Chapter 4

Finite Difference Operators

4.1 Introduction

The lecture defines and inter-relates the most common finite difference operators i.e. forward, backward, shift, central and the averaging operator. Use of the operators to prove finite difference identities is also done.

4.1.1 Finite differences

We consider a function $f(x)$ known for a certain set of equally spaced values of x such that

$$x = x_0 + rh, \qquad r = 0, 1, \ldots, n$$

and $h > 0$. This generates a set of $n+1$ pairs i.e. (x_r, f_r).

The x_r are called pivotal points and $f_r = f(x_r)$ are the pivotal values. The pairs can be written as;

$$\begin{array}{cc}
x_0 & f(x_0) \\
x_1 & f(x_1) \\
x_2 & f(x_2) \\
. & . \\
. & . \\
. & . \\
x_r & f(x_r)
\end{array}$$

We can form differences between consecutive pivotal values i.e. $f_{r+1} - f_r$ where

$$x_{r+1} = x_{r+h} \text{ for } r = 0, 1, 2, \ldots$$

The differences $f_{r+1} - f_r$ are called the first differences. We similarly define higher order differences.

4.1.2 Finite difference operators

The most common finite difference operators are; the forward difference operator, the backward difference operator, the central difference operator, the averaging operator and the shift operator.

4.2 The Forward Difference Operator Δ

defn The forward difference operator is denoted by Δ and defined by the difference equation;

$$\Delta f_r = f_{r+1} - f_r \tag{4.1}$$

Thus, the 1^{st} difference column in a difference table consists of the elements,

$$\Delta f_0, \quad \Delta f_1, \quad \Delta f_2 \ldots\ldots$$

The second differences are got by differentiating the first differences i.e

$$\Delta f_{r+1} - \Delta f_r \tag{4.2}$$

But we can easily show that Δ is a linear operator i.e.

$$\Delta(\alpha f_r + \beta g_r) = \alpha\Delta f_r + \beta\Delta g_r$$

Thus equation (4.2), becomes,

$$\Delta f_{r+1} - \Delta f_r = \Delta(f_{r+1} - f_r) = \Delta(\Delta f_r).$$

Example 4.2.1 Let $y = f(x)$. Then

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
\Delta y_n &= y_{n+1} - y_n
\end{aligned}$$

Example 4.2.2 Compute $\Delta^2 y_1$

Solution :

$$\begin{aligned}
\Delta^2 y_1 &= \Delta\Big(\Delta y_1\Big) \\
&= \Delta\Big(y_2 - y_1\Big) \\
&= \Delta y_2 - \Delta y_1 \\
&= (y_3 - y_2) - (y_2 - y_1) \\
&= y_3 - 2y_2 + y_1
\end{aligned}$$

■

Remark 4.2.1 In general,

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \tag{4.3}$$

Remark 4.2.2 If we write $\Delta^2 f_r$ for $\Delta(\Delta f_r)$, then the second differences are the quantities,

$$\Delta^2 f_0, \Delta^2 f_1, \Delta^2 f_2, \ldots\ldots$$

Remark 4.2.3 We similarly get third differences as consisting of the elements.

$$\Delta^3 f_r = \Delta(\Delta^2 f_r)$$

or

$$\Delta^3 f_r = \Delta^2(\Delta f_r) = \Delta^2 f_{r+1} - \Delta^2 f_r$$

Generally, we have that the n^{th} order differences consist of the elements/quantities,

$$\Delta^n f_r = \Delta^{n-1} f_{r+1} - \Delta^{n-1} f_r \tag{4.4}$$

$$\Delta^n f_r = \Delta(\Delta^{n-1} f_r) \tag{4.5}$$

Thus $\Delta^k f_r$ involves information at the pivoted points $x_r, x_{r+1}, \ldots, x_{r+k}$. Thus,

1.) Δf_0 involves information at x_0 and x_1

2.) $\Delta^2 f_0$ involves information at x_0, x_1 and x_2

3.) $\Delta^3 f_0$ involves information at x_0, x_1, x_2 and x_3

and so on. In fact this is how the name forward comes about.

Example 4.2.3

Table 4.1: Showing a table of forward differences in between used points.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
y_0				
	Δy_0			
y_1		$\Delta^2 y_0$		
	Δy_1		$\Delta^3 y_0$	
y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
	Δy_2		$\Delta^3 y_1$	
y_3		$\Delta^2 y_2$		
	Δy_3			
y_4				

or

Table 4.2: Forward Differences Table at point of reference.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	
y_2	Δy_2	$\Delta^2 y_2$		
y_3	Δy_3			
y_4				

Example 4.2.4 Given the value of

$$y = 46,\ 66,\ 81,\ 93,\ 101$$

Determine the forward differences of y.

Solution :

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
46				
	20			
66		-5		
	15		2	
81		-3		-3
	12		-1	
93		-4		
	8			
101				

Table 4.3: Forward difference in between the two used points

or by

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
46	20	-5	2	-3
66	15	-3	-1	
81	12	-4		
93	8			
101				

Table 4.4: Forward difference placed at point

■

Example 4.2.5 Given the table

x	1.4	1.6	1.8	2	2.2
y	4.0552	4.953	6.0496	7.3891	9.025

Solution : *The forward difference operator table is given by*

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.4	4.0552				
		0.8978			
1.6	4.953		0.1988		
		1.0966		0.0441	
1.8	6.0496		0.2429		0.0094
		1.3395		0.0535	
2	7.3891		0.2964		
		1.6359			
2.2	9.025				

■

Example 4.2.6 Construct the difference table of $f(x) = \sin x$ for $x = 0^\circ\,(10^\circ)\,50^\circ$:

Solution :

x°	$f(x) = \sin x$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
0	0					
		0.1736				
10	0.1736		-0.0052			
		0.1684		-0.0052		
20	0.3420		-0.0104		0.0004	
		0.1580		-0.0048		0
30	0.5000		-0.0152		0.0004	
		0.1428		-0.0044		
40	0.6428		-0.0196			
		0.1232				
50	0.7660					

■

Example 4.2.7 Write down the forward difference table for $f = x^3$ for

$$x = 1,\ 1.01,\ 1.02,\ 1.03,\ 1.04,\ 1.05$$

Solution :

x	$f(x) = x^3$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
1	1.000					
		0.03				
1.01	1.030		0.001			
		0.031		0		
1.02	1.061		0.001		-0.001	
		0.032		-0.001		0.003
1.03	1.093		0		0.002	
		0.032		0.001		
1.04	1.125		0.001			
		0.033				
1.05	1.158					

■

Example 4.2.8 If $f(x) = x^3 - 2x^2 + 1$, construct a forward difference table by taking $x = 0, 1, 2, 3, 4$. Verify that the fourth differences are zero.

Solution :

x	$f(x) = x^3 - 2x^2 + 1$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		-1			
1	0		2		
		1		6	
2	1		8		0
		9		6	
3	10		14		
		23			
4	33				

From the table we can see that

$$\Delta^4 f(x) = 0$$

■

Example 4.2.9 If $f(x) = x^2 + x + 1$, construct a forward difference table by taking $x = 0, 1, 2, 3, 4, 5$. Hence find

1.) $\Delta f(0)$ 2.) $\Delta^2 f(1)$ 3.) $\Delta^3 f(2)$

Solution :

x	$f(x) = x^2 + x + 1$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		0
3	13		2	
		8		0
4	21		2	
		10		
5	31			

From the table we can see that $\Delta^2 f(x) = 2 =$ *constant and* $\Delta^3 f(x) = 0$

We note that

- *Given function is* $f(x) = x^2 + x + 1$. *The highest power is* 2, *which obviously means* $\Delta^2 f(x) =$ *constant and* $\Delta^3 f(x) = 0$
- *No need to show steps of calculations as shown in* $\Delta f(x)$, $\Delta^2 f(x)$ *and* $\Delta^3 f(x)$ *columns.*

■

Exercise 4.2.1 If $f(x) = x^2 - 3x + 1$, construct a forward difference table by taking $x = 0, 1, 2, 3, 4, 5$. Verify that the second differences are constants.

Exercise 4.2.2 If $f(x) = 2x^2+5$, construct a forward difference table by taking $x = 0, 2, 4, 6, 8$. Verify that the third differences are zero.

Exercise 4.2.3 If $f(x) = 2x^3 - x^2 + 3x + 1$, construct a forward difference table by taking $x = 0, 1, 2, 3, 4, 5$. Verify that the third differences are constant.

Example 4.2.10

Given $f(x) = x^4$, form a table of finite differences, for x from $x = -2$ in steps of one to 5.

Solution :

Table 4.5: Showing a finite difference table for $f(x) = x^4$

x_r	f_r	Δf_r	$\Delta^2 f_r$	$\Delta^3 f_r$	$\Delta^4 f_r$	$\Delta^5 f_r$
-2	16					
		-15				
-1	1		14			
		-1		-12		
0	0		2		24	
		1		12		0
1	1		14		24	
		15		36		0
2	16		50		24	
		65		60		0
3	81		110		24	
		175		84		
4	256		194			
		369				
5	625					

■

Note 4.2.1 We note that the fourth differences are constants and fifth differences are all zeros. We state the following theorem to support this observation.

Theorem 4.2.1
Let $P_n(x)$ be a polynomial of degree $n(n \geq 1$ integer). Then

1. $\Delta^r P_n(x)$ is a polynomial of degree $n - r(r \geq n$ integer).

2. $\Delta^n P_n(x) =$ constant.

3. $\Delta^{n+1} P_n(x) = 0$.

Thus, parts (2.) and (3.) of the theorem confirm the validity of the results of the difference table (4.4) in which fourth order differences are all constants equal to 24 and fifth differences are all zero.

In fact the theorem can help us check the behaviour of a function (not polynomial) on an interval. If the n^{th} order differences are fairly constant on an interval for a particular function, then such a function more or less behaves like a polynomial of degree n in that interval.

Exercise 4.2.4 For a polynomial

$$P_6(x) = x^6 - 6x^2 + 1,$$

Construct a table of forward differences on the interval $[4, -4]$ with step size 1.

Example 4.2.11 By constructing a forward difference table, find the 7^{th} and 8^{th} terms of a sequence $8,\ 14,\ 22,\ 32,\ 44,\ 58,\ \ldots$

Solution : *Let*

$$f(1) = 8,\ f(2) = 14,\ f(3) = 22,\ f(4) = 32,\ f(5) = 44,\ f(6) = 58$$

We have to find $f(7)$ and $f(8)$.

We prepare the following forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	8			
		6		
2	14		2	
		8		0
3	22		2	
		10		0
4	32		2	
		12		0
5	44		2	
		14		0
6	58		2	
		16		0
7	74		2	
		18		
8	92			

- *To find $f(7)$, extra 2 is written in Δ^2 column. The entry in $\Delta f(x)$ is $14{+}2 = 16$ is added. The entry in $f(x)$ is $58 + 16 = 74$ is added. Thus $f(7) = 74$.*
- *To find $f(8)$, extra 2 is written in Δ^2 column. The entry in $\Delta f(x)$ is $16{+}2 = 18$ is added. The entry in $f(x)$ is $74 + 18 = 92$ is added. Thus $f(8) = 92$.*

Thus the $7th$ and $8th$ terms of series are 74 and 92 respectively. ■

Exercise 4.2.5 By constructing a forward difference table, find the 6^{th} and 7^{th} terms of a sequence 6, 11, 18, 27, 38, ...

