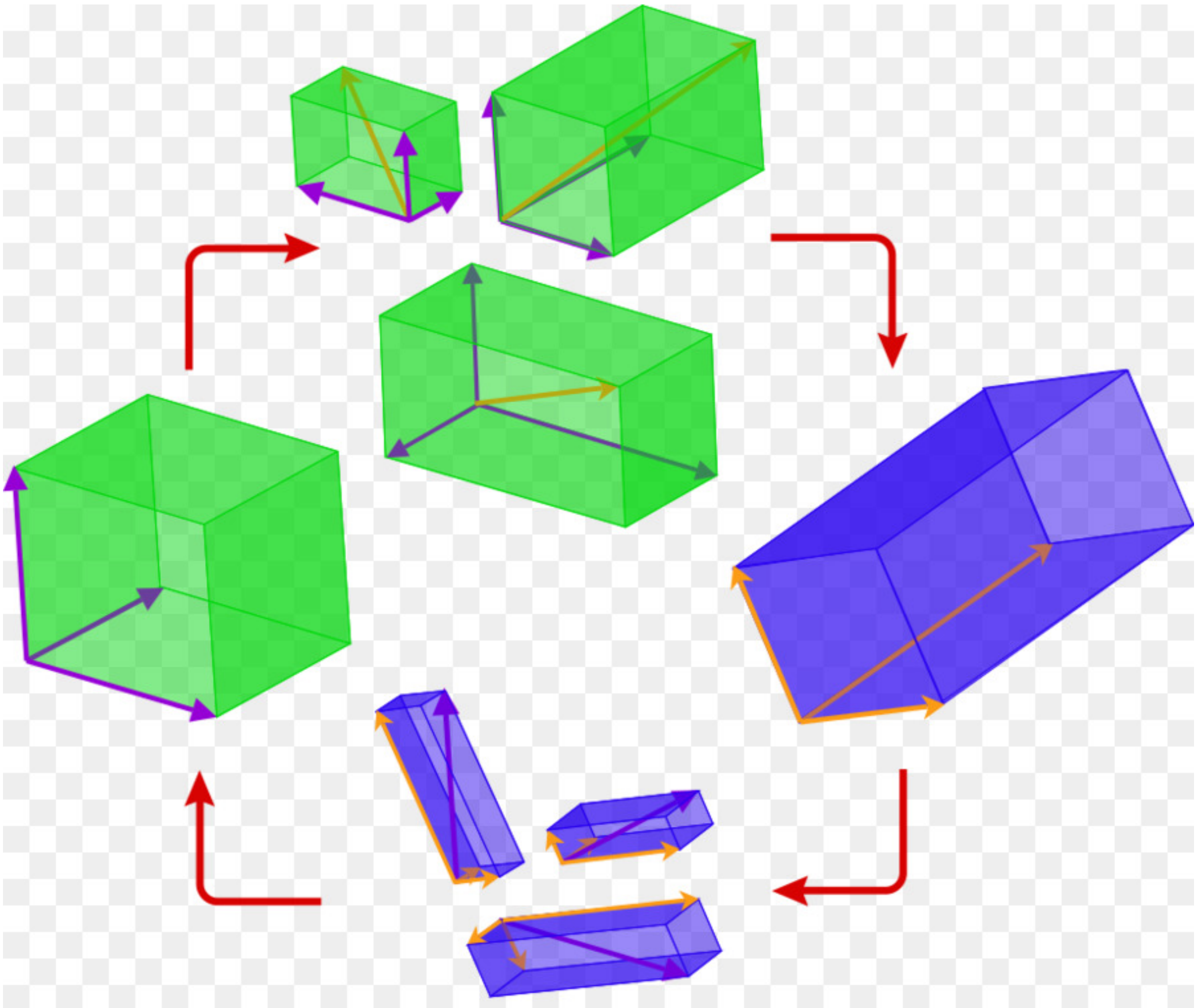
MTH2206: LINEAR ALGEBRA
EASTER SEMESTER 2025



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Chapter 1

Matrices

1.1 Definitions

Definition 1.1.1 An $m \times n$ matrix is a rectangular array of $m \times n$ numbers arranged in m horizontal rows and n vertical columns enclosed in brackets. These numbers in the matrix are called elements (entries) of the matrix. Matrices are always denoted by capital letters(block letters) while matrix entries are denoted by small letters. A matrix A can be generally written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \cdots & \\ \vdots & & \cdots & \\ \vdots & & \cdots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \quad \text{or} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \cdots & \\ \vdots & & \cdots & \\ \vdots & & \cdots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

Or more briefly (for notation) we have that $A = (a_{ij})$ or $B = (b_{ij})$ where a_{ij} (respectively b_{ij}) is an entry in the i^{th} row and j^{th} column of A (respectively B).

Definition 1.1.2 Square Matrix:
An $m \times n$ matrix A denoted by $A_{m\times n}$ or A_{mn} is said to be square if $m = n$ (that is the number of rows is equal to the number of columns).

Definition 1.1.3 Equality of Matrices:
If two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ then they are said to be equal if and only if all corresponding entries of A and B are the same that is

$$a_{ij} = b_{ij}, \ \forall \ 1 \leq i \leq m, 1 \leq j \leq n.$$

Example 1.1.1 Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$A = B \ \text{ if and only if } \ \begin{array}{ll} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{array}$$

Example 1.1.2 The following matrices are all different. Explain!

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$$

Definition 1.1.4 Matrix Transpose:
Let $A = (a_{ij})$ be an $m \times n$ matrix, then the transpose of A denoted by A^t or A^T or A' is an $n \times m$ matrix obtained from matrix A by interchanging its rows with columns.

In the (a_{ij}) notation we have that $A^T = (a_{ij})^T$ where $(a_{ij})^T = (a_{ji}) \;\; \forall \; i, j$. We can also write $A^T = (a_{ji})$.

Note 1.1.1 The transpose of matrix A is determined by interchanging the rows with columns of A that is if A is an $n \times m$ matrix then the transpose of A, denoted by A^T, is an $m \times n$ matrix that is obtained by interchanging the rows and columns of A. So, the first row of A^T is the first column of A, the second row of A^T is the second column of A, etc. Likewise, the first column of A^T is the first row of A, the second column of A^T is the second row of A, etc.

Example 1.1.3 The transpose of a general 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ is } A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Example 1.1.4 The transpose of the matrix A given by,

$$A = \begin{pmatrix} 2 & 3 & 7 \\ 5 & 6 & 8 \\ 0 & 1 & -1 \end{pmatrix} \text{ is } A^T = \begin{pmatrix} 2 & 5 & 0 \\ 3 & 6 & 1 \\ 7 & 8 & -1 \end{pmatrix}$$

Definition 1.1.5 Symmetric Matrices:
A *square* matrix A is said to be symmetric if its transpose matrix A^T is equal to A that is $A^T = A$.

Example 1.1.5 Matrix A in Example 1.1.4 is not symmetric since $A \neq A^T$.

Example 1.1.6 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$. Hence $A^T = A$, implying that A is symmetric.

Example 1.1.7 The matrix $A = \begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$ is symmetric.

Definition 1.1.6 Anti-Symmetric Matrices:
A matrix A is said to be anti-symmetric (sometimes called skew-symmetric) if $A^T = -A$.

Example 1.1.8 Let

$$B = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix} \text{ then } B^T = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Thus $B^T = -B$ and B is anti-symmetric (skew symmetric).

Remark 1.1.1 Symmetric and Skew Symmetric Matrices

1.) Given any matrix A, the matrices AA^T and A^TA are symmetric.

2.) Let A be a square matrix. The matrix $A + A^T$ is symmetric.

3.) Let A be a square matrix. The matrix $A - A^T$ is skew symmetric.

Definition 1.1.7 Identity or Unit matrix:
An identity matrix is an $(n \times n)$ *square* matrix whose leading diagonal is composed of 1's and all other off diagonal elements are zeros. An identity matrix is of the form,

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{cases} a_{ij} = 1, & \forall\ i = j \\ \\ a_{ij} = 0, & \forall\ i \neq j \end{cases}$$

usually denoted as I_n or I with a nice property that $I \cdot A = A \cdot I = A$ where A is an $n \times n$ matrix.

Example 1.1.9 The following is a 2×2 identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 1.1.10 The following is a 3×3 identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 1.1.11 The following is a 1×1 identity matrix

$$I_1 = 1$$

Example 1.1.12 The following is a 4×4 identity matrix

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 1.1.13 Check the following matrix is Identity matrix?

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution : *No, It's not an identity matrix, because it is of the order* 3×4, *which is not a square matrix.* ■

1.2 Other Special Types of Matrices

1.2.1 Diagonal Matrix

Definition 1.2.1 A *square* matrix $A = (a_{ij})$ is said to be a diagonal if

$$a_{ij} = 0 \ \forall \ i \neq j.$$

This means that off diagonal elements are all equal to zero.

Definition 1.2.2 A diagonal matrix is an $n \times n$ matrix in which the only nonzero entries lie on the diagonal.

Example 1.2.1 The matrices $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are diagonal matrices.

1.2.2 Tri-Diagonal Matrix

Definition 1.2.3 A *square* matrix $A = (a_{ij})$ is said to be a tri-diagonal if $a_{ij} = 0$ for $|i-j| \geq 2$, that is if every element in the i^{th} row and j^{th} column is zero when the absolute difference between i and j is greater or equal to two.

Example 1.2.2 A general 3×3 tri-diagonal matrix is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

Example 1.2.3 The matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \\ 0 & 1 & -1 \end{bmatrix}$ is a tri-diagonal matrix.

1.2.3 Triangular Matrix

Definition 1.2.4 A triangular matrix is a *square* matrix whose **non-zero** elements lie on the diagonal, and all the zero entries are either above or below the diagonal.

Example 1.2.4 The matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ are triangular matrices.

1.2.3.1 Upper Triangular Matrix

Definition 1.2.5 An upper triangular matrix is a type of triangular matrix $A = (a_{ij})$ where

$$a_{ij} = 0 \ \forall \ i > j.$$

Definition 1.2.6 An upper triangular matrix is a matrix in which any non-zero entries lie on or above the diagonal.

Example 1.2.5 A general $n \times n$ upper triangular matrix is of the form,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Example 1.2.6 Matrix B in Example 1.2.4 on page (p. 4) is an upper triangular matrix.

Example 1.2.7 Examples of upper triangular matrices are

$$C = \begin{bmatrix} 1 & \frac{1}{2} & 3 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 2 & 6 & 7 \\ 0 & 6 & 6 & 8 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 1.2.8 The matrix below is an upper triangular matrix

$$A = \begin{bmatrix} 6 & 1 & 4 & 3 & 2 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

1.2.3.2 Lower Triangular Matrix

Definition 1.2.7 Let $A = (a_{ij})$ be a *square* matrix then the matrix A is said to be lower triangular if

$$a_{ij} = 0, \ \forall \ i < j.$$

Definition 1.2.8 A lower triangular matrix is a matrix in which any nonzero entries lie on or below the diagonal.

Example 1.2.9 They generally take the form,

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

Example 1.2.10 See matrix A in Example 1.2.4 on page (p. 4) is a lower triangular matrix.

Example 1.2.11 Examples of lower triangular matrices are

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 5 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 9 & 7 \end{bmatrix}, \quad H = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 6 & 2 & 4 & 0 & 0 \\ 2 & 4 & 5 & 8 & 4 \end{bmatrix}$$

Example 1.2.12 The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

is neither upper nor lower Triangular matrix because it is not a square matrix.

1.2.4 Idempotent Matrix

Definition 1.2.9 A matrix $A = (a_{ij})$ is said to be idempotent if

$$A^2 = AA = A$$

Example 1.2.13 $A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$ is an idempotent matrix.

Example 1.2.14 Examples of idempotent matrices are idempotent matrices are:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 1.2.1 Show that the matrix $\begin{bmatrix} 1 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is not an idempotent matrix.

Exercise 1.2.2 Is the matrix $A = \frac{1}{2}\begin{bmatrix} 1-\cos\theta & \sin\theta \\ \sin\theta & 1+\cos\theta \end{bmatrix}$ idempotent? Justify your answer.

1.2.5 Invertible or Non-Singular Matrix

Definition 1.2.10 A *square* matrix $A = (a_{ij})$ is non-singular or invertible if and only if $\exists$ an $n \times n$ matrix denoted by A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

Note 1.2.1 I is the identity matrix, and A^{-1} will later be called the inverse of A.

Remark 1.2.1 If A^{-1} does not exist then we say that A is singular or non invertible.

Example 1.2.15 Consider the matrices A, B, C and I_4, as well as their transposes, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Identify the diagonal of each matrix, and state whether each matrix is diagonal, upper triangular, lower triangular, or none of the above.

We first compute the transpose of each matrix.

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix} \quad B^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 5 & 6 & 0 \end{bmatrix}$$

Note that $I_4^T = I_4$.

The diagonals of A and A^T are the same, consisting of the entries $1, 4$ and 6. The diagonals of B and B^T are also the same, consisting of the entries $3, 7$ and -1. Finally, the diagonals of C and C^T are the same, consisting of the entries $1, 4$ and 6.

The matrix A is upper triangular; the only nonzero entries lie on or above the diagonal. Likewise, A^T is lower triangular.

The matrix B is diagonal. By their definitions, we can also see that B is both upper and lower triangular. Likewise, I_4 is diagonal, as well as upper and lower triangular.

Finally, C is upper triangular, with C^T being lower triangular.

Remark 1.2.2 Make note of the definitions of diagonal and triangular matrices. We specify that a diagonal matrix must be square, but triangular matrices don't have to be. ("Most" of the time, however, the ones we study are.) Also, as we mentioned before in the example, by definition a diagonal matrix is also both upper and lower triangular. Finally, notice that by definition, the transpose of an upper triangular matrix is a lower triangular matrix, and vice-versa.

Example 1.2.16 For the matrix $A = \begin{bmatrix} 6 & -7 & 2 & 6 \\ 0 & -8 & -1 & 0 \\ 0 & 0 & 1 & -7 \end{bmatrix}$, A is upper triangular and A^T is lower triangular.

Example 1.2.17 The matrix $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is upper and lower triangular, it is diagonal, it is both symmetric and skew symmetric. It's got it all.

1.3 Operations on Matrices

1.3.1 Addition and Scalar Multiplication of matrices

Definition 1.3.1 Let A and B be matrices of the same size. The sum of A and B, written $A+B$ is the matrix whose ij-th entry is $a_{ij}+b_{ij}$.
Let $A=(a_{ij}),\quad B=(b_{ij})$. Then

$$C = A + B \;=\; (c_{ij}) = (a_{ij}+b_{ij}). \tag{1.1}$$

Note 1.3.1 Note that $c_{ij} = b_{ij} + a_{ij}$ and so $A+B = B+A$.

Remark 1.3.1 Also if $\underline{\underline{0}}$ is the zero matrix, then

$$\underline{\underline{0}} + A = A + \underline{\underline{0}} = A$$

for any matrix A.

Definition 1.3.2 Scalar multiplication of a matrix;
Let $A=(a_{ij})$ be an $m\times n$ matrix. Multiplying through matrix A by a scalar α, you get

$$\alpha A \;=\; \alpha(a_{ij}) = (\alpha a_{ij}). \tag{1.2}$$

This is equivalent to multiplying each entry of matrix A with the scalar α.
Alternatively: Let k be a scalar. Then kA is a matrix whose ij-entry is ka_{ij} Therefore

$$kA = (ka_{ij}).$$

Example 1.3.1 For example, $(-1)A = (-a_{ij})$ and

$$A + (-1)A = A - A = \underline{\underline{0}}$$

Example 1.3.2 Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 5 & -2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Find

1.) $A + 2B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 10 & -4 \\ 4 & 4 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 12 & -1 \\ 3 & 4 & 0 \end{pmatrix}$

2.) $3A' - B' = \begin{pmatrix} 3 & -3 \\ 6 & 0 \\ 9 & 6 \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ 5 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 1 & -2 \\ 11 & 7 \end{pmatrix}$

3.)

$$(3A-B)' \;=\; \left[\begin{pmatrix} 3 & 6 & 9 \\ -3 & 0 & 6 \end{pmatrix} - \begin{pmatrix} -1 & 5 & -2 \\ 2 & 2 & -1 \end{pmatrix}\right]' = \begin{pmatrix} 4 & 1 & 11 \\ -5 & -2 & 7 \end{pmatrix}' = \begin{pmatrix} 4 & -5 \\ 1 & -2 \\ 11 & 7 \end{pmatrix}$$

Note that $3A' - B' = (3A-B)'$.

Remark 1.3.2 Subtraction of two matrices A and B is defined using addition and scalar multiplication that is: $A - B = A + (-1)B$.

Theorem 1.3.1 Let A and B be matrices of the same size whose entries are in a field $\mathbb{F}$ and s, t be scalars. Then

$$(sA + tB)' = sA' + tB'.$$

Proof : *Let*

$$\begin{aligned} D &= sA + tB = (sa_{ij} + tb_{ij}) \\ \textit{Then}\ \ D' &= (sA + tB)' = (d_{ji}) \quad \textit{where} \\ d_{ji} &= sa_{ji} + tb_{ji} \\ \textit{recall that}\ A' &= (a_{ji})\ \textit{and}\ B' = (b_{ji}) \\ \textit{Then}\quad sA' + tB' &= (sa_{ji} + tb_{ji}) = d_{ji} \end{aligned}$$

Thus the ji-entry of D' is equal to the ji-entry of $SA'+tB'$ for all i and j. Therefore $D' = (SA + tB)' = SA' + tB'$. ■

Exercise 1.3.1 Solve the following matrix equation for a, b, c and d.

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

$$a = 5, b = -3, \quad c = 4, \quad d = 1$$

Example 1.3.3 Let $A = \begin{pmatrix} 1 & -3 \\ 2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 4 \\ -5 & 8 \end{pmatrix}$. Find $A + B + (A + B)'$

Note that $A + A' = \begin{pmatrix} 2 & -1 \\ -1 & 12 \end{pmatrix}$ and $B + B' = \begin{pmatrix} 14 & -1 \\ -1 & 16 \end{pmatrix}$
So that

$$A + B + (A + B)' = A + A' + B + B' = \begin{pmatrix} 16 & -2 \\ -2 & 28 \end{pmatrix}$$

Alternatively use

$$\begin{aligned} A + B &= \begin{pmatrix} 8 & 1 \\ -3 & 14 \end{pmatrix} \\ \Rightarrow\quad (A + B) + (A + B') &= \begin{pmatrix} 8 & 1 \\ -3 & 14 \end{pmatrix} + \begin{pmatrix} 8 & -3 \\ 1 & 14 \end{pmatrix} \\ &= \begin{pmatrix} 16 & -2 \\ -2 & 28 \end{pmatrix} \end{aligned}$$

Note that, the sums $A + A'$, $B + B'$ are symmetric matrices.

Theorem 1.3.2 Let A be a square matrix. Then $A + A'$ is symmetric.

Proof :

$$\begin{aligned} \textit{Let}\ \ A &= (a_{ij}). \quad \textit{Then}\ \ A' = (a_{ji}) \\ \textit{Now}\ \ A + A' &= (a_{ij} + a_{ji}) \end{aligned}$$

Clearly

$$a_{ij} + a_{ji} = b_{ji} = a_{ji} + a_{ji} \quad \forall\ i, j$$

it follows that $A + A'$ is symmetric. ■

Theorem 1.3.3 Let A, B and C be matrices and s and t scalars in a field $\mathbb{F}$. Then

1) $A + B = B + A$ (Addition of matrices obeys the commutativity law)

2) $(A + B) + C = A + (B + C)$ (Addition of matrices satisfies the Associative law).

Proof : *Let $A = (a_{ij})$ and $B = (b_{ij}), C = (c_{ij})$. Then by definition*

$$\begin{aligned} A + B &= (a_{ij} + b_{ij}) \\ (A + B) + C &= ((a_{ij} + b_{ij}) + c_{ij}) \end{aligned}$$

By associativity of scalars, $(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$

$$\begin{aligned} \textit{So} \quad (A + B) + C &= (a_{ij} + (b_{ij} + c_{ij})) \\ &= (a_{ij}) + (b_{ij} + c_{ij}) \\ &= A + (B + C) \end{aligned}$$

■

3) $A + 0 = A$ (Additive identity)

4) $A + (-A) = 0$ (Additive inverse)

5) $(st)A = s(tA)$ (commutative law)

6) $(s + t)A = sA + tA$ (distributive law)

7) $t(A + B) = tA + tB$ (distributive law)

8) $1 \cdot A = A$ (1 is a multiplicative identity)

Exercise 1.3.2 Prove Theorem 1.3.3

1.3.2 Multiplication of Matrix by Matrix

Definition 1.3.3 The product $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $A = [a_{jk}]$ times an $r \times p$ matrix $B = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $C = [c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + a_{j3}b_{3k} + \cdots + a_{jn}b_{nk} \qquad \begin{aligned} j &= 1, \cdots, m \\ k &= 1, \cdots, p. \end{aligned} \tag{1.3}$$

Example 1.3.4 (Please verify)

$$\begin{aligned} \text{Let} \quad A &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ \text{Then} \quad AB &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix} \\ \text{and} \quad BB' &= \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix} \end{aligned}$$

Example 1.3.5 *Matrix Multiplication*

$$\text{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Here $c_{11} = 3(2)+5(5)+(-1)(9) = 22$ and so on. The entry in the box is $c_{23} = 4(3)+0(7)+2(1) = 14$. The product **BA** is not defined.

Example 1.3.6 *Multiplication of a Matrix and a Vector*

1.)

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4(3) + 2(5) \\ 1(3) + 8(5) \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$$

2.) Whereas

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}$$

is undefined.

Example 1.3.7 *Products of Row and Column Vectors*

1.)

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [19]$$

2.)

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}.$$

Example 1.3.8

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

It is interesting that this also shows that $AB = 0$ does *not* necessarily imply $BA = 0$ or $A = 0$ or $B = 0$.

Example 1.3.9 *Computing Products Columnwise*
To obtain

$$\text{AB} = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

Calculate the columns

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

of AB and then write them as a single matrix, as shown in the first formula on the right.

Example 1.3.10 Let $A = \begin{bmatrix} 3+2i & 0 \\ -i & 2 \\ 1+i & 1-i \end{bmatrix}$, $B = \begin{bmatrix} -i & 2 \\ 0 & i \end{bmatrix}$ and $C = \begin{bmatrix} -1-i & 0 & -i \\ 3 & 2i & -5 \end{bmatrix}$.
Then

$$\begin{aligned} BC &= \begin{bmatrix} 5+i & 4i-11 \\ 3i-2 & -5i \end{bmatrix} \\ CA &= \begin{bmatrix} -23-25i & -1-i \\ 69-5i & -5+9i \end{bmatrix} \\ \text{and } \ (1+i)AB+(3-4i)C' &= \begin{bmatrix} 25-7i & 57+36i \\ -1-i & -8-6i \\ 6+3i & -15+26i \end{bmatrix} \end{aligned}$$

Theorem 1.3.4 Let A, B and C be conformable matrices over a field $\mathbb{F}$ and k a scalar. Then

1) $(AB)C = A(BC)$ (Associative property)

Proof : *Let $D = AB$ and $E = BC$*
Then by definition,

$$\begin{aligned} D &= (d_{ij}) \quad \textit{where} \ \ d_{ij} = \sum_k a_{ik} b_{kj} \\ \textit{and} \ \ E &= (e_{ij}) \quad \textit{where} \ \ e_{ij} = \sum_{k'} b_{ik'} c_{k'j} \\ \textit{Thus} \ \ F &= (f_{ij}) = DC = (AB)C \\ \textit{where} \ \ f_{ij} &= \sum_k d_{ik} c_{kj} = \sum_k \left[\sum_{i'} a_{ii'} b_{i'k} \right] c_{kj} \\ &= \sum_i a_{ii'} \sum_k b_{i'k} C_{kj} \\ &= \sum_i a_{ii'} e_{i'j} \\ \textit{Therefore} \ \ F &= (f_{ij}) = AE = A(BC) \end{aligned}$$

■

2) $A(B+C) = AB + AC$ (Distributive property)

3) $(B+C)A = BA + CA$

4) $k(AB) = (kA)B = A(kB)$

Remark 1.3.3 For $AB = \mathbf{0}$, a zero matrix, does not mean that $A = \mathbf{0}$ or $B = \mathbf{0}$ or both.

Example 1.3.11 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Then

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & -1 \end{bmatrix} \quad \text{and} \\ BA &= \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix} \end{aligned}$$

Note 1.3.2 In general, Matrix multiplication is not commutative; that is, $AB \neq BA$. In general, just because $AX = BX$, we cannot conclude that $A = B$. The commutativity $AB = BA$ is possible *only if* when the matrices are equal or when the matrices A and B commute.

Example 1.3.12 For conformable matrices A and B, consider

$$(A - B)(A + B) = A^2 - B^2 + AB - BA$$

Then

$$(A - B)(A + B) \quad = \quad A^2 - B^2 \tag{1.4}$$

if and only if A and B commute thus $AB = BA$. Note that the equality is always true for scalars.

Exercise 1.3.3 Let $X = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \Rightarrow XY' = \begin{bmatrix} 25 & 1 & 19 \\ 18 & 2 & 31 \end{bmatrix}$.

Find directly $Y'X$ and compare it with XY' above.

Definition 1.3.4 Let A be a square matrix. The powers of A are defined as

$$A^0 = I, \quad A^1 = A, \quad A^2 = AA, \quad A^2A, \ldots$$

Let $f(x)$ be a polynomial. Then

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \\ \text{and} \quad f(A) &= a_0I + a_1A + a_2A^2 + \ldots + a_nA^n \end{aligned}$$

If $f(A) = 0$, A is called the root or zero of the polynomial $f(x)$.

Example 1.3.13 Let $A = \begin{bmatrix} 2 & 2 \\ 3 & -1 \end{bmatrix}$ and $f(x) = x^2 - x - 8$. Then $A^2 = \begin{bmatrix} 10 & 2 \\ 3 & 7 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 26 & 18 \\ 27 & -1 \end{bmatrix}$. So

$$f(A) \quad = \quad A^2 - A - 8I = \begin{bmatrix} 10 & 2 \\ 3 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore A is the zero of $f(x)$.

Example 1.3.14 Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$. Find the matrix X such that $2A + 3X = -4A$..

We can use basic algebra techniques to manipulate this equation

$$X = -2A = \begin{bmatrix} -4 & 2 \\ -6 & -12 \end{bmatrix}.$$

1.3.3 Trace of a Square Matrix

Definition 1.3.5 Let A be an $n \times n$ square matrix, the trace of A denoted by $tr(A)$ is the sum of all diagonal elements of A. That is, trace$(A) = \sum_{i=1}^{n} a_{ii}$

Example 1.3.15 Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

Then $AA' = \begin{bmatrix} 5 & 1 \\ 1 & 26 \end{bmatrix} \Rightarrow tr(AA') = 31$, $A'A = \begin{bmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{bmatrix} \Rightarrow tr(A'A) = 31$

Theorem 1.3.5 Let A and B be conformable matrices. Then $tr(AB) = tr(BA)$.

$$\begin{aligned}
\text{Let } C &= AB = (C_{ij}) \text{ where } C_{ij} = \sum_k a_{ik}b_{kj} \\
\text{Then } tr(C) &= tr(AB) = \sum_i C_{ii} = \sum_i \sum_k a_{ik}b_{ki} \\
\text{Similarly let } D &= BA \\
\text{Then } tr(D) &= \sum_i \sum_k b_{ik}a_{ki} \quad \text{(why?)} = \sum_i \sum_k a_{ki}b_{ik} \quad \text{(Why?)} = tr(C) \\
\text{Thus } tr(AB) &= tr(BA)
\end{aligned}$$

1.4 Properties of Matrices Operations

1.4.1 Properties of Matrix Addition and Scalar Multiplication

1. $A + B = B + A$. i.e Matrix addition is commutative
2. $A + (B + C) = (A + B) + C$. i.e Associativity of matrix addition
3. $A + \underline{\underline{0}} = \underline{\underline{0}} + A = A$. where $\underline{\underline{0}}$ is a zero matrix, a matrix with zero entries.
4. $A + (-A) = \underline{\underline{0}}$ where $-A = (-1)A$

1.4.2 Properties of Matrix Multiplication

5. $A(BC) = (AB)C$. Associativity of matrix multiplication.
6. $A(B + C) = AB + AC$. Distributivity from the left.
7. $(A + B)C = AC + BC$. Distributivity from the right.
8. $\alpha(\beta A) = \alpha\beta(A)$. where α, β are scalars.
9. $\alpha(A + B) = \alpha A + \alpha B$.
10. $A(\alpha B) = \alpha(AB) = (\alpha A)B$.

1.4.3 Properties of Matrix Transpose

11. $(A^T)^T = A$.
12. $(A + B)^T = A^T + B^T$.
13. $(AB)^T = B^T A^T$.
14. $(\alpha A)^T = \alpha A^T$.

1.4.4 Properties of the Matrix Trace

Let A and B be $n \times n$ matrices. Then:

15. $\mathrm{tr}(A \pm B) = \mathrm{tr(A)} \pm \mathrm{tr(B)}$
16. $\mathrm{tr}(kA) = k\cdot\, \mathrm{tr(A)}$
17. $\mathrm{tr(AB)} = \mathrm{tr(BA)}$
18. $\mathrm{tr}(A^T) = \mathrm{tr(A)}$

1.5 Determinants

The determinant of an $n \times n$ matrix is the signed volume spanned by its column vectors. To compute the determinant of any square matrix, one can use any of the two methods, namely

1.) Permutations - Inversion technique/method and the

2.) Cofactor method.

1.5.1 Permutations and Inversion of a Permutation

Definition 1.5.1 Permutation
Let $S = \{1, 2, \ldots, n\}$ be a set containing the first n natural numbers. An ordered arrangement $< i_1 i_2 \ldots i_n >$ of elements of set S is called a permutation of S.

Example 1.5.1 The permutations for $S = \{1, 2, 3\}$ are

$$< 123 >, \; < 132 >, \; < 213 >, \; < 231 >, \; < 312 >, \text{ and } < 321 >$$

Exercise 1.5.1 Write down the permutations for $S = \{1, 2, 3, 4\}$

Note 1.5.1 A set of n elements has $n!$ permutations.

Note 1.5.2 We denote the set of all permutations of set S by S_n where n is the number of elements in the set S, and thus

$$\begin{aligned} S_1 &= \{< 1 >\} \\ S_2 &= \{< 12 > \; < 21 >\} \\ S_3 &= \{< 123 >< 132 >< 213 > \; < 231 >< 312 >< 321 >\} \end{aligned}$$

Definition 1.5.2 Inversion of Permutation

A permutation $< i_1 i_2 \ldots i_n >$ of set S is said to have an inversion if a larger integer i_t precedes (comes before) a smaller integer i_s. For example $< 12 >$ has no inversion, $< 21 >$ has one inversion because a larger number 2 comes before 1, $< 321 >$ has three inversions, $< 2, 3, 1 >$ has two inversions, and $< 1, 2, 3 >$ has no inversions.

Definition 1.5.3 Even and Odd Permutation

A permutation is even or odd depending on the total number of inversions, either even or odd.(*Here we do consider zero to be even*).

Example 1.5.2 The $< 1, 2 >$ is an even permutation since it has no inversion, where as $< 1, 3, 2 >$ is an odd permutation since it has one inversion.

Definition 1.5.4 Determinant
If $A = (a_{ij})$ is an $n \times n$ matrix, the determinant of A denoted as $|A|$ or $det(A)$ is defined by

$$|A| = \sum_{\sigma} \pm a_{1i_1} a_{2i_2} \ldots a_{ni_n}$$

where the σ denotes all permutations $< i_1 \; i_2 \ldots i_n >$ in the set $S = \{1, 2, \ldots, n\}$ (i.e the i_j to be substituted in the formula should be [in their order] got from each permutation).

Note 1.5.3 Also that the + sign in the summation is taken when permutation is even, or − when the permutation is odd.

Example 1.5.3 For $n = 2$ with $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $S_2 = \{< 1, 2 > < 2, 1 >\}$ the determinant of A is

$$|A| = \Sigma \pm a_{1i_1} a_{2i_2} = +a_{11}a_{22} - a_{12}a_{21} \tag{1.5}$$

Example 1.5.4 For $n = 3$, where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and

$$S_3 = \{< 123 > < 132 > < 213 > < 231 > < 312 >< 321 >\}, \quad \text{then}$$

$$\begin{aligned} |A| &= \Sigma \pm a_{1i_1} a_{2i_2} a_{3i_3} \\ &= +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned} \tag{1.6}$$

Example 1.5.5 The determinant of $A = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ is

$$|A| = +a_{11}a_{22} - a_{12}a_{21} = (1)(7) - (2)(3) = 1.$$

Example 1.5.6 Given $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 3 & 3 & 4 \end{pmatrix}$ compute $|A|$.

Solution

$$\begin{aligned} = |A| &= \Sigma \pm a_{1i_1} a_{2i_2} a_{3i_3} \\ &= +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ \Rightarrow |A| &= (1)(0)(4) - (1)(1)(3) - (2)(3)(4) + (2)(1)(3) + (0)(3)(3) - (0)(0)(3) = -21 \end{aligned}$$

1.5.2 Matrix Adjoint, Minors and Cofactors

Definition 1.5.5 If $A = (a_{ij})$ is an $n \times n$ then M_{ij} will denote $(n-1) \times (n-1)$ matrix obtained from A by deleting its i^{th} row and j^{th} column.
Its determinant which we denote by $|M_{ij}|$ is called the **minor** of the element a_{ij} of A.

Definition 1.5.6 The **Cofactor** of a_{ij} denoted by C_{ij} is given by

$$C_{ij} = (-1)^{i+j} |M_{ij}|$$

Example 1.5.7 Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}$. Find all the cofactors of the matrix A.

$$\begin{aligned} C_{11} &= (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = (1)(4) = 4 \\ C_{12} &= (-1)^{1+2} \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = (-1)(4) = -4 \end{aligned}$$

$$\begin{aligned}
C_{13} &= (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} = (1)(-4) = -4 \\
C_{21} &= (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = (-1)(4) = -4 \\
C_{22} &= (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = (1)(-4) = -4 \\
C_{23} &= (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = (-1)(-4) = 4 \\
C_{31} &= (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = (1)(-4) = -4 \\
C_{32} &= (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = (-1)(8) = 8 \\
C_{33} &= (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = (1)(-4) = -4
\end{aligned}$$

Then the cofactor matrix of A is

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 4 & -4 & -4 \\ -4 & -4 & 4 \\ -4 & 8 & -4 \end{pmatrix}$$

Definition 1.5.7 Determinant
Suppose that $A = (a_{ij})$ is an $n \times n$ matrix, and let C_{ij} denote the cofactor of the element a_{ij} with $i, j = 1, 2, \ldots, n$ then

(1)

$$|A| = \sum_{k=1}^{n} a_{k_j} C_{kj}$$

That is summing along the j^{th} column.

(2)

$$|A| = \sum_{k=1}^{n} a_{ik} C_{ik}$$

That is summing along the i^{th} row.

Example 1.5.8 Compute the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}$.

But we know (Example 1.5.7) that the cofactor matrix of A is $\begin{pmatrix} 4 & -4 & -4 \\ -4 & -4 & 4 \\ -4 & 8 & -4 \end{pmatrix}$ Summing along the

$$\begin{aligned}
1^{st} \text{ column } \Rightarrow |A| &= (1)(4) + (3)(-4) + (2)(-4) = -16 \\
2^{nd} \text{ column } \Rightarrow |A| &= (2)(-4) + (2)(-4) + (0)(8) = -16 \\
3^{rd} \text{ column } \Rightarrow |A| &= (3)(-4) + (1)(-4) + (2)(-4) = -16 \\
1^{st} \text{ row } \Rightarrow |A| &= (1)(4) + (2)(-4) + (3)(-4) = -16 \\
2^{nd} \text{ row } \Rightarrow |A| &= (3)(-4) + (2)(-4) + (1)(4) = -16 \\
3^{rd} \text{ row } \Rightarrow |A| &= (2)(-4) + (0)(8) + (2)(-4) = -16
\end{aligned}$$

Same value of determinant no matter which row or column you consider. But preferably a row or column with more zeros is better.

Definition 1.5.8 Adjoint
The transpose of the cofactor matrix of A is the **adjoint** of the matrix A. It is usually denoted by **adj**(A). For $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, the Cofactor Matrix is $\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & a_{22} & \dots & C_{2n} \\ . & & & \\ . & & & \\ . & & & \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$

Thus $\quad Adj(A) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ . & & & \\ . & & & \\ . & & & \\ C_{1n} & C_{2n} & \cdot & C_{nn} \end{bmatrix}$

Example 1.5.9 Compute the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$. The cofactor matrix of A is $\begin{bmatrix} 4 & -4 & -4 \\ -4 & -4 & 4 \\ -4 & 8 & -4 \end{bmatrix} \Rightarrow adj(A) = \begin{bmatrix} 4 & -4 & -4 \\ -4 & -4 & 8 \\ -4 & 4 & -4 \end{bmatrix}$

1.5.3 Properties of Determinants

1) $|A^T| = |A|$.

2) Interchanging two rows or columns in a matrix gives the negative (determinant) of the previous matrix.

3) If two rows or columns of a matrix are equal, then its determinant is equal to zero.

4) If any row or column in a matrix A is zero, then $|A| = 0$.

5) If any row or column is a constant multiple of another row or column, then $|A| = 0$.

6) A scalar k multiplying through a row or column of a matrix gives the determinant as $k|A|$.

Proof : *Let C_{ij} be the cofactor of a_{ij}. Then expanding by the first row, we have*

$$\begin{aligned} \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13} \\ &= k\left[a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\right] \\ &= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

Property (6) also states that a factor common to all elements of a row (or column) can be taken out as a factor of the determinant. ■

In general, a scalar α multiplying through all rows or all columns $|\alpha A| = \alpha^n |A|$

7) The value of the determinant remains unchanged if any row or column is replaced by a linear combination of any two rows or columns.

8) The determinant of a triangular or diagonal matrix is given by the product of its diagonal elements.

9) $|AB| = |A||B|$

10) $|A^{-1}| = \dfrac{1}{|A|}$; $\quad |A| \neq 0$ & A^{-1} is called the inverse of A.

11) Let B be obtained from A by

1.) multiplying a row (column) of A by a scalar k; then $|B| = k|A|$.

Proof :
If $R_i \leftarrow kR_j$ on A
Then every term in $|A|$ is multiplied by k is

$$\begin{aligned}
|B| &= \sum_{\sigma}(sgn(\sigma))a_{1i_1}a_{2i_2}\ldots(ka_{ji_j})\ldots a_{ni_n} \\
&= k\sum_{\sigma}(sgn(\sigma))a_{1i_1}a_{2i_2}\ldots a_{ji_j}\ldots a_{ni_1} \\
&= k|A|.
\end{aligned}$$

■

2.) adding a multiple of a row (column) of A to another; then $|B| = |A|$

$R_j \leftarrow R_j + cR_k$

$$\begin{aligned}
|B| &= \sum_{\sigma}(sgn(\sigma))a_{1\sigma(1)}a_{2\sigma(2)}\ldots(ca_{k\sigma(k)} + a_{j\sigma(j)})\ldots a_{n\sigma(n)} \\
&= c\sum_{\sigma} sgn(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\ldots\frac{a_m(\sigma_k)}{j^{th}}\ldots a_n\sigma(n) \\
& \quad \text{because } j^{th} \text{ row} + k^{th} \text{ row are the same.} \\
& \quad +\sum_{\sigma} sgna_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{n\sigma(n)}
\end{aligned}$$

3.) interchanging two rows (column) of A then $|B| = -|A|$

Proof : *Omitted but results can be demonstrated in case of* 3×3 *and* 4×4 *matrix* ■

Exercise 1.5.2 Compute the determinants of

1.) $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 3 & 3 & 4 \end{bmatrix}$ 2.) $B = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \\ 3 & 3 & 4 \end{bmatrix}$ 3.) $C = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 3 & 4 \end{bmatrix}$

$|A| = -21, |B| = 21$, and $|C| = -21$. Check and explain why these answers (Property 2).

Exercise 1.5.3 Using the properties of determinants explain why determinants of

1.) $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & -2 & 4 \end{pmatrix} = 0$

2.) $\begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} = -24$

3.) $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{pmatrix} = 27$

4.) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 24$

Example 1.5.10 Use the method of permutation to compute $|A|$

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix}$$

$$\begin{aligned} |A| &= +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= (-2)(5)(2) - (-2)(-7)(6) + (1)(-7)(1) - (1)(3)(2) + (4)(3)(6) - (4)(5)(1) \\ &= -137 + 72 = -65. \end{aligned}$$

Example 1.5.11 Compute the determinant using "linear combination of rows or columns" property.

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} \begin{array}{rcr} R_2 & \leftarrow & 2R_1 + R_2 \\ R_3 & \leftarrow & -5R_1 + R_3 \\ & = & \end{array} \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \begin{array}{rcl} R_3 & \leftarrow & \frac{13}{2}R_2 + R_3 \\ & = & \end{array}$$

$$= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \dfrac{17}{2} \end{vmatrix} = -17 \text{ triangular matrix, or factor scalar third row } \frac{1}{2}\begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 17 \end{vmatrix} = -17$$

Remark 1.5.1 For Gauss-Jordan row reduction (operation), any scalar multiplication is with the pivot.

Example 1.5.12 Compute the determinant using using "linear combination of rows or columns" property.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \begin{array}{rcc} R_1 & \leftarrow & R_1 \\ R_2 & \leftarrow & -\dfrac{1}{2}R_1 + R_2 \\ R_3 & \leftarrow & R_3 \\ R_4 & \leftarrow & R_4 \\ & = & \end{array} \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -\dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{2} \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \begin{array}{rcc} R_1 & \leftarrow & R_1 \\ R_2 & \leftarrow & R_2 \\ R_3 & \leftarrow & 4R_2 + R_3 \\ R_4 & \leftarrow & 2R_2 + R_4 \\ & = & \end{array}$$

$$= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -\dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{2} \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix} \begin{array}{rcr} R_1 & \leftarrow & R_1 \\ R_2 & \leftarrow & R_2 \\ R_3 & \leftarrow & R_3 \\ R_4 & \leftarrow & R_3 + R_4 \\ & = & \end{array} = \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -\dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{2} \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix} = 6 \quad \text{(why?)}$$

Example 1.5.13 We can combine Gauss-Jordan row-reduction and cofactor expansion to calculate determinants of large matrices.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$

expanding along column 1

$$\text{Now } 2\begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} \quad \text{(2 column interchange)}$$

$$= 2\left\{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\right\}$$

expanding along column 1

$$= 2(3-6) = -6.$$

$$\text{Also } \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 0 & -5 & -2 \\ 0 & -1 & 2 \end{vmatrix} \quad \text{by row reduction}$$

$$= \begin{vmatrix} -5 & -2 \\ -1 & 2 \end{vmatrix} \quad \text{expanding along column 1}$$

$$= -12$$

Therefore the required determinant $= -6 + 12 = 6$.

Example 1.5.14 Given that $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{bmatrix}$. Find the adj (A).

Since cofactors of A are

$$\begin{aligned}
C_{11} &= (-1)^{1+1}\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -11 \\
C_{12} &= (-1)^{1+2}\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix} = 29 \\
C_{13} &= (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 1 \\
C_{21} &= (-1)^{2+1}\begin{vmatrix} 0 & -2 \\ 2 & -3 \end{vmatrix} = -4 \\
C_{22} &= (-1)^{2+2}\begin{vmatrix} 1 & -2 \\ 5 & -3 \end{vmatrix} = 7 \\
C_{23} &= (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -2 \\
C_{31} &= (-1)^{3+1}\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix} = 2 \\
C_{32} &= (-1)^{3+2}\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = -10 \\
C_{33} &= (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1
\end{aligned}$$

The Matrix of cofactors is

$$\begin{bmatrix} -11 & 29 & 1 \\ -4 & 7 & -2 \\ 2 & -10 & 1 \end{bmatrix}$$

and thus the adjoint of A is

$$adj(A) = \begin{bmatrix} -11 & -4 & 2 \\ 29 & 7 & -10 \\ 1 & -2 & 1 \end{bmatrix}$$

The Matrix adjoint is useful in finding the inverse of a non-singular matrix.

Example 1.5.15 Given the matrix

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

Using the Permutation-inversion technique, and the cofactor minor technique.

The cofactors of A are

$$C_{11} = (-1)^2 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6 \qquad C_{21} = (-1)^3 \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = -6 \qquad C_{31} = (-1)^4 \begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} = -3$$

$$C_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0 \qquad C_{22} = (-1)^4 \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 4 \qquad C_{32} = (-1)^5 \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = -2$$

$$C_{13} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6 \qquad C_{23} = (-1)^5 \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} = 6 \qquad C_{33} = (-1)^6 \begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} = 9$$

The Matrix of cofactors is

$$\begin{bmatrix} 6 & 0 & -6 \\ -6 & 4 & 6 \\ -3 & -2 & 9 \end{bmatrix}$$

The determinant is given by

$$|A| = 4(6) + 3(0) + 2(-6) = 12$$

Example 1.5.16 Using properties of determinants, state the determinants of the following matrices

1.) $A = \begin{bmatrix} 2 & 1 & 6 \\ 7 & 4 & 2 \end{bmatrix}$: $|A|$ Does not exist (DNE) 0r is undefined because the matrix is not square.

2.) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$: $|A| = 24$

3.) $A = \begin{bmatrix} 9 & 8 & 6 \\ 7 & 4 & 0 \\ 9 & 8 & 6 \end{bmatrix}$: $|A| = 0$ because two of the rows of the matrix A are equal.

1.6 Matrix Inverses

Definition 1.6.1 Inverse of a matrix
The matrix B is said to be the inverse of matrix A if

$$AB = BA = I \tag{1.7}$$

I the identity matrix. We denote the inverse of A by A^{-1}.

Note 1.6.1 If matrix A has an inverse we say that A is invertible.

Theorem 1.6.1 If A is invertible, then its inverse is unique.

Proof : *Assume A is invertible. Suppose, by way of contradiction, that the inverse of A is not unique, i.e., let B and C be two distinct inverses of A.*

Then, by definition of inverse, we have

$$BA = \quad I \quad = AB \tag{1.8}$$

and

$$CA = \quad I \quad = AC \tag{1.9}$$

It follows that

$$B = BI$$

by definition of identity matrix,

$$B = B(AC)$$

by (1.9) above,

$$B = (BA)C$$

by associativity of matrix multiplication,

$$B = IC$$

by (1.8) above, and

$$B = C$$

by definition of identity matrix. Thus,

$$B = C$$

which contradicts the previous assumption that $B \neq C$. So it must be that case that the inverse of A is unique. ■

1.6.1 Inverse Methodology

The inverse of a matrix could be determined by any of the following techniques

(1) Direct method

(2) Adjoint method

(3) Method of elementary row operations.

1.6.2 Direct Method

Example 1.6.1 Given a matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, and if $A^{-1} = B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find the inverse B.

From definition of an inverse, equation (1.7)

$$\begin{aligned} AB &= I \\ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

To have

$$a = \frac{1}{3},\ b = -\frac{2}{3},\ c = \frac{1}{3},\ d = \frac{1}{3} \Rightarrow \quad B = A^{-1} = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

Note 1.6.2 *This method beomes increasingly difficult to use as the number of unknowns increase with the increasing order of the matrix which makes it not preferable to other methods.*

1.6.3 Adjoint Method

Theorem 1.6.2

$$adj(A).A = det(A)I$$

where I the identity matrix and $|A| \neq 0$.

