Chapter 4

Differentiation

4.1 Derivative of a Function

Definition 4.1.1 A derivative of a function $f(x)$ denoted as $f'(x)$ is said to exists if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{exists}$$

ie if

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

Definition 4.1.2 The derivative of a function $f(x)$ denoted by $f'(x)$ if exists, is given by

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} \tag{4.1}$$

Remark 4.1.1 A derivative of a function $f(x)$ `at a point` $x = a$ denoted as $f'(a)$ is said to exists if [substituting $x = a$ in Equation (4.1)]

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{4.2}$$

Definition 4.1.3 A derivative of a function $f(x)$ `at a point` $x = a$ denoted as $f'(a)$ is said to exists if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \text{exists} \quad \Rightarrow$$

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

Definition 4.1.4 The derivative of a function $f(x)$ at a point $x = a$ denoted as $f'(a)$ if exists, is given by

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \tag{4.3}$$

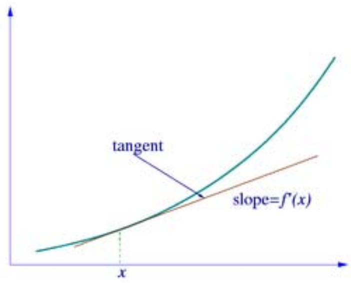


Figure 4.1: Illustration of the derivative of $f(x_0)$

This definition of the derivative is illustrated by the following figure.

The slope of the chord AB is

$$\frac{\Delta y}{h} = \frac{f(x_0+h)-f(x_0)}{h}$$

and indeed as $h \to 0$, this quotient tends to the slope of the tangent to the curve at $x = x_0$ which is $f'(x_0)$.

Example 4.1.1 Use the limit definition to compute to show that a derivative of a constant is zero.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\alpha-\alpha}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

Thus $f'(\alpha) = 0$

Note 4.1.1 This result shows that the derivative of a constant is 0 . Thus

$$\frac{d}{dx}(10) = \frac{d}{dx}(a) = \frac{d}{dx}(90) = 0$$

Example 4.1.2 Suppose $f(x) = x^2$, use the definition of a derivative to find $f'(x)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
\text{but } f(x+h) &= (x+h)^2\\
\text{therefore } \ f'(x) &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}\\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}\\
&= \lim_{h\to 0}\frac{2xh+h^2}{h}\\
&= \lim_{h\to 0}(2x+h) = 2x\\
\Rightarrow \ \frac{d}{dx}(x^2) &= 2x
\end{aligned}$$

Example 4.1.3 Use the formal definition of a derivative to compute $f'(x)$ for the function $f(x) = x^2 + 1$. We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h} \quad \text{bracketing } f(x+h) \text{ and } f(x) \\
&= \lim_{h\to 0} \frac{[x^2 + 2xh + h^2 + 1] - [x^2 + 1]}{h} \quad \text{the } f(x) \text{ terms have to cancel} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \quad \text{which of the six techniques of finding limits to use?} \\
&= \lim_{h\to 0} \frac{(2x+h)(h)}{h} \\
&= \lim_{h\to 0} (2x + h) = 2x \quad \text{can use method I now, of substitution}
\end{aligned}$$

Example 4.1.4 For $f(x) = x^n$ where $n \geq 1$ integer. Use the definition of a derivative to compute $\frac{d}{dx}(x^n)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
\text{since } f(x+h) &= (x+h)^n \\
\text{and } f(x) &= x^n \\
\Rightarrow f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}
\end{aligned}$$

We use Binomial theorem to expand $(x+h)^n$

$$\begin{aligned}
\Rightarrow f'(x) &= \lim_{h\to 0} \left(\frac{x^n + nx^{n-1}h + \frac{n(n-1)x^{n-2}(h)^2}{2!} + \cdots + h^n - x^n}{h} \right) \\
&= \lim_{h\to 0} \left[nx^{n-1} + \frac{hn(n-1)x^{n-2}}{2!} + ... + h^{n-1} \right] \Rightarrow \\
\frac{d}{dx}[x^n] &= nx^{n-1}
\end{aligned}$$

This result is an important differentiation formula . The formula is valid for all $n \in \mathbb{R}$ (real numbers)

Example 4.1.5 Suppose $f(x) = x$, using the definition of a derivative find $f'(x)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[x+h] - [x]}{h} \\
&= \lim_{h\to 0} \frac{h}{h} \\
&= \lim_{h\to 0} 1 \\
&= 1
\end{aligned}$$

The result shows that

$$\frac{d}{dx}[x] = 1$$

Example 4.1.6 Compute $f'(4)$ for the function given by $f(x) = x^2 + 2$

Using Equation (4.3), the derivative at a point,

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\
f'(4) &= \lim_{x\to 4} \frac{[x^2 + 2] - [18]}{x - 4} \\
&= \lim_{x\to 4} \frac{[x^2 - 16]}{x - 4} \\
&= \lim_{x\to 4} [x + 4] \\
&= 8
\end{aligned}$$

Example 4.1.7 Consider the function

$$f(x) = \frac{1}{x}\ , \quad x \neq 0$$

We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} \\
f'(x) &= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\
f'(x) &= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}
\end{aligned}$$

Example 4.1.8 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = (x+1)^{\frac{1}{3}}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{[(x+1)+h]^{\frac{1}{3}} - (x+1)^{\frac{1}{3}}}{h} \quad \text{Binomial expansion, fractional powers} \\
f'(x) &= \lim_{h\to 0} \frac{\left[(x+1)^{\frac{1}{3}} + \dfrac{1}{3}(x+1)^{\frac{-2}{3}}h + \dfrac{\frac{1}{3}\frac{-2}{3}(x+1)^{\frac{-5}{3}}h^2}{2!} + \cdots\right] - \left[(x+1)^{\frac{1}{3}}\right]}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{3}(x+1)^{\frac{-2}{3}}h + \dfrac{\frac{1}{3}\frac{-2}{3}(x+1)^{\frac{-5}{3}}h^2}{2!} + \cdots}{h} \\
f'(x) &= \lim_{h\to 0} \left[\frac{1}{3}(x+1)^{\frac{-2}{3}} + \frac{\frac{1}{3}\frac{-2}{3}(x+1)^{\frac{-5}{3}}h}{2!}\right] \\
&= \frac{1}{3}(x+1)^{\frac{-2}{3}}
\end{aligned}$$

Note 4.1.2 For any value of n, whether positive, negative, integer or non-integer, the value of the nth power of a binomial is given by:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \cdots + b^n$$

Example 4.1.9 Using the definition of a derivative, find

$$\frac{d}{dx}(\sin x)$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
\sin'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{(-\sin h)}{1}+\cos x\lim_{h\to 0}\frac{\cos h}{1}\\
&= \sin x(0)+\cos x(1)\\
&= \cos x
\end{aligned}$$

Example 4.1.10 Let a function f at a point $x=2$ be defined by $f(x)=(x+3)^{10}x$. Find

$$\left.\frac{df}{dx}\right|_{x=2}$$

The function $f(x)$ given to you is at only $x=2$, but we do not have the function as $x\to 2^+$ or the $f(x)$ as $x\to 2^-$, so we cannot compute the derivative since the function is not known.

Exercise 4.1 Use the definition of derivatives to compute $f'(x)$ given $f(x)=mx+c$

Example 4.1.11 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x)=\frac{1}{2}x-\frac{3}{5}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
f'(x) &= \lim_{h\to 0}\frac{\left(\frac{1}{2}(x+h)-\frac{3}{5}\right)-\left(\frac{1}{2}x-\frac{3}{5}\right)}{h}\\
f'(x) &= \lim_{h\to 0}\frac{\frac{1}{2}x+\frac{1}{2}h-\frac{3}{5}-\frac{1}{2}x+\frac{3}{5}}{h}
\end{aligned}$$

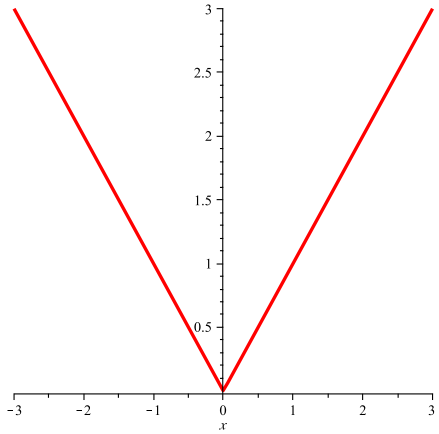
Algebraically and arithmetically simplify the expression in the numerator.

$$f'(x) = \lim_{h\to 0}\frac{\frac{1}{2}h}{h}$$

The term h now divides out and the limit can be calculated.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Example 4.1.12 Show that the function $y = |x|$ is not differentiable at $x = 0$.



Recall that

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

with the graph of $y = |x|$ on the left

Now check for differentiability at $x = 0$, i.e., compute $f'(0)$. Then

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \end{aligned}$$

Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

From the right of point $x = 0$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(x) - (0)}{x} = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$$

From the left of point $x = 0$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(-x) - (0)}{x} = \lim_{x \to 0} \frac{-x}{x} = \lim_{x \to 0} -1 = -1$$

Since the one-sided limits exist but are *not equal though finite*, $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist, and f is not differentiable at $x = 0$. This implies that the derivative of $f(x) = |x|$ does not exist at $x = 0$

Example 4.1.13 Compute the derivative of the function $f(x) = x|x|$.
The function is also given by

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

Convince your self that

$$f'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x = 0 \\ -2x, & x < 0 \end{cases} = 2|x|$$

Example 4.1.14 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = 5x^2 - 3x + 7$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{(5(x+h)^2 - 3(x+h) + 7) - (5x^2 - 3x + 7)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - 3x - 3h + 7 - 5x^2 + 3x - 7}{h}
\end{aligned}$$

Algebraically and arithmetically simplify the expression in the numerator.

$$f'(x) = \lim_{h\to 0} \frac{10xh + 5h^2 - 3h}{h}$$

The term h now divides out and the limit can be calculated.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}(10x + 5h - 3) \\
&= 10x - 3
\end{aligned}$$

Example 4.1.15 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = 4 - \sqrt{x+3}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\left(4 - \sqrt{(x+h)+3}\,\right) - \left(4 - \sqrt{x+3}\,\right)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+3} - \sqrt{(x+h)+3}}{h}
\end{aligned}$$

Eliminate the square root terms in the numerator of the expression by multiplying by the conjugate of the numerator divided by itself

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+3} - \sqrt{x+h+3}}{h} \, \frac{\sqrt{x+3} + \sqrt{x+h+3}}{\sqrt{x+3} + \sqrt{x+h+3}} \\
f'(x) &= \lim_{h\to 0} \frac{(x+3) - (x+h+3)}{h\left(\sqrt{x+3} + \sqrt{x+h+3}\right)} \\
f'(x) &= \lim_{h\to 0} \frac{-h}{h\left(\sqrt{x+3} + \sqrt{x+h+3}\right)}
\end{aligned}$$

The term h now divides out and the limit can be calculated.

$$f'(x) = \lim_{h\to 0} \frac{-1}{\left(\sqrt{x+3} + \sqrt{x+h+3}\right)} = \frac{-1}{\left(\sqrt{x+3} + \sqrt{x+3}\right)} = \frac{-1}{2\sqrt{x+3}}$$

Example 4.1.16 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \frac{x+1}{2-x}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)+1}{2-(x+h)} - \frac{x+1}{2-x}}{h} \\ f'(x) &= \lim_{h\to 0} \frac{[(x+h+1)(2-x)-(x+1)(2-x-h)]}{(2-x-h)(2-x)}\,\frac{1}{h} \end{aligned}$$

Algebraically and arithmetically simplify the expression in the numerator. It is important to note that the denominator of this expression should be left in factored form so that the term h can be easily eliminated later.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{2x+2h+2-x^2-xh-x-\{2x-x^2-xh+2-x-h\}}{(2-x-h)(2-x)h} \\ f'(x) &= \lim_{h\to 0} \frac{3h}{(2-x-h)(2-x)h} \end{aligned}$$

The term h now divides out and the limit can be calculated

$$f'(x) = \lim_{h\to 0} \frac{3}{(2-x-h)(2-x)} = \frac{3}{(2-x)(2-x)} = \frac{3}{(2-x)^2}$$

Example 4.1.17 Compute $f'(0)$ for the function given by $f(x) = 3x$

Using Equation (4.3), the derivative at a point,

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ f'(0) &= \lim_{x\to 0} \frac{[3x]-[0]}{x-0} \\ &= \lim_{x\to 0} \frac{[3x]}{x} \\ &= \lim_{x\to 0} [3] \\ &= 3 \end{aligned}$$

Exercise 4.2 Determine $f'(\pi)$ for the functions

(a).

$$f(x) = \cos x$$

(b).

$$f(x) = \frac{x^2-1}{2x}$$

(c).

$$f(x) = x^3(x-1)$$

Example 4.1.18 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = x^{\frac{2}{3}}$$

This problem may be more difficult than it first appears.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^{\frac{2}{3}}-x^{\frac{2}{3}}}{h}$$

Note that $A-B$ can be written as the difference of cubes,

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

so that

$$\begin{aligned} A-B = \left(A^{\frac{1}{3}}\right)^3 - \left(B^{\frac{1}{3}}\right)^3 &= \left(A^{\frac{1}{3}} - B^{\frac{1}{3}}\right)\left(A^{\frac{2}{3}} + A^{\frac{1}{3}}B^{\frac{1}{3}} + B^{\frac{2}{3}}\right) \\ \Rightarrow \left(A^{\frac{1}{3}} - B^{\frac{1}{3}}\right) &= \frac{A-B}{\left(A^{\frac{2}{3}} + A^{\frac{1}{3}}B^{\frac{1}{3}} + B^{\frac{2}{3}}\right)} \end{aligned}$$

This will help explain the next step.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{(x+h)^{\frac{2}{3}}-x^{\frac{2}{3}}}{h} = \lim_{h\to 0}\frac{\{(x+h)^2\}^{\frac{1}{3}}-\{x^2\}^{\frac{1}{3}}}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h\left\{(x+h)^{\frac{4}{3}}+(x+h)^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} \\ &= \lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h\left\{(x+h)^{\frac{4}{3}}+(x+h)^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} \\ &= \lim_{h\to 0}\frac{h(2x+h)}{h\left\{(x+h)^{\frac{4}{3}}+(x+h)^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} \\ &= \lim_{h\to 0}\frac{2x+h}{\left\{(x+h)^{\frac{4}{3}}+(x+h)^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} \\ &= \frac{2x}{\left\{x^{\frac{4}{3}}+x^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} = \frac{2x}{3x^{\frac{4}{3}}} = \frac{2}{3x^{\frac{1}{3}}} \end{aligned}$$

Exercise 4.3 Use the limit definition to compute the derivative, $f'(1)$, for

$$f(x) = x^{\frac{2}{3}}$$

Exercise 4.4 Use the limit definition to compute the derivative, $f'(0)$, for

$$f(x) = x^{\frac{2}{3}}$$

What do you realise.