Solution : 6, 11, 18, 27, 38, **51**, **66**, ... ■

Example 4.2.12 Let a function $f(x)$ is given at the points $(0,7),(4,43),(8,367)$ then find the forward difference of the function at $x=4$.

Solution :

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
\Delta f(4) &= f(4+h) - f(4) \\
&= f(8) - f(4) \\
&= 367 - 43 \\
&= 324
\end{aligned}$$

■

Example 4.2.13 Find $\Delta f(x)$ for the function x^2+2x+3 with $h=2$

Solution :

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
&= \left[(x+2)^2 + 2(x+2) + 3\right] - \left[x^2 + 2x + 3\right] \\
&= x^2 + 4x + 4 + 2x + 4 + 3 - x^2 - 2x - 3 \\
&= 4x + 8 \\
\Delta^2 f(x) &= \left[4(x+2) + 8\right] - \left[4x + 8\right] \\
&= 4x + 8 - 4x \\
&= 8 \\
\Delta^3 f(x) &= 8 - 8 \\
&= 0
\end{aligned}$$

■

Example 4.2.14 If $f(x) = e^x$, construct a forward difference table by taking

$$x = 0.1,\ 0.15,\ 0.2,\ 0.25,\ 0.3,\ 0.35,\ 0.4,\ 0.45,\ 0.5$$

Solution :

x	$f(x) = e^x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8
0.1	1.10517								
		0.05666							
0.15	1.16183		0.00291						
		0.05957		0.00015					
0.2	1.2214		0.00306		-0.00001				
		0.06263		0.00014		0.00005			
0.25	1.28403		0.0032		0.00004		-0.00011		
		0.06583		0.00018		-0.00006		0.00023	
0.3	1.34986		0.00338		-0.00002		0.00012		-0.00047
		0.06921		0.00016		0.00006		-0.00024	
0.35	1.41907		0.00354		0.00004		-0.00012		
		0.07275		0.0002		-0.00006			
0.4	1.49182		0.00374		-0.00002				
		0.07649		0.00018					
0.45	1.56831		0.00392						
		0.08041							
0.5	1.64872								

■

4.2.1 Differences as Related to Ordinates

By definition of the forward difference operator, we have:

$$\begin{aligned}
\Delta f_r &= f_{r+1} - f_r \\
\Delta^2 f_r &= \Delta f_{r+1} - \Delta f_r \\
&= (f_{r+2} - f_{r+1}) - (f_{r+1} - f_r) \\
&= f_{r+2} - 2f_{r+1} + f_r \\
\Delta^3 f_r &= \Delta^2 f_{r+1} - \Delta^2 f_r \\
&= (f_{r+3} - 2f_{r+2} + f_{r+1}) - (f_{r+2} - 2f_{r+1} + f_r) \\
&= f_{r+3} - 3f_{r+2} + 3f_{r+1} - f_r
\end{aligned}$$

Exercise 4.2.6 Show that,

$$\Delta^4 f_r = f_{r+4} - 4f_{r+3} + 6f_{r+2} - 4f_{r+1} + f_r$$

Example 4.2.15

Table 4.6: Showing a table of forward differences.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
y_0						
	Δy_0					
y_1		$\Delta^2 y_0$				
	Δy_1		$\Delta^3 y_0$			
y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
	Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
	Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
	Δy_4		$\Delta^3 y_3$			
y_5		$\Delta^2 y_4$				
	Δy_5					
y_6						

or

Table 4.7: Forward Differences Table at point of reference.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$	$\Delta^6 y_0$
y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	$\Delta^5 y_1$	
y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_2$		
y_3	Δy_3	$\Delta^2 y_3$	$\Delta^3 y_3$			
y_4	Δy_4	$\Delta^2 y_4$				
y_5	Δy_5					
y_6						

Note 4.2.2 There is an approximate equality between Δ operator and derivative.

4.3 The Backward Difference Operator ∇

Definition 4.3.1 This is denoted by ∇ and defined by the difference equation,

$$\nabla f_r = f_r - f_{r-1} \tag{4.6}$$

Exercise 4.3.1 Show that

$$\nabla^n f_r = \nabla^{n-1} f_r - \nabla^{n-1} f_{r-1}$$

before you continue reading.

Note 4.3.1 We also note that this operator has a nice property of linearity i.e

$$\nabla(\alpha f_r + \beta g_r) = \alpha \nabla f_r + \beta \nabla g_r$$

where α and β are real scalars.

Example 4.3.1

Table 4.8: Showing a table of backward differences.

y_0						
	∇y_1					
y_1		$\nabla^2 y_2$				
	∇y_2		$\nabla^3 y_3$			
y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
	∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$	
y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$
	∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$	
y_4		$\nabla^2 y_5$		$\nabla^4 y_6$		
	∇y_5		$\nabla^3 y_6$			
y_5		$\nabla^2 y_6$				
	∇y_6					
y_6						

or

Table 4.9: Backward Differences Table

y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
y_0						
y_1	∇y_1					
y_2	∇y_2	$\nabla^2 y_2$				
y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$

Example 4.3.2 Construct the backward difference table for the value of y for x and y given below.

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

Solution : *The backward differences for y is given by*

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Note that, the forward and backward give the same table, difference is at where you position. Compare this example and Example 4.2.11. To see a difference we employ Table 4.9 for the Backward difference operator.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
1901	66	20			
1911	81	15	-5		
1921	93	12	-3	2	
1931	101	8	-4	-1	-3

■

Exercise 4.3.2 Construct a table of backward differences for the polynomial in Exercise 4.2.4.

Exercise 4.3.3 Construct a table of backward differences for the function $f(x) = e^x$ on the interval $[-2, 2]$ with step size 0.5.

From your results what polynomial function fairly approximate e^x on this interval?

Exercise 4.3.4 What do you think are the uses of finite difference tables?

Example 4.3.3 Construct the backward difference table for the data given below:

x	1	2	3	4	5	6	7	8
y	8	14	22	32	44	58	74	92

Solution :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1	8			
		6		
2	14		2	
		8		0
3	22		2	
		10		0
4	32		2	
		12		0
5	44		2	
		14		0
6	58		2	
		16		0
7	74		2	
		18		
8	92			

or

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$	$\nabla^7 y$
1	8							
2	14	6						
3	22	8	2					
4	32	10	2	0				
5	44	12	2	0	0			
6	58	14	2	0	0	0		
7	74	16	2	0	0	0	0	
8	92	18	2	0	0	0	0	0

■

Exercise 4.3.5 Construct a backward difference table for the following data

x	0	10	20	30
y	0	0.174	0.347	0.518

Exercise 4.3.6 Construct a backward difference table for $y = f(x) = x^3 + 2x + 1$ for

$$x = 1, 2, 3, 4, 5.$$

Example 4.3.4 By constructing a difference table and using the second order differences as constant, find the sixth term of the series $8, 12, 19, 29, 42 \ldots$ 58

4.3.1 Relation between Δ and ∇

1.)

$$(1+\Delta)\,(1-\nabla) \equiv 1 \tag{4.7}$$

Proof:

$$\begin{aligned}(1+\Delta)(1-\nabla)f(x) &= (1+\Delta)[f(x)-f(x)+f(x-h)]\\ &= (1+\Delta)f(x-h)\\ &= f(x-h)+f(x)-f(x-h)\\ &= f(x)\end{aligned}$$

Therefore,

$$(1+\Delta)(1-\nabla) \equiv 1$$

$\blacksquare$

2.)

$$\Delta\nabla \equiv \nabla\Delta \tag{4.8}$$

Proof:

$$\Delta\nabla f(x) = \Delta[f(x)-f(x-h)] = f(x+h)-2f(x)+f(x-h)$$

and

$$\begin{aligned}\nabla\Delta f(x) = \nabla[f(x+h)-f(x)] &= [f(x+h)-f(x)]-[f(x)-f(x-h)]\\ &= f(x+h)-2f(x)+f(x-h)\end{aligned}$$

Therefore,

$$\Delta\nabla \equiv \nabla\Delta$$

$\blacksquare$

3.)

$$\Delta\nabla \equiv \Delta-\nabla \tag{4.9}$$

Proof: *Consider the function $f(x)$.*

$$\begin{aligned}\Delta f(x) &= f(x+h)-f(x)\\ \nabla f(x) &= f(x)-f(x-h)\end{aligned}$$

But

$$\begin{aligned}\Delta-\nabla)(f(x)) &= \Delta f(x)-\nabla f(x)\\ &= [f(x+h)-f(x)]-[f(x)-f(x-h)]\\ &= \Delta f(x)-\Delta f(x-h)\\ &= \Delta[f(x)-f(x-h)]\\ &= \Delta[\nabla f(x)]\end{aligned}$$

Therefore

$$\Delta-\nabla = \Delta\nabla$$

$\blacksquare$

4.3.2 Properties of Forward and Backward Finite Difference Operators

1. **Constant rule**: If c is a constant, then

   (a)
   $$\Delta c = 0 \tag{4.10}$$

   Proof: *Let* $f(x) = c$

   Therefore
   $$f(x+h) = c$$
   (where h *is the interval of difference)*
   $$\Delta c = \Delta f(x) = f(x+h) - f(x) = c - c = 0$$
   ■

   (b)
   $$\nabla c = 0 \tag{4.11}$$

2. **Linearity**: if a and b are constants,

   (a)
   $$\Delta(af + bg) = a\,\Delta f + b\,\Delta g \tag{4.12}$$

   Proof:
   $$\begin{aligned}\Delta[f(x) + g(x)] &= [f(x+h) + g(x+h)] - [f(x) + g(x)]\\ &= f(x+h) + g(x+h) - f(x) - g(x)\\ &= f(x+h) - f(x) + g(x+h) - g(x)\\ &= \Delta f(x) + \Delta g(x)\end{aligned}$$
   and
   $$\begin{aligned}\Delta[cf(x)] &= cf(x+h) - cf(x)\\ &= c[f(x+h) - f(x)]\\ &= c\Delta f(x)\end{aligned}$$
   Thus the proof ■

   (b)
   $$\nabla(af + bg) = a\,\nabla f + b\,\nabla g \tag{4.13}$$

3. **Product rule**:

   (a)
   $$\Delta(fg) = f\,\Delta g + g\,\Delta f + \Delta f\,\Delta g \tag{4.14}$$

   Proof:
   $$\begin{aligned}\Delta[f(x)g(x)] &= f(x+h)g(x+h) - f(x)g(x)\\ &= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)\\ &= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]\\ &= f(x+h)\Delta g(x) + g(x)\Delta f(x)\end{aligned}$$
   ■

Also, it can be shown that

$$\begin{aligned}\Delta[f(x)g(x)] &= f(x)\Delta g(x) + g(x+h)\Delta f(x)\\ &= f(x)\Delta g(x) + g(x)\Delta f(x) + \Delta f(x)\Delta g(x)\end{aligned}$$

(b)

$$\nabla(fg) = f\,\nabla g + g\,\nabla f - \nabla f\,\nabla g \tag{4.15}$$

4. **Quotient rule**:

(a)

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)} \tag{4.16}$$

Proof:

$$\begin{aligned}\Delta\left(\frac{f(x)}{g(x)}\right) &= \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\\ &= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\\ &= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\\ &= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)}\\ &= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}\end{aligned}$$

■

(b)

$$\nabla\left(\frac{f}{g}\right) = \frac{1}{g}\det\begin{bmatrix}\nabla f & \nabla g\\ f & g\end{bmatrix}\left(\det\begin{bmatrix}g & \nabla g\\ 1 & 1\end{bmatrix}\right)^{-1}$$

or

$$\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g\cdot(g - \nabla g)} \tag{4.17}$$

5. **Summation rules**:

(a)

$$\sum_{n=a}^{b}\Delta f(n) = f(b+1) - f(a) \tag{4.18}$$

(b)

$$\sum_{n=a}^{b}\nabla f(n) = f(b) - f(a-1) \tag{4.19}$$

4.4 The Shift Operator E

Definition 4.4.1 The shift operator is denoted by E and defined by

$$Ef_r = f_{r+1} \tag{4.20}$$

A shift operator is like a jump forward.

