



# Introduction to Algorithms

Part I. Recursion, Dynamic Programming

B9 - Introduction to Algorithms

M-ALG-100

# Overview of the module

Day 1 Concept of Algorithm, Cryptography, recursion, Knapsack, Shortest Path

Day 2 Complexity, Graph problems, Theory

# Organisation of the module

- ▶ Theoretical course
- ▶ Small coding exercises,
- ▶ Paper + pen exercises
- ▶ Project : explained tomorrow

# Organisation

- ▶ The exercices will be in python
- ▶ Please clone the following repository :  
`https://github.com/nlehir/ALG01.git`
- ▶ Third party libs : **matplotlib**, **numpy**, **pygraphviz**, **graphviz**.
- ▶ Optional but useful : **ipdb** (python debug) or another debugger

## Objective of the course

- ▶ This course is more about the mathematical nature of algorithms.
- ▶ It is not about optimizing the code itself, but about the mathematical reason why some methods are faster than others at solving problems.

# Day 1

What is an algorithm ?

## Cryptography

First examples

Public-key cryptography and symmetric key algorithm

RSA

## Recursion

Principle and examples

Shortcomings

## Two famous problems

The Knapsack problem

The Shortest Path problem

# What is an algorithm ?

- ▶ How could we define it ?

# What is an algorithm ?

- ▶ **Proposed definition** "A method to solve a problem based on a sequence of elementary operations, arranged in a determined order"



## Simple example

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- ▶ Please propose the simplest possible algorithm to complete this task (without worrying about the formalization yet)

## Simple example : solution

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- ▶ Could you think of a better (faster) solution ?

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- ▶ Could you think of a better (faster) solution ?
- ▶ We could split the folders stack in two parts, then check the first folder of the lower part of the stack (**dichotomic search**)

## Simple example : speed

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- ▶ At each dichotomic cut, how many folders can we remove from the stack ?

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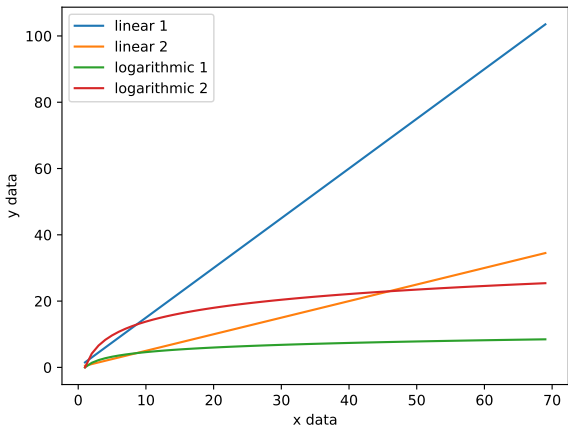
- ▶ Why is the **dichotomic seach** faster than the **linear search** ?
- ▶ At each dichotomic cut, how many folders can we remove from the stack ?
- ▶ Is  $S$  is the size of the stack, the new size after the cut is (roughly)  $\frac{S}{2}$ . Hence, around how many checks are needed, if  $n$  is the **initial number of folders** ?



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- ▶ We need at most  $\log_2 n$  checks (backboard)

## Logarithms and linear functions



## Simple example : comparison

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## Simple example : comparison

- ▶ How many checks does the **linear search need** at most ? ( $n$  is still the initial number of folders)
- ▶ It needs  $n$  checks.
- ▶ So we have to algorithms that perform the same task, but one of them is faster (  $\log n$  versus  $n$  )

## Simple search example

### Exercise 1: **Comparison between linear and dichotomic search**

If the stack contains 10 folders, how faster is the dichotomic search compared to the linear search ?

## Simple search example

### Exercise 1: **Comparison between linear and dichotomic search**

And the stack contains 50 folders ?

## Simple search example

**Exercice 1 : Comparison between linear and dichotomic search**

What if the stack contains 10000 folders ?

