Deep Dive 1

Minerva University

CS113: Linear Algebra

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February 8, 2023

Deep Dive 1

Concept Exploration

Let's start concept exploration with an explanation of the equation Ax = b. Let's imagine we have a set of the following linear equations:

$2x + y = 15$

$5x + 3y = 10$

We can take the coefficients of these equations to construct the matrix A

[2, 1]

[5, 3]

Variables will form matrix x

[x]

[y]

So that when we multiply the matrix A by the matrix x we will get the matrix b which represents the equation values:

[15]

[10]

So the set of linear equations presented above can be also written in the following way:

[2, 1] * [x] = [15]

[5, 3]   [y]   [10]

Another important concept worth explaining is an identity matrix. The identity matrix is a square matrix of size n x n, with ones on the main diagonal and zeros elsewhere. Here is an example of a 3x3 identity matrix:

[1, 0, 0]

[0, 1, 0]

[0, 0, 1]

Determinant.

A Determinant is a scalar value that is a function of the entries of a square matrix. To determine if the linear system of equations has a unique solution we will have to find the determinant. If the determinant is not 0 there is a unique solution and we will be able to find it. Referring to the previous concept explained, if the determinant is not zero, matrix b exists. If the determinant is equal to 0 it means that the unique solution does not exist thus the system has either zero or multiple solutions. The zero determinant is the result of a row that consists of zeros in the identity matrix. It causes the determinant to be zero because of the way it is calculated.

Here are the steps we will have to take to calculate the determinant:

Let’s start with the “base case”. The base case here is the algorithm for the 2x2 matrix

To find the determinant of the 2*2 matrix we use the following formula

$$a * d - c * b.$$

The visualization of multiplication is provided below in figure 2. As you can see since the red line is below the blue line we do “red ”multiplication first and only after that, we do “blue ”multiplication.

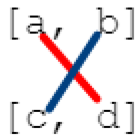

Figure 2: the figure demonstrates the application of the diagonal multiplication to find the determinant of

*the matrix 2*2.*

If we want to find the determinant of a more complex matrix we use the method of cofactors. The basic idea behind this method is to expand the determinant of the matrix along its first row or column, and then use the formula for the determinant of a 2x2 matrix to find the determinant of each minor of the matrix. The cofactor will be each element of any chosen row or column. Although we can choose any row or column it is important to stick to it. It is the most convenient to choose the top row which we will do. The minor of an element is obtained by deleting its corresponding row and column from the matrix. Figure 1 illustrates how to find a minor when we use the method of cofactors. We find the determinant of the minor and multiply it by the element in the first row that does not belong to any of the same columns as the minor (it belongs to the column we “excluded”)

We do it for every element in the row, so we will perform this as many times as the length of the row

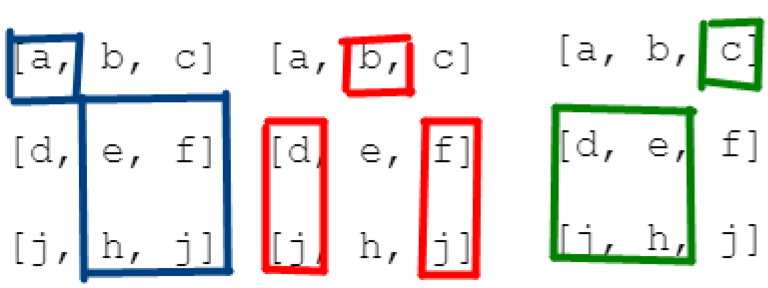

Figure 1: the figure represents the algorithm for finding a determinant for a 3x3 matrix. Color coding is used for better visual perception.

As we found the values for all 3 cases we add them together and get our determinant.

We can generalize the described algorithm to the following formula:

$$det(A) = a * (e * j - f * h) + b * (d * j - j * f) + c * (d * h - e * j)$$

As the size of matrices increases, the same algorithm is applied recursively. Firstly we get to our base case 2x2 matrix and from that, we find the determinant for 3x3, 4x4… 10x10 matrices, and so on.

Invertibility.

The invertibility of the matrix is the property of a matrix to have a unique inverse matrix. It is related to the concept of determinant because only the matrix whose determinant is not zero can have the inverse matrix. So what is an inverse matrix?

As we already know we can reduce a matrix to the Reduced Row Echelon Form using basic row operations. Row operations include swapping rows, adding rows, and multiplying a row by a scalar. All those operations do not change the linear equations that rows represent but help us to get the matrix to RREF. However, there is another type of manipulation to perform to reach the same RREF. We can multiply the matrix by the elementary matrices. An elementary matrix is a square matrix that is obtained by performing a single elementary row operation on an identity matrix.