Remark 4.4.1 E is called the shifting operator . It is also called the **displacement** operator.

Example 4.4.1

Table 4.10: Shift Differences Table

y	Ey	E^2y	E^3y	E^4y	E^5y
y_0	y_1	y_2	y_3	y_4	y_5
y_1	y_2	y_3	y_4	y_5	
y_2	y_3	y_4	y_5		
y_3	y_4	y_5			
y_4	y_5				
y_5					

Note 4.4.1 By Definition 4.4.1 we can define higher order shift operators as,

$$E(Ef_r) = Ef_{r+1} = f_{r+2}.$$

We usually denote

$$E(Ef_r) = E^2f_r$$

and,

$$E^kf_r = E(E^{k-1}f_r) = f_{r+k}.$$

Remark 4.4.2 For k negative, we get what is known as the backward shift. For example $E^{-1}f_r$ is a backward shift defined by,

$$E^{-1}f_r = f_{r-1}$$

We also note that k can be fractional.

Example 4.4.2 Find $E^{\frac{1}{2}}f_r$ and $E^{-5}f_r$.

Solution : *By definitions of the forward and backward shifts i.e.*

$$Ef_r = f_{r+1}$$

and

$$E^{-1}f_r = f_{r-1}$$

we have,

$$E^{\frac{1}{2}}f_r = f_{r+\frac{1}{2}}$$

and

$$E^{-5}f_r = f_{r-5}.$$

■

Example 4.4.3 Given the set of values

$$y = -2,\ 2,\ 5,\ 6,\ 8,\ 12,\ 13$$

then

1.) $E(2) = 5$ 2.) $E(8) = 12$ 3.) $E^3(-2) = 6$ 4.) $E^{-1}(13) = 12$

Example 4.4.4 Given $y_0 = 1,\ y_1 = 11,\ y_2 = 21,\ y_3 = 28$ and $y_4 = 29$.

1.)
$$E^4 y_0 = y_4 = 29$$

2.)
$$E^3 y_0 = y_3 = 28$$

3.)
$$E^2 y_0 = y_2 = 21$$

4.)
$$E y_0 = y_1 = 11$$

4.4.1 Relation between Δ and E

$$\Delta = E - 1 \tag{4.21}$$
$$E = \Delta + 1 \tag{4.22}$$

Proof: *Consider the function $f(x)$.*

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= Ef(x) - f(x)\\ &= (E-1)f(x)\end{aligned}$$

Therefore,

$$\Delta \equiv E - 1$$

■

4.4.2 Relation between ∇ and E

$$\nabla = 1 - E^{-1} \tag{4.23}$$
$$E = (1 - \nabla)^{-1} \tag{4.24}$$

Proof: *Consider the function $f(x)$.*

$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h)\\ &= f(x) - E^{-1}f(x)\\ &= (1 - E^{-1})f(x)\end{aligned}$$

Therefore,

$$\nabla \equiv 1 - E^{-1}$$

■

Remark 4.4.3 Property equ (4.23) can be also expressed as

$$\begin{aligned}\nabla &= 1 - E^{-1} \\ &= 1 - \frac{1}{E} \\ &= \frac{E-1}{E}\end{aligned}$$

hence

$$\nabla = \frac{E-1}{E} \tag{4.25}$$

4.4.3 Relation between Δ, ∇ and E

Example 4.4.5 Show that

$$\Delta = E\nabla \tag{4.26}$$

Solution : *Since*

$$\begin{aligned}E\nabla f_r &= E(\nabla f_r) \\ &= E(f_r - f_{r-1}) \\ &= Ef_r - Ef_{r-1} \text{ (because of linearity of E)} \\ &= f_{r+1} - f_r \textit{ (By definition of E)} \\ &= \Delta f_r\end{aligned}$$

Hence shown. ■

Remark 4.4.4 Relation equ (4.26) can be rewritten as

$$\begin{aligned}\Delta &= E\nabla \\ E^{-1}\Delta &= E^{-1}E\nabla \\ E^{-1}\Delta &= \nabla\end{aligned}$$

Therefore

$$\nabla = E^{-1}\Delta \tag{4.27}$$

Proof: *Consider the function $f(x)$.*

$$\nabla f(x) = f(x) - f(x-h) = \Delta f(x-h) = \Delta E^{-1} f(x)$$

Therefore

$$\nabla = \Delta E^{-1}$$

■

4.4.4 Differences in terms of pivoted values

From the relation

$$\Delta^k f_r = (E-1)^k f_r,$$

We can easily show that

$$\Delta^k f_r = \begin{pmatrix} k \\ 0 \end{pmatrix} f_{r+k} - \begin{pmatrix} k \\ 1 \end{pmatrix} f_{r+k-1} + \begin{pmatrix} k \\ 2 \end{pmatrix} f_{r+k-2} + \dots$$

Try and show this before you continue. Recall: $\begin{pmatrix} k \\ r \end{pmatrix} = \dfrac{k!}{r!(k-r)!}$.

Example 4.4.6 Express $\Delta^3 f_0$ in terms of pivoted values.

Solution : *Since*

$$\Delta^k f_r = \binom{k}{0} f_{r+k} - \binom{k}{1} f_{r+k-1} + \binom{k}{2} f_{r+k-2} + \dots$$

Therefore,

$$\begin{aligned}\Delta^3 f_0 &= \binom{3}{0} f_{3+0} - \binom{3}{1} f_{3+0-1} + \binom{3}{2} f_{3+0-2} - \binom{3}{3} f_{3+0-3} \\ &= \binom{3}{0} f_3 - \binom{3}{1} f_2 + \binom{3}{2} f_1 - \binom{3}{3} f_0 \\ &= f_3 - 3f_2 + 3f_1 - f_0\end{aligned}$$

■

Example 4.4.7 Construct the shift difference table for the value of y given below.

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

Solution :

x	y	Ey	E^2y	E^3y	E^4y
1891	46	66	81	93	101
1901	66	81	93	101	
1911	81	93	101		
1921	93	101			
1931	101				

■

4.5 The Central Difference Operator δ

Definition 4.5.1 The central difference operator is denoted by δ and defined by the equation

$$\delta f_r = f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}} \tag{4.28}$$

Example 4.5.1

$$\begin{aligned}\delta f_{r+\frac{1}{2}} &= f_{r+\frac{1}{2}+\frac{1}{2}} - f_{r+\frac{1}{2}-\frac{1}{2}}\\ \delta f_{r+\frac{1}{2}} &= f_{r+1} - f_r\end{aligned} \tag{4.29}$$

Note 4.5.1 Central difference is the difference between the upper (forward) and lower (backward) values.

Example 4.5.2

1.) $\delta f_{\frac{1}{2}} = f_1 - f_0$ 2.) $\delta f_{\frac{3}{2}} = f_2 - f_1$

Example 4.5.3

$$\begin{aligned}\delta^2 f_r &= \delta f_{r+\frac{1}{2}} - \delta f_{r-\frac{1}{2}}\\ &= (f_{r+1} - f_r) - (f_r - f_{r-1})\\ &= f_{r+1} - 2f_r + f_{r-1}\end{aligned}$$

With higher order differences defined by

$$\delta^k f_r = \delta^{k-1} f_{r+\frac{1}{2}} - \delta^{k-1} f_{r-\frac{1}{2}}.$$

Exercise 4.5.1 Determine the expressions for

1.) $\delta^3 f_r$ 2.) $\delta^4 f_r$

Example 4.5.4

Table 4.11: Showing a table of central differences using Eqn (4.29), subtractions

y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
y_0						
	$\delta y_{\frac{1}{2}}$					
y_1		$\delta^2 y_1$				
	$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$			
y_2		$\delta^2 y_2$		$\delta^4 y_2$		
	$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$		$\delta^5 y_{\frac{5}{2}}$	
y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
	$\delta y_{\frac{7}{2}}$		$\delta^3 y_{\frac{7}{2}}$		$\delta^5 y_{\frac{7}{2}}$	
y_4		$\delta^2 y_4$		$\delta^4 y_4$		
	$\delta y_{\frac{9}{2}}$		$\delta^3 y_{\frac{9}{2}}$			
y_5		$\delta^2 y_5$				
	$\delta y_{\frac{11}{2}}$					
y_6						

Example 4.5.5 Draw the central difference operator table for $f(x) = 2^x$, $0(1)6$

Table 4.12: Showing a table of central differences.

