Chapter 3

Numerical Differentiation

3.1 Introduction

3.1.1 Why Numerical techniques for finding derivatives

A large number of physical problems involve functions which are known only through experimental measurements. In order to find derivatives of such functions one is forced to use methods based on the available discrete data and the methods are purely numerical.
Sometimes, some functions with known analytical expressions, their derivatives may be so complex and computationally involved in their evaluation that one must resort to numerical methods, though less accurate, but have less involved expressions.
Throughout numerical weather prediction, you often need to calculate the gradient of a function at a number of points. As we rarely know the equation which defines the function, we need to calculate numerically an estimate of the gradient. This method requires the use of finite difference schemes. In this module we will be considering finite difference schemes to approximate the gradient of a function in one variable. The problem can be stated more specifically in these terms: how to estimate the gradient of a function in $f(x)$, at a point a.
Numerical differentiation is the process of finding the numerical value of a derivative of a given function at a given point. In general, numerical differentiation is more difficult than numerical integration. This is because while numerical integration requires only good continuity properties of the function being integrated, numerical differentiation requires more complicated properties such as Lipschitz classes.

3.1.2 Analytic definition of a derivative as compared to a numerical definition

Analytically, we define the derivative of $f(x)$ at $x = a$ denoted $f'(x)$ as the limiting process,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

However, a numerical approximation to $f'(x)$ involves a difference such as

$$\Phi(h) = \frac{f(x+h) - f(x)}{h}$$

In geometrical terms, this can be represented as in Figure 3.1.

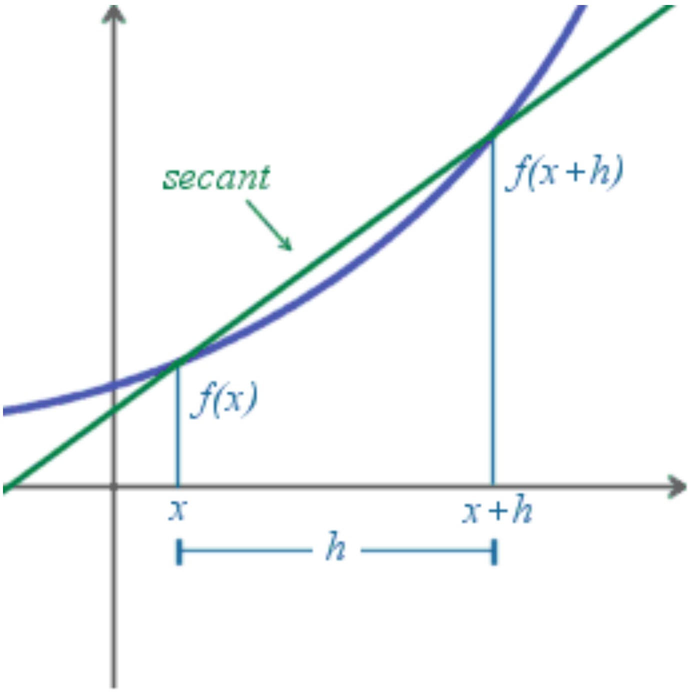

Figure 3.1: Geometrical interpretation of a derivative

Note 3.1.1 $f'(x)$ is the slope of the tangent T and, $\Phi(h)$ is the slope of the line L. As $h \to 0, \Phi(h) \to f'(x)$ suggesting that h should be small for a good approximation. However, in computing $\Phi(h)$ the two terms $f(x)$ and $f(x+h)$ are close in value. So there is likely to be a loss of significant digits.

3.2 Forward Difference Approximation

The forward difference approximation for a derivative is given by,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

The formula can geometrically be represented as in Figure (3.2)

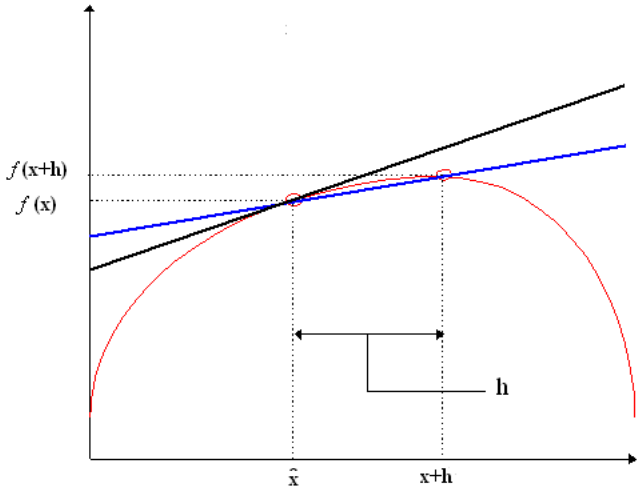

Figure 3.2: Geometrical representation of the forward difference approximation

What the above expression actually calculates is the gradient of the line which intersects the points $x, f(x)$ and $x+h, f(x+h)$ as you can see from the figure above. In the figure, the function is shown with a red line (a curve). The blue line (a lower straight line) represents the approximation to the gradient and the black line (the upperline) is the actual gradient of the function at $x = a$.

Example 3.2.1 Given a function $f(x) = x^2$. Numerically approximate (estimate) $f'(2)$ with step length below. Compare your results with the analytic/exact value of $f'(2)$.

1.) $h = 0.1$

Solution : *Since*

$$f'(x) \simeq \frac{f(x+h) - f(x)}{h}$$

But $f(x) = x^2$ *and* $x = 2, \quad h = 0.1$

$$\text{therefore} \quad \mathrm{f(x+h)} = (2+0.1)^2 = (2.1)^2 = 4.41$$

and

$$f(x) = 2^2 = 4$$

therefore

$$f'(2) \simeq \frac{4.41 - 4}{0.1} = \frac{0.41}{0.1} = 4.1$$

However, the exact value of

$$f'(2) = (2)(2) = 4$$

Since $f'(x) = 2x$. *This yields an error of* 0.1. ■

2.) $h = 0.01$

Solution : *Since* $h = 0.01$, *therefore*

$$f(x+h) = (2.01)^2 = 4.0401$$

and

$$f(x) = 2^2 = 4$$

therefore

$$f'(2) \simeq \frac{4.0401 - 4}{0.01} = \frac{0.0401}{0.01} = 4.01$$

The error committed is 0.01 *i.e smaller than in part (1).* ■

3.) $h = 0.001$.

Solution : *For* $h = 0.001$,

$$f(x+h) = (2.001)^2 = 4.004001$$

and

$$f(x) = 2^2 = 4$$

therefore

$$f'(2) \simeq \frac{4.004001 - 4}{0.001} = \frac{0.004001}{0.001} = 4.001$$

The error committed in this case is 0.001. ■

3.2.1 Analytical derivation of the forward difference approximation

We derive the forward difference approximation analytically from Taylor series expansion of $f(x+h)$. By Taylor series we have,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(x) + \dots \tag{3.1}$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(c_1) \ : \ x \leq c_1 \leq x+h \tag{3.2}$$

If we rearrange equation (3.2), we get,

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2}f''(c_1) \tag{3.3}$$

Equation (3.3) is the forward difference form which we can write in better notation as:

$$f'(x) \simeq \frac{f(x+h) - f(x)}{h} + \quad \text{Truncation} \quad \text{Error}$$

Note 3.2.1

1.) Error $0(h)$ means that the error $\to 0$ (goes to zero) near $x = a$ at the rate of Ah where A is a constant (i.e. halving h, halves the error).