## Simple example : comparison

- ▶ We say that they have a different **complexity**, which will be the topic of the course



## Simple example : comparison

- ▶ We say that they have a different **complexity**, which will be the topic of the course
- ▶ The complexity is an **approximation** : we are interested in **orders of magnitude**

## Time Complexity

- ▶ We will study the complexity of algorithms.
- ▶ More specifically we will focus on the **time complexity** of algorithms. It is an order of magnitude of the number elementary operations required to solve a problem, given a method (ie: given an algorithm)

## Time Complexity

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  - ▶  $10 \times 2^n$  is of the same order of magnitude as  $500 \times 2^n$
  - ▶ but  $3^n$  is **NOT** the same order of magnitude as  $2^n$



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- ▶ It is a measure of the memory used by the algorithm in order to run.
- ▶ We will also discuss it tomorrow.

## Formalization

- ▶ Let us define how we should **specify** an algorithm. It needs :
  - ▶ inputs
  - ▶ outputs
  - ▶ preconditions
  - ▶ postconditions

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  - ▶ inputs : stack of folder
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  - ▶ preconditions : the folders stack is sorted in the alphabetical order
  - ▶ postconditions :



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- ▶ In the case of our folders checking algorithm, this means :
  - ▶ inputs : stack of folder
  - ▶ outputs : boolean ( **True** if the stack contains the given name  $X$ , **False** otherwise)
  - ▶ preconditions : the folders stack is sorted in the alphabetical order
  - ▶ postconditions : the output is **True** if and only if the folders stack contains a folder whose name is  $X$ .

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- ▶ its correctness
- ▶ the termination : the algorithm should end after a **finite number** of computation steps

# Cryptography

- ▶ We will study some cryptography algorithm as they will provide us examples that show why the complexity of an algorithm is a important aspect of it.
- ▶ Please note that this section is not a cryptography course, but rather a course to focus on some mathematical aspects of the involved algorithms.

## First example

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$$A \mapsto F, \quad B \mapsto P,$$

$$C \mapsto A, \quad D \mapsto \dots$$



Figure: Example permutation

# Ciphering

## Exercise 2 : First ciphering example

- ▶ **cd ./crypto\_intro**
- ▶ Please modify the file **crypto\_intro/cipher\_1.py** so that the function *cipher\_1(s)* produces a random key and ciphers the text *s*, which is a string.
- ▶ "cipher" means "chiffre" in french



# Breaking the code

## Exercise 2 : First ciphering example part II

- ▶ Please modify the file **crypto\_intro/decipher\_1.py** in order to attempt to find the key from a **coded message** and an **extract**.

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### Exercise 2 : First ciphering example part II

- ▶ Please modify the file **crypto\_intro/decipher\_1.py** in order to attempt to find the key from a **coded message** and an **extract**.
- ▶ Is it working ?

# Breaking the code

## Exercise 2 : First ciphering example part II

- ▶ Please modify the file **crypto\_intro/decipher\_1.py** in order to attempt to find the key from a **coded message** and an **extract**
- ▶ Is it working ?
- ▶ Why is it taking such a long time ?

## Number of permutations

- ▶ How many keys are possible ?

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- ▶ How many keys are possible ?
- ▶ Let us count them.

## Number of permutations

- ▶ How many keys are possible ?
- ▶  $26! = 403291461126605635584000000$
- ▶ It is the number of **permutations**.

## Necessary time

**Exercise 3 :** Please evaluate the time that would be necessary on your machine to evaluate all possible keys.

## Necessary time

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- ▶ So I need 0.036 millisecond to try 1 key.
- ▶ Which means  $\simeq 1.45 \times 10^{25}$  seconds for  $26!$  permutations.
- ▶ Or  $\simeq 4.6 \times 10^{17}$  years.

## First example

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It is vulnerable to **statistical attacks**.

## Second example

- ▶ Let us do another example

C H A Q U E F O I S Q U U N H O M M E  
 B V A B V A B V A B V A B V A B  
 E D B S G F ... N G

Figure: Second coding method

## Second example

**Exercise 4:** Please modify the file **crypto\_intro/cipher\_2.py** so that the function *cipher\_2(s)* ciphers the text in the same way

## Second example : Breaking the code

- ▶ Please modify the file **crypto\_intro/decipher\_2.py** in order to attempt to find the key from a **coded message** and an **extract**



## Second example

- ▶ Please modify the file **crypto\_intro/decipher\_2.py** so that the function *cipher\_2(s)* ciphers the text in the same way
- ▶ Use a sentence with 50 characters. For key sizes does the algorithm break the code ?

