

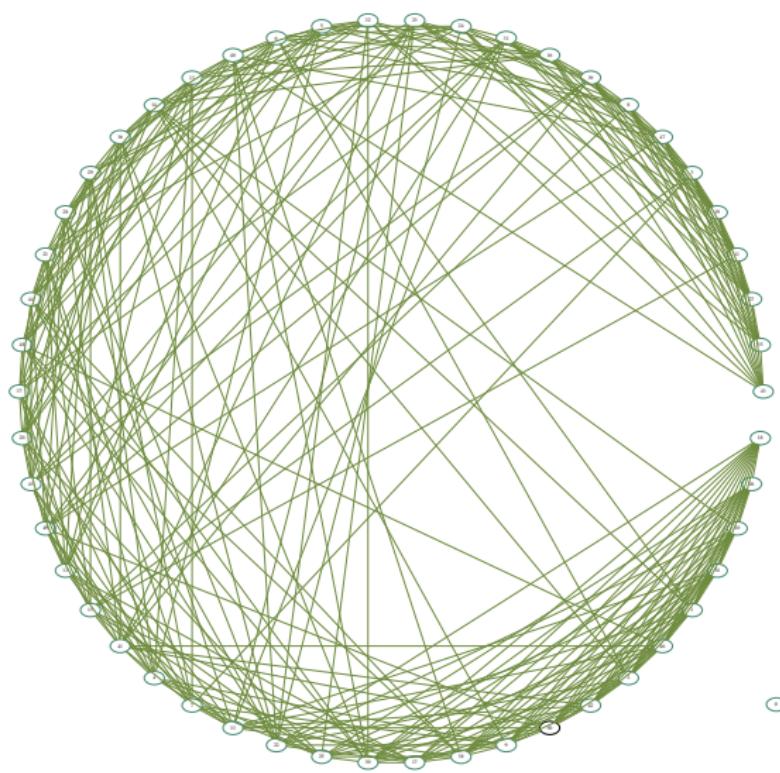


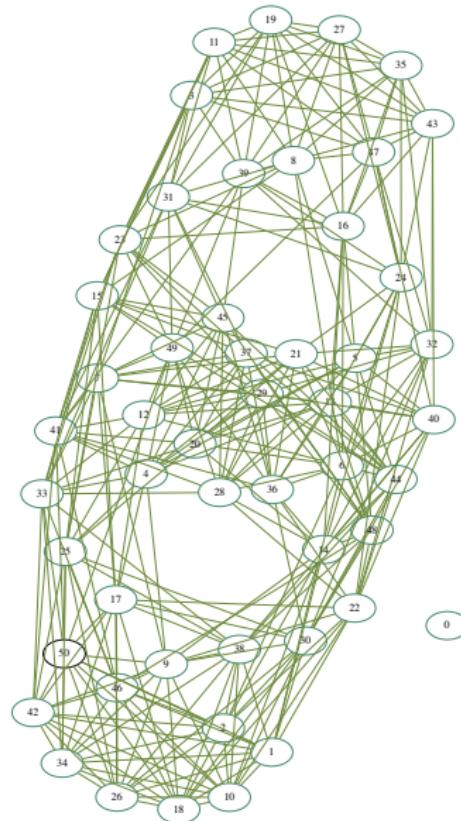
# Algorithms, Matching

Part 1. Networks and Matchings

B9 - Algorithms Matching

M-ALG-102





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## └ Introduction

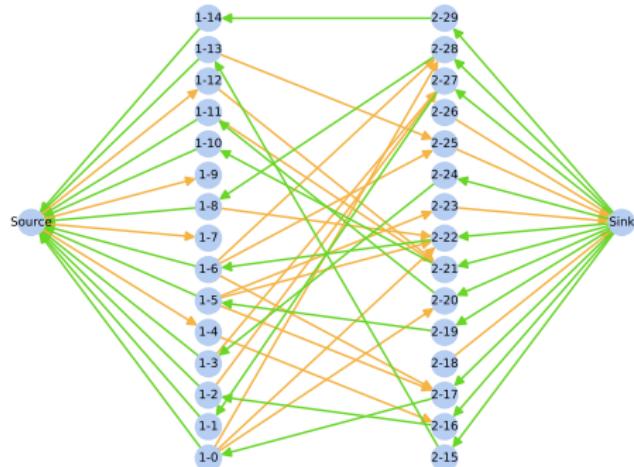
*Le réseau national  
après déclassement*



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## └ Introduction

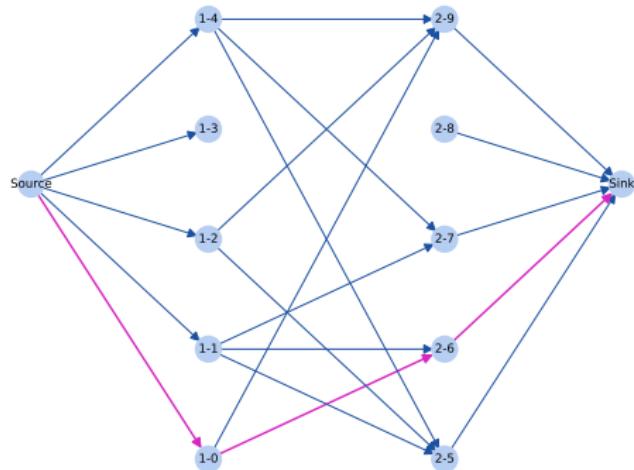
residual graph step 12

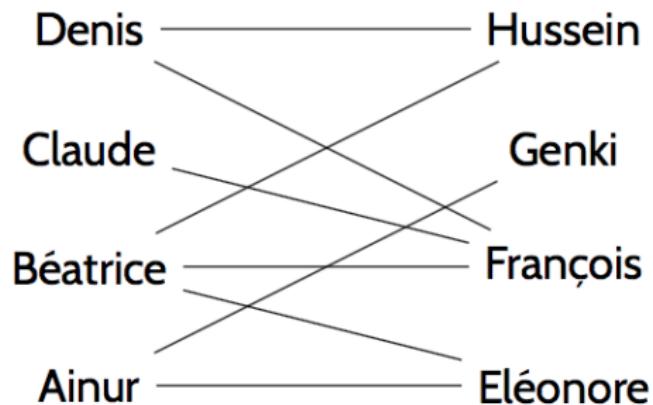


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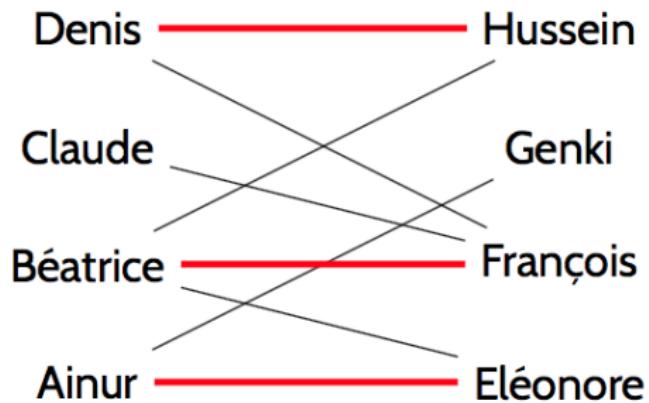
## └ Introduction

augmenting path step 1

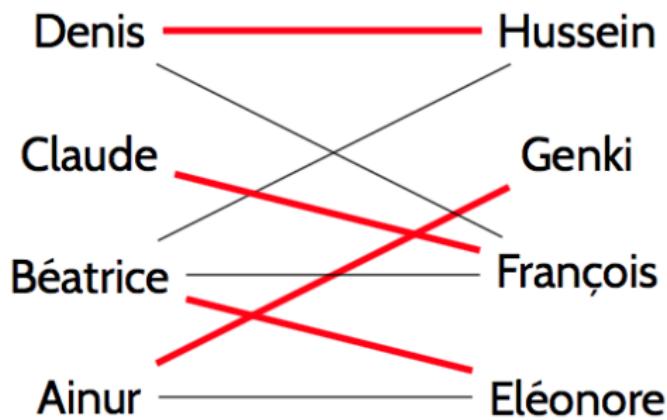


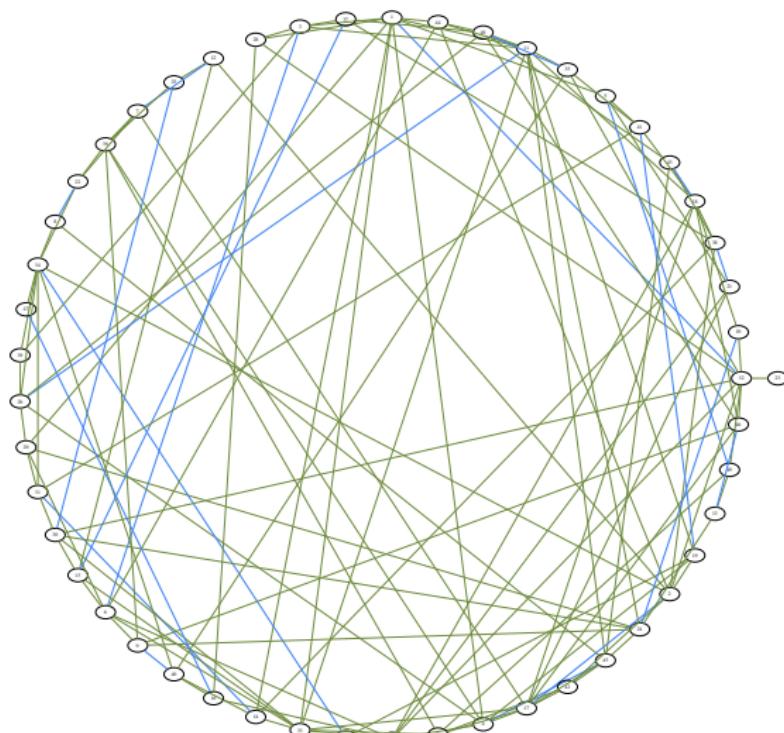


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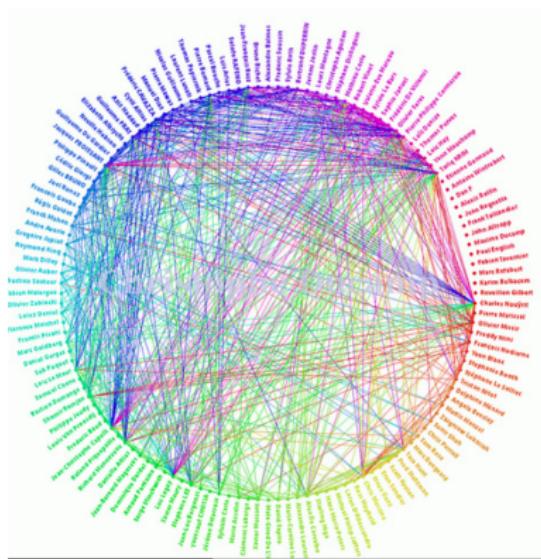


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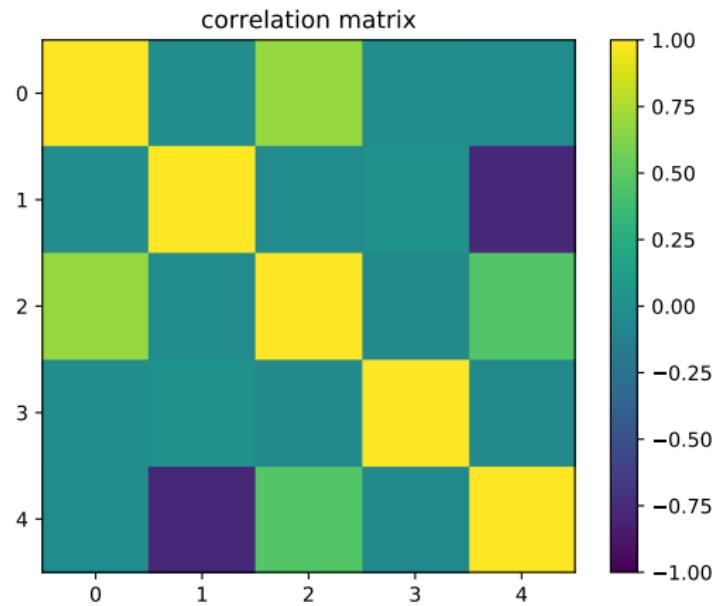


Matching size: 21  
Algo step: 128  
Nb nodes: 50



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## └ Introduction



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└ Introduction

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- ▶ The content of the course will sometimes be mathematical.
- ▶ Don't hesitate to ask if you need more reminders.

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└ Introduction

## Overview of the module

Day 1 Networks, the matching problem and the maximum flow problem

Day 2 Data clustering and representation

# Organisation of the module

- ▶ Course and exercises in python 3
- ▶ Small coding exercises, also paper + pen
- ▶ Project : explained tomorrow
- ▶ Please clone the following repository  
<https://github.com/nlehir/ALG02>

...

└ Introduction

# Libs

Day 1 networkx, numpy, matplotlib

Day 2 numpy, matplotlib, pandas, sklearn

# Day 1

## The matching problem

- Definition of the problem
- Experimental solutions
- Brute force algorithm
- Greedy algorithm

## The Maximum flow problem

- Presentation of the problem
- Solution with the Ford-Fulkerson algorithm
- Connection with the matching problem
- More results on the two problems

## Introductory example 1 : Max Flow

*Le réseau national  
après déclassement*

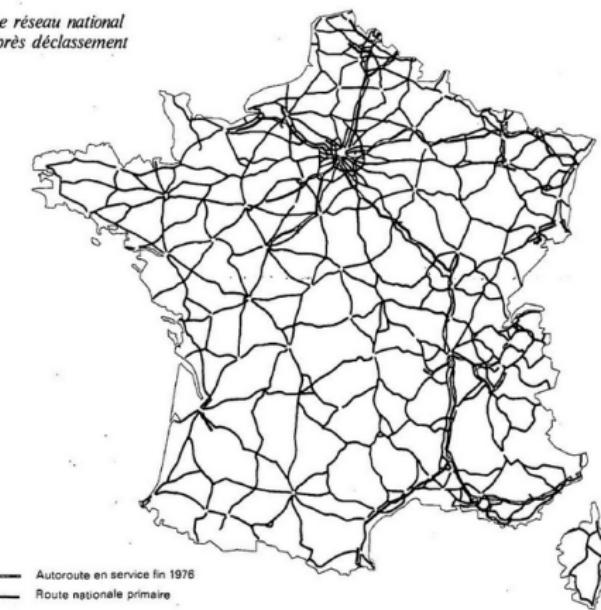


Figure: Problem 1 : transporting merchandise through a network

## Introductory example 2 : Maximum matching (Optimal allocation)

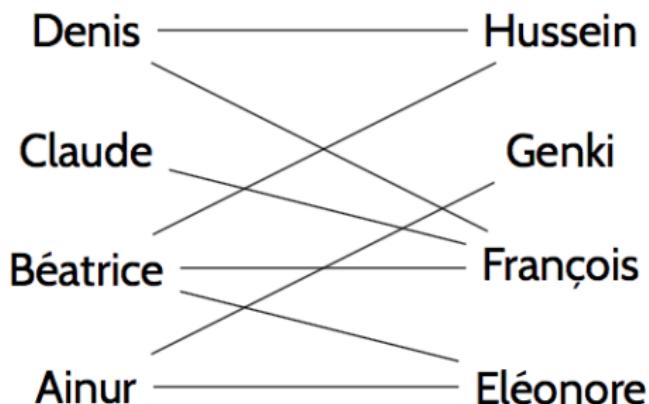


Figure: Problem 2 : Building the largest possible number of teams of 2 persons.

## Introductory example 2

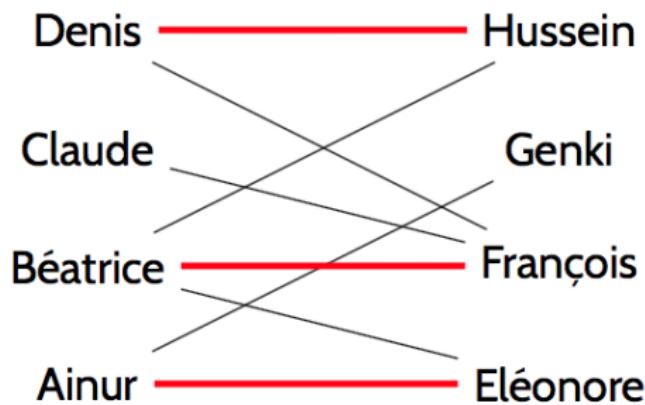


Figure: Problem 2 : not optimal allocation

## Introductory example 2

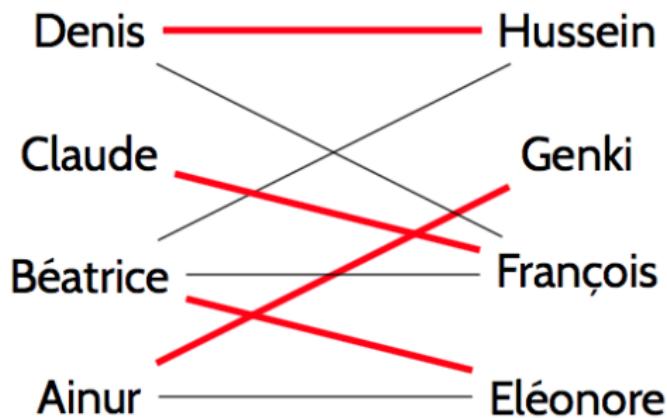


Figure: Problem 2 : optimal allocation

...

└ Introduction

## Other examples

- ▶ Assigning students to internships

## Other examples

- ▶ Assigning students to internships
- ▶ Assigning machines to a task

...

└ Introduction

# Summary

- ▶ Today we will work on **connecting the two problems.**

...

└ Introduction

## Summary

- ▶ Today we will work on **connecting the two problems**.
- ▶ Under some restrictions, the two problems **equivalent**.

...

└ The matching problem

└ Definition of the problem

## Reminders on graphs

- ▶ A graph is defined by ?

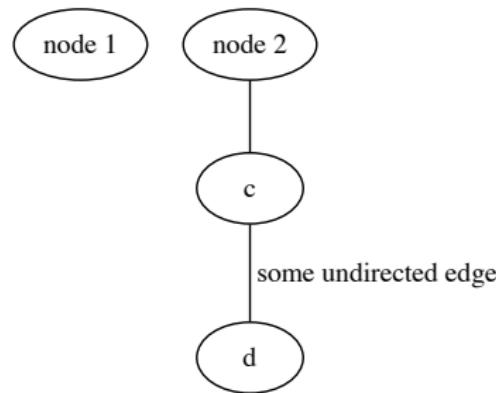
...

- └ The matching problem

- └ Definition of the problem

## Reminders on graphs

- ▶ A graph is defined by set of **vertices** (or **nodes**)  $V$  and a set of **edges**  $E$ .



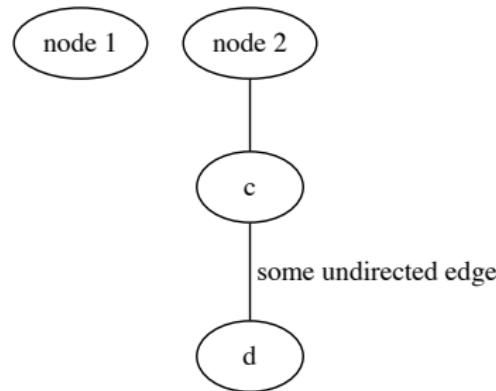
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- └ The matching problem

- └ Definition of the problem

## Reminders on graphs

- ▶ It can be **undirected**, as this one :



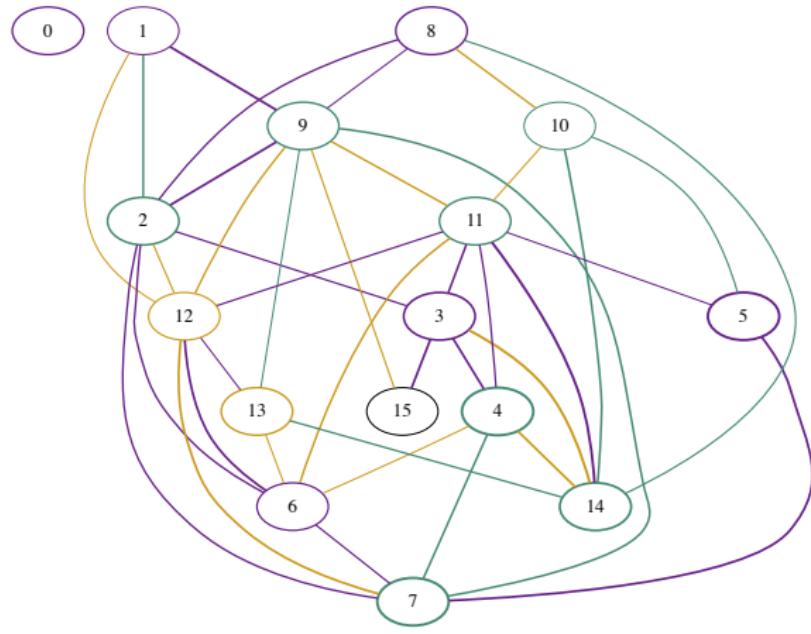
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- The matching problem

- Definition of the problem

## Reminders on graphs

### Undirected graph



## Reminders on graphs

- ▶ Or **directed**, as this one. (it is then called a **digraph**)

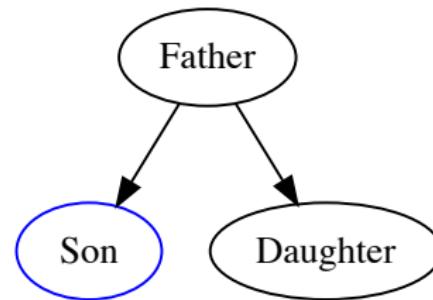


Figure: Digraph (graphviz demo)

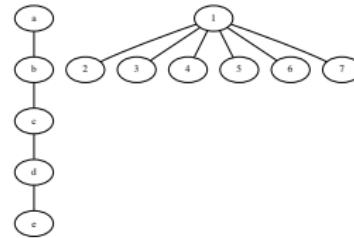
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- └ The matching problem

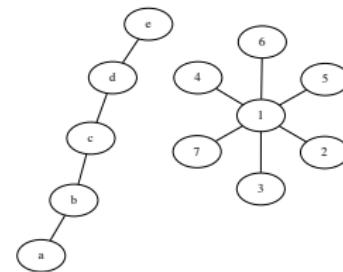
- └ Definition of the problem

## Useful tool : graphviz

- ▶ A tool to visualize graphs
- ▶ Several **generator programs** : dot, neato



(a) Image generated with **dot**



(b) Image generated with **neato**

...

- └ The matching problem

- └ Definition of the problem

## Warm up question

Given an **undirected** graph with  $n$  nodes, how many edges can we build ?

Notation of a graph :  $G(V, E)$

- ▶  $V$  : set of  $n$  vertices
- ▶  $E$  : set of edges

...