2.) The magnitude of the truncation error i.e $E_{truc} = -\frac{h}{2}f'(c_1)$. Indeed $\left| -\frac{h}{2}f'(c_1) \right|$ is the bound on the truncation error.

3.3 Backward Difference Approximation

The backward difference approximation for the derivative is given by,

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \frac{h}{2} f''(c_2)$$

The formula can be geometrically represented as in Figure (3.3)

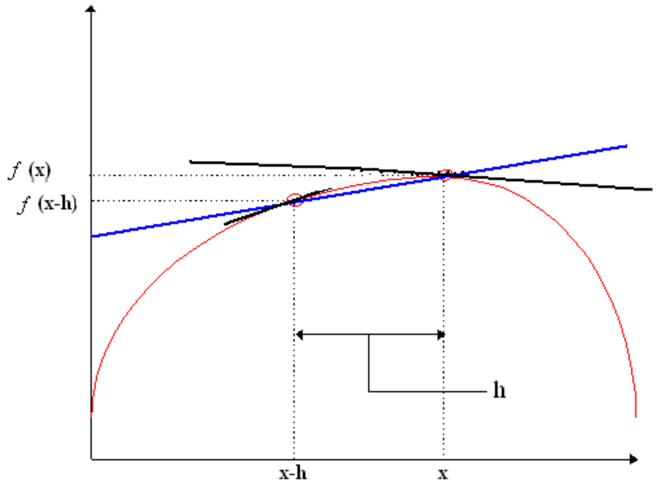

Figure 3.3: Geometrical representation of the backward difference approximation

Example 3.3.1 For the function $f(x) = x^2$, approximate $f'(2)$ using the backward difference approximation with step length

1.) $h = 0.1$

Solution : *Since the backward difference approximation is,*

$$f'(x) \simeq \frac{f(x) - f(x-h)}{h}$$

But $f(x) = x^2, x = 2$ *and* $h = 0.1$, *therefore*

$$f(x) = f(2) = 2^2 = 4$$

$$f(x-h) = f(2-0.1) = f(1.9) = (1.9)^2 = 3.61$$

therefore

$$f'(2) \simeq= \frac{4 - 3.6100}{0.1} = \frac{0.3900}{0.1} = 3.900$$

Since the exact value of

$$f'(2) = (2)(2) = 4,$$

therefore the absolute error committed by using the numerical formula is 0.1. *This is not very bad.* ■

2.) $h = 0.01$

Solution : *Since* $h = 0.01$ *and*

$$f(x) = f(2) = 2^2 = 4$$

$$f(x-h) = f(2-0.01) = f(1.99) = (1.99)^2 = 3.9601,$$

therefore

$$f'(2) \simeq \frac{4-3.9601}{0.01} = \frac{0.0399}{0.01} = 3.99$$

Error committed is 0.01. *This is better than the previous.* ∎

3.) $h = 0.001$

Solution : *Since* $h = 0.001$ *and* $f(x) = 4$

$$f(x-h) = f(2-0.001) = f(1.999) = (1.999)^2 = 3.996001,$$

therefore

$$f'(2) \simeq \frac{4-3.996001}{0.001} = \frac{0.003999}{0.001} = 3.999$$

Error committed is 0.00001. *This error is smaller than any of the previous two. In fact the zero error means the exact value of the derivative is generated.* ∎

Note 3.3.1 We note that the smaller h is the better is the numerical approximate to the derivative.

3.3.1 Analytical derivation of the backward difference approximation

Using the Taylor series expansion

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f^{(3)}(x) + \ldots \tag{3.4}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(c_2) \ : \ x-h \leq c_1 \leq x+h \tag{3.5}$$

Rearranging equation (3.5) i.e

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(c_2),$$

we get,

$$f'(x) = \frac{f(x)-f(x-h)}{h} + \frac{h}{2}f''(c_2) \ : \ x-h \leq c_2 \leq x$$

Which is the backward difference approximation.

The approximation could also be written as,

$$f'(x) = \frac{f(x)-f(x-h)}{h} + E_{trunc}$$

or

$$f'(x) = \frac{f(x)-f(x-h)}{h} + 0(h)$$

3.4 The Central Difference Approximation

We state the central difference approximation to the first derivative as;

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \frac{h^2}{3!} f^3(c)$$

The formula is geometrically represented as seen in the Figure (figure:central)

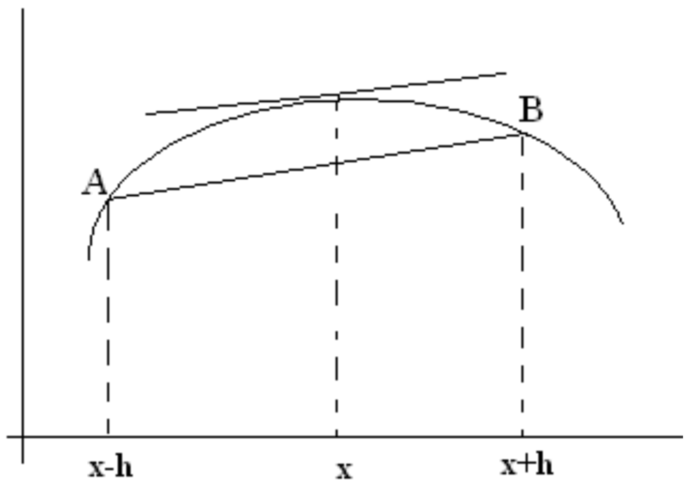

Figure 3.4: Geometrical interpretation of the central difference approximation

The slope of the line AB is the central difference approximation

$$f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h}.$$

From Figure 5.4, it is clear that the gradient of AB is closer to the gradient of the tangent at $x = a$. That is, the two lines are almost parallel. This is expected to be a better approximation than the forward and the backward. Indeed it is as we shall see in the following example.

Example 3.4.1
Approximate $f'(2)$ for $f(x) = x^2$ using the central difference approximation with step size

1.) $h = 0.1$

Solution : *Since $a = 2, h = 0.1$, but*

$$f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h}$$

$$= \frac{f(2.1) - f(1.9)}{2(0.1)}$$

$$= \frac{(2.1)^2 - (1.9)^2}{0.2} = 4.000000000$$

Since the exact value is,

$$f'(2) = 2(2) = 4$$

therefore the error is zero to 9 places of decimals.

This could not be achieved with a forward or backward formula using same step size. This is because the central formula is of higher order i.e $\mathcal{O}(h^2)$ ■

2.) $h = 0.01$

Solution : *with* $h = 0.01$,

$$f'(q) \simeq \frac{f(2.01) - f(1.99)}{2(0.01)}$$

$$= \frac{(2.01)^2 - (1.99)^2}{0.02} = 4.000000000.$$

This also generates a zero error. ■

3.) $h = 0.001$

Solution : *with* $h = 0.001$,

$$f'(2) \simeq \frac{f(2.01) - f(1.999)}{2(0.001)}$$

$$= \frac{(2.001)^2 - (1.999)^2}{0.002} = 4.00000000$$

Which also generates a zero error. ■

3.4.1 Analytical derivation of the central difference approximation

Using Taylor series expansion of $f(x+h)$ and $f(a-h)$ about $x = a$, we have;

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(c_1) \tag{3.6}$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f^{(3)}(c_2) \tag{3.7}$$

equation (3.6) minus equation (3.7) we get;

$$f(x+h) = f(x-h) = 2f'(x)h + \frac{h^3}{6}(f^{(3)}(c_1) - f^{(3)}(c_2)) \tag{3.8}$$

Now,

$$\frac{f^{(3)}(c_1) + f^{(3)}(2)}{2} = f^{(3)}(c_3), \quad c_1 \leq c_3 \leq c_2$$

This is derived from the intermediate value theorem.
So

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \frac{h^2}{3!}f^{(3)}(c_3)$$

i.e, the error is of order $\mathcal{O}(h^2)$.

