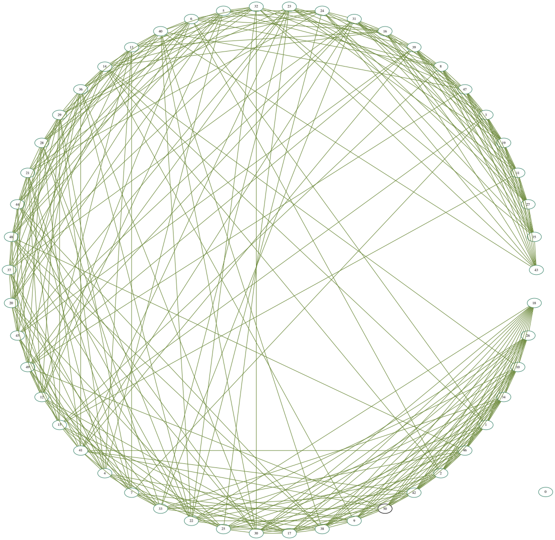
EPITECH.
PARIS GRADUATE SCHOOL OF DIGITAL INNOVATION

# Algorithms, Matching

## Part 1. Networks and Matchings

B9 - Algorithms Matching

M-ALG-102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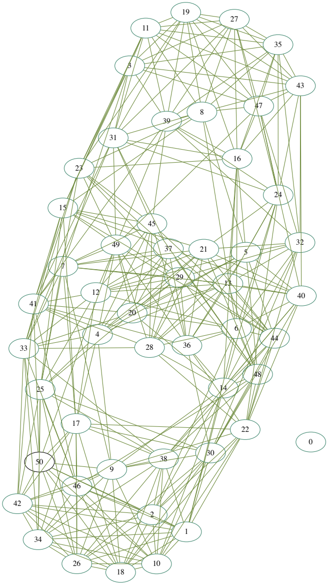
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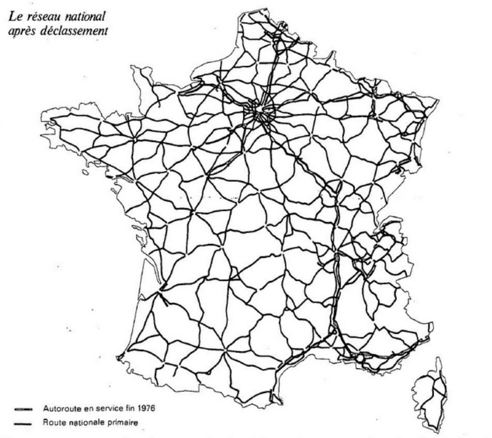
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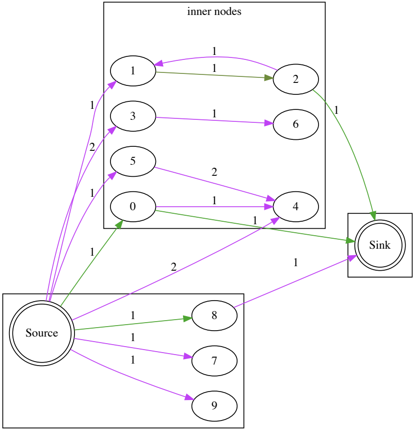
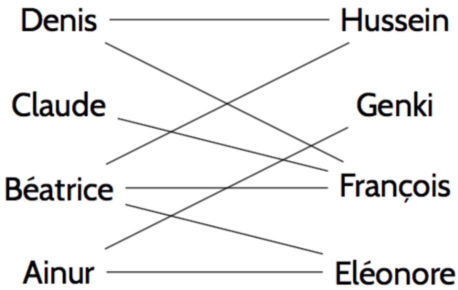
après déclassement
Autoroute en service fin 1976
Route nationale primaire




Residual graph
Algorithm step: 4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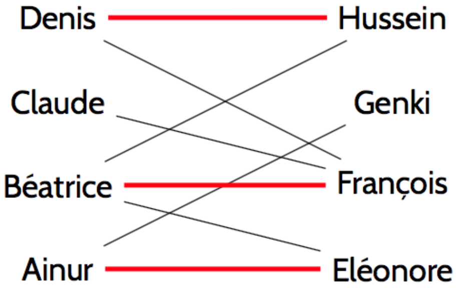
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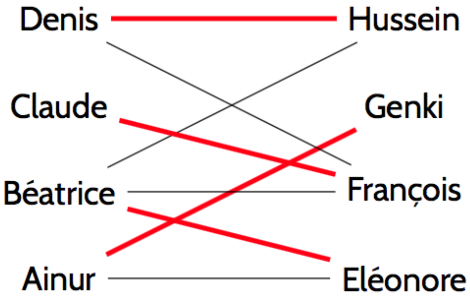
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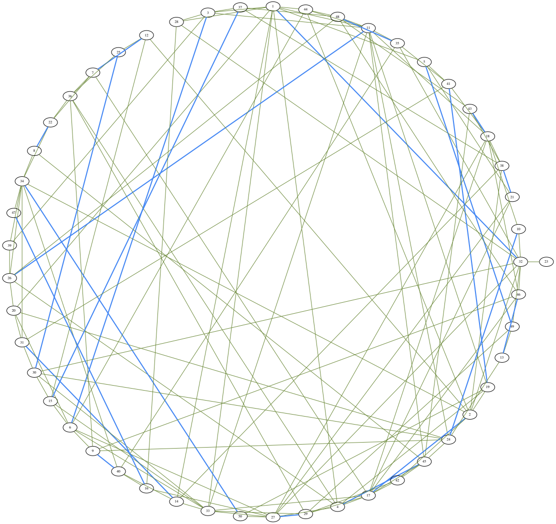
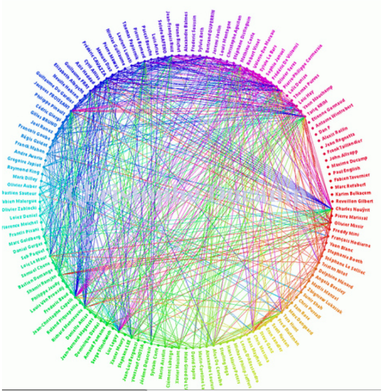
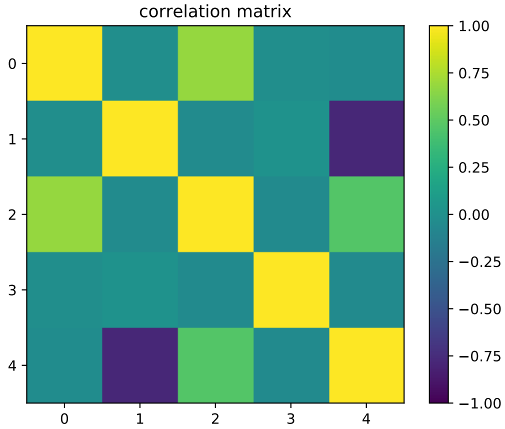


Matching size: 21
Algo step: 128
Nb nodes: 50




correlation matrix
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−0.25
−0.50
−0.75
−1.00

- The content of the course will sometimes be mathematical.
- Don't hesitate to ask if you need more reminders.

# Overview of the module

- Day 1 Networks, the matching problem and the maximum flow problem
- Day 2 Data clustering and representation

## Organisation of the module

- Course and exercises in python 3
- Small coding exercises, also paper + pen
- Project : explained tomorrow
- Please clone the following repository
  `https://github.com/nlehir/ALGO2`

# Libs

- Day 1 graphviz (helper file on the github)
- Day 2 numpy, matplotlib, pandas, sklearn

# Day 1

## The matching problem

- Definition of the problem
- Experimental solutions
- Brute force algorithm
- Greedy algorithm

## The Maximum flow problem

- Presentation of the problem
- Solution with the Ford-Fulkerson algorithm
- Connection with the matching problem
- More results on the two problems

# Introductory example 1 : Max Flow

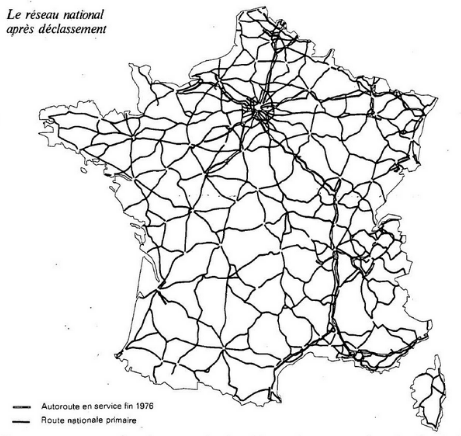



Figure: Problem 1 : transporting merchandise through a network

# Introductory example 2 : Maximum matching (Optimal allocation)

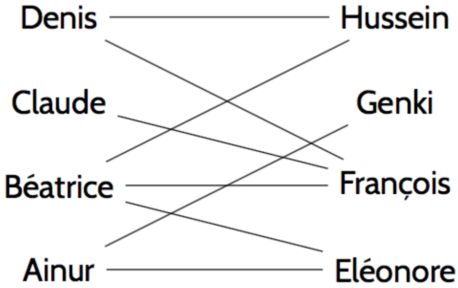



Figure: Problem 2 : Building the largest possible number of teams of 2 persons.

# Introductory example 2

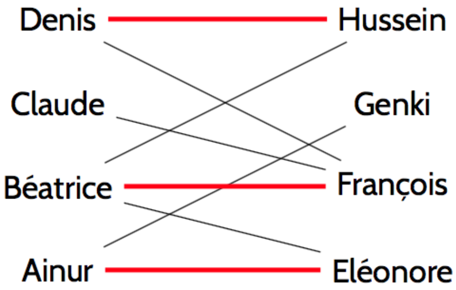



Figure: Problem 2 : not optimal allocation

## Introductory example 2

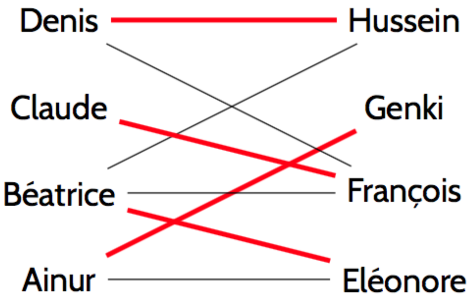



Figure: Problem 2 : optimal allocation

# Other examples

- Assigning students to internships

# Other examples

- Assigning students to internships
- Assigning machines to a task

# Summary

- Today we will work on **connnecting the two problems**.

## Summary

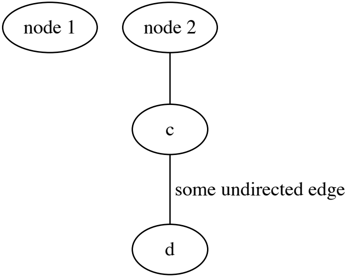
- Today we will work on **connnecting the two problems**.
- Under some restrictions, the two problems **equivalent**.

