Chapter 1

Functions

1.1 Interval Notation

In the definitions of Open, closed and Half closed intervals we shall assume that a and b are real numbers such that $a < b$.

1.1.1 Open Intervals

An Open Interval, written in interval notation as (a, b), is defined as the set of all numbers x such that $a < x < b$: where $a < b$ that is x lies between a and b. This can be represented in set-builder notation as $(a, b) = \{x : a < x < b\}$. The points a and b are themselves not included.

Example 1.1.1 *Express the following open intervals in set-builder notation:*

(a) $(2, 4)$

(b) $(-1, 3)$

These open intervals can be expressed in set-builder notation as:

(a) $(2, 4) = \{x : 2 < x < 4\}$

(b) $(-1, 3) = \{x : -1 < x < 3\}$

1.1.2 Closed Intervals

A Closed Interval, written in interval notation as $[a, b]$ is defined as the set of all numbers x such that $a \leq x \leq b$. Closed intervals can be represented in set-builder notation as: $[a, b] = (x : a \leq x \leq b)$ x lies between and includes a and b.

Example 1.1.2 *Express the following closed intervals in set builder notation.*

(a) $[m, n]$

(b) $[-1, 3]$

These closed intervals can be expressed in set builder notation as:

(a) $[m, n] = \{x : m \leq x \leq n\}$

(b) $[-1, 3] = \{x : -1 \leq x \leq 3\}$

1.1.3 Half-open Intervals

There are two types of half open intervals. The left half open interval $(a, b]$ and the right half open interval $[a, b)$.

The left half open interval is defined as the set of numbers x such that $a < x \leq b$. This can be represented in set-builder notation as

$$(a, b] = \{x : a < x \leq b\}$$

where $a < b$.

The right half open interval is defined as the set of number x such that $a \leq x < b$, where $a < b$. This can be represented in set-builder notation as

$$[a, b) = \{x : a \leq x < b\}$$

Example 1.1.3 *Express the following half open intervals in set-builder notation.*

(a) $(-2, 5]$

(b) $[-a, a)$

The half open intervals above can be expressed in set builder notation as:

(a) $(-2, 5] = \{x : -2 < x \leq 5\}$

(b) $[-a, a) = \{x : -a \leq x < a\}$

1.2 Infinite Intervals

We have four types of infinite intervals.

1.2.1 Open Infinite Intervals

Let a be a real number. The Open Infinite Interval (a, ∞) is the set of numbers x such that $a < x < \infty$. This is represented in set - builder notation by $(a, \infty) = \{x : a < x < \infty\}$. This can be displayed on the real line graph by the space between a and infinity but not including the point a.

Similarly, the open interval $(-\infty, a)$, is the set of numbers x such that $\infty < x < a$. This is represented in set builder notation by $(-\infty, a) = \{x : -\infty < x < a)$. On the real line graph, this can be displayed by the space between negative infinity and a, but not including a.

1.2.2 A closed-infinite interval $[a, \infty)$.

These are the set of numbers x such that $a \leq x < \infty$. This is expressed in set-builder notation by $(a, \infty) = \{x : a \leq x < \infty\}$. This can be displayed on the real line graph by the space between a and infinity including the point a.

1.2.3 Infinite Half Open Interval $(-\infty, a]$.

This is the set of numbers x such that $-\infty < x \leq a$. This is represented in set-builder notation by $(-\infty, a] = (x : -\infty < x \leq a)$. This can be displayed on the real line graph by the space between negative infinity and a including the point a.

1.2.4 An Infinite-Infinite Interval $(-\infty, \infty)$.

This is the set of numbers x such that $-\infty < x < \infty$. This is represented in set builder notation by $(-\infty, \infty) = \{x : -\infty < x < \infty\}$. This can be displayed on the number line by all this points between negative infinity and positive infinity. This represents the entire number line.

Example 1.2.1 *Express the following intervals in set - builder notation.*

(a) $(2, 4)$

(b) $[2, 4)$

(c) $(-4, 3]$

The above intervals can be expressed in set-builder notation as

(a) $(2, 4) = \{x : 2 < x < 4\}$

(b) $[2, 4) = \{x : 2 \leq x < 4\}$

(c) $(-4, 3] = \{x : -4 < x \leq 3\}$

Example 1.2.2 *Write the following sets in interval notation:*

(a) $A \;=\; \{x : -\infty < x < 2\}$

(b) $B \;=\; \{x : -1 < x < 4\}$

(c) $C \;=\; \{x : -\infty < x \leq 3\}$

(d) $D \;=\; \{x : 2 \leq x < \infty\}$

These sets can be represented in interval notation as follows:-

(a) The set of points do not include 2*, but continues to the left of* 2 *up to negative infinity, that is* $(-\infty, 2)$.

(b) The set of points that lies between -1 *and* -4*, that is,* $(-1, 4)$.

(c) The set of points that includes 3 *and continues to the left of* 3 *up to negative infinity, that is,* $(-\infty, 3]$

(d) The set of points that includes 2 *and continues to the right of* 2 *up to infinity that is,* $[2, \infty)$.

1.3 Inequalities

The inequality $a < b$ means $b - a$ is positive. The inequality $a \leq b$ means that either $a = b$ or $a < b$.

The second inequality is the most misunderstood by beginning students. Many students are quite happy with $3 < 5$ but fear $3 \leq 5$ or $4 \geq 4$ may be incorrect. All the three statements are correct.

Theorem 1.3.1 Let x, y, z and c be real numbers. Then

(i) Exactly one of the following holds: $x = y$, or $x < y$ or $x > y$

(ii) If $x < y$ and $y < z$, then $x < z$.

(iii) If $x < y$, then $x + c < y + c$.

(iv) If $x < y$ and $c > 0$, then $cx < cy$.

(v) If $x < y$ and $c < 0$, then $cx > cy$.

These inequalities are used to solve inequalities.

Note 1.3.1 Note that multiplying by a negative number reverses the direction of the inequality. The above properties can be proved. Properties (ii) and () are proved for you. The other proofs are left as an exercise.

Example 1.3.1 *Prove that if* $a < b$ *and* $b < c$ *then* $a < c$.
We need to recall that $a < b$ *means* $b - a$ *is positive and* $b < c$ *means* $c - b$ *is positive. The sum of two positive numbers is a positive number. Thus* $(b - a) + (c - b) = c - a$ *is positive and therefore* $a < c$.

1.3.1 Solving Inequalities

The process of finding a solution requires that the inequality be written in a form that shows the values of the variable that ascertain the truth of the initial inequality. Inequalities may involve sums differences, products and quotients.

Example 1.3.2 Solve the inequality $2x - 9 > -3$.

Add $+9$ to both sides of the inequality.
$2x - 9 + 9 > -3 + 9$; (using property of inequalities: if $a < b$ then $a + c < b + c$)
gives $2x > 6$.
Multiply both sides of the inequality by $\frac{1}{2}$ to obtain the coefficient of x as 1 :
$\frac{1}{2} \cdot 2x > 6 \cdot \frac{1}{2}$; (Using property of inequalities: if $a > b$ and $c > 0$ then $ac > bc$) from which $x > 3$.

Example 1.3.3 Solve the inequality $2(3 + 2x) - 4 > 7x$.

Distribute 2 to eliminate the brackets $6 + 4x - 4 > 7x$.
Combine the like terms on the left hand side (LHS) of the inequality. $4x + 2 > 7x$
Add $-4x + 4x + 2 > 7x - 4x$; (property of inequalities: if $a > b$ then $a + c > b + c$) then $2 > 3x$.
Multiply by $\frac{1}{3}$ to get a coefficient of $x = 1$. That is, $2 \cdot \frac{1}{3} > \frac{1}{3} \cdot 3x$ from which $\frac{2}{3} > x$.

Example 1.3.4 Solve the inequality $x^2 - 7x + 12 > 0$. This is a quadratic inequality.

We find the need to factor the inequality as $(x - 4)(x - 3) > 0$. We want the values of x that make $(x - 4)(x - 3)$ positive. Lets look at case 1
- (both numbers are positive). When $(x - 4) > 0$ and $(x - 3) > 0$. If $x - 4 > 0$ then $x > 4$. If $x - 3 > 0$ then $x > 3$.
We choose a solution that satisfies both inequalities. We select

$$x > 4$$

Case 2: (both numbers negative); $(x - 4) < 0$ and $(x - 3) < 0$.
If $x - 4 < 0$ then $x < 4$.
If $x - 3 < 0$ then $x < 3$.

The solution of this case is: All x such that $x < 4$ and $x < 3$. Both inequalities will be satisfied by $x < 3$

From case 1 we have $x > 4$ and from case 2 we have $x < 3$. The solution is therefore $x < 3$ or $x > 4$

Example 1.3.5 Solve

$$\frac{4}{x + 1} < 2.$$

Multiply both sides of the inequality by $(x+1)$ to eliminate the fraction. $(x+1)$ could be a positive $(x+1) > 0$, or negative that is, $(x+1) < 0$ number. We therefore consider both cases.
Case 1: $(x+1) > 0$ that is, $x > -1$.
Multiply both sides of $\frac{1}{x+1} < 2$ by a positive number $(x-1) > 0$, gives

$$\begin{aligned} 4 &< 2(x+1) \\ 4 &< 2x+2 \\ 2 &< 2x \\ 1 &< x \end{aligned}$$

The solution must satisfy both $x > -1$ and $1 < x$.
The intersection gives the solution as $x > 1$.
Case 2: $(x+1) < 0$ that is, $x < -1$.
Multiplying both sides of $\frac{4}{x+1} < 2$ by a negative number $(x+1) < 0$ gives

$$\begin{aligned} 4 &> 2(x+1) \\ 4 &> 2x+2 \\ 2 &> 2x \\ 1 &> x \end{aligned}$$

The solution must satisfy both $x < -1$ and $1 > x$.
The solution is $x < -1$. From the case 1 we have the solution $x > 1$ and in case 2 we have the solution $x < -1$. The whole solution is therefore $x > 1$ or $x < -1$.

Example 1.3.6 *Prove that if $a < b$ then $a + c < b + c$.*
$a < b$ means $b - a$ is positive. But $b - a$ may be rewritten as $(b + c) - (a + c)$. This is a positive number. Hence $a + c < b + c$.

Example 1.3.7 Solve $3 + x < 8$.

Add -3 to both sides of the inequality to obtain $x < 5$. Therefore the solution set is $\{x \in \mathbb{R} : x < 5\}$ which is the same as $(-1, 5)$.

Example 1.3.8 Solve $x^2 + x + 1 > 7$.

Add -7 to both sides to obtain

$$\begin{aligned} x^2 + x - 6 &> 0 \\ \text{i.e. } (x+3)(x-2) &> 0 \end{aligned}$$

But when is a product of two real numbers positive?

(i) $(x+3) > 0$ **and** $(x-2) > 0$. Then $x > -3$ **and** $x > 2$, implying that $x > 2$, OR

(ii) $(x+3) < 0$ **and** $(x-2) < 0$. Then $x < -3$ **and** $x < 2$, implying $x < -3$.

Therefore the solution set of the inequality is the set $(-\infty, -3) \cup (2, \infty)$.