Corollary 1.6.1 If $|A| \neq 0$ then

$$A^{-1} = \frac{1}{|A|} \cdot adj A \tag{1.10}$$

Proof : *From Theorem 1.6.2, $adj(A).A = det(A)I$. Dividing through by $|A|$,*

$$\frac{adj(A)}{|A|}A = I$$

Since $|A| \neq 0 \Rightarrow \quad A^{-1}$ exists. Multiplying through by A^{-1} we get

$$\frac{adj(A)}{|A|}AA^{-1} = IA^{-1} \quad \Rightarrow \quad \frac{adj(A)}{|A|} = IA^{-1} = A^{-1} \Rightarrow \quad A^{-1} = \frac{adj A}{|A|}$$

for $|A| \neq 0$ ■

Remark 1.6.1 From Corollary (1.6.1), equation (1.10), A^{-1} exists if and only if

1) A is a square matrix

2) $|A| \neq 0$

Example 1.6.2 Find the inverse of $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

The co-factors are given by

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = -1$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = -5$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = -3$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = 3$$

The cofactor matrix is

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -5 \\ 1 & -3 & 3 \end{bmatrix} \quad \Rightarrow \quad adj(A) = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & -3 \\ 1 & -5 & 3 \end{bmatrix}$$

With

$$|A| = (3)(-1) + (0)(1) + (-1)(1) = -4$$

Then

$$A^{-1} = \frac{adj A}{|A|} = \frac{1}{-4} \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & -3 \\ 1 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 3/4 \\ -1/4 & 5/4 & -3/4 \end{bmatrix}$$

Theorem 1.6.3 An $n \times n$ matrix is non singular if and only if $|A| \neq 0$.

Proof : *Suppose that A is non singular, then A^{-1} exists such that*

$$AA^{-1} = A^{-1}A = I$$

But

$$|AA^{-1}| = |A^{-1}||A| = |I| = 1$$

Then $|A| \neq 0$. ■

Example 1.6.3 Find the inverse of

$$A = \begin{bmatrix} 4 & -2 & 2 \\ 8 & 2 & -6 \\ 4 & -2 & 6 \end{bmatrix}$$

$$\left\{ |A| = 96 \ , \ A^{-1} = \frac{1}{96} \begin{bmatrix} 0 & 8 & 8 \\ -72 & 16 & 40 \\ -24 & 0 & 24 \end{bmatrix} \right\}$$

1.6.4 Method of Elementary Row Operation(Gauss-Jordan Elimination Method)

Definition 1.6.2 An elementary row operation on $A = (a_{ij})$ is anyone of the following :

1.) Interchanging any two rows of a matrix.

2.) Multiplying any row of A by a non zero constant.

3.) Replacing any row by a linear combination of the **row itself** and any other row of A.

Remark 1.6.2 Linear combination of rows involve summing and subtraction of rows. But *not* their product or quotient.

1.6.4.1 Process of computing the inverse using the elementary row operations.

Key Idea 1.1 For A an $n \times n$ matrix

1). Form the $n \times 2n$ matrix. i.e, $(A : I_n)$.

2). Apply elementary row operations to $(A : I_n)$.

3). Reduce $(A : I_n)$ to a matrix of the form $(I_n : B)$, then B will be the inverse matrix A.

Example 1.6.4 Compute A^{-1} using the elementary row operation method, given that

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

The $n \times 2n$ matrix $(A : I)$ is given by

$$(A : I) = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now the aim is to shift the identity matrix to side of A.

1.) Gauss-Jordan Elimination (Coefficients $-\dfrac{a_{ij}}{a_{ii}}$ multiplied on only the pivot):

$$(A : I) = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

- We begin with first column (**whole**) and we need $a_{21} = 0, a_{31} = 0$ [Operations for first column we apply the pivot in R_1].
$R_1 \to R_1, \quad R_2 \to R_2, \quad \dfrac{1}{3}R_1 + R_3 \to R_3$

$$(A : I) = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 1 \end{array}\right]$$

•In second column, have to make $a_{12}=0, a_{32}=0$
[But when looking for operations for second column, we only apply the pivot inR_2]
Let $-2R_2+R_1 \to R_1, \quad R_2 \to R_2, \quad -\frac{5}{3}R_2+R_3 \to R_3$

$$(A:I)=\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{4}{3} & \frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right]$$

•In third column, have to make $a_{13}=0, a_{23}=0$
[But when looking for operations for third column, only use the pivot in R_3]
Let $-\frac{3}{4}R_3+R_1 \to R_1, \quad \frac{3}{4}R_3+R_2 \to R_2, \quad R_3 \to R_3$

$$(A:I)=\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & \frac{3}{4} & -\frac{3}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & -\frac{4}{3} & \frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right]$$

• To have (I,B) by $\frac{1}{3}R_1 \to R_1,\ R_2 \to R_2,\ \frac{3}{4}R_3 \to R_3$ to have

$$(I:B)=\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/12 & -3/12 & -3/12 \\ 0 & 1 & 0 & 1/4 & -1/4 & 3/4 \\ 0 & 0 & 1 & -1/4 & 5/4 & -3/4 \end{array}\right]$$

$$\Rightarrow \quad B=A^{-1}=\begin{bmatrix} 3/12 & -3/12 & -3/12 \\ 1/4 & -1/4 & 3/4 \\ -1/4 & 5/4 & -3/4 \end{bmatrix}=\begin{bmatrix} 1/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 3/4 \\ -1/4 & 5/4 & -3/4 \end{bmatrix}$$

2.) Gauss Elimination (Coefficients can be multiplied on any entry):

$$(A:I)=\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

•We begin with first column (**whole**) and we need $a_{21}=0, a_{31}=0$
[Operations for first column we apply the pivot in R_1].
$R_1 \to R_1, \quad R_2 \to R_2, \quad R_1+3R_3 \to R_3$

$$(A:I)=\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 5 & 1 & 1 & 0 & 3 \end{array}\right]$$

•In second column, have to make $a_{12}=0, a_{32}=0$
[But when looking for operations for second column, we only apply the pivot inR_2]
Let $R_1-2R_2 \to R_1, \quad R_2 \to R_2, \quad 5R_2-R_3 \to R_3$

$$(A:I)=\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & -1 & 5 & -3 \end{array}\right]$$

•In third column, have to make $a_{13} = 0, a_{23} = 0$
[But when looking for operations for third column, only use the pivot in R_3]
Let $4R_1 + R_3 \to R_1, \quad 4R_2 - R_3 \to R_2, \quad R_3 \to R_3$

$$(A:I) = \left[\begin{array}{ccc|ccc} 12 & 0 & 0 & 3 & -3 & -3 \\ 0 & 4 & 0 & 1 & -1 & 3 \\ 0 & 0 & 4 & -1 & 5 & -3 \end{array}\right]$$

• To have (I, B) by $\frac{1}{12}R_1 \to R_1, \ \frac{1}{4}R_2 \to R_2, \ \frac{1}{4}R_3 \to R_3$ to have

$$(I:B) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/12 & -3/12 & -3/12 \\ 0 & 1 & 0 & 1/4 & -1/4 & 3/4 \\ 0 & 0 & 1 & -1/4 & 5/4 & -3/4 \end{array}\right]$$

$$\Rightarrow \quad B = A^{-1} = \begin{bmatrix} 3/12 & -3/12 & -3/12 \\ 1/4 & -1/4 & 3/4 \\ -1/4 & 5/4 & -3/4 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 3/4 \\ -1/4 & 5/4 & -3/4 \end{bmatrix}$$

Note 1.6.3 Always prove your inverse solution by checking whether, $AA^{-1} = I$

Definition 1.6.3 A matrix that has undergone Gaussian elimination is said to be in echelon form.

Note 1.6.4 Gauss-Jordan is a special case of Gaussian elimination.

Example 1.6.5 Compute A^{-1} given $A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

$$(A:I) = \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

1.) Gauss-Jordan Elimination (Coefficients multiplied on only the pivot):

$$(A:I) = \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

•The first column looks okay since the non-zero term is only in a_{11}.
Let $R_1 \to R_1, \quad R_2 \to R_2, \quad R_3 \to R_3$

$$(A:I) = \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

•In second column, have to make $a_{12} = 0, a_{32} = 0$
[But when looking for operations for second column,only use R_2]
Let $-R_2 + R_1 \to R_1, \quad R_2 \to R_2, \quad -R_2 + R_3 \to R_3$

$$(A:I) = \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right]$$

•In third column, have to make $a_{13} = 0, a_{23} = 0$
[But when looking for operations for third column,only use R_3]
Let $-R_3 + R_1 \to R_1, \quad R_2 \to R_2, \quad R_3 \to R_3$

$$(A : I) = \begin{bmatrix} 2 & 0 & 0 & | & 1 & 0 & -1 \\ 0 & 2 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 2 & | & 0 & -1 & 1 \end{bmatrix}$$

• To have (I, B) we use $\frac{1}{2}R_1 \to R_1, \ \frac{1}{2}R_2 \to R_2, \ \frac{1}{2}R_3 \to R_3$ to have

$$(I : B) = \begin{bmatrix} 1 & 0 & 0 & | & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & | & 0 & 1/2 & 0 \\ 0 & 0 & 1 & | & 0 & -1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow \quad B = A^{-1} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

2.) Gauss Elimination (Coefficients can be multiplied on any entry):

$$(A : I) = \begin{bmatrix} 2 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & 0 & | & 0 & 1 & 0 \\ 0 & 2 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

•The first column looks okay since the non-zero term is only in a_{11}.
Let $R_1 \to R_1, \quad R_2 \to R_2, \quad R_3 \to R_3$

$$(A : I) = \begin{bmatrix} 2 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & 0 & | & 0 & 1 & 0 \\ 0 & 2 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

•In second column, have to make $a_{12} = 0, a_{32} = 0$
[But when looking for operations for second column,only use R_2]
Let $R_1 - R_2 \to R_1, \quad R_2 \to R_2, \quad R_3 - R_2 \to R_3$

$$(A : I) = \begin{bmatrix} 2 & 0 & 2 & | & 1 & -1 & 0 \\ 0 & 2 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 2 & | & 0 & -1 & 1 \end{bmatrix}$$

•In third column, have to make $a_{13} = 0, a_{23} = 0$
[But when looking for operations for third column,only use R_3]
Let $R_1 - R_3 \to R_1, \quad R_2 \to R_2, \quad R_3 \to R_3$

$$(A : I) = \begin{bmatrix} 2 & 0 & 0 & | & 1 & 0 & -1 \\ 0 & 2 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 2 & | & 0 & -1 & 1 \end{bmatrix}$$

• To have (I, B) we use $\frac{1}{2}R_1 \to R_1, \ \frac{1}{2}R_2 \to R_2, \ \frac{1}{2}R_3 \to R_3$ to have

$$(I : B) = \begin{bmatrix} 1 & 0 & 0 & | & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & | & 0 & 1/2 & 0 \\ 0 & 0 & 1 & | & 0 & -1/2 & 1/2 \end{bmatrix} \Rightarrow B = A^{-1} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

Example 1.6.6 Determine the inverse A^{-1} of

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

using the Gauss-Jordan row reduction technique.

$$[A|I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ 3R_1 + R_2 & \rightsquigarrow & R_2 \\ -R_1 + R_3 & \rightsquigarrow & R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{rcl} -0.5R_2 + R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ -R_2 + R_3 & \rightsquigarrow & R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -1 & 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \quad \begin{array}{rcl} -\frac{3}{10}R_3 + R_1 & \rightsquigarrow & R_1 \\ \frac{7}{5}R_3 + R_2 & \rightsquigarrow & R_2 \\ R_3 & \rightsquigarrow & R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & \frac{7}{10} & -\frac{1}{5} & -\frac{3}{10} \\ 0 & 2 & 0 & -\frac{13}{5} & -\frac{2}{5} & \frac{7}{5} \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \quad \begin{array}{rcl} -R_1 & \rightsquigarrow & R_1 \\ \frac{1}{2}R_2 & \rightsquigarrow & R_2 \\ -\frac{1}{5}R_3 & \rightsquigarrow & R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{10} & \frac{1}{5} & \frac{3}{10} \\ 0 & 1 & 0 & -\frac{13}{10} & -\frac{1}{5} & \frac{7}{10} \\ 0 & 0 & 1 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

The last three columns constitute A^{-1}. Check:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $AA^{-1}A^{-1}A = I$

Remark 1.6.3 For sure $AA^{-1} = I$.

Example 1.6.7 Compute A^{-1} given $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ using the Gaussian elimination row reduction (not Gauss-Jordan elimination).

$$(A : I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 - 2R_1 & \rightsquigarrow & R_2 \\ R_3 - 4R_1 & \rightsquigarrow & R_3 \end{array}$$

$$(A : I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \qquad \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ R_3 + R_2 & \rightsquigarrow & R_3 \end{array}$$

$$(A : I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right] \qquad \begin{array}{rcl} 2R_3 + R_1 & \rightsquigarrow & R_1 \\ R_3 - R_2 & \rightsquigarrow & R_2 \\ R_3 & \rightsquigarrow & R_3 \end{array}$$

$$(A : I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right] \qquad \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ -R_3 & \rightsquigarrow & R_3 \end{array}$$

$$(I : B) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\Rightarrow \quad B = A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

Exercise 1.6.1 For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & -1 \\ 2 & 3 & 0 \end{bmatrix}$, determine

1.) $|A|$, using

(a) Permutation-inversions technique

(b) Cofactors.

$$|A| = -1$$

2.) A^{-1}, using

(a) $A^{-1} = \dfrac{1}{|A|} adj(A)$

(b) Row reduction (Gauss-Jordan elimination or Gauss elimination).

$$A^{-1} = \begin{bmatrix} -3 & -6 & 11 \\ 2 & 4 & -7 \\ 1 & 1 & -2 \end{bmatrix}$$

1.6.5 Properties of Matrix Inverse

1.) If A^{-1} is invertible, then
$$(A^{-1})^{-1} = A$$

2.) If AB is invertible, then
$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof : *Let $x = (AB)^{-1}$ then*
$$(AB)x = (AB)(AB)^{-1} = I \tag{1.11}$$
But also
$$(AB)x = A(Bx) \tag{1.12}$$
Thus (1.11) will become
$$\begin{aligned} (AB)x &= I \\ A(Bx) &= I \\ A^{-1}A(Bx) &= A^{-1}I \\ I(Bx) &= A^{-1} \\ Bx &= A^{-1} \\ B^{-1}Bx &= B^{-1}A^{-1} \\ x &= B^{-1}A^{-1} \end{aligned}$$
$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$ ■

3.) If A^T is invertible, then
$$(A^T)^{-1} = (A^{-1})^T$$

Proof : *Assume A is invertible, then A^{-1} exists and we have,*
$$\begin{aligned} (A^{-1})^T A^T &= (AA^{-1})^T = I^T = I \quad \textit{and} \\ A^T(A^{-1})^T &= (A^{-1}A)^T = I^T = I \end{aligned}$$
so A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. ■

4.) For αA invertible for any nonzero scalar α, then $(\alpha A)^{-1} = \dfrac{1}{\alpha}A^{-1}$.

5.) The inverse of a diagonal matrix is obtained by inverting the diagonal elements.

6.) If a product AB is not invertible, then A or B is not invertible.

7.) If A or B are not invertible, then AB is not invertible.

Exercise 1.6.2 Use both the methods of inverses and properties of inverses to determine A^{-1} for $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$. $A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/4 \end{bmatrix}$

Exercise 1.6.3 Find the inverse of $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

$$\left[A^{-1} = \begin{bmatrix} 1/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 1/4 \\ -1/4 & 5/4 & -3/4 \end{bmatrix} \right].$$

Example 1.6.8 Find the inverse A^{-1} for the matrix

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

1.) Using the Cofactor-Adjoint technique.

From Example 1.5.15, the cofactor matrix of A was given as $\begin{bmatrix} 6 & 0 & -6 \\ -6 & 4 & 6 \\ -3 & -2 & 9 \end{bmatrix}$

$$\Rightarrow \; adj(A) = \begin{bmatrix} 6 & -6 & -3 \\ 0 & 4 & -2 \\ -6 & 6 & 9 \end{bmatrix}$$

$$\Rightarrow \; A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{12} \begin{bmatrix} 6 & -6 & -3 \\ 0 & 4 & -2 \\ -6 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & -1/4 \\ 0 & 1/3 & -1/6 \\ -1/2 & 1/2 & 3/4 \end{bmatrix}$$

2.) Using Gaussian elimination row-reduction

$$[A : I] \quad = \quad \left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{r} R_1 \rightsquigarrow R_1 \\ R_1 - 4R_2 \rightsquigarrow R_2 \\ R_1 - 2R_3 \rightsquigarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & -9 & -2 & 1 & -4 & 0 \\ 0 & 3 & -2 & 1 & 0 & -2 \end{array}\right] \qquad \begin{array}{r} 3R_1 + R_2 \rightsquigarrow R_1 \\ R_2 \rightsquigarrow R_2 \\ 3R_3 + R_2 \rightsquigarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 12 & 0 & 4 & 4 & -4 & 0 \\ 0 & -9 & -2 & 1 & -4 & 0 \\ 0 & 0 & -8 & 4 & -4 & -6 \end{array}\right] \qquad \begin{array}{r} 2R_1 + R_3 \rightsquigarrow R_1 \\ 4R_2 - R_3 \rightsquigarrow R_2 \\ R_3 \rightsquigarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 24 & 0 & 0 & 12 & -12 & -6 \\ 0 & -36 & 0 & 0 & -12 & 6 \\ 0 & 0 & -8 & 4 & -4 & -6 \end{array}\right] \qquad \begin{array}{r} \frac{1}{24}R_1 \rightsquigarrow R_1 \\ -\frac{1}{36}R_2 \rightsquigarrow R_2 \\ -\frac{1}{8}R_3 \rightsquigarrow R_3 \end{array}$$

$$[I : B] \quad = \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & -1/4 \\ 0 & 1 & 0 & 0 & 1/3 & -1/6 \\ 0 & 0 & 1 & -1/2 & 1/2 & 3/4 \end{array}\right]$$

$$\Rightarrow \; A^{-1} = \begin{bmatrix} 1/2 & -1/2 & -1/4 \\ 0 & 1/3 & -1/6 \\ -1/2 & 1/2 & 3/4 \end{bmatrix}$$

Note 1.6.5 The value for an inverse is always the same for both methods.

1.7 Matrices Chapter Examples

Example 1.7.1 Compute the determinant of each of the following matrices. Indicate clearly the method being used.

1.) Method of cofactors and adjoints

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 7 \\ -3 & 2 & 0 & -1 & -6 \\ 2 & -2 & -1 & 1 & 4 \\ 0 & 0 & 0 & 4 & 3 \end{bmatrix}$$

The determinant can be got by

$$\begin{aligned} \det A &= -7 \det \begin{bmatrix} 2 & 0 & 0 & -3 \\ -3 & 2 & 0 & -1 \\ 2 & -2 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \textit{(along the second row)} \\ &= -7(4) \det \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 0 \\ 2 & -2 & -1 \end{bmatrix} \quad \textit{(along the fourth row)} \\ &= -7(4)(2) \det \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} \quad \textit{(along the first row)} \\ &= -7(4)(2)(-2) = 112 \end{aligned}$$

2.)

$$A = \begin{bmatrix} 1 & b & b^2 \\ b & b^2 & b^3 \\ b^2 & b^3 & b^4 \end{bmatrix}$$

(where b is any real number).

We can do row operations: Add $-b$ times the first row to the second row and $-b^2$ times the first row plus the second row. This gives

$$\begin{bmatrix} 1 & b & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $\det A = 0$. *Thus, the original matrix is not invertible, since its determinant is zero.*

Example 1.7.2 Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

1.) Find $\det A$ and give a reason for your answer.

$\det A = 0$ *because two rows are equal.*

2.) Find the cofactor C_{11} and then find $\det B$.

The cofactor $C_{11} = -1$. *Then* $\det B = \det A - C_{11} = 1$.

3.) Find detC for any value of x. You could use linearity in row 1.

$\det C = xC_{11} + \det B = -x + 1$. *Check this answer (zero), for* $x = 1$ *when* $C = A$.

Example 1.7.3 Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and assume that $\det(A) = 10$. Find

1.) $\det(3A)$

2.) $\det(2A^{-1})$

3.) $\det(2A^2)$

4.) $\det\left(3(A^T)^{-1}\right)$

Notice that A *is a* 3×3 *matrix. Therefore,*

$$\begin{aligned} \det(3A) &= 3^3 \det(A) = 270 \\ \det\left(2A^{-1}\right) &= 2^3 \det\left(A^{-1}\right) = \frac{2^3}{\det(A)} = \frac{8}{10} \\ \det\left(2A^2\right) &= 2^3 \det\left(A^2\right) = 2^3 \det\left(A\right)^2 = 800 \\ \det\left(3(A^T)^{-1}\right) &= 3^3 \det\left((A^T)^{-1}\right) = \frac{3^3}{\det\left(A^T\right)} = \frac{3^3}{\det\left(A\right)} = \frac{27}{10} \end{aligned}$$

5.) $\det \begin{bmatrix} a & g & d \\ b & h & e \\ g & i & f \end{bmatrix}$

Notice that we have

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow[\textit{Interchange}]{R_2 \to R_3} B = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} \quad \textit{and} \quad B^T = \begin{bmatrix} a & g & d \\ b & h & e \\ g & i & f \end{bmatrix}$$

As a result, we have $\det(B^T) = \det(B) = (-1)\det(A) = -10$.

Example 1.7.4 Specify whether the matrix has an inverse without trying to compute the inverse $A = \begin{bmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{bmatrix}$.

Solution : $|A| = -12 \neq 0 \;\Rightarrow\; \exists\, A^{-1}$ ■

Example 1.7.5 Suppose $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$. Which one of the following statements is true ?

A. A^{-1} does not exist.

B. The third row of A^{-1} is $[-1 \quad -1 \quad 1]$.

C. The second row of A^{-1} is $[1 \quad 2 \quad -1]$.

D. The first row of A^{-1} is $[2 \quad 0 \quad -1]$.

E. The second column of A^{-1} is $[0 \quad 2 \quad -1]^T$.

F. All of B, C, D, E are true.

B

Example 1.7.6 Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & x \end{bmatrix}$. For which value(s) of x is A invertible?

A. $x \neq -1$

B. $x \neq 1$

C. $x \neq 0$

D. $x = -1$

E. $x = 1$

F. $x \neq \pm 1$

A

Example 1.7.7 If three $n \times n$ matrices A, B and C satisfy $AB - BA = C$, then ABA is always equal to :

A. $A^2B - C$

B. $A^2B - CA$

C. $BA^2 + CA$

D. A^2B

E. $A^2B + AC$

F. $A^2B + BC$

C

Example 1.7.8 Let

$$A = \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix}.$$

For which values of k does $AB = BA$ hold? $k = 9$

Example 1.7.9 For the matrix $A = \begin{bmatrix} 2 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$ Compute

1.) $|A|$, using

(a) Permutation-inversions scheme

(b) Cofactors.

$|A| = -12$

2.) A^{-1}, using

(a) $A^{-1} = \frac{1}{|A|} adj(A)$

(b) Row reduction.

$$A^{-1} = \begin{bmatrix} -1/3 & 1/4 & 1/3 \\ 1/3 & 1/6 & -7/12 \\ 1/3 & -1/3 & 1/16 \end{bmatrix}$$

1.8 Matrices Chapter Exercises

Exercise 1.8.1 Define the subtraction of two matrices A and B.

Exercise 1.8.2 If A, B and C are conformable matrices then show that

$$A + (B + C) = (A + B) + C.$$

Exercise 1.8.3 If A is an $m \times p$, B is a $p \times q$, and C is a $q \times n$, then show that

$$(AB)C = A(BC).$$

Exercise 1.8.4 If A is an $m \times p$, B is an $m \times p$, and C is a $p \times n$, then show that

$$(A + B)C = AC + BC.$$

Exercise 1.8.5 If $r, s \in \mathbb{R}$ then show that

1.) $r(sA) = (rs)A$ 2.) $(r + s)A = rA + sA$ 3.) $A(rB) = r(AB)$

where A is an $m \times p$ matrix and B is a $p \times n$ matrix

Exercise 1.8.6 Given that A and B matrices and $r \in \mathbb{R}$ show that,

1.) $(A')' = A$ 2.) $(rA)' = rA'$ 3.) $(A + B)' = A' + B'$

Exercise 1.8.7 Show that there exists a unique $m \times n$ matrix 0 such that $A + 0 = A$ for any $m \times n$ matrix A.

Exercise 1.8.8 Show that for each $m \times n$ matrix A, there exists a unique $m \times n$ matrix $-A$ such that $A + (-A) = 0$.

Exercise 1.8.9 Let A and B be arbitrary $n \times n$ matrices. Is it *true* that

$$(A - B)(A + B) = A^2 - B^2$$

Exercise 1.8.10 Prove that $(AB)' = B'A'$ for A, B conformable matrices.

Exercise 1.8.11 If A is an idempotent matrix. Prove that

1.) $A^n = A \quad \forall\, n \in \mathbb{Z}_+$.

2.) $I - A$ is idempotent where I is an identity matrix of same order as A. (Is $A - I$ also idempotent?).

Exercise 1.8.12 If A and B are idempotent matrices. Is AB also idempotent? if not under what condition(s) will AB be idempotent.

Exercise 1.8.13 If A and B are invertible $n \times n$ matrices, Is it true that the inverse of $A + B$ is $A^{-1} + B^{-1}$?

Exercise 1.8.14 Let A be an $n \times n$ square invertible matrix. Prove that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Exercise 1.8.15 Find the matrix A such that

$$adj(A) = \begin{bmatrix} 2 & 4 \\ -5 & 7 \end{bmatrix}$$

Exercise 1.8.16 Let A be a 5×5 matrix with determinant 6. What is the determinant of A^{-1} (determinant of the inverse of A)?

A. 6.
B. 0.
C. 1/6.
D. 30.
E. 25/6.
F. 1.
G. Insufficient information to solve the question.

C

Exercise 1.8.17 Let A be a 5×5 matrix with determinant 6, and let B be a 5×5 matrix with determinant 4. What is the determinant of AB?

A. 24.
B. 10.
C. 0.
D. 1/6.
E. 1/4.
F. 1.
G. Insufficient information to solve the question.

A

Exercise 1.8.18 Suppose A is an $n \times n$ matrix, prove that $\det[adj(A)] = (\det A)^{n-1}$

Exercise 1.8.19 What is the size of the matrix B if the product AB has been computed and A is $m \times n$ matrix?

Exercise 1.8.20 For A and b matrices show that $r(A + B) = rA + rB$ for r a real number. What is known about the sizes of A and B?

Exercise 1.8.21 If A and B are symmetric matrices of the same order, show that $A^T B^T + AB$ is also symmetric.

Exercise 1.8.22 State any two axioms of inverse of a matrix.

Exercise 1.8.23 For a symmetric matrix A and a skew-symmetric matrix B, show that the matrix $AB - BA$ is also symmetric.

$$(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = -BA - A(-B) = AB - BA.$$

Exercise 1.8.24 *True* or *False*? Every nonzero square matrix has an inverse. *False*

Exercise 1.8.25 Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 2 & k^2+3 & 3 \\ 5 & 1 & 2k \end{bmatrix}$. Find the values of k such that $tr(A) = 7$.

$$(k + 3)(k - 1) = 0 \ \Rightarrow \ k = -3 \text{ or } k = 1$$

Exercise 1.8.26 Let A be a 5×5 matrix with determinant 6, and let B be a 5×5 matrix with determinant 4. What is the determinant of $A + B$?

A. 24.
B. 10.
C. 0.
D. 1/6.
E. 1/4.
F. 1.
G. Insufficient information to solve the question.

G

Exercise 1.8.27 Let A be a 4×4 matrix with determinant 3. What is the determinant of $-A$?

A. 3.
B. -3.
C. 1/3.
D. 0.
E. 1.
F. -1.
G. Insufficient information to solve the question.

A

Exercise 1.8.28 Let A be a 4×4 matrix with determinant 3. What is the determinant of $2A$?

A. 6.
B. 3.
C. 24.
D. 48.
E. 12.
F. 196,608.
G. Insufficient information to solve the question.

D

Exercise 1.8.29 Let A be a 4×4 matrix with determinant 3. What is the determinant of A^T (the transpose of A)?

A. 3^T.
B. 3.
C. 12.
D. 1/3.
E. 27.
F. 81.
G. Insufficient information to solve the question.

B

Exercise 1.8.30 Let A be a 4×4 matrix with determinant 3. Let B be the matrix formed by swapping the second and third rows of A. What is $\det(B)$?

A. 3.
B. 0.
C. 1/3.
D. -3.
E. 6.
F. 2.
G. Insufficient information to solve the question.

D

Exercise 1.8.31 If A is a 3×5 matrix, then the determinant of A is

A. A number (possibly non-zero).

B. 3^5.

C. Zero.

D. 5^3.

E. A 5×3 matrix.

F. Undefined.

G. A 3×5 matrix.

F

Exercise 1.8.32 Find the row reduced echelon form of the matrix below:

$$A = \begin{bmatrix} 1 & -2 & -4 & 3 \\ 2 & 5 & -2 & 9 \\ 1 & 7 & 2 & 6 \\ 0 & 5 & -2 & 9 \\ 1 & -2 & -4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5/4 \\ 0 & 0 & 1 & -11/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 1.8.33 Find the determinant of the matrix below. Specify whether the matrix has an inverse without trying to compute the inverse. $\begin{bmatrix} 2 & -2 & -2 & -2 \\ -2 & 2 & 3 & 0 \\ -2 & -2 & 2 & 0 \\ 1 & -1 & -3 & -1 \end{bmatrix}$

Row reduce the given matrix to an upper triangular matrix

$$\begin{bmatrix} 2 & -2 & -2 & -2 \\ -2 & 2 & 3 & 0 \\ -2 & -2 & 2 & 0 \\ 1 & -1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & -2 & -2 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

As a result, the determinant is -32 which is not 0. Therefore, the given matrix is invertible.

Exercise 1.8.34 Find the inverse of $A = \begin{bmatrix} 4 & -2 & 2 \\ 8 & 2 & -6 \\ 4 & -2 & 6 \end{bmatrix}$; $\left[A^{-1} = \frac{1}{96} \begin{bmatrix} 0 & 8 & 8 \\ -72 & 16 & 40 \\ -24 & 0 & 24 \end{bmatrix} \right]$

Exercise 1.8.35 Show that $\det(A) = 0$ where $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

Note that $A = -A^T$. Then we get

$$\begin{aligned} \det(A) &= \det((-1)A^T) \\ &= (-1)^3 \det(A^T) \\ &= (-1)^3 \det(A) \\ &= -\det(A), \text{ i.e.,} \end{aligned}$$

$$2\det(A) = 0 \quad \rightarrow \quad \det(A) = 0.$$

OR Calculate the determinant with respect to any column or any row of your choice.

Exercise 1.8.36 Find the inverse of the following matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

$$A^{-1} = \tfrac{1}{6} \begin{bmatrix} 6 & -6 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

Exercise 1.8.37

1.) An $n \times n$ matrix A is called **orthogonal** if $AA^T = I$. If A is orthogonal show that

$$\det(A) = \pm 1.$$

By the properties of determinant,

$$\det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = \det(A)^2 = \det(I) = 1.$$

So, $\det(A) = \pm 1$.

2.) An $n \times n$ matrix A is called **skew-symmetric** if $A^T = -A$. Show that if A is skew-symmetric and n is an odd positive integer, then A is not invertible.

By the properties of determinant,

$$\det(A^T) = \det(-A) \Rightarrow \det(A) = \det(-A) \Rightarrow \det(A) = (-1)^n \det(A) \Rightarrow \det(A) = -\det(A).$$

So, we get $\det(A) = 0$ *which implies that A is not invertible. Note that $-A$ means that every row of A is multiplied by -1.*

3.) Let A and B be two non-singular symmetric matrices that commute. Show that $A^{-1}B$ and $A^{-1}B^{-1}$ are symmetric.

Exercise 1.8.38 Specify whether the matrix has an inverse without trying to compute the inverse

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

We use the definition of determinant. We calculate the determinant across the 2nd rows and 3rd column.

$$\begin{aligned}
\begin{vmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{vmatrix} &= (-1)\begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{vmatrix} \\
&= (-1)\left((-1)\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix}\right) \\
&= (-1)\left((-1)\left((-1)\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}\right)\right) \\
&= (-1)(-1)(-1)[(1)-(1)] \\
&= 0
\end{aligned}$$

Since we have the determinant is 0, the matrix is not invertible.

Exercise 1.8.39 Find the inverse of the matrix A using the inverse formula (cofactor-adjoint) where $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ 3 & -1 & -1 \end{bmatrix}$

Exercise 1.8.40 Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 12 & 9 \\ 3 & 6 & 10 & 15 \\ 4 & 8 & 12 & 14 \end{pmatrix}$$

by using row operations (Hint: Recall the determinant of an upper triangular matrix).

Exercise 1.8.41 Let$A = \begin{bmatrix} 1 & 1 & a & 1 \\ 0 & a-1 & 2a & 1 \\ 0 & 0 & (a-1)(a^2-4) & a \\ 0 & 0 & 0 & a \end{bmatrix}$. Determine those values of a for which A is invertible. *Triangular. if and only if* $a \neq 0$ *and* $a \neq 1$ *and* $a \neq 2$ *and* $a \neq -2$.

Exercise 1.8.42 Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and assume that $\det(A) = 2$. Find

1.) $\det(-2A)$

2.) $\det\left(\frac{1}{2}A^{-1}\right)$

3.) $\det\left(2A^2\right)$

4.) $\det\left(3(A^T)^{-1}\right)$

5.) $\det\begin{bmatrix} a & g & d \\ b & h & e \\ c+2a & i+2g & f+2d \end{bmatrix}$ $\det(C) = \det(B^T) = \det(B) = (-1)\det(A) = -2$

Exercise 1.8.43 Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ -4 & 4 \end{bmatrix}.$$

Verify directly that $A(AB) = A^2B$.

Exercise 1.8.44 Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

We apply the row operations to A to have $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \det A = 1$

Exercise 1.8.45 Find x, assuming

$$\det \begin{bmatrix} x^2 & x & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} = 0$$

Calculate the determinant according to the third row: $x = 0$ *or* $x = 2$

Exercise 1.8.46 Let A and B be 4×4 matrices with $det(A) = -1$ and $det(B) = 2$. Find the determinant $\det(B^{-1}AB)$.

We have

$$\begin{aligned} \det(B^{-1}AB) &= \det(B^{-1}) \det(A) \det(B) \\ &= \frac{1}{\det(B)} \det(A) \det(B) \\ &= \det(A) \\ &= -1 \end{aligned}$$

Exercise 1.8.47 State whether the following are true or false. If true, explain why, if false, give a numerical example to illustrate.

1.) If A and B are 2 by 2 matrices, then $\det(A + B)$ is always equal to $\det A + \det B$.

False: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow 0 + 0 \neq 1$

2.) If a 13×13 matrix A satisfies then $A^2 = 0$, then A is not invertible.

True. Assume A invertible, $A = A^{-1}A^2 = 0 \Rightarrow \det A = 0 \Rightarrow$, *$A$ invertible*

Exercise 1.8.48 If $C = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and D is a $3 \times m$ matrix then the second row of the matrix CD is

A. not defined unless $m = 2$.

B. the same as the first row of D.

C. the same as the second row of D.

D. the sum of the first and the third row of D.

E. the sum of twice the second row of D and the third row of D.

F. twice the first row of D.

D

Exercise 1.8.49 Which of the following statements are **false**?

1.) For all invertible $n \times n$ matrices A and B, $\det(A^{-1}BA) = \det B$

2.) For all invertible $n \times n$ matrices A and B, $det(A^{-1}B^{-1}AB) = 1$

3.) For all $n \times n$ matrices A and B, $\left(A^TB^T\right)^T = AB$

4.) For all invertible $n \times n$ matrices A and B, $(ABA^{-1})^{-1} = A^{-1}B^{-1}A$

5.) For all $n \times n$ matrices A and B, $\det(A^TB) = \det(B^TA)$

A. (1) and (3)

B. (2) and (3)

C. (3) and (4)

D. (2) and (4)

E. (2) and (5)

F. (1) and (5)

(1), True:

(2), True:

(3), False : $\left(A^TB^T\right)^T = BA.$

(4), False : $(ABA^{-1})^{-1} = AB^{-1}A^{-1}.$

(5), True :

Its possible that $AB \neq BA$ *and* $A^{-1}B^{-1}A \neq AB^{-1}A^{-1}$

C

Exercise 1.8.50 Let A and B denote matrices, not necessarily square, and which have more than 1 row and more than 1 column, and let x denote a column vector (i.e., a $k \times 1$ matrix for some k).

State whether each of the following is (always) true, or is (possibly) false

- If you say the statement may be false, you must give an explicit example - with numbers! (Hint: Try an example with 2 or 3 rows or columns.)

- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, any by giving other valid proof.

1.) If A is $m \times n$ and rank$A = m$, then the system $Ax = 0$ has a unique solution.

$$\textit{False}: \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2.) If $AB = 0$ then either $A = 0$ or $B = 0$.

$$\textit{False}: \quad A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

3.) If B has a column of zeros then AB has a column of zeros. *True*

Write B with column entries where the j^{th} column is zeros.

$$B = [c_1 \quad c_2 \quad \cdots \quad 0 \quad c_{j+1} \quad \cdots \quad c_n]$$

Then

$$AB = [Ac_1 \quad Ac_2 \quad \cdots \quad A \cdot 0 \quad Ac_{j+1} \quad \cdots \quad Ac_n] = [Ac_1 \quad Ac_2 \quad \cdots \quad 0 \quad Ac_{j+1} \quad \cdots \quad Ac_n]$$

That is, if the j^{th} column of B is zero, then the j^{th} column of AB is zero as well.

Exercise 1.8.51 If two $n \times n$ matrices A and B satisfy $A^T = B^{-1}$ and $B^T = -B^{-1}$ then $(ABA)^T$ is always

A. $-B^3$ C. $-B^{-3}$ E. B^3

B. B^2A D. B^{-3} F. AB^2

C

Exercise 1.8.52 Determine the value(s) of λ for which the matrix

$$A = \begin{bmatrix} \lambda & -1 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda \end{bmatrix}$$

is invertible.

Invertible if $|A| \neq 0$, since already a square matrix. $\lambda \neq 0$ and $\lambda \neq \sqrt{2}$ and $\lambda \neq -\sqrt{2}$

Exercise 1.8.53 Assume that B is a 3×3 matrix with the property that $B^2 = B$. Which of the following statements about the matrix B **must** be true:

A. AB is invertible

B. $\det(B) = 0$

C. $\det(B^5) = \det(B)$

D. None of the above must be true

C

Two examples of matrices that satisfy $B^2 = B$ are $B = I_3$ and $B = 0_{3\times3}$ where $0_{3\times3}$ is the 3×3 matrix with all zero entries. So (A) is false because the 0 matrix satisfies the property but is not invertible. Similarly (B) is false because the identity matrix satisfies the property but the determinant of the identity is 1 not 0. (C) is true because

$$B^5 = (BB)(BB)B = (B)(B)B = (BB)B = (B)B = BB = B$$

so $B^5 = B$ and hence in particular $\det(B^5) = \det(B)$.

Exercise 1.8.54 *True* or *False*? If A and B are both invertible $n \times n$ matrices, then AB is invertible.

True: One way to see this is to note that if A and B are invertible then each of their determinants are nonzero. But then

$$det(AB) = det(A)det(B)$$

is also nonzero since its a product of two non zero numbers. Hence since $det(AB)$ is nonzero, it follows that AB is invertible.

Exercise 1.8.55 *True* or *False*? Let A and B be $n \times n$ matrices. Assume that

$$AB = I_n.$$

Then,

$$BA = I_n$$

True: Since $AB = I_n$ it follows from the invertible matrix theorem (the theorem that gives all the many equivalences for a matrix being invertible) that A and similarly B are invertible. Moreover, the equation $AB = I_n$ says that $B = A^{-1}$ Hence in this case $BA = (A^{-1})A = I_n$.

Exercise 1.8.56 Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

Exercise 1.8.57 If E is a 3×3 matrix of the form

$$E = \begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}.$$

Given $det(E) = 5$, compute the determinant of the following matrix

$$F = \begin{bmatrix} x & y & z \\ 1 & 8 & 3 \\ -3+4x & 7+4y & 2+4z \end{bmatrix}.$$

Exercise 1.8.58 Let the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}$$

1.) Compute X^{-1} $\qquad X^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$

2.) Compute XYX^{-1}

$$\begin{aligned} XYX^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 7 \\ -1 & 9 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -11 & 9 \\ -11 & 10 \end{bmatrix} \end{aligned}$$

3.) Compute $\det(XYX^{-1})$

$$\det \begin{bmatrix} -11 & 9 \\ -11 & 10 \end{bmatrix} = (-11)(10) - (9)(-11) = -110 + 99 = -11$$

4.) What is the relationship between $\det(Y)$, and $\det(XYX^{-1})$ and why?

$$\det(Y) = det \begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix} = (-3)(2) - (5)(1) = -6 - 5 = -11$$

So $\det(XYX^{-1}) = \det(Y)$. This is *not a coincidence. In fact,*

$$\begin{aligned} \det(XYX^{-1}) &= \det(X)\det(Y)\det(X^{-1}) = \det(X)\det(X^{-1})\det(Y) \\ &= \det(XX^{-1})\det(Y) = \det(I_2)\det(Y) = 1 * \det(Y) = \det(Y) \end{aligned}$$

Exercise 1.8.59 What is the reduced row echelon form of the matrix

$$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix}?$$

A. $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

E. none of the preceding.

Exercise 1.8.60 What is the determinant of the matrix $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 4 \end{bmatrix}$

A. -120 C. 0 E. 120

B. -24 D. 24

Exercise 1.8.61 What is the first row of the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

A. $[0 \quad -2 \quad 1]$ C. $[0 \quad 6 \quad -3]$ E. The inverse does not exist.

B. $[0 \quad -3 \quad -6]$ D. $[2 \quad 1 \quad 0]$

Exercise 1.8.62 If $\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = 3$, find $\begin{vmatrix} b & f-2n & n & 5j-b \\ a & e-2m & m & 5i-a \\ c & g-2o & o & 5k-c \\ d & h-2p & p & 5l-d \end{vmatrix}$

Exercise 1.8.63 Find the row echelon form of the following matrices.

$$A = \begin{bmatrix} 2 & -4 & 3 \\ -6 & 12 & -9 \\ 4 & -8 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -4 & 2 \\ -3 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 7 \end{bmatrix}$$

The row echelon form

$$A = \begin{bmatrix} 2 & -4 & 3 \\ -6 & 12 & -9 \\ 4 & -8 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 \\ -4 & 2 \\ -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 14 \\ 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 14 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -5 & -10 & -17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 17/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2/5 \end{bmatrix}$$

Exercise 1.8.64 Find the reduced row echelon form of the following matrix

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 4 & 8 & 12 \\ 0 & 8 & 16 & 34 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 8 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 1.8.65

Let

$$A = \begin{bmatrix} 4 & -5 & 3 \\ 5 & 7 & -2 \\ -3 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 0 & -1 \\ -1 & 5 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 5 \\ 4 & -3 \\ 1 & 0 \end{bmatrix}.$$

Calculate AC, BC and CB.

Exercise 1.8.66 Let

$$A = \begin{bmatrix} 7 & 0 \\ -1 & 5 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ -4 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 7 \\ -3 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 2 \end{bmatrix}.$$

Calculate the following matrices if they exist:

1.) AB	3.) CD	5.) DBC	7.) A^TA
2.) CA	4.) DC	6.) BDB	8.) AA^T

For those that don't exist, explain why.

CA is not defined since we cannot multiply a 2×1 matrix by a 3×2 matrix.

BD is not defined since we cannot multiply a 2×2 matrix by a 1×2 matrix.

Consequently BDB is not defined either.

Exercise 1.8.67 Let

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad N = \begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}, \quad \text{and } T = \begin{bmatrix} 5 & 2 \\ 1 & -2 \end{bmatrix}.$$

Show that $MN = MT$, even though N is not the same matrix as T.

Exercise 1.8.68 Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

1.) Calculate the products AD and DA and explain how the columns and rows of A change when we multiply A by D on the left and on the right.

2.) Find all the diagonal 3×3 matrices M such that $AM = MA$.

To have $AM = MA$ we need $a = b, a = c, 3b = 3c$ and $4b = 4c$. Hence the diagonal matrices that commute with A (i.e. $AM = MA$) are of the form $M = \lambda I_3, \lambda \in \mathbb{R}$.

Exercise 1.8.69 We can interpret vectors of $\mathbb{R}^n$ as $n \times 1$ matrices. Let $\mathbf{u}$ and $\mathbf{v}$ be the following vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

1.) Write down $\mathbf{u}^T$ and $\mathbf{v}^T$.

2.) What are the dimensions of the products $\mathbf{u}^T\mathbf{v}$ and $\mathbf{v}^T\mathbf{u}$?

3.) Are these products equal? Why?

4.) What are the dimensions of the products $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$?

5.) Are these products equal? Why?

Note 1.8.1

1. We call $\mathbf{u}^T\mathbf{v}$ the scalar product (or inner product) of the vectors $\mathbf{u}$ and $\mathbf{v}$.

2. The product $\mathbf{u}\mathbf{v}^T$ is called the outer product.

Exercise 1.8.70 An upper triangular matrix is a matrix such that all of its elements under the main diagonal are zeros. A lower triangular matrix is a matrix such that all of its elements over the main diagonal are zeros.

1.) What is the condition for an upper triangular matrix to be invertible?

2.) What is the condition for a lower triangular matrix to be invertible?

To have an invertible matrix, it has to be square (of dimensions $n \times n$). If an upper triangular matrix has only nonzero elements in the main diagonal, then it is already in row echelon form and all n diagonal positions are pivots, hence it is invertible. If there is a zero on the diagonal, in position (k, k), say, then it will stay zero after bringing the matrix to row echelon form and the columns 1 to k can only contain at most as many pivots as there are rows above the k^{th}, that is, $k-1$. This prevents the existence of n pivots, so in this case the matrix is not invertible.

A matrix is invertible exactly if its transpose is invertible. The transpose of a lower triangular matrix is an upper triangular matrix with the same diagonal entries, so (by the previous part) a lower triangular matrix is invertible exactly when it is square and its diagonal entries are all nonzero.

Exercise 1.8.71 *True* or *False*? If you say the statement may be false, you must give an explicit example - with numbers!. If you say the statement is always true, you must give a clear explanation.

1.) If A and B are $m \times n$ matrices then both AB^T and A^TB are defined.

2.) If $AB = C$ and C has 2 columns, then A has two columns.

3.) Multiplying B on the left with a diagonal matrix scales the rows of B.

4.) If $BC = BD$ then $C = D$.

5.) If $AC = 0$ then either $A = 0$ or $C = 0$.

6.) If A and B are $n \times n$ matrices then $(A + B)(A - B) = A^2 - B^2$.

7.) An elementary $n \times n$ matrix has either n or $n + 1$ non-zero entries.

8.) The transpose of an elementary matrix is an elementary matrix.

9.) An elementary matrix must be square.

10.) Every square matrix is a product of elementary matrices.

11.) If A is a 3×3 matrix with 3 pivot positions then there exist elementary matrices $E_1, \dots, E_p$, such that $E_1 \dots E_pA = I$.

12.) If $AB = I$ then A is invertible.

13.) If A and B are invertible then $A^{-1}B = B^{-1}A$.

14.) If A is invertible and $r \neq 0$, then $(rA)^{-1} = rA^{-1}$.

Answers to Exercise 1.8.71

(1) *True*	(4) *False*	(7) *True*	(10) *False*	(13) *False*
(2) *False*	(5) *False*	(8) *True*	(11) *True*	
(3) *True*	(6) *False*	(9) *True*	(12) *False*	(14) *False*

Exercise 1.8.72 1.) Calculate the following determinants by using cofactor expansions:

$$a = \begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}, \quad b = \begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}, \quad c = \begin{vmatrix} 4 & 0 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 & 0 \\ 2 & 6 & 3 & 0 & 0 \\ 5 & -8 & -4 & -3 & 0 \\ 5 & 0 & 0 & 0 & 2 \end{vmatrix}$$

.

