Chapter 6

Solution to Non-Linear Equations

6.1 Introduction

6.1.1 Motivation

Given a second degree polynomial, we can accurately compute it's root, that is

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

But what if $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \;\Rightarrow\; x =?$, $\sin x + x = 0 \;\Rightarrow\; x =?$

6.1.2 Problem Description

- Given a non-linear equation $f(x) = 0$, find a $x^\star$ such that $f(x^\star) = 0$. Thus, $x^\star$ is a root of $f(x) = 0$.
- Galois theory in math tells us that only polynomials of degree ≤ 4 can be solved with close forms using $+, -, \times, \div$ and taking roots.
- General non-linear equations can be solved with iterative methods.
- Basically, we try to guess the location of a root, and approximate it iteratively.
- Unfortunately, this process can go wrong, leading to another root or even diverge.

6.1.3 Methods to be discussed

- There are two types of methods, bracketing and open. The bracketing methods require an interval that is known to contain a root, while the open method does not.
- Commonly seen bracketing methods include the *Bisection method* and the *regula falsi method*, and the open methods are *Newton's method*, the *secant method*, and the *successive substitution*.

6.2 Newton Raphson's Method

6.2.1 Derivation of the Newton's method

If we are given a non-linear equation $f(x) = 0$ and we are to apply the Newton Raphson's method, we linear approximate the graph of $y = f(x)$ by a straight line passing through the point (x_0, f_0) and tangential to the graph of $y = f(x)$. Take the slope of this line to be p. Geometrically this is given in figure below.

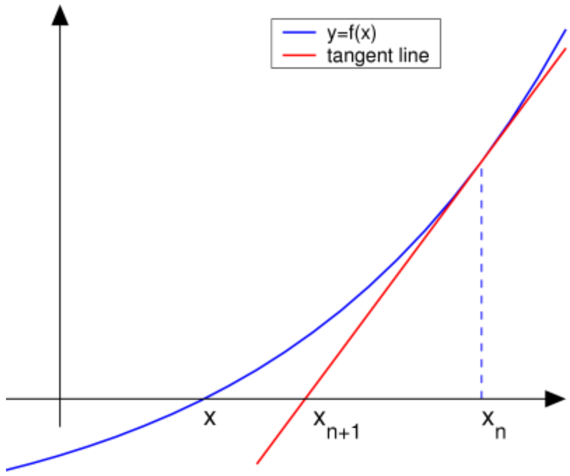


Figure 6.1: Geometrical representation of a tangent to a curve at a point

The equation of the line with slope p and passing through the point (x_0, f_0) is

$$\frac{y - f_0}{x - x_0} = p \tag{6.1}$$

However, we know that p is the slope of the tangent to $y = f(x)$ at (x_0, f_0). This is given by;

$$p = f'(x_0) = f'_0 \tag{6.2}$$

Substituting equation (6.2) in equation (6.1) we get

$$\frac{y - f_0}{x - x_0} = f'_0$$

$$y - f_0 = (x - x_0)f'_0 \tag{6.3}$$

From figure 6.1, line of equation 13.3 cuts the x-axis at the point $(x_1, 0)$ i.e when $x = x_1$ and $y = 0$.

Substituting in equation (6.3) we get,

$$0 - f_0 = (x_1 - x_0)f'_0$$

Making x_1 the subject, we get,

$$x_1 = x_0 - \frac{f_0}{f'_0} \quad \text{or}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'x_0} \tag{6.4}$$

equation (6.4) is actually the Newton's method for obtaining the next iterate x_1 from the previous iterate x_0. The equation (6.4) is generalized and written;

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \tag{6.5}$$

since the linear approximation of the curve is done at each of the iterates $x_n, x_{n+1}, x_{n+2}, \ldots$ as reflected in figure above.

Example 6.2.1 Use Newton Raphson's method to find the root of

$$x^2 - 3 = 0 \text{ on } [1, 2]$$

.

$$\begin{aligned}
& & f(x_n) &= x_n^2 - 3 \\
&\text{therefore} & f'(x_n) &= 2x_n \\
&\text{But Raphson's formula is} & x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}
\end{aligned}$$

Substituting in the Raphson's formula we get

$$\begin{aligned}
x_{n+1} &= x_n - \frac{(x_n^2 - 3)}{2x_n} \\
&= \frac{2x_n^2 - x_n^2 + 3}{2x_n} \\
&= \left(\frac{x_n^2 + 3}{2x_n}\right)
\end{aligned}$$

Taking the initial guess/approximation as $x_0 = 2$, but you could also consider $x_0 = 1$ you come up with the same answer.

$$\begin{aligned}
\Rightarrow x_1 &= \frac{x_0^2 + 3}{2x_0} = \frac{2^2 + 3}{2(2)} = 1.7500000 \\
x_2 &= \frac{x_1^2 + 3}{2x_1} = \frac{(1.75)^2 + 3}{2(1.75)} = 1.7321000 \\
x_3 &= \frac{x_2^2 + 3}{2x_2} = \frac{(1.7321)^2 + 3}{2(1.7321)} = 1.7320508 \\
x_4 &= \frac{x_3^2 + 3}{2x_3} = \frac{(1.7320508)^2 + 3}{2(1.7320508)} = 1.7320508
\end{aligned}$$

Thus the root is 1.7320508

Example 6.2.2 Use Newton-Raphson Method to find the only real root of the equation

$$x^3 - x - 1 = 0$$

correct to 9 decimal places.

Since $f(1) = -1$ and $f(2) = 5$, the function has a root in the interval $[1, 2]$ since the function changes sign between $[1, 2]$. Let us make an initial guess $x_0 = 1.5$.

$$f(x) = x^3 - x - 1 \Rightarrow f(x_n) = x_n^3 - x_n - 1$$
$$f'(x) = 3x^2 - 1 \Rightarrow f'(x_n) = 3x_n^2 - 1$$

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{(x_n^3 - x_n - 1)}{(3x_n^2 - 1)} \\
x_{n+1} &= \frac{2x_n^3 + 1}{3x_n^2 - 1} \\
x_1 &= \frac{2x_0^3 + 1}{3x_0^2 - 1} = \frac{2(1.5)^3 + 1}{3(1.5)^2 - 1} = 1.347826087 \\
x_2 &= \frac{2x_1^3 + 1}{3x_1^2 - 1} = \frac{2(1.347826087)^3 + 1}{3(1.347826087)^2 - 1} = 1.325200399 \\
x_3 &= \frac{2x_2^3 + 1}{3x_2^2 - 1} = \frac{2(1.325200399)^3 + 1}{3(1.325200399)^2 - 1} = 1.324718174 \\
x_4 &= \frac{2x_3^3 + 1}{3x_3^2 - 1} = \frac{2(1.324718174)^3 + 1}{3(1.324718174)^2 - 1} = 1.324717957 \\
x_5 &= \frac{2x_4^3 + 1}{3x_4^2 - 1} = \frac{2(1.324717957)^3 + 1}{3(1.324717957)^2 - 1} = 1.324717957
\end{aligned}$$

The approximated root of

$$x^3 - x - 1 = 0$$

to 9 decimal places is **1.324717957**

Remark 6.2.1 The more the decimal places the better the approximation.

Example 6.2.3 Repeat Example 6.2.2

$$x^3 - x - 1 = 0$$

using 4 *decimal places.*

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^3 - x_n - 1)}{(3x_n^2 - 1)} = \frac{2x_n^3 + 1}{3x_n^2 - 1} \\
x_1 &= \frac{2x_0^3 + 1}{3x_0^2 - 1} = \frac{2(1.5)^3 + 1}{3(1.5)^2 - 1} = 1.3478 \\
x_2 &= \frac{2x_1^3 + 1}{3x_1^2 - 1} = \frac{2(1.3478)^3 + 1}{3(1.3478)^2 - 1} = 1.3252 \\
x_3 &= \frac{2x_2^3 + 1}{3x_2^2 - 1} = \frac{2(1.3252)^3 + 1}{3(1.3252)^2 - 1} = 1.3247 \\
x_4 &= \frac{2x_3^3 + 1}{3x_3^2 - 1} = \frac{2(1.3247)^3 + 1}{3(1.3247)^2 - 1} = 1.3247
\end{aligned}$$

The approximated root of

$$x^3 - x - 1 = 0$$

to 4 decimal places is **1.3247**

Example 6.2.4 Repeat Example 6.2.2

$$x^3 - x - 1 = 0$$

using 1 *decimal places.*

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^3 - x_n - 1)}{(3x_n^2 - 1)} = \frac{2x_n^3 + 1}{3x_n^2 - 1} \\
x_1 &= \frac{2x_0^3 + 1}{3x_0^2 - 1} = \frac{2(1.5)^3 + 1}{3(1.5)^2 - 1} = 1.3 \\
x_2 &= \frac{2x_1^3 + 1}{3x_1^2 - 1} = \frac{2(1.3)^3 + 1}{3(1.3)^2 - 1} = 1.3
\end{aligned}$$

The approximated root of

$$x^3 - x - 1 = 0$$

to 1 decimal places is **1.3**

Remark 6.2.2 The fewer the decimal places the faster the iteration scheme converges.

Example 6.2.5 Find the root of the function

$$y = e^{-x} - x$$

in the vicinity of $x = 0.5$ correct to 4 decimal places.

$$\begin{aligned}
f(x) = e^{-x} - x &\Rightarrow f(x_n) = e^{-x_n} - x_n \\
f'(x) = -e^{-x} - 1 &\Rightarrow f'(x_n) = -e^{-x_n} - 1
\end{aligned}$$

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{(e^{-x_n} - x_n)}{(-e^{-x_n} - 1)} = \frac{-x_n e^{-x_n} - x_n - e^{-x_n} + x_n}{(-e^{-x_n} - 1)} = \frac{(x_n + 1)\, e^{-x_n}}{1 + e^{-x_n}} \\
x_1 &= \frac{(x_0 + 1)\, e^{-x_0}}{1 + e^{-x_0}} = \frac{([0.5] + 1)\, e^{-0.5}}{1 + e^{-0.5}} = 0.5663 \\
x_2 &= \frac{(x_1 + 1)\, e^{-x_1}}{1 + e^{-x_1}} = \frac{([0.5663] + 1)\, e^{-0.5663}}{1 + e^{-0.5663}} = 0.5671 \\
x_3 &= \frac{(x_2 + 1)\, e^{-x_2}}{1 + e^{-x_2}} = \frac{([0.5671] + 1)\, e^{-0.5671}}{1 + e^{-0.5671}} = 0.5671
\end{aligned}$$

The approximated root of

$$e^{-x} - x = 0$$

to 4 *decimal places* is **0.5671**

Example 6.2.6 *Pitfall - no real root*
Newton's method will fail (the sequence $\{x_n\}$ does not converge) on solving

$$f(x) = x^2 - 4x + 5 = 0$$

since f does not have any real root.

Example 6.2.7 Find the root of

$$f(x) = 3x + \sin x - e^x = 0$$

in the interval $[0, 1]$ numerically by the famous Newton Raphson's method

Solution
Since $f(0) = -1 < 0$ and $f(1) = 3 + \sin(1) - e > 0$, so there is a real root in $[0, 1]$. Using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(3x_n + \sin x_n - e^{x_n})}{(3 + \cos x_n - e^{x_n})}$$

$$= \frac{3x_n + x_n \cos x_n - x_n e^{x_n} - 3x_n - \sin x_n + e^{x_n}}{3 + \cos x_n - e^{x_n}}$$

$$= \frac{x_n \left[\cos x_n - e^{x_n}\right] - \sin x_n + e^{x_n}}{3 + \cos x_n - e^{x_n}}$$

to have the iterations

$$x_1 = \frac{x_0 \left[\cos x_0 - e^{x_0}\right] - \sin x_0 + e^{x_0}}{3 + \cos x_0 - e^{x_0}} = \frac{0 \left[\cos 0 - e^0\right] - \sin 0 + e^0}{3 + \cos 0 - e^0} = 0.33333$$

$$\begin{aligned} x_2 &= \frac{x_1 \left[\cos x_1 - e^{x_1}\right] - \sin x_1 + e^{x_1}}{3 + \cos x_1 - e^{x_1}} \\ &= \frac{0.33333 \left[\cos 0.33333 - e^{0.33333}\right] - \sin 0.33333 + e^{0.33333}}{3 + \cos 0.33333 - e^{0.33333}} = 0.36017 \end{aligned}$$

$$\begin{aligned} x_3 &= \frac{x_2 \left[\cos x_2 - e^{x_2}\right] - \sin x_2 + e^{x_2}}{3 + \cos x_2 - e^{x_2}} \\ &= \frac{0.36017 \left[\cos 0.36017 - e^{0.36017}\right] - \sin 0.36017 + e^{0.36017}}{3 + \cos 0.36017 - e^{0.36017}} = 0.36042 \end{aligned}$$

$$\begin{aligned} x_4 &= \frac{x_3 \left[\cos x_3 - e^{x_3}\right] - \sin x_3 + e^{x_3}}{3 + \cos x_3 - e^{x_3}} \\ &= \frac{0.36042 \left[\cos 0.36042 - e^{0.36042}\right] - \sin 0.36042 + e^{0.36042}}{3 + \cos 0.36042 - e^{0.36042}} = 0.36042 \end{aligned}$$

The approximated root of

$$3x + \sin x - e^x = 0$$

to 5 *decimal places* is **0.36042**

Example 6.2.8 *Pitfall - Alternating or oscillating solutions*
Use Newton's Method to find the only real root of the equation

$$f(x) = e^x - 2x = 0$$

with an initial guess of $x_0 = 1$.

$$\begin{aligned} f(x) = e^x - 2x &\Rightarrow f(x_n) = e^{x_n} - 2x_n \\ f'(x) = e^x - 2 &\Rightarrow f'(x_n) = e^{x_n} - 2 \end{aligned}$$

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - 2x_n}{e^{x_n} - 2} = \frac{e^{x_n}(x_n - 1)}{e^{x_n} - 2} \\ x_1 &= \frac{e^{x_0}(x_0 - 1)}{e^{x_0} - 2} = \frac{e^1(1-1)}{e^1 - 2} = 0 \\ x_2 &= \frac{e^{x_1}(x_1 - 1)}{e^{x_1} - 2} = \frac{e^0(0-1)}{e^0 - 2} = 1 \\ x_3 &= \frac{e^{x_2}(x_2 - 1)}{e^{x_2} - 2} = \frac{e^1(1-1)}{e^1 - 2} = 0 \\ x_4 &= \frac{e^{x_3}(x_3 - 1)}{e^{x_3} - 2} = \frac{e^0(0-1)}{e^0 - 2} = 1 \\ x_5 &= \frac{e^{x_4}(x_4 - 1)}{e^{x_4} - 2} = \frac{e^1(1-1)}{e^1 - 2} = 0 \\ x_6 &= \frac{e^{x_5}(x_5 - 1)}{e^{x_5} - 2} = \frac{e^0(0-1)}{e^0 - 2} = 1 \\ &\vdots \\ &\vdots \end{aligned}$$

Its an alternating solutions, the solutions oscillate near the local maxima or local minima.