└ The matching problem

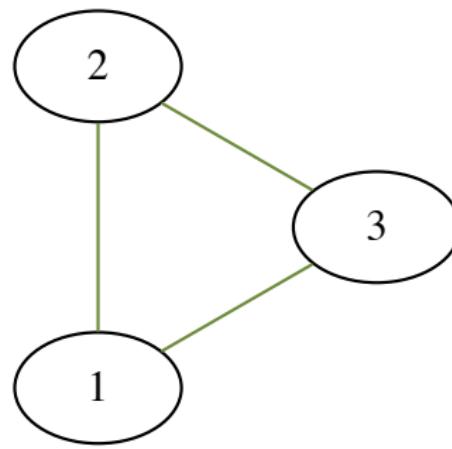
└ Definition of the problem



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- └ The matching problem

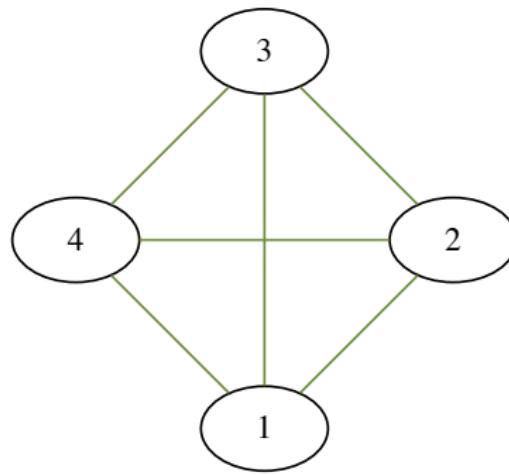
- └ Definition of the problem



...

- └ The matching problem

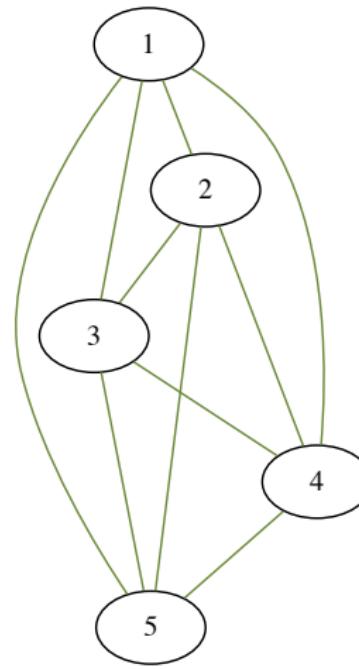
- └ Definition of the problem



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- └ The matching problem

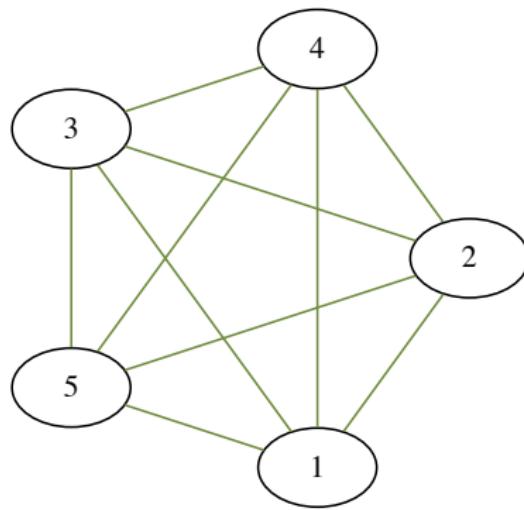
- └ Definition of the problem



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- └ The matching problem

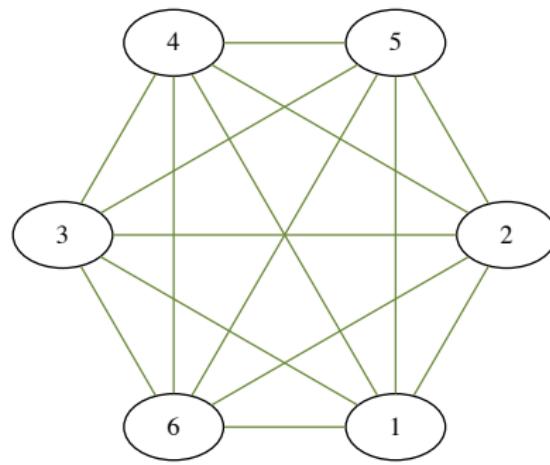
- └ Definition of the problem



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- └ The matching problem

- └ Definition of the problem



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- └ The matching problem

- └ Definition of the problem

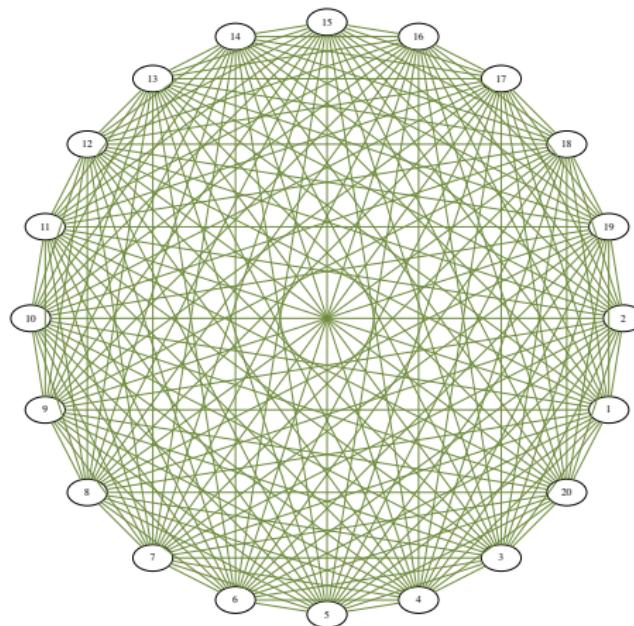


Figure: We cannot count anymore

...

- └ The matching problem

- └ Definition of the problem

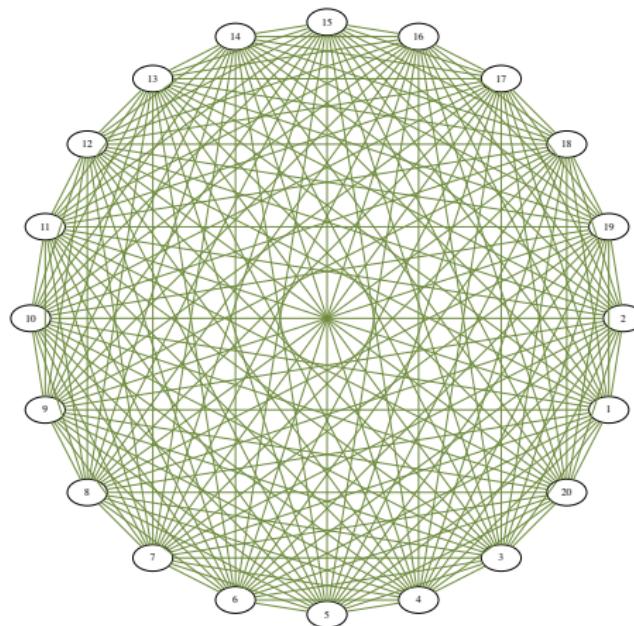


Figure: We cannot count anymore

...

└ The matching problem

└ Definition of the problem

## What if the graph is directed ?

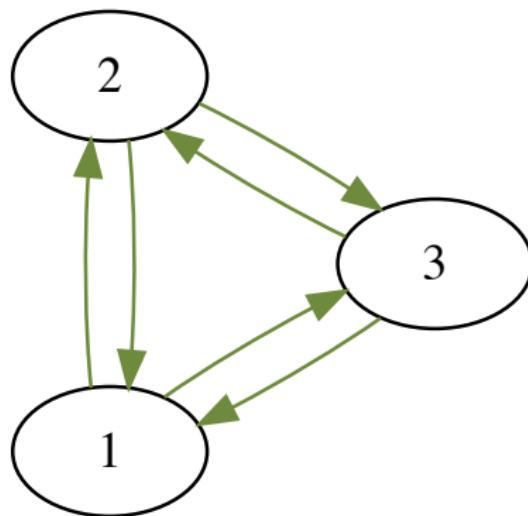


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- └ The matching problem

- └ Definition of the problem

## What if the graph is directed ?

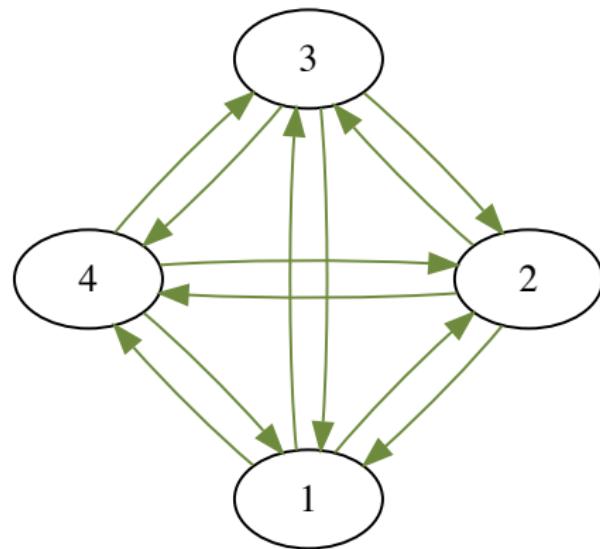


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- └ The matching problem

- └ Definition of the problem

## What if the graph is directed ?

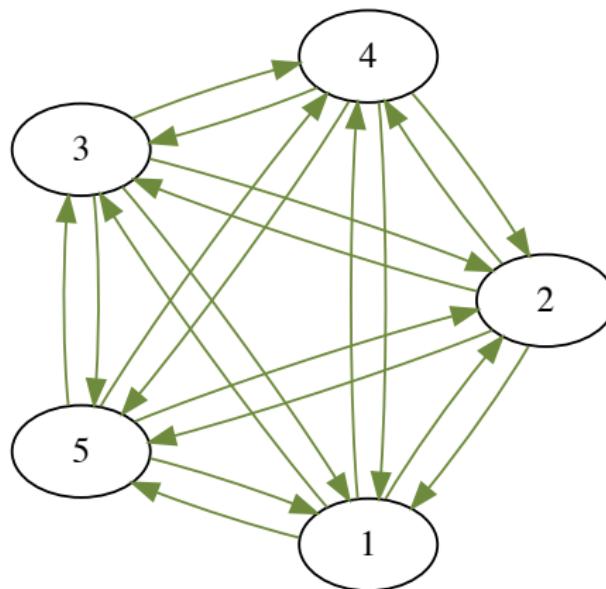


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- └ The matching problem

- └ Definition of the problem

## What if the graph is directed ?

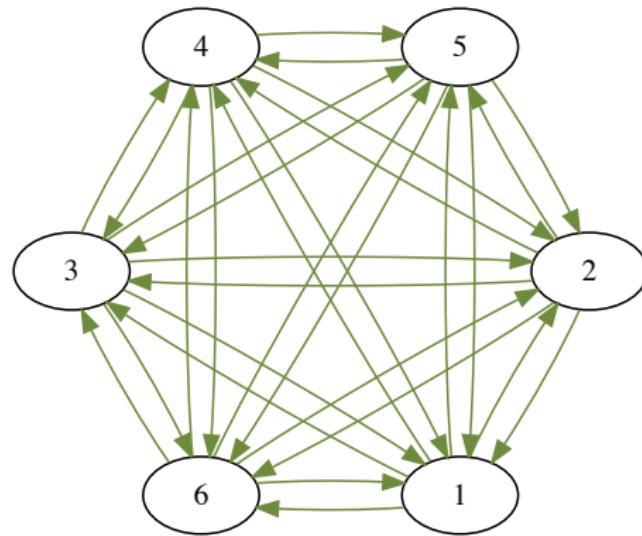


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- └ The matching problem

- └ Definition of the problem

## What if the graph is directed ?

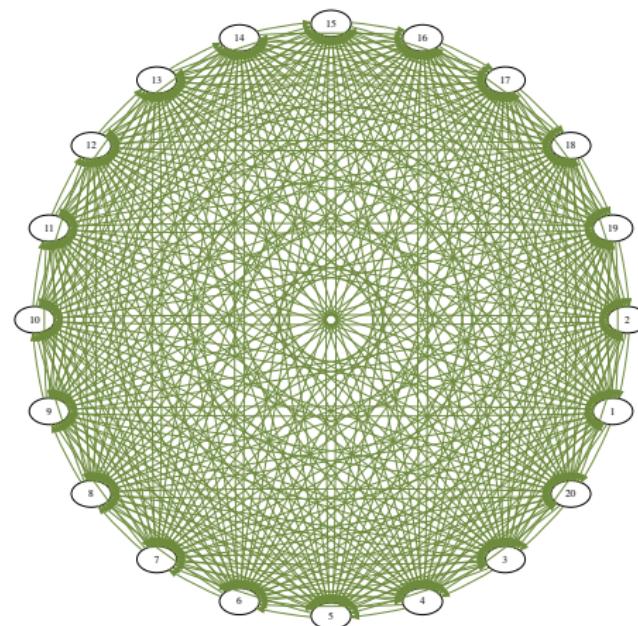


...

- └ The matching problem

- └ Definition of the problem

## What if the graph is directed ?



...

└ The matching problem

└ Definition of the problem

## Warm up question

Given an **directed** graph with  $n$  nodes, how many edges can we build ?

...

└ The matching problem

└ Definition of the problem

## Warm up question

Given an **directed** graph with  $n$  nodes, how many edges can we build ?

$$n(n - 1) \quad (1)$$

## Warm up question

Given an **directed** graph with  $n$  nodes, how many edges can we build ?

$$n(n - 1) \quad (2)$$

So if the graph is **undirected**, we can build :

$$\frac{n(n - 1)}{2} \quad (3)$$

edges.

...

- └ The matching problem

- └ Definition of the problem

## Remark

$\frac{n(n-1)}{2}$  is also the number of subsets of size 2 in a set of size  $n$ .

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} \quad (4)$$

...

└ The matching problem

  └ Definition of the problem

## Famous graph problem

- ▶ Do you know some famous **graph problems** ?

...

└ The matching problem

└ Definition of the problem

## Famous graph problem

- ▶ Do you know some famous **graph problems** ?
- ▶ Dominating set

...

└ The matching problem

└ Definition of the problem

## Famous graph problem

- ▶ Do you know some famous **graph problems** ?
- ▶ Dominating set
- ▶ Maximum clique

...

└ The matching problem

  └ Definition of the problem

## Famous graph problem

- ▶ Do you know some famous **graph problems** ?
- ▶ Dominating set
- ▶ Maximum clique
- ▶ Coloring

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└ The matching problem

└ Definition of the problem

## Matching problem

Let us now focus on the **matching problem**

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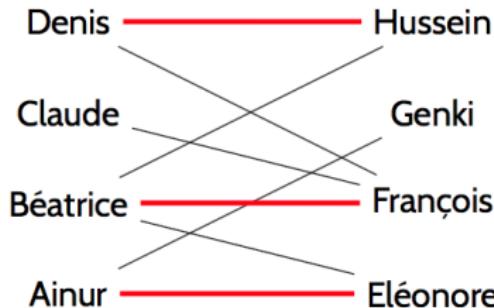
- The matching problem

- Definition of the problem

## Back to our problem

Given a **undirected** graph  $G = (V, E)$ , we want a **matching**  $M$ , which means:

- ▶ A subset of edges  $M \subset E$



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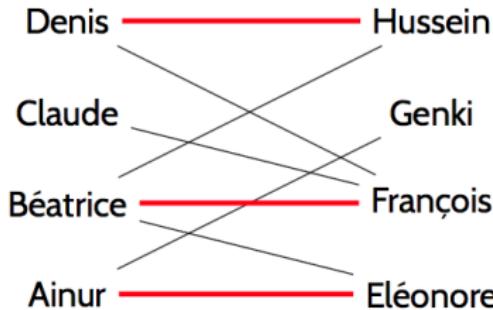
- The matching problem

- Definition of the problem

## Back to our problem

Given a **undirected** graph  $G = (V, E)$ , we want a **matching**, which means:

- ▶ A subset of edges  $M \subset E$
- ▶ Such that no pairs of edges of  $M$  are incident
- ▶ Equivalently, each node in the graph has **at most** one edge connected



...

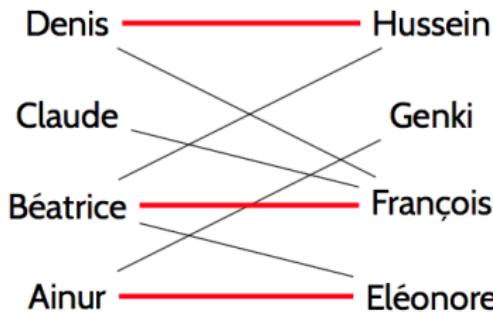
- The matching problem

- Definition of the problem

## Back to our problem

Given **undirected** a graph  $G = (V, E)$ , we want a **matching**, which means:

- ▶ A subset of edges  $M \subset E$
- ▶ Equivalently, each node in the graph has **at most** one edge connected
- ▶ Such that no pairs of edges of  $M$  are incident



...

└ The matching problem

  └ Definition of the problem

## Maximum matching

- ▶ The **size** of a matching is the number of edges it contains.

...

└ The matching problem

  └ Definition of the problem

## Maximum matching

- ▶ The **size** of a matching is the number of edges it contains.
- ▶ We want to find the matching of maximum size in a given graph.

...

- └ The matching problem

- └ Definition of the problem

## Example 1

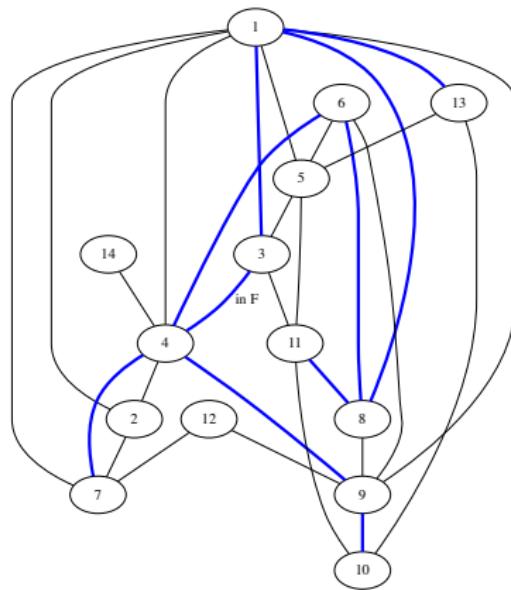


Figure: Is this a matching ?

...

- └ The matching problem

- └ Definition of the problem

## Example 2

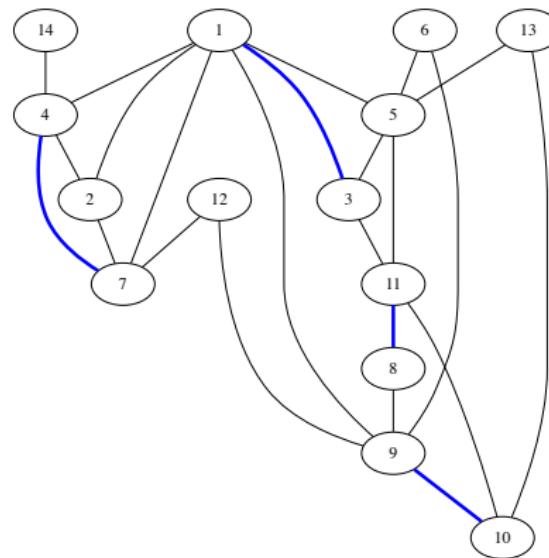


Figure: Is this a matching ?

...

- The matching problem

- Definition of the problem

## Example 3

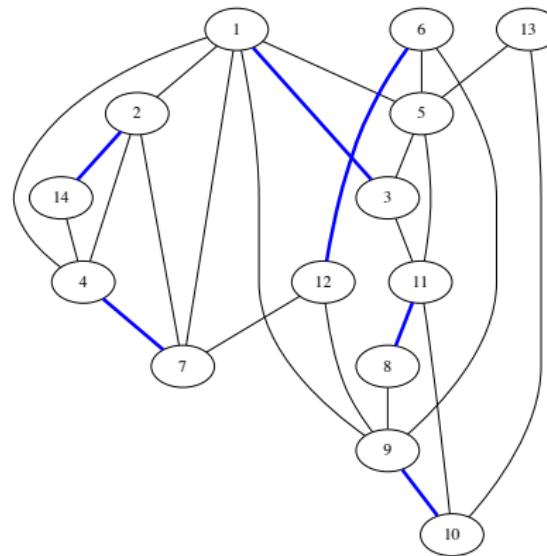


Figure: Is this an optimal matching ?

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- └ The matching problem

- └ Definition of the problem

## Example 4

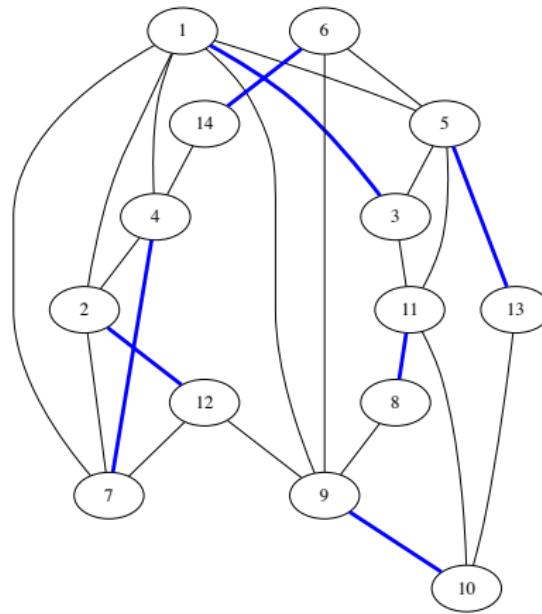


Figure: Is this an optimal matching ?

...

- └ The matching problem

- └ Definition of the problem

## Example 5

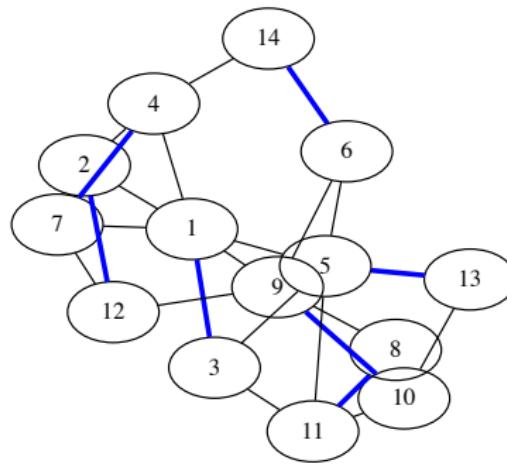


Figure: With neato

...

└ The matching problem

  └ Definition of the problem

## Optimal matching

**Exercice 1:** Given a graph of size  $n$ , what is maximum size possible for a **matching** ?

...

- └ The matching problem

- └ Definition of the problem

## Optimal matching

**Exercice 1:** Given a graph of size  $n$ , what is maximum size possible for a **matching** ?

- ▶ If  $n$  is even :  $\frac{n}{2}$
- ▶ Else  $n$  is odd :  $\frac{n-1}{2}$

...

└ The matching problem

  └ Definition of the problem

## Optimal matching

**Exercice 1:** Can you think of a graph that contains a matching of size  $n$ ? (assuming  $n$  is even)

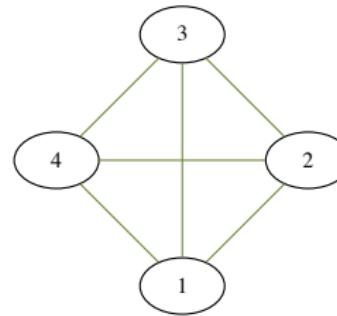
...

- └ The matching problem

- └ Definition of the problem

## Optimal

**Exercice 1:** Can you think of a graph that contains a matching of size  $n$  ? (assuming  $n$  is even)



**Figure:** The complete graph works

...