We can create an elementary matrix that will correspond to each row operation. Table 1 illustrates it with examples.

Row operation	Original Matrix	Multiplication by the elementary matrix	Result
Swap rows	[1,2] [3,4]	[1,2] * [0,1] [3,4] [1,0]	[3, 4] [2, 1]
Multiply by a constant	[1,5] [8,9]	[1,5] * [2,0] [8,9] [0,-1]	[1*2, 5*2] = [2,10] [8*(-1), 9*(-1)] [-8,-9]
Add multiples of one row to another	[3,7] [6,2]	[3,7] * [1, 2] [6,2] [3 1]	[3+2*6,7+2*2] = [15,11] [6+3*3,2+7*3] [15, 23]

Table 1: the table shows how the multiplication by elementary matrices can be used as equivalents to row operations

So we can apply all these multiplications to convert a matrix to its RREF. The thing that interests us most here is the elementary matrices we use. If we multiply all these matrices together in the order we performed row operations we will get an inverse matrix. Basically, it is the matrix that if multiplied by a matrix A results in the identity matrix. The inverse matrix is usually denoted as A^{-1} . Based on this reasoning we can derive the following equation:

$$A * A^{-1} = I$$

(I is an identity matrix)

Also the multiplication of the matrix and its inverse are commutative, so

$$A * A^{-1} = I = A^{-1} * A$$

If the inverse matrix exists we can be sure about 2 things:

- The matrix has a unique solution (thus the determinant is not 0)
- The unique solution can be found by multiplying the inverse matrix by the augmented side of the main matrix. $A^{-1} * b$ where b is the augmented side of the matrix.

Of course finding all elementary matrices and multiplying them is a bit challenging process. Luckily, there is an easier way to determine if the matrix has an inverse.

We can augment matrix A with the identity matrix, and perform row operations to convert the main matrix to RREF and in that case, we will get an inverse matrix on the augmented side.

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Part 1: Minervalia

The internal consumption rates of the first row represent the internal demand for the agricultural resource from each sector. The second row is the demand for energy from each economical sector. If we look at the cell [3,4] (1-based indexing), the value is 0.1. It means that education needs 0.1 unit of real estate to function. Following this logic, let's find the internal needs of the economy for the agricultural sector ($A_{internal}$).

$$A_{internal} = 0.14a + 0.04n + 0.1r + 0.03e + 0t\ .$$

This is how much each internal section in Minervalia consumes agricultural products. We can see that agriculture itself is the largest consumer of its own goods. The total agriculture needed for internal demands can be computed by adding all the elements in column A. If we also add the external demand to it we will see the total amount of the goods needed for the economy to function. External demand can be found in the last column of the given table, it shows the outside demand for each resource. Let’s expand our first example and find the total need for agriculture (X_a). It is basically the same equation but we also ad 51.1 which signified the outside demand.

$$X_a = 0.14a + 0.04n + 0.1r + 0.03e + 0t + 51.1$$

Similarly as we used X_a for the total amount of agriculture, we will use X_n for the total amount of energy, X_r for the total amount of real estate, X_e for the total amount of education and X_t for the total amount of entertainment.

The next step is to produce a series of equations for each resource as we did it for the Agricultural sector.

$X_a = 0.14x_a + 0.4x_n + 0.1x_r + 0.03x_e + 0x_t + 51.1$

$X_n = 0.2x_a + 0.19x_n + 0.18x_r + 0.11x_e + 0.16x_t + 141.1$

$X_r = 0.15x_a + 0.15x_n + 0.2x_r + 0.1x_e + 0.07x_t + 0$

$X_e = 0.04x_a + 0.16x_n + 0.04x_r + 0.12x_e + 0.14x_t + 131.1$

$X_t = 0x_a + 0.01x_n + 0.03x_r + 0.2x_e + 0.05x_t + 82.3$

Now we need to have all the variables on the left and numbers on the right

$$- X_a + 0.14x_a + 0.04x_n + 0.1x_r + 0.03x_e + 0x_t = - 51.1$$

$$- X_n + 0.2x_a + 0.19x_n + 0.18x_r + 0.11x_e + 0.16x_t = - 141.2$$

$$- X_r + 0.15x_a + 0.15x_n + 0.2x_r + 0.1x_e + 0.07x_t = 0$$

$$- X_e + 0.04x_a + 0.16x_n + 0.04x_r + 0.12x_e + 0.14x_t = - 131.6$$

$$- X_t + 0x_a + 0.01x_n + 0.03x_r + 0.2x_e + 0.05x_t = - 82.3$$

Since X_a represents the total we can substitute it with a 100 and subtract from 100 the value of x_a. We repeat this for each equation and get the following set of equations:

$$- 0.86x_a + 0.04x_n + 0.1x_r + 0.03x_e + 0x_t = - 51.1$$

$$0.2x_a - 0.81x_n + 0.18x_r + 0.11x_e + 0.16x_t = - 141.2$$

$$0.15x_a + 0.15x_n - 0.8x_r + 0.1x_e + 0.07x_t = 0$$

$$0.04x_a + 0.16x_n + 0.04x_r - 0.88x_e + 0.14x_t = -131.6$$

$$0x_a + 0.01x_n + 0.03x_r + 0.2x_e - 0.95x_t = -82.3$$

Part 2: Matrix:

If I rewrite my system in the form Mx = d I will get the following**:**

$$\begin{bmatrix} -0.86, & 0.04, & 0.1, & 0.03, & 0 \\ 0.2, & -0.81, & 0.18, & 0.11, & 0.16 \\ 0.15, & 0.15, & -0.8, & 0.1, & 0.07 \\ 0.04, & 0.16, & 0.04, & -0.88, & 0.14 \\ 0, & 0.01, & 0.03, & 0.2, & -0.95 \end{bmatrix} \begin{bmatrix} x_a \\ x_n \\ x_r \\ x_e \\ x_t \end{bmatrix} = \begin{bmatrix} -51.1 \\ -141.2 \\ 0 \\ -131.6 \\ -82.3 \end{bmatrix}$$

$\overline{d}$ represents the total number of units needed to satisfy external demand

$\overline{x}$ represents the total number of goods needed to satisfy both internal and external demand.

The set of linear equations defined in question 1 can be represented by a matrix M demonstrated in figure 1

```
[-0.86,  0.04,   0.1,  0.03,  0    | -51.1 ]
[0.2,   -0.81,  0.18,  0.11,  0.16 | -141.2]
[0.15,   0.15,  -0.8,   0.1,  0.07 | 0     ]
[0.04,   0.16,  0.04, -0.88,  0.14 |-131.6 ]
[0,      0.01,  0.03,   0.2, -0.95 |-82.3  ]
```

Figure 1: the figure demonstrates the matrix M.

Part 3: Solutions

Does the system have a feasible solution? In other words, is there a set of production levels that would satisfy both internal and external demand? Is this solution unique?

Way 1: Determinant

As it was described in detail in the concept exploration, a determinant should be a non-zero value for the matrix to have a solution. If there is no solution or there are multiple solutions the determinant will be zero.

Way 2: Reduced Row Echelon Form (RREF)

Another way to determine if the system has a unique solution is to perform the row operations to reduce the matrix to Reduced Row Echelon Form (RREF). Matrixes in RREF form satisfy the following requirements: In each row, the leftmost nonzero entry must be 1, while the column that contains this entry must contain zero entries (*Reduced Row-Echelon Form*, n.d.). If RREF will have an identity matrix on the left-hand side we have a unique solution. An example of an identity matrix is demonstrated in figure 4.

$$\begin{bmatrix} 1, & 0, & 0, & 0, & 0 & | a \\ 0, & 1, & 0, & 0, & 0 & | b \\ 0, & 0, & 1, & 0, & 0 & | c \\ 0, & 0, & 0, & 1, & 0 & | d \\ 0, & 0, & 0, & 0, & 1 & | h \end{bmatrix}$$

Figure 4: the figure represents the identity matrix of a size 5x5 on the non-augmented side. The augmented side consists of a vector [a,b,c,d, h] where each letter represents a number.

In an alternative case when we will not get an identity matrix in RREF we will get a row that consists of zeros. It means that there are either zero solutions to the system or there are infinite solutions to the system. Let's get deeper into both cases.

Zero solutions:

An example of a case when there are no solutions is illustrated in figure 5. Take a closer look at the last row. Basically, we have an equation $0 * x_t = h$ where h is any non-zero value. It is an obvious contradiction because everything that is multiplied by zero will result in zero and it is impossible for this equation to be true. Because of this contradiction, we conclude that there are no solutions to the system.

$$\left[\begin{array}{ccccc|c} 1, & 0, & 0, & 0, & 0 & a \\ 0, & 1, & 0, & 0, & 0 & b \\ 0, & 0, & 1, & 0, & 0 & c \\ 0, & 0, & 0, & 1, & 0 & d \\ 0, & 0, & 0, & 0, & 0 & h \end{array}\right]$$

Figure 5: the figure represents a matrix with no solutions in RREF. a,b,c,d, h represent non-zero numbers