# Reminders on graphs

- A graph is defined by ?

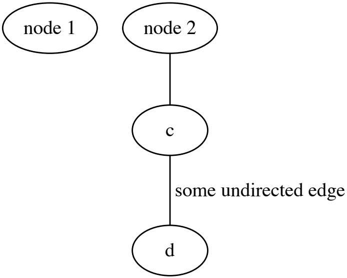
# Reminders on graphs

- A graph is defined by set of **vertices** (or **nodes** ) $V$ and a set of **edges** $E$.



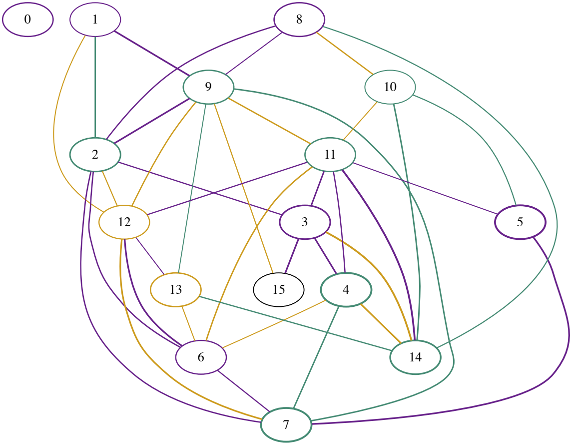
# Reminders on graphs

- It can be **undirected**, as this one :



# Reminders on graphs

## Undirected graph



# Reminders on graphs

- Or **directed**, as this one. (it is then called a **digraph**)

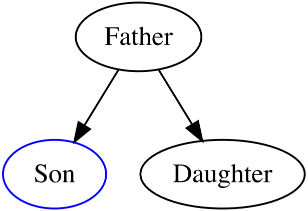
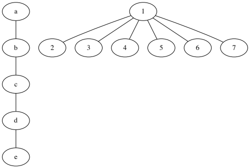


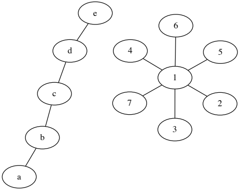

Figure: Digraph (graphviz demo)

# Useful tool : graphviz

- A tool to visualize graphs
- Several **generator programs** : dot, neato




(a) Image generated with **dot**




(b) Image generated with **neato**

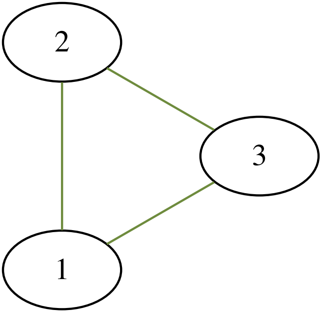
# Warm up question

Given an **undirected** graph with $n$ nodes, how many edges can we build ?

Notation of a graph : $G(V, E)$

- $V$ : set of $n$ vertices
- $E$ : set of edges


1
2


2
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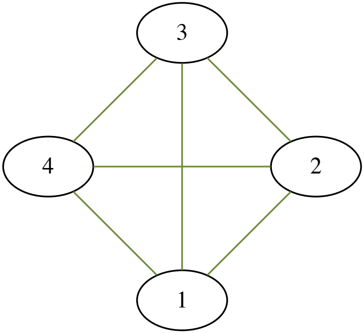
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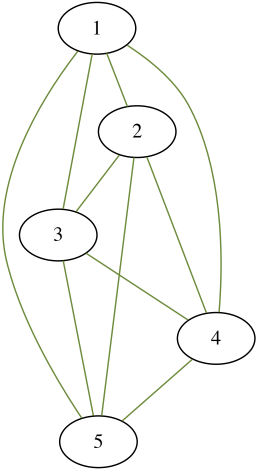
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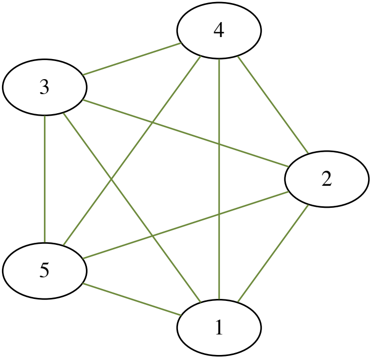
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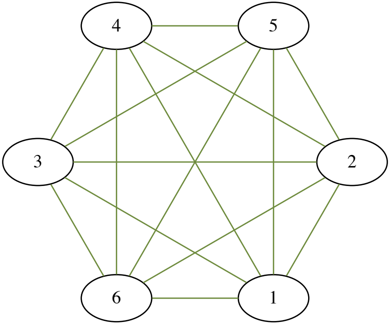
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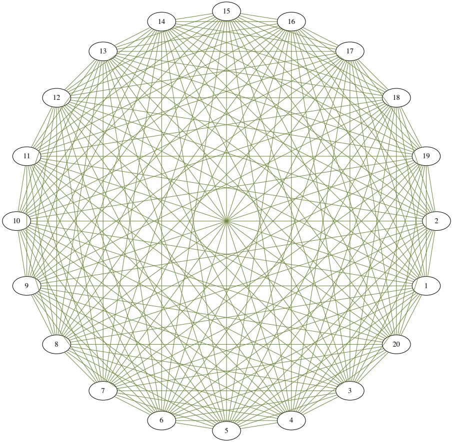



Figure: We cannot count anymore

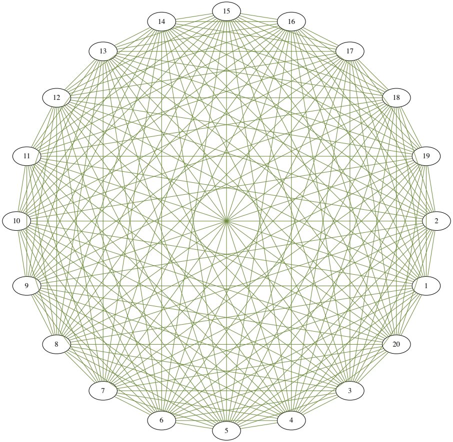
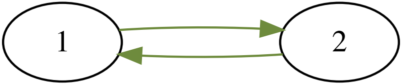



Figure: We cannot count anymore

# What if the graph is directed ?



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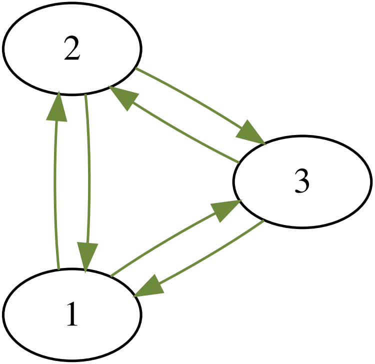

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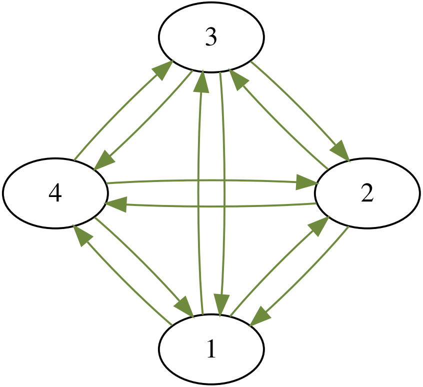

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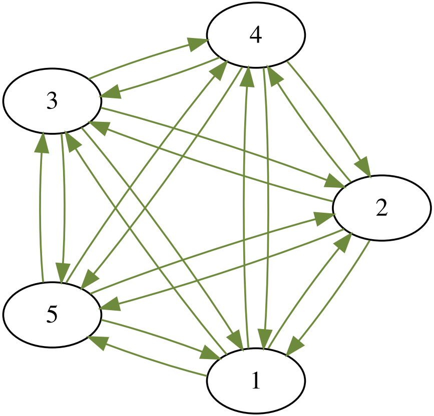

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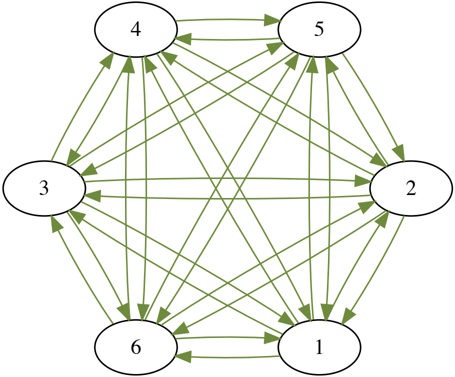

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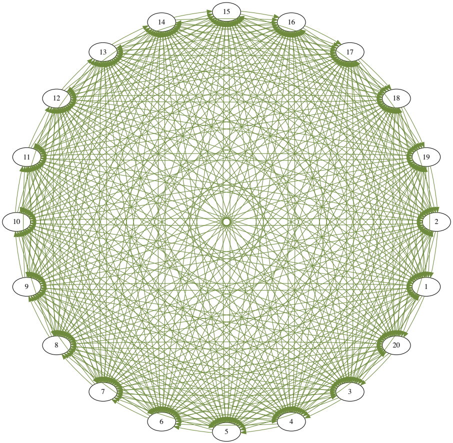

# Warm up question

Given an **directed** graph with $n$ nodes, how many edges can we build ?

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Given an **directed** graph with $n$ nodes, how many edges can we build ?

$$n(n-1) \tag{1}$$

# Warm up question

Given an **directed** graph with $n$ nodes, how many edges can we build ?

$$n(n-1) \tag{2}$$

So if the graph is **undirected**, we can build :

$$\frac{n(n-1)}{2} \tag{3}$$

edges.

# Remark

$\frac{n(n-1)}{2}$ is also the number of subsets of size 2 in a set of size $n$.