## Second example

- ▶ Please modify the file **crypto\_intro/decipher\_2.py** so that the function *cipher\_2(s)* ciphers the text in the same way
- ▶ Use a sentence with 100 characters. For which values does the algorithm break the code ?
- ▶ What is the number of keys that are to be tried, as a function of the size of the key ?

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- ▶ What is the number of keys that are to be tried, as a function of the size of the key ?  $26^{\text{key size}}$
- ▶ So probably you won't be able to break the code for  $k \geq 7$  or so.

# The problem of complexity

- Complexity is key for security problems. However, it is important in most other fields as well. It is an issue you have to understand and master for your algorithm to be efficient - or to work at all.

## Private and public keys

- ▶ Before diving into complexity we will study a more complex cryptosystem
- ▶ It will allow us to study a more complex algorithm

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- ▶ As opposed to symmetric key algorithms



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- ▶ This is called a symmetric key algorithm
- ▶ However would there be an advantage of using **two** keys ?

## Public keys and private keys

- ▶ **Public key** : used to cipher a text
- ▶ **Private key** : used to decipher a text

## Public keys and private keys

- ▶ **Public key** : used to cipher a text
- ▶ **Private key** : used to decipher a text
- ▶ There is no need to transmit the private key on the network.
- ▶ Whereas in a symmetric context, one needs a secure canal to transmit the key.

## Asymmetric cryptosystem

How many keys do we need to generate for each case to enable  $n$  persons to communicate ?

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- ▶ Symmetric : each subset of 2 persons must have 1 key.
- ▶ Asymmetric : each person must have 1 public key and 1 private key.

## Asymmetric cryptosystem

How many keys do we need to generate for each case ?

- ▶ Symmetric : each subset of 2 persons must have 1 key :  
$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}.$$
- ▶ Asymmetric : each person must have 1 public key and 1 private key :  $2n$ .



## Examples :

- ▶ Symmetric : AES
- ▶ Asymmetric : RSA, ssh, sftp

# RSA

- ▶ RSA is based on **modular exponentiation**.
- ▶ Let  $M$  be a message to cipher. We assume  $M$  is an integer.  
Let  $C$  be the code (also an integer)

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- ▶ Let  $M$  be a message to cipher. We assume  $M$  is an integer. Let  $C$  be the code (also an integer)
- ▶ We work **modulo an integer**  $n$  (hence the name **modular** exponentiation)
- ▶ e.g.  $17 \equiv 1 \pmod{4}$
- ▶  $25 \equiv 0 \pmod{5}$

# RSA

- ▶ RSA is based on **modular exponentiation**.
- ▶  $M$  : message to cipher.  $C$  : code.
- ▶ **Public key** :  $(n, a)$
- ▶ **Private key** :  $b$  ( $a$  and  $b$  must be carefully chosen)
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# RSA

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- ▶  $D \equiv C^b \pmod n$
- ▶ In order for the algorithm to work, we must have  $D \equiv M \pmod n$ .
- ▶ Which means :  $M^{ab} \equiv M \pmod n$

# RSA

- ▶  $M$  : message to cipher.  $C$  : code.
- ▶ Public key :  $(n, a)$ , Private key :  $b$
- ▶  $M^{ab} \equiv M \pmod n$
- ▶ The construction of  $n$ ,  $a$ , and  $b$  comes from **number theory** (Fermat theorem, Gauss theorem)



## RSA : construction of the keys

- ▶ Choose  $p$  and  $q$  prime numbers
- ▶  $n = pq$
- ▶  $\phi = (p - 1)(q - 1)$
- ▶ Choose  $a$  coprime with  $\phi$  (entiers premiers entre eux)
- ▶ Choose  $b$  inverse of  $a$  modulo  $\phi$ , which means

$$ab \equiv 1 \pmod{\phi} \quad (1)$$

## Setting up a RSA system

### Exercise 5 : Building RSA I : choosing keys

- ▶ **cd** to the **./rsa** directory
- ▶ Please modify **rsa\_functions.py** so that when calling **generate\_rsa\_keys()** from **cipher\_rsa.py**, a public key and a private key are created.
- ▶ You can change the prime numbers used.

