Nama	:	Teosofi Hidayah Agung
NRP	:	5002221132

1. Given sets of matrices as follows

$$M = \left\{ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \middle| \, a_{ij} = 0, \text{ if } i > j \right\}.$$

Determine whether M with usual matrix addition and multiplication operations forms a ring!

Proof:

By definition, M is an upper triangular matrix and can be rewritten as follows

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

We known $M \subseteq \mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times n}$ is ring. Then we can check M is subring of $\mathbb{R}^{n\times n}$ iff $\forall A, B \in M$ satisfying

(1) $A - B \in M$

(2) $AB \in M$

Lets check:

$$\begin{aligned} A - B &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{nn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \dots & a_{1n} - b_{1n} \\ 0 & a_{22} - b_{22} & \dots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} - b_{nn} \end{bmatrix} \in M \end{aligned}$$

and

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & \dots & \sum_{k=1}^{n} a_{1k}b_{kn} \\ 0 & a_{22}b_{22} & \dots & \sum_{k=2}^{n} a_{2k}b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}b_{nn} \end{bmatrix} \in M$$

Because (1) and (2) satisfied, so we can conclude that M is subring of $\mathbb{R}^{n\times n}$.
$\therefore$ M is ring $\blacksquare$

2. Is the set of three-dimension vectors i.e., $\mathbb{R}^3$ with the addition operation

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

and multiplication operation

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} := \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

forms a ring? Explain your reasoning!
Proof:
No, because is not satisfy the associative law for multiplication. In this case take $\vec{u} = [1, 2, 3]^T, \vec{v} = [1, 1, 1]^T, \vec{w} = [1, 2, 1]^T$, then we can check

$$(\vec{u} \times \vec{v}) \times \vec{w} = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \times \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-3 \\ -1+3 \\ 1-2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 \\ 1-1 \\ -2-2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

$$
\begin{aligned}
\vec{u} \times (\vec{v} \times \vec{w}) &= \begin{bmatrix}1\\2\\3\end{bmatrix} \times \left(\begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}1\\2\\1\end{bmatrix} \right) = \begin{bmatrix}1\\2\\3\end{bmatrix} \times \begin{bmatrix}1-2\\-1+1\\2-1\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \times \begin{bmatrix}-1\\0\\1\end{bmatrix} \\
&= \begin{bmatrix}2-0\\-(1+3)\\0+2\end{bmatrix} = \begin{bmatrix}2\\-4\\2\end{bmatrix}
\end{aligned}
$$

Thus $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$ or the multiplication isn't associative.

$\therefore$ The set is not ring under that operation.

As known from Elementary Linear Algebra course (ELA). the multiplication operation from the question is "cross product". The definition of multiplication can be rewritten as follows

$$
\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} \times \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} := \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix}M_{11}\\M_{12}\\M_{13}\end{bmatrix}
$$

and M_{ij} is *minor* of the entry in the i^{th} row and j^{th} column (also called the (i, j) minor).
Then the one of property *cross product* is

$$
\begin{aligned}
(\vec{u} \times \vec{v}) \times \vec{w} &= (u \cdot w)v - (v \cdot w)u \\
\vec{u} \times (\vec{v} \times \vec{w}) &= (u \cdot w)v - (u \cdot v)w
\end{aligned}
$$

and from that property we can conclude $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$

3. Is the set of three-dimension vectors i.e., $\mathbb{R}^3$ with the addition operation

$$
\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} + \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} := \begin{bmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{bmatrix}
$$

and multiplication operation

$$
\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} \times \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} := \begin{bmatrix}u_1v_1\\u_2v_2\\u_3v_3\end{bmatrix}
$$

forms a ring? Explain your reasoning!
Proof:

The set and operation similarly to **Contoh 6.1.6**. As we known

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

or $\mathbb{R}^3$ is ordered pairs (External Direct Product).
Because $\mathbb{R}$ is ring under addition and multiplication. Thus conclude $\mathbb{R}^3$ also ring $\blacksquare$