In other words, Newton's Method fails to produce a solution. Why is this? Because there is no solution to be found!

Mathematicians are often very happy when, after a great deal of work, they are just able to say that a solution to a problem exists. This is because once they know it exists, there might be some nice method, such as Newton's Method, to actually compute the solution.

Example 6.2.9 *Pitfall - Diverging sequence*
When Newton's method is applied to

$$f(x) = xe^{-x}$$

with $x_0 = 2$, it produces

n	x_n	$f(x_n)$
0	2.00000	2.70671×10^{-1}
1	4.00000	7.32626×10^{-2}
2	5.33333	2.57491×10^{-3}
3	6.56410	9.25597×10^{-4}
$\vdots$	$\vdots$	$\vdots$
20	24.96488	3.59105×10^{-10}
21	26.00660	1.31995×10^{-10}
22	27.04659	4.85206×10^{-11}

Note that,

- The sequence $\{x_n\}$ diverges to ∞ slowly;
- However, $f(x_n)$ goes to zero rapidly as x_n gets larger in a finite precision environment, and could be mistaken as a zero of f.

Definition 6.2.1 *Convergence of Newton's Method*
If the iterations are getting closer and closer to the correct answer the method is said to *converge*.

That is, the Newton's method is said to converge if $|x_{n+1} - x_n| \to 0$.

Remark 6.2.3 However, Newton's method *will not converge* if

1.) If $f'(x_n) = 0$ for some n

2.) If $\lim\limits_{n\to\infty} x_n$ does not exist

Example 6.2.10 *Pitfall - Interval size*
If the numerical When Newton's method applied to

$$f(x) = x^3 - x - 3$$

The algorithm given by

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - x_n - 3}{3x_n^2 - 1} \\ x_{n+1} &= \frac{2x_n^3 + 3}{3x_n^2 - 1} \end{aligned}$$

with $x_0 = \mathbf{0}$, it produces

$$\begin{aligned} x_1 &= -3.00000 \\ x_2 &= -1.96154 \\ x_3 &= -1.14718 \\ x_4 &= -0.00658 \\ x_5 &= -3.00039 \\ x_6 &= -1.96182 \\ x_7 &= -1.14743 \end{aligned}$$

The sequence will not converge. But if the algorithm starts with $x_0 = \mathbf{2}$, then it produces

$$\begin{aligned} x_1 &= 1.7272727 \\ x_2 &= 1.6736912 \\ x_3 &= 1.6717026 \\ x_4 &= 1.6716999 \\ x_5 &= 1.6716999 \end{aligned}$$

The sequence converges to the root 1.6716999 correct to *seven decimal places.*

This examples illustrates again that the starting point x_0 must be close enough to the zero of f.

Example 6.2.11 *Pitfall - Choice of x_0 and $f'(x) = 0$*
Use the Newton Method to find a non-zero solution of

$$x = 2\sin x \tag{6.6}$$

Let $f(x) = x - 2\sin x$. Then $f'(x) = 1 - 2\cos x$, and the Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n} = \frac{2\,(\sin x_n - x_n \cos x_n)}{1 - 2\cos x_n}$$

Let x_0= **1.1**. The next six estimates, to 3 decimal places, are:

$x_1 = 8.453$ $\qquad$ $x_3 = 203.384$ $\qquad$ $x_5 = -87.471$

$x_2 = 5.256$ $\qquad$ $x_4 = 118.019$ $\qquad$ $x_6 = -203.637$

Things don't look good, and they get worse. It turns out that $x_{35} < -64000000$. We could be stubborn and soldier on. Miracles happen-but not often. (One happens here, around $n = 212$.)

The trouble was caused by the choice of x_0, let us consider x_0= **1.5**. Here are the next six estimates, to 19 *decimal places* - to indicate how fast the convergence is

$x_1 = 2.0765582006304348291$ $\qquad$ $x_4 = 1.8954942764727706570$

$x_2 = 1.9105066156590806258$ $\qquad$ $x_5 = 1.8954942670339809987$

$x_3 = 1.8956220029878460925$ $\qquad$ $x_6 = 1.8954942670339809471$

The next iterate x_7 agrees with x_6 in the first 19 *decimal places*, indeed in the first 32, and the true root is equal to x_6 to 32 *decimal places.*

Remark 6.2.4 Note that choosing

$$x_0 = \frac{\pi}{3} \approx 1.0472$$

leads to immediate disaster, since then the denominator $f'(x) = 1 - 2\cos x_0 = 0$ and therefore x_1 does not exist.

Comment 6.2.1 The remedy to this is to rewrite the Equation (6.6) into other forms say

$$\frac{2}{x} - \frac{1}{\sin x} = 0 \text{ or } \frac{\sin x}{x} = \frac{1}{2}$$

works nicely - to have $f'(x)$ harder to be too small or almost zero for $\forall\ x \in (0, \pi)$ - as we shall see in the section of successive substitution, Section 6.6

Example 6.2.12 Consider the problem of finding the positive number x with $\cos x = x^3$ We can rephrase that as finding the zero of

$$f(x) = \cos x - x^3 \;\Rightarrow\; f'(x) = -\sin x - 3x^2$$

Since $\cos x \leq 1$ for all x and $x^3 > 1$ for $x > 1$, we know that our zero lies in $[0, 1]$. We try a starting value of $x_0 = 0.5$.

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos x_n - x_n^3}{-\sin x_n - 3x_n^2} = \frac{x_n \sin x_n + \cos x_n + 2x_n^3}{\sin x_n + 3x_n^2} \\
x_1 &= \frac{x_0 \sin x_0 + \cos x_0 + 2x_0^3}{\sin x_0 + 3x_0^2} = \frac{(0.5)\sin(0.5) + \cos(0.5) + 2(0.5)^3}{\sin(0.5) + 3(0.5)^2} = 1.1121416
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{x_1 \sin x_1 + \cos x_1 + 2x_1^3}{\sin x_1 + 3x_1^2} \\
&= \frac{(1.1121416)\sin(1.1121416) + \cos(1.112141637097) + 2(1.1121416)^3}{\sin(1.1121416) + 3(1.1121416)^2} = 0.9096727
\end{aligned}$$

$$\begin{aligned}
x_3 &= \frac{x_2 \sin x_2 + \cos x_2 + 2x_2^3}{\sin x_2 + 3x_2^2} \\
&= \frac{(0.9096727)\sin(0.9096727) + \cos(0.9096727) + 2(0.9096727)^3}{\sin(0.9096727) + 3(0.9096727)^2} = 0.8672638
\end{aligned}$$

$$\begin{aligned}
x_4 &= \frac{x_3 \sin x_3 + \cos x_3 + 2x_3^3}{\sin x_3 + 3x_3^2} \\
&= \frac{(0.8672638)\sin(0.8672638) + \cos(0.8672638) + 2(0.8672638)^3}{\sin(0.8672638) + 3(0.8672638)^2} = 0.8654771
\end{aligned}$$

$$\begin{aligned}
x_5 &= \frac{x_4 \sin x_4 + \cos x_4 + 2x_4^3}{\sin x_4 + 3x_4^2} \\
&= \frac{(0.8654771)\sin(0.8654771) + \cos(0.8654771) + 2(0.8654771)^3}{\sin(0.8654771) + 3(0.8654771)^2} = 0.8654740
\end{aligned}$$

$$\begin{aligned}
x_6 &= \frac{x_5 \sin x_5 + \cos x_5 + 2x_5^3}{\sin x_5 + 3x_5^2} \\
&= \frac{(0.8654740)\sin(0.8654740) + \cos(0.8654740) + 2(0.8654740)^3}{\sin(0.8654740) + 3(0.8654740)^2} = 0.8654740
\end{aligned}$$

Therefore, the solution is

$$x = 0.8654740$$

correct to 7 decimal places

Example 6.2.13 Use Newton's Method to estimate the point of intersection of $y = e^{-x^2}$ and $y = x$.

$$f(x) = x - e^{-x^2} \; , \; \Rightarrow \; x_{n+1} = x_n - \frac{x_n - e^{-x_n^2}}{1 + 2x_n e^{-x_n^2}}$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.500000	-0.27880	1.77880	-0.15673	0.65673
2	0.65673	0.00706	1.85331	0.00381	0.65292
3	0.65292	0.00000	1.85261	0.00000	0.65292
4	0.65292				

The solution exists and it is 0.65292

Exercise 6.2.1 Let $f(x) = x^2 - a$. Show that the Newton Method leads to the recurrence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Exercise 6.2.2 Newton's equation $y^3 - 2y - 5 = 0$ has a root near $y = 2$. Starting with $y_0 = 2$, compute y_1, y_2, and y_3, the next three Newton-Raphson estimates for the root.

Exercise 6.2.3 Find the root of the equation

$$x^2 + x - 1 = 0$$

in the interval $[0, 1]$, giving your answer correct to 4 decimal places.

Exercise 6.2.4 Show that the cubic equation

$$2x^3 + 3x^2 - 3x - 5 = 0$$

has a real root in the interval $[1, 2]$. Approximate this root correct to five decimal places using Newton Raphson's method.

Exercise 6.2.5 Use Newton's method to approximate the root of the equation

$$g(x) = x^3 - 2\sin x$$

on $[0.5, 2]$.

6.2.2 Roots of Positive Numbers

Suppose that our interest is to find the r^{th} root of a real positive number A. If x is the value of this root, then x is related to A by the equation,

$$A^{\frac{1}{r}} = x \text{ or } x^r = A \text{ or } x^r - A = 0$$

Let $f(x) = x^r - A$, then x is the root of the nonlinear equation,

$$f(x) = x^r - A = 0, \Rightarrow f(x_n) = x_n^r - A, \Rightarrow f'(x_n) = rx_n^{r-1}$$

Applying the Newton Raphson's method, we have;

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{(x_n^r - A)}{rx_n^{r-1}} \\ &= \frac{rx_n^r - x_n^r + A}{rx_n^{r-1}} \\ &= \frac{(r-1)x_n^r + A}{rx_n^{r-1}} \end{aligned}$$

$$x_{n+1} = \frac{1}{r}\left\{(r-1)x_n + \frac{A}{x_n^{r-1}}\right\} \tag{6.7}$$

Equation (6.7) is a general formula from which we can obtain quadratically convergent iterative processes for finding approximations to arbitrary roots of numbers.

Note 6.2.1 The root, or answer interested in is the x. And also what is $f(x)$ is the function $f(x) = 0$

Example 6.2.14 When $r = 2$, the square root of a number A we have

$$x = \sqrt{A} \Rightarrow x^2 = A$$

Thus for $r = 2$ in the general formula, we get,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$$

Which is the Newton's square root algorithm for extracting roots of positive numbers.

Example 6.2.15 Use Newton's square root algorithm to find the square root of 5 correct to *six decimal places.*

$$\sqrt{5} = x \Rightarrow 5^{\frac{1}{2}} = x \Rightarrow x^2 - 5 = 0 \Rightarrow f(x) = x^2 - 5 = 0$$

Or substituting in the general formula $r = 2$ and $A = 5$ we get,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$$

or, using the Newton Raphson formula directly,

$$\begin{aligned} f(x) &= x^2 - 5 \\ f'(x) &= 2x \end{aligned}$$

to have

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{x_n^2 - 5}{2x_n} \\ x_{n+1} &= \frac{2x_n^2 - x_n^2 + 5}{2x_n} \\ x_{n+1} &= \frac{x_n^2 + 5}{2x_n} \\ x_{n+1} &= \frac{1}{2}\left(x_n + \frac{5}{x_n}\right) \end{aligned}$$

Starting with $x_0 = 2.0$ we get,

$$\begin{aligned} x_1 &= \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(2.000000 + \frac{5}{2.000000}\right) = 2.250000 \\ x_2 &= \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = \frac{1}{2}\left(2.250000 + \frac{5}{2.250000}\right) = 2.236111 \\ x_3 &= \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right) = \frac{1}{2}\left(2.236111 + \frac{5}{2.236111}\right) = 2.236068 \\ x_4 &= \frac{1}{2}\left(x_3 + \frac{5}{x_3}\right) = \frac{1}{2}\left(2.236068 + \frac{5}{2.236068}\right) = 2.236068 \end{aligned}$$

The value of $x_2 = 2.236111$ is correct only to one decimal place since it agrees with the previous iterate $x_1 = 2.250000$ only in one decimal place.

However, $x_3 = 2.236068$ is correct to three decimal places since it is in agreement with the previous iterate x_2 in exactly three places of decimal.

But $x_4 = 2.236068$ is exactly the same as x_3. In fact they are exactly the same up to nine decimal places.

This means that $x_4 = 2.236068$ is the value of the root correct to nine decimal places. Thus x_4, must also be correct up to six decimal places.

Hence, the value of the root that you state as being correct to six decimal places or nine decimal places is $x_4 = 2.236068$. Compare with the value obtained from calculator.

Example 6.2.16 Use Newton's method to approximate $\sqrt{2}$

Solution : *For* $f(x) = x^2 - 2,\ f'(x) = 2x$. *such that*

$$\begin{aligned}
x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\
&= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\
&= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\
&= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = 1.5 \\
x_2 &= \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.416666667.
\end{aligned}$$

Continuing in this way, we find that

$$\begin{aligned}
x_1 &= 1.5 \\
x_2 &\approx 1.416666667 \\
x_3 &\approx 1.414215686 \\
x_4 &\approx 1.414213562 \\
x_5 &\approx 1.414213562.
\end{aligned}$$

Since we obtained the same value for x_4 *and* x_5, *it is unlikely that the value* x_n *will change on any subsequent application of Newton's method. We conclude that* $\sqrt{2} \approx 1.414213562$.. ■

Exercise 6.2.6 Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2.

Solution :

$$\begin{aligned}
x_1 &= 2 \\
x_2 &= 1.75
\end{aligned}$$

■

6.2.3 Reciprocals of Numbers

The reciprocal of a number A

$$\begin{aligned}
A^{-1} = x \Rightarrow x^{-1} &= A \\
\Rightarrow \frac{1}{x} &= A \\
\Rightarrow f(x) &= Ax - 1 = 0 \text{ wrong choice, } f'(x) = c \\
\Rightarrow f(x) &= \frac{1}{x} - A = 0 \;\Rightarrow\; f'(x) = -\frac{1}{x^2} \\
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} &\;\Rightarrow\; x_{n+1} = x_n - \frac{\frac{1}{x_n} - A}{-\frac{1}{x_n^2}} \\
\Rightarrow x_{n+1} &= x_n(2 - Ax_n)
\end{aligned}$$

Alternatively, if we have that $r = -1$ then $x^{-1} = A$ (positive number), this means $x = \dfrac{1}{A}$ (reciprocal of A) with $r = -1$ in the general formula in equation (6.7) then we get,

$$x_{n+1} = x_n(2 - Ax_n)$$

This formula is quadratically convergent and can suitably be applied to calculate the reciprocal of numbers.