## Setting up a RSA system

### Exercise 5 : Building RSA II : ciphering the text

- ▶ Please modify **rsa\_functions.py** so that when calling **cipher\_rsa()** from **cipher\_rsa.py**, a public key and a private key are created and the text stored in `texts` is coded and stored in `./cryptoed_messages`.

## Setting up a RSA system

**Exercise 5 :** Building RSA III : deciphering the text

- ▶ Please modify **rsa\_functions.py** so that when calling **decipher\_rsa()** from **decipher\_known\_rsa.py**, the generated public key private key are used to decipher the crypted text.

# Breaking RSA

## Exercise 5 : Trying to break RSA

- ▶ Modify **rsa\_functions.py** so that when calling **find\_private\_key()** from **decipher\_unknown\_rsa.py** the **secret private key is found** from the public key and used to decipher the crypted message.

## Conclusion on RSA

It is extremely hard to break RSA if  $n$  is sufficiently large, because you need to find the decomposition of  $n$  in **prime numbers**. This is another important example of a algorithmic that is too **complex** to be solved.

# Classic algorithmic methods

- ▶ We will study classical programming paradigms
- ▶ Recursivity
- ▶ Dynamic programming

# Recursion

- ▶ How would you define recursion ?



# Recursion

- ▶ How would you define recursion ?
- ▶ **Proposed definition** : a method to solve a problem based on smaller instances of the same problem.

## First Recursion example

### Exercise 6 : Facorial recursion

- ▶ **cd recursion**
- ▶ Please modify **factorial\_rec.py** so that it computes the factorial
- ▶  $n! = 1 \times 2 \times \dots \times n$

# Recursion

A recursive function always has :

- ▶ a base case
- ▶ a recursive case

## Warning

- ▶ Decrease does not mean terminate !
- ▶ What happens with the example **bad\_recursion.py** ?
- ▶ In python, you can see the recursion limit with **sys.getrecursionlimit()**

## Second example : exponentiation

- ▶ We will study the case of **exponentiation** (that we used in RSA)
- ▶ Given an integer  $a$ , and another integer  $n$ , we want to compute  $a^n$ .
- ▶ If we had to code it ourselves, we would naively do a method similar to **normal\_exponentiation.py**

## Fast exponentiation

- ▶ There is a faster method that uses recursion : **fast exponentiation** (backboard)

## Fast exponentiation

**Exercise 7 :** Using recursion to perform fast exponentiation

- ▶ Modify **fast\_exponentiation.py** so that it performs the fast exponentiation algorithm.

## Fast vs normal exponentiation

- ▶ Compute  $5^{300000}$  with normal exponentiation and fast exponentiation : which one is faster ?



## Fast vs normal exponentiation

- ▶ Compute  $5^{300000}$  with normal exponentiation and fast exponentiation : which one is faster ?
- ▶ Why is fast exponentiation faster ?

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## Fast vs normal exponentiation

- ▶ Let us compute the number of operations performed in fast exponentiation.
- ▶ Say we compute  $a^n$ .
- ▶ We call the function  $d$  times, where  $2^d = n$
- ▶ This means that  $d = \log_2(n)$ .
- ▶ We say that fast exponentiation has a **logarithmic complexity**, and we denote it  $\mathcal{O}(\log n)$

## Remark on fast exponentiation

- ▶ The fact that fast exponentiation is logarithmic is not related to recursivity.

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- ▶ The fact that fast exponentiation is logarithmic is not related to recursivity.
- ▶ If we write the binary decomposition of  $n$  :

$$n = \sum_{k=0}^d \alpha_k 2^k \quad (2)$$

- ▶ Then :

$$a^n = (a)^{\alpha_0} (a^2)^{\alpha_1} (a^{2^2})^{\alpha_2} \dots (a^{2^d})^{\alpha_d} \quad (3)$$

## Shortcomings of recursion

- ▶ Recursion can be an elegant way to write algorithms but when not made carefully, the memory usage can explode.
- ▶ Let's compute for instance the 100e term of the Fibonacci sequence.

$$f_{n+2} = f_{n+1} + f_n \quad (4)$$

## Non optimized Fibonacci

### Exercise 8: Memory and Fibonacci

- ▶ What happens with the function **bad\_fibonacci.py** ?



# Non optimized Fibonacci

## Exercise 8: Memory and Fibonacci

- ▶ What happens with the function **bad\_fibonacci.py** ?
- ▶ Let's look at an example.

