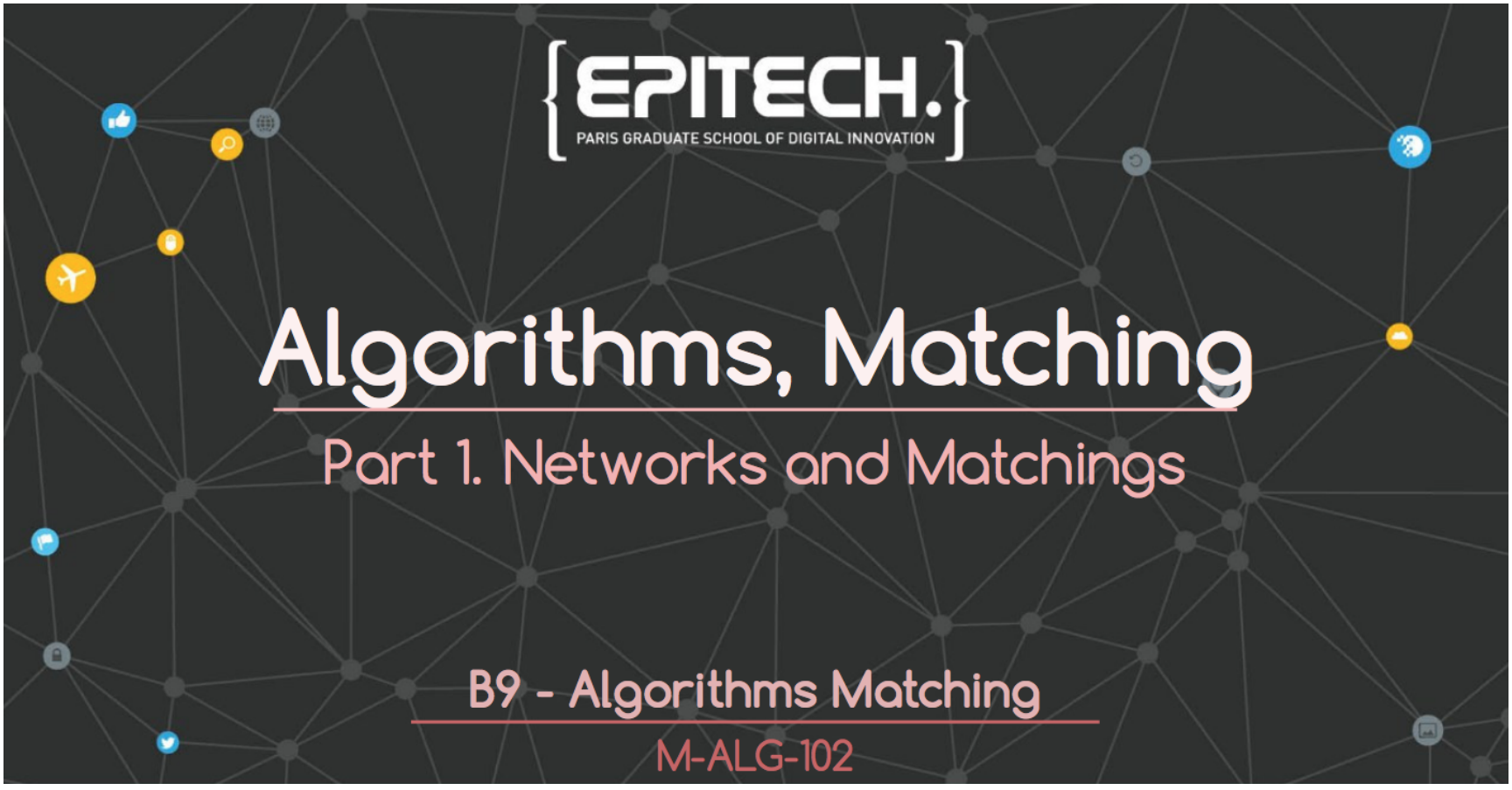
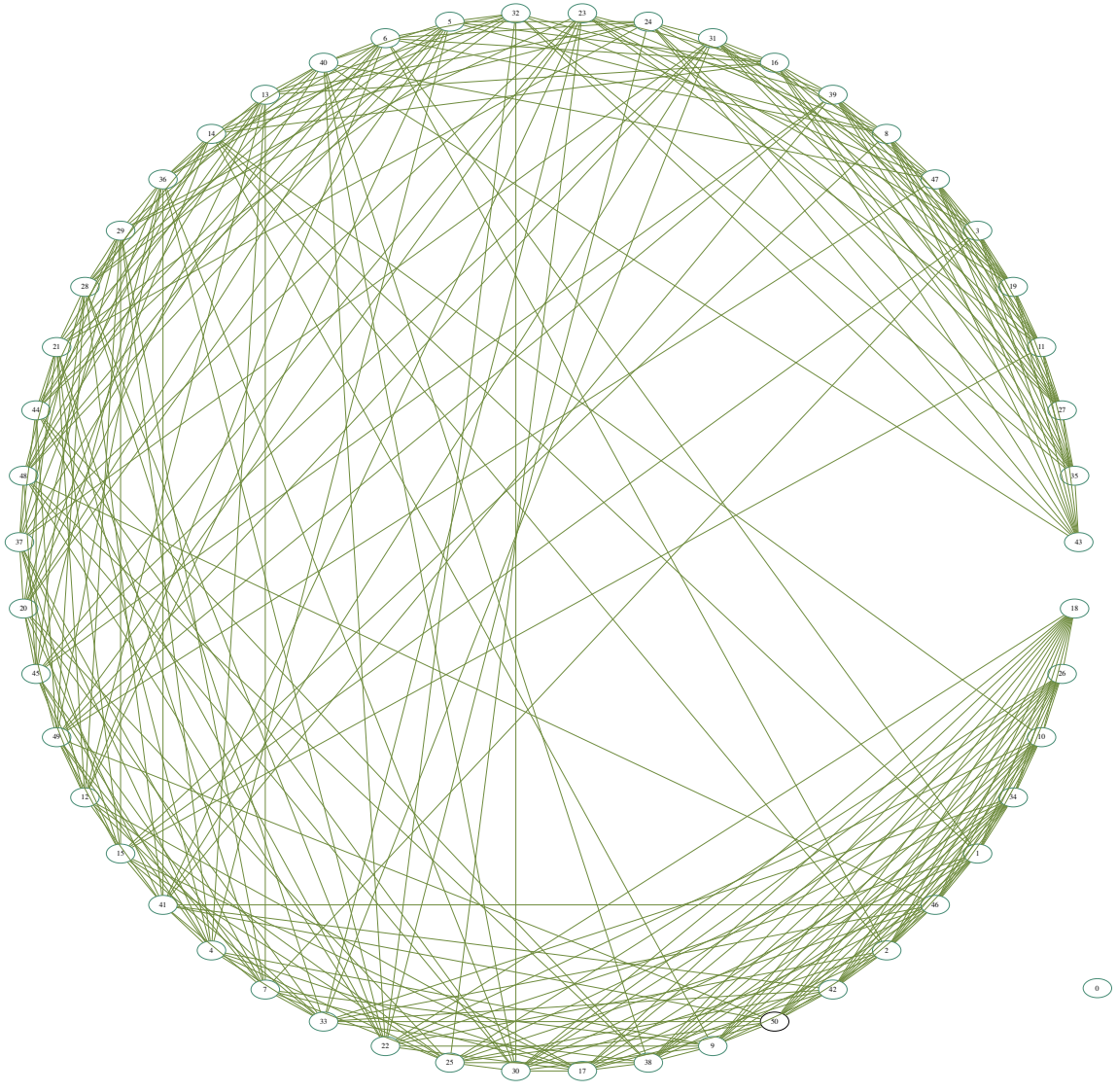

EPITECH.
PARIS GRADUATE SCHOOL OF DIGITAL INNOVATION
Algorithms, Matching
Part 1. Networks and Matchings
B9 - Algorithms Matching
M-ALG-102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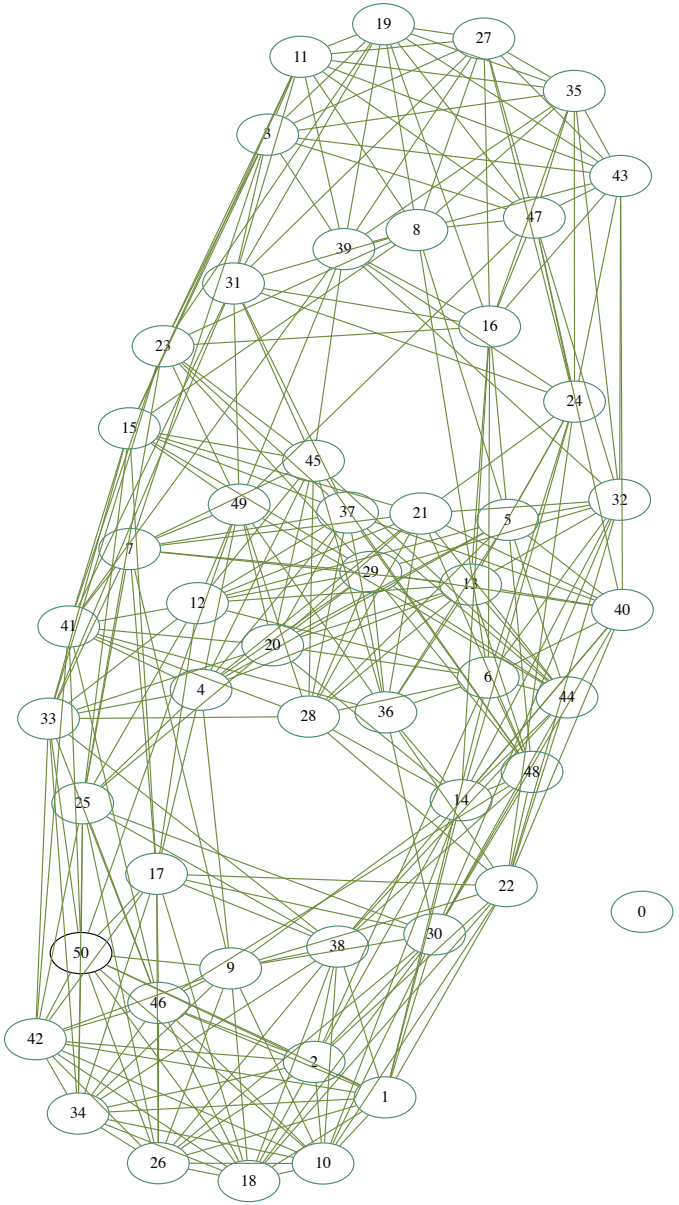
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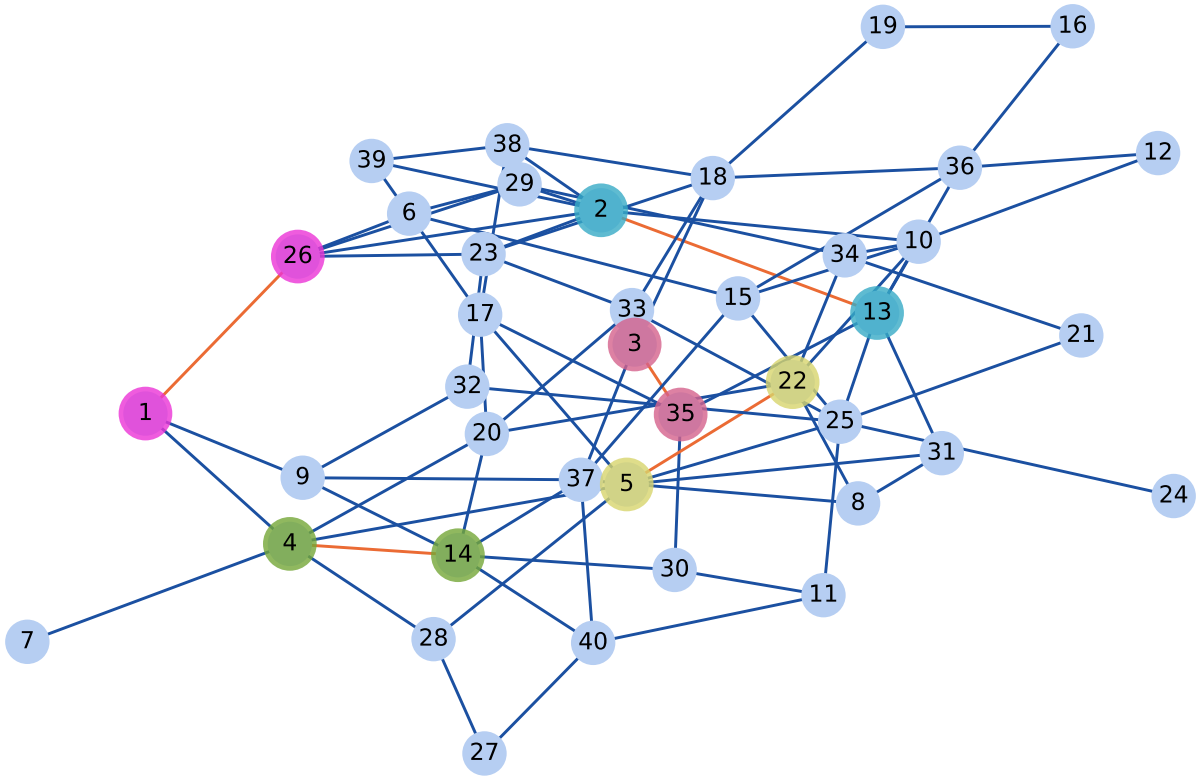
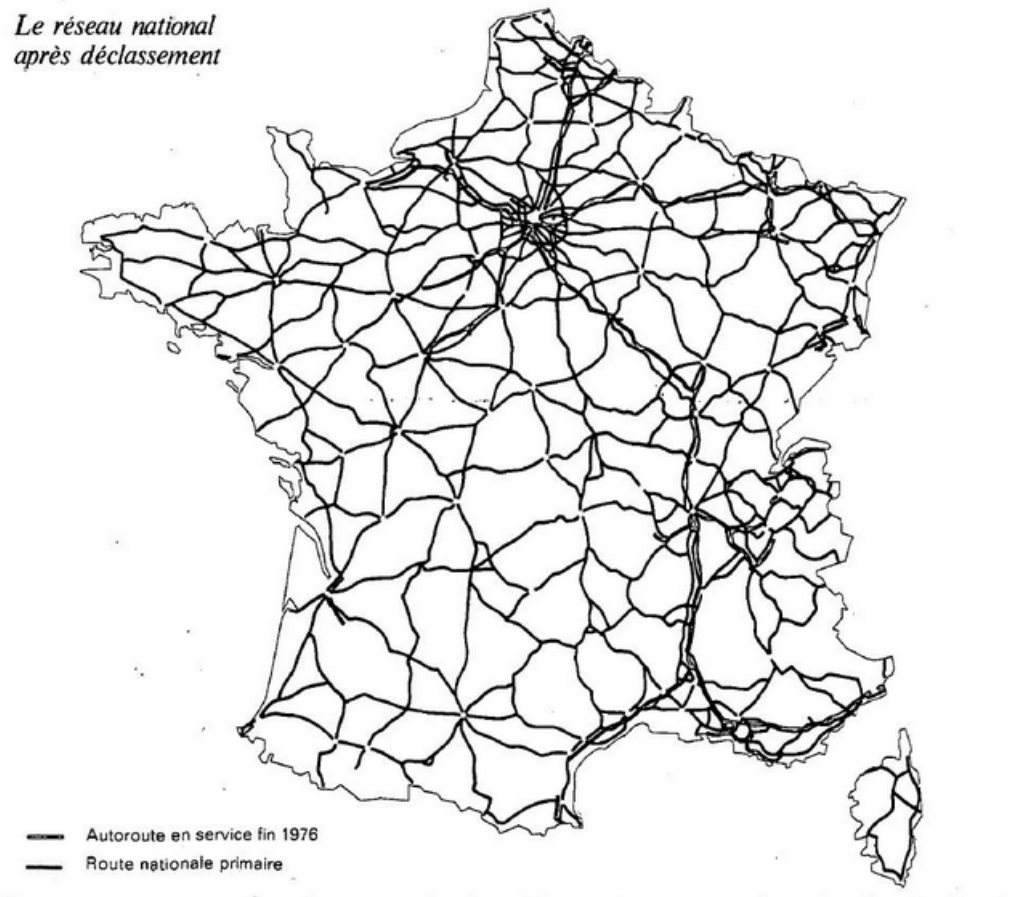
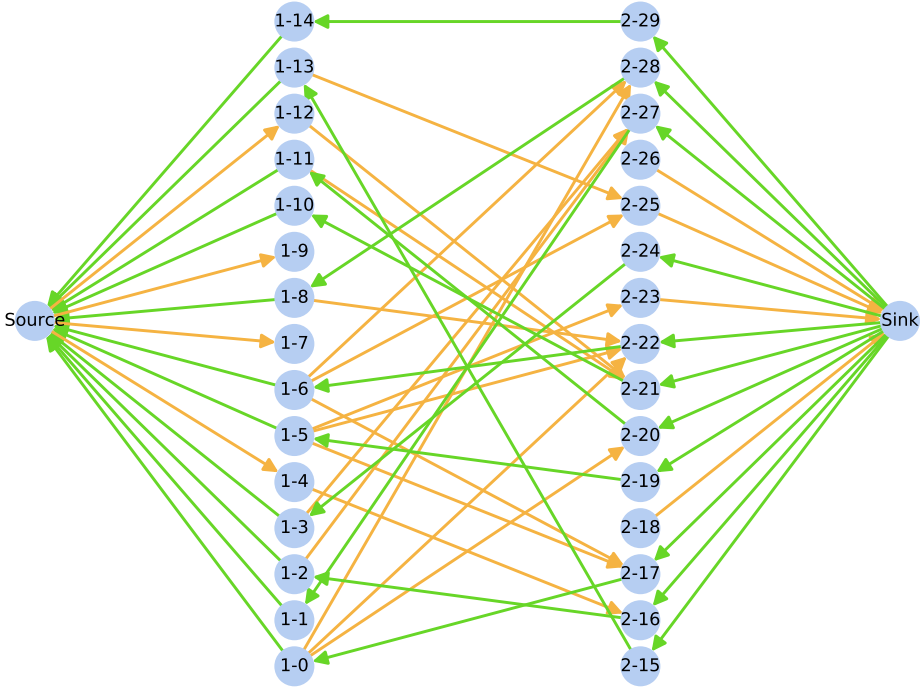
Matching size: 5
Algo step: 19
Nb nodes: 40

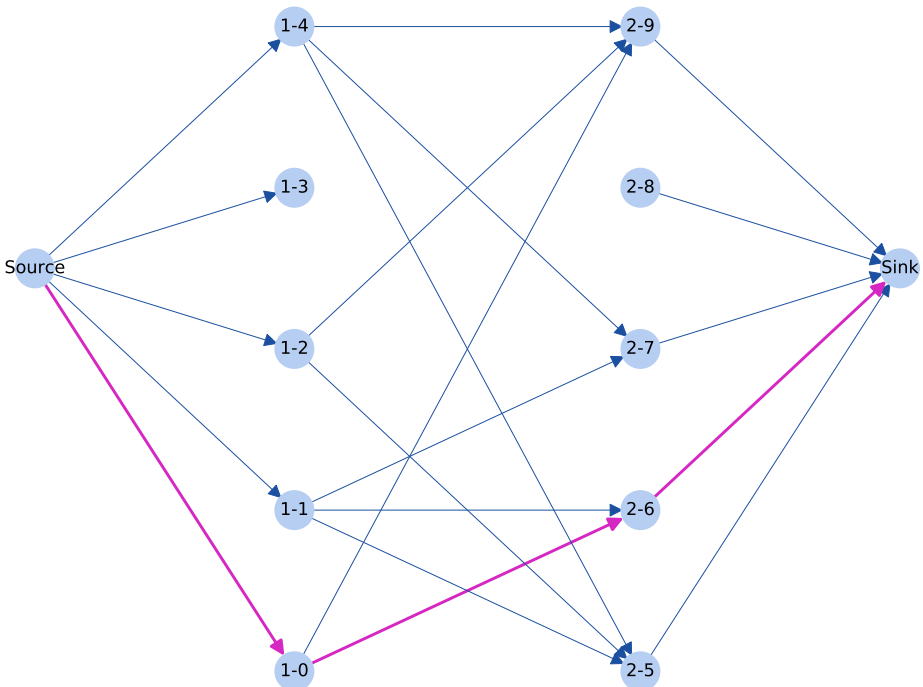


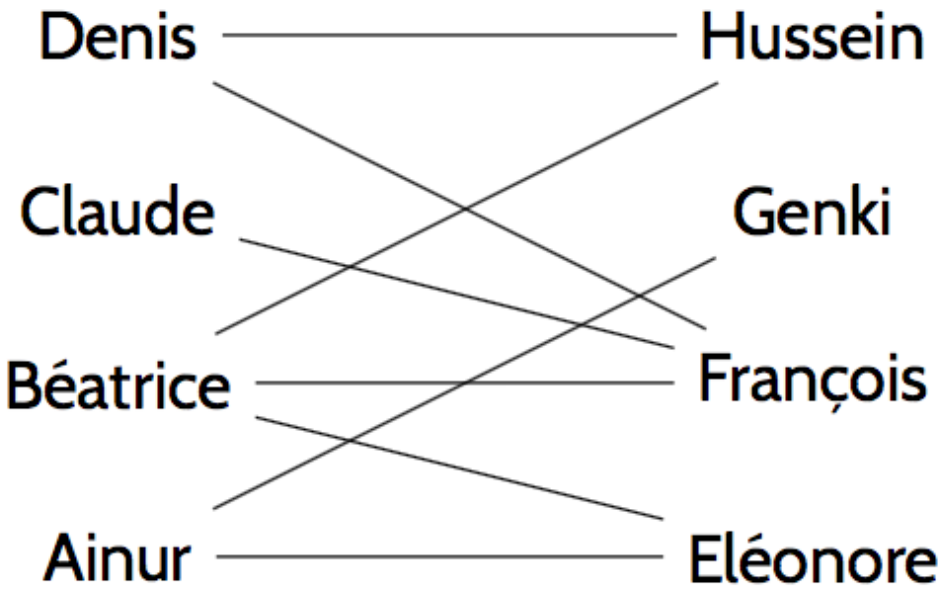

Le réseau national
après déclassement
Autoroute en service fin 1976
Route nationale primaire

residual graph step 12



augmenting path step 1

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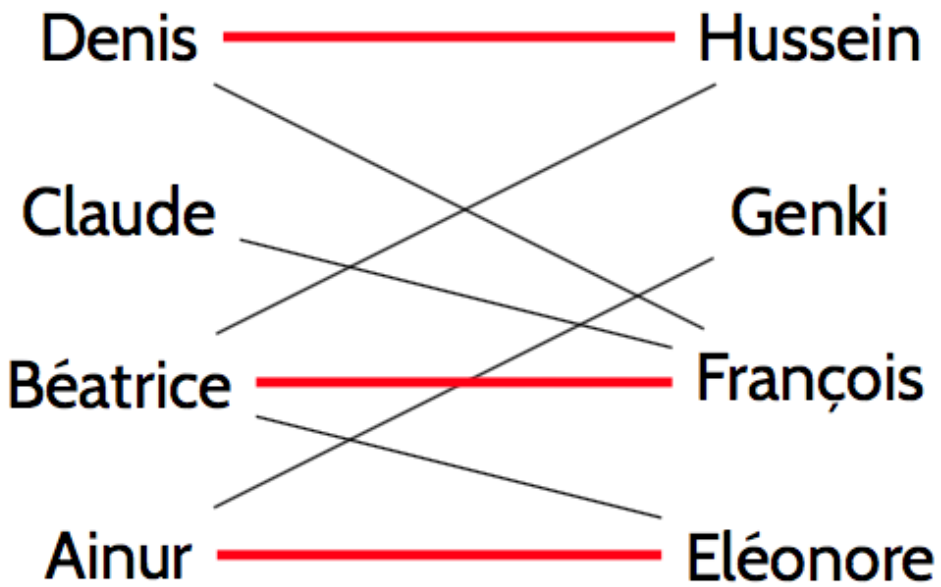
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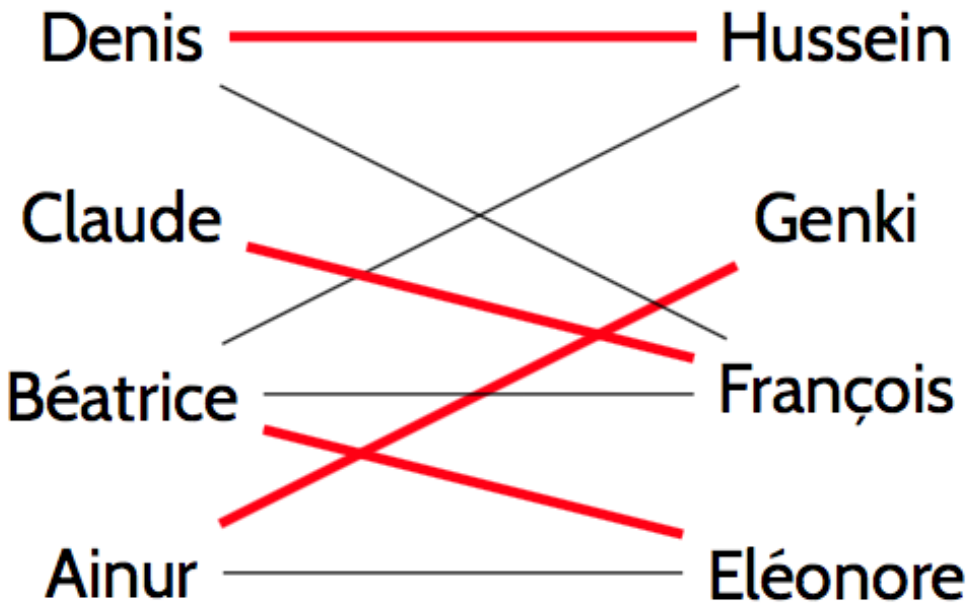
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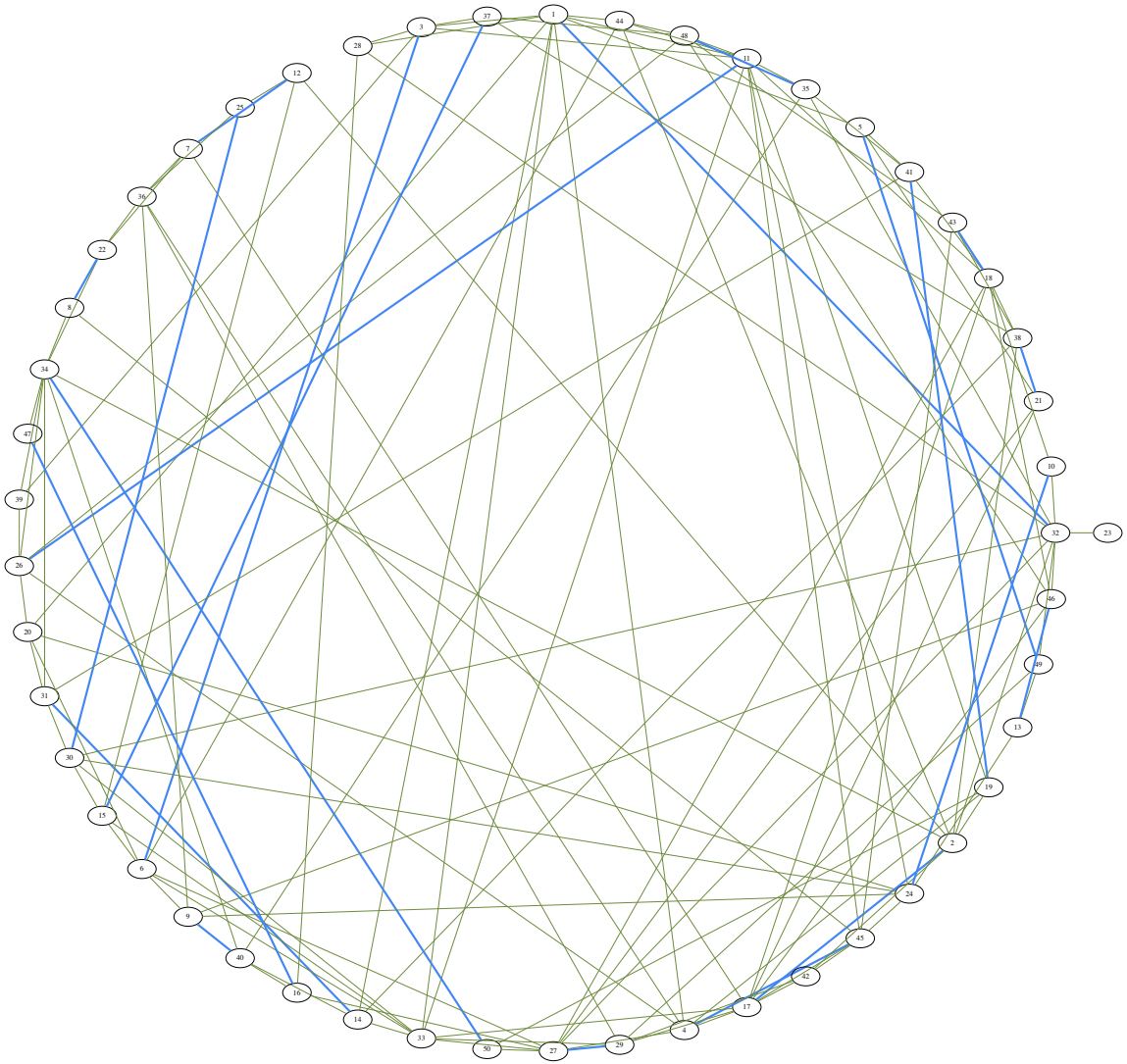
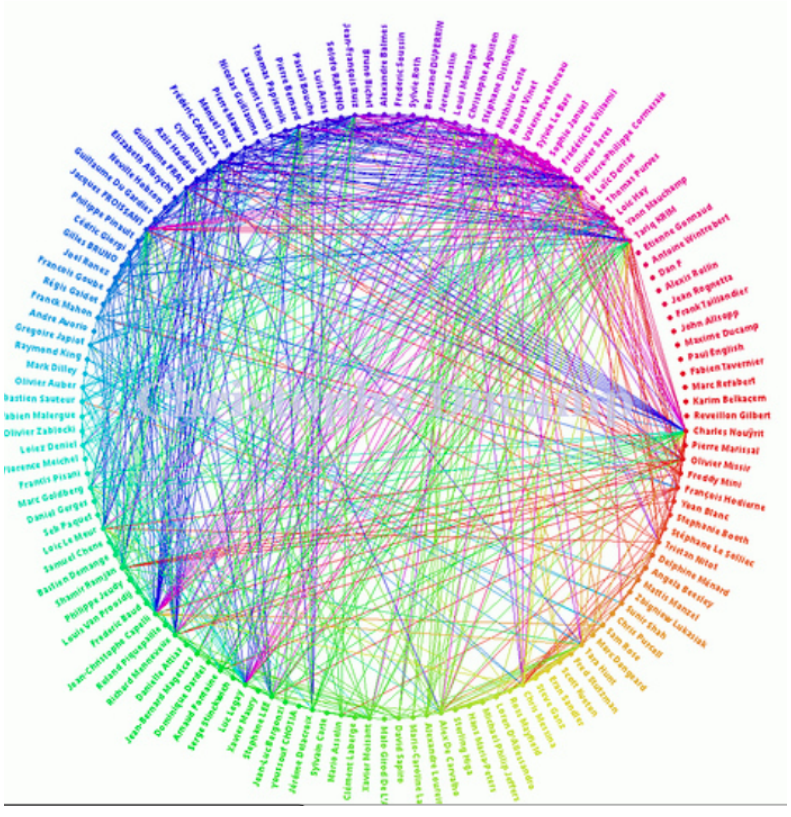
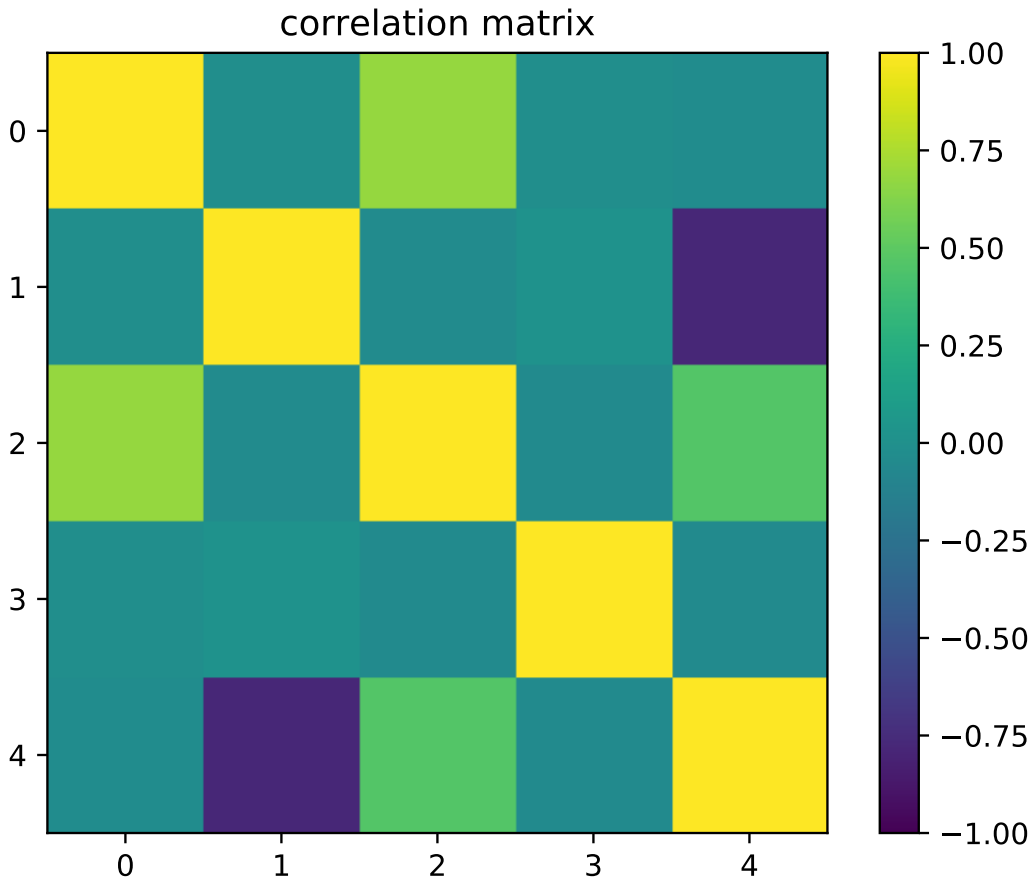


Matching size: 21
Algo step: 128
Nb nodes: 50




correlation matrix
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−0.75
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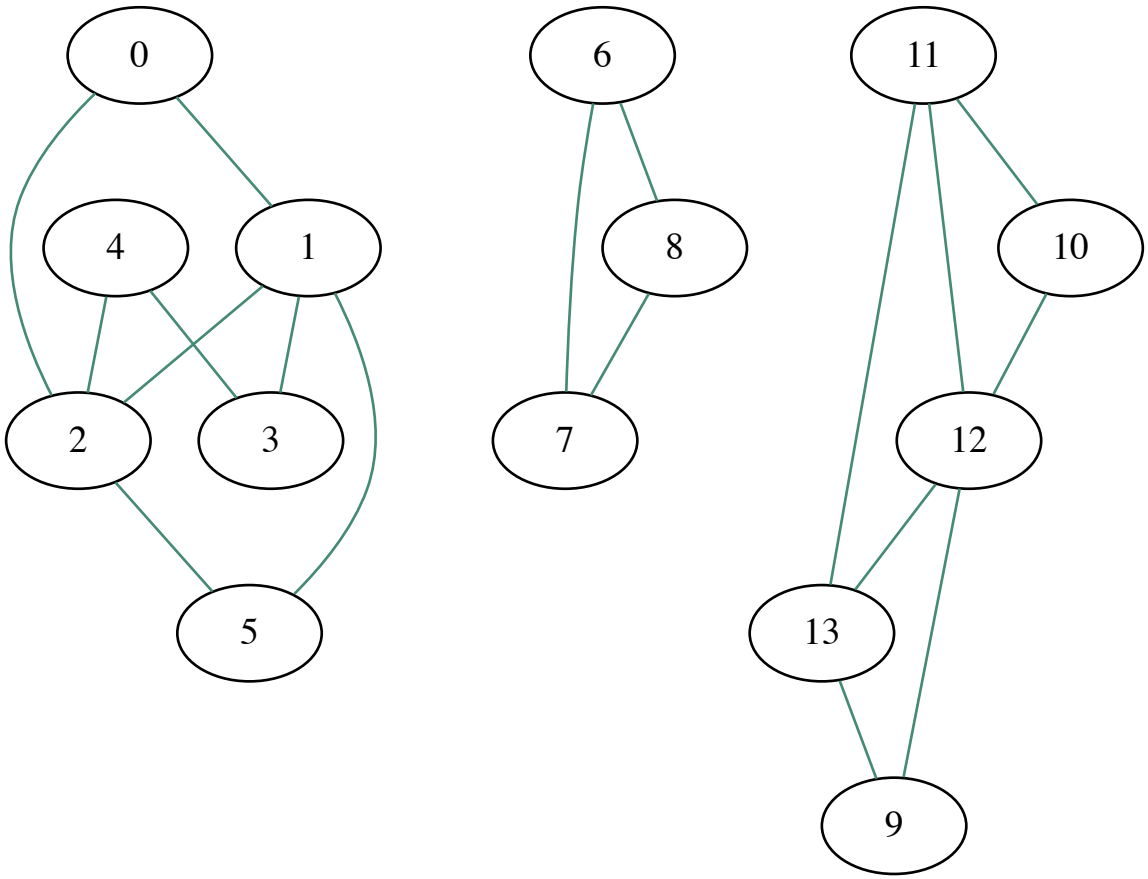
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9

- The content of the course will sometimes be mathematical.
- Don't hesitate to ask if you need more reminders.

Overview of the module

Day 1 Networks, the matching problem and the maximum flow problem

Day 2 Data clustering and representation

Organisation of the module

- Course and exercises in python 3
- Small coding exercises, also paper + pen
- Project : explained tomorrow
- Please clone the following repository
  `https://github.com/nlehir/ALGO2`

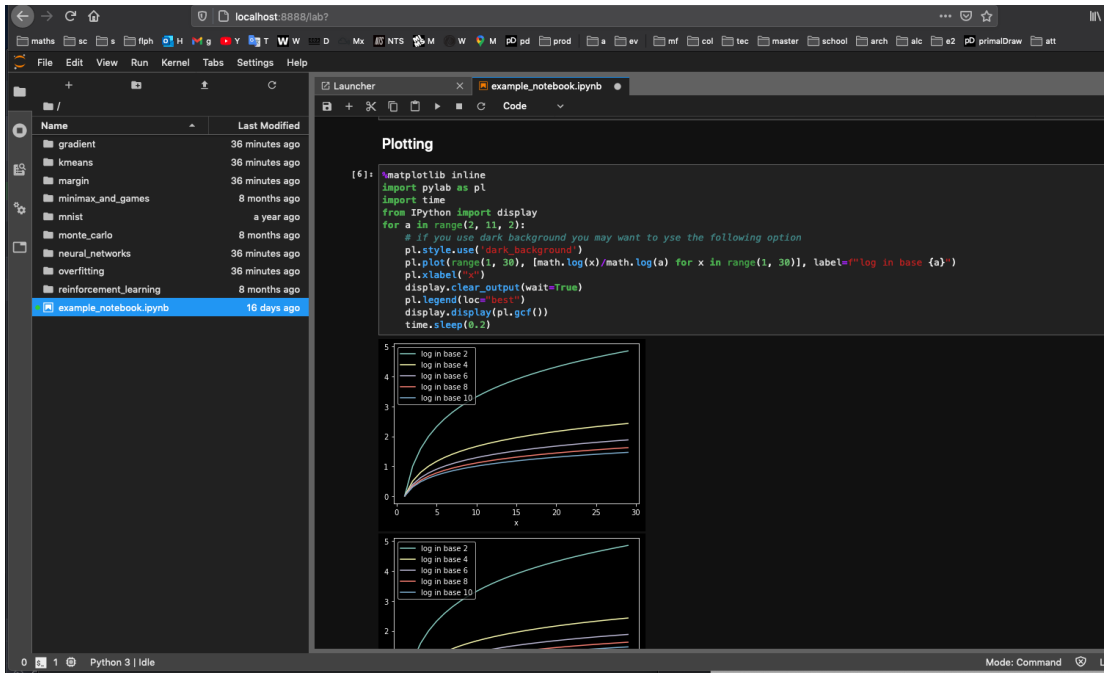
Libs

Day 1 networkx, numpy, matplotlib

Day 2 numpy, matplotlib, pandas, sklearn, seaborn

Organization

- In order to make installation easier and to make the course more interactive, you may use **docker** and **jupyter notebooks**. Please see the **README.md** in the github repo.



Organization

There is a memo for mathematical objects in the repo.

weighted L1 :
let α_1 and α_2 be real numbers ($\in \mathbb{R}$).

$$d(M_1, M_2) = \alpha_1|x_1 - x_2| + \alpha_2|y_1 - y_2| \tag{4}$$

2.2 Distances in three dimensions

We consider two points M_1 and M_2 in the 3D space $\mathbb{R}^3$ with coordinates (x_1, y_1, z_1), and (x_2, y_2, z_2), respectively.

