Tugas Analisis Vektor

Pembahasan Soal Bu Nur

Anggota Kelompok 10

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Semester Genap 2023/2024

SOAL & PEMBAHASAN

1. **[SOAL]**
   Diketahui fungsi vektor $\vec{F}(x,y) = 2xy^3\hat{i} + (1+3xy^2)\hat{j}$

   (a) Buktikan bahwa $\vec{F}(x,y)$ adalah medan vektor konservatif pada bidang-xy. $\vec{F}$ dikatakan medan vektor konservatif saat $\nabla \times \vec{F} = 0$.

   (b) Jika gaya $\vec{F}(x,y)$ adalah konservatif maka $\vec{F} = \nabla\phi$ dan $\phi(x,y)$ disebut potensial $\vec{F} = \nabla\phi$.

   [PEMBAHASAN]

   (a)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3 & 1+3xy^2 & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(6xy^2 - 6xy^2) = 0$$

   (b)

$$2xy^3\hat{i} + (1+3x^2y^2)\hat{j} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

   - $$\frac{\partial\phi}{\partial x} = 2xy^3$$
     $$\int \partial\phi = \int 2xy^3\,dx$$
     $$\phi = x^2y^3 + f(y)$$

   - $$\frac{\partial\phi}{\partial y} = 3x^2y^2 + f'(y)$$
     $$1 + 3x^2y^2 = 3x^2y^2 + f'(y)$$
     $$f'(y) = 1$$
     $$f(y) = y$$

   maka : $$\boxed{\phi = x^2y^3 + y}$$

2. **[SOAL]**

Diberikan $I = \oint_C (x-y)\,dx + (x+y)\,dy$. Jika C kurva tertutup yang membatasi daerah $y = x^3$ dan $x = y^2$

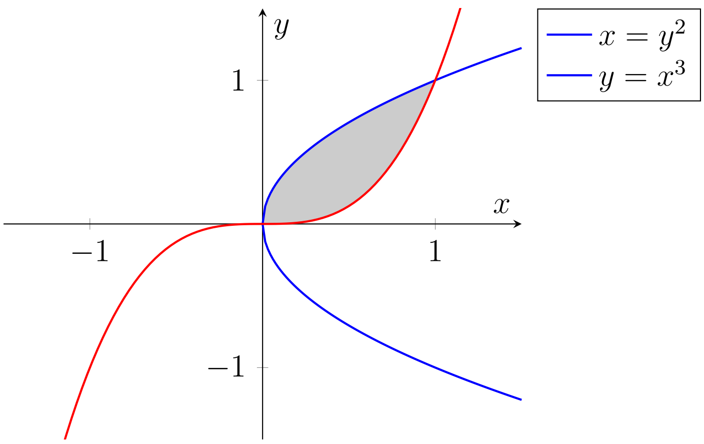
(a) Hitung integral tersebut tanpa menggunakan teorema Green.

(b) Hitung integral tersebut dengan menggunakan teorema Green.

[PEMBAHASAN]

- $I = \oint_C (x-y)\,dx + (x+y)\,dy$
- C kurva tertutup yang membatasi daerah $y = x^3$ dan $x = y^2$

(a) Hitung integral tersebut tanpa menggunakan teorema green



- $\mathbf{y = x^3}$
  $x = t,\ dx = dt$
  $y = t^3,\ dy = 3t^2$

$$\begin{aligned}
\int_0^1 (t - t^3)\,dt + (t + t^3)3t^2\,dt &= \int_0^1 t + 2t^3 + 3t^5\,dt \\
&= \frac{1}{2}t^2 + \frac{1}{2}t^4 + \frac{1}{2}t^6 \Big|_0^1 \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
&= \mathbf{\frac{3}{2}}
\end{aligned}$$

- $\mathbf{x = y^2}$
  $x = t^2,\ dx = 2t\,dt$
  $y = t,\ dy = dt$

$$\begin{aligned}
\int_1^0 (t^3 - t)2t\,dt + (t^2 + t)\,dt &= \int_1^0 2t^3 - t^2 + t\,dt \\
&= \frac{1}{2}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 \Bigg|_1^0 \\
&= \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{2}\right)\Bigg|_1^0 \\
&= -\frac{1}{2} + \frac{1}{3} - \frac{1}{2} \\
&= -\mathbf{\frac{2}{3}}
\end{aligned}$$

Hasil : $\mathbf{\frac{3}{2}} + \left(-\mathbf{\frac{2}{3}}\right) = \mathbf{\frac{5}{6}}$

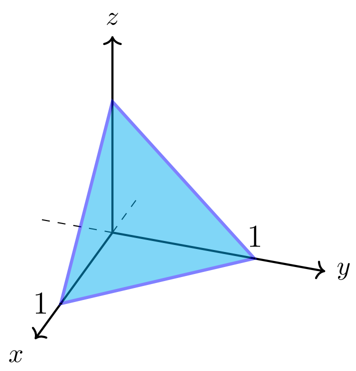
(b) Hitung integral tersebut dengan menggunakan teorema Green.