## Fibonacci and memoization

### Exercise 9 : Optimizing Fibonacci

- ▶ Modify **memoized\_fibonacci.py** so that it uses memoization to compute the sequence without uselessly computing several times the same terms.

## Remark

- ▶ In python, you can also use a **generator** in order to perform this kind of task.
- ▶ See **smarter\_fibonacci.py**

# The Knapsack problem

- ▶ We will apply the concept of recursion to a classical problem :  
**The Knapsack problem**

## The general Knapsack problem

- ▶ We will apply the concept of recursion to a classical problem :  
**The Knapsack problem**
- ▶ We have a bag of maximal capacity. It can not contain more than a certain weight, say  $W$ .
- ▶ We have several **objects**  $i$  each with a certain **weight**  $w_i$  and **value**  $v_i$ .

## The general Knapsack problem

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**The Knapsack problem**
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- ▶ We have several **objects**  $i$  each with a certain **weight**  $w_i$  and **value**  $v_i$ .
- ▶ We want to load the maximum possible value in the bag (which means respecting the weight constrain)

## The Knapsack problem : restricted variant

- ▶ We will focus on a **restricted variant**. The **value equals the size**.
- ▶ Each object  $i$  has a value  $v_i$ .
- ▶ The question is : "is it possible to fill the bag with a value exactly  $V$  ?"

## The Knapsack problem : restricted variant

- ▶ We will first focus on a **restricted variant**. The **value equals the weight**  $w_i = v_i$ .
- ▶ Each object  $i$  has a value  $v_i$ .
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## The Knapsack problem : restricted variant

- ▶ The question is : "is it possible to fill the bag with a value exactly  $V$  ?"
- ▶ **example** : values = [1, 8, 3, 7]
  - ▶ Can we fill the bag with the value 16 ?
  - ▶ Can we fill the bag with the value 10 ?
  - ▶ Can we fill the bag with the value 5 ?

## The Knapsack problem : restricted variant

### Exercice 10 : Reformulation of the problem

- ▶ Each object  $i$  has a value  $v_i$ . We have  $n$  objects.
- ▶ "is it possible to fill the bag with a value exactly  $V$  ?"
- ▶ We have a list of values

$$L = [v_1, \dots, v_n] \quad (5)$$

- ▶ Please try to reformulate the problem in terms of **sublists** of  $L$ .

## Solving the problem

### Exercise 11: A recursive solution

- ▶ Modify **knapsack\_recursive.py** so that it searches for a sublist of total value  $V$  **in a recursive way**.

## Breaking down an instance of the problem

- ▶ Let us analyze of this solution works on an example.

## Optimization and decision

- ▶ We say that our solution solves a **decision problem**. The answer provided is "yes" or "no".
- ▶ Given a constraint, how could we transpose our solution to an **optimization problem** ? Which means optimizing the total value put inside de bag.

## Optimization and decision

- ▶ We say that our solution solves a **decision problem**. The answer provided is "yes" or "no".
- ▶ Given a constraint, how could we transpose our solution to an **optimization problem** ? Which means optimizing the total value put inside de bag.
- ▶ We could search for the maximum  $V$  such that there exists a sublist of total value  $V$ .

- ...
- └ Two famous problems
  - └ The Knapsack problem

## Back to the knapsack : exhaustive search

We could also write the program in a non recursive way, by exploiting the correspondence with binary numbers : how ?

## Back to the knapsack : exhaustive search

We could also write the program in a non recursive way, by exploiting the correspondence with binary numbers : how ?

If  $x_i$  is a boolean coding the fact that object  $i$  is selected, the value of the selected sublist is :

$$\sum_{i=1}^n x_i v_i \quad (6)$$



- ...
- └ Two famous problems
  - └ The Knapsack problem

## Exhaustive search

$$\sum_{i=1}^n x_i v_i \tag{7}$$

How many vectors  $(x_1, \dots, x_n)$  are possible ?

- ...
- └ Two famous problems
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## Exhaustive search

$$\sum_{i=1}^n x_i v_i \quad (8)$$

How many vectors  $(x_1, \dots, x_n)$  are possible ?  $2^n$

This is called **exponential complexity**

- ...
- └ Two famous problems
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## Exhaustive search

$$\sum_{i=1}^n x_i v_i \tag{9}$$

How many vectors  $(x_1, \dots, x_n)$  are possible ?  $2^n$

This is called **exponential complexity**. If  $n$  is large, can we use this solution ?