L2

$$d(M_1, M_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{5}$$

L1

$$d(M_1, M_2) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| \tag{6}$$

L∞

$$d(M_1, M_2) = \max(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|) \tag{7}$$

weighted L1 :
let α_1, α_2 and α_3 be real numbers ($\in \mathbb{R}$).

$$d(M_1, M_2) = \alpha_1|x_1 - x_2| + \alpha_2|y_1 - y_2| + \alpha_3|z_1 - z_2| \tag{8}$$

3 LIKELIHOOD / VRAISEMBLANCE

— Observations : $(x_1, \ldots, x_n)$

Day 1

The matching problem

- Definition of the problem
- Experimental solutions
- Brute force algorithm
- Greedy algorithm

The Maximum flow problem

- Presentation of the problem
- Solution with the Ford-Fulkerson algorithm
- Connection with the matching problem
- More results on the two problems

Introductory example 1 : Max Flow

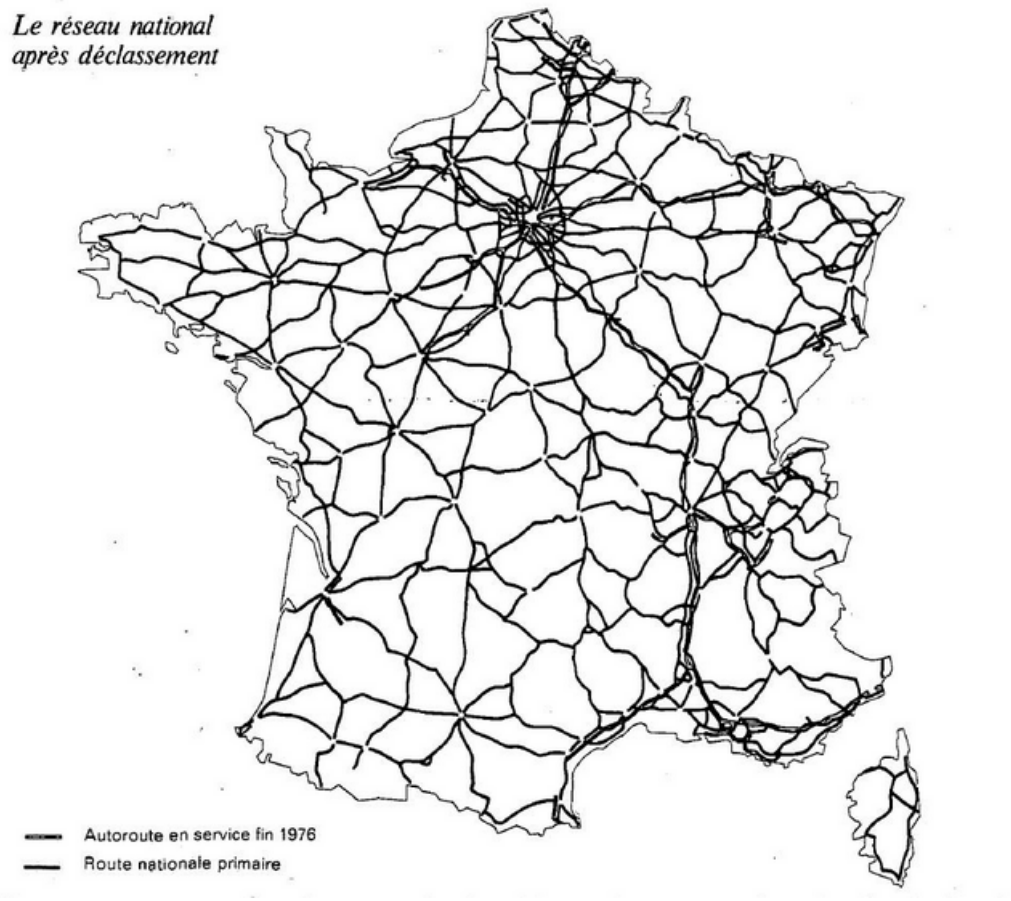


Figure: Problem 1 : transporting merchandise through a network

Introductory example 2 : Maximum matching (Optimal assignment, problème d'affectation)

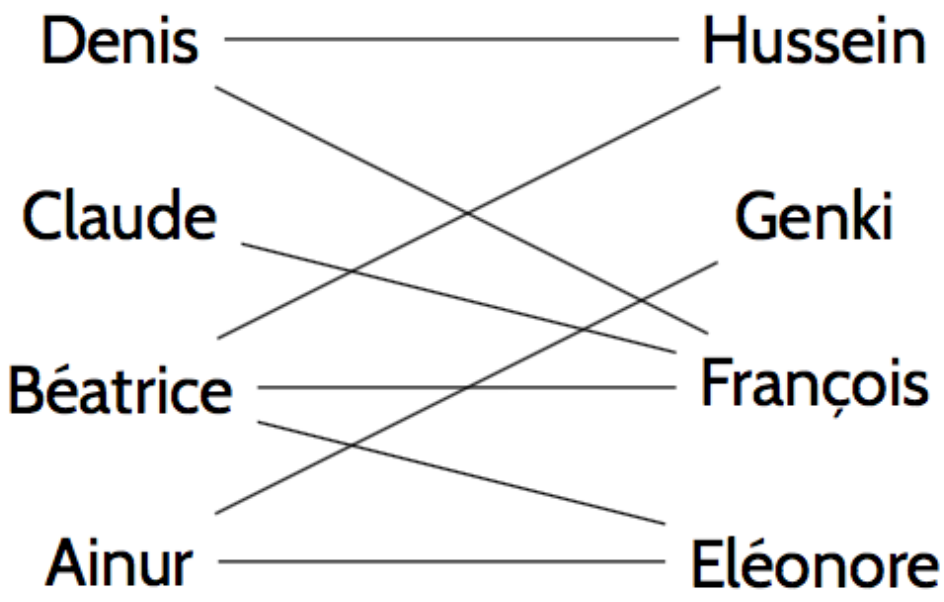


Figure: Problem 2 : Building the largest possible number of teams of 2 persons.

Introductory example 2

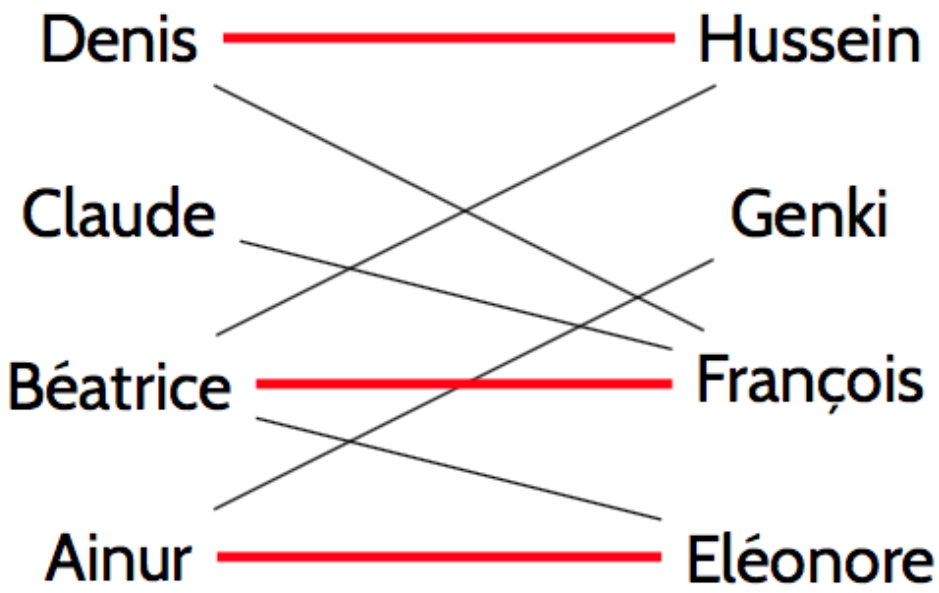


Figure: Problem 2 : not optimal assignment

Introductory example 2

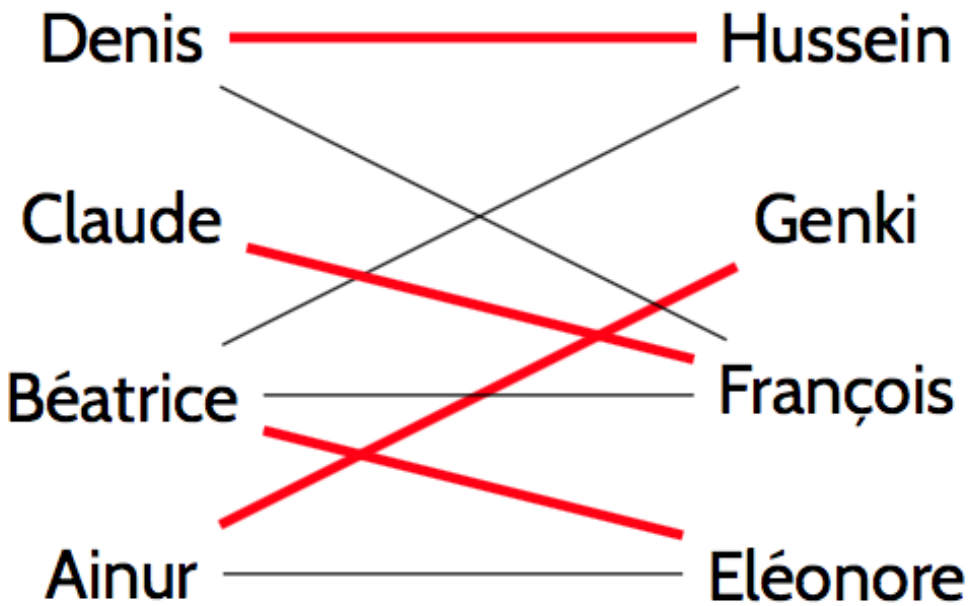


Figure: Problem 2 : optimal assignment

Other examples

- Assigning students to internships

Other examples

- Assigning students to internships
- Assigning machines to a task

Summary

- Today we will work on **connnecting the two problems**.

Summary

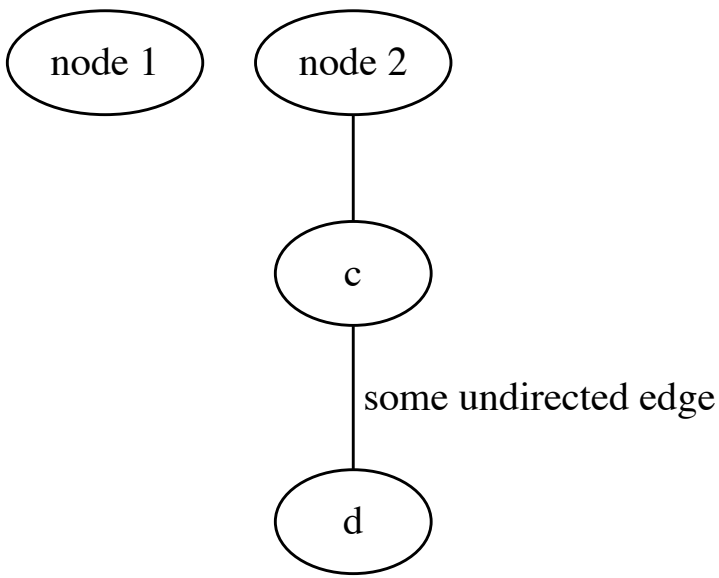
- Today we will work on **connnecting the two problems**.
- Under some restrictions, the two problems **equivalent**.

Reminders on graphs

- A graph is defined by ?

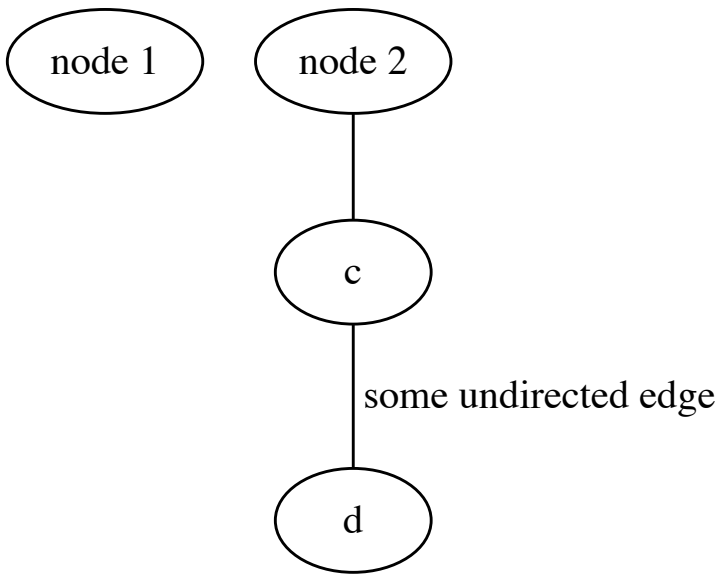
Reminders on graphs

- A graph is defined by set of **vertices** (or **nodes**) V and a set of **edges** E.



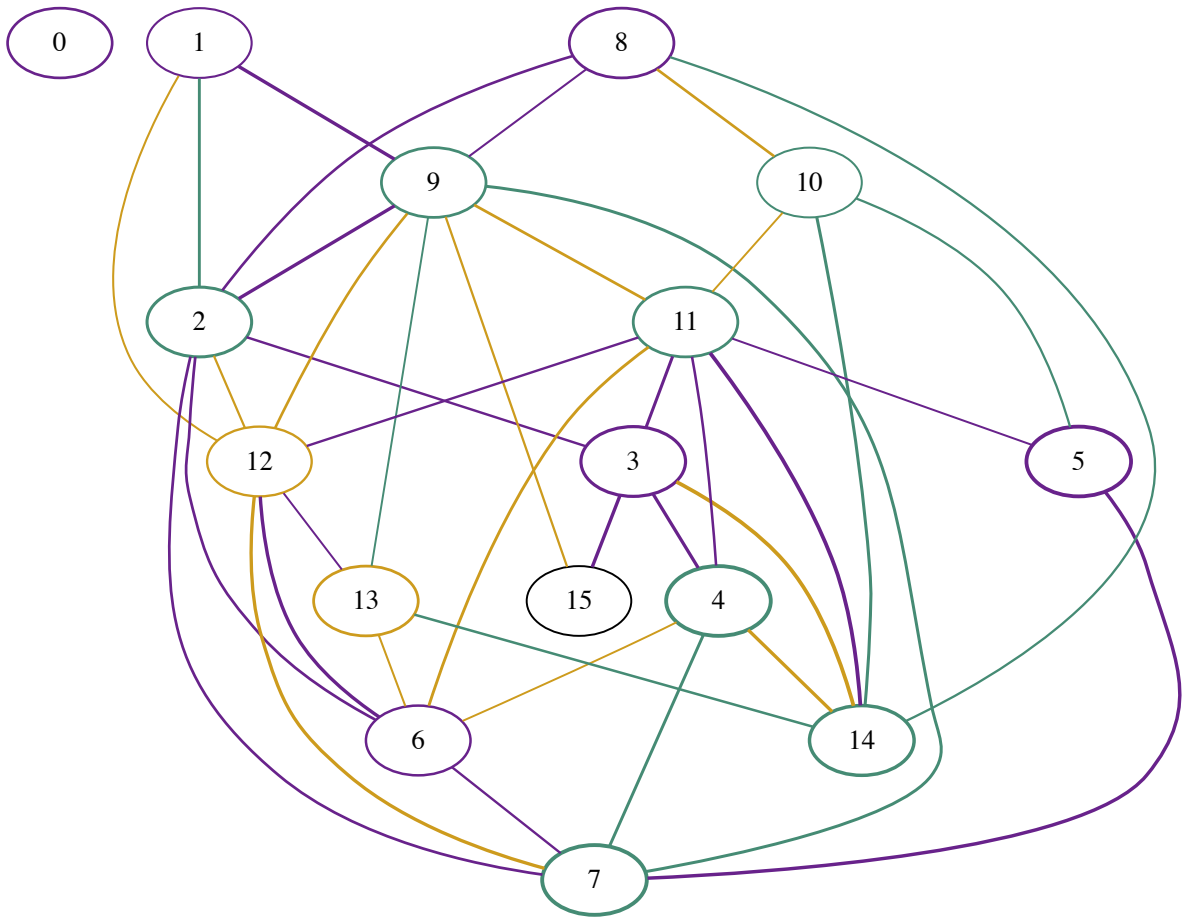
Reminders on graphs

- It can be **undirected**, as this one :



Reminders on graphs

Undirected graph



Reminders on graphs

- Or **directed**, as this one (it is then called a **digraph**)

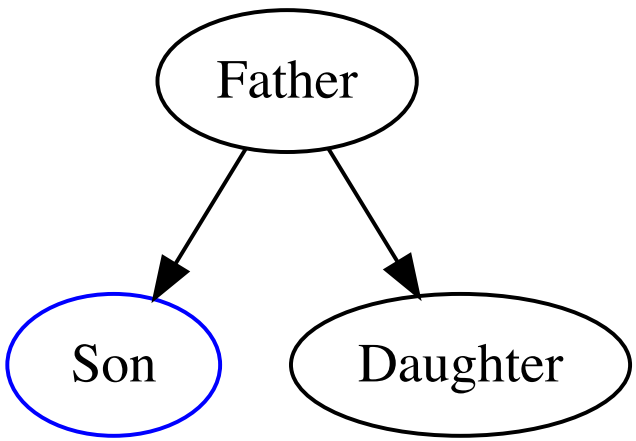
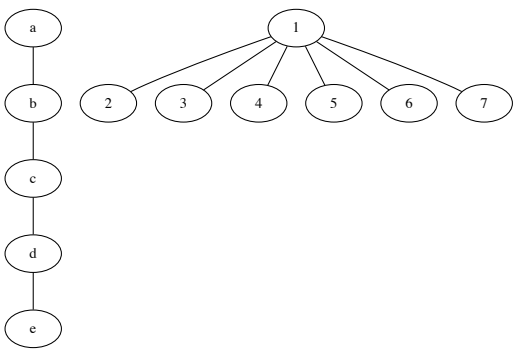


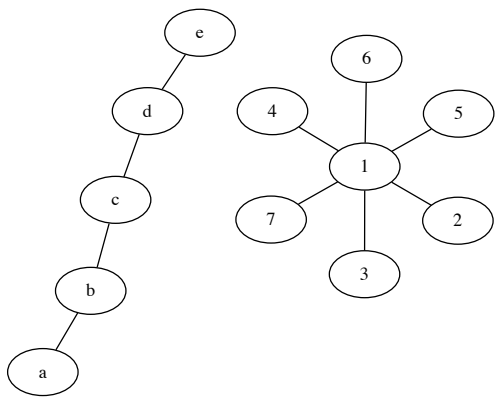
Figure: Digraph (generated with networkx)

Other available tool : graphviz

- A tool to visualize graphs
- Several **generator programs** : dot, neato



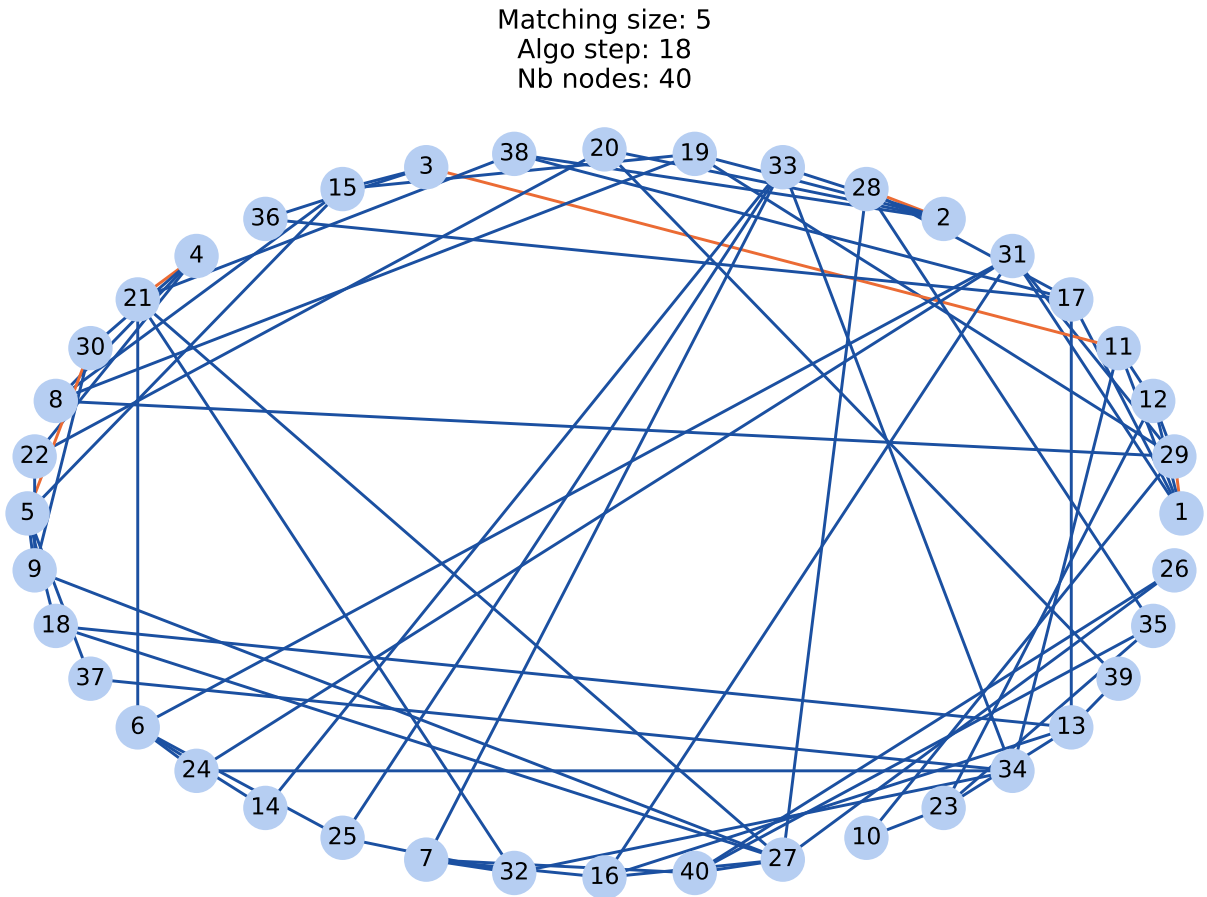
(a) Image generated with **dot**



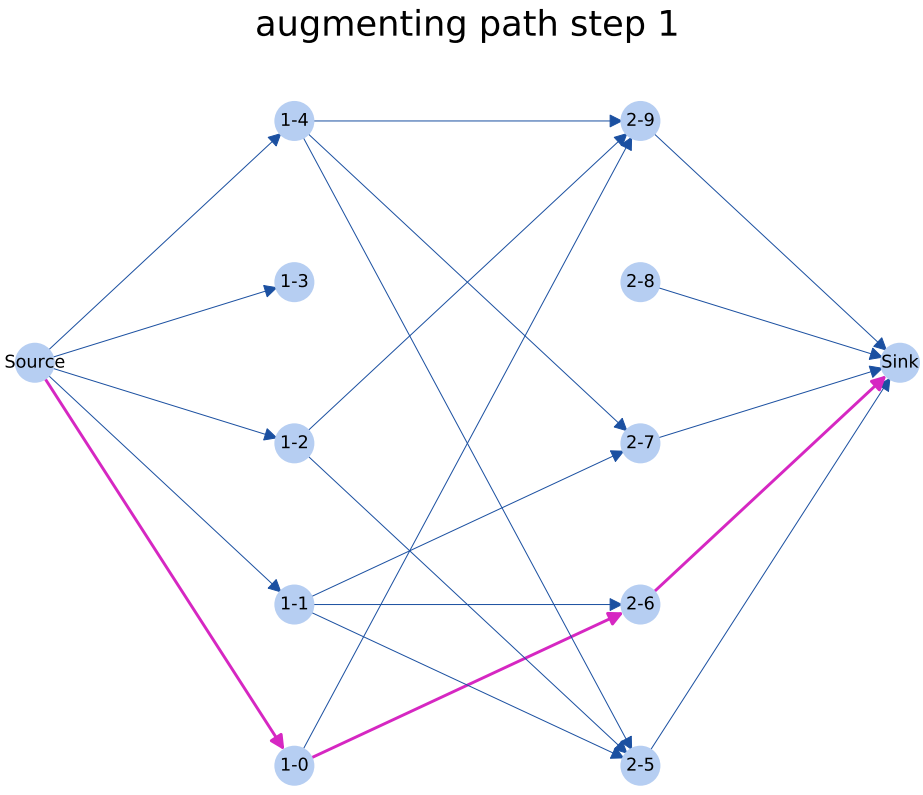
(b) Image generated with **neato**

Networkx

We will use networkx.



Networkx

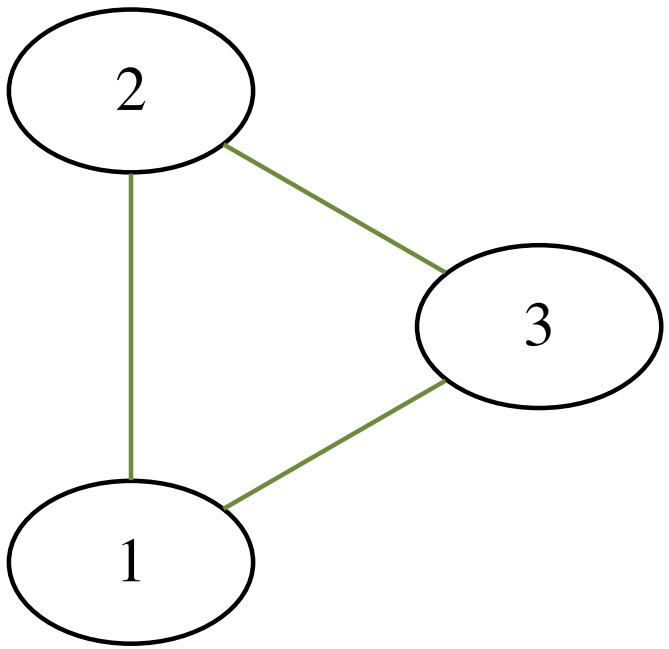


Warm up question

Given an **undirected** graph with n nodes, how many edges can we build ?
Notation of a graph : $G(V, E)$

- V : set of n vertices
- E : set of edges


1
2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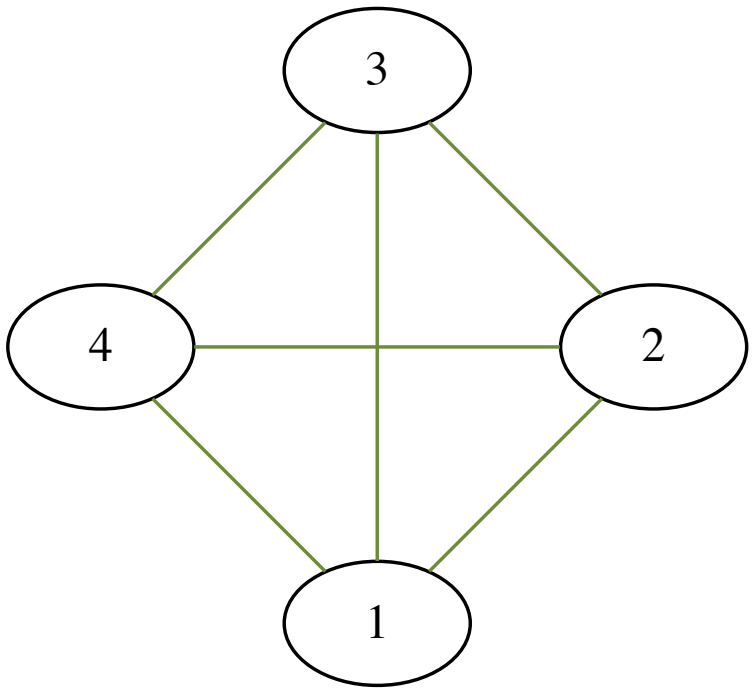
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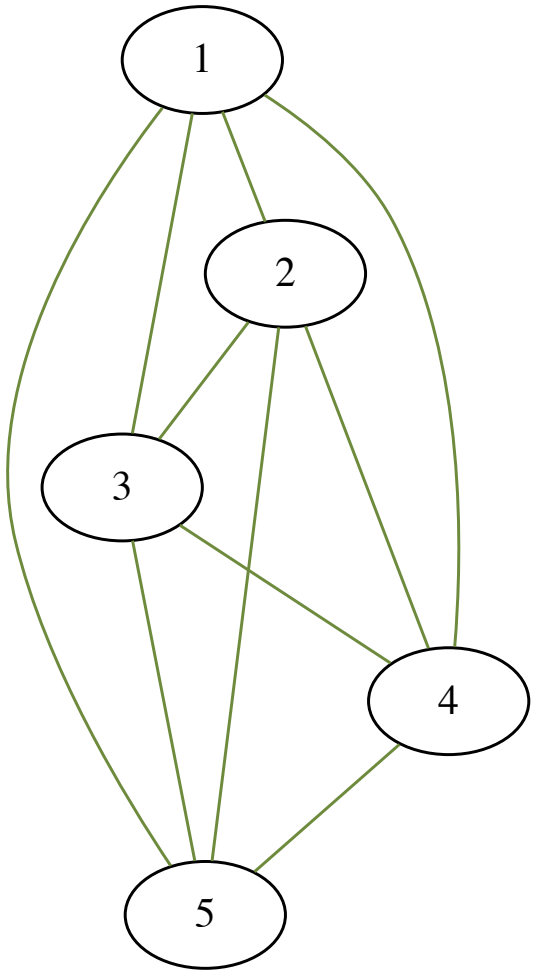
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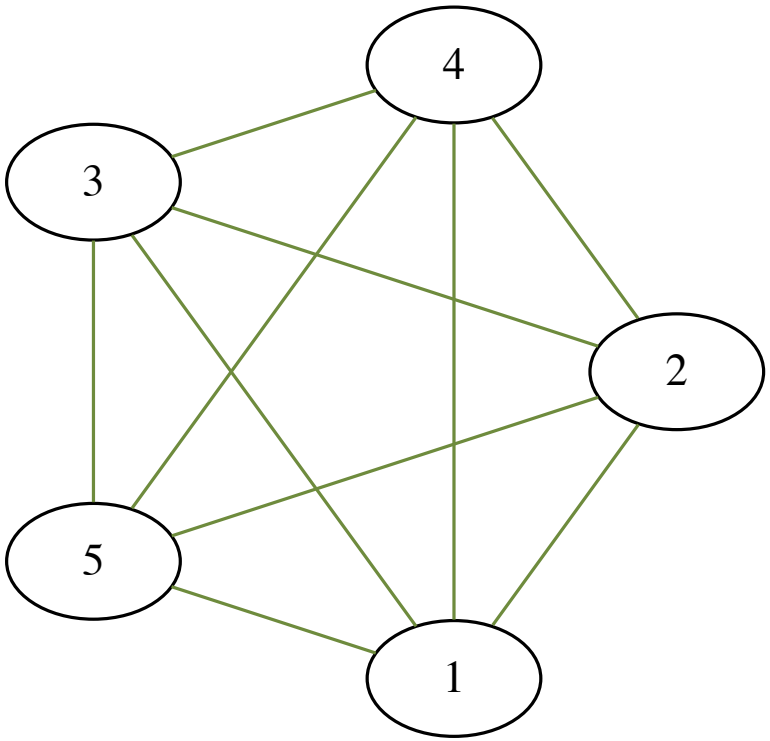
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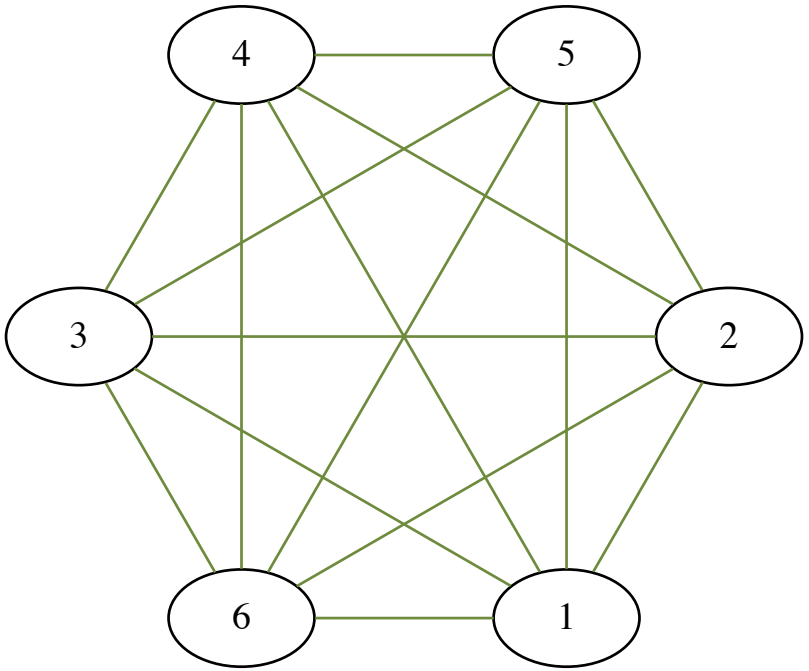
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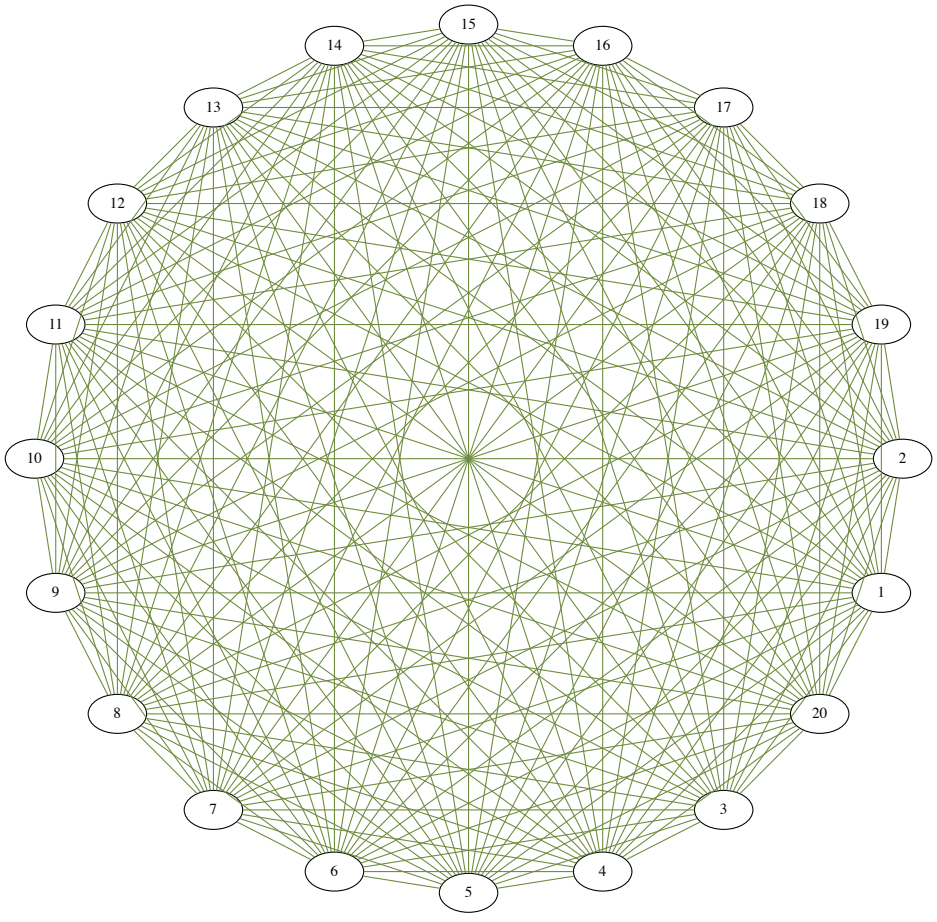
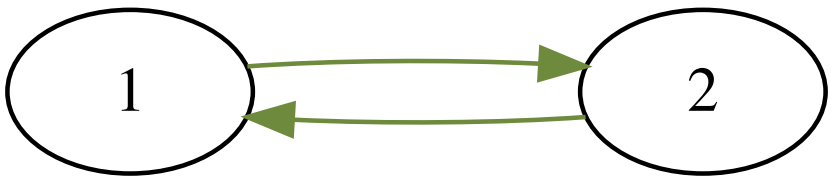
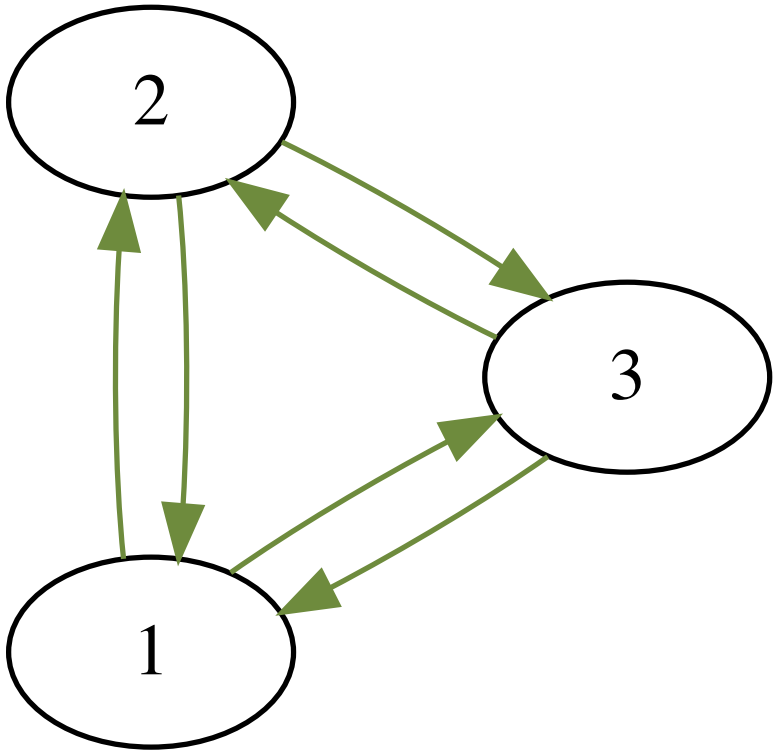


Figure: We cannot count anymore

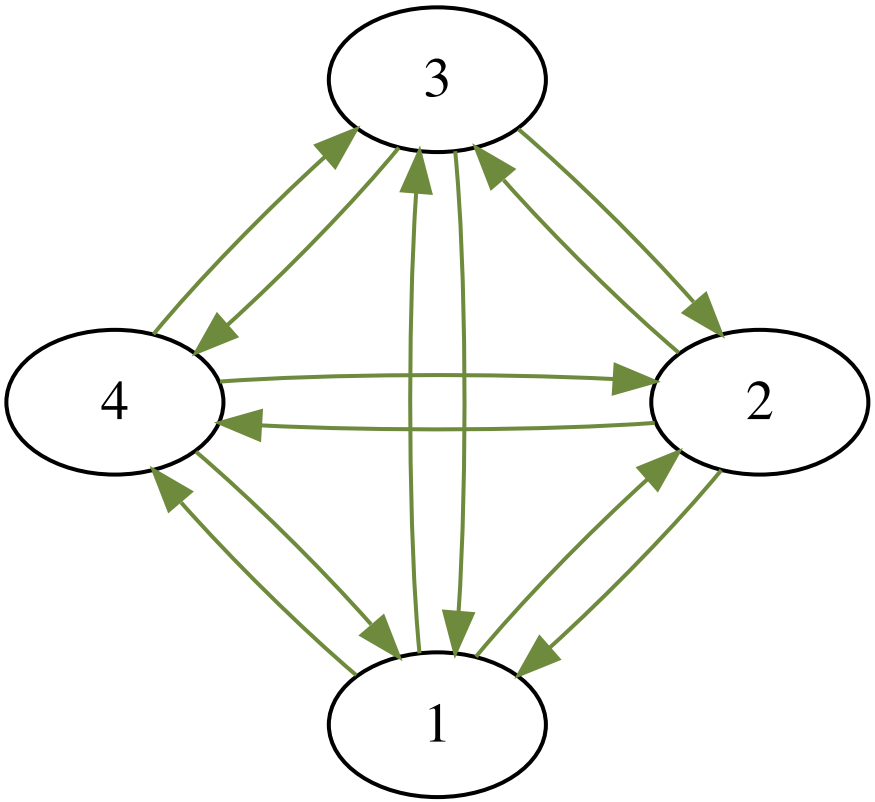
What if the graph is directed ?



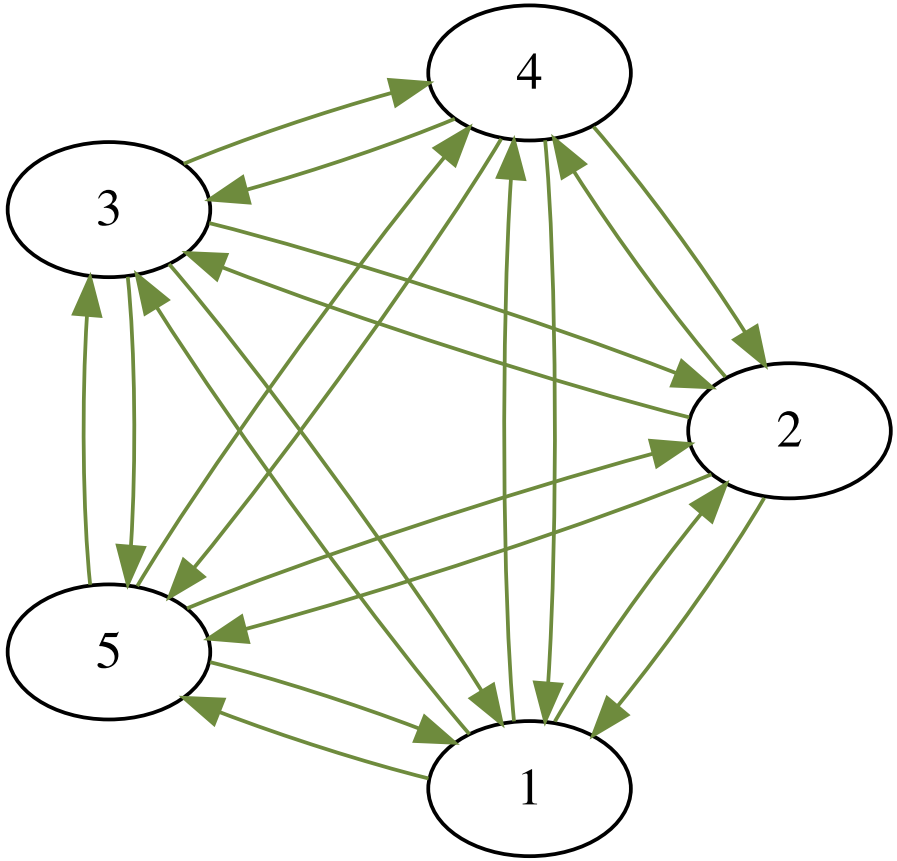
What if the graph is directed ?



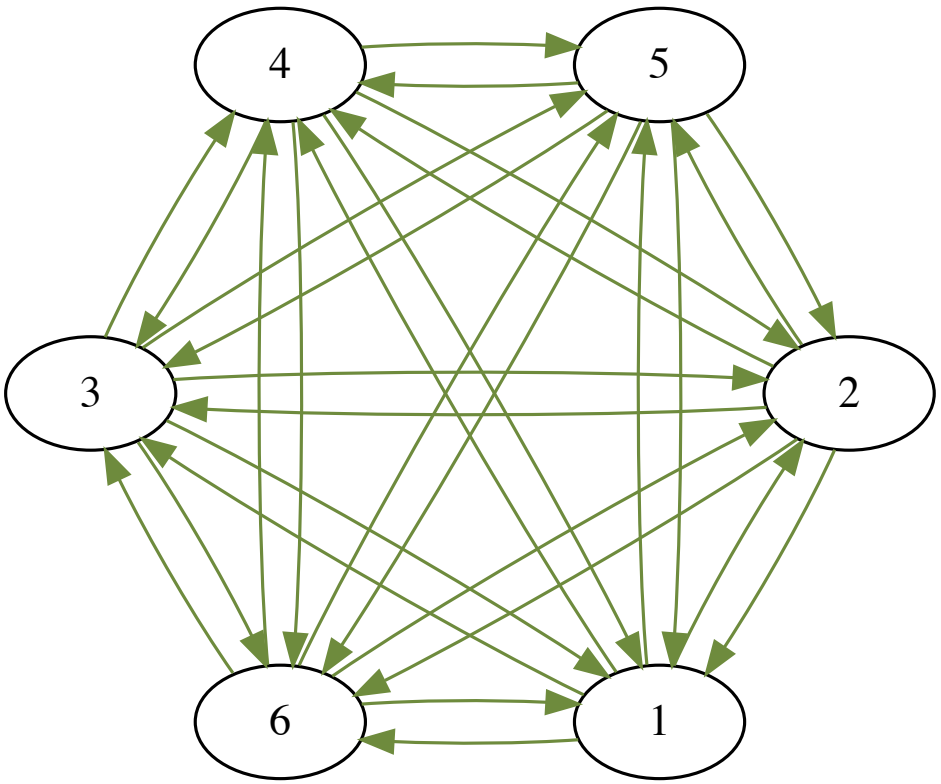
What if the graph is directed ?