Example 1.3.9 Solve the problem

$$\frac{x^2 - 5x + 4}{x+2} < 0$$

or

$$\frac{(x-1)(x-4)}{x+2} < 0$$

In this case there are several ways in which this inequality could be less than or equal to zero. (that is, Negative). These are:

(i) $(x-1)$ is positive; $(x-4)$ is positive while $x+2$ is negative.

(ii) $(x-1)$ is positive; $(x-4)$ is negative while $(x+2)$ is positive.

(iii) $(x-1)$ is negative; $(x+2)$ is positive while $(x+2)$ is positive.

(iv) $(x-1)$ is negative; $(x-4)$ is negative while $(x+2)$ is negative.

The solution of this type of inequality involves considering each of the cases (i) to (iv) above. The final solution is obtained by combining the results from the cases, that have solutions.

Example 1.3.10 Solve

$$\frac{(x-1)(x-4)}{x+2} < 0$$

Consider Case 1: If

$$\frac{\text{(positive) (positive)}}{\text{(negative)}}$$

Then: $(x-1) > 0$ [that is, $x > 1$] `and` $(x-4) > 0$ [that is, $x > 4$] `and` $(x+2) < 0$ [that is, $x < -2$] A number can not simultaneously be greater than 4 and less than -2. There is therefore no solution in this case.

Case 2:

$$\frac{\text{(positive) (negative)}}{\text{(positive)}}$$

Then

$$\begin{aligned}
(x-1) &> 0 \quad \text{that is, } x > 1 \;\text{ and}\\
(x-4) &< 0 \quad \text{that is, } x < 4 \;\text{ and}\\
(x+4) &> 0 \quad \text{that is, } x > -2
\end{aligned}$$

The solution is $1 < x < 4$.

Case 3: If

$$\frac{\text{(negative) (positive)}}{\text{(Positive)}}.$$

Then

$$\begin{aligned}
(x-1) &< 0 \quad \text{that is, } x < 1 \text{ and}\\
(x-4) &> 0 \quad \text{that is, } x > 4 \text{ and}\\
(x+2) &> 0 \quad \text{that is, } x > -2
\end{aligned}$$

The solution must satisfy all these inequalities $x < 1; x > 4; x > -2$. A number cannot be less than one and at the same time greater than 4. There is no solution in this case.

Case 4: If

$$\frac{\text{(negative) (negative)}}{\text{(negative)}}.$$

Then

$$\begin{aligned}
(x-1) &< 0 \quad \text{that is, } x < 1, \;\text{ and}\\
(x-4) &< 0 \quad \text{that is, } x < 4, \;\text{ and}\\
(x+2) &< 0 \quad \text{that is, } x < -2
\end{aligned}$$

The solution that satisfies all inequalities is $x < -2$.

The solution of the problem is obtained by combining the cases that produced solutions in this case, cases 2 and 4. Therefore the solution is $x < -2$ or $1 < x < 4$.

Example 1.3.11 Another group of inequalities are of the form $(x^3 + 2x^2 - 3x) > 0$ or $x(x+3)(x-1) > 0$. In this case there are several ways of producing a positive product. These could be:

1. x is positive; $(x+3)$ is positive and $(x-1)$ positive
2. x is positive; $(x+3)$ is negative and $(x-1)$ is negative

3. x is negative; $(x+3)$ is negative and $(x-1)$ is positive.
4. x is negative $(x+3)$ is positive and $(x-1)$ is negative.

Example 1.3.12 Solve

$$x(x+3)(x-1) > 0$$

Consider case 1: If (positive) (positive) (positive)
Then $x > 0$ and $(x+3) > 0$ and $(x-1) > 0$. Hence: $x > 0$ and $x > -3$ and $x > 1$. This yields the solution $x > 1$.

Case 2: If (positive) (negative) (negative)
Then $x > 0$ and $(x+3) < 0$ and $(x-1) < 0$ Hence: $x > 0$ and $x < -3$ and $x < 1$.
These inequalities have no solution.

Case 3: If (negative) (negative) (positive)
Then $x < 0$ and $(x+3) < 0$ and $(x-1) > 0$ Hence: $x < 0$ and $x > -3$ and $x > 1$
These inequalities have no solution.

Case 4: If (negative) (positive) negative)
Then $x < 0$ and $(x+3) > 0$ and $(x-1) < 0$ Hence: $x < 0$ and $x > -3$ and $x < 1$. The solution is $0 < x < 1$ Thus combining the solutions in cases 1 and 2. The solution to the problem is $x > 1$ or $-3 < x < 0$.

Example 1.3.13 Solve the inequality

$$\frac{(x-1)(x+2)}{(x-3)} < 0$$

Example 1.3.14 Solve the inequality

$$(x+2)(x-3) > 0$$

To be positive, both terms have to be positive,

$$\begin{aligned} (x+2) &\ > 0 \ \Rightarrow x > -2 \ \text{ and} \\ (x-3) &\ > 0 \ \Rightarrow x > 3 \end{aligned}$$

$$x > 3 \equiv (3, \infty)$$

"**Or**", both terms are negative

$$\begin{aligned} (x+2) &\ < 0 \ \Rightarrow x < -2 \ \text{ and} \\ (x-3) &\ < 0 \ \Rightarrow x < 3 \end{aligned}$$

$$x < -2 \equiv (-\infty, -2)$$

Thus the general solution from both options is

$$(-\infty, -2) \cup (3, \infty)$$

Example 1.3.15 Solve the inequality: $(x-1)(x-2)(x-5) \leq 0$
If one is negative and two are positive, or all negative. If have $\leq$ stick to $\leq$, and if $<$, stick to $<$ signs.

$$(-\infty, 1] \cup [2, 5]$$

Exercise 1.1 Solve for x.

1. $2x + 3 < 9$
2. $-4 \leq 2(x+2) < 10$
3. $(x-3)(x+1) < 0$
4. $(x+7)(2x-4) > 0$
5. $x^2 + x < 0$
6. $x^2 + x + 7 > 19$
7. $x^4 - 9x^2 < 0$

1.4 Absolute values in calculus

The absolute value of a real number x is denoted by $|x|$ and is given by

$$|x| = \begin{cases} x\ , & x \geq 0 \\ -x\ , & x < 0 \end{cases}$$

The absolute value of a real number x is a measure of how far the real number x is from 0, the origin of the number line. For this reason it is always a positive quantity and sometimes it is referred to as the magnitude of the number. Alternatively, the absolute value of x may be defined as $|x| = \sqrt{x^2}$. (Note that $\sqrt{x}$ stands for the positive square root of x). The distance between two real numbers a and b is the number $|a-b| = |b-a|$.

Example 1.4.1 Simplify the norm (abolute value) $|x-5|$

$$|x-5| = \begin{cases} (x-5)\ , & x \geq 5 \\ \\ -(x-5)\ , & x < 5 \end{cases}$$

Example 1.4.2 Solve the inequality $|x+3| = 4$. We know that

$$|x+3| = \begin{cases} (x+3)\ , & x \geq -3 \\ \\ -(x+3)\ , & x < -3 \end{cases}$$

$x + 3 = 4, \Rightarrow\ x = 1$ or $-(x+3) = 4, \Rightarrow\ x = -7$

Theorem 1.4.1 *The inequality $|x| \leq a$ is equivalent to the double inequality $-a \leq x \leq a$. The inequality $|y| \geq b$ holds for $y \leq -b$ or $y \geq b$.*

Theorem 1.4.2 *For any real numbers x and y,*

(a) $|x| \geq 0$,

(b) $|xy| = |x||y|$,

(c) $|x+y| \leq |x| + |y|$.

Note 1.4.1 Think of absolute as in two parts.

Example 1.4.3 Solve the inequality

$$|x+2| < 3$$

Method I:

$$\begin{aligned} |x+2| < 3 &\Rightarrow -3 < x+2 < 3 \\ x+2 < 3 &\Rightarrow x < 1 \\ x+2 > -3 &\Rightarrow x > -5 \end{aligned}$$

$$x \in (-5, 1)$$

Method II: Splitting the absolute into two, this is the more general one, and more reliable.

$$|x+2| < 3$$

$$\begin{array}{ccccc} (x+2) & & 3 & : & x \geq -2 \\ & < & & & \\ -(x+2) & & 3 & : & x < -2 \end{array}$$

$$\begin{aligned} (x+2) < 3 &\Rightarrow x < 1 \\ -(x+2) < 3 &\Rightarrow x > -5 \end{aligned}$$

$$x \in (-5, 1)$$

Example 1.4.4 Solve the inequality

$$|2x+3| \geq 8$$

Splitting the absolute into two for $|x+2| \geq 8$

$$\begin{array}{ccccc} (2x+3) & & 8 & : & x \geq -\dfrac{3}{2} \\ & \geq & & & \\ -(2x+3) & & 8 & : & x < -\dfrac{3}{2} \end{array}$$

$$(2x+3) \geq 8 \quad \Rightarrow \quad x \geq \frac{5}{2}$$

$$-(2x+3) \geq 8 \quad \Rightarrow \quad x \leq -\frac{11}{2}$$

$$\left(-\infty, -\frac{11}{2}\right] \cup \left[\frac{5}{2}, \infty\right)$$

Example 1.4.5 Solve the inequality

$$|-2x-4| \geq 9$$

$$\left(-\infty, -\frac{13}{2}\right] \cup \left[\frac{5}{2}, \infty\right)$$

Example 1.4.6 Solve the inequality

$$x+2 < |x^2-4|$$

This can be rewritten as $|x^2-4| > x+2$
Splitting the absolute into two for $|x^2-4| > x+2$

$$\begin{array}{rcccl} (x^2-4) & & x+2 & : & x \in (-\infty, -2] \cup [2, \infty) \\ & > & & & \\ -(x^2-4) & & x+2 & : & x \in (-2, 2) \end{array}$$

$$\begin{array}{rcccccl} (x^2-4) > x+2 & \Rightarrow & x^2-x-6 > 0 & \Rightarrow & (x+2)(x-3) > 0 & \Rightarrow & (-\infty, -2) \cup (3, \infty) \\ -(x^2-4) > x+2 & \Rightarrow & x^2+x-2 < 0 & \Rightarrow & (x+1)(x-2) < 0 & \Rightarrow & (-1, 2) \end{array}$$

$$[(-\infty, -2] \cup [2, \infty) \cap (-\infty, -3) \cup (3, \infty)] \quad \text{or} \quad [(-2, 2) \cap (-1, 2)]$$
$$\Rightarrow \quad (-\infty, -2) \cup (-1, 2) \cup (3, \infty)$$

Example 1.4.7 Solve the equation

$$4|2x-1| - 2 = 10$$

$$\begin{aligned} 4|2x-1| - 2 &= 10 \\ 4|2x-1| &= 12 \\ |2x-1| &= 3 \end{aligned}$$