Example 6.2.17 Use Newton's reciprocal algorithm to find the reciprocal of 3.

$$x_{n+1} = x_n(2 - Ax_n) \;\Rightarrow\; x_{n+1} = x_n(2 - 3x_n)$$

or, using the formula, since finding

$$3^{-1} = x \;\Rightarrow\; 3x = 1 \;\Rightarrow\; 3x - 1 = 0 \;\Rightarrow\; f(x) = 3x - 1$$

would be a wrong choice as x_{n+1} will equal to a constant

$$3^{-1} = x \;\Rightarrow\; 3 = x^{-1} \;\Rightarrow\; \frac{1}{x} - 3 = 0 \;\Rightarrow\; f(x) = \frac{1}{x} - 3$$

such that

$$f(x) = \frac{1}{x} - 3, \;\; f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n(2 - 3x_n)$$

Let $x_0 = 0.5$.

$$\begin{aligned}
x_1 &= x_0[2 - 3(x_0)] = 0.50000\Big[2 - 3(0.50000)\Big] = 0.25000 \\
x_2 &= x_1[2 - 3(x_1)] = 0.25000\Big[2 - 3(0.25000)\Big] = 0.31250 \\
x_3 &= x_2[2 - 3(x_2)] = 0.31250\Big[2 - 3(0.31250)\Big] = 0.33203 \\
x_4 &= x_3[2 - 3(x_3)] = 0.33203\Big[2 - 3(0.33203)\Big] = 0.33333 \\
x_5 &= x_4[2 - 3(x_4)] = 0.33333\Big[2 - 3(0.33333)\Big] = 0.33333
\end{aligned}$$

Thus $x_5 = 0.33333$ is the value of the reciprocal of 3 i.e $\dfrac{1}{3}$ correct to *five decimal places.*

6.2.4 Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(xn) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.

2. The approximations x_0, x_1, x_2, $\ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see Figure below).

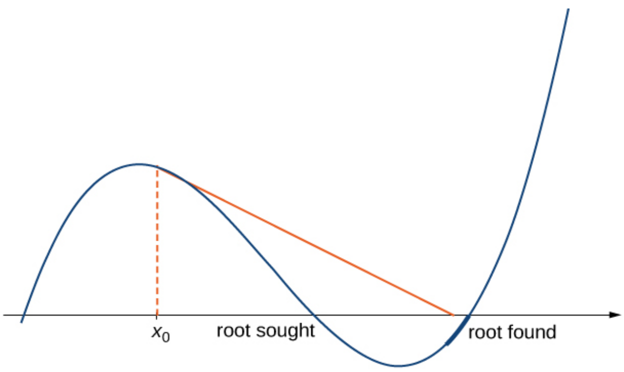


Figure 6.2: If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.

3. The approximations may fail to approach a root entirely. In Example 6.2.18 , we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

Example 6.2.18 Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence $x_1, x_2, \ldots$ fails to approach a root of f.

Solution : *For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 2$. Therefore,*

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0, x_1, x_2, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$.

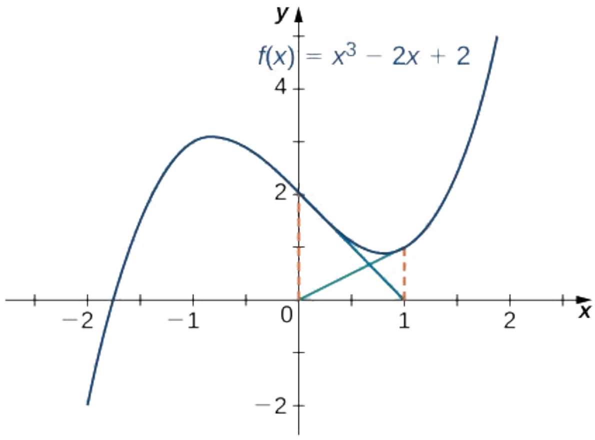


Figure 6.3: The approximations continue to alternate between 0 and 1 and never approach the root of f.

Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation. ■

Exercise 6.2.7 For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2.

Solution :

$$x_1 \approx -1.842105263$$
$$x_2 \approx -1.772826920$$

■

Note 6.2.2 From Example 6.2.18, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

6.2.5 Advantages of Newton Raphson's Method

- A faster method for converging on a single root of a function is the Newton- Raphson method.
- Perhaps it is the most widely used method of all locating formulas.
- Because the convergence rate is high, no worry for initial guess, interval size, and number of decimal places required.
- Requires only one guess

6.2.6 Limitations of Newton Raphson's Method

Good though it is, the method has some limitations.

- The Newton Raphson's method may also fail if $f(x)$ has a point of inflection in the neighborhood of the root.
- Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of f at each approximation.
- Frequently, $f'(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.
- If in the immediate neighborhood of a root of $f(x), f'(x)$ vanishes or is very small, the Newton Raphson's method will not converge. The reason for this failure is; since $f'(x)$ is very small, the quantity, $-\frac{f(x_n)}{f'(x_n)}$ becomes very large. In case it is zero - the denominator, then function is not defined. The consequence is that we are thrown away from the root we are approximating.

Exercise 6.2.8 Use Newton's square root algorithm to find the square root of 2 correct to 6 decimal places.

Exercise 6.2.9 Use cube root Newton's algorithm to find the cube root of 7 correct to four decimal places.

Exercise 6.2.10 Use Newton's reciprocal algorithm to find

1.) the reciprocal of the square root of 2.

2.) The reciprocal of the cube root of 4.

Exercise 6.2.11 State the advantages and disadvantages of the Newton's method for nonlinear equations.

Exercise 6.2.12 Define the Newton-Raphson method formula for finding the root of a nonlinear equation $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Exercise 6.2.13 With any simple example, write short notes on the "NRM" for solving nonlinear equations.

Example 6.2.19

1.) The convergence of the Newton-Raphson method technique highly depends on the initial guess. Discuss.

Solution : *Yes, when the initial guess is in the interval given, the iterations converge faster than otherwise.* ■

2.) Use Newton Raphson method to estimate one of the solutions of $x^2 - 4 = 0$ using $x_0 = 6$ to 2 decimal places.

Solution :

x_n	x_{n+1}
$x_0 = 6$	$x_1 = 3.33$
$x_1 = 3.33$	$x_2 = 2.27$
$x_2 = 2.27$	$x_3 = 2.01$
$x_3 = 2.01$	$x_4 = 2.00$
$x_4 = 2.00$	$x_5 = 2.00$

■

3.) Newton's Raphson's method is one of the popular schemes for solving a non-linear equation $f(x) = 0$. Prove that the Newton Raphson's method for finding the square root of a positive number A is given by,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$$

Use the scheme above to approximate the square root of $5(\sqrt{5})$ to three decimal places with $x_0 = 2$.

Solution :

x_n	x_{n+1}
$x_0 = 2$	$x_1 = 2.25$
$x_1 = 2.25$	$x_2 = 2.236$
$x_2 = 2.236$	$x_3 = 2.236$
$x_3 = 2.236$	$x_4 = 2.236$

■

6.3 Bisection Method (Interval Halving)

6.3.1 Background

The bisection method is one of the bracketing methods for finding roots of equations.

6.3.2 Implementation

Given a function $f(x)$ and an interval which might contain a root, perform a predetermined number of iterations using the bisection method.

6.3.3 Limitations

Investigate the result of applying the bisection method over an interval where there is a discontinuity. Apply the bisection method for a function using an interval where there are distinct roots. Apply the bisection method over a "large" interval.

6.3.4 Explanation on the bisection method

The bisection method takes a similar geometrical approach with the Regula falsi algorithm. You need two initial guesses x_0 and x_1 to the root $x^\star$ of the nonlinear equation $f(x) = 0$, such that

$$f(x_0)f(x_1) < 0$$

The next approximation is obtained by getting the arithmetic mean of the previous two. That is

$$\boxed{x_{n+1} = \frac{x_n + x_{n-1}}{2}} \tag{6.8}$$

However, the pair x_n, x_{n+1} to be used to get x_{n+2} must satisfy the condition

$$f(x_n)f(x_{n-1}) < 0 \tag{6.9}$$

Masenge (1989) called this method a trivial simplification of the regula falsi.

In mathematics, the bisection method is a root-finding algorithm which works by repeatedly dividing an interval in half and then selecting the subinterval in which the root exists.

Example 6.3.1 Use the Bisection method to estimate the root of $x^2 = 3$ on $(1, 2)$.

The non linear function $x^2 - 3 = 0 \;\Rightarrow\; f(x) = x^2 - 3$

Let

$$\begin{aligned} x_0 &= 1 \Rightarrow f_0 = -2.000 < 0 \\ x_1 &= 2 \Rightarrow f_1 = 1.000 > 0 \end{aligned}$$

For the Bisection scheme,

$$\begin{aligned}
x_{n+1} &= \frac{x_n + x_{n-1}}{2} \\
x_2 &= \frac{x_1 + x_0}{2} = \frac{1+2}{2} = 1.500 \;\Rightarrow\; f_2 = -0.75 < 0 \\
x_3 &= \frac{x_2 + x_1}{2} = \frac{1.500 + 2.000}{2} = 1.750 \;\Rightarrow\; f_3 = 0.062 > 0 \\
x_4 &= \frac{x_3 + x_2}{2} = \frac{1.750 + 1.500}{2} = 1.625 \;\Rightarrow\; f_4 = -0.359 < 0 \\
x_5 &= \frac{x_4 + x_3}{2} = \frac{1.625 + 1.750}{2} = 1.688 \;\Rightarrow\; f_5 = -0.151 < 0 \\
x_6 &= \frac{x_5 + x_3}{2} = \frac{1.688 + 1.750}{2} = 1.719 \;\Rightarrow\; f_6 = -0.045 < 0 \\
x_7 &= \frac{x_6 + x_3}{2} = \frac{1.719 + 1.750}{2} = 1.735 \;\Rightarrow\; f_7 = 0.010 > 0 \\
x_8 &= \frac{x_7 + x_6}{2} = \frac{1.735 + 1.719}{2} = 1.727 \;\Rightarrow f_8 = -0.017 < 0 \\
x_9 &= \frac{x_8 + x_7}{2} = \frac{1.727 + 1.735}{2} = 1.731 \;\Rightarrow\; f_9 = -0.004 < 0 \\
x_{10} &= \frac{x_9 + x_7}{2} = \frac{1.731 + 1.735}{2} = 1.733 \;\Rightarrow\; f_{10} = 0.003 > 0 \\
x_{11} &= \frac{x_{10} + x_9}{2} = \frac{1.733 + 1.731}{2} = 1.732 \;\Rightarrow\; f_{11} = 0.000
\end{aligned}$$

Since $f_{11} = 0.000$, then $x_{11} = 1.732$ it is the root or zero of the function $x^2 - 3 = 0$, that is the approximated square root of 3 correct to three decimal places.

Example 6.3.2 Compute the numerical root of

$$3x + \sin x - e^x = 0$$

using the Bisection method on the interval $[0, 0.5]$ correct to 4 decimal places.

Let

$$x_0 = 0 \Rightarrow f_0 = -1.0000 < 0$$
$$x_1 = 0.5 \Rightarrow f_1 = 0.3307 > 0$$

For the Bisection scheme,

$$\begin{aligned}
x_{n+1} &= \frac{x_n + x_{n-1}}{2} \\
x_2 &= \frac{x_1 + x_0}{2} = \frac{0 + 0.5}{2} = 0.2500 \;\Rightarrow\; f_2 = -0.2866 < 0 \\
x_3 &= \frac{x_2 + x_1}{2} = \frac{0.2500 + 0.5000}{2} = 0.3750 \;\Rightarrow\; f_3 = 0.0363 > 0 \\
x_4 &= \frac{x_3 + x_2}{2} = \frac{0.3750 + 0.2500}{2} = 0.3125 \;\Rightarrow\; f_4 = -0.1219 < 0 \\
x_5 &= \frac{x_4 + x_3}{2} = \frac{0.3125 + 0.3750}{2} = 0.3438 \;\Rightarrow\; f_5 = -0.0418 < 0 \\
x_6 &= \frac{x_5 + x_3}{2} = \frac{0.3438 + 0.3750}{2} = 0.3594 \;\Rightarrow\; f_6 = -0.0026 < 0 \\
x_7 &= \frac{x_6 + x_3}{2} = \frac{0.3594 + 0.3750}{2} = 0.3672 \;\Rightarrow\; f_7 = 0.0169 > 0 \\
x_8 &= \frac{x_7 + x_6}{2} = \frac{0.3672 + 0.3594}{2} = 0.3633 \;\Rightarrow f_8 = 0.0072 > 0 \\
x_9 &= \frac{x_8 + x_6}{2} = \frac{0.3633 + 0.3594}{2} = 0.3614 \;\Rightarrow\; f_9 = 0.0024 > 0 \\
x_{10} &= \frac{x_9 + x_6}{2} = \frac{0.3614 + 0.3594}{2} = 0.3604 \;\Rightarrow\; f_{10} = -0.0001 < 0 \\
x_{11} &= \frac{x_{10} + x_9}{2} = \frac{0.3604 + 0.3614}{2} = 0.3609 \;\Rightarrow\; f_{11} = 0.0012 > 0 \\
x_{12} &= \frac{x_{11} + x_{10}}{2} = \frac{0.3609 + 0.3604}{2} = 0.3607 \;\Rightarrow\; f_{12} = 0.0007 > 0 \\
x_{13} &= \frac{x_{12} + x_{10}}{2} = \frac{0.3607 + 0.3604}{2} = 0.3606 \;\Rightarrow\; f_{13} = 0.0004 > 0 \\
x_{14} &= \frac{x_{13} + x_{10}}{2} = \frac{0.3606 + 0.3604}{2} = 0.3605 \;\Rightarrow\; f_{14} = 0.0002 > 0 \\
x_{15} &= \frac{x_{14} + x_{10}}{2} = \frac{0.3605 + 0.3604}{2} = 0.3604 \;\Rightarrow\; f_{15} = -0.0001 < 1 \\
x_{16} &= \frac{x_{15} + x_{14}}{2} = \frac{0.3604 + 0.3605}{2} = 0.3604
\end{aligned}$$

So one of the roots of

$$3x + \sin x - e^x = 0$$

is approximately

$$0.3604$$

The exact solution is in fact 0.36044

Remark 6.3.1 The Bisection method takes long to converge.

6.3.5 Advantages of the Bisection method

- The method is simple.
- The method is always convergent.

