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1. Tunjukkan bahwa:

   (a) $\nabla \times \left(r^2\ \bar{r}\right) = 0$
   Jawab:

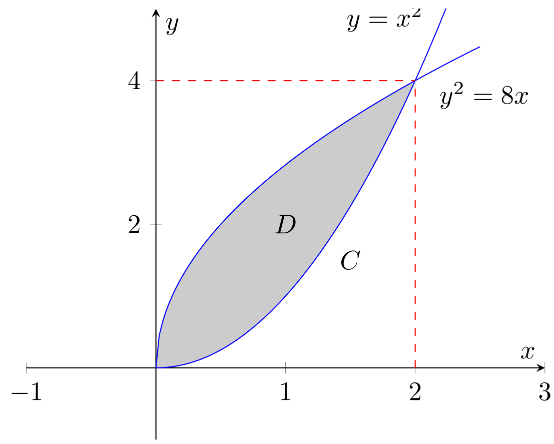
$$\begin{aligned}
\nabla \times \left(r^2\ \bar{r}\right) &= \nabla \times \left(r^2\ x\vec{i} + r^2\ y\vec{j} + r^2\ z\vec{k}\right) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r^2x & r^2y & r^2z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(r^2z) - \frac{\partial}{\partial z}(r^2y)\right)\vec{i} - \left(\frac{\partial}{\partial x}(r^2z) - \frac{\partial}{\partial z}(r^2x)\right)\vec{j} + \left(\frac{\partial}{\partial x}(r^2y) - \frac{\partial}{\partial y}(r^2x)\right)\vec{k} \\
&= (0-0)\,\vec{i} - (0-0)\,\vec{j} + (0-0)\,\vec{k} \\
&= 0\vec{i} + 0\vec{j} + 0\vec{k} \\
&= 0 \blacksquare
\end{aligned}$$

   (b) $\nabla \cdot \bar{r}\, f(r) = 3f(r) + |r|\,\dfrac{df}{dr}$
   Jawab:

$$\begin{aligned}
\nabla r\, f(r) &= (\nabla r) f(r) + r(\nabla f(r)) \\
&= 3f(r) + r\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) \\
&= 3f(r) + r\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\vec{i} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\vec{j} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\vec{k}\right) \\
&= 3f(r) + r\left(\frac{\partial r}{\partial x}\vec{i} + \frac{\partial r}{\partial y}\vec{j} + \frac{\partial r}{\partial z}\vec{k}\right)\frac{\partial f}{\partial r} \\
&= 3f(r) + r\,(\nabla r)\,\frac{\partial f}{\partial r} \\
&= 3f(r) + r\left(\frac{r}{r}\right)\frac{\partial f}{\partial r} \\
&= 3f(r) + |r|\frac{df}{dr} \blacksquare
\end{aligned}$$

2. Diberikan integral berikut $\oint_C (2xy - x^2)dx + (x+y^2)dy$ jika C adalah kurva tertutup yang dibatasi oleh $y = x^2$, dan $8x = y^2$.

   (a) Hitung integral tersebut dengan menggunakan teorema Green.
   Jawab:
   Daerah D yang dibatasi kurva C adalah sebagai berikut:



Sehingga nilai integralnya dengan menggunakan teorema Green adalah

$$
\begin{aligned}
\oint_C (2xy - x^2)dx + (x + y^2)dy &= \iint_D \left(\frac{\partial}{\partial x}(x + y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right) dA \\
&= \iint_D (1 - 2x) dA \\
&= \int_0^2 \int_{2\sqrt{2}\sqrt{x}}^{x^2} (1 - 2x) dy dx \\
&= \int_0^2 [y - 2xy]_{2\sqrt{2}\sqrt{x}}^{x^2} dx \\
&= \int_0^2 \left[x^2 - 2x(x^2) - 2\sqrt{2}\sqrt{x} + 4x\sqrt{2}\sqrt{x}\right] dx \\
&= \int_0^2 \left[x^2 - 2x^3 - 2\sqrt{2}\sqrt{x} + 4\sqrt{2}x\sqrt{x}\right] dx \\
&= \int_0^2 \left[x^2 - 2x^3 - 2\sqrt{2}\sqrt{x} + 4\sqrt{2}x^{3/2}\right] dx \\
&= \left[\frac{x^3}{3} - \frac{x^4}{2} - \frac{4\sqrt{2}}{3}x^{3/2} + \frac{8\sqrt{2}}{5}x^{5/2}\right]_0^2 \\
&= \frac{8}{3} - 8 - \frac{16}{3} + \frac{64}{5} \\
&= \frac{32}{15}
\end{aligned}
$$

(b) Hitung integral tersebut secara langsung tanpa menggunakan teorema Green.
Jawab:
Misalkan C_1 adalah kurva $y = x^2$ dan C_2 adalah kurva $8x = y^2$.

(i) Hitung integral di sepanjang C_1.

$$
\begin{aligned}
\int_{C_1} (2xy - x^2)dx + (x + y^2)dy &= \int_0^2 \left(2x(x^2) - x^2\right) dx + \left(x + x^2\right) 2xdx \\
&= \int_0^2 \left(2x^3 - x^2 + x + 2x^2\right) dx \\
&= \int_0^2 \left(2x^3 + x^2 + x\right) dx \\
&= \left[\frac{1}{2}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^2 \\
&= \frac{1}{2}(16) + \frac{1}{3}(8) + \frac{1}{2}(4) \\
&= 8 + \frac{8}{3} + 2 \\
&= \frac{32}{3}
\end{aligned}
$$

(ii) Hitung integral di sepanjang C_2.