## A heuristic for the general Knapsack problem

- ▶ A very important notion regarding algorithms is that of a **heuristic**.

## A heuristic for the general Knapsack problem

- ▶ A very important notion regarding algorithms is that of a **heuristic**.
- ▶ A heuristic is an **approximate solution** that is **possible, or easier to compute**.
- ▶ It is sometimes necessary when it is too hard to find the optimal solution, which, as we will see, happens in some real world situations (such as the Knapsack problem).

## Heuristic for the Knapsack problem

### Exercise 12 : Finding a heuristic

- ▶ Each object  $i$  has value  $v_i$  and weight  $w_i$ .
- ▶ We want to put the maximum value in the bag, keeping the total weight smaller than  $W$ .
- ▶ Can you propose a **heuristic** that gives an approximate solution to the Knapsack problem ?

## Heuristic for the Knapsack problem

**Exercise 12:** Finding a heuristic II : heuristics and bad solutions

- ▶ Can you find a situation where the solution given by the heuristic is bad ?

## Shortest path problem

- ▶ We will now study a famous **graph problem** : the shortest path.



- ...
- └ Two famous problems
  - └ The Shortest Path problem

## The Shortest Path problem

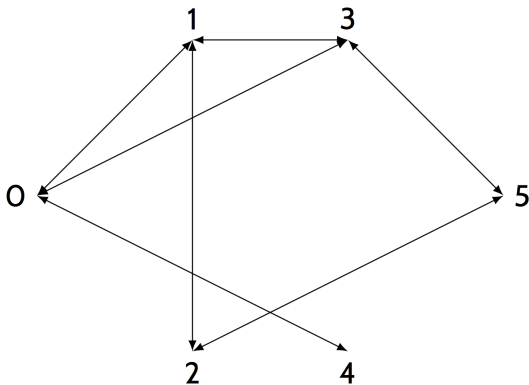
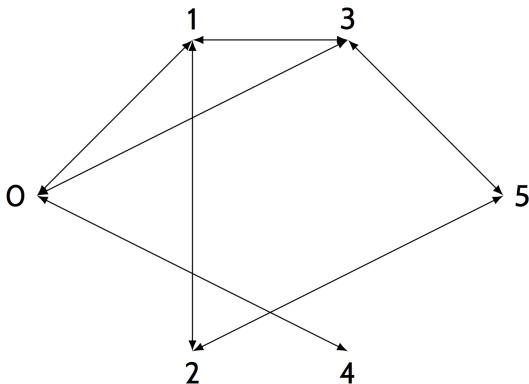


Figure: Toy graph

## The Shortest Path problem



**Figure:** We will progressively build the list of all shortest paths from 0 to all points

- ...
- └ Two famous problems
  - └ The Shortest Path problem

## Reminders on graphs

- ▶ A graph is defined by ?

## Reminders on graphs

- ▶ A graph is defined by set of vertices  $V$  and a set of edges  $E$ .

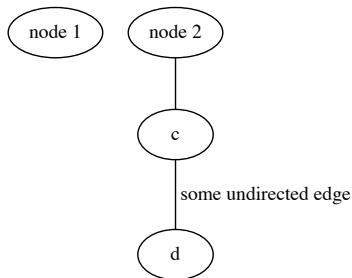


Figure: Simple graph (graphviz demo)

## Reminders on graphs

- It can be **undirected**, as this one :

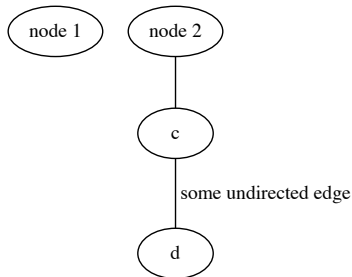


Figure: Simple graph (graphviz demo)

