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Trivial and Nontrivial Eigenvectors for Latin Squares in Max-Plus Algebra

Fazal Abbas [1,*], Mubasher Umer [2,*], Umar Hayat [2] and Ikram Ullah [2]

[1] Department of Mathematics and Computer Sciences, Stetson University, DeLand, FL 32723, USA
[2] Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan; umar.hayat@qau.edu.pk (U.H.); ikram.ullah@math.qau.edu.pk (I.U.)
* Correspondence: fabbas1@stetson.edu (F.A.); mumer@math.qau.edu.pk (M.U.)

Abstract: A square array whose all rows and columns are different permutations of the same length over the same symbol set is known as a Latin square. A Latin square may or may not be symmetric. For classification and enumeration purposes, symmetric, non-symmetric, conjugate symmetric, and totally symmetric Latin squares play vital roles. This article discusses the Eigenproblem of non-symmetric Latin squares in well known max-plus algebra. By defining a certain vector corresponding to each cycle of a permutation of the Latin square, we characterize and find the Eigenvalue as well as the possible Eigenvectors.

Keywords: max-plus algebra; Eigenproblem; permutations; latin squares

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1. Introduction

The time evolution of discrete event dynamic systems may be represented using equations that combine the minimum, maximum, and addition operations. The union of two sets of equations determines min-max-plus systems: one set includes the addition and minimization, and the other set contains the addition and maximization.

Max-plus algebra has a wide range of applications in mathematics and other fields such as optimization, mathematical physics, algebraic geometry, and combinatorics [1]. Furthermore, communications networks, machine scheduling, control theory, parallel processing systems, manufacturing systems, stenography, and traffic control all use max-plus algebra [2–6]. Work on characteristic problems and equation problems is also available. For instance, Olsder and Ross [7] proved and formulated the Cayley–Hamilton theorem and Cramer's rule in max algebra. Schutter and Moor [8] corrected the error in the aforementioned derivation of [7]. Wang and Tao [9] discussed the problem of global optimization in max-plus linear systems and found the conditions for the unique and optimal solutions. Other than this, Marotta et al. [10] proposed a framework for Timed Event Graphs (TEG) using tropical algebra. Helena and Ján [11] proposed a work that transforms the weak solvability versions into sub-Eigenvector problems or inexact two-sided max-plus linear systems. Their work finds efficient conditions (necessary and sufficient) for interval system solvability. Wang et al. [12] investigated in an analytic way the ordered structures of polynomial idempotent algebras over the max-plus algebra. In [13], a max-plus system is used to describe the Dutch railway system. The impact propagation of processing time variations is studied by using max-plus algebra [14]. The study [15] shows how max-plus algebra is useful in a dynamic programming algorithm.

Latin squares have many types, such as reduced, idempotent, unipotent, semisymmetric, and diagonal. This article considers the Eigenproblem of non-symmetric Latin squares in max-plus algebra. The Eigenproblem for a square matrix A is to determine a real number λ and a vector v in such a way that $Av = \lambda v$. Similar problems are studied for other matrices such as Monge matrices [16], inverse Monge matrices [17], and circulant matrices [18]. In [19], a power technique is designed to compute the Eigenvalue as well as the Eigenvectors for the similar systems. Umer et al. [20] efficiently developed a technique to solve Eigenproblems in max-plus algebra. In [21], the authors solve Eigenproblem by taking a permutation in Latin squares. To study Eigenproblems in detail, the readers are referred to [19,22–25].

In this article, we determine the Eigenvalue λ and Eigenvectors (both trivial and non-trivial) by considering the vectors corresponding to each cycle in a permutation in a Latin square.

The remaining paper is organized as follows. We recall notions related to the permutation in Section 2. Section 3 contains some basic notions. In Section 4, we present Latin squares and calculate Eigenvalues along with Eigenvectors for these matrices. The whole work is concluded in Section 5.