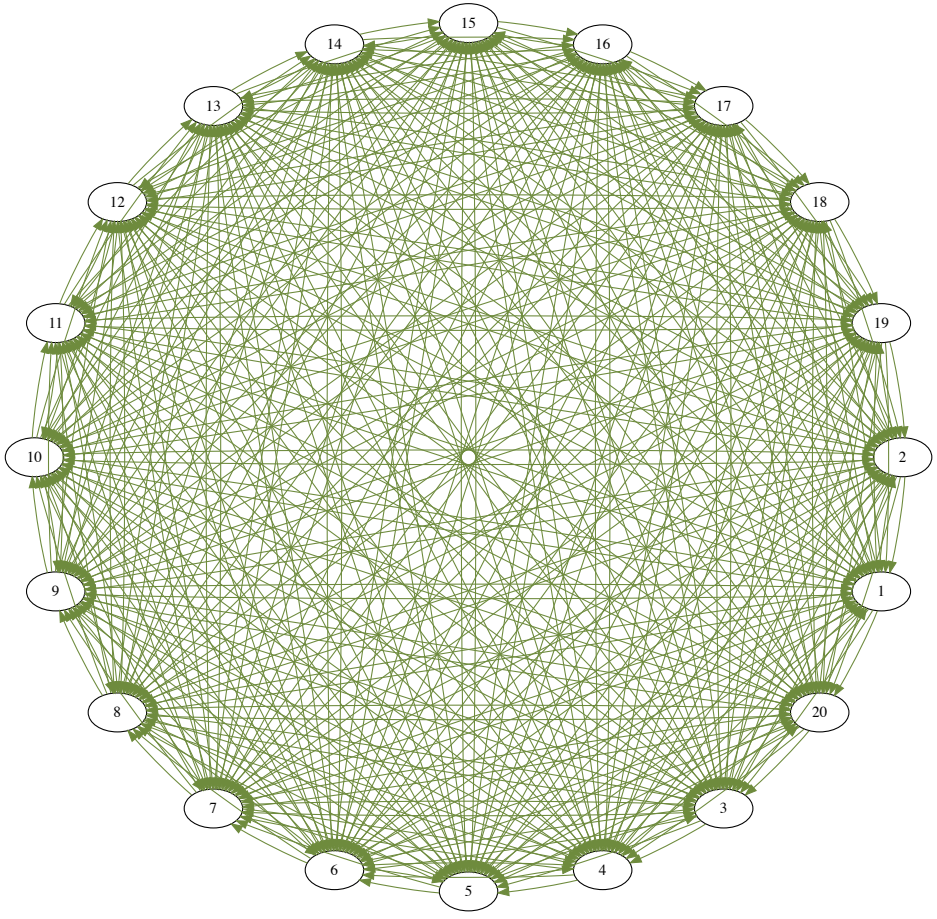
What if the graph is directed ?



What if the graph is directed ?



What if the graph is directed ?



Warm up question

Given an **directed** graph with n nodes, how many edges can we build ?

Warm up question

Given an **directed** graph with n nodes, how many edges can we build ?

$$n(n-1) \tag{1}$$

Warm up question

Given an **directed** graph with n nodes, how many edges can we build ?

$$n(n-1) \tag{2}$$

So if the graph is **undirected**, we can build :

$$\frac{n(n-1)}{2} \tag{3}$$

edges.

Remark

$\frac{n(n-1)}{2}$ is also the number of subsets of size 2 in a set of size n.

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} \tag{4}$$

Famous graph problem

- Do you know some famous **graph problems ?**

Famous graph problem

- Do you know some famous **graph problems ?**
- Dominating set

Famous graph problem

- Do you know some famous **graph problems ?**
- Dominating set
- Maximum clique

Famous graph problem

- Do you know some famous **graph problems ?**
- Dominating set
- Maximum clique
- Coloring

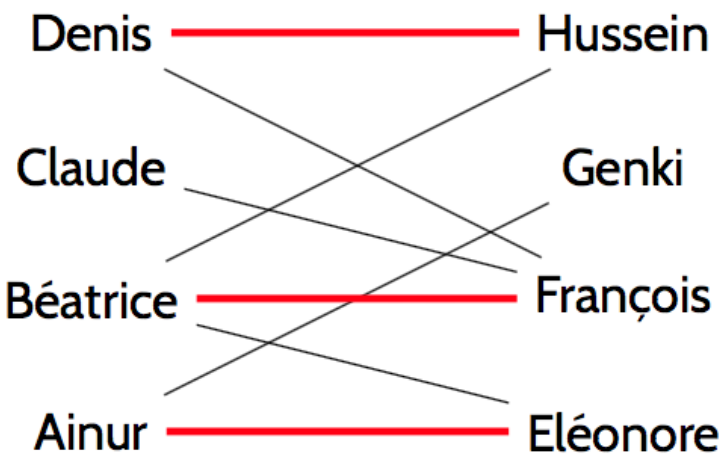
Matching problem

Let us now focus on the **matching problem** (problème du **couplage**)

Back to our problem

Given a **undirected** graph $G = (V, E)$, we want a **matching** M, which means:

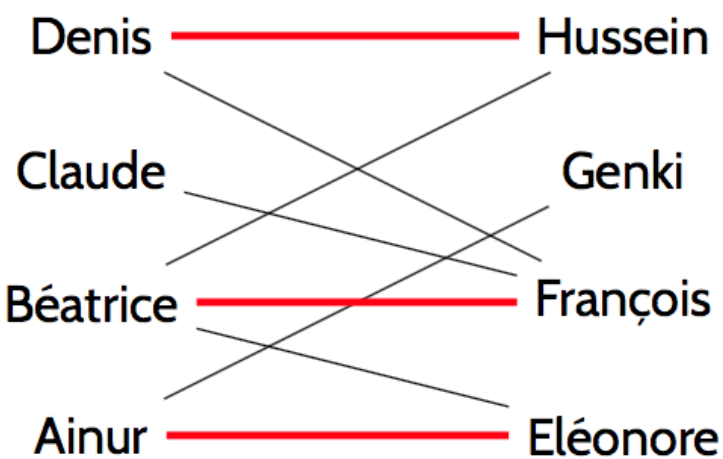
- A subset of edges $M \subset E$



Back to our problem

Given a **undirected** graph $G = (V, E)$, we want a **matching**, which means:

- A subset of edges $M \subset E$
- Such that no pairs of edges of M are incident
- Equivalently, each node in the graph is **at most** in one edge of M.



Back to our problem

Given **undirected** a graph $G = (V, E)$, we want a **matching**, which means:

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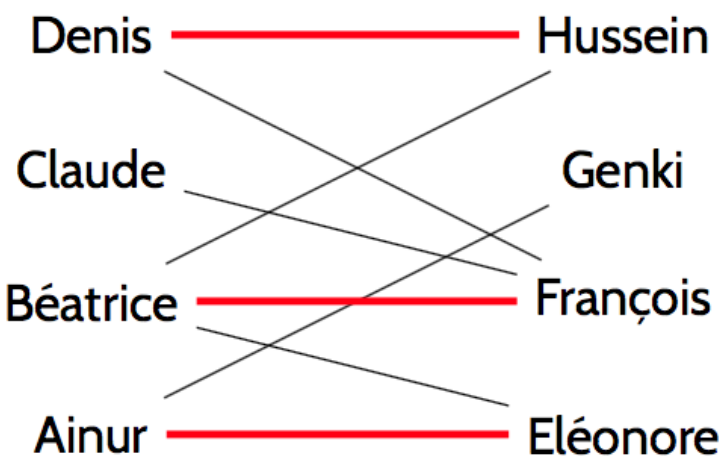
- Equivalently, each node in the graph is **at most** in one edge of M.
- No pairs of edges of M are incident



Maximum matching

- The **size** of a matching is the number of edges it contains.

Maximum matching

- The **size** of a matching is the number of edges it contains.
- We want to find the matching of maximum size in a given graph.

Example 1

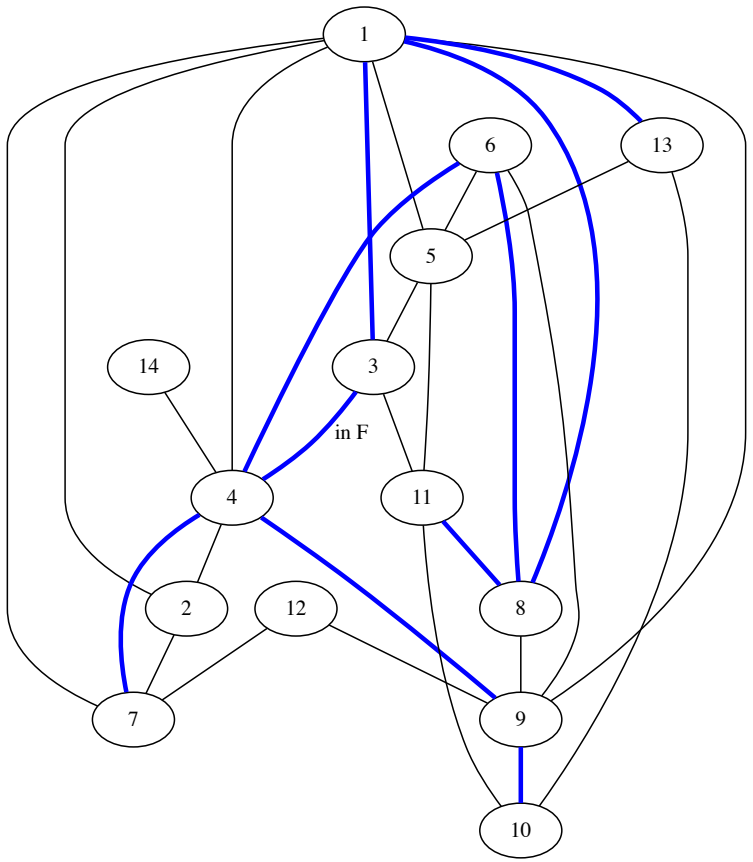


Figure: Is this a matching ?

Example 2

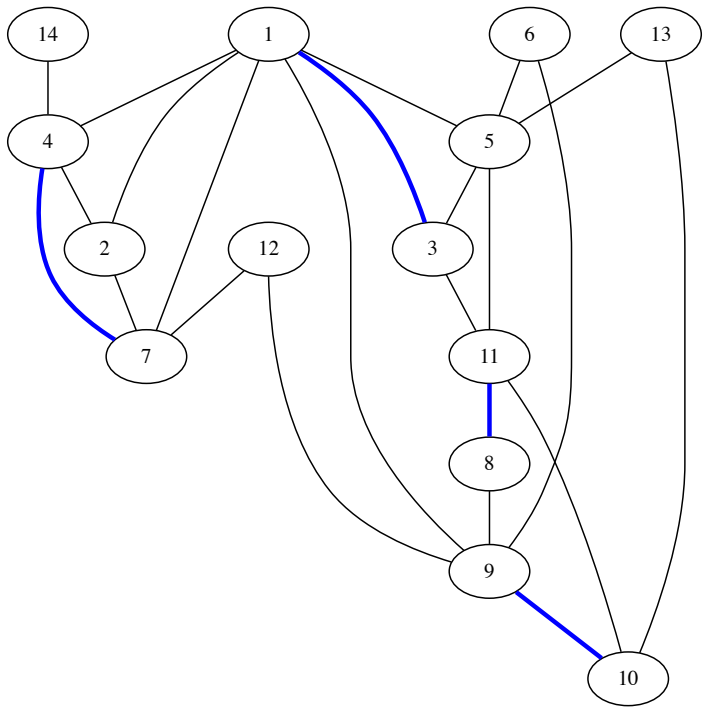


Figure: Is this a matching ?

Example 3

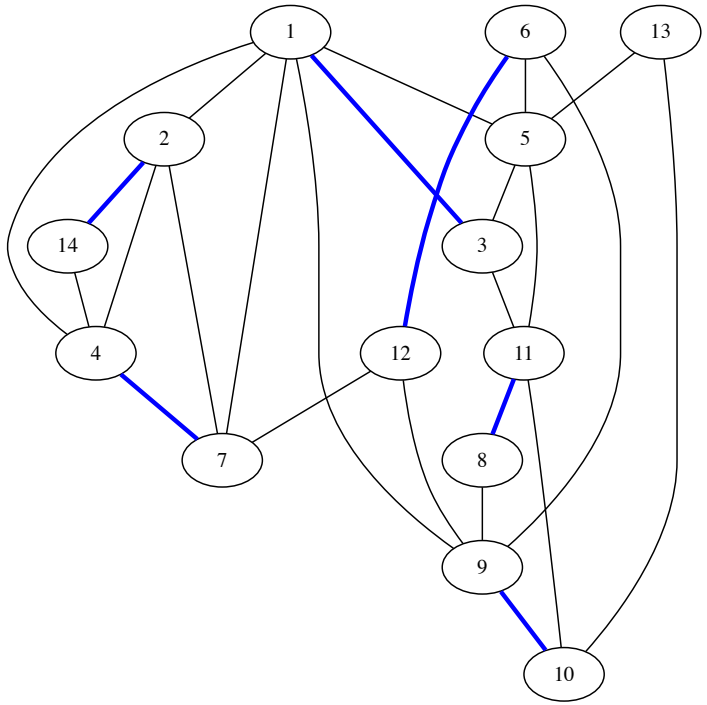


Figure: Is this an optimal matching ?

Example 4

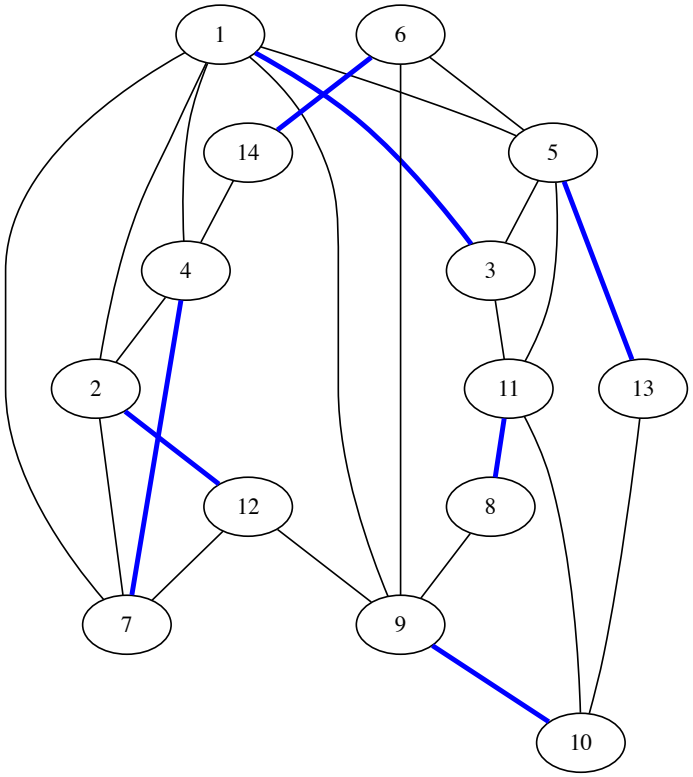


Figure: Is this an optimal matching ?

Example 5

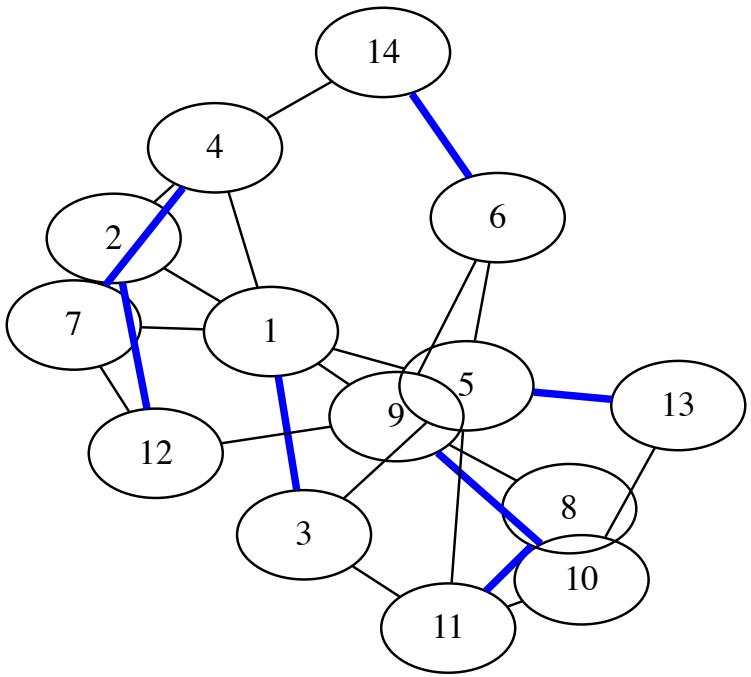


Figure: With neato

Optimal matching

Exercice 1 : Given a graph of size n, what is maximum size possible for a **matching ?**

Optimal matching

Exercice 1 : Given a graph of size n, what is maximum size possible for a **matching ?**

- If n is even : $\frac{n}{2}$
- Else n is odd : $\frac{n-1}{2}$

Optimal matching

Exercice 1 : Can you think of a graph with n nodes that contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal

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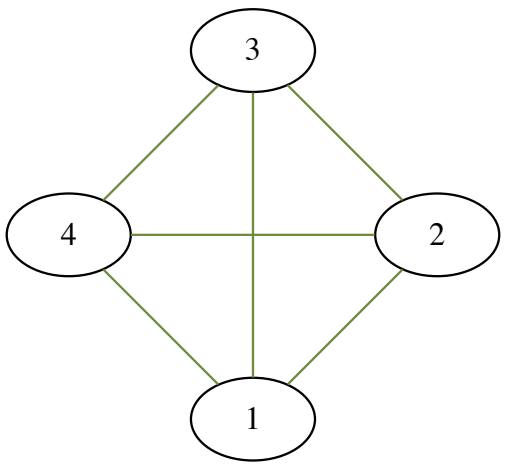

Figure: The complete graph works

Optimal matching

Exercice 1 : Can you think of a graph with n nodes that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal matching

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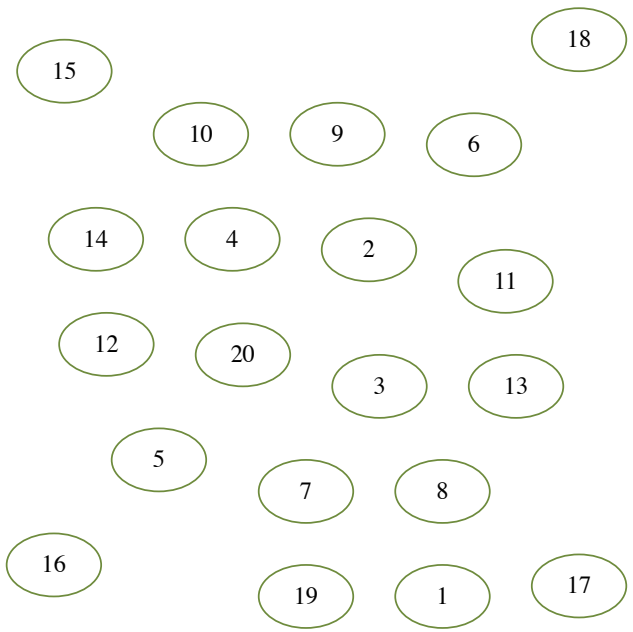


Optimal matching

Exercice 1 : Can you think of a **non trivial** graph that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal matching

Exercice 2 : Can you think of a **non trivial** graph that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

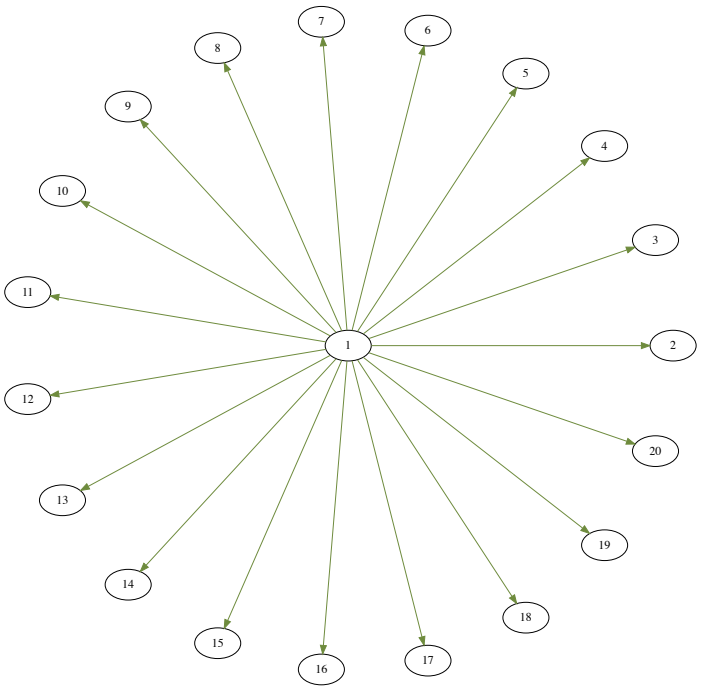


Figure: Star graph

Experiments

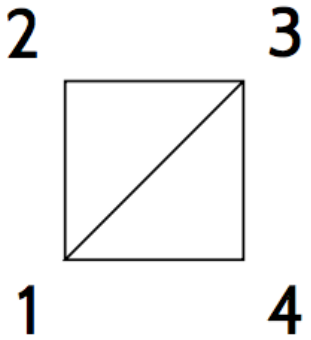
How would you code a graph ?

Experiments

How would you code a graph ?

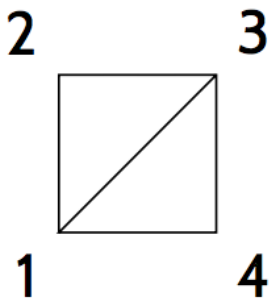
- list of sets of size 2 (for an undirected graph)
- a dictionary of successors (directed of undirected)

Coding a graph : as a list




```
g1 = [{1,2},{1,3},{2,3},{3,4},{1,4}]
```

Coding a graph : as a dictionary



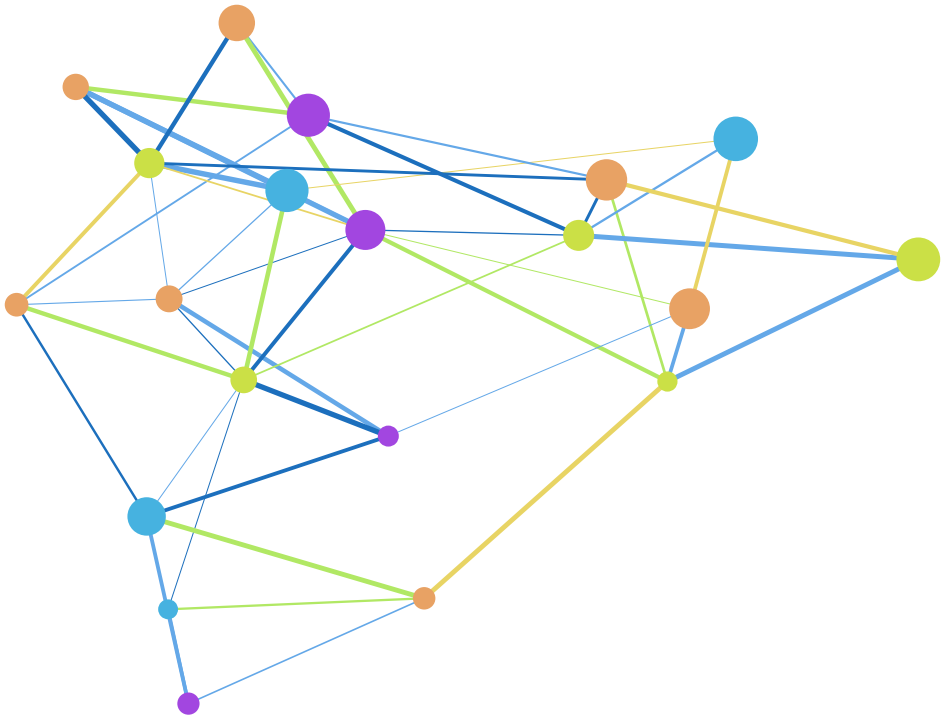

```
g1 = { 1:{2,3,4}, 2:{1,3}, 3:{1,2,4}, 4:{1,3} }
```

Random graph

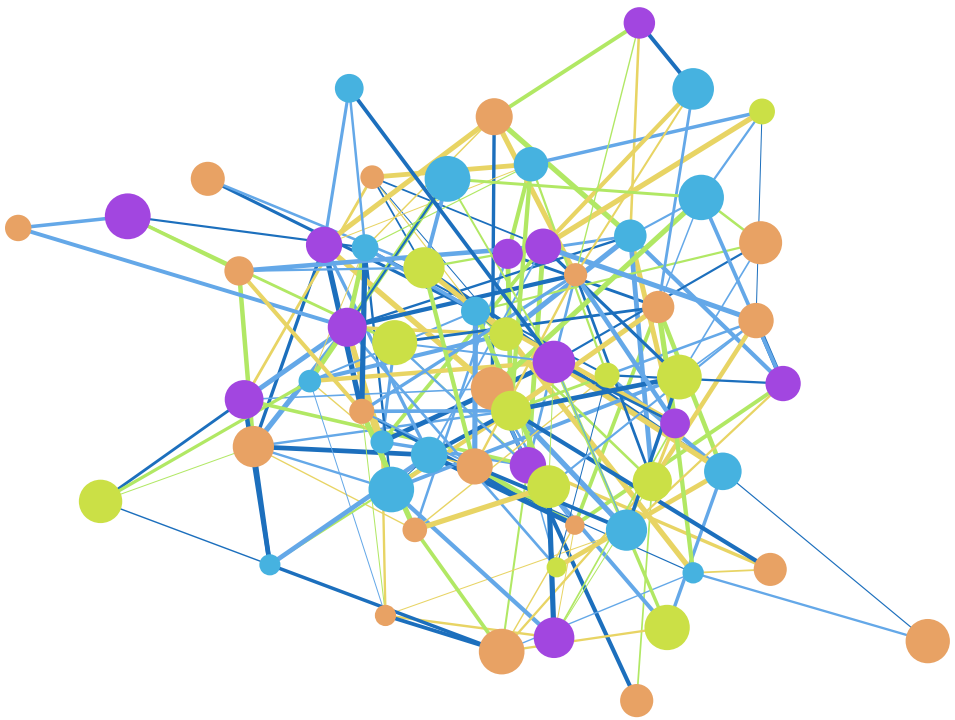
Exercice 3 : **cd other_graphs/** and please use **random_undirected_graph.py** to build a graph with 20 vertices and 50 edges.

- You will need to install **networx**, or use the notebook.

Example undirected graph obtained



Example undirected graph obtained

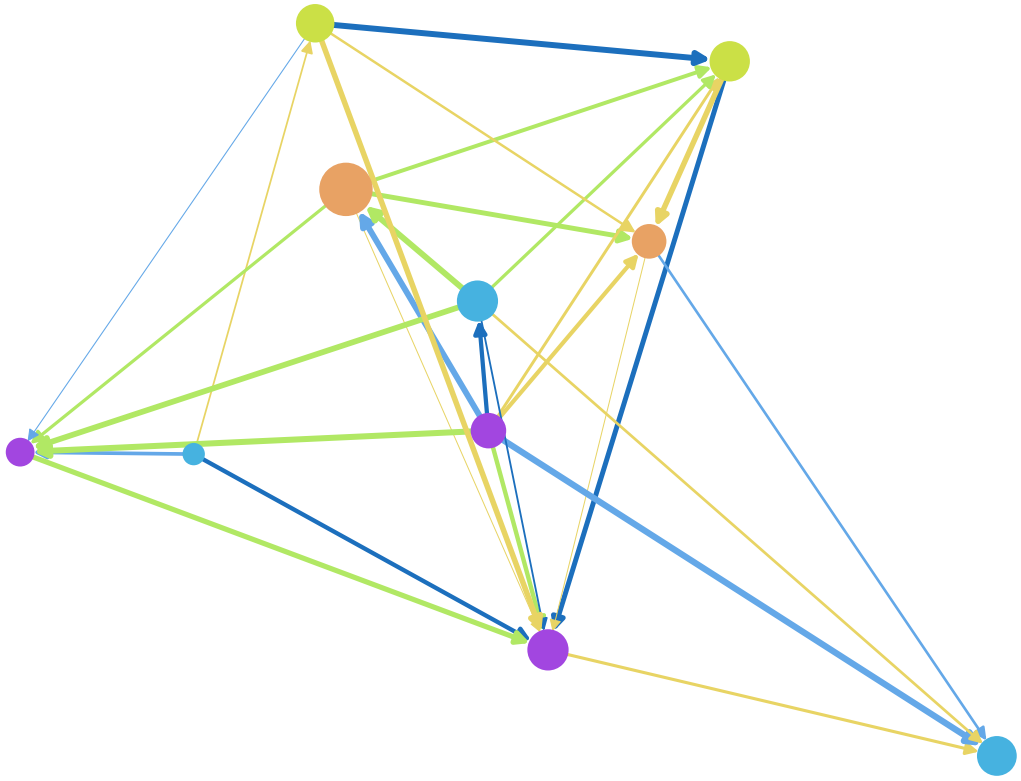


Example undirected graph obtained

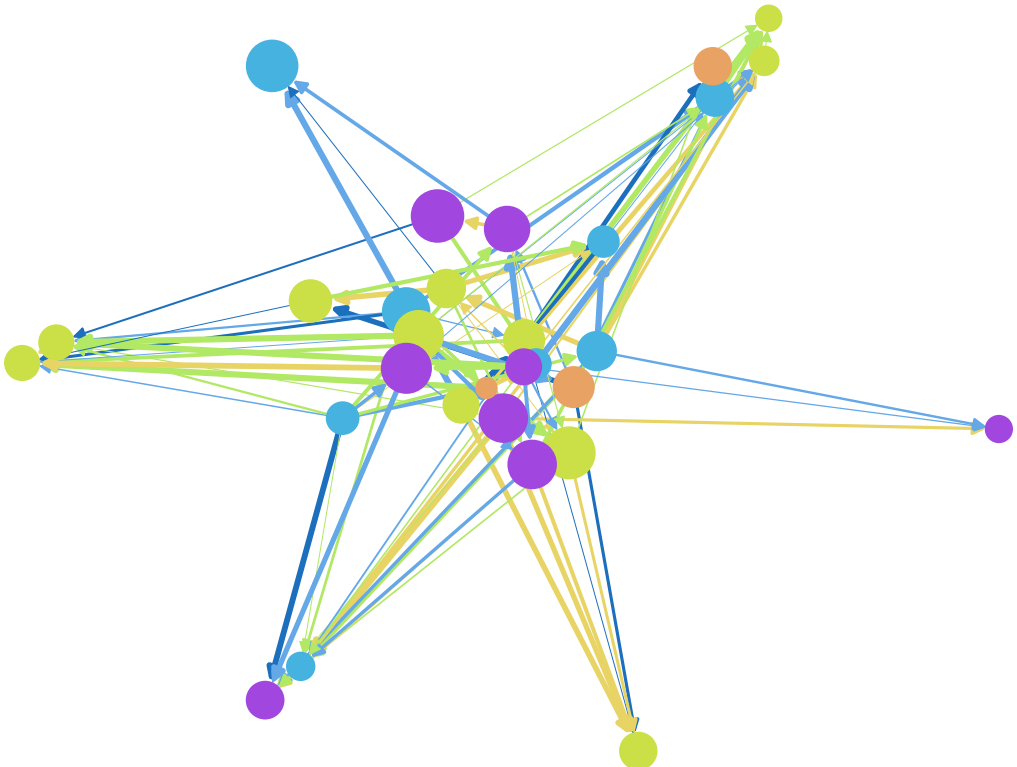
Exercice 4 : Random directed graph.

- Please use **random_directed_graph.py** to build a **directed** graph with a chosen number of vertices and **directed edges**.

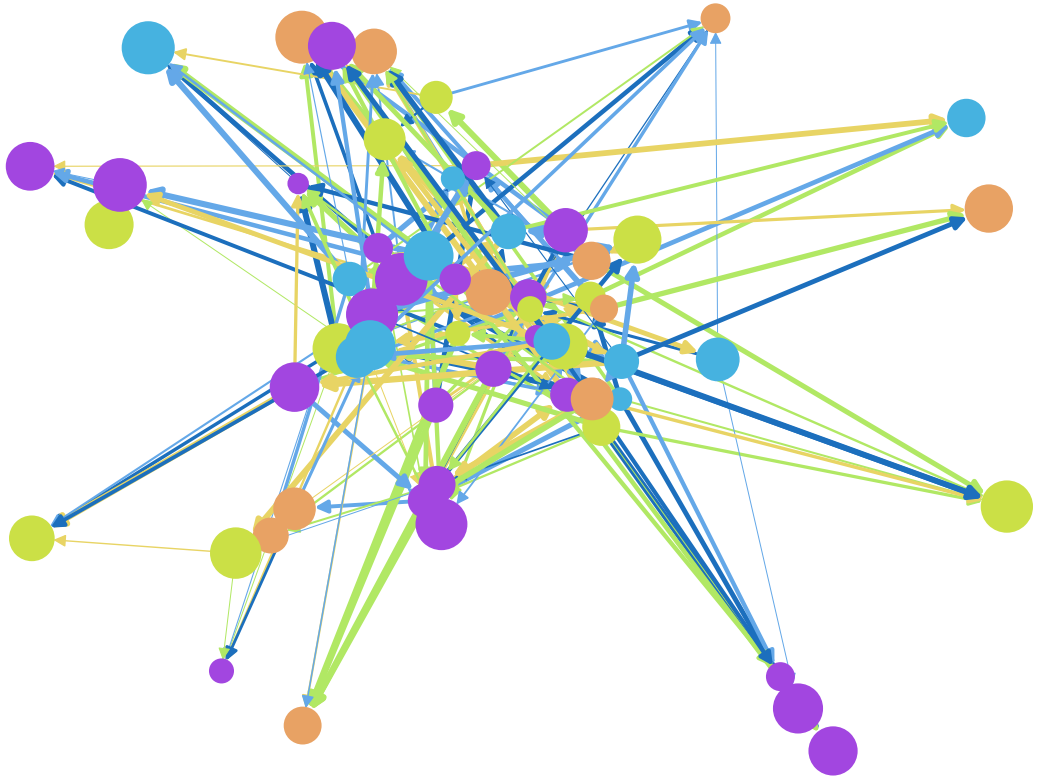
Example directed graph



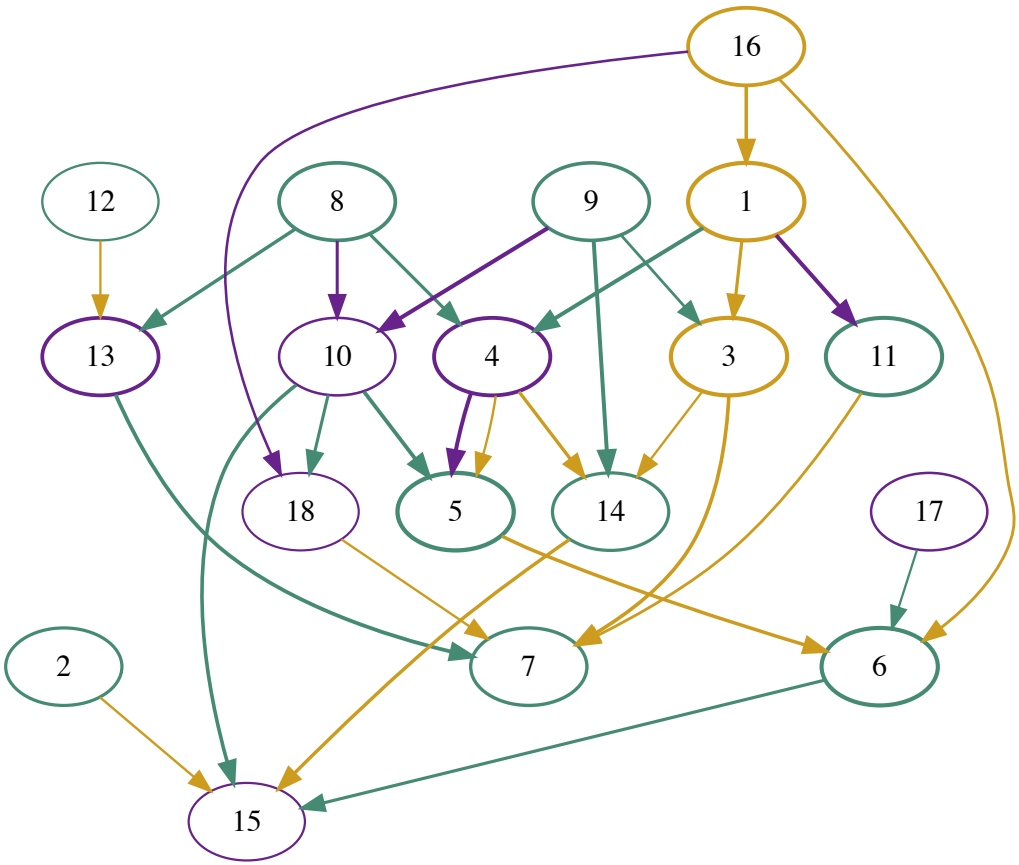
Example directed graph



Example directed graph



Example directed graph



Manual matching

Exercice 5 : Please manually find an **optimal matching** in your **undirected** graph.

Big graph

We could not manually find an optimal matching in this graph :

Summary

- We have defined the matching problem.
- When the size of the problem is large, we can not manually find an optimal matching.

Brute force approach

Exercice 6 : Enumeration

- Given a graph, what would a brute force approach on the matching problem be ?

Brute force approach

Exercice 6 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
  - 3) Return the biggest one obtained.

Brute force approach

Exercice 6 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
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If the graph contains n nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?

Brute force approach

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  - 3) Return the biggest one obtained.

If the graph contains n nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?
You can give a rough approximation.

Brute force approach

Exercice 6 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
  - 3) Return the biggest one obtained.

If the graph contains n nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?
It is a **polynomial** number of computations : so it is ok.

Notion of complexity

- The **time complexity** of an algorithm is a measure of the **number of elementary** operations needed for the algorithm to terminate with respect to the input size.

Brute force search

Exercice 7 : Complexity of brute force

- 1) Enumerate all possible subsets in the set of the edges.
- 2) Check if each subset is a matching.
- 3) Return the biggest one obtained.

What is the complexity of step 1 ?

Brute force search

Exercice 7 : Complexity of brute force

- 1) Enumerate all possible subsets in the set of the edges.
- 2) Check if each subset is a matching.
- 3) Return the biggest one obtained.

What is the complexity of step 1 ?

The number of subsets is $2^{\frac{n(n-1)}{2}}$ (in the worst case), which is exponential. If p is the numberof edges, we can also write it as 2^p.

Brute force search

Exercice 7 : **Complexity of brute force**

Assume that checking a subset requires 1 microsecond. How long should we wait in order to check all possible matching in a graph with 100 nodes ?

Other example of complexities

- linear search
- dichotomic search

Summary II

- For the matching problem on a large graph, we can neither
  - manually find an optimal matching
  - perform the exhaustive search (brute force algorithm)

Algorithms

- Hence, we need different algorithms to solve the problem.

Algorithms

- Hence, we need different algorithms to solve the problem.
- Let us first introduce some theoretical notions.

Notion of maximal and maximum matching

We will say that a matching M of cardinality (number of elements) $|M|$ is:

- **Maximum** if is has the maximum possible number of edges (is is thus optimal)

Notion of maximal and maximum matching

We will say that a matching M of cardinality $|M|$ is:

- **Maximum** if it has the maximum possible number of edges (is is thus optimal)
- **Maximal** if the set of edges obtained by adding any edge to it is **not a matching**. This means that $M \cup \{e\}$ is not a matching for any e not in M.
- $\cup$ means union of sets.

is being a **maximal** matching the same thing as beeing a **maximum** matching ?

Maximum implies maximal

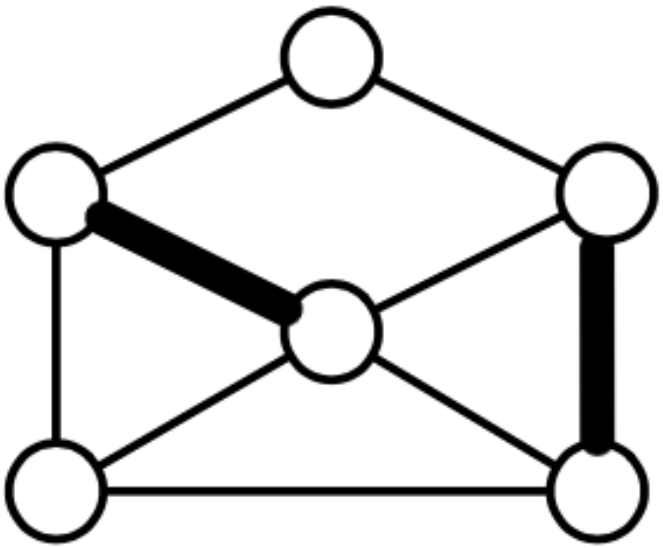
Let us show that a maximum matching is maximal.