Splitting the absolute into two for $|2x-1|=3$

$$\begin{matrix} (2x-1) & \multirow{2}{*}{=} & 3 & : & x \geq \frac{1}{2} \\ -(2x-1) & & 3 & : & x < \frac{1}{2} \end{matrix}$$

$$\begin{aligned} (2x-1) = 3 \quad &\Rightarrow \quad x = 2 \\ -(2x-1) = 3 \quad &\Rightarrow \quad x = -1 \end{aligned}$$

$$x=-1,\ x=2$$

Example 1.4.8 Solve for the inequality

$$|3x+1| < 2|x-6|$$

Whenever we have absolutes, we represent it as into two parts,

$$\begin{matrix} (3x+1)_{x\geq -\frac{1}{3}} & \multirow{2}{*}{<} & 2(x-6)_{x\geq 6} \\ -(3x+1)_{x<-\frac{1}{3}} & & -2(x-6)_{x<6} \end{matrix}$$

Taking all possible combinations of the terms,

$$\begin{aligned} [(3x+1)] < [2(x-6)] \quad &\Rightarrow x < -13 : \text{ for region } x \geq 6 \ \Rightarrow \ \text{no solution} \\ [(3x+1)] < [-2(x-6)] \quad &\Rightarrow \ x < \frac{11}{5} \ : \ \text{in region } x \in [-\frac{1}{3}, 6) \ \Rightarrow \ x \in \left[-\frac{1}{3}, \frac{11}{5}\right) \\ [-(3x+1)] < [2(x-6)] \quad &\Rightarrow \ x > \frac{11}{5} \ : \ \text{in no region} \ \Rightarrow \ \text{no solution} \\ [-(3x+1)] < [-2(x-6)] \quad &\Rightarrow x > -13 : (\text{for region}) \ x < -\frac{1}{3} \ \Rightarrow \ x \in \left(-13, -\frac{1}{3}\right) \end{aligned}$$

$$x \in \left[-\frac{1}{3}, \frac{11}{5}\right) \cup \left(-13, -\frac{1}{3}\right)$$

Example 1.4.9 Solve the equation

$$|x-2| + |x+5| \leq 0$$

Splitting the absolutes into two for;

$$\begin{matrix} (x-2)_{x\geq 2} & & (x+5)_{x\geq -5} & \\ & + & & \leq 0 \\ -(x-2)_{x<2} & & -(x+5)_{x<-5} & \end{matrix}$$

Taking all possible combinations of the terms,

$$\begin{aligned}
[(x-2)]+[(x+5)] &\leq 0 \;\Rightarrow x \leq -\frac{3}{2} : \text{ in region } x \geq 2 \;\Rightarrow\; \text{no solution} \\
[(x-2)]+[-(x+5)] &\leq 0 \;\Rightarrow\; \text{no solution } : \text{ in no region } \Rightarrow\; \text{no solution} \\
[-(x-2)]+[(x+5)] &\leq 0 \;\Rightarrow\; \text{no solution } : \text{ in region } x \in [-5,2) \;\Rightarrow\; \text{no solution} \\
[-(x-2)]+[-(x+5)] &\leq 0 \;\Rightarrow x \geq -\frac{3}{2} : \text{ in region (for) } x < -5 \;\Rightarrow\; \text{no solution}
\end{aligned}$$

No solution for the problem above.

Example 1.4.10 Solve the equation

$$|2x-2| = x+1$$

$$x = 3,\; x = \frac{1}{3}$$

Example 1.4.11 Solve for$|x-5| \leq 2$

$$|x-5| = \begin{cases} (x-5)\ , & x \geq 5 \\ -(x-5)\ , & x < 5 \end{cases}$$

(i) $(x-5) \leq 2$ and $x \geq 5$, $\quad\Rightarrow\quad x \leq 7$ and $x \geq 5$, $\;\Rightarrow\quad 5 \leq x \leq 7$

(ii) $-(x-5) \leq 2$ and $x < 5$, $\quad\Rightarrow\quad x \geq 3$ and $x < 5$, $\;\Rightarrow\quad 3 \leq x \leq 4$

(iii) Form (i) and also (ii) (the union), we realise $3 \leq x \leq 7$

Alternatively, $|x-5| \leq 2 \Rightarrow \quad -2 \leq (x-5) \leq 2 \;\Rightarrow\; 3 \leq x \leq 7$ by adding a 5 everywhere

Example 1.4.12 Solve for$|x+3| \leq 4$

$$|x+3| = \begin{cases} (x+3)\ , & x \geq -3 \\ -(x+3)\ , & x < -3 \end{cases}$$

(i) $(x+3) \leq 4$ and $x \geq -3, \quad \Rightarrow \quad x \leq 1$ and $x \geq -3, \; \Rightarrow \quad -3 \leq x \leq 1$

(ii) $-(x+3) \leq 4$ and $x < -3, \quad \Rightarrow \quad x \geq -7$ and $x < -3, \; \Rightarrow \quad -7 \leq x < -3$

(iii) Form (i) and also (ii) (the union), we realise $-7 \leq x \leq 1$

Alternatively, $|x+3| \leq 4 \quad \Rightarrow \quad -4 \leq (x+3) \leq 4$

$$-7 \leq x \leq 1$$

Example 1.4.13 Solve $|3-2x| \geq 1$.

$$|3-2x| = \begin{cases} (3-2x)\ , & x \leq \frac{3}{2} \\ \\ -(3-2x)\ , & x > \frac{3}{2} \end{cases}$$

(i) $(3-2x) \geq 1$ and $x \leq \frac{3}{2}, \quad \Rightarrow \quad x \leq 1$ and $x \leq \frac{3}{2}, \; \Rightarrow \quad x \leq 1$

(ii) $-(3-2x) \geq 1$ and $x < \frac{3}{2}, \quad \Rightarrow \quad x \geq 2$ and $x > \frac{3}{2}, \quad \Rightarrow \quad x \geq 2$

(iii) Form (i) and also (ii) (the union), we realise $x \in (-\infty, 1] \cup [2, \infty)$

Alternatively, Either $3-2x \geq 1$ or $3-2x \leq -1$. That is, either $x \leq 1$ or $x \geq 2$.

$$x \in (-\infty, 1] \cup [2, \infty)$$

Example 1.4.14 Solve $|2x+3| < 6$

$$-\frac{9}{2} < x < \frac{3}{2}$$

Example 1.4.15 Solve $|2x-3| > 5$.

$$x < -1 \text{ or } x > 4$$

Example 1.4.16 Find the absolute-value inequality statement that corresponds to the inequality

$$-2 < x < 4$$

I first look at the endpoints. Negative two and four are six units apart. Half of six is three. So I want to adjust this inequality so that it relates to -3 and 3, instead of to -2 and 4. To accomplish this, I will adjust the ends by subtracting 1 from all three "sides":

$$\begin{aligned} -2 &< x < 4 \\ -2-1 &< x-1 < 4-1 \\ -3 &< x-1 < 3 \end{aligned}$$

Since the last line above is in the "less than" format, the absolute-value inequality will be of the form "absolute value of something is less than 3". I can convert this nicely to

$$|x-1| < 3$$

Example 1.4.17 Find the absolute-value inequality statement that corresponds to the inequalities

$$x < 19 \text{ or } x > 24$$

I first look at the endpoints. Nineteen and 24 are five units apart. Half of five is 2.5. So I want to adjust the inequality so it relates to −2.5 and 2.5, instead of relating to 19 and 24. Since $19 - (-2.5) = 21.5$ and $24 - 2.5 = 21.5$, I need to subtract 21.5 all around:

$$\begin{aligned} x < 19 &\text{ or } x > 24 \\ x - 21.5 < 19 - 21.5 &\text{ or } x - 21.5 > 24 - 21.5 \\ x - 21.5 < -2.5 &\text{ or } x - 21.5 > 2.5 \end{aligned}$$

Since the last line above is the "greater than" format, the absolute-value inequality will be of the form "absolute value of something is greater than or equal to 2.5". I can convert this nicely to:

$$|x - 21.5| > 2.5$$

Exercise 1.2 Solve for x in the following:

1. $2x + 7 > 4x - 5$
2. $-4 \leq 2(x + 2) < 12$
3. $(x + 8)(4x - 6) > 0$
4. $x^2 + x < 0$
5. $|x - 8| = 2$
6. $|x + 4| - |x - 1| < 4$
7. $|x + 2| + |x - 5| \geq 10.$

1.4.1 Sketching absolute functions

Example 1.4.18 Graph the curve

$$f(x) = \frac{|x + 2|x^2}{|x|}$$

Now its not only one point to think of, but now both $x = -2$ and $x = 0$. We need to have different functions for

$$\begin{aligned} x &< -2 \\ -2 \leq x &\leq 0 \\ x &> 2 \end{aligned}$$

$$f(x) = \begin{cases} \frac{-(x+2)x^2}{-(x)}, & \text{if } x < -2 \\ \frac{(x+2)x^2}{-(x)}, & \text{if } -2 \leq x \leq 0 \\ \frac{(x+2)x^2}{(x)}, & \text{if } x > 0 \end{cases}$$

and sketch those function in the different ranges.

Example 1.4.19 Sketch the graph

$$f(x) = |x-2||x-4|$$

Example 1.4.20 Graph the function

$$f(x) = \frac{x\left(|4x-3| - |x+6| + |x|\right)}{3|x|}$$

Hence or otherwise, solve the inequality

$$\frac{x\left(|4x-3| - |x+6| + |x|\right)}{3|x|} > 0$$

Example 1.4.21 Let

$$g(x) = \begin{cases} x+2\ , & x \geq 0 \\ x-2\ , & x < 0 \end{cases}$$

Sketch the graph of $g(x)$.

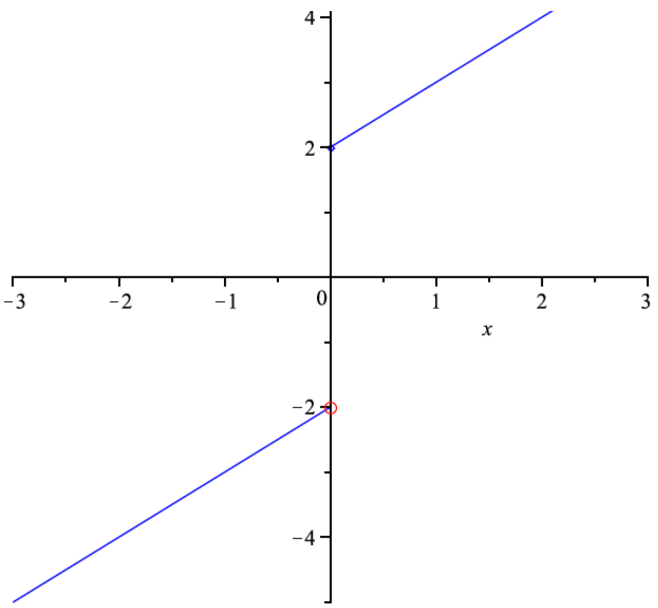

Figure 1.1: A piecewise function sketch

Does $\lim_{x \to 0} g(x)$ exist? Justify your answer.

1.5 Functions

Definition 1.5.1 *A function is a set of ordered pairs of elements such that not two ordered pairs of a set have the same first element.*

Functions can be considered as mappings. A function is a special kind of relation in which all ordered pairs have unique first elements. Every element of the first set is assigned to an element of the second set. Usually subsets of the set of real numbers $\mathbb{R}$ are considered.