2.) Calculate the following determinants by using row operations as well as cofactor expansions:

$$d = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}, \quad e = \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}$$

Solutions to Exercise 1.8.72

Exercise 1.8.72 (1) Expanding along the first row, we have

$$\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix} = 4\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} - 3\begin{vmatrix} 6 & 2 \\ 9 & 3 \end{vmatrix} = 4(15-14) - 3(18-18) = 1.$$

Expanding first along the third row, and then again along the first row, we have

$$b = \begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} = 2\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix} = 10\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 10 \cdot 1 = 10.$$

Expanding successively along the first rows, we have

$$c = \begin{vmatrix} 4 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 0 \\ 2 & 6 & 3 & 0 & 0 \\ 5 & -8 & -4 & -3 & 0 \\ 5 & 0 & 0 & 0 & 2 \end{vmatrix} = 4\begin{vmatrix} -1 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 \\ -8 & -4 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 & 0 \\ -4 & -3 & 0 \\ 0 & 0 & 2 \end{vmatrix} - 12\begin{vmatrix} -3 & 0 \\ 0 & 2 \end{vmatrix} = 72.$$

Exercise 1.8.72 (2) We compute

$$\begin{aligned} d &= \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix} \begin{matrix} R_3 \rightsquigarrow R_3 + R_1 \\ R_4 \rightsquigarrow R_4 - 3R_1 \\ = \end{matrix} \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 4 \\ 1 & 5 & 5 \\ 2 & 7 & 3 \end{vmatrix} \begin{matrix} R_2 \rightsquigarrow R_2 - R_1 \\ R_3 \rightsquigarrow R_3 - 2R_1 \end{matrix} \\ &= \begin{vmatrix} 1 & 5 & 4 \\ 0 & 0 & 1 \\ 0 & -3 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -3 & -5 \end{vmatrix} = 3. \end{aligned}$$

For e, we have

$$\begin{aligned} e &= \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ = \end{matrix} - \begin{vmatrix} 1 & 3 & 0 & -3 \\ -3 & -2 & 1 & -4 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} \begin{matrix} R_2 \rightsquigarrow R_2 + 3R_1 \\ R_3 \rightsquigarrow R_3 + 3R_1 \\ R_4 \rightsquigarrow R_4 - 3R_1 \end{matrix} \\ &= \begin{vmatrix} 1 & 3 & 0 & -3 \\ 0 & 7 & 1 & -13 \\ 0 & 13 & -2 & -1 \\ 0 & -13 & 0 & 13 \end{vmatrix} = -\begin{vmatrix} 7 & 1 & -13 \\ 13 & -2 & -1 \\ -13 & 0 & 13 \end{vmatrix} = 13\begin{vmatrix} 7 & 1 & -13 \\ 13 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} \\ &= 13\left(-\begin{vmatrix} 13 & -1 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 7 & -13 \\ 1 & -1 \end{vmatrix}\right) \\ &= 13(12 - 2 \cdot 6) = 0. \end{aligned}$$

Exercise 1.8.73 Calculate the determinants of the following elementary matrices:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

To which elementary row operation does each of these matrices correspond?

Solutions to Exercise 1.8.73

E_1 *corresponds to adding* k *times the second row to the third row, it has determinant* 1.

E_2 *corresponds to multiplying the second row by* k, *it has determinant* k.

E_3 *corresponds to interchanging the first and second rows, it has determinant* -1.

E_4 *corresponds to interchanging the first and third rows, it has determinant* -1.

Exercise 1.8.74 Given

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$$

Compute the following determinants

$$t = \begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}, \quad s = \begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}.$$

1.) $t = 7$. *Indeed, this matrix was obtained from the original matrix by adding the second row to the first, which does not change the value of the determinant.*

2.) $s = 14$. *this matrix was obtained from the original one by multiplying the second row by* 2, *which multiplies the determinant by* 2, *and then adding the first to the second line, which does not change the value of the determinant.*

Exercise 1.8.75 Let A, B, C, D be $n \times n$ matrices.

1.) Using the echelon form, show that

$$\det = \begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix} = (\det A)(\det D).$$

2.) Using the formula for determinants of the transpose matrix, show that

$$\det = \begin{bmatrix} A & \mathbf{0} \\ C & D \end{bmatrix} = (\det A)(\det D).$$

3.) Decide whether the following identity is true or false

$$\det = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det D) - (\det B)(\det C).$$

Justify your answer.

Solutions for Exercise 1.8.75

Exercise 1.8.75 (1) Let us consider the following cases:

- *If* $\det A = 0$ *or* $\det D = 0$*, then* A *or* D *has less than* n *pivots, and the matrix we are interested in has less than* $2n$ *pivots. Therefore, its determinant is null.*
- *If* $\det A \neq 0$ *and* $\det D \neq 0$*, then we put the big matrix in echelon form by using the elementary operations (1) to add to some row a multiple of other row, and (2) interchange two rows.*

  Let A' *be one matrix in echelon form equivalent to* A *obtained this way doing* r *changes of rows, and let* D' *one matrix in echelon form equivalent to* D *obtained this way with* s *interchanges.*

  Then the matrix we are interested in is transformed this way doing $r+s$ *row changes in the echelon form*

$$M = \begin{bmatrix} A' & B^* \\ 0 & D' \end{bmatrix}$$

where B^* *is some* $n \times n$ *matrix. Then*

$$\begin{aligned} \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} &= (-1)^{r+s}(\textit{product of pivots from } M) \\ &= (-1)^r(\textit{product of pivots from } A')(-1)^s(\textit{product of pivots from } D') \\ &= (\det A)(\det D). \end{aligned}$$

Exercise 1.8.75 (2)

$$\det \begin{bmatrix} A & \mathbf{0} \\ C & D \end{bmatrix} = \det \begin{bmatrix} A^T & C^T \\ \mathbf{0} & D^T \end{bmatrix} = (\det A^T)(\det D^T) = (\det A)(\det D).$$

Exercise 1.8.75 (3) We can show with a counterexample that the equality is not always true. Take for example

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Where

$$\det(A) \cdot \det(D) - \det(B) \cdot \det(C) = 1 \cdot 1 - 4 \cdot 1 = -3$$

and

$$\begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1$$

Exercise 1.8.76 Show that if A and B are non-singular matrices then A^{-1} and AB are also non-singular.

Exercise 1.8.77 Using both the method of elementary row operations (either Gauss-Jordan or Gaussian elimination, any of your choice) and the adjoint method, find the inverse of the matrix A if,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise 1.8.78 Let A, B, C, D be $n \times n$ matrices, with A invertible.

1.) Find matrices X and Y that give LU decomposition as follows:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & \mathbf{0} \\ X & I_n \end{bmatrix} \begin{bmatrix} A & B \\ \mathbf{0} & Y \end{bmatrix}$$

Show that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \cdot \det(D - CA^{-1}B).$$

2.) Show that if $AC = CA$, then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB).$$

Solutions to Exercise 1.8.78

Exercise 1.8.78 (1) By block-multiplication

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & \mathbf{0} \\ X & I_n \end{bmatrix} \begin{bmatrix} A & B \\ \mathbf{0} & Y \end{bmatrix} = \begin{bmatrix} A & B \\ XA & XB + Y \end{bmatrix}$$

and deduce

$$X = CA^{-1}, \quad Y = D - CA^{-1}B.$$

To compute the determinant, we use the results from Exercise 1.8.75:

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det \begin{bmatrix} I_n & \mathbf{0} \\ X & I_n \end{bmatrix} \det \begin{bmatrix} A & B \\ \mathbf{0} & Y \end{bmatrix} = \det(A)\det(Y)$$

$$= \det(A)\det(D - CA^{-1}B).$$

Exercise 1.8.78 (2) If $AC = CA$ (and A is invertible), then $ACA^{-1} = C$. By the formula of the determinant of a product of matrices

$$\det(A)\det(D - CA^{-1}B) = \det(AD - ACA^{-1}B) = \det(AD - CB).$$

Exercise 1.8.79 Show that

1.) If A is an invertible matrix, then $\det(A^{-1}) = 1/\det A$.

$$(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1, \quad \textit{and then} \quad \det A^{-1} = 1/\det A$$

2.) If A and P are square matrices, with P invertible, then $\det(PAP^{-1}) = \det A$.

$$\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1}) = \det(PP^{-1})(\det A) = (\det I)(\det A) = \det A$$

3.) If U is a square matrix such that $U^TU = 1$, then $\det U = \pm 1$.

$$1 = \det I = \det(U^TU) = (\det U^T)(\det U) = (\det U)^2, \quad \textit{then} \quad \det U = \pm 1$$

4.) If A is a square matrix such that $\det(A^4) = 0$, then A cannot be invertible.

$$0 = \det A^4 = (\det A)^4, \textit{ then } \det A = 0, \textit{ which implies } A \textit{ is not invertible.}$$

Exercise 1.8.80 Given the *Vandermonde matrices*

$$T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}, \quad V(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \end{bmatrix}.$$

1.) By row operations, show that $\det T = (b-a)(c-a)(c-b)$.

With different a, b and c, we have

$$T \sim \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \sim \begin{bmatrix} a & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{bmatrix} \sim \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{bmatrix},$$

from which

$$\det T = (b-a)(c-a)\det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{bmatrix} = (a-b)(b-c)(c-a).$$

2.) Let $f(t) = \det V(t)$, with a_1, a_2 and a_3 different numbers. Show that $f(t)$ is a polynomial of degree three (show that the coefficient of t^3 is non-zero). Find the three solutions of the equation $f(t) = 0$.

Expanding along the first row of $V(t)$ we see that $f(t)$ is of the form

$$f(t) = d_0 - d_1 t + d_2 t^2 - d_3 t^3,$$

where d_0, d_1, d_2 and d_3 are functions of a_1, a_2 and a_3. In particular,

$$d_3 = \det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = (a_1-a_2)(a_2-a_3)(a_3-a_1) \neq 0.$$

The three solutions of the equation $f(t) = 0$ are $t = a_1, t = a_2$ and $t = a_3$, since when replacing t by a_1, a_2 or a_3 in $V(t)$, two rows of the matrix are the same, and its determinant is zero.

Exercise 1.8.81 Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 0 & 0 \end{bmatrix}.$$

1.) Calculate the inverse of A by reducing the matrix $[A\ I]$.

2.) Calculate the inverse of A by calculating the adjoint matrix and the determinant. For the adjoint matrix beware of indices!

3.) Which of the two methods is more efficient (i.e., the fastest) for large matrices?

Solutions to Exercise 1.8.81

Exercise 1.8.81 (1) - (2) The inverse of A is

$$\begin{bmatrix} 0 & 1/11 & -3/11 & 5/11 \\ 0 & -4/11 & 12/11 & -9/11 \\ -1 & 16/11 & -15/11 & 14/11 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 0 & 1 & -3 & 5 \\ 0 & -4 & 12 & -9 \\ -11 & 16 & -15 & 14 \\ 11 & -11 & 11 & -11 \end{bmatrix}$$

The determinant of A is $\det(A) = 11$ *and the adjoint matrix is*

$$\begin{bmatrix} 0 & 1 & -3 & 5 \\ 0 & -4 & 12 & -9 \\ -11 & 16 & -15 & 14 \\ 11 & -11 & 11 & -11 \end{bmatrix}.$$

So one has

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Exercise 1.8.81 (3) The number of operations needed to reduce $[A\ I]$ has order n^3. Using the formula with the adjoint matrix and the determinant one computes n^2 determinants of $(n-1)\times(n-1)$ matrices and one determinant of dimension $n \times n$.

The number of operations needed to compute a determinant of dimension $m \times m$ is $\frac{2}{3}m^3$ operations. To compute the formula one needs $n^2\frac{2}{3}(n-1)^3 + \frac{2}{3}n^3 \approx n^5$ operations!

For big enough matrices the formula wich adjoint matrix and the determinant is definitely less efficient.

Exercise 1.8.82

1.) Let A be $n \times n$ matrix and M_{ij} from A by deleting its ith row and jth column. Define

(a) the minor of the element a_{ij} of A

(b) the cofactor of a_{ij}, denoted by C_{ij}.

2.) (a) The determinant of a matrix A is related to the adjoint of A by the relationship, $(adj A)\cdot A = (\det A)I$ where I is the identity matrix. Prove that if $|A| \neq 0$, then $A^{-1} = adj A/|A|$ Hence find A^{-1} for

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 6 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Find all the values of ψ for which the inverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \psi & 0 \\ 1 & 2 & \psi \end{bmatrix} \text{ exists.}$$

Exercise 1.8.83 Prove that if a matrix B is invertible then so is its inverse (B^{-1}).

Exercise 1.8.84 Prove that if two rows of a matrix A are equal then $|A| = 0$.

Exercise 1.8.85 A square matrix A is called skew-symmetric if $A^T = -A$. Prove that if A and B are $n \times n$ skew-symmetric matrices, then $A + B$ is skew-symmetric.

Exercise 1.8.86 With relevant examples where necessary, define the following :

1.) A Matrix

2.) A symmetric matrix

3.) A Triangular matrix

4.) An Idempotent matrix

5.) A non-singular(invertible) matrix

6.) a Tri-diagonal matrix

Exercise 1.8.87 Use properties of determinants to find the determinants of the following matrices.

1.) $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ c & 0 & d & 0 \\ e & f & g & h \end{bmatrix}$

2.) $\begin{bmatrix} 3 & 2 & 1 & 6 \\ 4 & 5 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & 10 & 11 \end{bmatrix}$

3.) $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -5 & -3 & -2 \\ 1 & -2 & 0 & 3 \\ 6 & 10 & 18 & 8 \end{bmatrix}$

4.) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 9 \\ 3 & 0 & 1 \end{bmatrix}$

5.) $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$

Exercise 1.8.88 An $n \times n$ matrix is said to be idempotent if $A^2 = A$. Show that A is idempotent, where

$$A = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Prove that if A is idempotent so is $(I_n - A)$.

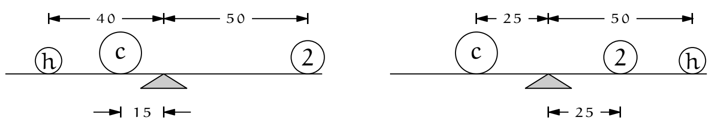
Chapter 2

Simultaneous Linear Systems

2.1 Solving Linear Systems

Systems of linear equations are common in science and mathematics. These two examples from high school science give a sense of how they arise.

Example 2.1.1 The first example is from Statics. Suppose that we have three objects, one with a mass known to be 2 kg and we want to find the unknown masses. Suppose further that experimentation with a meter stick produces these two balances.



For the masses to balance we must have that the sum of moments on the left equals the sum of moments on the right, where the moment of an object is its mass times its distance from the balance point. That gives a system of two equations.

$$\begin{aligned} 40h + 15c &= 100 \\ 25c &= 50 + 50h \end{aligned}$$

Example 2.1.2 The second example of a linear system is from Chemistry. We can mix, under controlled conditions, toluene C_7H_8 and nitric acid HNO_3 to produce trinitrotoluene $C_7H_5O_6N_3$ along with the byproduct water (conditions have to be very well controlled - trinitrotoluene is better known as TNT). In what proportion should we mix them? The number of atoms of each element present before the reaction

$$x\, C_7H_8 \;+\; y\, HNO_3 \quad\longrightarrow\quad z\, C_7H_5O_6N_3 \;+\; w\, H_2O$$

must equal the number present afterward. Applying that in turn to the elements C, H, N, and O gives this system.

$$\begin{aligned} 7x &= 7z \\ 8x + 1y &= 5z + 2w \\ 1y &= 3z \\ 3y &= 6z + 1w \end{aligned}$$

Example 2.1.3 A very simple supply and demand model might look like:

$$\begin{aligned}\text{Demand}: &\quad Q = \alpha_1 P + \alpha_2 X + e_d \\ \text{Supply}: &\quad Q = \beta_1 P + e_s\end{aligned}$$

It takes two equations to describe the supply and demand equilibrium

- The two equilibrium values, for price and quantity, $P^\star$ and $Q^\star$, respectively, are determined at the same time
- In this model the variables P and Q are called endogenous variables because their values are determined within the system we have created

The endogenous variables P and Q are dependent variables and both are random variables.

The income variable X has a value that is determined outside this system - Such variables are said to be exogenous, and these variables are treated like usual "x" explanatory variables. The terms e_d, e_s are the error terms

Note 2.1.1 All examples come down to solving a system of equations.

Remark 2.1.1 In each system, the equations involve only the first power of each variable. This chapter shows how to solve any such system.

2.2 A Linear System of Equations

A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where b and the coefficients $a_1, a_2, \ldots, a_n$ are real numbers, usually known in advance for example $4x_1 + 5x_2 - x_3 = 10$ is a linear equation in three variables.

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same set of variables (unknowns). The general form of a linear system in the variables $x_1, x_2, \ldots, x_n$ is

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= .. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= .. \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

This could be written in a short form as

$$A\boldsymbol{x} = \boldsymbol{b}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is called the **Coefficient matrix**,

$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ the column vector of unknowns, $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)^T$ the righthand column vector.

The matrix $\bar{A} = (A : \boldsymbol{b}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{pmatrix}$ is called the **augmented matrix** of the system $A\boldsymbol{x} = \boldsymbol{b}$.

Remark 2.2.1 m does not need to be equal to n, and the coefficients $a_{ij}, i, j = 1, 2, \ldots, n$, do not need to be integers, nor even "nice" real numbers.

Definition 2.2.1 A solution of the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is a list $(s_1, s_2, s_3, \ldots, s_n)$ of numbers that make each equation in the system true when $(s_1, s_2, s_3, \ldots, s_n)$ are substituted for $(x_1, x_2, x_3, \ldots, x_n)$ respectively for example $(x, y) = (1, 1)$ is a solution to the linear system

$$\begin{array}{rcrcl} x & + & y & = & 2 \\ 2x & + & 3y & = & 5 \end{array}$$

There are three ways of solving a system of linear equations that is;

1). Row reduction to Echelon form

2). Crammer's rule.

3). Using the direct method $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

2.3 Row Reduction to Echelon Form

Definition 2.3.1 A matrix is said to be in **Row Echelon Form** if it satisfies all three of the following conditions.

1.) If there are any rows of all zeros then they are at the bottom of the matrix.

2.) If a row does not consist of all zeros then its first non-zero entry (i.e. the left most nonzero entry) is a 1. This 1 is called a leading 1.

3.) In any two successive rows, neither of which consists of all zeroes, the leading 1 of the lower row is to the right of the leading 1 of the higher row.

Definition 2.3.2 A matrix is in **"Reduced" Row Echelon Form (RREF)** if the three conditions $(1) - (3)$ in Definition 2.3.1 hold and in addition, we have

4.) If a column contains a leading one, then all the other entries in that column are zero.

Example 2.3.1 Examples of matrices in row echelon form, **but not** in RREF

1.) $\begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2.) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

3.) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

4.) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

5.) $\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6.) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Example 2.3.2 The following examples are **not** in Echelon form

1.) $\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, second row, a zero row to be last row.

2.) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$, The leading entry in Row 1 is to the right of the leading entry in Row 2.

3.) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, The leading entries in Row 2 and 3 are in the same column.

4.) $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \end{bmatrix}$ 5.) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ 6.) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Example 2.3.3 Examples of matrices in Reduce Roe echelon Form - RREF

1.) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 3.) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 5.) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

2.) $\begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 4.) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 6.) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Remark 2.3.1 Further we notice from these definitions that a matrix that is in reduced row-echelon form is also in row Echelon form while a matrix in row-Echelon form may or may not be in reduced row- Echelon form. This therefore means that given any $m \times n$ matrix we may change it to Row Echelon form by applying elementary row operations.

Recall the following definition,

Definition 2.3.3 An elementary row operation is any one of the following

1) Interchanging two rows

2) Multiplying a row by a nonzero constant

3) Adding or subtracting a multiple of a row to another row

2.3.1 Methodology of Row Reduction for Solutions

To solve a linear system $A\boldsymbol{x} = \boldsymbol{b}$ we,

1. Form the augmented matrix $\bar{A} = (A : \boldsymbol{b})$
2. Reduce the matrix $\bar{A} = (A : \boldsymbol{b})$ to Row Echelon Form and
3. Use back substitution to get the solution.

Key Idea 2.1

Elementary Row Operations

1.) Add a scalar multiple of one row to another row, and replace the latter row with that sum

2.) Multiply one row by a nonzero scalar

3.) Swap the position of two rows

Example 2.3.4 Row reduce the matrix A by Gaussian elimination

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

1.) We begin with **first column** (*whole*) and we skip **the first entry**, and we make others zeros in that column, that is, we need $a_{21} = 0, a_{31} = 0$ and all below it say $a_{41} = 0, a_{51} = 0, a_{61} = 0$ if exists in the matrix or system given to you.

Operations for first column we apply the pivot in R_1.

That is,

$$R_1 \to R_1, \quad 2R_1 - 3R_2 \to R_2, \quad R_1 + 3R_3 \to R_3$$

to have

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 5 & 1 \end{pmatrix}$$

2.) In **second column**, we skip **the first two entries**, and we make others zeros in that column, that is we need $a_{32} = 0$ and all below it say $a_{42} = 0, a_{52} = 0, a_{62} = 0$ if exists in the matrix or system given to you.

But when looking for operations for second column, we only apply the pivot in R_2 called the pivot

Let

$$R_1 \to R_1, \quad R_2 \to R_2, \quad 5R_2 - R_3 \to R_3$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -6 \end{pmatrix}$$

3.) In **third column**, we skip **the first three entries**, and we make others zeros in that column, that is we would need $a_{43} = 0, a_{53} = 0, a_{63} = 0$ if are in the matrix

For the matrix given, the terms a_{43}, a_{53}, a_{63} do not exist. Therefore the end of raw reductions. Thus the reduced matrix is

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -6 \end{pmatrix}$$

Example 2.3.5 Perform the Gaussian elimination row reduction on the matrix below.

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 4 & 2 & 3 & 2 \\ 5 & 1 & 1 & -2 \\ 6 & 4 & -1 & -2 \\ 1 & 5 & 1 & -1 \\ 3 & 1 & 4 & 3 \end{bmatrix}$$

1.) For the first column, skip the first entry and make others zeros.

We can apply the following row reduction operations

$$\begin{array}{rcrcl} & & R_1 & \rightsquigarrow & R_1 \\ 2R_1 & - & R_2 & \rightsquigarrow & R_2 \\ 5R_1 & - & 2R_3 & \rightsquigarrow & R_3 \\ 3R_1 & - & R_4 & \rightsquigarrow & R_4 \\ R_1 & - & 2R_5 & \rightsquigarrow & R_5 \\ 3R_1 & - & 2R_6 & \rightsquigarrow & R_6 \end{array}$$

Which generates

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & 4 & -5 & 2 \\ 0 & 13 & -7 & 14 \\ 0 & 5 & -2 & 8 \\ 0 & -7 & -3 & 4 \\ 0 & 7 & -11 & 0 \end{bmatrix}$$

2.) For the second column, skip the first two entries and make others zeros.

Applying the following row reduction operations (are not unique, you can think of others to use)

$$\begin{array}{rcrcl} & & R_1 & \rightsquigarrow & R_1 \\ & & R_2 & \rightsquigarrow & R_2 \\ 13R_2 & - & 4R_3 & \rightsquigarrow & R_3 \\ 5R_2 & - & 4R_4 & \rightsquigarrow & R_4 \\ 7R_2 & + & 4R_5 & \rightsquigarrow & R_5 \\ 7R_2 & - & 4R_6 & \rightsquigarrow & R_6 \end{array}$$

to generate

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & 4 & -5 & 2 \\ 0 & 0 & -37 & -30 \\ 0 & 0 & -17 & -22 \\ 0 & 0 & -47 & 30 \\ 0 & 0 & 9 & 14 \end{bmatrix}$$

3.) For the third column, skip the first three entries and make others zeros.

Applying the following row reduction operations

$$\begin{array}{rcrcl} & & R_1 & \rightsquigarrow & R_1 \\ & & R_2 & \rightsquigarrow & R_2 \\ & & R_3 & \rightsquigarrow & R_3 \\ 17R_3 & - & 37R_4 & \rightsquigarrow & R_4 \\ 47R_3 & - & 37R_5 & \rightsquigarrow & R_5 \\ 9R_3 & + & 37R_6 & \rightsquigarrow & R_6 \end{array}$$

to generate

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & 4 & -5 & 2 \\ 0 & 0 & -37 & -30 \\ 0 & 0 & 0 & 304 \\ 0 & 0 & 0 & 2520 \\ 0 & 0 & 0 & 248 \end{bmatrix}$$

4.) For the fourth column, skip the first four entries and make others zeros.

Applying the following row reduction operations

$$\begin{array}{rcrcl} & & R_1 & \rightsquigarrow & R_1 \\ & & R_2 & \rightsquigarrow & R_2 \\ & & R_3 & \rightsquigarrow & R_3 \\ & & R_4 & \rightsquigarrow & R_4 \\ 2520R_4 & - & 304R_5 & \rightsquigarrow & R_5 \\ 248R_4 & - & 304R_6 & \rightsquigarrow & R_6 \end{array}$$

to generate

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & 4 & -5 & 2 \\ 0 & 0 & -37 & -30 \\ 0 & 0 & 0 & 304 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since done with all the rows in the matrix, we have completed the row reductions.

2.4 Existence of a Solution to a Linear System

A linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$ will either have;

1.) Unique solution (one solution)

2.) No solution

3.) Infinitely many solutions.

Let our $\bar{A}$ be the **row reduced** form (Echelon) of the augmented matrix of the given system. Let r be the number of non zero rows of the "reduced" augmented matrix and n be the number of unknowns of the linear system, the following are the conditions for the system to have no solution, unique solution and infinitely many solutions.

2.4.1 Unique Solution

This occurs when $n = r$ meaning non of the rows of the matrix is zero in the row reduced form of the augmented matrix $\bar{A}$.

Example 2.4.1 Solve the linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ 3x_1 & - & 2x_2 & + & 3x_3 & = & 8 \\ -2x_1 & + & 4x_2 & - & 3x_3 & = & -3 \end{array}$$

Reducing the augmented matrix by row reduction

$$\left[\begin{array}{rrrcr} 1 & 1 & 1 & | & 6 \\ 3 & -2 & 3 & | & 8 \\ -2 & 4 & -3 & | & -3 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ 3R_1 - R_2 & \rightsquigarrow & R_2 \\ 2R_1 + R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrrcr} 1 & 1 & 1 & | & 6 \\ 0 & 5 & 0 & | & 10 \\ 0 & 6 & -1 & | & 9 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ 6R_2 - 5R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrrcr} 1 & 1 & 1 & | & 6 \\ 0 & 5 & 0 & | & 10 \\ 0 & 0 & 5 & | & 15 \end{array}\right]$$

Therefore, by backward substitution,

$$\begin{array}{rcrcrcrcl} & & & & 5x_3 & = & 15 & \Rightarrow & x_3 = 3 \\ & & 5x_2 & & & = & 10 & \Rightarrow & x_2 = 2 \\ x_1 & + & x_2 & + & x_3 & = & 6 & \Rightarrow & x_1 = 1 \end{array}$$

such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example 2.4.2 Solve the linear system

$$\begin{array}{rcrcr} 3x & + & 4y & = & -2 \\ -3x & - & y & = & 5 \end{array}$$

Reducing the augmented matrix by row reduction

$$\left[\begin{array}{rrcr} 3 & 4 & | & -2 \\ -3 & -1 & | & 5 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_1 + R_2 & \rightsquigarrow & R_2 \\ & = & \end{array}$$

$$\left[\begin{array}{rrcr} 3 & 4 & | & -2 \\ 0 & 3 & | & 3 \end{array}\right]$$

Therefore, by backward substitution,

$$\begin{array}{rcrcrcl} & & 3y & = & 3 & \Rightarrow & y = 1 \\ 3x & + & 4y & = & -2 & \Rightarrow & x = -2 \end{array}$$

such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example 2.4.3 Solve the linear system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & 0 \\ -x_1 & + & 4x_2 & + & 3x_3 & = & 0 \end{array}$$

Reducing the augmented matrix by row reduction

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 0 \\ -1 & 4 & 3 & 0 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ 2R_1 - R_2 & \rightsquigarrow & R_2 \\ R_1 + R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ 5R_2 + R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

Therefore, by backward substitution,

$$\begin{array}{rcrcrcccl} & & & & 4x_3 & = & 0 & \Rightarrow & x_3 = 0 \\ & & -x_2 & & & = & 0 & \Rightarrow & x_2 = 0 \\ x_1 & + & x_2 & + & x_3 & = & 0 & \Rightarrow & x_1 = 0 \end{array}$$

such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A unique **trivial** (all zeros) solution.

Example 2.4.4 Solve the linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 2x_2 & - & 3x_3 & = & 0 \end{array}$$

Raw reducing the augmented matrix, we generate

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 3 & 1 & 1 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_1 - 3R_2 & \rightsquigarrow & R_2 \\ 2R_1 - R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ 2R_2 - 5R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & -21 & 0 \end{array}\right]$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 2.4.5 Solve the linear system

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 2x_3 & = & 32 \\ x_1 & - & 2x_2 & + & x_3 & = & -5 \\ x_1 & - & x_2 & - & x_3 & = & 1 \end{array}$$

Reducing the augmented matrix by row reduction

$$\left[\begin{array}{rrr|r} 2 & 3 & 2 & 32 \\ 1 & -2 & 1 & -5 \\ 1 & -1 & -1 & 1 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_1 - 2R_2 & \rightsquigarrow & R_2 \\ R_1 - 2R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrr|r} 2 & 3 & 2 & 32 \\ 0 & 7 & 0 & 42 \\ 0 & 5 & 4 & 30 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ 5R_2 - 7R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrr|r} 2 & 3 & 2 & 32 \\ 0 & 7 & 0 & 42 \\ 0 & 5 & -28 & 0 \end{array}\right]$$

Therefore, by backward substitution,

$$\begin{array}{rcrcrcrcl} & & & & -28x_3 & = & 0 & \Rightarrow & x_3 = 0 \\ & & 7x_2 & & & = & 42 & \Rightarrow & x_2 = 6 \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & 32 & \Rightarrow & x_1 = 7 \end{array}$$

such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix}$$

Exercise 2.4.1 In the exercises, a matrix A and a vector $\vec{b}$ are given. Solve the equation $A\vec{x} = \vec{b}$.

1.) $A = \begin{bmatrix} 1 & 2 & 12 \\ 0 & 1 & 6 \\ -3 & 0 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -17 \\ -5 \\ 20 \end{bmatrix}$ $\qquad \vec{x} = \begin{bmatrix} -7 \\ 1 \\ -1 \end{bmatrix}$

2.) $A = \begin{bmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 2 & -8 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -69 \\ 10 \\ -102 \end{bmatrix}$ $\qquad \vec{x} = \begin{bmatrix} -9 \\ 10 \\ -4 \end{bmatrix}$

3.) $A = \begin{bmatrix} 5 & 0 & -2 \\ -8 & 1 & 5 \\ -2 & 0 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 33 \\ -70 \\ -15 \end{bmatrix}$ $\qquad \vec{x} = \begin{bmatrix} 3 \\ -1 \\ -9 \end{bmatrix}$

Example 2.4.6 Rewrite the system in matrix form and solve it by Gaussian Elimination (Gauss-Jordan elimination)

$$\begin{array}{rcrcrcr} -5x_1 & + & x_2 & + & x_3 & = & -13 \\ 9x_1 & + & 2x_2 & - & 5x_3 & = & 17 \\ x_1 & + & 5x_2 & + & x_3 & = & 5 \end{array}$$

The solution is given by

$$\begin{array}{rcl} x_1 & = & 3 \\ x_2 & = & 0 \\ x_3 & = & 2 \end{array}$$

2.4.2 No Solution

When $n = r$ in a row reduced and there is a row whose only non zero element is in the last column of $\bar{A}$, then the linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has no solution. Indeed in the row reduced form, such a system is said to be inconsistent.

Example 2.4.7 Solve the linear system

$$\begin{aligned} x_1 - 3x_2 + 5x_3 &= 3 \\ x_2 + 3x_3 &= 2 \\ x_2 + 3x_3 &= 5 \end{aligned}$$

Reducing the augmented matrix by row reduction

$$\left[\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 5 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ R_2 - R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Operations started with the second column, as the first was already in a reduced form.

By backward substitution,

$$0x_3 \ = \ 3 \ \Rightarrow \ x_3 = \frac{3}{0}$$

Does not exist, since cannot divide any number by zero. The equation is algebraically undefined, thus the linear system is inconsistent. Therefore, no solutions.

Example 2.4.8 Solve the linear system

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 2 \\ 2x & + & 3y & + & 3z & = & 3 \\ -3x & - & 4y & - & 5z & = & -5 \end{array}$$

Row reducing the augmented matrix $\bar{A} = (A, \boldsymbol{b})$ by row reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 3 & 3 & 3 \\ -3 & -4 & -5 & -5 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ 2R_1 - R_2 & \rightsquigarrow & R_2 \\ 3R_1 + R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 1 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ 2R_2 - R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

By backward substitution, $0x_3 = 1 \quad \Rightarrow \quad x_3 = \dfrac{1}{0}$. The equation is algebraically undefined, thus the linear system is inconsistent. Therefore, no solutions.

Exercise 2.4.2 Find the solution to the linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & 0 \end{array}$$

No solution exists. This system is inconsistent

Example 2.4.9 Solve

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & x_3 & = & 4 \\ 2x_1 & + & 2x_2 & + & x_3 & = & -1 \\ 4x_1 & + & 8x_2 & + & 3x_3 & = & 2 \end{array}$$

The augmented matrix is

Row reducing the augmented matrix $\bar{A} = (A, \boldsymbol{b})$ by row reduction

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2 & 2 & 1 & -1 \\ 4 & 8 & 3 & 2 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 - 2R_1 & \rightsquigarrow & R_2 \\ R_3 - 4R_1 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & -4 & -1 & -9 \\ 0 & -4 & -1 & -14 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ R_3 - R_2 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & -4 & -1 & -9 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

By backward substitution,

$$0x_3 = -5 \quad \Rightarrow \quad x_3 = \frac{-5}{0}$$

The equation is algebraically undefined, thus the linear system is inconsistent. It has no solution.

2.4.3 Infinitely Many Solutions

If in the row reduced form of $\bar{A}$, we have $n > r$, then the system has more than one solution thus it has infinitely many solutions depending on the choice of different parameters. Such a linear system is consistent.

Example 2.4.10 Solve the linear system

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 1 \\ 4x_1 & - & 2x_2 & + & 8x_3 & = & 1 \\ 3x_1 & - & x_2 & + & 7x_3 & = & 0 \\ 6x_1 & - & 4x_2 & + & 10x_3 & = & 3 \end{array}$$

Row reducing the augmented matrix $\bar{A} = (A, \boldsymbol{b})$ by row reduction

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 4 & -2 & 8 & 1 \\ 3 & -1 & 7 & 0 \\ 6 & -4 & 10 & 3 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ 4R_1 - R_2 & \rightsquigarrow & R_2 \\ 3R_1 - R_3 & \rightsquigarrow & R_3 \\ 6R_1 - R_4 & \rightsquigarrow & R_4 \\ & = & \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & -2 & -4 & 3 \\ 0 & -2 & -4 & 3 \\ 0 & -2 & -4 & 3 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ R_2 - R_3 & \rightsquigarrow & R_3 \\ R_2 - R_4 & \rightsquigarrow & R_4 \\ & = & \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & -2 & -4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving using backward substitution,

$$-\ 2x_2 \ -\ 4x_3 \ =\ 3$$

One equations ($r = 1$) with two unknowns ($n = 2$) (impossible to solve) we have to let one known, so let

$$x_3 = p, \ \Rightarrow \ -2x_2 - 4p = 3 \ \Rightarrow \ x_2 = \frac{3 + 4p}{-2} = -\frac{3}{2} - \frac{4p}{2}$$

Using Row 1, by backward substitution,

$$x_1 - x_2 + x_3 = 1 \ \Rightarrow \ x_1 - \left[-\frac{3}{2} - \frac{4p}{2}\right] + p = 1 \ \Rightarrow \ x_1 = -\frac{1}{2} - 3p$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - 3p \\ -\frac{3}{2} - \frac{4p}{2} \\ p \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 0 \end{bmatrix} + p \begin{bmatrix} -3 \\ -\frac{4}{2} \\ 1 \end{bmatrix}$$

Since p can take on many different values, then the system has infinitely many solutions.

Example 2.4.11 Solve the linear system

$$\begin{array}{rcrcrcrcrcr} x_1 & - & 3x_2 & & & + & 5x_4 & & & = & 4 \\ -x_1 & + & 3x_2 & + & x_3 & - & 3x_4 & & & = & -11 \\ -x_1 & + & 3x_2 & & & - & 5x_4 & + & x_5 & = & -3 \\ 3x_1 & - & 9x_2 & & & + & 15x_4 & & & = & 12 \end{array}$$

Reducing the augmented matrix by Gaussian elimination, we get,

$$\left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 5 & 0 & 4 \\ -1 & 3 & 1 & -3 & 0 & -11 \\ -1 & 3 & 0 & -5 & 1 & -3 \\ 3 & -9 & 0 & 15 & 0 & 12 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_1 + R_2 & \rightsquigarrow & R_2 \\ R_1 + R_3 & \rightsquigarrow & R_3 \\ 3R_1 - R_4 & \rightsquigarrow & R_4 \\ & = & \end{array}$$

$$\left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

By backward substitution

$$x_5 = 1 \tag{2.1}$$

and

$$x_3 + 2x_4 = -7$$

One equation yet two unknowns, we let one be known say, let

$$x_4 = s \tag{2.2}$$
$$\Rightarrow x_3 = -7 - 2s \tag{2.3}$$

Using the first equation

$$\begin{aligned} x_1 - 3x_2 + 5x_4 &= 4 \\ x_1 - 3x_2 + 5s &= 4 \\ x_1 - 3x_2 &= 4 - 5s \end{aligned}$$

One equation with two unknowns, again we let, say let

$$x_2 = p \tag{2.4}$$
$$\Rightarrow x_1 = 4 - 5s + 3p \tag{2.5}$$

such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 - 5s + 3p \\ p \\ 7 - 2s \\ s \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 7 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + p \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

You could have realized that $s\&p$ could take on many values, thus giving us many answers.

Note 2.4.1 So there are many types of equations, some have infinitely many solutions, others one (unique solution), yet others do not have any solution.

Example 2.4.12 Solve

$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & - & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & & & - & x_4 & = & -1 \\ x_1 & - & 6x_2 & + & 3x_3 & - & 8x_4 & = & 7 \end{array}$$

Reducing the augmented matrix we get,

$$\left[\begin{array}{rrrrcr} 3 & 2 & 1 & -4 & | & 1 \\ 2 & 3 & 0 & -1 & | & -1 \\ 1 & -6 & 3 & -8 & | & 7 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ 2R_1 - 3R_2 & \rightsquigarrow & R_2 \\ R_1 - 3R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrrrcr} 3 & 2 & 1 & -4 & | & 1 \\ 0 & -5 & 2 & -5 & | & 5 \\ 0 & 20 & -8 & 20 & | & -20 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ 4R_2 + R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrrrcr} 3 & 2 & 1 & -4 & | & 1 \\ 0 & -5 & 2 & -5 & | & 5 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right]$$

By backward substitution, we have

$$-5x_2 + 2x_3 - 5x_4 = 5$$

One equation with three unknowns, thus we have to let some two variables to be known, so we let

$$\begin{array}{rrcl} & x_4 & = & t \\ & x_3 & = & s \\ \Rightarrow & x_2 & = & -1 - t + \dfrac{2}{5}s \end{array}$$

Using the first row, $3x_1 + 2x_2 + x_3 - 4x_4 = 1$ to have

$$x_1 \quad = \quad \frac{1}{3} + \frac{4}{3}t - \frac{1}{3}s + \frac{2}{3} + \frac{2}{3}t - \frac{4}{15}s = 1 + 2t - \frac{3}{5}s$$

with $s, t \in \mathbb{R}$.

The solutions in parametric vector form is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 + 2t - \dfrac{3}{5}s \\ -1 - t + \dfrac{2}{5}s \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\dfrac{3}{5} \\ \dfrac{2}{5} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}; \quad s, t \in \mathbb{R}$$

Infinitely many solutions, since for every different values of t and s, we generate a new solution.

Example 2.4.13 Find conditions that must be satisfied for the system to be consistent

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

Reducing the augmented matrix is

$$\left[\begin{array}{rrrcc} 1 & -2 & 5 & | & b_1 \\ 4 & -5 & 8 & | & b_2 \\ -3 & 3 & -3 & | & b_3 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 - 4R_1 & \rightsquigarrow & R_2 \\ 3R_1 + R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrrcc} 1 & -2 & 5 & | & b_1 \\ 0 & 3 & -12 & | & b_2 - 4b_1 \\ 0 & -3 & 12 & | & b_3 + 3b_1 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ R_2 + R_3 & \rightsquigarrow & R_3 \\ & = & \end{array}$$

$$\left[\begin{array}{rrrcc} 1 & -2 & 5 & | & b_1 \\ 0 & 3 & -12 & | & b_2 - 4b_1 \\ 0 & 0 & 0 & | & b_3 + 3b_1 + b_2 - 4b_1 \end{array}\right]$$

Which simplifies to

$$\left[\begin{array}{rrrcc} 1 & -2 & 5 & | & b_1 \\ 0 & 3 & -12 & | & b_2 - 4b_1 \\ 0 & 0 & 0 & | & b_3 + b_2 - b_1 \end{array}\right]$$

Thus the system is consistent only iff

$$b_2 + b_3 - b_1 = 0 \ \Rightarrow \ b_1 = b_2 + b_3$$

In which case the solution is

$$\begin{aligned} x_2 &= \frac{1}{3}(b_2 - 4b_1 + 12s), \quad s \in \mathbb{R} \\ x_1 &= b_1 - 5s + 2x_2 \\ &= b_1 - 5s + \frac{2}{3}b_2 - \frac{8}{3}b_1 + 8s \\ &= \frac{-5}{3}b_1 + \frac{2}{3}b_2 + 3s \end{aligned}$$

or equivalently

$$\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} -5/3 \\ -4/3 \end{array}\right] b_1 + \left[\begin{array}{c} 2/3 \\ 1/3 \end{array}\right] b_2 + \left[\begin{array}{c} 3 \\ 12/3 \end{array}\right] s; \quad s \in \mathbb{R}$$

Exercise 2.4.3 Find the solution to the linear system

$$\begin{array}{rrrrcr} & x_2 & - & x_3 & = & 3 \\ x_1 & & + & 2x_3 & = & 2 \\ & -3x_2 & + & 3x_3 & = & -9 \end{array}$$

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{c} 2 - 2t \\ 3 + t \\ t \end{array}\right]$$

Example 2.4.14 Solve the linear system

$$\begin{array}{ccccccccc} x_1 & + & x_2 & - & 2x_3 & + & x_4 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & = & 0 \end{array}$$

Gaussian elimination reduced the augmented matrix to

$$\left[\begin{array}{cccccc} 1 & 1 & -2 & 1 & | & 0 \\ 1 & 2 & 1 & 3 & | & 0 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 - R_1 & \rightsquigarrow & R_2 \\ & = & \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 1 & -2 & 1 & | & 0 \\ 0 & 1 & 3 & 2 & | & 0 \end{array}\right]$$

To solve, by backward substitutions, one equation with three unknowns, we let

$$x_3 = p \tag{2.6}$$

$$x_4 = s \tag{2.7}$$

Therefore, by the first equation,

$$x_2 = -3p - 2s \tag{2.8}$$

$$x_1 = 5p + 2s \tag{2.9}$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5p + 2s \\ -3p - 2s \\ p \\ s \end{bmatrix} = p \begin{bmatrix} 5 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

signifying infinitely many solutions.

Example 2.4.15

$$\begin{array}{ccccccc} 5x_1 & - & x_2 & + & 3x_3 & = & 0 \\ 4x_1 & - & 3x_2 & + & 7x_3 & = & 0 \end{array}$$

Reducing by Gaussian elimination,

$$\left[\begin{array}{ccccc} 5 & -1 & 3 & | & 0 \\ 4 & -3 & 7 & | & 0 \end{array}\right] \begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ 4R_1 - 5R_2 & \rightsquigarrow & R_2 \\ & = & \end{array}$$

$$\left[\begin{array}{ccccc} 5 & -1 & 3 & | & 0 \\ 0 & 11 & -23 & | & 0 \end{array}\right]$$

Let x_3 be the free variable, that is, let

$$x_3 = t$$

to have general solution as

$$[x_1, x_2, x_3] = \left[-\frac{2}{11}t,\ \frac{23}{11}t,\ t\right],\ t \in \mathbb{R}$$

Example 2.4.16 Solve the linear system of equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 7x_3 & = & 0 \\ -2x_1 & - & 3x_2 & + & 9x_3 & = & 0 \\ -2x_2 & & & + & 10x_3 & = & 0 \end{array}$$

The reduced augmented matrix is given by

$$\left[\begin{array}{rrr|r} 1 & 2 & -7 & 0 \\ -2 & -3 & 9 & 0 \\ 0 & -2 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & -7 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & -7 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving by backward substitution and we let $x_3 = t$ the general solution is $(3t, 5t, t), t \in \mathbb{R}$ or in its vector parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

Exercise 2.4.4 Solve the following linear system of equations

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & + & x_3 & - & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & & & - & x_4 & = & -1 \\ x_1 & - & 6x_2 & + & 3x_3 & - & 8x_4 & = & 7 \end{array}$$

2.5 Computer Algebra Systems

The linear systems in this chapter are small enough that their solution by hand is easy. But large systems are easiest, and safest, to do on a computer. There are special purpose programs such as LINPACK for this job. Another popular tool is a general purpose computer algebra system, including both commercial packages such as Maple, Mathematica, or MATLAB, or free packages such as Sage.

For example, in the Topic on Networks, we need to solve this.

$$\begin{array}{rcl}
i_0 - i_1 - i_2 & = & 0 \\
i_1 - i_3 - i_5 & = & 0 \\
i_2 - i_4 + i_5 & = & 0 \\
i_3 + i_4 - i_6 & = & 0 \\
5i_1 + 10i_3 & = & 10 \\
2i_2 + 4i_4 & = & 10 \\
5i_1 - 2i_2 + 50i_5 & = & 0
\end{array}$$

We could do this by hand but it would take a while and be error-prone. Using a computer is better.

We illustrate by solving that system under Sage.

```
sage: var('i0,i1,i2,i3,i4,i5,i6')
(i0, i1, i2, i3, i4, i5, i6)
sage: network_system=[i0-i1-i2==0, i1-i3-i5==0,
....:          i2-i4+i5==0,, i3+i4-i6==0, 5*i1+10*i3==10,
....:          2*i2+4*i4==10, 5*i1-2*i2+50*i5==0]
sage: solve(network_system, i0,i1,i2,i3,i4,i5,i6)
[[i0 == (7/3), i1 == (2/3), i2 == (5/3), i3 == (2/3),
      i4 == (5/3), i5 == 0, i6 == (7/3)]]
```

Magic.

Here is the same system solved under Maple. We enter the array of coefficients and the vector of constants, and then we get the solution.

```
> A:=array( [[1,-1,-1,0,0,0,0],
             [0,1,0,-1,0,-1,0],
             [0,0,1,0,-1,1,0],
             [0,0,0,1,1,0,-1],
             [0,5,0,10,0,0,0],
             [0,0,2,0,4,0,0],
             [0,5,-2,0,0,50,0]] );
> u:=array( [0,0,0,0,10,10,0] );
> linsolve(A,u);
        7   2   5   2   5       7
     [ -,  -,  -,  -,  -,  0,  - ]
        3   3   3   3   3       3
```

If a system has infinitely many solutions then the program will return a parametrization.

Exercise 2.5.1

1.) Use the computer to solve the two problems that opened this chapter.

(a) This is the Statics problem.

$$\begin{aligned} 40h + 15c &= 100 \\ 25c &= 50 + 50h \end{aligned}$$

Solution : *The commands*

```
> A:=array( [[40,15],
             [-50,25]] );
> u:=array([100,50]);
> linsolve(A,u);
```

yield the answer $[1, 4]$. ■

(b) This is the Chemistry problem.

$$\begin{aligned} 7h &= 7j \\ 8h + 1i &= 5j + 2k \\ 1i &= 3j \\ 3i &= 6j + 1k \end{aligned}$$

Solution :

```
> A:=array( [[7,0,-7,0],
             [8,1,-5,2],
             [0,1,-3,0],
             [0,3,-6,-1]] );
> u:=array([0,0,0,0]);
> linsolve(A,u);
```

prompts the reply $[t_1, 3t_1, t_1, 3t_1]$. ■

2.) Use the computer to solve these systems below, or conclude 'many solutions' or 'no solutions'.