Counter Example

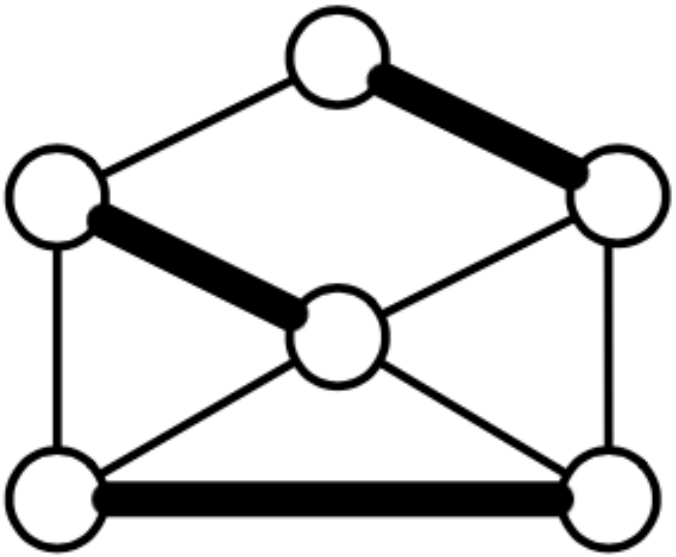
However, a matching that is maximal is **not necessary Maximum**.

Counter Example

However, a matching that is maximal is **not necessary Maximum**.
Can you find an example ?



(a) A maximal matching not maximum



(b) A maximum matching

Greedy algorithm

Can you propose a greedy algorithm to address the maximum matching problem ?

Greedy algorithm

Result : Matching M
$M \leftarrow \emptyset$;
for $e \in E$ **do**
  if $M \cup \{e\}$ *is a matching* **then**
    $M \leftarrow M \cup \{e\}$
  end
end
return M

Algorithme 0 : Greedy algorithm to find a matching

Greedy algorithm

- What is the type of matching algorithm returned by this algorithm ?
- What is the complexity of this algorithm ? (as a function of the number of nodes n of the graph)

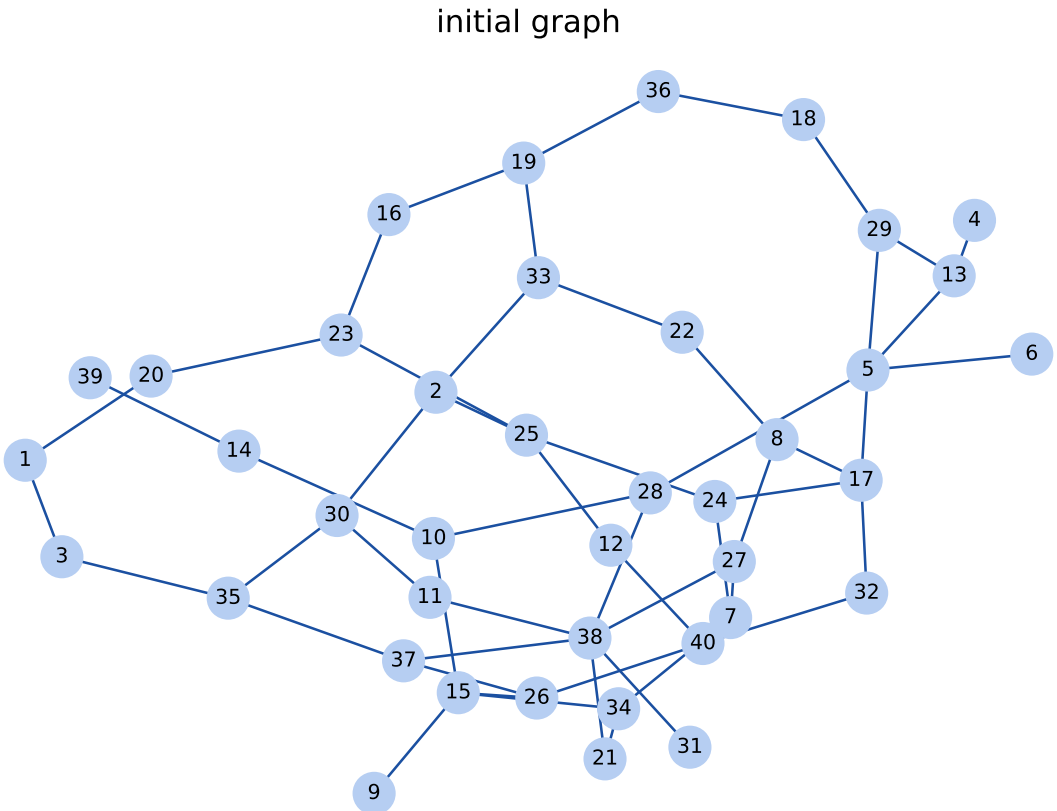
Greedy algorithm

- The greedy algorithm returns a **maximal** matching (proof)
- Its complexity is smaller than $\mathcal{O}(np)$ (n nodes, p edges) (proof)
- smaller than **cubic** in the number of nodes : $\mathcal{O}(n^3)$

Greedy algorithm

- We will implement the greedy algorithm to find a maximal matching.

Exercice 8 : **cd matching_greedy/** and use **generate_graph.py** to build a graph with a least 30 nodes. The images are stored in **images/**, data stored in **data/**

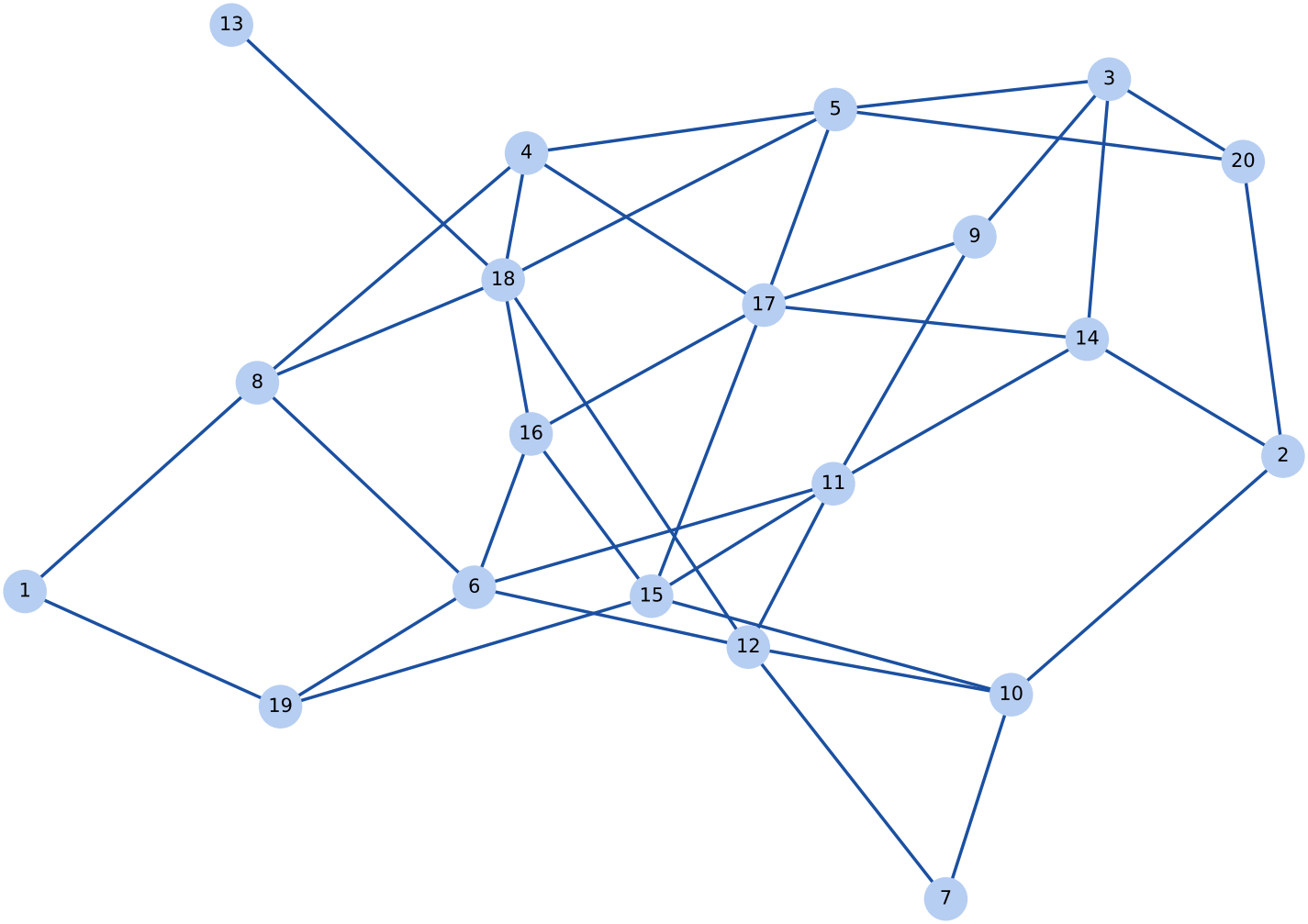


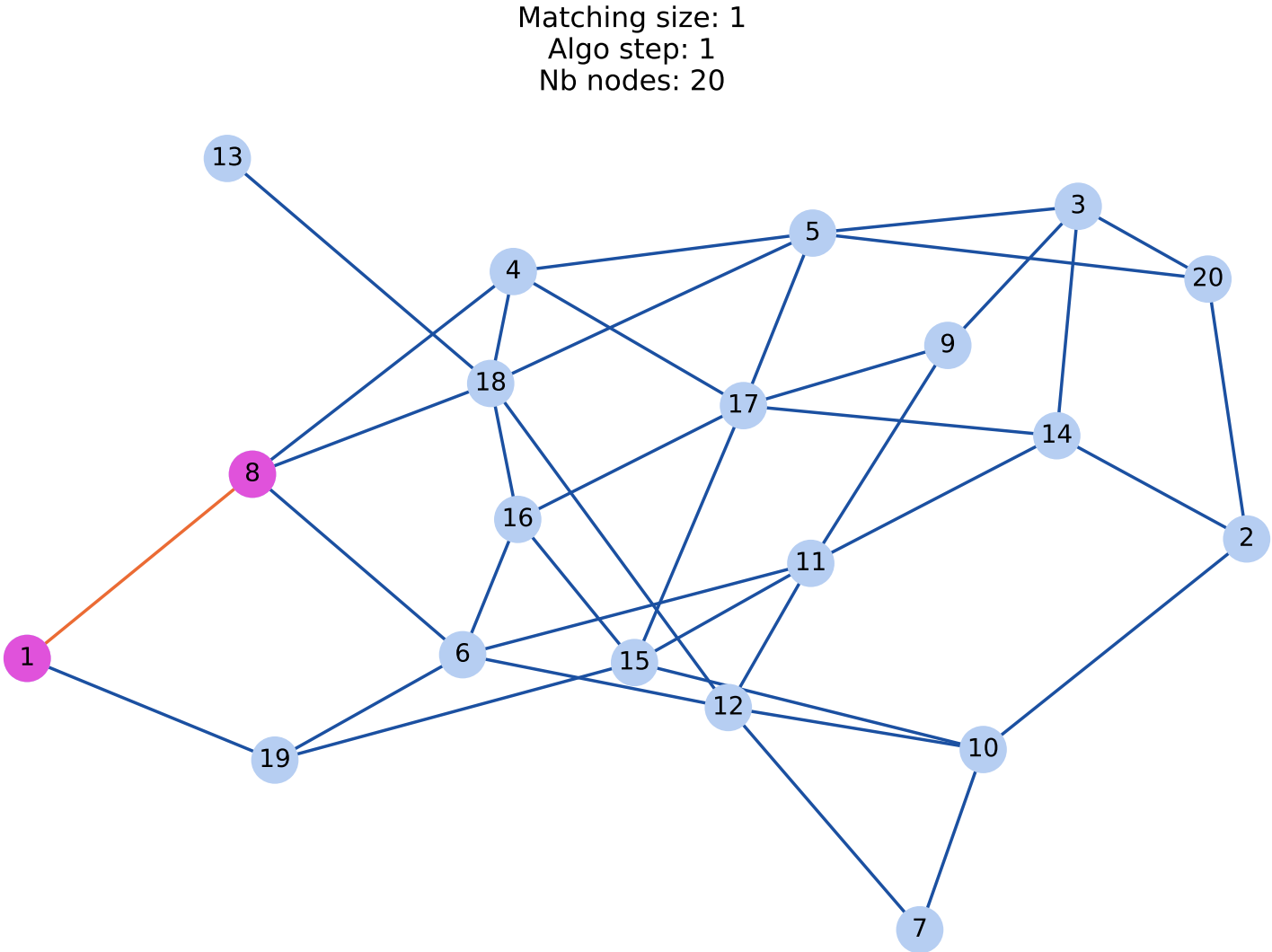
Implementing the greedy algorithm

Exercice 8 : Implement the greedy algorithm on this graph.

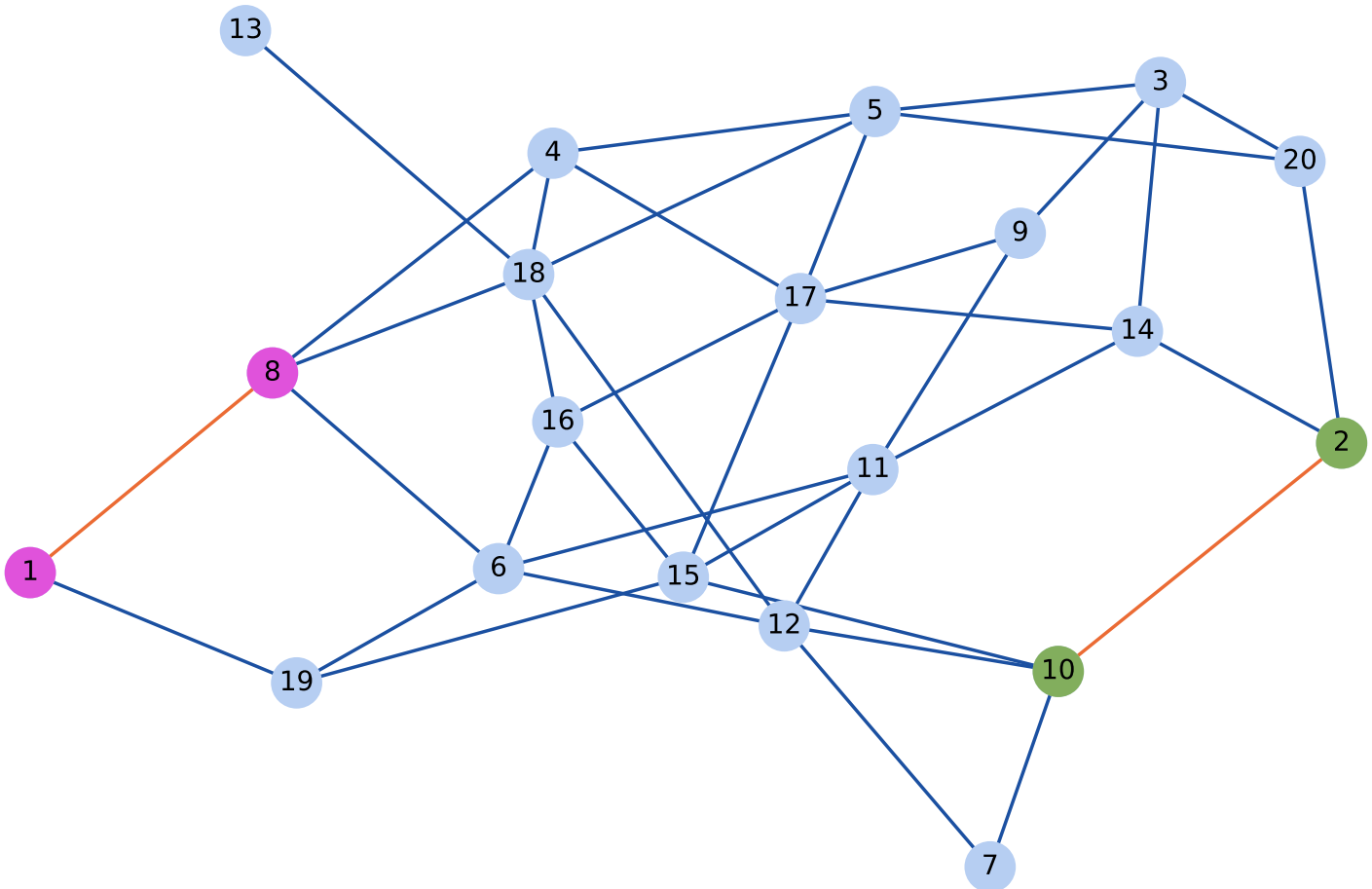
- Use the functions in **matching_functions.py** and call them from **apply_matching_algorithm.py**
- More details in the file.

initial graph

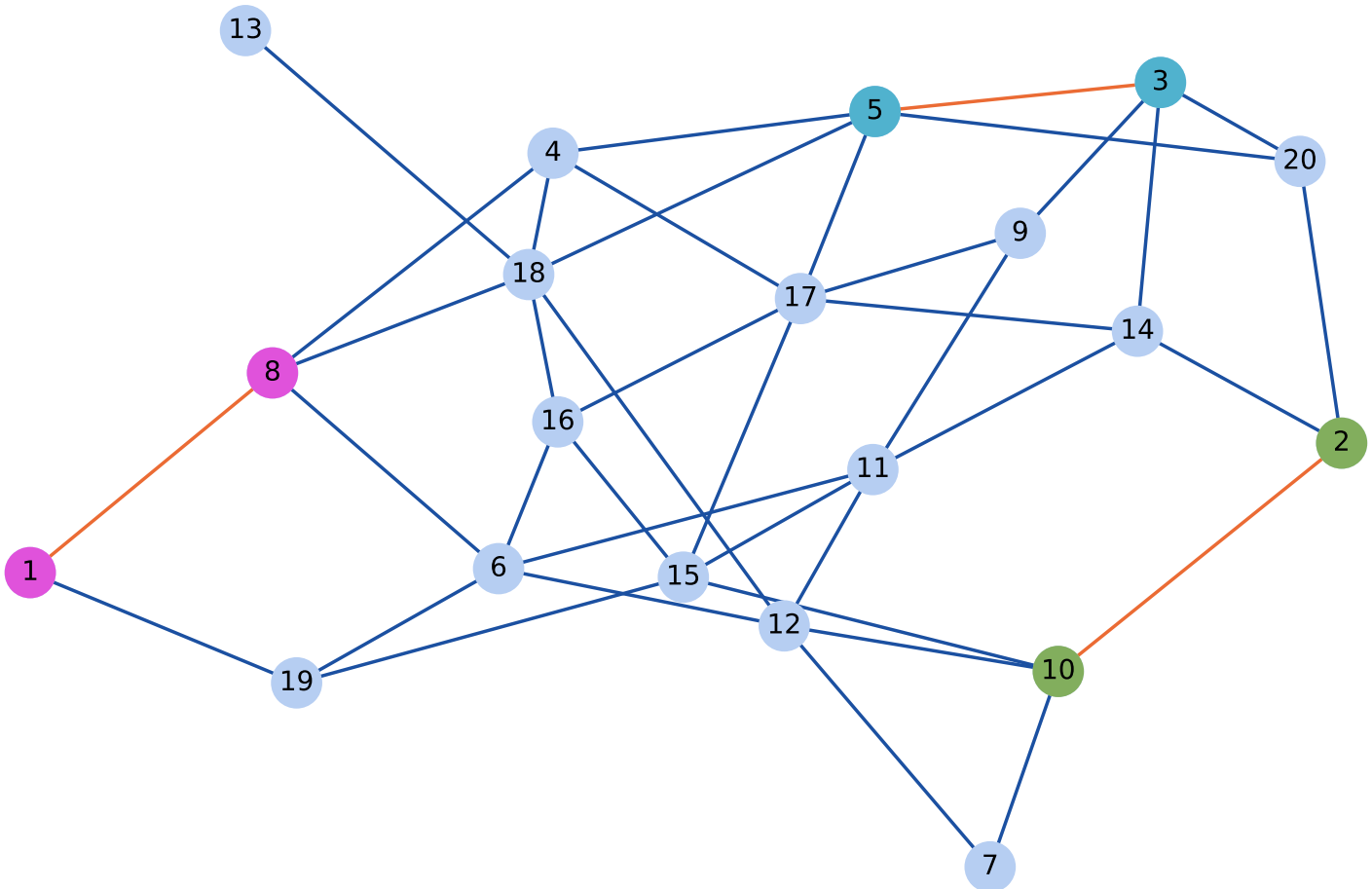



Matching size: 1
Algo step: 1
Nb nodes: 20
13
3
5
4
20
9
18
17
14
8
16
2
11
1
6
15
12
10
19
7

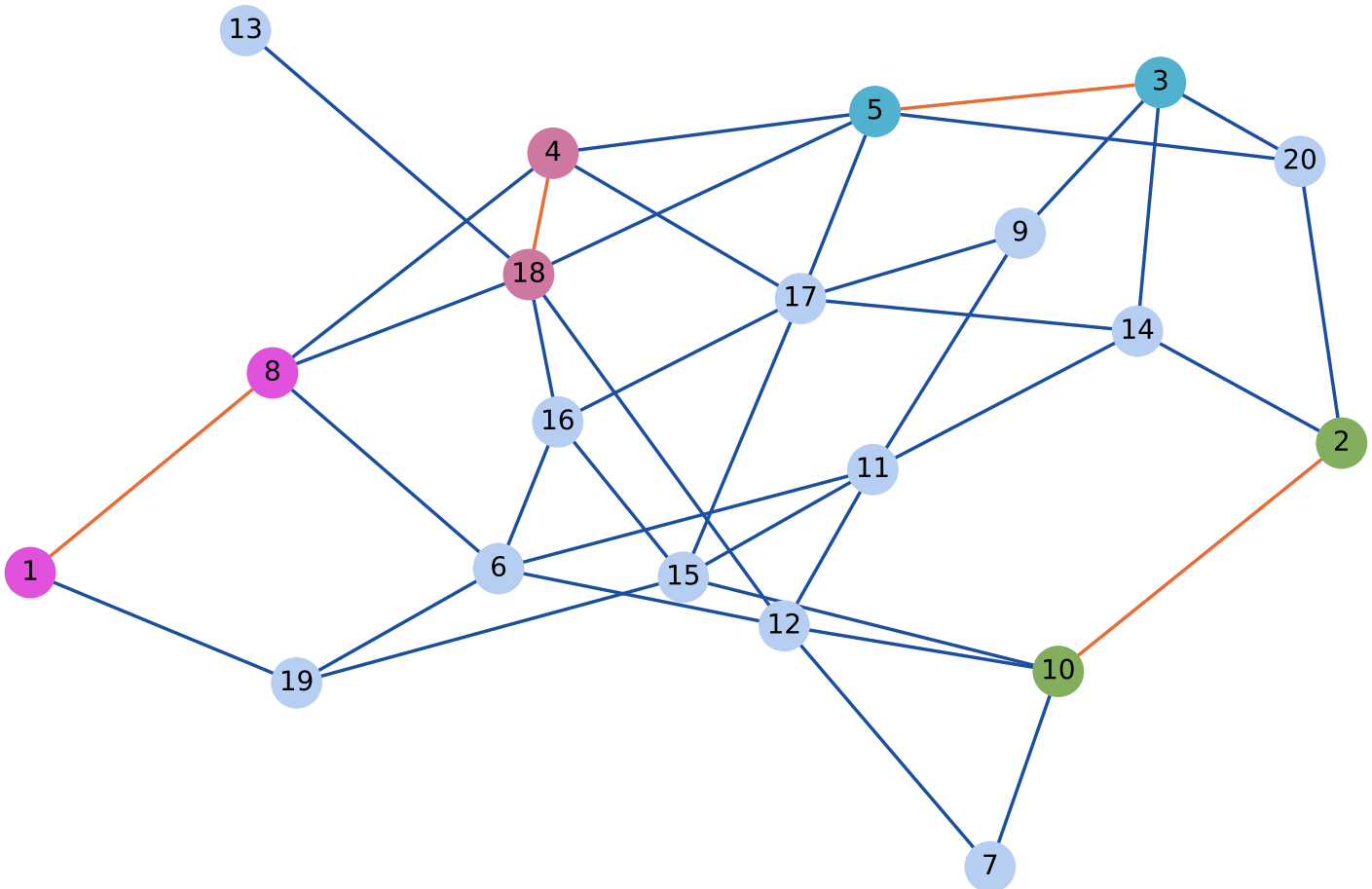
Matching size: 2
Algo step: 3
Nb nodes: 20



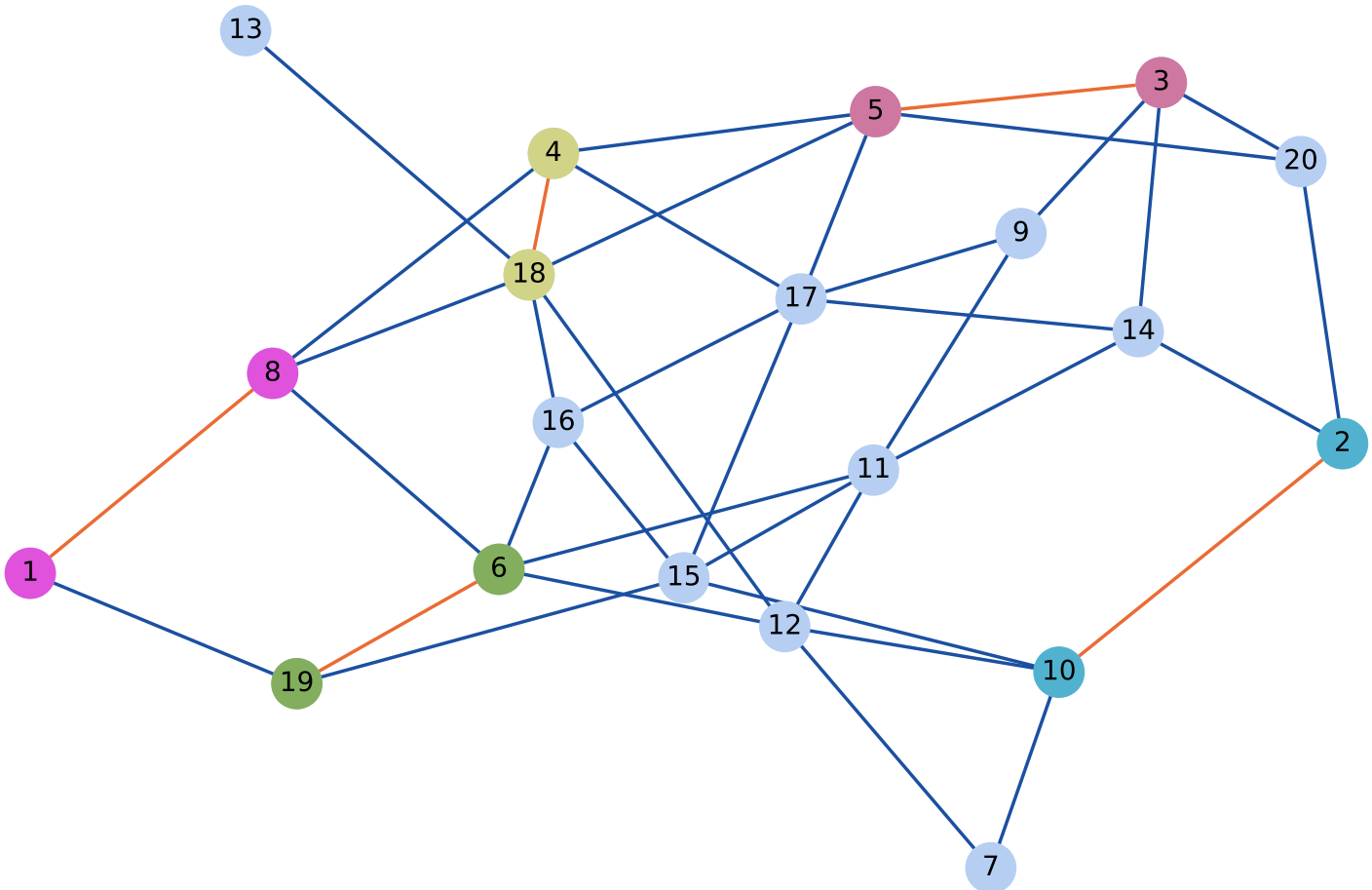
Matching size: 3
Algo step: 6
Nb nodes: 20



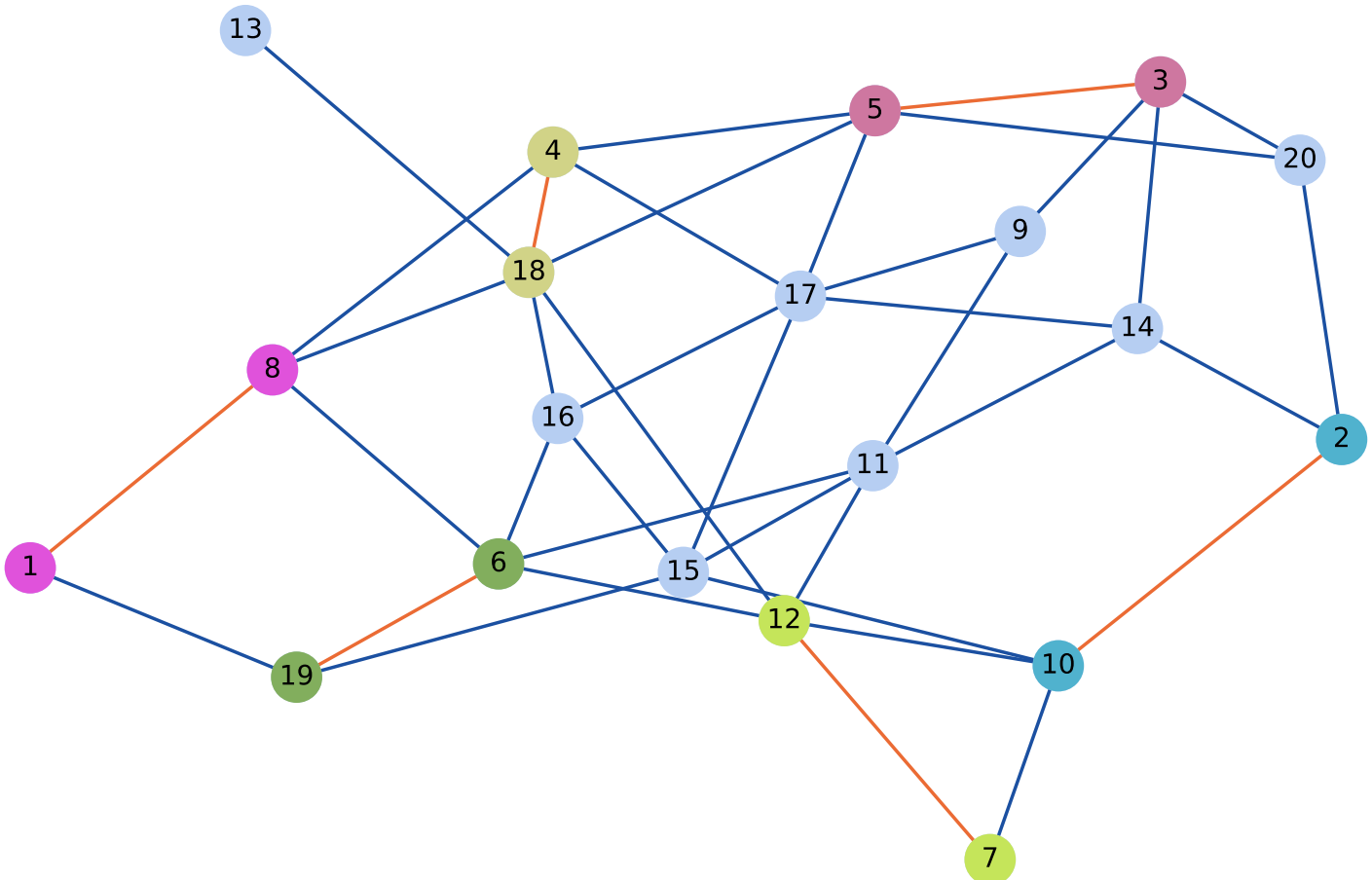
Matching size: 4
Algo step: 11
Nb nodes: 20



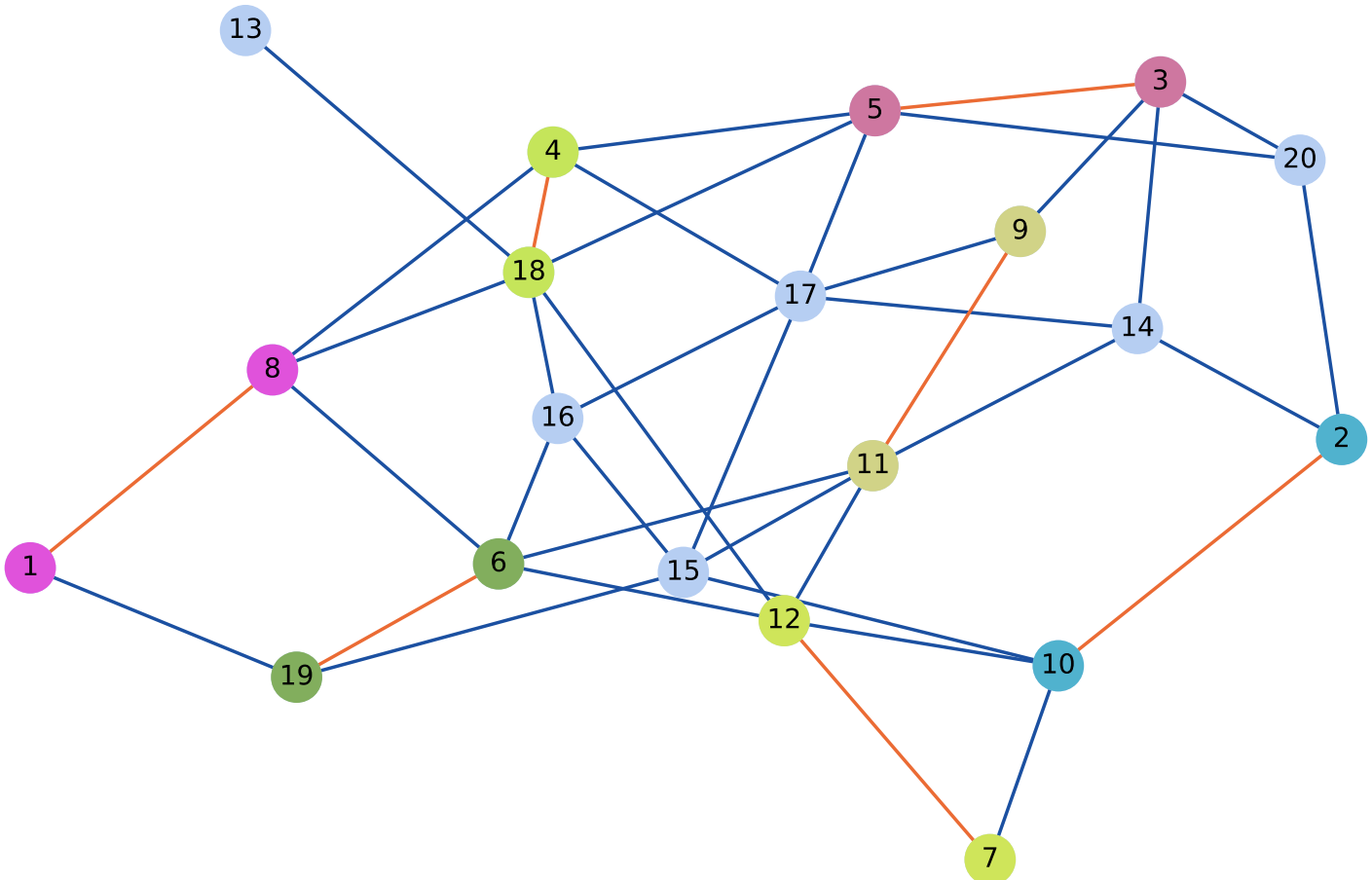
Matching size: 5
Algo step: 17
Nb nodes: 20



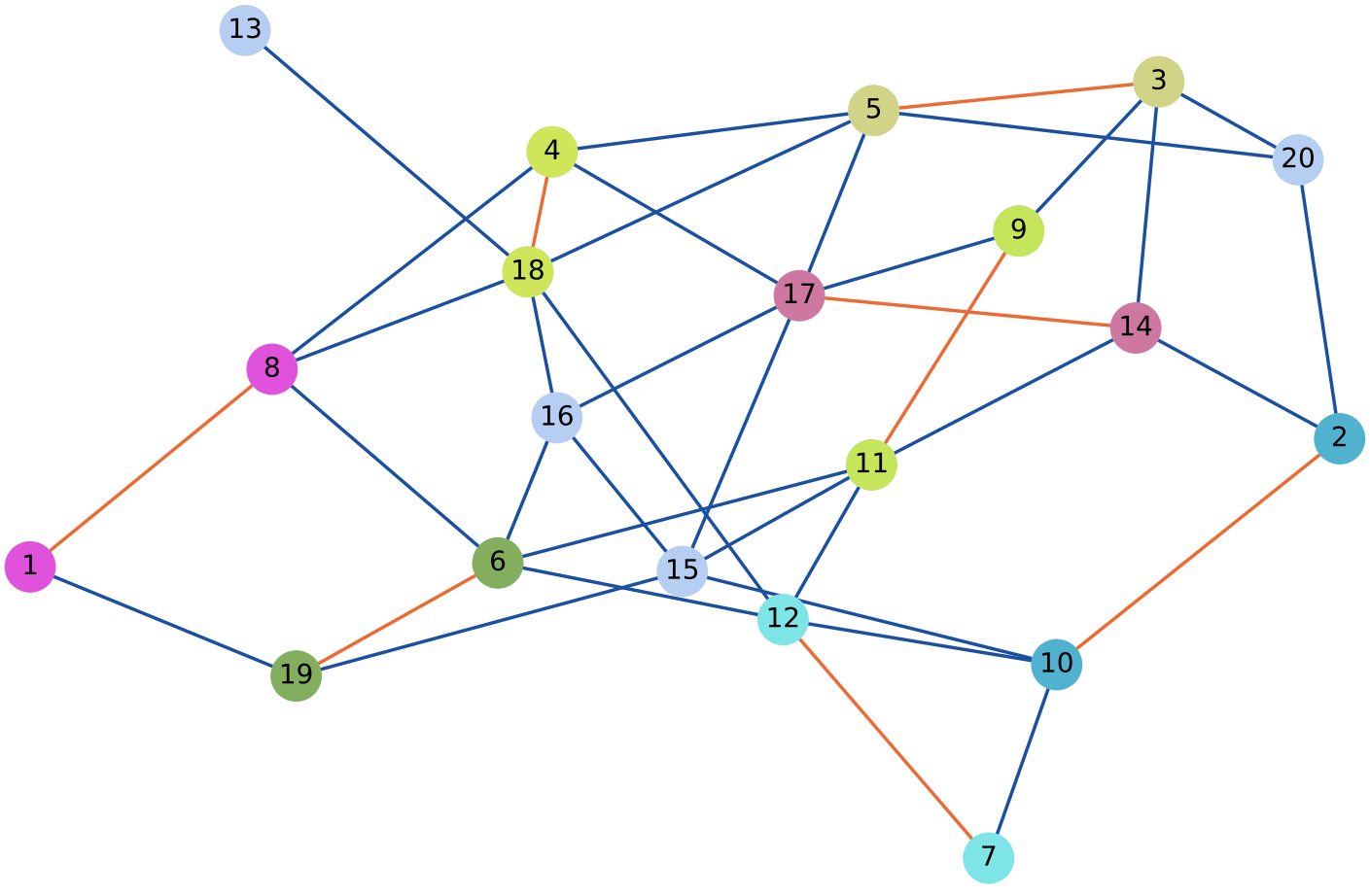
Matching size: 6
Algo step: 22
Nb nodes: 20



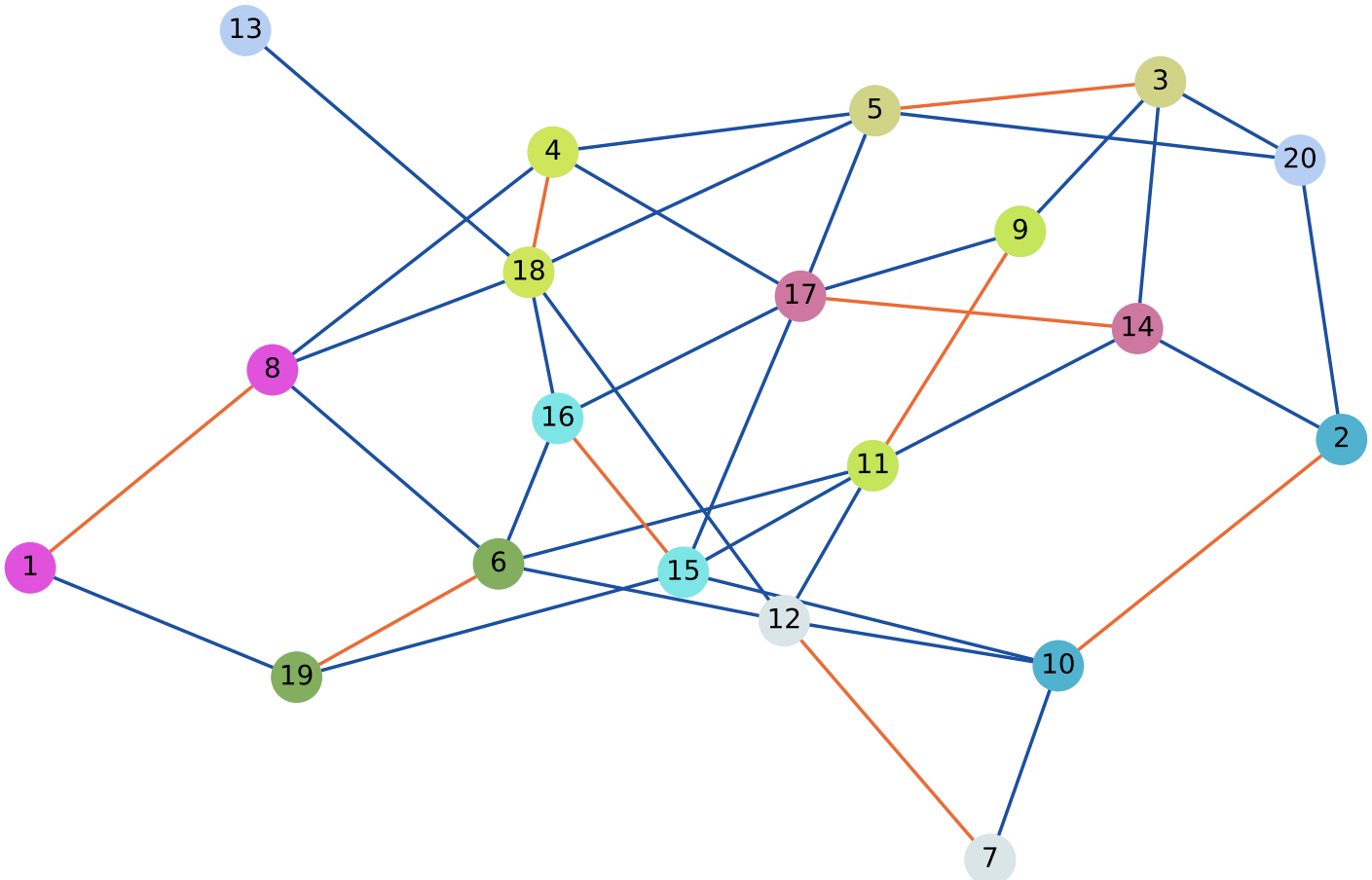
Matching size: 7
Algo step: 25
Nb nodes: 20