└ The matching problem

  └ Definition of the problem

## Optimal matching

**Exercice 1:** Can you think of a graph that does **not** contains a matching of size  $n$  ? (assuming  $n$  is even)

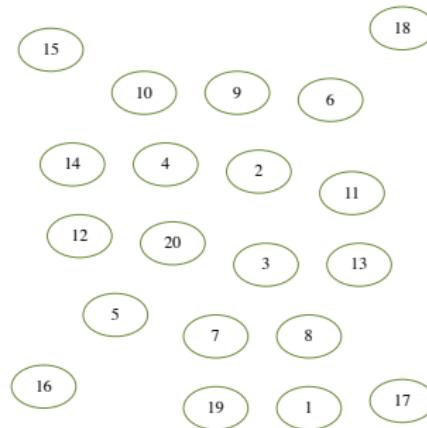
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- The matching problem

- Definition of the problem

## Optimal matching

**Exercice 1:** Can you think of a graph that does **not** contains a matching of size  $n$  ? (assuming  $n$  is even)



...

└ The matching problem

  └ Definition of the problem

## Optimal matching

**Exercice 1:** Can you think of a **non trivial** graph that does **not** contains a matching of size  $n$  ? (assuming  $n$  is even)

...

- The matching problem

- Definition of the problem

## Optimal matching

Exercice 2: Can you think of a **non trivial** graph that does **not** contains a matching of size  $n$ ? (assuming  $n$  is even)

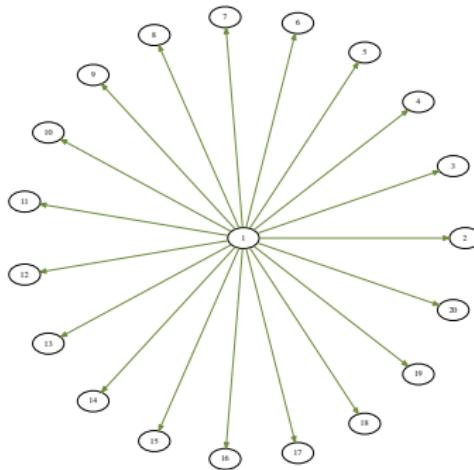


Figure: Star graph

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- └ The matching problem
- └ Experimental solutions

## Experiments

How would you code a graph ?

...

- └ The matching problem
- └ Experimental solutions

## Experiments

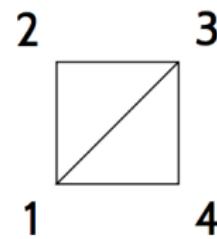
How would you code a graph ?

- ▶ list of sets of size 2 (for an undirected graph)
- ▶ a dictionary of successors (directed or undirected)

...

- └ The matching problem
- └ Experimental solutions

## Coding a graph : as a list

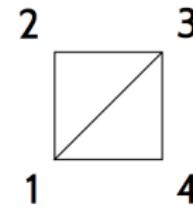


```
g1 = [{1,2},{1,3},{2,3},{3,4},{1,4}]
```

...

- └ The matching problem
- └ Experimental solutions

## Coding a graph : as a dictionary



```
g1 = { 1:{2,3,4}, 2:{1,3}, 3:{1,2,4}, 4:{1,3} }
```

...

- └ The matching problem
- └ Experimental solutions

## Random graph

Exercice 3: **cd other\_graphs/** and please use  
**random\_undirected\_graph.py** to build a graph with 20 vertices  
and 50 edges.

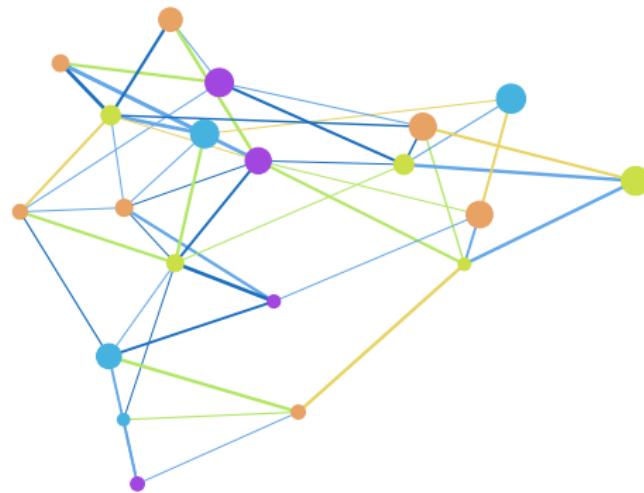
- ▶ You will need to install **networkx**

...

- └ The matching problem

- └ Experimental solutions

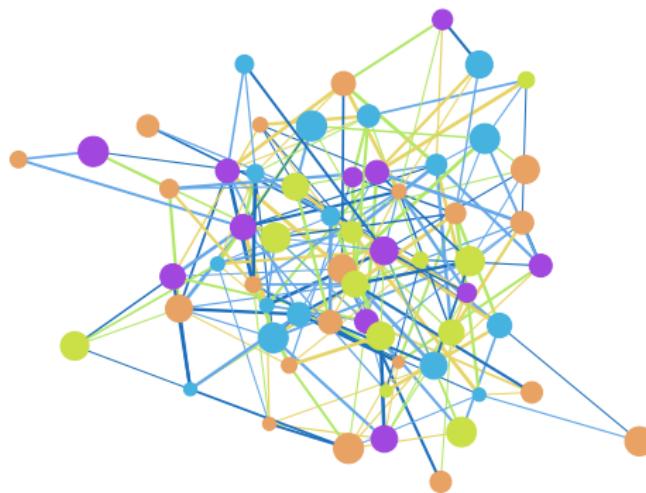
## Example undirected graph obtained



...

- └ The matching problem
- └ Experimental solutions

## Example undirected graph obtained



...

- └ The matching problem
- └ Experimental solutions

## Example undirected graph obtained



...

- └ The matching problem
- └ Experimental solutions

**Exercice 4 :** Random directed graph.

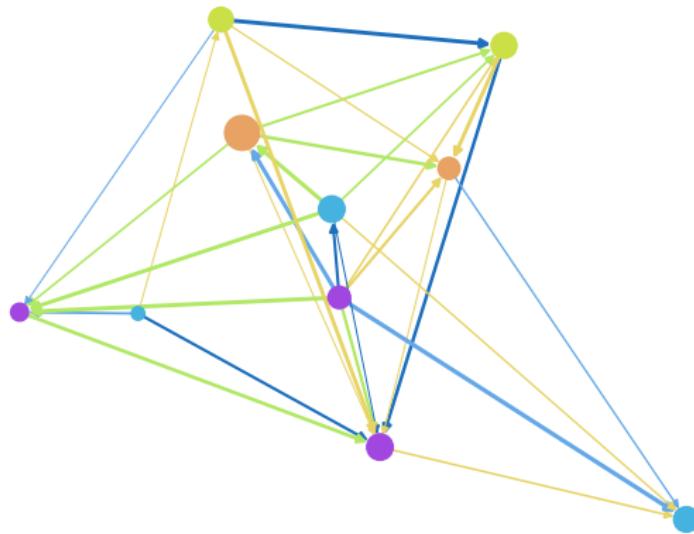
- ▶ Please use **random\_directed\_graph.py** to build a **directed** graph with a chosen number of vertices and **directed edges**.

...

- └ The matching problem

- └ Experimental solutions

## Example directed graph

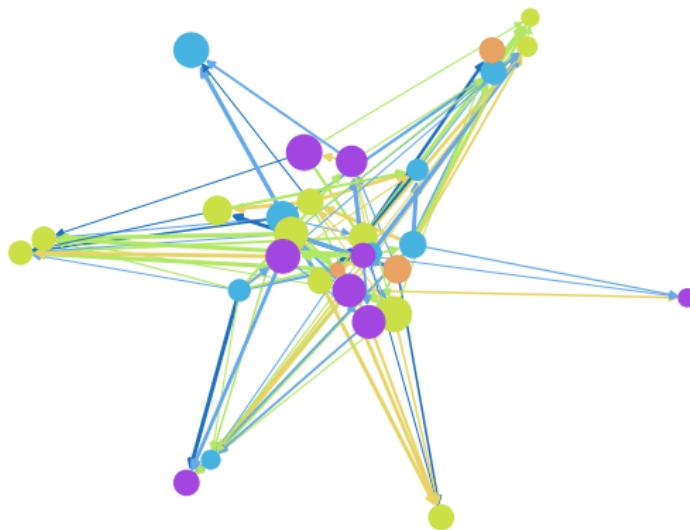


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- └ The matching problem

- └ Experimental solutions

## Example directed graph

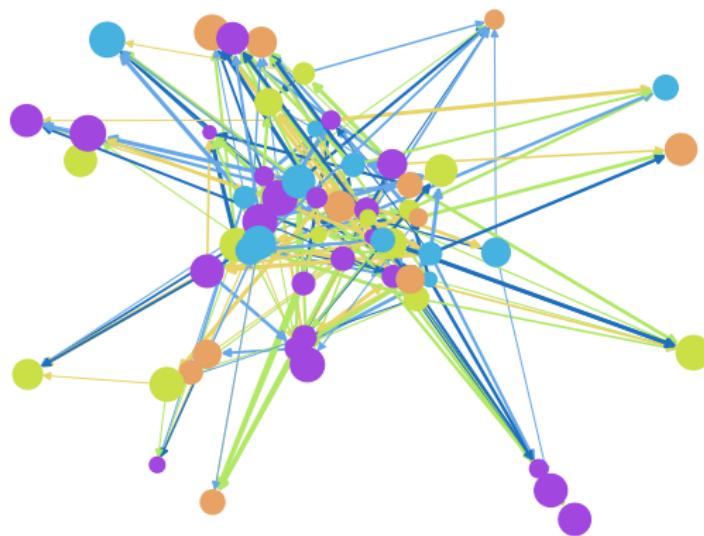


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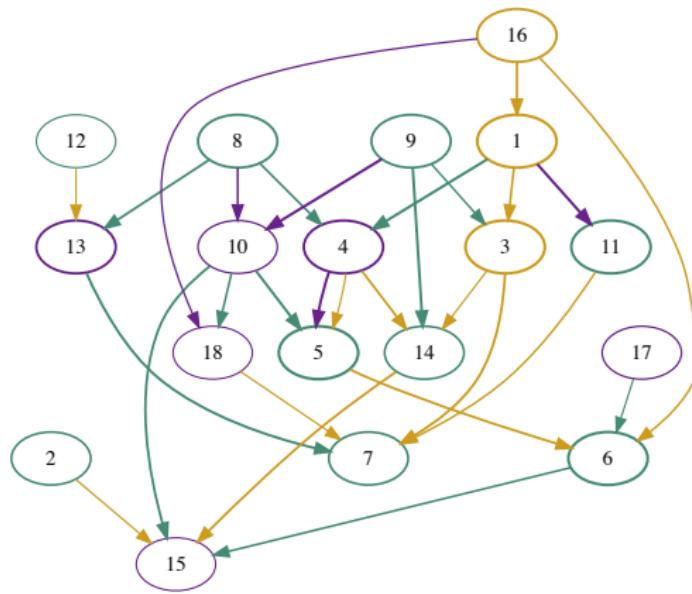
- └ The matching problem

- └ Experimental solutions

## Example directed graph



## Example directed graph



...

- └ The matching problem
- └ Experimental solutions

## Manual matching

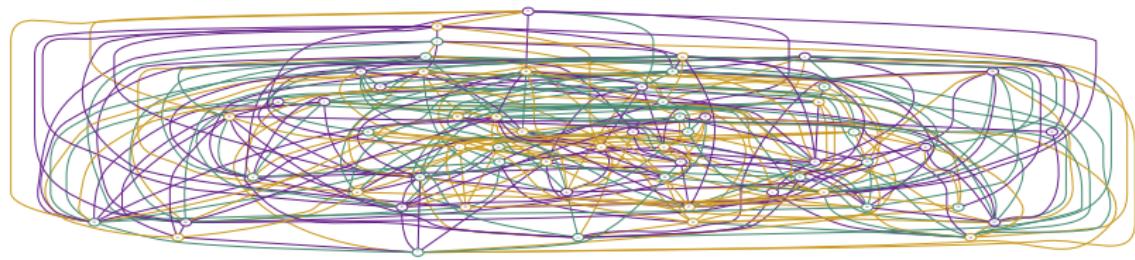
Exercice 5 : Please manually find an **optimal matching** in your **undirected** graph.

...

- └ The matching problem
- └ Experimental solutions

## Big graph

We could not manually find an optimal matching in this graph :



...

- └ The matching problem
  - └ Brute force algorithm

## Summary

- ▶ We have defined the matching problem.
- ▶ When the size of the problem is large, we can not manually find an optimal matching.

...

- └ The matching problem
  - └ Brute force algorithm

## Brute force approach

### Exercice 6 : Enumeration

- ▶ Given a graph, what would a brute force approach on the matching problem be ?

## Brute force approach

### Exercice 6 : Exhaustive search

- ▶ Given a graph what would a brute force approach on the matching problem be ?
  - ▶ 1) Enumerate all possible subsets in the set of the edges.
  - ▶ 2) Check if each subset is a matching.
  - ▶ 3) Return the biggest one obtained.

## Brute force approach

### Exercice 6 : Exhaustive search

- ▶ Given a graph what would a brute force approach on the matching problem be ?
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If the graph contains  $n$  nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?

## Brute force approach

### Exercice 6 : Exhaustive search

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  - ▶ 2) Check if each subset is a matching.
  - ▶ 3) Return the biggest one obtained.

If the graph contains  $n$  nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?

You can give a rough approximation.

## Brute force approach

### Exercice 6 : Exhaustive search

- ▶ Given a graph what would a brute force approach on the matching problem be ?
  - ▶ 1) Enumerate all possible subsets in the set of the edges.
  - ▶ 2) Check if each subset is a matching.
  - ▶ 3) Return the biggest one obtained.

If the graph contains  $n$  nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?

It is a **polynomial** number of computations : so it is ok.

...

- └ The matching problem
  - └ Brute force algorithm

## Notion of complexity

- ▶ The **time complexity** of an algorithm is a measure of the **number of elementary** operations needed for the algorithm to terminate with respect to the input size.

...

- └ The matching problem
  - └ Brute force algorithm

## Brute force search

### Exercice 7 : Complexity of brute force

- ▶ 1) Enumerate all possible subsets in the set of the edges.
- ▶ 2) Check if each subset is a matching.
- ▶ 3) Return the biggest one obtained.

What is the complexity of step 1 ?

## Brute force search

Exercice 7 : Complexity of brute force

- ▶ 1) Enumerate all possible subsets in the set of the edges.
- ▶ 2) Check if each subset is a matching.
- ▶ 3) Return the biggest one obtained.

What is the complexity of step 1 ?

The number of subsets is  $2^{\frac{n(n-1)}{2}}$  (in the worst case), which is exponential.

...

- └ The matching problem
  - └ Brute force algorithm

## Brute force search

**Exercice 7:** Complexity of brute force Assume that checking a subset requires 1 microsecond. How long should we wait in order to check all possible matching in a graph with 100 nodes ?

...

- └ The matching problem
  - └ Brute force algorithm

## Other example of complexities

- ▶ linear search
- ▶ dichotomic search

...

└ The matching problem

  └ Greedy algorithm

## Summary II

- ▶ For the matching problem on a large graph, we can neither
  - ▶ manually find an optimal matching
  - ▶ perform the exhaustive search (brute force algorithm)

...

└ The matching problem

  └ Greedy algorithm

# Algorithms

- ▶ Hence, we need **algorithms** to solve the problem.

...

└ The matching problem

  └ Greedy algorithm

# Algorithms

- ▶ Hence, we need **algorithms** to solve the problem.
- ▶ Let us introduce some theoretical notions.

...

- └ The matching problem
  - └ Greedy algorithm

## Notion of maximal and maximum matching

We will say that a matching  $M$  of cardinality (number of elements)  $|M|$  is:

- ▶ **Maximum** if it has the maximum possible number of edges (it is thus optimal)

...

- └ The matching problem

- └ Greedy algorithm

## Notion of maximal and maximum matching

We will say that a matching  $M$  of cardinality  $|M|$  is:

- ▶ **Maximum** if it has the maximum possible number of edges (it is thus optimal)
- ▶ **Maximal** if the set of edges obtained by adding any edge to it is **not a matching**. This means that  $M \cup \{e\}$  is not a matching for any  $e$  not in  $M$ .
- ▶  $\cup$  means union of sets.

...

└ The matching problem

└ Greedy algorithm

**Exercice 8:** Question : is being a **maximal** matching the same thing as beeing a **maximum** matching ?

...

└ The matching problem

  └ Greedy algorithm

## Maximum implies maximal

Let us show that a maximum matching is maximal.

...

└ The matching problem

  └ Greedy algorithm

## Counter Example

However, a matching that is maximal is **not necessarily Maximum**.

...

└ The matching problem

  └ Greedy algorithm

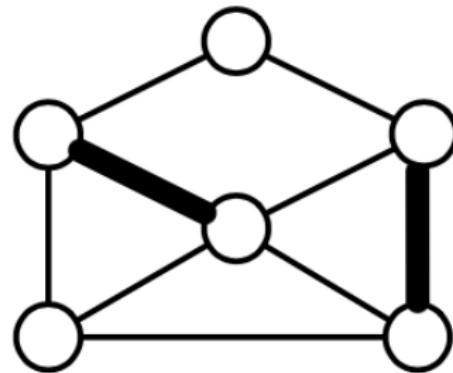
## Counter Example

However, a matching that is maximal is **not necessary Maximum**.  
Can you find an example ?

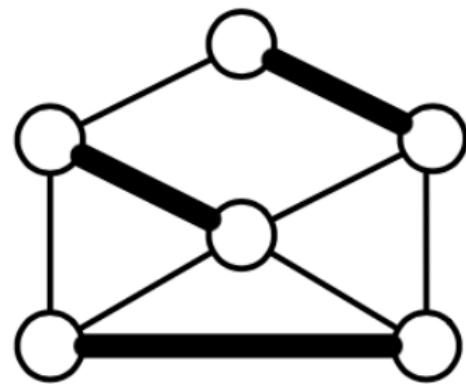
...

- └ The matching problem

- └ Greedy algorithm



(a) A maximal matching not maximum



(b) A maximum matching

...

└ The matching problem

  └ Greedy algorithm

## Greedy algorithm

Can you propose a greedy algorithm to address the maximum matching problem ?

## Greedy algorithm

**Result:** Matching  $M$

$M \leftarrow \emptyset;$

**for**  $e \in E$  **do**

**if**  $M \cup \{e\}$  is a matching **then**

$M \leftarrow M \cup \{e\}$

**end**

**end**

return  $M$

**Algorithm 0:** Greedy algorithm to find a matching

...

- └ The matching problem
  - └ Greedy algorithm

## Greedy algorithm

- ▶ What is the type of matching algorithm returned by this algorithm ?
- ▶ What is the complexity of this algorithm ? (as a function of the number of nodes  $n$  of the graph)

...

- └ The matching problem

- └ Greedy algorithm

## Greedy algorithm

- ▶ The greedy algorithm returns a **maximal** matching (proof)
- ▶ Its complexity is **cubic** :  $\mathcal{O}(n^3)$

...

└ The matching problem

  └ Greedy algorithm

## Greedy algorithm

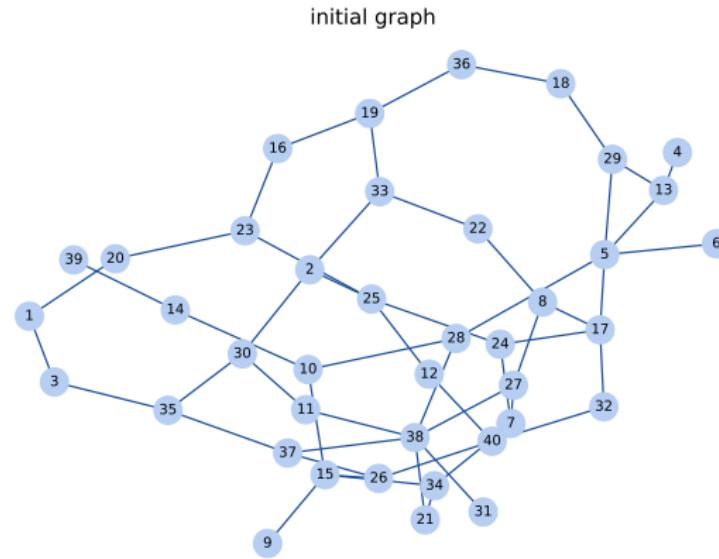
- ▶ We will implement the greedy algorithm to find a maximal matching.

...

- The matching problem

- Greedy algorithm

**Exercice 9:** `cd matching_greedy/` and use `generate_graph.py` to build a graph with at least 30 nodes. The images are stored in `images/`, data stored in `data/`



## Implementing the greedy algorithm

Exercice 9 : Implement the greedy algorithm on this graph.

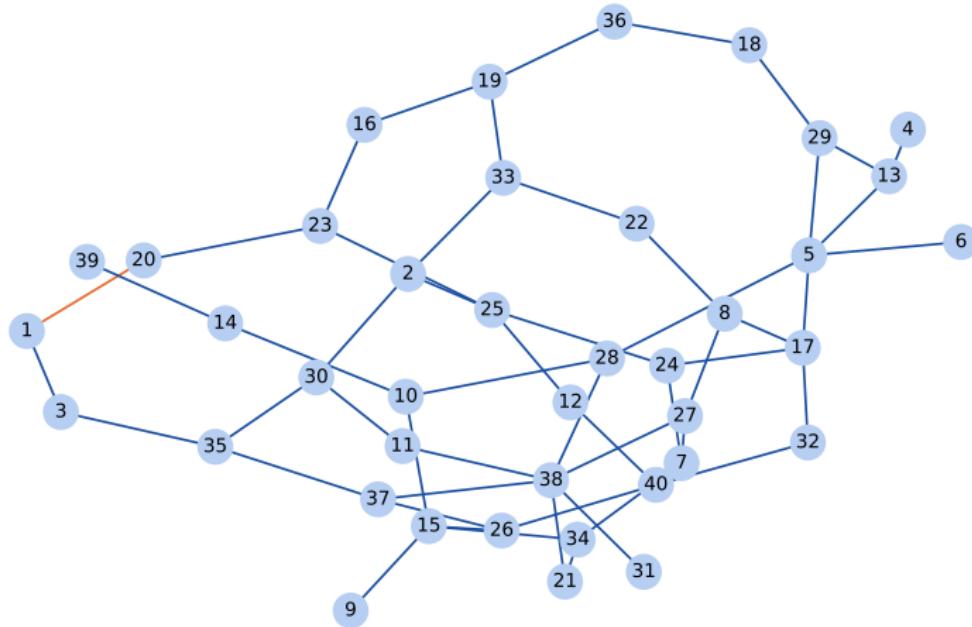
- ▶ Use the functions in **matching\_functions.py** and call them from **apply\_matching\_algorithm.py**
- ▶ More details in the file.

...

- The matching problem

- Greedy algorithm

Matching size: 1  
Algo step: 1  
Nb nodes: 40



...