Example 4.1.19 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \cos 3x$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\cos 3(x+h) - \cos 3x}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\cos(3x+3h) - \cos 3x}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\{\cos 3x \cos 3h - \sin 3x \sin 3h\} - \cos 3x}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\cos 3x(\cos 3h - 1) - \sin 3x \sin 3h}{h}
\end{aligned}$$

Since

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Recall the following two well-known trigonometry limits (La'Hopital rule):

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \ , \ \lim_{h\to 0} \frac{\sin h}{h} = 1 \ \Rightarrow \ \lim_{h\to 0} \frac{\sin 3h}{h} = 3$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\cos 3x(\cos 3h - 1) - \sin 3x \sin 3h}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\cos 3x(\cos 3h - 1)}{h} - \lim_{h\to 0} \frac{\sin 3x \sin 3h}{h} \\
f'(x) &= \cos 3x \lim_{h\to 0} \frac{(\cos 3h - 1)}{h} - \sin 3x \lim_{h\to 0} \frac{\sin 3h}{h}
\end{aligned}$$

Terms without h were factored out, as

$$\lim_{x\to a} \alpha f(x) = \alpha \lim_{x\to a} f(x)$$

$$\begin{aligned}
f'(x) &= \cos 3x \lim_{h\to 0} \frac{(\cos 3h - 1)}{h} - \sin 3x \lim_{h\to 0} \frac{\sin 3h}{h} \\
f'(x) &= \cos 3x \ (0) - \sin 3x \ (3) \\
f'(x) &= -3 \sin 3x
\end{aligned}$$

Example 4.1.20 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \frac{x-1}{x^2+3x}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{(x+h)-1}{(x+h)^2+3(x+h)} - \dfrac{x-1}{x^2+3x}}{h} \end{aligned}$$

Get a common denominator for the expression in the numerator.

$$= \lim_{h\to 0} \frac{(x^3+3x^2+x^2h+3xh-x^2-3x-x^3-2x^2h-xh^2-3x^2-3xh+x^2+2xh+h^2+3x+3h)}{(x^2+2xh+h^2+3x+3h)(x^2+3x)} \; \frac{1}{h}$$

Algebraically and arithmetically simplify the expression in the numerator. The terms $x^3, 2x^2, -3x$, and $3xh$ will subtract out. It is important to note that the denominator of this expression should be left in factored form so that the term h can be easily eliminated later.

$$= \lim_{h\to 0} \frac{-x^2h+2xh+h^2+3h}{(x^2+2xh+h^2+3x+3h)(x^2+3x)h}$$

Factor h from the numerator.

$$= \lim_{h\to 0} \frac{h(-x^2+2x+h+3)}{(x^2+2xh+h^2+3x+3h)(x^2+3x)h}$$

The term h now divides out and the limit can be calculated.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{-x^2+2x+h+3}{(x^2+2xh+h^2+3x+3h)(x^2+3x)} \\ f'(x) &= \frac{-x^2+2x+3}{(x^2+3x)(x^2+3x)} \\ f'(x) &= \frac{2x+3-x^2}{(x^2+3x)^2} \end{aligned}$$

Example 4.1.21 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \sqrt{x^3 - x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^3 - (x+h)} - \sqrt{x^3 - x}}{h}$$

Eliminate the square root terms in the numerator of the expression by multiplying by the conjugate of the numerator divided by itself.

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)^3 - (x+h)} - \sqrt{x^3 - x}}{h} \frac{\sqrt{(x+h)^3 - (x+h)} + \sqrt{x^3 - x}}{\sqrt{(x+h)^3 - (x+h)} + \sqrt{x^3 - x}}$$

Algebraically and arithmetically simplify the expression in the numerator. It is important to note that the denominator of this expression should be left in factored form so that the term h can be easily eliminated later.

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - (x^3 - x)}{h\sqrt{(x+h)^3 - (x+h)} + \sqrt{x^3 - x}}$$

$$f'(x) = \lim_{h \to 0} \frac{[x^3 + 3xh^2 + 3x^2h + h^3 - x - h] - (x^3 - x)}{h\sqrt{(x+h)^3 - (x+h)} + \sqrt{x^3 - x}}$$

$$f'(x) = \lim_{h \to 0} \frac{3xh^2 + 3x^2h + h^3 - h}{h\sqrt{(x+h)^3 - (x+h)} + \sqrt{x^3 - x}}$$

Factor h from the numerator.

$$f'(x) = \lim_{h \to 0} \frac{h\left[3xh + 3x^2 + h^2 - 1\right]}{h\sqrt{(x+h)^3 - (x+h)} + \sqrt{x^3 - x}}$$

The term h now divides out and the limit can be calculated.

$$f'(x) = \lim_{h \to 0} \frac{3xh + 3x^2 + h^2 - 1}{\sqrt{(x+h)^3 - (x+h)} + \sqrt{x^3 - x}}$$

$$f'(x) = \frac{3x^2 - 1}{\sqrt{x^3 - x} + \sqrt{x^3 - x}}$$

$$f'(x) = \frac{3x^2 - 1}{2\sqrt{x^3 - x}}$$

Example 4.1.22 Assume a piecewise function $f(x)$ defined as

$$f(x) = \begin{cases} 2+\sqrt{x}, & \text{if } x \geq 1 \\ \frac{1}{2}x + \frac{5}{2}, & \text{if } x < 1 \end{cases}$$

Show whether or not $f(x)$ is differentiable at $x = 1$, i.e., use the limit definition of the derivative to compute $f'(1)$.

To compute $f'(1)$: Lets first compute $f(1) = 2 + \sqrt{1} = 3$, then

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \end{aligned}$$

Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

From the right of point $x = 1$

$$\begin{aligned} \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x\to 1} \frac{(2+\sqrt{x}\ ) - (3)}{x - 1} \\ &= \lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{(\sqrt{x} - 1)}{(\sqrt{x} - 1)(\sqrt{x} + 1)} \\ &= \lim_{x\to 1} \frac{1}{(\sqrt{x} + 1)} = \frac{1}{2} \end{aligned}$$

From the left of point $x = 1$

$$\begin{aligned} \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x\to 1} \frac{\left(\frac{1}{2}x + \frac{5}{2}\right) - (3)}{x - 1} \\ &= \lim_{x\to 1} \frac{\frac{x-1}{2}}{x - 1} \\ &= \lim_{x\to 1} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Since the one-sided limits exists and are *equal*

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \frac{1}{2}$$

does exist, and thus f is differentiable at $x = 1$.

Example 4.1.23 Assume that

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } \ x \neq 0 \\ 0, & \text{if } \ x = 0 \end{cases}$$

Show that f is differentiable at $x = 0$, i.e., use the limit definition of the derivative to compute $f'(0)$.

To have from right and from left, we use the Squeeze law to create functions from left and right.

$$x^2 \sin\left(\frac{1}{x}\right) = \begin{cases} x^2, & \text{if } \ x > 0 \\ -x^2, & \text{if } \ x < 0 \end{cases}$$

The derivative at $x = 0$ is

$$f'(0) = 0$$

Show the solution above.

Remark 4.1.2 What follows is a common *incorrect* attempt to solve this problem using another method. Since $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, it follows, using the product rule and chain rule, that

$$f'(x) = x^2 \cos\left(\frac{1}{x}\right)\left\{\frac{-1}{x^2}\right\} + 2x \sin\left(\frac{1}{x}\right) = -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$$

for $x \neq 0$. Then

$$\begin{aligned} f'(0) &= \lim_{x \to 0} f'(x) \\ &= \lim_{x \to 0} \left\{ -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)\right\} \end{aligned}$$

Because the term $-\cos\left(\frac{1}{x}\right)$ oscillates between 1 and -1 as h approaches zero, this limit does not exist.

An *incorrect* conclusion would be that $f'(0)$ does not exist, i.e., f is not differentiable at $x = 0$. If f' were continuous at $x = 0$, this would be a valid method to compute $f'(0)$.

Example 4.1.24 Use the limit definition to compute the derivative, $f'(x)$, for a piecewise function

$$f(x) = |x^2 - 3x|$$

First rewrite $f(x)$. That is,

$$f(x) = |x^2 - 3x| = \begin{cases} (x^2 - 3x), & x \in (-\infty, 0] \cup [3, \infty) \\ -(x^2 - 3x), & 0 < x < 3 \end{cases}$$

In other words, the region for positive, $+(x^2 - 3x)$

$$\begin{array}{rcl} (x^2 - 3x) \geq 0 & \Rightarrow & x(x-3) \geq 0 \Rightarrow \\ \text{either (both positives) } x \geq 0 \ \& \ (x-3) \geq 0 & \text{or} & \text{(both negatives) } x \leq 0 \ \& \ (x-3) \leq 0 \\ \text{either } \ x \geq 0 \ \& \ x \geq 3 & \text{or} & x \leq 0 \ \& \ x \leq 3 \\ \text{either } \ x \geq 3 & \text{or} & x \leq 0 \\ \Rightarrow & & x \in (-\infty, 0] \cup [3, \infty) \end{array}$$

and the region for negative, $-(x^2 - 3x)$

$$\begin{array}{rcl} (x^2 - 3x) < 0 & \Rightarrow & x(x-3) < 0 \Rightarrow \\ \text{either (one negative) } x > 0 \ \& \ (x-3) < 0 & \text{or} & \text{(the other negative) } x < 0 \ \& \ (x-3) > 0 \\ \text{either } \ x > 0 \ \& \ x < 3 & \text{or} & x < 0 \ \& \ x > 3 \\ \text{either } \ (0, 3) & \text{or} & \text{no solution} \\ \Rightarrow & & x \in (0, 3) \end{array}$$

This can be summarized as

$$f(x) = \begin{cases} (x^2 - 3x), & \text{if } \ x \leq 0 \\ -(x^2 - 3x), & \text{if } \ 0 < x < 3 \\ (x^2 - 3x), & \text{if } \ x \geq 3 \end{cases}$$

1. Check for differentiability at $x = 0$, i.e., compute $f'(0)$. Then

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \end{aligned}$$

   Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

   From the right of point $x = 0$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(3x - x^2) - (0)}{x} = \lim_{x \to 0} \frac{3x - x^2}{x} = \lim_{x \to 0} \ 3 - x = 3$$

   From the left of point $x = 0$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(x^2 - 3x) - (0)}{x} = \lim_{x \to 0} \frac{x^2 - 3x}{x} = \lim_{x \to 0} \ x - 3 = -3$$

   Since the one-sided limits exist but are *not equal*, $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist, and f is not differentiable at $x = 0$.

2. Now check for differentiability at $x = 3$, i.e., compute $f'(3)$: $f(3) = (3)^2 - 3(3) = 0$ Then
$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ f'(3) &= \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} \end{aligned}$$
Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

From the right of point $x = 3$
$$\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{(x^2-3x)-(0)}{x-3} = \lim_{x\to 3} \frac{x^2-3x}{x-3} = \lim_{x\to 3} x = 3$$
From the left of point $x = 3$
$$\lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{(3x-x^2)-(0)}{x-3} = \lim_{x\to 3} \frac{3x-x^2}{x-3} = \lim_{x\to 3} -x = -3$$
Since the one-sided limits exist but are *not equal*, $f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$ does not exist, and f is not differentiable at $x = 3$. Since the one-sided limits exist but are *not equal*, $f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$ does not exist, and f is not differentiable at $x = 3$.

3. Assume that $x < 0$. Then
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-3(x+h)-(x^2-3x)}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2-3h}{h} = \lim_{h\to 0} \frac{h(2x+h-3)}{h} = \lim_{h\to 0} (2x+h-3) = 2x-3. \end{aligned}$$

4. Assume that $x > 3$. Then it is also true (the same function of $f(x) = (x^2 - 3x)$) that
$$f'(x) = 2x - 3$$

5. Assume that $0 < x < 3$. Then
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)-(x+h)^2-(3x-x^2)}{h} \\ &= \lim_{h\to 0} \frac{3x+3h-(x^2+2xh+h^2)-3x+x^2}{h} = \lim_{h\to 0} \frac{3x+3h-x^2-2xh-h^2-3x+x^2}{h} \\ &= \lim_{h\to 0} \frac{3h-2xh-h^2}{h} = \lim_{h\to 0} \frac{h(3-2x-h)}{h} = \lim_{h\to 0} (3-2x-h) = 3-2x. \end{aligned}$$

6. Summarizing, the derivative of f (the function f is not differentiable at $x = 0$ or $x = 3$) is
$$f'(x) = \begin{cases} 2x-3, & \text{if } x < 0 \\ \text{DNE}, & \text{if } x = 0 \\ 3-2x, & \text{if } 0 < x < 3 \\ \text{DNE}, & \text{if } x = 3 \\ 2x-3, & \text{if } x > 3 \end{cases}$$

Example 4.1.25 Assume that

$$f(x) = \begin{cases} \dfrac{1}{4}x^3 - \dfrac{1}{2}x^2, & \text{if } x \geq 2 \\ \\ \dfrac{-3x+6}{x^2+2}, & \text{if } x < 2 \end{cases}$$

Determine if f is differentiable at $x = 2$, i.e., determine if

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \quad \text{exists}$$

From right, for region $x > 2$

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\left\{\frac{1}{4}x^3 - \frac{1}{2}x^2 - 0\right\}}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{4}x^2(x-2)}{(x-2)} = \lim_{x \to 2} \frac{1}{4}x^2 = 1$$

From left, for region $x < 2$

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\left\{\frac{-3x+6}{x^2+2} - 0\right\}}{x - 2} = \lim_{x \to 2} \frac{-3x+6}{(x-2)(x^2+2)} = \lim_{x \to 2} \frac{-3(x-2)}{(x-2)(x^2+2)} = -\frac{1}{2}$$

Since the derivatives are not equal, the derivative does not exist.