Infinite solutions:

An example of a case when there is an infinite number of solutions is illustrated in figure 6. Take a closer look at the last row. Basically, we have an equation $0 * x_t = 0$. In this equation x_t can have any value because anything multiplied by zero will result in zero.

$$\left[\begin{array}{ccccc|c} 1, & 0, & 0, & 0, & 0 & a \\ 0, & 1, & 0, & 0, & 0 & b \\ 0, & 0, & 1, & 0, & 0 & c \\ 0, & 0, & 0, & 1, & 0 & d \\ 0, & 0, & 0, & 0, & 0 & 0 \end{array}\right]$$

Figure 6: the figure represents a matrix with an infinite number of solutions in RREF. a,b,c,d, h represent non-zero numbers

I used Sage Math to find determinants ad reduce the matrix M in the RREF.

As you can see in figure 7, the determinant is a non-zero value. This can also be confirmed by the pattern we have on RREF.

As we can see, REEF is an identity matrix, so we can find the values of each variable.

```
-1929708981/5000000000

[                 1                 0                 0                 0                 0|180770215990/1929708981]
[                 0                 1                 0                 0                 0|544228566350/1929708981]
[                 0                 0                 1                 0                 0|   72029685980/643236327]
[                 0                 0                 0                 1                 0|449206945810/1929708981]
[                 0                 0                 0                 0                 1|274296206000/1929708981]
```

Figure 7: the first output row in the figure is the determinant of the matrix M. The next row is teh matrix M in RREF form. The figure was produced using SageMath and the code used can be found in Appendix A.

The solutions are

$x_a = 93.68$

$x_n = 282$

$x_r = 111.98$

$x_e = 232.78$

$x_t = 142.14$

It means that the total needed amount of agriculture is 93.68 units, energy 282 units, real estate 111,98 units, electricity 232.78 units, entertainment 142.14 units.

Part 4: Adjusted Parameters

i) **infinitely many solutions:**

We will reach indefinitely many solutions if one of the vectors will be a zero vector and the corresponding value on the augmented side will be zero as well. It will result in a zero row and in a matrix with a zero row the determinant cannot have a different value rather than zero because we will always face the multiplication by zero. In practice, it will mean that since we have 1 non-pivot column, the solutions lie on a line defined by the free variable from the last column. As we can see in figure 8 the determinant is still zero which means that there is either no solutions or indefinitely many of them. That is why it is important not only to find the determinant but also RREF to see what exact case we have. By looking at the last row we can see that there are indefinitely many solutions. It consists of zeros on both sides which leads to the equation

$x_t * 0 = 0$

So x_t can be any number since any number multiplied by zero will result in zero. That is why we have infinitely many solutions here.

```
determinant =  0
[      0    1/25    1/10   3/100       0|-511/10]
[      0 -81/100    9/50  11/100    4/25| -706/5]
[      0    3/20    -4/5    1/10   7/100|      0]
[      0    4/25    1/25  -22/25    7/50| -658/5]
[      0       0       0       0       0|      0]

[              0              1              0              0              0|-154360290/437719]
[              0              0              1              0              0|-126367045/437719]
[              0              0              0              1              0|-118544160/437719]
[              0              0              0              0              1|-944073950/437719]
[              0              0              0              0              0|               0]
```

Figure 8: the first output row in the figure is the determinant of the matrix M. The next row is the matrix

M in RREF form with the augmented vector changed to a zero vector. The figure was produced using SageMath and the code used can be found in Appendix A.

This can also be explained by the fact that if the row fully consists of zeros we can disregard it. In that case, we will have a matrix that has 5 variables and only 4 rows. Since the number of rows is smaller than the number of variables there cannot be one solution.

ii) no solutions.

There will be no solutions if one of the columns will be equal to zero. You can see an example of it in figure 9. In RREF we can observe the pattern in the last row where X*0 = 1. This is the reason why there are no solutions. This equation is impossible since anything multiplied by zero will result in zero. It means that the lines defined by the linear equations the matrix consists of are parallel, so they will never intersect

```
determinant =  0
[      0    1/25    1/10   3/100       0|-511/10]
[      0 -81/100    9/50  11/100    4/25| -706/5]
[      0    3/20    -4/5    1/10   7/100|      0]
[      0    4/25    1/25  -22/25    7/50| -658/5]
[      0   1/100   3/100     1/5  -19/20|-823/10]

[0 1 0 0 0|0]
[0 0 1 0 0|0]
[0 0 0 1 0|0]
[0 0 0 0 1|0]
[0 0 0 0 0|1]
```

Figure 9: the first output row in the figure is the determinant of the matrix M. The first column consists of zeros. The figure was produced using SageMath and the code used can be found in Appendix A.

iii)unique solution:

we already have a unique solution in our original matrix. The matrix in RREF corresponds to the identity matrix so the number or picot column is the same as the number of variables which allows us to find the value for each variable.