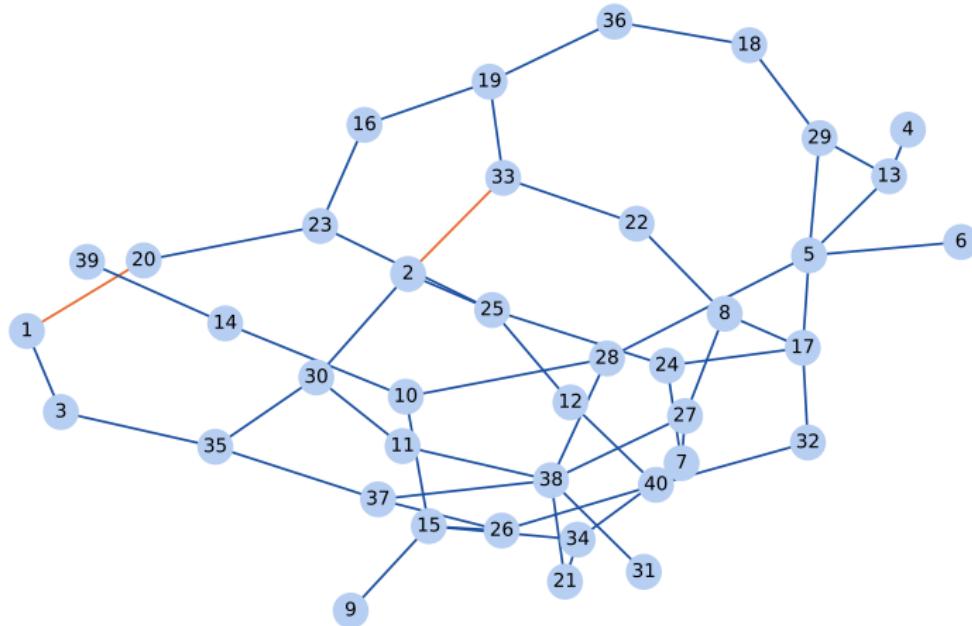
- The matching problem

- Greedy algorithm

Matching size: 2

Algo step: 3

Nb nodes: 40



...

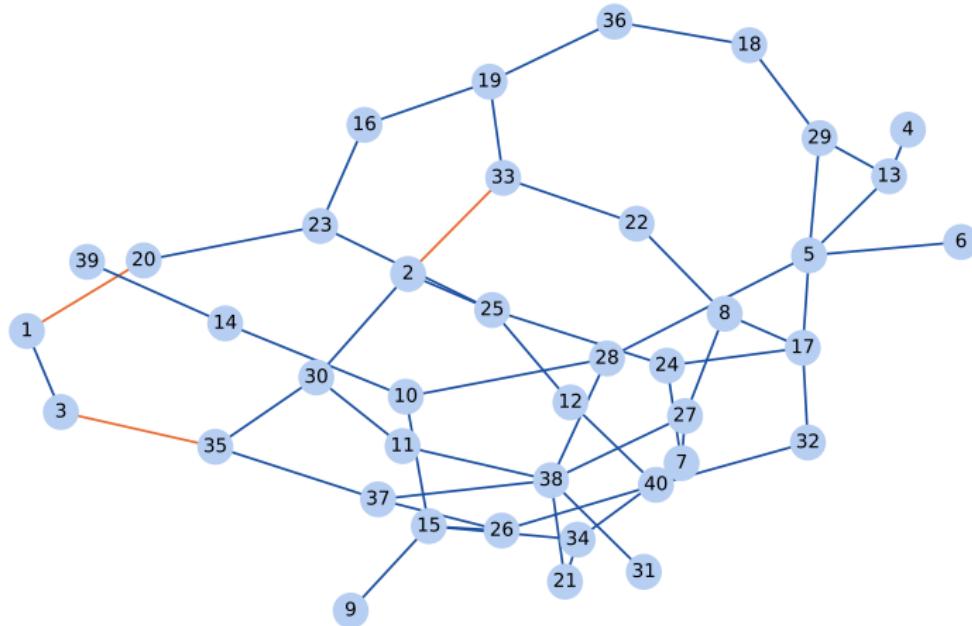
- The matching problem

- Greedy algorithm

Matching size: 3

Algo step: 6

Nb nodes: 40



...

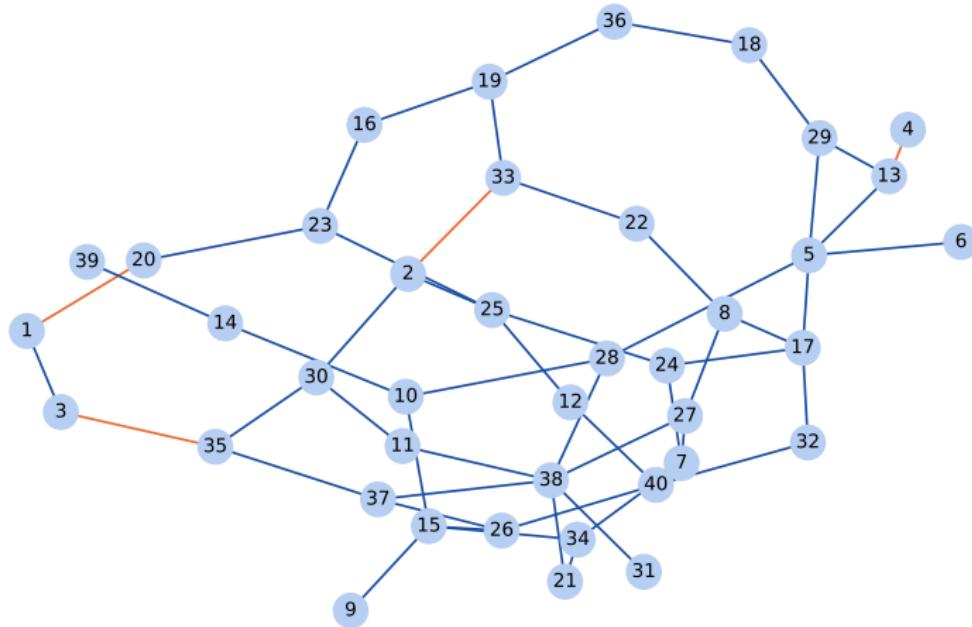
- The matching problem

- Greedy algorithm

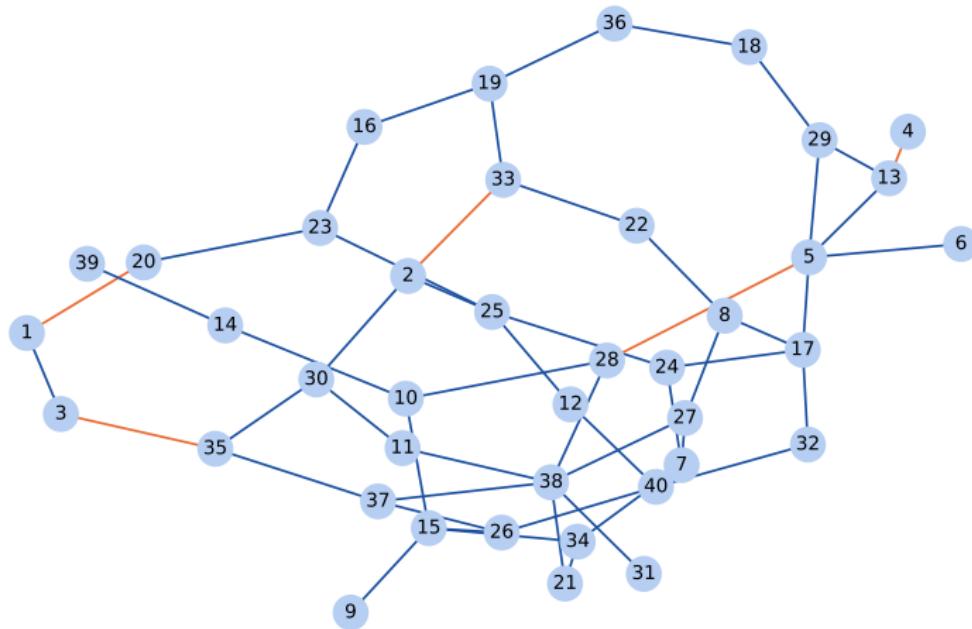
Matching size: 4

Algo step: 7

Nb nodes: 40



Matching size: 5  
Algo step: 8  
Nb nodes: 40

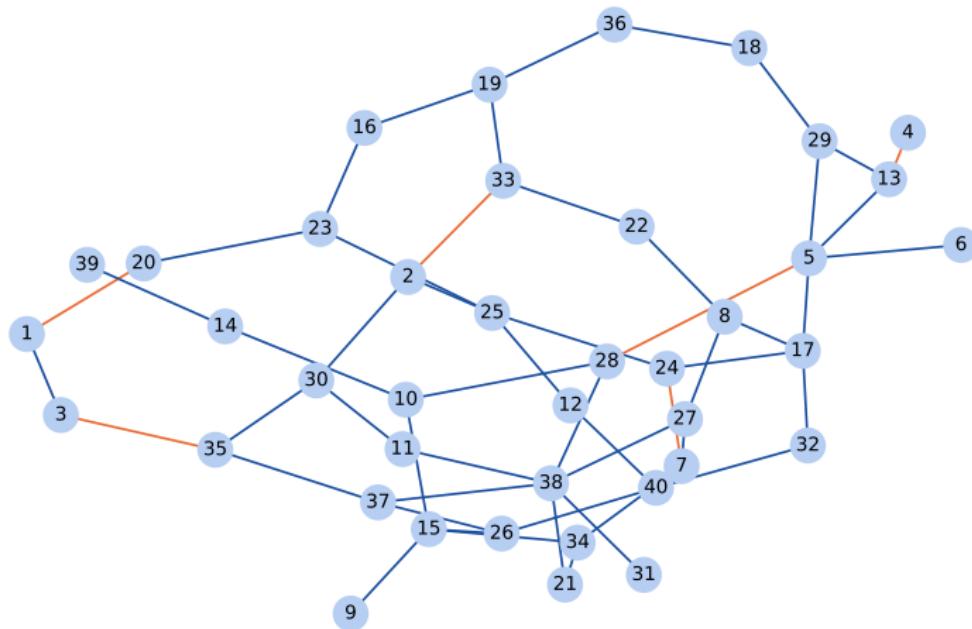


...

- The matching problem

- Greedy algorithm

Matching size: 6  
Algo step: 13  
Nb nodes: 40

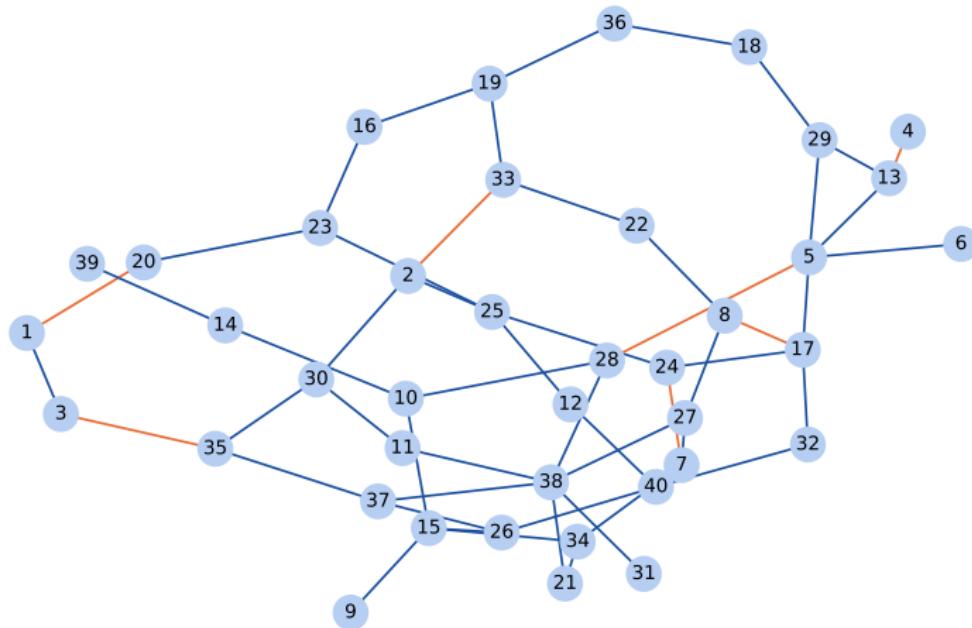


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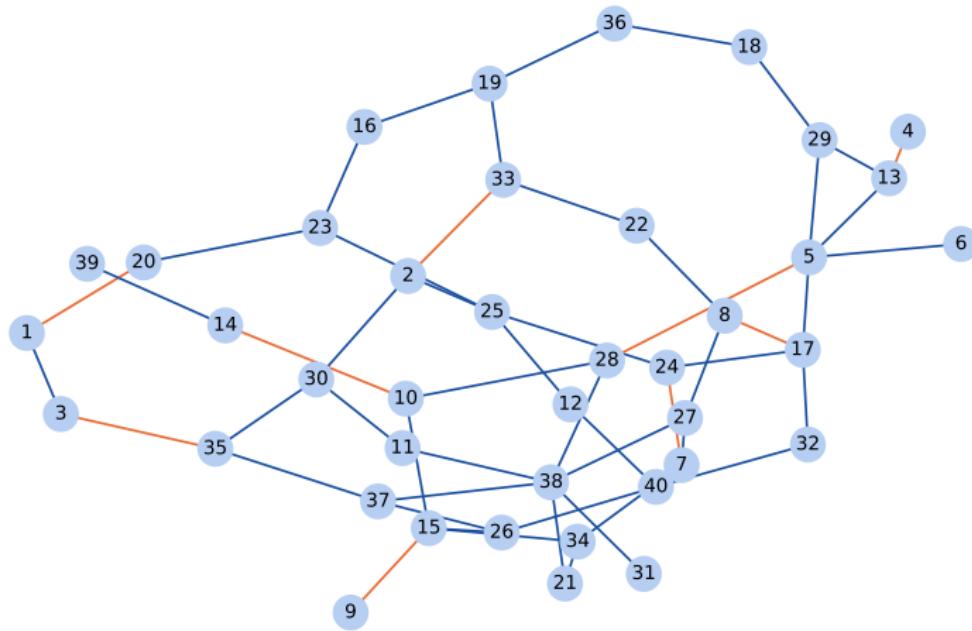
- The matching problem

- Greedy algorithm

Matching size: 7  
Algo step: 17  
Nb nodes: 40



Matching size: 9  
Algo step: 23  
Nb nodes: 40



...

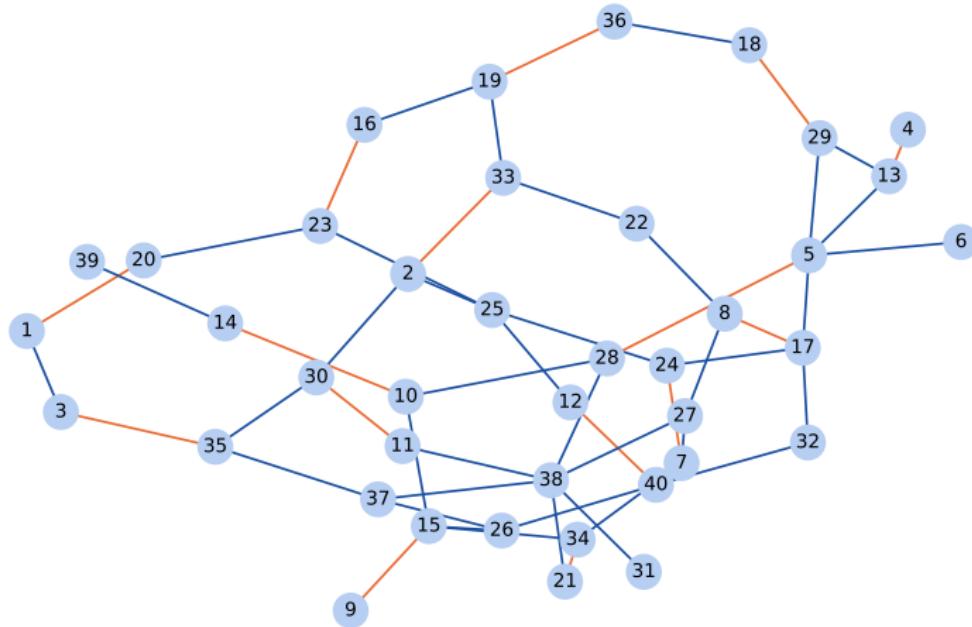
- The matching problem

- Greedy algorithm

Matching size: 15

Algo step: 41

Nb nodes: 40



...

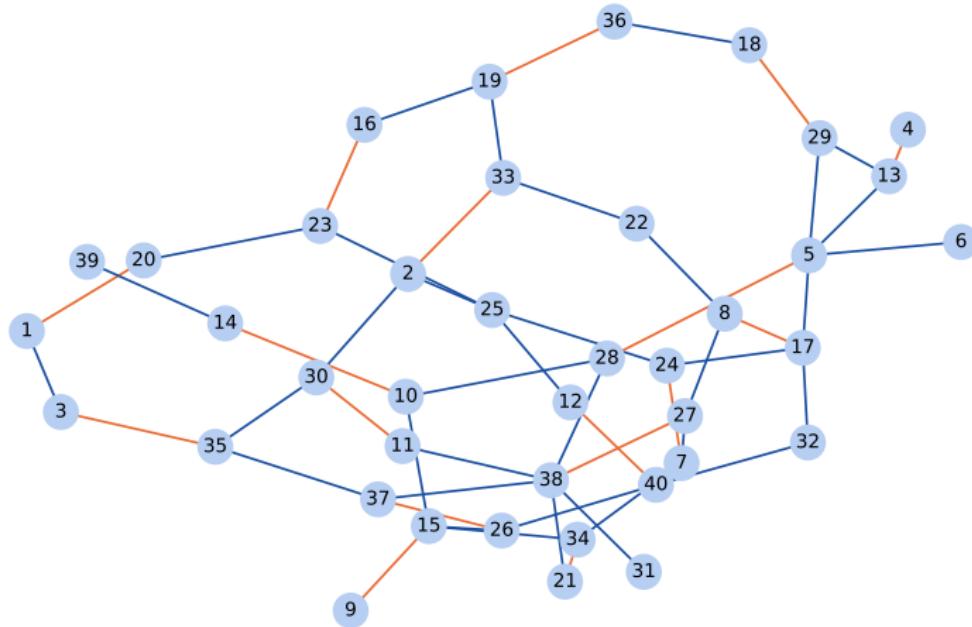
- The matching problem

- Greedy algorithm

Matching size: 17

Algo step: 48

Nb nodes: 40

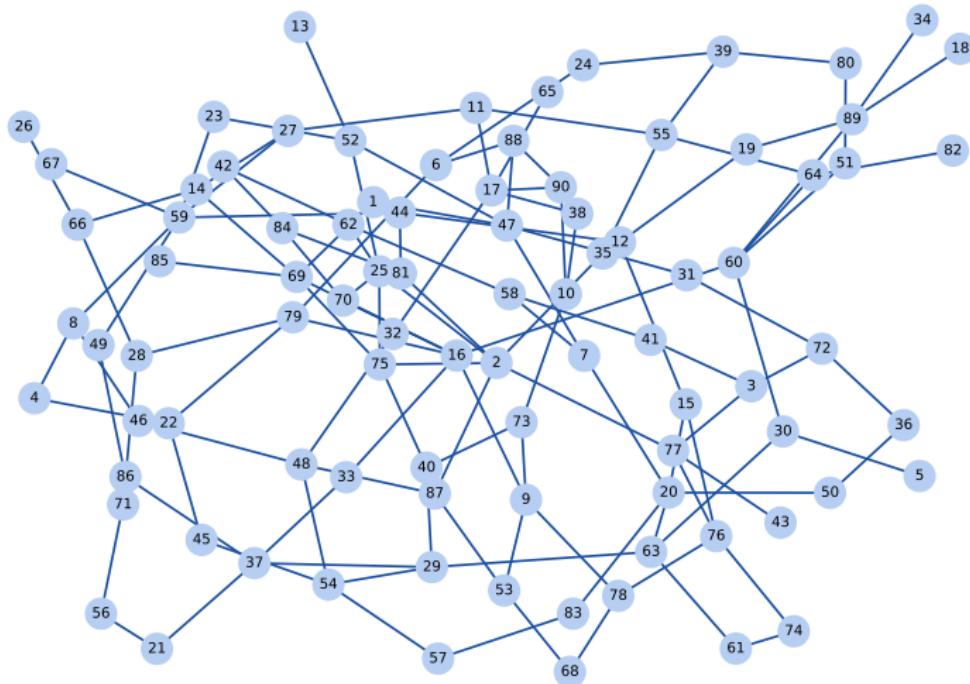


...

The matching problem

└ Greedy algorithm

initial graph



...

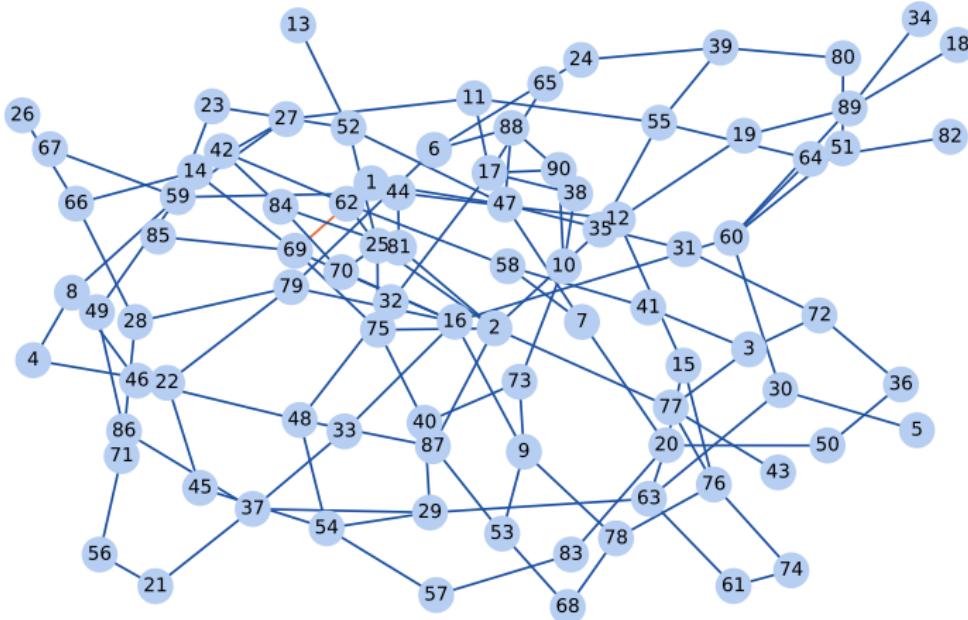
## The matching problem

### Greedy algorithm

Matching size: 1

Algo step: 1

Nb nodes: 90



...

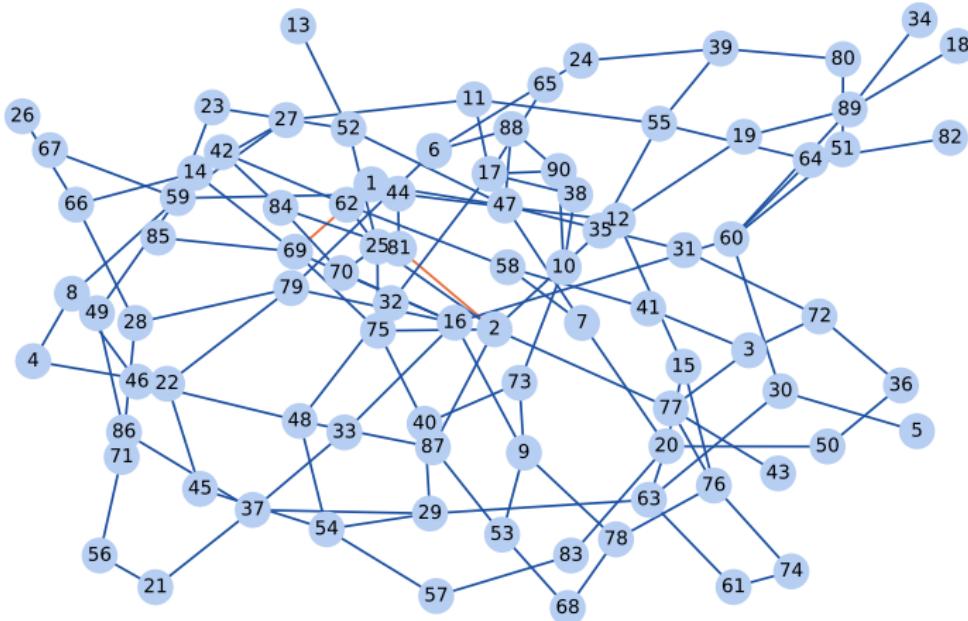
- The matching problem

- Greedy algorithm

Matching size: 2

Algo step: 3

Nb nodes: 90



...