A function involves two sets and a rule of correspondence between them. The rule of correspondence specifies how to pair the elements of the one set with those in the other.

A function is a relation in which for each member of the first set there is a single corresponding member of the second set. A rule exists between two quantities.

A function has to satisfy The Vertical Line Test.

1.5.1 Domain and Range as a set of Ordered Pairs.

The domain (D) is the set of all values that the first elements, in the ordered pair can take. The Range (R) is the set of all values that the second elements, in the ordered pair, can take.

Example 1.5.1 The following table defines a function.

x	-3	5	6	7
y	-2	8	9	5

(a) List the ordered pairs of the function

(b) State the domain of the function

(c) State the range of the function

(a) The ordered pairs of the function are: $(-3,-2);(5,8);(6,9)\text{and}(7,5)$.

(b) The Domain are the set of the first elements of the ordered pairs. $D=\{-3,5,6,7\}$

(c) The Range are the set of the second elements of the ordered pairs. $R=\{-2,8,9,5\}$

Example 1.5.2 State the domain and range of the function $f=\{(-2,4)(-1,4)(0,4)(2,4)\}$

The Domain $D=\{-2,-1,0,2\}$, are the first elements in the ordered pairs. The Range $R=\{4\}$, is the second element in the ordered pairs.

Example 1.5.3 A mapping defined by

$$\{(-1,2),\quad (4,6),\quad (5,0),\quad (4,3)\}$$

is not a function, since the first element 4 appears twice (more than once).

Note 1.5.1 A function can be given in the form of ordered pairs such as

$$\{(-1,1),\quad (4,11),\quad (5,13),\quad (1,5)\}$$

or by a formula i.e $f(x) = 2x + 3$

Definition 1.5.2 The domain of a function is the set of "input" or argument values for which the function is defined. That is, the function provides an "output" or "value" for each member of the domain.

Definition 1.5.3 The range is the codomain or the image of the function

Example 1.5.4 Determine the domain and range of the given function:

$$y = \frac{(x^2 + x - 2)}{(x^2 - x - 2)}$$

The domain is all the values that x is allowed to take on. The only problem I have with this function is that I need to be careful not to divide by zero. So the only values that x can not take on are those which would cause division by zero. So I'll set the denominator equal to zero and solve; my domain will be everything else.

$$\begin{aligned} x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x = 2 \text{ or } x = -1 & \end{aligned}$$

Then the domain is "all x not equal to -1 or 2".

Since the graph will eventually cover all possible values of y, then the range is "all real numbers".

Example 1.5.5 Determine the domain and range of the given function:

$$y = -\sqrt{-2x + 3}$$

The domain is all values that x can take on. The only problem I have with this function is that I cannot have a negative inside the square root. So I'll set the insides greater-than-or-equal-to zero, and solve. The result will be my domain:

$$\begin{aligned} -2x + 3 &\geq 0 \\ -2x &\geq -3 \\ 2x &\leq 3 \\ x &\leq \frac{3}{2} = 1.5 \end{aligned}$$

Then the domain is "all $x \leq \frac{3}{2}$".

The range requires a graph. I need to be careful when graphing radicals:The range is "$y < 0$".

Example 1.5.6 Determine the domain and range of the given function:

$$y = -x^4 + 4$$

This is just a garden-variety polynomial. There are no denominators (so no division-by-zero problems) and no radicals (so no square-root-of-a-negative problems). There are no problems with a polynomial. There are no values that I can't plug in for x. When I have a polynomial, the answer is always that the domain is "all x".

The range will vary from polynomial to polynomial, and they probably won't even ask, but when they do, I look at the picture: The graph goes only as high as y = 4, but it will go as high as I like. Then: The range is "all $y \geq 4$".

Example 1.5.7 Find the domain of function f defined by

$$f(x) = \frac{1}{(x-1)}$$

The domain x is such that $x \neq 1$, that is

$$x \in (-\infty, 1) \cup (1, +\infty)$$

Example 1.5.8 what is the domain of the function

$$f(x) = \sqrt{x^2 - 1} \quad ?$$

Well, what could go wrong here? No division is indicated at all, so there is no risk of dividing by 0. But we are taking a square root, so we must insist that $x^2 - 1 \geq 0$ to avoid having complex numbers come up. That is, a preliminary description of the 'domain' of this function is that it is the set of real numbers x so that $x^2 - 1 \geq 0$.

But we can be clearer than this: we know how to solve such inequalities. Often it's simplest to see what to *exclude* rather than *include*: here we want to *exclude* from the domain any numbers x so that $x^2 - 1 < 0$ from the domain.

We recognize that we can factor

$$x^2 - 1 = (x-1)(x+1) = (x-1)\,(x-(-1))$$

This is negative exactly on the interval $(-1, 1)$, so this is the interval we must prohibit in order to have just the domain of the function. That is, the domain is the union of two intervals:

$$(-\infty, -1] \cup [1, +\infty)$$

Exercise 1.3 Find the domain of the function

$$f(x) = \frac{x-2}{x^2+x-2}$$

That is, find the largest subset of the real line on which this formula can be evaluated meaningfully.

Exercise 1.4 Find the domain of the function

$$f(x) = \frac{x-2}{\sqrt{x^2+x-2}}$$

Exercise 1.5 Find the domain of the function

$$f(x) = \sqrt{x(x-1)(x+1)}$$

Example 1.5.9 Compute the domain of the fuction

$$|x-3.2| + |x-5.2| = 2$$

Note 1.5.2 The domain is where the function lives (are the x values), and the range is what the function can be (is the $f(x)$). To get domain, make sure the denominator is not equal to zero, and cannot have a square root of a negative number.

Example 1.5.10 Find the domain and range of the following function

(i)

$$f(x) = \frac{3}{x^2-1}$$

$$f(x) = \frac{3}{x^2-1} = \frac{3}{(x-1)(x+1)}$$

so function defined every where otherthan at $x = 1$ and $x = -1$. Thus the domain, $D_f = \mathbb{R} - \{-1, 1\}$, the domain are all possible solutions, which is all numbers, i.e $R_f = \mathbb{R}$

(ii) $f(x) = \sqrt{x-4}$, We know that we want only positive entries in the square root sign, that is, $x - 4 \geq 0$, meaning the domain is only values of x, such that $x \geq 4$

(iii) Find the domain of

$$f(x) = \sqrt{x^2-1}$$

$$(-\infty, -1) \cup (1, +\infty)$$

(iv)

$$f(x) = \sqrt{\frac{2x}{x+2}}$$

This is just a mix of the two conditions, make sure you dont have zero below, and no negative numbers in the square root. That is $x \neq -2$, and $\frac{2x}{x-2} \geq 0$, $x \geq -2$

Example 1.5.11 Find the domain and range of the function

$$f(x) = \sqrt{\frac{x(x-2)}{(4-3x)(6-2x)}}$$

The domain is

$$x \neq \frac{4}{3},\ x \neq 3,\ \frac{x(x-2)}{(4-3x)(6-2x)} \geq 0$$

The range is $\mathbb{R}$

Example 1.5.12 Find the range of values of the given function

$$f(x) = \frac{x}{|x|}$$

if $x \neq 0$ while $f(0) = 0$

Example 1.5.13 Let

$$\begin{aligned} f(x) &= \frac{7x}{x^2-16} \\ g(x) &= \sqrt{x} \end{aligned}$$

Find $(fog)(x)$ and give its domain.

$$fog(x) = \frac{7\sqrt{x}}{x-16}\ ,\quad \text{Domain } := \mathbb{R}^+ - 16$$

1.5.2 Function Notation

The notation $f(x)$ represents the second element in the ordered pair that has x as its first element. This ordered pair can be represented as $(x, f(x))$. We read $f(x)$ as "f of x" or "f at x" since $f(x)$ gives the value of f at x. This relation can also be represented as:

$$x \Rightarrow f(x)$$

Often we replace $f(x)$ by y, also called the dependent variable, and write $y = f(x)$. With this notation, x is called the independent variable. For example $f(x) = 3x + 2$ may be written as $y = 3x + 2$.

1.5.3 Evaluation of Functions

The evaluation of a function is illustrated by examples. A rule describing a function can be represented by an equation or formula, a table or a graph, which are recorded as ordered pairs. The evaluation of a function is to find a number that is paired with a given number.

Example 1.5.14 To evaluate the function $f(x) = 3x + 2$ at $x = 2$, we need the number $f(2)$. Since $f(x) = 3x + 2$, replace x by 2 in the expression for the function and simplify. Thus $f(2) = 3 \times 2 + 2 = 6 + 2 = 8$. Therefore the ordered pair is $(2, 8)$. We write $f(2) = 8$ or $2 \to 8$. This is the y value in the expression by $y = 3x + 2$ when $x = 2$.

Example 1.5.15 Let $f(x) = x^2 + 6x + 2$ Find the value of

(a) $f(0)$ (c) $f(1)$ (e) $f(n-2)$

(b) $f(-2)$ (d) $f(x+1)$

(a) $f(0) = 0^2 + 6(0) + 2 = 2$. Thus $0 \to 2$ or as an ordered pair, $(0, 2)$.

(b) $f(-2) = (-2)^2 + 6(-2) + 2 = -6$. Thus $-2 \to -6$ or as an ordered pair, $(-2, -6)$.

(c) $f(1) = 1^2 + 6(1) + 2 = 9$. Thus $1 \to 9$ or as an ordered pair, $(1, 9)$.

(d)

$$\begin{aligned} f(x+1) &= (x+1)^2 + 6(x+1) + 2 \\ &= (x^2 + 2x + 1 + 6x + 6 + 2) \\ &= x^2 + 8x + 9. \end{aligned}$$

Thus $(x + 1) \to x^2 + 8x + 9$ or as an ordered pair, $((x + 1),\ (x^2 + 8x + 9))$.

(e)

$$\begin{aligned} f(n-2) &= (n-2)^2 + 6(n-2) + 2 \\ &= n^2 - 4n + 4 + 6n - 12 + 2 \\ &= n^2 + 2n - 6 \text{ or as an ordered pair, } ((n-2), (n2 + 2n - 6) \end{aligned}$$

Thus $(n-2) \to n^2 + 2n - 6$

Example 1.5.16 Let $f(x) = 2x^2 - 1$. Let $h > 0$. Find the value of $\frac{f(x+h)-f(x)}{h}$.

