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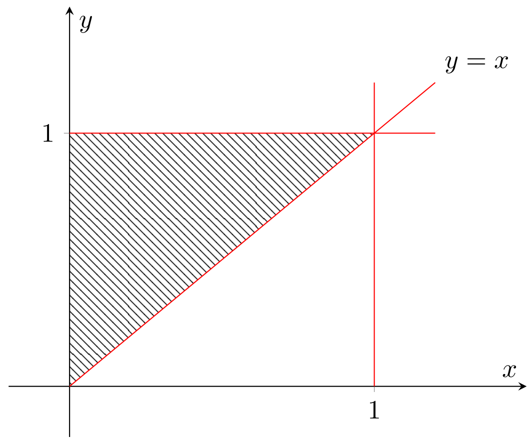
19. Let X and Y be continuous random variables with joint pdf of the form

$$f(x, y) = k(x + y) \quad 0 \le x \le y \le 1$$

and zero otherwise.

(a) Find the value of k.
Solution:
The region of pdf can be seen in the following figure.



So the value of k can be found by integrating the pdf over that region.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy &= 1 \\
\int_{0}^{1}\int_{0}^{y} k(x + y)\, dx\, dy &= 1 \\
k\int_{0}^{1}\left[\frac{x^2}{2} + xy\right]_0^y dy &= 1 \\
k\int_{0}^{1}\left(\frac{y^2}{2} + y^2\right) dy &= 1 \\
k\left[\frac{y^3}{6} + \frac{y^3}{3}\right]_0^1 &= 1 \\
k\left(\frac{1}{6} + \frac{1}{3}\right) &= 1 \\
k\left(\frac{1}{6} + \frac{2}{6}\right) &= 1 \\
k\left(\frac{1}{2}\right) &= 1 \\
k &= 2
\end{aligned}$$

(b) Find the marginals, $f_1(x)$ and $f_2(y)$.
Solution:

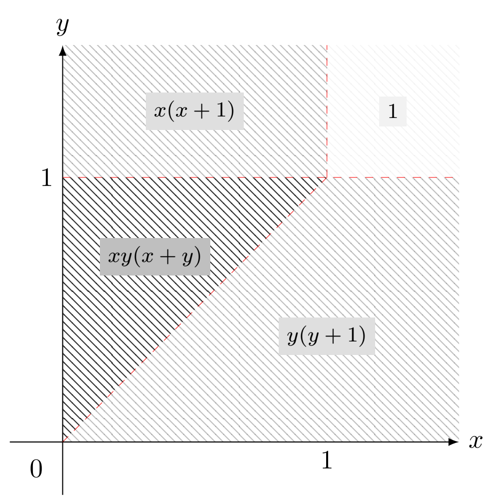
$$
\begin{aligned}
f_1(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \int_x^1 2(x+y)\,dy \\
&= \int_x^1 2x+2y\,dy \\
&= \left[2xy+y^2\right]_x^1 \\
&= 2x+1-3x^2 \\
&= 1-3x^2
\end{aligned}
$$

$$
\begin{aligned}
f_2(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\
&= \int_0^y 2(x+y)\,dx \\
&= \int_0^y 2x+2y\,dx \\
&= \left[x^2+2xy\right]_0^y \\
&= y^2+2y^2 \\
&= 3y^2
\end{aligned}
$$

(c) Find the joint CDF $F(x,y)$.
Solution:

$$
\begin{aligned}
F(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv \\
&= \int_0^y\int_0^x 2(u+v)\,du\,dv \\
&= \int_0^y \left[u^2+2uv\right]_0^x\,dv \\
&= \int_0^y \left[x^2+2xv\right]\,dv \\
&= \left[x^2v+xv^2\right]_0^y \\
&= x^2y+xy^2 \\
&= xy(x+y), \quad 0\le x\le y\le 1
\end{aligned}
$$

The other region of $F(x,y)$ can presented as following figure.