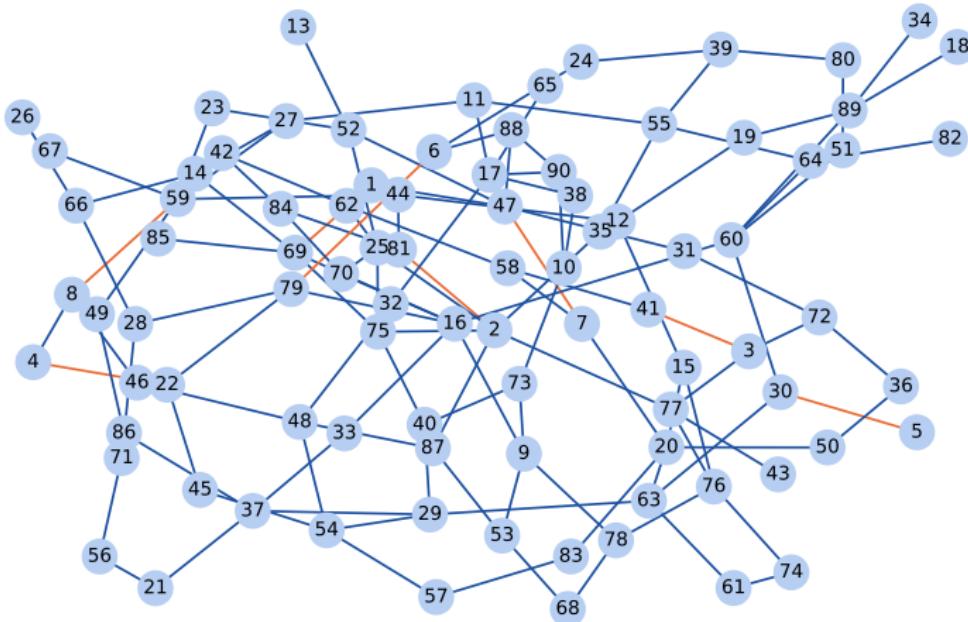
## The matching problem

### Greedy algorithm

Matching size: 8

Algo step: 21

Nb nodes: 90



...

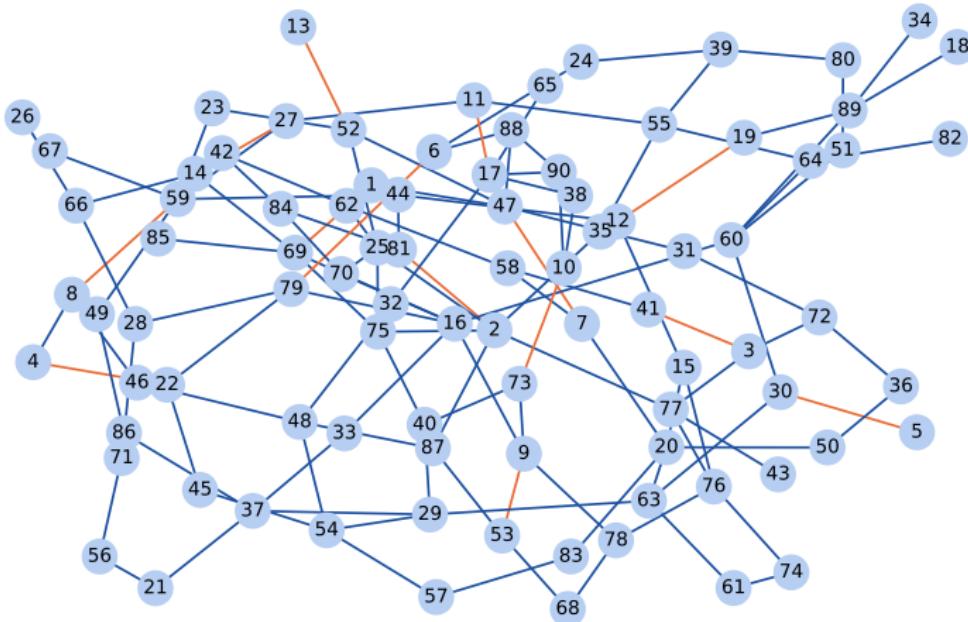
- The matching problem

- Greedy algorithm

Matching size: 14

Algo step: 38

Nb nodes: 90



...

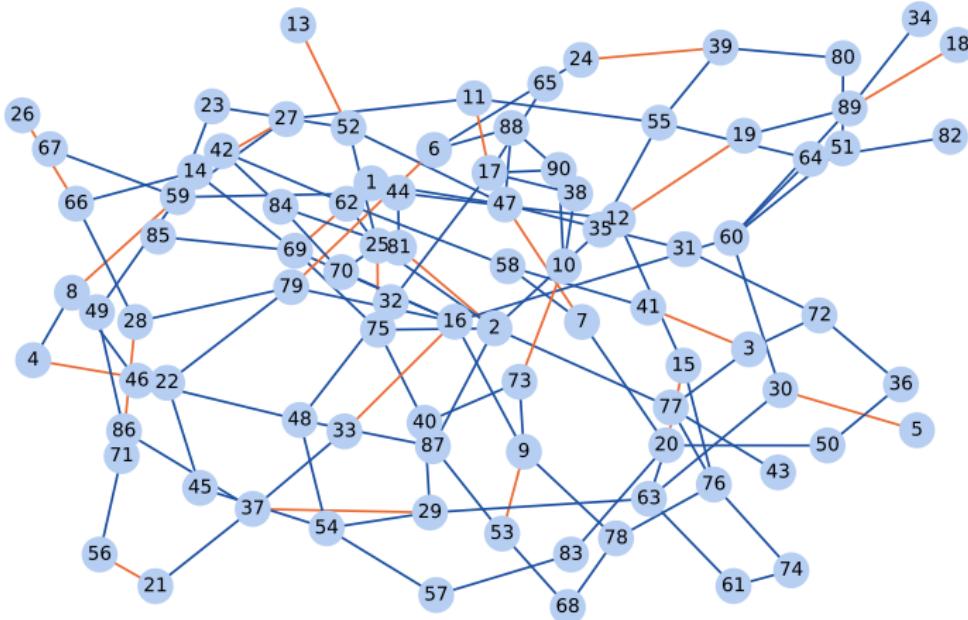
## The matching problem

### Greedy algorithm

Matching size: 23

Algo step: 72

Nb nodes: 90



...

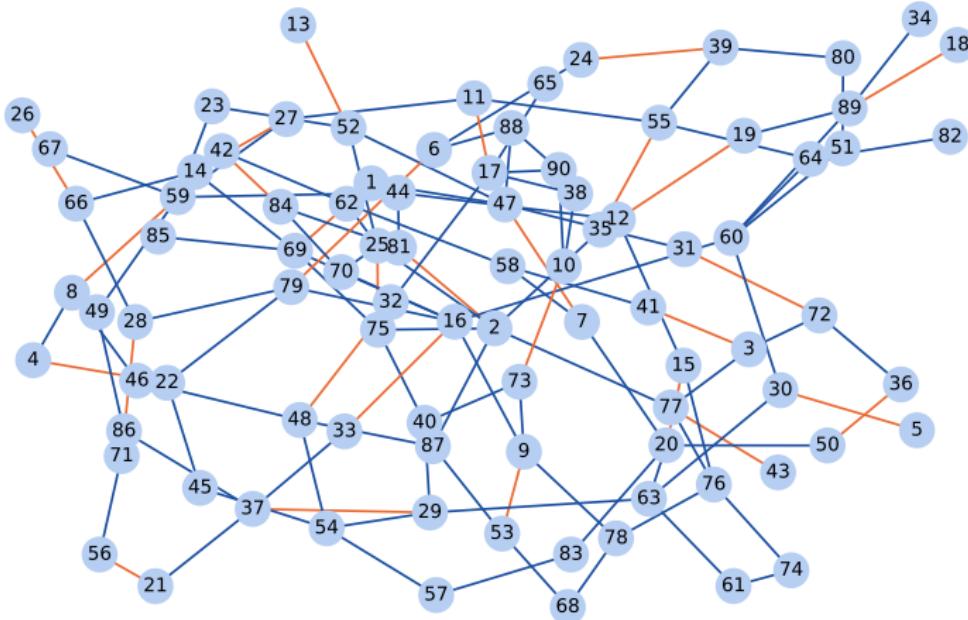
## The matching problem

### Greedy algorithm

Matching size: 29

Algo step: 94

Nb nodes: 90



...

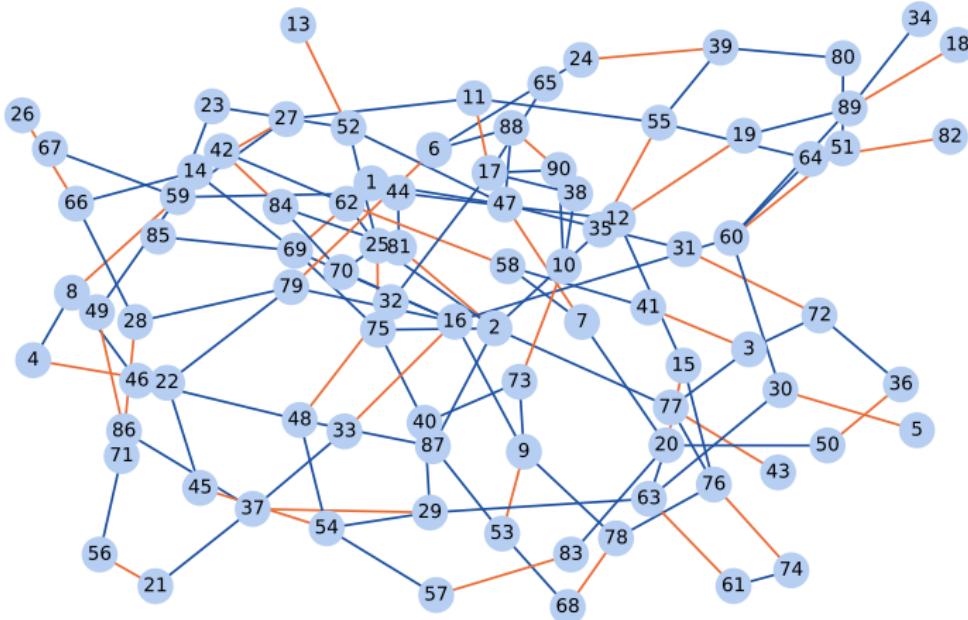
## The matching problem

### Greedy algorithm

Matching size: 39

Algo step: 128

Nb nodes: 90

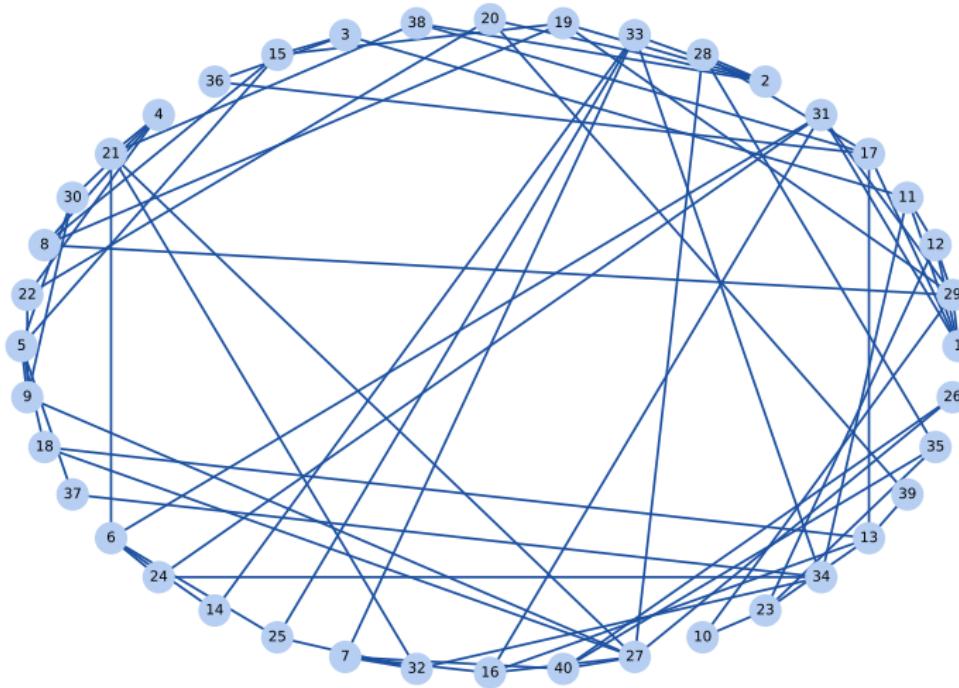


...

- The matching problem

- Greedy algorithm

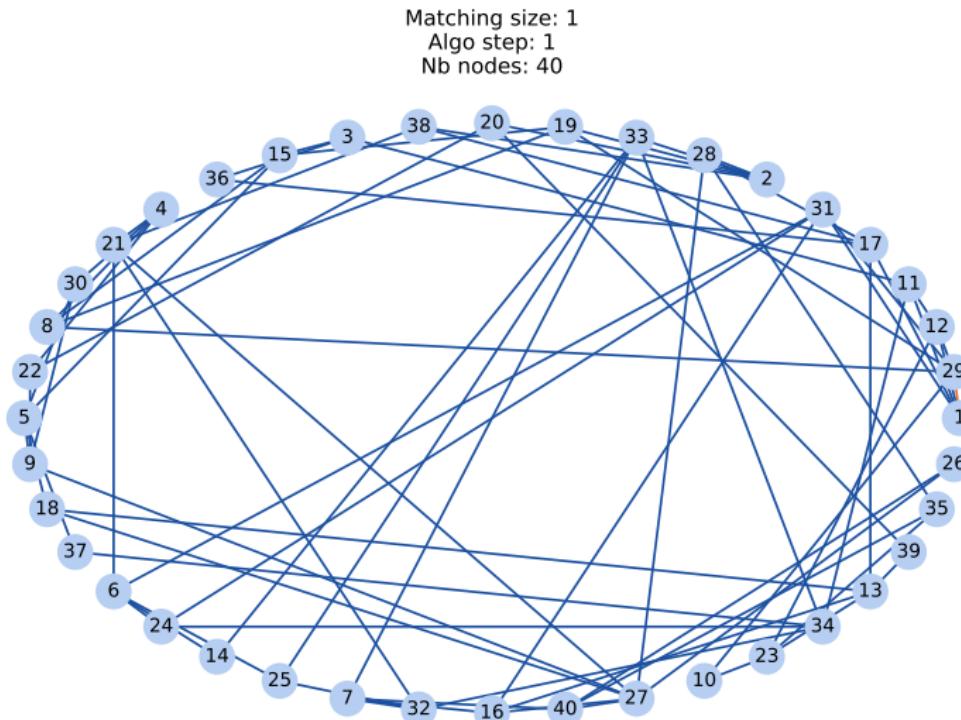
initial graph



...

- The matching problem

- Greedy algorithm



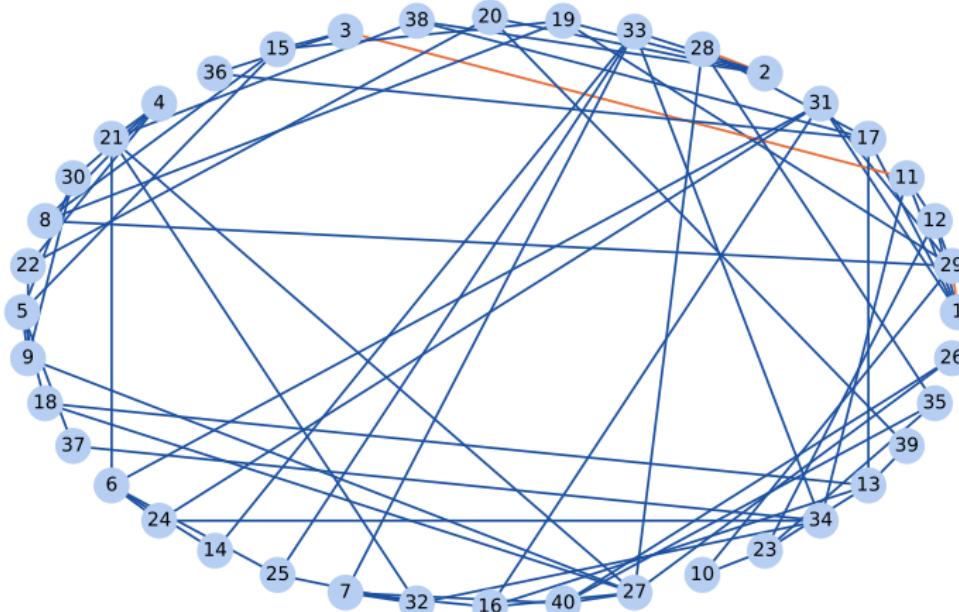
#### └ The matching problem

## └ Greedy algorithm

Matching size: 3

Algo step: 1

Nb nodes: 40



...

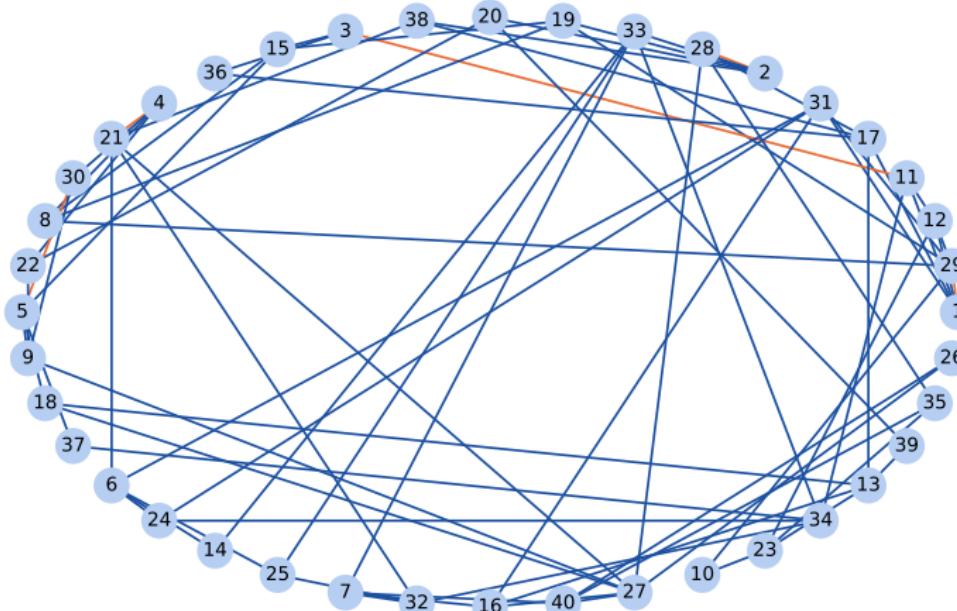
- The matching problem

- Greedy algorithm

Matching size: 5

Algo step: 18

Nb nodes: 40



...

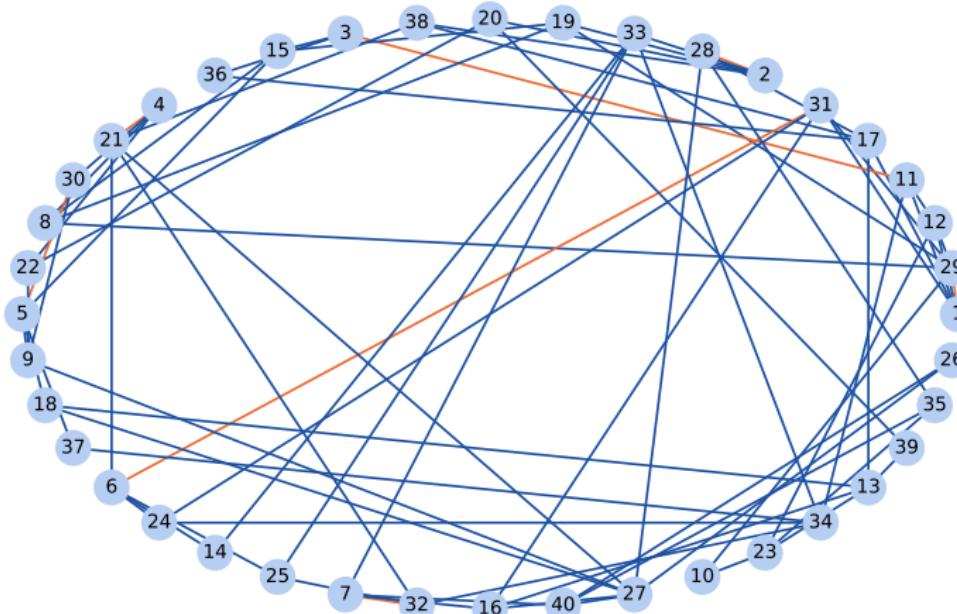
- The matching problem

- Greedy algorithm

Matching size: 7

Algo step: 28

Nb nodes: 40



...