6.3.6 Disadvantages of the Bisection method

- It requires the values of $a = x_0$ and $b = x_1$.
- The convergence of interval halving is very slow (slow at converging to the root $x^\star$). For example, a simple non linear equation

$$x^3 - x - 2$$

  will converge to about 1.521 after 15-fifteen iterations.
- The method fails in case of approximating a double root or a root of even multiplicity.
- A faster method for converging on a single root of a function is the Newton-Raphson iteration Method.

Example 6.3.3 Using the Bisection algorithm, find the root of

$$\cos x - xe^x = 0$$

correct to 3 decimal places on the interval $0 \leq x \leq 1$

Let $x_0 = 0 \Rightarrow f_0 = 1.000 > 0$ and $x_1 = 1 \Rightarrow f_1 = -2.178 < 0$, for the Bisection scheme,

$$\begin{aligned}
x_{n+1} &= \frac{x_n + x_{n-1}}{2} \\
x_2 &= \frac{x_1 + x_0}{2} = \frac{0+1}{2} = 0.500 \Rightarrow f_2 = 0.053 > 0 \\
x_3 &= \frac{x_2 + x_1}{2} = \frac{0.500 + 1.000}{2} = 0.750 \Rightarrow f_3 = -0.856 < 0 \\
x_4 &= \frac{x_3 + x_2}{2} = \frac{0.750 + 0.500}{2} = 0.625 \Rightarrow f_4 = -0.357 < 0 \\
x_5 &= \frac{x_4 + x_2}{2} = \frac{0.625 + 0.500}{2} = 0.562 \Rightarrow f_5 = -0.140 < 0 \\
x_6 &= \frac{x_5 + x_2}{2} = \frac{0.562 + 0.500}{2} = 0.531 \Rightarrow f_6 = -0.041 < 0 \\
x_7 &= \frac{x_6 + x_2}{2} = \frac{0.531 + 0.500}{2} = 0.516 \Rightarrow f_7 = 0.005 > 0 \\
x_8 &= \frac{x_7 + x_6}{2} = \frac{0.516 + 0.531}{2} = 0.524 \Rightarrow f_8 = -0.019 < 0 \\
x_9 &= \frac{x_8 + x_7}{2} = \frac{0.524 + 0.516}{2} = 0.520 \Rightarrow f_9 = -0.007 < 0 \\
x_{10} &= \frac{x_9 + x_7}{2} = \frac{0.520 + 0.516}{2} = 0.518 \Rightarrow f_{10} = -0.001 < 0 \\
x_{11} &= \frac{x_{10} + x_7}{2} = \frac{0.518 + 0.516}{2} = 0.517 \Rightarrow f_{11} = 0.002 > 0 \\
x_{12} &= \frac{x_{11} + x_{10}}{2} = \frac{0.517 + 0.518}{2} = 0.518 \Rightarrow f_{12} = -0.001 < 0 \\
x_{13} &= \frac{x_{12} + x_{11}}{2} = \frac{0.518 + 0.517}{2} = 0.518
\end{aligned}$$

The scheme converges at $x = 0.518$. Analytically, the root is 0.5177

Example 6.3.4 Estimate the zero to the non linear equation

$$f(x) = x^3 - 5x^2 - 2x + 10$$

using numerical Bisection method, if the graphical methods found the real root between $x = 1$ and $x = 3$:

Let

$$\begin{aligned} x_0 = 1 \Rightarrow f_0 = 4.0000 > 0 \\ x_1 = 3 \Rightarrow f_1 = -14.0000 < 0 \end{aligned}$$

By *interval halving*,

$$\begin{aligned}
x_{n+1} &= \frac{x_n + x_{n-1}}{2} \\
x_2 &= \frac{x_1 + x_0}{2} = \frac{3+1}{2} = 2.0000 \;\Rightarrow\; f_2 = -6.0000 < 0 \\
x_3 &= \frac{x_2 + x_0}{2} = \frac{2.0000 + 1.0000}{2} = 1.5000 \;\Rightarrow\; f_3 = -0.8750 < 0 \\
x_4 &= \frac{x_3 + x_0}{2} = \frac{1.5000 + 1.0000}{2} = 1.2500 \;\Rightarrow\; f_4 = 1.6406 > 0 \\
x_5 &= \frac{x_4 + x_3}{2} = \frac{1.2500 + 1.5000}{2} = 1.3750 \;\Rightarrow\; f_5 = 0.3965 > 0 \\
x_6 &= \frac{x_5 + x_3}{2} = \frac{1.3750 + 1.5000}{2} = 1.4375 \;\Rightarrow\; f_6 = -0.2366 < 0 \\
x_7 &= \frac{x_6 + x_5}{2} = \frac{1.4375 + 1.3750}{2} = 1.4063 \;\Rightarrow\; f_7 = 0.0802 > 0 \\
&\vdots
\end{aligned}$$

It is evident that the functional values $f(x_i)$ are approaching zero as the number of iterations is increased.

After more six iteration the approximated root of 1.40625 compares favorably with the exact value of $\sqrt{2}$

Example 6.3.5 The following polynomial has a root within the interval $3.75 \leq x \leq 5.00$

$$f(x) = x^3 - x^2 - 10x - 8 = 0$$

If a tolerance of 0.01(1%) is required, find this root using bisection method.

The Bisection algorithm is given by

$$x_{n+1} = \frac{x_n + x_{n-1}}{2} \text{ alternatively } x_m = \frac{x_s + x_e}{2}$$

where x_m is the mid point, x_s starting point of the two, and x_e end point of the two

Iteration i	x_s	x_m	x_e	$f(x_s)$	$f(x_m)$	$f(x_e)$	$f(x_s)f(x_m)$	$f(x_m)f(x_e)$	error ϵ_d
1	3.7500	4.3750	5.0000	-6.8281	12.8496	42.0000	-	+	$-$
2	3.7500	4.0625	4.3750	-6.8281	1.9182	12.8496	-	+	0.31250
3	3.7500	3.9063	4.0625	-6.8281	-2.7166	1.9182	+	-	0.15625
4	3.9063	.9844	4.0625	-2.7166	-0.4661	1.9182	+	-	0.07813
5	3.9844	4.0234	4.0625	-0.4661	0.7092	1.9182	-	+	0.03906
6	3.9844	4.0039	4.0234	-0.4661	0.1174	0.7092	-	+	0.01953
7	3.9844	3.9941	4.0039	-0.4661	-0.1754	0.1174	+	-	0.00977
8	3.9941	3.9990	4.0039	-0.1754	-0.0293	0.1174	+	-	0.00488
9	3.9990	4.0015	4.0039	-0.0293	0.0440	0.1174	-	+	0.00244
10	3.9990	4.0002	4.0015	-0.0293	0.0073	0.0440	-	+	0.00122
11	3.9990	3.9996	4.0002	-0.0293	-0.0110	0.0073	+	-	0.00061
12	3.9996	3.9999	4.0002	-0.0110	-0.0018	0.0073	+	-	0.00031
13	3.9999	4.0001	4.0002	-0.0018	0.0027	0.0073	-	+	0.00015
14	3.9999	4.0000	4.0001	-0.0018	0.0005	0.0027	-	+	0.00008
15	3.9999	4.0000	4.0000	-0.0018	-0.0007	0.0005	+	-	0.00004

Example 6.3.6 Consider finding the root of

$$f(x) = e^{-x}(3.2\sin x - 0.5\cos x)$$

on the interval $[3, 4]$, this time with $\epsilon_{step} = 0.001$, $\epsilon_{abs} = 0.001$.

Iteration i	x_s	x_e	$f(x_s)$	$f(x_e)$	x_m	$f(x_m)$	ϵ_{abs}
1	3.0000	4.0000	0.047127	-0.038372	3.5000	-0.019757	0.5
2	3.0000	3.5000	0.047127	-0.019757	3.2500	0.0058479	0.25
3	3.2500	3.5000	0.0058479	-0.019757	3.3750	-0.0086808	0.125
4	3.2500	3.3750	0.0058479	-0.0086808	3.3125	-0.0018773	0.0625
5	3.2500	3.3125	0.0058479	-0.0018773	3.2812	0.0018739	0.0313
6	3.2812	3.3125	0.0018739	-0.0018773	3.2968	-0.000024791	0.0156
7	3.2812	3.2968	0.0018739	-0.000024791	3.2890	0.00091736	0.0078
8	3.2890	3.2968	0.00091736	-0.000024791	3.2929	0.00044352	0.0039
9	3.2929	3.2968	0.00044352	-0.000024791	3.2948	0.00021466	0.002
10	3.2948	3.2968	0.00021466	-0.000024791	3.2958	0.000094077	0.001
11	3.2958	3.2968	0.000094077	-0.000024791	3.2963	0.000034799	0.0005

Remark 6.3.2 Thus, after the 11th iteration, we note that the final interval, $[3.2958, 3.2968]$ has a width less than 0.001 and $|f(3.2968)| = 0.000025 < 0.001$ and therefore we chose $x_e = 3.2968$ to be our approximation of the root.
Although it is $x_m = 3.2963$ in the last interval, but it is not the better approximation, since scheme has not yet converged and yet $|f(3.2963)| = 0.000035$ a much more deviation from the exact value 0.00000 compared to $f(3.2968)$

Remark 6.3.3 The Bisection scheme has not yet converged, but iteration terminated because of the required error achieved.

Example 6.3.7 The root of

$$e^x - 2 = 0$$

is known to exist in $[0, 2]$. Use 8 iterations to find an approximate value of the root (or find an approximate value of the root to within a tolerance of ϵ)

Iteration i	x_s	x_m	x_e	$f(x_s)$	$f(x_e)$	$f(x_m)$
1	0.0000	1.0000000	2.000000	-1.0000	0.7183	5.3891
2	0.0000	0.5000000	1.000000	-1.0000	-0.3513	0.7183
3	0.5000	0.7500000	1.000000	-0.3513	0.1170	0.7183
4	0.5000	0.6250000	0.750000	-0.3513	-0.1318	0.1170
5	0.6250	0.6875000	0.750000	-0.1318	-0.0113	0.1170
6	0.6875	0.7187500	0.750000	-0.0113	0.0519	0.1170
7	0.6875	0.7031250	0.718750	-0.0113	0.0201	0.0519
8	0.6875	0.6953125	0.703125	-0.0113	0.0043	0.0201

Exercise 6.3.1 Will the Bisection Method applied to $f(x) = \tan x$ and initial interval $[a, b] = [1, 2]$ converge to a root? Why or why not? To which value, if any, will the Bisection Method converge?

Exercise 6.3.2 If $g(x) = \cos x - 2x$, and $[a_1, b_1] = [0, 1]$, use the Bisection Method to compute x_3. Show your work.

Exercise 6.3.3 Consider the equations

(a) $x^5 + x = 1$

(b) $\sin x = 6x + 5$

(c) $\ln x + x^2 = 3$

Apply two steps of the Bisection Method to find an approximate root within 1/8 of the true root.

Exercise 6.3.4 Approximate to 2 decimal places the roots of the following equations using the bisection method.

(a) $x^2 = 3$

(b) $x^3 = 2$

(c) $x^4 = 2$

Exercise 6.3.5 The function $h(x) = x \sin x$ occurs in the study of damped forced oscillation. Find the value of x that lies in the interval $[0, 2]$ where the function takes on the value $h(x) = 1$. Use interval bisection.

Exercise 6.3.6 If $a = 0.1$ and $b = 1.0$, how many steps of the bisection method are needed to determine a root in this interval with an error of at most $\frac{1}{2} \times 10^{-8}$?

Exercise 6.3.7 Consider obtaining the root of:

$$f(x) = \frac{e^x + 1 + \sin x}{(x-2)}.$$

Show that $f(1.9) < 0, f(2.1) > 0$ and use the bisection method to obtain the root.

Exercise 6.3.8 Find the real root of the equation

$$x^3 - x^2 - x + 1 = 0$$

using the bisection algorithm.

Exercise 6.3.9 The bisection method generates intervals $[a_0, b_0], [a_1, b_1]$, and so on, which of these inequalities are true for the root r that is being calculated?

(a) $|r - a_n| \leq 2|r - b_n|$

(b) $|r - a_n| \leq 2^{-n-1}(b_0 - a_0)$

(c) $|r - b_n| \leq 2^{-n-1}(b_0 - a_0)$

(d) $0 \leq r - a_n 2^{-n}(b_0 - a_0)$

(e) $|r - \frac{1}{2}(a_n + b_n)| \leq 2^{-n-2}(b_0 - a_0)$

Example 6.3.8 Find all the real solutions to the cubic equation

$$x^3 + 4x^2 - 10 = 0$$

in the interval $[1, 2]$.

Example 6.3.9 Use Newton's method to find the roots of the cubic polynomial $x^3 - 3x + 2 = 0$ in the interval

(a) $[0, 2]$

(b) $[-3, -1]$

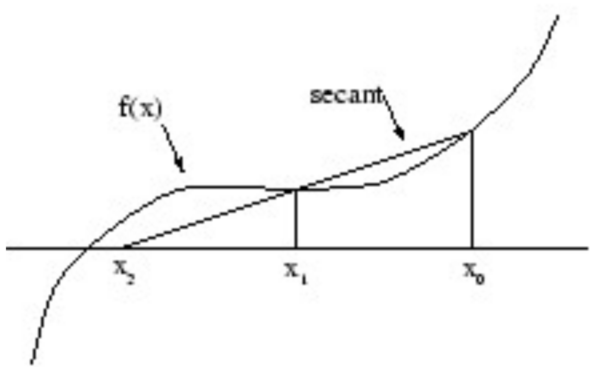
6.4 Secant Method (Chords Method)

The Secant method needs two points near the root before the algorithm can be applied. Thus it is of the form

$$x_{r+1} = f(x_r, x_{r-1}).$$

6.4.1 Derivation of the Secant method

We linear approximate the graph of $y = f(x)$ in figure (10.1), by a chord passing through the points A and B. The equation of this chord is,



The equation of the Chord is,

$$y = f(x_1) = \frac{[f(x_0) - f(x_1)](x - x_1)}{x_0 - x_1}$$

This Chord cuts the x-axis at x_2 i.e.

$$-f(x_1) = \frac{f(x_0) - f(x_1)}{(x_0 - x_1)}(x_2 - x_1)$$

$$\text{giving,}\ \ x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\text{or in general we have that,}\quad x_{n+1} = g(x_n, x_{n-1})$$

That is

$$x_{n+1} = x_n - \frac{f(x_n)\,[x_n - x_{n-1}]}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} = \frac{x_n\,[f(x_n) - f(x_{n-1})] - f(x_n)\,[x_n - x_{n-1}]}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} = \frac{x_n f(x_n) - x_n f(x_{n-1}) - x_n f(x_n) + x_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$\boxed{x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}} \tag{6.10}$$

The error in the $(n+1)^{th}$ iterate is related to the error in the n^{th} iterate e_n by the relation $e_{n+1} \simeq A e_n^k$ where $k \simeq 1.618\ldots$ and A is a constant. This relation suggests that the method has order of convergence 1.618.