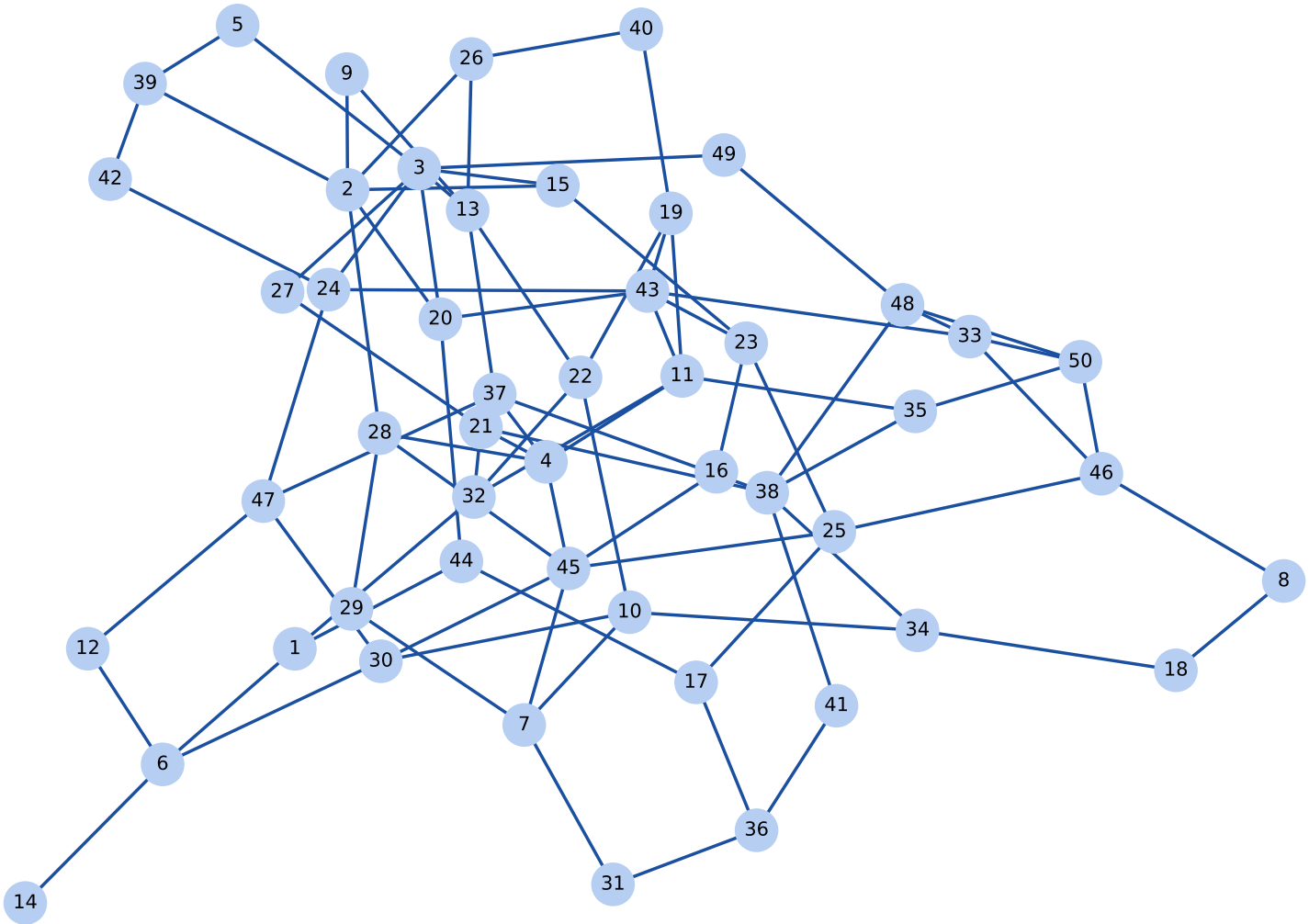
Matching size: 8
Algo step: 34
Nb nodes: 20



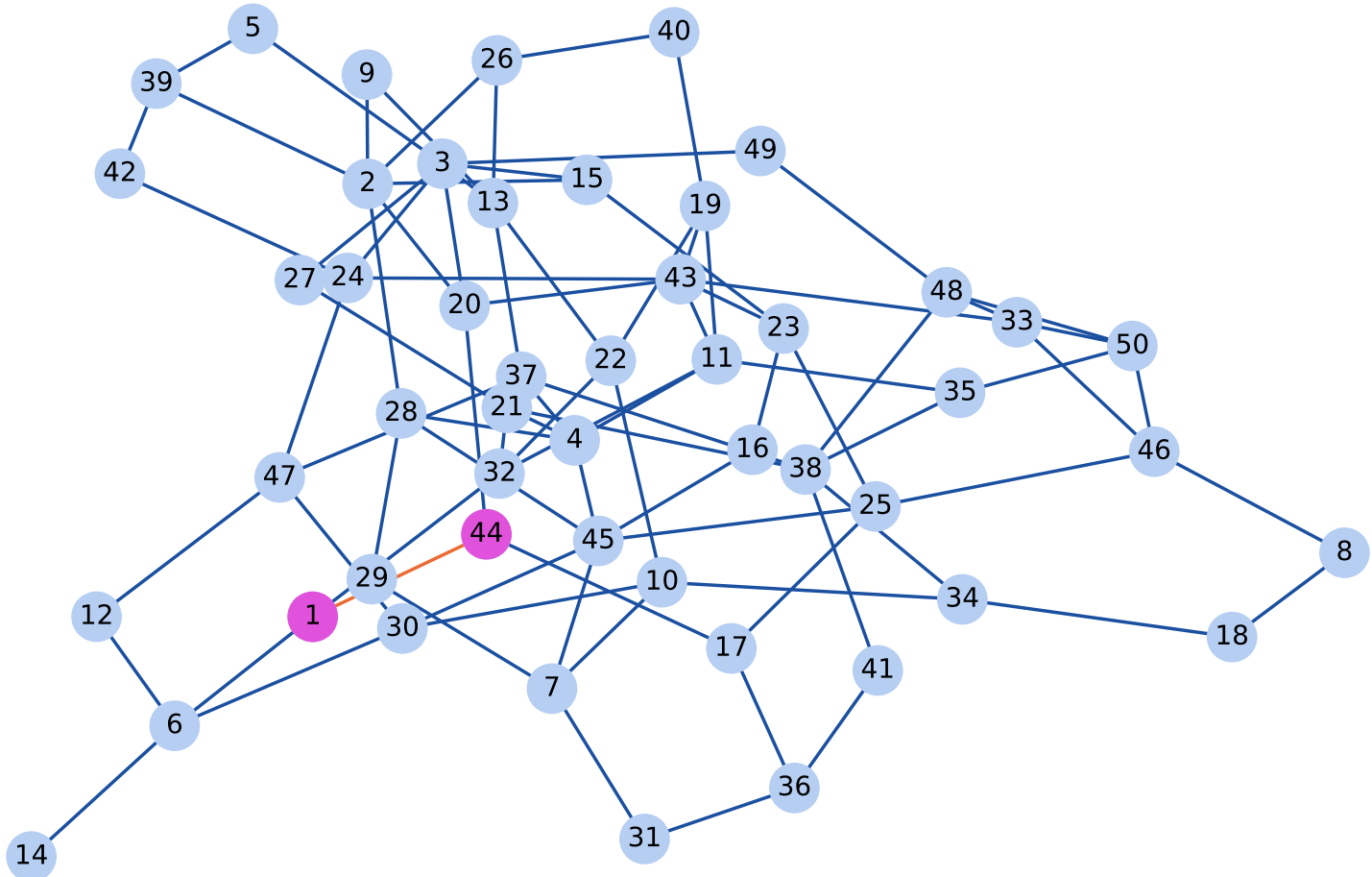
Matching size: 9
Algo step: 36
Nb nodes: 20



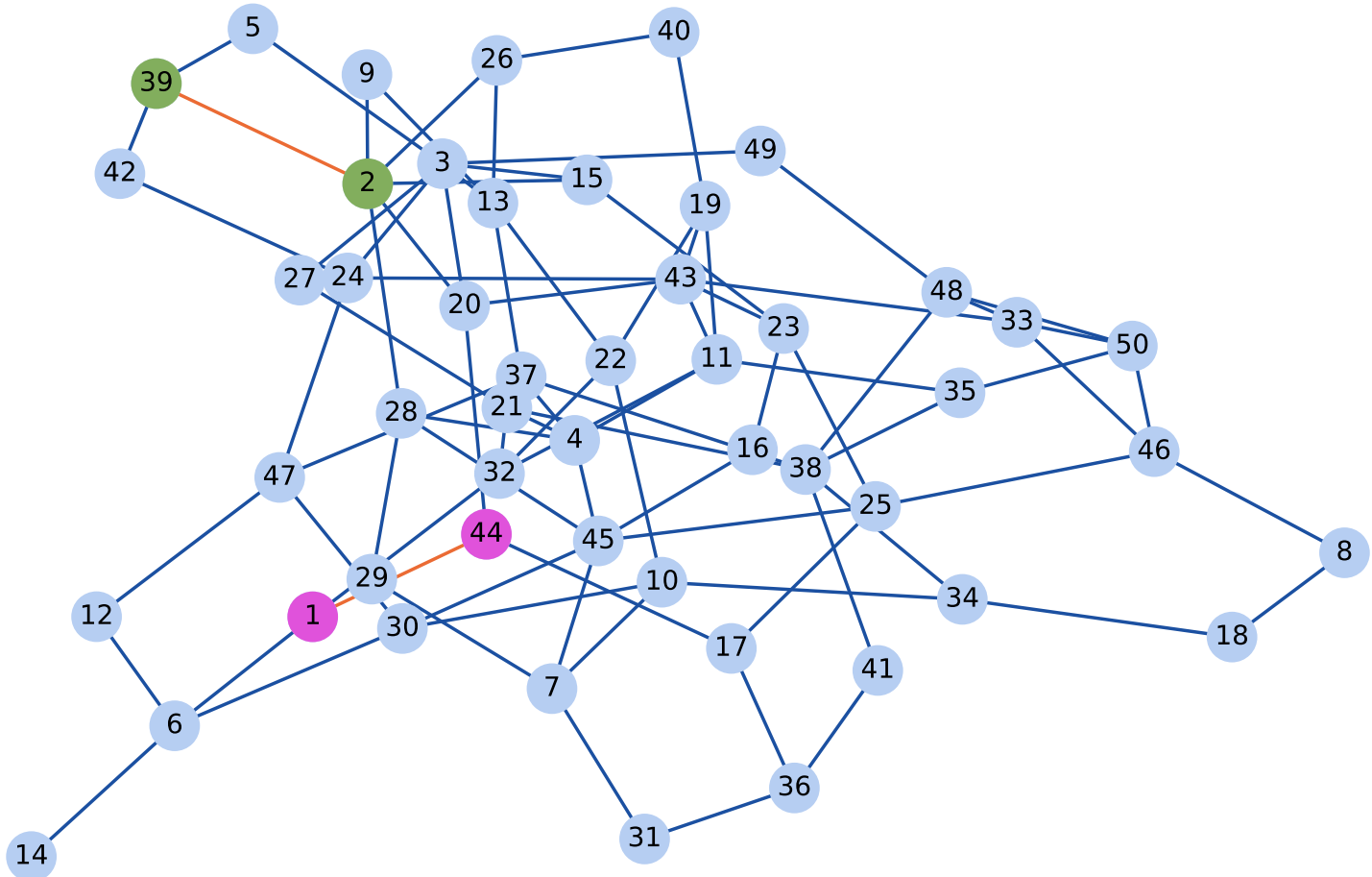
initial graph



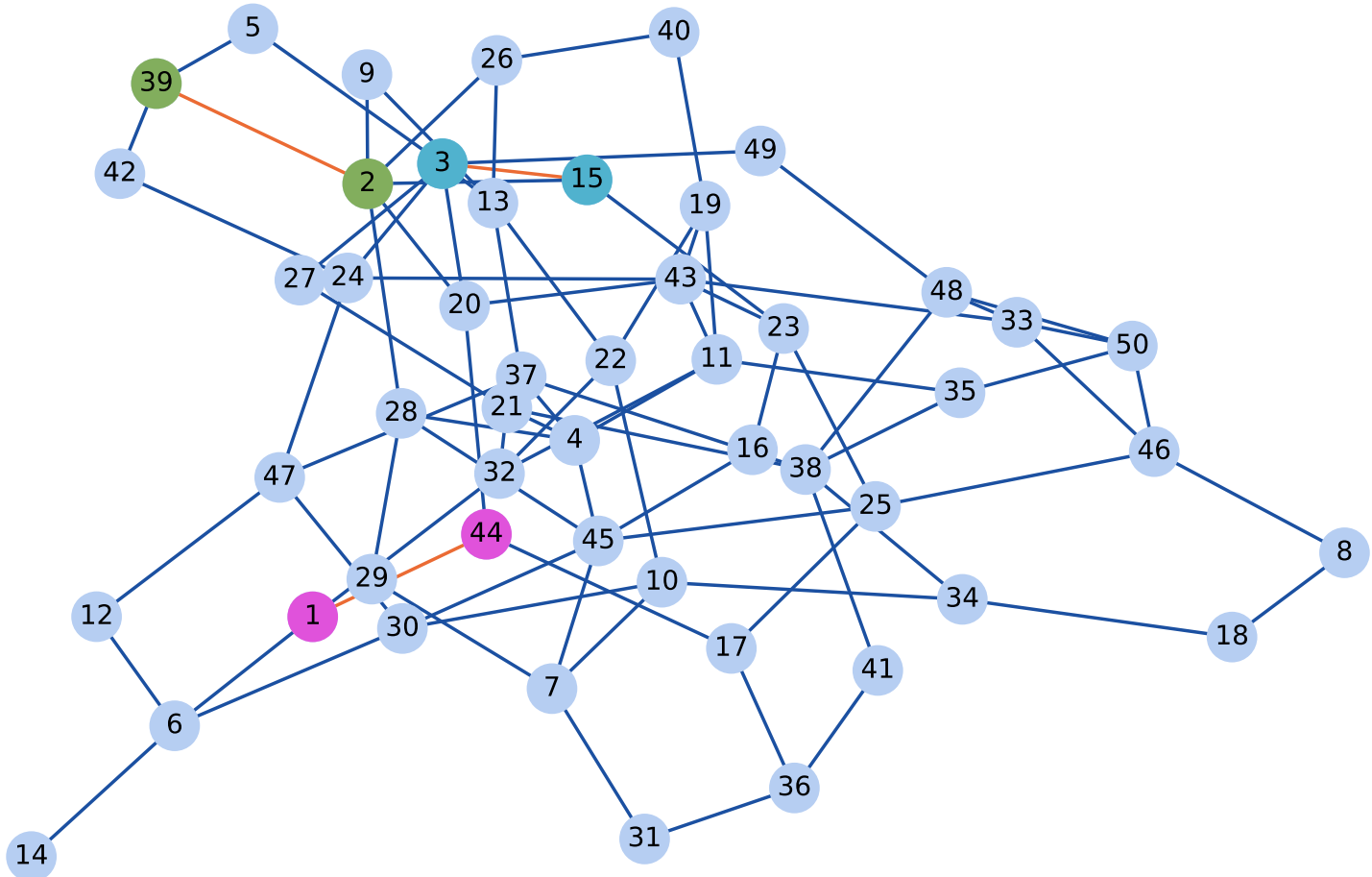
Matching size: 1
Algo step: 1
Nb nodes: 50

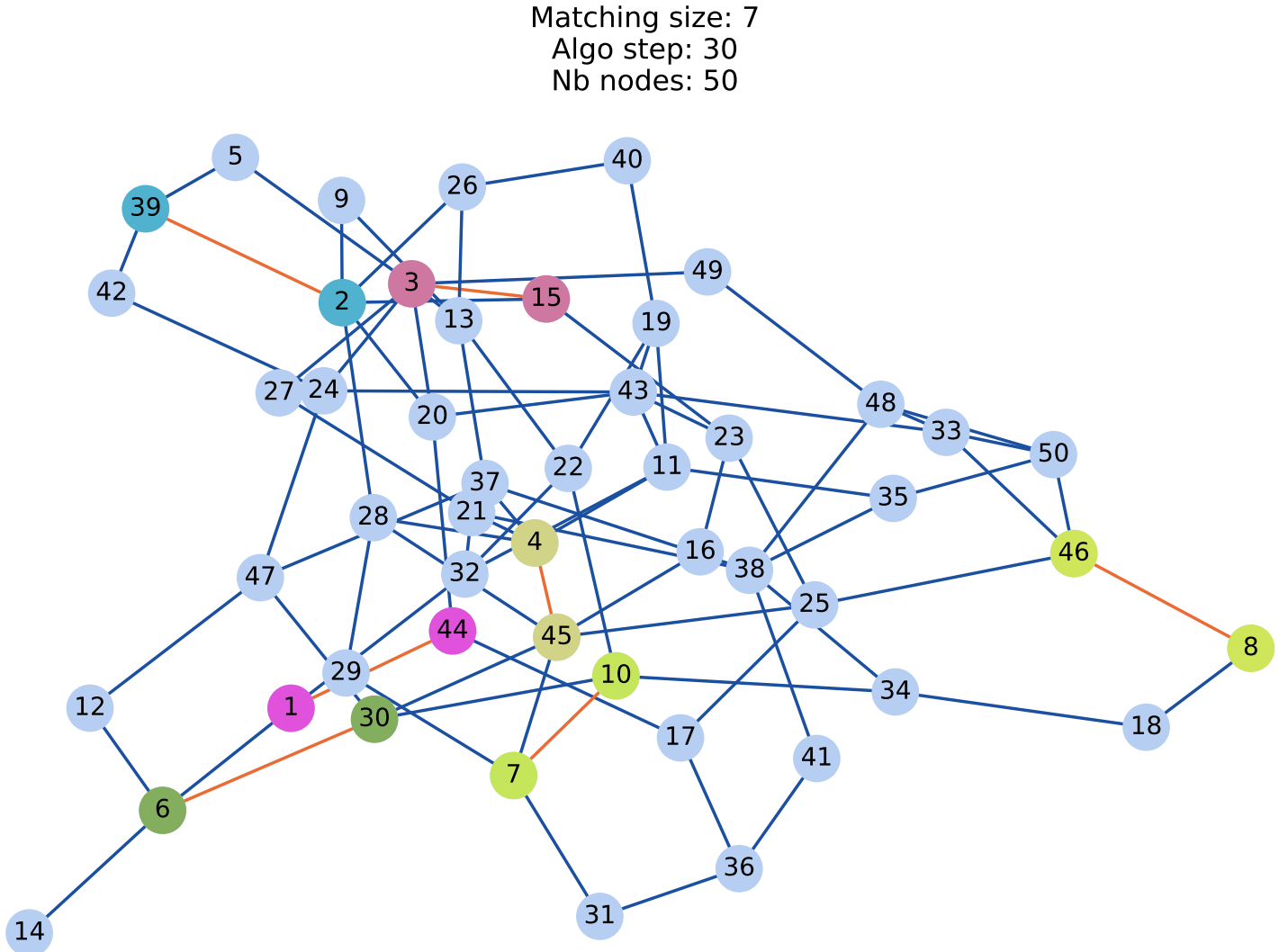


Matching size: 2
Algo step: 4
Nb nodes: 50



Matching size: 3
Algo step: 10
Nb nodes: 50




Matching size: 7
Algo step: 30
Nb nodes: 50
5
40
39
9
26
49
42
3
2
15
13
19
27
24
43
48
20
23
33
50
22
11
37
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32
38
25
44
45
8
29
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12
1
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7
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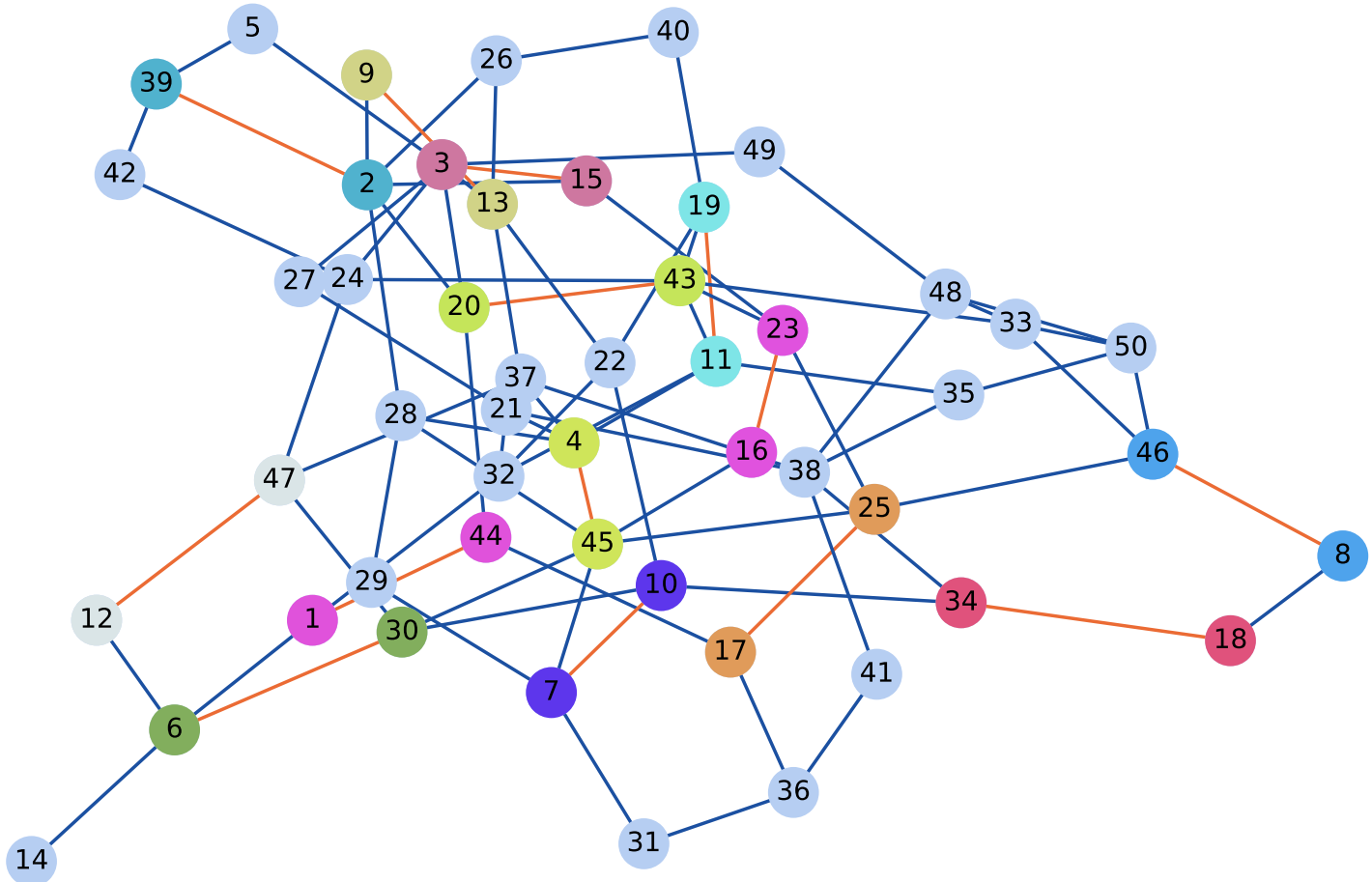
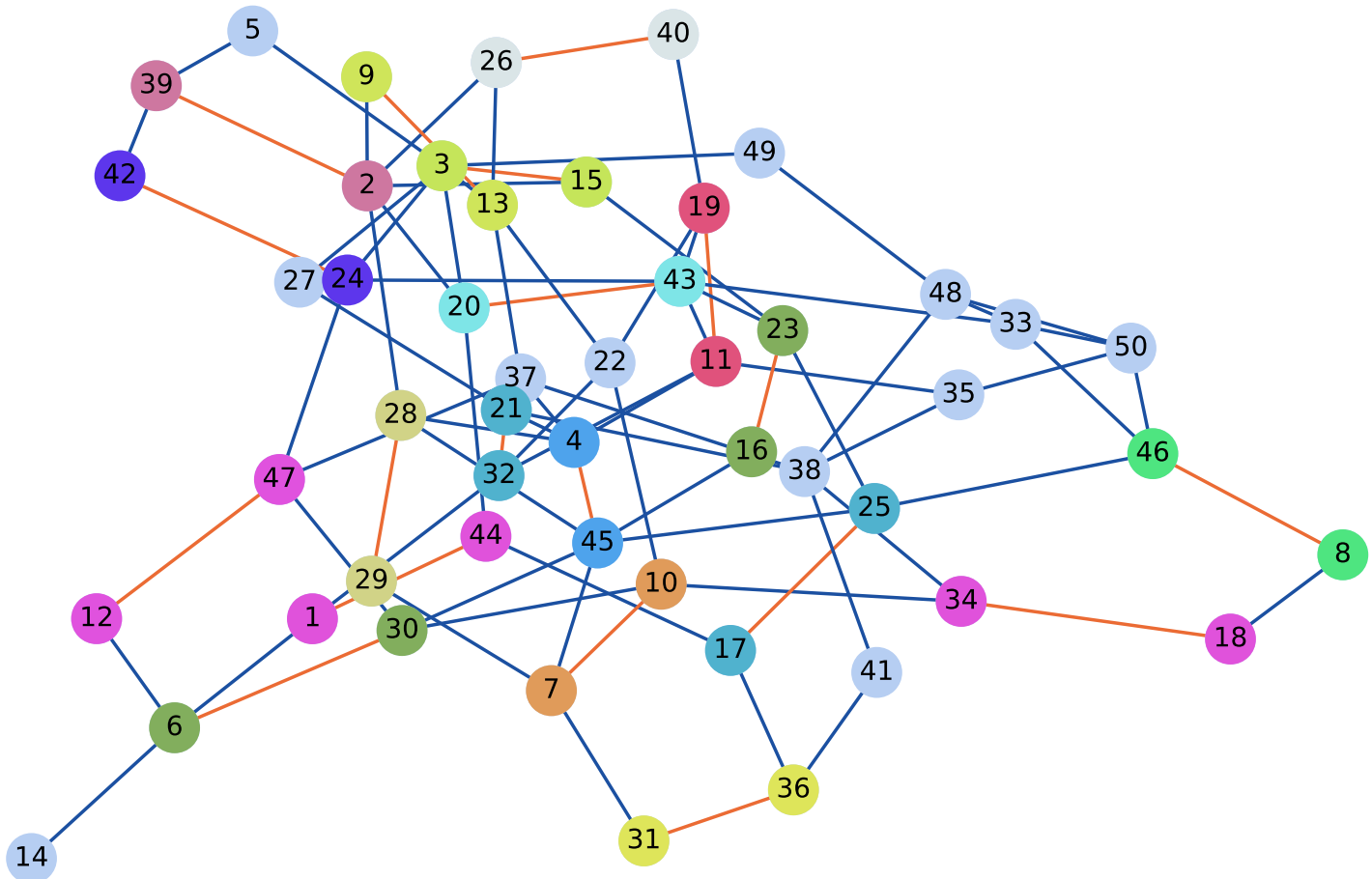
36
14
31

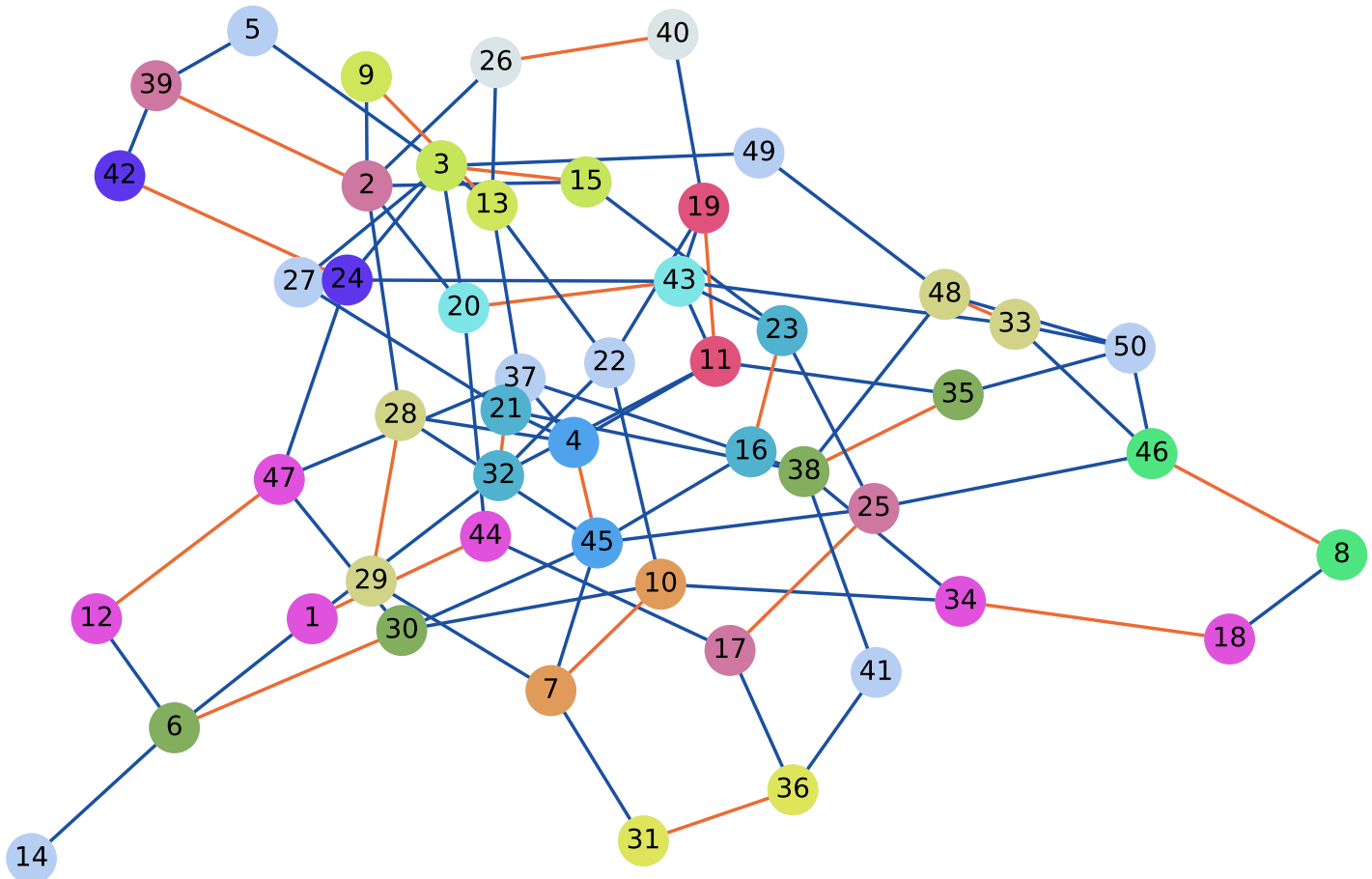
Matching size: 14
Algo step: 54
Nb nodes: 50



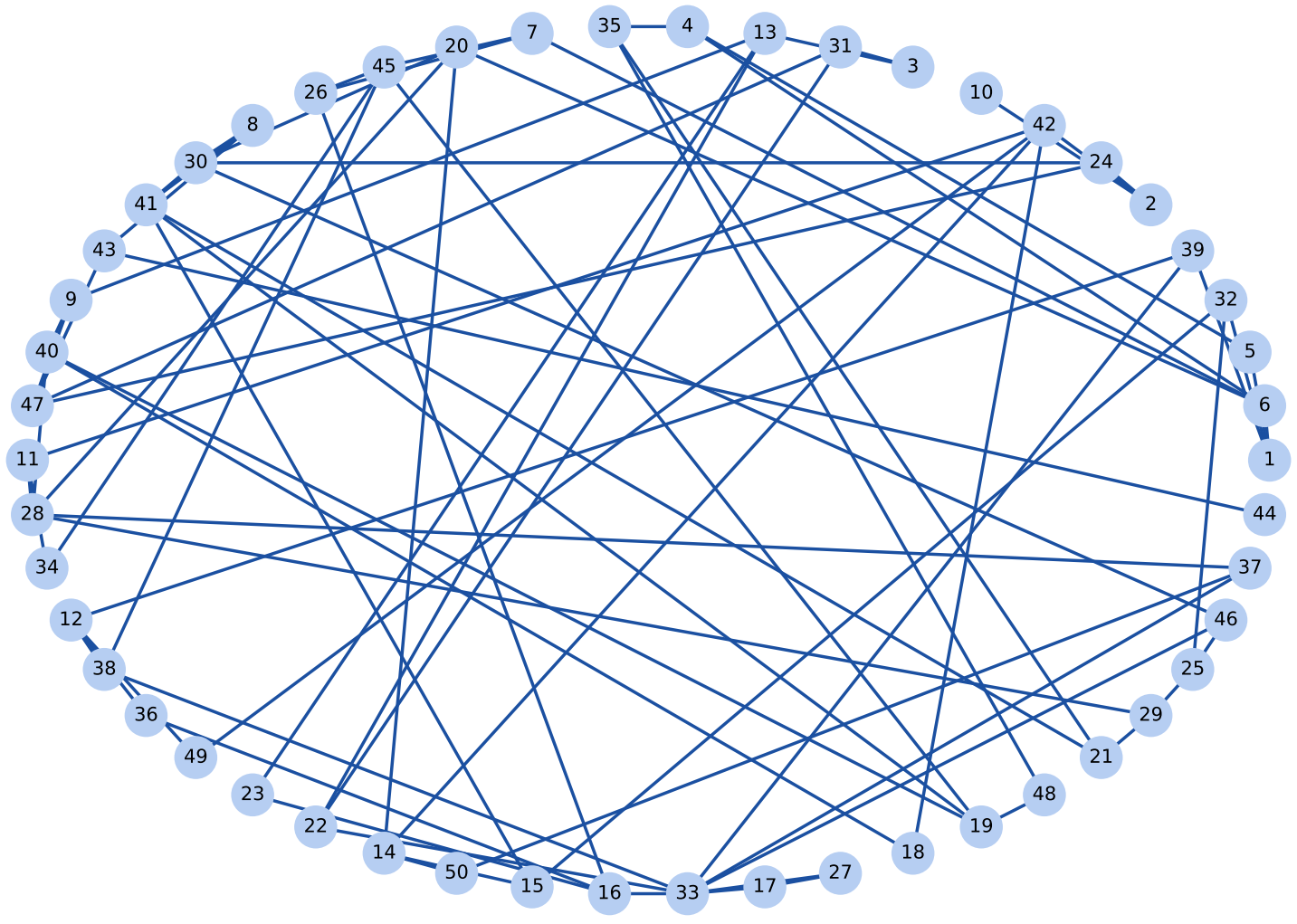
Matching size: 19
Algo step: 72
Nb nodes: 50