(a)

$$\begin{aligned} 2x + 2y &= 5 \\ x - 4y &= 0 \end{aligned}$$

Solution :

```
> A:=array( [[2,2],
             [1,-4]] );
> u:=array([5,0]);
> linsolve(A,u);
```

gives the expected answer of $\left[2, \frac{1}{2}\right]$. *The answer is* $x = 2$ *and* $y = \frac{1}{2}$ *The others are similar.* ■

(b)

$$\begin{aligned} -x + y &= 1 \\ x + y &= 2 \end{aligned}$$

Solution : *The answer is* $x = \frac{1}{2}$ *and* $y = \frac{3}{2}$. ■

(c)

$$\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$$

Solution : *This system has infinitely many solutions. In the first subsection, with z as a parameter, we got $x = \dfrac{(43 - 7z)}{4}$ and $y = \dfrac{(13 - z)}{4}$. Maple responds with $[-12 + 7t_1, t_1, 13 - 4t_1]$, for some reason preferring y as a parameter.* ■

(d)

$$\begin{aligned} -x - y &= 1 \\ -3x - 3y &= 2 \end{aligned}$$

Solution : *There is no solution to this system. When the array A and vector u are given to Maple and it is asked to* `linsolve(A,u)`*, it returns no result at all, that is, it responds with no solutions.* ■

(e)

$$\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ x \qquad + z &= 5 \\ x + y - z &= 10 \end{aligned}$$

Solution : *The solutions is $(x, y, z) = (5, 5, 0)$.* ■

(f)

$$\begin{aligned} 2x \qquad + z + w &= 5 \\ y \qquad - w &= -1 \\ 3x \qquad - z - w &= 0 \\ 4x + y + 2z + w &= 9 \end{aligned}$$

Solution : *There are many solutions. Maple gives $[1, -1 + t_1, 3 - t_1, t_1]$.* ■

Exercise 2.5.2

1.) Use the computer to solve these systems

(a)

$$\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned}$$

Solution : *This system has infinitely many solutions. Maple responds with $[6 - 2t_1, t_1]$.* ■

(b)

$$\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned}$$

Solution : *The solution set has only one member. Maple has no trouble finding it $[0, 1]$.* ■

(c)

$$\begin{aligned} x_1 \qquad + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned}$$

Solution : *This system's solution set is infinite. Maple gives*

$$[t_1, -t_1 + 3, -t_1 + 4]$$

■

(d)

$$\begin{aligned} 2a + b - c &= 2 \\ 2a + c &= 3 \\ a - b &= 0 \end{aligned}$$

Solution : *There is a unique solution. Maple gives* $[1, 1, 1]$. ■

(e)

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + y + w &= 4 \\ x - y + z + w &= 1 \end{aligned}$$

Solution : *This system has infinitely many solutions: Maple generates* $[3 - 2t_1 + t_2, t_1, t_2, -2 + 3t_1 - 2t_2]$. ■

(f)

$$\begin{aligned} x + z + w &= 4 \\ 2x + y - w &= 2 \\ 3x + y + z &= 7 \end{aligned}$$

Solution : *The solution set is empty and Maple replies to the* ***`linsolve(A,u)`*** *command with no returned solutions.* ■

2.) What does the computer give for the solution of the general 2×2 system?

$$\begin{aligned} ax + cy &= p \\ bx + dy &= q \end{aligned}$$

Solution : *In response to this prompting*

```
> A:=array( [[a,c],
             [b,d]] );
> u:=array([p,q]);
> linsolve(A,u);
```

Maple thought for perhaps twenty seconds and gave this reply.

$$\left[-\frac{-d\,p + q\,c}{-b\,c + a\,d}, \frac{-b\,p + a\,q}{-b\,c + a\,d}\right]$$

■

2.6 Input-Output Analysis

An economy is an immensely complicated network of interdependence. Changes in one part can ripple out to affect other parts. Economists have struggled to be able to describe, and to make predictions about, such a complicated object and mathematical models using systems of linear equations have emerged as a key tool. One is Input-Output Analysis, pioneered by W. Leontief, who won the 1973 Nobel Prize in Economics.

Consider an economy with many parts, two of which are the steel industry and the auto industry. These two interact tightly as they work to meet the demand for their product from other parts of the economy, that is, from users external to the steel and auto sectors. For instance, should the external demand for autos go up, that would increase in the auto industry's usage of steel. Or, should the external demand for steel fall, then it would lead lower steel's purchase of autos. The type of Input-Output model that we will consider takes in the external demands and then predicts how the two interact to meet those demands.

We start with a listing of production and consumption statistics. (These numbers, giving dollar values in millions, are describing the 1958 U.S. economy. Today's statistics would be different, both because of inflation and because of technical changes in the industries.)

	used by steel	*used by auto*	*used by others*	*total*
value of steel	5,395	2,664		25,448
value of auto	48	9,030		30,346

For instance, the dollar value of steel used by the auto industry in this year is 2,664 million. Note that industries may consume some of their own output.

We can fill in the blanks for the external demand. This year's value of the steel used by others is 17,389 and this year's value of the auto used by others is 21,268. With that, we have a complete description of the external demands and of how auto and steel interact, this year, to meet them.

Now, imagine that the external demand for steel has recently been going up by 200 per year and so we estimate that next year it will be 17,589. We also estimate that next year's external demand for autos will be down 25 to 21,243. We wish to predict next year's total outputs.

That prediction isn't as simple as adding 200 to this year's steel total and subtracting 25 from this year's auto total. For one thing, a rise in steel will cause that industry to have an increased demand for autos, which will mitigate to some extent the loss in external demand for autos. On the other hand, the drop in external demand for autos will cause the auto industry to use less steel and so lessen somewhat the upswing in steel's business. In short, these two industries form a system, and we need to predict where the system as a whole will settle.

We have these equations.

$$\begin{aligned}
\text{next year's production of steel} &= \text{next year's use of steel by steel} \\
&\quad + \text{next year's use of steel by auto} \\
&\quad + \text{next year's use of steel by others} \\
\text{next year's production of autos} &= \text{next year's use of autos by steel} \\
&\quad + \text{next year's use of autos by auto} \\
&\quad + \text{next year's use of autos by others}
\end{aligned}$$

On the left side put the unknowns s be next years total production of steel and a for next year's total output of autos. At the ends of the right sides go our external demand estimates for next year $17,589$ and $21,243$. For the remaining four terms, we look to the table of this year's information about how the industries interact.

For next year's use of steel by steel, we note that this year the steel industry used 5395 units of steel input to produce $25,448$ units of steel output. So next year, when the steel industry will produce s units out, we expect that doing so will take $s \cdot (5395)/(25,448)$ units of steel input - this is simply the assumption that input is proportional to output. (We are assuming that the ratio of input to output remains constant over time; in practice, models may try to take account of trends of change in the ratios.)

Next year's use of steel by the auto industry is similar. This year, the auto industry uses 2664 units of steel input to produce 30346 units of auto output. So next year, when the auto industry's total output is a, we expect it to consume $a \cdot (2664)/(30346)$ units of steel.

Filling in the other equation in the same way gives this system of linear equations.

$$\begin{aligned}
\frac{5,395}{25,448} \cdot s + \frac{2,664}{30,346} \cdot a + 17,589 &= s \\
\frac{48}{25,448} \cdot s + \frac{9,030}{30,346} \cdot a + 21,243 &= a
\end{aligned}$$

Gauss's Method

$$\begin{aligned}
\frac{20,053}{25,448}s - \frac{2,664}{30,346}a &= 17,589 \\
-\frac{48}{25,448}s + \frac{21,316}{30,346}a &= 21,243
\end{aligned}$$

gives $s = 25,698$ and $a = 30,311$.

Looking back, recall that above we described why the prediction of next year's totals isn't as simple as adding 200 to last year's steel total and subtracting 25 from last year's auto total. In fact, comparing these totals for next year to the ones given at the start for the current year shows that, despite the drop in external demand, the total production of the auto industry will rise. The increase in internal demand for autos caused by steel's sharp rise in business more than makes up for the loss in external demand for autos.

One of the advantages of having a mathematical model is that we can ask "What if ...?" questions. For instance, we can ask "What if the estimates for next year's external demands are somewhat off?" To try to understand how much the model's predictions change in reaction to changes in our estimates, we can try revising our estimate of next year's external steel demand from 17,589 down to 17,489, while keeping the assumption of next year's external demand for autos fixed at 21,243. The resulting system

$$\begin{aligned} \frac{20,053}{25,448}s - \frac{2,664}{30,346}a &= 17,489 \\ -\frac{48}{25,448}s + \frac{21,316}{30,346}a &= 21,243 \end{aligned}$$

when solved gives $s = 25,571$ and $a = 30,311$. This is **sensitivity analysis**. We are seeing how sensitive the predictions of our model are to the accuracy of the assumptions.

Naturally, we can consider larger models that detail the interactions among more sectors of an economy; these models are typically solved on a computer. Naturally also, a single model does not suit every case and assuring that the assumptions underlying a model are reasonable for a particular prediction requires the judgments of experts. With those caveats however, this model has proven in practice to be a useful and accurate tool for economic analysis.

Exercise 2.6.1 *Hint: these systems are easiest to solve on a computer.*

1.) With the steel-auto system given above, estimate next year's total productions in these cases.

(a) Next year's external demands are: up 200 from this year for steel, and unchanged for autos.

Solution : *With the external use of steel as* 17 789 *and the external use of autos as* 21 243*, we get* $s = 25\,952$*,* $a = 30\,312$*.* ■

(b) Next year's external demands are: up 100 for steel, and up 200 for autos.

Solution : $s = 25,857$, $a = 30,596$ ■

(c) Next year's external demands are: up 200 for steel, and up 200 for autos.

Solution : $s = 25\,984$, $a = 30\,597$ ■

2.) In the steel-auto system, the ratio for the use of steel by the auto industry is 2 664/30 346, about 0.0878. Imagine that a new process for making autos reduces this ratio to .0500.

(a) How will the predictions for next year's total productions change compared to the first example discussed above (i.e., taking next year's external demands to be 17,589 for steel and 21,243 for autos)?

Solution : *Octave gives* $s = 24\,244$*,* $a = 30\,307$ ■

(b) Predict next year's totals if, in addition, the external demand for autos rises to be 21,500 because the new cars are cheaper.

Solution : *Octave gives* $s = 24\,267$*,* $a = 30\,673$ ■

3.) This table gives the numbers for the auto-steel system from a different year, 1947. The units here are billions of 1947 dollars.

	used by steel	*used by auto*	*used by others*	*total*
value of steel	6.90	1.28		18.69
value of autos	0	4.40		14.27

(a) Solve for total output if next year's external demands are: steel's demand up 10% and auto's demand up 15%.

Solution : *These are the equations.*

$$\begin{aligned} (11.79/18.69)s - (1.28/4.27)a &= 11.56 \\ -(0/18.69)s + (9.87/4.27)a &= 11.35 \end{aligned}$$

Octave gives $s = 20.66$ and $a = 16.41$. ■

(b) How do the ratios compare to those given above in the discussion for the 1958 economy?

Solution : *These are the ratios.*

1947	*by steel*	*by autos*
use of steel	0.63	0.09
use of autos	0.00	0.69

1958	*by steel*	*by autos*
use of steel	0.79	0.09
use of autos	0.00	0.70

■

(c) Solve the 1947 equations with the 1958 external demands (note the difference in units; a 1947 dollar buys about what \$1.30 in 1958 dollars buys). How far off are the predictions for total output?

Solution : *Octave gives (in billions of 1947 dollars) $s = 24.82$ and $a = 23.63$. In billions of 1958 dollars that is $s = 32.26$ and $a = 30.71$.* ■

4.) Predict next year's total productions of each of the three sectors of the hypothetical economy shown below

	used by farm	*used by rail*	*used by shipping*	*used by others*	*total*
value of farm	25	50	100		800
value of rail	25	50	50		300
value of shipping	15	10	0		500

if next year's external demands are as stated.

(a) 625 for farm, 200 for rail, 475 for shipping

(b) 650 for farm, 150 for rail, 450 for shipping

5.) This table gives the interrelationships among three segments of an economy.

	used by food	*used by wholesale*	*used by retail*	*used by others*	*total*
value of food	0	2,318	4,679		11,869
value of wholesale	393	1,089	22,459		122,242
value of retail	3	53	75		116,041

We will do an Input-Output analysis on this system.

(a) Fill in the numbers for this year's external demands.

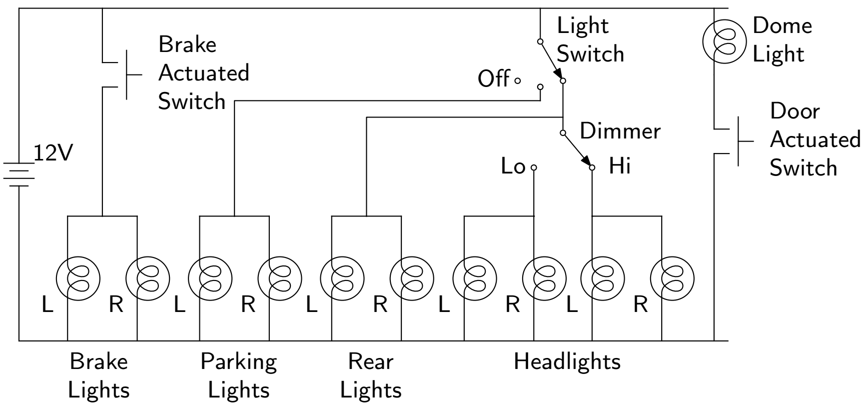
(b) Set up the linear system, leaving next year's external demands blank.

(c) Solve the system where we get next year's external demands by taking this year's external demands and inflating them 10%. Do all three sectors increase their total business by 10%? Do they all even increase at the same rate?

(d) Solve the system where we get next year's external demands by taking this year's external demands and reducing them 7%. (The study from which these numbers come concluded that because of the closing of a local military facility, overall personal income in the area would fall 7%, so this might be a first guess at what would actually happen.)

2.7 Analyzing Networks

The diagram below shows some of a car's electrical network. The battery is on the left, drawn as stacked line segments. The wires are lines, shown straight and with sharp right angles for neatness. Each light is a circle enclosing a loop.



The designer of such a network needs to answer questions like: How much electricity flows when both the hi-beam headlights and the brake lights are on? We will use linear systems to analyze simple electrical networks.

For the analysis we need two facts about electricity and two facts about electrical networks.

The first fact about electricity is that a battery is like a pump, providing a force impelling the electricity to flow, if there is a path. We say that the battery provides a **potential** to flow. For instance, when the driver steps on the brake then the switch makes contact and so makes a circuit on the left side of the diagram, so the battery's force creates a current flowing through that circuit to turn on the brake lights.

The second electrical fact is that in some kinds of network components the amount of flow is proportional to the force provided by the battery. That is, for each such component there is a number, it's **resistance**, such that the potential is equal to the flow times the resistance. Potential is measured in **volts**, the rate of flow is in **amperes**, and resistance to the flow is in **ohms**; these units are defined so that volts = amperes · ohms.

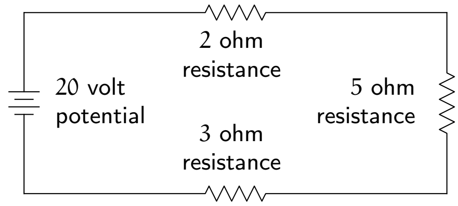
Components with this property, that the voltage-amperage response curve is a line through the origin, are **resistors**. For example, if a resistor measures 2 ohms then wiring it to a 12 volt battery results in a flow of 6 amperes. Conversely, if electrical current of 2 amperes flows through that resistor then there must be a 4 volt potential difference between it's ends. This is the **voltage drop** across the resistor. One way to think of the electrical circuits that we consider here is that the battery provides a voltage rise while the other components are voltage drops.

The two facts that we need about networks are Kirchhoff's Laws.

Current Law. For any point in a network, the flow in equals the flow out.

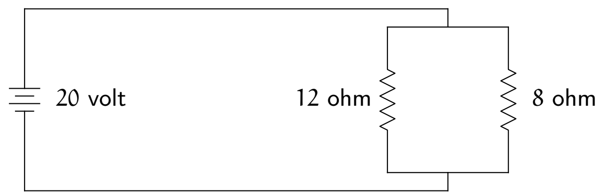
Voltage Law. Around any circuit the total drop equals the total rise.

We start with the network below. It has a battery that provides the potential to flow and three resistors, drawn as zig-zags. When components are wired one after another, as here, they are in **series**.



By Kirchhoff's Voltage Law, because the voltage rise is 20 volts, the total voltage drop must also be 20 volts. Since the resistance from start to finish is 10 ohms (the resistance of the wire connecting the components is negligible), the current is $(20/10) = 2$ amperes. Now, by Kirchhoff's Current Law, there are 2 amperes through each resistor. Therefore the voltage drops are: 4 volts across the 2 oh m resistor, 10 volts across the 5 ohm resistor, and 6 volts across the 3 ohm resistor.

The prior network is simple enough that we didn't use a linear system but the next one is more complicated. Here the resistors are in **parallel**.



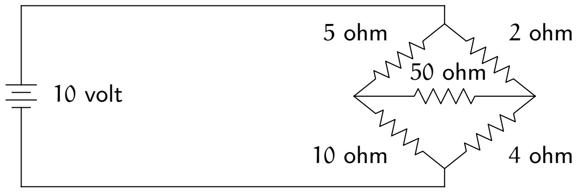
We begin by labeling the branches as below. Let the current through the left branch of the parallel portion be i_1 and that through the right branch be i_2, and also let the current through the battery be i_0. Note that we don't need to know the actual direction of flow - if current flows in the direction opposite to our arrow then we will get a negative number in the solution.

The Current Law, applied to the point in the upper right where the flow i_0 meets i_1 and i_2, gives that $i_0 = i_1 + i_2$. Applied to the lower right it gives $i_1 + i_2 = i_0$. In the circuit that loops out of the top of the battery, down the left branch of the parallel portion, and back into the bottom of the battery, the voltage rise is 20 while the voltage drop is $i_1 \cdot 12$, so the Voltage Law gives that $12i_1 = 20$. Similarly, the circuit from the battery to the right branch and back to the battery gives that $8i_2 = 20$. And, in the circuit that simply loops around in the left and right branches of the parallel portion (taken clockwise, arbitrarily), there is a voltage rise of 0 and a voltage drop of $8i_2 - 12i_1$ so the Voltage Law gives that $8i_2 - 12i_1 = 0$.

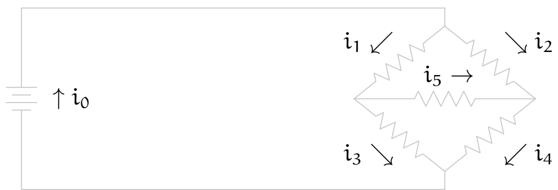
$$\begin{array}{rrrl}
i_0 - & i_1 - & i_2 = & 0 \\
-i_0 + & i_1 + & i_2 = & 0 \\
 & 12i_1 & = & 20 \\
 & & 8i_2 = & 20 \\
 & -12i_1 + & 8i_2 = & 0
\end{array}$$

The solution is $i_0 = 25/6$, $i_1 = 5/3$, and $i_2 = 5/2$, all in amperes. (Incidentally, this illustrates that redundant equations can arise in practice.)

Kirchhoff's laws can establish the electrical properties of very complex networks. The next diagram shows five resistors, wired in a **series-parallel** way.



This network is a **Wheatstone bridge**. To analyze it, we can place the arrows in this way.



Kirchhoff's Current Law, applied to the top node, the left node, the right node, and the bottom node gives these.

$$\begin{aligned} i_0 &= i_1 + i_2 \\ i_1 &= i_3 + i_5 \\ i_2 + i_5 &= i_4 \\ i_3 + i_4 &= i_0 \end{aligned}$$

Kirchhoff's Voltage Law, applied to the inside loop (the i_0 to i_1 to i_3 to i_0 loop), the outside loop, and the upper loop not involving the battery, gives these.

$$\begin{aligned} 5i_1 + 10i_3 &= 10 \\ 2i_2 + 4i_4 &= 10 \\ 5i_1 + 50i_5 - 2i_2 &= 0 \end{aligned}$$

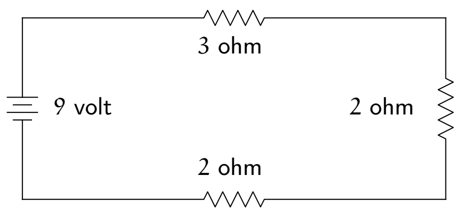
Those suffice to determine the solution $i_0 = 7/3$, $i_1 = 2/3$, $i_2 = 5/3$, $i_3 = 2/3$, $i_4 = 5/3$, and $i_5 = 0$.

We can understand many kinds of networks in this way. For instance, the exercises analyze some networks of streets.

Exercise 2.7.1

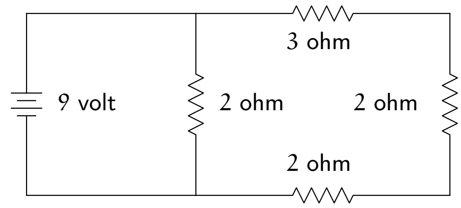
1.) Calculate the amperages in each part of each network.

(a) This is a simple network.



Solution : *The total resistance is 7 ohms. With a 9 volt potential, the flow will be 9/7 amperes. Incidentally, the voltage drops will then be: 27/7 volts across the 3 ohm resistor, and 18/7 volts across each of the two 2 ohm resistors.* ■

(b) Compare this one with the parallel case discussed above.



Solution : *One way to do this network is to note that the 2 ohm resistor on the left has a voltage drop of 9 volts (and hence the flow through it is 9/2 amperes), and the remaining portion on the right also has a voltage drop of 9 volts, and so we can analyze it as in the prior item. We can also use linear systems.*

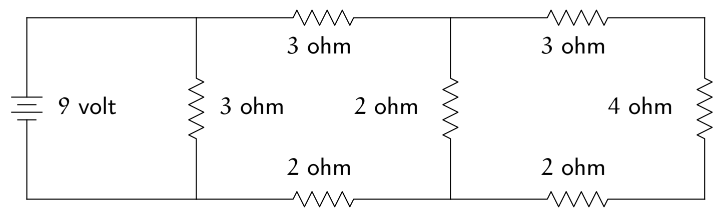
Using the variables from the diagram we get a linear system

$$\begin{aligned} i_0 - i_1 - i_2 \quad &= 0 \\ i_1 + i_2 - i_3 &= 0 \\ 2i_1 \quad &= 9 \\ 7i_2 \quad &= 9 \end{aligned}$$

which yields the unique solution $i_1 = 81/14$, $i_1 = 9/2$, $i_2 = 9/7$, *and* $i_3 = 81/14$.

Of course, the first and second paragraphs yield the same answer. Essentially, in the first paragraph we solved the linear system by a method less systematic than Gauss's Method, solving for some of the variables and then substituting. ■

(c) This is a reasonably complicated network.

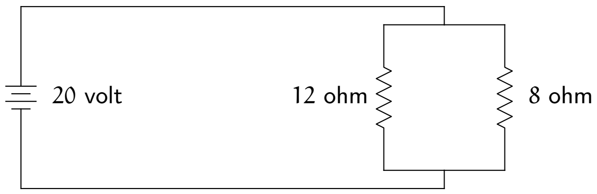


Solution : *Using these variables, one linear system that suffices to yield a unique solution is this.*

$$\begin{aligned} i_0 - i_1 - i_2 \qquad\qquad &= 0 \\ i_2 - i_3 - i_4 \qquad &= 0 \\ i_3 + i_4 - i_5 \quad &= 0 \\ i_1 \qquad + i_5 - i_6 &= 0 \\ 3i_1 \qquad\qquad &= 9 \\ 3i_2 \quad + 2i_4 + 2i_5 \quad &= 9 \\ 3i_2 + 9i_3 \quad + 2i_5 \quad &= 9 \end{aligned}$$

(The last three equations come from the circuit involving i_0-i_1-i_6, the circuit involving i_0-i_2-i_4-i_5-i_6, and the circuit with i_0-i_2-i_3-i_5-i_6.) Octave gives $i_0 = 4.35616$, $i_1 = 3.00000$, $i_2 = 1.35616$, $i_3 = 0.24658$, $i_4 = 1.10959$, $i_5 = 1.35616$, $i_6 = 4.35616$. ■

2.) In the first network that we analyzed, with the three resistors in series, we just added to get that they acted together like a single resistor of 10 ohms. We can do a similar thing for parallel circuits. In the second circuit analyzed,



the electric current through the battery is 25/6 amperes. Thus, the parallel portion is **equivalent** to a single resistor of $20/(25/6) = 4.8$ ohms.

(a) What is the equivalent resistance if we change the 12 ohm resistor to 5 ohms?

Solution : *Using the variables from the earlier analysis,*

$$\begin{array}{rrrl} i_0 - & i_1 - & i_2 = & 0 \\ -i_0 + & i_1 + & i_2 = & 0 \\ & 5i_1 & = & 20 \\ & & 8i_2 = & 20 \\ & -5i_1 + & 8i_2 = & 0 \end{array}$$

The current flowing in each branch is then is $i_2 = 20/8 = 2.5$, $i_1 = 20/5 = 4$, *and* $i_0 = 13/2 = 6.5$, *all in amperes. Thus the parallel portion is acting like a single resistor of size* $20/(13/2) \approx 3.08$ *ohms.* ■

(b) What is the equivalent resistance if the two are each 8 ohms?

Solution : *A similar analysis gives that is* $i_2 = i_1 = 20/8 = 4$ *and* $i_0 = 40/8 = 5$ *amperes. The equivalent resistance is* $20/5 = 4$ *ohms.* ■

(c) Find the formula for the equivalent resistance if the two resistors in parallel are r_1 ohms and r_2 ohms.

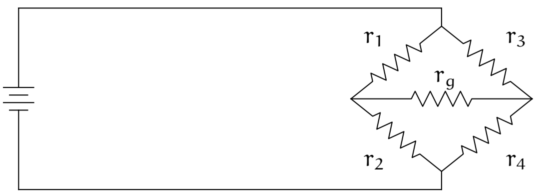
Solution : *Another analysis like the prior ones gives is* $i_2 = 20/r_2$, $i_1 = 20/r_1$, *and* $i_0 = 20(r_1 + r_2)/(r_1 r_2)$, *all in amperes. So the parallel portion is acting like a single resistor of size* $20/i_1 = r_1 r_2/(r_1 + r_2)$ *ohms. (This equation is often stated as: the equivalent resistance r satisfies* $1/r = (1/r_1) + (1/r_2)$*.)* ■

3.) For the car dashboard example that opens this Topic, solve for these amperages (assume that all resistances are 2 ohms).

(a) If the driver is stepping on the brakes, so the brake lights are on, and no other circuit is closed.

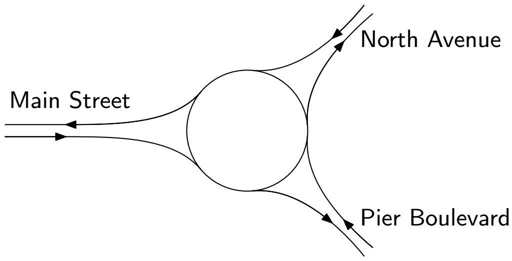
(b) If the hi-beam headlights and the brake lights are on.

4.) Show that, in this Wheatstone Bridge,



r_2/r_1 equals r_4/r_3 if and only if the current flowing through r_g is zero. (In practice, we place an unknown resistance at r_4. At r_g we place a meter that shows the current. We vary the three resistances r_1, r_2, and r_3 (typically they each have a calibrated knob) until the current in the middle reads 0, and then the above equation gives the value of r_4.)

5.) Consider this traffic circle.



This is the traffic volume, in units of cars per five minutes.

	North	*Pier*	*Main*
into	100	150	25
out of	75	150	50

We can set up equations to model how the traffic flows.

(a) Adapt Kirchhoff's Current Law to this circumstance. Is it a reasonable modeling assumption?

Solution : *An adaptation is: in any intersection the flow in equals the flow out. It does seem reasonable in this case, unless cars are stuck at an intersection for a long time.* ■

(b) Label the three between-road arcs in the circle with a variable. Using the (adapted) Current Law, for each of the three in-out intersections state an equation describing the traffic flow at that node.

Solution : *Because* 50 *cars leave via Main while* 25 *cars enter,* $i_1 - 25 = i_2$. *Similarly Pier's in/out balance means that* $i_2 = i_3$ *and North gives* $i_3 + 25 = i_1$. *We have this system.*

$$\begin{aligned} i_1 - i_2 \qquad &= \ \ 25 \\ i_2 - i_3 &= 0 \\ -i_1 \qquad + i_3 &= -25 \end{aligned}$$

■

(c) Solve that system.

Solution : *The row operations $\rho_1 + \rho_2$ and $rho_2 + \rho_3$ lead to the conclusion that there are infinitely many solutions. With i_3 as the parameter,*

$$\begin{pmatrix} 25 + i_3 \\ i_3 \\ i_3 \end{pmatrix} \quad : i_3 \in \Re$$

of course, since the problem is stated in number of cars, we might restrict i_3 to be a natural number. ■

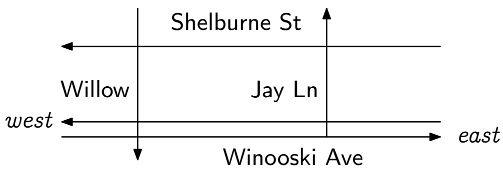
(d) Interpret your solution.

Solution : *If we picture an initially-empty circle with the given input/output behavior, we can superimpose a z_3-many cars circling endlessly to get a new solution.* ■

(e) Restate the Voltage Law for this circumstance. How reasonable is it?

Solution : *A suitable restatement might be: the number of cars entering the circle must equal the number of cars leaving. The reasonableness of this one is not as clear. Over the five minute time period we could find that a half dozen more cars entered than left, although the problem statement's into/out table does satisfy this property. In any event, it is of no help in getting a unique solution since for that we would need to know the number of cars circling endlessly.* ■

6.) This is a network of streets.



We can observe the hourly flow of cars into this network's entrances, and out of its exits.

	east Winooski	*west Winooski*	*Willow*	*Jay*	*Shelburne*
into	80	50	65	–	40
out of	30	5	70	55	75

(Note that to reach Jay a car must enter the network via some other road first, which is why there is no 'into Jay' entry in the table. Note also that over a long period of time, the total in must approximately equal the total out, which is why both rows add to 235 cars.) Once inside the network, the traffic may flow in different ways, perhaps filling Willow and leaving Jay mostly empty, or perhaps flowing in some other way. Kirchhoff's Laws give the limits on that freedom.

(a) Determine the restrictions on the flow inside this network of streets by setting up a variable for each block, establishing the equations, and solving them. Notice that some streets are one-way only. (*Hint:* this will not yield a unique solution, since traffic can flow through this network in various ways; you should get at least one free variable.)

Solution : *Here is a variable for each unknown block; each known block has the flow shown. We apply Kirchhoff's principle that the flow into the intersection of Willow and Shelburne must equal the flow out to get* $i_1 + 25 = i_2 + 125$. *Doing the intersections from right to left and top to bottom gives these equations.*

$$\begin{array}{rcl} i_1 - i_2 & = & 10 \\ -i_1 + i_3 & = & 15 \\ i_2 + i_4 & = & 5 \\ -i_3 - i_4 + i_6 & = & -50 \\ i_5 - i_7 & = & -10 \\ -i_6 + i_7 & = & 30 \end{array}$$

The row operation $\rho_1 + \rho_2$ *followed by* $\rho_2 + \rho_3$ *then* $\rho_3 + \rho_4$ *and* $\rho_4 + \rho_5$ *and finally* $\rho_5 + \rho_6$ *result in this system.*

$$\begin{array}{rcl} i_1 - i_2 & = & 10 \\ -i_2 + i_3 & = & 25 \\ i_3 + i_4 - i_5 & = & 30 \\ -i_5 + i_6 & = & -20 \\ -i_6 + i_7 & = & -30 \\ 0 & = & 0 \end{array}$$

Since the free variables are i_4 *and* i_7 *we take them as parameters.*

$$\begin{aligned} i_6 &= i_7 - 30 \\ i_5 &= i_6 + 20 = (i_7 - 30) + 20 = i_7 - 10 \\ i_3 &= -i_4 + i_5 + 30 = -i_4 + (i_7 - 10) + 30 = -i_4 + i_7 + 20 \qquad () \\ i_2 &= i_3 - 25 = (-i_4 + i_7 + 20) - 25 = -i_4 + i_7 - 5 \\ i_1 &= i_2 + 10 = (-i_4 + i_7 - 5) + 10 = -i_4 + i_7 + 5 \end{aligned}$$

Obviously i_4 *and* i_7 *have to be positive, and in fact the first equation shows that* i_7 *must be at least* 30. *If we start with* i_7, *then the* i_2 *equation shows that* $0 \le i_4 \le i_7 - 5$. ■

(b) Suppose that someone proposes construction for Winooski Avenue East between Willow and Jay, and traffic on that block will be reduced. What is the least amount of traffic flow that can we can allow on that block without disrupting the hourly flow into and out of the network?

Solution : *We cannot take* i_7 *to be zero or else* i_6 *will be negative (this would mean cars going the wrong way on the one-way street Jay). We can, however, take* i_7 *to be as small as* 30, *and then there are many suitable* i_4*'s. For instance, the solution*

$$(i_1, i_2, i_3, i_4, i_5, i_6, i_7) = (35, 25, 50, 0, 20, 0, 30)$$

results from choosing $i_4 = 0$. ■

Matrices for Networks. Matrices have various engineering applications, as we shall see. For instance, they can be used to characterize connections in electrical networks, in nets of roads, in production processes, etc., as follows.

1.) **Nodal Incidence Matrix**. The network in Fig. 2.1 consists of six branches (connections) and four nodes (points where two or more branches come together). One node is the reference node (grounded node, whose voltage is zero). We number the other nodes and number and direct the branches. This we do arbitrarily. The network can now be described by a matrix $A = [a_{jk}]$, where

$$a_{jk} = \begin{cases} + \ \text{ if branch } k \text{ leaves node } (j) \\ -1 \ \text{ if branch } k \text{ enters node } (j) \\ 0 \ \text{ if branch } k \text{ does not touch node } (j) \end{cases}$$

A is called the nodal incidence matrix of the network. Show that for the network in Fig. 2.1 the matrix **A** has the given form.

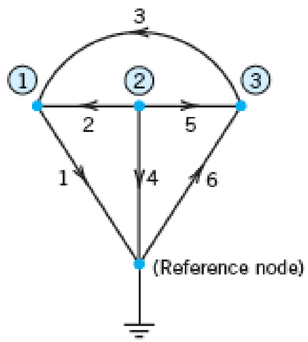


Figure 2.1: Network and nodal incidence matrix

$$\begin{array}{rc} \text{Branch} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{r} \text{Node } (1) \\ \text{Node } (2) \\ \text{Node } (3) \end{array} & \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

2.) Find the nodal incidence matrices of the networks in Fig. 2.2.

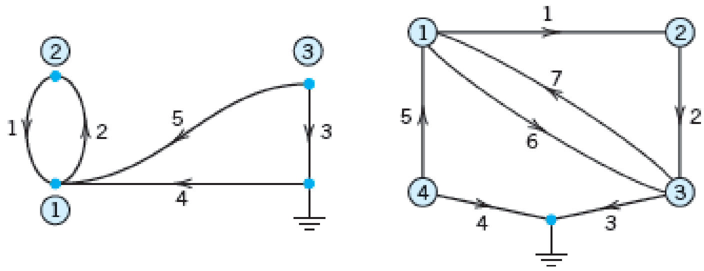


Figure 2.2: Electrical networks

3.) Sketch the three networks corresponding to the nodal incidence matrices

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix}.$$

4.) **Mesh Incidence Matrix.** A network can also be characterized by the mesh incidence matrix $M = [m_{jk}]$, where

$$m_{jk} = \begin{cases} +1 & \text{if branch k is in mesh } \boxed{\text{j}} \text{ and and has the same orientation} \\ -1 & \text{if branch k is in } \boxed{\text{j}} \text{ and has the opposite orientation} \\ 0 & \text{if branch k is not in mesh} \boxed{\text{j}} \end{cases}$$

and a mesh is a loop with no branch in its interior (or in its exterior). Here, the meshes are numbered and directed (oriented) in an arbitrary fashion. Show that for the network in Fig. 157, the matrix **M** has the given form, where Row 1 corresponds to mesh 1, etc.

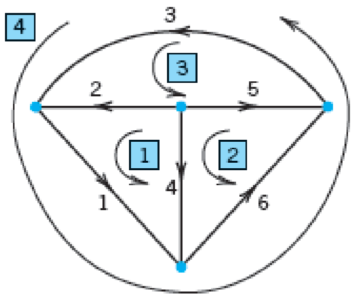


Figure 2.3: Network and matrix **M**

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Example 2.7.1 Weight Watching. Matrix Times Vector
Suppose that in a weight-watching program, a person of 185 lb burns 350 cal/hr in walking (3 mph), 500 in bicycling (13 mph), and 950 in jogging (5.5 mph). Bill, weighing 185 lb, plans to exercise according to the matrix shown. Verify the calculations (W=Walking, B = Bicycling, J = Jogging)

$$\begin{array}{c} \\ \text{Mon} \\ \text{Wed} \\ \text{Fri} \\ \text{Sat} \end{array} \begin{array}{c} \begin{array}{ccc} \text{W} & \text{B} & \text{J} \end{array} \\ \begin{bmatrix} 1.0 & 0 & 0.5 \\ 1.0 & 1.0 & 0.5 \\ 1.5 & 0 & 0.5 \\ 2.0 & 1.5 & 1.0 \end{bmatrix} \end{array} \begin{bmatrix} 350 \\ 500 \\ 950 \end{bmatrix} = \begin{bmatrix} 825 \\ 1325 \\ 1000 \\ 2400 \end{bmatrix} \begin{array}{c} \text{Mon} \\ \text{Wed} \\ \text{Fri} \\ \text{Sat} \end{array}$$

2.8 Algebra in Real World Problems

Example 2.8.1 A three-digit number has two properties. The tens-digit and the ones-digit add up to 5. If the number is written with the digits in the reverse order, and then subtracted from the original number, the result is 792. Use a system of equations to find all of the three-digit numbers with these properties.

Solution : *Let a be the hundreds digit, b the tens digit, and c the ones digit. Then the first condition says that $b + c = 5$. The original number is $100a + 10b + c$, while the reversed number is $100c + 10b + a$. So the second condition is*

$$792 = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c$$

So we arrive at the system of equations

$$\begin{aligned} b + c &= 5 \\ 99a - 99c &= 792 \end{aligned}$$

Using equation operations, we arrive at the equivalent system

$$\begin{aligned} a - c &= 8 \\ b + c &= 5 \end{aligned}$$

We can vary c and obtain infinitely many solutions. However, c must be a digit, restricting us to ten values $(0 - 9)$. Furthermore, if $c > 1$, then the first equation forces $a > 9$, an impossibility. Setting $c = 0$, yields 850 as a solution, and setting $c = 1$ yields 941 as another solution. ■

Example 2.8.2 Find all of the six-digit numbers in which the first digit is one less than the second, the third digit is half the second, the fourth digit is three times the third and the last two digits form a number that equals the sum of the fourth and fifth. The sum of all the digits is 24. (From The MENSA Puzzle Calendar for January 9, 2006.)

Solution : *Let $abcdef$ denote any such six-digit number and convert each requirement in the problem statement into an equation.*

$$\begin{aligned} a &= b - 1 \\ c &= \frac{1}{2}b \\ d &= 3c \\ 10e + f &= d + e \\ 24 &= a + b + c + d + e + f \end{aligned}$$

In a more standard form this becomes

$$\begin{aligned} a - b &= -1 \\ -b + 2c &= 0 \\ -3c + d &= 0 \\ -d + 9e + f &= 0 \\ a + b + c + d + e + f &= 24 \end{aligned}$$

Using equation operations, this system can be converted to the equivalent system

$$\begin{aligned} a + \frac{16}{75}f &= 5 \\ b + \frac{16}{75}f &= 6 \\ c + \frac{8}{75}f &= 3 \\ d + \frac{8}{25}f &= 9 \\ e + \frac{11}{75}f &= 1 \end{aligned}$$

Clearly, choosing $f = 0$ will yield the solution $abcde = 563910$. Furthermore, to have the variables result in single-digit numbers, none of the other choices for $f(1, 2, \ldots, 9)$ will yield a solution. ■

Example 2.8.3 The upwind speed $v(t)$ of a rocket at time t is approximated by $v(t) = at^2 + bt + c$, $0 \leq t \leq 100$ where a, b and c are constants. It has been found that the speed at times $t = 3$, $t = 6$, and $t = 9$ seconds are respectively, $64, 133$ and 208 miles per second respectively. Find the speed at time $t = 15$ seconds. (Use Gaussian elimination method.)

Solution : *Since $v(3) = 64, v(6) = 133$ and $v(9) = 208$, we get the following system of linear equations*

$$\begin{aligned} 9a + 3b + c &= 64, \\ 36a + 6b + c &= 133, \\ 81a + 9b + c &= 208. \end{aligned}$$

We solve the above system of linear equations by Gaussian elimination method.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 20 \\ 1 \end{bmatrix}$$

Therefore, we get

$$v(t) = \frac{1}{3}t^2 + 20t + 1$$

Hence,

$$v(15) = \frac{1}{3}(15)^2 + 20(15) + 1 = 376$$

■

Example 2.8.4 Sales Figures in Matrix Form

Sales figures for three products I, II, III in a store on Monday (Mon), Tuesday (Tues), may for each week be arranged in a matrix

$$A = \begin{array}{c} \begin{array}{ccccccc} \text{Mon} & \text{Tue} & \text{Wed} & \text{Thur} & \text{Fri} & \text{Sat} & \text{Sun} \end{array} \\ \left[\begin{array}{ccccccc} 40 & 33 & 81 & 0 & 21 & 47 & 33 \\ 0 & 12 & 78 & 50 & 50 & 96 & 90 \\ 10 & 0 & 0 & 27 & 43 & 78 & 56 \end{array}\right] \end{array} \cdot \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array}$$

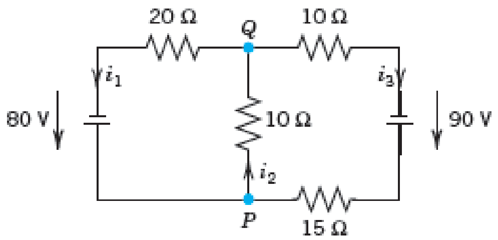
If the company has 10 stores, we can set up 10 such matrices, one for each store. Then, by adding corresponding entries of these matrices, we can get a matrix showing the total sales of each product on each day. Can you think of other data which can be stored in matrix form? For instance, in transportation or storage problems? Or in listing distances in a network of roads?

Definition 2.8.1

1.) ***Kirchhoff's Current Law (KCL)***. At any point of a circuit, the sum of the inflowing currents equals the sum of the outflowing currents.

2.) ***Kirchhoff's Voltage Law (KVL)***. In any closed loop, the sum of all voltage drops equals the impressed electromotive force.

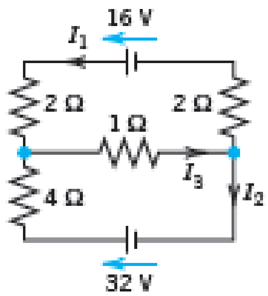
Example 2.8.5 Node P gives the first equation, node Q the second, the right loop the third, and the left loop the fourth, as indicated in the figure.



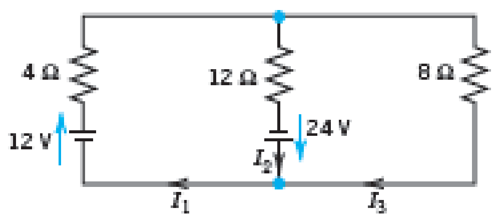
$$\begin{aligned} \text{Node } P &: \quad i_1 - i_2 + i_3 = 0 \\ \text{Node } Q &: \quad -i_1 + i_2 - i_3 = 0 \\ \text{Right loop} &: \quad 10i_2 + 25i_3 = 90 \\ \text{Left loop} &: \quad 20i_1 + 10i_2 = 80 \end{aligned}$$

Exercise 2.8.1 Using Kirchhoff's laws and showing the details, find the currents:

1.)



2.)



Example 2.8.6 Models of markets. Determine the equilibrium solution ($D_1 = S_1, D_2 = S_2$) of the two-commodity market with linear model (D, S, P = demand, supply, price; index 1 =first commodity, index 2 =second commodity)

$$\begin{aligned} D_1 &= 40 - 2P_1 - P_2, \qquad & S_1 &= 4P_1 - P_2 + 4, \\ D_2 &= 5P_1 - 2P_2 + 16, \qquad & S_2 &= 3P_2 - 4. \end{aligned}$$

Example 2.8.7 Balancing a chemical equation $x_1C_3H_8 + x_2O_2 \to x_3CO_2 + x_4H_2O$ means finding integer x_1, x_2, x_3, x_4 such that the numbers of atoms of carbon (C), hydrogen (H), and oxygen (O) are the same on both sides of this reaction, in which propane C_3H_8 and O_2 give carbon dioxide and water. Find the smallest positive integers $x_1, \ldots, x_4$

2.9 Linear Systems Chapter Examples

Example 2.9.1 Solve the following system.

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & z & = & 1 \\ x & - & 2y & + & 4z & = & -3 \\ -x & + & 2y & - & z & = & -3 \end{array}$$

using

1.) Gaussian elimination

Reducing the augmented matrix by Gaussian elimination,

$$(A:b) = \left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 1 & -2 & 4 & -3 \\ -1 & 2 & -1 & -3 \end{array}\right]$$

$$\begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_1 - 3R_2 & \rightsquigarrow & R_2 \\ R_1 + 3R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 0 & 8 & -13 & 10 \\ 0 & 8 & -4 & -8 \end{array}\right]$$

$$\begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ R_2 - R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 0 & 8 & -13 & 10 \\ 0 & 0 & -9 & 18 \end{array}\right]$$

using backward substitution

$$-9z = 18 \quad \Rightarrow \quad z = -2$$

and

$$8y - 13z = 10 \quad \Rightarrow \quad 8y - 13(-2) = 10 \quad \Rightarrow \quad 8y = 10 - 26 \quad \Rightarrow \quad 8y = -16 \quad \Rightarrow \quad y = -2$$

and

$$3x + 2y - z = 1 \quad \Rightarrow \quad 3x + 2(-2) - (-2) = 1 \quad \Rightarrow \quad 3x = 3 \quad \Rightarrow \quad x = 1$$

Therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

2.) Gauss-Jordan elimination

Reducing the augmented matrix by Gauss-Jordan elimination,

$$(A:b) \;=\; \left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 1 & -2 & 4 & -3 \\ -1 & 2 & -1 & -3 \end{array}\right]$$

$$\begin{aligned} R_1 &\rightsquigarrow R_1 \\ -\frac{1}{3}R_1 + R_2 &\rightsquigarrow R_2 \\ \frac{1}{3}R_1 + R_3 &\rightsquigarrow R_3 \end{aligned}$$

$$\sim \left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 0 & -8/3 & 13/3 & -10/3 \\ 0 & 8/3 & -4/3 & -8/3 \end{array}\right]$$

$$\begin{aligned} R_1 &\rightsquigarrow R_1 \\ R_2 &\rightsquigarrow R_2 \\ R_2 + R_3 &\rightsquigarrow R_3 \end{aligned}$$

$$\sim \left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 0 & -8/3 & 13/3 & -10/3 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

using backward substitution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

3.) Row Echelon form: See Definition 2.3 on page (61) If condition(4) is satisfied, it's the Reduced Row Echelon Form (RREF).