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 1 \\ &= 2(x^2 + 2hx + h^2) - 1 \\ &= 2x^2 + 2hx + 2h^2 - 1 \\ f(x+h) - f(x) &= (2x^2 + 4hx + 2h^2 - 1) - (2x^2 - 1) \\ &= 4hx + 2h^2 \end{aligned}$$

$$\text{Therefore,} \quad \begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{4hx + 2h^2}{h} \\ &= 4x + 2h \end{aligned}$$

Example 1.5.17 Given the function $f(x) = 2x + 3$ and $a = 2$. Compute and simplify the value of $\frac{f(x)-f(a)}{x-a}$, with $x \neq a$

Since $f(x) = 2x + 3$ and $a = 2, f(a) = f(2) = 7$. In addition $(x - a) = (x - 2)$ and therefore

$$\frac{f(x) - f(a)}{x - a} = \frac{2x + 3 - 7}{x - 2} = \frac{2x - 4}{x - 2} = 2\frac{(x-2)}{(x-2)} = 2$$

1.6 Types of functions

1.6.1 Equal Functions

Let $f_1(x)$ and $f_2(x)$ be functions that are defined on the same domain, D. If for each element x of D, $f_1(x) = f_2(x)$, then the two functions are equal and we write $f_1(x) = f_2(x)$. Two functions $f_1(x)$ and $f_2(x)$ are equal if and only if $f_1(x)$ and $f_2(x)$ have the same domains and $f_1(x) = f_2(x)$ for all x in this common domain.

Example 1.6.1 Prove that the functions:

(a) $f_1(x) = 2x + 1$ and $f_2(x) = \frac{4x}{2} + 1$ are equal, where the domain for both functions is the set of real numbers R.

(b) $f_1(x) = \sin x$ and $f_2(x) = \cos\left(\frac{\pi}{2} - x\right)$ are equal, where the domain of for both functions is the set of real numbers.

(a) Let a be a real number. Then $f_1(a) = 2a+1 = \frac{4}{2}a+1 = \frac{4a}{2}+1 = f_2(a)$. Since a was arbitrary, $f_1(x) = f_2(x)$for all real numbers. The two functions are therefore equal.

(b) Let a be a real number. Then $f_2(a) = \cos\left(\frac{\pi}{2} - a\right) = \cos\left(\frac{\pi}{2}\right)\cos a + \sin\left(\frac{\pi}{2}\right)\sin a = 0 \times \cos a + 1 \times \sin a = \sin a$. Here we have used the fact that $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$.

1.6.2 Identity Function

A function f that associates each member of the domain with itself is called the identity function. The identity function is defined by the equation $f : x \to x$ or $f(x) = x$. The domain and the range of the identity function is the set of real numbers.

The identity function is a special kind of one-to-one and onto function.

1.6.3 Constant Function

A function f that associates each real number in a set A the same fixed number k in the set B is called a constant function. The constant function is defined by the formula $f : x \to k : \quad f(x) = k$(a single fixed number).

1.6.4 One-to-One Function

If f is a function of A into B is said to be one-to-one $(1-1)$ if no two elements of A correspond to the same element in B. Each element of the domain has a different image in the Range.

A function is one-to-one if and only if each element in the domain is mapped into a unique element of the co-domain (range).

A one to one function is a function in which every element in the range of the function corresponds with one and only one element in the domain.

For a $1-1$ function, If $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

Example 1.6.2 If $f(a) = f(b)$ implies that $a = b$, then f is $1-1$, show whether or not $g(x) = 3x - 2$ is one-on-one?

$$\begin{aligned}
\text{see if } g(a) &= g(b) \text{ implies } a = b \\
3a - 2 &= 3b - 2 \\
3a &= 3b \\
a &= b
\end{aligned}$$

Thus g is $1-1$.

Example 1.6.3 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is injective.

Example 1.6.4 The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is not injective, because (for example) $g(1) = 1 = g(-1)$. However, if g is redefined so that its domain is the non-negative real numbers $[0, +\infty)$, then g is injective.

A one-to-one function is one in which each x has only one y and each y has at most one x to form ordered pairs.

Example 1.6.5 Let the function $f : R \to R$ be defined by the equation $f(x) = 3x + 2$ where R is the set of real numbers. Each real number will be mapped onto a unique image by the function $f(x) = 3x + 2$. Hence f is a one-to-one function.

Example 1.6.6 Let the function $f : R \to R$ be defined by the formula $f(x) = 3x^2 + 2$ Verify whether or not f is a one-to-one function.

The negative values of R are mapped onto the same elements as the corresponding positive elements. For example, when $x = -2, f(-2) = 14$ and $f(2) = 14$. The images of two real numbers -2 and 2 are the same number equal to 14. It follows that f is NOT a one-to-one function.

Definition 1.6.1 A one to one function is also called an injective function.

Example 1.6.7 Which functions below are one to one ?

Function #1 $\{(2, 27), (3, 28), (4, 29), (5, 30)\}$

Function #2 $\{(11, 14), (12, 14), (16, 7), (18, 13)\}$

Function #3 $\{(3, 12), (4, 13), (6, 14), (8, 1)\}$

Relation #1 and Relation #3 are both one-to-one functions.

Example 1.6.8 A diagramatical example of $1 - 1$

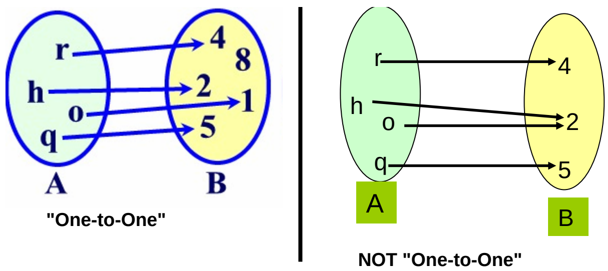

Figure 1.2: 1-1 and not 1-1 respectively

Theorem 1.6.1 The Horizontal Line Test: If a function is one to one, then the function not only passes the vertical line test, but it also passes the horizontal line test.

Definition 1.6.2 The Horizontal Line Test : If a horizontal line only intersects with the graph of a function once, then this function is one-to-one. If a horizontal line intersects the graph of the function more than once, then this function is not one to one.

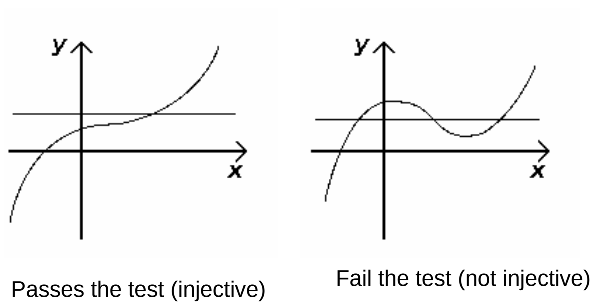

Figure 1.3: A 1-1 and not 1-1 by Horizontal test

Example 1.6.9 Which functions below are one to one ?

Function #1 $\{(2, 1), (4, 5), (6, 7), (8, 9)\}$, Function #2 $\{(3, 4), (8, 5), (6, 7), (22, 4)\}$

Function #3 $\{(-3, 4), (21, -5), (0, 0), (8, 9)\}$, Function #4 $\{(9, 19), (34, 5), (6, 17), (8, 19)\}$

Relation #1 and Relation #3 are both one-to-one functions.

1.6.5 Many-to-One Function

If f is a relation of A into B. If more than one element of A is mapped into the same element of B then f is a many-to-one function.

1.6.6 Onto-Functions

If f is a function of A into B. Sometimes the range, $f(A)$ does not exhaust all the elements of the set B called the co-domain. The range is therefore a subset of the co-domain. If each member of the co-domain is an image of at least one member of A then f is a function of A ONTO B and is therefore an onto function.

Definition 1.6.3 A function f from A to B is called *onto* if for all b in B there is an a in A such that $f(a) = b$. All elements in B are used. Such functions are referred to as **surjective**.

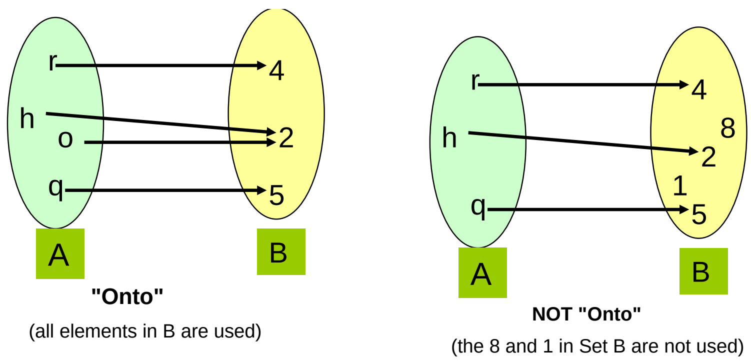

Figure 1.4: An "onto" and "Not onto" functions

A function is said to be onto if all in range is an image (is a result of the transformation - mapping)

Example 1.6.10 Is $f(x) = 3x - 4$ onto where $f : \mathbb{R} \to \mathbb{R}$?

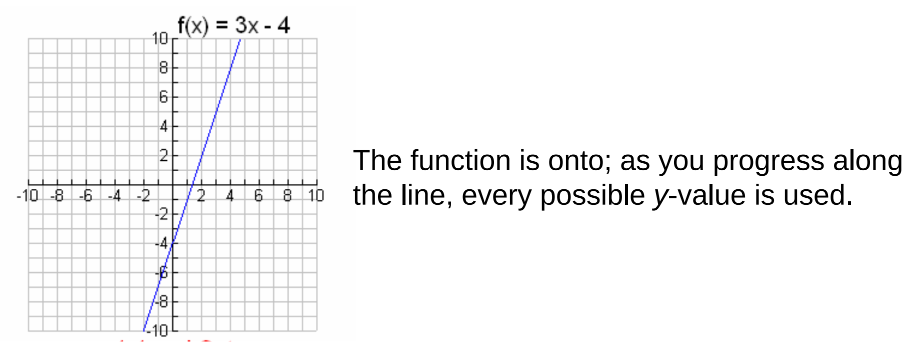

Figure 1.5: An onto function

Example 1.6.11 Is $g(x) = x^2 - 2$ onto where $g : \mathbb{R} \to \mathbb{R}$??

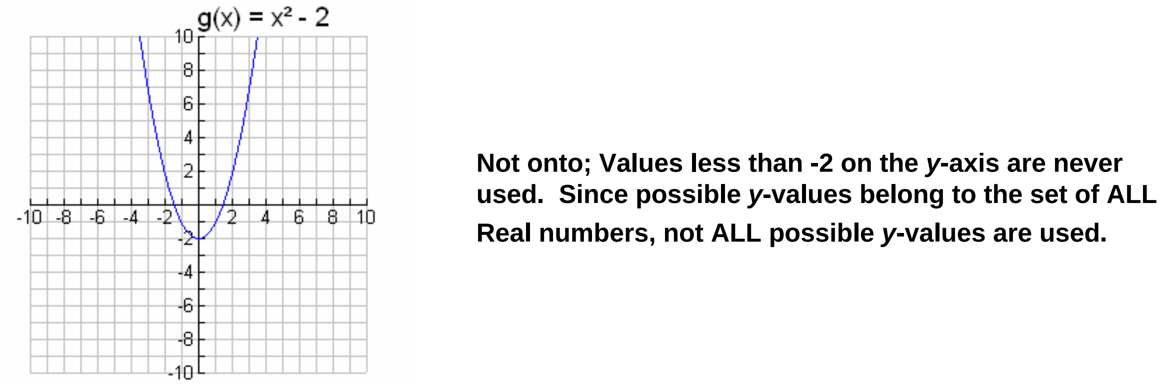

Figure 1.6: A Not onto function

1.6.7 Bijective Function

A Bijective, is a function that is both $1-1$ and onto.