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
0	1						
		1					
1	2		1				
		2		1			
2	4		2		1		
		4		2		1	
3	8		4		2		1
		8		4		2	
4	16		8		4		
		16		8			
5	32		16				
		32					
6	64						

4.5.1 Relation between δ and E

From,

$$\delta f_r = f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}}$$

and

$$E^{\frac{1}{2}} f_r = f_{r+\frac{1}{2}}$$

and

$$E^{-\frac{1}{2}} f_r = f_{r-\frac{1}{2}},$$

we get

$$\delta f_r = (E^{\frac{1}{2}} - E^{-\frac{1}{2}}) f_r$$

and hence conclude that;

$$\delta = E^{1/2} - E^{-1/2} \tag{4.30}$$

$$\delta = E^{1/2}(1 - E^{-1}) \tag{4.31}$$

4.5.2 Relation between δ and Δ

$$\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} \tag{4.32}$$

Proof: *We know that*

$$\Delta = E - 1 \text{ and } \delta = E^{1/2} - E^{-1/2}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} \\
&= \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{4}} \\
&= \frac{1}{2}\left(E + E^{-1} - 2\right) + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{\left(E + E^{-1} - 2\right)}{4}} \\
&= \frac{E}{2} + \frac{E^{-1}}{2} - 1 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{\frac{\left(E^{1/2} + E^{-1/2}\right)^2}{4}} \\
&= \frac{E}{2} + \frac{E^{-1}}{2} - 1 + \left(E^{1/2} - E^{-1/2}\right)\frac{\left(E^{1/2} + E^{-1/2}\right)}{2} \\
&= \frac{E}{2} + \frac{E^{-1}}{2} - 1 + \frac{\left(E - E^{-1}\right)}{2} \\
&= \frac{E}{2} + \frac{E^{-1}}{2} - 1 + \frac{E}{2} - \frac{E^{-1}}{2} \\
&= E - 1 \\
&= \Delta
\end{aligned}$$

■

4.6 Averaging Operator μ

Definition 4.6.1 The averaging operator is denoted by u and defined by the equation,

$$\mu f_r = \frac{1}{2}\left\{f_{r+\frac{1}{2}} + f_{r-\frac{1}{2}}\right\} \tag{4.33}$$

or alternatively given by

$$\mu f_{r+\frac{1}{2}} = \frac{1}{2}\left\{f_{r+1} + f_r\right\} \tag{4.34}$$

Example 4.6.1

Table 4.13: Showing a table of Averaging Operator

y	μy	$\mu^2 y$	$\mu^3 y$	$\mu^4 y$	$\mu^5 y$	$\mu^6 y$
y_0						
	$\mu y_{\frac{1}{2}}$					
y_1		$\mu^2 y_1$				
	$\mu y_{\frac{3}{2}}$		$\mu^3 y_{\frac{3}{2}}$			
y_2		$\mu^2 y_2$		$\mu^4 y_2$		
	$\mu y_{\frac{5}{2}}$		$\mu^3 y_{\frac{5}{2}}$		$\mu^5 y_{\frac{5}{2}}$	
y_3		$\mu^2 y_3$		$\mu^4 y_3$		$\mu^6 y_3$
	$\mu y_{\frac{7}{2}}$		$\mu^3 y_{\frac{7}{2}}$		$\mu^5 y_{\frac{7}{2}}$	
y_4		$\mu^2 y_4$		$\mu^4 y_4$		
	$\mu y_{\frac{9}{2}}$		$\mu^3 y_{\frac{9}{2}}$			
y_5		$\mu^2 y_5$				
	$\mu y_{\frac{11}{2}}$					
y_6						

Example 4.6.2 Draw the central difference operator table for $f(x) = 2^x$, $0(1)6$

Table 4.14: Showing a table of Averaging.

x	y	μy	$\mu^2 y$	$\mu^3 y$	$\mu^4 y$	$\mu^5 y$	$\mu^6 y$
0	1						
		1.5					
1	2		2.25				
		3		3.375			
2	4		4.5		5.0625		
		6		6.75		3.0375	
3	8		9		1.0125		6.834375
		12		13.5		10.63125	
4	16		18		20.25		
		24		27			
5	32		36				
		48					
6	64						

4.7 Relations between Multi Operators

1.)

$$\mu\delta \equiv \frac{\Delta + \nabla}{2} \tag{4.35}$$

Proof:

$$\begin{aligned}
\frac{\Delta + \nabla}{2} &= \frac{1}{2}\big[[\Delta f(x) + \nabla f(x)\big] \\
&= \frac{1}{2}\big[f(x+h) - f(x) + f(x) - f(x-h)\big] \\
&= \frac{1}{2}\big[f(x+h) - f(x-h)\big] \\
&= \frac{1}{2}\big[E - E^{-1}\big] \\
&= \mu\delta f(x)
\end{aligned}$$

Thus,

$$\mu\delta \equiv \frac{\Delta + \nabla}{2}$$

■

2.)

$$\Delta\nabla \equiv \nabla\Delta \ \equiv \ \delta^2 \tag{4.36}$$

Proof:

$$\begin{aligned}
\Delta\nabla f(x) &= \Delta[f(x) - f(x-h)] \\
&= f(x+h) - 2f(x) + f(x-h)
\end{aligned}$$

Again,

$$\begin{aligned}
\nabla\Delta f(x) &= f(x+h) - 2f(x) + f(x-h) \\
&= (E - 2 + E^{-1})f(x) \\
&= \left(E^{1/2} - E^{-1/2}\right)^2 f(x) \\
&= \delta^2 f(x)
\end{aligned}$$

Hence,

$$\Delta\nabla \equiv r\Delta \equiv \left(E^{1/2} - E^{-1/2}\right)^2 \equiv \delta$$

■

3.)

$$\mu\delta \equiv \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2} \tag{4.37}$$

Proof:

$$
\begin{aligned}
\left[\frac{\Delta E^{-1}}{2}+\frac{\Delta}{2}\right] f(x) &= \frac{1}{2}\left[\Delta f(x-h)+\Delta f(x)\right] \\
&= \frac{1}{2}\big[f(x)-f(x-h)+f(x+h)-f(x)\big] \\
&= \frac{1}{2}\big[f(x+h)-f(x-h)\big] \\
&= \frac{1}{2}\big[E-E^{-1}\big] f(x) \\
&= \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)\left(E^{1/2}-E^{-1/2}\right) f(x) \\
&= \mu\delta f(x)
\end{aligned}
$$

Hence

$$\frac{\Delta E^{-1}}{2}+\frac{\Delta}{2} \equiv \mu\delta$$

■

4.)

$$E^{\frac{1}{2}} \equiv \mu+\frac{\delta}{2} \tag{4.38}$$

Proof:

$$\left(\mu+\frac{\delta}{2}\right) f(x)=\left\{\frac{1}{2}\left[E^{1/2}+E^{-1/2}\right]+\frac{1}{2}\left[E^{1/2}-E^{-1/2}\right]\right\} f(x)=E^{1/2} f(x)$$

Thus

$$E^{\frac{1}{2}} \equiv \mu+\frac{\delta}{2}$$

■

5.)

$$1+\delta^2\mu^2 \equiv \left(1+\frac{\delta^2}{2}\right)^2 \tag{4.39}$$

Proof:

$$\delta\mu f(x)=\frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)\left(E^{1/2}-E^{-1/2}\right) f(x)=\frac{1}{2}\left[E-E^{-1}\right] f(x)$$

Therefore,

$$
\begin{aligned}
\left(1+\delta^2\mu^2\right) f(x) &= \left[1+\frac{1}{4}\left(E-E^{-1}\right)^2\right] f(x) \\
&= \left[1+\frac{1}{4}\left(E^2-2+E^{-2}\right)\right] f(x) \\
&= \frac{1}{4}\left(E+E^{-1}\right)^2 f(x) \\
&= \left[1+\frac{1}{2}\left(E^{1/2}-E^{-1/2}\right)^2\right]^2 f(x) \\
&= \left[1+\frac{\delta^2}{2}\right] f(x)
\end{aligned}
$$

Hence

$$1 + \delta^2\mu^2 \equiv \left(1 + \frac{\delta^2}{2}\right)^2$$

■

6.)

$$\Delta \equiv \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}} \tag{4.40}$$

Proof:

$$\begin{aligned}
&\left[\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}\right] f(x) \\
&= \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 f(x) + \left[\left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{1}{4}\left(E^{1/2} - E^{-1/2}\right)^2}\right] f(x) \\
&= \frac{1}{2}\left[E + E^{-1} - 2\right] f(x) + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)\left(E^{1/2} + E^{-1/2}\right) f(x) \\
&= \frac{1}{2}\left[E + E^{-1} - 2\right] f(x) + \frac{1}{2}\left(E - E^{-1}\right) f(x) \\
&= (E - 1) f(x)
\end{aligned}$$

Hence,

$$\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}} \equiv E - 1 \equiv \Delta$$

■

	E	Δ	∇	δ
E	E	$\Delta + 1$	$(1 - \nabla)^{-1}$	$1 + \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$
Δ	$E - 1$	Δ	$(1 - \nabla)^{-1} - 1$	$\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$
∇	$1 - E^{-1}$	$1 - (1 - \nabla)^{-1}$	∇	$-\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1 + \Delta)^{-1/2}$	$\nabla(1 + \nabla)^{-1/2}$	δ
μ	$\frac{E^{1/2} - E^{-1/2}}{2}$	$\left(1 + \frac{\Delta}{2}\right)(1 + \Delta)^{-1/2}$	$\left(1 + \frac{\nabla}{2}\right)(1 + \nabla)^{-1/2}$	$1 + \frac{\delta^2}{4}$

Table 4.15: It is shown that any operator can be expressed with the help of another operator.

4.8 Finite Difference Operators Examples

Example 4.8.1 Compute

$$\Delta^2\left[\frac{1}{x^2+5x+6}\right]$$

using $h=1$.

Solution :

$$\begin{aligned}
\Delta^2\left[\frac{1}{x^2+5x+6}\right] &= \Delta^2\left[\frac{1}{(x+3)(x+2)}\right]\\
&= \Delta^2\left[\frac{1}{(x+3)(x+2)}\right]\\
&= \Delta^2\left[\frac{1}{x+2}-\frac{1}{x+3}\right] \quad , \ \textit{partial fraction}\\
&= \Delta\Delta\left[\frac{1}{x+2}-\frac{1}{x+3}\right]\\
&= \Delta\left[\Delta\left(\frac{1}{x+2}\right)-\Delta\left(\frac{1}{x+3}\right)\right]\\
&= \Delta\left[\left(\frac{1}{x+3}-\frac{1}{x+2}\right)-\left(\frac{1}{x+4}-\frac{1}{x+3}\right)\right]\\
&= \Delta\left[\frac{2}{x+3}-\frac{1}{x+2}-\frac{1}{x+4}\right]\\
&= \Delta\left(\frac{2}{x+3}\right)-\Delta\left(\frac{1}{x+2}\right)-\Delta\left(\frac{1}{x+4}\right)\\
&= \left(\frac{2}{x+4}-\frac{2}{x+3}\right)-\left(\frac{1}{x+3}-\frac{1}{x+2}\right)-\left(\frac{1}{x+5}-\frac{1}{x+4}\right)\\
&= \frac{3}{x+4}-\frac{3}{x+3}+\frac{1}{x+2}-\frac{1}{x+5}
\end{aligned}$$

■

Example 4.8.2 Evaluate

$$\Delta\tan^{-1}x$$

Solution :

$$\begin{aligned}
\Delta\tan^{-1}x &= \tan^{-1}(x+h)-\tan^{-1}x\\
&= \tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)\\
&= \tan^{-1}\left(\frac{h}{1+x^2+xh}\right)
\end{aligned}$$

Applied the identity

$$\tan^{-1}A-\tan^{-1}B=\tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

■

Example 4.8.3 Evaluate

$$\Delta \tan^{-1} x$$

Solution :

$$\begin{aligned}\Delta \tan^{-1} x &= \tan^{-1}(x+h) - \tan^{-1} x\\ &= \tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)\\ &= \tan^{-1}\left(\frac{h}{1+x^2+xh}\right)\end{aligned}$$

