v^{k^\perp} \leq A\sin(k \cdot y). \tag{16}$$

Write $\ell(s)$ for the left hand side. As above, $\ell(s) = \pm A$ if and only if $s = s_{\pm A}$, although in this case, we have $s_A < s_{-A}$. We now assume that $s_A \leq s < s_{-A}$. Practically the same calculation as above can be performed to find

$$\begin{aligned} F_{v,A,k}(s) &= 2\pi(\pi - 2\arcsin(A^{-1}\ell(s))) \\ &= 2\pi\left(\pi + 2\arcsin\left(A^{-1}\left(\sqrt{s-(v^k)^2} + v^{k^\perp}\right)\right)\right). \end{aligned}$$

□

We can proceed to working with non-constant v and A. For measurable $v\colon \mathbb{T}^2 \to \mathbb{R}^2$, $A\colon \mathbb{T}^2 \to \mathbb{R}$, we write

$$F^\lambda_{v,A,k}(s;x) := F^\lambda_{v(x),A(x),k}(s) \text{ and } F_{v,A,k}(s;x) := F_{v(x),A(x),k}(s) = \lim_{\lambda\to\infty} F^\lambda_{v,A,k}(s;x).$$

We also define the distribution function

$$F^\lambda_{v,A,k}(s) := \left|\left\{x \in \mathbb{T}^2 : |v(x) + A(x)w_k(x)|^2 \leq s\right\}\right|$$

and its pointwise limit

$$F_{v,A,k}(s) := \lim_{\lambda\to\infty} F^\lambda_{v,A,k}(s).$$

As above, $F_{v,A,k}$ will be our main focus for now.

Lemma 2. Suppose there exists a partition $\{C_{ij}\}_{i,j=1}^N$ of $[0,2\pi)^2$ into $N \times N$ squares of side length $h = 2\pi/N$, such that v and A are constant on each C_{ij} (upon identifying $\mathbb{T}^2$ with $[0,2\pi)^2$ in the natural way), with values v_{ij} and A_{ij} respectively. Then

$$F_{v,A,k}(s) = \frac{1}{4\pi^2}\sum_{i,j=1}^N h^2 F_{v_{ij},A_{ij},k}(s)$$

Proof. We have that

$$\begin{aligned} F_{v,A,k}(s) &= \lim_{\lambda\to\infty} F^\lambda_{v,A,k}(s) \\ &= \lim_{\lambda\to\infty}\sum_{i,j=1}^N \left|\left\{x \in C_{ij} : |v_{ij} + A_{ij}w_k(\lambda x)|^2 \leq s\right\}\right|. \end{aligned}$$

So it suffices to show that

$$\lim_{\lambda\to\infty}\left|\left\{x\in C_{ij}:|v_{ij}+A_{ij}w_k(\lambda x)|^2\le s\right\}\right|=\frac{h^2}{4\pi^2}F_{v_{ij},A_{ij},k}(s).$$

$\square$

Theorem 1. Let $v\in C^1(\mathbb{T}^2;\mathbb{R}^2)$, $A\in C^1(\mathbb{T}^2)$, and $k\in\mathbb{Z}^2\setminus\{0\}$. Then

$$F_{v,A,k}(s)=\frac{1}{4\pi^2}\int_{\mathbb{T}^2}F_{v,A,k}(s;x)\,\mathrm{d}x.$$

Proof. Partition $\mathbb{T}^2$ into $N\times N$ squares C_{ij} of side length $h=2\pi/N$ (as above), with N to be determined later. Then for $x,y\in C_{ij}$, $\lambda\ge 1$, we want that for all $\varepsilon>0$, there exists $N_0\in\mathbb{N}$ such that $N\ge N_0$ implies

$$\left|F_{v,A,k}(s)-\sum_{i,j=1}^{N}\frac{h^2}{4\pi^2}F_{v,A,k}(s;x_{ij})\right|<\varepsilon.$$

We may then take $N\to\infty$ to conclude

$$\left|F_{v,A,k}(s)-\int_{\mathbb{T}^2}F_{v,A,k}(s;x)\,\mathrm{d}x\right|\le\varepsilon.$$

Now, as proved above,

$$\begin{aligned}
&F_{v,A,k}(s)-\sum_{i,j=1}^{N}\frac{h^2}{4\pi^2}F_{v,A,k}(s;x_{ij})\\
&=\lim_{\lambda\to\infty}\left|F^{\lambda}_{v,A,k}(s)-\sum_{i,j=1}^{N}\left|\left\{x\in C_{ij}:|v(x_{ij})+A(x_{ij})w_k(\lambda x)|^2\le s\right\}\right|\right|\\
&\le\lim_{\lambda\to\infty}\sum_{i,j=1}^{N}\left|\left|\left\{x\in C_{ij}:|v(x)+A(x)w_k(\lambda x)|^2\le s\right\}\right|-\left|\left\{x\in C_{ij}:|v(x_{ij})+A(x_{ij})w_k(\lambda x)|^2\le s\right\}\right|\right|\\
&\le 2N^2h^2=8\pi^2
\end{aligned}$$

(we want each term in the sum to be $o(h^2)$ to ensure convergence, right now it's $O(h^2)$) Each term in the sum is less than $\||v+Aw_k(\lambda\cdot)|^2-|v(x_{ij})+A(x_{ij})w_k(\lambda\cdot)|^2\|_{L^1(C_{ij})}$. $\square$

We obviously