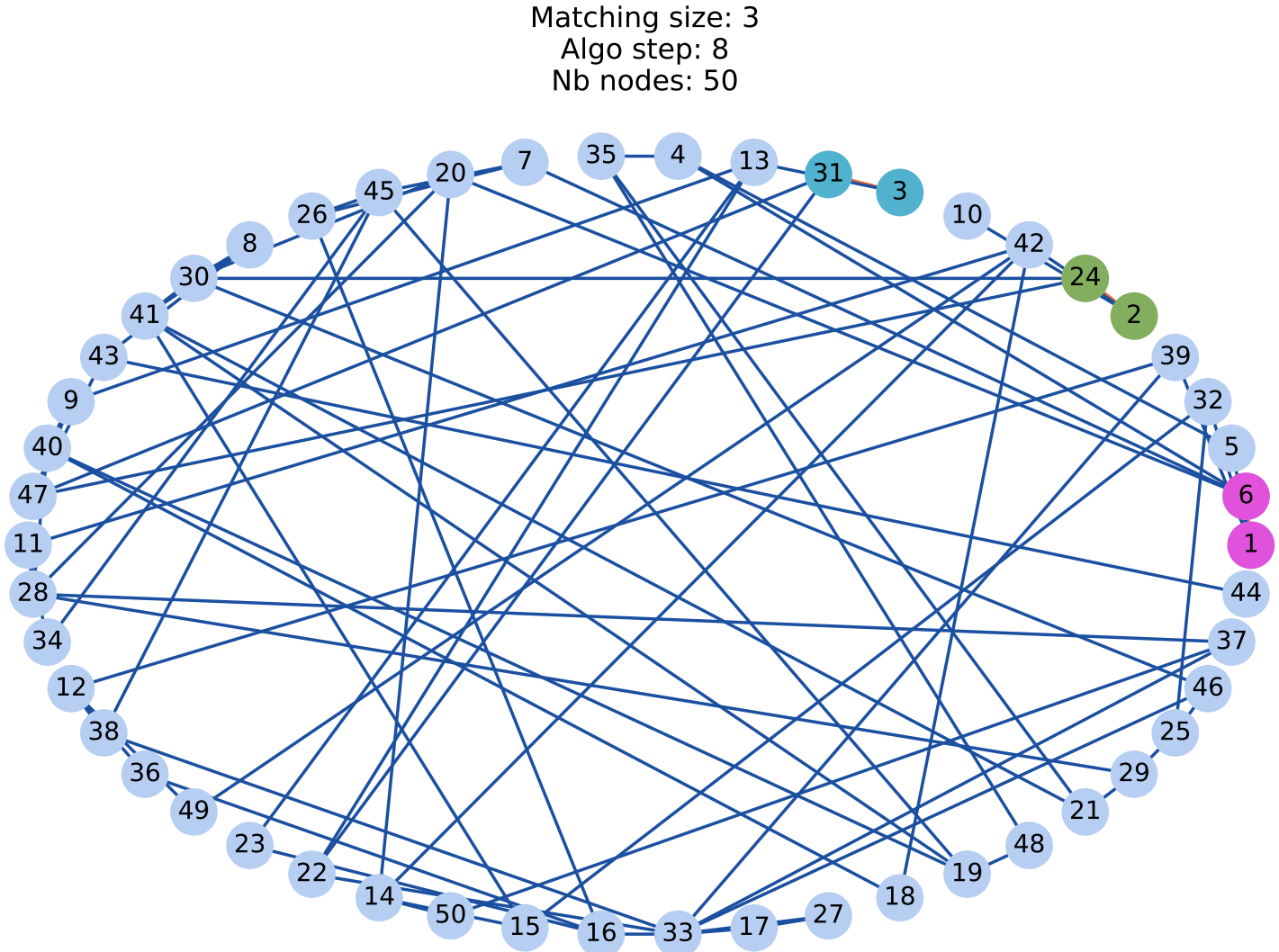


Matching size: 21
Algo step: 78
Nb nodes: 50



initial graph




Matching size: 3
Algo step: 8
Nb nodes: 50
35
4
13
31
3
10
42
24
2
39
32
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6
1
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37
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25
29
21
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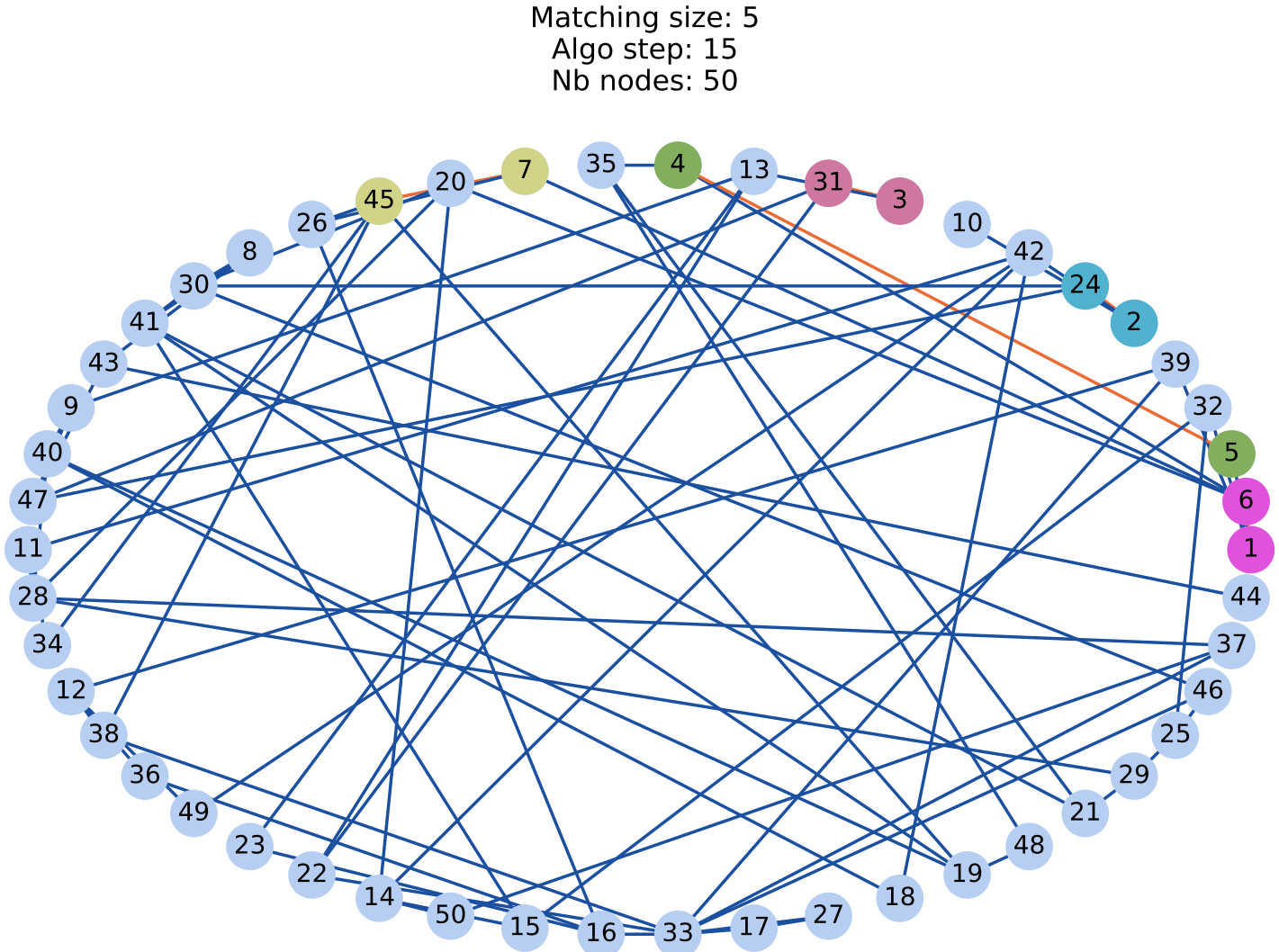
Matching size: 5
Algo step: 15
Nb nodes: 50
7
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4
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31
3
20
45
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8
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30
24
41
2
43
39
9
32
40
5
47
6
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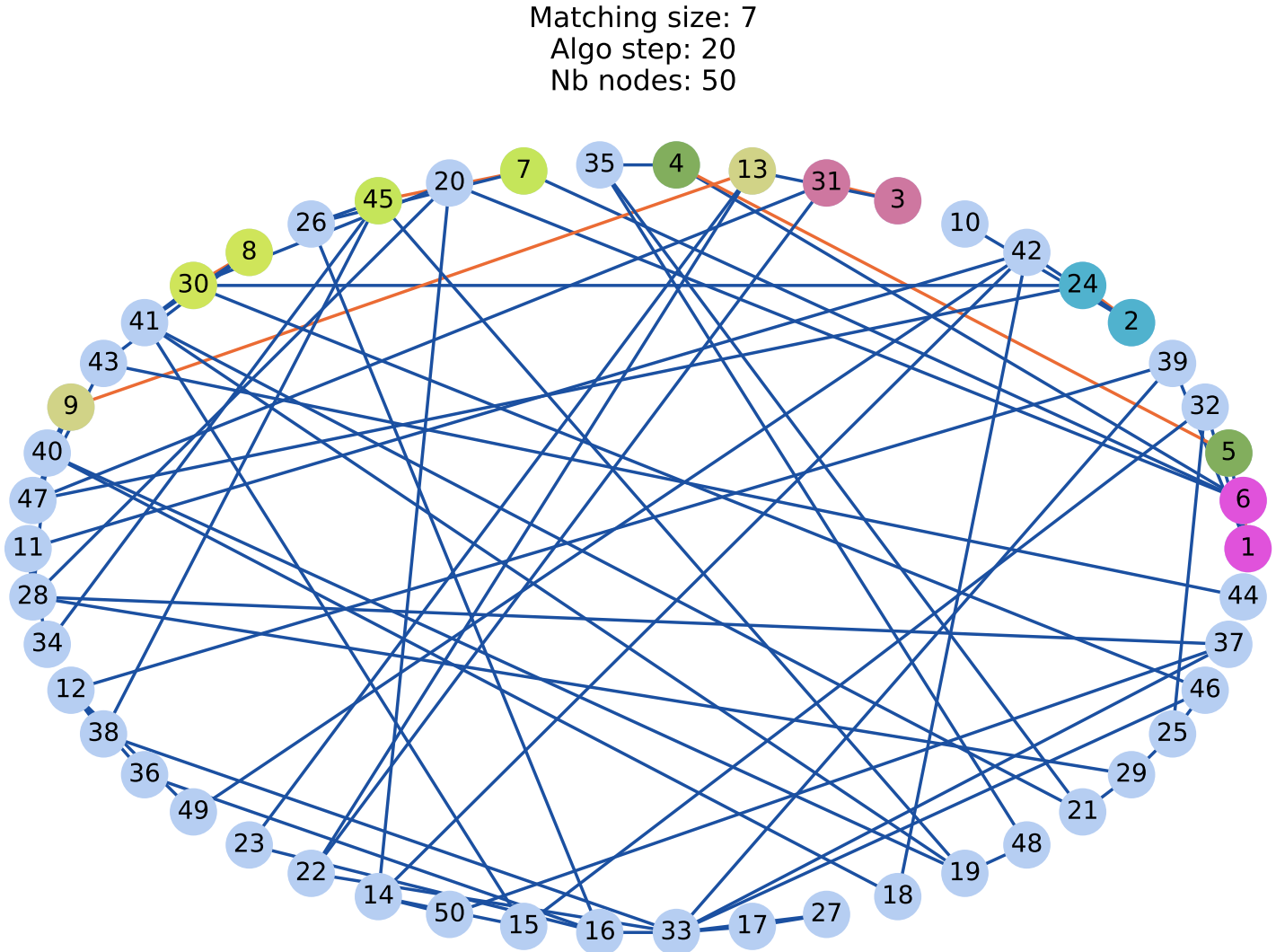
Matching size: 7
Algo step: 20
Nb nodes: 50
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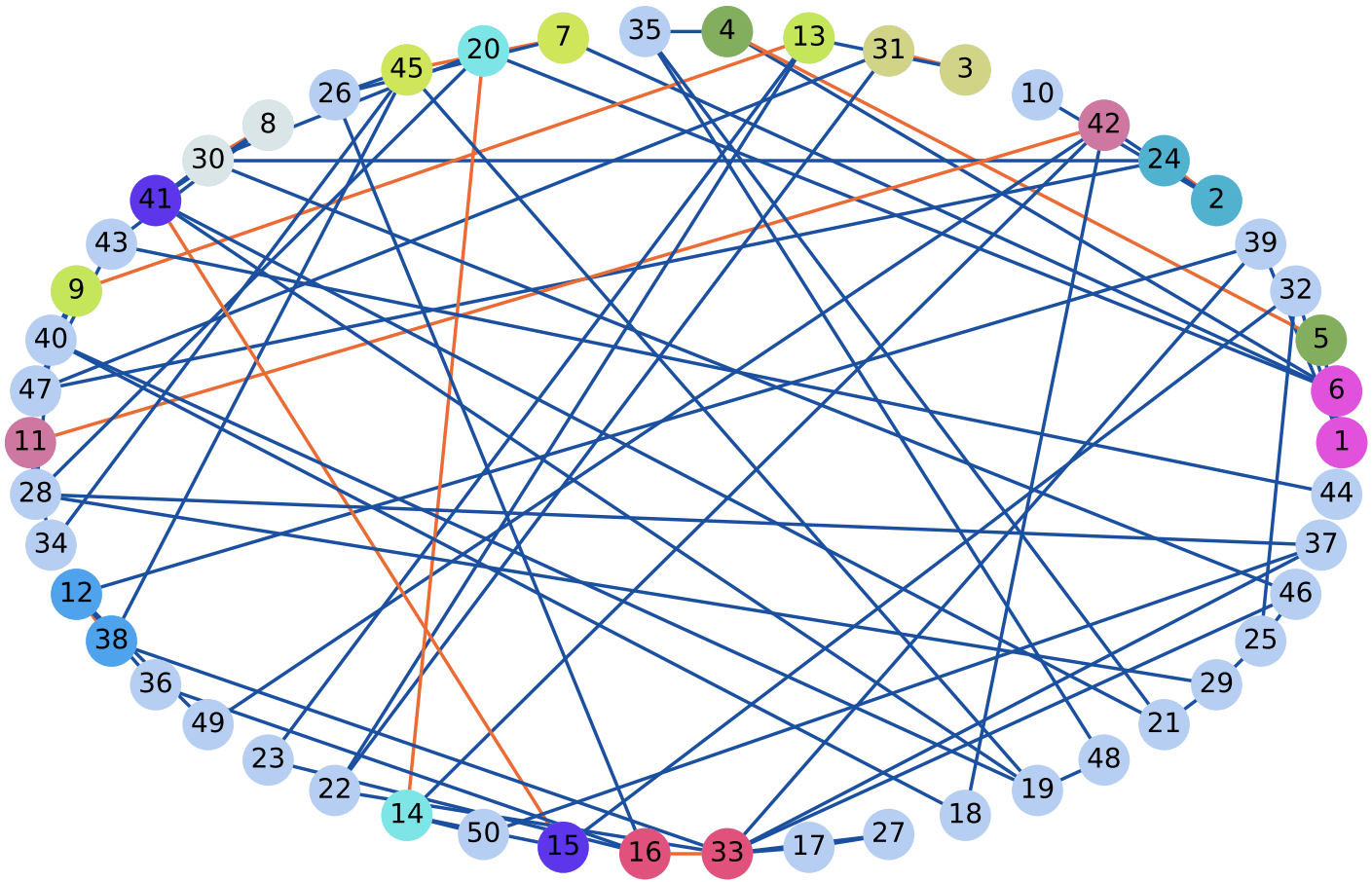
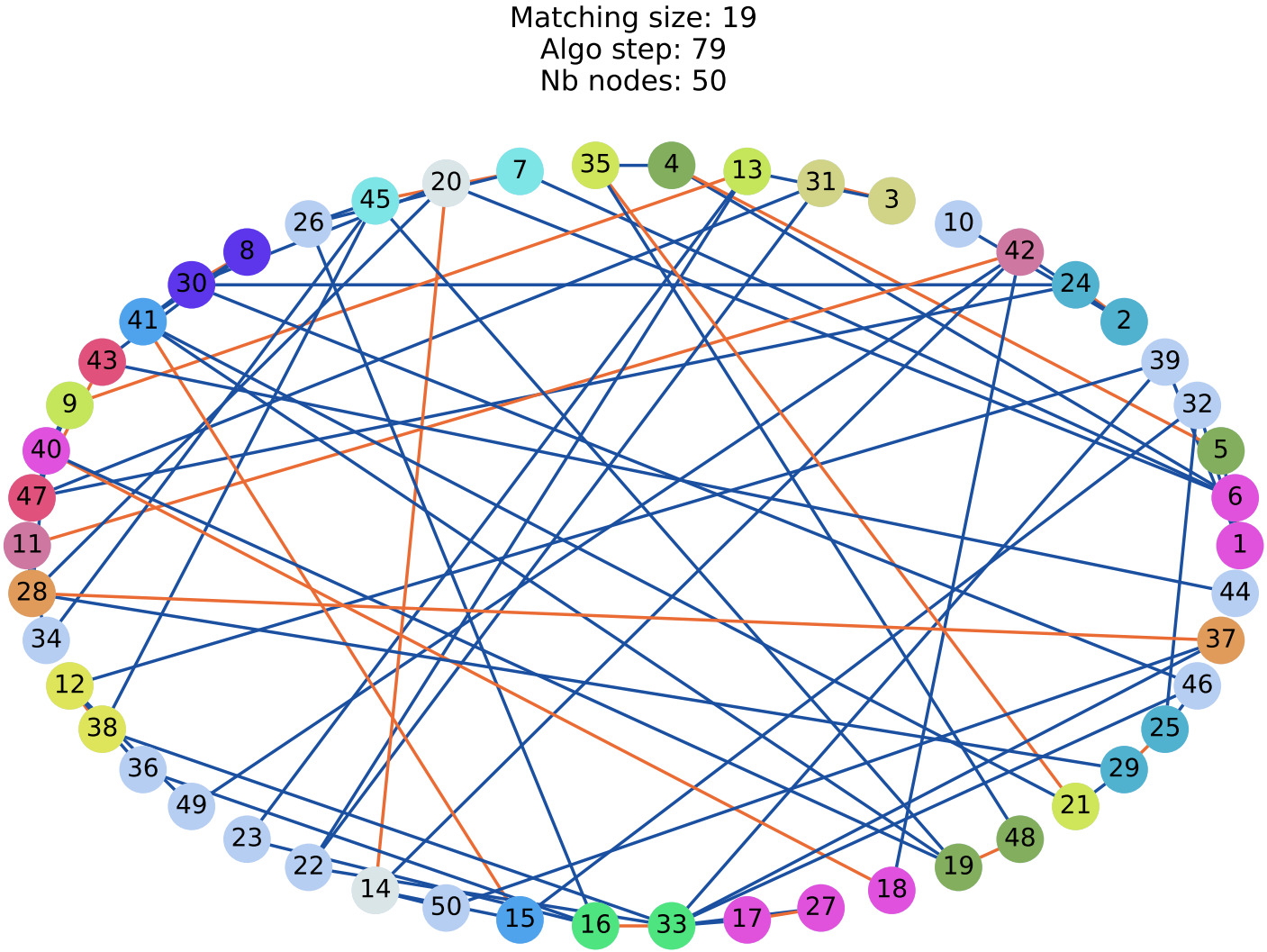
Matching size: 12
Algo step: 38
Nb nodes: 50




Matching size: 19
Algo step: 79
Nb nodes: 50
35
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Example

Exercice 9 : Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?

Example

Exercice 10 : Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?

Greedy matching

However, is $|M|$ is the cardinality of a matching returned by the greedy algorithm, and if $|M^*|$ is the cardinal of the real optimal matching, we have :

$$|M| \geq \frac{|M^*|}{2} \tag{5}$$

Changing the problem (for now)

We temporarily leave the maximum matching problem to focus on another problem : the **Maximum flow problem**

Max flow

Figure: Optimizing the quantity of merchandise transported from one place to another, respecting some constraints

Example

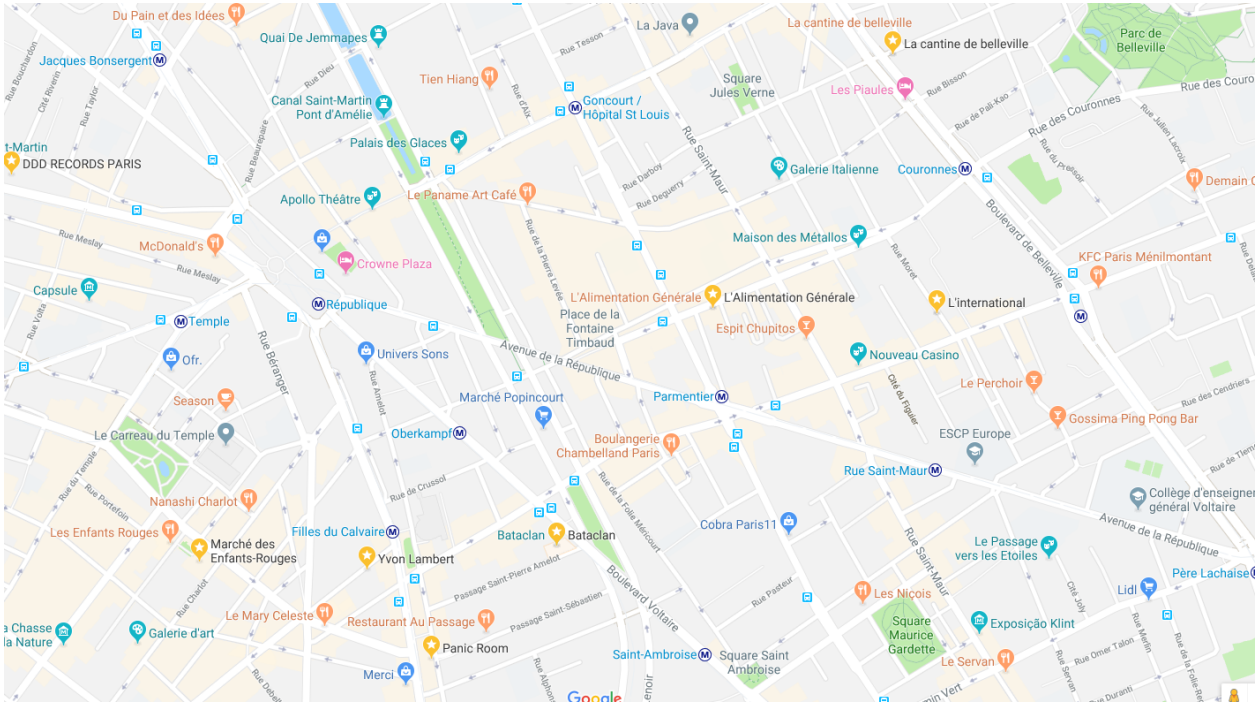

Figure: Optimizing the quantity of merchandise transported from one place to another, respecting some constraints

Formalizing the problem

We introduce the concept of **flow network (reseau de flot).**

Formalizing the problem

- A **Directed graph** $G = (E, V)$

Formalizing the problem

Flow network (reseau de flot) :

- A **Directed graph** $G = (E, V)$
- Each edge (u, v) must have a **capacity** $c(u, v) \geq 0$

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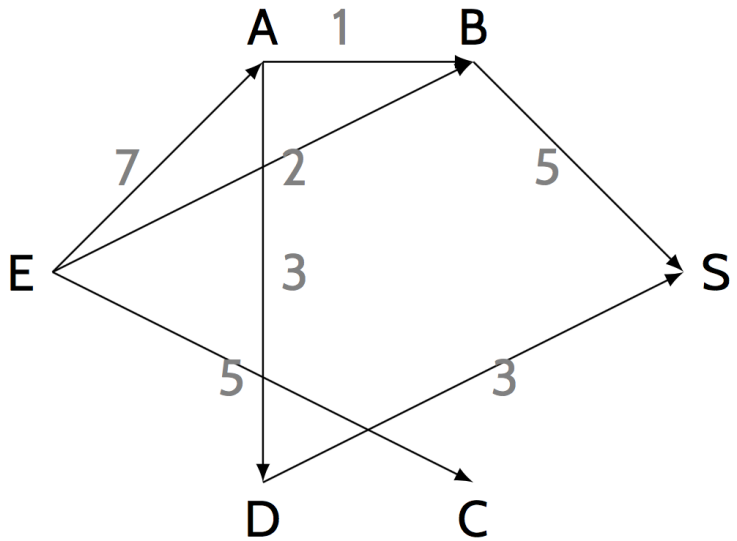

Figure: A **flow network (reseau de flot)** with capacities

Formalizing the problem

Flow network (reseau de flot) :

- A **Directed graph** $G = (E, V)$
- Each edge (u, v) must have a **capacity** $c(u, v) \geq 0$
- We define two special nodes : a **source** E and a **sink** S.
- A **flow** f is a function $f(u, v) \leq c(u, v)$ (+ additional constraints)

Formalizing the problem

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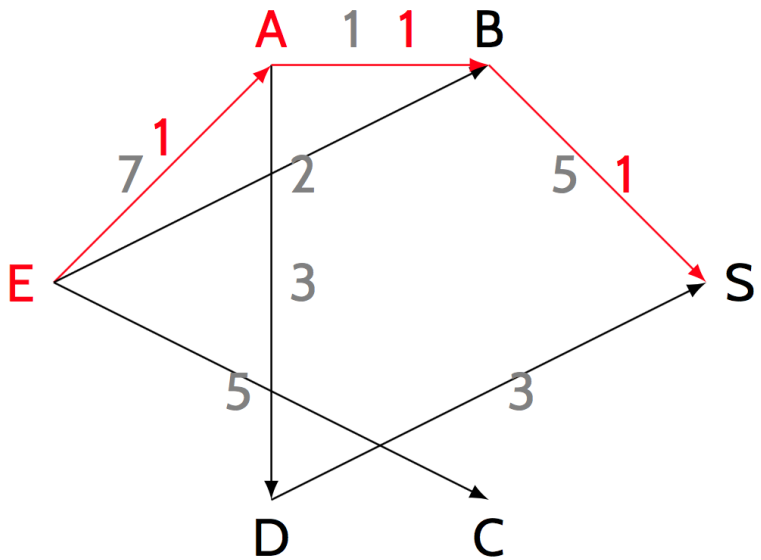

Formalizing the problem

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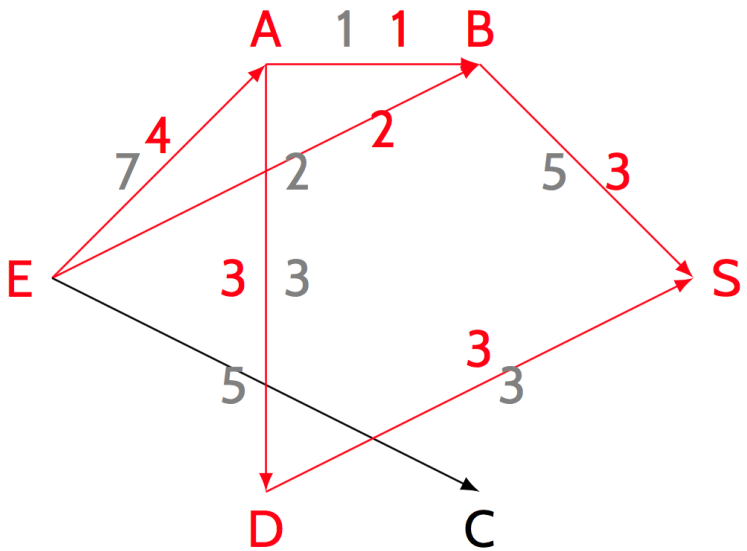

Conservation of the flow

We must have :

- antisymmetry : $f(v,u) = -f(u,v)$

conservation of the flow

we must have :

- antisymmetry : $f(v,u) = -f(u,v)$
- flow conservation : $\sum_{w \in V} f(u,w) = 0$ for any $u \notin \{e,s\}$

Other formulation of the flow conservation

Exercice 10 : Other formulation of the flow conservation

Let us show that for a flow f, we have for any node $u \notin \{e, s\}$:

$$\sum_{f(u,v)>0} f(u,v) = \sum_{f(v,u)>0} f(v,u) \tag{6}$$

Maximum flow

- The **value of the flow**, noted $|f|$, is $\sum_{v \in S} f(E, v)$
- The problem is that of finding a flow with **maximum value**

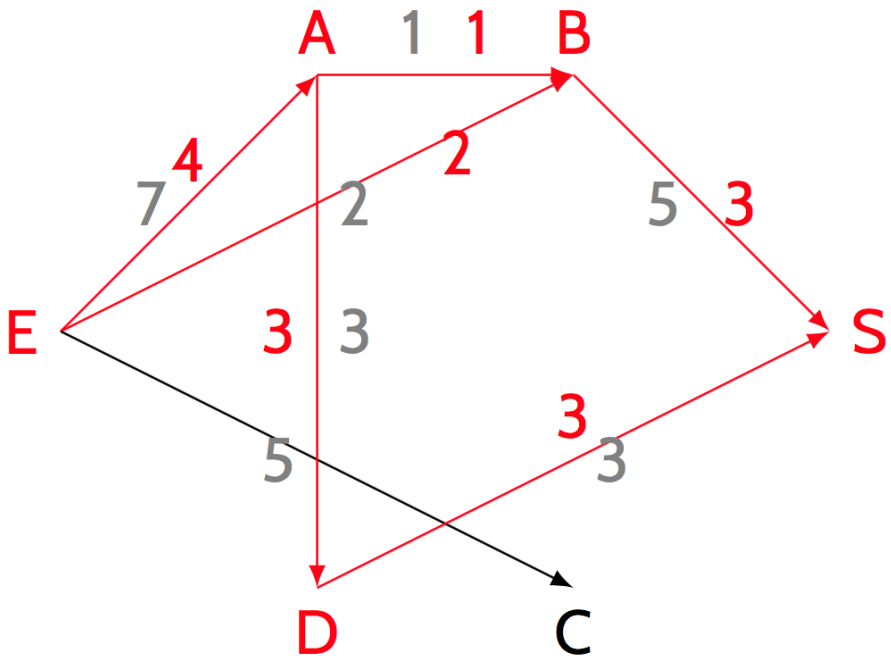

Figure: Max flow

Ford Fulkerson algorithm

We will introduce an algorithm to solve the problem. This algorithm :

- terminates
- is correct
- is polynomial

Ford Fulkerson algorithm

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- is correct
- is polynomial

So it is a good algorithm.

Ford Fulkerson algorithm

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- terminates
- is correct
- is polynomial

So it a good algorithm. **This section is going to be a little bit technical.**

Residual graph

- Given a graph with capacities $c(u, v)$ and a flow $f(u, v)$, we will define its **residual graph** that has a capacity $c_r(u, v)$:

$$c_r(u, v) = c(u, v) - f(u, v) \tag{7}$$

Example of residual graph

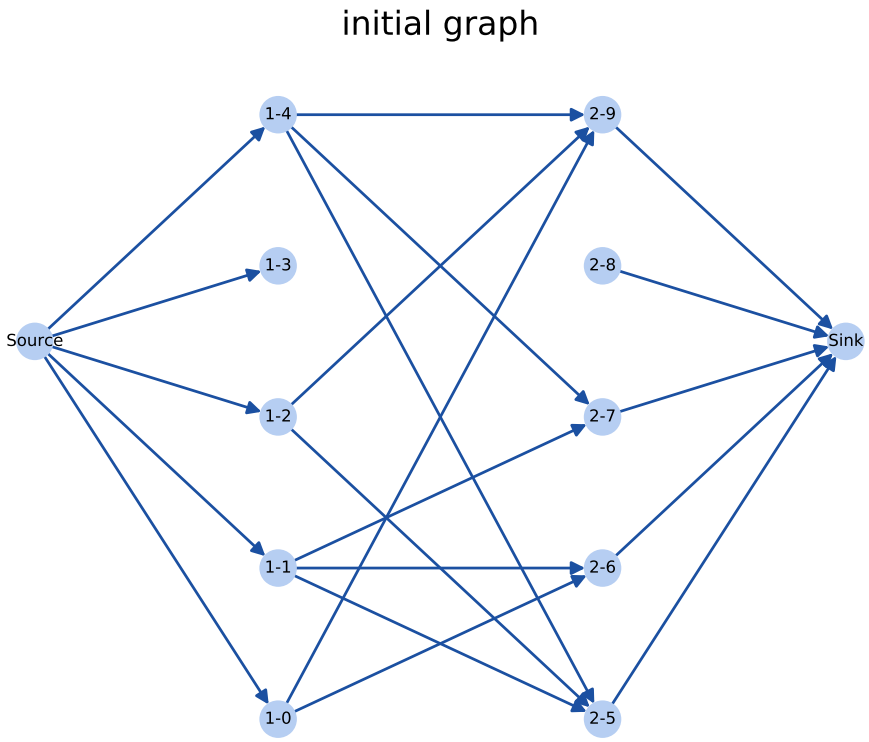
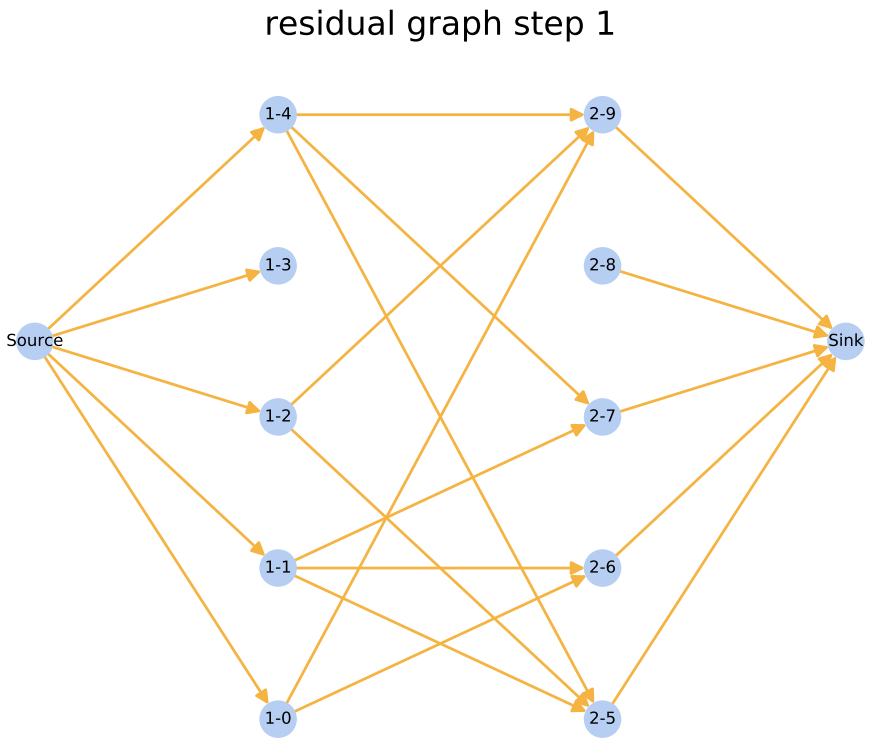


Figure: All initial capacities set to 1

Example of residual graph



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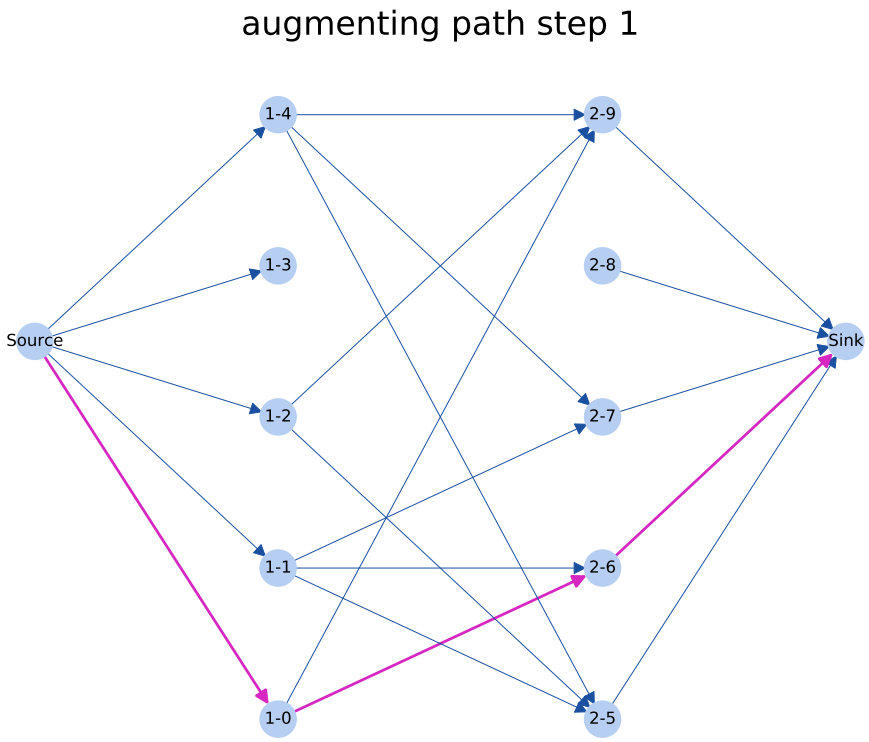

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Example of residual graph

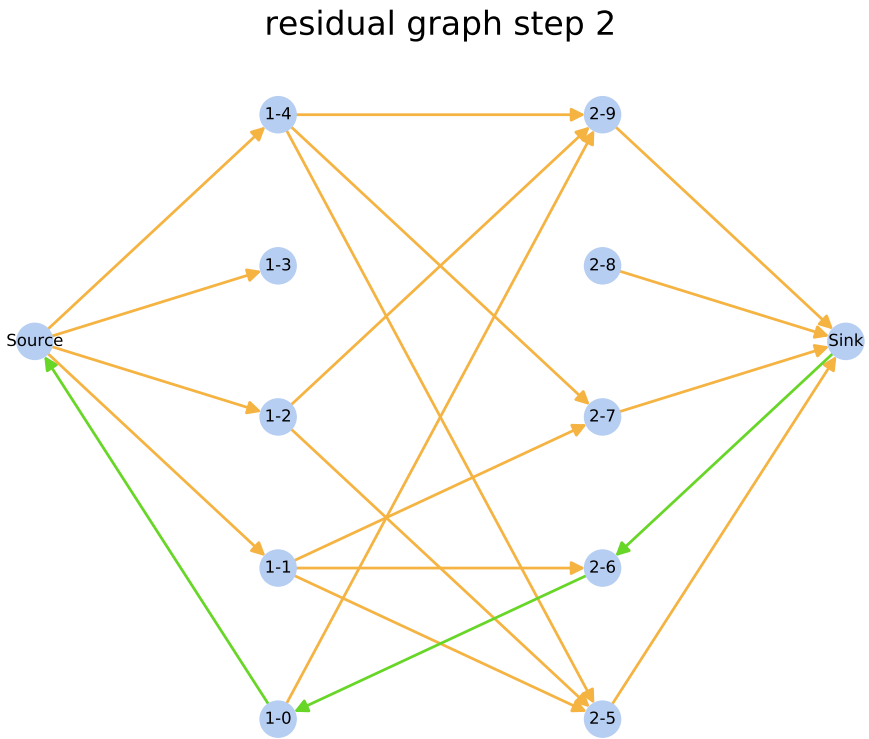


Figure: All initial capacities set to 1

Example of residual graph

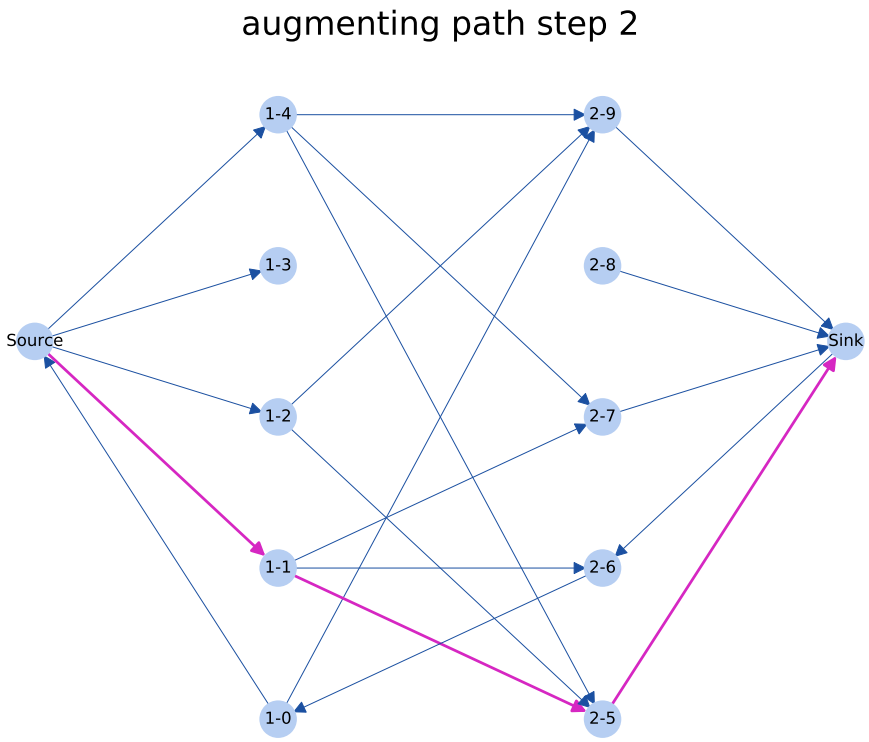
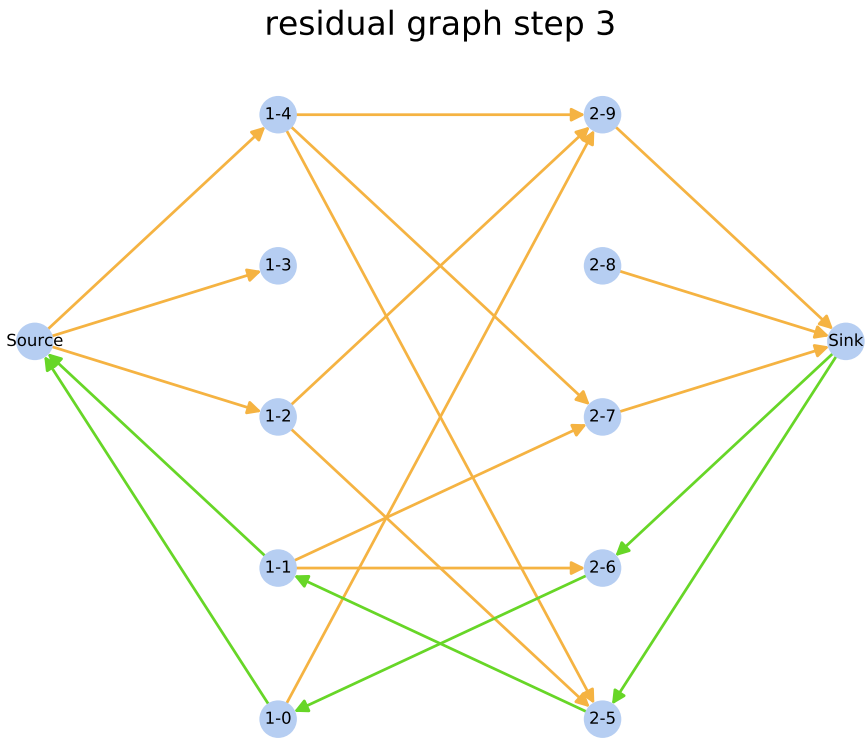


Figure: All initial capacities set to 1

Example of residual graph



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Example of residual graph

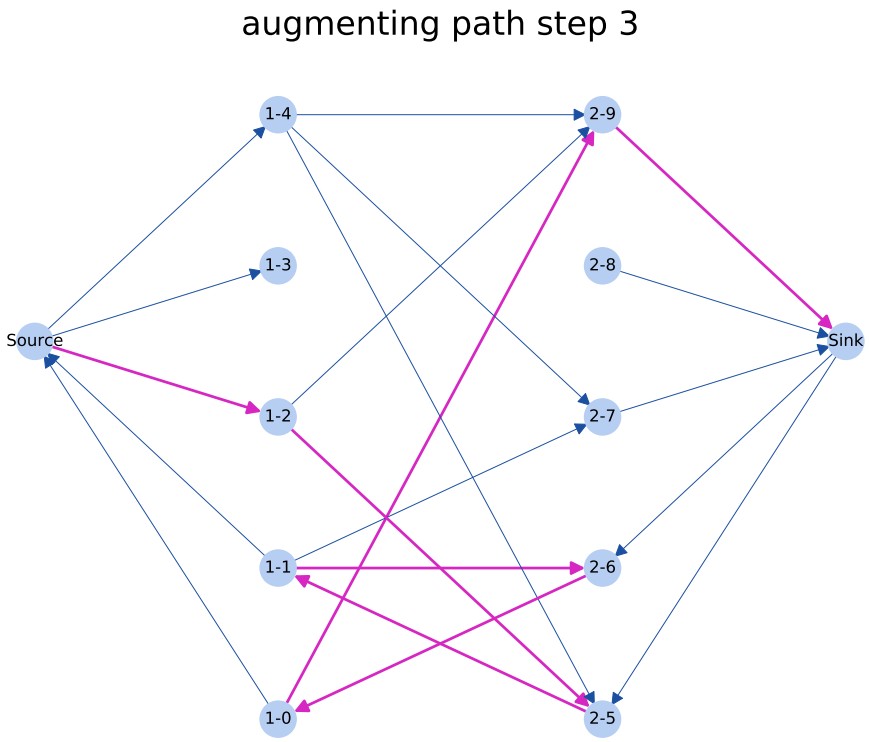

Figure: All initial capacities set to 1

Residual graph

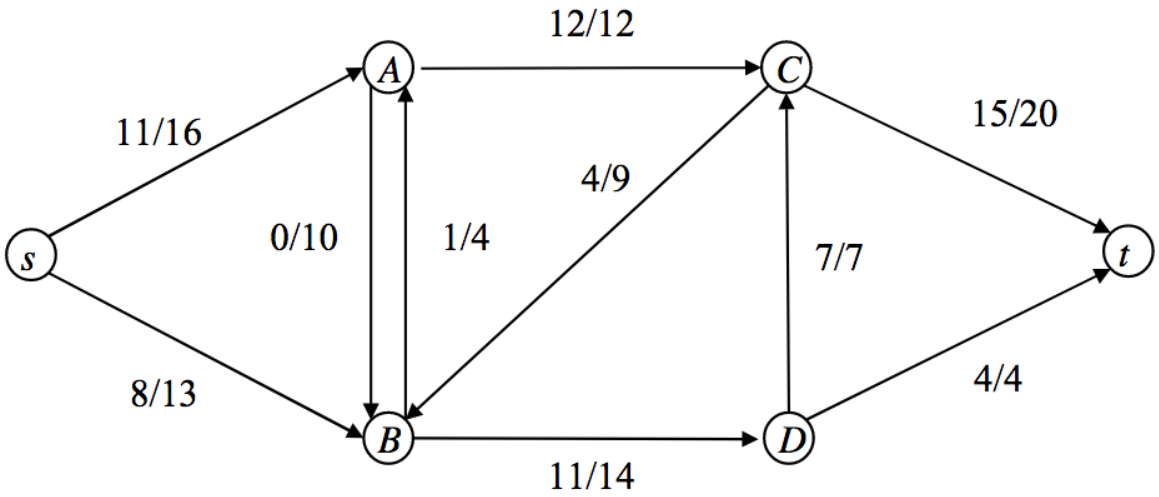

Figure: Another flow network

Residual graph

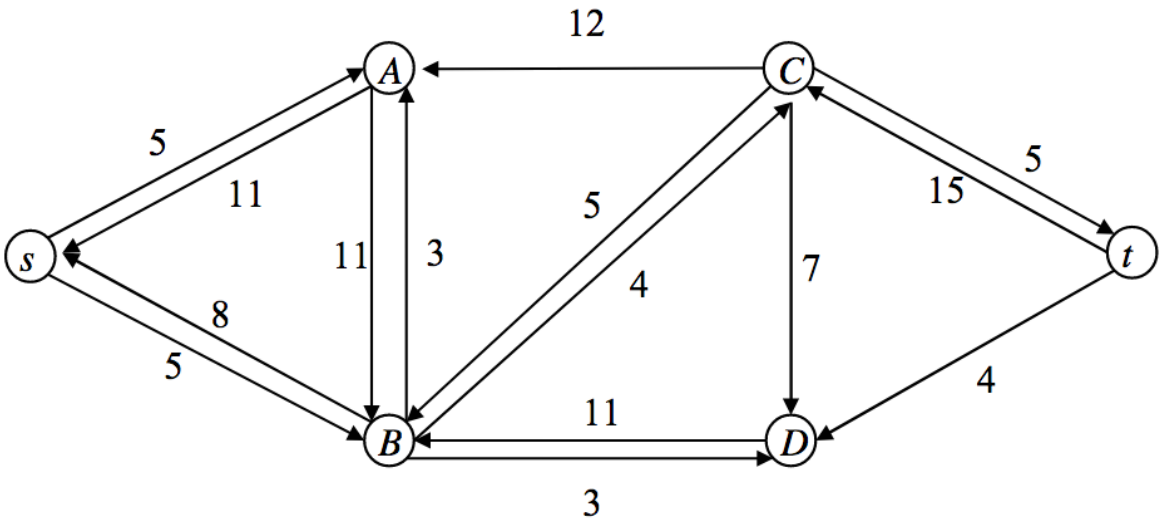

Figure: Residual graph

Augmenting path

An augmenting path is a path in the **residual graph** from the source to the sink with capacities > 0.