Applied the identity

$$\tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

■

Example 4.8.4 Forward differences can be written in a tabular form as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	$y_0 = f(x_0)$			
		$\Delta y_0 = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	$y_2 = f(x_2)$		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Example 4.8.5 Construct the forward difference table for the following x values and its corresponding f values.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
f	0.003	0.067	0.148	0.248	0.370	0.518	0.697

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

Example 4.8.6 Construct the forward difference table, where $f(x) = \dfrac{1}{x}, x = 1(0.2)2, 4D$.

x	$f(x) = \dfrac{1}{x}$	Δf first difference	$\Delta^2 f$ second difference	$\Delta^3 f$	$\Delta^4 f$	Δ^5
1.0	1.000					
		-0.1667				
1.2	0.8333		0.0477			
		-0.1190		-0.0180		
1.4	0.7143		0.0297		0.0082	-0.0045
		-0.0893		-0.0098		
1.6	0.6250		0.0199		0.0037	
		-0.0694		-0.0061		
1.8	0.5556		0.0138			
		-0.0556				
2.0	0.5000					

Example 4.8.7 Construct the forward difference table for the data

$$\begin{array}{rcrrrr} x & : & -2 & 0 & 2 & 4 \\ y = f(x) & : & 4 & 9 & 17 & 22 \end{array}$$

The forward difference table is as follows:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	4			
		$\Delta y_0 = 5$		
0	9		$\Delta^2 y_0 = 3$	
		$\Delta y_1 = 8$		$\Delta^3 y_0 = -6$
2	17		$\Delta^2 y_1 = -3$	
		$\Delta y_2 = 5$		
4	22			

Example 4.8.8 Forward Difference Table

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
x_0	f_0						
x_1	f_1	Δf_0	$\Delta^2 f_0$				
				$\Delta^3 f_0$			
x_2	f_2	Δf_1	$\Delta^2 f_2$		$\Delta^4 f_0$		
				$\Delta^3 f_1$		$\Delta^5 f_0$	
x_3	f_3	Δf_2	$\Delta^2 f_2$		$\Delta^4 f_1$		$\Delta^6 f_0$
				$\Delta^3 f_2$		$\Delta^5 f_1$	
x_4	f_4	Δf_3	$\Delta^2 f_3$		$\Delta^4 f_2$		
				$\Delta^3 f_3$			
x_5	f_5	Δf_4	$\Delta^2 f_4$				
		Δf_5					
x_6	f_6						

Example 4.8.9 Express $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the values of the function f.

$$\begin{aligned} \Delta^2 f_0 &= \Delta f_1 - \Delta f_0 = f_2 - f_1 - (f_1 - f_0) = f_2 - 2f_1 + f_0 \\ \Delta^3 f_0 &= \Delta^2 f_1 - \Delta^2 f_0 = \Delta f_2 - \Delta f_1(\Delta f_1 - \Delta f_0) \\ &= (f_3 - f_2) - (f_2 - f_1) - (f_2 - f_1) + (f_1 - f_0) \\ &= f_3 - 3f_2 + 3f_1 - f_0 \end{aligned}$$

In general,

$$\Delta^n f_0 = f_n -^n C_1 f_{n-1} +^n C_2 f_{n-2} -^n C_3 f_{n-3} + \cdots + (-1)^n f_0.$$

If we write y_n to denote f_n the above results takes the following forms:

$$\begin{aligned}
\Delta^2 y_0 &= y_2 - 2y_1 + y_0 \\
\Delta^3 y_0 &= y_3 - 3y_2 + 3y_1 - y_0 \\
\Delta^n y_0 &= y_n -^n C_1 y_{n-1} +^n C_2 y_{n-2} - 2 -^n C_3 y_{n-3} + \cdots + (-1)^n y_0
\end{aligned}$$

Example 4.8.10 Show that the value of y_n can be expressed in terms of the leading value y_0 and the leading differences $\Delta y_0, \Delta^2 y_0, \ldots, \Delta^n y_0$.

Solution : *(For notational convenience, we treat y_n as f_n and so on.)* *From the forward difference table we have*

$$\left.\begin{array}{lll}
\Delta f_0 = f_1 - f_0 & or & f_1 = f_0 + \Delta f_0 \\
\Delta f_1 = f_2 - f_1 & or & f_2 = f_1 + \Delta f_1 \\
\Delta f_2 = f_3 - f_2 & or & f_3 = f_2 + \Delta f_2
\end{array}\right\}$$

and so on. Similarly,

$$\left.\begin{array}{lll}
\Delta^2 f_0 = \Delta f_1 - \Delta f_0 & or & \Delta f_1 = \Delta f_0 + \Delta^2 f_0 \\
\Delta^2 f_1 = \Delta f_2 - \Delta f_1 & or & \Delta f_2 = \Delta f_1 + \Delta^2 f_1
\end{array}\right\}$$

and so on. Similarly, we can write

$$\left.\begin{array}{lll}
\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 & or & \Delta^2 f_1 = \Delta^2 f_0 + \Delta^3 f_0 \\
\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1 & or & \Delta^2 f_2 = \Delta^2 f_1 + \Delta^3 f_1
\end{array}\right\}$$

and so on. Also, we can write f_2 as

$$\begin{aligned}
f_2 &= (f_0 + \Delta f_0) + (\Delta f_0 + \Delta^2 f_0) \\
&= f_0 + 2\Delta f_0 + \Delta^2 f_0 \\
&= (1 + \Delta)^2 f_0
\end{aligned}$$

Hence

$$\begin{aligned}
f_3 &= f_2 + \Delta f_2 \\
&= (f_1 + \Delta f_1) + \Delta f_0 + 2\Delta^2 f_0 + \Delta^3 f_0 \\
&= f_0 + 3\Delta f_0 + 3\Delta^2 f_0 + \Delta^3 f_0 \\
&= (1 + \Delta)^3 f_0
\end{aligned}$$

That is, we can symbolically write

$$f_1 = (1 + \Delta) f_0, f_2 = (1 + \Delta)^2 f_0, f_3 = (1 + \Delta)^3 f_0$$

Continuing this procedure, we can show, in general

$$f_n = (1 + \Delta)^n f_0$$

Using binomial expansion, the above is

$$f_n = f_0 +^n C_1 \Delta f_0 +^n C_2 \Delta^2 f_0 + \cdots + \Delta^n f_0$$

Thus

$$f_n = \sum_{i=0}^{n} {}^nC_i \Delta^i f_0$$

■

Example 4.8.11 Construct the backward difference table for the data

$$\begin{array}{rcrrrr} x & : & -2 & 0 & 2 & 4 \\ y = f(x) & : & -8 & 3 & 1 & 12 \end{array}$$

Solution : *The backward difference table is as follows:*

x	$Y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
-2	-8			
		$\nabla y_1 = 3 - (-8) = 11$		
0	3		$\nabla^2 y_2 = -2 - 11 = -13$	
		$\nabla y_2 = 1 - 3 = -2$		$\nabla^3 y_3 = 13 - (-13) = 26$
2	1		$\nabla^2 y_3 = 11 - (2) = 13$	
		$\nabla y_3 = 12 - 1 = 11$		
4	12			

■

Example 4.8.12 Construct a forward difference table for the following data

x	0	10	20	30
y	0	0.174	0.347	0.518

Solution : *The Forward difference table is given below*

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		0.174		
10	0.174		-0.001	
		0.173		-0.001
20	0.347		-0.002	
		0.171		
30	0.518			

■

Example 4.8.13 Construct a forward difference table for $y = f(x) = x^3 + 2x + 1$ for $x = 1, 2, 3, 4, 5$

Solution : $y = f(x) = x^3 + 2x + 1$ *for* $x = 1, 2, 3, 4, 5$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4				
		9			
2	13		12		
		21		6	
3	34		18		0
		39		6	
4	73		24		
		63			
5	136				

■

Example 4.8.14 By constructing a difference table and using the second order differences as constant, find the sixth term of the series $8, 12, 19, 29, 42 \ldots$

Solution : *Let k be the sixth term of the series in the difference table*

First we find the forward differences.

x	y	Δ	Δ^2
1	8		
		4	
2	12		3
		7	
3	19		3
		10	
4	29		3
		13	
5	42		$k - 55$
		$k - 42$	
6	k		

Given that the second differences are constant

$\therefore k–55 = 3$

$k = 58$

$\therefore$ *the sixth term of the series is* 58

■

Example 4.8.15 Find

1.) Δe^{ax}

Solution :

$$\begin{aligned}\Delta e^{ax} &= e^{a(x+h)} - e^{ax} \\ &= e^{ax}.e^{h} - e^{ax} \\ &= e^{ax}\left[e^{h} - 1\right] \\ &\quad \left[\because a^{m+n} = a^{m}.a^{n}\right]\end{aligned}$$

■

2.) $\Delta^2 e^x$

Solution :

$$\begin{aligned}\Delta^2 e^{x} &= \Delta.\left[\Delta e^{x}\right] \\ &= \Delta\left[e^{x+h} - e^{x}\right] = \Delta\left[e^{x}e^{h} - e^{x}\right] \\ &= \Delta e^{x}\left[e^{h} - 1\right] \\ &= (e^{h} - 1)\Delta e^{x} \\ &= (e^{h} - 1).(e^{h} - 1).e^{x} \\ &= (e^{h} - 1)^2.e^{x}\end{aligned}$$

■

3.) $\Delta \log x$

Solution :

$$\begin{aligned}\Delta \log x &= \log(x+h) - \log x \\ &= \log \frac{x+h}{x} \\ &= \log\left(\frac{x}{x} + \frac{h}{x}\right) \\ &= \log\left(1 + \frac{h}{x}\right)\end{aligned}$$

■

Example 4.8.16 Evaluate $\Delta\left[\frac{5x+12}{x^2+5x+6}\right]$ by taking '1' as the interval of differencing.

Solution : $\Delta\left[\frac{5x+12}{x^2+5x+6}\right]$

By Partial fraction method $\frac{5x+12}{x^2+5x+6} = \frac{A}{x+3} + \frac{B}{x+2}$

$$A = \frac{5x+12}{x+2}[x=-3] = \frac{-15+12}{-1} = \frac{-3}{-1} = 3$$

$$B = \frac{5x+12}{x+3}[x=-2]$$

$$= \frac{2}{1} = 2$$

$$\frac{5x+12}{x^2+5x+6} = \left[\frac{3}{x+3} + \frac{2}{x+2}\right]$$

$$\Delta\left[\frac{5x+12}{x^2+5x+6}\right] = \Delta\left[\frac{3}{x+3} + \frac{2}{x+2}\right]$$

$$= \left[\frac{3}{x+1+3} - \frac{3}{x+3}\right] + \left[\frac{2}{x+1+2} - \frac{2}{x+2}\right]$$

$$= 3\left[\frac{1}{x+4} - \frac{1}{x+3}\right] + 2\left[\frac{1}{x+3} - \frac{1}{x+2}\right]$$

$$= \left[\frac{-3}{(x+4)(x+3)} - \frac{2}{(x+3)(x+2)}\right]$$

$$= \frac{-5x-14}{(x+2)(x+3)(x+4)}$$

■

Example 4.8.17 Evaluate $\Delta^2\left(\frac{1}{x}\right)$ by taking '1' as the interval of differencing.