Example 1.6.12 Example (1.6.10) is $1-1$ and Surjective, thus Bijective.

Example 1.6.13 Example (1.6.11) is not $1-1$ and not "onto"

Definition 1.6.4 Bijections are functions that are both injective and surjective.

Example 1.6.14 Let $f : [0,\infty) \to [0,\infty)$ be defined by $f(x) = \sqrt{x}$. This function is an injection and a surjection and so it is also a bijection.

Example 1.6.15 Describe the four functions below as injective, onto or bijective.

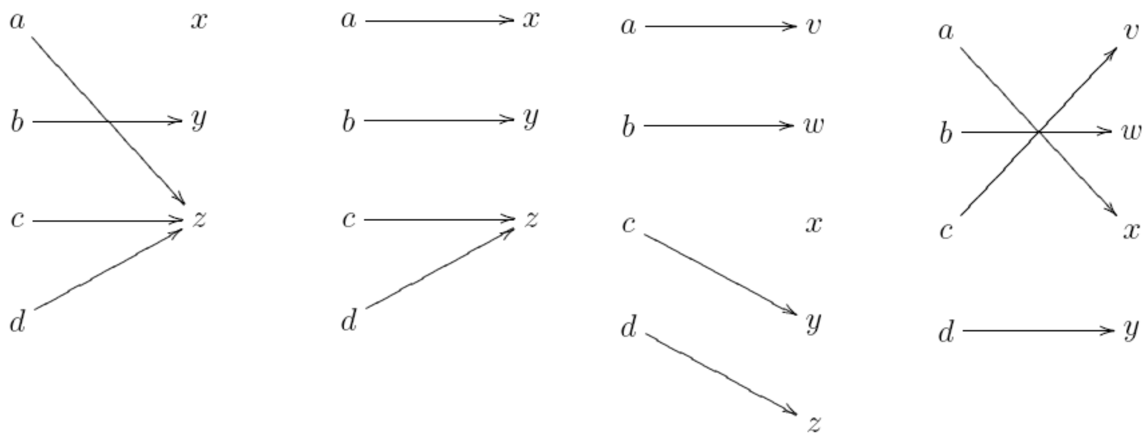

Figure 1.7: Mixed functions

(i) the function is neither injective nor surjective.

(ii) is a surjection, but not an injection

(iii) is an injection, but not a surjection.

(iv) is both a surjection and an injection, and therefore a bijection.

Example 1.6.16 Prove that the function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$ is injective.

Proof: Let $a, b \in \mathbb{N}$ be such that $f(a) = f(b)$. This implies $a^2 = b^2$ by the definition of f. Thus $a = b$ or $a = -b$. Since the domain of f is the set of natural numbers (positive integers), both a and b must be non-negative. Thus $a = b$. This shows

$$\forall a, \forall b,\ [f(a) = f(b) \Rightarrow a = b]$$

which shows f is injective.

Example 1.6.17 Prove that the function $g : \mathbb{N} \to \mathbb{N}$, defined by $g(n) = \frac{n}{3}$, is surjective.

Proof: Let $n \in \mathbb{N}$. Notice that $g(3n) = \frac{3n}{3} = n$.
Since $3n \in \mathbb{N}$, this shows n is in the range of g. Hence g is surjective.

Example 1.6.18 Prove that the function $g : \mathbb{N} \to \mathbb{N}$, defined by $g(n) = 0$, is not injective.

Proof: The numbers 1 and 2 are in the domain of g and are not equal, but $g(1) = g(2) = 0$. Thus g is not injective.

Example 1.6.19 Prove that the function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$, is not surjective.

Proof: The number 3 is an element of the codomain, $\mathbb{N}$. However, 3 is not the square of any integer. Therefore, there is no element of the domain that maps to the number 3, so f is not surjective.

1.6.8 Even Functions

A function f is called an even function if it maps the negative of x in the domain to the same image as x. This can be defined as $f(-x) = f(x)$ for all x in the domain.

Example 1.6.20 The function $f(x) = x^2 + 2$ is even since

$$f(-x) = (-x)^2 + 2 = x^2 + 2 = f(x)$$

1.6.9 Odd Functions

A function f is called an Odd function if it maps the negative of x to the negative of $f(x)$ for all x in the domain. That is, if $f(-x) = -f(x)$.

Example 1.6.21 The function $f(x) = x^3 + x$ is odd since

$$f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$$

Example 1.6.22 Determine whether the function $f(x) = x^2 - x$ is even, odd or neither.

$$\begin{aligned} f(-x) &= (-x)^2 - (-x) \\ &= x^2 + x \end{aligned}$$

$f(-x)$ is neither equal to $f(x)$ nor equal to $-f(x)$. Therefore the function f is neither even nor odd.

Example 1.6.23 The function $f(x) = x^2$ is even, $g(x) = x^3$ is an odd function but $h(x) = x + 1$ is neither odd nor even.

1.6.10 Inverse of functions

An inverse of a relation that satisfies the definition of a function is called an inverse function.

The inverse of a function f is denoted by f^{-1}.

Note 1.6.1 An inverse **exists if** the function is one-to-one and onto.

Let A and B be any sets and f be a function from A into B, that is, $f : A \to B$. The inverse function maps elements in B into those in A, that is $f^{-1} : B \to A$. The domain of the function f is the range of f^{-1} and the range of f is the domain of f^{-1}.

Definition 1.6.5 Suppose $f : A \to B$ is a bijection. Then the inverse of f, denoted

$$f^{-1} : B \to A$$

is the function defined by the rule

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

Finding the Inverse Function

If we are given a function that is one-to-one and onto then we can find its inverse. If a relation is defined by an equation, the inverse can be obtained by

1. interchanging x and y in the equation,

2. solving the new equation with y as the subject of the equation, that is, $y = ...$ and
3. replacing y by $f^{-1}(x)$.

This gives the inverse function.

Example 1.6.24 Find the inverse of the function $f(x) = 2x + 4$ if it exists.

This function is surely one-to-one and onto (verify this). It is therefore possible to find its inverse. The function $f(x) = 2x+4$ can be written as $y = 2x+4$. Now, interchanging x and y gives $x = 2y+4$. Making y the subject of the equation gives $y = \frac{x-4}{2}$. The inverse function $f^{-1}(x) = \frac{x-4}{2} = \frac{1}{2}x - 2$.

Example 1.6.25 Suppose $f : \mathbb{R} - (2) \to \mathbb{R} - (1)$ is defined by

$$f(x) = \frac{x}{x-2}$$

Then the function

$$f^{-1}(x) = \frac{2x}{x-1}$$

is the inverse of f. Show.

Exercise 1.6 Find the inverse of the function $f : \mathbb{R} - (2) \to \mathbb{R} - (1)$ defined by

$$f(x) = \frac{x-1}{x+2}$$

Exercise 1.7 Find the inverse of the function $f : \mathbb{R} \to (-\infty, 1)$ defined by $f(x) = 1 - e^x$.

Theorem 1.6.2 Let A and B are any sets and the function $f : A \to B$ be one-to-one and onto (the inverse function exists) $(f^{-1}of) : A \to A$ is the identity function on A similarly the composite function $(fof^{-1}) : B \to B$ is the identity function on B.

Proof: Let $a \in A$. Then $(f^{-1}of)(a) = f^{-1}(f(a)) = f^{-1}(b)$ for some $b \in B$. Now we have $f(a) = b$ which implies that $f^{-1}(b) = a$. It follows that $(f^{-1}of)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$ and therefore that $(f^{-1}of) : A \to A$ is the identity function on A.

Similarly if $b \in B$, then $(fof^{-1})(b) = f(f^{-1}(b)) = f(a)$ for some a satisfying $f^{-1}(b) = a$ and $f(a) = b$. Hence $(fof^{-1})(b) = f(f^{-1}(b)) = f(a) = b$. Therefore (fof^{-1}) is the identity function on B.

Theorem 1.6.3 *If a function is a bijection, then its inverse is also a bijection.*

Example 1.6.26 The function $f(x) = x^2$ does not have an inverse, since it is not a bijection. Whenever required to compute an inverse, need to first check whether it is bijective.

Exercise 1.8 Compute the inverse of the bijective function $h(x) : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \frac{3x+2}{x-1} , & x \neq 1 \\ 3 , & x = 1 \end{cases}$$

1.6.11 Operations of functions

(1) **Sums of functions**
Given two functions $g(x), f(x)$, their sum denoted by $(f+g)(x)$ is defined as $(f+g)(x) = f(x) + g(x)$
or $(g+f)(x) = g(x) + f(x)$

Example 1.6.27 Given that $f(x) = 3x$ and $g(x) = x^2$, then
$(f+g)(x) = f(x) + g(x) = 3x + x^2$
or $(g+f)(x) = g(x) + f(x) = x^2 + 3x$

Example 1.6.28 Given that $f(x) = 2x$ and $g(x) = 3x$, then

$(f+g)(x) = f(x) + g(x) = 2x + 3x = 5x$
or $(g+f)(x) = g(x) + f(x) = 3x + 2x = 5x$

Example 1.6.29 Given $f_1 = 5x + 2$ and $f_2 = x_2 + 4$. Find $(f_1 + f_2)(x)$.

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x) \\ &= 5x + 2 + x^2 + 4 \\ &= x^2 + 5x + 6 \end{aligned}$$

(2) **Difference of functions**
Given two functions $g(x), f(x)$, their difference denoted by $(f-g)(x)$ is defined as $(f-g)(x) = f(x) - g(x)$
or $(g-f)(x) = g(x) - f(x)$

Example 1.6.30 Given that $f(x) = 3x$ and $g(x) = x^2$, then
$(f-g)(x) = f(x) - g(x) = 3x - x^2$
or $(g-f)(x) = g(x) - f(x) = x^2 - 3x$

Example 1.6.31 Given that $f(x) = 2x$ and $g(x) = 3x$, then

$(f-g)(x) = f(x) - g(x) = 2x - 3x = -x$
or $(g-f)(x) = g(x) - f(x) = 3x - 2x = x$

Example 1.6.32 Let $f(x) = 3x^2 + 5$ and $f_2(x) = \frac{1}{x} + 2$. Evaluate $(f_1 - f_2)(x)$.

$$\begin{aligned}(f_1 - f_2)(x) &= f_1(x) - f_2(x) \\ &= \left[3x^2 + 5\right] - \left(\frac{1}{x} + 2\right) \\ &= 3x^2 - \frac{1}{x} + 3\end{aligned}$$

(3) **Product of functions**
Given two functions $g(x), f(x)$, their product denoted by $(f.g)(x)$ or simplfy $(fg)(x)$ is defined as $(f.g)\,(x) = f(x).g(x)$
or $(g.f)\,(x) = g(x).f(x)$

Example 1.6.33 Given that $f(x) = 3x$ and $g(x) = x^2$, then
$(f.g)\,(x) = f(x).g(x) = 3x.x^2 = 3x^3$
or $(g.f)\,(x) = g(x).f(x) = x^2.3x = 3x^3$

Example 1.6.34 Given that $f(x) = 2x$ and $g(x) = 3x$, then

$(f.g)\,(x) = f(x).g(x) = 2x.3x = 6x^2$
or $(g.f)\,(x) = g(x).f(x) = 3x.2x = 6x^2$

Example 1.6.35 If $f_1(x) = x^2 - 4$ and $f_2(x) = x - 2$ find $(f_1 \cdot f_2)(x)$.

$$\begin{aligned}(f_1 \cdot f_2)(x) &= f_1(x) \cdot f_2(x) \\ &= (x^2 - 4)(x - 2) \\ &= x^3 - 2x^2 - 4x + 8\end{aligned}$$

(4) **Quotient of functions**
Given two functions $g(x), f(x)$, their quotient denoted by $(f/g)(x)$ is defined as $(f/g)\,(x) = f(x)/g(x)$
or $(g/f)\,(x) = g(x)/f(x)$

Example 1.6.36 Given that $f(x) = 3x$ and $g(x) = x^2$, then
$(f/g)\,(x) = f(x)/g(x) = 3x/x^2 = 3/x$
or $(g.f)\,(x) = g(x)/f(x) = x^2/3x = x/3$

Example 1.6.37 Given that $f(x) = 2x$ and $g(x) = 3x$, then

$(f/g)\,(x) = f(x)/g(x) = 2x/3x = 2/3$
or $(g/f)\,(x) = g(x)/f(x) = 3x/2x = 3/2$ The quotient is sometimes referred to as a rational function.