Exercise 3.4.1 Derive the common approximations below

$$\left.\begin{aligned}
f'(x) &\approx \frac{f(x+h)-f(x)}{h} && (a)\\
f'(x) &\approx \frac{f(x)-f(x-h)}{h} && (b)\\
f'(x) &\approx \frac{f(x+h)-f(x-h)}{2h} && (c)\\
f'(x) &\approx \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h} && (d)\\
f'(x) &\approx \frac{-f(x+2h)+4f(x+h)-4f(x-h)+f(x+2h)}{4h} && (e)
\end{aligned}\right\} \tag{3.9}$$

Exercise 3.4.2

1.) For $f(x) = e^x$, approximate $f'(1)$ using forward, backward and central difference formulae with,

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$

Compare the results with the analytic/exact value of $f'(1)$. Comment on your results.

2.) Let $f(x)$ be given by the table below. The inherent round off error has the bound

$$|e_k| \leq 5 \times 10^{-5}$$

use the rounded values in your calculations.

x	1.100	1.190	1.199	1.200	1.201	1.210
$f(x)$	0.4536	0.3717	0.3633	0.3624	0.314	0.3630

(a) Find approximation for $f'(1.2)$ using the central difference formula with $h = 0.01$, and $h = 0.001$.

(b) Compare with $f'(1.2) = -\sin(1.2) = -0.932$.

(c) Find the total error bound for the three cases in part (2a).

3.) For $f(x) = \cos x$, Use the forward, backward and central difference formulae to approximate $f'(0.8)$ using $h = 0.001$. Compare your results with the analytic values.[Hint: either all in radians or all in degrees, Ans= -0.717]

4.) Repeat question three using the formula

$$f'(x_0) \simeq \frac{f_1 - f_{-1}}{2h} \tag{3.10}$$

Compare your results with analytic results. What can you say about the order of the truncation error in (3.10) in comparison to the forward, backward and central difference formulae.

3.5 Comparision

It is clear that the central difference gives a much more accurate approximation of the derivative compared to the forward and backward differences. Central differences are useful in solving partial differential equations. If the data values are available both in the past and in the future, the numerical derivative should be approximated by the central difference.

3.6 The Second Derivative Approximation

The most commonly used approximation is of a central difference form which we obtain from the following Taylor series approximations earlier considered in the previous lecture. Consider again the expansions,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(c)) \tag{3.11}$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(c_2) \tag{3.12}$$

addition of equations (3.11) and (3.12) gives,

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{12}\Big(f^{(4)}(c_1) + f^4(c_2)\Big) \tag{3.13}$$

and using,

$$\frac{f^{(4)}(c_1) + f^{(4)}(c_2)}{2} = f^{(4)}(c_3)$$

with $c_1 \leq c_3 \leq c_2$. This is derived from the intermediate value theorem and making $f''(x)$ the subject in equation (3.13), we get,

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} + \frac{h^2}{12}f^{(iv)}(c_3) \tag{3.14}$$

or

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} + \mathcal{O}(h^2) \tag{3.15}$$

Equation (3.15) is a second order approximation for the second order derivative. The formula is handy for approximating second order derivatives.

Example 3.6.1 Approximate $f''(1)$ for the function $f(x) = x^3$ with

1.) $h = 0.100$

Solution : *Since,*

$$f''(x) \simeq \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

But $x = 1, \quad h = 0.1,$ *therefore*

$$f''(1) \simeq \frac{f(0.9) - 2f(1) + f(1.1)}{(0.1)^2}$$

$$= \frac{(0.9)^3 - 2(1)^3 + (1.1)^3}{0.01}$$

$$= 6.00000000000005$$

However, the exact value is $f''(x) = 6x$*, therefore* $f''(1) = 6$*. Hence error committed is just* 5.0×10^{-14}*. This is really small.* ■

2.) $h = 0.010$

Solution : *With $h = 0.01$*

$$f''(1) \simeq \frac{f(0.99) - 2f(1) + f(1.01)}{(0.01)^2}$$

$$= \frac{(0.99)^3 - 2 + (1.01)^3}{0.0001}$$

$$= 6.00000000000000$$

Since the exact value is 6, the error committed is zero to 14 decimal places. ■

3.) $h = 0.001$.

Solution : *This also generates a zero error, since,*

$$f''(1) \simeq \frac{f(0.999) - 2f(1) + f(1.001)}{(0.001)^2}$$

$$= \frac{(0.999)^3 - 2 + (1.001)^3}{0.000001}$$

$$= 6.00000000000000$$

■

3.6.1 Error analysis in numerical differentiation

Addition of a rounding error term

When we use a formula such as

$$f'(x) = \frac{f(x+h) - f(x)}{h} + E_{trun}(h^2) \tag{3.16}$$

in general the difference $\frac{f(x+h)-f(x)}{h}$ will be evaluated with a rounding error. For example if f is evaluated to four decimal places with $h = 0.1$, there is a possible rounding error of

$$\frac{2 \times 0.00005}{0.1} = 0.001$$

in the difference

$$\frac{f(x+h) - f(x)}{h}.$$

This is because when a number is evaluated to n- decimal places, the maximum absolute error committed is $\frac{1}{2}10^{-n}$. Thus maximum error in

$$f(x+h) \text{ is } \frac{1}{2}10^{-4} = 0.00005$$

and maximum absolute error in

$$f(x) \text{ is } \frac{1}{2}10^{-4} = 0.00005.$$

But the error in the difference is a sum of the absolute errors in the individual numbers. So the maximum error in

$$f(x+h) - f(x) \text{ is } 2(\frac{1}{2}10^{-4}) \text{ or } 2(0.00005)$$

Thus, the error in

$$\frac{f(x+h)-f(x)}{h} \quad \text{is} \quad 2\frac{(0.00005)}{0.1}$$

assuming $h = 0.1$ is exact. Thus equation (3.16) becomes

$$f'(x) = \frac{y_1 - y_0}{h} + E_{truc} + E_{round}$$

where, y_1, y_0 are the rounded values of $f(x+h)$ and $f(x)$ respectively.

Optimum step size

We now have that:

$$\text{Total error} \;=\; \mathrm{E_{trunc}} \;+\; \mathrm{E_{round}}.$$

Thus there is no point in choosing h so that $|E_{trunc}|$ is small. If $|E_{round}|$ is large since the benefit is swamped.