(a) Row Echelon Form: For the Gaussian elimination or the Gauss-Jordan elimination, ensure that the leading entry is a 1, by dividing each row with the leading entry.

$$\left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 0 & 8 & -13 & 10 \\ 0 & 0 & -9 & 18 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 2/3 & -1/3 & 1/3 \\ 0 & 1 & -13/8 & 10/8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

or using the reduced Gauss-Jordan

$$\left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 0 & -8/3 & 13/3 & -10/3 \\ 0 & 0 & 3 & -6 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 2/3 & -1/3 & 1/3 \\ 0 & 1 & -13/8 & 10/8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

(b) Reduced Row Echelon Form (RREF) - Similar to process of inverses, of reducing the coefficient matrix to identity matrix.

$$(A:b) = \left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 1 & -2 & 4 & -3 \\ -1 & 2 & -1 & -3 \end{array}\right]$$

$$\begin{aligned} R_1 &\rightsquigarrow R_1 \\ -\frac{1}{3}R_1 + R_2 &\rightsquigarrow R_2 \\ \frac{1}{3}R_1 + R_3 &\rightsquigarrow R_3 \end{aligned}$$

$$\sim \left[\begin{array}{rrr|r} 3 & 2 & -1 & 1 \\ 0 & -8/3 & 13/3 & -10/3 \\ 0 & 8/3 & -4/3 & -8/3 \end{array}\right]$$

$$\begin{aligned} \frac{6}{8}R_2 + R_1 &\rightsquigarrow R_1 \\ R_2 &\rightsquigarrow R_2 \\ R_2 + R_3 &\rightsquigarrow R_3 \end{aligned}$$

$$\sim \left[\begin{array}{rrr|r} 3 & 0 & 9/4 & -3/2 \\ 0 & -8/3 & 13/3 & -10/3 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

$$\begin{aligned} -\frac{9}{12}R_3 + R_1 &\rightsquigarrow R_1 \\ -\frac{13}{9}R_3 + R_2 &\rightsquigarrow R_2 \\ R_3 &\rightsquigarrow R_3 \end{aligned}$$

$$\sim \left[\begin{array}{rrr|r} 3 & 0 & 0 & 3 \\ 0 & -8/3 & 0 & 16/3 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

$$\begin{aligned} \frac{1}{3}R_1 &\rightsquigarrow R_1 \\ -\frac{3}{8}R_2 &\rightsquigarrow R_2 \\ \frac{1}{3}R_3 &\rightsquigarrow R_3 \end{aligned}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

4.) Direct method

The inverse of the coefficient matrix is given by

$$A^{-1} = \begin{bmatrix} 1/4 & 0 & -1/4 \\ 1/8 & 1/6 & 13/24 \\ 0 & 1/3 & 1/3 \end{bmatrix}$$

The solution by the direct method is given by

$$X = A^{-1}b = \begin{bmatrix} 1/4 & 0 & -1/4 \\ 1/8 & 1/6 & 13/24 \\ 0 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example 2.9.2 Solve the following system.

$$\begin{array}{rcrcrcl} 3x_1 & + & 2x_2 & - & x_3 & = & 6 \\ -2x_1 & + & 2x_2 & + & x_3 & = & 3 \\ x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Example 2.9.3 Solve the following system.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 2 \\ x_1 & - & x_2 & + & 3x_3 & = & 4 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 3 \end{array}$$

1.) Gauss-Jordan elimination

Reducing the augmented matrix by Gauss-Jordan elimination (a special case of Gaussian elimination), we can refer to it as Gaussian elimination too,

$$(A:b) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 3 & 4 \\ 2 & 2 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ -R_1 + R_2 & \rightsquigarrow & R_2 \\ -2R_1 + R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

using backward substitution

$$-x_3 = -1 \quad \Rightarrow \quad x_3 = 1$$

and

$$-2x_2 + 2x_3 = 2 \quad \Rightarrow \quad -2x_2 + 2(1) = 2 \quad \Rightarrow \quad 2x_2 = 0 \quad \Rightarrow \quad x_2 = 0$$

and

$$x_1 + x_2 + x_3 = 2 \quad \Rightarrow \quad x_1 + (0) + (1) = 2 \quad \Rightarrow \quad x_1 = 1$$

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

2.) Row Echelon form: See Definition 2.3 on page (61) If condition(4) is satisfied, it's the Reduced Row Echelon Form (RREF).

(a) Row Echelon Form: For the Gaussian elimination or for the Gauss-Jordan elimination, ensure that the leading entry is a 1, by dividing each row with the leading entry.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(b) Reduced Row Echelon Form (RREF).

$$(A : b) \quad = \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 3 & 4 \\ 2 & 2 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} R_1 &\rightsquigarrow R_1 \\ -R_1 + R_2 &\rightsquigarrow R_2 \\ -2R_1 + R_3 &\rightsquigarrow R_3 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\begin{aligned} \frac{1}{2}R_2 + R_1 &\rightsquigarrow R_1 \\ R_2 &\rightsquigarrow R_2 \\ R_3 &\rightsquigarrow R_3 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\begin{array}{rcl} 2R_3 + R_1 & \rightsquigarrow & R_1 \\ 2R_3 + R_2 & \rightsquigarrow & R_2 \\ R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ -\frac{1}{2}R_2 & \rightsquigarrow & R_2 \\ -1R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

3.) Direct method

The inverse of the coefficient matrix is given by

$$A^{-1} = \begin{bmatrix} -7/2 & 1/2 & 2 \\ 5/2 & -1/2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

The solution by the direct method is given by

$$X = A^{-1}b = \begin{bmatrix} -7/2 & 1/2 & 2 \\ 5/2 & -1/2 & -1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 2.9.4 Solve the following system.

$$\begin{array}{rcrcrcl} x & - & 3y & + & z & = & 4 \\ -x & + & 2y & - & 5z & = & 3 \\ 5x & - & 13y & + & 13z & = & 8 \end{array}$$

Using the row reduction Gaussian elimination ( )

$$(A:b) = \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ -1 & 2 & -5 & 3 \\ 5 & -13 & 13 & 8 \end{array}\right]$$

$$\begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_1 + R_2 & \rightsquigarrow & R_2 \\ -5R_1 + R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & -4 & 7 \\ 0 & 2 & 8 & -12 \end{array}\right]$$

$$\begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ R_2 & \rightsquigarrow & R_2 \\ 2R_2 + R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & -4 & 7 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The system of equations is inconsistent. Therefore, no solutions.

Remark 2.9.1 Have applied the special Gaussian elimination, the Gauss-Jordan elimination. You could apply any form of row reductions, the Gaussian elimination isn't unique.

Exercise 2.9.1 Solve the following system.

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & 3x_3 & = & 0 \\ 4x_1 & + & 2x_2 & - & 6x_3 & = & 0 \\ x_1 & - & x_2 & + & x_3 & = & 0 \end{array}$$

Using the Gauss-Jordan row reduction (a special form of Gaussian elimination)

$$(A:b) = \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 4 & 2 & -6 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{rcl} R_1 & \rightsquigarrow & R_1 \\ -2R_1 + R_2 & \rightsquigarrow & R_2 \\ -\frac{1}{2}R_1 + R_3 & \rightsquigarrow & R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3/2 & 5/2 & 0 \end{array}\right]$$

$$R_2 \Leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 0 & -3/2 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Using backward substitution,

$$-\frac{3}{2}x_2 + \frac{5}{2}x_3 = 0$$

Since one equation, with two unknowns, let

$$x_3 = t, \quad \Rightarrow \quad x_2 = \frac{5}{3}t$$

such that

$$\begin{aligned}
2x_1 + x_2 - 3x_3 &= 0 \\
2x_1 + \frac{5}{3}t - 3t &= 0 \\
2x_1 &= 3t - \frac{5}{3}t \\
2x_1 &= \frac{4}{3}t \\
x_1 &= \frac{2}{3}t
\end{aligned}$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}t \\ \\ \frac{5}{3}t \\ \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \\ \frac{5}{3} \\ \\ 1 \end{bmatrix} t$$

Infinitely many solutions, since for a different value of t, we get a different solution, and all satisfying the system of equations. Say for $t = -3, -1, 0, 1, 3$ the corresponding solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ \\ -5 \\ \\ -3 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \\ -\frac{5}{3} \\ \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \\ \frac{5}{3} \\ \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ \\ 5 \\ \\ 3 \end{bmatrix}$$

and so so many others, whenever you let t be another different value.

Exercise 2.9.2 Solve the linear system of equations below:

$$\begin{array}{rcrcrcr}
x & & & + & z & = & -2 \\
2x & + & y & + & 3z & = & 3 \\
x & + & y & + & z & = & -1
\end{array}$$

using any row reduction technique (Gauss-Jordan, or Gaussian elimination or Echelon form or Reduced row Echelon form).

Applying the direct method, the solution is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 6 \end{bmatrix}$$

Example 2.9.5 Show that the following homogeneous system has nontrivial solutions.

$$\begin{array}{rcrcrcrcr}
x_1 & - & x_2 & + & 2x_3 & - & x_4 & = & 0 \\
2x_1 & + & 2x_2 & & & + & x_4 & = & 0 \\
3x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 0
\end{array}$$

Solution : *The reduction of the augmented matrix to Gauss-Jordan form is outlined below.*

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow[\substack{-2R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 0 & 4 & -4 & 3 & 0 \\ 0 & 4 & -4 & 2 & 0 \end{array}\right] \xrightarrow[-R_2+R_3\to R_3]{\substack{R_1\to R_1 \\ R_2\to R_2}} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 0 & 4 & -4 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Whose solution is by backward substitution given by $x_4 = 0$ *and*

$$\begin{aligned} 4x_2 - 4x_3 + 3x_4 &= 0 \\ 4x_2 - 4x_3 + 3(0) &= 0 \\ 4x_2 - 4x_3 &= 0 \end{aligned}$$

One equation with two unknowns, so let one be known. Let $x_3 = t$

$$\begin{aligned} 4x_2 - 4x_3 &= 0 \\ 4x_2 - 4t &= 0 \\ 4x_2 &= 4t \\ x_2 &= t \end{aligned}$$

Also

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ x_1 - (t) + 2(t) - (0) &= 0 \\ x_1 &= -t \end{aligned}$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix}$$

by letting $x_3 = t$*, hence for* $t = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

which is non-trivial. ∎

Example 2.9.6 Solve the linear system whose augmented matrix is given by

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ -2 & 3 & -1 & 5 & -3 \\ 3 & -1 & 4 & -1 & 7 \end{array}\right]$$

Using Gauss-Jordan elimination,

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ -2 & 3 & -1 & 5 & -3 \\ 3 & -1 & 4 & -1 & 7 \end{array}\right] \xrightarrow[\substack{2R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ 0 & 7 & 5 & 13 & 5 \\ 0 & -7 & -5 & -13 & -5 \end{array}\right] \xrightarrow[R_2+R_3\to R_3]{\substack{R_1\to R_1 \\ R_2\to R_2}} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ 0 & 7 & 5 & 13 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the solutions are all vectors x of the form

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 11 \\ 5 \\ -7 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 13 \\ 0 \\ -7 \end{bmatrix}$$

Example 2.9.7 Show that, for arbitrary values of s and t,

$$\begin{aligned} x_1 &= t - s + 1 \\ x_2 &= t + s + 2 \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

is a solution to the system

$$\begin{array}{rcrrrcr} x_1 & - & 2x_2 & +3x_3 & +x_4 & = & -3 \\ 2x_1 & - & x_2 & +3x_3 & -x_4 & = & 0 \end{array}$$

Simply substitute these values of $x_1,\ x_2,\ x_3$, and x_4 in each equation.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + x_4 &= (t - s + 1) - 2(t + s + 2) + 3s + t = -3 \\ 2x_1 - x_2 + 3x_3 - x_4 &= 2(t - s + 1) - (t + s + 2) + 3s - t = 0 \end{aligned}$$

Because both equations are satisfied, it is a solution for all choices of s and t.

2.10 Linear Systems Chapter Exercises

Exercise 2.10.1 Rewrite the system in matrix form and solve it by Gaussian Elimination (Gauss-Jordan elimination)

1.)

$$\begin{aligned} 2x - y &= 1 \\ y &= 3 \end{aligned}$$

(2, 3)

2.)

$$\begin{aligned} 3x + 4y &= -2 \\ -3x - y &= 5 \end{aligned}$$

(-2, 1)

3.)

$$\begin{aligned} \frac{x}{3} - \frac{4y}{5} &= \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} &= \frac{1}{2} \end{aligned}$$

$\left(3, -\dfrac{1}{2}\right)$

4.)

$$\begin{aligned} 6x + 3y &= 9 \\ 4x + 2y &= 12 \end{aligned}$$

No solution

5.)

$$\begin{aligned} x - y &= 0 \\ x + y &= 2 \\ -2x + y &= -2 \end{aligned}$$

No solution

6.)

$$\begin{aligned} x - \frac{1}{3}y + \frac{1}{2}z &= 1 \\ y - \frac{1}{2}z &= 4 \\ z &= -1 \end{aligned}$$

$\left(\dfrac{8}{3}, \dfrac{7}{2}, -1\right)$

7.)

$$\begin{aligned} 3x - y + z &= 3 \\ 2x - 4y + 3z &= 16 \\ x - y + z &= 5 \end{aligned}$$

$(-1, 0, 6)$

Exercise 2.10.2 (Review Exercises) In Exercises 1 - 8, take a trip down memory lane and solve the given system using Gaussian elimination. Classify each system as consistent (unique solution or infinitely many solutions), or inconsistent (no solutions).

1.) $\begin{cases} x+2y &= 5 \\ x &= 6 \end{cases}$

2.) $\begin{cases} 2y-3x &= 1 \\ y &= -3 \end{cases}$

3.) $\begin{cases} \dfrac{x+2y}{4} &= -5 \\ \dfrac{3x-y}{2} &= 1 \end{cases}$

4.) $\begin{cases} \dfrac{2}{3}x-\dfrac{1}{5}y &= 3 \\ \dfrac{1}{2}x+\dfrac{3}{4}y &= 1 \end{cases}$

5.) $\begin{cases} \dfrac{1}{2}x-\dfrac{1}{3}y &= -1 \\ 2y-3x &= 6 \end{cases}$

6.) $\begin{cases} x+4y &= 6 \\ \dfrac{1}{12}x+\dfrac{1}{3}y &= \dfrac{1}{2} \end{cases}$

7.) $\begin{cases} 3y-\dfrac{3}{2}x &= -\dfrac{15}{2} \\ \dfrac{1}{2}x-y &= \dfrac{3}{2} \end{cases}$

8.) $\begin{cases} \dfrac{5}{6}x+\dfrac{5}{3}y &= -\dfrac{7}{3} \\ -\dfrac{10}{3}x-\dfrac{20}{3}y &= 10 \end{cases}$

Exercise 2.10.3 In Exercises 1 - 11, put each system of linear equations into triangular form and solve the system if possible. Classify each system as consistent, or inconsistent.

1.) $\begin{cases} -5x+y = 17 \\ x+y = 5 \end{cases}$

2.) $\begin{cases} x+y+z = 3 \\ 2x-y+z = 0 \\ -3x+5y+7z = 7 \end{cases}$

3.) $\begin{cases} 4x-y+z = 5 \\ 2y+6z = 30 \\ x+z = 5 \end{cases}$

4.) $\begin{cases} 4x-y+z = 5 \\ 2y+6z = 30 \\ x+z = 6 \end{cases}$

5.) $\begin{cases} x+y+z = -17 \\ y-3z = 0 \end{cases}$

6.) $\begin{cases} x-2y+3z = 7 \\ -3x+y+2z = -5 \\ 2x+2y+z = 3 \end{cases}$

7.) $\begin{cases} 3x-2y+z = -5 \\ x+3y-z = 12 \\ x+y+2z = 0 \end{cases}$

8.) $\begin{cases} 2x-y+z = -1 \\ 4x+3y+5z = 1 \\ 5y+3z = 4 \end{cases}$

9.) $\begin{cases} x-y+z = -4 \\ -3x+2y+4z = -5 \\ x-5y+2z = -18 \end{cases}$

10.) $\begin{cases} 2x-4y+z = -7 \\ x-2y+2z = -2 \\ -x+4y-2z = 3 \end{cases}$

11.) $\begin{cases} 2x-y+z = 1 \\ 2x+2y-z = 1 \\ 3x+6y+4z = 9 \end{cases}$

Exercise 2.10.4 Given a linear system of two equations in two unknowns, give a geometric interpretation if the system has;

1.) no solution 2.) a unique solution 3.) infinitely many solutions

Exercise 2.10.5 Using Gaussian elimination, solve the following linear systems

1.) $\begin{array}{rcrcrcr} -2x_1 & + & x_2 & - & x_3 & = & 4 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 13 \\ 3x_1 & & & + & x_3 & = & -1 \end{array}$

3.) $\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & -2 \\ -x_1 & + & x_2 & - & 2x_3 & = & 3 \\ 2x_1 & - & x_2 & + & 3x_3 & = & -7 \end{array}$

2.) $\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & -2 \\ -x_1 & + & x_2 & - & 2x_3 & = & 3 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 1 \end{array}$

Exercise 2.10.6 Find the values of h and k such that the systems below have

1.) no solution 2.) a unique solution 3.) Many solutions

For

(a) $\begin{array}{rcrcr} x_1 & + & hx_2 & = & 1 \\ 2x_1 & + & 3x_2 & = & k \end{array}$

(b) $\begin{array}{rcrcr} x_1 & - & 3x_2 & = & 1 \\ 2x_1 & + & hx_2 & = & k \end{array}$

Exercise 2.10.7 Find a condition on g, h, c and k that makes the systems below consistent,

1.) $\begin{array}{rcrcrcr} x_1 & - & 4x_2 & + & 7x_3 & = & g \\ & & 3x_2 & - & 5x_3 & = & h \\ -2x_1 & + & 5x_2 & - & cx_3 & = & k \end{array}$

2.) $\begin{array}{rcrcrcr} 2x_1 & + & 5x_2 & - & 3x_3 & = & g \\ 4x_1 & + & 7x_2 & - & 4x_3 & = & h \\ -6x_1 & - & 3x_2 & + & x_3 & = & k \end{array}$

Exercise 2.10.8 Solve the linear system of equations below:

$$\begin{array}{rcrcrcr} x & & & + & z & = & -2 \\ 2x & + & y & + & 3z & = & 3 \\ x & + & y & + & z & = & -1 \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 6 \end{bmatrix}$$

Exercise 2.10.9 Find all solutions to the system of linear equations

$$\begin{array}{rcrcrcr} -4x & & & + & 5z & = & -2 \\ -3x & + & 3y & + & 5z & = & 3 \\ -x & + & 2y & + & 2z & = & -1 \end{array}$$

Using $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} -4 & 10 & -15 \\ 1 & -3 & 5 \\ -3 & 8 & -12 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 53 \\ -16 \\ 42 \end{bmatrix}$$

Exercise 2.10.10 Using Gaussian elimination, find all solutions to the following system of linear equations:

$$\begin{aligned} 2x_2 + 3x_3 + 4x_4 &= 1 \\ x_1 - 3x_2 + 4x_3 + 5x_4 &= 2 \\ -3x_1 + 10x_2 - 6x_3 - 7x_4 &= -4 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} - \frac{1}{3}\lambda \\ 0 \\ \frac{1}{3} - \frac{4}{3}\lambda \\ \lambda \end{bmatrix}, \lambda \in \mathbb{R}.$$

Exercise 2.10.11 Determine the values of k such that the linear system

$$\begin{aligned} 9x_1 + kx_2 &= 9 \\ kx_1 + x_2 &= -3 \end{aligned}$$

is consistent. *if and only if* $k \neq 3$

Exercise 2.10.12 Determine when the augmented matrix below represents a consistent linear system.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 2 & 1 & 5 & b \\ 1 & -1 & 1 & c \end{array}\right]$$

if and only if $b - 3a + c = 0$.

Exercise 2.10.13

(a) Solve the matrix equation $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) Is it possible to solve $A\vec{x} = \vec{b}$ for any given $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ where A is the matrix given in part (a) above? Explain. *not possible*

(c) Describe the set of all $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ for which $A\vec{x} = \vec{b}$ does have a solution.

if and only if $b_3 + b_2 - 2b_1 = 0$

Exercise 2.10.14 Consider the linear system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -1 & -2 & -2 & -2 \\ 3 & -2 & -2 & -2 & -2 \\ -3 & 2 & 1 & 1 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

1.) Solve the linear system.

2.) Write the general solution in parametric-vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -11 \\ -18 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} t : \qquad s, t \in \mathbb{R}$$

3.) Give a particular solution **p**. $\begin{bmatrix} -11 \\ -18 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

4.) Write the solution set for the homogeneous equation $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 4 \\ 8 \\ -3 \\ 0 \\ 1 \end{bmatrix} t : \qquad s, t \in \mathbb{R}$$

Exercise 2.10.15 The augmented matrix of a linear system has the form $\begin{bmatrix} a & 1 & 1 \\ 2 & a-1 & 1 \end{bmatrix}$. Determine the values of a such that the linear system is consistent.

if and only if $a \neq -1$

Exercise 2.10.16

1.) Write the augmented matrix corresponding the system below:

$$\begin{aligned} x_1 - 6x_2 - 4x_3 &= -5 \\ 2x_1 - 10x_2 - 9x_3 &= -4 \\ -x_1 + 6x_2 + 5x_3 &= 3. \end{aligned}$$

2.) Solve the system by applying the row reduction algorithm. If the system is consistent, find the general solution set. $(x_1, x_2, x_3) = (-1, 2, -2)$

Exercise 2.10.17 Solve the linear system using Cramer's Rule:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_2 + 3x_3 &= 5 \\ x_3 &= 1 \end{aligned}$$

$(1, 1, 1)$.

Exercise 2.10.18

1.) Write the given matrix equation below as system of linear equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$$

2.) Solve the system and write the general solution. $(x_1, x_2, x_3) = (-7/2, 15/2, -3)$

Exercise 2.10.19

1.) Solve the nonhomogeneous system $A\vec{x} = \vec{b}$ and write the solution in parametric vector form where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -3 \\ -1 & 2 & -4 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

$(x_1, x_2, x_3) = (-2, 7, 4)$

2.) Using the parametric vector form of the solution set, determine a particular solution $\mathbf{p}$.

$$p = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}$$

3.) Write the general solution for the system $A\vec{x} = \vec{0}$ in parametric vector form.

$$\boldsymbol{v}_h = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 2.10.20 Verify that if $ad - bc \neq 0$, then the system of equations

$$\begin{aligned} ax_1 + bx_2 &= r \\ cx_1 + dx_2 &= s \end{aligned}$$

has a unique solution. *Examine cases $a \neq 0$, $b \neq 0$*

Exercise 2.10.21 Write the augmented matrix corresponding the system below:

$$\begin{aligned} 3x_1 - x_2 + x_3 + 2x_4 &= -2 \\ x_1 + 2x_2 - x_3 + x_4 &= 1 \\ -x_1 - 3x_2 + 2x_3 - 4x_4 &= -6. \end{aligned}$$

Apply row reduction algorithm and solve the system. If the system is consistent, find the general solution set.

$$\begin{bmatrix} 1 \\ -5 \\ -10 \\ 0 \end{bmatrix} + \begin{bmatrix} -5/3 \\ 11/3 \\ 20/3 \\ 1 \end{bmatrix} \psi, \quad \psi \in \mathbb{R}$$

Exercise 2.10.22 Write the given matrix equation given below as a system of linear equations:

$$\begin{bmatrix} 1 & 2 & 13 \\ 1 & -1 & -2 \\ 2 & 4 & 26 \\ 2 & 1 & 11 \\ 3 & 3 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13 \\ 2 \\ -26 \\ -11 \\ -24 \end{bmatrix}$$

Solve the system and write the general solution in the parametric-vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} \psi : \quad \psi \in \Re$$

Exercise 2.10.23 Solve the nonhomogeneous system $A\vec{x} = \vec{b}$ and write the solution in parametric vector form where

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \psi + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \xi : \quad \psi, \xi \in \Re$$

Exercise 2.10.24 Determine when the augmented matrix below represents a consistent linear system.

$$\begin{bmatrix} 1 & -1 & 2 & 1 & a \\ -1 & 3 & 1 & 1 & b \\ 3 & -5 & 5 & 1 & c \\ 2 & -2 & 4 & 2 & d \end{bmatrix}$$

if and only if $d - 2a = 0$.

Exercise 2.10.25

1.) Solve the matrix equation $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ -1 & -4 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

2.) Is it possible to solve $A\vec{x} = \vec{b}$ for any given $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$? Explain

The coefficient matrix A has only 2 pivot positions. Therefore, it is NOT possible to solve $A\vec{x} = \vec{b}$ *for any given b.*

3.) Describe the set of all $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ for which $A\vec{x} = \vec{b}$ does have a solution.

if and only if $-b_1 + 2b_2 + b_3 = 0$.

Exercise 2.10.26 Solve the linear system using Cramer's Rule:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 2 \\ 3x_1 - 2x_2 + x_3 &= -1 \\ -5x_1 - 4x_2 + 2x_3 &= 3 \end{aligned}$$

$(-5/11, 36/11, 76/11)$.

Exercise 2.10.27 Determine if the system below is consistent or inconsistent (You don't need to find the solution set if it is consistent.)

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_1 - 2x_2 - 19x_3 &= 21 \\ x_2 + 6x_3 &= 3 \end{aligned}$$

Inconsistent

Exercise 2.10.28 Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}$ Is there a matrix C such that $CA = B$?

Let $C = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$ *so that* $CA = B$. *Then form a system of equations, solved by row reductions to have* $\begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ *Since we have a solution for the system above, we conclude that there exists a matrix* C *such that* $CA = B$.

Exercise 2.10.29 If the augmented matrix $[A|\vec{b}]$ of a system $A\vec{x} = \vec{b}$ is row-equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 1 & | & -2 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix},$$

which of the following statements is true?

A. The system is inconsistent

B. $X = (5, -2 - s, 1)$ is the solution for any value of s

C. $X = (5, -2, 1)$ is the unique solution of the system

D. $X = (5s, -2s, s)$ is a solution for any value of s

E. $X = (5t, -2 - s, s)$ is the solution for any value of s and t

F. $X = (5, -2, 1)$ is the unique solution to the system

F

Exercise 2.10.30 Determine the value(s) of a so that the following linear system has no solution.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= a \\ x_1 + x_2 + ax_3 &= 1 \\ 3x_1 + 4x_2 + (a^2 - 2)x_3 &= 1 \end{aligned}$$

Exercise 2.10.31 For a non-homogeneous system of 12 equations in 15 unknowns, answer the following three questions:

1.) Can the system be inconsistent?

2.) Can the system have infinitely many solutions?

3.) Can the system have exactly one solution?

A. No, Yes, No.

B. Yes, Yes, Yes.

C. Yes, Yes, No.

D. No, No, No.

E. Yes, No, Yes.

F. No, No, Yes.

C

Exercise 2.10.32 Suppose A is an $n \times n$ matrix and that,

$$\text{there is a vector } b \in \mathbb{R}^n \text{, for which } A\vec{x} = \vec{b} \text{ is inconsistent.}$$

State whether each of the following is (always) true, or is (possibly) false.

If you say the statement may be false, you must give an explicit example (with numbers!).

If you say the statement is true, you must give a clear explanation.

1.) The system $A\vec{x} = \vec{0}$ has a unique solution. *False*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$[A \mid \vec{b}\,] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

is inconsistent, yet

$$[A \mid \vec{0}\,] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has infinitely many solutions.

2.) The matrix A is not invertible. *True*

Since $A\vec{x} = \vec{b}$ is not consistent for all $b \in \mathbb{R}^n$, A is not invertible.
This follows theorem A invertible $\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent $\forall\ b \in \mathbb{R}^n$

Exercise 2.10.33 The reduced row-echelon form of the augmented matrix of a certain system is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & 1 & 0 & -1 & 1 & 20 \\ 0 & 0 & 0 & 1 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. *2 free variables*

Exercise 2.10.34 Which of the following statements is true for the linear system (in 4 variables)?

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & & & + & 6x_4 & = & 0 \\ & & & & x_3 & & & = & 2 \end{array}$$

A. The system has no solutions

B. $(-3s - 6t, s, 2, t)$ is a solution for any values of s and t

C. $(-9, 1, 2, 1)$ is the unique (only) solution of the system

D. $(3s + 6t, s, -2, t)$ is a solution for any values of s and t

E. $(-3s - 6t, s, 0, t)$ is a solution for any values of s and t

F. $(3s + 6t, s, 0, t)$ is a solution for any values of s and t

B

Exercise 2.10.35 For a nonhomogeneous system of 3013 equations in 2014 unknowns, answer the following three questions:

(i) Can the system be inconsistent?

(ii) Can the system have infinitely many solutions?

(iii) Can the system have a unique solution?

A. Yes, Yes, No.

B. No, No, Yes.

C. Yes, No, Yes.

D. No, Yes, Yes.

E. Yes, Yes, Yes.

F. No, No, No.

E

Exercise 2.10.36 Complete the following phrase to make a true statement: "A homogeneous linear system of 2011 linear equations in 1231 unknowns ......"

A. ...... is always consistent.

B. ...... always has a unique solution.

C. ...... may be inconsistent.

D. ...... which is consistent always has a unique solution.

E. ...... which is consistent never has a unique solution.

F. ...... is never consistent.

A

E could, if number of non zero rows of reduced A is 1231, which is possible.

Exercise 2.10.37 Solve the following system by using elementary row operations on the corresponding augmented matrix.

$$4y = 6$$
$$x - 6y = 3$$

$$\left[\begin{array}{rr|r} 0 & 4 & 6 \\ 1 & -6 & 3 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -6 & 3 \\ 0 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -6 & 3 \\ 0 & 1 & 3/2 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 12 \\ 0 & 1 & 3/2 \end{array}\right].$$

The solution is thus $x = 12, y = \dfrac{3}{2}$.

Exercise 2.10.38 Each statement below is True or False.

1. Every system of 2 equations in 2 unknowns has a unique solution.
2. The set of solutions of the system consisting of the single equation

$$2x - 3y = 0$$

   in the three variables x, y and z is a subspace of $\mathbb{R}^3$.

3. There is a linear system in 2 variables which is inconsistent.

Choose the correct sequence from the possibilities below.

A. True, True, False.

B. True, False, True.

C. True, False, False.

D. False, True, True.

E. False, False, True.

F. False, True, False.

D, 1 *false:* $x + y = 0, 2x + 2y = 0$

Exercise 2.10.39 State whether *True* or *False*. If a linear system is consistent, it must be homogeneous. *False:* $x = 1, y = 0$ eqns

Exercise 2.10.40 Consider the following system of equations:

$$\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 3 \\ x_1 & - & x_2 & - & x_3 & = & 1 \\ -x_1 & + & x_2 & & & = & 4 \end{array}$$

The above system of linear equations is:

A. inconsistent

B. consistent with infinitely many solutions

C. consistent with a unique solution

D. None of the above

C

Row reduction gives a pivot in every row so there is a solution. Moreover, since there is also a pivot in every column there are no free variables so the solution is unique.

Exercise 2.10.41 *True* or *False*? Let A be an $m \times n$ matrix. Then, the homogeneous equation $A\vec{x} = \vec{0}$ is consistent if and only if the augmented matrix $[A \mid \vec{0}]$ has a pivot in every row.

False: The homogeneous equation $A\vec{x} = \vec{0}$ is always consistent. This is because after row reducing you will never have a row of all zeros and a nonzero in the augmented part,(because the augmented part is all zeros). In fact the zero vector is always a solution to a homogeneous equation.

Exercise 2.10.42 Explain what is meant by the solution space of a linear homogeneous system.

Exercise 2.10.43 Solve the equation $\begin{bmatrix} -4 & x \\ -x & 4 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ over $\mathbb{R}$. $x = \pm\sqrt{17}$

Exercise 2.10.44 For a nonhomogeneous system of 2012 equations in 1999 unknowns, answer the following three questions:

1.) Can the system be inconsistent?

2.) Can the system have infinitely many solutions?

3.) Can the system have a unique solution?

A. Yes, Yes, No.

C. Yes, No, Yes.

E. Yes, Yes, Yes.

B. No, No, Yes.

D. No, Yes, Yes.

F. No, No, No.

Exercise 2.10.45 Which of the statements below is true for the following system?

$$\begin{array}{rcrcrcr} 2x & - & y & + & 2z & = & 0 \\ x & + & y & - & 2z & = & -2 \\ 3x & - & y & + & z & = & 4 \\ 2x & + & y & - & z & = & 0 \end{array}$$

A. It has no solutions

B. It has an infinite number of solutions

C. It has the trivial solution

D. It has the unique solution $(3, 4, 5)$

E. It has the solutions $\pm(4, 3, 1)$

F. It has the unique solution $(0, -1, 2)$

Exercise 2.10.46

1.) Find the point (x, y) of intersection of the two lines having equations

$$x + 4y = 7 \quad \text{and} \quad x - y = -1.$$

We must solve the system formed by the equations of the two lines. Starting from the augmented matrix, we have:

$$\left[\begin{array}{rr|r} 1 & 4 & 7 \\ 1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 4 & 7 \\ 0 & -5 & -8 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 4 & 7 \\ 0 & 1 & 8/5 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 3/5 \\ 0 & 1 & 8/5 \end{array}\right].$$

The solution is thus $x = \dfrac{3}{5}, y = \dfrac{8}{5}$.

2.) Do the three planes with equations

$$x_1 + 2x_2 + x_3 = 4, \qquad x_2 - x_3 = 1, \qquad x_1 + 3x_2 = 0$$

intersect?

We must solve the system whose augmented matrix is:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

The third line of the last matrix indicates that the system is inconsistent. Hence the three planes do not intersect.

Exercise 2.10.47 Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period. (You must justify all your answers.)

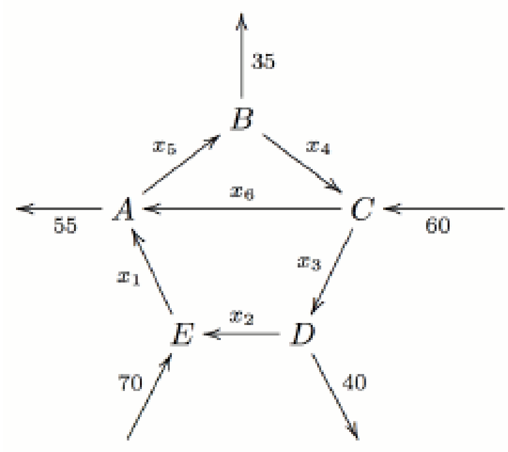


Figure 2.4:

(a) Write down a system of linear equations which describes the the traffic flow,together with all the constraints on the variables $x_i, i = 1, \ldots, 6$. (Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b).*)

(b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 1 & 55 \\ 0 & 1 & 0 & 0 & -1 & 1 & -15 \\ 0 & 0 & 1 & 0 & -1 & 1 & 25 \\ 0 & 0 & 0 & 1 & -1 & 0 & -35 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

(c) If $\overline{ED}$ were closed due to roadwork, find the minimum flow along $\overline{AB}$, using your results from (b).

Exercise 2.10.48 Let

$$A = \left[\begin{array}{cc} -3 & 1 \\ 6 & -2 \end{array}\right]$$

Prove that the equation $A\vec{x} = \vec{b}$ is not consistent for all $\vec{b}$ of $\mathbb{R}^2$. Find and describe the set of vectors b for which $A\vec{x} = \vec{b}$ is consistent.

Let $\vec{b} = \left[\begin{array}{c} b_1 \\ b_2 \end{array}\right]$. *The augmented matrix of the system is equivalent to the row echelon matrix*

$$\left[\begin{array}{cc|c} -3 & 1 & b_1 \\ 0 & 0 & 2b_1 + b_2 \end{array}\right]$$

Hence the equation $A\vec{x} = \vec{b}$ *is not consistent unless* $2b_1 + b_2 = 0$. *The same of* $\vec{b} \in \mathbb{R}^2$ *for which the equation is consistent is a line passing through the origin.*

Exercise 2.10.49 We consider the following two linear systems:

1.)

$$\begin{aligned} x + 2y &= k \\ 4x + hy &= 5 \end{aligned}$$

2.)

$$\begin{aligned} -3x + hy &= 1 \\ 6x + ky &= -3 \end{aligned}$$

Determine the values of h and k for which both of these systems

(a) have no solutions,

(b) have a single solution,

(c) have an infinite number of solutions.

For the first system,

if $h = 8$ *and* $k \neq \frac{5}{4}$, *there is no solution,*

if $h \neq 8$, *there is a unique solution,*

if $h = 8$ *and* $k = \frac{5}{4}$, *there are infinitely many solutions.*

For the second system,

if $k = -2h$, *there is no solution,*

if $k \neq -2h$, *there is a unique solution,*

there are no values of h *and* k *for which there would be infinitely many solutions.*

Exercise 2.10.50 Study the system

$$\begin{aligned} x + 2my + z &= 4m \\ 2mx + y + z &= 2 \\ x + y + 2mz &= 2m^2 \end{aligned}$$

with real parameter m. Determine, with proof, the values of m for this system to have

1.) infinitely many solutions 2.) no solution 3.) a unique solution

Exercise 2.10.51 Show that the linear system of equations

$$\begin{aligned} -x_2 + x_3 + 2x_4 &= a \\ x_1 + 2x_2 - x_3 + x_4 &= b \\ -3x_1 + x_2 + 4x_3 + 3x_4 &= c \\ 4x_1 - x_2 - 3x_3 + 2x_4 &= d \end{aligned}$$

has a solution if $2a + b = c + d$.

For the case $a = c = 1, b = 8$ and $d = 9$, find the solution of the above system of equations.

Chapter 3

Vector Spaces and Vector Subspaces

3.1 Vectors in $\mathbb{R}^n, \mathbb{C}^n$

Definition 3.1.1 Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be an ordered set of n numbers(tuples) from $\mathbb{R}$ or $\mathbb{C}$. Then $\boldsymbol{v}$ is called a **vector** and the numbers v_i are called components or coordinates. Then, n is the dimension of $\boldsymbol{v}$. We sometimes say that $\boldsymbol{v}$ is an n-dimensional vector in $\mathbb{R}$ or $\mathbb{C}$ and write $\boldsymbol{v} \in \mathbb{R}^n$ or $\boldsymbol{v} \in \mathbb{C}^n$.

Definition 3.1.2 Let $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^n$. Then $\boldsymbol{v}$ is said to be equal to $\boldsymbol{u}$ iff $u_i = v_i, \quad i = 1, 2, \ldots, n$, and we write $\boldsymbol{v} = \boldsymbol{u}$.

Example 3.1.1 Let $\boldsymbol{u} = (2, -7, 1), \quad \boldsymbol{v} = (-3, 0, 4)$ and $\boldsymbol{w} = (0, 5, -8)$. Find the components of the vector $\boldsymbol{x}$ that satisfies $2\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{x} = 7\boldsymbol{x} + \boldsymbol{w}$

Solution : *Note that* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x} \in \mathbb{R}^3$.

$$\begin{aligned}
2\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{x} &= 2(2, -7, 1) - (-3, 0, 4) + (x_1, x_2, x_3) \\
&= (7 + x_1, \quad -14 + x_2, \quad -2 + x_3) \\
\textit{Also} \;\; 7x + \boldsymbol{w} &= (7x_1, \quad 7x_2 + 5, \quad 7x_3 - 8) \\
\textit{So} \;\; 2\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{x} &= 7\boldsymbol{x} + \boldsymbol{w} \\
\Rightarrow \quad 7 + x_1 &= 7x_1 \quad \textit{or} \quad x_1 = \frac{7}{6}, \\
-14 + x_2 &= 7x_2 + 5 \quad \textit{or} \quad x_2 = \frac{-19}{6} \\
-2 + x_3 &= 7x_3 - 8 \quad \textit{or} \quad x_3 = 1 \\
\textit{Therefore} \quad \boldsymbol{x} &= \left(\frac{7}{6}, -\frac{19}{6}, 1\right).
\end{aligned}$$

■

3.2 Vector Arithmetic

Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ and $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ be vectors defined over a field $\mathbb{F}$. For our purpose, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

The sum of $\boldsymbol{v}$ and $\boldsymbol{u}$ is the vector obtained by adding corresponding components and is denoted by $\boldsymbol{v} + \boldsymbol{u}$, i.e $\boldsymbol{v} + \boldsymbol{u} = (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n)$. This is coordinate-wise addition of vectors.

Example 3.2.1 Let $\underline{\underline{0}}$ be the zero vector. For example, in $\mathbb{R}^3$, $\underline{\underline{0}} = (0, 0, 0)$.

3.3 Scalar Multiplication

Let $k \in \mathbb{F}$, (k is a scalar), we define the scalar product $k\boldsymbol{v}$ as the vector obtained by multiplying each component of $\boldsymbol{v}$ by k, i.e, $k\boldsymbol{v} = (kv_1, kv_2, \ldots, kv_n)$. Thus for $k = -1$ then

$$k\boldsymbol{v} = -\boldsymbol{v} = (-v_1, -v_2, \ldots, -v_n) \quad \text{and} \quad \boldsymbol{v} + k\boldsymbol{v} = 0.$$

3.3.1 Properties of Vector Addition

Let $\boldsymbol{v}, \boldsymbol{u}$ and $\boldsymbol{w}$ be vectors in $\mathbb{F}^n$, $\mathbb{F}$ a field, then

(a) $\boldsymbol{v} + \boldsymbol{u} \in \mathbb{F}^n$ (closure)

(b) $\boldsymbol{v} + \boldsymbol{u} = \boldsymbol{u} + \boldsymbol{v}$ (commutative law)

(c) $(\boldsymbol{v} + \boldsymbol{u}) + \boldsymbol{w} = \boldsymbol{v} + (\boldsymbol{u} + \boldsymbol{w})$ (Associative law)

(d) $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$ (existence of additive identity $\mathbf{0}$)

(e) $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$ (existence of additive inverse)

3.3.2 Properties of Scalar Multiplication (Product)

Let s, t be scalars in $\mathbb{F}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{F}^n$. Then

(f) $s\boldsymbol{v} \in \mathbb{F}^n$ (closure property)

(g) $(st)\boldsymbol{v} = s(t\boldsymbol{v})$ (associative law)

(h) $(s + t)\boldsymbol{v} = s\boldsymbol{v} + t\boldsymbol{v}$ and $s(\boldsymbol{v} + \boldsymbol{u}) = s\boldsymbol{v} + s\boldsymbol{u}$ (distributive law)

(i) $1\boldsymbol{v} = \boldsymbol{v}$ (existence of multiplicative identity)

Example 3.3.1 Prove that $\boldsymbol{v} + \boldsymbol{u} = \boldsymbol{u} + \boldsymbol{v}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{F}^n$.

Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ and $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$.

$$\begin{aligned} \text{Then} \quad \boldsymbol{v} + \boldsymbol{u} &= (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \quad (\text{since } v_i + u_i = u_i + v_i \text{ for } u_i,\ v_i \in \mathbb{F} \text{ a field}) \\ &= \boldsymbol{u} + \boldsymbol{v} \quad (\text{By definition}) \end{aligned}$$

Example 3.3.2 Let $\boldsymbol{u} = (2, -7, 1)$, $\boldsymbol{v} = (-3, 0, 4)$, $\boldsymbol{w} = (0, 5, -8)$. Find

(a) $3\boldsymbol{u} - 4\boldsymbol{v} = (18, -21, -13)$

(b) $2\boldsymbol{u} + 3\boldsymbol{v} - 5\boldsymbol{w}$

Example 3.3.3 Let $\boldsymbol{u} = (-1, 3, 2, 0)$, $\boldsymbol{v} = (2, 0, 4, -1)$, $\boldsymbol{w} = (7, 1, 1, 4)$ and $\boldsymbol{x} = (6, 3, 1, 2)$. Find scalars a, b, c, d such that $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} + d\boldsymbol{x} = (0, 5, 6, -3)$

$$\begin{array}{rcrcrcrcr} -a & + & 2b & + & 7c & + & 6d & = & 0 \\ 3a & & & + & c & + & 3d & = & 5 \\ 2a & + & 4b & + & c & + & d & = & 6 \\ & - & b & + & 4c & + & 2d & = & -3 \end{array}$$

a, b, c and d is a solution to the above system. We discussed methods for solving such systems in Chapter 2.2 of this lecture book.

3.4 Vector Spaces

3.4.1 Introduction

We have been knowing a vector from the Euclidean n space as a directed line segment e.g $\hat{u} = a\hat{i} + b\hat{j} + c\hat{k}$,where a, b, c have always been in $\Re$, but here we shall consider a vector as a point in space e.g $\hat{u} = (a, b, c)$ and also not only $a, b, c \in \Re$ but also in $\mathbb{Z}$, the integers and $\mathbb{C}$, the complex numbers.

We say $\hat{u} = (a, b, c)$, with $a, b, c \in \Re$ is a vector in a Real field/space i.e, we write $(a, b, c) \in \Re^3$ a vector in the real field with 3-dimension (3 entries or tuples). Further a vector would be seen as matrix or a function (polynomial) and that's only a couple of possibilities for vectors.

We say that $\hat{u} = (a, b, c, d)$, with $a, b, c, d \in \mathbb{C}$ is a vector in a complex field/space or $(a, b, c, d) \in \mathbb{C}^4$ a vector in the complex field [since entries are complex numbers] with 4-dimension. In general ;
$(a_1, a_2, a_3, \ldots, a_n) \in \Re^n$ if $a_1, a_2, a_3, \ldots, a_n \in \Re$.
$(a_1, a_2, a_3, \ldots, a_n) \in \mathbb{Z}^n$ if $a_1, a_2, a_3, \ldots, a_n \in \mathbb{Z}$.
$(a_1, a_2, a_3, \ldots, a_n) \in \mathbb{C}^n$ if $a_1, a_2, a_3, \ldots, a_n \in \mathbb{C}$.

3.4.2 Vector Space

Definition 3.4.1 Vector space
We define a vector space as a set of elements $\boldsymbol{V}(+, \cdot)$, the elements are called vectors together with two operations, addition $(+)$ and scalar multiplication $(\cdot)$ satisfying the following properties (axioms).

(a) If $X, Y, Z \in \boldsymbol{V}$

1) $X + Y \in \boldsymbol{V}$ (closure of addition)
2) $X + Y = Y + X$ (commutativity of addition)
3) $X + (Y + Z) = (X + Y) + Z$ (associativity of addition)
4) $\exists$ a zero i.e $\underset{\sim}{0} \in \boldsymbol{V}$ such that $X + \underset{\sim}{0} = \underset{\sim}{0} + X = X$
   i.e [there exists a vector $\underset{\sim}{0}$ in $\boldsymbol{V}$ whose entries when added to entries of X gives a vector in $\boldsymbol{V}$ whose entries are entries of X]
5) For each $X \in \boldsymbol{V}$, $\exists$ a unique vector $-X \in \boldsymbol{V}$ such that

$$X + (-X) = (-X) + X = 0$$

(b) If $X, Y, \in \boldsymbol{V}$and α, β are scalars then

1) $\alpha \cdot X \in \boldsymbol{V}$ (closure of multiplication)
2) $\alpha \cdot (X + Y) = \alpha \cdot X + \alpha \cdot Y$ (distributivity of multiplication)
3) $(\alpha + \beta) \cdot X = \alpha \cdot X + \beta \cdot X$.
4) $\alpha \cdot (\beta \cdot X) = \alpha\beta \cdot X$
5) $\exists$ a one i.e, $1 \in \mathbb{F}$ such that $1 \cdot X = X \cdot 1 = X$ (Multiplicative identity)

Note 3.4.1 To check whether a space is a vector space, we have to check all the properties. However if one of them is violated/not satisfied, then the space is not a vector space. Where $+$ and $\cdot$ are not usual addition and subtraction.

Remark 3.4.1 The $+$, and $\cdot$ signs are not necessarily ordinary addition and multiplication unless stated so.

Example 3.4.1 Show that $\mathbb{R}^3$ a 3-dimensional space is a vector space with the ordinary vector addition and multiplication.

Let $X, Y, Z \in V = \mathbb{R}^3$ i.e

$$X = (x_1, x_2, x_3), \quad Y = (y_1, y_2, y_3), \quad Z = (z_1, z_2, z_3) \; ; \; x_i, y_i, z_i \in \mathbb{R} \; \forall \; 1 \leq i \leq 3$$

(a) 1) Clearly

$$X + Y \;=\; (x_1 + y_1, x_2 + y_2, x_3 + y_3) \;\in\; \mathbb{R}^3$$

2)

$$\begin{aligned} LHS: \quad X + Y &= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\ &= x_1 + y_1, x_2 + y_2, x_3 + y_3 \\ RHS: \quad Y + X &= (y_1, y_2, y_3) + (x_1, x_2, x_3) \\ &= y_1 + x_1, y_2 + x_2, y_3 + x_3 \\ &= x_1 + y_1, x_2 + y_2, x_3 + y_3 \end{aligned}$$

Commutativity of addition holds since $X + Y = Y + X$.