Example 6.4.1 Use the Secant method to find the root near 2 of the equation

$$x^3 - 2x - 5 = 0$$

Start the iteration with $x_0 = 1.9, \Rightarrow f(x_0) = -1.941$ and $x_1 = 2.0 \Rightarrow f(x_1) = -1.000$.

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.9)(-1.000) - (2.0)(-1.941)}{-1.000 - {}^{-}1.941} = 2.10627, \quad f_2 = 0.13166$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(2.0)(0.13166) - (2.10627)(-1.000)}{0.13166 - {}^{-}1.000} = 2.09391, \quad f_3 = -0.00716$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(2.10627)(-0.00716) - (2.09391)(0.13166)}{-0.00716 - 0.13166} = 2.09455, \; f_4 = -0.00002$$

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = \frac{(2.09391)(-0.00002) - (2.09455)(-0.00716)}{-0.00002 - {}^{-}0.00716} = 2.09455$$

Thus since x_4 and x_5 are identical to 5 decimal places, so $x_5 = 2.09455$ is the value of the root correct to five decimal places.

Example 6.4.2 Estimate the root of

$$x^4 - x - 10 = 0$$

with the initial guess as 1.0 and 2.0 using the numerical Secant scheme. Take solutions to 5 decimal places.

Let

$$x_0 = 1.0 \Rightarrow f_0 = -10.0$$
$$x_1 = 2.0 \Rightarrow f_1 = 4.0$$

For the Secant rule

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.0)(4.0) - (2.0)(-10.0)}{4.0 - {}^{-}10.0} = 1.71429$$

$$\Rightarrow f_2 = -3.07780$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(2.0)(-3.07780) - (1.71429)(4.0)}{-3.07780 - 4.0} = 1.83853$$

$$\Rightarrow f_3 = -0.41283$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(1.71429)(-0.41283) - (1.83853)(-3.07780)}{-0.41283 - {}^{-}3.07780} = 1.85778$$

$$\Rightarrow f_4 = 0.05401$$

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = \frac{(1.83853)(0.05401) - (1.85778)(-0.41283)}{0.05401 - {}^{-}0.41283} = 1.85555$$

$$\Rightarrow f_5 = -0.00085$$

$$x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)} = \frac{(1.85778)(-0.00085) - (1.85555)(0.05401)}{-0.00085 - 0.05401} = 1.85558$$

$$\Rightarrow f_6 = -0.00011$$

$$x_7 = \frac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)} = \frac{(1.85555)(-0.00011) - (1.85558)(-0.00085)}{-0.00011 - {}^{-}0.00085} = 1.85558$$

So the iterative process converges at 1.85558

Note 6.4.1 If the function $f(x)$ has a trigonometric term, the calculators better be in *radians*.

Example 6.4.3 Approximate the root of

$$x - \sin x - \frac{1}{2} = 0$$

Let the initial guess be 1.0 and 2.0 using the Secant algorithm.

Let $x_0 = 1.0 \;\Rightarrow\; f_0 = -0.34147$ and $x_1 = 2.0 \;\Rightarrow\; f_1 = 0.59070$

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.0)(0.59070) - (2.0)(-0.34147)}{0.59070 -^{-} 0.34147} = 1.36632 \\
&\Rightarrow\; f_2 = -0.11285 \\
x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(2.0)(-0.11285) - (1.36632)(0.59070)}{-0.11285 - 0.59070} = 1.46796 \\
&\Rightarrow\; f_3 = -0.02676 \\
x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(1.36632)(-0.02676) - (1.46796)(-0.11285)}{-0.02676 -^{-} 0.11285} = 1.49955 \\
&\Rightarrow\; f_4 = 0.00209 \\
x_5 &= \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = \frac{(1.46796)(0.00209) - (1.49955)(-0.02676)}{0.00209 -^{-} 0.02676} = 1.49726 \\
&\Rightarrow\; f_5 = -0.00004 \\
x_6 &= \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)} = \frac{(1.49955)(-0.00004) - (1.49726)(0.00209)}{-0.00004 - 0.00209} = 1.49730 \\
&\Rightarrow\; f_6 = 0.00000 \\
x_7 &= \frac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)} = \frac{(1.49726)(0.00000) - (1.49730)(-0.00004)}{-0.00000 -^{-} 0.00004} = 1.49730
\end{aligned}$$

So the iterative process converges at 1.49730, and it is almost exact.

Example 6.4.4 Let the initial guess be -2.0 and -1.0, find the root of

$$(x^2 + 5x + 2)e^{-x} + 1 = 0$$

using the Secant technique approximated to 5 decimal places.

Let $x_0 = -2.0 \;\Rightarrow\; f_0 = -28.55622$ and $x_1 = -1.0 \;\Rightarrow\; f_1 = -4.43656$

$$
\begin{aligned}
x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(-2.0)(-4.43656) - (-1.0)(-28.55622)}{-4.43656 - ^{-}28.55622} = -0.81606 \\
&\Rightarrow\; f_2 = -2.19865 \\
x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(-1.0)(-2.19865) - (-0.81606)(-4.43656)}{-2.19865 - ^{-}4.43656} = -0.63535 \\
&\Rightarrow\; f_3 = -0.45933 \\
x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(-0.81606)(-0.45933) - (-0.63535)(-2.19865)}{-0.45933 - ^{-}2.19865} = -0.58763 \\
&\Rightarrow\; f_4 = -0.06695 \\
x_5 &= \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = \frac{(-0.63535)(-0.06695) - (-0.58763)(-0.45933)}{-0.06695 - ^{-}0.45933} = -0.57949 \\
&\Rightarrow\; f_5 = -0.00260 \\
x_6 &= \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)} = \frac{(-0.58763)(-0.00260) - (-0.57949)(-0.06695)}{-0.00260 - ^{-}0.06695} = -0.57916 \\
&\Rightarrow\; f_6 = -0.00001 \\
x_7 &= \frac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)} = \frac{(-0.57949)(-0.00001) - (-0.57916)(-0.00260)}{-0.00001 - ^{-}0.00260} = -0.57916
\end{aligned}
$$

The approximated zero of

$$(x^2 + 5x + 2)e^{-x} + 1 = 0$$

is

$$-0.57916$$

to five decimal places.

Example 6.4.5 Apply the Secant method to show that for the non linear equation

$$\cos x - xe^{x} = 0$$

the approximated roots are given by

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
1.0	2.0	0.83267	0.72878	0.56240	0.52478	0.51801	0.51776	0.51776

So the iterative process converges at 0.51776

Example 6.4.6 With initial conditions as $x_0 = 1.0$ and $x_1 = 2.0$, iterate with Secant method to show that for

$$x - e^{-x} = 0$$

the inexact solutions approximated to 5 decimal places are

x_0	x_1	x_2	x_3	x_4	x_5	x_6
1.0	2.0	0.48714	0.58378	0.56739	0.56714	0.56714

So one of the roots of $f(x) = x - e^{-x}$ is approximately 0.56714

Example 6.4.7 Find the zero of the non linear equation

$$e^{-x} = 3\log_{10} x$$

by the Secant method to 5 decimal places with the initial guess as 1.0 and 2.0

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
1.0	2.0	1.32394	1.22325	1.24759	1.24683	1.24682	1.24682

So one of the roots of $e^{-x} = 3\log_{10} x$ is approximately 1.24682.

Hint: The function $f(x)$ used is

$$f(x) = e^{-x} - 3\log_{10} x = 0$$

Example 6.4.8 Use the Secant method to find a solution to $x = \cos x$, and compare the approximations with those given by Newton's method with $x_0 = \pi/4$.

For the *Secant method* we need two initial approximations. Suppose we use $x_0 = 0.5$ and $x_1 = \pi/4$.

n	x_{n-1}	x_n	x_{n+1}	$\lvert x_{n+1} - x_n\rvert$
1	0.500000000	0.785398163	0.736384139	0.0490140246
2	0.785398163	0.736384139	0.739058139	0.0026740004
3	0.736384139	0.739085149	0.739085149	0.0000270101
4	0.739058139	0.739085149	0.739085133	0.0000000161

The *Newton's method* for $x = \cos x$ with $x_0 = \pi/4$ is given by

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	$\lvert x_{n+1} - x_n\rvert$
0	0.78539816	-0.078291	-1.707107	0.73953613	0.04586203
1	0.73953613	-0.000755	-1.673945	0.73908518	0.00045096
2	0.73908518	-0.000000	-1.673612	0.73908513	0.00000004
3	0.73908513	-0.000000	-1.673612	0.73908513	0.00000000

Observations:

- For Newton's method, an excellent approximation is obtained with $n = 2$
- Because of the agreement of x_2 and x_3 we could reasonably expect this result to be accurate to the places listed.
- Comparing results, we see that the Secant Method approximation x_4 is accurate to the tenth decimal place, whereas Newton's method obtained this accuracy by x_2.
- Here, the convergence of the Secant method is much faster than functional iteration but slightly slower than Newton's method.

6.4.2 Advantages and Disadvantages of the Secant method

The method;

1.) can work for double roots.

2.) has order of convergence of 1.618.

3.) is not always convergent.

The above are advantages or disadvantages depending on the comparison technique in question.

Remark 6.4.1 The Secant method and Newton's method are often used to refine an answer obtained by another technique (such as the Bisection Method).

Exercise 6.4.1 Find the real root of

$$f(x) = x^3 + x^2 - 3x - 3 = 0$$

using Secant technique correct to 2 decimal places.

Exercise 6.4.2 Show that there is a root of the equation

$$f(x) = 3x - \sin x - e^x = 0$$

in the interval $(0, 1)$. Estimate this root to 2 decimal places using the Secant method.

Exercise 6.4.3 Find the roots of the following equations, using the methods of Secant.

(a) $e^x = \cos x$

(b) $x^3 - 2x + 1 = 0$

(c) $\sin 2x - e^x - 1 = 0$

(d) $\ln(x - 1) = x^2$

Exercise 6.4.4 Use the Secant method to find the real root of the equation

$$x^3 + 2x^2 - x + 5 = 0.$$

Exercise 6.4.5 Consider obtaining the root of;

$$f(x) = \frac{e^x + \sin x}{(x - 2)}$$

Show that $f(1.9) < 0, \quad f(2.1) > 0$ and use the Secant method to obtain the root.

6.5 The Regula-Falsi method (Method of false position)

The regula falsi algorithm uses two points near the root before the algorithm is applied (a similar geometric approach like the Secant method-the regula falsi method is an associate of the Secant method) with the exception that,

$$f(x_r)f(x_{r-1}) < 0$$

at each stage of the algorithm. This is termed as *root bracketing*, like in the Bisection method. Thus to start the method, you need two points $x_0 = a_0$ and $x_1 = b_0$ near the root such that

$$f(a_0)f(b_1) < 0$$

That is $f(a_0)$ and $f(b_0)$ are of opposite signs, which implies by the intermediate value theorem that the function f has a root in the interval $[a_0, b_0]$, assuming continuity of the function f. The method proceeds by producing a sequence of shrinking intervals $[a_k, b_k]$ that all contain a root of f.

6.5.1 Geometric representation and derivation of the Regula falsi algorithm

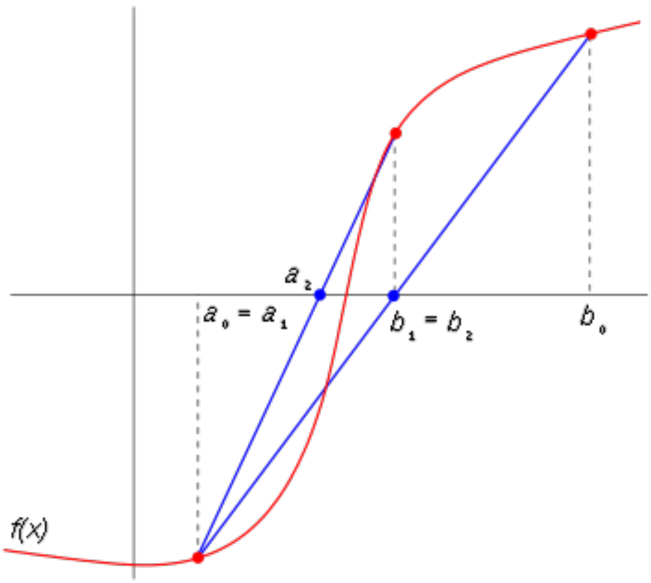


Figure 6.4: The first two iterations of the false position method

From Figure (6.4) above, we note that the produce $f(x_0)(f(x_1) < 0$, which is in conformity with the regula falsi. The equation of the Chord CD is,

$$y = f(x_1) = \frac{[f(x_0) - f(x_1)](x - x_1)}{x_0 - x_1}$$

This Chord cuts the x-axis at x_2 i.e.

$$-f(x_1) = \frac{f(x_0) - f(x_1)}{(x_0 - x_1)}(x_2 - x_1)$$

$$\text{giving, } \; x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

or in general we have that, $\quad x_{n+1} = g(x_n, x_{n-1})$

That is

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)\left[x_n - x_{n-1}\right]}{f(x_n) - f(x_{n-1})} \\
x_{n+1} &= \frac{x_n\left[f(x_n) - f(x_{n-1})\right] - f(x_n)\left[x_n - x_{n-1}\right]}{f(x_n) - f(x_{n-1})} \\
x_{n+1} &= \frac{x_n f(x_n) - x_n f(x_{n-1}) - x_n f(x_n) + x_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})} \\
x_{n+1} &= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}
\end{aligned}$$

$$\boxed{x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}} \tag{6.11}$$

Equation (6.11) is the popular Regula false/falsi position method with condition that at each stage of the algorithm,

$$f(x_n) f(x_{n-1}) < 0 \tag{6.12}$$

Example 6.5.1 Find the root between $(2, 3)$ of $x^3 - 2x - 5 = 0$, by using regula falsi method. Approximate values to 3 decimal places.