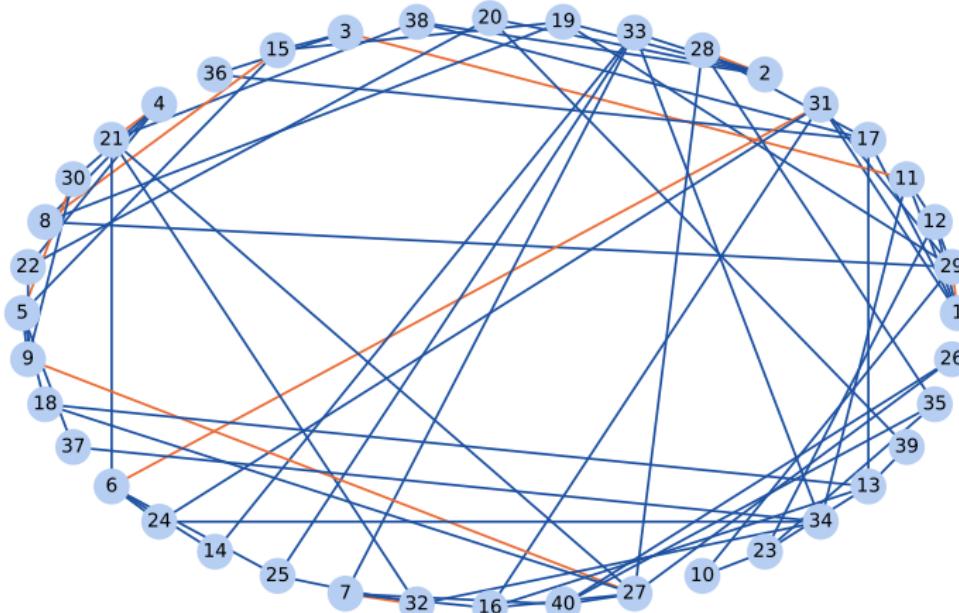
- The matching problem

- Greedy algorithm

Matching size: 9

Algo step: 35

Nb nodes: 40



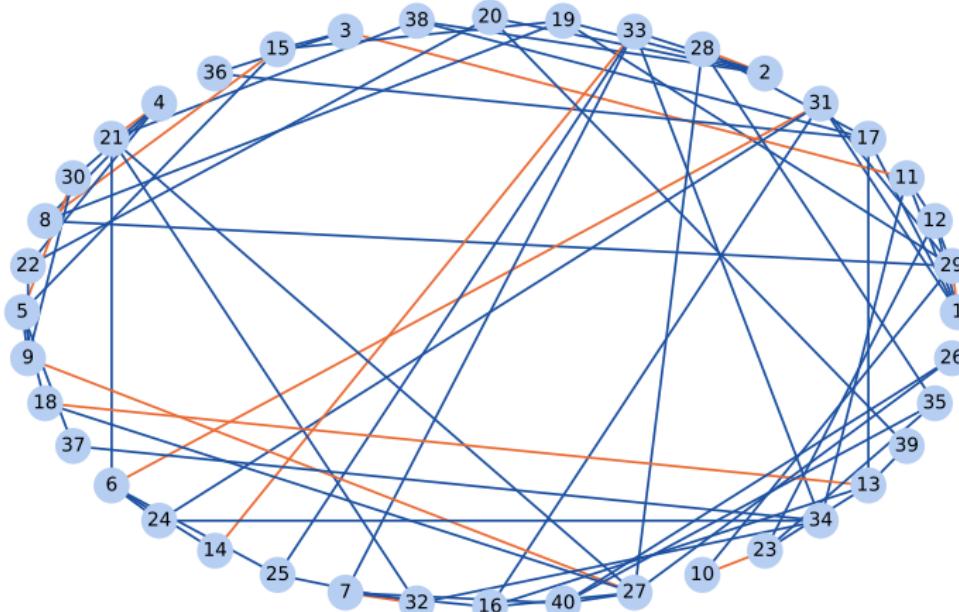
## └ The matching problem

### └ Greedy algorithm

Matching size: 12

Algo step: 49

Nb nodes: 40



...

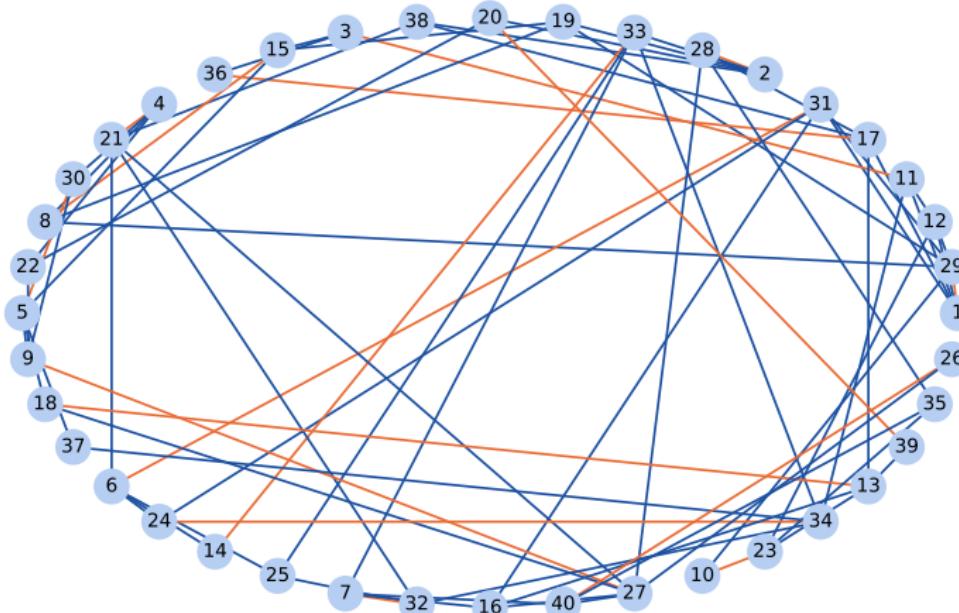
- The matching problem

- Greedy algorithm

Matching size: 16

Algo step: 68

Nb nodes: 40



## Example

**Exercice 10:** Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?

...

- └ The matching problem

- └ Greedy algorithm

## Example

**Exercice 11:** Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?



## Greedy matching

However, is  $|M|$  is the cardinality of a matching returned by the greedy algorithm, and if  $|M^*|$  is the cardinal of the real optimal matching, we have :

$$|M| \geq \frac{|M^*|}{2} \quad (5)$$

...

└ The Maximum flow problem

## Changing the problem (for now)

We temporarily leave the maximum matching problem to focus on another problem : the **Maximum flow problem**

...

- └ The Maximum flow problem
- └ Presentation of the problem

## Max flow



**Figure:** Optimizing the quantity of something transported from one place to another, under constraints

...

## The Maximum flow problem

### Presentation of the problem

# Example

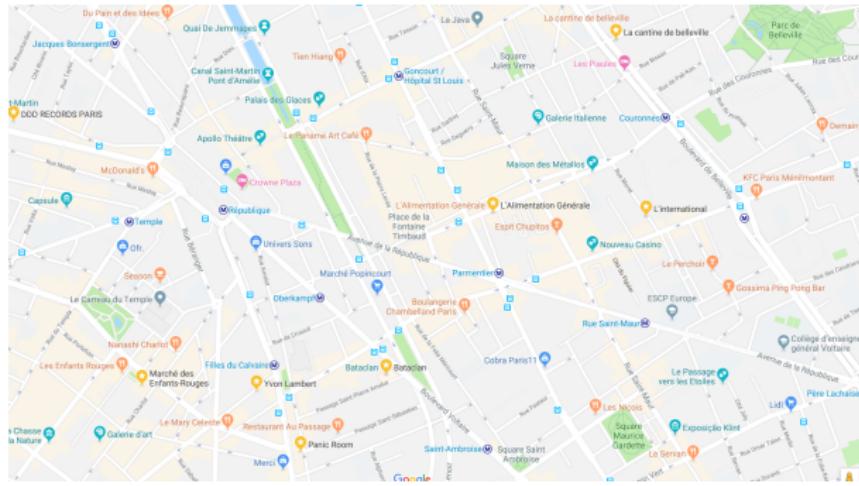


Figure: Optimizing the quantity of something transported from one place to another, under constraints

...

- └ The Maximum flow problem
- └ Presentation of the problem

## Formalizing the problem

We introduce the concept of **flow network (reseau de flot)**.

...

- └ The Maximum flow problem
  - └ Presentation of the problem

## Formalizing the problem

- ▶ A **Directed graph**  $G = (E, V)$

...

- The Maximum flow problem

- Presentation of the problem

## Formalizing the problem

We introduce the concept of **flow network (reseau de flot)**.

- ▶ A **Directed graph**  $G = (E, V)$
- ▶ Each edge  $(u, v)$  must have a **capacity**  $c(u, v) \geq 0$

...

- └ The Maximum flow problem
- └ Presentation of the problem

## Formalizing the problem

We introduce the concept of **flow network (reseau de flot)**.

- ▶ A **Directed graph**  $G = (E, V)$
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- ▶ We define two special nodes : a **source**  $E$  and a **sink**  $S$ .

## Formalizing the problem

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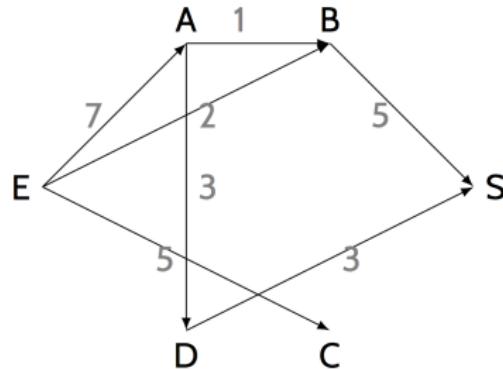


Figure: A **flow network (reseau de flot)** with capacities

...

- The Maximum flow problem

- Presentation of the problem

## Formalizing the problem

We introduce the concept of **flow network (reseau de flot)**.

- ▶ A **Directed graph**  $G = (E, V)$
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- ▶ We define two special nodes : a **source**  $E$  and a **sink**  $S$ .
- ▶ A **flow**  $f$  is a function  $f(u, v) \leq c(u, v)$  (+ additional constraints)

...

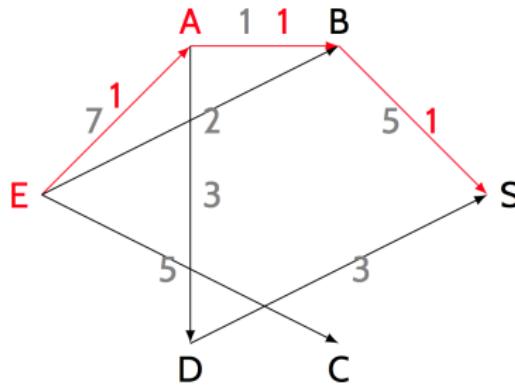
- The Maximum flow problem

- Presentation of the problem

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...

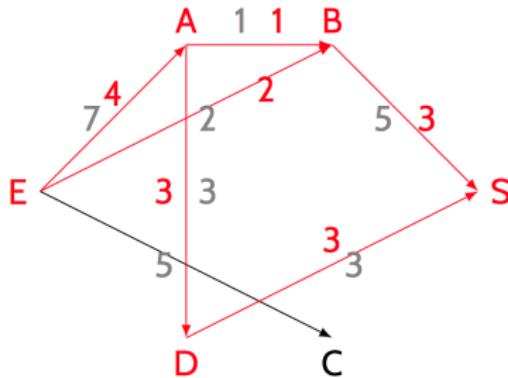
- The Maximum flow problem

- Presentation of the problem

## Formalizing the problem

We introduce the concept of **flow network (reseau de flot)**.

- ▶ Each edge  $(u, v)$  must have a **capacity**  $c(u, v) \geq 0$
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...

└ The Maximum flow problem

  └ Presentation of the problem

## Conservation of the flow

We must have :

- ▶ antisymmetry :  $f(v, u) = -f(u, v)$
- ▶ flow conservation

...

- └ The Maximum flow problem

- └ Presentation of the problem

## conservation of the flow

we must have :

- ▶ antisymmetry :  $f(v, u) = -f(u, v)$
- ▶ flow conservation :  $\sum_{w \in V} f(u, w) = 0$  for  $u \notin \{e, s\}$

...

- └ The Maximum flow problem

- └ Presentation of the problem

## Other formulation of the flow conservation

Let us show that for a flow  $f$ :

$$\sum_{f(u,v)>0} f(u,v) = \sum_{f(v,u)>0} f(v,u) \quad (6)$$

...

- The Maximum flow problem

- Presentation of the problem

## Maximum flow

- The **value of the flow**, noted  $|f|$ , is  $\sum_{v \in S} f(E, v)$
- The problem is that of finding a flow with **maximum value**

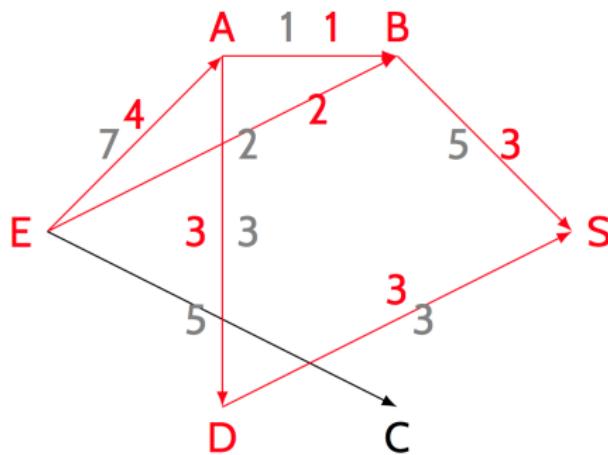


Figure: Max flow

## Ford Fulkerson algorithm

We will introduce an algorithm to solve the problem. This algorithm :

- ▶ terminates
- ▶ is correct
- ▶ is polynomial

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So it is a good algorithm.

## Ford Fulkerson algorithm

We will introduce an algorithm to solve the problem. This algorithm :

- ▶ terminates
- ▶ is correct
- ▶ is polynomial

So it a good algorithm. **This section is going to be a little bit technical.**

...

└ The Maximum flow problem

  └ Solution with the Ford-Fulkerson algorithm

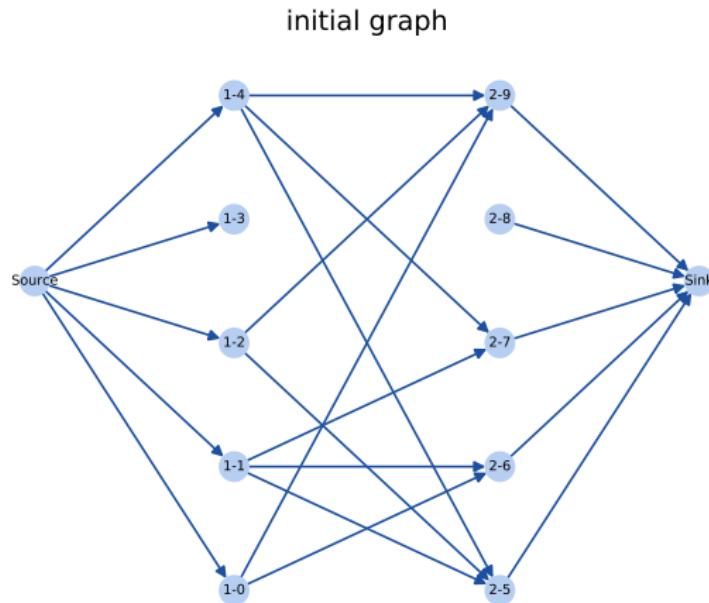
## Residual graph

- ▶ Given a graph with capacities  $c(u, v)$  and a flow  $f(u, v)$ , we will define its **residual graph** that has a capacity  $c_r(u, v)$  :

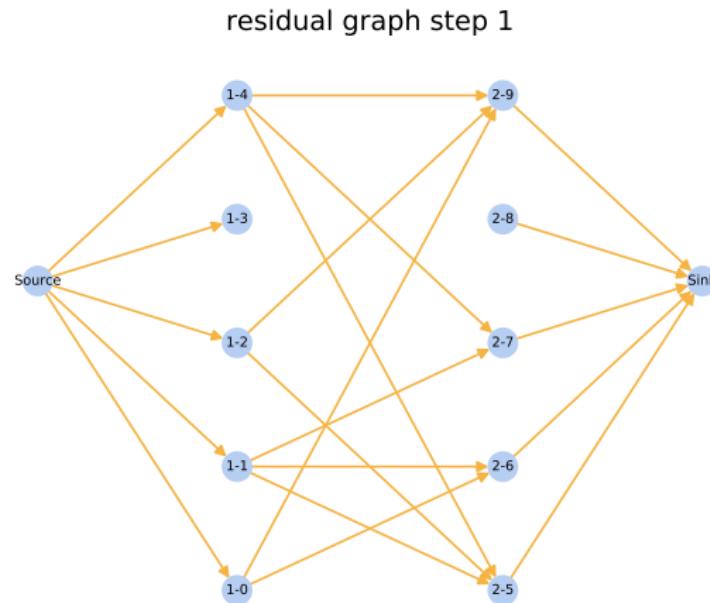
$$c_r(u, v) = c(u, v) - f(u, v) \quad (7)$$

- ...  
└ The Maximum flow problem  
  └ Solution with the Ford-Fulkerson algorithm

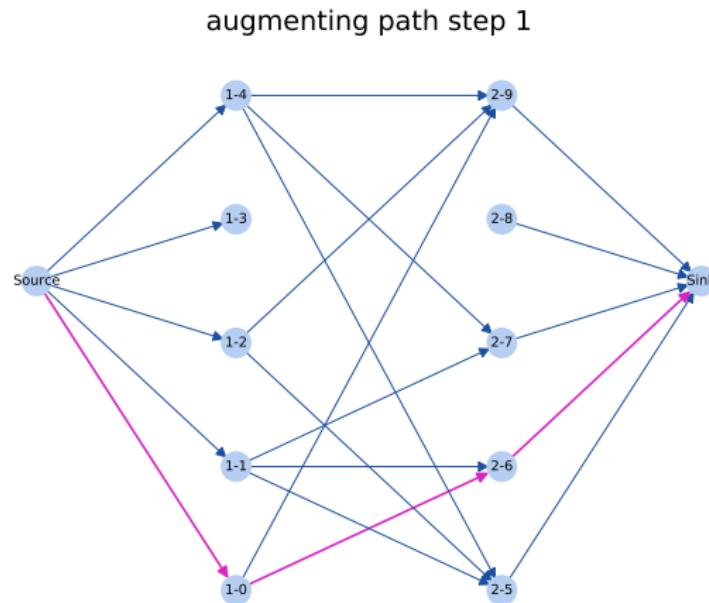
## Example of residual graph



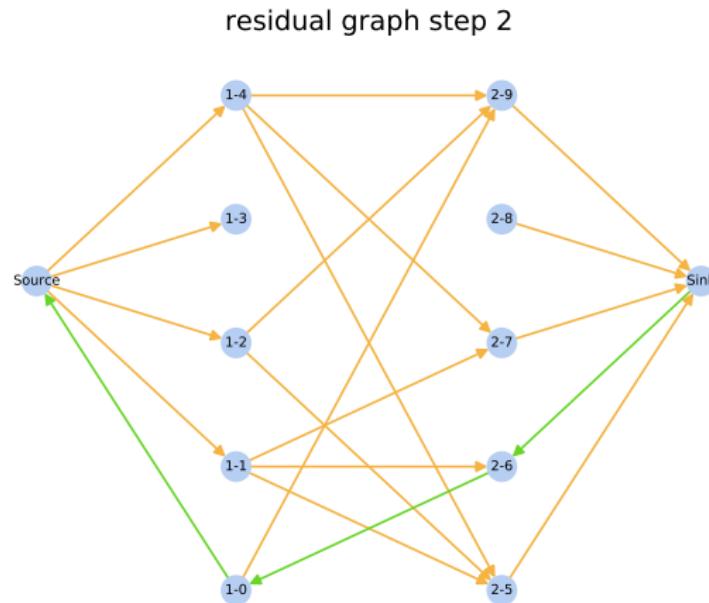
## Example of residual graph



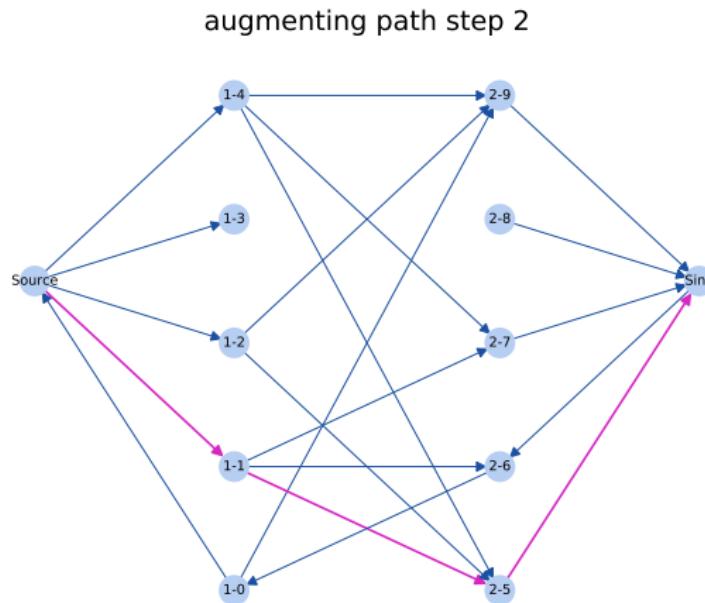
## Example of residual graph



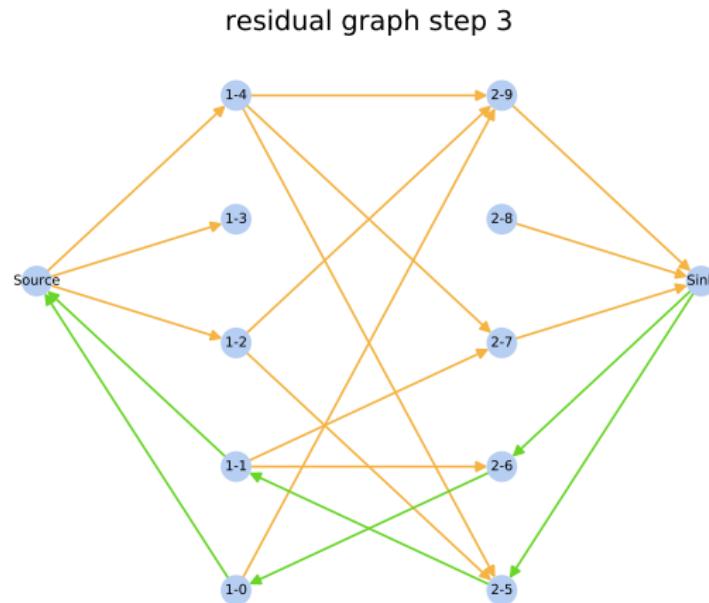
## Example of residual graph



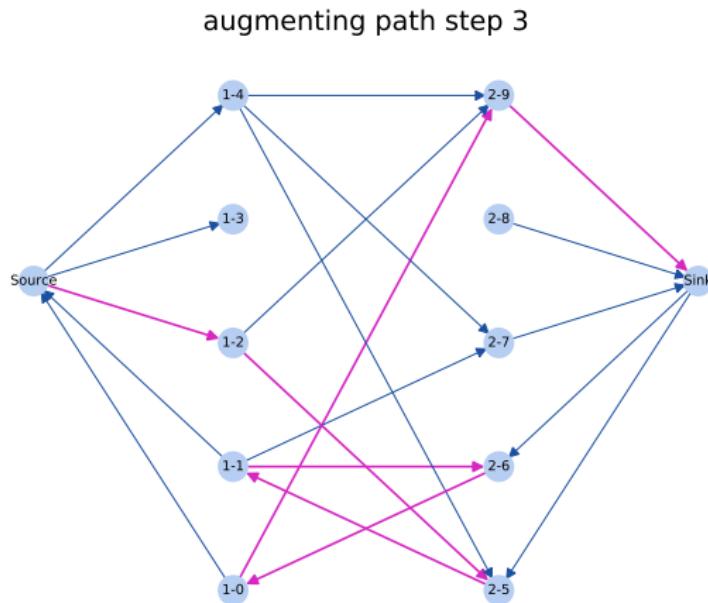
## Example of residual graph



## Example of residual graph



## Example of residual graph



## Residual graph

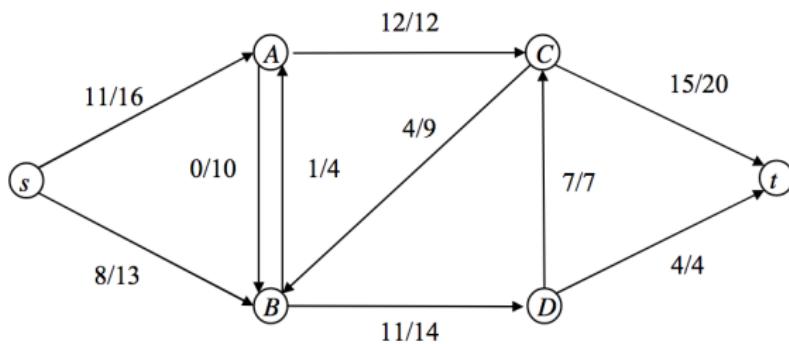


Figure: Another flow network

## Residual graph

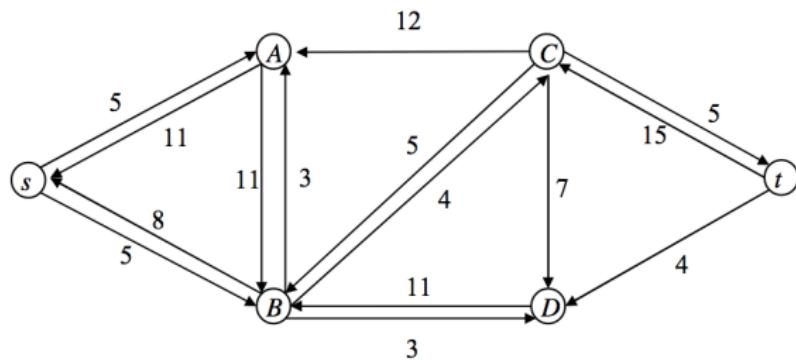


Figure: Residual graph

...