Remark 4.1.3 : Use of the limit definition of the derivative of f at $x = 2$ also leads to a correct solution to this problem.

Remark 4.1.4 : What follows is a common incorrect attempt to solve this problem using another method.

For $x > 2$

$$f'(x) = \frac{3}{4}x^2 - x$$

For $x < 2$

$$f'(x) = \frac{(x^2+2)(-6) - (-6x-6)(2x)}{(x^2+2)^2} = \frac{6x^2 + 12x - 12}{(x^2+2)^2}$$

Then

$$\lim_{x \to 2^+} f'(x) = \lim_{x \to 2^+} \left\{\frac{3}{4}x^2 - x\right\} = \frac{3}{4}(2)^2 - 2 = 1$$

and

$$\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^-} \frac{6x^2 + 12x - 12}{(x^2+2)^2} = \frac{6(2)^2 + 12(2) - 12}{((2)^2+2)^2} = 1$$

An *incorrect* conclusion would be that $f'(2) = 1$. If f' were continuous at $x = 2$, this would be a valid method to compute $f'(2)$.

Example 4.1.26 Given a piecewise function

$$f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ 2, & \text{if } 0 \leq x \leq 3 \\ 4 - x, & \text{if } x > 3 \end{cases}$$

Show that (using the definitions of differentiability)

$$f'(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x < 3 \\ -1, & \text{if } x > 3 \end{cases}$$

Exercise 4.5 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \dfrac{1+x}{2}, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \\ \sqrt{x}, & \text{if } x > 1 \end{cases}$$

1. Find $f'(1)$
2. Find $f''(1)$

Example 4.1.27 Show that the piecewise function

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is not differentiable at $x = 0$. $[0 \neq -1]$
It can also be seen from the graph, having a sharp turn at $x = 0$.

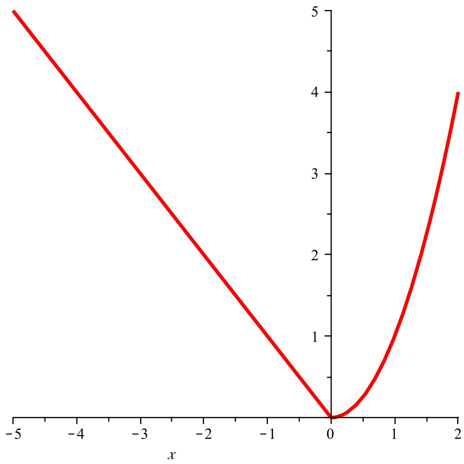


Figure 4.2: Sharp turn at $x = 0$, not differentiable

Example 4.1.28 Find the derivative of $y = |5x - 2|$ The function is given by

$$y = \begin{cases} (5x-2), & x \geq \dfrac{2}{5} \\ \\ -(5x-2), & x < \dfrac{2}{5} \end{cases}$$

Note that, we have not been asked $f'\left(\dfrac{2}{5}\right)$ [at a point], but $f'(x)$ [everywhere], but since we have a special point $x = \dfrac{2}{5}$, to differentiate everywhere, it is to differentiate at $x = \dfrac{2}{5}$, $x > \dfrac{2}{5}$ and $x < \dfrac{2}{5}$

1.) `Differentiable at` $x = \dfrac{2}{5}$:

Derivative is given by $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$$f'\left(\frac{2}{5}\right) = \lim_{x \to 2/5} \frac{f(x) - f\left(\frac{2}{5}\right)}{x - \frac{2}{5}}$$

Since finding limit of a piecewise function, we use the informal definition of limits:

From the right of point $x = \dfrac{2}{5}$

$$\lim_{x \to \frac{2}{5}^+} \frac{f(x) - f\left(\frac{2}{5}\right)}{x - \frac{2}{5}} = \lim_{x \to \frac{2}{5}} \frac{(5x-2) - 0}{x - \frac{2}{5}} = \lim_{x \to 2/5} \frac{5\left(x - \frac{2}{5}\right) - 0}{\left(x - \frac{2}{5}\right)} = \lim_{x \to 2/5} 5 = 5$$

From the left of point $x = \dfrac{2}{5}$

$$\lim_{x \to \frac{2}{5}^-} \frac{f(x) - f\left(\frac{2}{5}\right)}{x - \frac{2}{5}} = \lim_{x \to 2/5} \frac{-(5x-2) - 0}{x - \frac{2}{5}} = \lim_{x \to \frac{2}{5}} \frac{-5\left(x - \frac{2}{5}\right) - 0}{\left(x - \frac{2}{5}\right)} = \lim_{x \to \frac{2}{5}} -5 = -5$$

since

$$\lim_{x \to \frac{2}{5}^+} \frac{f(x) - f\left(\frac{2}{5}\right)}{x - \frac{2}{5}} \neq \lim_{x \to \frac{2}{5}^-} \frac{f(x) - f\left(\frac{2}{5}\right)}{x - \frac{2}{5}} \Rightarrow \lim_{x \to \frac{2}{5}} \frac{f(x) - f\left(\frac{2}{5}\right)}{x - \frac{2}{5}} = f'\left(\frac{2}{5}\right) \text{ DNE}$$

the function is not differentiable at $x = \frac{2}{5}$.

2.) `Differentiability for` $x > \frac{2}{5}$:

For the region $x > \frac{2}{5}$, $f(x) = y = (5x - 2)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[5(x+h) - 2] - [(5x-2)]}{h} \\ &= \lim_{h\to 0} \frac{5h}{h} = \lim_{h\to 0} \ 5 = 5 \end{aligned}$$

3.) `Differentiability for` $x < \frac{2}{5}$:

For the region $x < \frac{2}{5}$, $f(x) = y = -(5x - 2)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[-\{5(x+h) - 2\}] - [-(5x-2)]}{h} \\ &= \lim_{h\to 0} \frac{-5h}{h} = \lim_{h\to 0} \ -5 = -5 \end{aligned}$$

In summary, the derivative of the function y denoted as y' is given by

$$y' = \begin{cases} 5, & x > \frac{2}{5} \\ \text{DNE}, & x = \frac{2}{5} \\ -5, & x < \frac{2}{5} \end{cases}$$

Example 4.1.29 Given

$$f(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ \cos x, & x < 0 \end{cases}$$

Show that $f(x)$ is differentiable at $x = 0$, and $f'(0) = 0$

Remark 4.1.5 Even if f does have a derivative, it may not have a second derivative. For example, let

$$f(x) = \begin{cases} +x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

Calculation shows that f is a differentiable function whose derivative is

$$f'(x) = \begin{cases} +2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases} = 2|x|$$

an absolute function which does not have a derivative, thus $f''(x)$ does not exist.

Example 4.1.30 Given a function

$$y = \frac{|4x-3|}{4x-3}$$

Find the derivative of y

$$y = \begin{cases} \dfrac{(4x-3)}{4x-3}, & x \geq \dfrac{3}{4} \\ -\dfrac{(4x-3)}{4x-3}, & x < \dfrac{3}{4} \end{cases} = \begin{cases} 1, & x \geq \dfrac{3}{4} \\ -1, & x < \dfrac{3}{4} \end{cases}$$

1.) `Differentiability at` $x = \dfrac{3}{4}$:

$$\lim_{x \to \frac{3}{4}^-} \frac{f(x) - f\left(\dfrac{3}{4}\right)}{x - \frac{3}{4}} \neq \lim_{x \to \frac{3}{4}^+} \frac{f(x) - f(\frac{3}{4})}{x - \dfrac{3}{4}} \Leftrightarrow \lim_{x \to \frac{3}{4}} \frac{f(x) - f\left(\dfrac{3}{4}\right)}{x - \dfrac{3}{4}} \text{ DNE}$$

$y'\left(\dfrac{3}{4}\right)$ does not exist.

2.) `Differentiability in region` $x > \dfrac{3}{4}$: consider $f(x) = 1$

$$y' = 0$$

3.) `Differentiability in region` $x < \dfrac{3}{4}$: consider $f(x) = -1$

$$y' = 0$$

$$y' = \begin{cases} 0, & x > \dfrac{3}{4} \\ \text{DNE}, & x = \dfrac{3}{4} \\ 0, & x < \dfrac{3}{4} \end{cases}$$

Exercise 4.6 Using the definition of derivative, calculate the derivative of

$$f(x) = (3x+2)^{\frac{2}{3}}$$

Apply Binomial expansion at a certain point.

Exercise 4.7 Using the definition of derivative, calculate the derivative of

$$f(x) = \frac{5}{3x+4}$$

Exercise 4.8 Find the derivative of the function $\sqrt{x}$ at the point $x = 1$. $f'(1) = \frac{1}{2}$

Exercise 4.9 Prove that if $f(x) = x^n$ then

$$f'(x) = nx^{(n-1)}$$

4.2 Continuity Versus Differentiability

Theorem 4.2.1 *Let f be differentiable at x_0. Then f is continuous at x_0.*

Proof: What is known is that f is differentiable at x_0. That is

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad \text{exists say } \xi \tag{4.4}$$

Now that known, we need to prove that the function $f(x)$ is continuous. A function is said to be continuous if the limit exists and equal to function at that point.

Lets compute the limit at x_0 but using known information Equation (4.4)

$$\begin{aligned}
\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} [f(x) - f(x_0) + f(x_0)] \\
&= \lim_{x \to x_0} [f(x) - f(x_0)] + \lim_{x \to x_0} f(x_0) \\
&= \lim_{x \to x_0} \frac{[f(x) - f(x_0)]\,(x - x_0)}{x - x_0} + \lim_{x \to x_0} f(x_0) \\
&= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0}\right] \lim_{x \to x_0} (x - x_0) + \lim_{x \to x_0} f(x_0) \\
&= \xi \cdot [0] + \lim_{x \to x_0} f(x_0) \\
&= 0 + \lim_{x \to x_0} f(x_0) \\
&= \lim_{x \to x_0} f(x_0) \\
&= f(x_0)
\end{aligned}$$

Hence

$$\lim_{x \to x_0} f(x) = f(x_0)$$

so f is continuous at $x = x_0$. ∎

Remark 4.2.1 *A differentiable function is a continuous function but the reverse is not true*

Example 4.2.1 Show that the function $f(x) = |x|$ is continuous but not differentiable at $x = 0$. Check whether the function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is continuous at $x = 0$

1.) $f(0) = 0$

2.) $\lim_{x\to 0} f(x)$

Since a piecewise function, to compute the limit, we use the informal definition of limits.

$$\begin{aligned} \text{From the left}: \lim_{x\to 0^-} f(x) &= \lim_{x\to 0}(-x) = 0 \\ \text{From the right}: \lim_{x\to 0^+} f(x) &= \lim_{x\to 0}(x) = 0 \end{aligned}$$

Thus the limit exists and equal to 0, that is $\lim_{x\to 0} f(x) = 0$

3.)

$$\lim_{x\to 0} f(x) = f(0) = 0$$

Therefore, the function is continuous at $x = 0$.

Show that the function $y = |x|$ is not differentiable at $x = 0$.

Recall that

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Now check for differentiability at $x = 0$, i.e., compute $f'(0)$. Then

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \end{aligned}$$

Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

From the right of point $x = 0$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{(x) - (0)}{x} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

From the left of point $x = 0$

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{(-x) - (0)}{x} = \lim_{x\to 0} \frac{-x}{x} = \lim_{x\to 0} -1 = -1$$

Since the one-sided limits exist but are *not equal though finite*, $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist, and f is not differentiable at $x = 0$. This implies that the derivative of $f(x) = |x|$ does not exist at $x = 0$ as seen by the sharp curve at the point.

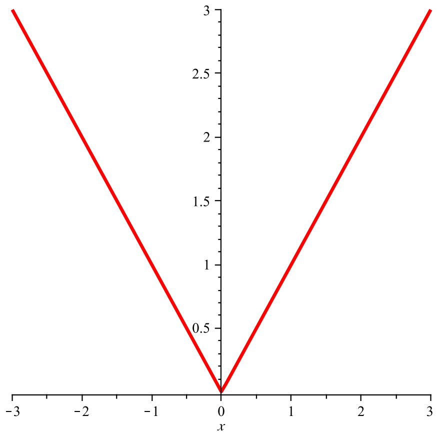


Figure 4.3: Graph of $y = |x|$

Remark 4.2.2 Sometimes a derivative may fail to exist at a point. In general, there are three reasons why a derivative at a point may not exist.

1. The graph of the function has a sharp turn or a cusp, e.g. $f(x) = |x|$ at $x = 0$.

2. The graph is not continuous at the point, one of the three properties of continuity fails

$$g(x) = \frac{x^2 + x}{x} \text{ at } x = 0$$

3. The graph has a vertical tangent line at the point, e.g. $h(x) = x^{\frac{1}{3}}$ at $x = 0$.

Note 4.2.1 So what is the derivative, after all? The derivative measures the steepness of the graph of a function at some particular point on the graph

Example 4.2.2 A function is defined by the following formula:

$$f(x) = \begin{cases} x^2+2, & x \le 1 \\ a\left[x - \left(\dfrac{1}{x}\right)\right] + b, & x > 1 \end{cases}$$

Find a and b such that f is continuous and differentiable. Plot the function, if possible.

