1. Determine the coefficients for x^5y^{13} and x^9y^9 in the expansion of $(3x-4y)^{18}$.
   Solution:
   The general term in the binomial expansion of $(a+b)^n$ is given by
   $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
   For the expansion of $(3x-4y)^{18}$, we have $a = 3x$, $b = -4y$, and $n = 18$.

   - For the term x^5y^{13}, we need to find k such that $n-k=5$ and $k=13$. Thus, $k = 18-5 = 13$.
   $$\text{Coefficient} = \binom{18}{13}(3)^{18-13}(-4)^{13} = \binom{18}{5}(3)^5(-4)^{13}$$
   - For the term x^9y^9, we need to find k such that $n-k=9$ and $k=9$. Thus, $k = 18-9 = 9$.
   $$\text{Coefficient} = \binom{18}{9}(3)^{18-9}(-4)^9 = \binom{18}{9}(3)^9(-4)^9$$

2. Compute
   $$\sum_{k=1}^{n} \binom{n}{k} 2^{n-k}$$
   Solution:
   We can use the binomial theorem, which states that
   $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
   Setting $a = 2$ and $b = 1$, we have
   $$(2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} 1^k$$
   This simplifies to
   $$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = 3^n$$

3. A bakery sells chocolate, cinnamon, and plain doughnuts and at a particular time has 6 chocolate, 6 cinnamon, and 3 plain. If a box contains 12 doughnuts, how many different options are there for a box of doughnuts?
   Solution:
   We can use the stars and bars method to solve this problem. Let x_1, x_2, and x_3 represent the number of chocolate, cinnamon, and plain doughnuts in the box, respectively. We need to find the number of non-negative integer solutions to the equation
   $$x_1 + x_2 + x_3 = 12$$
   subject to the constraints $0 \le x_1 \le 6$, $0 \le x_2 \le 6$, and $0 \le x_3 \le 3$.
   We can edited the constraints by
   $$0 \le 6 - x_1 \le 6, \quad 0 \le 6 - x_2 \le 6, \quad 0 \le 3 - x_3 \le 3$$
   This means we can rewrite the equation as
   $$(6-x_1) + (6-x_2) + (3-x_3) = 15 - 12 \iff (6-x_1) + (6-x_2) + (3-x_3) = 3$$
   That equation isn't affected by the constraints, so we can use the stars and bars method to find the number of non-negative integer solutions to this equation.
   $$\binom{3+2}{2} = \binom{5}{2} = 10$$

4. Determine the number of integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

which satisfy

$$1 \leq x_1 \leq 6, \quad 0 \leq x_2 \leq 7, \quad 4 \leq x_3 \leq 8, \quad 2 \leq x_4 \leq 6$$

Solution:
We can transform the variables to eliminate the lower bounds:

$$y_1 = x_1 - 1, \quad y_2 = x_2, \quad y_3 = x_3 - 4, \quad y_4 = x_4 - 2$$

This gives us the new equation:

$$y_1 + y_2 + y_3 + y_4 = 13$$

The new constraints are:

$$0 \leq y_1 \leq 5, \quad 0 \leq y_2 \leq 7, \quad 0 \leq y_3 \leq 4, \quad 0 \leq y_4 \leq 4$$

We can use the principle of inclusion-exclusion to count the number of solutions. First, we find the total number of non-negative integer solutions without constraints:

$$\binom{13+4-1}{4-1} = \binom{16}{3} = 560$$

Next, we subtract the cases that violate the constraints:

- For $y_1 > 5$: Let $y_1' = y_1 - 6$, then $y_1' + y_2 + y_3 + y_4 = 7$ with $0 \leq y_1'$, giving $\binom{10}{3} = 120$ solutions.
- For $y_2 > 7$: Let $y_2' = y_2 - 8$, then $y_1 + y_2' + y_3 + y_4 = 5$ with $0 \leq y_2'$, giving $\binom{8}{3} = 56$ solutions.
- For $y_3 > 4$: Let $y_3' = y_3 - 5$, then $y_1 + y_2 + y_3' + y_4 = 8$ with $0 \leq y_3'$, giving $\binom{11}{3} = 165$ solutions.
- For $y_4 > 4$: Let $y_4' = y_4 - 5$, then $y_1 + y_2 + y_3 + y_4' = 8$ with $0 \leq y_4'$, giving $\binom{11}{3} = 165$ solutions.