2. Related Notions to a Permutation

A permutation is a rearrangement of the objects of a set into a particular order. For example, all possible arrangements of a three element set $\{1,2,3\}$ are given as; $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$. In the current work, the symbol $\underline{n}$ stands for the set $\{1,\ldots,n\}$, i.e., $\underline{n} = \{1,\ldots,n\}$. In algebra, a bijective mapping $\tau : Z \to Z$ on a set Z is called a permutation. For example, a mapping τ with $\tau(1) = 3$, $\tau(2) = 1$, $\tau(3) = 2$ determines the rearrangement $(3,1,2)$. Let $X = \underline{n}$; then S_n denotes the group of all permutations of X, where the product is defined by the composition of mappings and the identity element is the identity mapping. A permutation can be represented by cyclic notation as $(r_1\, r_2 \ldots r_t)$, if $\tau(r_1) = r_2$, $\tau(r_2) = r_3, \ldots, \tau(r_t) = r_1$ and called a t-cycle. An element r_i is fixed by a permutation τ if $\tau(r_i) = r_i$. A complete factorization of a permutation τ rewrites τ and puts all 1-cycle of s for all s fixed by τ. The complete factorization of $(2\;4\;3\;6) \in S_6$ is given by $(2\;4\;3\;6)(1)(5)$.

3. Max-Plus Algebra

To study in max-plus algebra detail, readers are referred to [26,27]. Here we recall some basic notions of max-plus algebra. The max-plus semiring means the structure $\mathbb{R}_{max} = (\mathbb{R}_\epsilon, \oplus, \otimes)$, where $\mathbb{R}_\epsilon = \mathbb{R} \cup \{\epsilon\}$ for $\epsilon = -\infty$ and $\oplus, \otimes$ are binary operations on $\mathbb{R}$ introduced as:

$$\begin{aligned} r \oplus s &= max\{r,s\}, \\ r \otimes s &= r + s, \qquad \text{for all } r,s \in \mathbb{R}_\epsilon. \end{aligned}$$

In max-plus algebra, the collection of all matrices of order $r \times s$ is represented as $\mathbb{R}^{r\times s}_{max}$, while $\mathbb{R}^{r}_{max}$ denotes the set of all vectors of order $r \times 1$.

Suppose that $U = [u_{ij}]$, $V = [v_{ij}]$ are two matrices such that $U, V \in \mathbb{R}^{r\times s}_{max}$ and $\theta \in \mathbb{R}$, then:

$$\begin{aligned} U \oplus V &= [w_{ij}], \quad \text{where } w_{ij} = max\{u_{ij}, v_{ij}\}, \\ \theta \otimes U &= \theta \otimes [u_{ij}] = \theta + [u_{ij}], \end{aligned}$$

$$\text{for } i \in \underline{r},\ j \in \underline{s}.$$

If $U \in \mathbb{R}^{r\times t}_{max}$ and $V \in \mathbb{R}^{t\times s}_{max}$, then:

$$U \otimes V = [w_{ij}], \quad \text{where } w_{ij} = \bigoplus_{l=1}^{t} (u_{il} \otimes v_{lj}) = max\{u_{il} + v_{lj}\},$$

$$\text{for } l \in \underline{t}, i \in \underline{r},\ j \in \underline{s}.$$

A graph $\mathcal{G}$ for a matrix $U \in \mathbb{R}^{n\times n}_{max}$ is a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ and $\mathcal{E}$ represent all the vertices (nodes) and edges (arcs), respectively, such that the nodes i and j are joined by an arc in $\mathcal{G}$ if and only if $u_{ij} \neq \epsilon$. It is denoted by (j,i). The weight $w(i,j)$ of the arc

(j, i), is equal to u_{ij}. A path is a sequences of arcs $(j_1, j_2), (j_2, j_3), \ldots, (j_k, j_{k+1})$ denoted by $j_1 \to j_2 \to \ldots \to j_{k+1}$. A path is said to be an elementary path if no node occurs twice. An elementary closed path $j_1 \to j_2 \to \ldots \to j_k \to j_1$ is called a circuit. The length of a path P is the number of arcs on that path. It is denoted as $l(P)$. The sum of weight of each arc in a path $P = j_1 \to j_2 \to \ldots \to j_k$ is called the weight P_w of P. The weight obtained by dividing P_w by $l(p)$ is called the average weight of P. A circuit is said to be a critical circuit if the average weight of that circuit is maximum. A graph is said to be strongly connected if a path exists from each node i to each node j.