$x_a = 93.68, x_n = 282, x_r = 111.98, x_e = 232.78, x_t = 142.14$

Reflection

1. Give two examples of applying HCs to solve a problem. How did you apply the HC? Was the application successful? If yes, why? If not, how could you improve your application?

#audience: I successfully applied this HC by tailoring my communication style to meet the needs of my audience: a lost student that takes cs113. I explained my thought process in detail. I explained all key concepts before I use them. I added visualization of the process of determinant calculation to help a student to build a better intuition of the concept. I also did not use mathematical jargon and tried to make my language as simple as possible. I added a lot of figures to help a student to understand the content.

#algorithms: I successfully applied this HC by providing an algorithm to determine if the matrix has a unique solution. I described 2 ways and used algorithmic thinning to explain how the determinant is calculated. I gave an example of the recursive algorithm, defined the base case, and explained how it will work so that a student would be able to calculate the determinant of a square matrix of any size by themselves if they really want it and have time for it.

2. What mathematical ideas are you curious to know more about as a result of the material from this unit?

I am really interested in machine learning and I know that matrices are the basis of this. I really wanna know how exactly they are connected and how we can build simple machine learning models using matrices.

Appendix A

Code for figure 7: find RREF of the initial matrix

```
v1 = vector(QQ,[-0.86,  0.04,   0.1,  0.03,  0 ])
v2 = vector(QQ, [0.2,   -0.81,  0.18,  0.11,  0.16 ])
v3 = vector(QQ, [0.15,   0.15,  -0.8,   0.1,  0.07])
v4 = vector(QQ, [0.04,   0.16,  0.04, -0.88,  0.14])
v5 = vector(QQ, [0,      0.01,  0.03,   0.2, -0.95])
b = vector(QQ, [-51.1, -141.2,0, - 131.6, -82.3])
A = matrix([v1,v2,v3,v4,v5])
print(A.determinant())
A = A.augment(b,subdivide = True)
A.rref()
```

Code to find the values for the variables:

```
In [17]
a = 180770215990/1929708981
n = 544228566350/1929708981
r = 72029685980/643236327
e = 449206945810/1929708981
t = 274296206000/1929708981

print(float(a),float(n),float(r),float(e),float(t))
```

Run Code   Reset to Original Submission

```
Out [17]    93.67744969312551 282.0262390383724 111.98012760868215 232.7848137894944 142.1438199753085
```

Code for figure 8: RREF for the matrix with infinite solutions: the highlighted part signifies the part of the original system that was changed. In this case, one of the vectors was changed to a zero vector. Since we have a zero row the determinant cannot have a different value rather than zero

```
v1 = vector(QQ,[0,  0.04,   0.1,  0.03,  0 ])
v2 = vector(QQ, [0,   -0.81,  0.18,  0.11,  0.16 ])
v3 = vector(QQ, [0,   0.15,  -0.8,   0.1,  0.07])
v4 = vector(QQ, [0,   0.16,  0.04, -0.88,  0.14])
v5 = vector(QQ, [0,      0,  0,    0, 0])
b = vector(QQ, [-51.1, -141.2,0, - 131.6, 0])
A = matrix([v1,v2,v3,v4,v5])
print(  'determinant = ', A.determinant())
A = A.augment(b,subdivide = True)
print(A)
A.rref()
```

Code for figure 9: RREF for zero solution case. the highlighted part signifies the part of the original system that was changed. In this case the first column was changed to a zero vector.

```
v1 = vector(QQ,[0,  0.04,   0.1,  0.03,  0 ])
v2 = vector(QQ, [0,   -0.81,  0.18,  0.11,  0.16 ])
v3 = vector(QQ, [0,   0.15,  -0.8,   0.1,  0.07])
v4 = vector(QQ, [0,   0.16,  0.04, -0.88,  0.14])
v5 = vector(QQ, [0,      0.01,  0.03,   0.2, -0.95])
b = vector(QQ, [-51.1, -141.2,0, - 131.6, -82.3])
A = matrix([v1,v2,v3,v4,v5])
print(  'determinant = ', A.determinant())
A = A.augment(b,subdivide = True)
print(A)
A.rref()
```

References:

Reduced Row-Echelon Form. (n.d.).
https://people.math.carleton.ca/%7Ekcheung/math/notes/MATH1107/wk04/04_reduced_row-echelon_form.html