$$\begin{aligned}
\oint_C M\,dx + N\,dy &= \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\ dy \\
\oint_C (x - y)\,dx + (x + y)\,dy &= \iint_R 1 - (-1)\,dx\ dy = \iint_R 2\,dx\ dy \\
&= \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} 2\,dx\ dy = \int_{x=0}^1 2y \Bigg|_{x^3}^{\sqrt{x}} dx \\
&= \int_{x=0}^1 2\sqrt{x} - 2x^3\,dx = 2\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^4\right)\Bigg|_0^1 \\
&= 2\left(\frac{2}{3} - \frac{1}{4}\right) = 2\left(\frac{8-3}{12}\right) \\
&= \mathbf{\frac{5}{6}}
\end{aligned}$$

Hasil : $\mathbf{\frac{5}{6}}$

3. **[SOAL]**
   Gunakan teorema Stokes untuk menghitung $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ dengan $\boldsymbol{F} = xy\boldsymbol{i} + yz\boldsymbol{j} + zx\boldsymbol{k}$ dan C adalah segitiga dengan titik sudut $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ terorientasikan searah dengan putaran jarum jam bila dilihat dari atas.

   [PEMBAHASAN]



Teorema Stokes menyatakan bahwa $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot n \, dS$.

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & zx \end{vmatrix} = (0-y)\boldsymbol{i} - (z-0)\boldsymbol{j} + (0-x)\boldsymbol{k} = -y\boldsymbol{i} - z\boldsymbol{j} - x\boldsymbol{k}$$

Persamaan bidang S adalah $x + y + z = 1$. Normal vektor bidang S adalah $\boldsymbol{n} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$. Sehingga

$$dS = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} dA = \frac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})dA$$

Kemudian akhirnya didapatkan

$$\begin{aligned}
\iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot n \, dS &= \iint_S (-y\boldsymbol{i} - z\boldsymbol{j} - x\boldsymbol{k}) \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \frac{1}{\sqrt{3}} dA \\
&= \frac{1}{\sqrt{3}} \iint_S (-y - z - x) dA \\
&= \frac{1}{\sqrt{3}} \int_0^1 \int_0^{1-x} (-x - y - (1 - x - y)) \, dydx \\
&= \frac{1}{\sqrt{3}} \int_0^1 \int_0^{1-x} (-1) dydx \\
&= \frac{-1}{\sqrt{3}} \int_0^1 1 - x \, dx \\
&= \frac{1}{\sqrt{3}} \int_0^1 x - 1 \, dx \\
&= \frac{1}{\sqrt{3}} \left(\frac{x^2}{2} - x \right) \Big|_0^1 \\
&= \frac{1}{\sqrt{3}} \left(\frac{1}{2} - 1 \right) = -\frac{1}{2\sqrt{3}}
\end{aligned}$$

4. **[SOAL]**

   Hitung fluks medan vektor $\iint_S \boldsymbol{F} \cdot n \, dS$ jika diberikan vektor gaya $\boldsymbol{F} = 4x\boldsymbol{i} + 2y^2\boldsymbol{j} + z^2\boldsymbol{k}$ menembus permukaan tertutup S yang dibatasi oleh $x^2 + y^2 = 4$ dan $z = 3$.

   [PEMBAHASAN]

- Dengan Teorema Gauss

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V (\nabla\cdot\vec{F})\,dV.$$

- Divergensi $\vec{F}$:

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2).$$

$$= 4 - 4y + 2z.$$

- Integral volume:

  V dibatasi $x^2+y^2 = 4 \rightarrow r = 2$, $z = 0$ sampai $z = 3$. Ubah ke koordinat kutub.

$$\begin{aligned}
\iiint_V (4-4y+2z)\,dV &= \iiint_V (4-4(r\sin\theta+2z)\,r\,dr\,d\theta\,dz \\
&= \int_0^3\int_0^{2\pi}\int_0^2 (4r-4r^2\sin\theta+2zr)\,dr\,d\theta\,dz \\
&= \int_0^3\int_0^{2\pi}\left[2r^2-\frac{4r^3\sin\theta}{3}+2r^2\right]_0^2 d\theta\,dz \\
&= \int_0^3\int_0^{2\pi}\left[8-\frac{32}{3}\sin\theta+4z\right] d\theta\,dz \\
&= \int_0^3\left[16\pi+\frac{32}{3}\cos\theta+4z\theta\right]_0^{2\pi} dz \\
&= \int_0^3 [16\pi+0+8\pi z]\;dz \\
&= \left[16\pi z+4\pi z^2\right]_0^3 \\
&= 48\pi+36\pi = 84\pi
\end{aligned}$$