Solution :

$$\Delta^2\left(\frac{1}{x}\right) = \Delta\left(\Delta\left(\frac{1}{x}\right)\right)$$

Now

$$\Delta\left[\frac{1}{x}\right] = \frac{1}{x+1} - \frac{1}{x}$$

$$\Delta^2\left(\frac{1}{x}\right) = \Delta\left(\frac{1}{1+x} - \frac{1}{x}\right)$$

$$= \Delta\left(\frac{1}{1+x}\right) - \Delta\left(\frac{1}{x}\right)$$

Similarly

$$\Delta^2\left(\frac{1}{x}\right) = \frac{2}{x(x+1)(x+2)}$$

Generally

$$\Delta^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x(x+1)(x+2)\dots(x+n)}$$

■

Example 4.8.18 Prove that $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$ taking '1' as the interval of differencing.

Solution : *We know that* $f(4) - f(3) = \Delta f(3)$

$$\begin{aligned}
&f(4) - f(3) = \Delta f(3) \\
&= \Delta[f(2) + \Delta f(2)] \ \textit{i.e.}\ [f(3) - f(2) = \Delta f(2)] \\
&= \Delta f(2) + \Delta^2 f(2) \\
&= \Delta f(2) + \Delta^2[f(1) + \Delta f(1)] \\
&f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)
\end{aligned}$$

■

Example 4.8.19 Given $U_0 = 1, U_1 = 11, U_2 = 21, U_3 = 28$ and $U_4 = 29$ find $\Delta^4 U_0$

Solution :

$$\begin{aligned}
\Delta^4 U_0 = (E-1)^4 U_0 & \\
&= \left(e^4 - 4E^3 + 6E + 1\right) U_0 \\
&= E^4 U_0 - 4E^3 U_0 + 6E^2 U_0 - 4EU_0 + U_0 \\
&= U_4 - 4U_3 + 6U_2 - 4U_1 + U_0 \\
&= 29 - 4(28) + 6(21) - 4(11) + 1. \\
&= 156 - 156 = 0
\end{aligned}$$

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

■

Example 4.8.20 Given that $y_3 = 2, y_4 = -6, y_5 = 8, y_6 = 9$ and $y_7 = 17$ Calculate $\Delta^4 y_3$

Solution : *Given that* $y_3 = 2, y_4 = -6, y_5 = 8, y_6 = 9$ *and* $y_7 = 17$

$$\begin{aligned}
\Delta^4 y_3 &= (E-1)^4 y_3 \\
&= \left(E^4 - 4E^3 + 6E^2 - 4E + 1\right) y_3 \\
&= E^4 y_3 - 4E^3 y_3 + 6E^2 y_3 - 4Ey_3 + y_3 \\
&= y_7 - 4y_6 + 6y_5 - 4y_4 + y_3 \\
&= 17 - 4(9) + 6(8) - 4(-6) + 2 \\
&= 17 - 36 + 48 + 24 + 2 = 55
\end{aligned}$$

■

Example 4.8.21 From the following table find the missing value

x	2	3	4	5	6
$f(x)$	45.0	49.2	54.1	–	67.4

Solution : *Since only four values of $f(x)$ are given, the polynomial which fits the data is of degree three. Hence fourth differences are zeros.*

(ie)

$$\begin{aligned}
\Delta^4 y_0 = 0, \ \therefore (E-1)^4 y_0 &= 0 \\
(E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 &= 0 \\
E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4Ey_0 + y_0 &= 0 \\
y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 &= 0 \\
67.4 - 4y_3 + 6(54.1) - 4(4.2) + 45 &= 0 \\
240.2 = 4y_3 \ \therefore y_3 &= 60.05
\end{aligned}$$

■

Example 4.8.22 Estimate the production for 1964 and 1966 from the following data

Solution :

Year	1961	1962	1963	1964	1965	1966	1967
Production	200	220	260	–	350	–	430

Since five values are given, the polynomial which fits the data is of degree four.

Hence $\Delta^5 y_k = 0$ *(ie)* $(E-1)^5 y_k = 0$

i.e; $(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_k = 0$

$$E^5 y_k - 5E^4 y_k + 10E^3 y_k - 10E^2 y_k + 5Ey_k - y_k = 0 \tag{4.41}$$

Put $k = 0$ in (4.41)

$$\begin{aligned}
E^5y_0 - 5E^4y_0 + 10E^3y_0 - 10E^2y_0 + 5Ey_0 - y_0 &= 0\\
y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 &= 0\\
y_5 - 5(350) + 10y_3 - 10(260) + 5(220) - 200 &= 0\\
y_5 + 10y_3 &= 3450 \qquad (4.42)
\end{aligned}$$

Put $k = 1$ in (4.41)

$$\boxed{\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 2 & 1 & & & \\
 & & 1 & 3 & & 3 & 1 & & \\
 & 1 & 4 & & 6 & & 4 & 1 & \\
1 & 5 & & 10 & & 10 & 5 & & 1
\end{array}}$$

$$\begin{aligned}
E^5y_1 - 5E^4y_1 + 10E^3y_1 - 10E^3y_1 - y_1 &= 0\\
y_6 - 5y_5 + 10y_4 - 10y_3 - y_1 &= 0\\
430 - 5y_5 + 10(350) - 10y_3 + 5(260) - 220 &= 0\\
5y_5 + 10y_3 &= 5010 \qquad (4.43)\\
(4.43) - (4.42) \Rightarrow 4y_5 &= 1560\\
y_5 &= 390\\
\textit{From } 390 + 10y_3 &= 3450\\
10y_3 &= 3450 - 390\\
y_3 &\cong 306
\end{aligned}$$

■

Example 4.8.23 Evaluate $\Delta(\log ax)$.

Solution :

$$\begin{aligned}
\Delta(\log ax) &= \log a(x+h) - \log(ax)\\
&= \log\left(\frac{ax+ah}{ax}\right)\\
&= \log\left(\frac{ax}{ax} + \frac{ah}{ax}\right)\\
&= \log\left(1 + \frac{ah}{ax}\right)\\
\therefore \Delta(\log ax) &= \log\left(1 + \frac{h}{x}\right)
\end{aligned}$$

■

Example 4.8.24 If $y = x^3 - x^2 + x - 1$ calculate the values of y for $x = 0, 1, 2, 3, 4, 5$ and form the forward differences table.

Solution :

When $x = 0, y = 0 + 0 + 0 - 1 \Rightarrow y = -1$
When $x = 1, y = 1^3 - 1^2 + 1 - 1 \Rightarrow y = 0.$
When $x = 2, y = 2^3 - 2^2 + 2 - 1 \Rightarrow y = 8 - 4 + 1 \Rightarrow y = 5$
When $x = 3, y = 3^3 - 3^2 + 3 - 1 \Rightarrow y = 27 - 9 + 2 \Rightarrow y = 20$
When $x = 4, y = 4^3 - 4^2 + 4 - 1 \Rightarrow y = 64 - 16 + 3 \Rightarrow y = 51$
When $x = 5, y = 5^3 - 5^2 + 5 - 1 \Rightarrow y = 125 - 25 + 4 \Rightarrow y = 104.$
Hence, the forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	-1					
1	0	1	4	6		
2	5	5	10	6	0	
3	20	15	16	6	0	0
4	51	31	22			
5	104	53				

■

Example 4.8.25 If $h = 1$ then prove that $(E^{-1}\Delta)x^3 = 3x^2 - 3x + 1$.

Solution : *Given* $h = 1$
LHS$= (E^{-1}\Delta)x^3$
$=\Delta(E^{-1}(x^3))$

$$
\begin{aligned}
&= \Delta(x-h)^3 \qquad [\because E^{-1}f(x) = f(x-nh)] \\
&= \Delta(x-1)^3 \qquad [\because h = 1] \\
&= (x-1+1)^3 - (x-1)^3 \\
&\qquad [\because \Delta f(x) = f(x+h) - f(x)] \\
&= x^3 - (x-1)^3 \\
&= x^3 - (x^3 - 3x^2 + 3x - 1) \\
&\qquad [\because (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3] \\
&= x^3 - x^3 + 3x + 1 \\
&= 3x^2 - 3x + 1 \\
&= RHS
\end{aligned}
$$

Hence proved. ■

Example 4.8.26 If $f(x) = x^2 + 3x$ than show that $\Delta f(x) = 2x + 4$

Solution : *Given* $f(x) = x^2 + 3h$

$$
\begin{aligned}
LHS &= \Delta f(x) \\
&= f(x+h) - f(x) \\
&= [(x+h)^2 + 3(x+h)] - [x^2 + 3x] \\
&= \cancel{x^2} + h^2 + 2xh + \cancel{3x} + 3h - \cancel{x^2} - \cancel{3x} \\
&= h^2 + 2xh + 3h
\end{aligned}
$$

when

$$\begin{aligned} h = 1, LHS &= 1^2 + 2x(1) + 3(1) \\ &= 1 + 2x + 3 \\ &= 2x + 4 = RHS. \end{aligned}$$

Hence proved ■

Example 4.8.27 Evaluate $\Delta[1/(x+1)(x+2)]$ by taking '1' as the interval of differencing

Solution : *By partial fraction method*

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

$$\Rightarrow \frac{1}{(x+1)\cancel{(x+2)}} = \frac{A(x+2)+B(x+1)}{(x+1)\cancel{(x+2)}}$$

$$\Rightarrow 1 = A(x+2) + B(x+1)$$

$$When\ x = -1, 1 = A[-1+2] \Rightarrow 1 = A$$

$$When\ x = -2, 1 = B[-2+1] \Rightarrow 1 = -B$$

$$\Rightarrow B = -1$$

$$\therefore \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\therefore \Delta\left[\frac{1}{(x+1)(x+2)}\right] = \Delta\left[\frac{1}{x+1} - \frac{1}{x+2}\right]$$

$$= \left(\frac{1}{x+1+1} - \frac{1}{x+1}\right) - \left(\frac{1}{x+1+2} - \frac{1}{x+2}\right)$$

$$[\because \Delta f(x) = f(x+1) - f(x)]$$

$$= \left(\frac{1}{x+2} - \frac{1}{x+1}\right) - \left(\frac{1}{x+3} - \frac{1}{x+2}\right) \ Where\ h = 1$$

$$= \frac{1}{x+2} - \frac{1}{x+1} - \frac{1}{x+3} + \frac{1}{x+2}$$

$$= \frac{2}{x+2} - \frac{1}{x+1} - \frac{1}{x+3}$$

$$= \frac{2(x+1)(x+3) - 1(x+2)(x+3) - 1(x+1)(x+2)}{(x+1)(x+2)(x+3)}$$

$$\frac{2(x^2+4x+3) - (x^2+5x+6) - (x^2+3x+2)}{(x+1)(x+2)(x+3)}$$

$$= \frac{2x^2 + 8x + \cancel{6} - x^2 - 5x - \cancel{6} - x^2 - 3x - 2}{(x+1)(x+2)(x+3)}$$

$$= \frac{\cancel{2x^2} + \cancel{8x} - \cancel{8x} - \cancel{2x^2} - 2}{(x+1)(x+2)(x+3)}$$

$$= \frac{-2}{(x+1)(x+2)(x+3)}$$

$$\therefore = \Delta\left[\frac{1}{(x+1)(x+2)}\right] = \frac{-2}{(x+1)(x+2)(x+3)}$$