Let us take $x_0 = 2$ and $x_1 = 3$.

$$\begin{aligned} f(x) &= x^3 - 2x - 5 \\ f(x_0) &= f(2) = 2^3 - 2(2) - 5 = -1 < 0 \text{ (negative)} \\ f(x_1) &= f(3) = 3^3 - 2(3) - 5 = 16 > 0 \text{ (positive)} \end{aligned}$$

The first approximation to root is x_2 and is given by

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\ x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2f(3) - (3)f(2)}{f(3) - f(2)} = \frac{2(16) - (3)(-1)}{16 - (-1)} = \frac{35}{17} = 2.059 \\ &\Rightarrow f(2.059) = -0.389 < 0 \end{aligned}$$

The root lies between $x_2 = 2.059$ and $x_1 = 3$

Taking $x_2 = 2.059$ and $x_1 = 3$. we have the second approximation to the root given by

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\ x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(3)f(2.059) - 2.059f(3)}{f(2.059) - f(3)} = \frac{(3)(-0.389) - 2.059(16)}{(-0.389) - 16} = 2.081 \\ &\Rightarrow f(2.081) = -0.150 < 0 \end{aligned}$$

The root lies between $x_3 = 2.081$ and $x_1 = 3$

Taking $x_3 = 2.081$ and $x_1 = 3$. we have the third approximation to the root given by

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\ x_4 &= \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} = \frac{(3)f(2.081) - 2.081f(3)}{f(2.081) - f(3)} = \frac{(3)(-0.150) - 2.081(16)}{(-0.15) - 16} = 2.090 \\ &\Rightarrow f(2.090) = -0.051 < 0 \end{aligned}$$

The root lies between $x_4 = 2.090$ and $x_1 = 3$

Taking $x_4 = 2.090$ and $x_1 = 3$. we have the fourth approximation to the root given by

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\ x_5 &= \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)} = \frac{(3)f(2.090) - 2.090f(3)}{f(2.090) - f(3)} = \frac{(3)(-0.051) - (2.090)(16)}{(-0.051) - 16} = 2.091 \end{aligned}$$

If we need a root to 3 decimal places, we could still continue with the iterations (the root lies between $x_5 = 2.091$ and $x_1 = 3$), but if to 2 decimal places, the required root is 2.09

Example 6.5.2 Approximate a root of

$$3x + \sin x - e^x = 0$$

using the regula falsi scheme of solving non linear equations $f(x) = 0$ to 3 decimal places on the interval $[0, 0.5]$.

If we sketch the function $f(x)$ it's clear that there is a root between 0 and 0.5 and also another root between 1.5 and 2.0. Now let us consider the function $f(x)$ in the interval $[0, 0.5]$ where $f(0) \times f(0.5)$ is less than zero and use the regula-falsi scheme to obtain the zero of $f(x) = 0$.

Let $x_0 = 0.0 \;\Rightarrow\; f_0 = -1.000 < 0$ and $x_1 = 0.5 \;\Rightarrow\; f_1 = 0.331 > 0$

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}\\
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(0.0)(0.331) - (0.5)(-1.000)}{0.331 -^- 1.000} = 0.376, \;\Rightarrow\; f_2 = 0.039 > 0\\
x_3 &= \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = \frac{(0.0)(0.039) - (0.376)(-1.000)}{0.039 -^- 1.000} = 0.362, \;\Rightarrow\; f_3 = 0.004 > 0\\
x_4 &= \frac{x_0 f(x_3) - x_3 f(x_0)}{f(x_3) - f(x_0)} = \frac{(0.0)(0.004) - (0.362)(-1.000)}{0.004 -^- 1.000} = 0.361, \;\Rightarrow\; f_4 = 0.001 > 0\\
x_5 &= \frac{x_0 f(x_4) - x_4 f(x_0)}{f(x_4) - f(x_0)} = \frac{(0.0)(0.001) - (0.361)(-1.000)}{0.001 -^- 1.000} = 0.360, \;\Rightarrow\; f_5 = -0.001 < 0\\
x_5 &= \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)} = \frac{(0.361)(-0.001) - (0.360)(0.001)}{-0.001 - 0.001} = 0.360
\end{aligned}$$

So one of the roots of

$$3x + \sin x - e^x = 0$$

is approximately 0.360.

Example 6.5.3 Estimate the zero of the equation

$$x \cos\left(\frac{x}{x-2}\right) = 0$$

with the initial guess of $x_0 = 1$ and $x_1 = 1.5$ to 3 decimal places using the popular regula falsi technique.

Let

$$\begin{aligned} x_0 = 1.0 &\Rightarrow f_0 = 0.540 > 0 \\ x_1 = 1.5 &\Rightarrow f_1 = -1.485 < 0 \end{aligned}$$

For the regula falsi

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.0)(-1.485) - (1.5)(0.540)}{-1.485 - 0.540} = 1.133, \ \Rightarrow \ f_2 = 0.296 > 0 \\
x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(1.5)(0.296) - (1.133)(-1.485)}{0.296 -^- 1.485} = 1.194, \ \Rightarrow \ f_3 = 0.107 > 0 \\
x_4 &= \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} = \frac{(1.5)(0.107) - (1.194)(-1.485)}{0.107 -^- 1.485} = 1.214, \ \Rightarrow \ f_4 = 0.032 > 0 \\
x_5 &= \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)} = \frac{(1.5)(0.032) - (1.214)(-1.485)}{0.032 -^- 1.485} = 1.220, \ \Rightarrow \ f_5 = 0.008 > 0 \\
x_6 &= \frac{x_1 f(x_5) - x_5 f(x_1)}{f(x_5) - f(x_1)} = \frac{(1.5)(0.008) - (1.220)(-1.485)}{0.008 -^- 1.485} = 1.222, \ \Rightarrow \ f_6 = 0.000 > 0 \\
x_7 &= \frac{x_1 f(x_6) - x_6 f(x_1)}{f(x_6) - f(x_1)} = \frac{(1.5)(0.000) - (1.222)(-1.485)}{0.000 -^- 1.485} = 1.222
\end{aligned}$$

So one of the roots of

$$x \cos\left(\frac{x}{x-2}\right) = 0$$

is approximately

$$1.222$$

to 3 decimal places.

Example 6.5.4 Use the Regula-Falsi method to compute a real root of the equation

$$x^3 - 9x + 1 = 0,$$

(a) if the root lies between 2 and 4

(b) if the root lies between 2 and 3.

Comment on the results.

Let $x_0 = 2.0 \ \Rightarrow \ f_0 = -9 < 0$ and $x_1 = 4.0 \ \Rightarrow \ f_1 = 29 > 0$, For the regula falsi

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 2.4736, \ \Rightarrow \ f_2 = -6.12644 < 0 \\
x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 2.73989, \ \Rightarrow \ f_3 = -3.090707 < 0 \\
x_4 &= \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} = 2.86125, \ \Rightarrow \ f_4 = -1.326868 < 0 \\
&\vdots
\end{aligned}$$

Let $x_0 = 2.0 \ \Rightarrow \ f_0 = -9 < 0$ and $x_1 = 3.0 \ \Rightarrow \ f_1 = 1 > 0$, For the regula falsi

$$\begin{aligned}
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 2.90000, \ \Rightarrow \ f_2 = -0.7110 < 0 \\
x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 2.94156, \ \Rightarrow \ f_3 = -0.0207 < 0 \\
x_4 &= \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} = 2.94275, \ \Rightarrow \ f_4 = -0.0011896 < 0 \\
&\vdots
\end{aligned}$$

We observe that the value of the root as a third approximation is evidently different in both the cases, while the value of x_4, when the interval considered is (2, 3), is closer to the root. Important observation: The initial interval (x_0, x_1) in which the root of the equation lies should be sufficiently small.

Example 6.5.5 Use Regula-Falsi method to find a real root of the equation

$$\ln x - \cos x = 0$$

accurate to four decimal places after *three* successive approximations.

Let

$$\begin{aligned} x_0 &= 1.0 \Rightarrow f_0 = -0.540302 < 0 \\ x_1 &= 2.0 \Rightarrow f_1 = 1.109 > 0 \end{aligned}$$

For the regula falsi

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\ x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.0)(1.109) - (2.0)(-0.540302)}{1.109 -^- 0.540302} = 1.3275 \\ &\Rightarrow \ f_2 = 0.0424 > 0 \\ &\text{Next iteration to be for } x_0 \text{ and } x_2 \text{ since } f_0 f_2 < 0\text{, change signs} \\ x_3 &= \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = \frac{(1.0)(0.0424) - (1.3275)(-0.540302)}{0.0424 -^- 0.540302} = 1.3037 \\ &\Rightarrow \ f_3 = 0.001248 > 0 \\ &\text{Next iteration to be for } x_0 \text{ and } x_3 \text{ since } f_0 f_3 < 0\text{, change signs} \\ x_4 &= \frac{x_0 f(x_3) - x_3 f(x_0)}{f(x_3) - f(x_0)} = \frac{(1.0)(0.001248) - (1.3037)(-0.540302)}{0.001248 -^- 0.540302} = 1.3030 \end{aligned}$$

The required real root is 1.3030

Note 6.5.1 The iterations have not converged, but we were interested in up to the third iteration.

Note 6.5.2 For Matlab program for the non linear equation, we write $\ln x$ as $\log x$, where as $\log x$ as $\log_{10} x$

Example 6.5.6 Use the method of false position to solve

$$e^x + 2^{-x} + 2\cos x - 6 = 0 \quad 1 \le x \le 2$$

correct your approximations to 4 decimal places.

Let

$$\begin{aligned} x_0 &= 1.0 \Rightarrow f_0 = -1.7011 < 0 \\ x_1 &= 2.0 \Rightarrow f_1 = 0.8068 > 0 \end{aligned}$$

For the regula falsi

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.0)(0.8068) - (2.0)(-1.7011)}{0.8068 -^- 1.7011} = 1.6783, \;\Rightarrow\; f_2 = -0.5457 < 0 \\
x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(2.0)(-0.5457) - (1.6783)(0.8068)}{-0.5457 - 0.8068} = 1.8081, \;\Rightarrow\; f_3 = -0.0858 < 0 \\
x_4 &= \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} = \frac{(2.0)(-0.0858) - (1.8081)(0.8068)}{-0.0858 - 0.8068} = 1.8265, \;\Rightarrow\; f_4 = -0.0118 < 0 \\
x_5 &= \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)} = \frac{(2.0)(-0.0118) - (1.8265)(0.8068)}{-0.0118 - 0.8068} = 1.8290, \;\Rightarrow\; f_5 = -0.0016 < 0 \\
x_6 &= \frac{x_1 f(x_5) - x_5 f(x_1)}{f(x_5) - f(x_1)} = \frac{(2.0)(-0.0016) - (1.8290)(0.8068)}{-0.0016 - 0.8068} = 1.8293, \;\Rightarrow\; f_6 = -0.0003 < 0 \\
x_7 &= \frac{x_1 f(x_6) - x_6 f(x_1)}{f(x_6) - f(x_1)} = \frac{(2.0)(-0.0003) - (1.8294)(0.8068)}{-0.0003 - 0.8068} = 1.8294, \;\Rightarrow\; f_7 = 0.0001 > 0 \\
x_8 &= \frac{x_6 f(x_7) - x_7 f(x_6)}{f(x_7) - f(x_6)} = \frac{(1.8293)(0.0001) - (1.8294)(-0.0003)}{0.0001 -^- 0.0003} = 1.8294
\end{aligned}$$

So the root of

$$e^x + 2^{-x} + 2\cos x - 6 = 0$$

is approximately 1.8294 to 4 decimal places. Analytically, the exact value would be 1.82938.

Remark 6.5.1 Look back at how x_8 was computed, how and why it was x_6 and x_7 paired together, and not x_7 with x_1.

Example 6.5.7 Solve the equation

$$2x\cos 2x - (x-2)^2 = 0 \quad 2 \le x \le 3$$

correct your approximations to 4 decimal places. Perform only four iterations.

Let

$$\begin{aligned} x_0 &= 2.0 \Rightarrow f_0 = -2.6146 < 0 \\ x_1 &= 3.0 \Rightarrow f_1 = 4.7610 > 0 \end{aligned}$$

For the regula falsi

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\ x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(2.0)(4.7610) - (3.0)(-2.6146)}{4.7610 -^{-} 2.6146} = 2.3545 \\ &\Rightarrow \ f_2 = -0.1416 < 0 \\ x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(3.0)(-0.1416) - (2.3545)(4.7610)}{-0.1416 - 4.7610} = 2.3731 \\ &\Rightarrow \ f_3 = 0.0212 > 0 \\ x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(2.3545)(0.0212) - (2.3731)(-0.1416)}{0.0212 -^{-} 0.1416} = 2.3707 \\ &\Rightarrow \ f_4 = 0.0001 > 0 \\ x_5 &= \frac{x_2 f(x_4) - x_4 f(x_2)}{f(x_4) - f(x_2)} = \frac{(2.3545)(0.0001) - (2.3707)(-0.1416)}{0.0001 -^{-} 0.1416} = 2.3707 \end{aligned}$$

Remark 6.5.2 The scheme converges after the fourth iteration and converges to 2.3707.

In fact, one of the classical root of the non linear equation

$$2x\cos 2x - (x-2)^2 = 0$$

is

$$2.37069$$

Remark 6.5.3 In every next iteration, the previous point must be part of it.

Example 6.5.8 Using Regula-Falsi algorithm, approximate the real root of

$$f(x) = x^3 - 2x - 2 = 0$$

Now since, $f(1) = -3 < 0$ and $f(2) = 2 > 0$ and $f(x)$ is continuous for all real x, there exists $x^\star \in (1, 2)$ such that $f(x^\star) = 0$ (the intermediate value theorem).

Let $x_0 = 1.0 \;\Rightarrow\; f_0 = -3.00000 < 0$ and $x_1 = 2.0 \;\Rightarrow\; f_1 = 2.00000 > 0$,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.0)(2.00000) - (2.0)(-3.00000)}{2.00000 -^{-} 3.00000} = 1.60000, \;\Rightarrow\; f_2 = -1.10400 < 0$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(2.0)(-1.10400) - (1.60000)(2.00000)}{-1.10400 - 2.00000} = 1.74227$$

$$\Rightarrow\; f_3 = -0.19587 < 0$$

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} = \frac{(2.0)(-0.19587) - (1.74227)(2.00000)}{-0.19587 - 2.00000} = 1.76526$$

$$\Rightarrow\; f_4 = -0.02972 < 0$$

$$x_5 = \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)} = \frac{(2.0)(-0.02972) - (1.76526)(2.00000)}{-0.02972 - 2.00000} = 1.76870$$

$$\Rightarrow\; f_5 = -0.00438 < 0$$

$$x_6 = \frac{x_1 f(x_5) - x_5 f(x_1)}{f(x_5) - f(x_1)} = \frac{(2.0)(-0.00438) - (1.76870)(2.00000)}{-0.00438 - 2.00000} = 1.76921$$

$$\Rightarrow\; f_6 = -0.00061 < 0$$

$$x_7 = \frac{x_1 f(x_6) - x_6 f(x_1)}{f(x_6) - f(x_1)} = \frac{(2.0)(-0.00061) - (1.76921)(2.00000)}{-0.00061 - 2.00000} = 1.76928$$

$$\Rightarrow\; f_7 = -0.00009 < 0$$

$$x_8 = \frac{x_1 f(x_7) - x_7 f(x_1)}{f(x_7) - f(x_1)} = \frac{(2.0)(-0.00009) - (1.76928)(2.00000)}{-0.00009 - 2.00000} = 1.76929$$

$$\Rightarrow\; f_8 = -0.00002 < 0$$

$$x_9 = \frac{x_1 f(x_8) - x_8 f(x_1)}{f(x_8) - f(x_1)} = \frac{(2.0)(-0.00002) - (1.76929)(2.00000)}{-0.00002 - 2.00000} = 1.76929$$

The numerical solution is therefore 1.76929 to five decimal points.