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} \tag{4}$$

# Famous graph problem

- Do you know some famous **graph problems ?**

# Famous graph problem

- Do you know some famous **graph problems ?**
- Dominating set

# Famous graph problem

- Do you know some famous **graph problems ?**
- Dominating set
- Maximum clique

# Famous graph problem

- Do you know some famous **graph problems ?**
- Dominating set
- Maximum clique
- Coloring

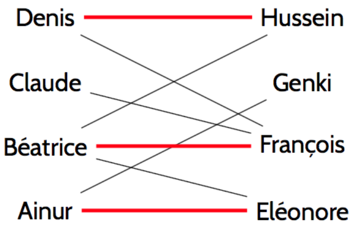
# Matching problem

Let us now focus on the **matching problem**

# Back to our problem

Given a **undirected** graph $G = (V, E)$, we want a **matching** $M$, which means:

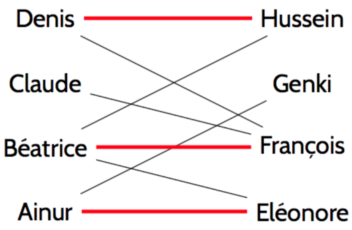
- A subset of edges $M \subset E$



# Back to our problem

Given a **undirected** graph $G = (V, E)$, we want a **matching**, which means:

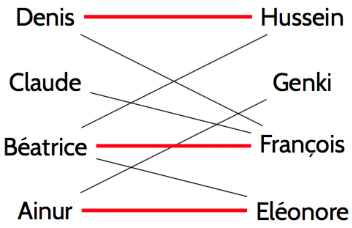
- A subset of edges $M \subset E$
- Such that no pairs of edges of $M$ are incident
- Equivalently, each node in the graph has **at most** one edge connected



# Back to our problem

Given **undirected** a graph $G = (V, E)$, we want a **matching**, which means:

- A subset of edges $M \subset E$
- Equivalently, each node in the graph has **at most** one edge connected
- Such that no pairs of edges of $M$ are incident



# Maximum matching

- The **size** of a matching is the number of edges it contains.

# Maximum matching

- The **size** of a matching is the number of edges it contains.
- We want to find the matching of maximum size in a given graph.

# Example 1

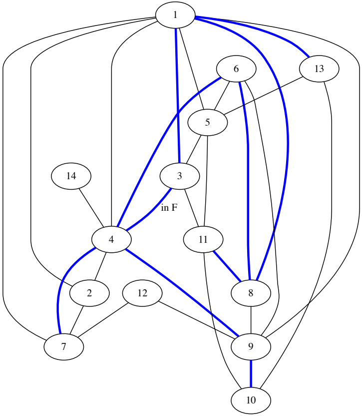



Figure: Is this a matching ?

# Example 2

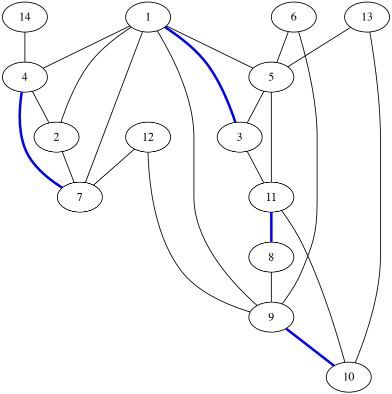



Figure: Is this a matching ?

# Example 3

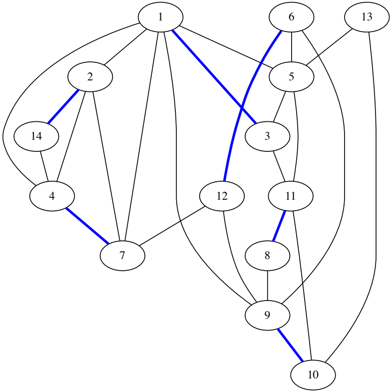



Figure: Is this an optimal matching ?

# Example 4

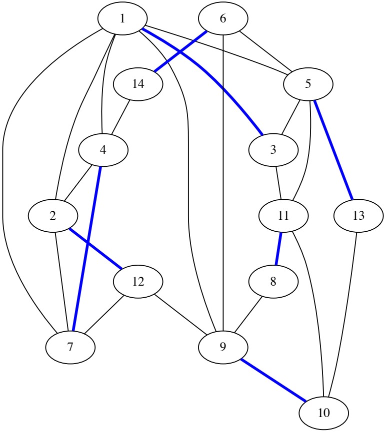



Figure: Is this an optimal matching ?

# Example 5

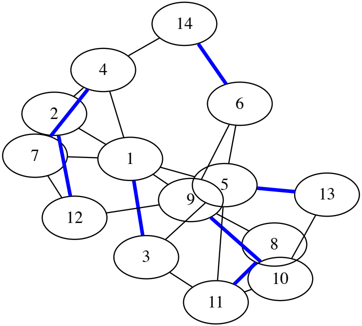



Figure: With neato

# Optimal matching

Exercice 1 : Given a graph of size $n$, what is maximum size possible for a **matching ?**

# Optimal matching

Exercice 1 : Given a graph of size $n$, what is maximum size possible for a **matching ?**

- If $n$ is even : $\frac{n}{2}$
- Else $n$ is odd : $\frac{n-1}{2}$

# Optimal matching

Exercice 1 : Can you think of a graph that contains a matching of size $n$ ? (assuming $n$ is even)

# Optimal

Exercice 1 : Can you think of a graph that contains a matching of size $n$ ? (assuming $n$ is even)

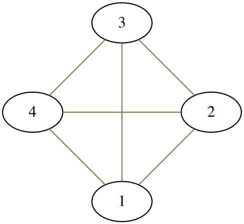



Figure: The complete graph works

# Optimal matching

Exercice 1 : Can you think of a graph that does **not** contains a matching of size $n$ ? (assuming $n$ is even)

# Optimal matching

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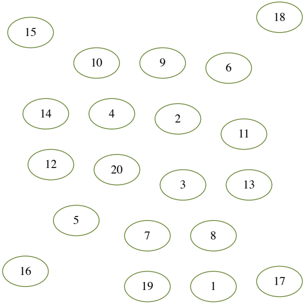


# Optimal matching

Exercice 1 : Can you think of a graph that does **not** contains a matching of size $n$ ? (assuming $n$ is even)

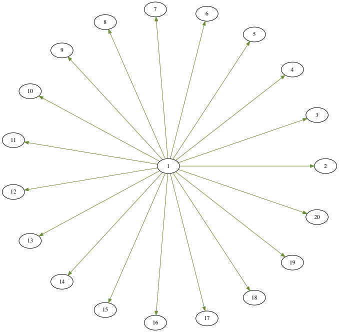



Figure: Star graph

# Experiments

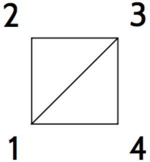
How would you code a graph ?

# Experiments

How would you code a graph ?

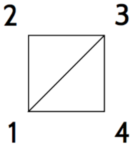
- list of sets of size 2 (for an undirected graph)
- a dictionary of successors (directed of undirected)

# Coding a graph : as a list




```
g1 = [{1,2},{1,3},{2,3},{3,4},{1,4}]
```

# Coding a graph : as a dictionary




```
g1 = { 1:{2,3,4}, 2:{1,3}, 3:{1,2,4}, 4:{1,3} }
```

# Random graph

Exercice 2 : **cd other_graphs/** and please use **random_graph.py** to build a graph with 20 vertices and 50 edges.

- You will need to install **graphviz**

# Graphviz installation help

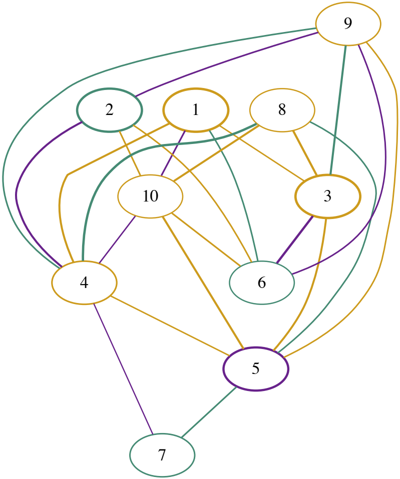
**On windows :**

- download the msi file from graphviz.org
- open the msi file
- update the PATH with the location of the graphviz lib that was just installed
- if not already installed, install **pip**
- **pip install graphviz**
- it might be necessary to restart your computer

**On mac :**

- use **homebrew** (package manager for mac) and / or **pip**

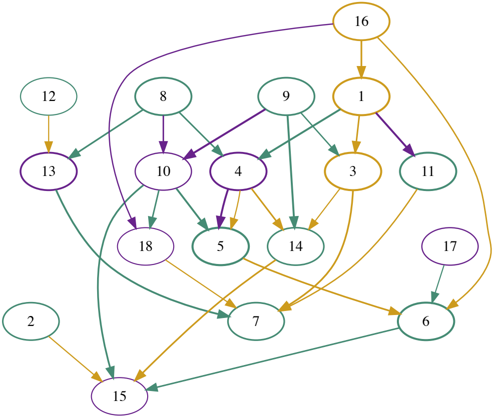
# Example undirected graph obtained



Exercice 3 : and please use **random_graph.py** to build a random directed graph.

- **cd other_graphs** and please use **directed_random_graph** to build a graph with a chosen number of vertices and **directed edges**.

# Directed graph

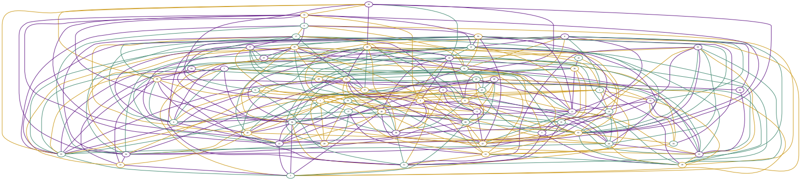


# Manual matching

Exercice 4 : Please manually find an **optimal matching** in your **undirected** graph.

# Big graph

We could not manually find an optimal matching in this graph :



# Summary

- We have defined the matching problem.
- When the size of the problem is large, we can not manually find an optimal matching.

# Brute force approach

Exercice 5 : Enumeration

- Given a graph, what would a brute force approach on the matching problem be ?

# Brute force approach

**Exercice 5 :** Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
  - 3) Return the biggest one obtained.

# Brute force approach

Exercice 5 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
  - 3) Return the biggest one obtained.

If the graph contains $n$ nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?

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If the graph contains $n$ nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?
You can give a rough approximation.

# Brute force approach

Exercice 5 : Exhaustive search

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  - 2) Check if each subset is a matching.
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If the graph contains $n$ nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?

It is a **polynomial** number of computations : so it is ok.

# Notion of complexity

- The **time complexity** of an algorithm is a measure of the **number of elementary** operations needed for the algorithm to termiate with respect to the input size.

# Brute force search

Exercice 6 : Complexity of brute force

- 1) Enumerate all possible subsets in the set of the edges.
- 2) Check if each subset is a matching.
- 3) Return the biggest one obtained.

What is the complexity of step 1 ?

# Brute force search

Exercice 6 : Complexity of brute force

- 1) Enumerate all possible subsets in the set of the edges.
- 2) Check if each subset is a matching.
- 3) Return the biggest one obtained.

What is the complexity of step 1 ?
The number of subsets is $2^{\frac{n(n-1)}{2}}$ (in the worst case), which is exponential.

# Brute force search

Exercice 6 : Complexity of brute force Assume that checking a subset requires 1 microsecond. How long should we wait in order to check all possible matching in a graph with 100 nodes ?