In the following equation, the starting point describes the time evolution of a system:

$$z_i(l+1) = max\{u_{i1} + z_1(l), u_{i2} + z_2(l), \ldots, u_{in} + z_n(l)\}.$$

The above system can compactly be written over max-plus algebra as:

$$z(l+1) = U \otimes z(l), \tag{1}$$

where

$$z(l) = \begin{pmatrix} z_1(l) \\ z_2(l) \\ . \\ . \\ . \\ z_n(l) \end{pmatrix} \in \mathbb{R}_\epsilon^n,$$

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \ldots & u_{1n} \\ u_{21} & u_{22} & u_{23} & \ldots & u_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & u_{n3} & \ldots & u_{nn} \end{pmatrix} \in \mathbb{R}_\epsilon^{n \times n}.$$

A real value λ is an Eigenvalue of a matrix U, if there exists a vector z, such that:

$$U \otimes z = \lambda \otimes z.$$

The corresponding vector z is called the Eigenvector of U. If $\mathcal{G}$ of a matrix U is strongly connected, then U is irreducible. It is well known that there exists a unique Eigenvalue for an irreducible matrix U. Let λ be an Eigenvalue of a matrix U; then, define the matrix U_λ, as $U_\lambda = -\lambda \otimes U$. Define also:

$$z^*(l+1) = U_\lambda \otimes z^*(l). \tag{2}$$

The following Algorithm 1 computes the Eigenvalue and Eigenvectors of a matrix in max-plus algebra [20]. It works as:

Algorithm 1 Eigenvalue and Eigenvectors in Max-Plus Algebra

1. Determine the Eigenvalue λ as the maximal circuit mean in $\mathcal{G}$.
2. Define the matrix $U_\lambda = -\lambda \otimes U$.
3. Consider an initial vector $z^*(0) \neq \boldsymbol{\epsilon}$.
4. Iterate (2), until we obtain $z^*(m) = z^*(n)$ for positive integer values $m > n \geq 0$.
5. Determine the Eigenvector as

$$v = z^*(n) \oplus \ldots \oplus z^*(m-1).$$

4. Latin Squares in Max-Plus Algebra

In this section, we solve the Eigenproblem for Latin squares. A Latin square is a square matrix of order n with elements from n independent variables over $\mathbb{R}^+$ in such a way that each row and each column is a different permutation of the n variables [28]. In the following, an example of a Latin square of order 5 is given:

$$L = \begin{pmatrix} 5 & 2 & 4 & 3 & 1 \\ 4 & 1 & 3 & 2 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 1 & 3 & 5 & 4 & 2 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}.$$

Here we consider Latin squares of size n in max-plus algebra. In max-plus algebra, we have two kinds of Latin squares: (1) Latin squares L with elements $\underline{n} = \{1, \ldots, n\}$; (2) Latin squares L with entries $\underline{n}_\epsilon = \{1, \ldots, n-1, \epsilon\}$.