3)

$$\begin{aligned} LHS: \quad X + (Y + Z) &= (x_1, x_2, x_3) + [(y_1, y_2, y_3) + (z_1, z_2, z_3)] \\ &= (x_1, x_2, x_3) + (y_1 + z_1, y_2 + z_2, y_3 + z_3) \\ &= x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), x_3 + (y_3 + z_3) \\ &= x_1 + y_1 + z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3 \\ RHS: \quad (X + Y) + Z &= (x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (x_3 + y_3) + z_3 \\ &= x_1 + y_1 + z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3 \end{aligned}$$

Associativity of addition holds as $X + (Y + Z) = (X + Y) + Z$.

4) In $\mathbb{R}^3$ $\exists$ $\mathbf{0}$ where $\mathbf{0} = (0, 0, 0)$ such that for any $X \in V = \mathbb{R}^3$

$$\begin{aligned} LHS: \quad X + \mathbf{0} &= (x_1, x_2, x_3) + (0, 0, 0) \\ &= (x_1 + 0, x_2 + 0, x_3 + 0) \\ &= (x_1, x_2, x_3) \\ RHS: \quad \mathbf{0} + X &= (0, 0, 0) + (x_1, x_2, x_3) \\ &= (0 + x_1, 0 + x_2, 0 + x_3) \\ &= (x_1, x_2, x_3) \end{aligned}$$

Satisfied as $X + \mathbf{0} = \mathbf{0} + X = X$

5) For each $X \in \mathbb{R}^3$, $\exists$ a unique vector $-X \in \mathbb{R}^3$ such that

$$-X = -(x_1, x_2, x_3) = (-x_1, -x_2, -x_3)$$

Such that

$$\begin{array}{llcl}
LHS: & X+-X & = & (x_1,x_2,x_3)+(-x_1,-x_2,-x_3)\\
 & & = & x_1-x_1,x_2-x_2,x_3-x_3\\
 & & = & (0,0,0)\\
RHS: & -X+X & = & (-x_1,-x_2,-x_3)+(x_1,x_2,x_3)\\
 & & = & -x_1+x_1,-x_2+x_2,-x_3+x_3\\
 & & = & (0,0,0)
\end{array}$$

So $X+-X=-X+X=\mathbf{0}$

(b) Let $X,Y,Z\in\mathbb{R}^n,\alpha,\beta\in\mathbb{F}$

1) Also

$$\alpha\cdot X=(\alpha x_1,\alpha x_2,\alpha x_3)\ \in\ \mathbb{R}^3$$

2)

$$\begin{array}{llcl}
LHS: & \alpha\cdot(X+Y) & = & \alpha\cdot((x_1,x_2,x_3)+(y_1,y_2,y_3))\\
 & & = & \alpha\cdot(x_1+y_1,x_2+y_2,x_3+y_3)\\
 & & = & \alpha(x_1+y_1,x_2+y_2,x_3+y_3)\\
RHS: & \alpha\cdot X+\alpha\cdot Y & = & \alpha\cdot(x_1,x_2,x_3)+\alpha\cdot(y_1,y_2,y_3)\\
 & & = & (\alpha x_1,\alpha x_2,\alpha x_3)+(\alpha y_1,\alpha y_2,\alpha y_3)\\
 & & = & (\alpha x_1+\alpha y_1,\alpha x_2+\alpha y_2,\alpha x_3+\alpha y_3)\\
 & & = & \alpha(x_1+y_1,x_2+y_2,x_3+y_3)
\end{array}$$

Property (b2) satisfied since

$$\alpha\cdot(X+Y)=\alpha\cdot X+\alpha\cdot Y$$

3)

$$\begin{array}{llcl}
LHS: & (\alpha+\beta)\cdot X & = & (\alpha+\beta)(x_1,x_2,x_3)\\
 & & = & ((\alpha+\beta)x_1,(\alpha+\beta)x_2,(\alpha+\beta)x_3)\\
 & & = & \alpha x_1+\beta x_1,\alpha x_2+\beta x_2,\alpha x_3+\beta x_3\\
RHS: & \alpha\cdot X+\beta\cdot X & = & \alpha\cdot(x_1,x_2,x_3)+\beta\cdot(x_1,x_2,x_3)\\
 & & = & \alpha(x_1,x_2,x_3)+\beta(x_1,x_2,x_3)\\
 & & = & \alpha x_1+\beta x_1,\alpha x_2+\beta x_2,\alpha x_3+\beta x_3
\end{array}$$

Property (b3) satisfied since

$$(\alpha+\beta)\cdot X=\alpha\cdot X+\beta\cdot X$$

4)

$$\begin{array}{llcl}
LHS: & \alpha\cdot(\beta\cdot X) & = & \alpha\cdot(\beta\cdot(x_1,x_2,x_3))\\
 & & = & \alpha\cdot(\beta x_1,\beta x_2,\beta x_3)\\
 & & = & (\alpha\beta x_1,\alpha\beta x_2,\alpha\beta x_3)\\
RHS: & \alpha\beta\cdot X & = & \alpha\beta\cdot(x_1,x_2,x_3)\\
 & & = & \alpha\beta(x_1,x_2,x_3)\\
 & & = & (\alpha\beta x_1,\alpha\beta x_2,\alpha\beta x_3)
\end{array}$$

Property (b4) satisfied since

$$\alpha \cdot (\beta \cdot X) = \alpha\beta \cdot X$$

5)

$$\begin{aligned} LHS: \quad 1 \cdot X &= 1(x_1, x_2, x_3) \\ &= (x_1, x_2, x_3) \\ RHS: \quad X \cdot 1 &= (x_1, x_2, x_3) \cdot 1 \\ &= (x_1, x_2, x_3)(1) \\ &= (x_1, x_2, x_3) \end{aligned}$$

Property (b5) satisfied since

$$1 \cdot X = X \cdot 1 = X$$

Thus with usual addition and scalar multiplication, $\mathbb{R}^3$ is a vector space.

Example 3.4.2 Let $\boldsymbol{V}$ be a set of ordered pairs of real numbers $\mathbb{R}^2$ with usual $+$ and $\cdot$ defined by

$$s \cdot X = (u_1,\ su_2)X$$

Prove that $\mathbb{R}^2(+, \cdot)$ is not a vector space.

Checking property (b3)

$$(\alpha + \beta) \cdot X = \alpha \cdot X + \beta \cdot X$$

$$\begin{aligned} LHS: \quad (\alpha + \beta) \cdot X &= (u_1,\ (\alpha + \beta)u_2)\, X \\ &= (u_1,\ \alpha u_2 + \beta u_2)\, X \end{aligned}$$

$$\begin{aligned} RHS: \quad \alpha \cdot X + \beta \cdot X &= (u_1,\ \alpha u_2)\, X + (u_1,\ \beta u_2)\, X \\ &= (2u_1,\ \alpha u_2 + \beta u_2)\, X \end{aligned}$$

Thus $(\alpha + \beta) \cdot X \neq \alpha \cdot X + \beta \cdot X$ and therefore $\mathbb{R}^2(+, \cdot)$ is not a vector space under the specified operation.

Remark 3.4.2 Showing that something is not a vector space can be tricky because it is completely possible that only one of the axioms fails. In this case because we are dealing with the standard/usual addition all the axioms involving the addition of objects from $\boldsymbol{V}$ will be valid. Also, in this case of all the axioms involving the scalar multiplication are valid **except** one.

Example 3.4.3 Let V be a set of ordered pairs of real numbers $\mathbb{R}^2$ with $+$ and $\cdot$ defined by

$$\begin{aligned} X + Y &= X - Y \\ \alpha \cdot X &= \alpha X \end{aligned}$$

Prove that $\mathbb{R}^2(+, \cdot)$ is not a vector space.

Using (a2)

$$X + Y = Y + X$$

$$LHS: \quad X + Y = X - Y$$

$$RHS: \quad Y + X = Y - X$$

Hence $X + Y \neq Y + X$, thus $\mathbb{R}^2(+, \cdot)$ with the above addition and scalar multiplication is not a vector space.

Example 3.4.4 Let V be a set of all triples of real numbers (x, y, z) i.e $\mathbb{R}^3$ with the operations defined by

$$\begin{aligned}(x_1, x_2, x_3) + (y_1, y_2, y_3) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3)\\ \alpha \cdot (x_1, x_2, x_3) &= (0, 0, 0)\end{aligned}$$

Let $X, Y, Z \in V = \mathbb{R}^3$ i.e

$$X = (x_1, x_2, x_3), \quad Y = (y_1, y_2, y_3), \quad Z = (z_1, z_2, z_3)$$

(a) 1) Clearly

$$X + Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in \mathbb{R}^3$$

2)

$$\begin{aligned}LHS: \quad X + Y &= (x_1, x_2, x_3) + (y_1, y_2, y_3)\\ &= x_1 + y_1, x_2 + y_2, x_3 + y_3\\ RHS: \quad Y + X &= (y_1, y_2, y_3) + (x_1, x_2, x_3)\\ &= y_1 + x_1, y_2 + x_2, y_3 + x_3\\ &= x_1 + y_1, x_2 + y_2, x_3 + y_3\end{aligned}$$

Commutativity of addition holds since $X + Y = Y + X$.

3)

$$\begin{aligned}LHS: \quad X + (Y + Z) &= (x_1, x_2, x_3) + [(y_1, y_2, y_3) + (z_1, z_2, z_3)]\\ &= (x_1, x_2, x_3) + (y_1 + z_1, y_2 + z_2, y_3 + z_3)\\ &= x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), x_3 + (y_3 + z_3)\\ &= x_1 + y_1 + z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3\\ RHS: \quad (X + Y) + Z &= (x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (x_3 + y_3) + z_3\\ &= x_1 + y_1 + z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3\end{aligned}$$

Associativity of addition holds as $X + (Y + Z) = (X + Y) + Z$.

4) In $\mathbb{R}^3$ $\exists$ $\mathbf{0}$ where $\mathbf{0} = (0, 0, 0)$ such that for any $X \in V = \mathbb{R}^3$

$$\begin{aligned}LHS: \quad X + \mathbf{0} &= (x_1, x_2, x_3) + (0, 0, 0)\\ &= (x_1 + 0, x_2 + 0, x_3 + 0)\\ &= (x_1, x_2, x_3)\\ RHS: \quad \mathbf{0} + X &= (0, 0, 0) + (x_1, x_2, x_3)\\ &= (0 + x_1, 0 + x_2, 0 + x_3)\\ &= (x_1, x_2, x_3)\end{aligned}$$

Satisfied as $X + \mathbf{0} = \mathbf{0} + X = X$

5) For each $X \in \mathbb{R}^3$, $\exists$ a unique vector $-X \in \mathbb{R}^3$ such that

$$-X = -(x_1, x_2, x_3) = (-x_1, -x_2, -x_3)$$

Such that

$$\begin{array}{rrcl} LHS: & X+-X &=& (x_1,x_2,x_3)+(-x_1,-x_2,-x_3) \\ && = & x_1-x_1,x_2-x_2,x_3-x_3 \\ && = & (0,0,0) \\ RHS: & -X+X &=& (-x_1,-x_2,-x_3)+(x_1,x_2,x_3) \\ && = & -x_1+x_1,-x_2+x_2,-x_3+x_3 \\ && = & (0,0,0) \end{array}$$

So $X+-X=-X+X=\mathbf{0}$

(b) Let $X,Y,Z\in\mathbb{R}^n,\alpha,\beta\in\mathbb{F}$

1) Also

$$\alpha\cdot X=(0,0,0)\ \in\ \mathbb{R}^3$$

Property (b1) satisfied.

2)

$$\begin{array}{rrcl} LHS: & \alpha\cdot(X+Y) &=& \alpha\cdot((x_1,x_2,x_3)+(y_1,y_2,y_3) \\ && = & \alpha\cdot(x_1+y_1,x_2+y_2,x_3+y_3) \\ && = & (0,0,0) \\ RHS: & \alpha\cdot X+\alpha\cdot Y &=& \alpha\cdot(x_1,x_2,x_3)+\alpha\cdot(y_1,y_2,y_3) \\ && = & (0,0,0)+(0,0,0)=(0+0,0+0,0+0) \\ && = & (0,0,0) \end{array}$$

Property (b2) satisfied since

$$\alpha\cdot(X+Y)=\alpha\cdot X+\alpha\cdot Y$$

3)

$$\begin{array}{rrcl} LHS: & (\alpha+\beta)\cdot X &=& (\alpha+\beta)\cdot(x_1,x_2,x_3) \\ && = & (0,0,0) \\ RHS: & \alpha\cdot X+\beta\cdot X &=& \alpha\cdot(x_1,x_2,x_3)+\beta\cdot(x_1,x_2,x_3) \\ && = & (0,0,0)+(0,0,0)=(0+0,0+0,0+0) \\ && = & (0,0,0) \end{array}$$

Property (b3) satisfied since

$$(\alpha+\beta)\cdot X=\alpha\cdot X+\beta\cdot X$$

4)

$$\begin{array}{rrcl} LHS: & \alpha\cdot(\beta\cdot X) &=& \alpha\cdot(\beta\cdot(x_1,x_2,x_3)) \\ && = & \alpha\cdot(0,0,0) \\ && = & (0,0,0) \\ RHS: & \alpha\beta\cdot X &=& \alpha\beta\cdot(x_1,x_2,x_3) \\ && = & (0,0,0) \end{array}$$

Property (b4) satisfied since

$$\alpha\cdot(\beta\cdot X)=\alpha\beta\cdot X$$

5)

$$\begin{array}{rrcl} LHS: & 1\cdot X & = & 1\cdot(x_1,x_2,x_3) \\ & & = & (0,0,0) \\ RHS: & X\cdot 1 & = & (x_1,x_2,x_3)\cdot 1 \\ & & = & (0,0,0) \end{array}$$

Property (b5) not satisfied since

$$1\cdot X = X\cdot 1 \neq X$$

Thus with defined addition and scalar multiplication, $\mathbb{R}^3$ is not a vector space.

Exercise 3.4.1 Let V be a set of all polynomials P_n of degree less or equal to n, with usual addition and multiplication, show that V is a vector space.

Exercise 3.4.2 Let V be a set of all ordered triples of real numbers (x,y,z) with $+$ and $\cdot$ defined by

$$\begin{array}{rcl} (x,y,z)+(x',y',z') & = & (x+x',y+y',z+z') \\ \alpha\cdot(x,y,z) & = & (\alpha x,y,z) \end{array}$$

Where α is a scalar, show that V is not a vector space. (b3)

Exercise 3.4.3 Let V be a set of real valued functions defined on $\Re$. Let f,g be members of V with $+$ and $\cdot$ defined by

$$\begin{array}{rcl} (f+g)x & = & f(x)+g(x) \\ (\alpha\cdot f)x & = & \alpha f(x) \end{array}$$

Prove that V is a vector space.

Exercise 3.4.4 Let V be a set of all $m\times n$ matrices with the usual addition and scalar multiplication. Is V a vector space ?

Example 3.4.5 Let V be a set of all ordered triples of real numbers (x,y,z) with $+$ and $\cdot$ defined by

$$\begin{array}{rcl} (x,y,z)+(x',y',z') & = & (x+x',y+y',z+z') \\ \alpha\cdot(x,y,z) & = & (1,0,1) \end{array}$$

Where α is a scalar. Is V a vector space ?

$$\begin{array}{rcl} (b2)\text{ fails} & : & (1,0,1)\neq(2,0,2) \\ (b3)\text{ fails} & : & (1,0,1)\neq(2,0,2) \\ (b5)\text{ fails} & : & (1,0,1)\neq(x,y,z) \end{array}$$

However, (a1), (a2),(a3), (a4), (a5) satisfied. Similarly (b1) and (b4) holds.

Exercise 3.4.5 Verify that $V=\{0\}$ is a vector space!

Example 3.4.6 Show that $\boldsymbol{V} = \mathbb{R}^n$ an n-dimensional space is a vector space with usual vector addition and multiplication.

Let $X, Y, Z \in \boldsymbol{V}$ i.e

$$X = (x_1, x_2, x_3, \ldots, x_n), \quad Y = (y_1, y_2, y_3, \ldots, y_n), \quad Z = (z_1, z_2, z_3, \ldots, z_n)$$

(a) 1) Clearly

$$X + Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n) \in \mathbb{R}^n$$

2)

$$\begin{aligned} X + Y &= (x_1, x_2, x_3, \ldots, x_n) + (y_1, y_2, y_3, \ldots, y_n) \\ &= x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n \\ &= y_1 + x_1, y_2 + x_2, y_3 + x_3, \ldots, y_n + x_n \\ &= (y_1, y_2, y_3, \ldots, y_n) + (x_1, x_2, x_3, \ldots, x_n) \\ &= Y + X \end{aligned}$$

Commutativity of addition holds.

3)

$$\begin{aligned} X + (Y + Z) &= x_1, x_2, x_3, \ldots, x_n + (y_1, y_2, y_3, \ldots, y_n + z_1, z_2, z_3, \ldots, z_n) \\ &= x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), x_3 + (y_3 + z_3), \ldots, x_n + (y_n + z_n) \\ &= x_1 + y_1 + z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3, \ldots, x_n + y_n + z_n \\ &= (x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (x_3 + y_3) + z_3, \ldots, (x_n + y_n) + z_n \\ &= (x_1, x_2, x_3, \ldots, x_n + y_1, y_2, y_3, \ldots, y_n) + (z_1, z_2, z_3, \ldots, z_n) \\ &= (X + Y) + Z \end{aligned}$$

Associativity of addition holds.

4) In $\mathbb{R}^n \ \exists \ \mathbf{0}$ where $\mathbf{0} = (0, 0, 0, \ldots, 0_n)$ such that for any $X \in \boldsymbol{V} = \mathbb{R}^n$

$$\begin{aligned} X + \mathbf{0} &= (x_1, x_2, x_3, \ldots, x_n) + (0, 0, 0, \ldots, 0_n) \\ &= (x_1 + 0, x_2 + 0, x_3 + 0, \ldots, x_n + 0_n) \\ &= (x_1, x_2, x_3, \ldots, x_n) \\ &= X \end{aligned}$$

5) For each $X \in \mathbb{R}^n$, $\exists$ a unique vector $-X \in \mathbb{R}^n$ such that

$$-X = -(x_1, x_2, \ldots, x_n) = (-x_1, -x_2, -x_3, \ldots, -x_n)$$

Such that

$$\begin{aligned} X + -X &= (x_1, x_2, x_3, \ldots, x_n) + (-x_1, -x_2, -x_3, \ldots, -x_n) \\ &= (x_1 - x_1, x_2 - x_2, x_3 - x_3, \ldots, x_n - x_n) \\ &= (-x_1 + x_1, -x_2 + x_2, -x_3 + x_3, \ldots, -x_n + x_n) \\ &= (0, 0, 0, \ldots, 0_n) \\ &= -X + X \\ &= \mathbf{0} \end{aligned}$$

(b) 1) Also

$$\alpha \cdot X = (\alpha x_1, \alpha x_2, \alpha x_3, \dots, \alpha x_n) \ \in \ \mathbb{R}^n$$

2)

$$\begin{aligned}
\alpha \cdot (X+Y) &= \alpha \cdot ((x_1, x_2, x_3, \dots, x_n) + (y_1, y_2, y_3, \dots, y_n)) \\
&= \alpha \cdot (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n) \\
&= \alpha \cdot (x_1, x_2, x_3, \dots, x_n) + \alpha \cdot (y_1, y_2, y_3, \dots, y_n) \\
&= \alpha \cdot X + \alpha \cdot Y
\end{aligned}$$

3)

$$\begin{aligned}
(\alpha+\beta) \cdot X &= (\alpha+\beta)(x_1, x_2, x_3, \dots, x_n) \\
&= ((\alpha+\beta)x_1, (\alpha+\beta)x_2, (\alpha+\beta)x_3, \dots, (\alpha+\beta)x_n) \\
&= \alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, \alpha x_3 + \beta x_3, \dots, \alpha x_n + \beta x_n \\
&= \alpha(x_1, x_2, x_3, \dots, x_n) + \beta(x_1, x_2, x_3, \dots, x_n) \\
&= \alpha \cdot X + \beta \cdot X
\end{aligned}$$

4)

$$\begin{aligned}
\alpha \cdot (\beta \cdot X) &= \alpha \cdot (\beta \cdot (x_1, x_2, x_3, \dots, x_n)) \\
&= \alpha \cdot (\beta x_1, \beta x_2, \beta x_3, \dots, \beta x_n) \\
&= \alpha\beta(x_1, x_2, x_3, \dots, x_n) \\
&= \alpha\beta \cdot X
\end{aligned}$$

5)

$$1 \cdot X = 1(x_1, x_2, x_3, \dots, x_n) = (x_1, x_2, x_3, \dots, x_n) = X$$

Hence $\boldsymbol{V} = \mathbb{R}^n$ with usual(ordinary) addition and multiplication is a vector space.

Remark 3.4.3 $0X = \mathbf{0}$ for all $X \in V$.

Note that the 0 on the left hand side is a scalar, whereas the 0 on the right hand side is a vector.

Proof : *For $X \in V$ we have*

$$0X = (0+0)X = 0X + 0X,$$

using distributivity. Adding the additive inverse of $0X$ to both sides we obtain

$$\mathbf{0} = 0X - 0X = (0X + 0X) - 0X = 0X.$$

■

Remark 3.4.4 $(-1)X = -X$ for every $X \in V$.

Proof : *For $X \in V$ we have*

$$X + (-1)X = 1X + (-1)X = (1 + (-1))X = 0X = 0,$$

which shows that (-1)X is the additive inverse $-X$ of X. ■

3.5 Vector Subspaces

Definition 3.5.1 Vector subspace
If we let $V(+,\cdot)$ be a vector space, and let $W \subset V$. If $W(+,\cdot)$ is a vector space with respect to the operations in $V(+,\cdot)$, then we say that W is a vector subspace of V. [i.e if W is a subset and also *satisfies all the properties of a vector space*, then its a vector subspace].

Example 3.5.1 Let W be a set of vectors of the form $(\alpha, \beta, 0)$ - last entry is a zero, with α, β all members of real numbers ($\alpha, \beta \in \mathbb{R}$) under usual addition and multiplication. Is W a subspace of $\mathbb{R}^3$.

It is clear that $W \subset \mathbb{R}^3$. We now check whether W is a vector space.

Let $X, Y, Z \in V = \mathbb{R}^3$ i.e

$$X = (x_1, x_2, 0), \quad Y = (y_1, y_2, 0), \quad Z = (z_1, z_2, 0)$$

(a) 1) Clearly

$$X + Y = (x_1 + y_1, x_2 + y_2, 0 + 0) = (x_1 + y_1, x_2 + y_2, 0) \in \mathbb{R}^3$$

2)

$$\begin{aligned}
LHS: \quad X + Y &= (x_1, x_2, 0) + (y_1, y_2, 0) \\
&= (x_1 + y_1, x_2 + y_2, 0) \\
RHS: \quad Y + X &= (y_1, y_2, 0) + (x_1, x_2, 0) \\
&= (y_1 + x_1, y_2 + x_2, 0 + 0) \\
&= (x_1 + y_1, x_2 + y_2, 0)
\end{aligned}$$

Commutativity of addition holds since $X + Y = Y + X$.

3)

$$\begin{aligned}
LHS: \quad X + (Y + Z) &= (x_1, x_2, 0) + [(y_1, y_2, 0) + (z_1, z_2, 0)] \\
&= (x_1, x_2, 0) + (y_1 + z_1, y_2 + z_2, 0 + 0) \\
&= x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), 0 + (0 + 0) \\
&= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, 0) \\
RHS: \quad (X + Y) + Z &= (x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (0 + 0) + 0 \\
&= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, 0)
\end{aligned}$$

Associativity of addition holds as $X + (Y + Z) = (X + Y) + Z$.

4) In $W = \mathbb{R}^3 \ \exists \ \mathbf{0}$ where $\mathbf{0} = (0, 0, 0)$ such that for any $X \in V = \mathbb{R}^3$

$$\begin{aligned}
LHS: \quad X + \mathbf{0} &= (x_1, x_2, 0) + (0, 0, 0) \\
&= (x_1 + 0, x_2 + 0, 0 + 0) \\
&= (x_1, x_2, 0) \\
RHS: \quad \mathbf{0} + X &= (0, 0, 0) + (x_1, x_2, 0) \\
&= (0 + x_1, 0 + x_2, 0 + 0) \\
&= (x_1, x_2, 0)
\end{aligned}$$

Satisfied as $X + \mathbf{0} = \mathbf{0} + X = X$

5) For each $X \in \mathbb{R}^3$, $\exists$ a unique vector $-X \in \mathbb{R}^3$ such that

$$-X = -(x_1, x_2, x_3) = (-x_1, -x_2, -0)$$

Such that

$$\begin{aligned} LHS: \quad X + -X &= (x_1, x_2, 0) + (-x_1, -x_2, -0) \\ &= (x_1 - x_1, x_2 - x_2, 0 - 0) \\ &= (0, 0, 0) \\ RHS: \quad -X + X &= (-x_1, -x_2, -0) + (x_1, x_2, 0) \\ &= (-x_1 + x_1, -x_2 + x_2, -0 + 0) \\ &= (0, 0, 0) \end{aligned}$$

So $X + -X = -X + X = \mathbf{0}$

(b) 1) Also

$$\alpha \cdot X = (\alpha x_1, \alpha x_2, \alpha 0) \in \mathbb{R}^3$$

2)

$$\begin{aligned} LHS: \quad \alpha \cdot (X + Y) &= \alpha \cdot ((x_1, x_2, 0) + (y_1, y_2, 0)) \\ &= \alpha \cdot (x_1 + y_1, x_2 + y_2, 0 + 0) \\ &= \alpha(x_1 + y_1, x_2 + y_2, 0) \\ RHS: \quad \alpha \cdot X + \alpha \cdot Y &= \alpha \cdot (x_1, x_2, 0) + \alpha \cdot (y_1, y_2, 0) \\ &= (\alpha x_1, \alpha x_2, 0) + (\alpha y_1, \alpha y_2, 0) \\ &= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, 0 + 0) \\ &= \alpha(x_1 + y_1, x_2 + y_2, 0) \end{aligned}$$

Property (b2) satisfied since

$$\alpha \cdot (X + Y) = \alpha \cdot X + \alpha \cdot Y$$

3)

$$\begin{aligned} LHS: \quad (\alpha + \beta) \cdot X &= (\alpha + \beta)(x_1, x_2, 0) \\ &= ((\alpha + \beta)x_1, (\alpha + \beta)x_2, (\alpha + \beta)0) \\ &= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, 0) \\ RHS: \quad \alpha \cdot X + \beta \cdot X &= \alpha \cdot (x_1, x_2, 0) + \beta \cdot (x_1, x_2, 0) \\ &= \alpha(x_1, x_2, 0) + \beta(x_1, x_2, 0) \\ &= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, 0) \end{aligned}$$

Property (b3) satisfied since

$$(\alpha + \beta) \cdot X = \alpha \cdot X + \beta \cdot X$$

4)

$$\begin{aligned} LHS: \quad \alpha \cdot (\beta \cdot X) &= \alpha \cdot (\beta \cdot (x_1, x_2, 0)) \\ &= \alpha \cdot (\beta x_1, \beta x_2, 0) \\ &= (\alpha\beta x_1, \alpha\beta x_2, 0) \\ RHS: \quad \alpha\beta \cdot X &= \alpha\beta \cdot (x_1, x_2, 0) \\ &= \alpha\beta(x_1, x_2, 0) \\ &= (\alpha\beta x_1, \alpha\beta x_2, 0) \end{aligned}$$

Property (b4) satisfied since

$$\alpha \cdot (\beta \cdot X) = \alpha\beta \cdot X$$

5)

$$\begin{aligned} LHS: \quad 1 \cdot X &= 1(x_1, x_2, 0) \\ &= (x_1, x_2, 0) \\ RHS: \quad X \cdot 1 &= (x_1, x_2, 0) \cdot 1 \\ &= (x_1, x_2, 0)(1) \\ &= (x_1, x_2, 0) \end{aligned}$$

Property (b5) satisfied since

$$1 \cdot X = X \cdot 1 = X$$

Hence W with standard vector addition and multiplication is a vector space. Since a subset and a vector space, its a Vector subspace.

Theorem 3.5.1 If we let $V(+, \cdot)$ be a vector space, and let W a non empty subset of V. Then W is a subspace of V iff, $\exists\, 0 \in W$ and

1). If $X, Y \in W \Rightarrow X + Y \in W$

2). If $\alpha \in \mathbb{F}, X \in W \Rightarrow \alpha X \in W$

Remark 3.5.1 This imply that now to show a subspace only need to show subset and (1),(2) above only (Closed under addition and Scalar multiplication).

Example 3.5.2 Let W be a set of all 2×3 matrices of the form $\begin{bmatrix} \alpha & \beta & 0 \\ 0 & \lambda & \gamma \end{bmatrix}$, where $\alpha, \beta, \lambda, \gamma \in \Re$. Show that W is a vector subspace of all 2×3.

Clearly W is a subset of V a set of all 2×3 matrices.
Let

$$X = \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix}, \; Y = \begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix} \in W, \; a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{R}$$

So $X, Y \in W$ and
Then

1).

$$X + Y = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & 0 \\ 0 & c_1 + c_2 & d_1 + d_2 \end{bmatrix} \in W$$

Since 2×3 and $a_{13} = a_{21} = 0$

2).

$$\alpha X = \begin{bmatrix} \alpha a_1 & \alpha b_1 & 0 \\ 0 & \alpha c_1 & \alpha d_1 \end{bmatrix} \in W$$

Since 2×3 and $a_{13} = a_{21} = 0$

Property 1). and 2). are satisfied, thus W is a vector subspace of V.

Example 3.5.3 Let W be a set containing all vectors of the form $(a, b, 1)$ with $a, b \in \Re$. Show that W is not a vector subspace of $\mathbb{R}^3$.

Clearly W is a subset of V a set of all $\mathbb{R}^3$. Let

$$X = (a_1, b_1, 1) \quad , Y = (a_2, b_2, 1) \quad \in W,\ a_1, b_1, a_2, b_2 \in \mathbb{R}$$

Then

1).

$$X + Y = (a_1 + a_2, b_1 + b_2, 2) \notin \quad W$$

Since last entry is not a 1 but a 2.

Thus W is not a vector subspace of $\mathbb{R}^3$ since Property 1) does not hold.

Example 3.5.4 Show whether $W \subset \mathbb{R}^3$ defined by

$$W = \Big\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\Big\}$$

is a vector subspace of $\mathbb{R}^3$.

Let $X, Y \in W$. Then $X = (-x_2 - x_3,\ x_2,\ x_3)$ and $Y = (-y_2 - y_3,\ y_2,\ y_3)$. Then

1).

$$X + Y \quad = \quad (-x_2 - x_3,\ x_2,\ x_3) + (-y_2 - y_3,\ y_2,\ y_3) = (-x_2 - y_2 - x_3 - y_3,\ x_2 + y_2,\ x_3 + y_3) \in W$$

Since $-x_2 - y_2 - x_3 - y_3 = (x_2 + y_2) + (x_3 + y_3) = 0$

2).

$$\alpha X \quad = \quad \alpha\,(-x_2 - x_3,\ x_2,\ x_3) = (-\alpha x_2 - \alpha x_3,\ \alpha x_2,\ \alpha x_3) \in W$$

Since $\alpha x_1 + \alpha y_1 + \alpha z_1 = \alpha(x_1 + y_1 + z_1) = \alpha(0) = 0$

Exercise 3.5.1 Let V be a set of all ordered pairs of real numbers (x, y) with the operations defined by

$$\begin{aligned}(x_1, x_2) + (y_1, y_2) &= (x_1 - y_1,\ x_2 - y_2)\\ \alpha \cdot (x_1, x_2) &= (\alpha x_1,\ \alpha x_2)\end{aligned}$$

Is V a vector space?

Exercise 3.5.2 *True* or *False*? Every vector space is finite dimensional.

Exercise 3.5.3 Show that the set of *singular* 2×2 matrices under the usual operations is *not* a vector space.

$$\textit{Let}\ \ X = \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 \\ y_1 & y_2 \end{bmatrix}.\ \ \textit{Then}\ \ X + Y \notin V$$

Exercise 3.5.4 Prove that $\mathbb{R}$ is a vector space with addition and scalar multiplications given by $x + y = (x + y)^{\frac{1}{3}}$ and $\alpha \cdot x = (\ \sqrt[3]{a}\ )\, x$. $X = x, Y = y$. *Binomial Expansions*

3.6 End of chapter Questions

Exercise 3.6.1 Let $\mathscr{P}_2$ be a set of all polynomials of degree less or equal to 2. show that $\mathscr{P}_2$ is a vector subspace of all polynomials of degree ≤ 3.

Exercise 3.6.2 Prove that a set of all real valued continuous functions is a subspace of V the space of all real valued functions.

Exercise 3.6.3 The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with usual addition and multiplication is a not a vector subspace.

Exercise 3.6.4 The set $V = \mathbb{R}^3$ with the standard vector addition and scalar multiplication defined as

$$\mu(x_1,\ x_2,\ x_3) = (0,\ 0,\ \mu x_3).$$

Is $V(.,+)$ a vector space?

Exercise 3.6.5 Suppose that the set V, a the set of positive real numbers (i.e. $x > 0$) with addition and scalar multiplication defined as follows, $x + y = xy, c \cdot x = x^c$. Prove that $V(.,+)$ is not a vector space.

Exercise 3.6.6 Let the set V be the points on a plane through the origin in $\mathbb{R}^3$ with the standard addition and scalar multiplication. Prove that $V(.,+)$ is not a vector space.

Exercise 3.6.7 Determine if the given set is a subspace of the given vector space,

(a) Let W be the set of all points, (x, y), from $\mathbb{R}^2$ in which $x > 0$. Is this a subspace of $\mathbb{R}^2$?

(b) Let W be the set of all points from $\mathbb{R}^3$ of the form $(x_1, 0, x_3)$. Is this a subspace of R^3?

(c) Let W be the set of all points from $\mathbb{R}^3$ of the form $(1, x_2, x_3)$. Is this a subspace of R^3?

Exercise 3.6.8 Determine if the given set is a subspace of the given vector space,

(a) Let W be the set of matrices of the form, $\begin{bmatrix} 0 & a \\ b & c \\ d & e \end{bmatrix}$. Is W a subspace of $M_{3\times 2}$?

(b) Let W be the set of matrices of the form, $\begin{bmatrix} 2 & a_1 \\ 0 & a_2 \end{bmatrix}$. Is W a subspace of $M_{2\times 2}$?

Exercise 3.6.9 Let $\mathscr{P}_2$ be a set of all polynomials of degree less or equal to 2 whose coefficients are integers. show that $\mathscr{P}_2$ is not a vector subspace of all polynomials of degree ≤ 2.

Exercise 3.6.10 If A is a 3×5 matrix, then the transpose of A is

A. A 3×5 matrix.

B. A number (possibly non-zero).

C. Zero.

D. Undefined.

E. A 5×3 matrix.

F. A subspace of $\mathbb{R}^3$.

G. A subspace of $\mathbb{R}^5$.

E

Exercise 3.6.11 Let V be an vector space, and let W be a subset of V. What does it mean when we say that W is closed under addition?

A. Whenever X and Y are in V, then $X + Y$ is in V.

B. Whenever X and Y are in W, then $X + Y$ is in V.

C. Whenever X and Y are in W, then $X + Y$ is in W.

D. Whenever X and Y are in V, then $X + Y$ is in W.

E. If $X + Y$ is in W, then X and Y are in W.

F. $W(X + Y) = WX + WY$ for every two vectors X and Y.

G. Every vector in W is the sum of two vectors in W.

C

Exercise 3.6.12 Let V be a vector space, and let W be a subset of V. What does it mean when we say that W is closed under scalar multiplication?

A. Whenever X is in V and c is a scalar, then cX is in V.

B. Whenever X is in V and c is a scalar, then cX is in W.

C. Whenever X is in W and c is a scalar, then cX is in W.

D. Whenever X is in W and c is a scalar, then cX is in V. *Wrong*

E. If cX is in W and X is in W, then c is a scalar.

F. If cX is in W and c is a scalar, then X is in W. *Partially Correct*

G. $W(cX) = cWX$ for every vector X and scalar c.

C

Exercise 3.6.13 If A is a 3×5 matrix, then the row-reduced echelon form of A is

A. A 3×5 matrix.

B. A number (possibly non-zero).

C. Zero.

D. Undefined.

E. A 5×3 matrix.

F. A subspace of $\mathbb{R}^3$.

G. A subspace of $\mathbb{R}^5$.

A, G

Exercise 3.6.14 Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices are $(1, 0, -2), (1, 2, 4)$, and $(7, 1, 0)$.

We need calculate the determinant $\begin{vmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 0 \end{vmatrix}$ *then take the absolute value to get the volume.*
So, the volume of the parallelepiped is $|22| = 22$.

Exercise 3.6.15 If A is a 3×5 matrix, then the inverse of A is

A. A 3×5 matrix.

B. A number (possibly non-zero).

C. Zero.

D. Undefined. *Correct, Only square matrices can be invertible.*

E. A 5×3 matrix. F. A subspace of $\mathbb{R}^3$. G. A subspace of $\mathbb{R}^5$.

D

Exercise 3.6.16 Find the volume of the parallelepiped S formed by the triple of vectors in $\mathbb{R}^3$

$$x = (1, 1, 1), \; y = (2, 3, 4), \; z = (1, 1, 5).$$

$|\det([x \; y \; z])| = 4 \; units^3$

Exercise 3.6.17 Which of the following subsets of $\mathbb{R}^4$ are closed under (the standard operation of) multiplication by scalars?

(a) $\{(a, b, c, d) \mid abc = 0\}$ *closed*

(b) $\{(a, b, c, d) \mid a = 1, b = 0 \;\text{ and }\; a + d = 0\}$ *not closed*

(c) $\{(a, b, c, d) \mid a > 1 \;\text{ and }\; b < 1\}$ *not closed*

(d) $\{(a, b, c, d) \mid a > 0 \;\text{ and }\; b < 0\}$ *not closed*

(e) $\{(a, b, c, d) \mid a - b + 2c = 0\}$ *closed*

Exercise 3.6.18 Which of the following are subspaces of $\mathbb{F}(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$?

(a) $U = \{f \in \mathbb{F}(\mathbb{R}) \mid f(-1)f(1) = 0\}$ *not closed under addition*

(b) $V = \{f \in \mathbb{F}(\mathbb{R}) \mid f(1) + f(2) = 0\}$ *yes, subspace*

(c) $S = \{f \in \mathbb{F}(\mathbb{R}) \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$ *yes, subspace*

(d) $T = \{f \in \mathbb{F}(\mathbb{R}) \mid f(1) > 0\}$ *not closed under scalar multiplication*

Exercise 3.6.19 If we give $X = \mathbb{R}^2$ the non-standard operations

$$(x, y) \oplus (x', y') = (x + x' - 1, y + y' + 2) \quad \text{(vector addition)}$$

and

$$k \odot (x, y) = (kx - k + 1, ky + 2k - 2) \quad \text{(multiplication by scalars)}.$$

(a) What is the zero vector for X? $\vec{0} = (1, -2)$

(b) If $\vec{v} = (x, y)$ is in X then what is $-\vec{v}$? $-\vec{v} = (-x + 2, -y - 4)$

(c) Check that X is a vector space.

Exercise 3.6.20 Let

$$W = (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0 \text{ and } z \geq 0.$$

Then,

A. W is a subspace of $\mathbb{R}^3$.

B. $(0, 0, 0) \notin W$ and W is not closed under multiplication by scalars.

C. W is closed under addition but W is not closed under multiplication by scalars.

D. W is closed under addition and W is closed under multiplication by scalars.

E. W is not closed under addition but W is closed under multiplication by scalars.

F. None of the other statements is true.

Its (C). W is closed under addition, but for any $(1, 0, 0) \in W$*, but* $-1(1, 0, 0) = (-1, 0, 0) \notin W$

Exercise 3.6.21 Which two of the following are subspaces of $\mathbb{F}(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$?

$$\begin{aligned} S &= \{f \in \mathbb{F}(\mathbb{R}) \mid f(1)f(2) = 0\} \\ T &= \{f \in \mathbb{F}(\mathbb{R}) \mid f(-x) = 2f(x), \forall x \in \mathbb{R}\} \\ U &= \{f \in \mathbb{F}(\mathbb{R}) \mid f(1) > 1\} \\ V &= \{f \in \mathbb{F}(\mathbb{R}) \mid f(6) = 0\} \end{aligned}$$

A. T and U.

B. T and V .

C. S and T.

D. S and V .

E. S and U.

F. U and V .

B

S is not a subspace since for

$$f(x) = x - 1, g(x) = x - 2 \ \in \ S$$

but $f + g \notin S$ since $(f + g)(x) = 2x - 3 \neq 0$ for $x = 1, 2$.
U is not a subspace since $\vec{0} \notin U$. So only T and V are subspaces.

Exercise 3.6.22 Which two of the following are subsets of $\mathbb{F}(\mathbb{R})$ are subspaces of $\mathbb{F}(\mathbb{R})$?

$$\begin{aligned} S &= \{f \in \mathbb{F}(\mathbb{R}) \mid f(-x) + f(x) = 0\} \\ T &= \{f \in \mathbb{F}(\mathbb{R}) \mid f(-1)f(0) = 0\} \\ U &= \{f \in \mathbb{F}(\mathbb{R}) \mid f(0) \geq 0\} \\ V &= \{f \in \mathbb{F}(\mathbb{R}) \mid f(0) + f(1) = 0\} \end{aligned}$$

A. V and S.

B. U and S .

C. S and T.

D. V and T .

E. U and T.

F. U and V .

A

Exercise 3.6.23 Which of the following are subspaces of $\mathbb{R}^3$?

I. $\{(x, y, z) \mid x - 2y = 0\}$

II. $\{(x, y, z) \mid xyz = 0\}$

III. $\{(x, y, z) \mid y = 2z\}$

IV. $\{(x, y, z) \mid x = y + 3 = z\}$

A. (I) and (II)

B. (I) and (III)

C. (II) and (IV)

D. (II) and (III)

E. (I), (III) and (IV)

F. (III) and (IV)

(II) is not closed under addition.
(IV) does not contain a zero
(I) and (III) are subspaces (in fact planes through the origin). B

Exercise 3.6.24 Which of the following are subspaces of $M_{2\times 2}$?

$$\begin{aligned}
U &= \left\{ \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in M_{2\times 2} \mid x, y, z \in R \right\} \\
V &= \left\{ \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \in M_{2\times 2} \mid y \in R \right\} \\
W &= \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_{2\times 2} \mid xw - zy = 0 \right\} \\
S &= \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \in M_{2\times 2} \mid x, y, z \in R \right\}
\end{aligned}$$

A. Only U and V

B. Only U and W

C. Only U, V and W

D. Only V, W and S

E. Only W and S

F. Only U, V and S

U is closed under addition and scalar multiplication *True* ✓

V is closed under addition and scalar multiplication *True* ✓

W is not a subspace because $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$ *and* $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$ *but*

$$M_1 + M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W$$

That is W is not closed under addition. *False* X

S is closed under addition and scalar multiplication *True* ✓

F

Exercise 3.6.25 Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0 \text{ and } z \geq 0\}$. Which one of the following statements is true?

A. $(0, 0, 0) \in X$ and X is closed under multiplication by scalars.

B. $(0, 0, 0) \notin X$ but X is closed under addition. *False*: $(0, 0, 0) \in X$

C. X is closed under addition but X is not closed under multiplication by scalars.

D. X is closed under addition and X is closed under multiplication by scalars.

E. X is not closed under addition but X is closed under multiplication by scalars.

F. None of the other statements is true.

C

Exercise 3.6.26 Let $X = \{(a, b, c) \in \mathbb{R}^3 \mid bc = 0\}$. Then,

A. X is closed under addition and X is closed under multiplication by scalars

B. X is closed under addition but X is not closed under multiplication by scalars

C. X is not closed under addition but X is closed under multiplication by scalars

D. $(0, 0, 0) \notin X$ but X is closed under addition

E. $(0, 0, 0) \in X$ but X is not closed under multiplication by scalars

F. None of the other statements is true.

C

Note that

- *Let* $u = (0, 1, 0)$ *and* $v = (0, 0, 1)$. *Then* $u, v \in X$. *But* $u + v = (0, 1, 1) \notin X$. *Hence* X *is not closed under addition.*
- *If* $u = (a, b, c) \in X \quad \Rightarrow \quad bc = 0$ *and* $k \in \mathbb{R}$, *then* $ku = (ka, kb, kc)$. *Checking whether in* X. $(kb)(kc) = k^2 bc = k^2(0) = 0 \quad \Rightarrow \quad ku \in X$. *Hence* X *is closed under scalar multiplication.*
- *Note that* $(0, 0, 0) \in X$. *So part D is false.*

Exercise 3.6.27 *True* or *False*? If you say the statement may be false, you must give an explicit example - with numbers!. If you say the statement is always true, you must give a clear explanation.

(a) If v and w are vectors in $\mathbb{R}^2$ and X is a subspace of $\mathbb{R}^3$ with $v + 2w \in X$, then both v and w belong to X.

False: Neither v nor w belong to X

(b) $\left\{ \begin{bmatrix} a & b \\ a & c \end{bmatrix} \in M_{22} \mid a, b, c \in \mathbb{R}\} \right\}$ is a subspace of M_{22}

True: Some of the matrices could be $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \cdots$

Exercise 3.6.28 Which of the following are subspaces of $\mathbb{R}^3$?

(1) $\{(x, x+y, x+2y) \in \mathbb{R}^3 \mid x, y \in R\}$

(2) $\{(x, y, z) \in \mathbb{R}^3 \mid x-2 = y-3 = z\}$

(3) $\{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$

(4) $\{(x, y, z) \in \mathbb{R}^3 \mid x-y-z = 0\}$

A. (1) and (2)

B. (1), (3) and (4)

C. (3) and (4)

D. (1) and (3)

E. (1) and (4)

F. (2) and (3)

E

Note that

(1), True: Closed under addition and scalar multiplication. Also a zero exists.

(2), False: Does not contain a zero, so is not a subspace of $\mathbb{R}^3$

(3), False: Not closed under addition. Contains the vectors $(1,0,0)$ *and* $(0,1,1)$*. Their sum equal to* $(1,1,1)$ *not in the set. Hence not a subspace of* $\mathbb{R}^3$

(4), True: Closed under addition and scalar multiplication. Also a zero exists.

Exercise 3.6.29 Which of the following are subspaces of $M_{22}(\mathbb{R})$?

A. $\left\{ \begin{bmatrix} a & 1 \\ b & b \end{bmatrix} \in M_{22} \mid a, b \in R \right\}$

B. $\left\{ \begin{bmatrix} a & b \\ 2a & c \end{bmatrix} \in M_{22} \mid a, b, c \in R \right\}$

C. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \mid ab = 1;\ a, d \in R \right\}$

D. $\left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in M_{22} \mid a, b, c \text{ integers.} \right\}$

E. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \mid cd = 0;\ a, b \in R \right\}$

F. None of the above.

B

Hints:

A. False ✗: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin A$

B. True ✓: *Closed under addition and scalar multiplication. Zero exists.*

C. False ✗: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin C$

D. False ✗: *Contains* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *but* $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin D$

E. False ✗: *Contains* $E_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ *and* $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *but their their sum* $E_1 + E_2 \notin E$*. Not closed under addition.*

F. False ✗: *Since negates B*

Exercise 3.6.30 Let M_{22} denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} 0 & a \\ b & 2a \end{bmatrix} \in M_{22} \mid a, b \in \mathbb{R} \right\}.$$

(a) Either check that U is closed under addition.

(b) Give a matrix $A \in M_{22}$ such that $A \notin U$. You may assume that U is a subspace of M_{22}.