Example 6.5.9 Use the method of False Position to find a solution to $x = \cos x$, and compare the approximations with those given by the Newton's method and the Secant Method. Let the initial approximations be $x_0 = 0.5$ and $x_1 = \frac{\pi}{4}$

	False Position	**Secant**	**Newton-Raphson**
n	x_n	x_n	x_n
0	0.5	0.5	0.5
1	0.7853981635	0.7853981635	0.7853981635
2	0.7363841388	0.7363841388	0.7395361337
3	0.7390581392	0.7390581392	0.7390851781
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	0.7390851332
5	0.7390851332	0.7390851332	
7	0.7390851332		

Note 6.5.3 Note that the False Position and Secant approximations agree through x_3 and that the method of False Position requires an additional iteration to obtain the same accuracy as the Secant method.

Remark 6.5.4

- The added insurance of the method of False Position commonly requires more calculation than the Secant method, . . .
- just as the simplification that the Secant method provides over Newton's method usually comes at the expense of additional iterations.

Example 6.5.10 The function

$$f(x) = x^2 e^x - 1$$

has a root in the interval $[0, 1]$ since $f(0)f(1) < 0$. The results from the false position and secant methods, both started with $x_0 = 0$ and $x_1 = 1$, are shown in the table

Iterates	**False position**	**Secant**
x_2	0.3679	0.3679
x_3	0.5695	0.5695
x_4	0.6551	0.7974
x_5	0.6868	0.6855
x_6	0.6978	0.7012
x_7	0.7016	0.7035

It appears from these results that the secant method gives the correct result $x = 0.7035$ a little more quickly.

6.5.2 Order of Convergence of the Regula algorithm

The error in the $(n+1)^{th}$ iterate (denoted e_{n+1}) is related to the error in the n^{th} iterate e_n by the equation,

$$e_{n+1} \simeq A e_n^k$$

where $k = 1$. This suggests that although the regula falsi uses the same formula as the Secant method, the order of convergence of the regula falsi is one compared to $\simeq 1.618$ for the Secant. Thus, the method is slower at converging to the root compared to the Secant method. However, with the condition $f(x_r)f(x_{r-1}) < 0$ at each stage, ensures that the regula falsi is always convergent which is not the case with Secant method.

6.5.3 Advantages and disadvantages of the regula falsi algorithm

1.) The regula falsi algorithm is always convergent.

2.) The order of convergence of the method is one.

The two basic points on the advantages and disadvantages.

Whether it is an advantage or a disadvantage it all depends on the comparison in question. For instance in comparison with the Secant method, it is disadvantageous that the regula falsi has order of convergence one. While it is advantageous that it is always convergent.

Exercise 6.5.1 Find the approximate value of the real root of

$$x \log_{10} x = 1.2$$

by regula falsi method

Exercise 6.5.2 Find the root of the

$$xe^x = 3$$

by regula falsi method and correct to the three decimal places

Exercise 6.5.3 Find a root which lies between 1 and 2 of

$$f(x) = x^3 + 2x^2 + 10x - 20$$

(Leonardo's Equation) using the regula falsi method

Exercise 6.5.4 Use the regula false algorithm to find the root of

$$f(x) = x^2 - 4x + 2 = 0$$

that lies in the interval $(0, 1)$ and state your answer correct to three decimal places.

Exercise 6.5.5 Verify that $x = 3$ is a solution of $x = g(x)$ where

$$g(x) = \frac{18x}{(x^2 + 9)}.$$

Use the regula false to approximate this root.

Exercise 6.5.6 Consider the equation

$$f(x) = \frac{e^x + 1 + \sin x}{(x - 2)} = 0$$

whose root you would want to find. Show that

$$f(1.9) < 0, f(2.1) > 0$$

and use the regula false algorithm to compute this root.

Exercise 6.5.7 Approximate to three decimal places the roots of the following equations using the regula false algorithm.

(a) $x^3 = 2$

(b) $x^2 = 3$

(c) $x^4 = 2$

(d) $x^5 = 3$

Exercise 6.5.8

(a) Derive the regula falsi algorithm by clearly giving its geometrical illustration.

(b) What advantages and disadvantages does the Secant method enjoy over other methods so far considered for solving nonlinear equations.

Exercise 6.5.9 Use both the Falsi Position and Bisection to solve the same equation

$$f(d) = 2552 - 30d^2 + d^3 = 0$$

and see if there is a difference in the number of steps falsi Position takes to converge versus Bisection.

Exercise 6.5.10
Find the solution of $x^3 + x - 4 = 0$ in the interval $[1, 4]$ with accuracy 10^{-3}. Apply

(a) Newton Raphson method

(b) Bisection algorithm

(c) Regular Falsi scheme

Exercise 6.5.11 Approximate the solution of $x^3 - x - 1 = 0$ in the interval $[1, 2]$ with accuracy 10^{-4} with the Bisection method.

Exercise 6.5.12 Use Newton's method to solve the equation $x^3 - x - 1 = 0$

Exercise 6.5.13 The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^{-6} using the Regular Falsi method.

Summary 6.1 Increased number of decimal points and/or increased size of interval given both increase the number of iterations as they increase.

However, increased number of decimal points and/or increased size of interval given both improve on the accuracy of the scheme.

6.6 Successive Substitution (Fixed Point Method)

6.6.1 Background knowledge

Successive substitution is one of the iterative techniques for solving nonlinear equations. Iterative techniques start with an initial value/guess x_0 to the root α and then using a suitable recurrence relation we generate a sequence of approximations $\{x_k\}_{k=o}^{\infty}$. If the sequence $\{x_0, x_1, \ldots\}$ converges, then it does so on the required root. Iterative techniques are written in the form

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots,$$

if the next iterate x_{n+1} depends on the previous one x_n.

$$\texttt{or } x_{n+1} = g_r(x_n, x_{n-1}), n = 1, 2, \ldots,$$

if the next iterate depends on the previous two i.e. x_n and x_{n-1}.

6.6.2 Successive Substitutions

In the method, we seek the roots of,

$$f(x) = 0. \tag{6.13}$$

We try to split $f(x)$ in the form,

$$f(x) = x - g(x) \tag{6.14}$$

However, this splitting may not be unique. But not all the different splittings may be useful to us. We can determine the type of splitting which is useful to a numerical analyst. Now, instead of solving equation (6.13) we now solve $x = g(x)$. The scheme for solving this problem is given by the algorithm;

$$x_{n+1} = g(x_n), \; n = 0, 1, 2, \ldots$$

Thus we start with a suitable value x_0, and generate the sequence of approximations

$$\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_1) \\ x_3 &= g(x_2) \\ x_4 &= g(x_3) \\ &\vdots \\ x_{n+1} &= g(x_n) \\ &\vdots \\ &\vdots \end{aligned}$$

That is, the sequence is $\{x_1, x_2, \ldots, x_n, \ldots\}$

Example 6.6.1 Find the real root of the equation

$$x^2 - 2x - 3 = 0$$

in the interval $[2, 4]$.

Splitting

$$f(x) = x^2 - 2x - 3 = 0$$

in the form

$$f(x) = x - g(x) = 0.$$

1.) Can have the form

$$x = g_1(x) = \frac{3}{x-2}$$

Taking the initial approximation $x_0 = 4$, we get the iterates

$$\begin{aligned}
x_{n+1} &= \frac{3}{x_n - 2} \\
x_1 &= \frac{3}{x_0 - 2} = \frac{3}{(4.000000 - 2)} = 1.500000 \\
x_2 &= \frac{3}{x_1 - 2} = \frac{3}{(1.500000 - 2)} = -6.000000 \\
x_3 &= \frac{3}{x_2 - 2} = \frac{3}{(-6.000000 - 2)} = -0.375000 \\
x_4 &= \frac{3}{x_3 - 2} = \frac{3}{(-0.375000 - 2)} = -1.263158 \\
x_5 &= \frac{3}{x_4 - 2} = \frac{3}{(-1.263158 - 2)} = -0.919355 \\
x_6 &= \frac{3}{x_5 - 2} = \frac{3}{(-0.919355 - 2)} = -1.027624 \\
x_7 &= \frac{3}{x_6 - 2} = \frac{3}{(-1.027624 - 2)} = -0.990876 \\
&\vdots
\end{aligned}$$

According to the behavior of the iterates, there is no hope for convergence in the interval $[2, 4]$. Hence such a rearrangement-choice of $g(x)$- is not good.

2.) Splitting $f(x) = 0$ in the form,

$$x = g_2(x) = \frac{x^2 - 3}{2}$$

giving the iterative scheme,

$$\begin{aligned}
x_{n+1} &= \frac{x_n^2 - 3}{2} \\
x_1 &= \frac{x_0^2 - 3}{2} = 6.500000 \\
x_2 &= \frac{x_1^2 - 3}{2} = 19.625000 \\
x_3 &= \frac{x_2^2 - 3}{2} = 191.070312 \\
&\vdots
\end{aligned}$$

This shows that the iterates are obviously diverging. Hence such a rearrangement of $g_2(x)$ is not good too.

3.) Splitting $f(x) = 0$ in the form,

$$x = g_3(x) = \sqrt{(2x+3)}$$

Giving the iteration formula,

$$x_{n+1} = \sqrt{(2x_n+3)}$$

Thus with $x_0 = 4$ we get,

$$\begin{aligned}
x_1 &= \sqrt{(2x_0+3)} = 3.316625\\
x_2 &= \sqrt{(2x_1+3)} = 3.103748\\
x_3 &= \sqrt{(2x_2+3)} = 3.034386\\
x_4 &= \sqrt{(2x_3+3)} = 3.011440\\
x_5 &= \sqrt{(2x_4+3)} = 3.003811\\
x_6 &= \sqrt{(2x_5+3)} = 3.001270\\
x_7 &= \sqrt{(2x_6+3)} = 3.000423\\
x_8 &= \sqrt{(2x_7+3)} = 3.000141\\
x_9 &= \sqrt{(2x_8+3)} = 3.000047\\
x_{10} &= \sqrt{(2x_9+3)} = 3.000016\\
x_{11} &= \sqrt{(2x_{10}+3)} = 3.000005\\
x_{12} &= \sqrt{(2x_{11}+3)} = 3.000002\\
x_{13} &= \sqrt{(2x_{12}+3)} = 3.000001\\
x_{14} &= \sqrt{(2x_{13}+3)} = 3.000000\\
x_{15} &= \sqrt{(2x_{14}+3)} = 3.000000
\end{aligned}$$

In fact this is an arrangement which is giving a sequence of iterates which are converging to the root α. The sequence is converging to the root $x = 3.000000$.

Note 6.6.1 One would actually wonder as to whether you have to keep trying the splitting until you get one which converges to the root. We can test and know the splitting which gives a convergence sequence of approximations before starting to compute the iterates. This wonderful criterion is called the convergence criterion for the iterative scheme of the form

$$x_{n+1} = g(x_n).$$

However, before stating the criterion, lets first formerly state what is meant by an iterative scheme

$$x_{n+1} = g(x_n)$$

being convergent.

Definition 6.6.1
An iterative scheme/process. $x_{n+1} = g(x_n)$ is convergent if,

$$\lim_{r\to\infty} |x_{n+1} - x_n| = 0,$$

otherwise we say that the scheme is divergent.

6.6.3 Convergence criterion

Definition 6.6.2

1.) Let the function $g(x)$ be continuous in a small interval $[a, b]$ containing a simple (single) root of the function $f(x)$.

2.) Let also $g(x)$ be differentiable in the open interval (a, b).

If there exists a real number L such that

- $0 \leq |g'(x)| \leq L \leq 1 \ \forall \ \ x \in (a, b)$.
- $a \leq g(x) \leq b \ \forall \ \ x \in (a, b)$.

then for an arbitrary starting value x_0 taken from (a, b) the iteration formula $x_{n+1} = g(x_n)$ will converge.

Remark 6.6.1 The rate of convergence of the iteration will depend on the smallness of the constant L relative to unity.

Example 6.6.2

1.) Given the function $f(x) = x^3 - \sin x = 0,\ [0, 1]$, using successive substitution technique, we can generate

$$x_{n+1} = (\sin x_n)^{\frac{1}{3}} \ \& \ x_{n+1} = \frac{\sin x_n}{x_n^2}$$

Which of the two methods converge? Why?

Since for

$$g(x) = (\sin x)^{\frac{1}{3}},\ g'(x) = \frac{1}{3}(\sin)^{\frac{-2}{3}} \cos x \ \Rightarrow \ 0 < g'(x) < 1$$

which implies, $g(x) = (\sin x)^{\frac{1}{3}}$ is the converging formula.

2.) Use the converging formula in (ii) above, to approximate the root of

$$f(x) = x^3 - \sin x = 0$$

in $[0, 1]$ to 3 decimal places.

Hint: Let $x_0 = 1$ and use radians.

n	$(\sin x)^{\frac{1}{3}}$
1	1.0
2	0.944
3	0.932
4	0.929
5	0.929

Note 6.6.2 Sometimes this method is known as the fixed point method, i.e. a fixed point is a number $x^\star$ such that $x^\star = g(x^\star)$. Thus a root of $f(x) = 0$ is a fixed point of the scheme $x = g(x)$.

Example 6.6.3 Given

$$f(x) = x^3 - 7x + 2 = 0$$

in $[0,1]$. Find a sequence that $\{x_n\}$ that converges to the root of $f(x) = 0$ in $[0,1]$.

Rewrite $f(x) = 0$ as $x = \frac{1}{7}(x^3 + 2)$. Then

$$g(x) = \frac{1}{7}(x^3 + 2) \Rightarrow g'(x) = \frac{3}{7}x^2 < \frac{3}{7} < 1 \; \forall \; x \in [0,1]$$

Hence by the convergence criterion, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \frac{1}{7}(x_n^3 + 2)$$

converges to a root of $x^3 - 7x + 2 = 0$

Example 6.6.4 Approximate

$$f(x) = x^3 - x - 1 = 0$$

on $(1,2)$ to *nine decimal places* by the successive substitution method.

Note that $f(-1) = -1$ and $f(2) = 5$. Therefore by the IVT (Intermediate Value Theorem of Calculus) a root exists on $(0,2)$.

