Diberikan persamaan $y^2 u_{xx} - x^2 u_{yy} = 0$ dengan $x, y \neq 0$. Transformasikan PDP tersebut ke bentuk kanoniknya.

1. Dari soal diperoleh $A = y^2$, $B = 0$, $C = -x^2$, dan $D = E = F = G = 0$, sehingga

$$B^2 - 4AC = 0 - 4 \cdot y^2 \cdot (-x^2) = 4x^2y^2 > 0 \rightarrow \text{PDP hiperbolik}$$

2. Persamaan karakteristiknya adalah

   (a) Persamaan karakteristik pertama:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{B + \sqrt{B^2 - 4AC}}{2A} \\
\frac{dy}{dx} &= \frac{\sqrt{4x^2y^2}}{2y^2} \\
\frac{dy}{dx} &= \frac{x}{y} \\
\int y\,dy &= \int x\,dx \\
\frac{y^2}{2} &= \frac{x^2}{2} + c_1 \\
\underbrace{\frac{y^2}{2} - \frac{x^2}{2}}_{\phi_1(x,y)} &= c_1
\end{aligned}$$

   (b) Persamaan karakteristik kedua:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{B - \sqrt{B^2 - 4AC}}{2A} \\
\frac{dy}{dx} &= \frac{-\sqrt{4x^2y^2}}{2y^2} \\
\frac{dy}{dx} &= \frac{-x}{y} \\
\int y\,dy &= \int -x\,dx \\
\frac{y^2}{2} &= -\frac{x^2}{2} + c_2 \\
\underbrace{\frac{y^2}{2} + \frac{x^2}{2}}_{\phi_2(x,y)} &= c_2
\end{aligned}$$

   sehingga diperoleh $\xi(x,y) = \dfrac{y^2}{2} - \dfrac{x^2}{2}, \quad \eta(x,y) = \dfrac{y^2}{2} + \dfrac{x^2}{2}.$

3. Dapatkan turunan ξ, η terhadap x, y

$$\begin{aligned} \xi_x &= -x & \eta_x &= x \\ \xi_y &= y & \eta_y &= y \\ \xi_{xx} &= -1 & \eta_{xx} &= 1 \\ \xi_{xy} &= 0 & \eta_{xy} &= 0 \\ \xi_{yy} &= 1 & \eta_{yy} &= 1 \end{aligned}$$

4. Subtitusi hasil ke persamaan berikut

$$\begin{aligned} C^* = A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ &= 3(-3)^2 + 10(-3)(1) + 3(1)^2 = 0 \\ B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ &= 2(y^2)(-x)(x) + 0 + 2(-x^2)(y)(y) \\ &= -4x^2y^2 = 4(\xi^2 - \eta^2) \\ D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\ &= y^2(-1) + 0 + (-x^2) + 0 + 0 \\ &= -y^2 - x^2 = -2\eta \\ E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\ &= y^2 + 0 + (-x^2) + 0 + 0 \\ &= y^2 - x^2 = 2\xi \\ F^* &= F = 0 \\ G^* &= G = 0 \end{aligned}$$

oleh karena itu didapat:

$$\begin{aligned} A^*w_{\xi\xi} + B^*w_{\xi\eta} + C^*w_{\eta\eta} + D^*w_\xi + E^*w_\eta + F^*w + G^* &= 0 \\ 4(\xi^2 - \eta^2)w_{\xi\eta} - 2\eta w_\xi + 2\xi w_\eta &= 0 \\ w_{\xi\eta} &= \frac{\eta w_\xi - \xi w_\eta}{\xi^2 - \eta^2} \end{aligned}$$

5. Solusi dari bentuk kanonik adalah $w = f(\xi) + g(\eta)$, sehingga misalkan

$$\begin{aligned} w_\xi &= f'(\xi) \implies w_{\xi\eta} = \frac{\eta f'(\xi) - \xi g'(\eta)}{\xi^2 - \eta^2} \\ w_\eta &= g'(\eta) \\ w_{\xi\eta} &= 0 \implies \frac{f'(\xi)}{\xi} = \frac{g'(\eta)}{\eta} = k \end{aligned}$$

Karenanya diperlorek

$$\begin{aligned}\frac{f(\xi)}{\xi} = k &\implies f(\xi) = k\frac{\xi^2}{2} + C_1 \\ \frac{g(\eta)}{\eta} = k &\implies g(\eta) = k\frac{\eta^2}{2} + C_2\end{aligned}$$

Maka didapatkan solusi

$$\begin{aligned} w &= f(\xi) + g(\eta) \\ &= k\frac{\xi^2}{2} + C_1 + k\frac{\eta^2}{2} + C_2 \\ &= k\left[(\frac{y^2}{2} - \frac{x^2}{2})^2 + (\frac{y^2}{2} + \frac{x^2}{2})^2\right] + C_1 + C_2 \\ w(x,y) &= k\left(\frac{x^4 + y^4}{4}\right) + C \end{aligned}$$