(Remember that you must justify your answers.)

In general terms,

$$a \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 2a \end{bmatrix}$$

Think of more two ways. Also $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin U$

Exercise 3.6.31 Recall the vector space $\mathscr{P}_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ of polynomial functions of degree at most 2, and define

$$W = \{p \in \mathscr{P}_2 \mid p(1) = 0\}.$$

Explain why W is a subspace of $\mathscr{P}_2$

Exercise 3.6.32 Which of the following are subspaces of M_{22}?

$$\begin{aligned} U &= \left\{ \begin{bmatrix} x & x \\ y & x-y \end{bmatrix} \in M_{22} \mid x, y \in \mathbb{R} \right\}, \\ V &= \left\{ \begin{bmatrix} x & x+y \\ y & 2y \end{bmatrix} \in M_{22} \mid x, y \in \mathbb{R} \right\}, \\ W &= \left\{ \begin{bmatrix} x & x \\ y & xy \end{bmatrix} \in M_{22} \mid x, y \in \mathbb{R} \right\}. \end{aligned}$$

A. U only

C. W only

E. U and W only

B. V only

D. U and V only

F. V and W only

D, W *not closed under* $+$

Exercise 3.6.33 Let $U = \{(x, y, z, w) \in \mathbb{R}^4 \mid xyw = 0\}$. Then,

A. $(0, 0, 0, 0) \in U$ but U is not closed under multiplication by scalars

B. U is closed under addition and U is closed under multiplication by scalars

C. U is closed under addition but U is not closed under multiplication by scalars

D. U is not closed under addition but U is closed under multiplication by scalars

E. $(0, 0, 0, 0) \notin U$ but U is closed under addition

F. None of the other statements is true.

D

Exercise 3.6.34 For any subsets U, W of vector space V, define

$$U + W = \{\vec{u} + \vec{w} \mid \vec{u} \in U, \vec{w} \in W\}.$$

(a) Prove: if U, W are subspaces of V, then $U + W$ is also a subspace.

(b) Let U be the vector subspace of $\mathbb{R}^4$ with basis $(1, 1, 1, 2)$ and $(1, -1, 1, -1)$. Let W be the set of vectors (x, y, z, w) that satisfy the system

$$\begin{aligned} x + y - z - w &= 0 \\ x - y - z + w &= 0. \end{aligned}$$

Show that $U + W$ is a vector space.

Exercise 3.6.35 For each of the following sets, determine if it is a vector (sub)space:

(a) The set of vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with the property $2x_1 - x_2 = 0,\ 3x_3 - x_4 = 0$.

(b) The set of vectors $(x_1, x_2, x_3) \in \mathbb{R}^3$ with the property $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$.

(c) The set of vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with the property

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

(d) The set of all vectors of the form

$$(a + b - 1, \quad 2a + 3c - 1, \quad b - c, \quad a + b + c + 2) \in \mathbb{R}^4$$

where a, b, and c are arbitrary real numbers.

Exercise 3.6.36 Let $V \subseteq \mathbb{R}^2$ be the set of vectors in the first quadrant:

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq 0, y \geq 0 \right\}.$$

(a) If u and v are in V, is the sum $u + v$ necessarily in V? *Yes*

(b) To show that V is not a subspace of $\mathbb{R}^2$, find a vector w and a scalar $c \in \mathbb{R}$ such that cw is not in V.

One may take the vector $w = (1, 1)$ in V and the scalar $c = -1$, so that $cw = (-1, -1)$ is not in V.

Exercise 3.6.37 Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 5y - 4z = 0\}$. Explain very briefly why W is a subspace of R^3.

Exercise 3.6.38 Show that the solution space of $A\boldsymbol{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$ where A is an n-square matrix and that the solution set B of $A\boldsymbol{x} = \boldsymbol{b}$ is not a subspace of $\mathbb{R}^n$ if $\boldsymbol{b} \neq \mathbf{0}$. Show that either $B = \emptyset$ or B has infinitely many elements.

Exercise 3.6.39 If V is a vector space of all $n \times n$ matrices and W a subspace of V consisting of diagonal square matrices, show that W is a subspace of V.

3.7 Vector Spaces Chapter Examples

Chapter 4

Linear Dependance and Independence

4.1 Linear Combination

Definition 4.1.1 Linear combination
We define a vector $\boldsymbol{x} \in V$ to be a linear combination of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots, \boldsymbol{x}_n \in V$ iff X can be expressed in the form,

$$\boldsymbol{x} = \alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2 + \alpha_3\boldsymbol{x}_3 + \ldots + \alpha_n\boldsymbol{x}_n \tag{4.1}$$

where α_i are scalars.

Example 4.1.1 Write the vector $(2,1,5,-5) \in \mathbb{R}^4$ as a linear combination of the vectors $(1,2,1,-1),(1,0,2,-3),(1,1,0,-2) \in \mathbb{R}^4$

We solve for the constants α_1, α_2, and α_3 such that

$$(2,1,5,-5) = \alpha_1(1,2,1,-1) + \alpha_2(1,0,2,-3) + \alpha_3(1,1,0,-2)$$

Which implies

$$\begin{array}{rcrcrcr} \alpha_1 & + & \alpha_2 & + & \alpha_3 & = & 2 \\ 2\alpha_1 & & & + & \alpha_3 & = & 1 \\ \alpha_1 & + & 2\alpha_2 & & & = & 5 \\ -\alpha_1 & - & 3\alpha_2 & - & 2\alpha_3 & = & -5 \end{array} \sim \left[\begin{array}{ccccr} 1 & 1 & 1 & | & 2 \\ 0 & 2 & 1 & | & 3 \\ 0 & 0 & 3 & | & -3 \\ 0 & 0 & 0 & | & 0 \end{array}\right] \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Thus the vector $(2,1,5,-5)$ can be expressed as a linear combination of the vectors

$$(2,1,5,-5) = 1(1,2,1,-1) + 2(1,0,2,-3) + -1(1,1,0,-2)$$

So it can be expressed in terms of those vectors, thus its a linear combination of the given vectors.

Example 4.1.2 Write a polynomial $\mathscr{P} = t^2 + t + 2$ over $\Re$ as a linear combination of the polynomials $\mathscr{P}_1 = 2t^2 + t,\ \mathscr{P}_2 = 3t^2 + 2t + 2$.

We solve for the constants α_1, α_2 such that $\mathscr{P} = \alpha_1\mathscr{P}_1 + \alpha_2\mathscr{P}_2$.

$$t^2 + t + 2 = \alpha_1(2t^2 + t) + \alpha_2(3t^2 + 2t + 2)$$

Equating coefficients generates

$$\begin{array}{rcrcr} 2\alpha_1 & + & 3\alpha_2 & = & 1 \\ \alpha_1 & + & 2\alpha_2 & = & 1 \\ & & 2\alpha_2 & = & 2 \end{array} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Indeed $t^2+t+2 = -1(2t^2+t)+1(3t^2+2t+2)$. So it can be expressed in terms of those vectors.

Example 4.1.3 Write the vector $\boldsymbol{X} = (2, -5, 3) \in \mathbb{R}^3$ as a linear combination of the vectors

$$\boldsymbol{v}_1 = (1, -3, 2),\ \boldsymbol{v}_2 = (2, -4, -1),\ \boldsymbol{v}_3 = (1, -5, 7) \in \mathbb{R}^3.$$

We solve for the constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$(2, -5, 3) = \alpha_1(1, -3, 2) + \alpha_2(2, -4, -1) + \alpha_3(1, -5, 7)$$

Generates

$$\begin{array}{rcrcrcr} \alpha_1 & + & 2\alpha_2 & + & \alpha_3 & = & 2 \\ -3\alpha_1 & - & 4\alpha_2 & - & 5\alpha_3 & = & -5 \\ 2\alpha_1 & - & \alpha_2 & + & 7\alpha_3 & = & 3 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

An inconsistent system, so no scalars $\alpha_1, \alpha_2, \alpha_3$ satisfy the relation. So $\boldsymbol{X}$ can not be expressed as a linear combination of the vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Exercise 4.1.1 Write the polynomial $t^2 + 1$ over $\Re$ as a linear combination of polynomials

$$t^2 + 2t + 1, t^2 - 1, t + 2$$

Exercise 4.1.2 Check whether the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ can be written as a linear combination of matrices

$$B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},\ B_3 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ B_4 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

[0, -2, 1, -2]

Exercise 4.1.3 Write the vector $\vec{v} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$. [2,4,1]

Exercise 4.1.4 Determine if the vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Not

4.2 Spanning Sets

Definition 4.2.1 Spanning sets
Let V be a vector space and S be a set with vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots, \boldsymbol{x}_n \in V$. Then S spans V or V is spanned by set S if *every element* of V can be written as a linear combination of elements in S.

In other words if every vector e belonging to space V is a linear combination of all vectors in the set S.
If $S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots, \boldsymbol{x}_n\}$, $V = \{e_1, e_2, e_3, \ldots, e_n\}$, $S \subset V$ and

$$e_i = \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \alpha_3 \boldsymbol{x}_3 + \ldots + \alpha_n \boldsymbol{x}_n$$

then set S spans V.

Example 4.2.1 Show that S spans V, where $S = \{(1,2,1), (1,0,2), (1,1,0)\}$ and V is $\mathbb{R}^3$.

Let $(a, b, c) \in V = \mathbb{R}^3$. To show that we can find constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$(a, b, c) = \alpha_1(1,2,1) + \alpha_2(1,0,2) + \alpha_3(1,1,0)$$

Generating

$$\begin{array}{rcrcrcl} \alpha_1 & + & \alpha_2 & + & \alpha_3 & = & a \\ 2\alpha_1 & & & + & \alpha_3 & = & b \\ \alpha_1 & + & 2\alpha_2 & & & = & c \end{array} \sim \left[\begin{array}{ccccc} 1 & 1 & 1 & | & a \\ 0 & 2 & 1 & | & 2a - b \\ 0 & 0 & 3 & | & 4a - b - 2c \end{array}\right]$$

$\Rightarrow \quad \alpha_1, \alpha_2, \alpha_3$ are defined, the system is consistent, thus S spans V.

Example 4.2.2 Let $V = \mathscr{P}_2$ is a space of all polynomials of degree less or equal to 2. Let

$$S = \{t^2 + 2t + 1,\ t^2 + 2\}$$

Does S span $\mathscr{P}_2$?

Let $at^2 + bt + c \in \mathscr{P}_2$ [You could have also chosen $2t^2 + t + 3, 2t + 2, t, t^2 - 2t + c$ any provided its a polynomial of degree less or equal to 2, but we usually consider the general case since for spanning, if *every element* can be expressed as a linear combination]. Such that

$$at^2 + bt + c = \alpha(t^2 + 2t + 1) + \beta(t^2 + 2)$$

Equating coefficients

$$\begin{array}{rcrcl} \alpha & + & \beta & = & a \\ 2\alpha & & & = & b \\ \alpha & + & 2\beta & = & c \end{array}$$

With solutions

$$\alpha = \frac{b}{2} \tag{4.2}$$

$$\beta = a - \frac{b}{2} \tag{4.3}$$

$$\alpha = c - 2\beta = c - 2\left(a - \frac{b}{2}\right) = c - 2a - b \tag{4.4}$$

The system is inconsistent (there are two values of α), $\Rightarrow$ the coefficients do not exist, thus S does not span $\mathscr{P}_2$.

4.3 Linear Dependence and Independence

Definition 4.3.1 Linear Dependence
Let $S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots, \boldsymbol{x}_k\} \; \forall \; x_i \;\in\; V$, $i = 1, 2, \cdots, k$
S is said to be linearly dependant (l.d) iff $\exists$ scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ **not all zero** such that $\alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2 + \alpha_3\boldsymbol{x}_3 + \ldots + \alpha_k\boldsymbol{x}_k = \mathbf{0}$

Definition 4.3.2 Linear independence
Let $S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots, \boldsymbol{x}_k\} \; \forall \; x_i \;\in\; V$
S is said to be linearly independent (l.i) iff

$$\alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2 + \alpha_3\boldsymbol{x}_3 + \ldots + \alpha_k\boldsymbol{x}_k \;=\; \mathbf{0} \tag{4.5}$$

Then

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_k \;=\; 0 \tag{4.6}$$

Example 4.3.1 Show whether the set $\{(1, 0, 1, 2), (0, 1, 1, 2), (1, 1, 1, 3)\}$ is linearly independent.

$$\alpha_1(1, 0, 1, 2) + \alpha_2(0, 1, 1, 2) + \alpha_3(1, 1, 1, 3) = (0, 0, 0, 0)$$

To have

$$\begin{array}{rcrcrcl} \alpha_1 & & & + & \alpha_3 & = & 0 \\ & & \alpha_2 & + & \alpha_3 & = & 0 \\ \alpha_1 & + & \alpha_2 & + & \alpha_3 & = & 0 \\ 2\alpha_1 & + & 2\alpha_2 & + & 3\alpha_3 & = & 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since all $\alpha_i's$ are zero, the vectors in the set are linearly independent.

Example 4.3.2 Show whether the set $\{(1, 2, -1), (1, -2, 1), (-3, 2, -1), (2, 0, 0)\}$ is linearly dependent.

$$\alpha_1(1, 2, -1) + \alpha_2(1, -2, 1) + \alpha_3(-3, 2, -1) + \alpha_4(2, 0, 0) = \mathbf{0}$$

In terms of a system of equations,

$$\begin{array}{rcrcrcrcl} \alpha_1 & + & \alpha_2 & - & 3\alpha_3 & + & 2\alpha_4 & = & 0 \\ 2\alpha_1 & - & 2\alpha_2 & + & 2\alpha_3 & & & = & 0 \\ -\alpha_1 & + & \alpha_2 & - & \alpha_3 & & & = & 0 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 4 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

To have infinitely many solutions, not only zeros, thus linearly dependent.

Exercise 4.3.1 Show whether the set $\{t^2 + t + 2, 2t^2 + t, 3t^2 + 2t + 2\}$ is linearly dependent.

Exercise 4.3.2 Which of the following sets of $\mathbb{R}^3$ are linearly dependent ?

(a) $\{(1, 2, -1), (3, 2, 5)\}$

(b) $\{(4, 2, 1), (2, 6, -5), (1, -2, 3)\}$

(c) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(d) $(1, 1, 0), (0, 2, 3), (1, 2, 3), (3, 6, 6)$

Exercise 4.3.3 Prove that $\{2, 4\sin^2 x, \cos^2 x\}$ is linearly dependent.

Theorem 4.3.1 Let $S_1 \subseteq V,\ S_2 \subseteq V\ \ (S_1\ S_2)$ infinite, and $S_1\ \subseteq\ S_2$ then

(a) If S_1 is l.d, so is S_2.

(b) If S_2 is l.i, so is S_1.

Note : (Geometrical interpretation in $\mathbb{R}^2 \& \mathbb{R}^3$)
If $S = \{x_1, x_2, x_3, \ldots, x_k\}\ \forall\ x_i\ \in\ \mathbb{R}^2$ the vectors lie on the same line.
If $S = \{x_1, x_2, x_3, \ldots, x_k\}\ \forall\ x_i\ \in\ \mathbb{R}^3$ the vectors lie on the same plane.

Theorem 4.3.2 Let $S = \{x_1, x_2, x_3, \ldots, x_n\}\ \forall\ x_k \in S$.
S is said to be linearly dependent (l.d) iff one of the vectors belonging to S is a linear combination of other vectors in S.

> **Proof :** ($\Rightarrow$) *If S is linearly dependent then there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zeros such that*
>
> $$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \tag{4.7}$$
>
> *Assume $\alpha_1 \neq 0$ then we can write equation (4.7) in the form,*
>
> $$x_1 = \beta_1 x_2 + \beta_2 x_3 + \ldots + \beta_{n-1} x_n \tag{4.8}$$
>
> *where $\beta_i = -\frac{\alpha_{i+1}}{\alpha_1}$. Equation (4.8) shows that x_1 is a linear combination of other vectors $\{x_2, x_3, x_4, \ldots, x_n$ in S.*
>
> ($\Leftarrow$) *If one of the vectors say x_1 is a linear combination of the other vectors thus, $x_1 = \beta_1 x_2 + \beta_2 x_3 + \ldots + \beta_{n-1} x_n$ then $x_1 - \beta_1 x_2 - \beta_2 x_3 - \ldots - \beta_{n-1} x_n = 0$ in which the coefficient of x_1 is $1 \neq 0$ implying that the vectors are linearly dependent.* ■

Theorem 4.3.3 If a finite set S of vectors contain a **zero** vector, then the vectors in S are linearly dependent (l.d)

> **Proof :** *Let the set of vectors in S be given by $S = \{x_1, 0, x_3, \ldots, x_n\}$ since S is finite and contains a zero vector. For the set to be linearly independent or dependent we consider,*
>
> $$\alpha_1 x_1 + \alpha_2 \cdot 0 + \ldots + \alpha_n x_n = 0 \tag{4.9}$$
>
> *Equation (4.9) is true for $\alpha_2 \neq 0$ thus the vectors in S are linearly dependent.* ■

Theorem 4.3.4 If S contains **two** vectors, then its linearly dependent (l.d) iff one of the vectors is a scalar multiplication of the other (similar to the definition of parallel vectors).

Theorem 4.3.5 If $S = \{x_1, x_2, x_3, \ldots, x_r\}\ \in\ \mathbb{R}^n$ if $r > n$, then the set is linearly dependent (l.d)
[This applies only to $\mathbb{R}^n$ not $\mathbb{Z}^n, \mathbb{C}^n$]

4.4 Linear Dependence and Independence Chapter Examples

Example 4.4.1 Express vector $\mathbf{v} = \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}$ as a linear combination of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$

Solution : *For vector $\boldsymbol{v}$ to be a linear combination of the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we need to find a scalars α_1 and α_2 such that (see definition above)*

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 = \boldsymbol{v}$$

Substitute $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}$ by their components to obtain the equation

$$\alpha_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}$$

Solving the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 3 & 0 & 6 \\ 0 & 4 & 12 \end{array}\right] \xrightarrow[\substack{-3R_1+R_2 \to R_2 \\ R_3 \to R_3}]{R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 12 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 4 & 12 \\ 0 & 0 & 0 \end{array}\right]$$

The solutions are $\alpha_2 = 3$ and $\alpha_1 = 2$. Hence,

$$\begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

■

Example 4.4.2 Show that the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ cannot be expressed as a linear combination of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Solution : *We need to show that we cannot find α_1 and α_2 such that*

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The equivalent system is given by

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= 1 \\ 2\alpha_1 + 4\alpha_2 &= 3 \end{aligned}$$

Solve the above system using any method (I will apply Gauss-Jordan).

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 3 \end{array}\right] \xrightarrow[-2R_1+R_2 \to R_2]{R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The system has no solution; therefore vector $\boldsymbol{v}$ cannot be expressed as a linear combinations of the given vectors u_1 and u_2. ■

Example 4.4.3 Show that the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ span the vector space $\mathbb{R}^2$

Solution : *A vector in the space $\mathbb{R}^2$ is of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ where a and b can take any value in the set of real numbers $\mathbb{R}$.*

To show that the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ span $\mathbb{R}^2$, we need to show that any vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ is a linear combination of the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. We therefore need to show that we can find the scalars α_1 and α_2 such that

$$\alpha_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

for any values of a and b.

Scalar multiply and add the vectors on the right side in the above equation

$$\begin{bmatrix} \alpha_1 - \alpha_2 \\ 5\alpha_1 + 2\alpha_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solve the above for α_1 and α_2 using method. Here we use the method of Gaussian elimination.

$$\left[\begin{array}{rr|c} 1 & -1 & a \\ 5 & 2 & b \end{array}\right] \xrightarrow[-5R_1+R_2 \to R_2]{R_1 \to R_1} \left[\begin{array}{rr|c} 1 & -1 & a \\ 0 & 7 & b-5a \end{array}\right]$$

To have the solution as

$$\alpha_2 = \frac{b-5a}{7}, \quad \alpha_1 = \frac{b+2a}{7}$$

We have proved that any vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ may be expressed as a linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and therefore $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ span $\mathbb{R}^2$. ■

Example 4.4.4 Show that the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ DO NOT span the vector space $\mathbb{R}^2$

Solution : *We need to show that we cannot find scalars α_1 and α_2 for any real numbers a and b such that*

$$\alpha_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Scalar multiply and add the terms on the right side

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -2\alpha_1 - 4\alpha_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Reducing the augmented matrix

$$\left[\begin{array}{rr|c} 1 & 2 & a \\ -2 & -4 & b \end{array}\right] \xrightarrow[2R_1+R_2 \to R_2]{R_1 \to R_1} \left[\begin{array}{rr|c} 1 & 2 & a \\ 0 & 0 & 2a+b \end{array}\right]$$

The last equation shows the there are no solutions if a and b are such that $2a+b \neq 0$. Hence the given vectors do not span $\mathbb{R}^2$. ■

Example 4.4.5 Show that the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ span the vector space $\mathbb{R}^3$

Solution : *Vectors $\boldsymbol{u}_1, \boldsymbol{u}_2$ and $\boldsymbol{u}_3$ span $\mathbb{R}^3$ if we can show that any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ is a linear combination of the vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$. We therefore need to show that we can find the scalars α_1, α_2 and α_3 such that*

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

for any values of a , b and c.

Scalar multiply and add the vectors on the right side in the above equation

$$\begin{bmatrix} \alpha_1 \\ 2\alpha_1 + \alpha_2 + 2\alpha_3 \\ -\alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Use any method to solve the above for α_1, α_2 and α_3.

$$\alpha_1 = a, \ \alpha_2 = \frac{b - 2c - 2a}{3} \quad \text{and} \quad \alpha_3 = \frac{b + c - 2a}{3}$$

Therefore, any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ may be expressed as a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ and therefore these 3 vectors span $\mathbb{R}^3$. ■

Example 4.4.6

1.) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Determine if A can be expressed as a linear combination of the set

$$\{M_1, M_2\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Solution : *We want to see if scalars α_1, α_2 can be found such that $\alpha_1 M_1 + \alpha_2 M_2 = A$.*

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Reducing the augmented matrix for the system given

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3 \\ R_4 \to R_4}]{\substack{R_1 \to R_1 \\ R_2 \to R_2}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solution to this equation is given by

$$\begin{aligned} 1 &= \alpha_1 \\ 2 &= \alpha_2 \end{aligned}$$

and it follows that A is a linear combination of $\{M_1, M_2\}$. ■

2.) Let $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Determine if B can be expressed as a linear combination of the set

$$\{M_1, M_2\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Solution : *Again we write $\alpha_1 M_1 + \alpha_2 M_2 = B$ and see if a solution can be found for α_1, t.*

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

That can be reduced by

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3 \\ R_4 \to R_4}]{\substack{R_1 \to R_1 \\ R_2 \to R_2}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

Clearly no values of α_1 and α_2 can be found such that this equation holds. Therefore B is not a linear combination of $\{M_1, M_2\}$. ■

Example 4.4.7 Let $S = \{x^2 + 1,\ x - 2,\ 2x^2 - x\}$. Show that S is a spanning set for $\mathbb{P}_2$, the set of all polynomials of degree at most 2.

Solution : *Let $p(x) = ax^2 + bx + c$ be an arbitrary polynomial in $\mathbb{P}_2$. To show that S is a spanning set, it suffices to show that $p(x)$ can be written as a linear combination of the elements of S. In other words, can we find r, s, t such that:*

$$\alpha_1(x^2 + 1) + \alpha_2(x - 2) + \alpha_3(2x^2 - x) = ax^2 + bx + c = p(x)$$

If a solution $\alpha_1, \alpha_2, \alpha_3$ can be found, then this shows that for any such polynomial $p(x)$, it can be written as a linear combination of the above polynomials and S is a spanning set.

$$\begin{aligned} \alpha_1(x^2 + 1) + \alpha_2(x - 2) + \alpha_3(2x^2 - x) &= ax^2 + bx + c \\ \alpha_1 x^2 + \alpha_1 + \alpha_2 x - 2\alpha_2 + 2\alpha_3 x^2 - \alpha_3 x &= ax^2 + bx + c \\ (\alpha_1 + 2\alpha_3)x^2 + (\alpha_2 - \alpha_3)x + (\alpha_1 - 2\alpha_2) &= ax^2 + bx + c \end{aligned}$$

For this to be true, the following must hold:

$$\begin{array}{ccccccc} \alpha_1 & & & + & 2\alpha_3 & = & a \\ & & \alpha_2 & - & \alpha_3 & = & b \\ \alpha_1 & - & 2\alpha_2 & & & = & c \end{array}$$

To check that a solution exists, set up the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 \\ -R_1 + R_3 \to R_3}]{R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 0 & -2 & -2 & -a + c \end{array}\right] \xrightarrow[2R_2 + R_3 \to R_3]{\substack{R_1 \to R_1 \\ R_2 \to R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 0 & 0 & -4 & -a + 2b + c \end{array}\right]$$

to have the solution as

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a + 2b + \frac{1}{2}c \\ \frac{1}{4}a - \frac{1}{4}c \\ \frac{1}{4}a - b - \frac{1}{4}c \end{bmatrix}$$

or reduced by RREF, the Reduced Row Enchlon Form

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}a + 2b + \frac{1}{2}c \\ 0 & 1 & 0 & \frac{1}{4}a - \frac{1}{4}c \\ 0 & 0 & 1 & \frac{1}{4}a - b - \frac{1}{4}c \end{array}\right]$$

Clearly a solution exists for any choice of a, b, c. Hence S is a spanning set for $\mathbb{P}_2$. ■

Example 4.4.8 Let $S \subseteq \mathbb{P}_2$ be a set of polynomials given by

$$S = \left\{x^2 + 2x - 1, 2x^2 - x + 3\right\}$$

Determine if S is linearly independent.

Solution : *To determine if this set S is linearly independent, we write*

$$\alpha_1(x^2 + 2x - 1) + \alpha_2(2x^2 - x + 3) = 0x^2 + 0x + 0$$

If it is linearly independent, then $\alpha_1 = \alpha_2 = 0$ will be the only solution. We proceed as follows.

$$\begin{aligned} \alpha_1(x^2 + 2x - 1) + \alpha_2(2x^2 - x + 3) = & \ 0x^2 + 0x + 0 \\ \alpha_1 x^2 + 2\alpha_1 x - \alpha_1 + 2\alpha_2 x^2 - \alpha_2 x + 3\alpha_2 = & \ 0x^2 + 0x + 0 \\ (\alpha_1 + 2\alpha_2)x^2 + (2\alpha_1 - \alpha_2)x - \alpha_1 + 3\alpha_2 = & \ 0x^2 + 0x + 0 \end{aligned}$$

It follows that

$$\begin{aligned} \alpha_1 + 2\alpha_2 = & \ 0 \\ 2\alpha_1 - \alpha_2 = & \ 0 \\ -\alpha_1 + 3\alpha_2 = & \ 0 \end{aligned}$$

The augmented matrix and resulting are given by

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ -1 & 3 & 0 \end{array}\right] \xrightarrow[\substack{-2R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3}]{R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & 5 & 0 \end{array}\right] \xrightarrow[R_2+R_3 \to R_3]{\substack{R_1 \to R_1 \\ R_2 \to R_2}} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence the solution is $\alpha_1 = \alpha_2 = 0$ and the set is linearly independent. ■

Example 4.4.9 Determine if the set S given below is independent.

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$$

Solution : *To determine if S is linearly independent, we look for solutions to*

$$\alpha_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that this equation has nontrivial solutions (show it), for example $\alpha_1 = 2$, $\alpha_2 = 3$ and $\alpha_3 = -1$. Therefore S is linearly dependent. ■

Example 4.4.10 Verify that

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

span $M_{2\times 2}$.

Solution : *An arbitrary vector in* $M_{2\times 2}$ *is of the form*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If we write

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = A$$

such that

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then equating the elements of the matrices on each side of the equation yields the system

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= a, \\ \alpha_2 + \alpha_3 + \alpha_4 &= b, \\ \alpha_3 + \alpha_4 &= c, \\ \alpha_4 &= d. \end{aligned}$$

To form an augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right]$$

which is already in a reduced form. Solving this by back substitution gives

$$\alpha_1 = a - b, \quad \alpha_2 = b - c, \quad \alpha_3 = c - d, \quad \alpha_4 = d.$$

Hence, we have

$$A = (a-b)A_1 + (b-c)A_2 + (c-d)A_3 + dA_4.$$

Consequently every vector in $M_{2\times 2}$ *can be written as a linear combination of* $A_1, A_2, A_3,$ *and* A_4*, and therefore these matrices do indeed span* $M_{2\times 2}$*.* ■

Exercise 4.4.1 The most natural spanning set for $M_{2\times 2}$ is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

a fact that we leave to the reader as an exercise.

Example 4.4.11 Decide whether each subset of $\mathbb{R}^3$ is linearly dependent or linearly independent.

1.) $\left\{ \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 14 \end{pmatrix} \right\}$

Solution : *Considering*

$$\alpha_1 \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + \alpha_3 \begin{pmatrix} 4 \\ -4 \\ 14 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives rise to this linear system.

$$\begin{array}{rcrcrcl} \alpha_1 & + & 2\alpha_2 & + & 4\alpha_3 & = & 0 \\ -3\alpha_1 & + & 2\alpha_2 & - & 4\alpha_3 & = & 0 \\ 5\alpha_1 & + & 4\alpha_2 & + & 14\alpha_3 & = & 0 \end{array}$$

Gauss-Jordan method (you could apply any other technique, say Gaussian elimination, Row Enchelon, Reduced row Enchelon form)

$$\left[\begin{array}{rrr|r} 1 & 2 & 4 & 0 \\ -3 & 2 & -4 & 0 \\ 5 & 4 & 14 & 0 \end{array}\right] \xrightarrow[\substack{3R_1+R_2 \to R_2 \\ -5R_1+R_3 \to R_3}]{R_1 \to R_1} \left[\begin{array}{rrr|r} 1 & 2 & 4 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \xrightarrow[(3/4)R_2+R_3 \to R_3]{\substack{R_1 \to R_1 \\ R_2 \to R_2}} \left[\begin{array}{rrr|r} 1 & 2 & 4 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yields a free variable. Let $\alpha_3 = t$, solving by backward substitution

$$8\alpha_2 + 8\alpha_3 = 0 \quad \Rightarrow \quad 8\alpha_2 + 8t = 0 \quad \Rightarrow \quad \alpha_2 = -t$$

such that

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = 0 \quad \Rightarrow \quad \alpha_1 - 2t + 4t = 0 \quad \Rightarrow \quad \alpha_1 = 2t$$

Therefore,

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} t$$

so there are infinitely many solutions.

For an example of a particular dependence we can set α_3 to be, say, 1. Then we get $\alpha_2 = -1$ and $\alpha_1 = -2$.

Therefore, linearly dependent. ■

2.) $\left\{ \begin{pmatrix} 1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix} \right\}$

Solution : *The linear system that arises here*

$$\begin{array}{rcl} \alpha_1 + 2\alpha_2 + 3\alpha_3 & = & 0 \\ 7\alpha_1 + 7\alpha_2 + 7\alpha_3 & = & 0 \\ 7\alpha_1 + 7\alpha_2 + 7\alpha_3 & = & 0 \end{array}$$

is reduced by Gauss Jordan

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 7 & 7 & 7 & 0 \\ 7 & 7 & 7 & 0 \end{array}\right] \xrightarrow[\substack{-7R_1+R_2\to R_2 \\ -7R_1+R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -14 & 0 \\ 0 & -7 & -14 & 0 \end{array}\right] \xrightarrow[-R_2+R_3\to R_3]{\substack{R_1\to R_1 \\ R_2\to R_2}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

has infinitely many solutions. Therefore, linearly dependent. ■

3.) $\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}$

Solution : *The system*

$$\begin{array}{rcrcl} & & \alpha_2 & = & 0 \\ & & & = & 0 \\ -\alpha_1 & + & 4\alpha_2 & = & 0 \end{array}$$

reducing by Gauss-Jordan,

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & 0 \end{array}\right) \xrightarrow{R_1\leftrightarrow R_3} \left(\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

has only the solution

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is linearly independent.

We could also have gotten the answer by inspection-the second vector is obviously not a multiple of the first, and vice versa. ■

4.) $\left\{ \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \\ -1 \end{pmatrix} \right\}$

Solution : *The system*

$$\begin{array}{rcrcrcrcl} 9\alpha_1 & + & 2\alpha_2 & + & 3\alpha_3 & + & 12\alpha_4 & = & 0 \\ 9\alpha_1 & & & + & 5\alpha_3 & + & 12\alpha_4 & = & 0 \\ & + & \alpha_2 & - & 4\alpha_3 & - & \alpha_4 & = & 0 \end{array}$$

which can be reduced as It is linearly dependent. The linear system

$$\left[\begin{array}{cccc|c} 9 & 2 & 3 & 12 & 0 \\ 9 & 0 & 5 & 12 & 0 \\ 0 & 1 & -4 & -1 & 0 \end{array}\right] \xrightarrow[\substack{-R_1+R_2\to R_2 \\ R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{cccc|c} 9 & 2 & 3 & 12 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1 & -4 & -1 & 0 \end{array}\right] \xrightarrow[(1/2)R_2+R_3\to R_3]{\substack{R_1\to R_1 \\ R_2\to R_2}}$$

$$\left[\begin{array}{cccc|c} 9 & 2 & 3 & 12 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & -3 & -1 & 0 \end{array}\right]$$

Infinetly many solutions. Let $\alpha_4 = t$, then

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \frac{31}{27}t \\ -\frac{1}{3}t \\ -\frac{1}{3}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{31}{27} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} t$$

for example, if $t = 27$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 31 \\ -9 \\ -9 \\ 27 \end{bmatrix}$$

Therefore, it is linearly dependent. ■

Example 4.4.12 Determine whether the following subsets of $\mathscr{P}_3$ is linearly dependent or is independent?

1.) $\{3 - x + 9x^2, 5 - 6x + 3x^2, 1 + 1x - 5x^2\}$

Setting up the relation

$$\alpha_1(3 - x + 9x^2) + \alpha_2(5 - 6x + 3x^2) + \alpha_3(1 + 1x - 5x^2) = 0 + 0x + 0x^2$$

gives a linear system

$$\begin{aligned} 3\alpha_1 + 5\alpha_2 + \alpha_3 &= 0 \\ -\alpha_1 - 6\alpha_2 + \alpha_3 &= 0 \\ 9\alpha_1 + 3\alpha_2 - 5\alpha_3 &= 0 \end{aligned}$$

Gauss-Jordan reduced as

$$\left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ -1 & -6 & 1 & 0 \\ 9 & 3 & -5 & 0 \end{array}\right] \xrightarrow[\substack{(1/3)R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ 0 & -\frac{13}{3} & \frac{4}{3} & 0 \\ 0 & -12 & -8 & 0 \end{array}\right] \xrightarrow[-(36/13)R_2+R_3\to R_3]{\substack{R_1\to R_1 \\ R_2\to R_2}} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ 0 & -\frac{13}{3} & \frac{4}{3} & 0 \\ 0 & 0 & -\frac{152}{13} & 0 \end{array}\right]$$

with only one solution:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, it's linearly independent.

2.) $\{-x^2, 1+4x^2\}$

Solution : *This set is independent. We can see this by inspection, straight from the definition of linear independence. Obviously neither is a multiple of the other.* ■

3.) $\{2+x+7x^2, 3-x+2x^2, 4-3x^2\}$

Solution : *This set is linearly independent. The linear system reduces in this way*

$$\left[\begin{array}{rrr|r} 2 & 3 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 7 & 2 & -3 & 0 \end{array}\right] \xrightarrow[-(7/2)R_1+R_3\to R_3]{-(1/2)R_1+R_2\to R_2} \xrightarrow{-(17/5)R_2+R_3\to R_3} \left[\begin{array}{rrr|r} 2 & 3 & 4 & 0 \\ 0 & -5/2 & -2 & 0 \\ 0 & 0 & -51/5 & 0 \end{array}\right]$$

to show that there is only the solution $\alpha_1 = 0,\ \alpha_2 = 0,$ *and* $\alpha_3 = 0.$ ■

4.) $\{8+3x+3x^2, x+2x^2, 2+2x+2x^2, 8-2x+5x^2\}$

Solution : *The linear system*

$$\left[\begin{array}{rrrr|r} 8 & 0 & 2 & 8 & 0 \\ 3 & 1 & 2 & -2 & 0 \\ 3 & 2 & 2 & 5 & 0 \end{array}\right] \xrightarrow[\substack{-(3/8)R_1+R_2\to R_2 \\ -(3/8)R_1+R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{rrrr|r} 8 & 0 & 2 & 8 & 0 \\ 0 & 1 & \frac{5}{4} & -5 & 0 \\ 0 & 2 & \frac{5}{4} & 2 & 0 \end{array}\right] \xrightarrow[-2R_2+R_3\to R_3]{\substack{R_1\to R_1 \\ R_2\to R_2}} \left[\begin{array}{rrrr|r} 8 & 0 & 2 & 8 & 0 \\ 0 & 1 & \frac{5}{4} & -5 & 0 \\ 0 & 0 & -\frac{5}{4} & 12 & 0 \end{array}\right]$$

This set is linearly dependent since infinetely many solutions. ■

Example 4.4.13 Determine whether the vector v is a linear combination of vectors in S?

1.)

$$v = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Solution : *For v to be a linear combination of $S = \{v_1, v_2, \ldots, v_n\}$, then we can express v in terms of S,*

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = v.$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

to form an augmented matrix which can be reduced by Gauss-Jordan as

$$\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

such that, the solution is

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which exists. Therefore, v can be expressed as a linear combination of S. Infact,

$$(2)\begin{pmatrix}1\\0\\0\end{pmatrix} + (1)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}$$

■

2.)

$$v = x - x^3, \quad S = \left\{x^2, \ 2x + x^2, \ x + x^3\right\}$$

Solution : *Expressing v as a linear combination of S, we have that*

$$\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n &= v \\ \alpha_1\left(x^2\right) + \alpha_2\left(2x + x^2\right) + \alpha_3\left(x + x^3\right) &= x - x^3 \end{aligned}$$

To form a linear system

$$\begin{array}{ccccccc} & & 2\alpha_2 & + & \alpha_3 & = & 1 \\ \alpha_1 & + & \alpha_2 & & & = & 0 \\ & & & & \alpha_3 & = & -1 \end{array}$$

Which can be solved by Gauss-Jordan reducing the augmented matrix.

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

such that, the solution is

$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

which exists. Therefore, v can be expressed as a linear combination of S. Infact,

$$(-1)\left(x^2\right) + (1)\left(2x + x^2\right) + (-1)\left(x + x^3\right) = x - x^3$$

■

3.)

$$v = \begin{pmatrix}0 & 1\\4 & 2\end{pmatrix}, \quad S = \left\{\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}, \ \begin{pmatrix}2 & 0\\2 & 3\end{pmatrix}\right\}$$

Solution : *Expressing v as a linear combination of S, we have that*

$$\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n &= v \\ \alpha_1\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + \alpha_2\begin{bmatrix}2 & 0\\2 & 3\end{bmatrix} &= \begin{bmatrix}0 & 1\\4 & 2\end{bmatrix} \end{aligned}$$

To form a linear system

$$\begin{array}{ccccc} \alpha_1 & + & 2\alpha_2 & = & 0 \\ & & & = & 1 \\ \alpha_1 & + & 2\alpha_2 & = & 4 \\ \alpha_1 & + & 3\alpha_2 & = & 2 \end{array}$$

Which can be solved by Gauss-Jordan reducing the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-R_1+R_2\to R_2 \\ -R_1+R_3\to R_3 \\ R_4\to R_4}]{R_1\to R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{array}\right]$$

No solution. Therefore impossible to express it as a linear combination. ■

Example 4.4.14 Determine whether the set of vectors in S

1.)

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

spans $\mathbb{R}^3$

Solution : *For $S = \{v_1, v_2, \dots, v_n\}$, to span $\mathbb{R}^3$, if there exists any $v \in \mathbb{R}^3$ such that S,*

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = v.$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

to form an augmented matrix which can be reduced by Gauss-Jordan as

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 1 & c \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & c \\ 0 & 0 & b \end{array}\right]$$

such that, the no solution, the system is inconsistent, and therefore, S does not span $\mathbb{R}^3$. ■

2.)

$$S = \left\{x^2, \quad 2x + x^2, \quad x + x^3\right\}$$

spans $\mathscr{P}_3$

Solution : *For set S to span $\mathscr{P}_3$,*

$$\alpha_1 \left(x^2\right) + \alpha_2 \left(2x + x^2\right) + \alpha_3 \left(x + x^3\right) = a + bx + cx^2 + dx^3$$

To form a linear system

$$\begin{array}{rcrcrcl} & & & & & = & a \\ & & 2\alpha_2 & + & \alpha_3 & = & b \\ \alpha_1 & + & \alpha_2 & & & = & c \\ & & & & \alpha_3 & = & d \end{array}$$

Which can be solved by Gauss-Jordan reducing the augmented matrix.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & a \\ 0 & 2 & 1 & b \\ 1 & 1 & 0 & c \\ 0 & 0 & 1 & d \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & c \\ 0 & 2 & 1 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 1 & d \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 1 & 0 & c \\ 0 & 2 & 1 & b \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & a \end{array}\right]$$

No solution. Therefore, the set S does not span $\mathscr{P}_3$. ■

3.)

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} \right\}$$

spans $M_{2\times 2}$

Solution : *For S to span $M_{2\times 2}$*

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

To form a linear system

$$\begin{array}{ccccc} \alpha_1 & + & 2\alpha_2 & = & a \\ & & & = & b \\ \alpha_1 & + & 2\alpha_2 & = & c \\ \alpha_1 & + & 3\alpha_2 & = & d \end{array}$$

Which can be solved by Gauss-Jordan reducing the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 0 & b \\ 1 & 2 & c \\ 1 & 3 & d \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cc|c} 1 & 2 & a \\ 1 & 3 & d \\ 1 & 2 & c \\ 0 & 0 & b \end{array}\right] \xrightarrow[\substack{-R_1+R_2\to R_2 \\ -R_1+R_3\to R_3 \\ R_4\to R_4}]{R_1\to R_1} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & -a+d \\ 0 & 0 & -a+c \\ 0 & 0 & b \end{array}\right]$$

No solution. Therefore S does not span $M_{2\times 2}$. ■

Example 4.4.15 Which of these sets spans $\mathbb{R}_3$?

1.)

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

Solution : *For any $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ this equation*

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

to have an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 2 & 0 & b \\ 0 & 0 & 3 & c \end{array}\right]$$

has the solution

$$\alpha_1 = a, \alpha_2 = \frac{b}{2}, \text{ and } \alpha_3 = \frac{c}{3}$$

Yes the set spans $\mathbb{R}^3$ since the solutions for $\alpha_1, \alpha_2, \alpha_3$ exists (either a unique solution or infinetly many solutions). ■

2.)

$$\left\{\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

Solution : *The equation*

$$\alpha_1\begin{pmatrix}2\\0\\1\end{pmatrix}+\alpha_2\begin{pmatrix}1\\1\\0\end{pmatrix}+\alpha_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

gives rise to this

$$\left[\begin{array}{ccc|c}2&1&0&a\\0&1&0&b\\1&0&1&c\end{array}\right]\xrightarrow[\substack{R_2\to R_2\\-(1/2)R_1+R_3\to R_3}]{R_1\to R_1}\left[\begin{array}{ccc|c}2&1&0&a\\0&1&0&b\\0&-\frac{1}{2}&1&\frac{a}{2}+c\end{array}\right]\xrightarrow[(1/2)R_2+R_3\to R_3]{\substack{R_1\to R_1\\R_2\to R_2}}\left[\begin{array}{ccc|c}2&1&0&a\\0&1&0&b\\0&0&1&\frac{a}{2}-\frac{b}{2}+c\end{array}\right]$$

so that, given any a, b and c, we can compute that

$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}\frac{a}{2}-\frac{b}{2}\\b\\\frac{a}{2}-\frac{b}{2}+c\end{bmatrix}$$

Therefore, since solutions exists, the set spans $\mathbb{R}^3$. ■

3.)

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}3\\0\\0\end{pmatrix}\right\}$$

Solution : *No, the set does not span $\mathbb{R}^3$* ■

4.)

$$\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}3\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\5\end{pmatrix}\right\}$$

Solution : *Yes. The equation*

$$\alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix}+\alpha_2\begin{pmatrix}3\\1\\0\end{pmatrix}+\alpha_3\begin{pmatrix}-1\\0\\0\end{pmatrix}+\alpha_4\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

leads to this reduction.

$$\left[\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\1&0&0&5&z\end{array}\right]\xrightarrow{-R_1+R_3\to R_3}\xrightarrow{3R_2+R_3\to R_3}\left[\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\0&0&1&6&-x+3y+z\end{array}\right]$$

We have infinitely many solutions. We can, for example, set α_4 to be t and solve for α_3, α_2, and α_1 in terms of a, b, and c by the usual methods of back-substitution. Therefore, the set spans $\mathbb{R}^3$ ■

5.)

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} \right\}$$

Solution : *No. The set does not span $\mathbb{R}^3$. The equation*

$$\alpha_1 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \alpha_4 \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

leads to this reduction.

$$\left[\begin{array}{cccc|c} 2 & 3 & 5 & 6 & a \\ 1 & 0 & 1 & 0 & b \\ 1 & 1 & 2 & 2 & c \end{array}\right] \xrightarrow[-(1/2)R_1+R_3\to R_3]{-(1/2)R_1+R_2\to R_2} \xrightarrow{-(1/3)R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 2 & 3 & 5 & 6 & a \\ 0 & -\frac{3}{2} & -\frac{3}{2} & -3 & -\frac{1}{2}a+b \\ 0 & 0 & 0 & 0 & -\frac{1}{3}a-\frac{1}{3}b+c \end{array}\right]$$

No solution. This shows that not every three-tall vector can be so expressed. Only the vectors satisfying the restriction that

$$-\frac{1}{3}a - \frac{1}{3}b + c = 0$$

are in the span. ■

Example 4.4.16

1.) Are the vectors $v_1 = (4, 1, -2)$, $v_2 = (-3, 0, 1)$, and $v_3(1, -2, 1)$ linearly independent or linearly dependent?

Solution : *Considering*

$$\alpha_1 \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives rise to this linear system.

$$\begin{array}{rcrcrcl} 4\alpha_1 & - & 3\alpha_2 & + & \alpha_3 & = & 0 \\ \alpha_1 & & & - & 2\alpha_3 & = & 0 \\ -2\alpha_1 & + & \alpha_2 & + & \alpha_3 & = & 0 \end{array} \tag{4.10}$$

Gauss-Jordan method (you could apply any other technique, say Gaussian elimination, Row Enchelon, Reduced row Enchelon form)

$$\left[\begin{array}{ccc|c} 4 & -3 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right] \xrightarrow[\substack{-(1/4)R_1+R_2\to R_2 \\ (1/2)R_1+R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{ccc|c} 4 & -3 & 1 & 0 \\ 0 & \frac{3}{4} & -\frac{9}{4} & 0 \\ 0 & -\frac{1}{2} & \frac{3}{2} & 0 \end{array}\right] \xrightarrow[(2/3)R_2+R_3\to R_3]{\substack{R_1\to R_1 \\ R_2\to R_2}} \left[\begin{array}{ccc|c} 4 & -3 & 1 & 0 \\ 0 & \frac{3}{4} & -\frac{9}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yields a free variable. Let $\alpha_3 = t$, solving by backward substitution, we have the solution

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} t$$

so there are infinitely many solutions.

The homogeneous system (4.10) above has nontrivial solutions. This shows that the vectors: v_1, v_2, and v_3 are dependent.

Therefore, the set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent. ■

2.) Are the vectors $v_1 = (2, 5, 3), v_2 = (1, 1, 1)$, and $v_3 = (4, -2, 0)$ linearly independent or linearly dependent?