Example 1.6.38 If $f_1(x) = x^2 - 4$ and $f_2(x) = x - 2$. Evaluate $(f_1/f_2)(x)$.

$$\begin{aligned}(f_1/f_2)(x) &= f_1(x) \div f_2(x)\\ &= (x^2-4) \div (x-2)\\ &= \frac{(x+2)(x-2)}{(x-2)}\\ &= (x+2)\end{aligned}$$

The domain of this function in the set of real numbers excludes $x = 2$ since for $x = 2$ we have $f_2(x) = 0$.

(5) **Composite functions**
Given two functions $g(x), f(x)$, their composite function denoted by $fog(x)$ is defined as $fog\,(x) = f\,(g(x))$
or $gof\,(x) = g\,(f(x))$

Example 1.6.39 Given that $f(x) = 3x$ and $g(x) = x^2$, then
$fog\,(x) = f\,(x^2) = 3x^2$
$gof\,(x) = g\,(3x) = (3x)^2 = 9x^2$

Example 1.6.40 Given that $h(x) = 2x$ and $g(x) = 5x$, then
$hog\,(x) = h\,(5x) = 2(5x) = 10x$
$goh\,(x) = g\,(2x) = 5(2x) = 10x$

Example 1.6.41 Let $f_1(x) = x^2+3x+2$ and $f_2(x) = x+3$. Compute and simplify (if possible) $(f_2 o f_1)(x)$ and $(f_1 o f_2)(x)$.

$$\begin{aligned}(f_2 o f_1)(x) &= f_2(f_1(x))\\ &= f_2(x^2+3x+2)\\ &= (x^2+3x+2)+3\\ &= x^2+3x+5\end{aligned}$$

To compute $(f_1 o f_2)(x)$, substitute f_2 into f_1

$$\begin{aligned}(f_1 o f_2)(x) = f_1(f_2(x))&\\ &= f_1(x+3)\\ &= (x+3)^2+3(x+3)+2\\ &= x^2+6x+9+3x+9+2\\ &= x^2+9x+20\end{aligned}$$

Note that in this case commutativity does not hold that is,

$$(f_2 o f_1)(x) \neq (f_1 o f_2)(x).$$

Example 1.6.42 Let $f_1(x) = \frac{1}{x}$ and $f_2(x) = x^2$. Compute and simplify (if possible) $(f_2 o f_1)(x)$ and $(f_1 o f_2)(x)$.

$$\begin{aligned}
(f_2 o f_1)(x) &= f_2(f_1(x)) \\
&= f_2\left(\frac{1}{x}\right) \\
&= \frac{1}{x^2} \\
\text{whereas} \quad (f_1 o f_2)(x) &= f_1(f_2(x)) \\
&= f_1(x^2) \\
&= \frac{1}{x^2}.
\end{aligned}$$

Note that in this case the composite functions commute, that is,

$$(f_2 o f_1)(x) = (f_1 o f_2)(x)$$

Theorem 1.6.4 *The composition of two injective functions is injective.*

Domain of Composite, Difference, addition and product of functions

Definition 1.6.6 If we denote a domain as D, we define the domains

(i) $D_{f+g} = D_{f-g} = D_{fg} = D_f \cap D_g$

(ii) $D_{f/g} = (D_f \cap D_g) \setminus \{x : g(x) = 0\}$

(iii) $D_{fog} = \{x \in D_g : g(x) \in D_f\}$. Since $g(x)$ is a domain of fog, then domain of g is domain of the composite, where $g(x)$ is in domain of fog.

Note 1.6.2 Finding the domain of a composite function consists of two steps:

Step 1. Find the domain of the ”inside” (input) function. If there are any restrictions on the domain, keep them.

Step 2. Construct the composite function. Find the domain of this new function. If there are restrictions on this domain, add them to the restrictions from Step 1. If there is an overlap, use the more restrictive domain (or the intersection of the domains). The composite may also result in a domain unrelated to the domains of the original functions.

Example 1.6.43 Let function $f(x) = \sqrt{x+1}$, function $g(x) = \frac{1}{x}$, and function $h(x) = x + 3$. Find an equation defining each function and state the domain.

(1) $f + g$

$$f + g = \sqrt{x+1} + \frac{1}{x}$$

$$D_f : x \geq -1 \ , \quad D_g : \ x \neq 0 \qquad D_{f+g} : \ [-1, 0) \cup (0, +\infty)$$

(2) $f - g$

$$f - g = \sqrt{x+1} - \frac{1}{x}$$

$$D_f : x \geq -1 \ , \quad D_g : \ x \neq 0 \qquad D_{f-g} : \ [-1, 0) \cup (0, +\infty)$$

(3) $f \cdot g$

$$f \cdot g = \sqrt{x+1} \cdot \left(\frac{1}{x}\right) = \frac{\sqrt{x+1}}{x}$$

$$D_f : x \geq -1 \ , \quad D_g : \ x \neq 0 \qquad D_{f \cdot g} : \ [-1, 0) \cup (0, +\infty)$$

(4) f/h

$$f/h = \frac{\sqrt{x+1}}{x+3}$$

$$D_f : x \geq -1 \ , \quad D_h : \ \Re, \quad h(x) \neq 0 \ \rightarrow \ x \neq -3 \qquad D_{f/h} : \ [-1, +\infty)$$

Example 1.6.44 Given

$$f(x) = x^2 + 2 \text{ and } \ g(x) = \sqrt{3 - x}$$

The composite functions

$$\begin{aligned} fog(x) = f\left(g(x)\right) &= \left(\sqrt{3-x}\right)^2 + 2 = 5 - x \\ gof(x) = g\left(f(x)\right) &= \sqrt{3 - (x^2 + 2)} = \sqrt{1 - x^2} \end{aligned}$$

The domain for $g(x) = \sqrt{3-x}$ is $x \leq 3$.
The domain for $f(g(x)) = 5 - x$ is all real numbers, but you must keep the domain of the

inside function. So the domain for the composite function is also $x \leq 3$.

The domain for $f(x) = x^2 + 2$ is all real numbers.
The domain for the composite function $g(f(x)) = \sqrt{1 - x^2}$ is $-1 \leq x \leq 1$. The input function $f(x)$ has no restrictions, so the domain of $g(f(x))$ is determined only by the composite function. So the domain is $-1 \leq x \leq 1$.

Example 1.6.45 Find fog and gof and the domain of each, where

$$f(x) = \frac{1-x}{3x} \quad \text{and} \quad g(x) = \frac{1}{1+3x}$$

fog: Step 1. What is the domain of the inside function $g(x)$ $\qquad x \neq -1/3$ Keep this!!
Step 2. The composite

$$f(g(x)) = \frac{1 - \left(\frac{1}{1+3x}\right)}{3\left(\frac{1}{1+3x}\right)} = \frac{\frac{3x}{1+3x}}{\frac{3}{1+3x}} = x$$

This function puts no additional restrictions on the domain, so the composite domain is $x \neq -1/3$.

gof: Step 1. What is the domain of the inside function $f(x)$? $\qquad x \neq 0$ Keep this!!
Step 2. The composite

$$g(f(x)) = \frac{1}{1 + 3\frac{1-x}{3x}} = \frac{1}{\frac{1}{x}} = x$$

This function puts <u>no additional</u> restrictions on the domain, so the composite domain is $x \neq 0$.

Example 1.6.46 Find fog and gof and the domain of each, where

$$f(x) = \frac{3x}{x-1} \quad \text{and} \quad g(x) = \frac{2}{x}$$

fog: Step 1. What is the domain of the inside function $g(x)$ $\qquad x \neq 0$ Keep this!!
Step 2. The composite

$$f(g(x)) = \frac{3\left(\frac{2}{x}\right)}{\left(\frac{2}{x}\right) - 1} = \frac{\frac{6}{x}}{\frac{2-x}{x}} = \frac{6}{2-x} \qquad \text{Domain :} \quad x \neq 2$$

Combine this domain with the domain from Step 1: the composite domain is $x \neq 0$ and $x \neq 2$

gof: Step 1. What is the domain of the inside function $f(x)$? $x \neq 1$ Keep this!!
Step 2. The composite

$$g(f(x)) = \frac{2}{\frac{3x}{x-1}} = \frac{2(x-1)}{3x} \quad \text{Domain :} \quad x \neq 0$$

Combine this domain with the domain from Step 1: the composite domain is $x \neq 1$ and $x \neq 0$

Example 1.6.47 Find fog and gof and the domain of each, where

$$f(x) = \sqrt{x-2} \text{ and } g(x) = \sqrt{x^2-1}$$

fog: Step 1. What is the domain of the inside function $g(x)$ $x \geq 1$ or $x \leq -1$ Keep this!!
Step 2. The composite

$$f(g(x)) = \sqrt{\sqrt{x^2-1} - 2}$$

The domain of this function is where

$$\sqrt{x^2-1} \geq 2 \Rightarrow x^2 - 1 \geq 4 \Rightarrow x^2 \geq 5 \Rightarrow x \geq \sqrt{5} \text{ or } x \leq -\sqrt{5}$$

This function has a more restrictive domain than $g(x)$, so the composite domain is

$$D_{fog} := x \geq \sqrt{5} \text{ or } x \leq -\sqrt{5}$$

We can also show that

$$D_{gof} := x \geq 3$$

Exercise 1.9 Find fog and gof and the domain of each for the following functions.

