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12. Misalkan $r = x\vec{i} + y\vec{j} + z\vec{k}$ dan $r = |r|$ periksalah kebenaran persamaan berikut ini

(a) $\nabla \cdot r = 3$
Jawab:

$$\begin{aligned}\nabla r &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \\ &= 1 + 1 + 1 = 3 \blacksquare\end{aligned}$$

(b) $\nabla^2 r^3 = 12r$
Jawab:

$$\begin{aligned}\nabla^2 r^3 &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)^2 \left(\sqrt{x^2+y^2+z^2}\right)^3 \\ &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(\frac{\partial\left(x^2+y^2+z^2\right)^{3/2}}{\partial x}\vec{i} + \frac{\partial\left(x^2+y^2+z^2\right)^{3/2}}{\partial y}\vec{j} + \frac{\partial\left(x^2+y^2+z^2\right)^{3/2}}{\partial z}\vec{k}\right) \\ &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(3x\left(x^2+y^2+z^2\right)^{1/2}\vec{i} + 3y\left(x^2+y^2+z^2\right)^{1/2}\vec{j} + 3z\left(x^2+y^2+z^2\right)^{1/2}\vec{k}\right) \\ &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(3x\vec{i} + 3y\vec{j} + 3z\vec{k}\right)\sqrt{x^2+y^2+z^2} \\ &= \left[(3+3+3)\sqrt{x^2+y^2+z^2}\right] + \left[\frac{3x^2}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{3y^2}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{3z^2}{\sqrt{x^2+y^2+z^2}}\vec{k}\right] \\ &= [9r] + \frac{1}{r}\left(3x^2\vec{i} + 3y^2\vec{j} + 3z^2\vec{k}\right) \\ &= 9r + \frac{3r^2}{r} = 12r \blacksquare\end{aligned}$$

(c) $\nabla \cdot r\, r = 4r$
Jawab:

$$\begin{aligned}\nabla \cdot r\, r &= r(\nabla \cdot r) + r(\nabla r) \\ &= 3r + r(\nabla\sqrt{x^2+y^2+z^2}) \\ &= 3r + r\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}\right) \\ &= 3r + r\left(\frac{r}{r}\right) = 4r \blacksquare\end{aligned}$$

(d) $\nabla r = r/r$ **Jawab**:

$$\begin{aligned}\nabla r &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\sqrt{x^2+y^2+z^2} \\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k} \\ &= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r} = r/r \blacksquare\end{aligned}$$

(e) $\nabla \left(\frac{1}{r}\right) = -r/r^3$

Jawab:

$$\begin{aligned}
\nabla \left(\frac{1}{r}\right) &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(x^2+y^2+z^2)^{-1/2} \\
&= \frac{-x}{2\sqrt{(x^2+y^2+z^2)^3}}\vec{i} + \frac{-y}{2\sqrt{(x^2+y^2+z^2)^3}}\vec{j} + \frac{z}{2\sqrt{(x^2+y^2+z^2)^3}}\vec{k} \\
&= \frac{-x\vec{i} - y\vec{j} - z\vec{k}}{r^3} = -r/r^3 \blacksquare
\end{aligned}$$

(f) $\nabla \times r = 0$

Jawab:

$$\begin{aligned}
\nabla \times r &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(x\vec{i} + y\vec{j} + z\vec{k}\right) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\vec{i} - 0\vec{j} + 0\vec{k} = 0 \blacksquare
\end{aligned}$$

(g) $\nabla \ln r = r/r^2$

Jawab:

$$\begin{aligned}
\nabla \ln r &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(\frac{1}{2}\ln(x^2+y^2+z^2)\right) \\
&= \frac{x}{x^2+y^2+z^2}\vec{i} + \frac{y}{x^2+y^2+z^2}\vec{j} + \frac{z}{x^2+y^2+z^2}\vec{k} \\
&= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r^2} = r/r^2 \blacksquare
\end{aligned}$$

(h) $\nabla r\, f(r) = 3f(r) + |r|\frac{df}{dr}$

Jawab:

$$\begin{aligned}
\nabla r\, f(r) &= (\nabla r) f(r) + r(\nabla f(r)) \\
&= 3f(r) + r\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) \\
&= 3f(r) + r\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\vec{i} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\vec{j} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\vec{k}\right) \\
&= 3f(r) + r\left(\frac{\partial r}{\partial x}\vec{i} + \frac{\partial r}{\partial y}\vec{j} + \frac{\partial r}{\partial z}\vec{k}\right)\frac{\partial f}{\partial r} \\
&= 3f(r) + r\,(\nabla r)\,\frac{\partial f}{\partial r} \\
&= 3f(r) + r\left(\frac{r}{r}\right)\frac{\partial f}{\partial r} \\
&= 3f(r) + |r|\frac{df}{dr} \blacksquare
\end{aligned}$$