1.) To be continuous at $x = 1$

1.) $f(1) = 1^2 + 2 = 3$

2.) $\lim\limits_{x\to 1} f(x)$

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1}(x^2+2) = 3 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1}\left(a\left[x-\left(\frac{1}{x}\right)\right]+b\right) = b \end{aligned}$$

3.) For continuity, $\lim\limits_{x\to 1} f(x) = f(1) \quad \Rightarrow \quad b = 3$

2.) To be differentiable at $x = 1$

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ exists } \quad \Rightarrow \quad f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \text{ exists}$$

Finding a limit for a piecewise function, we check from left and from right, and to be equal since a derivative exists.

i) From the left of point $x = 1$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{(x^2+2)-(3)}{x-1} = \lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1} \; x+1 = 2$$

ii) From the right of point $x = 1$

$$\begin{aligned} \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to 1}\frac{a\left[x-\left(\frac{1}{x}\right)\right]+b-3}{x-1} = \lim_{x\to 1}\frac{a\left[x-\left(\frac{1}{x}\right)\right]+3-3}{x-1} \\ &= \lim_{x\to 1}\frac{a\left[x-\left(\frac{1}{x}\right)\right]}{x-1} = \lim_{x\to 1}\frac{\frac{a\,(x^2-1)}{x}}{x-1} = \lim_{x\to 1}\frac{a\,(x+1)}{x} = 2a \end{aligned}$$

limits to equal, $2 = 2a \quad \Rightarrow \quad a = 1$

Example 4.2.3 We wish to determine the values of the parameters k and m for which the function below is differentiable at $x = 3$:

$$f(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ 5 - mx, & 3 < x \le 5 \end{cases}$$

For a function to be differentiable at a domain value,

1. the function must be *continuous* there.

2. the derivative exists (the pieces must match with the same slope).

1.) To be continuous at $x = 3$

i) $f(3) = k\sqrt{3+1} = 2k$

ii) $\lim_{x\to 3} f(x)$ exists, and since a piecewise function,

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3}\left(k\sqrt{x+1}\right) = 2k\\ \lim_{x\to 3^+} f(x) &= \lim_{x\to 3}(5-mx) = 5-3m\end{aligned}$$

iii) The limits should be equal and equal to $f(3)$ to be continuous.

$$5 - 3m = 2k \tag{4.5}$$

2.) To be differentiable at $x = 3$

$$f'(3) = \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} \quad \text{exists}$$

Finding a limit for a piecewise function, we check from left and from right, and to be equal since a derivative exists.

i) From the left of point $x = 3$

$$\begin{aligned}\lim_{x\to 3^-}\frac{f(x)-f(3)}{x-3} &= \lim_{x\to 3}\frac{(k\sqrt{x+1})-(2k)}{x-3} = \frac{0}{0} \quad \Rightarrow \quad \text{La'Hopital}\\ &= \lim_{x\to 3}\frac{\frac{1}{2}k(x+1)^{-\frac{1}{2}}}{1} = \lim_{x\to 3}\frac{\frac{1}{2}k}{\sqrt{x+1}} = \frac{1}{4}k\end{aligned}$$

ii) From the right of point $x = 3$

$$\begin{aligned}\lim_{x\to 3^+}\frac{f(x)-f(3)}{x-3} &= \lim_{x\to 3}\frac{(5-mx)-(2k)}{x-3} = \frac{(5-3m)-2k}{0}\\ &= \frac{0:\ \text{Eqn (4.5)}}{0} \quad \Rightarrow \quad \text{La'Hopital}\\ &= \lim_{x\to 3}\frac{-m}{1} = -m\end{aligned}$$

iii) For derivative to exist, the limits must be equal,

$$\frac{1}{4}k = -m \tag{4.6}$$

To be differentiable, it has to be continuous and derivative exists. Thus solving the simultaneous equations (4.5) and (4.6)

$$\begin{aligned}5-3m &= 2k\\ \frac{k}{4} &= -m\\ \Rightarrow \quad k = 4,& \quad m = -1\end{aligned}$$

Example 4.2.4 Let

$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Show that f is continuous for all values of x. Show that f is differentiable for all values of x, but that the derivative, f', is not continuous at $x = 0$.

First show that f is continuous for all values of x. Describe f using functional composition. Let

$$g(x) = \frac{1}{x}, h(x) = \cos x, \text{ and } k(x) = x^2$$

Function h is well-known to be continuous for all values of x.

Function k is a polynomial and is therefore continuous for all values of x.

Function g is the quotient of functions continuous for all values of x, and is therefore continuous for all values of x except $x = 0$, that x which makes the denominator zero. Thus, for all values of x except $x = 0$

$$f(x) = k(x)h(g(x)) = x^2 \cos(g(x)) = x^2 \cos\left(\frac{1}{x}\right)$$

is a continuous function (the product and functional composition of continuous functions). `Continuity of` f `at` $x = 0$. Function f is defined at $x = 0$ since

1.) The limit $\lim_{x \to 0} \cos\left(\dfrac{1}{x}\right)$ does not exist since the values of $\cos\left(\dfrac{1}{x}\right)$ oscillate between -1 and $+1$ as x approaches zero. However, for $x \neq 0$

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq +1$$

so that

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2.$$

Since

$$\lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} x^2,$$

it follows from the Squeeze Principle that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

2.) The function is defined at $x = 0$

$$f(0) = 0.$$

3.)

$$\lim_{x \to 0} f(x) = 0 = f(0),$$

all three conditions are satisfied, and f is continuous at $x = 0$. Thus, f is continuous for all values of x.

Now show that f is differentiable for all values of x. For $x \neq 0$ we can differentiate f using the product rule and the chain rule. That is, for $x \neq 0$ the derivative of f is

$$\begin{aligned} f'(x) &= x^2\ D\Big\{\cos\left(\frac{1}{x}\right)\Big\} + D\{x^2\}\ \cos\left(\frac{1}{x}\right) \\ &= x^2\Big\{-\sin\left(\frac{1}{x}\right)\ D\Big\{\frac{1}{x}\Big\}\Big\} + \{2x\}\cos\left(\frac{1}{x}\right) \\ &= -x^2\sin\left(\frac{1}{x}\right)\Big\{\frac{-1}{x^2}\Big\} + 2x\cos\left(\frac{1}{x}\right) \\ &= \sin\left(\frac{1}{x}\right) + 2x\cos\left(\frac{1}{x}\right) \end{aligned}$$

Use the limit definition of the derivative to differentiate f at $x = 0$. Then

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h) - 0}{h} \\ &= \lim_{h\to 0} \frac{(h)^2\cos\left(\frac{1}{h}\right)}{h} \\ &= \lim_{h\to 0}\ h\cos\left(\frac{1}{h}\right) \end{aligned}$$

Use the Squeeze Principle to evaluate this limit. For $h \neq 0$

$$-1 \leq \cos\left(\frac{1}{h}\right) \leq +1.$$

If $h > 0$, then

$$-h \leq h\cos\left(\frac{1}{h}\right) \leq h.$$

If $h < 0$, then

$$-h \geq h\cos\left(\frac{1}{h}\right) \geq h.$$

In either case,

$$\lim_{h\to 0}\ (-h) = 0 = \lim_{h\to 0}\ h,$$

and it follows from the Squeeze Principle that

$$f'(0) = \lim_{h\to 0}\ h\cos\left(\frac{1}{h}\right) = 0.$$

Thus, f is differentiable for all values of x. Check to see if f' is continuous at $x = 0$. The function f' is defined at $x = 0$ since

1.) However,

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left[\sin\left(\frac{1}{x}\right) + 2x\cos\left(\frac{1}{x}\right)\right]$$

does not exist since the values of $\sin\left(\frac{1}{x}\right)$ oscillate between -1 and $+1$ as x approaches zero.

2.)

$$f'(0) = 0$$

Thus, condition (a) is violated, and the derivative, f', is not continuous at $x = 0$.

Note 4.2.2 The continuity of function f for all values of x also follows from the fact that f is differentiable for all values of x.

4.3 Differentiation Theorems

Let $f(x)$ and $g(x)$ be differentiable and α a scalar, then $\alpha f(x)$, $(f \pm g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$ are all differentiable functions such that

1.) *Constant rule*: if $f(x)$ is constant, then

$$f'(x) = 0 \tag{4.7}$$

2.) The *scalar multiplication*

$$(\alpha f)'(x) = \alpha f'(x) \tag{4.8}$$

3.) The *sum rule* of differentiation

$$(f+g)'(x) = f'(x) + g'(x) \tag{4.9}$$

4.) The *product rule* of differentiation

$$(fg)'(x) = f'(x)g(x) + g'(x)f(x) \tag{4.10}$$

5.) The *quotient rule* of differentiation

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \tag{4.11}$$

Example 4.3.1 Prove the sum rule of differentiation equation (4.9)

Proof :

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x)
\end{aligned}$$

■

Example 4.3.2 Prove the product rule $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$ of differentiation Equation (4.10).

Proof :

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)\left[f(x+h) - f(x)\right] + f(x)\left[g(x+h) - g(x)\right]}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)\left[f(x+h) - f(x)\right]}{h} + \lim_{h\to 0} \frac{f(x)\left[g(x+h) - g(x)\right]}{h} \\
&= \lim_{h\to 0} g(x+h) \lim_{h\to 0} \frac{\left[f(x+h) - f(x)\right]}{h} + \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{\left[g(x+h) - g(x)\right]}{h} \\
&= g(x)f'(x) + f(x)g'(x)
\end{aligned}$$

■

Exercise 4.10 Prove the difference rule of differentiation $(f-g)'(x) = f'(x) - g'(x)$

Exercise 4.11 Use the formal definition of derivatives to prove $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$, the quotient rule Eqn (4.11)

Example 4.3.3 Use the product rule theorem Eqn (4.10) to find $\frac{d}{dx}(e^{x^2}\sin x)$

$$\begin{aligned} \text{Let } f(x) &= e^{x^2} \Rightarrow f'(x) = 2xe^{x^2} \\ \text{and } g(x) &= \sin x \Rightarrow g'(x) = \cos x \end{aligned}$$

$\Rightarrow$ By the theorem, we have that,

$$\begin{aligned} (fg)'(x) &= f'(x)g(x) + g'(x)f(x) \\ &= 2xe^{x^2}\sin x + \cos x e^{x^2} \\ &= e^{x^2}(2x\sin x + \cos x) \end{aligned}$$

Example 4.3.4 Use the quotient rule of the theorem to compute

$$\frac{d}{dx}\left(\frac{\sin^2 x}{1-e^{-x}}\right)$$

Let

$$\begin{aligned} f(x) = \sin^2 x &\Rightarrow f'(x) = 2\sin x\cos x \\ g(x) = 1 - e^{-x} &\Rightarrow g'(x) = e^{-x} \end{aligned}$$

By the quotient theorem, Eqn (4.11) we have that,

$$\begin{aligned} \left(\frac{f}{g}\right)'(x) &= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \\ &= \frac{2\sin x\cos x(1-e^{-x}) - e^{-x}\sin^2 x}{(1-e^{-x})^2} \end{aligned}$$

Example 4.3.5

$$\frac{d}{dx}\left(\frac{1}{x-2}\right) = \frac{\frac{d}{dx}1\cdot(x-2) - 1\cdot\frac{d}{dx}(x-2)}{(x-2)^2} = \frac{0\cdot(x-2) - 1\cdot 1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

Example 4.3.6

$$\begin{aligned} \frac{d}{dx}\left(\frac{x-1}{x-2}\right) &= \frac{(x-1)'(x-2) - (x-1)(x-2)'}{(x-2)^2} = \frac{1\cdot(x-2) - (x-1)\cdot 1}{(x-2)^2} \\ &= \frac{(x-2)-(x-1)}{(x-2)^2} = \frac{-1}{(x-2)^2} \end{aligned}$$

Example 4.3.7

$$\frac{d}{dx}\left(\frac{5x^3+x}{2-x^7}\right) = \frac{(5x^3+x)' \cdot (2-x^7) - (5x^3+x) \cdot (2-x^7)}{(2-x^7)^2}$$
$$= \frac{(15x^2+1) \cdot (2-x^7) - (5x^3+x) \cdot (-7x^6)}{(2-x^7)^2}$$

and there's hardly any point in simplifying the last expression, unless someone gives you a good reason. In general, it's not so easy to see how much may or may not be gained in 'simplifying', and we won't make ourselves crazy over it.

Note 4.3.1 One way that the product rule can be useful is in postponing or eliminating a lot of algebra. For example, to evaluate

$$\frac{d}{dx}\left((x^3+x^2+x+1)(x^4+x^3+2x+1)\right)$$

we *could* multiply out and then take the derivative term-by-term as we did with several polynomials above. This would be at least mildly irritating because we'd have to do a bit of algebra. Rather, just apply the product rule *without* feeling compelled first to do any algebra:

$$\frac{d}{dx}\left((x^3+x^2+x+1)(x^4+x^3+2x+1)\right)$$
$$= (x^3+x^2+x+1)'(x^4+x^3+2x+1) + (x^3+x^2+x+1)(x^4+x^3+2x+1)'$$
$$= (3x^2+2x+1)(x^4+x^3+2x+1) + (x^3+x^2+x+1)(4x^3+3x^2+2)$$

Now if we were somehow still obliged to multiply out, then we'd still have to do some algebra. But *we can take the derivative without multiplying out*, if we want to, by using the product rule.

For that matter, once we see that there is a *choice* about doing algebra either *before* or *after* we take the derivative, it might be possible to make a choice which minimizes our computational labor. This could matter.