## Reminders on graphs

### Undirected graph

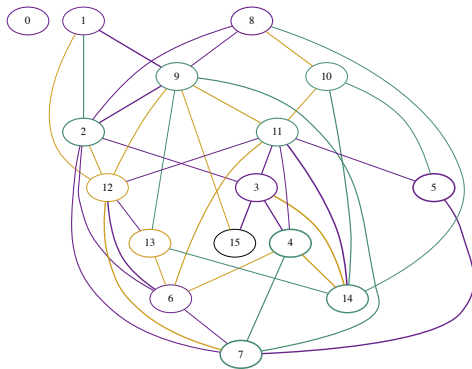


Figure: Undirected random graph generated with python

## Reminders on graphs

- Or **directed**, as this one. (it is then called a **digraph**)

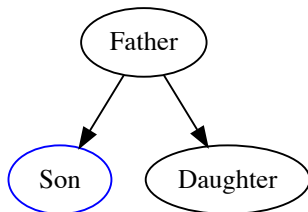
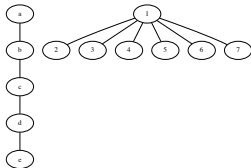


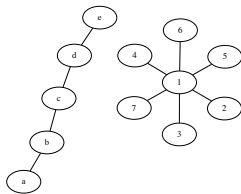
Figure: Digraph (graphviz demo)

## Useful tool : graphviz

- ▶ A tool to visualize graphs
- ▶ Several **generator programs** : dot, neato



(a) Image generated with **dot**



(b) Image generated with **neato**



- ...
- └ Two famous problems
  - └ The Shortest Path problem

## Back to the shortest path problem

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  - ▶ a set of edges
  - ▶ or a set of neighbors for each node (we will use this solution in the exercises)

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- ▶ the shortest path problem is considered an easy problem in terms of algorithmic complexity.
- ▶ It has solutions that are polynomial in the size of the graph and rather intuitive (Dijkstra algorithm)
- ▶ We will develop more tomorrow on what "polynomial complexity" is, but roughly speaking, it is way smaller than exponential.

- ...
- └ Two famous problems
  - └ The Shortest Path problem

## Paths and graphs

**Exercise 13:** Building all the paths in a graph

Modify **build\_all\_paths.py** in order to build all the paths in the graph, under a certain length.

- ...
- └ Two famous problems
  - └ The Shortest Path problem

## Paths and graphs

**Exercise 14:** Build all the paths to a destination

Modify **build\_paths\_to\_destination.py** in order to build all the shortest paths to a fixed destination, under a certain length.

- ...
- └ Two famous problems
  - └ The Shortest Path problem

## Paths and graphs

**Exercise 14:** Build all the paths to a destination II

Modify **build\_paths\_to\_destination\_no\_loops** in order to build all the shortest paths to a destination, under a certain length, **avoiding loops**.

- ...
- └ Two famous problems
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## Complexity

If we were using a  $n \times n$  chessboard, how many paths would have to be tested to find the path from  $(0,0)$  to  $(n,n)$  ?



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If we were using a  $n \times n$  chessboard, how many paths would have to be tested to find the path from  $(0,0)$  to  $(n,n)$  ?

A number of order  $4^{2n}$  : this is an **exponential complexity**, it takes way too long to compute.

- ...
- └ Two famous problems
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## Other approach

- ▶ We need another approach.

## Path existence

**Exercise 15:** Recursion and paths of fixed length

Please modify **path\_existence.py** in order to recursively check if there exists a path of length  $l$  from 0 to a destination.

- ...
- └ Two famous problems
  - └ The Shortest Path problem

## Shortest paths

**Exercise 16:** Recursion and shortest path

Modify **one\_shortest\_path.py** in order to recursively build one shortest path from 0 to a destination.

- ...
- └ Two famous problems
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## Shortest paths

**Exercise 16:** Recursion and shortest path

Modify **all\_shortest\_paths.py** in order to recursively build all shortest paths from 0 to a destination.

- ...
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## Shortest paths : complexity

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If we were using a  $n \times n$  chessboard, how many paths would have to be tested to find the path from  $(0,0)$  to  $(n,n)$  ?  
A number of order  $(2n)^2$  which is a **polynomial complexity** : it is ok to compute it.

- ...
- └ Two famous problems
  - └ The Shortest Path problem

## Conclusion

We experimentally saw that some algorithms (e.g. polynomial ones) run way faster than others (exponential ones). This is the key phenomenon behind algorithmic complexity.

Tomorrow we will discuss more examples and more theoretical notions about this.