Orthogonal

Theorem 1

Suppose W is a subspace of an inner product space V. Then:

(a) $W^\perp$ *is a subspace of* V.

(b) *Only the zero vector,* $\mathbf{0}$, *is common to both* W *and* $W^\perp$.

(c) $(W^\perp)^\perp = W$. *In other words, the orthogonal complement of* $W^\perp$ *is* W.

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In this case, assume (a) has been proven, thus use (a) to help for proofing (b) and (c).

Proof

Hence W and $W^{\perp}$ is a subspace of V, its obvious that $\mathbf{0}$ is common to both W and $W^{\perp}$.

Assume there is a vector $\mathbf{v} \neq \mathbf{0}$ that is common to both W and $W^{\perp}$. Then $\mathbf{v} \in W$ and $\mathbf{v} \in W^{\perp}$. Since $\mathbf{v} \in W$, then $\langle \mathbf{v}, \mathbf{v} \rangle = 0$. This implies that $\mathbf{v} = \mathbf{0}$, which is a contradiction. Therefore, only the zero vector, $\mathbf{0}$, is common to both W and $W^{\perp}$. $\square$

Proof

Let any $\mathbf{u} \in W$, then $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in W^{\perp}$. Thus by definition, any $\mathbf{u}$ is orthogonal to any $\mathbf{v} \in W^{\perp}$ or can be written as $\mathbf{u} \in (W^{\perp})^{\perp}$.

$$W \subseteq (W^{\perp})^{\perp} \tag{1}$$

Next consider $\mathbf{w} \in (W^{\perp})^{\perp}$ but $\mathbf{w} \notin W$. By definition, exist $\mathbf{v}_0 \in W^{\perp}$ such that is not orthogonal to $\mathbf{w}$. Other words, $\langle \mathbf{v}_0, \mathbf{w} \rangle = 0$ because $\mathbf{w} \in (W^{\perp})^{\perp}$. So its contradict with the assumption that $\mathbf{w} \notin W$. Therefore, $\mathbf{w} \in W$.

$$(W^{\perp})^{\perp} \subseteq W \tag{2}$$

Therefore, $(W^{\perp})^{\perp} = W$. $\square$