Example 4.3.8 Suppose we want to differentiate $y = x^2 \cos 3x$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(-3x\sin 3x + 2\cos 3x)$$

Example 4.3.9 Suppose we want to differentiate

$$y = x^3(4-x)^{1/2}$$

$$f(x) = x^3 \quad\Rightarrow\quad f'(x) = 3x^2$$
$$g(x) = (4-x)^{1/2} \quad\Rightarrow\quad g'(x) = -\frac{1}{2}(4-x)^{-1/2}$$

$$\begin{aligned}
(fg)'(x) &= f'(x).g(x) + g'(x)f(x)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= (3x^2)(4-x)^{1/2} - \frac{x^3}{2(4-x)^{1/2}}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(4-x)^{1/2}\cdot 3x^2}{1}\cdot\frac{2(4-x)^{1/2}}{2(4-x)^{1/2}} - \frac{x^3}{2(4-x)^{1/2}} = \frac{6x^2(4-x)-x^3}{2(4-x)^{1/2}} = \frac{x^2(24-7x)}{2(4-x)^{1/2}}
\end{aligned}$$

Example 4.3.10 Suppose we want to differentiate $y = (1-x^3)\mathrm{e}^{2x}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= (1-x^3)\cdot 2\mathrm{e}^{2x} + \mathrm{e}^{2x}\cdot(-3x^2)\\
&= \mathrm{e}^{2x}(2-3x^2-2x^3)
\end{aligned}$$

Example 4.3.11 Find the derivative of each of the following:

1.) $x \tan x$

$$x \sec^2 x + \tan x$$

2.) $x^2 \mathrm{e}^{-x}$

$$x(2-x)\mathrm{e}^{-x}$$

3.) $5\mathrm{e}^{-2x} \sin 3x$

$$5\mathrm{e}^{-2x}(3\cos 3x - 2\sin 3x)$$

4.) $3x^{1/2} \cos 2x$

$$\frac{3}{2}x^{-1/2}(\cos 2x - 4x \sin 2x)$$

5.) $2x^6(1+x)^5$

$$2x^5(1+x)^4(6+11x)$$

6.) $x^{-2}(1+x^2)^{1/2}$

$$-x^{-3}((1+x^2)^{-1/2}(2+x^2)$$

7.) $x\mathrm{e}^x \sin x$

$$\mathrm{e}^x[(1+x)\sin x + x\cos x]$$

8.) $7x^{3/2}\mathrm{e}^{-4x} \cos 2x$

$$\frac{7}{2}x^{1/2}\mathrm{e}^{-4x}(3\cos 2x - 8x\cos 2x - 4x \sin x)$$

Example 4.3.12 Compute the derivative of $y = (2x^2+6x)(2x^3+5x^2)$

$$y' = (2x^2+6x)(6x^2+10x) + (2x^3+5x^2)(4x+6) = 20x^4 + 88x^3 + 90x^2$$

Exercise 4.12 Find $\dfrac{d}{dx}f(x)$ given that

1.)

$$f(x) = 2x^{\frac{1}{2}} - x^3 + 2$$

2.)

$$f(x) = \alpha x^3 - \beta x^2 + \lambda x - \theta$$

where α, β, λ and θ are constants.

Exercise 4.13 Find $\dfrac{dy}{dx}$ of

1.)

$$y = xe^x$$

2.)

$$y = (x^3+3)^5 \sin x$$

3.)

$$\frac{1}{x^2} + \frac{1}{y} = \frac{1}{5}$$

4.)

$$\sqrt[3]{\left(\frac{x+1}{x-4}\right)}$$

4.4 Other techniques of differentiation

Other than the sum, difference, product, quotient or constant differentiation, other forms of differentiation include

1.) Chain rule
2.) Implicit differentiation
3.) Parametric differentiation
4.) Logarithmic differentiation

4.4.1 Chain Rule - Composite differentiation

Theorem 4.4.1 Let $g(x)$ be differentiable at x and $h(g)$ be differentiable at $g(x)$, then $h \circ g(x) = h(g(x))$ is differentiable at x and if $f(x) = h(g(x))$, then

$$\begin{aligned} (h \circ g)'(x) = \frac{d}{dx}[h(g(x))] &= \frac{dh}{dg} \cdot \frac{dg}{dx} \\ (h \circ g)'(x) &= h'(g(x)) \cdot g'(x) \end{aligned} \tag{4.12}$$

Proof :

$$\begin{aligned} \frac{d}{dx}[h(g(x))] &= \lim_{h \to 0} \frac{h(g(x+h)) - h(g(x))}{h} \\ &= \lim_{h \to 0} \frac{h(g(x+h)) - h(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\ Let\ \Delta g &= g(x+h) - g(x) \\ \lim_{h \to 0} \Delta g &= 0 = \lim_{h \to 0} \frac{h(g+h) - h(g)}{\Delta g} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \end{aligned}$$

■

Note 4.4.1 The chain rule can only be used when you can express a function f given as a composite of two functions h and g.

Example 4.4.1 Using the chain rule find $f'(x)$ for

$$f(x) = \frac{1}{(4x^2 - x)^5}$$

This can be decomposed as the composite of two functions:

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) &= 4x^2 - x, \\ h(g) &= \frac{1}{g^5} \end{aligned}$$

Their derivatives are:

$$\frac{dg}{dx} = 8x - 1 \quad , \quad \frac{dh}{dg} = -5g^{-6} = \frac{-5}{g^6}$$

The derivative function is therefore:

$$\frac{df}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx} = \frac{-5}{g^6} \cdot (8x - 1) = -\frac{5(8x - 1)}{(4x^2 - x)^6}$$

Example 4.4.2 Differentiate

$$f(x) = (3x+1)^2$$

In a short form, its

$$\begin{aligned} f'(x) &= 2(3x+1)(3) \\ f'(x) &= 6(3x+1) \end{aligned}$$

To use the chain rule, this can be decomposed as the composite of two functions:

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) &= 3x+1, \\ h(g) &= g^2 \end{aligned}$$

Their derivatives are:

$$\frac{dg}{dx} = 3, \qquad \frac{dh}{dg} = 2g$$

The derivative function is therefore:

$$\frac{df}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx} = 2g(3) = 6g = 6(3x+1)$$

Example 4.4.3 Differentiate

$$y = \sqrt{13x^2 - 5x + 8}$$

$$\begin{aligned} y' &= \frac{1}{2}(13x^2 - 5x + 8)^{-\frac{1}{2}}(26x - 5) \\ y' &= \frac{26x - 5}{2\sqrt{13x^2 - 5x + 8}} \end{aligned}$$

Example 4.4.4 Differentiate

$$y = (1 - 4x + 7x^5)^{30}$$

$$\begin{aligned} y' &= 30(1 - 4x + 7x^5)^{29}(-4 + 35x^4) \\ y' &= 30(35x^4 - 4)(1 - 4x + 7x^5)^{29} \end{aligned}$$

Example 4.4.5 Differentiate

$$y = (4x + x^{-5})^{\frac{1}{3}}$$

$$y' = \frac{1}{3}(4x + x^{-5})^{-\frac{2}{3}}(4 - 5x^{-6})$$

Example 4.4.6 Differentiate

$$y = \left(\frac{8x - x^6}{x^3}\right)^{-\frac{4}{5}}$$

First, begin by simplifying the expression before we differentiate it

$$y = \left(8x^{-2} - x^3\right)^{-\frac{4}{5}}$$

$$y' = -\frac{4}{5}\left(8x^{-2} - x^3\right)^{-\frac{9}{5}}\left(-16x^{-3} - 3x^2\right)$$

Example 4.4.7 The derivative of

$$f(x) = x^4 + \sin(x^2) - \ln(x)e^x + 7$$

$$\begin{aligned} f'(x) &= 4x^{(4-1)} + \frac{d\,(x^2)}{dx}\cos(x^2) - \frac{d\,(\ln x)}{dx}e^x - \ln x\frac{d\,(e^x)}{dx} + 0 \\ &= 4x^3 + 2x\cos(x^2) - \frac{1}{x}e^x - \ln(x)e^x. \end{aligned}$$

Here the second term was computed using the chain rule and third using the product rule. The known derivatives of the elementary functions $x^2, x^4, \sin x, \ln(x)$ and e^x, as well as the constant 7, were also used.

Example 4.4.8 For concreteness, consider the function

$$f(x) = e^{\sin x^2}$$

This can be decomposed as the composite of three functions:

$$\begin{aligned} f(x) &= h(g(p(x))) \\ p(x) &= x^2, \\ g(p) &= \sin p, \\ h(g) &= e^g \end{aligned}$$

Their derivatives are:

$$\frac{dp}{dx} = 2x, \quad \frac{dg}{dp} = \cos p, \quad \frac{dh}{dg} = e^g$$

The derivative function is therefore:

$$\begin{aligned} \frac{df}{dx} &= \frac{dh}{dg}\cdot\frac{dg}{dp}\cdot\frac{dp}{dx} \\ \frac{df}{dx} &= e^{\sin x^2}\cdot\cos x^2\cdot 2x \end{aligned}$$

Exercise 4.14 Differentiate

$$y = \sin(5x)$$

Exercise 4.15 Differentiate

$$y = e^{5x^2+7x-13}$$

Exercise 4.16 Differentiate

$$y = 3\tan\sqrt{x}$$
$$y' = \frac{3\sec^2\sqrt{x}}{2\sqrt{x}}$$

Exercise 4.17 Differentiate

$$y = \cos^2(x^3)$$

Example 4.4.9 Use the chain rule to differentiate

$$f(x) = (x^3 + 5x)^7$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = x^3 + 5x, \quad & h(g) = g^7 \\ f'(x) &= 7(x^3 + 5x)^6 \cdot (3x^2 + 5) \end{aligned}$$

Example 4.4.10 Use the chain rule to differentiate

$$f(x) = \sqrt{5\cos x}$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = 5\cos x, \quad & h(g) = g^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(5\cos x)^{-\frac{1}{2}} \cdot 5(-\sin x) \end{aligned}$$

Example 4.4.11 Use the chain rule to differentiate

$$f(x) = 7e^{x^2-5}$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = x^2 - 5, \quad & h(g) = 7e^g \\ f'(x) &= 7e^{x^2-5} \cdot (2x) \end{aligned}$$

Example 4.4.12 Use the chain rule to differentiate

$$f(x) = -3\tan(5x^4)$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = 5x^4, \quad & h(g) = -3\tan g \\ f'(x) &= -3\sec^2(5x^4) \cdot (20x^3) \end{aligned}$$

Example 4.4.13 Use the chain rule to differentiate

$$f(x) = \frac{8}{4 + \sin x}$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = 4 + \sin x, \qquad & h(g) = \frac{8}{g} \\ f'(x) &= -8(4 + \sin x)^{-2} \cdot \cos x \end{aligned}$$

Example 4.4.14 Find the derivative of $f(x) = \sin(5x)$ using the chain rule.

$$f'(x) = 5 \cdot \Big[\cos(5x)\Big] = 5\cos(5x)$$

Example 4.4.15 Find the derivative of

$$f(t) = \left(t^2 - \frac{2}{t^3}\right)^2$$

We will use the Chain rule. Set $f = y(u(t))$

$$u = t^2 - \frac{2}{t^3} \text{ and } y = u^2$$

The Chain rule implies

$$\frac{df}{dt} = \frac{du}{dt}\frac{dy}{du} = \left(2t + \frac{6}{t^4}\right) 2u = 2\left(2t + \frac{6}{t^4}\right)\left(t^2 - \frac{2}{t^3}\right)$$

Exercise 4.18 Use the chain rule to find $f'(x)$ for

1.)

$$f(x) = (4 - x)^{\frac{1}{2}}$$

$$h(g) = g^{\frac{1}{2}}$$

2.)

$$f(x) = (7x^2 - 5x)^3$$

$$h(g) = g^3$$

3.)

$$f(x) = \frac{1}{(3x - 2)}$$

$$h(g) = g^{-1} = \frac{1}{g}$$

4.)

$$f(x) = 5 + \cos^3 x$$

5.)

$$f(x) = \sqrt[3]{1 + \tan x}$$

Exercise 4.19 Given $f = x^2 - 3$ and $g = 4x + 7$, compute

1.) $(f \circ g)'(x)$

2.) $(g \circ f)'(x)$

3.) $(f \circ g)'(5)$

4.) $(g \circ f)'(-4)$

4.4.2 Differentiation of implicit functions

Implicit differentiation is nothing more than a special case of the well-known chain rule for derivatives. The majority of differentiation problems in first-year calculus involve functions y written *explicitly* as functions of x. For example, if

$$y = 3x^2 - \sin(7x+5)$$

then the derivative of y is

$$y' = 6x - 7\cos(7x+5)$$

However, some functions y are written *implicitly* as functions of x. A familiar example of this is the equation

$$x^2 + y^2 = 25$$

which represents a circle of radius five centered at the origin. Suppose that we wish to find the slope of the line tangent to the graph of this equation at the point $(3, -4)$

How could we find the derivative of y in this instance ? One way is to first write y explicitly as a function of x. Thus,

$$x^2 + y^2 = 25 \;\Rightarrow\; y^2 = 25 - x^2$$

and

$$y = \pm\sqrt{25 - x^2}$$

where the positive square root represents the top semi-circle and the negative square root represents the bottom semi-circle. Since the point $(3, -4)$ lies on the bottom semi-circle given by

$$y = -\sqrt{25 - x^2}$$

i.e,

$$y' = \frac{x}{\sqrt{25 - x^2}}$$

Thus, the slope of the line tangent to the graph at the point $(3, -4)$ is

$$m = y' = \frac{3}{\sqrt{25 - (3)^2}} = \frac{3}{4}$$

Unfortunately, not every equation involving x and y can be solved explicitly for y

With Implicit differentiation, we differentiate both sides with respect to x, and make y' the subject.