$$
F(x,y) = \begin{cases}
0 & \text{for } x<0 \text{ or } y<0 \\
xy(x+y) & \text{for } 0\le x\le y\le 1 \\
x(x+1) & \text{for } y>1, \quad 0\le x\le 1 \\
y(y+1) & \text{for } x\ge 0, \quad 0\le y\le x\le 1 \\
1 & \text{for } x>1 \text{ or } y>1
\end{cases}
$$

(d) Find the conditional pdf $f(y|x)$.
Solution:

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f_1(x)} \\ &= \frac{2x+2y}{1-3x^2}, \quad 0 \le x \le y \le 1 \end{aligned}$$

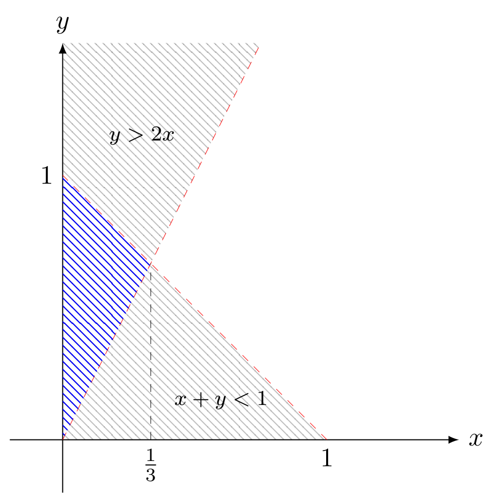
(e) Find the conditional pdf $f(x|y)$.
Solution:

$$\begin{aligned} f(x|y) &= \frac{f(x,y)}{f_2(y)} \\ &= \frac{2x+2y}{3y^2}, \quad 0 \le x \le y \le 1 \end{aligned}$$

29. Suppose X and Y are continuous random variables with joint pdf given by $f(x,y) = 24xy$ if $0 < x$, $0 < y$, $x + y < 1$ and zero otherwise.

(a) Are X and Y independent? Why or why not?
Solution:
X and Y dependent because "support set" isn't cartesian product.

(b) Find $P[Y > 2X]$.
Solution:
The probabilty and pdf region can be seen in the following figure.



Hence, the probability can be calculated as following.

$$
\begin{aligned}
P[Y > 2X] &= \int_0^{1/3} \int_{2x}^{1-x} 24xy \, dy \, dx \\
&= \int_0^{1/3} \left[12xy^2\right]_{2x}^{1-x} \, dx \\
&= \int_0^{1/3} \left[12x(1-x)^2 - 12x(2x)^2\right] \, dx \\
&= \int_0^{1/3} \left[12x(1-2x+x^2) - 48x^3\right] \, dx \\
&= \int_0^{1/3} \left[12x - 24x^2 + 12x^3 - 48x^3\right] \, dx \\
&= \int_0^{1/3} \left[12x - 24x^2 - 36x^3\right] \, dx \\
&= \left[6x^2 - 8x^3 - 9x^4\right]_0^{1/3} \\
&= 6\left(\frac{1}{3}\right)^2 - 8\left(\frac{1}{3}\right)^3 - 9\left(\frac{1}{3}\right)^4 \\
&= 6\left(\frac{1}{9}\right) - 8\left(\frac{1}{27}\right) - 9\left(\frac{1}{81}\right) \\
&= \frac{6}{9} - \frac{8}{27} - \frac{9}{81} \\
&= \frac{2}{3} - \frac{8}{27} - \frac{1}{9} \\
&= \frac{18 - 8 - 3}{27} \\
&= \frac{7}{27}
\end{aligned}
$$

(c) Find the marginal pdf of X.

$$
\begin{aligned}
f_1(x) &= \int_{-\infty}^{\infty} f(x,y) \, dy \\
&= \int_0^{1-x} 24xy \, dy \\
&= \left[12xy^2\right]_0^{1-x} \\
&= 12x(1-x)^2 \\
&= 12x(1-2x+x^2) \\
&= 12x - 24x^2 + 12x^3
\end{aligned}
$$