Let $L = [l_{ij}]$ be a Latin square of order n; then, we can define a permutaion symbol τ_p for each $p \in \underline{n}$ as, $\tau_p(i) = j$ such that $l_{ij} = p$. In this article, we represent a permutation symbol in a complete factorization by using the cycle notation. Considering the Latin square given above, we have $l_{15} = l_{22} = l_{33} = l_{41} = l_{54} = 1$. Therefore, the permutation symbol τ_1 is given by $\tau_1(1) = 5$, $\tau_1(2) = 2$, $\tau_1(3) = 3$, $\tau_1(4) = 1$, $\tau_1(5) = 4$. Hence, the permutation symbol τ_1 is presented in the cyclic notation as (1 5 4)(2)(3). Similarly, we obtain $\tau_2 =$ (1 2 4 5 3), $\tau_3 =$ (1 4 2 3 5), $\tau_4 =$ (1 3 2)(4)(5), and $\tau_5 =$ (1)(2 5)(3 4).

Let $\tau \in S_n$ be a permutation in complete factorization notation, such that:

$$\tau = c_1 c_2 \ldots c_r,$$

where c_i is a cycle of length less or equal than n for each $i \in \{1, \ldots, r\}$. Let $c_i = (a_1\ a_2 \ldots a_k)$ be a cycle for some $i \in \{1, \ldots, r\}$. Then, we define a vector of length n corresponding to the cycle c_i, such that each entry of this vector corresponding to a_i-th position contains s, while all other entries are equal to t, for $s, t \in \mathbb{R}$. This is denoted by $V^n_{c_i}(s,t)$. In particular, for a cycle $c_i = (a_1\ a_2 \ldots a_k)$, $V^n_{c_i}(1,0)$ is a vector of length n, such that each entry of this vector corresponding to a_i-th position is 1, while all other entries are equal to 0.

Example 1. *Consider a permutation $\tau \in S_5$ given below:*

$$\tau = (1\,3\,4)(2)(5).$$

Here $c_1 = (1\,3\,4)$, $c_2 = (2)$, and $c_3 = (5)$ are three cycles. The vectors $V^5_{c_i}(1,0)$ corresponding to cycles c_i for $i = 1,2,3$ are given by:

$$V^5_{c_1}(1,0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad V^5_{c_2}(1,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad V^5_{c_3}(1,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Similarly, one can obtain the vector $V^5_{c_i}(s,t)$ corresponding to cycles c_i for $i = 1,2,3$. In Lemma 1, we write $V^n_c(s+1,s)$ as a multiple of $V^n_c(1,0)$. We will use this Lemma to prove our main result.

Lemma 1. *Let $\tau \in S_n$ be a permutation and $c = (a_1\ a_2 \ldots a_k)$ be a cycle in τ. Then:*

$$V^n_c(s+1,s) = s \otimes V^n_c(1,0).$$

Proof. Let $V^n_c(1,0) = [v_1 \ldots v_n]^T$; then, $s \otimes V^n_c(1,0) = [v_1 + s \ldots v_n + s]^T$. Since we have 1 at a_i-th position for all $i \in \{1, \ldots, k\}$ and 0 at the remaining positions in $V^n_c(1,0)$, therefore $s \otimes V^n_c(1,0) = V^n_c(s+1,s)$. □

Two vectors x_1, x_2 are linearly dependent if there exists some $l \in \mathbb{R}$ with $x_1 = l \otimes x_2$. If two vectors are not linearly dependent, then they are linearly independent. It is well known that there exists an Eigenvector corresponding to each critical circuit in a digraph $\mathcal{G}(A)$. Therefore, the number of critical circuits in a digraph $\mathcal{G}(A)$ represents the number of linearly-independent Eigenvectors of A.

In [21], the authors showed that for a Latin square L, the maximal entry is the Eigenvalue λ and the number of cycles in the permutation symbol τ_λ represents the number of linearly-independent Eigenvectors of L. Now, we prove the following result to compute the Eigenvectors corresponding to each cycle in the permutation symbol τ_λ.

Theorem 1. *Let $\tau_\lambda \in S_n$ be a permutation symbol of the Eigenvalue λ for a Latin square L of size n and $c = (a_1\ a_2 \ldots a_k)$ be a cycle in τ_λ, then:*

$$v = V_c^n(1,0).$$

is the Eigenvector of L.