Example 4.4.16 Differentiate

$$x^2 + y^2 = 25$$

$$\begin{aligned}
x^2 + y^2 &= 25 \\
D(x^2) + D(y^2) &= D(25) \\
2x + 2yy' &= 0 \\
2yy' &= -2x \\
y' &= \frac{-2x}{2y} = \frac{-x}{y}
\end{aligned}$$

Thus, the slope of the line tangent to the graph at the point $(3, -4)$ is

$$m = y' = \frac{-x}{y} = \frac{-(3)}{(-4)} = \frac{3}{4}$$

Example 4.4.17 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$x^3 + y^3 = 4$$

Begin with $x^3 + y^3 = 4$. Differentiate both sides of the equation, getting

$$\begin{aligned} x^3 + y^3 &= 4 \\ D(x^3 + y^3) &= D(4) \\ D(x^3) + D(y^3) &= D(4) \end{aligned}$$

Remember to use the chain rule on $D(y^3)$

$$3x^2 + 3y^2 y' = 0$$

so that (Now solve for y')

$$3y^2 y' = -3x^2$$

and

$$y' = \frac{-x^2}{y^2}$$

Example 4.4.18 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$(x-y)^2 = x + y - 1$$

Begin with $(x-y)^2 = x + y - 1$. Differentiate both sides of the equation, getting

$$\begin{aligned} (x-y)^2 &= x + y - 1 \\ D(x-y)^2 &= D(x + y - 1) \\ D(x-y)^2 &= D(x) + D(y) - D(1) \\ 2(x-y)D(x-y) &= 1 + y' - 0 \\ 2(x-y)(1-y') &= 1 + y' \end{aligned}$$

Now solve for y'

$$\begin{aligned} y'[-2(x-y) - 1] &= 1 - 2(x-y) \\ y' &= \frac{1 - 2(x-y)}{-2(x-y) - 1} = \frac{2y - 2x + 1}{2y - 2x - 1} \end{aligned}$$

Example 4.4.19 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$y = \sin(3x + 4y)$$

Begin with $y = \sin(3x + 4y)$. Differentiate both sides of the equation, getting

$$\begin{aligned} y &= \sin(3x + 4y) \\ D(y) &= D\left(\sin(3x + 4y)\right) \\ y' &= \cos(3x + 4y) D\left(3x + 4y\right) \\ y' &= \cos(3x + 4y)\left(3 + 4y'\right) \end{aligned}$$

Now solve for y'

$$\begin{aligned} y'\left[1 - 4\cos(3x + 4y)\right] &= 3\cos(3x + 4y) \\ y' &= \frac{3\cos(3x + 4y)}{1 - 4\cos(3x + 4y)} \end{aligned}$$

Example 4.4.20 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$y = x^2y^3 + x^3y^2$$

Begin with $y = x^2y^3 + x^3y^2$. Differentiate both sides of the equation, getting

$$\begin{aligned} y &= x^2y^3 + x^3y^2 \\ D(y) &= D\left(x^2y^3 + x^3y^2\right) \\ y' &= D(x^2y^3) + D(x^3y^2) \\ y' &= x^2(3y^2y') + (2x)y^3 + x^3(2yy') + (3x^2)y^2 \\ y' &= 3x^2y^2y' + 2xy^3 + 2x^3yy' + 3x^2y^2 \end{aligned}$$

Now solve for y'

$$y' = \frac{2xy^3 + 3x^2y^2}{1 - 3x^2y^2 - 2x^3y}$$

Example 4.4.21 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$e^{xy} = e^{4x} - e^{5y}$$

$$y' = \frac{4e^{4x} - ye^{xy}}{xe^{xy} + 5e^{5y}}$$

Example 4.4.22 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$\cos^2 x + \cos^2 y = \cos(2x + 2y)$$

$$y' = \frac{[\cos x \sin x - \sin(2x + 2y)]}{[\sin(2x + 2y) - \cos y \sin y]}$$

Example 4.4.23 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$x = \sqrt{x^2 + y^2}$$

$$y' = \frac{\sqrt{x^2 + y^2} - x}{y}$$

Example 4.4.24 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$\frac{x - y^3}{y + x^2} = x + 2$$

$$y' = \frac{1 - y - 3x^2 - 4x}{3y^2 + x + 2}$$

Example 4.4.25 Assume that y is a function of x. Find $y' = \dfrac{dy}{dx}$ for

$$(x^2+y^2)^3 = 8x^2y^2$$

$$y' = \frac{16xy^2 - 6x(x^2+y^2)^2}{6y(x^2+y^2)^2 - 16x^2y}$$

Example 4.4.26 For the function, $x^2+y^3 = 5y$, find $\dfrac{dy}{dx}$

$$\begin{aligned}
\texttt{Since } x^2+y^3 &= 5y\\
\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^3) &= \frac{d}{dx}(5y)\\
\Rightarrow 2x + 3y^2\frac{dy}{dx} &= 5\frac{dy}{dx}\\
\Rightarrow \frac{dy}{dx}(3y^2-5) &= -2x\\
\Rightarrow \frac{dy}{dx} &= \left(\frac{-2x}{3y^2-5}\right)
\end{aligned}$$

Example 4.4.27 Find $\dfrac{dy}{dx}$ for $x^2y - 2x^3y^2 = 4$

We differentiate term by term with respect to x i.e

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(x^2y) - 2\frac{d}{dx}(x^3y^2) &= \frac{d}{dx}(4)\\
\Rightarrow 2xy + x^2\frac{dy}{dx} - 2(3x^2y^2 + 2x^3y\frac{dy}{dx}) &= 0\\
\Rightarrow 2xy + x^2\frac{dy}{dx} - 6x^2y^2 - 4x^3y\frac{dy}{dx} &= 0\\
\Rightarrow \frac{dy}{dx}(x^2 - 4x^3y) &= 6x^2y^2 - 2xy\\
\texttt{therefore } \frac{dy}{dx} &= \frac{6x^2y^2 - 2xy}{x^2 - 4x^3y}\\
&= \frac{6xy^2 - 2y}{x - 4x^2y}
\end{aligned}$$

Exercise 4.20

$$2y + 5 - x^2 - y^3 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x}{2-3y^2}$$

Exercise 4.21

$$y^4 + x^5 - 7x^2 - 5x^{-1} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-5x^4 + 14x - 5x^{-2}}{4y^3}$$

Exercise 4.22

$$y^4 + 2x^2y^2 + 6x^2 = 7 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-xy^2 - 3x}{y^3 + x^2y}$$

4.4.3 Parametric equations

To differentiate parametric equations, we must use the *chain rule.*

The equations of a plane curve $f(x, y) = 0$ may be given by equations of the type $x = x(t)$ and $y = y(t)$, where t is the variable called a parameter. These equations are called parametric equations of the curve.

Example 4.4.28 Consider the parametric equations

$$x = \cos t, \ y = \sin t \qquad \text{for } 0 \leq t \leq 2\pi$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

Example 4.4.29 If $x = 2at^2$ and $y = 4at$, find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} = (4a) \cdot \frac{1}{4at} = \frac{1}{t} \\ \frac{dx}{dy} &= \frac{dx}{dt} \cdot \frac{dt}{dy} = (4at) \cdot \frac{1}{4a} = t \end{aligned}$$

Example 4.4.30 Finding the second derivative is a little trickier.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{dt}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

or Example (4.4.29)

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx} = \left(\frac{-1}{t^2}\right)\frac{1}{4at}$$

Exercise 4.23 Given

$$x = t^3 - t \qquad y = 4 - t^2$$

Compute

1.) dx/dy

2.) dy/dx

3.) $(dy/dx)^2$

4.) d^2y/dx^2

Example 4.4.31

$$x = t + \cos t, \ \ y = \sin t$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos t}{1 - \sin t} \\ \frac{d^2y}{dx^2} &= \frac{-\sin t + 1}{(1 - \sin t)^3} = \frac{1}{(1 - \sin t)^2} \end{aligned}$$

Example 4.4.32 Find the second derivative if $x = t - t^2$, $y = t - t^3$

$$\frac{d^2y}{dx^2} = \frac{1 - 3t^2}{(1 - 2t)^2}$$

Example 4.4.33 Find the equation of a curve whose parametric equations are

1.) $x = t$ and $y = 5t + 6$

2.) $x = \alpha \cos\theta$ and $y = \beta \sin\theta$

To solve, just eliminate the parameter

1.) We eliminate the parameter t from the equations i.e $y = 5x + 6$

2.)

$$\begin{aligned} x^2 &= \alpha^2 \cos^2\theta \\ \Rightarrow \frac{x^2}{\alpha^2} &= \cos^2\theta \qquad (4.13) \\ \texttt{and } y^2 &= \beta^2 \sin^2\theta \\ \Rightarrow \frac{y^2}{\beta^2} &= \sin^2\theta \qquad (4.14) \end{aligned}$$

Adding equations (4.13) to (4.14) we have

$$\begin{aligned} \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} &= \cos^2\theta + \sin^2\theta \\ \Rightarrow \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} &= 1 \end{aligned}$$

Which is an ellipse

4.4.3.1 Derivatives of parametrically defined curves

If we let $x = x(t)$ and $y = y(t)$ be parametric equations of $f(x, y) = 0$ this means that,

$$\begin{aligned} \frac{dx}{dt} &= x'(t), \quad \frac{dy}{dt} = y'(t) \\ \texttt{but } \frac{dy}{dx} &= \frac{dy}{dt}.\frac{dt}{dx} \ (\texttt{Chain rule }) \\ &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(x)}{x'(t)} \ \texttt{provided } (x'(t) \neq 0) \end{aligned}$$

Example 4.4.34 Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \quad , \quad y = t^2$$

$$\frac{dy}{dx} = \frac{2}{5t^3 - 12t}$$

Example 4.4.35 Given

$$x(t) = t^3 \ , \quad y(t) = t^4$$

$$\frac{dy}{dx} = \frac{4}{3}t \qquad \frac{d^2y}{dx^2} = \frac{4}{9t^2}$$

Example 4.4.36 Compute dy/dx for the following parametric equations

1.) $x = t^2$ and $y = 4t^2 + 5$

Solution : *Since* $x(t) = t^2 \quad \Rightarrow x'(t) = 2t$

$$\begin{aligned} \textit{and } y &= 4t^2 + 5 \ \Rightarrow y'(t) = 8t \\ \textit{therefore } \frac{dy}{dx} &= \frac{y'(t)}{x'(t)} = \frac{8t}{2t} = 4 \end{aligned}$$

■

2.) $x = t$ and $y = \dfrac{t^3}{3} + \dfrac{1}{7t^2}$

Solution : $x = t \ \Rightarrow x'(t) = 1$

$$\begin{aligned} \textit{and } y &= \frac{t^3}{3} + \frac{1}{7}t^{-2} \ \Rightarrow y'(t) = t^2 - \frac{2}{7}t^{-3} \\ \textit{therefore } \frac{dy}{dx} &= \frac{y'(t)}{x'(t)} = t^2 - \frac{2}{7t^3} \end{aligned}$$

■

3.) $x = e^t \cos t$ and $y = e^t \sin t$ $(0 \leq t \leq \pi)$

Solution : *Since* $x = e^t \cos t, \quad y = e^t \sin t \ (0 \leq t \leq \pi)$

$$\begin{aligned} x'(t) &= e^t \cos t - e^t \sin t \\ y'(t) &= e^t \sin t + e^t \cos t \\ \Rightarrow \frac{dy}{dx} &= \frac{e^t(\cos t + \sin t)}{e^t(\cos t - \sin t)} \\ &= \frac{(\cos t + \sin t)}{(\cos t - \sin t)} \end{aligned}$$

This holds only when $\cos t \neq \sin t$ ■

Exercise 4.24 Given an equation

$$\begin{aligned} y &= \sin 2p \\ x &= cosp \end{aligned}$$

Compute the ordinary differential equations

1.) dx/dy 2.) $(dy/dx)^2$

4.4.4 Logarithmic differentiation

Logarithmic differentiation is a powerful technique for differentiating functions . However, the method is "uneconomical " for simple functions like polynomial functions.

This is done by

1.) Let y, be the function to differentiate

2.) Take logs on both sides, take $\log_e = \ln$

3.) Differentiate both sides

4.) Make dy/dx the subject

5.) Substitute back the y value.

We here present some common suitable forms for the logarithmic differentiation.