Example 3.6.2 Suppose

$$f(x+h) = y_1 + e_1$$

and

$$f(x) = y_0 + e_0$$

and $|e_1|, |e_0| < e$ (small number). Find the optimum choice for h when using the approximation

$$f'(x) \simeq \frac{f(x+h)-f(x)}{h}$$

Solution : *For the approximation,*

$$f'(x) \simeq \frac{f(x+h)-f(x)}{h}$$

Maximum absolute error in $f(x+h)$ is e and that in $f(x)$ is also e. So

$$|E_{round}| < \frac{e+e}{h} = \frac{2e}{h}$$

Now

$$|E_{trunc}| = \frac{h}{2}|f''(c_1)| \leq \frac{h}{2}M_2$$

say, where $M_2 = \max|f''(x)|, \quad a \leq x \leq a+h$ Thus,

$$|\text{Total error}| \leq \frac{\mathrm{2e}}{\mathrm{h}} + \frac{\mathrm{h}}{2}\mathrm{M_2}.$$

The optimum choice for h is

$$\frac{d}{dh}\left|\textit{Total error}\right| = 0$$

$$-\frac{2e}{h^2} + \frac{M_2}{2} = 0$$

$$h = 2\sqrt{\frac{e}{M_2}}$$

■

Exercise 3.6.1

1.) For $f(x) = x^2 + x$, approximate $f''(1)$ using a step length,

(a) $h = 0.4$ (b) $h = 0.04$ (c) $h = 0.004$

2.) Given the function $f(x) = e^x$, approximate $f''(x)$ at $x = 2$ using step length h of magnitude.

(a) 0.1 (b) 0.01 (c) 0.001

3.) Let $f(x) = \cos x$. Use the formula considered in this Lecture for approximating $f''(x)$, with $h = 0.01$ to calculate approximations for $f'(0.8)$. Compare with the true value.

4.) Given that

$$f''(x_0) \simeq \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2}$$

Use the formula to approximate $f''(1)$ for $f(x) = e^x$ using $h = 0.5$. Compare your answer with the analytic answer and the answer obtained when using formula equation (3.15). What do you notice and suggest?

5.) Show that for the central difference formula

$$f'(x) = \frac{f(x+h) - f(x)}{2h} + \frac{h^2}{3!} f^{(3)}(c),$$

(a) $|\text{Total error}| \leq \frac{e}{h} + \frac{Mh^2}{6}$

(b) and Optimum $h = \left(\frac{3e}{M}\right)^{\frac{1}{3}}$

(c) for $f(x) = \cos x$, $x = 0.8$ and $h = 0.0001$, show that, $|E_{trunc}| \simeq 0.1 \times 10^{-8}$. And if $e = 0.5 \times 10^{-9}$, show that, $|E_{round}| < 0.5 \times 10^{-5}$ and Optimum $h = 0.0011$.

Chapter 4

Finite Difference Operators

4.1 Introduction

The lecture defines and inter-relates the most common finite difference operators i.e. forward, backward, shift, central and the averaging operator. Use of the operators to prove finite difference identities is also done.

4.1.1 Finite differences

We consider a function $f(x)$ known for a certain set of equally spaced values of x such that

$$x = x_0 + rh, \qquad r = 0, 1, \ldots, n$$

and $h > 0$. This generates a set of $n + 1$ pairs i.e. (x_r, f_r).

The x_r are called pivotal points and $f_r = f(x_r)$ are the pivotal values. The pairs can be written as;

$$\begin{array}{cc}
x_0 & f(x_0) \\
x_1 & f(x_1) \\
x_2 & f(x_2) \\
. & . \\
. & . \\
. & . \\
x_r & f(x_r)
\end{array}$$

We can form differences between consecutive pivotal values i.e. $f_{r+1} - f_r$ where

$$x_{r+1} = x_{r+h} \text{ for } r = 0, 1, 2, \ldots$$

The differences $f_{r+1} - f_r$ are called the first differences. We similarly define higher order differences.

Example 4.1.1 Masenge (1989) used in his book the example of tabulating the function $f(x) = \ln(1 + x)$ in the interval $[2.00, 2.10]$. This was tabulated as,

		Differences		
x_r	f_r	1^{st}	2^{nd}	3^{rd}
2.00	1.098612			
		0.006645		
2.02	1.105257		-0.000044	
		0.00601		0.00000
2.04	1.111858		-0.000044	
		0.006557		0.000002
2.06	1.118415		-0.000042	
		0.006515		-0.000001
2.08	1.124930		-0.000043	
		0.006472		
2.10	1.31402			

Definition 4.1.1
We say that differences are converging if elements of higher order differences decrease rapidly in magnitude. Such a behavior is common from tables of well-behaved functions like polynomials.

4.1.2 Finite difference operators

The most common finite difference operators are; the forward difference operator, the backward difference operator, the central difference operator, the averaging operator and the shift operator.

4.2 The Forward Difference Operator Δ

The forward difference operator is denoted by Δ and defined by the difference equation;

$$\Delta f_r = f_{r+1} - f_r \tag{4.1}$$

Thus, the 1^{st} difference column in a difference table consists of the elements,

$$\Delta f_0, \quad \Delta f_1, \quad \Delta f_2 \ldots\ldots$$

The second differences are got by differentiating the first differences i.e

$$\Delta f_{r+1} - \Delta f_r \tag{4.2}$$

But we can easily show that Δ is a linear operator i.e.

$$\Delta(\alpha f_r + \beta g_r) = \alpha\Delta f_r + \beta\Delta g_r$$

Thus equation (4.2), becomes,

$$\Delta f_{r+1} - \Delta f_r = \Delta(f_{r+1} - f_r) = \Delta(\Delta f_r).$$

Example 4.2.1 Let $y = f(x)$. Then

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
\Delta y_n &= y_{n+1} - y_n
\end{aligned}$$

Example 4.2.2 Compute $\Delta^2 y_1$

Solution :

$$\begin{aligned}
\Delta^2 y_1 &= \Delta\Big(\Delta y_1\Big) \\
&= \Delta\Big(y_2 - y_1\Big) \\
&= \Delta y_2 - \Delta y_1 \\
&= (y_3 - y_2) - (y_2 - y_1) \\
&= y_3 - 2y_2 + y_1
\end{aligned}$$

■

Remark 4.2.1 In general,

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \tag{4.3}$$

Remark 4.2.2 If we write $\Delta^2 f_r$ for $\Delta(\Delta f_r)$, then the second differences are the quantities,

$$\Delta^2 f_0, \Delta^2 f_1, \Delta^2 f_2, \ldots\ldots$$

Remark 4.2.3 We similarly get third differences as consisting of the elements.

$$\Delta^3 f_r = \Delta(\Delta^2 f_r)$$

or

$$\Delta^3 f_r = \Delta^2(\Delta f_r) = \Delta^2 f_{r+1} - \Delta^2 f_r$$

Generally, we have that the n^{th} order differences consist of the elements/quantities,

$$\Delta^n f_r = \Delta^{n-1} f_{r+1} - \Delta^{n-1} f_r \qquad (4.4)$$

$$\Delta^n f_r = \Delta(\Delta^{n-1} f_r) \qquad (4.5)$$

Thus $\Delta^k f_r$ involves information at the pivoted points $x_r, x_{r+1}, \ldots, x_{r+k}$. Thus,

1.) Δf_0 involves information at x_0 and x_1

2.) $\Delta^2 f_0$ involves information at x_0, x_1 and x_2

3.) $\Delta^3 f_0$ involves information at x_0, x_1, x_2 and x_3

and so on. In fact this is how the name forward comes about.