■

Example 4.8.28 Find the missing entry in the following table

x	0	1	2	3	4
y_x	1	3	9	–	81

Solution : *Since only four values of $f(x)$ are given, the polynomial which fits the data is of degree* 3.
Hence four differences are zero.

$$\begin{aligned}
\Delta^4 (y_0) &= 0 \\
(E-1)^4 (y_0) &= 0 \\
\left(E^4 - 4E^3 + 6E^2 - 4E + 1\right)(y_0) &= 0 \\
y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 &= 0 \\
81 - 4(y_3) + 6(9) - 4(3) + 1 &= 0 \\
81 - 4y_3 + 54 - 12 + 1 &= 0 \\
81 + 54 - 11 &= 4y_3 \\
124 = 4y_3 \Rightarrow y_3 &= \frac{124}{4} = 31. \\
y_3 &= 31
\end{aligned}$$

■

Example 4.8.29 Following are the population of a district

Year (x)	1881	1891	1901	1911	1921	1931
Population (y)Thousands	363	391	421	–	467	501

Find the population of the year 1911

Solution : *Since only five values of $f(x)$ are given, the polynomial which fits the data is of degree* 4.
Hence fifth differences are zeros.

$$\begin{aligned}
\therefore \Delta^5 y_0 &= 0 \\
(E-1)^5 y_0 &= 0 \\
(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0 &= 0 \\
y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 &= 0 \\
501 - 5(467) + 10y_3 - 10(421) + 5(391) - 363 &= 0 \\
501 - 2335 + 10y_3 - 4210 + 1955 - 363 &= 0 \\
10y_3 - 4352 &= 0 \\
10y_3 &= 4352 \\
y_3 = \frac{4352}{10} &= 435.2
\end{aligned}$$

Since the population is given in thousands, population of the year 1911

$$= 435.2 \times 1000 = 435200$$

■

Example 4.8.30 Find the missing entries from the following.

x	0	1	2	3	4	5
$y = f(x)$	0	–	8	15	–	35

Solution : *Let the missing entries by y_1 and y_4*

Since only four values of $f(x)$ are given, the polynomial which fits the data is of degree 3.

Hence fourth differences are zero

$$\therefore \Delta^4 y_k = 0 \Rightarrow (E-1)^4 y_k = 0$$
$$(E^4 - 4E^3 + 6E^2 - 4E + 1)y_k = 0 \tag{4.44}$$

Put $k = 0$ in (1) we get,

$$\begin{aligned}
(E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 &= 0\\
y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 &= 0\\
y_4 - 4(15) + 6(8) - 4(y_1) + 0 &= 0\\
y_4 - 60 + 48 - 4y_1 &= 0\\
y_4 - 4y_1 - 12 &= 0\\
y_4 - 4y_1 &= 12
\end{aligned} \tag{4.45}$$

Put $k = 1$ in (4.44) we get

$$\begin{aligned}
(E^4 - 4E^3 + 6E^2 - 4E + 1)y_1 &= 0\\
y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 &= 0\\
35 - 4y_4 + 6(15) - 4(8) + y_1 &= 0\\
35 - 4y_4 + 90 - 32 + y_1 &= 0\\
-4y_4 + y_1 &= -93
\end{aligned} \tag{4.46}$$

$$\begin{aligned}
(4.45) \times 4 \to 4y_4 - 16y_1 &= 48\\
(4.46) \to -4y_4 + y_1 &= -93\\
\textbf{Adding}, -15y_1 &= -45\\
y_1 = \frac{-45}{-15} &= 3\\
y_1 &= 3.
\end{aligned}$$

Subtracting $y_1 = 3$ in (4.45) we get,

$$\begin{aligned}
y_4 - 4(3) &= 12\\
y_4 - 12 &= 12\\
y_4 &= 12 + 12\\
y_4 &= 24
\end{aligned}$$

Hence the missing entries are 3 and 24 ■

Example 4.8.31 $\Delta\left[x(x+1)(x+2)\right]$

Solution :

$$\begin{aligned}
&\Delta\left[x(x+1)(x+2)\right]\\
&=\Delta x(x^2+3x+2)\\
&=\Delta(x^3+3x^2+2x)\\
&=(x+h)^3+3(x+h)^2+2(x+h)-(x^3+3x^2+2x)\\
&=x^3+3x^2h+3xh^2+h^3+3(x^2+2hx+h^2)+2x+2h-x^3-3x^2-2x\\
&=3x^2h+3xh+h^3+3x^2+6hx+3h^2+2h-3x^2\\
&=3x^2h+9hx+h^3+3h^2+2h\\
h&=1\\
&=3x^2+9x+6\\
&=3(x^2+3x+2)\\
&=3(x^2+2x+x+2)\\
&=3\left[x(x+2)+1(x+2)\right]\\
&\Delta\left[x(x+1)(x+2)\right]=3(x+1)(x+2), h=1
\end{aligned}$$

■

Example 4.8.32 $\Delta\left(\dfrac{1}{1+x^2}\right)$

Solution :

$$\begin{aligned}
&\Delta\left(\frac{1}{1+x^2}\right)\\
&=\frac{1}{1+(x+h)^2}-\frac{1}{1+x^2}\\
&=\frac{1+x^2-\left[1+(x+h)^2\right]}{\left[1+x^2\right]\left[1+(x+h)^2\right]}\\
&=\frac{1+x^2-1-x^2-2hx-h^2}{\left[1+x^2\right]\left[1+(x+h)^2\right]}\\
&=\frac{-2hx-h^2}{\left[1+x^2\right]\left[1+(x+h)^2\right]}\\
put\quad h&=1\\
&=\frac{-2x-1}{\left[1+x^2\right]\left[1+(x+h)^2\right]}\\
&\Delta\left(\frac{1}{1+x^2}\right)=\frac{-(2x+1)}{\left[1+x^2\right]\left[1+(x+h)^2\right]}, h=1
\end{aligned}$$

■

Example 4.8.33 $\Delta [x!]$

Solution :

$$\begin{aligned}
&\Delta [x!] \\
&= (x+h)! - x! \\
put \quad h &= 1 \\
&= (x+1)! - x! \\
&= (x+1)x! - x! \\
&= (x+1-1)x! \\
&\Delta [x!] = x(x!)
\end{aligned}$$

■

Example 4.8.34 $\Delta \cos (ax+b)$

Solution :

$$\begin{aligned}
&\Delta \cos (ax+b) \\
&= \cos [a(x+h)+b] - \cos (ax+b) \\
&= 2\sin \left(\frac{a(x+h)+b+ax+b}{2}\right) \sin \left(\frac{ax+b-a(x+h)-b}{2}\right) \\
&= 2\sin \left(\frac{ax+ah+b+ax+b}{2}\right) \sin \left(\frac{ax-ax-ah}{2}\right) \\
&= 2\sin \left(ax+b+\frac{ah}{2}\right) \sin \left(-\frac{ah}{2}\right) \\
&\Delta \cos (ax+b) = 2\sin \left(\frac{ah}{2}\right) \cos \left\{ax+b+\frac{1}{2}(ah+\pi)\right\}
\end{aligned}$$

■

Example 4.8.35 $\Delta(\cot 2^x)$

Solution :

$$\begin{aligned}
&\Delta(\cot 2^x) \\
\Delta(\cot 2^x) &= \cot 2^{x+h} - \cot 2^x \\
&= \frac{\cos 2^{x+h}}{\sin 2^{x+h}} - \frac{\cos 2^x}{\sin 2^x} \\
&= \frac{\cos 2^{x+h} \sin 2^x - \cos 2^x \sin 2^{x+h}}{\sin 2^x \sin 2^{x+h}} \\
&= \frac{\sin (2^x - 2^{x+h})}{\sin 2^x \sin 2^{x+h}} \\
\Delta(\cot 2^x) &= \frac{\sin \{(1-2^h)2^x\}}{\sin 2^x \sin 2^{x+h}}
\end{aligned}$$

■

Example 4.8.36 $\Delta \tan^{-1} ax$

Solution :

$$\begin{aligned}
&\Delta \tan^{-1} ax \\
\Delta \tan^{-1} ax &= \tan^{-1} a(x+h) - \tan^{-1} ax \\
&= \tan^{-1} \left\{ \frac{ax+h-ax}{1+a(x+h)ax} \right\} \\
&= \tan^{-1} \left\{ \frac{h}{1+a^2x^2+a^2xh} \right\} \\
put \quad h &= 1 \\
\Delta \tan^{-1} ax &= \tan^{-1} \left\{ \frac{h}{1+a^2x^2+a^2x} \right\}
\end{aligned}$$

■

Example 4.8.37 $\Delta(x+\cos x)$

Solution :

$$\begin{aligned}
&\Delta(x+\cos x) \\
&= \Delta x + \Delta \cos x \\
&= x+h+\cos(x+h) - x - \cos x \\
&= h + \cos(x+h) - \cos x \\
&= h - 2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x-x-h}{2}\right) \\
&= h - 2\sin\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right) \\
\Delta(x+\cos x) &= h - 2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)
\end{aligned}$$

■

Example 4.8.38 $\Delta\left(\tan^{-1} x - \frac{2^x}{(x+1)!}\right), h = 1$

Solution :

$$\begin{aligned}
&\Delta\left(\tan^{-1} x - \frac{2^x}{(x+1)!}\right) \\
&= \tan^{-1}(x+h) - \frac{2^{x+h}}{(x+h+1)!} - \tan^{-1} x + \frac{2^x}{(x+1)!} \\
&= \tan^{-1}(x+h) - \tan^{-1} x - \frac{2^{x+h}}{(x+h+1)!} + \frac{2^x}{(x+1)!} \\
put \quad h &= 1 \\
&= \tan^{-1}(x+1) - \tan^{-1} x - \frac{2^{x+1}}{(x+2)!} + \frac{2^x}{(x+1)!} \\
&= \tan^{-1} \frac{x+1-x}{1+(x+1)x} - \frac{2^{x+1}}{(x+2)(x+1)!} + \frac{2^x}{(x+1)!} \\
&= \tan^{-1} \frac{1}{1+x^2+x} - \frac{2^x}{(x+1)!}\left[\frac{2}{(x+2)} - \frac{1}{1}\right]
\end{aligned}$$

$$= \tan^{-1}(\frac{1}{1+x^2+x}) - \frac{2^x}{(x+1)!}\left[\frac{2-x-2}{x+2}\right]$$
$$= \tan^{-1}(\frac{1}{1+x^2+x}) - \frac{2^x}{(x+1)!}\left[\frac{-x}{x+2}\right]$$
$$= \tan^{-1}\left(\frac{1}{1+x^2+x}\right) + \frac{2^x x}{(x+2)!}$$
$$\Delta\left(\tan^{-1}x - \frac{2^x}{(x+1)!}\right) = \tan^{-1}\left(\frac{1}{1+x^2+x}\right) + \frac{2^x x}{(x+2)!}$$