1. $f(x) = x + 3 \quad g(x) = \sqrt{9-x^2}$

$$fog(x) = \sqrt{9-x^2} + 3 \quad \text{Domain :} \quad -3 \leq x \leq 3$$
$$gof(x) = \sqrt{-x^2-6x} \quad \text{Domain :} \quad -6 \leq x \leq 0$$

2. $f(x) = \sqrt{x+3} \quad g(x) = 2x - 5$

$$fog(x) = \sqrt{2x-2} \quad \text{Domain :} \quad x \geq 1$$
$$gof(x) = 2\sqrt{x+3} - 5 \quad \text{Domain :} \quad x \geq -3$$

3. $f(x) = \frac{-3}{x} \quad g(x) = \frac{x}{x-2}$

$$fog(x) = -\frac{3(x-2)}{x} \quad \text{Domain :} \quad x \neq 2 \text{ and } x \neq 0$$

$$gof(x) = \frac{3}{3+2x} \quad \text{Domain :} \quad x \neq 0 \text{ and } x \neq -3/2$$

4. $f(x) = x^2 + 2 \quad g(x) = \sqrt{x-5}$

$$\begin{aligned} fog(x) &= x - 3 & \text{Domain : } & x \geq 5 \\ gof(x) &= \sqrt{x^2 - 3} & \text{Domain : } & x \geq \sqrt{3} \text{ or } x \leq -\sqrt{3} \end{aligned}$$

5. $f(x) = \frac{2}{x-3} \quad g(x) = \frac{5}{x+2}$

$$fog(x) = -\frac{2(x+2)}{3x+1} \qquad \text{Domain : } \quad x \neq -2 \text{ and } x \neq -1/3$$

$$gof(x) = \frac{5(x-3)}{2x-4} \qquad \text{Domain : } \quad x \neq 3 \text{ and } x \neq 2$$

Example 1.6.48 Let $f(x) = \sqrt{x+3}$ and $g(x) = \sqrt{16 - x^2}$, find

(i) $D_f = [-3, \infty)$

(ii) $D_g = [-4, 4]$

(iii) $D_f \cap D_g = [-3, \infty) \cap [-4, 4] = [-3, 4]$

(iv) $D_{f+g} = D_{f-g} = D_{fg} = [-3, 4]$

(v) $D_{f/g} = (D_f \cap D_g) \setminus \{x : g(x) = 0\} = [-3, 4] \setminus \{-4, 4\} = [-3, 4)$

Example 1.6.49 Given the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 5$

(i) $(fog)(x) = f(g(x)) = f(x^2 + 5) = \sqrt{x^2 + 5}$

(ii) $(gof)(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 5 = x + 5$

(iii) $D_{fog} = \{x \in D_g : g(x) \in D_f\}$. Since $D_g = \mathbb{R}$ and $g(x) \in D_f$, so $D_{fog} = \mathbb{R}$

(iv) $D_{gof} = \{x \in D_f : f(x) \in D_g\}$. Since $D_f = x \geq 0 = \mathbb{R}^+ = [0, \infty)$ and $f(x) \in D_g$, so $D_{fog} = [0, \infty)$

Example 1.6.50 Find $(fog)(-2)$ given

$$f(x) = -3x + 2, \; g(x) = |x - 4|$$

Ans= -16 [Note that we only take $|x - 4| = -(x - 4)$ since $x = -2 < 4$]

Example 1.6.51 Find $(fog)(x)$ and the domain of fog, given

$$f(x) = \frac{(x-1)}{(x+2)}, \quad g(x) = \frac{(x+1)}{(x-2)}$$

Ans: $(fog)(x) = 3/(3x - 3)$. The domain of fog is: $(-\infty, 1) \cup (1, 2) \cup (2, +\infty)$

Example 1.6.52 Let function $f(x) = \sqrt{x+1}$, function $g(x) = \frac{1}{x}$, and function $h(x) = x + 3$. Find $gofoh$ and state the domain.

$$gofoh(x) = \frac{1}{\sqrt{x+4}} \quad ; \quad x > -4$$

1.6.12 Other functions

Definition 1.6.7 A function is said to be **piecewise** defined if its defined by applying different formulas to the different parts of its domain.

Example 1.6.53 An example of a piecewise function in t,

$$g(t) = \begin{cases} t-1\ , & t \leq -3 \\ 2t^3\ , & -3 < t \leq 9 \\ 4-6t, & t > 9 \end{cases}$$

Example 1.6.54 An example of a piecewise function in x,

$$f(x) = \begin{cases} x^2+1\ , & x \geq 0 \\ -x-9\ , & x < 0 \end{cases}$$

Definition 1.6.8 A function f is said to be a **power function** if it has the form $f(x) = x^n$, for some $n \in \mathbb{N}$.

Definition 1.6.9 A function f is said to be a **quadratic function** if it can be written in the form

$$f(x) = Ax^2 + Bx + C$$

Definition 1.6.10 A **polynomial function** f is one that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are constants and n is a positive integer.
This integer n is then called the *degree of the polynomial*, provided $a_n \neq 0$.

Definition 1.6.11 A function g is said to be a **rational function** if it is of the type

$$g(x) = \frac{f(x)}{h(x)}$$

where both f and h are polynomial functions.

Definition 1.6.12 A function f is said to be **periodic**, with period T, if

$$f(x + nT) = f(x)$$

for all x in its domain and for $n = 1, 2, \ldots$

That means the function repeats itself as the rate of T intervals of x. For example the function $f(x) = \sin x$ is periodic with period 2π since $\sin(x + 2n\pi) = \sin x$ for all n = 1, 2, . .

Definition 1.6.13 A function f is said to be strictly **monotonic increasing** in the interval (a, b) if, for all pairs of numbers x_1, x_2 in (a, b),

$$f(x_1) < f(x_2) \text{ when } x_1 < x_2$$

Definition 1.6.14 The **floor function** or **greatest integer function** is the function defined as follows: for a real number x, the floor of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x.

The floor of x is sometimes referred to as the integer part or integral value of x. (Is the integer just below - on the left)

Example 1.6.55

(i) $\lfloor 5 \rfloor = 5$

(ii) $\lfloor 1.78 \rfloor = 1$

(iii) $\lfloor 0 \rfloor = 0$

(iv) $\lfloor -1.6 \rfloor = -2$

(v) $\lfloor -3.5 \rfloor = -4$

(vi) $\lfloor -99.9 \rfloor = -100$

Example 1.6.56

$$\lfloor 1.7 \rfloor = 1, \ \lfloor \pi \rfloor = 3 \text{ and } \lfloor -3.2 \rfloor = -4$$

The floor of x satisfies the following inequality.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

Definition 1.6.15 The **ceiling function** or **smallest integer function** is the function defined as follows: for a real number x, the ceiling of x, denoted $\lceil x \rceil$, is the smallest integer not less than x. (The integer just above - on the right)

Example 1.6.57

(i) $\lceil 5 \rceil = 5$

(ii) $\lceil 1.78 \rceil = 2$

(iii) $\lceil 0 \rceil = 0$

(iv) $\lceil -1.6 \rceil = -1$

(v) $\lceil -3.5 \rceil = -3$

(vi) $\lceil -99.9 \rceil = -99$

Example 1.6.58

$$\lceil 1.7 \rceil = 2,\ \lceil \pi \rceil = 4 \text{ and } \lceil -3.2 \rceil = -3$$

The ceiling of x satisfies the following inequality.

$$x \leq \lceil x \rceil < x + 1$$

Definition 1.6.16 The **sign function** is the function, denoted *sgn*, defined as

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Note that for any real number $x = \operatorname{sgn}(x)|x|$. So if $x \neq 0$,

$$\operatorname{sgn}(x) = \frac{x}{|x|}$$

The sgn function is usually referred to as the *signum* function.

Example 1.6.59 Graph the curve

$$f(x) = \frac{|x-3|}{|x-1|}\ ,\ x \in \mathbb{R}$$

Note that, we cannot solve and play around any mathematical expression with norms (absolute, modulus), so we remove it by considering the positive and negative options for each modulus

$$f_1(x) = \frac{+(x-3)}{+(x-1)} :\ x \geq 3, x \geq 1 \ \Rightarrow\ f_1(x) = \frac{(x-3)}{(x-1)} :\ x \in [3, \infty)$$

$$f_2(x) = \frac{+(x-3)}{-(x-1)} :\ x \geq 3, x < 1 \ \Rightarrow\ f_2(x) = -\frac{(x-3)}{(x-1)} :\ x \texttt{ DNE}$$

$$f_3(x) = \frac{-(x-3)}{+(x-1)} :\ x < 3, x \geq 1 \ \Rightarrow\ f_3(x) = -\frac{(x-3)}{(x-1)} :\ x \in [1, 3)$$

$$f_4(x) = \frac{-(x-3)}{-(x-1)} :\ x < 3, x < 1 \ \Rightarrow\ f_4(x) = \frac{(x-3)}{(x-1)} :\ x \in (-\infty, 1)$$

We sketch each function f_1, f_2, f_3, f_4 within its region/domain. For example, for

$$f_1(x) = \frac{(x-3)}{(x-1)} :\ x \in [3, \infty) \ \Rightarrow$$

x	3	4	5	6
$f_1(x)$	0	$\frac{1}{3}$	$\frac{2}{4}$	$\frac{3}{5}$

Sketching the functions f_1, f_2, f_3, f_4 will result into the graph

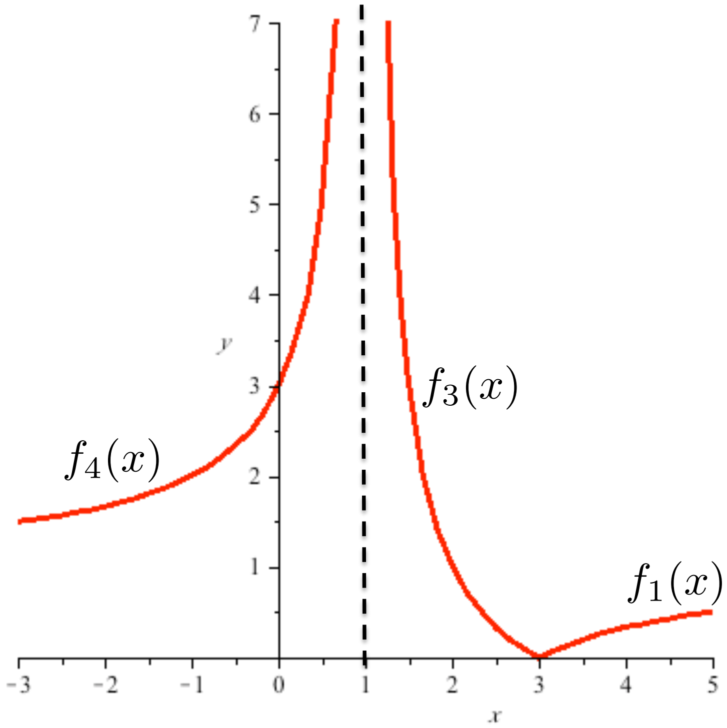

Figure 1.8: A graph of $f(x) = \frac{|x-3|}{|x-1|}$

Exercise 1.10 Graph the curve

$$f(x) = \frac{|x+6|}{|x+2|}$$

Exercise 1.11 Graph the curve

$$f(x) = \frac{|x+3| - 4|x-1|}{|x-2|}$$

Exercise 1.12 Compute the domain of the function $f(x)$ where

$$f(x) = \frac{a\sqrt{x+b}}{x+c}$$

where $a, b, c \in \mathbb{R}$