Example 4.2.3

Table 4.1: Showing a table of forward differences in between used points.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
y_0				
	Δy_0			
y_1		$\Delta^2 y_0$		
	Δy_1		$\Delta^3 y_0$	
y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
	Δy_2		$\Delta^3 y_1$	
y_3		$\Delta^2 y_2$		
	Δy_3			
y_4				

or

Table 4.2: Forward Differences Table at point of reference.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	
y_2	Δy_2	$\Delta^2 y_2$		
y_3	Δy_3			
y_4				

Example 4.2.4 Given the value of

$$y = 46,\ 66,\ 81,\ 93,\ 101$$

Determine the forward differences of y.

Solution :

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
46				
	20			
66		-5		
	15		2	
81		-3		-3
	12		-1	
93		-4		
	8			
101				

Table 4.3: Forward difference in between the two used points

or by

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
46	20	-5	2	-3
66	15	-3	-1	
81	12	-4		
93	8			
101				

Table 4.4: Forward difference placed at point

■

Example 4.2.5 Given the table

x	1.4	1.6	1.8	2	2.2
y	4.0552	4.953	6.0496	7.3891	9.025

Solution : *The forward difference operator table is given by*

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.4	4.0552				
		0.8978			
1.6	4.953		0.1988		
		1.0966		0.0441	
1.8	6.0496		0.2429		0.0094
		1.3395		0.0535	
2	7.3891		0.2964		
		1.6359			
2.2	9.025				

■

Example 4.2.6 Construct the difference table of $f(x) = \sin x$ for $x = 0^\circ\,(10^\circ)\,50^\circ$:

Solution :

x°	$f(x) = \sin x$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
0	0					
		1736				
10	0.1736		-52			
		1684		-52		
20	0.3420		-104		4	
		1580		-48		0
30	0.5000		-152		4	
		1428		-44		
40	0.6428		-196			
		1232				
50	0.7660					

■

Example 4.2.7 Write down the forward difference table for $f = x^3$ for

$$x = 1,\ 1.01,\ 1.02,\ 1.03,\ 1.04,\ 1.05$$

Solution :

x	$f(x) = x^3$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
1	1					
		0.03				
1.01	1.03		0.001			
		0.031		-0.001		
1.02	1.061		0		0.002	
		0.031		0.001		-0.003
1.03	1.092		0.001		-0.001	
		0.032		0		
1.04	1.124		0.001			
		0.033				
1.05	1.157					

■

Example 4.2.8 If $f(x) = x^3 - 2x^2 + 1$, construct a forward difference table by taking $x = 0, 1, 2, 3, 4$. Verify that the fourth differences are zero.

Solution :

x	$f(x) = x^3 - 2x^2 + 1$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		-1			
1	0		2		
		1		6	
2	1		8		0
		9		6	
3	10		14		
		23			
4	33				

From the table we can see that

$$\Delta^4 f(x) = 0$$

■

Example 4.2.9 If $f(x) = x^2 + x + 1$, construct a forward difference table by taking $x = 0, 1, 2, 3, 4, 5$. Hence find

1.) $\Delta f(0)$ 2.) $\Delta^2 f(1)$ 3.) $\Delta^3 f(2)$

Solution :

x	$f(x) = x^2 + x + 1$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		0
3	13		2	
		8		0
4	21		2	
		10		
5	31			

From the table we can see that $\Delta^2 f(x) = 2 =$ *constant and* $\Delta^3 f(x) = 0$

We note that

- *Given function is* $f(x) = x^2 + x + 1$. *The highest power is* 2, *which obviously means* $\Delta^2 f(x) =$ *constant and* $\Delta^3 f(x) = 0$
- *No need to show steps of calculations as shown in* $\Delta f(x)$, $\Delta^2 f(x)$ *and* $\Delta^3 f(x)$ *columns.*

■

Exercise 4.2.1 If $f(x) = x^2 - 3x + 1$, construct a forward difference table by taking $x = 0, 1, 2, 3, 4, 5$. Verify that the second differences are constants.

Exercise 4.2.2 If $f(x) = 2x^2+5$, construct a forward difference table by taking $x = 0, 2, 4, 6, 8$. Verify that the third differences are zero.

Exercise 4.2.3 If $f(x) = 2x^3 - x^2 + 3x + 1$, construct a forward difference table by taking $x = 0, 1, 2, 3, 4, 5$. Verify that the third differences are constant.

Example 4.2.10

Given $f(x) = x^4$, form a table of finite differences, for x from $x = -2$ in steps of one to 5.

Solution :

Table 4.5: Showing a finite difference table for $f(x) = x^4$

x_r	f_r	Δf_r	$\Delta^2 f_r$	$\Delta^3 f_r$	$\Delta^4 f_r$	$\Delta^5 f_r$
-2	16					
		-15				
-1	1		14			
		-1		-12		
0	0		2		24	
		1		12		0
1	1		14		24	
		15		36		0
2	16		50		24	
		65		60		0
3	81		110		24	
		175		84		
4	256		194			
		369				
5	625					

■

Note 4.2.1 We note that the fourth differences are constants and fifth differences are all zeros. We state the following theorem to support this observation.

Theorem 4.2.1
Let $P_n(x)$ be a polynomial of degree $n(n \geq 1$ integer). Then

1. $\Delta^r P_n(x)$ is a polynomial of degree $n - r(r \geq n$ integer)
2. $\Delta^n P_n(x) =$ constant
3. $\Delta^{n+1} P_n(x) = 0$.

Thus, parts (2.) and (3.) of the theorem confirm the validity of the results of the difference table (4.4) in which fourth order differences are all constants equal to 24 and fifth differences are all zero.

In fact the theorem can help us check the behaviour of a function (not polynomial) on an interval. If the n^{th} order differences are fairly constant on an interval for a particular function, then such a function more or less behaves like a polynomial of degree n in that interval.

Exercise 4.2.4 For a polynomial

$$P_6(x) = x^6 - 6x^2 + 1,$$

Construct a table of forward differences on the interval $[4, -4]$ with step size 1.

Example 4.2.11 By constructing a forward difference table, find the 7^{th} and 8^{th} terms of a sequence $8,\ 14,\ 22,\ 32,\ 44,\ 58,\ \ldots$

Solution : *Let*

$$f(1) = 8,\ f(2) = 14,\ f(3) = 22,\ f(4) = 32,\ f(5) = 44,\ f(6) = 58$$

We have to find $f(7)$ and $f(8)$.

We prepare the following forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	8			
		6		
2	14		2	
		8		0
3	22		2	
		10		0
4	32		2	
		12		0
5	44		2	
		14		0
6	58		2	
		16		0
7	74		2	
		18		
8	92			

- *To find $f(7)$, extra 2 is written in Δ^2 column. The entry in $\Delta f(x)$ is $14{+}2 = 16$ is added. The entry in $f(x)$ is $58 + 16 = 74$ is added. Thus $f(7) = 74$.*
- *To find $f(8)$, extra 2 is written in Δ^2 column. The entry in $\Delta f(x)$ is $16{+}2 = 18$ is added. The entry in $f(x)$ is $74 + 18 = 92$ is added. Thus $f(8) = 92$.*

Thus the $7th$ and $8th$ terms of series are 74 and 92 respectively. ■

Exercise 4.2.5 By constructing a forward difference table, find the 6^{th} and 7^{th} terms of a sequence 6, 11, 18, 27, 38, ...

Solution : 6, 11, 18, 27, 38, **51**, **66**, ... ■

Example 4.2.12 Let a function $f(x)$ is given at the points $(0, 7), (4, 43), (8, 367)$ then find the forward difference of the function at $x = 4$.