Proof. To prove the result, we have to show that $L \otimes V_c^n(1,0) = \lambda \otimes V_c^n(1,0)$. Since $c = (a_1\ a_2 \ldots a_k)$, therefore $\tau_\lambda(a_1) = a_2$, $\tau_\lambda(a_2) = a_3$, …, $\tau_\lambda(a_{k-1}) = a_k$, and $\tau_\lambda(a_k) = a_1$. Therefore, after multiplying a_j-th row with $V_c^n(1,0)$, we obtain $\lambda + 1$ at the a_j-th position for all $j \in \{1, \ldots, k\}$, while by multiplying the remaining rows, we obtain λ at the remaining positions. Hence, by Lemma 1:

$$\begin{aligned} L \otimes V_c^n(1,0) &= V_c^n(\lambda+1, \lambda) \\ &= \lambda \otimes V_c^n(1,0). \end{aligned}$$

which completes the proof. □

If "L" is a Latin square of order n, then there are at most n possible Eigenvectors. Furthermore, $0_n = [0, \ldots, 0]^T$ (n entries of zeros) is the trivial Eigenvector and other vectors are nontrivial Eigenvectors. For a Latin square of order "n", if there is only one cycle of length "n" in the permutation symbol τ_λ, then there exists only trivial Eigenvector and if there are more than one cycle in the permutation symbol τ_λ, then there exist nontrivial Eigenvectors. Using this concept, we propose an algorithm to find the Eigenvalue and Eigenvectors of a Latin square in max-plus algebra. The Algorithm 2 contains the following steps:

Algorithm 2 Eigenvalue and Eigenvectors for Latin Squares in Max-Plus Algebra

1. Determine the Eigenvalue as $\lambda = max(L)$.
2. Determine the permutation symbol τ_λ.
3. For each cycle c in a permutation symbol τ_λ, determine the corresponding vector $V_c^n(1,0)$.
4. Each $V_c^n(1,0)$ is the required Eigenvector.

Consider the following examples to illustrate the Algorithm 2. In these examples, we consider Latin squares with entries in $\underline{n}_\epsilon$.

Example 2. *Consider a Latin square:*

$$L = \begin{bmatrix} 2 & 1 & 4 & 3 & \epsilon \\ 1 & 3 & \epsilon & 4 & 2 \\ 3 & 4 & 2 & \epsilon & 1 \\ \epsilon & 2 & 3 & 1 & 4 \\ 4 & \epsilon & 1 & 2 & 3 \end{bmatrix}.$$

Here $max(L) = 4 = \lambda$. *The permutation symbol for the Eigenvalue* λ *is given by:*

$$\tau_\lambda = (1\,3\,2\,4\,5).$$

We have only one cycle in this permutation symbol, i.e., $c = (1\,3\,2\,4\,5)$. *Therefore, the Eigenvector corresponding to this cycle is given by:*

$$V_c^5(1,0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

which is a trivial Eigenvector.

Example 3. *Now consider a Latin square:*

$$L = \begin{bmatrix} 2 & 1 & 3 & 4 & \epsilon \\ 1 & 3 & \epsilon & 2 & 4 \\ \epsilon & 2 & 4 & 3 & 1 \\ 4 & \epsilon & 2 & 1 & 3 \\ 3 & 4 & 1 & \epsilon & 2 \end{bmatrix}.$$

The Eigenvalue λ *is computed as* $\lambda = max(L) = 4$. *The permutation symbol for* λ *is given by:*

$$\tau_\lambda = (1\,4)\;(2\,5)\;(3).$$

Here, $c_1 = (1\,4)$, $c_2 = (2\,5)$, $c_3 = (3)$. *Therefore, the Eigenvector corresponding to the cycle* c_1 *is given as:*

$$V_{c_1}^5(1,0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

To verify whether $V_{c_1}^5(1,0)$ *is the correct Eigenvector or not, we check:*

$$L \otimes V_{c_1}^5(1,0) = \begin{pmatrix} 5 \\ 4 \\ 4 \\ 5 \\ 4 \end{pmatrix} = \lambda \otimes V_{c_1}^5(1,0),$$

which shows that $V_{c_1}^5(1,0)$ *is the correct Eigenvector. Similarly, the Eigenvectors corresponding to the cycles* c_2 *and* c_3 *are given as:*

$$V_{c_2}^5(1,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad V_{c_3}^5(1,0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

respectively. All these three vectors are non-trivial Eigenvectors.