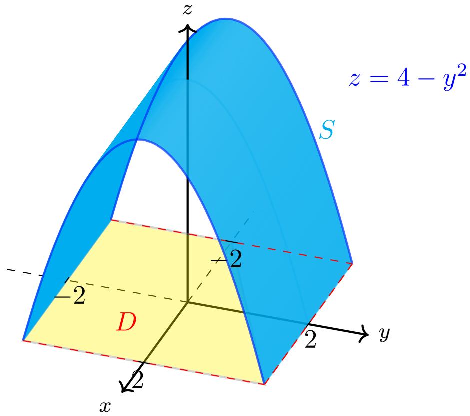
$$
\begin{aligned}
\int_{C_2} (2xy - x^2)dx + (x + y^2)dy &= \int_0^4 \left(2\left(\frac{y^2}{8}\right) y - \left(\frac{y^2}{8}\right)^2\right) \frac{y}{4}dy + \left(\left(\frac{y^2}{8}\right) + y^2\right) dy \\
&= \int_0^4 \left(\frac{y^4}{16} - \frac{y^5}{256} + \frac{y^3}{8} + y^2\right) dy \\
&= \left[\frac{1}{5}\frac{y^5}{16} - \frac{1}{6}\frac{y^6}{256}\right]_0^4 + \left[\frac{1}{4}\frac{y^4}{8} + \frac{1}{3}y^3\right]_0^4 \\
&= \frac{1}{5}\frac{4^5}{16} - \frac{1}{6}\frac{4^6}{256} + \frac{1}{4}\frac{4^4}{8} + \frac{1}{3}4^3 \\
&= \frac{592}{15}
\end{aligned}
$$

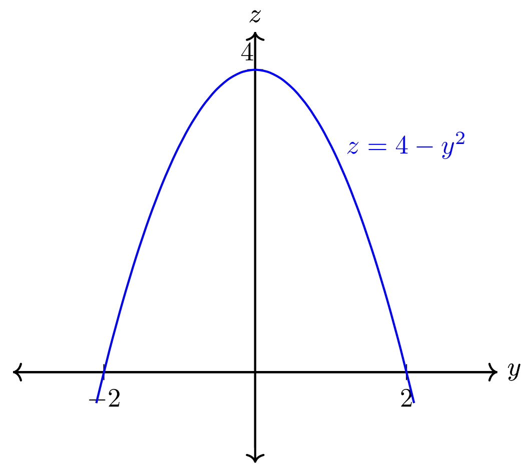
Sehingga nilai integralnya adalah

$$
\begin{aligned}
\oint_C (2xy - x^2)dx + (x + y^2)dy &= \oint_{C_2 - C1} (2xy - x^2)dx + (x + y^2)dy = \\
&= \frac{592}{15} - \frac{32}{3} \\
&= \frac{32}{15}
\end{aligned}
$$

3. Diberikan vektor gaya $\bar{A} = (2xy + z)\vec{i} + y^2\vec{j} + (x + 3y)\vec{k}$ dan S adalah permukaan yang dibatasi oleh $z = 4 - y^2$, $x = 0$, $x = 2$ dan bidang xy.

   (a) Gambarkan geometri batasan permukaan benda.
   Jawab:





(b) Hitunglah $\iint_S \bar{A} \cdot \bar{n}\, dS$ dengan menggunakan teorema Gauss.

Jawab:

$$
\begin{aligned}
\iint_S \bar{A} \cdot \bar{n}\, dS &= \iiint_V \nabla \cdot \bar{A}\, dV \\
&= \iiint_V \left(\frac{\partial}{\partial x}(2xy + z) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(x + 3y) \right) dV \\
&= \iiint_V (2y + 2y + 0) dV \\
&= \int_0^2 \int_0^2 \int_0^{4-y^2} 4y\, dz\, dy\, dx \\
&= \int_0^2 \int_0^2 4y(4 - y^2) dy\, dx \\
&= \int_0^2 \int_0^2 16y - 4y^3\, dy\, dx \\
&= \int_0^2 \left[8y^2 - y^4 \right]_0^2 dx \\
&= \int_0^2 \left[8(2)^2 - (2)^4 \right] dx \\
&= \int_0^2 [8(4) - 16]\, dx \\
&= \int_0^2 16\, dx = 32
\end{aligned}
$$

(c) Hitunglah $\iint_S \bar{A} \cdot \bar{n}\, dS$ secara langsung tanpa menggunakan teorema Gauss.

Jawab:

Diketahui $g(x, y, z) = 4 - y^2$ dan misalkan D adalah daerah yang berada dibawah permukaan.

$$
\begin{aligned}
\iint_S \bar{A}\cdot\bar{n}\,dS &= \iint_D \left(-Pg_x - Qg_y + R\right)\,dS \\
&= \iint_D \Big(-(2xy+z)0 - y^2(-2y) + (x+3y)0\Big)\,dS \\
&= \iint_D 2y^3\,dS \\
&= \int_0^2 \int_0^2 2y^3\,dy\,dx \\
&= \int_0^2 \left[\frac{1}{2}y^4\right]_0^2 dx \\
&= \int_0^2 \left[\frac{1}{2}(2)^4\right] dx \\
&= \int_0^2 8\,dx = 16
\end{aligned}
$$