# Other example of complexities

- linear search
- dichotomic search

# Summay II

- For the matching problem on a large graph, we can neither
  - manually find an optimal matching
  - perform the exhaustive search (brute force algorithm)

# Algorithms

- Hence, we need **algorithms** to solve the problem.

# Algorithms

- Hence, we need **algorithms** to solve the problem.
- Let us introduce some theoretical notions.

# Notion of maximal and maximum matching

We will say that a matching $M$ of cardinality (number of elements) $|M|$ is:

- **Maximum** if is has the maximum possible number of edges (is is thus optimal)

# Notion of maximal and maximum matching

We will say that a matching $M$ of cardinality $|M|$ is:

- **Maximum** if is had the maximum possible number of edges (is is thus optimal)
- **Maximal** if the set of edges obtained by adding any edge to it is **not a matching**. This means that $M \cup \{e\}$ is not a matching for any $e \notin M$.

Exercice 7 : Question Is being a **maximal** matching the same thing as beeing a **maximum** matching ?

# Maximum implies maximal

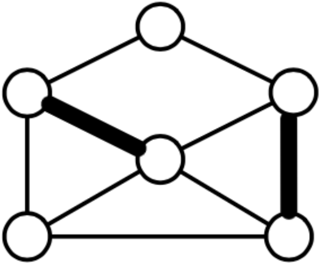
Let us show that a maximum matching is maximal.

# Counter Example

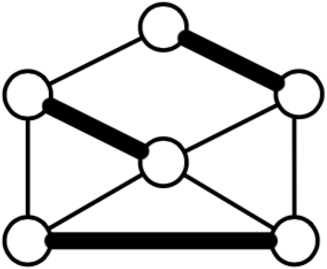
However, a matching that is maximal is **not necessary Maximum**.

# Counter Example

However, a matching that is maximal is **not necessary Maximum**.
Can you find an example ?



(a) A maximal matching not maximum



(b) A maximum matching

# Greedy algorithm

Can you propose a greedy algorithm to address the maximum matching problem ?

# Greedy algorithm

**Result:** Matching M

$M \leftarrow \emptyset$;

**for** $e \in E$ **do**

  **if** $M \cup \{e\}$*is a matching* **then**

    $M \leftarrow M \cup \{e\}$

  **end**

**end**

return $M$

**Algorithm 0:** Greedy algorithm to find a matching

# Greedy algorithm

- What is the type of matching algorithm returned by this algorithm ?
- What is the complexity of this algorithm ? (as a function of the number of nodes $n$ of the graph)

# Greedy algorithm

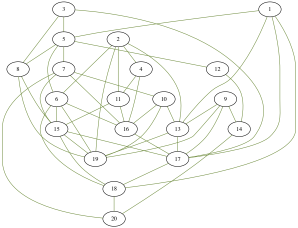
- The greedy algorithm returns a **maximal** matching (proof)
- Its complexity is **cubic** : $\mathcal{O}(n^3)$

# Greedy algorithm

- We will implement the greedy algotithm to find a maximal matching.

# Implementing the greedy algorithm

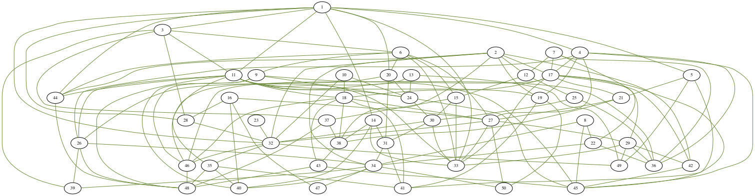
Exercice 8 : **cd matching_greedy/** and use **generate_graph.py** to build a graph with a least 30 nodes. The images are stored in **images/**, data stored in **data/**

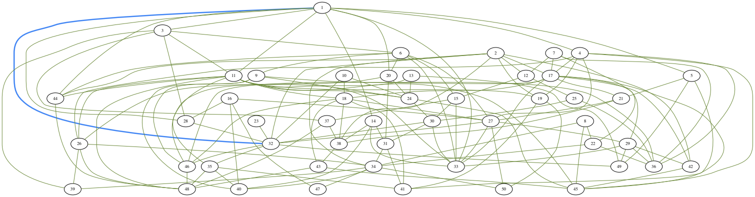


# Implementing the greedy algorithm

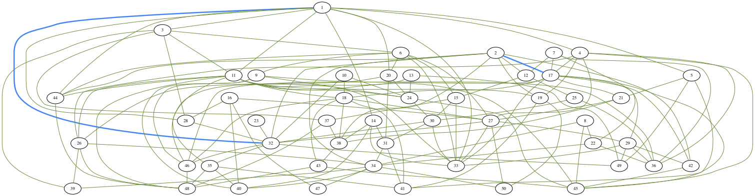
Exercice 8 : **cd matching_greedy/** and use **generate_graph.py** to build a graph with a least 30 nodes.
And implement the greedy algorithm on this graph.

- Use the functions in **matching_functions.py** and call them from **match_graphs**
- edit the lines below "CHANGE HERE" to pergorm the greedy algorithm.

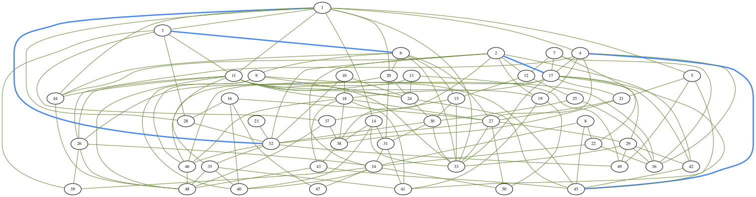


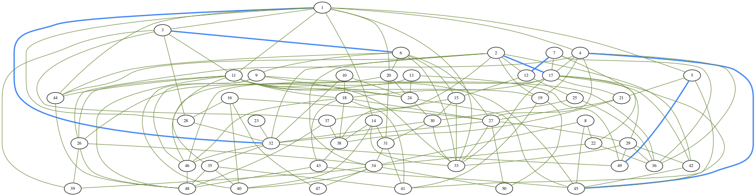

Matching size: 1
Algo step: 1
Nb nodes: 50



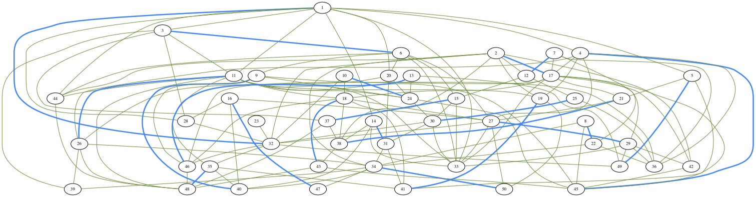
Matching size: 2
Algo step: 11
Nb nodes: 50



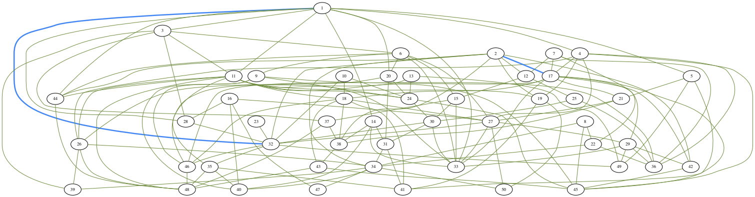
Matching size: 4
Algo step: 24
Nb nodes: 50



Matching size: 6
Algo step: 39
Nb nodes: 50



Matching size: 21
Algo step: 128
Nb nodes: 50



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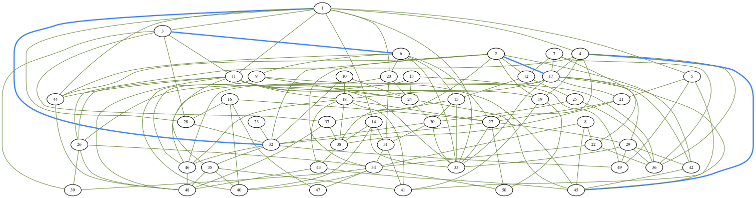
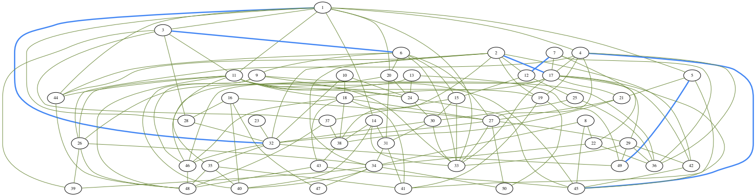


Matching size: 4
Algo step: 24
Nb nodes: 50



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Algo step: 39
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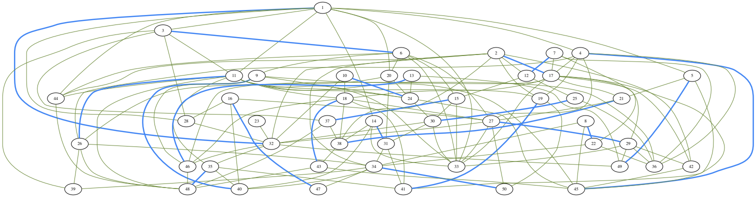
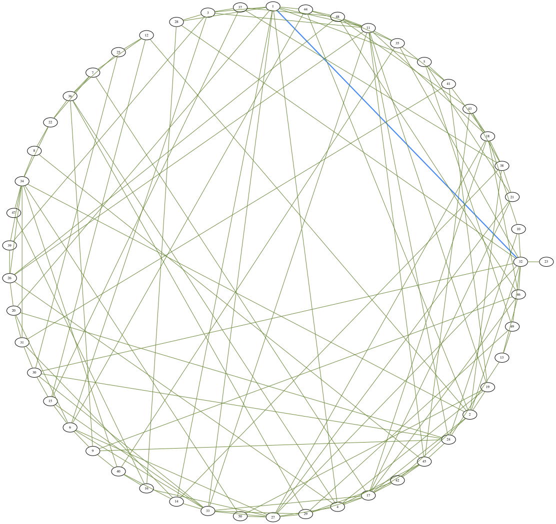
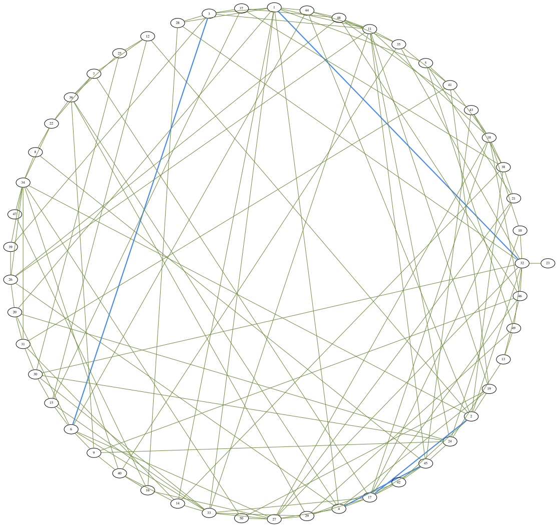