1.
$$y = u(x)v(x)$$

where $u(x)$ and $v(x)$ are quite big expressions .On differentiating, we take logarithms to base e on both sides i.e

$$\begin{aligned} \ln y &= \ln u(x) + \ln v(x) \\ \Rightarrow \frac{1}{y}\frac{dy}{dx} &= \frac{1}{u(x)}.u'(x) + \frac{1}{v(x)}v'(x) \\ \Rightarrow \frac{dy}{dx} &= y(\frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)}) \\ &= v(x)u(x)(\frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)}) \\ &= v(x)u'(x) + u(x)v'(x) \end{aligned}$$

2.
$$y = \frac{u(x)v(x)}{h(x)g(x)}$$

Taking logs to both sides, we have

$$\begin{aligned} \ln y &= \ln u(x) + \ln v(x) - \ln h(x) - \ln g(x) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)} - \frac{h'(x)}{h(x)} - \frac{g'(x)}{g(x)} \\ \Rightarrow \frac{dy}{dx} &= y[(\frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)} - \frac{h'(x)}{h(x)} - \frac{g'(x)}{g(x)})]. \end{aligned}$$

3.
$$y = (u(x))^{v(x)}$$

Taking logs on both sides we have,

$$\begin{aligned} \ln y &= v(x)\ln u(x) \\ \frac{1}{y}\frac{dy}{dx} &= v'(x)\ln u(x) + v(x)\frac{u'(x)}{u(x)} \end{aligned}$$

Example 4.4.37 Find the derivative of

$$\frac{(x^2+1)^3(x+1)^4}{x\sqrt{(x-1)}\sqrt{(x+3)}}$$

Using logarithmic differentiation,

$$\begin{aligned}
\text{Let } y &= \frac{(x^2+1)^3(x+1)^4}{x\sqrt{(x-1)}\sqrt{(x+3)}} \\
\Rightarrow \ln y &= 3\ln(x^2+1) + 4\ln(x+1) - \ln x - \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+3) \\
\Rightarrow \frac{1}{y}\frac{dy}{dx} &= \left\{\frac{6x}{x^2+1} + \frac{4}{x+1} - \frac{1}{x} - \frac{1}{2(x-1)} - \frac{1}{2(x+3)}\right\} \\
\text{therefore } \frac{dy}{dx} &= \left[\frac{6x}{x^2+1} + \frac{4}{x+1} - \frac{1}{x} - \frac{1}{2(x-1)} - \frac{1}{2(x+3)}\right] y \\
&= \left[\frac{6x}{x^2+1} + \frac{4}{x+1} - \frac{1}{x} - \frac{1}{2(x-1)} - \frac{1}{2(x+3)}\right] \frac{(x^2+1)^3(x+1)^4}{x\sqrt{(x-1)}\sqrt{(x+3)}}
\end{aligned}$$

Example 4.4.38 Given

$$y = (\sin x)^{\cos x}$$

find the derivative dy/dx

$$\begin{aligned}
\text{Since } y &= (\sin x)^{cosx} \\
\Rightarrow \ln y &= \cos x \ln(\sin x) \\
\Rightarrow \frac{1}{y}\frac{dy}{dx} &= -\sin x \ln \sin x + \frac{\cos^2 x}{\sin x} \\
\Rightarrow \frac{dy}{dx} &= \left(\frac{\cos^2 x}{\sin x} - \sin x \ln \sin x\right)(\sin x)^{\cos x}
\end{aligned}$$

Note 4.4.2 In this case it is only logarithmic differentiation which is applicable

Example 4.4.39 Differentiate

$$y = x^x$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply*! The function must first be revised before a derivative can be taken. Begin with

$$y = x^x$$

Apply the natural logarithm to both sides of this equation getting

$$\begin{aligned}
\ln y &= \ln x^x \\
\ln y &= x \ln x
\end{aligned}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule on the right-hand side. Thus, differentiating, we get

$$\frac{1}{y}y' = x\frac{1}{x} + (1)\ln x = 1 + \ln x$$

Multiply both sides of this equation by y (making y' the subject), getting

$$y' = y(1+\ln x) = x^x(1+\ln x)$$

Example 4.4.40 Differentiate the function

$$y = \frac{(x+2)(x-6)^3(x+4)^2}{(x-3)}$$

$$\begin{aligned}
\ln y &= \ln\left[\frac{(x+2)(x-6)^3(x+4)^2}{(x-3)}\right] = \ln(x+2) + \ln(x-6)^3 + \ln(x+4)^2 - \ln(x-3) \\
\frac{\frac{dy}{dx}}{y} &= \frac{1}{(x+2)} + \frac{3(x-6)^2}{(x-6)^3} + \frac{2(x+4)}{(x+4)^2} - \frac{1}{(x-3)} \\
&= \frac{1}{(x+2)} + \frac{3}{(x-6)} + \frac{2}{(x+4)} - \frac{1}{(x-3)} \\
\frac{dy}{dx} &= y\left[\frac{1}{(x+2)} + \frac{3}{(x-6)} + \frac{2}{(x+4)} - \frac{1}{(x-3)}\right] \\
&= \left(\frac{(x+2)(x-6)^3(x+4)^2}{(x-3)}\right)\left[\frac{1}{(x+2)} + \frac{3}{(x-6)} + \frac{2}{(x+4)} - \frac{1}{(x-3)}\right]
\end{aligned}$$

Example 4.4.41 Differentiate

$$y = x^{(e^x)}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation getting

$$\ln y \;=\; \ln x^{(e^x)} = e^x \ln x$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule on the right-hand side.

$$\begin{aligned}
\frac{1}{y}y' &= e^x\left\{\frac{1}{x}\right\} + e^x \ln x \\
\frac{1}{y}y' &= \frac{e^x}{x} + \left\{\frac{x}{x}\right\}e^x \ln x = \frac{e^x}{x} + \frac{xe^x \ln x}{x} = \frac{e^x + xe^x \ln x}{x} = \frac{e^x(1 + x\ln x)}{x}
\end{aligned}$$

Multiply both sides of this equation by y, getting (by combining the powers of x)

$$y' = y\frac{e^x(1+x\ln x)}{x} = x^{(e^x)}\frac{e^x(1+x\ln x)}{x^1} = x^{(e^x-1)}e^x(1+x\ln x)$$

Example 4.4.42 Differentiate

$$y = (3x^2+5)^{\frac{1}{x}}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln(3x^2+5)^{1/x} = \left(\frac{1}{x}\right)\ln(3x^2+5) = \frac{\ln(3x^2+5)}{x}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the quotient rule and the chain rule on the right-hand side. Thus,

$$\frac{1}{y}y' = \frac{x\left\{\frac{1}{3x^2+5}\right\}(6x) - \ln(3x^2+5)(1)}{x^2}$$

Get a common denominator and combine fractions in the numerator.

$$\frac{1}{y}y' = \frac{\dfrac{6x^2}{3x^2+5} - \ln(3x^2+5)\left\{\dfrac{3x^2+5}{3x^2+5}\right\}}{\dfrac{x^2}{1}}$$

Dividing by a fraction is the same as multiplying by its reciprocal.

$$\begin{aligned}\frac{1}{y}y' &= \frac{6x^2-(3x^2+5)\ln(3x^2+5)}{3x^2+5}\frac{1}{x^2}\\ \frac{1}{y}y' &= \frac{6x^2-(3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)}\end{aligned}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{6x^2-(3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)} = (3x^2+5)^{\frac{1}{x}}\,\frac{6x^2-(3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)^1}$$

Combine the powers of $(3x^2+5)$

$$y' = \frac{(3x^2+5)^{\left(\frac{1}{x}-1\right)}\left\{6x^2-(3x^2+5)\ln(3x^2+5)\right\}}{x^2}$$

Example 4.4.43 Differentiate

$$y = (\sin x)^{x^3}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln(\sin x)^{x^3} = x^3\ln(\sin x)$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side.

$$\frac{1}{y}y' = x^3\left\{\frac{1}{\sin x}\right\}\cos x + (3x^2)\ln(\sin x)$$

Get a common denominator and combine fractions on the right-hand side.

$$\frac{1}{y}y' = \frac{x^3\cos x}{\sin x} + 3x^2\ln(\sin x)\left\{\frac{\sin x}{\sin x}\right\} = \frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{\sin x}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{\sin x} = (\sin x)^{x^3}\frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{(\sin x)^1}$$

Combine the powers of $(\sin x)$.

$$y' = (\sin x)^{(x^3-1)}\left\{x^3\cos x + 3x^2\sin x\ln(\sin x)\right\}$$

Example 4.4.44 Differentiate

$$y = 7x(\cos x)^{\frac{x}{2}}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\ln y = \ln\left((7x)(\cos x)^{\frac{x}{2}}\right) = \ln(7x) + \ln(\cos x)^{\frac{x}{2}} = \ln(7x) + \left(\frac{x}{2}\right)\ln(\cos x)$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x . Use the product rule and the chain rule on the right-hand side.

$$\frac{1}{y}y' = \left\{\frac{7}{7x}\right\} + \left(\frac{x}{2}\right)\left\{\frac{1}{\cos x}\right\}(-\sin x) + \left(\frac{1}{2}\right)\ln(\cos x) = \frac{1}{x} - \frac{x\sin x}{2\cos x} + \frac{\ln(\cos x)}{2}$$

Get a common denominator and combine fractions on the right-hand side.

$$\frac{1}{y}y' = \frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}$$

Multiply both sides of this equation by y, getting

$$\begin{aligned}
y' &= y\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}\\
y' &= 7x(\cos x)^{\frac{x}{2}}\left[\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}\right]\\
y' &= 7(\cos x)^{x/2}\left[\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2(\cos x)^1}\right]\\
y' &= \left(\frac{7}{2}\right)(\cos x)^{\left(\frac{x}{2}-1\right)}\left\{2\cos x - x^2\sin x + x\cos x\ln(\cos x)\right\}
\end{aligned}$$

Example 4.4.45 Differentiate

$$y = (\sin x)^{x^3}$$

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln(\sin x)^{x^3} = x^3\ln(\sin x)$$

Differentiate both sides of this equation.

$$\frac{1}{y}y' = x^3\left\{\frac{1}{\sin x}\right\}\cos x + (3x^2)\ln(\sin x)$$

Into a common denominator

$$\frac{1}{y}y' = \frac{x^3\cos x}{\sin x} + 3x^2\ln(\sin x)\left\{\frac{\sin x}{\sin x}\right\} = \frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{\sin x}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{\sin x} = (\sin x)^{x^3}\frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{(\sin x)^1}$$

Combine the powers of $(\sin x)$.

$$y' = (\sin x)^{(x^3-1)}\left\{x^3\cos x + 3x^2\sin x\ln(\sin x)\right\}$$

Example 4.4.46 Differentiate

$$y = \sqrt{x}^{\sqrt{x}}\ e^{x^2}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\begin{aligned} \ln y &= \ln\left(\sqrt{x}^{\sqrt{x}} e^{x^2}\right) \\ &= \ln\left(\sqrt{x}^{\sqrt{x}}\right) + \ln\left(e^{x^2}\right) \\ &= \sqrt{x}\ln(\sqrt{x}) + x^2\ln(e) \\ &= \sqrt{x}\ln(\sqrt{x}) + x^2(1) \\ &= \sqrt{x}\ln(\sqrt{x}) + x^2 \end{aligned}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side.

$$\frac{1}{y}y' = \sqrt{x}\left\{\frac{1}{\sqrt{x}}\right\}(1/2)x^{-1/2} + (1/2)x^{-1/2}\ln(\sqrt{x}) + 2x = \frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}} + 2x$$

Get a common denominator and combine fractions on the right-hand side.

$$y' = \frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}} + 2x\left\{\frac{2\sqrt{x}}{2\sqrt{x}}\right\} = \frac{1+\ln(\sqrt{x})+4x^{1+1/2}}{2\sqrt{x}} = \frac{1+\ln(\sqrt{x})+4x^{3/2}}{2\sqrt{x}}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{1+\ln(\sqrt{x})+4x^{3/2}}{2\sqrt{x}} = \sqrt{x}^{\sqrt{x}} e^{x^2}\frac{1+\ln(\sqrt{x})+4x^{3/2}}{2\sqrt{x}^1}$$

Combine the powers of $\sqrt{x}$.

$$y' = (1/2)\sqrt{x}^{(\sqrt{x}-1)} e^{x^2}\left\{1+\ln(\sqrt{x})+4x^{3/2}\right\}$$

Example 4.4.47 Differentiate

$$y = x^{\ln x}(\sec x)^{3x}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\begin{aligned} \ln y &= \ln\left(x^{\ln x}(\sec x)^{3x}\right) \\ &= \ln x^{(\ln x)} + \ln(\sec x)^{3x} \\ &= (\ln x)(\ln x) + 3x\ln(\sec x) \\ &= (\ln x)^2 + 3x\ln(\sec x) \end{aligned}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side.

$$\frac{1}{y}y' = 2(\ln x)\left\{\frac{1}{x}\right\} + 3x\left\{\frac{1}{\sec x}\right\}(\sec x \tan x) + (3)\ln(\sec x)$$

Divide out a factor of $\sec x$.

$$\frac{1}{y}y' = \frac{2\ln x}{x} + 3x\tan x + 3\ln(\sec x)$$

Get a common denominator and combine fractions on the right-hand side.

$$\begin{aligned}\frac{1}{y}y' &= \frac{2\ln x}{x} + 3x\tan x\left\{\frac{x}{x}\right\} + 3\ln(\sec x)\left\{\frac{x}{x}\right\}\\ &= \frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x}\end{aligned}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x} = x^{\ln x}(\sec x)^{3x}\frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x^1}$$

Combine the powers of x.

$$y' = x^{(\ln x - 1)}(\sec x)^{3x}\left\{2\ln x + 3x^2\tan x + 3x\ln(\sec x)\right\}$$

Example 4.4.48 Consider the function

$$f(x) = \frac{x^5e^x(4x+3)}{5^{\ln x}(3-x)^2}$$

Find an equation of the line tangent to the graph of f at $x = 1$.

First note that

$$f(1) = \frac{(1)^5e^1\,(4(1)+3)}{5^{\ln 1}(3-1)^2} = \frac{7}{4}e^1$$

so that the tangent line passes through the point

$$(x, y) = \left(1, \frac{7}{4}e^1\right)$$

Then getting the tangent, by differentiating

$$\begin{aligned}\ln f(x) &= 5\ln x + x + \ln(4x+3) - (\ln 5)\ln x - 2\ln(3-x)\\ \frac{1}{f(x)}f'(x) &= \frac{5}{x} + 1 + \frac{4}{4x+3} - \frac{\ln 5}{x} + \frac{2}{3-x}\\ f'(x) &= f(x)\left\{\frac{5}{x} + 1 + \frac{4}{4x+3} - \frac{\ln 5}{x} + \frac{2}{3-x}\right\}\end{aligned}$$

The slope of the line tangent to the graph of f at $x = 1$ is

$$\begin{aligned}f'(1) &= f(1)\left\{\frac{5}{1} + 1 + \frac{4}{4+3} - \frac{\ln 5}{1} + \frac{2}{3-1}\right\}\\ &= \frac{7}{4}e^1\left(7 + \frac{4}{7} - \ln 5\right)\\ &= \frac{7}{4}e^1\left(\frac{53}{7} - \ln 5\right)\end{aligned}$$

Thus, the equation of the line tangent to the graph of f at $x = 1$ is

$$\begin{aligned}
\frac{y - \frac{7}{4}e^1}{x-1} &= \frac{7}{4}e^1\left(\frac{53}{7} - \ln 5\right) \\
y &= \frac{7}{4} + \frac{7}{4}e^1\left(\frac{53}{7} - \ln 5\right)(x-1)
\end{aligned}$$

Example 4.4.49 Consider the function

$$f(x) = \pi^2 + 2^x + x^2 + x^{\frac{1}{x}}$$

Determine the slope of the line perpendicular to the graph of f at $x = 1$.