■

Example 4.8.39 $\Delta\left(\dfrac{x^2}{\cos 2x}\right)$

Solution :

$$\Delta\left(\frac{x^2}{\cos 2x}\right)$$
$$= \frac{(x+h)^2}{\cos\{2(x+h)\}} - \frac{x^2}{\cos 2x}$$
$$= \frac{(x+h)^2}{\cos\{2x+2h\}} - \frac{x^2}{\cos 2x}$$
$$= \frac{x^2+2xh+h^2}{\cos(2x+2h)} - \frac{x^2}{\cos 2x}$$
$$= \frac{x^2\cos 2x + 2xh\cos 2x + h^2\cos 2x - x^2\cos(2x+2h)}{\cos 2x\cos(2x+2h)}$$
$$= \frac{x^2\{\cos 2x - \cos(2x+2h)\} + 2xh\cos 2x + h^2\cos 2x}{\cos 2x\cos(2x+2h)}$$
$$= \frac{x^2.2\sin\left(\frac{2x+2x+2h}{2}\right)\sin\left(\frac{2x+2h-2x}{2}\right) + 2xh\cos 2x + h^2\cos 2x}{\cos 2x\cos(2x+2h)}$$
$$= \frac{2x^2\sin(2x+h)\sin h + 2xh\cos 2x + h^2\cos 2x}{\cos 2x\cos(2x+2h)}$$
$$= \frac{2x^2\sin(2x+h)\sin h + h(2x+h)\cos 2x}{\cos 2x\cos(2x+2h)}$$

■

Example 4.8.40 $\Delta^2 x^3$

Solution :

$$\Delta^2 x^3$$
$$= \Delta(\Delta x^3)$$
$$= \Delta\left[(x+h)^3 - x^3\right]$$
$$= \Delta\left[x^3 + 3x^2h + 3xh^2 + h^3 - x^3\right]$$
$$= \Delta\left[3x^2h + 3xh^2 + h^3\right]$$
$$= 3(x+h)^2h + 3(x+h)h^2 + h^3 - 3x^2h - 3xh^2 - h^3$$

$$
\begin{aligned}
&= 3(x^2+2hx+h^2)h + 3xh^2 + 3h^3 - 3x^2h - 3xh^2 \\
&= 3x^2h + 6h^2x + 3h^3 + 3xh^2 + 3h^3 - 3x^2h - 3xh^2 \\
&= 6h^2x + 6h^3 \\
&= 6h^2(x+h)
\end{aligned}
$$

■

Example 4.8.41 $\Delta^2\left[\dfrac{a^{2x}+a^{4x}}{(a^2-1)^2}\right]$

Solution :

$$
\begin{aligned}
&\Delta^2\left[\frac{a^{2x}+a^{4x}}{(a^2-1)^2}\right] \\
&= \Delta\left[\frac{a^{2x+2h}+a^{4x+4h}}{(a^2-1)^2} - \frac{a^{2x}+a^{4x}}{(a^2-1)^2}\right] \\
&= \frac{1}{(a^2-1)^2}\Delta\left[a^{2x+2h}+a^{4x+4h}-a^{2x}-a^{4x}\right]
\end{aligned}
$$

$$put \quad h = 1$$

$$
\begin{aligned}
&= \frac{1}{(a^2-1)^2}\Delta\left[a^{2x+2}+a^{4x+4}-a^{2x}-a^{4x}\right] \\
&= \frac{1}{(a^2-1)^2}\Delta\left[a^{2x}(a^2-1)+a^{4x}(a^4-1)\right] \\
&= \frac{1}{(a^2-1)^2}\Delta\left[a^{2x}(a^2-1)+a^{4x}(a^2-1)(a^2+1)\right] \\
&= \frac{1}{(a^2-1)^2}\Delta\left[(a^2-1)\{a^{2x}+a^{4x}(a^2+1)\}\right] \\
&= \frac{1}{(a^2-1)^2}\Delta(a^2-1)\left[a^{2x}+a^{4x}(a^2+1)\right] \\
&= \frac{1}{(a^2-1)}\Delta\left[a^{2x}+a^{4x}(a^2+1)\right] \\
&= \frac{1}{(a^2-1)}\left[a^{2x+2h}+(a^2+1)a^{4x+4h}-a^{2x}-a^{4x}(a^2+1)\right]
\end{aligned}
$$

$$put \quad h = 1$$

$$
\begin{aligned}
&= \frac{1}{(a^2-1)}\left[a^{2x+2}+(a^2+1)a^{4x+4}-a^{2x}-(a^2+1)a^{4x}\right] \\
&= \frac{1}{(a^2-1)}\left[a^{2x}(a^2-1)+a^{4x}\{(a^2+1)a^4-(a^2+1)\}\right] \\
&= \frac{1}{(a^2-1)}\left[a^{2x}(a^2-1)+a^{4x}(a^2+1)(a^4-1)\right] \\
&= \frac{1}{(a^2-1)}\left[a^{2x}(a^2-1)+a^{4x}(a^2+1)(a^2+1)(a^2-1)\right] \\
&= \frac{1}{(a^2-1)}(a^2-1)\left[a^{2x}+a^{4x}(a^2+1)^2\right]
\end{aligned}
$$

$$\Delta^2\left[\frac{a^{2x}+a^{4x}}{(a^2-1)^2}\right] = a^{2x}+a^{4x}(a^2+1)^2$$

■

Example 4.8.42 $\Delta^3\left[\frac{1}{(3x+1)(3x+4)(3x+7)}\right]$

Solution :

$$\Delta^3\left[\frac{1}{(3x+1)(3x+4)(3x+7)}\right]$$

$$=\Delta^2\left[\Delta\frac{1}{(3x+1)(3x+4)(3x+7)}\right]$$

$$=\Delta^2\left[\frac{1}{(3x+3h+1)(3x+3h+4)(3x+3h+7)}-\frac{1}{(3x+1)(3x+4)(3x+7)}\right]$$

put $h=1$

$$=\Delta^2\left[\frac{1}{(3x+4)(3x+7)(3x+10)}-\frac{1}{(3x+1)(3x+4)(3x+7)}\right]$$

$$=\Delta^2\left[\frac{3x+1-3x-10}{(3x+1)(3x+4)(3x+7)(3x+10)}\right]$$

$$=\Delta^2\left[\frac{-9}{(3x+1)(3x+4)(3x+7)(3x+10)}\right]$$

$$=\Delta\left[\Delta\frac{-9}{(3x+1)(3x+4)(3x+7)(3x+10)}\right]$$

$$=\Delta\left[\frac{-9}{(3x+3h+1)(3x+3h+4)(3x+3h+7)(3x+3h+10)}\right.$$
$$\left.+\frac{9}{(3x+1)(3x+4)(3x+7)(3x+10)}\right]$$

put $h=1$

$$=9\Delta\left[\frac{-1}{(3x+4)(3x+7)(3x+10)(3x+13)}+\frac{1}{(3x+1)(3x+4)(3x+7)(3x+10)}\right]$$

$$=9\Delta\left[\frac{-3x-1+3x+13}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}\right]$$

$$=9\Delta\left[\frac{12}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}\right]$$

$$=108\left[\frac{1}{(3x+3h+1)(3x+3h+4)(3x+3h+7)(3x+3h+10)(3x+3h+13)}\right.$$
$$\left.-\frac{1}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}\right]$$

put $h=1$

$$=108\left[\frac{1}{(3x+4)(3x+7)(3x+10)(3x+13)(3x+16)}\right.$$
$$\left.-\frac{1}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}\right]$$

$$=108\left[\frac{3x+1-3x+16}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)(3x+16)}\right]$$

$$=\frac{-1620}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)(3x+16)}$$

■

4.9 Finite Difference Operators Exercise

Exercise 4.9.1 Prove the following finite difference identities.

1.) $E^{-\frac{1}{2}}\Delta = \delta$

2.) $\nabla = E^{-1}\Delta$

3.) $\Delta = E^{\frac{1}{2}}\delta$

4.) $\delta = E^{\frac{1}{2}}\nabla$

5.) $\nabla = E^{-\frac{1}{2}}\delta$

6.) $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$

7.) $\Delta = \mu\delta + \dfrac{1}{2}\delta^2 = E^{\frac{1}{2}}\delta$

8.) $\Delta - \nabla = \Delta\nabla = \delta^2$

Exercise 4.9.2 From the fact that $\Delta = (E - 1)$, we can write

$$\Delta^k f_r = (E-1)^k f_r.$$

Hence show that the forward difference can be written in terms of pivotal values by the formula,

$$\Delta^k f_r = \begin{pmatrix} k \\ 0 \end{pmatrix} f_{r+k} - \begin{pmatrix} k \\ 1 \end{pmatrix} f_{r+k-1} + \begin{pmatrix} k \\ 2 \end{pmatrix} f_{r+k-2} + \dots$$

Hence express $\Delta^5 f_0$ in terms of pivotal values.

Exercise 4.9.3 Prove the following identity

$$u(f_r g_r) = u f_r u g_r + \frac{1}{4}\delta f_r \delta g_r$$

Exercise 4.9.4 Use mathematical induction to prove that,

$$\Delta^r f_k = \nabla^r f_{k+r} = \delta^r f_{k+\frac{r}{2}},$$

Exercise 4.9.5 Evaluate $\Delta ab^{cx}, h = 1$

Solution : $ab^{cx}(b^c - 1)$ ■

Exercise 4.9.6 Compute $\Delta(\log x)$

Solution : $\log\left(1 + \dfrac{1}{x}\right)$ ■

Exercise 4.9.7 Compute $\nabla(x^2 + 2x)$

Solution : $2x + 1, h = 1$ ■

Exercise 4.9.8 Compute $\Delta(e^{2x}\log 3x)$

Solution : $e^{2x}[e^{2h}\log\left\{1 + \dfrac{h}{x}\right\} + (e^{2h} - 1)\log 3x]$ ■

Exercise 4.9.9 Compute $\Delta^2(3e^x)$

Solution : $3(e^h - 1)^2 e^x$ ■

Exercise 4.9.10 Evaluate $\Delta^4(ae^x)$

Solution : $a(e - 1)^4 e^x, h = 1$ ■

Exercise 4.9.11 Evaluate $\left(\frac{\Delta^2}{E}\right) x^2$

Solution : $2h^2$ ■

Exercise 4.9.12 Evaluate $E^n(e^x)$

Solution : e^{x+nh} ■

Exercise 4.9.13 Evaluate $\Delta^2(\cos 2x)$

Solution : $-4\sin^2 h \cos(2x+2h)$ ■

Exercise 4.9.14 Evaluate Show that $e^x = \left(\frac{\Delta^2}{E}\right) e^x \frac{Ee^x}{\Delta^2 e^x}$