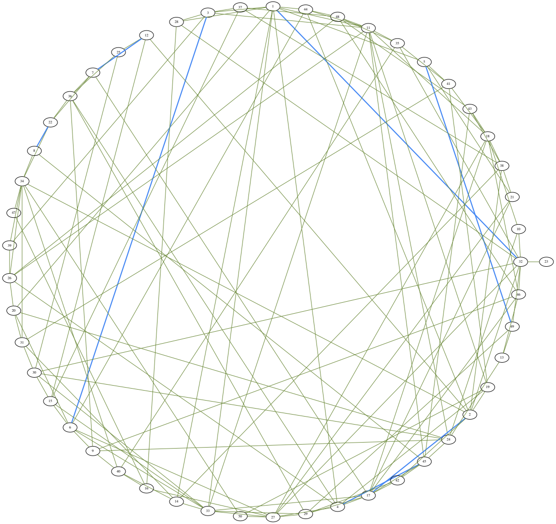
Matching size: 21
Algo step: 128
Nb nodes: 50



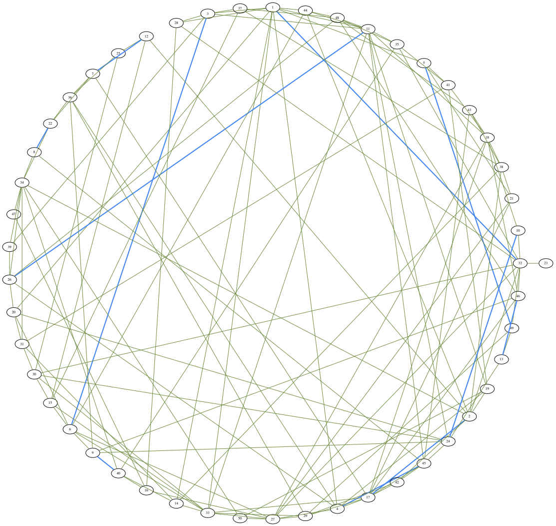
Matching size: 1
Algo step: 1
Nb nodes: 50



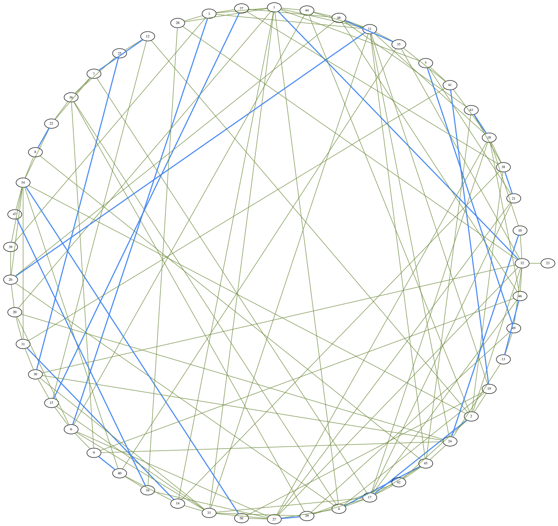

Matching size: 4
Algo step: 24
Nb nodes: 50




Matching size: 7
Algo step: 43
Nb nodes: 50



Matching size: 11
Algo step: 66
Nb nodes: 50



Matching size: 21
Algo step: 128
Nb nodes: 50

# Example

Exercice 9 : Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?

# Example

Exercice 10 : Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?

# Greedy matching

However, is $|M|$ is the cardinality of a matching returned by the greedy algorithm, and if $|M^*|$ is the cardinal of the real optimal matching, we have :

$$|M| \geq \frac{|M^*|}{2} \tag{5}$$

# Changing the problem (for now)

We temporarily leave the maximum matching problem to focus on another problem : the **Maximum flow problem**

# Max flow


Figure: Optimizing the quantity of something transported from one place to another, under constraints

# Example

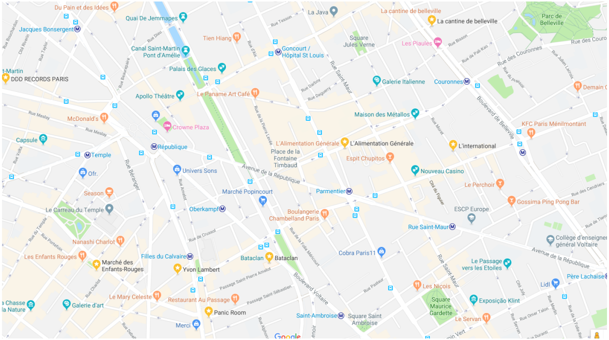



Figure: Optimizing the quantity of something transported from one place to another, under constraints

# Formalizing the problem

We introduce the concept of **flow network (reseau de flot).**

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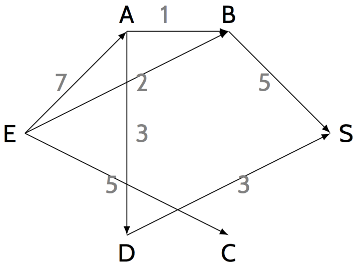


Figure: A **flow network (reseau de flot)** with capacities

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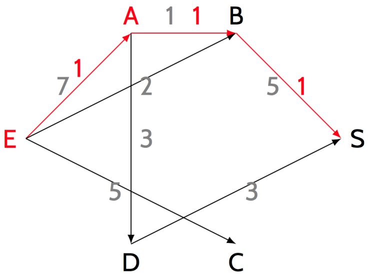

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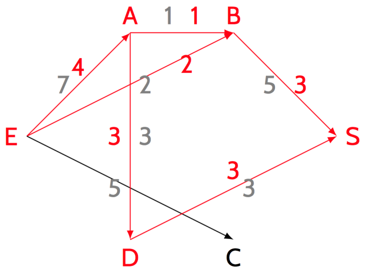

# Conservation of the flow

We must have :

- antisymetry : $f(v,u) = -f(u,v)$
- flow conservation

# conservation of the flow

we must have :

- antisymetry : $f(v,u) = -f(u,v)$
- flow conservation : $\sum_{w \in v} f(u,w) = 0$ for $u \notin \{e,s\}$

# Other formulation of the flow conservation

Let us show that for a flow $f$:

$$\sum_{f(u,v)>0} f(u,v) = \sum_{f(v,u)>0} f(v,u) \tag{6}$$

# Maximum flow

- The **value of the flow**, noted $|f|$, is $\sum_{v \in S} f(E, v)$
- The problem is that of finding a flow with **maximum value**

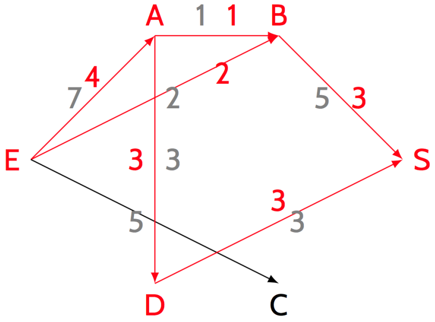


Figure: Max flow

# Ford Fulkerson algorithm

We will introduce an algorithm to solve the problem. This algorithm :

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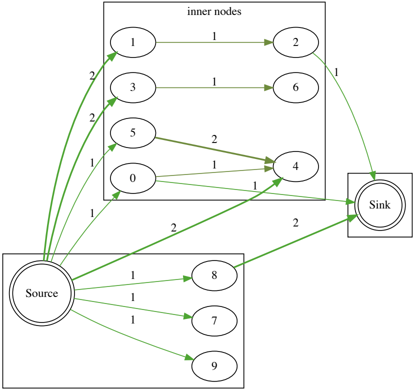
So it's great. **This section is going to be a little bit technical.**

# Residual graph

- Given a graph with capacities $c(u, v)$ and a flow $f(u, v)$, we will define its **residual graph** that has a capacity $c_r(u, v)$ :

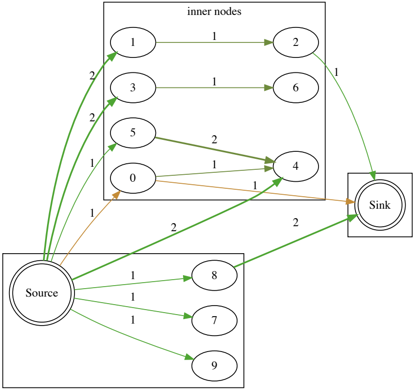
$$c_r(u, v) = c(u, v) - f(u, v) \tag{7}$$

# Example of residual graph



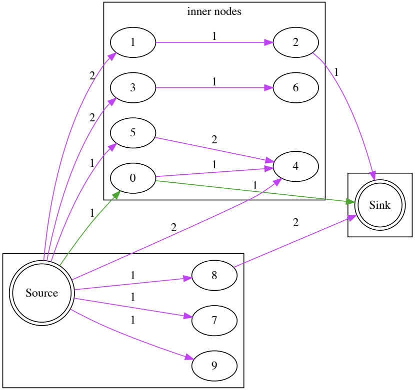

Initial Flow network

# Example of residual graph




Flow
Algorithm step: 1
flow value: 1

# Example of residual graph




Residual graph
Algorithm step: 2

# Residual graph

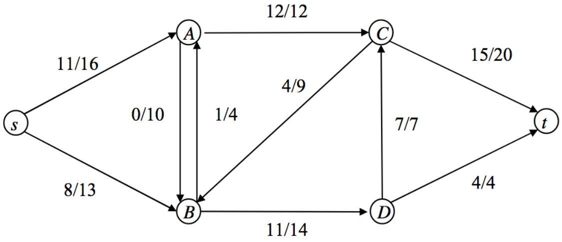



Figure: Another flow network

# Residual graph

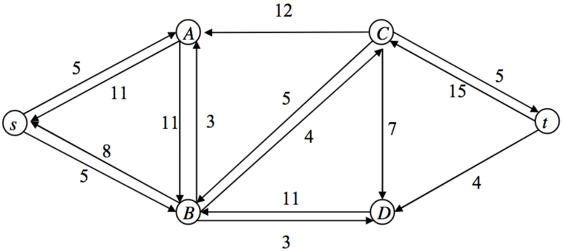
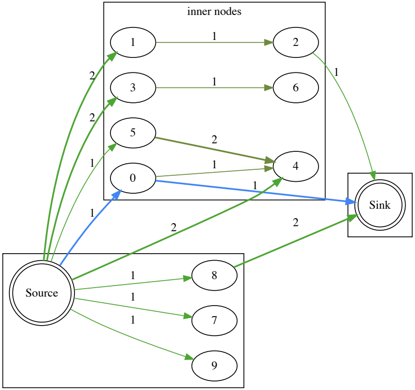



Figure: Residual graph

# Augmenting path

An augmenting path is a path in the **residual graph** from the source to the sink with capacities $> 0$.