Augmenting path

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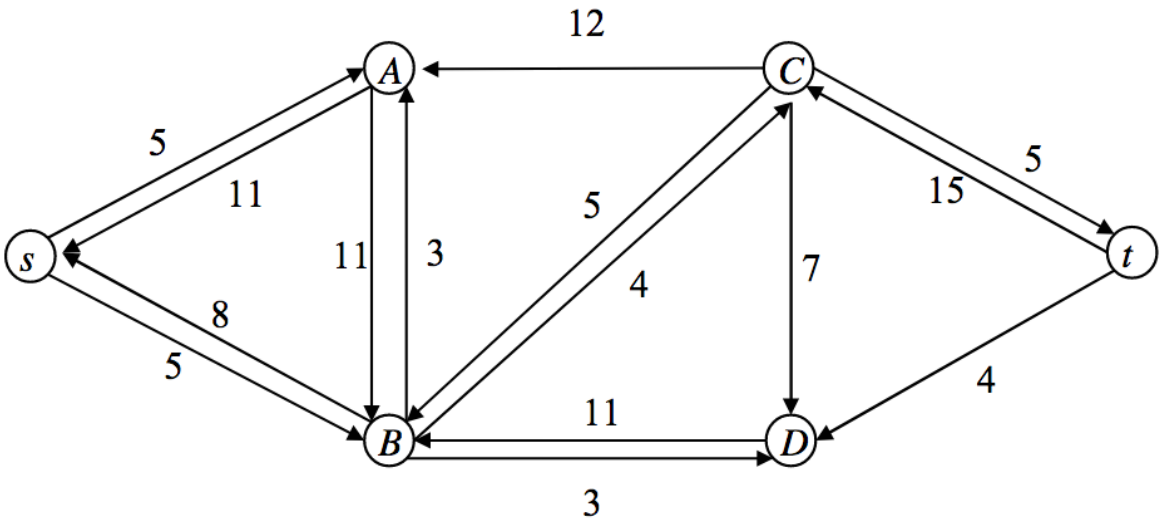


Figure: Residual graph

Augmenting path

An augmenting path is a path from the source to the sink with capacities > 0.
The Ford-Fulkerson algorithm uses augmenting paths until there are no more augmenting paths.

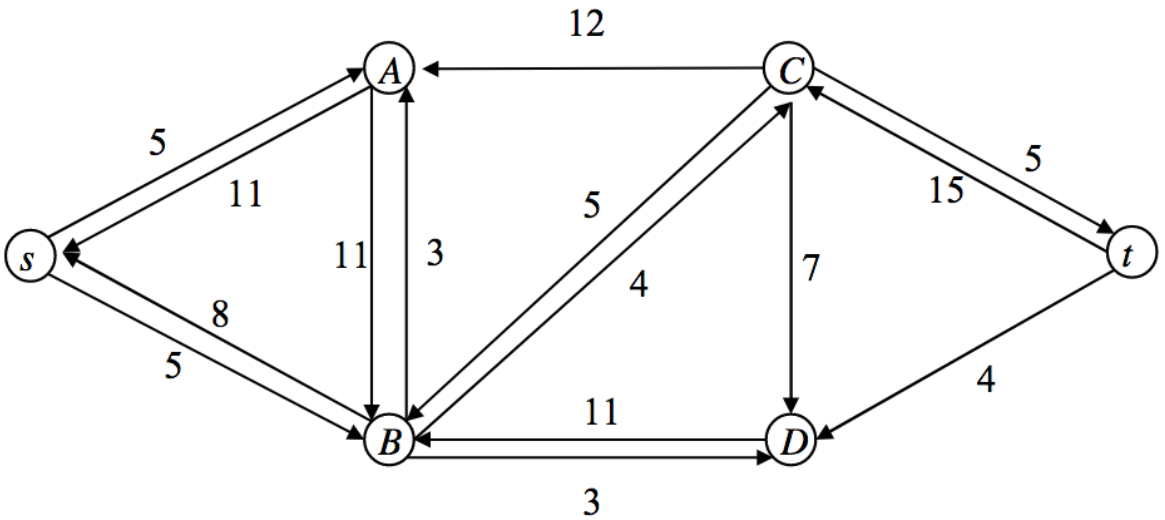


Figure: Residual graph

Ford Fulkerson algorithm

Can you deduce the algorithm from the previous remarks ?

Ford Fulkerson algorithm

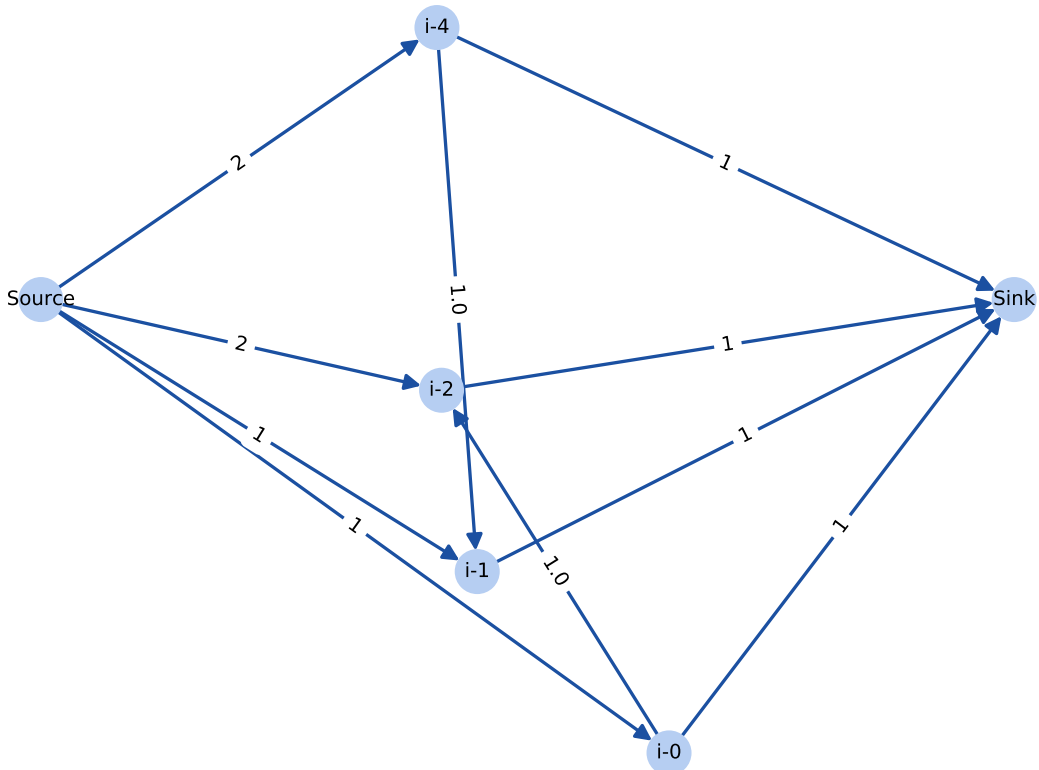
Result : Flow f
for $(u, v) \in E$ **do**
  $f(u, v) = 0$
end
while $\exists \rho$ *augmenting path* **do**
  augment f with ρ
end
return f

Algorithme 1 : Ford Fulkerson algorithm

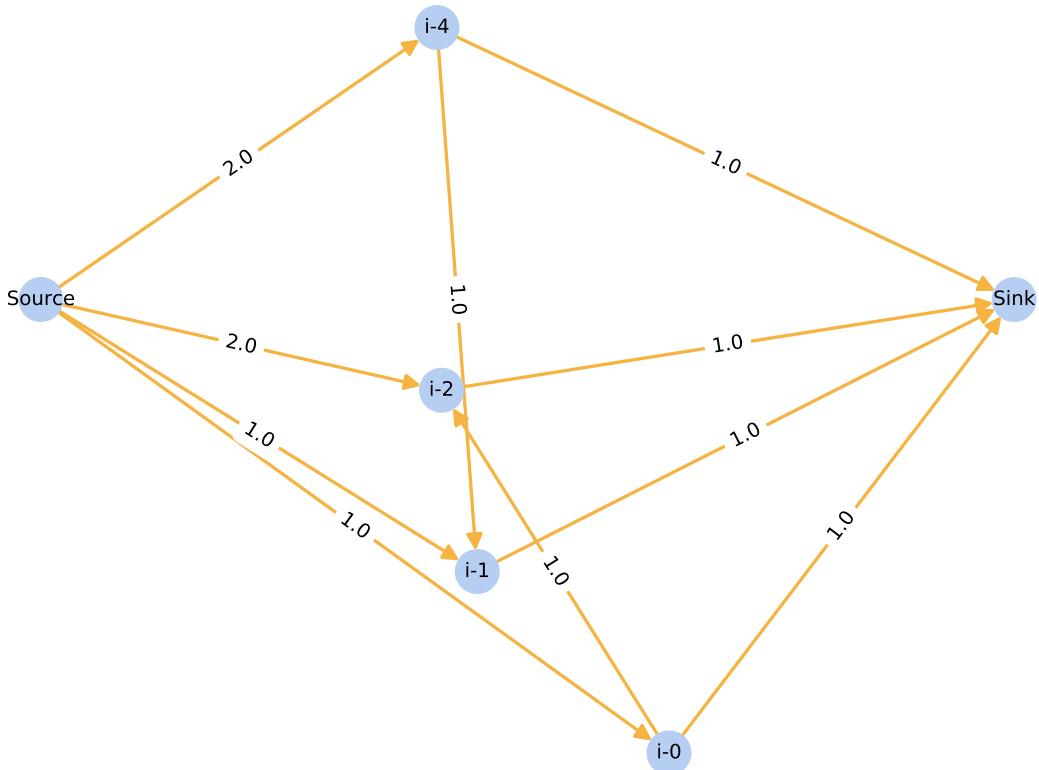
Ford-fulkerson algorithm

Let us some complete instances of the algorithm:

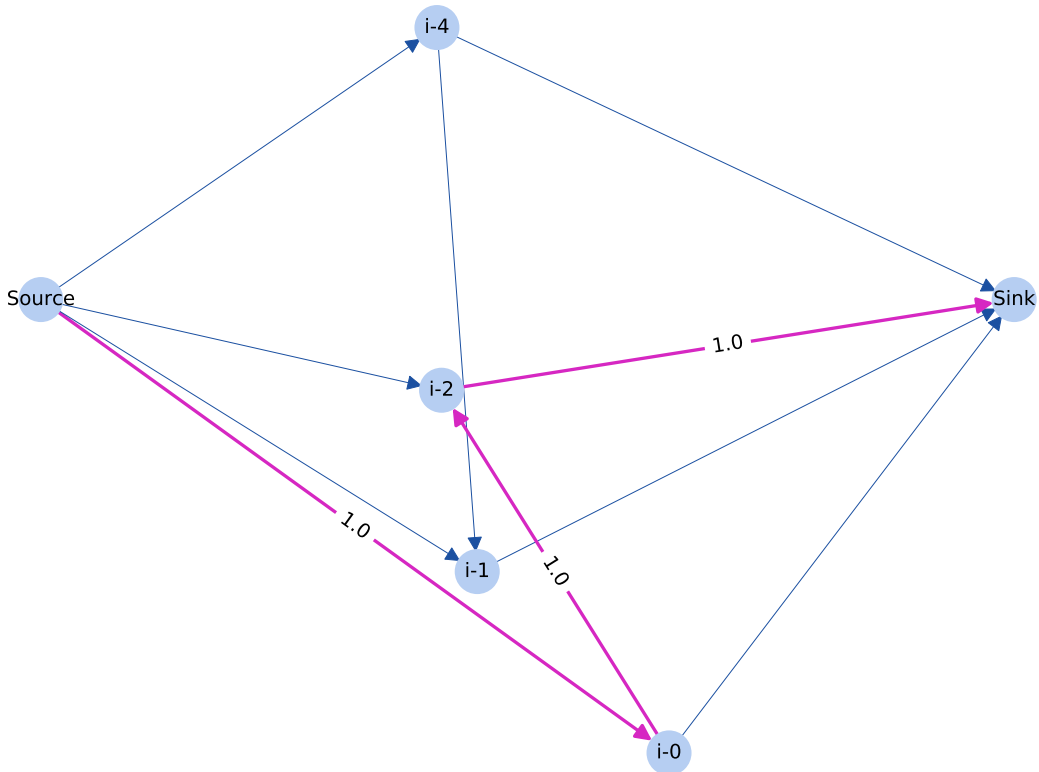
initial graph



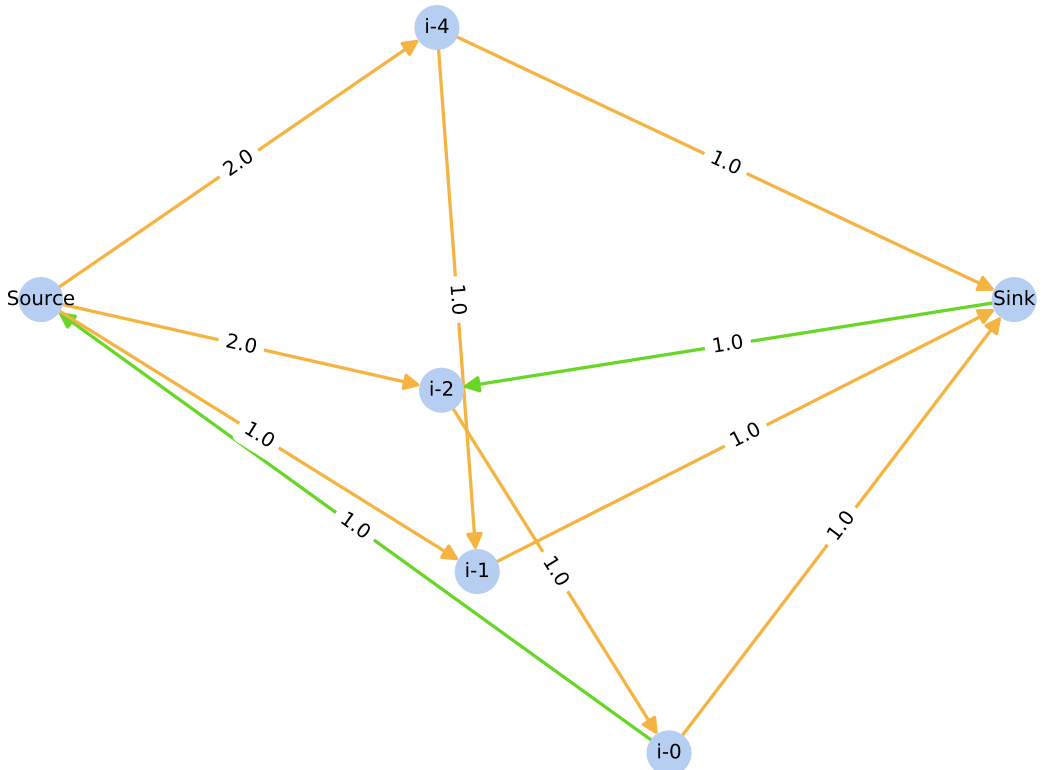
residual graph step 1



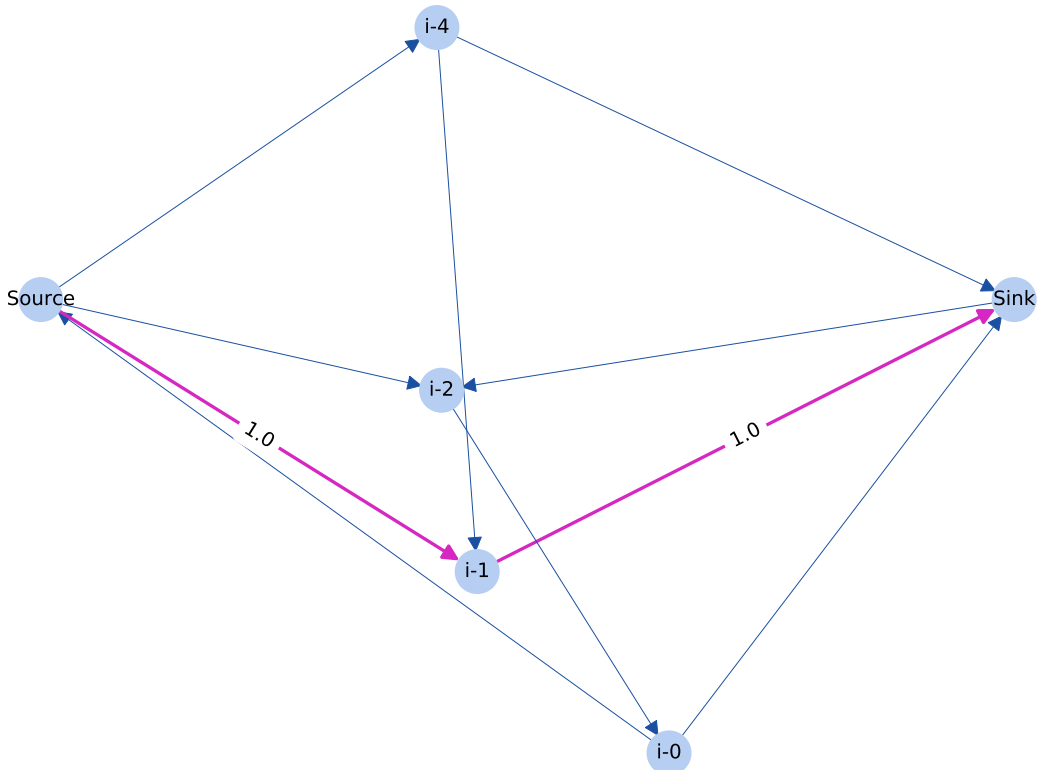
augmenting path step 1



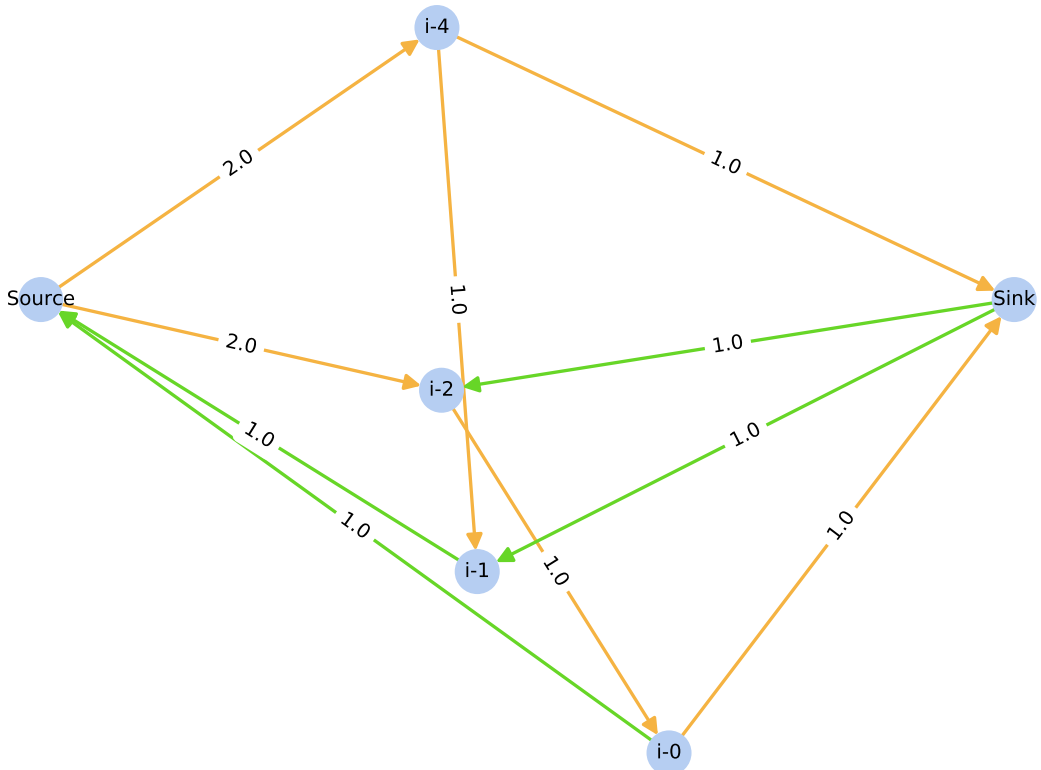
residual graph step 2



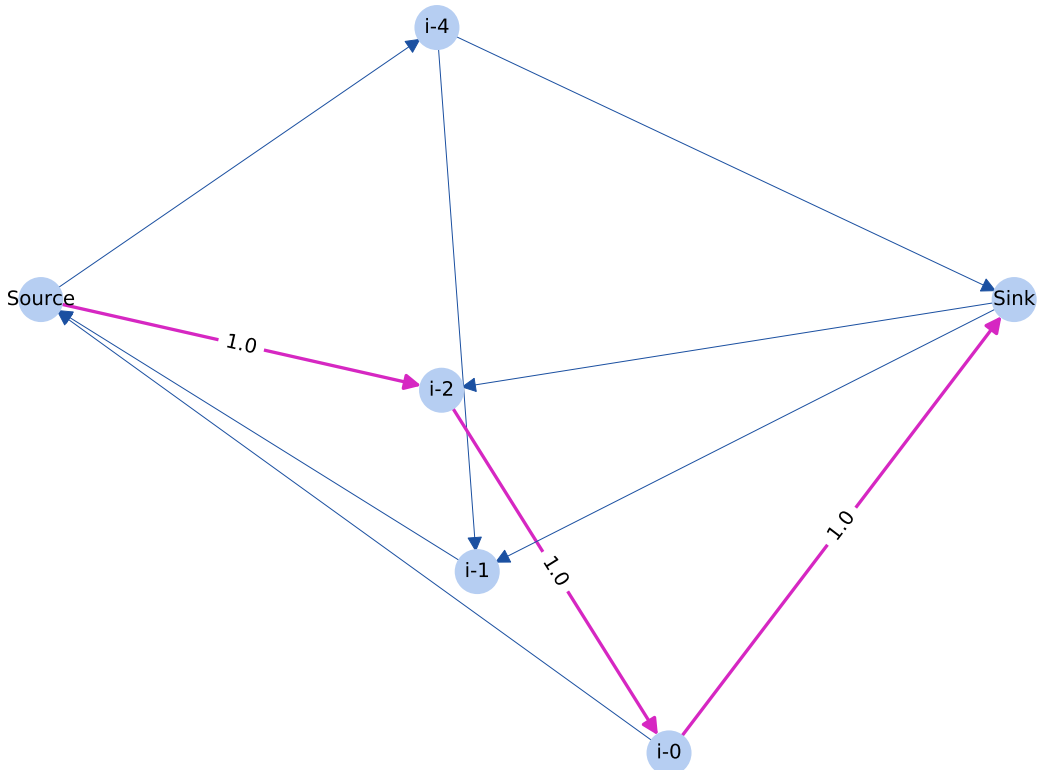
augmenting path step 2



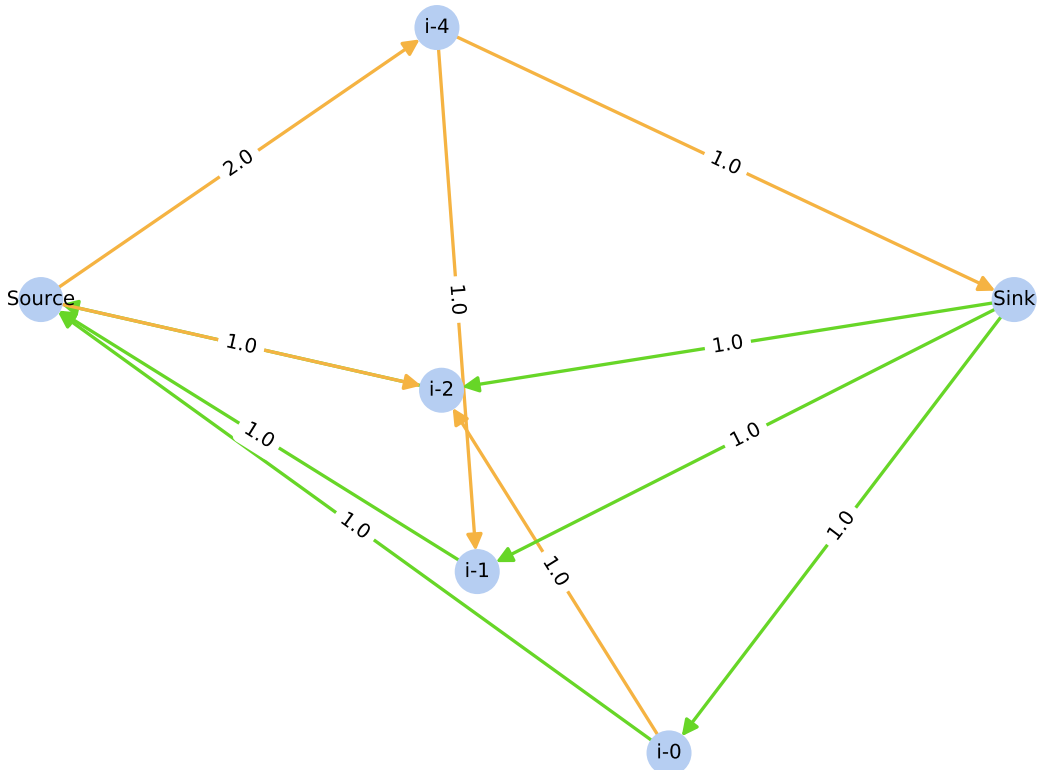
residual graph step 3



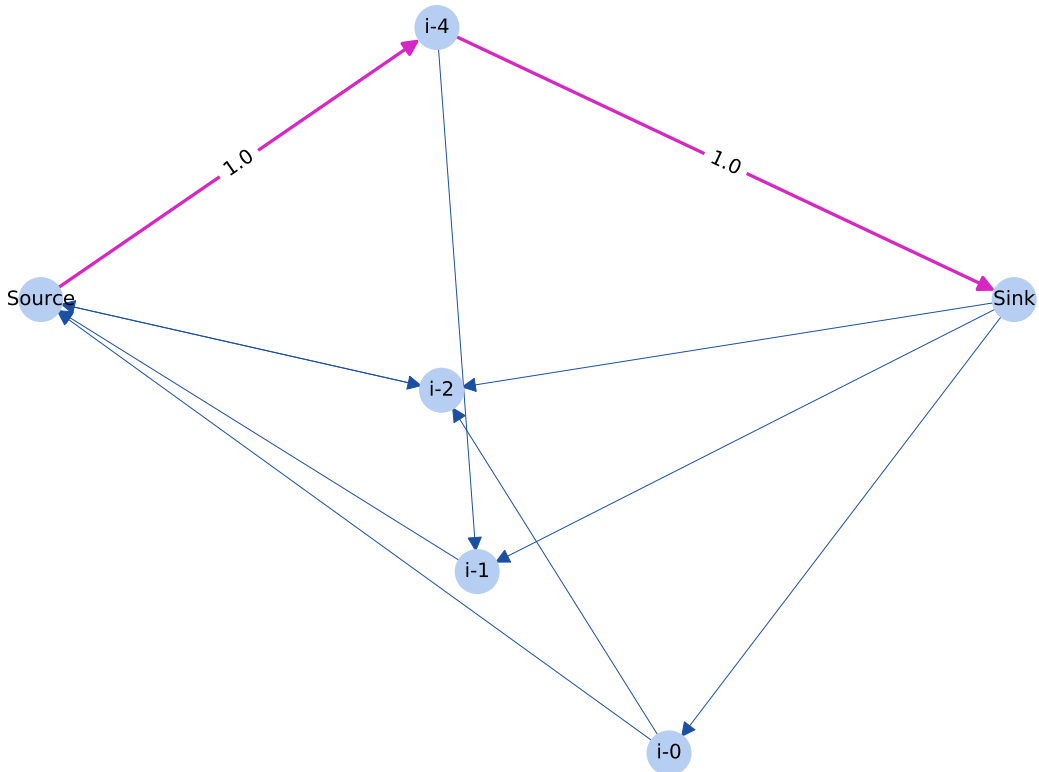
augmenting path step 3



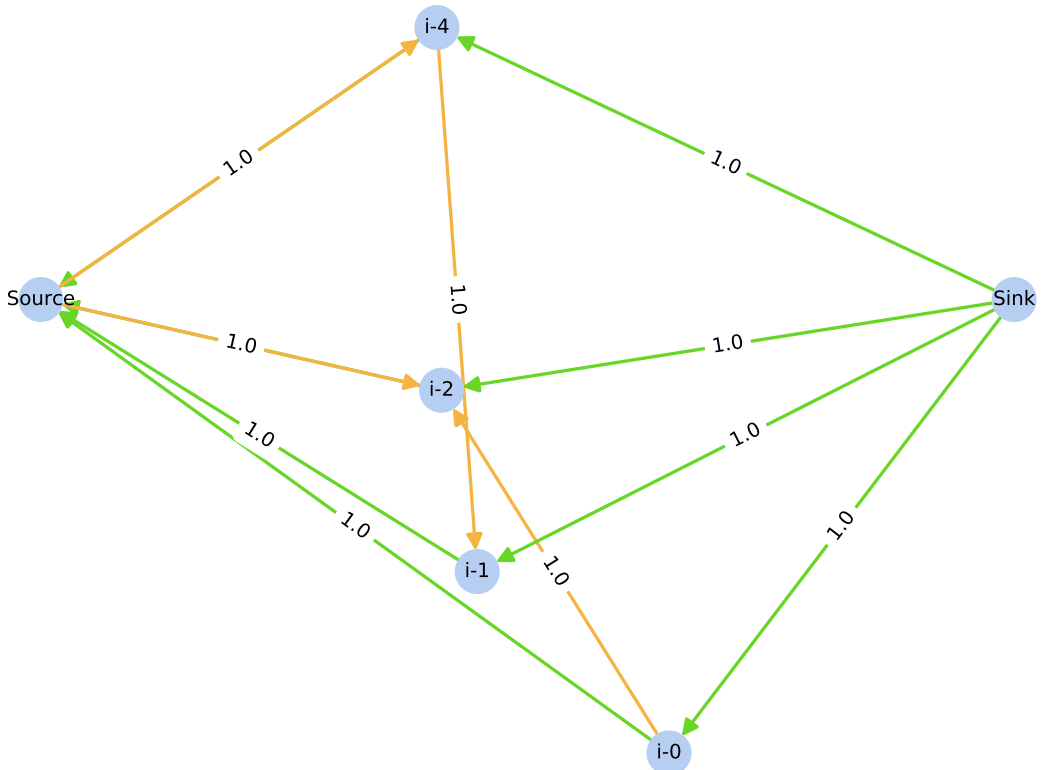
residual graph step 4



augmenting path step 4



residual graph step 5



initial graph

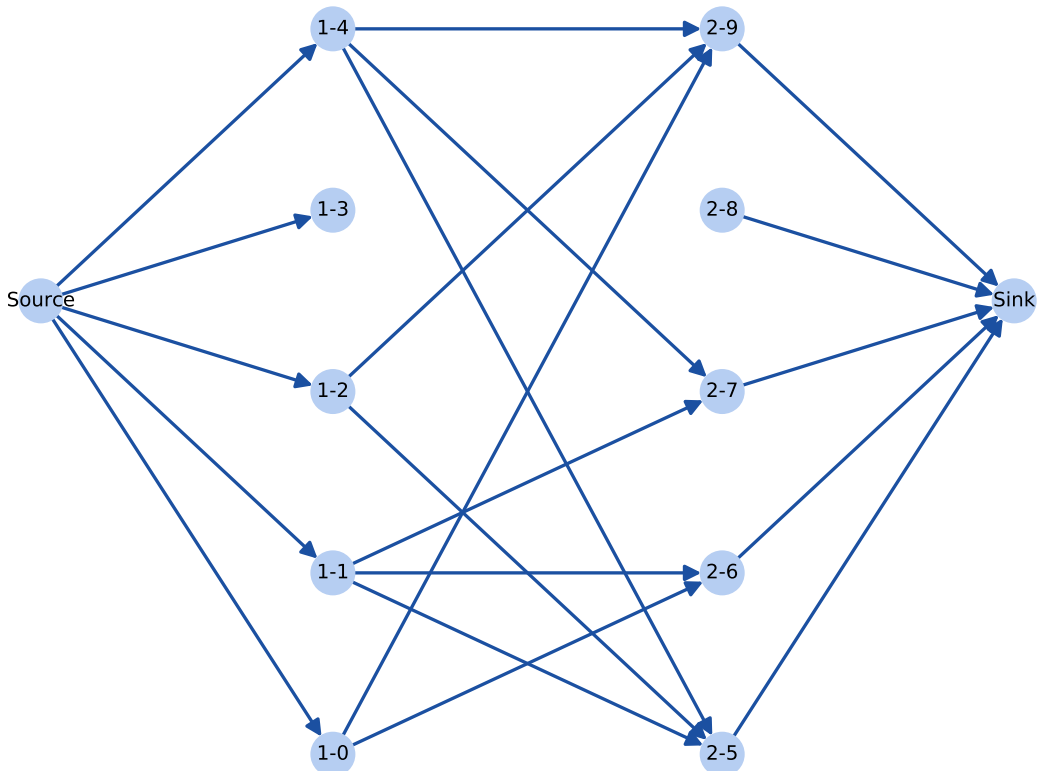
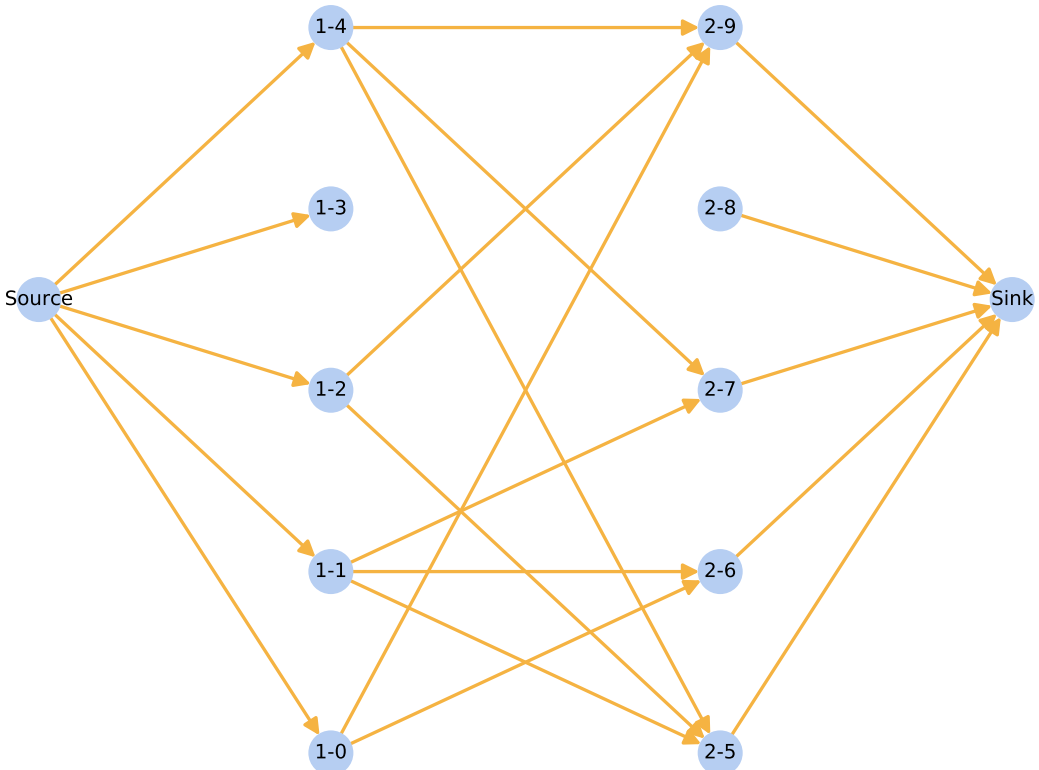
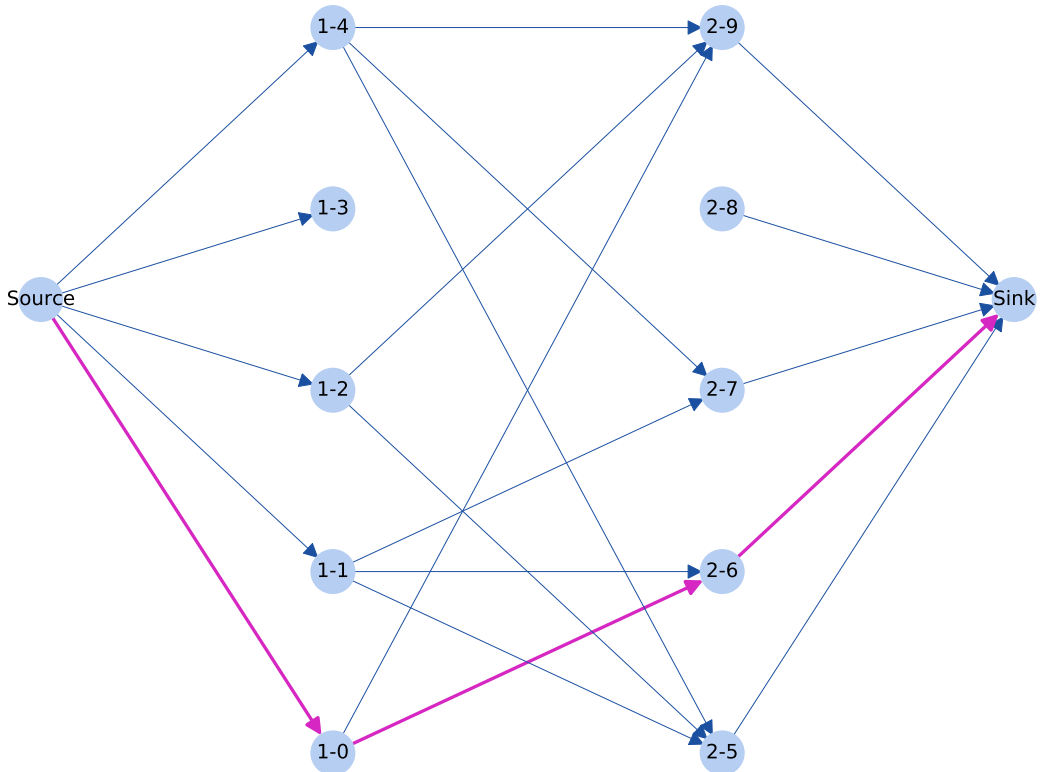


