Nama	:	Teosofi Hidayah Agung
NRP	:	5002221132

16. Buktikan bahwa $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1$
Bukti:
Menggunakan bentuk (II) fungsi gamma, didapatkan

$$\begin{aligned}
\Gamma(p)\Gamma(1-p) &= \int_0^\infty x^{2p-1}e^{-x^2}\,dx \int_0^\infty y^{2(1-p)-1}e^{-y^2}\,dy \\
&= \int_0^\infty x^{2p-1}e^{-x^2}\,dx \int_0^\infty y^{1-2p}e^{-y^2}\,dy \\
&= \int_0^\infty \int_0^\infty \left(\frac{x}{y}\right)^{2p-1} e^{-(x^2+y^2)}\,dx\,dy
\end{aligned}$$

Ubah kedalam koordinat kutub $x = r\cos(\theta)$ dan $y = r\sin(\theta)$.

- $r^2 = x^2 + y^2, \quad 0 \le r \le \infty$
- $\cot(\theta) = \dfrac{x}{y}, \quad 0 \le \theta \le 2\pi$

$$\begin{aligned}
&\Rightarrow \int_0^{2\pi} \int_0^\infty \cot^{2p-1}(\theta)e^{-r^2}\,|r|\,dr\,d\theta \\
&= 4\int_0^{\frac{\pi}{2}} \cot^{2p-1}(\theta)\,d\theta \int_0^\infty re^{-r^2}\,dr \\
&= 4\int_0^{\frac{\pi}{2}} \cot^{2p-1}(\theta)\,d\theta \lim_{p\to\infty}\left[\frac{1}{2}e^{-r^2}\right]_0^p \\
&= 2\int_0^{\frac{\pi}{2}} \cot^{2p-1}(\theta)\,d\theta\ (0-1) \\
&= 2\int_0^{\frac{\pi}{2}} \cot^{2p-1}(\theta)\,d\theta = \frac{\pi}{\sin p\pi}
\end{aligned}$$

17. Buktikan $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)$
Bukti:

$$\begin{aligned}
\beta(x,x) = \frac{\Gamma(x)\Gamma(x)}{\Gamma(x+x)} &= \frac{\Gamma(x)^2}{\Gamma(2x)} \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}(\theta)\cos^{2x-1}(\theta)\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (\sin(\theta)\cos(\theta))^{2x-1}\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \left(\frac{2\sin(\theta)\cos(\theta)}{2}\right)^{2x-1} d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin(2\theta)}{2}\right)^{2x-1} d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\right)^{2x-1} \sin^{2x-1}(2\theta)\,d\theta \\
&= \frac{1}{2^{2x-2}}\int_0^{\frac{\pi}{2}} \sin^{2x-1}(2\theta)\,d\theta
\end{aligned}$$

Subtitusi $t = 2\theta \Rightarrow dt = 2d\theta$.

$$= \frac{1}{2^{2x-2}} \int_0^{\pi} \sin^{2x-1}(t) \left(\frac{1}{2}\right) dt$$
$$= \frac{1}{2^{2x-2}} \left(\frac{1}{2}\right) \int_0^{\pi} \sin^{2x-1}(t)\, dt$$
$$= \frac{1}{2^{2x-2}} \left(\frac{1}{2}\right) 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1}(t)\, dt$$
$$= \frac{1}{2^{2x-2}} \int_0^{\frac{\pi}{2}} \sin^{2x-1}(t)\, dt$$
$$= \frac{1}{2^{2x-2}} \beta\left(x, \frac{1}{2}\right)$$

Sehingga kita memiliki persamaan $\beta(x,x) = \frac{1}{2^{2x-2}}\beta\left(x,\frac{1}{2}\right)$. dapat direpresentasikan oleh fungsi gamma

$$\frac{\Gamma(x)^2}{\Gamma(2x)} = \frac{1}{2^{2x-2}} \frac{\Gamma(x)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)}$$
$$\frac{\Gamma(x)^{\cancel{2}}}{\Gamma(2x)} = \frac{1}{2^{2x-2}} \frac{\cancel{\Gamma(x)}\sqrt{\pi}}{\Gamma\left(x+\frac{1}{2}\right)}$$
$$\therefore \quad \boxed{\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)}$$

19. Buktikan $\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin^3 x} - \frac{1}{\sin^2 x}\right)^{\frac{1}{4}} \cos x\, dx = \beta\left(\frac{1}{4}, \frac{5}{4}\right)$

    Bukti:
    Lakukan subtitusi $u = \sin x \Rightarrow du = \cos x\, dx$.