In this function the only terms that requires logarithmic differentiation is $x^{\frac{1}{x}}$ and 2^x

$$\begin{aligned}
\ln y &= \frac{\ln x}{x} \\
\frac{1}{y}y' &= \frac{1 - \ln x}{x^2} \\
y' &= \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2}
\end{aligned}$$

Now return to the original function $f(x) = \pi^2 + 2^x + x^2 + x^{\frac{1}{x}}$. Differentiating, we get

$$\begin{aligned}
f'(x) &= (0) + 2^x \ln 2 + 2x + \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2} \\
&= 2^x \ln 2 + 2x + \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2}
\end{aligned}$$

The slope of the line tangent to the graph of f at $x = 1$ is

$$\begin{aligned}
f'(1) &= 2^{(1)} \ln 2 + 2(1) + \frac{(1)^{\frac{1}{1}}(1 - \ln 1)}{1^2} \\
&= 3 + \ln 4
\end{aligned}$$

Thus, the slope of the line perpendicular to the graph of f at $x = 1$ is

$$m = \frac{-1}{3 + \ln 4}$$

Example 4.4.50 Differentiate $y = x^{(x^{(x^4)})}$

$$\begin{aligned}
y &= x^{(x^{(x^4)})} \\
\ln y &= x^{(x^4)} \ln x \\
\ln(\ln y) &= \ln\left(x^{(x^4)} \ln x\right) \\
&= \ln x^{(x^4)} + \ln(\ln x) = x^4 \ln x + \ln(\ln x)
\end{aligned}$$

Differentiate both sides of this equation.

$$\begin{aligned}
\ln(\ln y) &= x^4 \ln x + \ln(\ln x) \\
\left\{\frac{1}{\ln y}\right\}\left\{\frac{1}{y}\right\}y' &= x^4\left\{\frac{1}{x}\right\} + \left(4x^3\right)\ln x + \left\{\frac{1}{\ln x}\right\}\left\{\frac{1}{x}\right\} \\
\left\{\frac{1}{\ln y}\right\}\left\{\frac{1}{y}\right\}y' &= \left(x^3 + 4x^3 \ln x\right) + \frac{1}{x \ln x} \\
\left\{\frac{1}{\ln y}\right\}\left\{\frac{1}{y}\right\}y' &= \frac{x^4(1 + 4\ln x)\ln x + 1}{x \ln x}
\end{aligned}$$

$$y' = x^{(x^{(x^4)}+x^4-1)}\left\{x^4(1+4\ln x)\ln x+1\right\}$$

Example 4.4.51 Differentiate

$$y = \frac{(\ln x)^x}{2^{3x+1}}$$

$$\begin{aligned}
\ln y &= x\ln(\ln x) - (3x+1)\ln 2 \\
\frac{1}{y}y' &= \frac{1}{\ln x} + \ln(\ln x) - \ln 2^3 \\
y' &= \frac{(\ln x)^{(x-1)}\left\{1 + (\ln x)\ln(\ln x) - (\ln 8)\ln x\right\}}{2^{3x+1}}
\end{aligned}$$

Example 4.4.52 Differentiate

$$y = \frac{x^{2x}(x-1)^3}{(3+5x)^4}$$

$$\begin{aligned}
\ln y &= (2x)\ln x + 3\ln(x-1) - 4\ln(3+5x) \\
\frac{1}{y}y' &= 2 + 2\ln x + \frac{3}{x-1} - \frac{20}{3+5x} = \frac{2(\ln x)(x-1)(3+5x)+10x^2-9x+23}{(x-1)(3+5x)} \\
y' &= \frac{x^{2x}(x-1)^2\left\{2(\ln x)(x-1)(3+5x)+10x^2-9x+23\right\}}{(3+5x)^5}
\end{aligned}$$

Exercise 4.25 Differentiate

1.) $y = x^{x^x}$

2.) $y = 2^x$

3.) $y = 2^{\cot x}$

Exercise 4.26 Let $y = x^x$, find $\dfrac{dy}{dx}$.

Exercise 4.27 Differentiate the following functions with respect to x

1.) x^{x-1}

2.) $x^{\sin x}$

3.) $(x-1)^{\ln x}$

4.) $\dfrac{xe^x}{\sin x}$

5.) $\dfrac{(x^2-1)^3(x-1)^{1/2}}{(x-1)^3}$

(vi) $\dfrac{1}{(x+1)^6}$

Exercise 4.28 Find $\dfrac{dy}{dx}$ given that,

1.) $x = t^3 - 2,\ y = t^2 + 2$

2.) $x = \cos t,\ y = 6\sin t$

3.) $x = \dfrac{1}{t^2}\, y = 4t^3 + 8$

4.) $x = 2 + \sqrt{t},\ y = 2 - \sqrt{t}$

Exercise 4.29 Find $y'(x)$ given that

1.) $xy^2 - 3x^2y = 10$

2.) $(yx)^{\frac{1}{2}} + y^{\frac{1}{2}} = 0$

3.) $\cot y = 2x^3 + \cot(x+y)$

4.) $(x+y^2)^3 + x^2y = \alpha^2$

Exercise 4.30 For each of the functions given below determine the equation of the tangent at the points indicated.

1.) $f(x) = 3x^2 - 2x + 4$ at $x = 0$ and 3.

$$y = -2x + 4, \ y = 16x - 23$$

2.) $f(x) = 5x^3 + 12x^2 - 7x$ at $x = -1$ and 1.

$$y = -16x - 2, \ y = 32x - 22$$

3.) $f(x) = xe^x$ at $x = 0$.

$$y = x$$

4.) $f(x) = (x^2 + 1)^3$ at $x = -2$ and 1.

$$y = -300x - 475, \ y = 24x - 16$$

5.) $f(x) = \sin 2x$ at $x = 0$ and $\frac{\pi}{6}$.

$$y = 2x, \ y = x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

6.) $f(x) = 1 - 2x$ at $x = -3$, 0 and 2.

$$y = 1 - 2x, \ y = 1 - 2x, \ y = 1 - 2x$$

Exercise 4.31 Find the equation of each tangent of the function $f(x) = x^3 - 5x^2 + 5x - 4$ which is parallel to the line $y = 2x + 1$.

$$y = 2x - \frac{95}{27}, \ y = 2x - 13$$

Exercise 4.32 Find the equation of each tangent of the function $f(x) = x^3 + x^2 + x + 1$ which is perpendicular to the line $2y + x + 5 = 0$.

$$y = 2x + 2, \ y = 2x + \frac{22}{27}$$

Example 4.4.53 Suppose we wish to find the equation of the tangent and the equation of the normal to the curve

$$y = x + \frac{1}{x}$$

at the point where $x = 2$.

First of all we shall calculate the y coordinate at the point on the curve where $x = 2$:

$$y = 2 + \frac{1}{2} = \frac{5}{2}$$

Next we want the gradient of the curve at the point $x = 2$. We need to find dy/dx.

Noting that we can write y as $y = x + x^{-1}$ then

$$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Furthermore, when $x = 2$

$$\frac{dy}{dx} = 1 - \frac{1}{4} = \frac{3}{4}$$

This is the gradient of the tangent to the curve at the point $(2, \frac{5}{2})$. We know that the standard equation for a straight line is

$$\frac{y - y_1}{x - x_1} = m$$

With the given values we have

$$\frac{y - \frac{5}{2}}{x - 2} = \frac{3}{4}$$

Rearranging

$$\begin{aligned}
y - \frac{5}{2} &= \frac{3}{4}(x - 2) \\
4\left(y - \frac{5}{2}\right) &= 3(x - 2) \\
4y - 10 &= 3x - 6 \\
4y &= 3x + 4
\end{aligned}$$

So the equation of the tangent to the curve at the point where $x = 2$ is $4y = 3x + 4$.

Now we need to find the equation of the normal to the curve.

Let the gradient of the normal be m_2. Suppose the gradient of the tangent is m_1. Recall that the normal and the tangent are perpendicular and hence $m_1 m_2 = -1$. We know $m_1 = \frac{3}{4}$. So

$$\frac{3}{4} \times m_2 = -1$$

and so

$$m_2 = -\frac{4}{3}$$

So we know the gradient of the normal and we also know the point on the curve through which it passes, $\left(2, \frac{5}{2}\right)$.

As before,

$$\begin{aligned}
\frac{y - y_1}{x - x_1} &= m \\
\frac{y - \frac{5}{2}}{x - 2} &= -\frac{4}{3}
\end{aligned}$$

Rearranging

$$\begin{aligned}
3\left(y-\frac{5}{2}\right) &= -4(x-2) \\
3y-\frac{15}{2} &= -4x+8 \\
3y+4x &= 8+\frac{15}{2} \\
6y+8x &= 31
\end{aligned}$$

This is the equation of the normal to the curve at the given point.

Example 4.4.54 Consider the curve $xy = 4$. Suppose we wish to find the equation of the normal at the point $x = 2$. Further, suppose we wish to know where the normal meet the curve again, if it does.

Notice that the equation of the given curve can be written in the alternative form $y = \dfrac{4}{x}$. A graph of the function $y = \dfrac{4}{x}$ is shown in Figure (4.4).

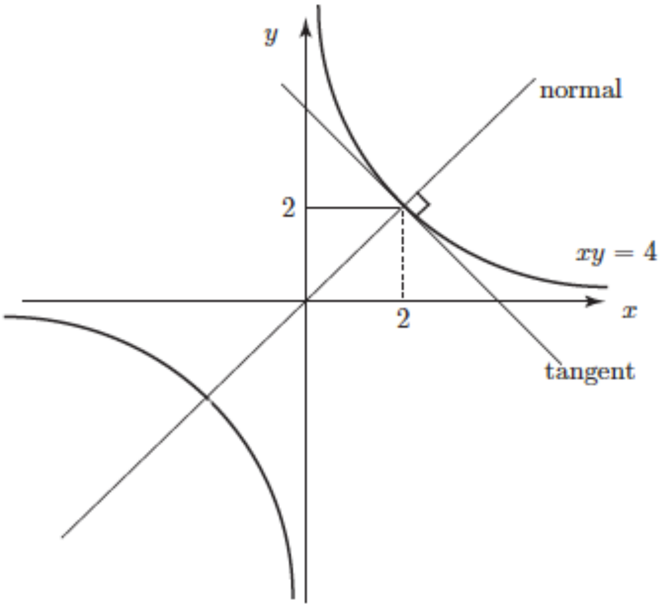


Figure 4.4: A graph of the curve $xy = 4$ showing the tangent and normal at $x = 2$.

From the graph we can see that the normal to the curve when $x = 2$ does indeed meet the curve again (in the third quadrant). We shall determine the point of intersection. Note that when $x = 2$, $y = \dfrac{4}{2} = 2$.

We first determine the gradient of the tangent at the point $x = 2$. Writing

$$y = \frac{4}{x} = 4x^{-1}$$

and differentiating, we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -4x^{-2} = -\frac{4}{x^2}$$

Now, when $x = 2$ $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{4}{4} = -1.$

So, we have the point $(2, 2)$ and we know the gradient of the tangent there is -1. Remember that the tangent and normal are at right angles and for two lines at right angles the product of their gradients is -1. Therefore we can deduce that the gradient of the normal must be $+1$. So, the normal passes through the point $(2, 2)$ and its gradient is 1.

As before, we use the equation of a straight line in the form:

$$\begin{aligned} \frac{y - y_1}{x - x_1} &= m \\ \frac{y - 2}{x - 2} &= 1 \quad \Rightarrow \quad y - 2 = x - 2 \quad \Rightarrow \quad y = x \end{aligned}$$

So the equation of the normal is $y = x$.

We can now find where the normal intersects the curve $xy = 4$. At any points of intersection both of the equations

$$xy = 4 \qquad \text{and} \qquad y = x$$

are true at the same time, so we solve these equations simultaneously. We can substitute $y = x$ from the equation of the normal into the equation of the curve:

$$\begin{aligned} xy &= 4 \\ x \cdot x &= 4 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

So we have two values of x where the normal intersects the curve. Since $y = x$ the corresponding y values are also 2 and -2. So our two points are $(2, 2)$, $(-2, -2)$. These are the two points where the normal meets the curve. Notice that the first of these is the point we started off with.

Exercise 4.33 For each of the functions given below determine the equations of the tangent and normal at each of the points indicated.

1.) $f(x) = x^2 + 3x + 1$ at $x = 0$ and 4.

At $x = 0$: $y = 3x + 1$, $y = -\frac{1}{3}x + 1$, At $x = 4$: $y = 11x - 15$, $y = -\frac{1}{11}x + \frac{323}{11}$

2.) $f(x) = 2x^3 - 5x + 4$ at $x = -1$ and 1.

At $x = -1$: $y = x + 8$, $y = -x + 6$, At $x = 1$: $y = x$, $y = -x + 2$

Exercise 4.34 Find the equation of each normal of the function $f(x) = \frac{1}{3}x^3 + x^2 + x - \frac{1}{3}$ which is parallel to the line $y = -\frac{1}{4}x + \frac{1}{3}$

Exercise 4.35 Find the x co-ordinate of the point where the normal to $f(x) = x^2 - 3x + 1$ at $x = -1$ intersects the curve again. $\frac{21}{5}$

Exercise 4.36 Find an equation of the line tangent to the graph of $y = x + e^x$ at $x = 0$.