13. (a) Buktikan bahwa $div(grad\, f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$, ($\nabla^2 f$ disebut laplacian)

Jawab:

$$\begin{aligned}
\nabla(\nabla f) &= \nabla\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \blacksquare
\end{aligned}$$

(b) Jika $\Phi = x^2z - 3xy^2z - xy^2$ maka tentukan $\nabla\Phi$, $|\nabla\Phi|$ dan *laplace* Φ pada titik $(1, 1, 0)$.
Jawab:

$$\begin{aligned}
\nabla\Phi &= \frac{\partial \Phi}{\partial x}\vec{i} + \frac{\partial \Phi}{\partial y}\vec{j} + \frac{\partial \Phi}{\partial z}\vec{k} \\
&= (2xz - 3y^2z - y^2)\vec{i} + (-6xyz - 2xy)\vec{j} + (x^2 - 3xy^2)\vec{k} \\
\boxed{\nabla\Phi_{(1,1,0)}} &= -\vec{i} - 2\vec{j} - 2\vec{k} \\
\boxed{|\nabla\Phi|_{(1,1,0)}} &= \sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \\
\nabla^2\Phi &= \frac{\partial^2 \Phi}{\partial x} + \frac{\partial^2 \Phi}{\partial y} + \frac{\partial^2 \Phi}{\partial z} \\
&= 2z + (-6xz - 2x) + 0 = -6xz - 2x - 2z \\
\boxed{\nabla^2\Phi_{(1,1,0)}} &= -2
\end{aligned}$$

14. Jika $F = r/r^p$ carilah $divF$. Apakah terdapat nilai p sehingga berlaku $div\, F = 0$.
Jawab:
Misalkan $r = x\vec{i} + y\vec{j} + z\vec{k}$ dan $r = |r| = \sqrt{x^2 + y^2 + z^2}$

$$\begin{aligned}
\nabla \cdot F &= \nabla(r/r^p) = (\nabla r)r^{-p} + r(\nabla r^{-p}) \\
&= 3r^{-p} + r(-p\, r^{-p-2}\, r) \\
&= 3r^{-p} - p\, r^{-p-2} \\
&= r^{-p}(3 - p)
\end{aligned}$$

Jika $\nabla F = 0$, maka

$$\begin{aligned}
r^{-p}(3 - p) &= 0 \\
p &= 3
\end{aligned}$$

15. Dapatkan derivatif berarah dari $\varphi = 4xz^2 - 3x^2y^2z$ pada $(2, -1, 2)$ dalam arah $2\vec{i} - 3\vec{j} + 6\vec{k}$.
Jawab:
Vektor gradiennya sebagai berikut

$$\begin{aligned}
\nabla\varphi &= (4z^2 - 6xy^2z)\vec{i} + (-6x^2yz)\vec{j} + (8xz - 3x^2y^2)\vec{k} \\
\nabla\varphi_{(2,-1,2)} &= -8\vec{i} + 48\vec{j} + 20\vec{k}
\end{aligned}$$

Akan dicari panjang vektor yang searah dengan $\vec{u} = 2\vec{i} - 3\vec{j} + 6\vec{k}$, hal ini dapat dihitung menggunakan konsep proyeksi vektor

$$
\begin{aligned}
D_{\vec{u}}\varphi(2,-1,2) &= \nabla\varphi \cdot \frac{\vec{u}}{|\vec{u}|} \\
&= (-8\vec{i} + 48\vec{j} + 20\vec{k}) \cdot \frac{1}{7}(2\vec{i} - 3\vec{j} + 6\vec{k}) \\
&= \frac{1}{7}(-16 - 144 + 120) = -\frac{40}{7}
\end{aligned}
$$

16. Suatu benda mempunyai massa m, berputar dalam suatu orbit melingkar dengan kecepatan sudur ω akan mengalami gaya sentrifugal yang diberikan oleh $F(x,y,z) = m\omega^2 r$.

    Tunjukkan bahwa $f(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$ adalah fungsi potensial untuk F.

    Jawab:

    Diketahui $r = x\vec{i} + y\vec{j} + z\vec{k}$, $|r| = \sqrt{x^2+y^2+z^2}$ dan $\nabla \times F = 0$ yang berakibat bahwa F medan konservatif, sehingga didapatkan bahwa fungsi potensial F adalah f yang dimana $F = \nabla f$

$$
\begin{aligned}
F(x,y,z) &= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \\
m\omega^2(x\vec{i} + y\vec{j} + z\vec{k}) &= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \\
&\Rightarrow f = \int m\omega^2 x dx = \frac{1}{2}m\omega^2 x^2 \\
&\Rightarrow f = \int m\omega^2 y dy = \frac{1}{2}m\omega^2 y^2 \\
&\Rightarrow f = \int m\omega^2 z dz = \frac{1}{2}m\omega^2 z^2
\end{aligned}
$$

    Sehingga dapat disimpulkan bahwa $f(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$ ■