Solution :

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) \\ \Delta f(4) &= f(4+h) - f(4) \\ &= f(8) - f(4) \\ &= 367 - 43 \\ &= 324 \end{aligned}$$

■

Example 4.2.13 Find $\Delta f(x)$ for the function $x^2 + 2x + 3$ with $h = 2$

Solution :

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) \\ &= \left[(x+2)^2 + 2(x+2) + 3\right] - \left[x^2 + 2x + 3\right] \\ &= x^2 + 4x + 4 + 2x + 4 + 3 - x^2 - 2x - 3 \\ &= 4x + 8 \\ \Delta^2 f(x) &= \left[4(x+2) + 8\right] - \left[4x + 8\right] \\ &= 4x + 8 - 4x \\ &= 8 \\ \Delta^3 f(x) &= 8 - 8 \\ &= 0 \end{aligned}$$

■

Example 4.2.14 If $f(x) = e^x$, construct a forward difference table by taking

$$x = 0.1,\ 0.15,\ 0.2,\ 0.25,\ 0.3,\ 0.35,\ 0.4,\ 0.45,\ 0.5$$

Solution :

x	$f(x) = e^x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8
0.1	1.10517								
		0.05666							
0.15	1.16183		0.00291						
		0.05957		0.00015					
0.2	1.2214		0.00306		-0.00001				
		0.06263		0.00014		0.00005			
0.25	1.28403		0.0032		0.00004		-0.00011		
		0.06583		0.00018		-0.00006		0.00023	
0.3	1.34986		0.00338		-0.00002		0.00012		-0.00047
		0.06921		0.00016		0.00006		-0.00024	
0.35	1.41907		0.00354		0.00004		-0.00012		
		0.07275		0.0002		-0.00006			
0.4	1.49182		0.00374		-0.00002				
		0.07649		0.00018					
0.45	1.56831		0.00392						
		0.08041							
0.5	1.64872								

■

4.2.1 Differences as Related to Ordinates

By definition of the forward difference operator, we have:

$$\begin{aligned}
\Delta f_r &= f_{r+1} - f_r \\
\Delta^2 f_r &= \Delta f_{r+1} - \Delta f_r \\
&= (f_{r+2} - f_{r+1}) - (f_{r+1} - f_r) \\
&= f_{r+2} - 2f_{r+1} + f_r \\
\Delta^3 f_r &= \Delta^2 f_{r+1} - \Delta^2 f_r \\
&= (f_{r+3} - 2f_{r+2} + f_{r+1}) - (f_{r+2} - 2f_{r+1} + f_r) \\
&= f_{r+3} - 3f_{r+2} + 3f_{r+1} - f_r
\end{aligned}$$

Exercise 4.2.6 Show that,

$$\Delta^4 f_r = f_{r+4} - 4f_{r+3} + 6f_{r+2} - 4f_{r+1} + f_r$$

Example 4.2.15

Table 4.6: Showing a table of forward differences.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
y_0						
	Δy_0					
y_1		$\Delta^2 y_0$				
	Δy_1		$\Delta^3 y_0$			
y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
	Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
	Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
	Δy_4		$\Delta^3 y_3$			
y_5		$\Delta^2 y_4$				
	Δy_5					
y_6						

or

Table 4.7: Forward Differences Table at point of reference.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$	$\Delta^6 y_0$
y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	$\Delta^5 y_1$	
y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_2$		
y_3	Δy_3	$\Delta^2 y_3$	$\Delta^3 y_3$			
y_4	Δy_4	$\Delta^2 y_4$				
y_5	Δy_5					
y_6						

Note 4.2.2 There is an approximate equality between Δ operator and derivative.

4.3 The Backward Difference Operator ∇

This is denoted by ∇ and defined by the difference equation,

$$\nabla f_r = f_r - f_{r-1} \tag{4.6}$$

Exercise 4.3.1 Show that

$$\nabla^n f_r = \nabla^{n-1} f_r - \nabla^{n-1} f_{r-1}$$

before you continue reading.

Note 4.3.1 We also note that this operator has a nice property of linearity i.e

$$\nabla(\alpha f_r + \beta g_r) = \alpha \nabla f_r + \beta \nabla g_r$$

where α and β are real scalars.

Example 4.3.1

Table 4.8: Showing a table of backward differences.

y_0						
	∇y_1					
y_1		$\nabla^2 y_2$				
	∇y_2		$\nabla^3 y_3$			
y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
	∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$	
y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$
	∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$	
y_4		$\nabla^2 y_5$		$\nabla^4 y_6$		
	∇y_5		$\nabla^3 y_6$			
y_5		$\nabla^2 y_6$				
	∇y_6					
y_6						

or

Table 4.9: Backward Differences Table

y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
y_0						
y_1	∇y_1					
y_2	∇y_2	$\nabla^2 y_2$				
y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$

Example 4.3.2 Construct the backward difference table for the value of for x and y given below.

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

Solution : *The backward differences for y is given by*

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Note that, the forward and backward give the same table, difference is at where you position. Compare this example and Example 4.2.11. To see a difference we employ Table 4.9 for the Backward difference operator.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
1901	66	20			
1911	81	15	-5		
1921	93	12	-3	2	
1931	101	8	-4	-1	-3

■

Exercise 4.3.2 Construct a table of backward differences for the polynomial in Exercise 4.2.4.

Exercise 4.3.3 Construct a table of backward differences for the function $f(x) = e^x$ on the interval $[-2, 2]$ with step size 0.5.

From your results what polynomial function fairly approximate e^x on this interval?

Exercise 4.3.4 What do you think are the uses of finite difference tables.

Example 4.3.3 Construct the backward difference table for the data given below:

x	1	2	3	4	5	6	7	8
y	8	14	22	32	44	58	74	92

Solution :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1	8			
		6		
2	14		2	
		8		0
3	22		2	
		10		0
4	32		2	
		12		0
5	44		2	
		14		0
6	58		2	
		16		0
7	74		2	
		18		
8	92			

or

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$	$\nabla^7 y$
1	8							
2	14	6						
3	22	8	2					
4	32	10	2	0				
5	44	12	2	0	0			
6	58	14	2	0	0	0		
7	74	16	2	0	0	0	0	
8	92	18	2	0	0	0	0	0

■

Exercise 4.3.5 Construct a backward difference table for the following data

x	0	10	20	30
y	0	0.174	0.347	0.518

Exercise 4.3.6 Construct a backward difference table for $y = f(x) = x^3 + 2x + 1$ for

$$x = 1, 2, 3, 4, 5.$$

Example 4.3.4 By constructing a difference table and using the second order differences as constant, find the sixth term of the series $8, 12, 19, 29, 42\ldots$ 58

4.3.1 Relation between Δ and ∇

1.)
$$(1+\Delta)\,(1-\nabla) \equiv 1 \tag{4.7}$$

Proof:

$$\begin{aligned}(1+\Delta)(1-\nabla)f(x) &= (1+\Delta)[f(x) - f(x) + f(x-h)]\\ &= (1+\Delta)f(x-h)\\ &= f(x-h) + f(x) - f(x-h)\\ &= f(x)\end{aligned}$$

Therefore,

$$(1+\Delta)(1-\nabla) \equiv 1$$

■

2.)
$$\Delta\nabla \equiv \nabla\Delta \tag{4.8}$$

Proof:

$$\Delta\nabla f(x) = \Delta[f(x) - f(x-h)] = f(x+h) - 2f(x) + f(x-h)$$

and

$$\begin{aligned}\nabla\Delta f(x) = \nabla[f(x+h) - f(x)] &= [f(x+h) - f(x)] - [f(x) - f(x-h)]\\ &= f(x+h) - 2f(x) + f(x-h)\end{aligned}$$