Now for the intersections:

- For $y_1 \geq 6$ and $y_2 \geq 8$: Let $y_1' = y_1 - 6$ and $y_2' = y_2 - 8$, then $y_1' + y_2' + y_3 + y_4 = -1$, which has no solutions.
- For $y_1 \geq 6$ and $y_3 \geq 5$: Let $y_1' = y_1 - 6$ and $y_3' = y_3 - 5$, then $y_1' + y_2 + y_3' + y_4 = 2$, giving $\binom{5}{3} = 10$ solutions.
- For $y_1 \geq 6$ and $y_4 \geq 5$: Let $y_1' = y_1 - 6$ and $y_4' = y_4 - 5$, then $y_1' + y_2 + y_3 + y_4' = 2$, giving $\binom{5}{3} = 10$ solutions.
- For $y_2 \geq 8$ and $y_3 \geq 5$: Let $y_2' = y_2 - 8$ and $y_3' = y_3 - 5$, then $y_1 + y_2' + y_3' + y_4 = 0$, giving $\binom{3}{3} = 1$ solution.
- For $y_2 \geq 8$ and $y_4 \geq 5$: Let $y_2' = y_2 - 8$ and $y_4' = y_4 - 5$, then $y_1 + y_2' + y_3 + y_4' = 0$, giving $\binom{3}{3} = 1$ solution.
- For $y_3 \geq 5$ and $y_4 \geq 5$: Let $y_3' = y_3 - 5$ and $y_4' = y_4 - 5$, then $y_1 + y_2 + y_3' + y_4' = 3$, giving $\binom{6}{3} = 20$ solutions.

And the other intersections are either impossible or yield no additional solutions.

Now we can apply the principle of inclusion-exclusion:

$$\text{Total solutions} = 560 - (120 + 56 + 165 + 165) + (0 + 10 + 10 + 1 + 1 + 20) = 560 - 506 + 42 = 96$$

5. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which exactly four integers are in their natural positions.
Solution:

The idea is we can choose 4 positions from the 8 to be fixed, and then we need to derange the remaining 4 integers (i.e., arrange them such that none of them are in their original positions). The number of ways to choose 4 positions from 8 is given by $\binom{8}{4} = 70$.

The number of derangements of n objects, denoted as D_n, can be calculated using the formula:

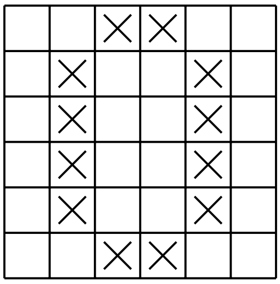
$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

For $n = 4$, we have:

$$D_4 = 4! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right) = 24 \cdot \frac{3}{8} = 9$$

Therefore, the total number of permutations is $70 \cdot 9 = 630$.

6. Determine the number of ways to place rooks on a 6×6 chessboard such that no two rooks can attack each other and none are placed on forbidden positions (marked with X):

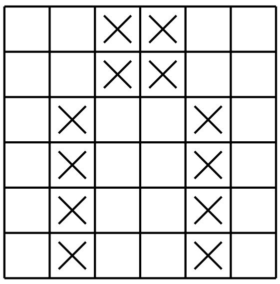


Solution:
Formula for the number of ways to place k rooks on an $n \times n$ chessboard such that no two rooks can attack each other is given by:

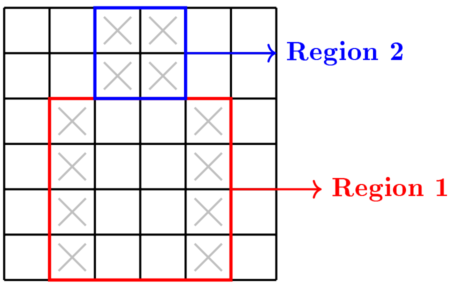
$$S = n! - r_1(n-1)! - r_2(n-2)! - \ldots + (-1)^k r_k(n-k)!$$

where r_i is the number of forbidden positions in the i-th row.

Now consider the rows and columns that can be swapped. Then we can change the forbidden positions to:



Then we can split by two region of forbidden positions



So we can calculate the number of ways to place rooks in each region separately.

- For one rook the possible positions such that is in forbidden positions is $r_1 = 12$.
- For two rooks the possible positions such that is in forbidden positions is $r_2 = 4 \cdot 8 + 2 + 2\binom{4}{2} = 34 + 12 = 46$.
- For three rooks the possible positions such that is in forbidden positions is $r_3 = 2 \cdot 8 + 4 \cdot 2 \cdot \binom{4}{2} = 16 + 48 = 64$.
- For four rooks the possible positions such that is in forbidden positions is $r_4 = 2 \cdot 2\binom{4}{2} = 24$.

For five and six rooks, there are no possible positions such that is in forbidden positions.

Then the number of ways to place rooks on the chessboard is given by:

$$S = 6! - r_1(6-1)! - r_2(6-2)! - r_3(6-3)! - r_4(6-4)! = 720 - 12 \cdot 120 - 46 \cdot 30 - 64 \cdot 6 - 24 \cdot 1 = 720 - 1440 + 1104 - 384 + 48 = 48$$