Remark 1. *The main purpose of this article is to present an alternative algorithm for the computation of Eigenvalues and Eigenvectors of a Latin square in max-plus algebra. Here, we give a computational comparison of Algorithm 1 with Algorithm 2. In the case of a Latin square,* *Algorithm 2 works quite easily when compared with Algorithm 1. This is because Algorithm 2 computes the Eigenvector by using the permutation symbol τ_λ, while in the case of Algorithm 1, it ends up in a periodic behaviour. When using Algorithm 1, one obtains an Eigenvector v as $v = z^*(n) \oplus \ldots \oplus z^*(m-1)$. Therefore, for large values of "m" and "n", its running time is more than Algorithm 2. Hence, the computation of an Eigenvector using the Algorithm 2 is easier than using the Algorithm 1.*

5. Conclusions

The Eigenproblem regarding Latin squares in max-plus algebra is solved in this work. We have defined a vector corresponding to a cycle in a permutation. Trivial and nontrivial Eigenvectors are characterized by considering the vectors corresponding to each cycle in a permutation symbol of the Eigenvalue. In the future, we will discuss the Eigenproblem of Latin squares with conjugate symmetry.

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References

1. Halburd, R.G.; Southall, N.J. Tropical nevanlinna theory and ultra-discrete equations. *Int. Math. Res. Not.* **2009**, *5*, 887–911.
2. Cuninghame-Green, R.A. Lecture notes in economics and mathematical systems. In *Minimax Algebra*; Springer: New York, NY, USA, 1979.
3. De Shutter, B. On the ultimate behavior of the sequence of consecutive powers of a matrix in the max-plus algebra. *Linear Algebra Its Appl.* **2000**, *307*, 103–117. [CrossRef]
4. Gaubert, S. Methods and applications of (max,+) linear algebra. In *Annual Symposium on Theoretical Aspects of Computer Science*; Springer: Berlin/Heidelberg, Germany, 1997; pp. 261–282.
5. Santoso, K.A.; Fatmawati; Suprajitno, H. On max-plus algebra and its application on image steganography. *Sci. World J.* **2018**, *2018*, 6718653. [CrossRef] [PubMed]
6. Cahyono, J.; Adzkiya, D.; Davvaz, B. A cryptographic algorithm using wavelet transforms over max-plus algebra. *J. King Saud-Univ.-Comput. Inf. Sci.* **2022**, *34*, 627–635. [CrossRef]
7. Olsder, G.J.; Roos, C. Cramér and Cayley-Hamilton in the max algebra. *Linear Algebra Its Appl.* **1988**, *101*, 87–108. [CrossRef]
8. De Schutter, B.; De Moor, B. A note on the characteristic equation in the max-plus algebra. *Linear Algebra Its Appl.* **1997**, *261*, 237–250. [CrossRef]
9. Tao, Y.; Wang, C. Global optimization for max-plus linear systems and applications in distributed systems. *Automatica* **2020**, *119*, 109104. [CrossRef]
10. Marotta, A.M.; Gonçalves, V.M.; Maia, C.A. Tropical lexicographic optimization: Synchronizing timed event graphs. *Symmetry* **2020**, *12*, 1597. [CrossRef]
11. Myšková, H.; Plavka, J. Polynomial and pseudopolynomial procedures for solving interval two-sided (max, plus)-linear systems. *Mathematics* **2021** *9*, 2951. [CrossRef]
12. Wang, C.; Xia, Y.; Tao, Y. Ordered structures of polynomials over max-plus algebra. *Symmetry* **2021**, *13*, 1137. [CrossRef]
13. Subiono. On Classes of Min-Max-Plus Systems and Their Application. Ph.D. Thesis, Delft University of Technology, Delft, The Netherlands, 2000.
14. Martnez-Olvera, C.; Mora-Vargas, J. A max-plus algebra approach to study time disturbance propagation within a robustness improvement context. *Math. Probl. Eng.* **2018**, *2018*, 1932361. [CrossRef]
15. Comet, J.-P. Application of max-plus algebra to biological sequence comparisons. *Theor. Comput. Sci.* **2003**, *293*, 189–217. [CrossRef]
16. Gavalec, M.; Plavka, J. Structure of the eigenspace of a Monge matrix in max-plus algebra. *Discret. Appl. Math.* **2008**, *10*, 596–606. [CrossRef]