Therefore,

$$\Delta\nabla \equiv \nabla\Delta$$

■

3.)
$$\Delta\nabla \equiv \Delta - \nabla \tag{4.9}$$

Proof: *Consider the function $f(x)$.*

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ \nabla f(x) &= f(x) - f(x-h)\end{aligned}$$

But

$$\begin{aligned}\Delta - \nabla)(f(x)) &= \Delta f(x) - \nabla f(x)\\ &= [f(x+h) - f(x)] - [f(x) - f(x-h)]\\ &= \Delta f(x) - \Delta f(x-h)\\ &= \Delta[f(x) - f(x-h)]\\ &= \Delta[\nabla f(x)]\end{aligned}$$

Therefore

$$\Delta - \nabla = \Delta\nabla$$

■

4.3.2 Properties of Forward and Backward Finite Difference Operators

1. **Constant rule**: If c is a constant, then

   (a)
   $$\Delta c = 0 \tag{4.10}$$

   Proof: *Let $f(x) = c$*

   Therefore
   $$f(x+h) = c$$
   (where h is the interval of difference)
   $$\Delta c = \Delta f(x) = f(x+h) - f(x) = c - c = 0$$

   ■

   (b)
   $$\nabla c = 0 \tag{4.11}$$

2. **Linearity**: if a and b are constants,

   (a)
   $$\Delta(af + bg) = a\,\Delta f + b\,\Delta g \tag{4.12}$$

   Proof:
   $$\begin{aligned}\Delta[f(x) + g(x)] &= [f(x+h) + g(x+h)] - [f(x) + g(x)]\\ &= f(x+h) + g(x+h) - f(x) - g(x)\\ &= f(x+h) - f(x) + g(x+h) - g(x)\\ &= \Delta f(x) + \Delta g(x)\end{aligned}$$

   and
   $$\begin{aligned}\Delta[cf(x)] &= cf(x+h) - cf(x)\\ &= c[f(x+h) - f(x)]\\ &= c\Delta f(x)\end{aligned}$$

   Thus the proof ■

   (b)
   $$\nabla(af + bg) = a\,\nabla f + b\,\nabla g \tag{4.13}$$

3. **Product rule**:

   (a)
   $$\Delta(fg) = f\,\Delta g + g\,\Delta f + \Delta f\,\Delta g \tag{4.14}$$

   Proof:
   $$\begin{aligned}\Delta[f(x)g(x)] &= f(x+h)g(x+h) - f(x)g(x)\\ &= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)\\ &= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]\\ &= f(x+h)\Delta g(x) + g(x)\Delta f(x)\end{aligned}$$

   ■

Also, it can be shown that

$$\begin{aligned}\Delta[f(x)g(x)] &= f(x)\Delta g(x) + g(x+h)\Delta f(x)\\ &= f(x)\Delta g(x) + g(x)\Delta f(x) + \Delta f(x)\Delta g(x)\end{aligned}$$

(b)

$$\nabla(fg) = f\,\nabla g + g\,\nabla f - \nabla f\,\nabla g \tag{4.15}$$

4. **Quotient rule**:

(a)

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)} \tag{4.16}$$

Proof:

$$\begin{aligned}\Delta\left(\frac{f(x)}{g(x)}\right) &= \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\\ &= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\\ &= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\\ &= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)}\\ &= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}\end{aligned}$$

■

(b)

$$\nabla\left(\frac{f}{g}\right) = \frac{1}{g}\det\begin{bmatrix}\nabla f & \nabla g\\ f & g\end{bmatrix}\left(\det\begin{bmatrix}g & \nabla g\\ 1 & 1\end{bmatrix}\right)^{-1}$$

or

$$\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g\cdot(g - \nabla g)} \tag{4.17}$$

5. **Summation rules**:

(a)

$$\sum_{n=a}^{b}\Delta f(n) = f(b+1) - f(a) \tag{4.18}$$

(b)

$$\sum_{n=a}^{b}\nabla f(n) = f(b) - f(a-1) \tag{4.19}$$

4.4 The Shift Operator E

Definition 4.4.1 The shift operator is denoted by E and defined by

$$Ef_r = f_{r+1} \tag{4.20}$$

A shift operator is like a jump forward.

Remark 4.4.1 E is called the shifting operator . It is also called the **displacement** operator.

Example 4.4.1

Table 4.10: Shift Differences Table

y	Ey	E^2y	E^3y	E^4y	E^5y
y_0	y_1	y_2	y_3	y_4	y_5
y_1	y_2	y_3	y_4	y_5	
y_2	y_3	y_4	y_5		
y_3	y_4	y_5			
y_4	y_5				
y_5					

Note 4.4.1 By Definition 4.4.1 we can define higher order shift operators as,

$$E(Ef_r) = Ef_{r+1} = f_{r+2}.$$

We usually denote

$$E(Ef_r) = E^2 f_r$$

and,

$$E^k f_r = E(E^{k-1} f_r) = f_{r+k}.$$

Remark 4.4.2 For k negative, we get what is known as the backward shift. For example $E^{-1}f_r$ is a backward shift defined by,

$$E^{-1}f_r = f_{r-1}$$

We also note that k can be fractional.

Example 4.4.2 Find $E^{\frac{1}{2}}f_r$ and $E^{-5}f_r$.

Solution : *By definitions of the forward and backward shifts i.e.*

$$Ef_r = f_{r+1}$$

and

$$E^{-1}f_r = f_{r-1}$$

we have,

$$E^{\frac{1}{2}}f_r = f_{r+\frac{1}{2}}$$

and

$$E^{-5}f_r = f_{r-5}.$$

■

Example 4.4.3 Given the set of values

$$y = -2,\ 2,\ 5,\ 6,\ 8,\ 12,\ 13$$

then

1.) $E(2) = 5$ 2.) $E(8) = 12$ 3.) $E^3(-2) = 6$ 4.) $E^{-1}(13) = 12$

Example 4.4.4 Given $y_0 = 1,\ y_1 = 11,\ y_2 = 21,\ y_3 = 28$ and $y_4 = 29$.