# Augmenting path




augmenting path: [0, 1, 11]
Algorithm step: 1
path capacity: 1.0

# Augmenting path

An augmenting path is a path in the residual graph from the source to the sink with capacities $> 0$.

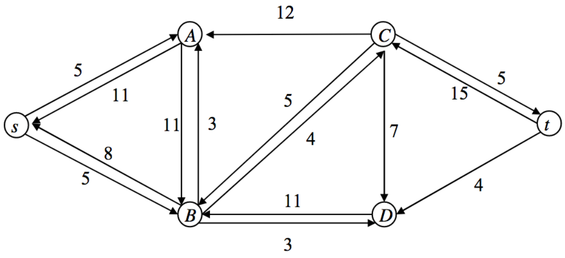



Figure: Residual graph

# Augmenting path

An augmenting path is a path from the source to the sink with capacities $> 0$.
The Ford-Fulkerson algorithm uses augmenting paths until there are no more augmenting paths.

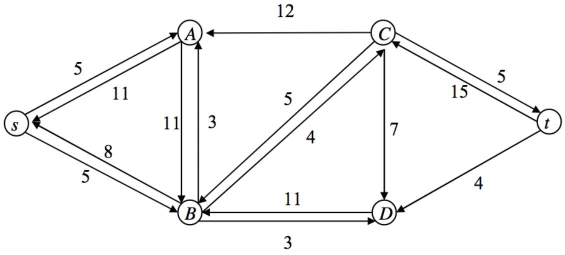



Figure: Residual graph

# Ford Fulkerson algorithm

Can you deduce the algorithm from the previous remarks ?

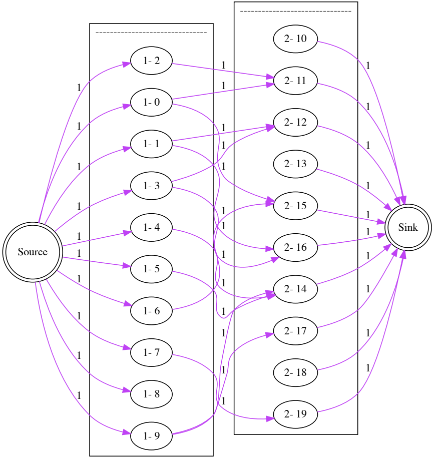
# Ford Fulkerson algorithm

**Result:** Flow $f$
**for** $(u, v) \in E$ **do**
  $f(u, v) = 0$
**end**
**while** $\exists \rho$ *augmenting path* **do**
  augment $f$ with $\rho$
**end**
return $f$

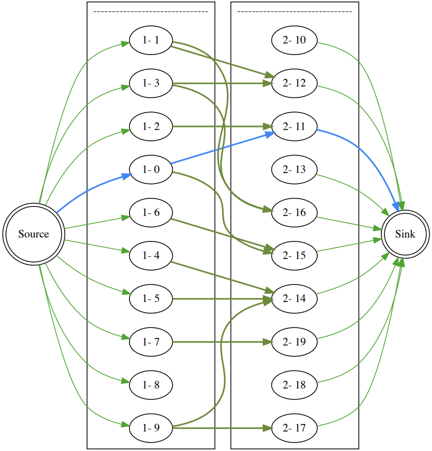
**Algorithm 1:** Ford Fulkerson algorithm

# Ford

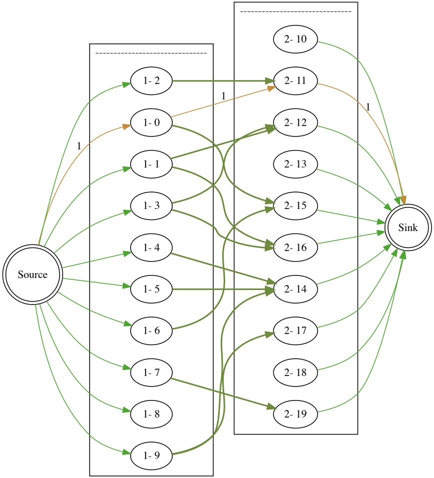
Let's do a complete instance of the algotithm:



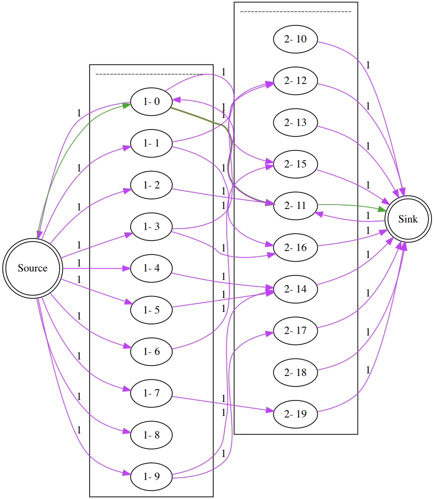

Residual graph
Algorithm step: 1



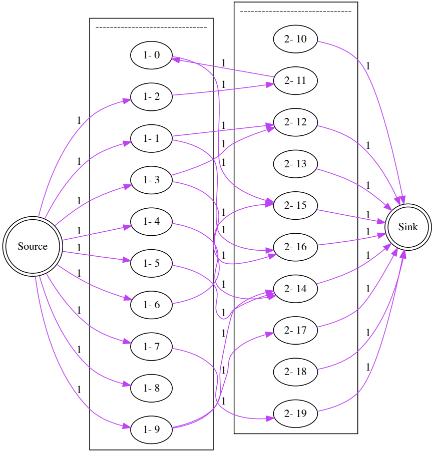

augmenting path: [0, 1, 12, 21]
Algorithm step: 1
path capacity: 1.0



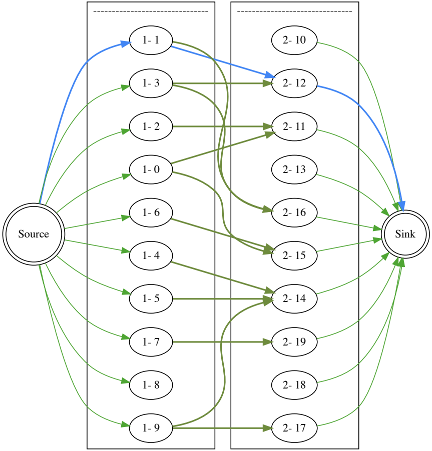

Flow
Algorithm step: 1
flow value: 1



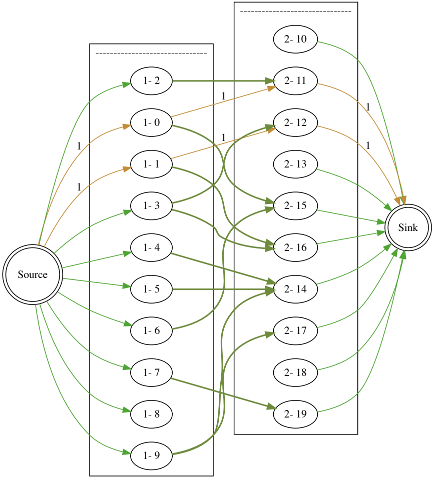

Residual graph
Algorithm step: 2



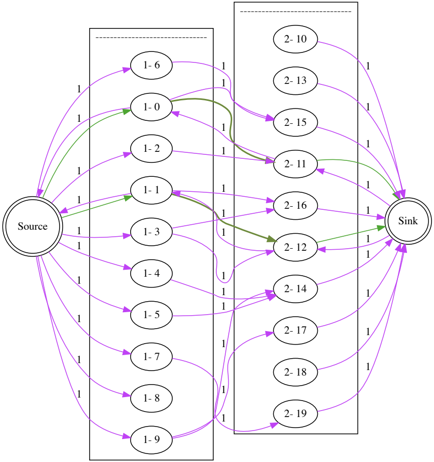

Simple residual graph in purple
Algorithm step: 2



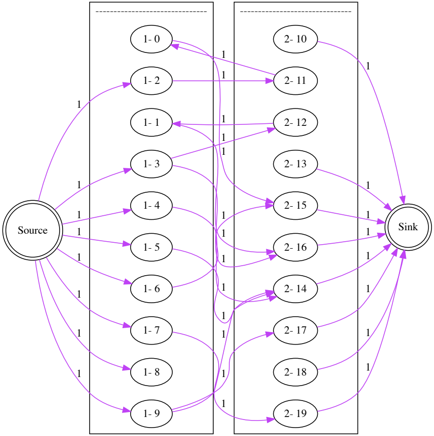

augmenting path: [0, 2, 13, 21]
Algorithm step: 2
path capacity: 1.0



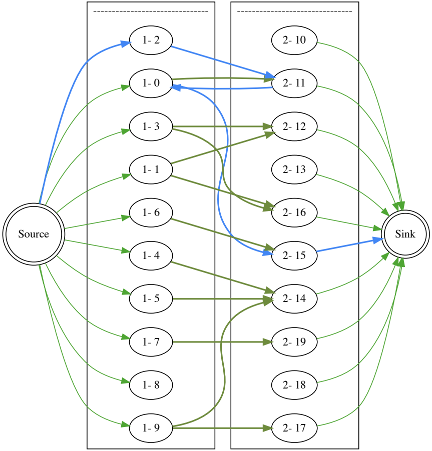

Flow
Algorithm step: 2
flow value: 2



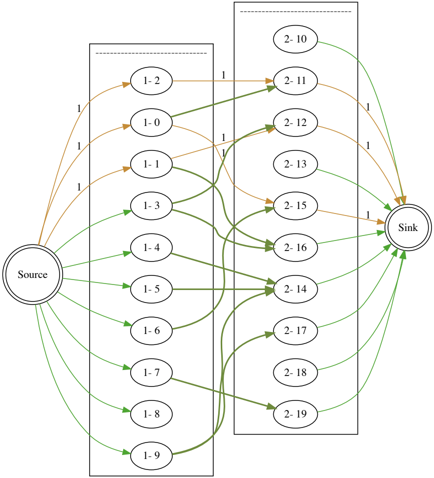

Residual graph
Algorithm step: 3



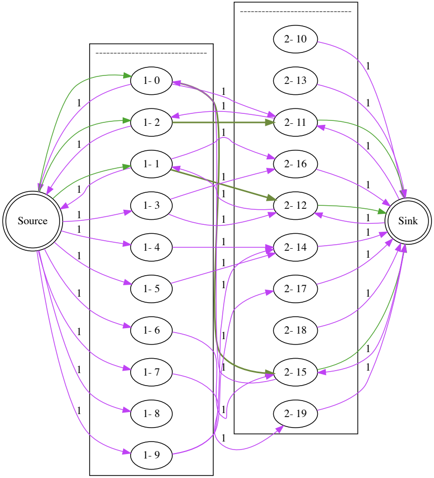

Simple residual graph in purple
Algorithm step: 3



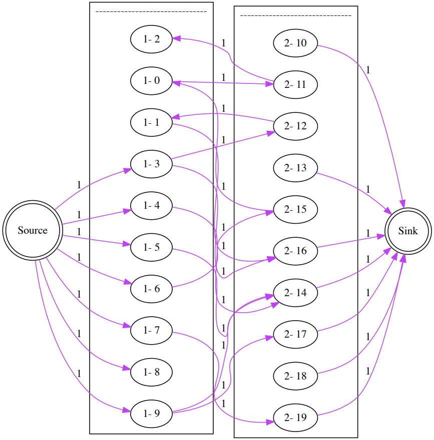

augmenting path: [0, 3, 12, 1, 16, 21]
Algorithm step: 3
path capacity: 1.0



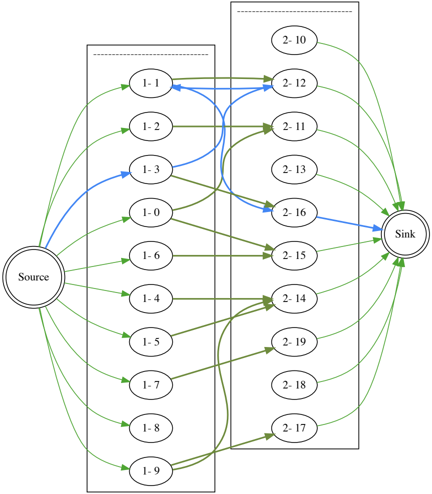

Flow
Algorithm step: 3
flow value: 3



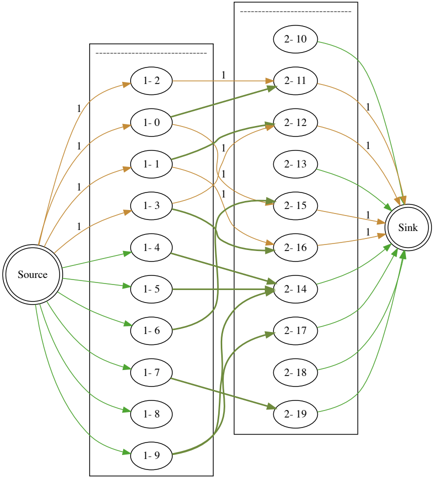

Residual graph
Algorithm step: 4



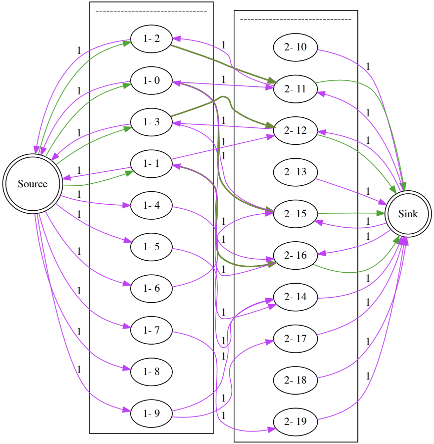

Simple residual graph in purple
Algorithm step: 4



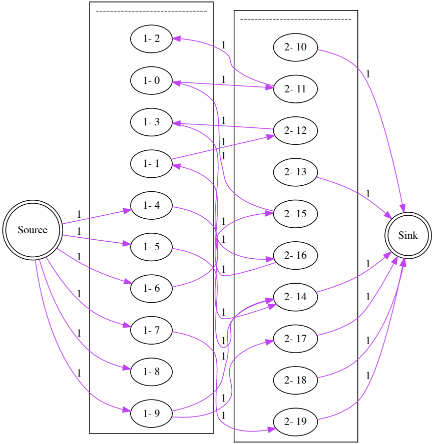

augmenting path: [0, 4, 13, 2, 17, 21]
Algorithm step: 4
path capacity: 1.0



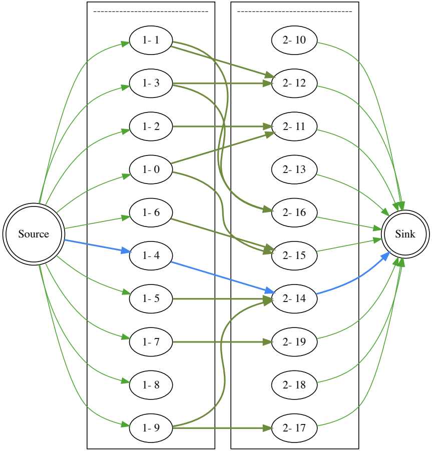

Flow
Algorithm step: 4
flow value: 4



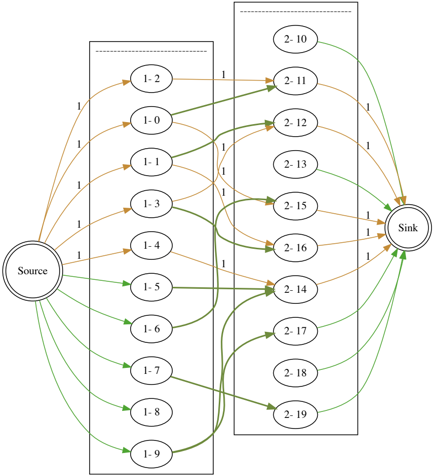

Residual graph
Algorithm step: 5



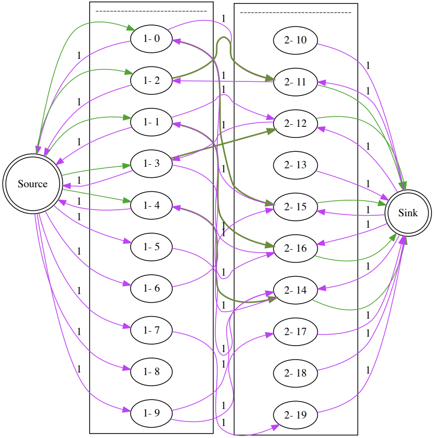

Simple residual graph in purple
Algorithm step: 5



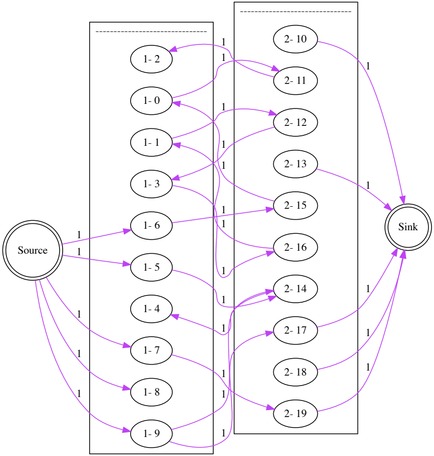

augmenting path: [0, 5, 15, 21]
Algorithm step: 5
path capacity: 1.0



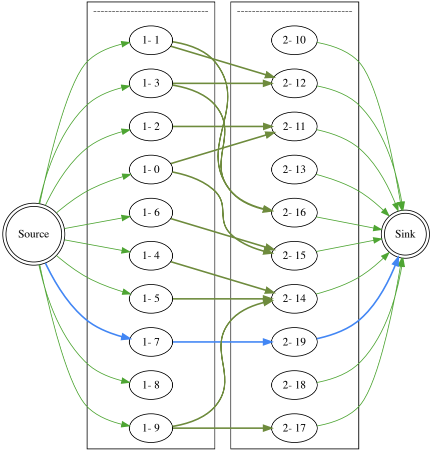

Flow
Algorithm step: 5
flow value: 5



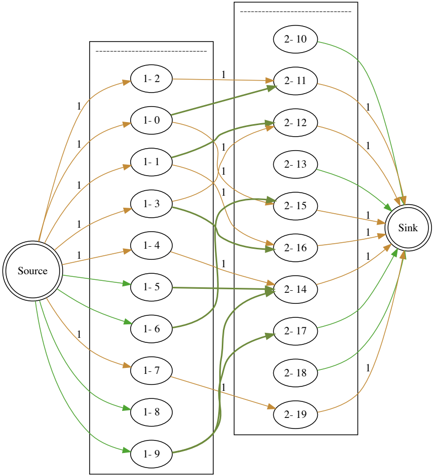

Residual graph
Algorithm step: 6



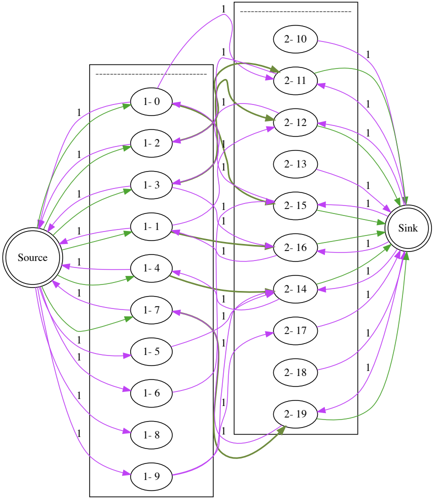

Simple residual graph in purple
Algorithm step: 6



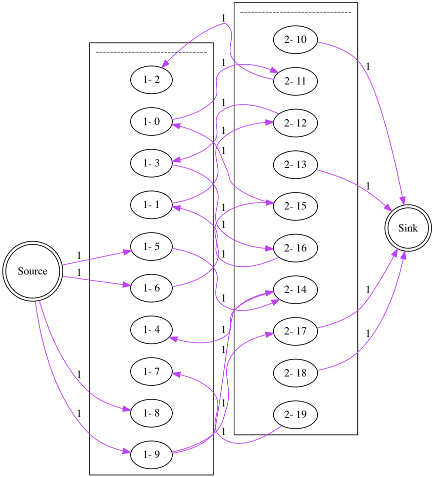

augmenting path: [0, 8, 20, 21]
Algorithm step: 6
path capacity: 1.0



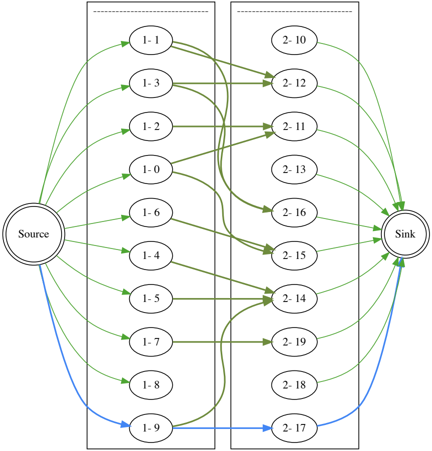

Flow
Algorithm step: 6
flow value: 6



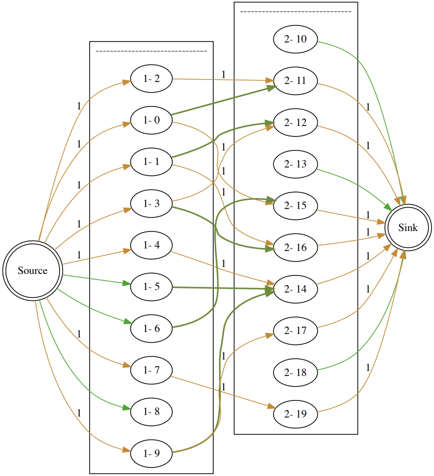

Residual graph
Algorithm step: 7



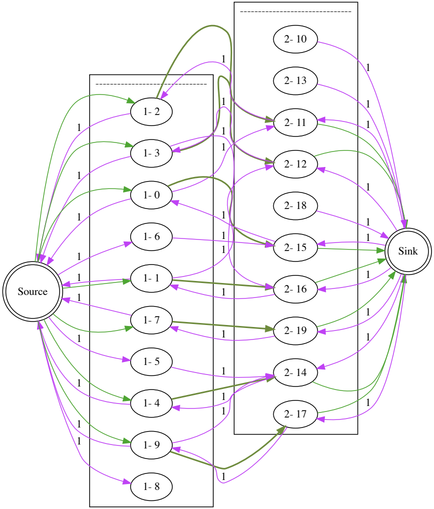

Simple residual graph in purple
Algorithm step: 7



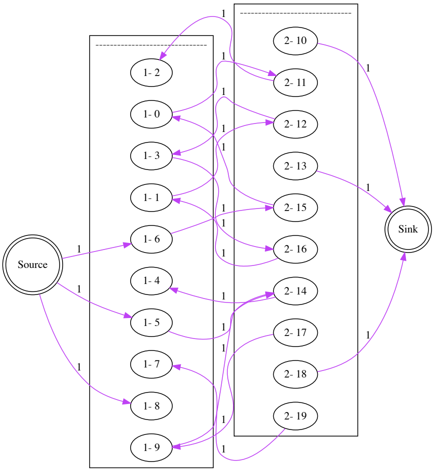

augmenting path: [0, 10, 18, 21]
Algorithm step: 7
path capacity: 1.0




Flow
Algorithm step: 7
flow value: 7




Residual graph
Algorithm step: 8




Simple residual graph in purple
Algorithm step: 8

# Ford Fulkerson algorithm

- We will implement the Ford Fulkerson algorithm (1956) on a general graph.

# Ford Fulkerson algorithm

Exercice 10 : We will implement the Ford Fulkerson algorithm (1956)

- **cd ford_fulkerson/** and edit **generate_flow_network** to generate a flow network.

# Algorithm

- We will now use the functions contained in **ford_functions.py** and call them from **apply_ford_fulkerson.py**

# Algorithm

Exercice 11 : step 1

- Modify **find_augmenting_paths()** in order to find the augmenting paths.

# Algorithm

Exercice 11 : step 2

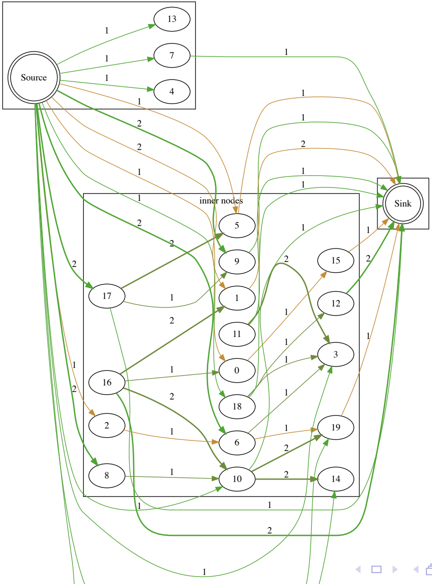
- now edit **augment_flow()**

# Algorithm

Exercice 11 : step 3

- finally, edit the computation of the value of the flow

- Now the algorithm should be able to run


Source
13
7
4
inner nodes
5
9
1
11
0
18
6
10
17
16
2
8
15
12
3
19
14
Sink

# Complexity

What is the complexity of Ford Fulkerson ?

# Complexity

What is the complexity of Ford Fulkerson ?