17. Imaev, A.A.; Judd, R.P. Computing an eigenvector of an inverse Monge matrix in maxplus algebra. *Discret. Appl. Math.* **2010**, *158*, 1701–1707. [CrossRef]
18. Tomaskova, H. Eigenproblem for circulant matrices in max-plus algebra. In Proceedings of the 12th WSEAS international conference on Mathematical Methods, Computational Techniques and Intelligent Systems, Sousse, Tunisia, 3–6 May 2010; World Scientific and Engineering Academy and Society (WSEAS): Sousse, Tunisia, 2010; pp. 158–161.
19. Subiono; van der Woude, J. Power algorithms for (max,+)- and bipartite (min,max,+)-systems. *Discret. Event Dyn. Syst.* **2000**, *10*, 369–389. [CrossRef]
20. Umer, M.; Hayat, U.; Abbas, F. An efficient algorithm for nontrivial eigenvectors in max-plus algebra. *Symmetry* **2019**, *11*, 738. [CrossRef]
21. Mufid, M.S.U. Subiono Eigenvalues and eigenvectors of latin squares in max-plus algebra. *J. Indones. Math. Soc.* **2014**, *20*, 37–45. [CrossRef]
22. Umer, M.; Hayat, U.; Abbas, F.; Agarwal, A.; Kitanov, P. An efficient algorithm for eigenvalue problem of Latin squares in a bipartite min-max-plus system. *Symmetry* **2020**, *12*, 311. [CrossRef]
23. Akian, M.; Gaubert, S.; Nitica, V.; Singer, I. Best approximation in maxplus semimodules. *Linear Algebra Its Appl.* **2011**, *435*, 3261–3296. [CrossRef]
24. Braker, J.G.; Olsder, G.J. The power algorithm in max algebra. *Linear Algebra Its Appl.* **1993**, *182*, 67–89. [CrossRef]
25. Garca-Planas, M.I.; Magret, M.D. Eigenvectors of permutation matrices. *Adv. Pure Math.* **2015**, *5*, 390–394. [CrossRef]
26. Hanniah, U. Subvektor Eigen Bilangan Bulat Dalam Aljabar Maks-Plus. 2020. Available online: https://digilib.uns.ac.id/dokumen/download/80035/NDMxNTQ2/Subvektor-Eigen-Bilangan-Bulat-Dalam-Aljabar-Maks-Plus-abstrak.pdf (accessed on 25 April 2022).
27. Rosyada, S.A.; Kurniawan, S.V.Y. Bases in min-plus algebra. In Proceedings of the International Conference of Mathematics and Mathematics Education (I-CMME 2021), Ankara, Turkey, 16–18 September 2021; Atlantis Press: Amsterdam, The Netherlands, 2021; pp. 313–316.
28. McKay, B.D.; Wanless, I.M. On the number of Latin squares. *Ann. Comb.* **2005**, *9*, 334–344. [CrossRef]