    • $x = 0 \Rightarrow u = \sin(0) = 0$
    • $x = 0 \Rightarrow u = \sin(\pi/2) = 1$

    $$\int_{u=0}^{u=1} \left(\frac{1}{u^3} - \frac{1}{u^2}\right)^{\frac{1}{4}} du = \int_0^1 \left(\frac{1-u}{u^3}\right)^{\frac{1}{4}} du = \int_0^1 (u)^{-\frac{3}{4}} (1-u)^{\frac{1}{4}}\, du$$

    • $m - 1 = -\dfrac{3}{4} \Rightarrow m = \dfrac{1}{4}$
    • $m - 1 = \dfrac{1}{4} \Rightarrow m = \dfrac{5}{4}$

    $$\Rightarrow \int_0^1 (u)^{-\frac{3}{4}} (1-u)^{\frac{1}{4}}\, du = \beta\left(\frac{1}{4}, \frac{5}{4}\right)$$
    $$\therefore \quad \boxed{\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin^3 x} - \frac{1}{\sin^2 x}\right)^{\frac{1}{4}} \cos x\, dx = \beta\left(\frac{1}{4}, \frac{5}{4}\right)}$$

21. Buktikan rumus Stirling berikut ini $n! = \sqrt{2\pi n}\, n^n e^{-n}$.
    Bukti:
    Diketahui bahwa $n! = \Gamma(n+1) = \int_0^{\infty} x^n e^{-x}\, dx$. Lalu dengan melakukan subtitusi $x = nt$,

didapatkan

$$\begin{aligned}\int_0^\infty (nt)^n e^{-nt}\, d(nt) &= \int_0^\infty n^n\, t^n\, e^{-n}\, e^{-t}\, n\, dt\\ &= n^{n+1}\int_0^\infty t^n\, e^{-nt}\, dt\\ &= n^{n+1}\int_0^\infty e^{n\ln(t)}\, e^{-nt}\, dt\\ &= n^{n+1}\int_0^\infty e^{n(\ln(t)-t)}\, dt\end{aligned}$$

Dengan menggunakan aproksimasi deret taylor $\boxed{\ln(t) - t \approx -1 - \dfrac{(t-1)^2}{2}}$. maka

$$\begin{aligned}n^{n+1}\int_0^\infty e^{n(\ln(t)-t)}\, dt &\approx n^{n+1}\int_0^\infty e^{n(-1-\frac{(t-1)^2}{2})}\, dt\\ &= n^{n+1}e^{-n}\int_0^\infty e^{-n\frac{(t-1)^2}{2}}\, dt\\ &= n^{n+1}e^{-n}\int_0^\infty e^{-\left(\frac{\sqrt{n}(t-1)}{\sqrt{2}}\right)^2}\, dt\end{aligned}$$

Selanjutnya dengan subtitusi $u = \frac{\sqrt{n}(t-1)}{\sqrt{2}} \Rightarrow du = \sqrt{\frac{n}{2}}dt$

$$n^{n+1}e^{-n}\int_{-\sqrt{\frac{n}{2}}}^\infty e^{-u^2}\sqrt{\frac{2}{n}}\, du = n^n e^{-n}\sqrt{2n}\int_{-\sqrt{\frac{n}{2}}}^\infty e^{-u^2}\, du$$

Perhatikan untuk n yang sangat besar ($n \to \infty$), maka $-\sqrt{\frac{n}{2}} \to -\infty$

$$\begin{aligned}n^n e^{-n}\sqrt{2n}\int_{-\infty}^\infty e^{-u^2}\, du &= n^n e^{-n}\sqrt{2n}(\sqrt{\pi}), \quad \left(\int_{-\infty}^\infty e^{-u^2}\, du = \sqrt{\pi}\right)\\ &= n^n e^{-n}\sqrt{2n\pi}\end{aligned}$$

Sehingga untuk n yang sangat besar terbukti bahwa

$$\boxed{n! \approx \sqrt{2\pi n}\, n^n e^{-n}}$$

24. Tunjukkan $\int_{-\infty}^\infty \frac{e^{ax}}{e^x+1}\, dx = \frac{\pi}{\sin\pi}$
Bukti:

$$\int_{-\infty}^\infty \frac{e^{ax}}{e^x+1}\, dx = \int_{-\infty}^\infty \frac{(e^x)^a}{e^x+1}\, dx$$

Subtitusi $u = e^x \Rightarrow du = e^x\, dx$.

$$\Rightarrow \int_0^\infty \frac{u^a}{u+1}\,\frac{1}{u}\, du = \int_0^\infty \frac{u^{a-1}}{u+1}\, du$$

Bentuk (iv) fungsi beta $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}}\, dx$. Sehingga

- $m - 1 = a - 1 \iff m = a$
- $m + n = 1 \iff n = 1 - a$

$$\begin{aligned}\int_0^\infty \frac{u^{a-1}}{u+1}\, du &= \beta(a, 1-a)\\ &= \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(a+1-a)}\\ &= \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi}\end{aligned}$$

$$\therefore \boxed{\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx = \frac{\pi}{\sin a\pi}}, \quad 0 < a < 1$$