└ The Maximum flow problem

└ Solution with the Ford-Fulkerson algorithm

## Augmenting path

An augmenting path is a path in the **residual graph** from the source to the sink with capacities  $> 0$ .

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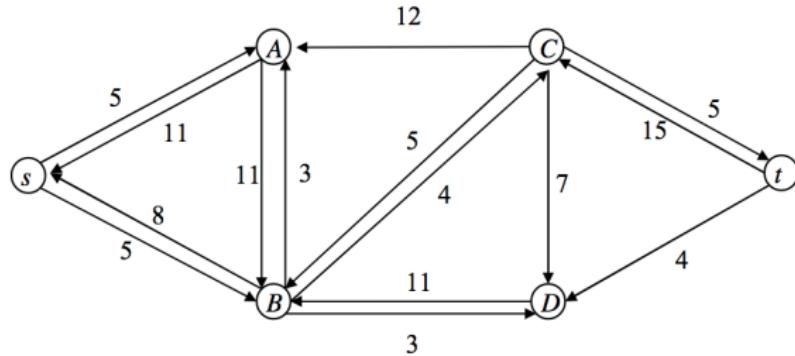


Figure: Residual graph

## Augmenting path

An augmenting path is a path from the source to the sink with capacities  $> 0$ .

The Ford-Fulkerson algorithm uses augmenting paths until there are no more augmenting paths.

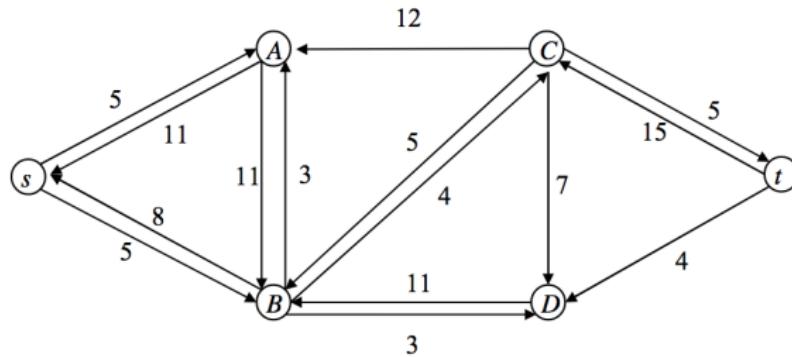


Figure: Residual graph

...

└ The Maximum flow problem

└ Solution with the Ford-Fulkerson algorithm

## Ford Fulkerson algorithm

Can you deduce the algorithm from the previous remarks ?

## Ford Fulkerson algorithm

**Result:** Flow  $f$

**for**  $(u, v) \in E$  **do**

  |  $f(u, v) = 0$

**end**

**while**  $\exists \rho$  augmenting path **do**

  | augment  $f$  with  $\rho$

**end**

return  $f$

**Algorithm 1:** Ford Fulkerson algorithm

...

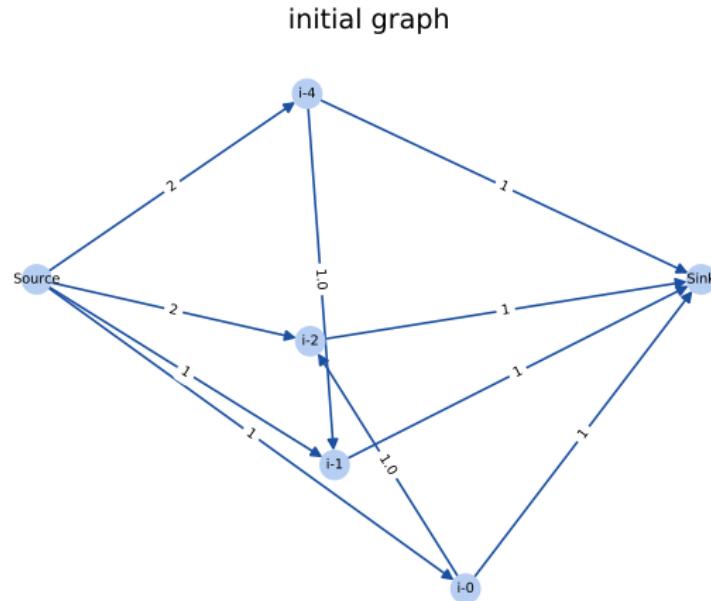
└ The Maximum flow problem

└ Solution with the Ford-Fulkerson algorithm

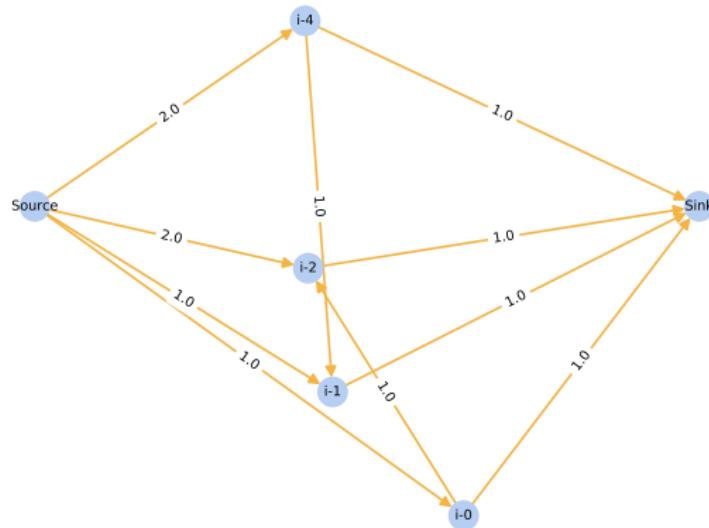
## Ford-Fulkerson algorithm

Let us some complete instances of the algorithm:

- ...  
└ The Maximum flow problem  
└ Solution with the Ford-Fulkerson algorithm



### residual graph step 1

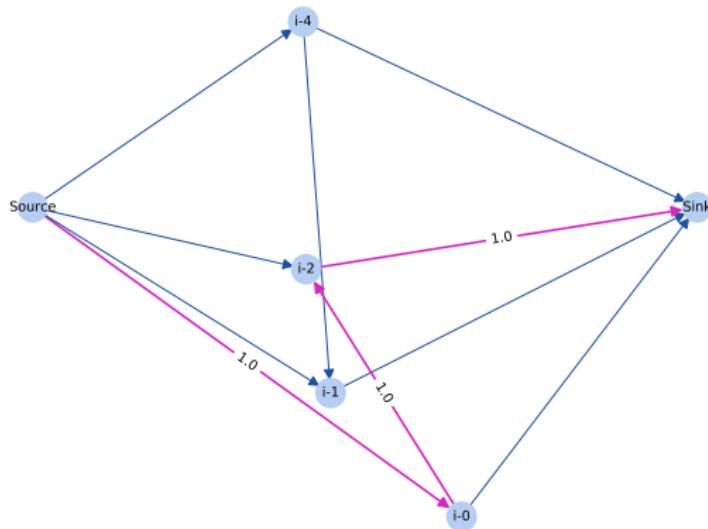


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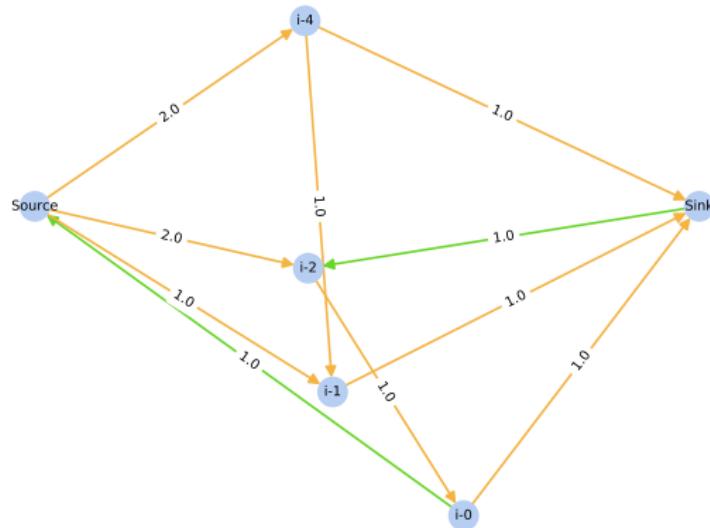
- └ The Maximum flow problem

- └ Solution with the Ford-Fulkerson algorithm

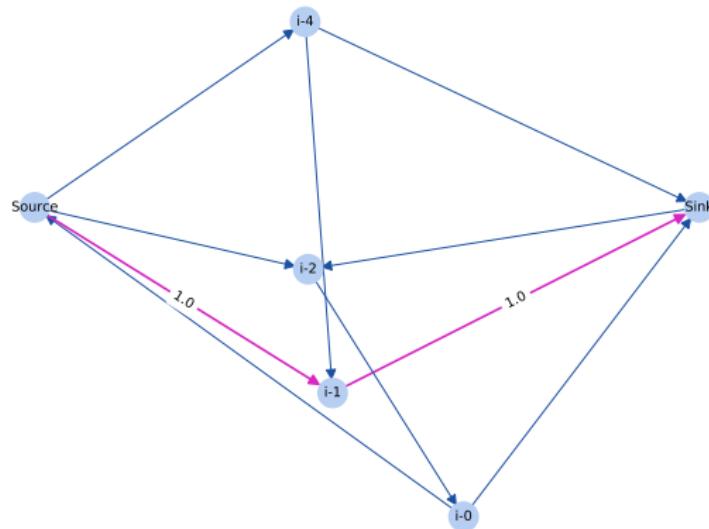
augmenting path step 1



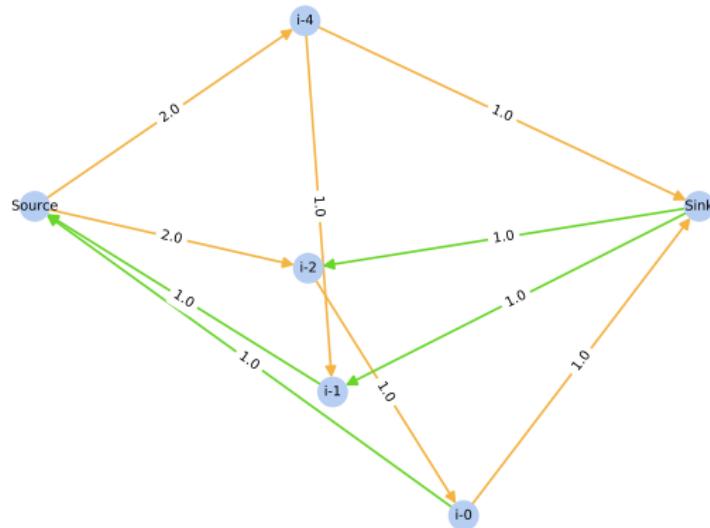
## residual graph step 2



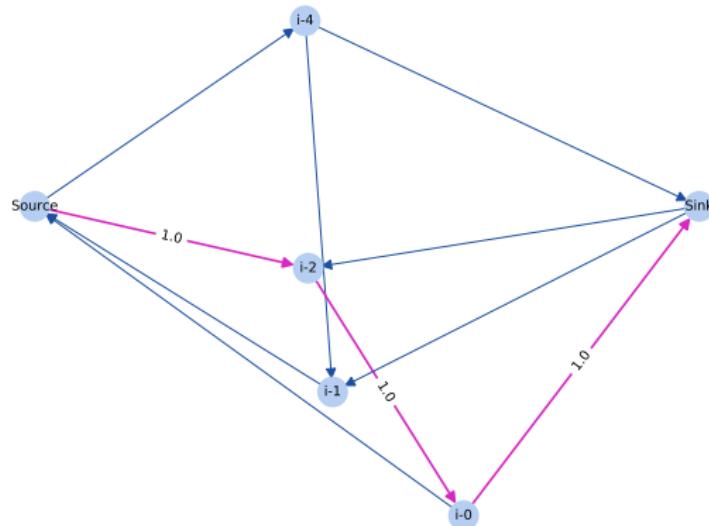
### augmenting path step 2



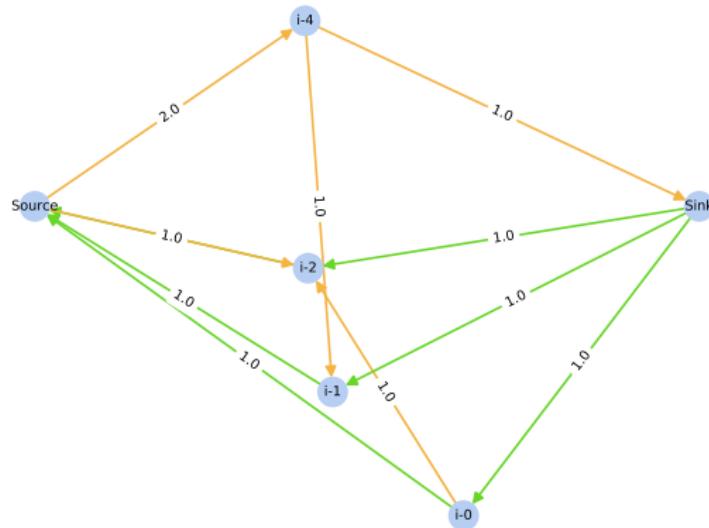
### residual graph step 3



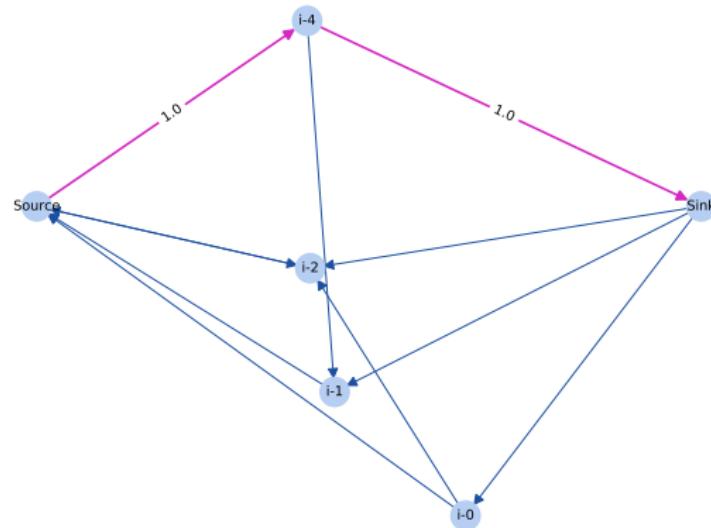
augmenting path step 3



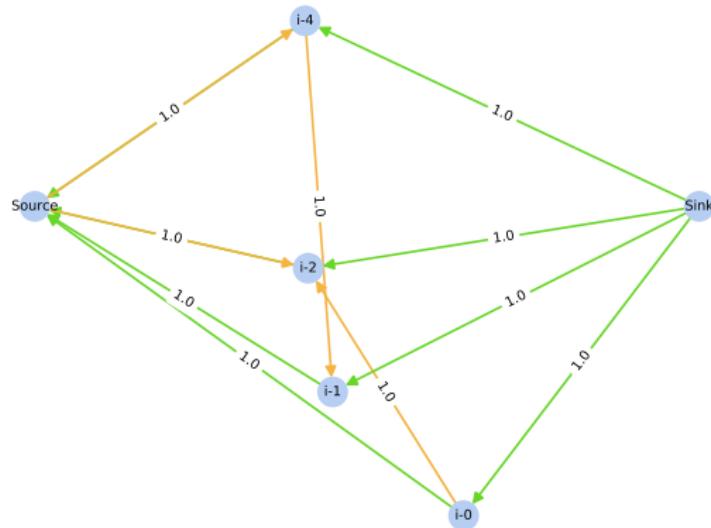
### residual graph step 4



### augmenting path step 4



### residual graph step 5



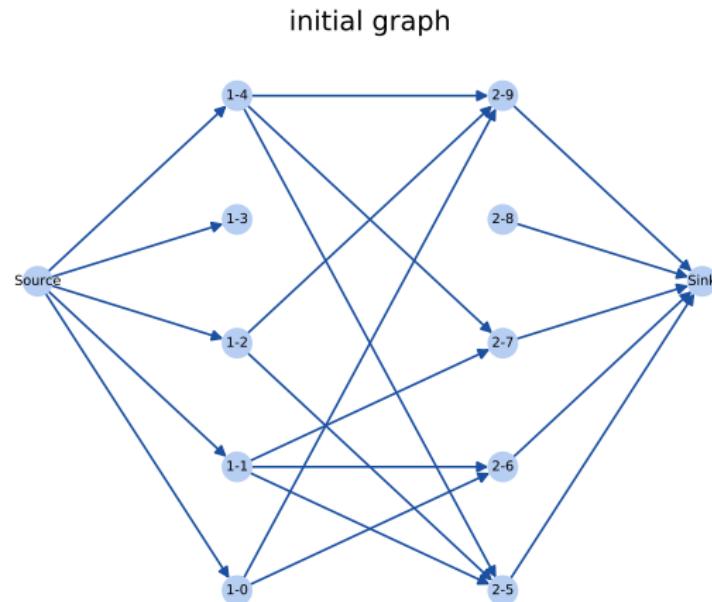
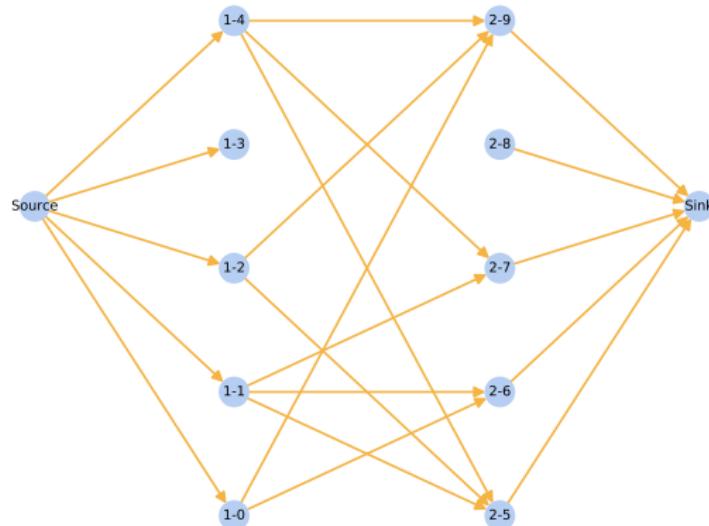
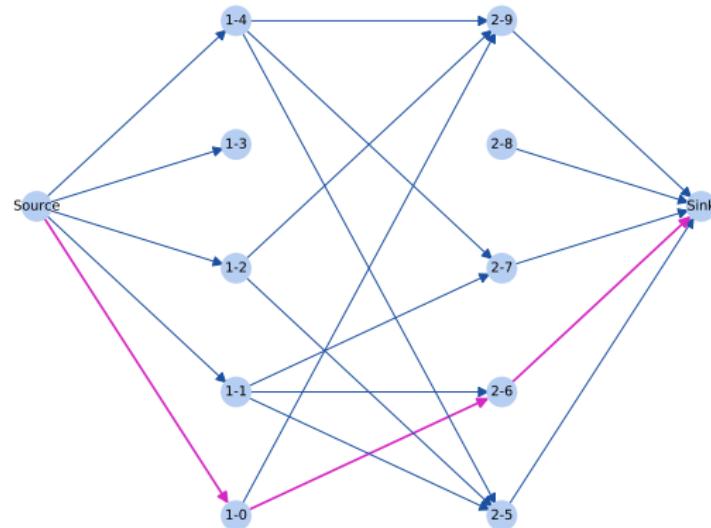