Figure: Initial capacity set to 1

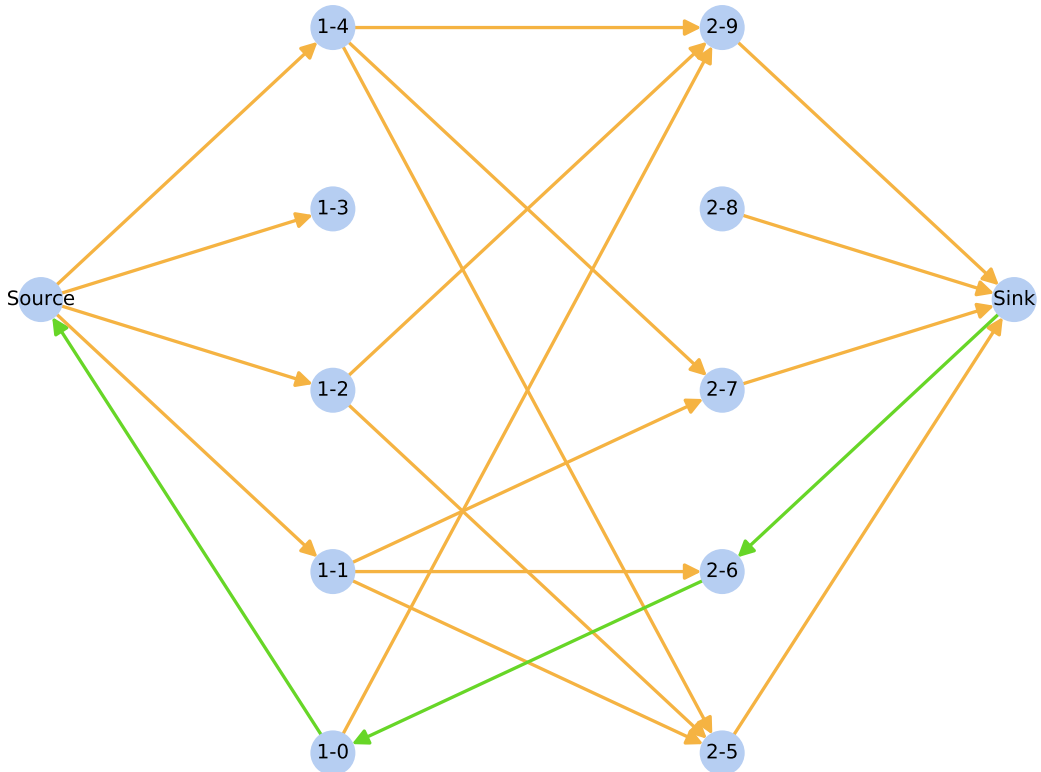
residual graph step 1



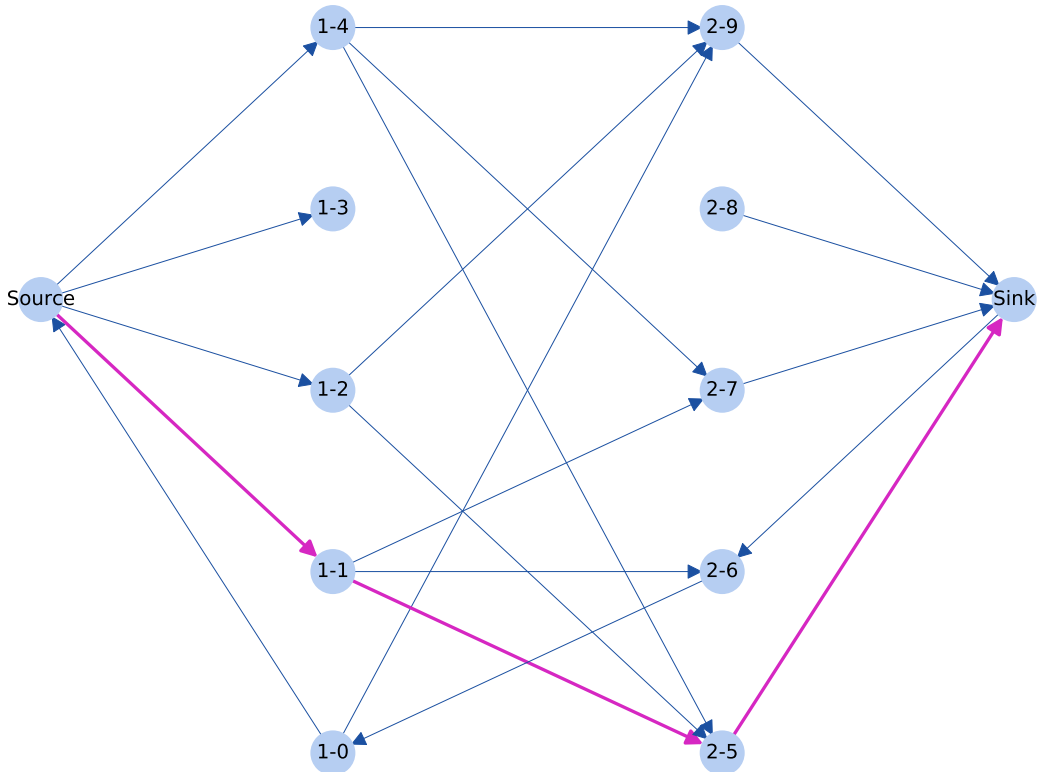
augmenting path step 1



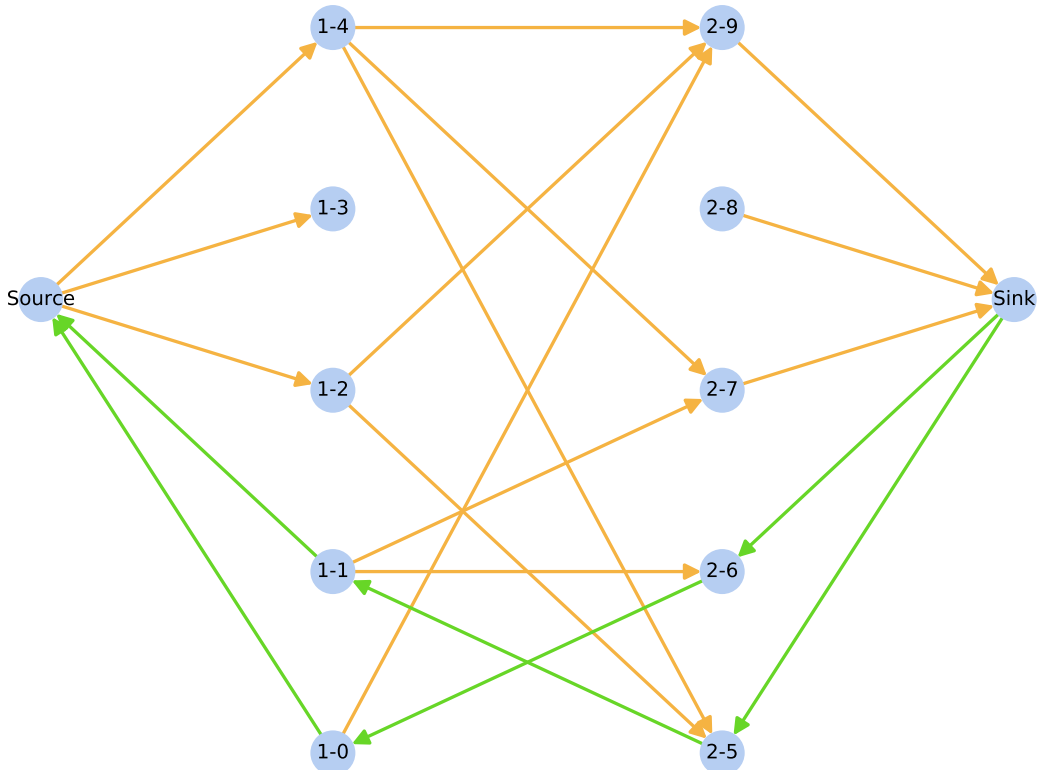
residual graph step 2



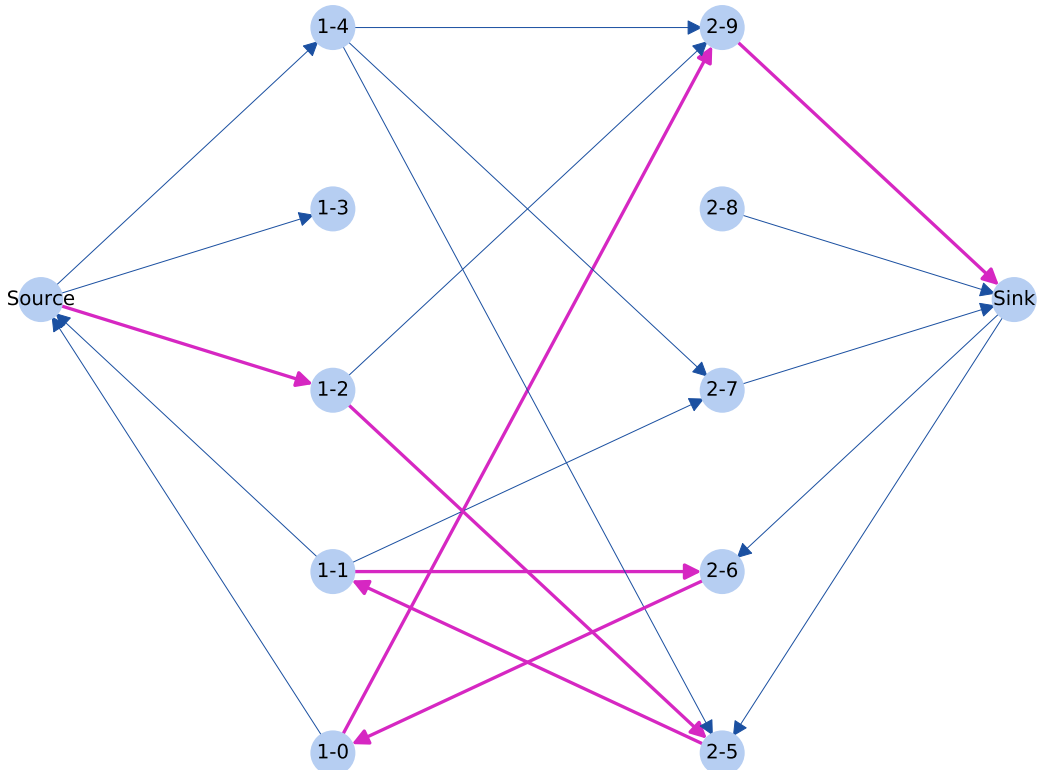
augmenting path step 2



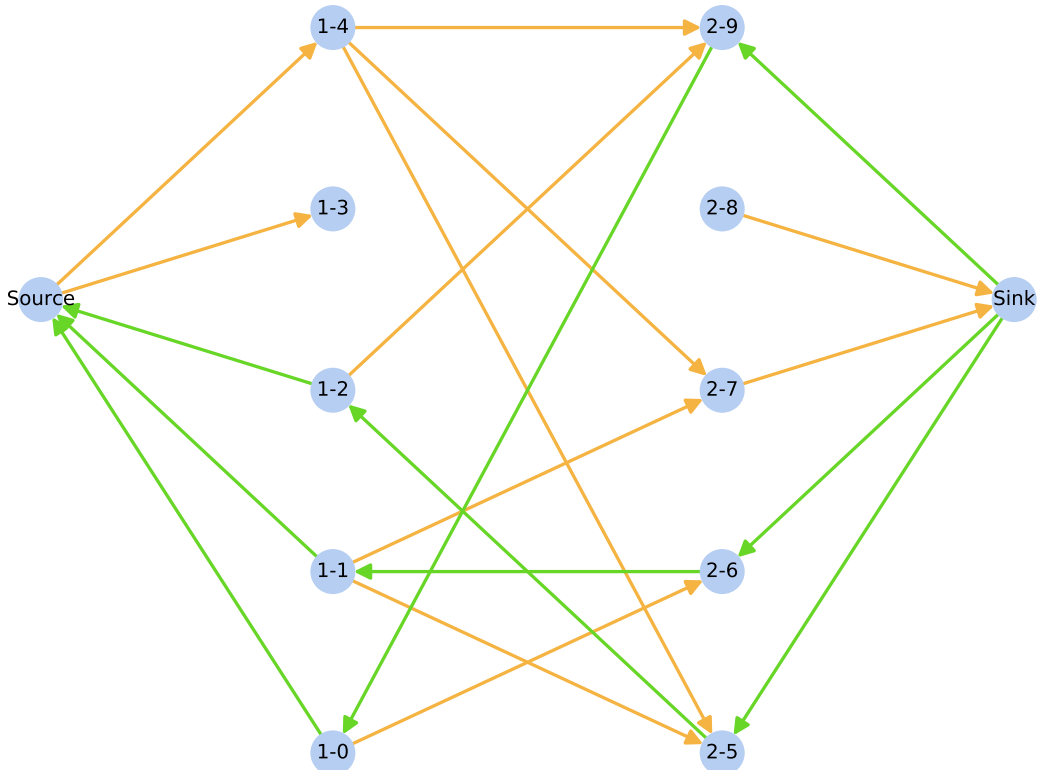
residual graph step 3



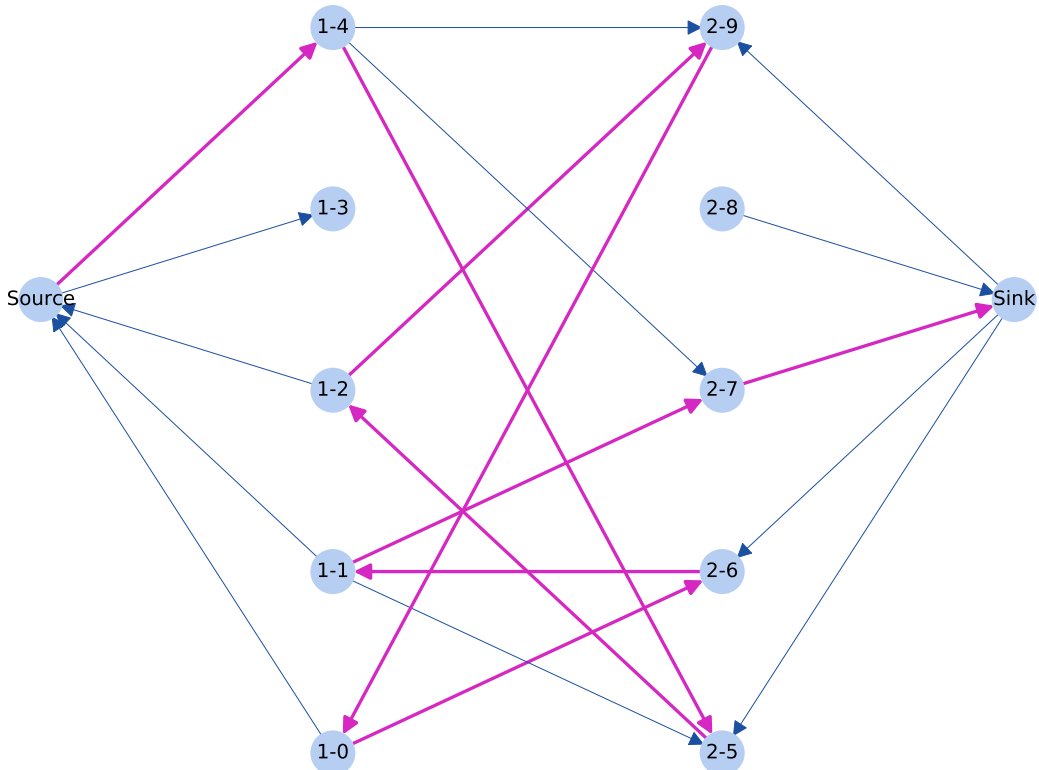
augmenting path step 3



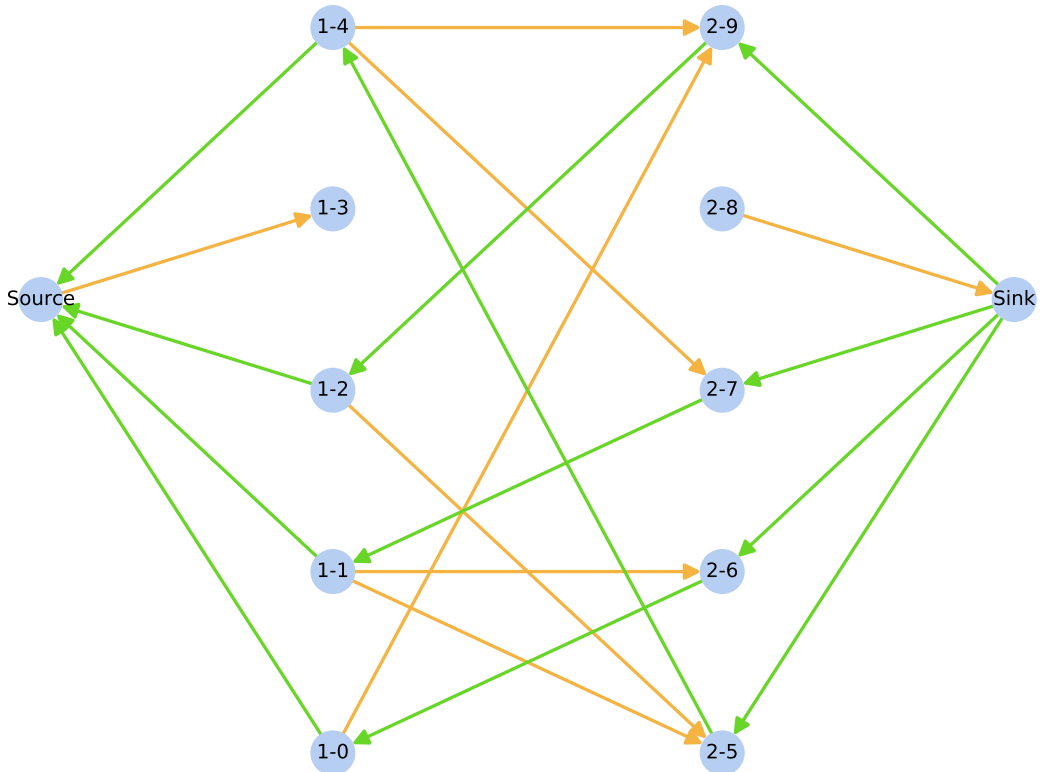
residual graph step 4



augmenting path step 4



residual graph step 5



Ford Fulkerson algorithm

- We will implement the Ford Fulkerson algorithm (1956) on a general graph.

Numpy exercise

Exercice 11 : **Numpy arrays.**
First, we will do an exercise to get more familiar with numpy.
Please follow the notebook **numpy_demo/numpy_arrays.py**.

Ford Fulkerson algorithm

Exercice 12 : We will implement the Ford Fulkerson algorithm (1956)

- **cd ford_fulkerson/** and edit **generate_flow_network.py** to generate a flow network.

Algorithm

- We will now use the functions contained in **ford_functions.py** and call them from **apply_ford_fulkerson.py**

Algorithm

Exercice 13 : step 1

- Modify **find_augmenting_paths()** in order to find the augmenting paths.

Algorithm

Exercice 13 : step 2

- now edit **augment_flow()**

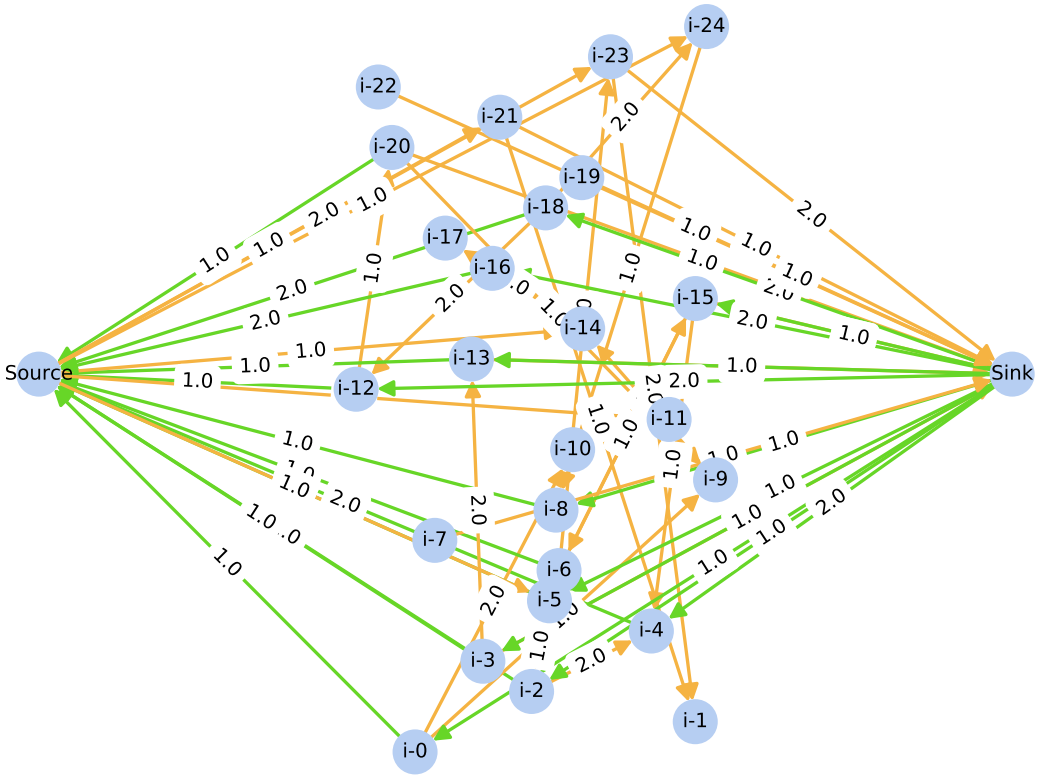
Algorithm

Exercice 13 : step 3

- finally, edit the computation of the value of the flow

- Now the algorithm should be able to run

residual graph step 15



Complexity

Complexity of Ford Fulkerson:

$$\mathcal{O}(|f^*| \times |E|) \tag{8}$$

Modification of Ford Fulkerson

What would we an intuitive and potentially faster modification of the algorithm ?

Modification of the algorithm

What would we an intuitive and potentially faster modification of the algorithm ?
Use the shortest augmenting path with strictly positive capacity.

Termination

- When the capacities are **integer numbers** or **rational numbers** Ford Fulkerson terminates.

Termination

- When the capacities are **integer numbers** or **rational numbers** Ford Fulkerson terminates.
- However, when the capacities are general **real numbers**, the algorithm might not terminate.

Link with the matching problem

- We now go back to the matching problem, in the case of a **bipartite graph** ("problème d'affectation")
- We will show that in that case, we can connect the two problems.

Bipartite graph

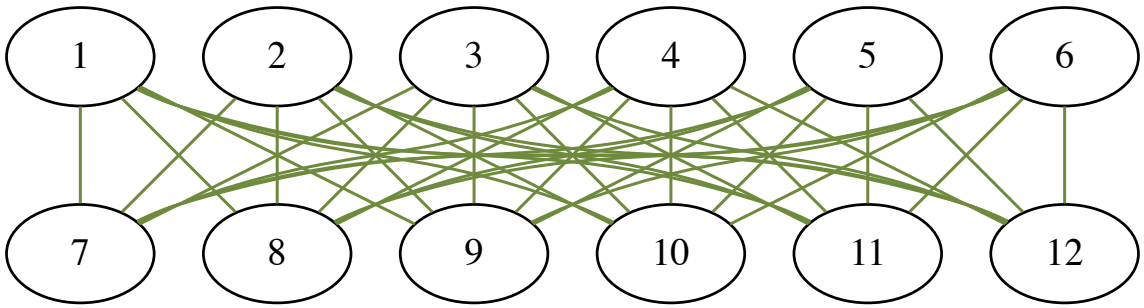


Figure: Complete bipartite graph (not all bipartite graphs are complete)

Matching problem

We now go back to the matching problem, in the case of a **bipartite graph**.

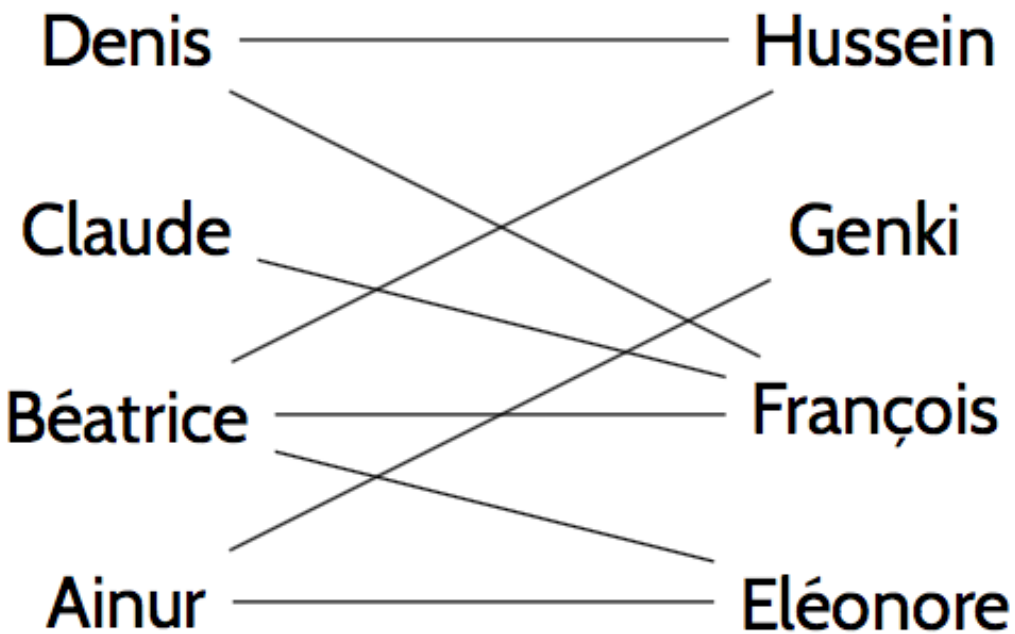


Figure: Bipartite graph

Equivalence between matching and flow

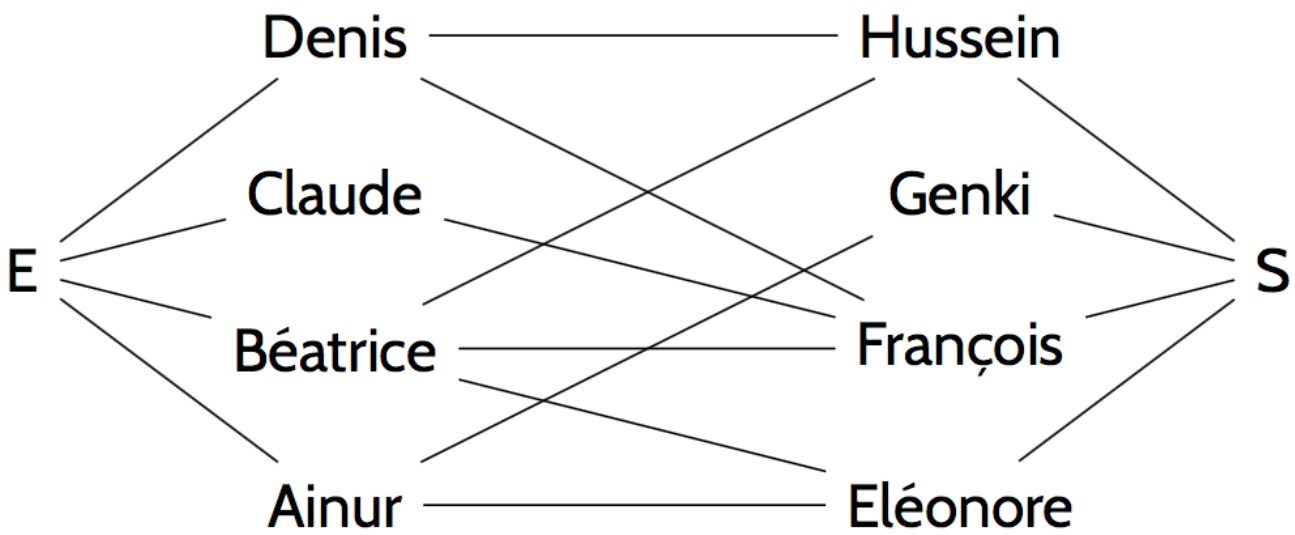


Figure: Introduce two more nodes. All edges have capacity 1. We consider **flows with integer values**

Ford Fulkerson for matching

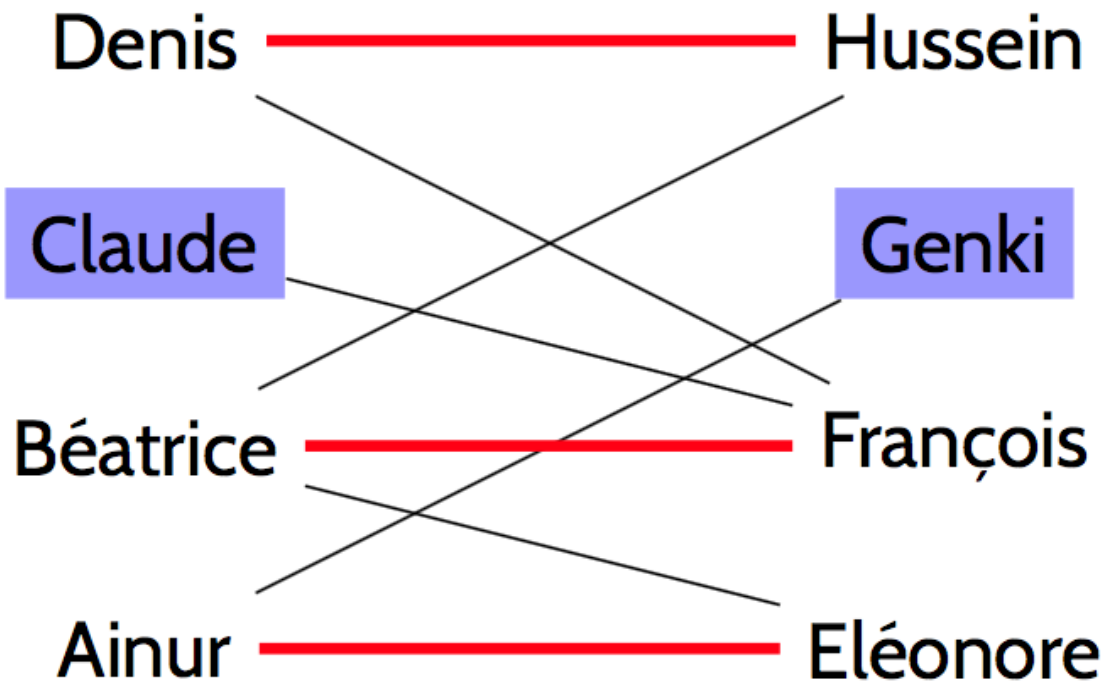


Figure: Non optimal solution

Ford Fulkerson for matching

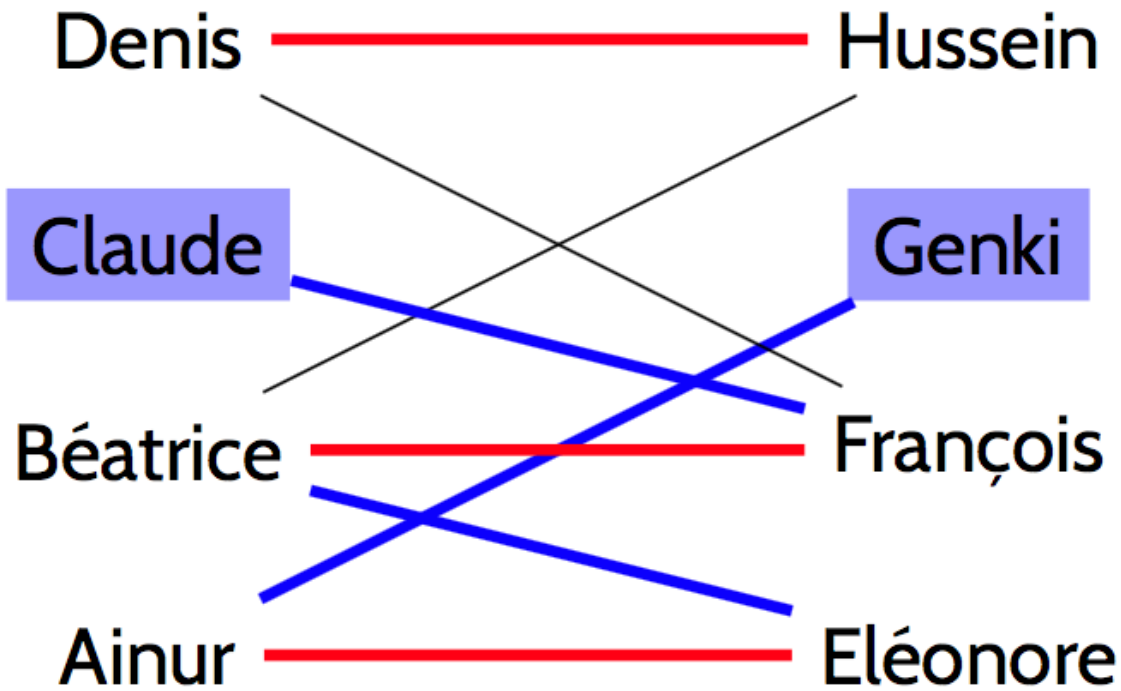


Figure: Optimal solution

Connection

Exercice 13 : Find a connection between the two problems

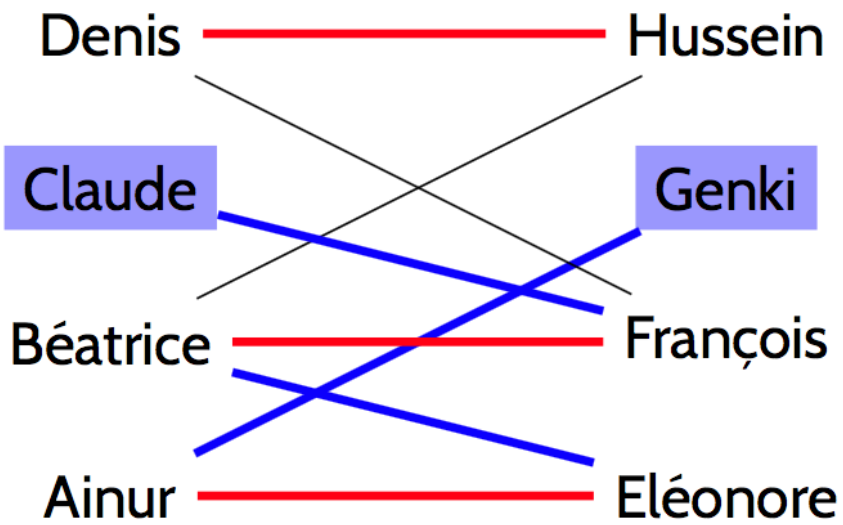


Figure: Optimal solution

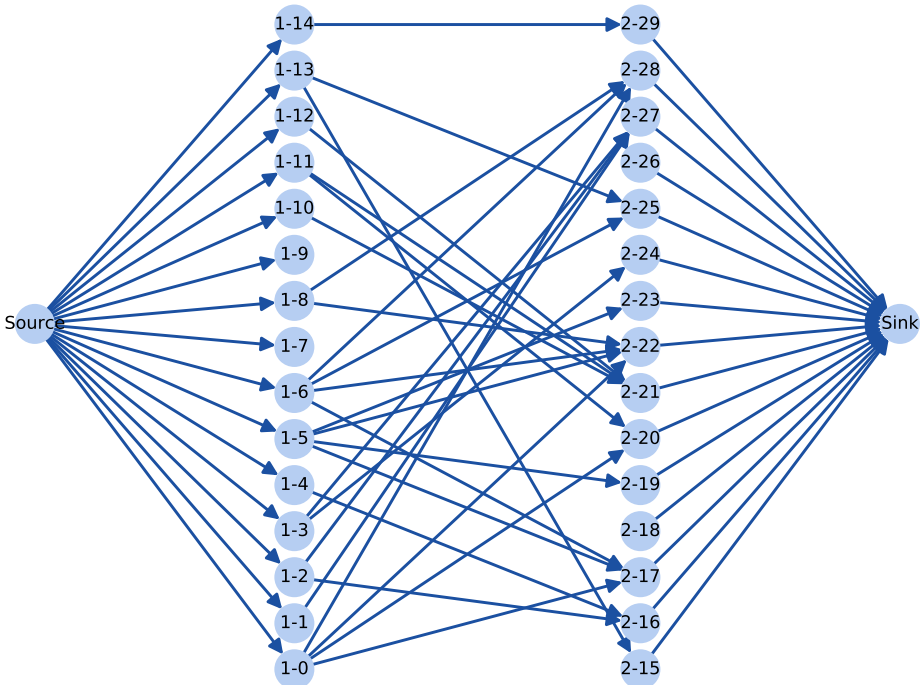
Exercice 14 : Ford Fulkerson and matching

- We will transpose Ford Fulkerson to a bipartite graph in order to find an optimal matching.
- **cd ford_matching/**
- edit **generate_matching_problem.py** in order to generate an instance of the problem.

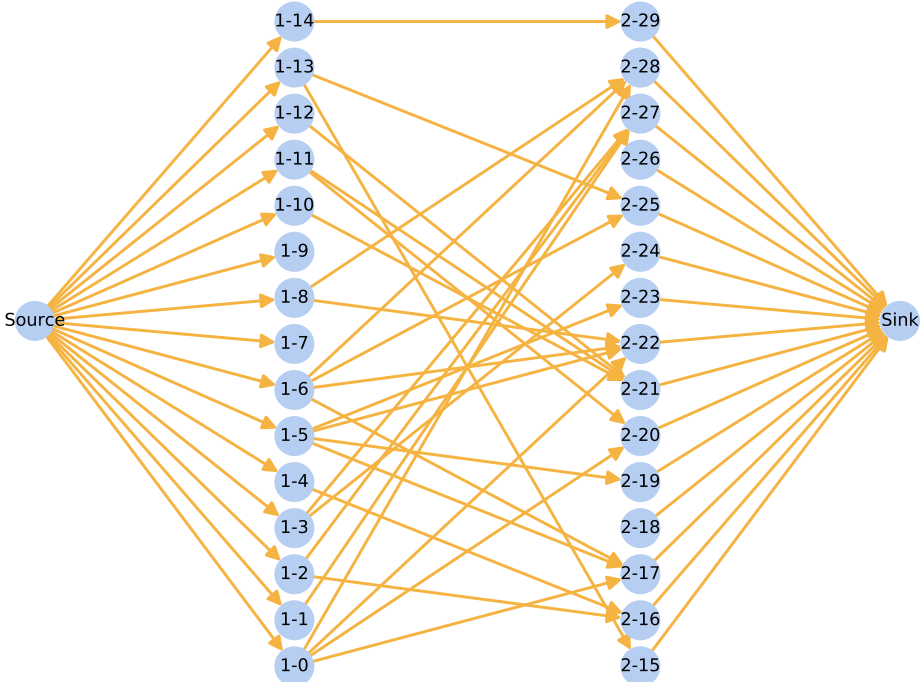
Exercice 14 : Ford Fulkerson and matching

- Apply the algorithm on an example generated by the previous function.
- Apply the algorithm to as an instance of your choice.

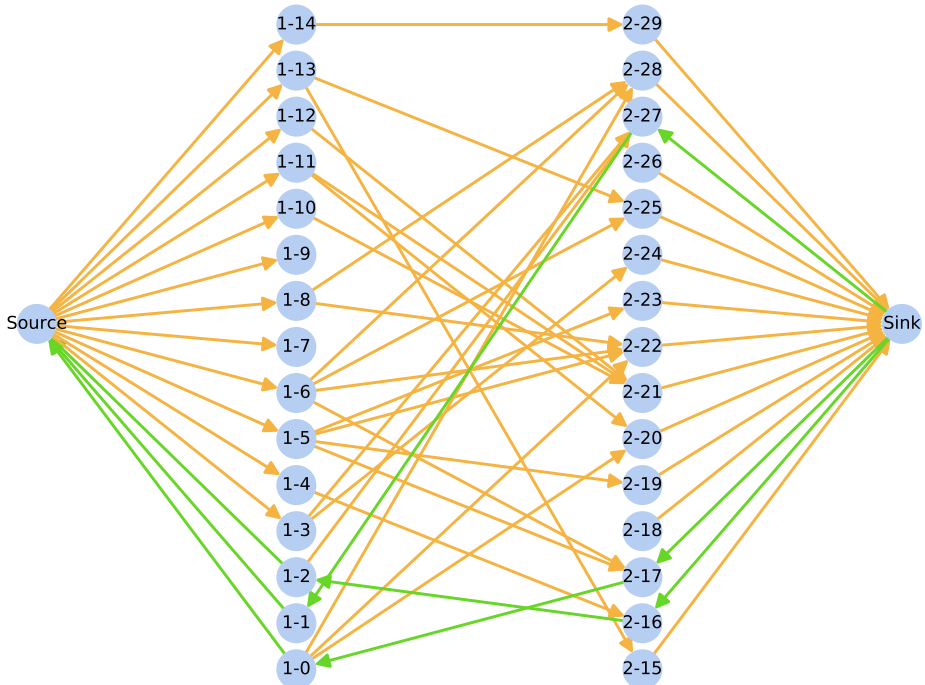
initial graph



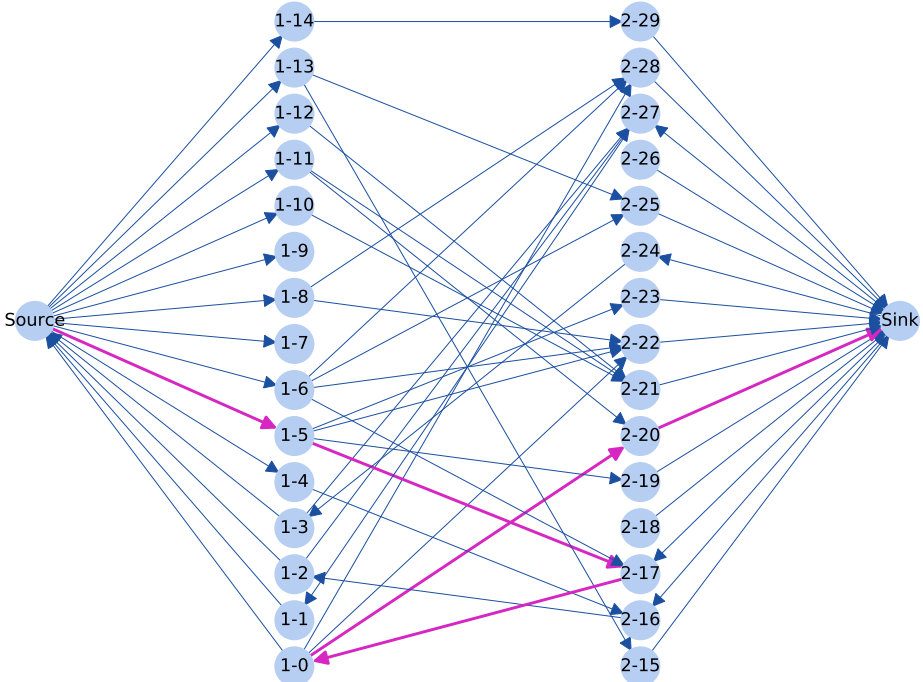
residual graph step 1



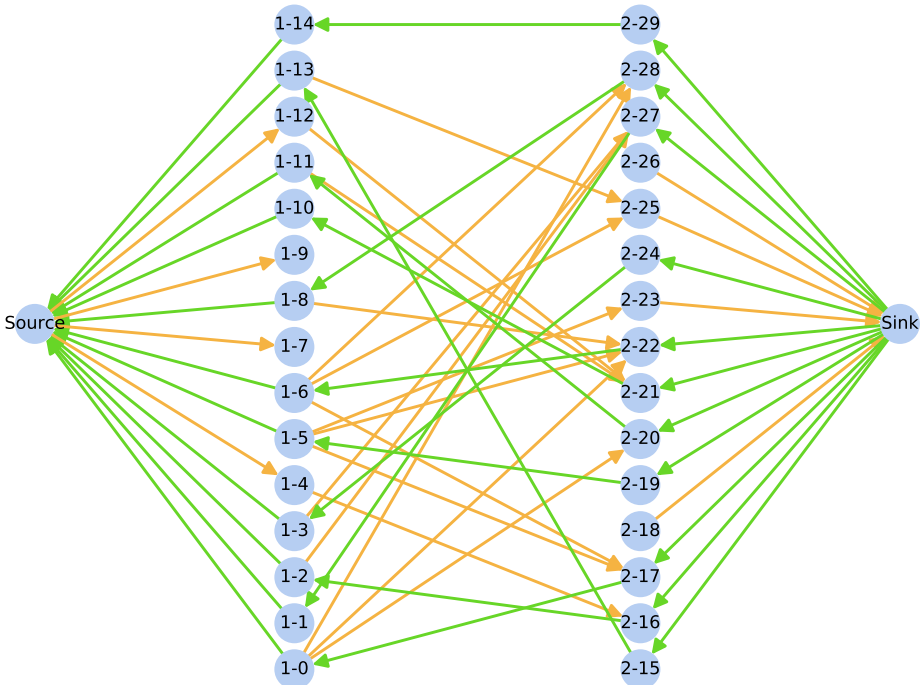
residual graph step 4



augmenting path step 5



residual graph step 12



Famous theorem

The maximum flow theorem is equivalent to another famous problem, the **minimum cut** theorem.

Perfect matching

In the case of a bipartite graph, what is the best matching possible ?

Perfect matching

In the case of a bipartite graph, what is the best matching possible ?

A matching where **all nodes are allocated.** It is called a **perfect** matching.

We must have that the two parts of the graph are of same cardinalty in order to have a perfect matching.

Hall's marriage theorem

This theorem gives a condition that is necessary and sufficient for the existence of a perfect matching in a bipartite graph : the "marriage condition".
If $G = (U, V, E)$ is bipartite, the condition means that :

$$\forall X \subset U, |N_G(X)| \geq |X| \tag{9}$$

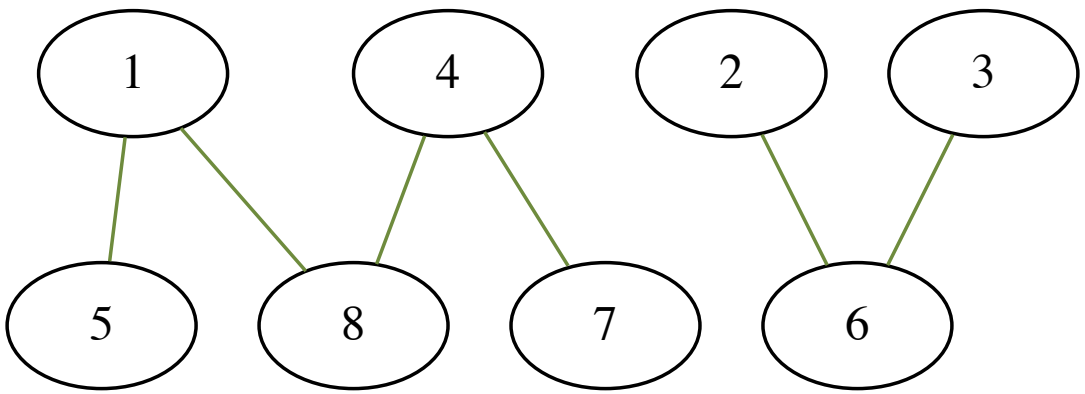
where $N_G(X)$ is the set of neighbors of X in G.

Hall's theorem

Exercice 15 : Application of the theorem.
Can you think of a graph that does not abide by the marriage condition and thus has **no perfect matching** ?

Illustration of Hall's theorem

Exercice 15 : Application of the theorem



Case of a non bipartite graph

In the case of a **non-bipartite**, we can not use the Ford-Fulkerson algorithm.
Other methods exist such as the **Blossom algorithm.**
(Edmonds-Karp)

Conclusion

Ford Fulkerson and its variants (Edmonds-Karp) are polynomial. As a result thay can run on datasets that are way bigger than exhaustive search algotirhms.