1.)
$$E^4 y_0 = y_4 = 29$$

2.)
$$E^3 y_0 = y_3 = 28$$

3.)
$$E^2 y_0 = y_2 = 21$$

4.)
$$E y_0 = y_1 = 11$$

4.4.1 Relation between Δ and E

$$\Delta = E - 1 \tag{4.21}$$
$$E = \Delta + 1 \tag{4.22}$$

Proof: *Consider the function $f(x)$.*

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= Ef(x) - f(x)\\ &= (E-1)f(x)\end{aligned}$$

Therefore,

$$\Delta \equiv E - 1$$

■

4.4.2 Relation between ∇ and E

$$\nabla = 1 - E^{-1} \tag{4.23}$$
$$E = (1 - \nabla)^{-1} \tag{4.24}$$

Proof: *Consider the function $f(x)$.*

$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h)\\ &= f(x) - E^{-1}f(x)\\ &= (1 - E^{-1})f(x)\end{aligned}$$

Therefore,

$$\nabla \equiv 1 - E^{-1}$$

■

Remark 4.4.3 Property equ (4.23) can be also expressed as

$$\begin{aligned}\nabla &= 1 - E^{-1}\\ &= 1 - \frac{1}{E}\\ &= \frac{E-1}{E}\end{aligned}$$

hence

$$\nabla = \frac{E-1}{E} \tag{4.25}$$

4.4.3 Relation between Δ, ∇ and E

Example 4.4.5 Show that

$$\Delta = E\nabla \tag{4.26}$$

Solution : *Since*

$$\begin{aligned}E\nabla f_r &= E(\nabla f_r)\\ &= E(f_r - f_{r-1})\\ &= Ef_r - Ef_{r-1} \text{ (because of linearity of E)}\\ &= f_{r+1} - f_r \textit{ (By definition of E)}\\ &= \Delta f_r\end{aligned}$$

Hence shown. ■

Remark 4.4.4 Relation equ (4.26) can be rewritten as

$$\begin{aligned}\Delta &= E\nabla\\ E^{-1}\Delta &= E^{-1}E\nabla\\ E^{-1}\Delta &= \nabla\end{aligned}$$

Therefore

$$\nabla = E^{-1}\Delta \tag{4.27}$$

Proof: *Consider the function $f(x)$.*

$$\nabla f(x) = f(x) - f(x-h) = \Delta f(x-h) = \Delta E^{-1} f(x)$$

Therefore

$$\nabla = \Delta E^{-1}$$

■

4.4.4 Differences in terms of pivoted values

From the relation

$$\Delta^k f_r = (E-1)^k f_r,$$

We can easily show that

$$\Delta^k f_r = \begin{pmatrix} k \\ 0 \end{pmatrix} f_{r+k} - \begin{pmatrix} k \\ 1 \end{pmatrix} f_{r+k-1} + \begin{pmatrix} k \\ 2 \end{pmatrix} f_{r+k-2} + \dots$$

Try and show this before you continue. Recall: $\begin{pmatrix} k \\ r \end{pmatrix} = \frac{k!}{r!(k-r)!}$.

Example 4.4.6 Express $\Delta^3 f_0$ in terms of pivoted values.

Solution : *Since*

$$\Delta^k f_r = \binom{k}{0} f_{r+k} - \binom{k}{1} f_{r+k-1} + \binom{k}{2} f_{r+k-2} + \ldots$$

Therefore,

$$\begin{aligned}
\Delta^3 f_0 &= \binom{3}{0} f_{3+0} - \binom{3}{1} f_{3+0-1} + \binom{3}{2} f_{3+0-2} - \binom{3}{3} f_{3+0-3} \\
&= \binom{3}{0} f_3 - \binom{3}{1} f_2 + \binom{3}{2} f_1 - \binom{3}{3} f_0 \\
&= f_3 - 3f_2 + 3f_1 - f_0
\end{aligned}$$

■

Example 4.4.7 Construct the shift difference table for the value of y given below.

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

Solution :

x	y	Ey	E^2y	E^3y	E^4y
1891	46	66	81	93	101
1901	66	81	93	101	
1911	81	93	101		
1921	93	101			
1931	101				

■

4.5 The Central Difference Operator δ

The central difference operator is denoted by δ and defined by the equation

$$\delta f_r = f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}} \tag{4.28}$$

Example 4.5.1

$$\begin{aligned} \delta f_{r+\frac{1}{2}} &= f_{r+\frac{1}{2}+\frac{1}{2}} - f_{r+\frac{1}{2}-\frac{1}{2}} \\ \delta f_{r+\frac{1}{2}} &= f_{r+1} - f_r \end{aligned} \tag{4.29}$$

Note 4.5.1 Central difference is the difference between the upper (forward) and lower (backward) values.

Example 4.5.2

1.) $\delta f_{\frac{1}{2}} = f_1 - f_0$

2.) $\delta f_{\frac{3}{2}} = f_2 - f_1$

Example 4.5.3

$$\begin{aligned} \delta^2 f_r &= \delta f_{r+\frac{1}{2}} - \delta f_{r-\frac{1}{2}} \\ &= (f_{r+1} - f_r) - (f_r - f_{r-1}) \\ &= f_{r+1} - 2f_r + f_{r-1} \end{aligned}$$

With higher order differences defined by

$$\delta^k f_r = \delta^{k-1} f_{r+\frac{1}{2}} - \delta^{k-1} f_{r-\frac{1}{2}}.$$

Exercise 4.5.1 Determine the expressions for

1.) $\delta^3 f_r$

2.) $\delta^4 f_r$

Example 4.5.4

Table 4.11: Showing a table of central differences using Eqn (4.29), subtractions

y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
y_0						
	$\delta y_{\frac{1}{2}}$					
y_1		$\delta^2 y_1$				
	$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$			
y_2		$\delta^2 y_2$		$\delta^4 y_2$		
	$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$		$\delta^5 y_{\frac{5}{2}}$	
y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
	$\delta y_{\frac{7}{2}}$		$\delta^3 y_{\frac{7}{2}}$		$\delta^5 y_{\frac{7}{2}}$	
y_4		$\delta^2 y_4$		$\delta^4 y_4$		
	$\delta y_{\frac{9}{2}}$		$\delta^3 y_{\frac{9}{2}}$			
y_5		$\delta^2 y_5$				
	$\delta y_{\frac{11}{2}}$					
y_6						

Example 4.5.5 Draw the central difference operator table for $f(x) = 2^x$, $0(1)6$

Table 4.12: Showing a table of central differences.

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
0	1						
		1					
1	2		1				
		2		1			
2	4		2		1		
		4		2		1	
3	8		4		2		1
		8		4		2	
4	16		8		4		
		16		8			
5	32		16				
		32					
6	64						

4.5.1 Relation between δ and E

From,

$$\delta f_r = f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}}$$

and

$$E^{\frac{1}{2}} f_r = f_{r+\frac{1}{2}}$$

and

$$E^{-\frac{1}{2}} f_r = f_{r-\frac{1}{2}},$$

we get

$$\delta f_r = (E^{\frac{1}{2}} - E^{-\frac{1}{2}}) f_r$$

and hence conclude that;

$$\begin{aligned} \delta &= E^{1/2} - E^{-1/2} && (4.30) \\ &= E^{1/2}(1 - E^{-1}) && (4.31) \end{aligned}$$

4.5.2 Relation between δ and Δ

$$\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} \qquad (4.32)$$

Proof: *We know that*

$$\begin{aligned} \Delta &= E - 1 \\ \delta &= E^{1/2} - E^{-1/2} \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} \\ &= \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{4}} \\ &= \frac{1}{2}\left(E + E^{-1} - 2\right) + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{\left(E + E^{-1} - 2\right)}{4}} \\ &= \frac{E}{2} + \frac{E^{-1}}{2} - 1 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{\frac{\left(E^{1/2} + E^{-1/2}\right)^2}{4}} \\ &= \frac{E}{2} + \frac{E^{-1}}{2} - 1 + \left(E^{1/2} - E^{-1/2}\right)\frac{\left(E^{1/2} + E^{-1/2}\right)}{2} \\ &= \frac{E}{2} + \frac{E^{-1}}{2} - 1 + \frac{\left(E - E^{-1}\right)}{2} \\ &= \frac{E}{2} + \frac{E^{-1}}{2} - 1 + \frac{E}{2} - \frac{E^{-1}}{2} \\ &= E - 1 \\ &= \Delta \end{aligned}$$

■