$$\mathcal{O}(|f^*| \times |E|) \tag{8}$$

# Modification of Ford Fulkerson

What would we an intuitive and potentially faster modification of the algorithm ?

# Edmonds Karp

What would we an intuitive and potentially faster modification of the algorithm ?
Use the shortest augmenting path with positive capacity.

# Termination

- When the capacities are **real numbers** or **rational numbers** Ford Fulkerson terminates.

# Termination

- When the capacities are **integer numbers** or **rational numbers** Ford Fulkerson terminates.
- However, when the capacities are general **real numbers**, the algorithm might not temrinate.

# Link with the matching problem

- We now go back to the matching problem, in the case of a **bipartite graph**.
- We will show that in that case, we can connect the two problems.

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- We will show that in that case, we can connect the two problems.

# Bipartite graph

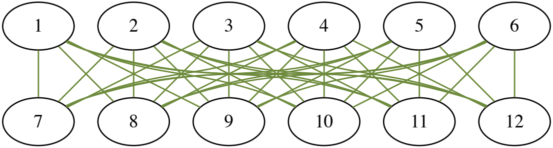



Figure: Complete bipartite graph (not all bipartite graphs are complete)

## Matching problem

We now go back to the matching problem, in the case of a **bipartite graph**.

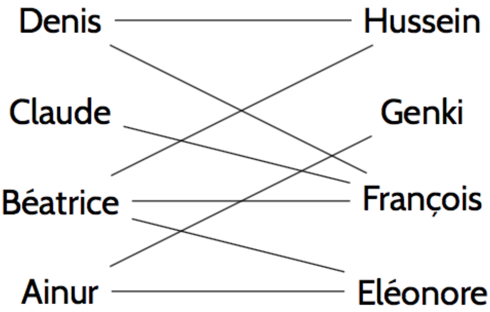



Figure: Bipartite graph

# Equivalence between matching and flow

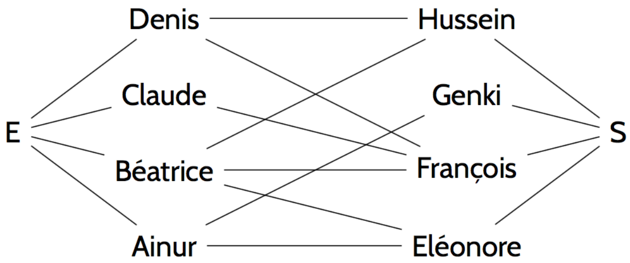



Figure: Introduce two more nodes. All edges have capacity 1. We consider **flows with integer values**

# Ford Fulkerson for matching

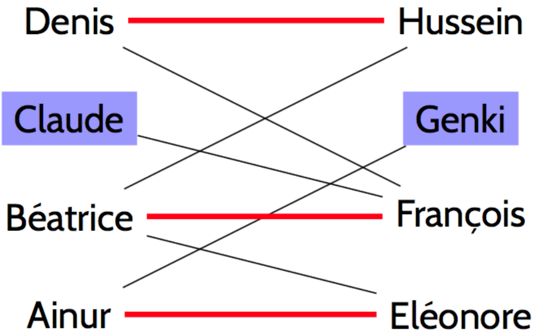



Figure: Non optimal solution

# Ford Fulkerson for matching

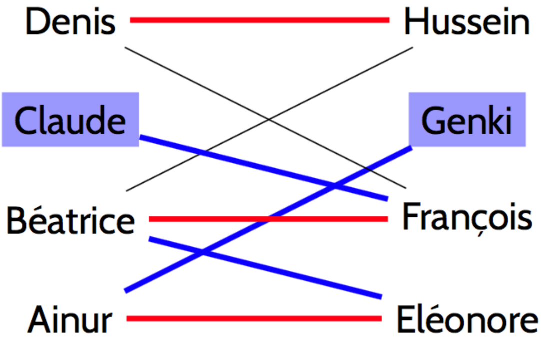



Figure: Optimal solution

# Connection

Exercice 11 : Find a connection between the two problems

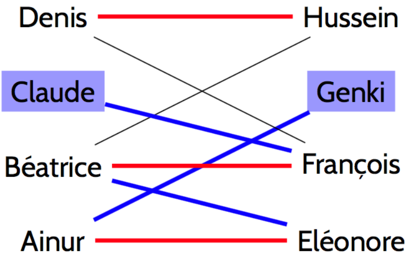



Figure: Optimal solution

Exercice 12 : Ford Fulkerson and matching

- We will transpose Ford Fulkerson to a bipartite graph in order to find an optimal matching.
- **cd ford_matching/**
- edit **generate_matching_problem.py** in order to generate an instance of the problem.

Exercice 12 : Ford Fulkerson and matching

- Apply the algorithm on an example generated by the previous function.
- Apply the algorithm to as an instance of your choice.

# Famous theorem

The maximum flow theorem is equivalent to another famous problem, the **minimum cut** theorem.

# Perfect matching

In the case of a bipartite graph, what is the best matching possible ?

# Perfect matching

In the case of a bipartite graph, what is the best matching possible ?
A matching where **all nodes are allocated.** It is called a **perfect** matching.
We must have that the two parts of the graph are of same cardinalty in order to have a perfect matching.

# Hall's marriage theorem

This theorem gives a condition that is necessary and sufficient for the existence of a perfect matching in a bipartite graph : the "marriage condition".
If $G = (U, V, E)$ is bipartite, the condition means that :

$$\forall X \subset U, |N_G(X)| \geq |X| \tag{9}$$

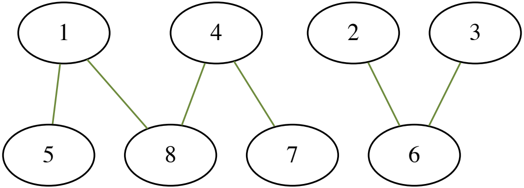
where $N_G(X)$ is the set of neighbors of $X$ in $G$.

# Hall's theorem

Exercice 13 : Application of the theorem. Can you think of a graph that does not abide by the marriage condition and thus has **no perfect matching** ?

# Illustration of Hall's theorem

Exercice 13 : Application of the theorem



# Conclusion

Ford Fulkerson and its variants (Edmonds-Karp) are polynomial. As a result thay can run on datasets that are way bigger than exhaustive search algotirhms.

See you tomorrow