Figure: simple/initial

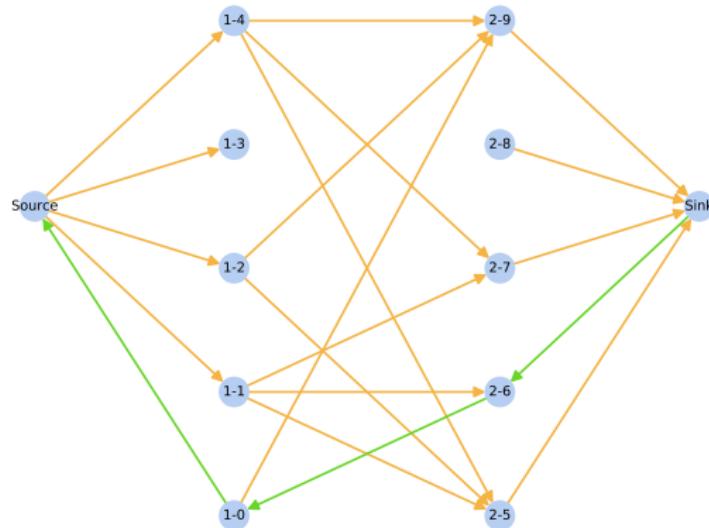
residual graph step 1



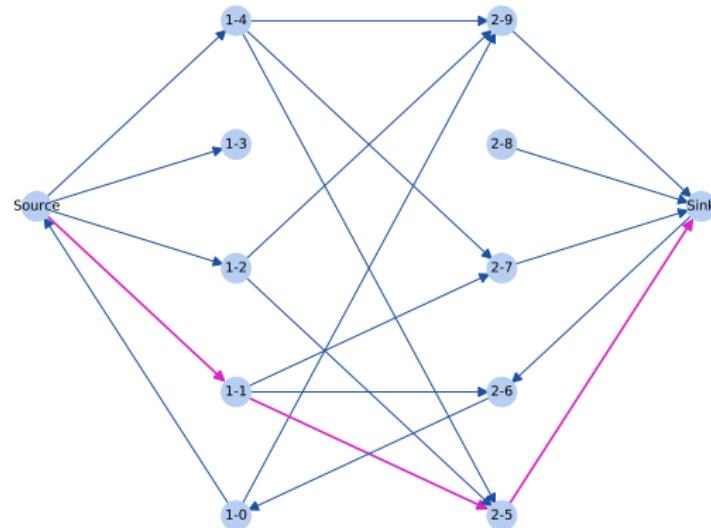
### augmenting path step 1



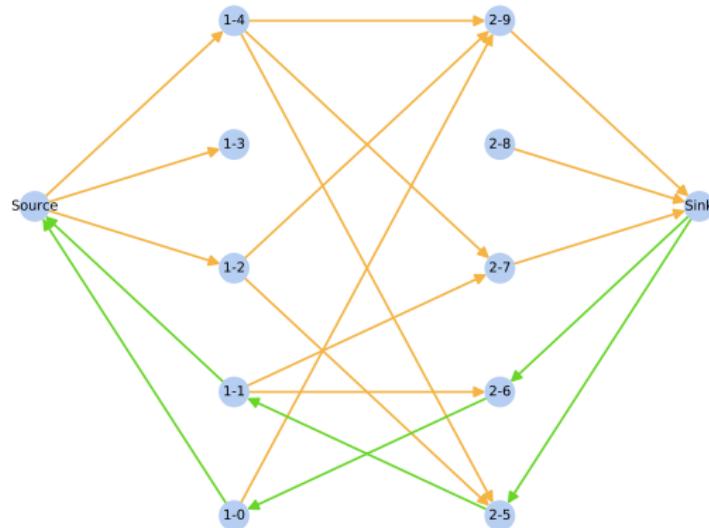
### residual graph step 2



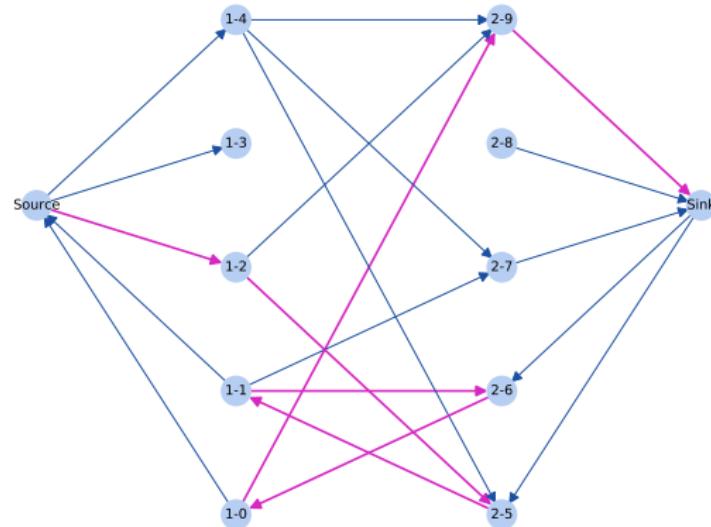
### augmenting path step 2



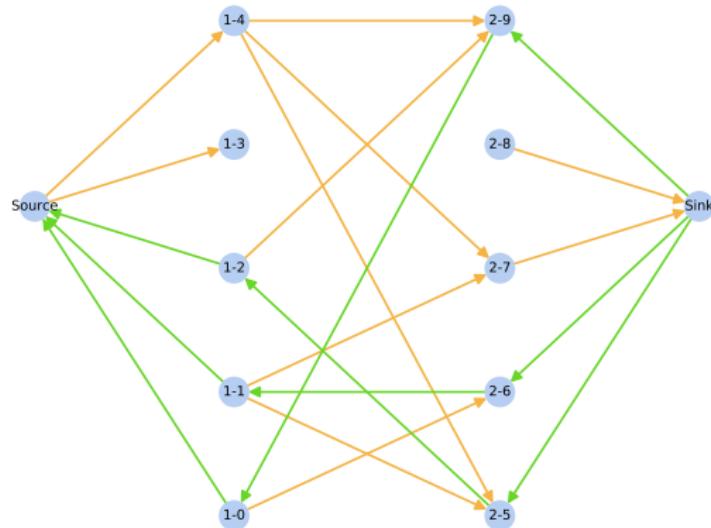
### residual graph step 3



### augmenting path step 3

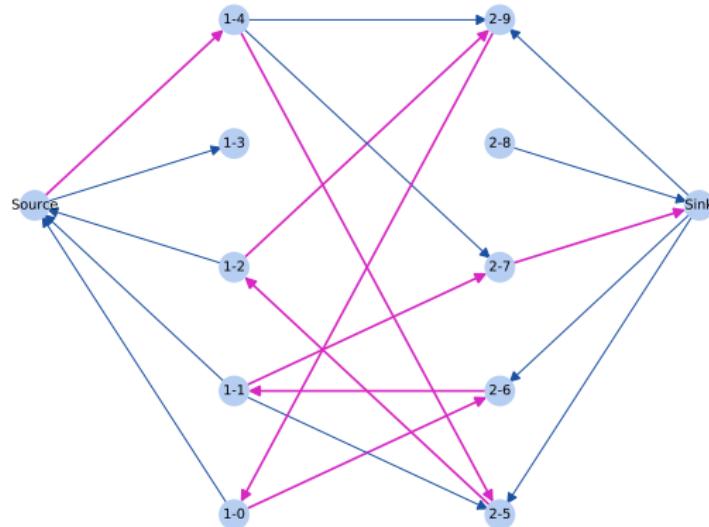


### residual graph step 4

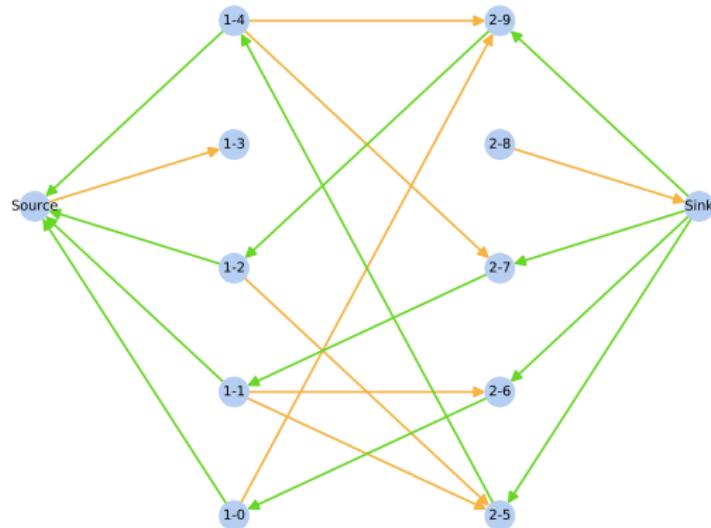


- ...  
└ The Maximum flow problem  
└ Solution with the Ford-Fulkerson algorithm

augmenting path step 4



### residual graph step 5



...

└ The Maximum flow problem

  └ Solution with the Ford-Fulkerson algorithm

## Ford Fulkerson algorithm

- ▶ We will implement the Ford Fulkerson algorithm (1956) on a general graph.

...

└ The Maximum flow problem

  └ Solution with the Ford-Fulkerson algorithm

## Ford Fulkerson algorithm

**Exercice 11:** We will implement the Ford Fulkerson algorithm (1956)

- ▶ **cd ford\_fulkerson/** and edit **generate\_flow\_network.py** to generate a flow network.

...

└ The Maximum flow problem

└ Solution with the Ford-Fulkerson algorithm

## Algorithm

- ▶ We will now use the functions contained in **ford\_functions.py** and call them from **apply\_ford\_fulkerson.py**

# Algorithm

Exercice 12 : step 1

- ▶ Modify **find\_augmenting\_paths()** in order to find the augmenting paths.

# Algorithm

Exercice 12 : step 2

- ▶ now edit **augment\_flow()**

...

└ The Maximum flow problem

  └ Solution with the Ford-Fulkerson algorithm

# Algorithm

Exercice 12 : step 3

- ▶ finally, edit the computation of the value of the flow

...

└ The Maximum flow problem

└ Solution with the Ford-Fulkerson algorithm

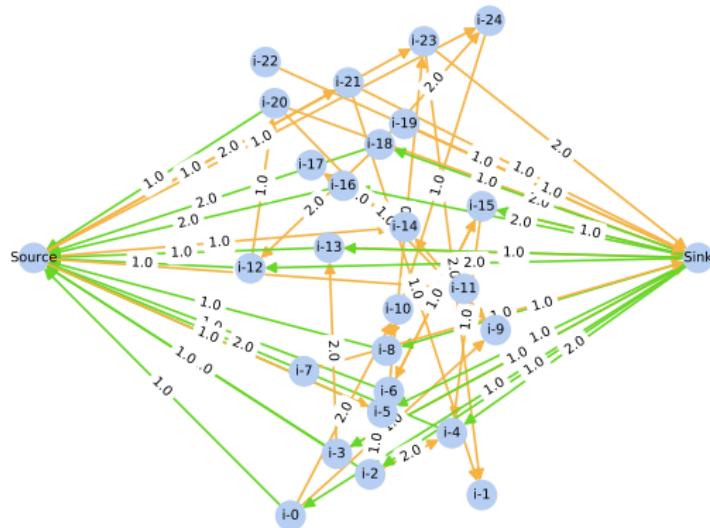
- ▶ Now the algorithm should be able to run

...

- The Maximum flow problem

- Solution with the Ford-Fulkerson algorithm

residual graph step 15



- ...
  - └ The Maximum flow problem
    - └ Solution with the Ford-Fulkerson algorithm

## Complexity

What is the complexity of Ford Fulkerson ?

...

└ The Maximum flow problem

  └ Solution with the Ford-Fulkerson algorithm

## Complexity

What is the complexity of Ford Fulkerson ?

$$\mathcal{O}(|f^*| \times |E|) \tag{8}$$

...

└ The Maximum flow problem

  └ Solution with the Ford-Fulkerson algorithm

## Modification of Ford Fulkerson

What would we an intuitive and potentially faster modification of the algorithm ?

...

└ The Maximum flow problem

  └ Solution with the Ford-Fulkerson algorithm

## Edmonds Karp

What would we an intuitive and potentially faster modification of the algorithm ?

Use the shortest augmenting path with positive capacity.

...

└ The Maximum flow problem

└ Solution with the Ford-Fulkerson algorithm

## Termination

- ▶ When the capacities are **real numbers** or **rational numbers** Ford Fulkerson terminates.

...

└ The Maximum flow problem

  └ Solution with the Ford-Fulkerson algorithm

## Termination

- ▶ When the capacities are **integer numbers** or **rational numbers** Ford Fulkerson terminates.
- ▶ However, when the capacities are general **real numbers**, the algorithm might not terminate.

...

└ The Maximum flow problem

└ Connection with the matching problem

## Link with the matching problem

- ▶ We now go back to the matching problem, in the case of a **bipartite graph**.
- ▶ We will show that in that case, we can connect the two problems.

...

└ The Maximum flow problem

└ Connection with the matching problem

## Link with the matching problem

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...

- └ The Maximum flow problem

- └ Connection with the matching problem

## Bipartite graph

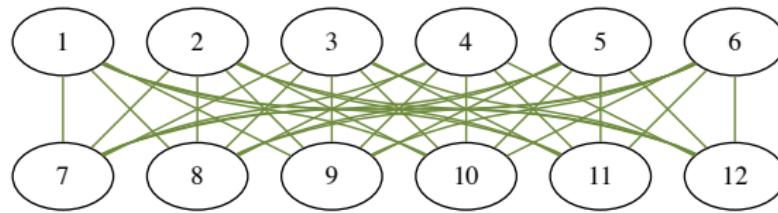


Figure: Complete bipartite graph (not all bipartite graphs are complete)

## Matching problem

We now go back to the matching problem, in the case of a **bipartite graph**.

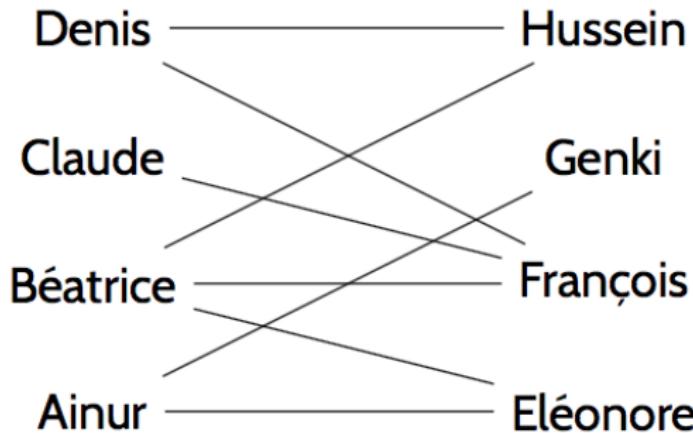


Figure: Bipartite graph

## Equivalence between matching and flow

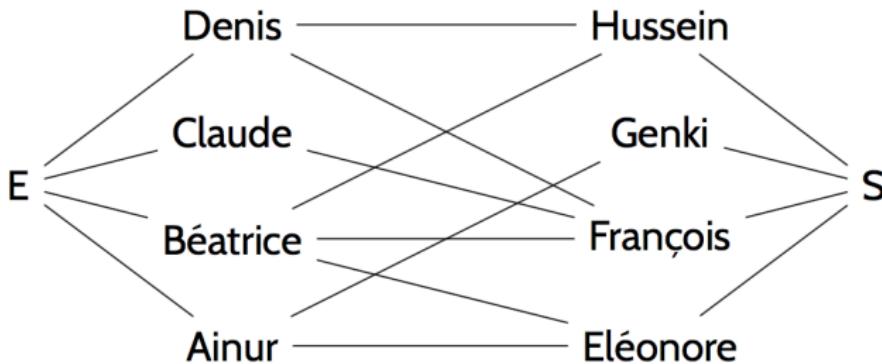


Figure: Introduce two more nodes. All edges have capacity 1. We consider **flows with integer values**

## Ford Fulkerson for matching

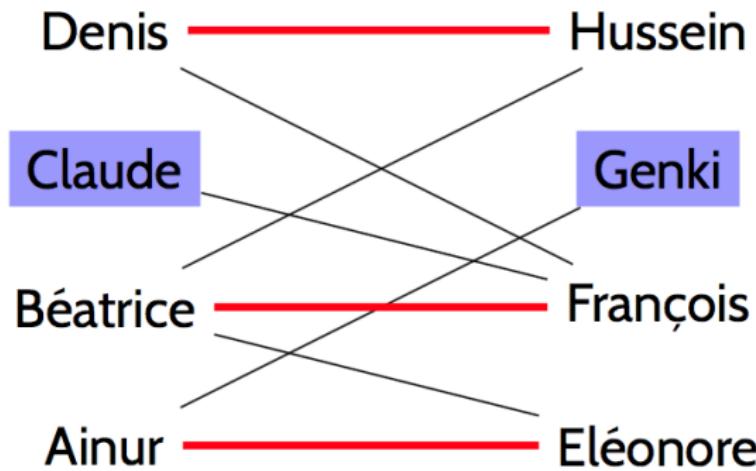


Figure: Non optimal solution

## Ford Fulkerson for matching

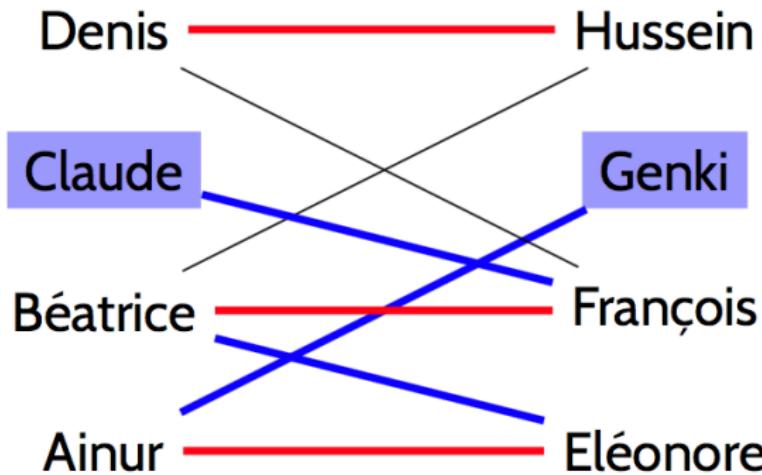


Figure: Optimal solution

## Connection

Exercice 12: Find a connection between the two problems

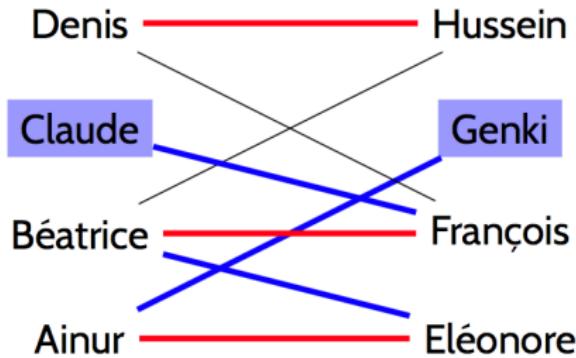


Figure: Optimal solution

### Exercice 13 : Ford Fulkerson and matching

- ▶ We will transpose Ford Fulkerson to a bipartite graph in order to find an optimal matching.
- ▶ **cd ford\_matching/**
- ▶ edit **generate\_matching\_problem.py** in order to generate an instance of the problem.

...

└ The Maximum flow problem

└ Connection with the matching problem

### Exercice 13: Ford Fulkerson and matching

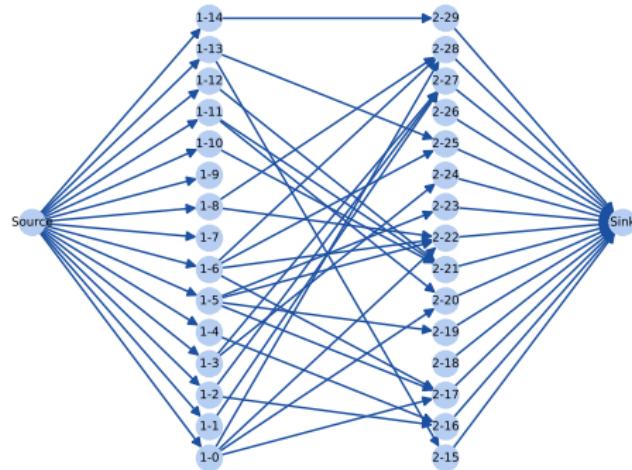
- ▶ Apply the algorithm on an example generated by the previous function.
- ▶ Apply the algorithm to as an instance of your choice.

...

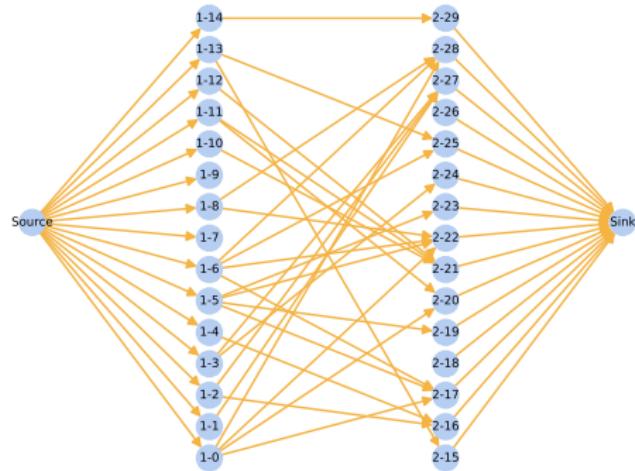
## The Maximum flow problem

### Connection with the matching problem

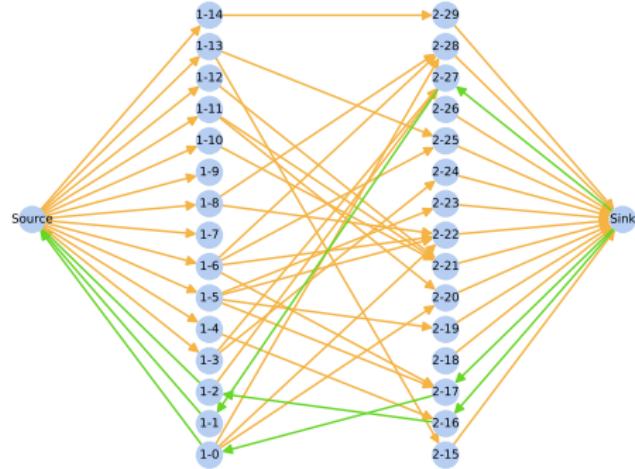
initial graph



### residual graph step 1



### residual graph step 4

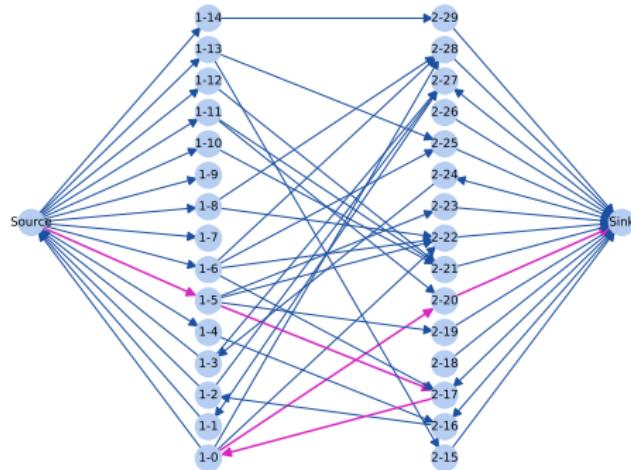


...

- The Maximum flow problem

- Connection with the matching problem

augmenting path step 5

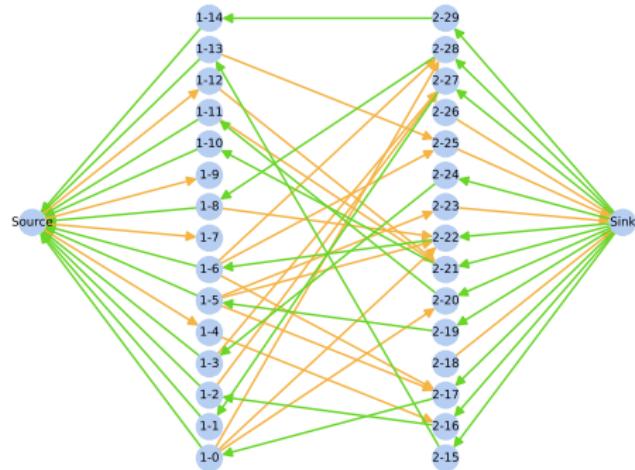


...

## The Maximum flow problem

### Connection with the matching problem

residual graph step 12



## Famous theorem

The maximum flow theorem is equivalent to another famous problem, the **minimum cut** theorem.

## Perfect matching

In the case of a bipartite graph, what is the best matching possible ?

## Perfect matching

In the case of a bipartite graph, what is the best matching possible ?

A matching where **all nodes are allocated**. It is called a **perfect** matching.

We must have that the two parts of the graph are of same cardinality in order to have a perfect matching.

## Hall's marriage theorem

This theorem gives a condition that is necessary and sufficient for the existence of a perfect matching in a bipartite graph : the "marriage condition".

If  $G = (U, V, E)$  is bipartite, the condition means that :

$$\forall X \subset U, |N_G(X)| \geq |X| \tag{9}$$

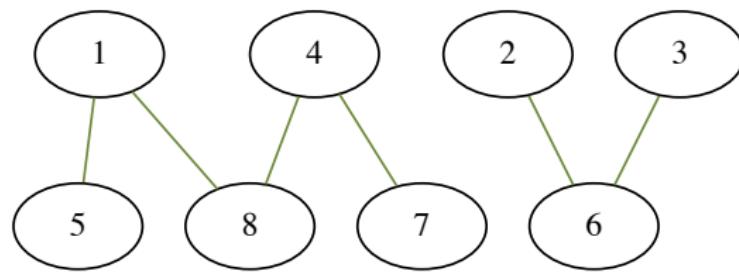
where  $N_G(X)$  is the set of neighbors of  $X$  in  $G$ .

## Hall's theorem

**Exercice 14:** Application of the theorem. Can you think of a graph that does not abide by the marriage condition and thus has **no perfect matching** ?

## Illustration of Hall's theorem

Exercice 14 : Application of the theorem



## Conclusion

Ford Fulkerson and its variants (Edmonds-Karp) are polynomial.  
As a result they can run on datasets that are way bigger than  
exhaustive search algorithms.

- ...  
└ The Maximum flow problem  
└ More results on the two problems

See you tomorrow