Set

$$x_{n+1} = g(x) = (1 + x)^{\frac{1}{3}}$$

Note that

$$g'(x) = \frac{1}{3}(1 + x)^{-\frac{2}{3}} \Rightarrow \frac{1}{3(1+2)^{2/3}} \le g'(x) \le \frac{1}{3(1+1)^{2/3}}$$

$$\Rightarrow 0 \le g'(x) \le \frac{1}{3(2^{2/3})} < 1 \; \forall \; x \in (1,2)$$

By the convergence criterion, the sequence

$$x_{n+1} = (1 + x_n)^{\frac{1}{3}}$$

will converge to a fixed point on $(1,2)$

Let the initial guess be $x_0 = 1.3$

$$\begin{aligned}
x_1 &= (1 + x_0)^{1/3} = (1 + 1.300000000)^{1/3} = 1.320006122 \\
x_2 &= (1 + x_1)^{1/3} = (1 + 1.320006122)^{1/3} = 1.323822354 \\
x_3 &= (1 + x_2)^{1/3} = (1 + 1.323822354)^{1/3} = 1.324547818 \\
x_4 &= (1 + x_3)^{1/3} = (1 + 1.324547818)^{1/3} = 1.324685639 \\
x_5 &= (1 + x_4)^{1/3} = (1 + 1.324685639)^{1/3} = 1.324711818 \\
x_6 &= (1 + x_5)^{1/3} = (1 + 1.324711818)^{1/3} = 1.324716791 \\
x_7 &= (1 + x_6)^{1/3} = (1 + 1.324716791)^{1/3} = 1.324717736 \\
x_8 &= (1 + x_7)^{1/3} = (1 + 1.324717736)^{1/3} = 1.324717915 \\
x_9 &= (1 + x_8)^{1/3} = (1 + 1.324717915)^{1/3} = 1.324717949 \\
x_{10} &= (1 + x_9)^{1/3} = (1 + 1.324717949)^{1/3} = 1.324717956 \\
x_{11} &= (1 + x_{10})^{1/3} = (1 + 1.324717956)^{1/3} = 1.324717957 \\
x_{12} &= (1 + x_{11})^{1/3} = (1 + 1.324717957)^{1/3} = 1.324717957 \\
x_{13} &= (1 + x_{12})^{1/3} = (1 + 1.324717957)^{1/3} = 1.324717957
\end{aligned}$$

It converges to a fixed point 1.324717957

Example 6.6.5 Find the root of

$$f(x) = e^{-x} - x = 0$$

starting with an initial guess of $x_0 = 0$ correct to one decimal places.

We use the substitution

$$x_{n+1} = e^{-x_n}$$

to have the solution as

$$x = 0.5$$

after six iterations

Remark 6.6.2 Correct to *one decimal place* is too much an approximation, but for faster convergence - not to have more than 14 iterations - it is the pay-off between accuracy and computational difficulties.

Find the root of

$$f(x) = e^{-x} - x = 0$$

using both the Newton-Raphson and Fixed point iteration

Newton-Raphson

n	x_n
0	0
1	0.500000000
2	0.566311003
3	0.567143165
4	0.567143290

Fixed Point Iteration with $x_{n+1} = e^{-x_n}$

n	x_n
0	0
1	1.000000
2	0.367879
3	0.692201
4	0.500473
5	0.606244
6	0.545396
7	0.579612
8	0.560115
9	0.571143
10	0.564879

The correct solution is $x = 0.56714329$. We realise that even at 10^{th} iteration, the fixed point method has not yet converged.

Exercise 6.6.1 Verify for the following example that $x = 3$ is a solution of $x = g(x)$.

1.) $g(x) = \dfrac{18x}{(x^2+9)}$

2.) $g(x) = x^3 - 24$

3.) $g(x) = \dfrac{2x^3}{(3x^2-9)}$

4.) $g(x) = \dfrac{81}{(x^2+18)}$

Starting with $x_0 = 3.1$, calculate the first few iteration and justify theoretically the apparent behavior.

Exercise 6.6.2 Consider the fixed iteration $x_k = (x_{k-1})$ for $g(x) = 2(x-1)^{\frac{1}{2}}$ for $x \geq 1$. Show that only one fixed point exists (at $x = 2$) and that $g'(2) = 1$. Compute iterations starting from

1.) $x_0 = 1.5$ and

2.) $x_0 = 2.5$ and show them on a plot of x and $g(x)$.

Exercise 6.6.3 By splitting $f(x) = x^3 - x - 1 = 0$ in the form $f(x) = x - g(x) = 0$ for finding the root $\alpha \in [1, 2]$,

1.) Get three different splitting and with their corresponding iterative formulae.

2.) Test using the convergence criterion which of the splitting lead to a convergent sequence.

3.) For the scheme leading to a convergent sequence, start with a suitable initial approximation and find the root correct to 3 decimal places.

6.7 Non-Linear Equations Chapter Examples

Example 6.7.1 Use the Bisection method to approximate the root of the non linear equation $x^3 + 4 = 0$ on the interval $[2, 4]$?

Solution : *Both* $f(2) = 12$ *and* $f(4) = 68$ *are of same signs* $f(2)f(4) > 0$ *which violates the Bisection method condition*

$$f(x_0)f(x_1) < 0$$

Since $f(2)f(4) > 0$*, there is no root between* $[2, 4]$*.* ■

Example 6.7.2 Show that the iteration equation for the secant method can be written in the following form:

$$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}.$$

Solution : *The Secant iteration is defined as*

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

We have

$$\begin{aligned} x_{n+1} &= \frac{(f(x_n) - f(x_{n-1}))x_n - f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})} \end{aligned}$$

■

Example 6.7.3 State any two advantages of Newton Raphson's scheme over the secant scheme. [2 marks]

Solution :

- *NRM scheme converges quadratically compared to the Secant scheme whose order of convergence is* 1.62
- *NRM requires on one initial guess compared to the Secant scheme which requires two initial guesses.*

■

Example 6.7.4 Estimate to 3 decimal places the numerical root of

$$x^2 - 2 = 0 \tag{6.15}$$

using

1.) Bisection algorithm with $x_0 = 1.25$ and $x_1 = 1.5$. [4 marks]

Solution : *We compute $f(1.25)$ and $f(1.5)$ and notice that $f(1.25)f(1.5) < 0$ $\Rightarrow$ there exists $x^* \in (1.25, 1.5)$ such that $f(x^*) = 0$. To compute x^*, we keep bisecting the interval in which the root lies.*

$$\begin{aligned}
\Rightarrow x_2 &= \frac{x_0 + x_1}{2}\ \frac{1.250 + 1.500}{2}\ 1.375\ f(x_2) < 0 \\
\Rightarrow x_3 &= \frac{x_2 + x_1}{2}\ \frac{1.375 + 1.500}{2}\ 1.436\ f(x_3) > 0 \\
\Rightarrow x_4 &= \frac{x_2 + x_3}{2}\ \frac{1.436 + 1.375}{2}\ 1.406\ f(x_4) < 0 \\
\Rightarrow x_5 &= \frac{x_3 + x_4}{2}\ \frac{1.436 + 1.406}{2}\ 1.421\ f(x_5) > 0 \\
\Rightarrow x_6 &= \frac{x_5 + x_6}{2}\ \frac{1.406 + 1.421}{2}\ 1.414 \\
\Rightarrow x_7 &= 1.414
\end{aligned}$$

Therefore, the root is $x = 1.414$ ∎

2.) Newton Raphson's method with $x_0 = 1.5$.

Solution : $f(x) = x^2 - 2 \Rightarrow f'(x) = 2x$.

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \frac{x_n^2 - 2}{2x_n} \\
x_1 &= 1.417 \\
x_2 &= 1.414 \\
x_3 &= 1.414
\end{aligned}$$

∎

3.) State one advantage and one disadvantage of the Bisection algorithm over the Newton Raphson's method.

Solution :

- *The bisection method is easy since it does not require one to find the derivative unlike the NRM.*
- *The bisection method is always convergent unlike the NRM which requires us to choose x_0 to be close to the root.*

and

- *The bisection method is slow to converge unlike the NRM.*
- *The bisection method is computationally involved since you need to test $f_n f_{n-1} < 0$ on every step.*

■

Example 6.7.5 Give one common feature in both the Bisection and Regula-Falsi numerical algorithms of solving non-linear equations.

Solution :

- *Both require on to check where the root lies i.e,* $f_n f_{n-1} < 0$.
- *Both need two initial guesses* x_0 *and* x_1.

■

Example 6.7.6 Compute the next two iterations for equation (6.15) using the Regular falsi scheme with $x_0 = 1.25$ and $x_1 = 1.5$ correct to 4 decimal places.

Solution : *Here* $f(1.25) = -0.438 < 0$ *and* $f(1.5) = 0.25 > 0$

$\therefore$ *Now, Root lies between* $x_0 = 1.25$ *and* $x_1 = 1.5$

The first approximation to root is x_2 *and is given by*

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\
&= \frac{(1.25)f(1.5) - (1.5)f(1.25)}{f(1.5) - f(1.25)} \\
&= \frac{(1.25)(0.25) - (1.5)(-0.438)}{0.25 - (-0.438)} \\
&= \frac{0.9695}{0.688} \\
&= 1.4091 \\
&\Rightarrow f(1.4091) = 1.4091^2 - 2 = -0.0145 < 0
\end{aligned}$$

The root lies between $x_2 = 1.4091$ *and* $x_1 = 1.5$

The second approximation to root is x_3 *and is given by*

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \\
x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\
&= \frac{(1.5)f(1.4091) - (1.4091)f(1.5)}{f(1.4091) - f(1.5)} \\
&= \frac{(1.5)(-0.0145) - (1.4091)(0.25)}{-0.0145 - (0.25)} \\
&= \frac{-0.374025}{-0.2645} \\
&= 1.4141
\end{aligned}$$

■

6.7.1 Applying The Fixed Point Method

Example 6.7.7 Show that there exists a root $\alpha \in (1.5, 2)$ for the function $f(x) = x^3 - 2x - 1$, and use the Fixed Point Method to approximate α. Use either $x = g_1(x)$ or $x = g_2(x)$ where $g_1(x) = \frac{1}{2}(x^3 - 1)$ and $g_2(x) = (2x+1)^{1/3}$ (justify your choice for the convergence of the Fixed Point Method). Use $x_0 = 1.75$ as an initial approximation of α until the accuracy of $\epsilon = 10^{-2}$ is achieved and verify the accuracy in this approximation.

To show that a root α exists in $(1.5, 2)$, we first note that $f(1.5) = (1.5)^3 - 2(1.5) - 1 = -0.625 < 0$ and $f(2) = (2)^3 - 2(2) - 1 = 3 > 0$. Therefore $f(1.5) \cdot f(2) < 0$ and thus a root α exists in $(1.5, 2)$.

We now want to choose either g_1 or g_2 to apply the Fixed Point Method to. Note that since $g_1(x) = \frac{1}{2}(x^3 - 1)$ then the derivative of $g_1(x)$ is $g_1'(x) = \frac{3}{2}x^2$. We note that $g_1'(x) > 0$ on the interval $[1.5, 2]$ and so $\mid g_1'(x) \mid = g_1'(x)$ on this interval. Furthermore, g_1' is continuous and is an increasing function on $[1.5, 2]$ and thus the maximum is achieved at the right endpoint of this interval, that is $\max_{1.5 \leq x \leq 2} \mid g_1'(x) \mid = \max_{1.5 \leq x \leq 2} g_1'(x) = \frac{3}{2}(2)^2 \mid = 6$.

$$\lambda = \max_{1.5 \leq x \leq 2} \mid g_1'(x) \mid = 6 > 1 \tag{6.16}$$

As we see, the choice of $x = g_1(x)$ does not satisfy the condition for convergence. Now note that since $g_2(x) = (2x+1)^{1/3}$ we have that $g_2'(x) = \frac{2}{3}(2x+1)^{-2/3}$. Once again we see that $g_2'(x) > 0$ on the interval $[1.5, 2]$ and so $\mid g_2'(x) \mid = g_2'(x)$ on this interval. Furthermore, g_2' is continuous and is a decreasing function on $[1.5, 2]$ and thus the maximum is achieved at the left endpoint of this interval, that is

$$\max_{1.5 \leq x \leq 2} \mid g_2'(x) \mid = \max_{1.5 \leq x \leq 2} \mid g_2'(x) \mid = \frac{2}{3}(2(1.5)+1)^{-2/3} = 0.264566\ldots < 1$$

and o $x = g_2(x)$ is a good choice for applying the Fixed Point method. We will now apply the fixed point method to $x = g_2(x) = (2x+1)^{1/3}$. The first iteration using the initial approximation $x_0 = 1.75$ is:

$$\begin{aligned} x_1 &= g(x_0) \\ x_1 &= (2(1.75)+1)^{1/3} \\ x_1 &= 1.650963\ldots \end{aligned} \tag{6.17}$$

The second iteration using the approximation $x_1 = 1.650963$ is:

$$\begin{aligned} x_2 &= g(x_1) \\ x_2 &= (2(1.650963)+1)^{1/3} \\ x_2 &= 1.626376\ldots \end{aligned} \tag{6.18}$$

The third iteration using the approximation $x_2 = 1.626376$ is:

$$\begin{aligned} x_3 &= g(x_2) \\ x_3 &= (2(1.626376)+1)^{1/3} \\ x_3 &= 1.620155\ldots \end{aligned} \tag{6.19}$$

Note that $\mid x_3 - x_2 \mid = \mid -0.006221 \mid = 0.006221 < 0.01 = 10^{-2} = \epsilon$. We will now verify that the root of interest α is between $x_3 - \epsilon$ and $x_3 + \epsilon$ to ensure that x_3 is within $\epsilon = 10^{-2}$ of α. We note that:

$$\begin{aligned} f(x_3 - \epsilon) &= f(1.620155 - 0.01) = f(1.610155) \\ = (1.610155)^3 - 2(1.610155) - 1 &= -0.045823\ldots < 0 \\ f(x_3 + \epsilon) &= f(1.620155 + 0.01) = f(1.630155) \\ = (1.630155)^3 - 2(1.630155) - 1 &= 0.071672\ldots > 0 \end{aligned} \tag{6.20}$$

Thus we have verified that the a root α exists in $[x_3 - \epsilon, x_3 + \epsilon] = [1.610155, 1.630155]$ and so we have verified that x_3 is within $\epsilon = 10^{-2}$ of α.

6.8 Non-Linear Equations Chapter Exercises

Exercise 6.8.1 Show that the Newton-Raphson iteration scheme for the function

$$f(x) = \frac{1}{x} - a,$$

where a is a constant, is given by

$$x_{n+1} = x_n(2 - ax_n).$$

What is the root of f? Do two iterations of the Newton-Raphson scheme for $a = 3$ and starting with $x_0 = 0.3$ to find x_1 and x_2. Estimate how many iterations would be needed to determine the root to 16 decimal places?

Exercise 6.8.2 Use the Newton Raphson method to estimate the reciprocal of 18 correct to 8 decimal places.