Solution : *Considering*

$$\alpha_1 \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives rise to this linear system.

$$\begin{array}{rcrcrcl} 2\alpha_1 & + & \alpha_2 & + & 4\alpha_3 & = & 0 \\ 5\alpha_1 & + & \alpha_2 & - & 2\alpha_3 & = & 0 \\ 3\alpha_1 & + & \alpha_2 & & & = & 0 \end{array} \tag{4.11}$$

Gauss-Jordan method (you could apply any other technique, say Gaussian elimination, Row Enchelon, Reduced row Enchelon form)

$$\left[\begin{array}{ccr|c} 2 & 1 & 4 & 0 \\ 5 & 1 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right] \xrightarrow[\substack{-(5/2)R_1+R_2 \to R_2 \\ -(3/2)R_1+R_3 \to R_3}]{R_1 \to R_1} \left[\begin{array}{ccr|c} 2 & 1 & 4 & 0 \\ 0 & -\frac{3}{2} & -12 & 0 \\ 0 & -\frac{1}{2} & -6 & 0 \end{array}\right] \xrightarrow[-(1/3)R_2+R_3 \to R_3]{\substack{R_1 \to R_1 \\ R_2 \to R_2}} \left[\begin{array}{ccr|c} 2 & 1 & 4 & 0 \\ 0 & -\frac{3}{2} & -12 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

such that, solving by backward substitution, we have the solution

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The homogeneous system (4.11) above has only trivial solutions. This shows that the vectors: v_1, v_2, and v_3 are independent.

Therefore, the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent. ■

3.) Determine whether the following set of vectors in $\mathbb{R}^3$ are linear independent or linearly dependent?

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 14 \end{pmatrix} \right\}$$

Solution : *Considering*

$$\alpha_1 \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + \alpha_3 \begin{pmatrix} 4 \\ -4 \\ 14 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives rise to this linear system.

$$\begin{array}{rcrcrcl} \alpha_1 & + & 2\alpha_2 & + & 4\alpha_3 & = & 0 \\ -3\alpha_1 & + & 2\alpha_2 & - & 4\alpha_3 & = & 0 \\ 5\alpha_1 & + & 4\alpha_2 & + & 14\alpha_3 & = & 0 \end{array} \tag{4.12}$$

Gauss-Jordan method (you could apply any other technique, say Gaussian elimination, Row Enchelon, Reduced row Enchelon form)

$$\left[\begin{array}{rrr|r} 1 & 2 & 4 & 0 \\ -3 & 2 & -4 & 0 \\ 5 & 4 & 14 & 0 \end{array}\right] \xrightarrow[\substack{3R_1+R_2\to R_2 \\ -5R_1+R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{rrr|r} 1 & 2 & 4 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \xrightarrow[(3/4)R_2+R_3\to R_3]{\substack{R_1\to R_1 \\ R_2\to R_2}} \left[\begin{array}{rrr|r} 1 & 2 & 4 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yields a free variable. Let $\alpha_3 = t$, solving by backward substitution, we have the solution

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t$$

so there are infinitely many solutions.

The homogeneous system (4.12) above has nontrivial solutions. This shows that the vectors: v_1, v_2, and v_3 are dependent.

Therefore, the set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent. ■

Example 4.4.17 Show that the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -5 \\ -15 \end{bmatrix}$ are linearly dependent.

Solution :

$$\alpha_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -5 \\ -15 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

■

The solution is

$$\alpha_2 = t, \ \alpha_1 = 5\alpha_2 = 5t$$

has many solutions and therefore vectors u_1 and u_2 given above are linearly dependent.

Example 4.4.18 Show that the vectors $\mathbf{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix}-2\\3\\0\end{bmatrix}$ are linearly independent.

Solution : *For linear independency and dependency,*

$$\alpha_1\begin{bmatrix}1\\0\\1\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\2\end{bmatrix} + \alpha_3\begin{bmatrix}-2\\3\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

It is a homogeneous system of equation. Using matrices, it may be written as

$$\begin{bmatrix}1 & 0 & -2\\0 & 1 & 3\\1 & 2 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

The above system of equations has a trivial solution $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

Therefore the vectors are lineraly independent. ∎

Example 4.4.19 Find the values of m for which the vectors $\mathbf{u}_1 = \begin{bmatrix}m\\4\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\-1\\8\end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix}0\\-1\\m\end{bmatrix}$ are linearly dependent.

Solution : *We use the equation of linearity given in the definition*

$$\alpha_1\begin{bmatrix}m\\4\\0\end{bmatrix} + \alpha_2\begin{bmatrix}1\\-1\\8\end{bmatrix} + \alpha_3\begin{bmatrix}0\\-1\\m\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

The above system of homogeneous equations in matrix form as

$$\begin{bmatrix}m & 1 & 0\\4 & -1 & -1\\0 & 8 & m\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

The augmented matrix is reduced by Gauss-Jordan elimination

$$\left[\begin{array}{ccc|c}m & 1 & 0 & 0\\4 & -1 & -1 & 0\\0 & 8 & m & 0\end{array}\right] \xrightarrow[\substack{-(4/m)R_1+R_2\to R_2\\R_3\to R_3}]{R_1\to R_1} \left[\begin{array}{ccc|c}m & 1 & 0 & 0\\0 & -\frac{4}{m}-1 & -1 & 0\\0 & 8 & m & 0\end{array}\right] \xrightarrow[-\frac{8}{-\frac{4}{m}-1}R_2+R_3\to R_3]{\substack{R_1\to R_1\\R_2\to R_2}}$$

Replacing the third row,

$$\begin{aligned}
-\frac{8}{-\frac{4}{m}-1}R_2 + R_3 &\to R_3\\
-\frac{8}{-\frac{4+m}{m}}R_2 + R_3 &\to R_3\\
\frac{8m}{4+m}R_2 + R_3 &\to R_3
\end{aligned}$$

Replacing a_{32},

$$\begin{aligned}\frac{8m}{4+m}R_2 + R_3 &\rightarrow R_3\\ \frac{8m}{4+m}\left(-\frac{4}{m}-1\right)+8 &\rightarrow R_3\\ 0 &\rightarrow R_3\end{aligned}$$

Replacing a_{33},

$$\begin{aligned}\frac{8m}{4+m}R_2 + R_3 &\rightarrow R_3\\ \frac{8m}{4+m}(-1)+m &\rightarrow R_3\\ \frac{-8m}{4+m}+m &\rightarrow R_3\\ \frac{-8m+4m+m^2}{4+m} &\rightarrow R_3\\ \frac{-4m+m^2}{4+m} &\rightarrow R_3\\ \frac{m^2-4m}{4+m} &\rightarrow R_3\end{aligned}$$

such that

$$\left[\begin{array}{ccc|c} m & 1 & 0 & 0\\ 0 & -\frac{4}{m}-1 & -1 & 0\\ 0 & 0 & \frac{m^2-4m}{4+m} & 0\end{array}\right]$$

For the vectors to be linearly dependent, the system of equations must have unique or infinitely many solutions and hence $a_{33} = 0 \quad \Leftrightarrow \quad \dfrac{m^2-4m}{4+m} = 0$

$$\begin{aligned}\frac{m^2-4m}{4+m} &= 0\\ -m^2+4m &= 0\\ m(-m+4) &= 0\end{aligned}$$

Solve the above form m

$m = 0$ *and* $m = 4$ *are the values for which the given vectors are linearly dependent.*
■

Example 4.4.20 Are the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

linearly dependent or independent?

Solution : *Write the equation of linearity given in the definition above*

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Write the above as a system of homogeneous equations

$$\begin{bmatrix} 1 & 5 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which can be Gauss-Jordan reduced as

$$\left[\begin{array}{rrr|r} 1 & 5 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[\substack{-R_1+R_2\to R_2 \\ R_3\to R_3 \\ R_1+R_4\to R_4}]{R_1\to R_1} \left[\begin{array}{rrr|r} 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 5 & -1 & 0 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_4} \left[\begin{array}{rrr|r} 1 & 5 & 0 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The row reduced form of the above system of equations is written as

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which corresponds to the system

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 0 \\ \alpha_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The equation of linearity has only the trivial solution

$$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$$

and therefore the vectors u_1, u_2 and u_3 are linearly independent. ■

Example 4.4.21 Consider the vectors $A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ in $M_{2\times 2}$. Determine whether or not

$$B = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

can be expressed as a linear combination of $\{A_1, A_2\}$.

> **Solution :** *We want to see if scalars α_1, α_2 can be found such that $\alpha_1 A_1 + \alpha_2 A_2 = B$.*
>
> $$\alpha_1 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$
>
> *Reducing the augmented matrix for the system given*
>
> $$\left[\begin{array}{rr|r} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -1 & 4 \end{array}\right] \xrightarrow[\substack{R_1+R_3\to R_3 \\ -3R_1+R_4\to R_4}]{\substack{R_1\to R_1 \\ -2R_1+R_2\to R_2}} \left[\begin{array}{rr|r} 1 & -2 & 3 \\ 0 & 5 & -5 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{array}\right] \xrightarrow[\substack{R_2+5R_3\to R_3 \\ R_2-R_4\to R_4}]{\substack{R_1\to R_1 \\ R_2\to R_2}} \left[\begin{array}{rr|r} 1 & -2 & 3 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$
>
> *The solution to this equation is given by*
>
> $$\begin{aligned} 1 &= \alpha_1 \\ -1 &= \alpha_2 \end{aligned}$$
>
> *and it follows that A is a linear combination of $\{A_1, A_2\}$.* ■

4.5 Linear Dependence and Independence Chapter Exercises

Exercise 4.5.1 Let S be the set of three vectors $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \right\}$. Let $W =< S >$ be the span of S. Is the vector $\begin{pmatrix} -1 \\ 8 \\ -4 \end{pmatrix}$ in W? Give an explanation for your answer.

Exercise 4.5.2 Which of the following belong to the span of S. Where

$$S = \{t^2 + 2t + 1,\ t^2 + 3,\ t - 1\}$$

(a) $-t^2 + t - 4$

(b) $2t^2 + 2t + 2$

(c) $t^2 + t + 1$

(d) $t^2 + t + 2$

Exercise 4.5.3 Determine if the following sets of vectors are linearly independent or linearly dependent

(a) $v_1 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, v_2 = \begin{pmatrix} 4 & 1 \\ 0 & -3 \end{pmatrix}$

(b) $v_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(c) $\mathscr{P}_1(x) = x - 3, \mathscr{P}_2(x) = x^2 + 2x, \mathscr{P}_3(x) = x^2 + 1$

(d) $\mathscr{P}_1(x) = 2x^2 - x + 7, \mathscr{P}_2(x) = x^2 + 4x + 2, \mathscr{P}_3(x) = x^2 - 2x + 4$

Exercise 4.5.4 For each of the following statements state whether it is false or True.

(a) If $S = \{v_1, v_2, v_3, v_4\}$ are vectors in $\mathbb{R}^4$, and $v_3 = 2v_1 + v_2$ then S is linearly dependent.

(b) If $S = \{v_1, v_2, v_3, v_4\}$ are vectors in $\mathbb{R}^4$, and $v_3 = 0$ then S is linearly independent.

(c) If the vectors v_1, v_2 lie on a line in $\mathbb{R}^2$ then they are linearly dependent.

(d) If $S = \{v_1, v_2, v_3, v_4\}$ are vectors in $\mathbb{R}^4$, and v_3 is not a linear combination of $\{v_1, v_2, v_4\}$ then S is linearly independent.

Exercise 4.5.5 Show that the natural basis span $\mathbb{R}^n$ $(n \geq 1)$. Check for $n = 2, 3, \&4$.

Exercise 4.5.6 Let $V = \mathbb{R}^3$, which of the following sets span V.

(a) $\{(1, 0, 2), (1, 1, 0), (1, 1, 1)\}$.

(b) $\{(1, 2, 3), (1, 1, 1)\}$.

(c) $\{(1, 1, 0), (2, 0, 1), (2, 1, 1), (0, 0, 1)\}$.

Exercise 4.5.7 Show whether the following $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \right\}$ spans M_2 the space of all 2×2 matrices with entries in $\mathbb{R}$.

Exercise 4.5.8 Given that $V = \mathscr{P}_3$, a space of polynomials of degree less or equal to 3. Check whether or not the set $S = \{t^3, t^3 + 1, t^3 + 2t^2 + 1\}$ spans V.

Exercise 4.5.9 Let V be a vector space, and let S be a subset of V. What does it mean when we say that S is linearly independent?

A. S is closed under both addition and scalar multiplication.

B. S is a basis.

Every basis is linearly independent, but not vice versa.

C. Every element in V is a linear combination of elements in S.

D. The number of elements of S is less than or equal to the dimension of V. *Partially Correct*

It is true that if S is linearly independent, then the number of elements in S is less than the dimension of V, but it is possible for S to have fewer elements than the dimension of V without S being linearly independent.

E. The only way to write $\mathbf{0}$ as a linear combination of elements of S is the zero combination (where one takes zero multiples of each element of S).

F. All the elements of S are distinct from each other.

G. S has nullity zero.

E

Exercise 4.5.10 Let V be a vector space, and let S be a subset of V. What does it mean when we say that S is linearly dependent?

A. S is closed under both addition and scalar multiplication.

B. Every element of S is a linear combination of other elements of S.

Only one of the elements of S needs to be able to be expressed as a combination of the others in order to establish linear dependence.

C. The number of elements of S is greater than the dimension of V. *Partially Correct*

D. There is a way to write $\mathbf{0}$ as a linear combination of elements of S other than the zero combination.

E. The span of S has smaller dimension than the dimension of V.

F. S depends on a linear transformation.

G. At least two of the elements of S are the same.

D

Exercise 4.5.11 If $T : U \to V$ is any linear transformation from U to V and $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a basis for U, then set $T(B) = \{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)\}$

A. spans V

B. spans U

C. is a basis for V

D. is linearly independent

E. spans the range of T

B

Exercise 4.5.12 Let V be a vector space, and let S be a subset of V. What does it mean when we say that S spans V?

A. S is a basis for V.

B. The elements of S are all distinct from each other.

C. Every vector in V can be expressed as a linear combination of vectors in S.

D. Every vector in V has exactly one representation as a linear combination of vectors in S.

E. S has at least as many elements as the dimension of V. *Partially Correct*

It is necessary for S to have at least as many elements as the dimension of V in order for S to span V, but it is not sufficient.

F. The rank of S is the same as the dimension of V. *Partially Correct*

This is true for finite-dimensional vector spaces, but not for infinite-dimensional ones.

C

Exercise 4.5.13 Let V be a five-dimensional vector space, and let S be a subset of V which spans V. Then S

A. Must have exactly five elements.

B. Must have at most five elements.

C. Must have infinitely many elements.

D. Must be linearly independent.

E. Must be a basis for V.

F. Must be linearly dependent.

G. Must consist of at least five elements.

G

Exercise 4.5.14 Let V be a five-dimensional vector space, and let S be a subset of V which is linearly independent. Then S

A. Must consist of at least five elements.

B. Must have exactly five elements.

C. Must have at most five elements.

D. Must have infinitely many elements.

E. Must span V.

F. Must be a basis for V.

G. Can have any number of elements (except zero).

C

Exercise 4.5.15 Let V be a five-dimensional vector space, and let S be a subset of V which is linearly dependent. Then S

A. Must consist of at least five elements.

B. Must have exactly five elements.

C. Must have at most five elements.

D. Must have infinitely many elements.

E. Must span V.

F. Must be a basis for V.

G. Can have any number of elements (except zero).

G

Exercise 4.5.16 Let V be a five-dimensional vector space, and let S be a subset of V consisting of three vectors. Then S

A. Must be linearly independent, but may or may not span V.

B. Must be linearly dependent, and must span V.

C. May or may not be linearly independent, and may or may not span V.

D. Must be linearly dependent, but may or may not span V.

E. Cannot span V, but can be linearly independent or dependent.

F. Must be linearly independent, but cannot span V.

G. Can span V, but only if it is linearly independent, and vice versa.

E

Example 4.5.1 Complete the definition: Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent if ...

$$\text{no } \boldsymbol{v}_i \text{ is in the span of } \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}.$$

OR

$$\text{they do not satisfy any nontrivial linear relation.}$$

Exercise 4.5.17 Let V be a three-dimensional vector space, and let S be a subset of V consisting of five vectors. Then S

A. Cannot span V, but can be linearly independent or dependent.

B. Must be linearly dependent, and must span V.

C. Must be linearly independent, but may or may not span V.

D. May or may not be linearly independent, and may or may not span V.

E. Must be linearly dependent, but may or may not span V.

F. Must be linearly independent, but cannot span V.

G. Can span V, but only if it is linearly independent, and vice versa.

E

Exercise 4.5.18 Determine if b is a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$ where

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}.$$

If b is a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$, express b as a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$. *Yes,* $b = 1 \cdot \boldsymbol{a}_1 + 1 \cdot \boldsymbol{a}_2 + 2 \cdot \boldsymbol{a}_3$

Exercise 4.5.19 Determine if the following sets of vector are linearly independent. If not, write one vector as a linear combination of other vectors in the set.

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

This set is linearly dependent by Theorem, i.e., zero vector is in the set.

(b) $\left\{ \begin{bmatrix} -5 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 36 \\ 12 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix} \right\}$

This set is linearly dependent by Theorem, i.e., more vectors than the entries in each vectors.

(c) $\left\{ \boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$

Linearly independent, ONLY the trivial solution for system $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 = 0$.

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} \right\}$

This set is linearly dependent by Theorem, i.e., the third vector is the addition of the first two vector.

(e) $\left\{ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} \right\}$

Linearly independent, ONLY the trivial solution for system $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3 = 0$.

Exercise 4.5.20 Determine if b is a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$ where

$$\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 6 \\ -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -17 \\ 17 \\ 7 \end{bmatrix}.$$

If b is a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$, express b as a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$. *Yes,* $b = 3 \cdot \boldsymbol{a}_1 - 1 \cdot \boldsymbol{a}_2 + 2 \cdot \boldsymbol{a}_3$

Exercise 4.5.21 Find the value(s) of h for which the following set of vectors

$$\left\{ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} h \\ 1 \\ -h \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2h \\ 3h+1 \end{bmatrix} \right\}$$

is linearly independent. iff $h \neq -1/2$ and $h \neq -1$

Exercise 4.5.22 Let $A = \begin{bmatrix} 3 & -1 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ -1 & -3 & 2 & -4 \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$. Denote the columns of A by $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$. Let $W = \text{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$.

(a) Is b in W?

Yes, b spans $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$, i.e $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 + x_4\boldsymbol{a}_4 = b$ has solutions - Infinitely many.

(b) If b is in W, then express b as a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and $\boldsymbol{a}_4$.

Example 4.5.2 Suppose $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a linearly independent set in $\mathbb{R}^n$. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2\}$ is also linearly independent.

Proof : *Let us assume that $\{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2\}$ is NOT linearly independent. Then $\alpha_1\boldsymbol{v}_1 + \alpha_2(\boldsymbol{v}_1 + \boldsymbol{v}_2) = 0$ has a nontrivial solution. Let us say this solution is (c_1, c_2) where $c_1 \neq 0$ or $c_2 \neq 0$. Then we see that*

$$\begin{aligned} 0 &= c_1\boldsymbol{v}_1 + c_2(\boldsymbol{v}_1 + \boldsymbol{v}_2) \\ &= c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 \\ &= (c_1 + c_2)\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 \end{aligned}$$

If $c_1 \neq 0$, then $c_1 + c_2 \neq 0$. If $c_2 \neq 0$, then $c_2 \neq 0$. This means that $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 = 0$ has a nontrivial solution. In other words, we deduce that $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly dependent. This is a contradiction.

Because we are given that $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly independent. This contradiction is coming from the assumption that $\{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2\}$ is NOT linearly independent. Therefore, $\{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2\}$ must be linearly independent. ■

Exercise 4.5.23 Let $A = \begin{bmatrix} 3 & -1 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 3 & -1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$. Denote the columns of A by $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$. Does $b = \text{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$?. T

Example 4.5.3 Suppose $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a linearly independent set in $\mathbb{R}^n$. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3\}$ is also linearly independent.

Proof : *Let us assume on the contrary that* $\{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3\}$ *is linearly dependent. Then there is a nontrivial solution for the vector equation*

$$\alpha_1\boldsymbol{v}_1 + \alpha_2(\boldsymbol{v}_1 + \boldsymbol{v}_2) + \alpha_3(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) = 0.$$

Let us denote this solution (c_1, c_2, c_3) *where not all* c_1, c_2, c_3 *are zero, i.e.,*

$$c_1\boldsymbol{v}_1 + c_2(\boldsymbol{v}_1 + \boldsymbol{v}_2) + c_3(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) = 0.$$

We simplify the last expression and we get $(c_1 + c_2 + c_3)\boldsymbol{v}_1 + (c_2 + c_3)\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = 0$. *Note that not all* $c_1+c_2+c_3, c_2+c_3, c_3$ *are zero (because if all of* $c_1+c_2+c_3, c_2+c_3, c_3$ *are zero, then all* c_1, c_2, c_3 *are zero which is not the case).*

In other words, the equation $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 = 0$ *has a nontrivial solution which implies that* $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ *is linearly dependent, a CONTRADICTION. Thus, we conclude that* $\{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3\}$ *must be linearly independent.* ■

Exercise 4.5.24 Determine if the following sets of vector are linearly independent. If not, write one vector as a linear combination of other vectors in the set.

(a) $\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}\right\}$

Linearly independent because not multiple of each other.

(b) $\left\{\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 2 \\ -10 \end{bmatrix}\right\}$

Linearly dependent because $\boldsymbol{v}_1 + \boldsymbol{v}_2 = (1/2)\boldsymbol{v}_3$.

(c) $\left\{\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 30 \\ -2 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}\right\}$

Linearly independent because $[\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3] \sim I$.

(d) $\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

Linearly dependent because of the zero vector.

(e) $\left\{\begin{bmatrix} -5 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 36 \\ 12 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix}\right\}$

Linearly dependent because more vectors than the entries.

Exercise 4.5.25 Let

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v}_4 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

(a) Show that $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly dependent.

(b) Show that $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

(c) Show that $\boldsymbol{v}_4$ can be written as a linear combination of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Exercise 4.5.26 Let $\boldsymbol{v}_1 = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ s \\ 1 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ s \end{bmatrix}$. For which value(s) of s, if any, $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly independent.

Given vectors are linearly independent for every values of s except the numbers $1, -1$ *and* 0.

Exercise 4.5.27 Mark each statement *True* or *False.* Justify your answer. Also give reasons for not otherwise. Let S be a set of m vectors in $\mathbb{R}^n$.

(a) If $m > n$ then S is linearly independent. *False.*

(b) If the zero vector is in S, then S is linearly dependent. *True.*

(c) If S is linearly independent and T is a subset of S, then T is linearly independent. *True.*

(d) If T is linearly dependent and T is a subset of S, then S is linearly dependent. *True.*

(e) The linear system $A\vec{x} = \vec{b}$ has a unique solution if and only if the column vectors of A are linearly independent. *True.*

Exercise 4.5.28 Let $\vec{v}_1 = (1,1,1), \vec{v}_2 = (1,2,3)$ and $\vec{v}_3 = (1,1,2)$.

(a) Show that the vectors above are linearly independent.

After reductions. There are three pivot positions and three vectors. Therefore, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

(b) Find the unique scalars(weights) c_2, c_2, c_3 such that $\vec{v} = (2,1,3)$ can be written as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$. $c_1 = 0, c_2 = -1,$ *and* $c_3 = 3$

Exercise 4.5.29 Determine whether the set of polynomials is linearly independent or linearly dependent:

$$\mathbf{p}_1(x) = 1, \quad \mathbf{p}_2(x) = -2 + 4x^2, \quad \mathbf{p}_3(x) = 2x, \quad \text{and} \quad \mathbf{p}_4(x) = -12x + 8x^3.$$

Linearly independent

Exercise 4.5.30 For which $a's \in \mathbb{R}$ the set of polynomials $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly independent in $\mathscr{P}_2$ where

$$\mathbf{p}_1(x) = a, \quad \mathbf{p}_2(x) = -2 + (a-4)x, \quad \mathbf{p}_3(x) = 1 + 2x + (a-1)x^2.$$

Every value of a except $0, 1, 4$

Exercise 4.5.31 Suppose that $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent and

$$\vec{w}_1 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3, \quad \vec{w}_2 = \vec{v}_2 + \vec{v}_3 \text{ and } \vec{w}_3 = \vec{v}_3.$$

Show that $T = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is linearly independent.

Let us assume that $\{\vec{v}_3, \vec{v}_2 + \vec{v}_3, \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$ is NOT linearly independent. Then

$$\alpha_1\vec{v}_3 + \alpha_2(\vec{v}_2 + \vec{v}_3) + \alpha_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = 0$$

has a nontrivial solution. Let us say this solution is (c_1, c_2, c_3) where $c_1 \neq 0$ or $c_2 \neq 0$ or $c_3 \neq 0$. Then we see that

$$\begin{aligned} 0 &= c_1\vec{v}_3 + c_2(\vec{v}_2 + \vec{v}_3) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) \\ &= c_1\vec{v}_3 + c_2\vec{v}_2 + c_2\vec{v}_3 + c_3\vec{v}_1 + c_3\vec{v}_2 + c_3\vec{v}_3 \\ &= (c_1 + c_2 + c_3)\vec{v}_3 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_1. \end{aligned}$$

If $c_1 \neq 0$, then $c_1 + c_2 + c_3 \neq 0$. If $c_2 \neq 0$, then $c_2 + c_3 \neq 0$. If $c_3 \neq 0$, then $c_3 \neq 0$. This means that

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = 0$$

has a nontrivial solution. In other words, we deduce that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. This is a contradiction. Because we are given that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. This contradiction is coming from the assumption that $\{\vec{v}_3, \vec{v}_2 + \vec{v}_3, \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$ is not linearly independent.

Therefore, $\{\vec{v}_3, \vec{v}_2 + \vec{v}_3, \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$ must be linearly independent.

Exercise 4.5.32 *True* or *False*? When **u** and **v** are nonzero vector, Span$\{\mathbf{u}, \mathbf{v}\}$ contains the line through **u** and the origin. *True*

Exercise 4.5.33 Which of the following statements are true? Explain your answers.

1.) The span of any one vector in $\mathbb{R}^3$ is a line through the origin. *False*

2.) The span of any two distinct vectors in $\mathbb{R}^2$ is all of $\mathbb{R}^2$. *False*

3.) A set of vectors $\{u, v, w\}$ in a vector space V spans V if every vector in V is a linear combination of $u, u + v$ and $u + v + w$. *True*

4.) Any spanning set for $M_{22}(\mathbb{R})$ contains at least four elements. *True*

5.) The set $\{(1, 1),\ (2, 3)\}$ spans $\mathbb{R}^2$. *True*

Exercise 4.5.34 For which value of s does the vector $(6, 3, s)$ belong to the subspace of $\mathbb{R}^3$ spanned by $(1, 2, 3)$ and $(0, 1, 2)$? $s = 0$

Exercise 4.5.35 Let
$$W = \left\{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0\right\}.$$

(a) Is W a subspace of $\mathbb{R}^3$? *Yes. For $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$ & $a\vec{u} \in W$*

(b) Find a spanning set for W.

$\{(x, y, z) \in W \Leftrightarrow z = y - x\}$, *So*

$$\begin{aligned} W &= \{(x, y, z) : z = y - x\} \\ &= \left\{(x, y, y - x) \in \mathbb{R}^3 \ : \ x, y \in \mathbb{R}\right\} \\ &= \{x(1, 0, -1) + y(0, 1, 1) \ : \ x, y \in \mathbb{R}\} \\ &= rpan \ \ \{(1, 0, -1), (0, 1, 1)\} \end{aligned}$$

Hence $\{(1, 0, -1), (0, 1, 1)\}$ *is a spanning set.*

Exercise 4.5.36 Consider the vector space $\mathbb{F}(\mathbb{R}) = \{f \mid f : R \to R\}$, with the standard operations. Recall that the zero vector in $\mathbb{F}(\mathbb{R})$ is the function that has the value 0 for all $x\mathbb{R}R$.
Let $U = \{f \in \mathbb{F}(\mathbb{R}) \mid f(1) = f(-1)\}$ be the subspace of functions which have the same value at $x = -1$ and $x = 1$ (you do not have to check that U is a subspace).
Define functions g, h, j, and $k \in \mathbb{F}(\mathbb{R})$ by

$$\begin{aligned} g(x) &= 2x^3 - x^2 - 2x + 1, \qquad & h(x) &= x^3 + x^2 - x + 1, \\ k(x) &= -x^3 + 5x^2 + x + 1, \qquad & \text{and} \quad j(x) &= x^3 - x. \end{aligned}$$

(a) Show that g and h belong to U. $g(1) = g(-1) = 0, h(1) = h(-1) = 2$

(b) Show that $k \in \text{rpan} \ \{g, h\}$. $\alpha_1 = -2, \alpha_2 = 3$

(c) Show that $j \notin \text{rpan} \ \{g, h\}$. α_1, α_2 *DNE*

(d) Show that $\text{rpan} \ \{g, h\} \neq \text{rpan} \ \{g, h, j\}$

$j \notin$ *rpan* $\{g, h\}$ *while* $j \in$ *rpan* $\{g, h, j\}$

Exercise 4.5.37 Find the value of t for which $(4, 6, 3, t)$ belongs to span

$$\{(1, 3, -4, 1), (2, 8, -5, -1), (-1, -5, 0, 2)\}.$$

A. 0

B. 4

C. 7

D. 11

E. 13

F. 15

E

Exercise 4.5.38 State whether the following is true or false. If true, explain why, if false, give a numerical example to illustrate.

The columns of a 3×4 matrix are always linearly dependent. *True. dim* $\mathbb{R}^3 = 3 < 4$

Exercise 4.5.39 For what value of α is the set of vectors $\{(1,1,1),(1,2,0),(2,3,\alpha)\}$ linearly dependent?

A. -1

B. 2

C. 0

D. 1

E. -1/2

F. -2

D

Exercise 4.5.40 If a, b and c are scalars and u, v and w are vectors in some vector space V, which of the following statements are always true?

I. The set $\{u, v, w\}$ of vectors is linearly independent if $au+bv+cw = 0$ *when* $a = b = c = 0$.

II. The set $\{u, v, w\}$ of vectors is linearly independent if $au+bv+cw = 0$ *only if* $a = b = c = 0$.

III. The set $\{u, v\}$ spans V if $\{u, v\}$ is linearly independent.

IV. The set $\{u, v, w\}$ spans V if every vector in V is a linear combination of $-2v, u+v+w$ and $v-3w$.

A. Only I & II are true.

B. Only II & IV are true.

C. Only II & III are true.

D. Only I & III & IV are true.

E. Only II & III & IV are true.

F. Only I is true.

B

Exercise 4.5.41 Let A be a 6×4 matrix. Answer the following questions:

(1) Can the system $A\vec{x} = \vec{0}$ have a non trivial solution?

(2) Can the columns of A span $\mathbb{R}^6$?

(3) Can the columns of A be linearly independent?

A. Yes, No, Yes.

B. Yes, Yes, Yes.

C. Yes, No, No.

D. No, No, Yes.

E. No, No, No.

F. No, Yes, Yes.

(1), True: It will always have infinitely many solutions.

(2), False: We need at least 6 *vectors to span* $\mathbb{R}^6$.

(3), True : When $\text{rank}A = 4$, *which is possible.*

A

Exercise 4.5.42 In a vector space V , suppose $\{u, v\}$ is linearly independent and w is such that $\{u, v, w\}$ is linearly dependent. Which of the following is ALWAYS true?

A. $u \in$ rpan $\{v, w\}$ *Could be false*

$$u = (1, 0), v = (0, 1) \text{ and } w = (1, 1)$$

B. $v \in$ rpan $\{u, w\}$ *Could be false*

$$u = (1, 0), v = (0, 1) \text{ and } w = (1, 1)$$

C. $w \in$ rpan $\{u, v\}$ *Could be false*

$$u = (1, 0), v = (0, 1) \text{ and } w = (0, 0)$$

D. $\{u, u + v, w\}$ is linearly independent *Could be false*

E. $\{u, w\}$ is linearly dependent *True*

Its a theorem. If $\{u, v\}$ is l.i, then $\{u, v, w\}$ is l.i $\Leftrightarrow w \notin$ rpan $\{u, v\}$

F. $\{v, w\}$ is linearly dependent *Could be false*

$$u = (1, 0), v = (0, 1) \text{ and } w = (1, 1)$$

Exercise 4.5.43 If $\{u, v, w\}$ is a set of vectors in a vector space V , and a, b, and c are scalars, which of the following statements are true?

I. If $a = b = c = 0$ implies $au + bv + cw = 0$, then $\{u, v, w\}$ is linearly independent.

II. If $au + bv + cw = 0$ implies $a = b = c = 0$, then $\{u, v, w\}$ is linearly independent.

III. If none of the vectors u, v or w is a multiple of any other vector in $\{u, v, w\}$, then $\{u, v, w\}$ is linearly independent.

IV. If $au + bv + cw = 0$ can occur only when $a = b = c = 0$, then $\{u, v, w\}$ is linearly independent.

A. (I) & (II)

B. (I) & (III)

C. (II) & (III)

D. (I) & (IV)

E. (II) & (IV)

F. (III) & (IV)

*I, False: The statement $a = b = c = 0 \Rightarrow au + bv + cw = 0$ holds for **any** vectors u, v, w (e.g $u = v = w = 0$) .*

II, True: Its the definition of linear independence.

III, False : $\{(1, 0), (0, 1), (1, 1)\}$ satisfies the given condition, but is linearly dependent.

IV, True : This is simply a restatement of the definition: "A implies B" means A can only occur when B does too.

E

Exercise 4.5.44 If V is vector space with u, v, w in V , and $\{u, v, w\}$ is linearly independent, show that $\{u, v\}$ must also be linearly independent. (Do not give an example: use the definition of independence to show that the statement is true in every vector space V.)

$\alpha_1 = \alpha_2 = \alpha_3 = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = 0 \quad \Rightarrow \quad \{u, v\}$ *is linearly independent.*

Exercise 4.5.45 Give an example of 3 vectors u, v, w in $\mathbb{R}^2$, such that $\{u, v, w\}$ spans $\mathbb{R}^2$, and $\{u, v\}$ also spans $\mathbb{R}^2$.

$u = (1, 0), v = (0, 1), w = (1, 1)$ *where w should be a linear combination of* $\{u, v\}$

Exercise 4.5.46 Suppose $\{u, v, w\}$ is a set of vectors in a vector space V . Which of the following statements are equivalent to

"$\{u, v, w\}$ is linearly independent."

I. None of the vectors u, v or w is a multiple of any other single vector in $\{u, v, w\}$.

II. If a, b, c are scalars, then $au + bv + cw = 0$ implies $a = b = c = 0$.

III. Both $\{u, v\}$ and $\{v, w\}$ are linearly independent.

IV. If $a = b = c = 0$, then $au + bv + cw = 0$.

V. None of the vectors u, v or w is a linear combination of the other vectors in $\{u, v, w\}$.

A. I & II	C. II & III	E. II & IV
B. I & III	D. II & V	F. III & IV

I, *False: e.g* $\{(1, 0, 0), (0, 1, 0), (1, 1, 0)\}$ *satisfy I but linearly dependant.*

II, *True* ✓*: Definition of linear independence*

III, *False: See example in I*

IV, *False: This is true for* ***any*** *vectors u, v, w and not necessarily linear independent.*

V, *True* ✓*: Linear independency infer trivial solutions of linear combination of vectors*

D

Exercise 4.5.47 Find the value(s) of t for which $(2, 6, 5, 2t)$ lies in the subspace of $\mathbb{R}^4$ spanned by $(1, 2, 2, 2), (3, 7, 6, 6)$ and $(1, 2, 1, 2)$.

A. $t = -4$ only.	C. $t = 0$ or 2.	E. $t = 2$ or 4.
B. $t = -2$ or -4.	D. $t = -2, 0$ or 4.	F. $t = 2$ only.

F

The system is consistent iff $2t - 4 = 0$

Exercise 4.5.48 Which of the following are subspaces of $M_{2\times 2}$?

$$\begin{aligned}
U &= \left\{ \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in M_{2\times 2} \mid x, y, z \in R \right\} \\
V &= \left\{ \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \in M_{2\times 2} \mid y \in R \right\} \\
W &= \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_{2\times 2} \mid xw - zy = 0 \right\} \\
S &= \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \in M_{2\times 2} \mid x, y, z \in R \right\}
\end{aligned}$$

A. Only U and V

B. Only U and W

C. Only U, V and W

D. Only V, W and S

E. Only W and S

F. Only U, V and S

Using the fact that, the span of vectors is always a subspace.

$U = rpan \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ *True* ✓

$V = rpan \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ *True* ✓

$W \neq rpan \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ *False* X

$S = rpan \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ *True* ✓

F

See Exercise 3.6.24 for a different approach.

Exercise 4.5.49 *True* or *False*? If you say the statement may be false, you must give an explicit example - with numbers!. If you say the statement is always true, you must give a clear explanation.

If V is a vector space and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subset V$ is linearly dependent, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is also linearly dependent. *False*

Let $V = \mathbb{R}^2, \boldsymbol{v}_1 = (1, 0), \boldsymbol{v}_2 = (0, 1),$ *and* $\boldsymbol{v}_3 = (0, 0)$. *Then* $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ *is linearly dependent but* $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ *is linearly independent.*

Exercise 4.5.50 Let A be a real $n \times n$ matrix. Give statements equivalent to

$$\text{“}\det A = 0\text{”},$$

for: the rows of A.

The rows of A are linearly dependent.

The rows of A do not span $\mathbb{R}^n$.

The rows of A are not basis of $\mathbb{R}^n$.

Exercise 4.5.51 Which of the following sets are linearly independent in

$$\mathbb{F}(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}?$$

$$\begin{aligned} S &= \left\{x, x^2\right\} \\ T &= \left\{1, x, x^2, (1-x)^2\right\} \\ U &= \left\{1, 2\cos^2 x, 3\sin^2 x\right\} \end{aligned}$$

A. S and T.

B. S and U.

C. T and U.

D. S only.

E. T only.

F. S, T and U.

D

Exercise 4.5.52 Consider the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}$$

Which of the following statements is true:

A. The columns are linearly dependent

B. The matrix has determinant -1

C. The matrix is not invertible

D. None of the above

D

Computation reveals that the matrix has determinant equal to 1*, so (B) is false. Moreover, since the determinant is non zero, (A) and (C) are false as well by the many equivalent characterizations of invertible matrices.*

Exercise 4.5.53 *True* or *False*? If for some matrix A, and some vectors $\vec{x}, \vec{b}$, we have $A\vec{x} = \vec{b}$, then $\vec{b}$ is in the span of the column vectors of A.

True: By definition if $A\vec{x} = \vec{b}$, then $\vec{b}$ is a linear combination of the column vectors of A. This is the same as $\vec{b}$ being in the span of the column vectors of A.

Exercise 4.5.54 Determine if the given set of vectors is linearly independent or linearly dependent.

(a) $\left\{\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$;

(b) $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$;

(c) $\left\{\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}\right\}$;

Exercise 4.5.55 Determine if the given set of vectors spans $\mathbb{R}^3$

(a) $\left\{\begin{bmatrix} \pi \\ 2\pi \\ -\pi \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}\right\}$;

(b) $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$;

Exercise 4.5.56 Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, w = \begin{bmatrix} 3 \\ 10 \\ h \end{bmatrix}.$$

(a) For which values of h can the vector w be generated as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?
$h = -9$

(b) In this case, what are the weights a_1, a_2 of the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively?
$a_1 = 6, a_2 = -7$

Exercise 4.5.57 Does the vector $v = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix}$ lie in the plane of $\mathbb{R}^3$ generated by the columns of the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & 1 \\ -2 & -8 \end{bmatrix}$? Justify your answer.

To see whether the vector $\vec{v}$ lies in the plane generated by the columns of A, we construct a new matrix $B = [A\ \vec{v}]$, its reduced row echelon form shows that

$$\vec{v} = -5\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 1 \\ -8 \end{bmatrix},$$

and hence $\vec{v}$ lies indeed in the plane.

Exercise 4.5.58 Show that the columns of the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 8 & 0 \\ 4 & 7 & 9 & 10 \end{bmatrix}$$

are linearly independent using the definition of linear independence.

By definition, the columns of M are linearly independent if and only if the equation $M\vec{x} = \vec{0}$ only has the trivial solution. The solution of this equation is obvious (given the lower triangular matrix) and given by $\vec{x} = \vec{0}$. Therefore the columns of the matrix are linearly independent. Note that the result remains valid if the lower triangular matrix is replaced by an upper triangular matrix.

Exercise 4.5.59 Is it possible to write the polynomial $f(t) = 1 + 4t + 7t^2$ as a linear combination of the polynomials $1 + t^2, t + t^2$ and $1 + 2t + t^2$?

If $f(t)$ can be written as linear combination of the three polynomials, there are three real x, y and z such as

$$1 + 4t + 7t^2 = x(1 + t^2) + y(t + t^2) + z(1 + 2t + t^2).$$

The existence of these three real numbers is related to the fact that the system of equations got by matching the coefficients of equal powers of t in both members of equation is consistent. We find that this system is

$$\begin{aligned} x + z &= 1 \\ y + 2z &= 4 \\ x + y + z &= 7 \end{aligned}$$

which is consistent. Therefore, $f(t)$ can be written as a linear combination of the three given polynomials. The solution is:

$$2(1 + t^2) + 6(t + t^2) - 1(1 + 2t + t^2) = 1 + 4t + 7t^2$$

Exercise 4.5.60 Show that the vectors $(1, 1, 1), (0, 1, 1)$ and $(0, 1, -1)$ generate $\mathbb{R}^3$. (Hint: you need to show that any vector (a, b, c) in $\mathbb{R}^3$ can be written as a linear combination of these vectors).

Exercise 4.5.61 Prove that if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ are vectors in a vector space V, the set W of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ is a subspace of V.

Exercise 4.5.62 Determine whether the following set of vectors in M_{41} is linearly independent or linearly dependant. $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\}$

Exercise 4.5.63 *True* or *False*? Every spanning set of a vector space contains a basis of the space.

Exercise 4.5.64 Show that this $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^3$.

Exercise 4.5.65 Show that $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$ is a linear combination of S by finding c_1 and c_2 giving a linear relationship.

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

Show that the pair α_1, α_2 is unique.

Exercise 4.5.66 Consider the vectors

$$A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$$

in $M_{2\times 2}$. Determine whether or not

$$B = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

can be expressed as a linear combination of $\{A_1, A_2\}$.

Chapter 5

Eigenvalues and Eigenvectors

Definition 5.0.1 Let A be an $n \times n$ matrix, $\boldsymbol{X}$ a nonzero $n \times 1$ column vector and λ a scalar. If

$$A\boldsymbol{X} \quad = \quad \lambda \boldsymbol{X}, \tag{5.1}$$

then $\boldsymbol{X}$ is an *eigenvector* of A and λ is an *eigenvalue* of A.

Definition 5.0.2 Given an $n \times n$ square matrix A, the real number λ is called an eigenvalue of A if $\exists$ a non zero vector $\boldsymbol{X}$ such that

$$A\boldsymbol{X} = \lambda \boldsymbol{X} \tag{5.2}$$

where $\boldsymbol{X}$ is an eigenvector of this matrix and λ is the eigenvalue, $\boldsymbol{X}$ is called an eigenvector of the matrix A associated with eigenvalue λ (Latent or Characteristic values of the matrix A).

Example 5.0.1 Let $A = I_n,\ \lambda = 1\ \ \forall\ \boldsymbol{X} \in \mathbb{R}^n$ since $IX = \lambda \boldsymbol{X}$.

Example 5.0.2 $A\boldsymbol{X} = \lambda \boldsymbol{X}$ $\ A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ has two eigenvalues $\pm\frac{1}{2}$ with corresponding eigenvectors $(1, 1)$ and $(1, -1)$

Definition 5.0.3 The algebraic multiplicity of an eigenvalue λ_0 of a matrix A is the number of factors of $(\lambda - \lambda_0)$ in the characteristic polynomial $det(\lambda I_n - A)$.

Definition 5.0.4 The eigenspace of A corresponding to an eigenvalue λ is the linear space (vector space) spanned by all eigenvectors of A associated with λ. This is the null space of $(A - \lambda I_n)$.

Definition 5.0.5 The geometric multiplicity of an eigenvalue λ of a matrix A is the dimension of the eigenspace of A corresponding to λ.

Definition 5.0.6 The sum of the algebraic multiplicities of all eigenvalues of an $n \times n$ matrix is equal to n.

Example 5.0.3 If we work over $\mathbb{R}$ the the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

has no eigenvectors.

Note 5.0.1 The word "eigen" is German for "proper" or "characteristic." Therefore, an *eigenvector* of A is a "characteristic vector of A." This vector tells us something about A.

Theorem 5.0.1 *Every matrix has an eigenvalue*
Suppose A is a square matrix. Then A has at least one eigenvalue.

5.1 Eigenvalues & Corresponding Eigenvectors.

Definition 5.1.1 For $A\boldsymbol{X} = \lambda\boldsymbol{X}$, $A(n \times n)$,

$$\begin{aligned} A\boldsymbol{X} &= \lambda\boldsymbol{X} \\ A\boldsymbol{X} - \lambda\boldsymbol{X} &= 0 \\ (A - \lambda I_n)\boldsymbol{X} &= 0 \end{aligned} \tag{5.3}$$

then $\boldsymbol{X}$ is the eigenvector of matrix A.

Definition 5.1.2 For a non trivial solution (or infinitely many solutions) to the homogeneous linear system,

$$\begin{aligned} |A - \lambda I_n| &= 0 \qquad (5.4) \\ \text{or } \det(A - \lambda I_n) &\equiv P(\lambda) \end{aligned}$$

$P(\lambda)$ is called the *characteristic polynomial* of A. λ is called the eigenvalue of matrix A.

Equation $P(\lambda) = 0$ is called the characteristic equation of A. Indeed

$$\begin{aligned} \Rightarrow \quad P(\lambda) &= \lambda^n + a_1\lambda^{n-1}\lambda + \ldots + a_n \\ (A - \lambda I_n)\boldsymbol{X} &= 0 \quad a_1, \ldots a_n \in \mathbb{R} \end{aligned}$$

$P(\lambda) = 0$ has n roots ($n \in \mathbb{R}$ or $n \in \mathbb{C}$) i.e the roots may be all real or all complex or both and may or may not be repeated.

Note 5.1.1 For complex roots, they occur in pairs i.e a complex root together with its conjugate.

Example 5.1.1 Compute the eigenvalues and eigenvectors for $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Using (5.4) for the eigenvalues

$$\begin{aligned} |A - \lambda I_n| &= 0 \\ \left| \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| &= 0 \\ \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(4-\lambda) - (-2)(1) &= 0 \\ 4 - \lambda - 4\lambda + \lambda^2 + 2 &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0 = P(\lambda) \\ (\lambda - 3)(\lambda - 2) &= 0 \end{aligned}$$

Therefore $\lambda_1 = 2$ and $\lambda_2 = 3$, the eigenvalues. Finding their corresponding eigenvectors

1.) For $\lambda_1 = 2$. Using (5.3) for eigen vectors

$$\begin{aligned}
(A - \lambda I_n)\boldsymbol{X} &= 0 \\
\left(\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\boldsymbol{X} &= 0 \\
\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

Forms a linear system of equations

$$\begin{aligned}
-x_1 + x_2 &= 0 \\
-2x_1 + 2x_2 &= 0
\end{aligned}$$

That can be reduced by a Gaussian elimination

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \xrightarrow[-2R_1+R_2 \to R_2]{R_1 \to R_1} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Solved to have

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) For $\lambda_2 = 3$

$$\begin{aligned}
(A - \lambda I_n)\boldsymbol{X} &= 0 \\
\left(\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\boldsymbol{X} &= 0 \\
\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

which is equivalent to the linear system of equations

$$\begin{aligned}
-2x_1 + x_2 &= 0 \\
-2x_1 + x_2 &= 0
\end{aligned}$$

Reducing the argumented matrix

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_2 \to R_2]{R_1 \to R_1} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

With solutions of the eigenvector

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{s}{2} \\ \\ s \end{bmatrix} = s\begin{bmatrix} \frac{1}{2} \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 5.1.2 For $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$. Find the eigenvalues & their corresponding eigenvectors.

1.) We first compute the characteristic polynomial.

$$\begin{aligned} |A - \lambda I_n| &= 0 \\ |A - \lambda I_3| &= 0 \\ \left| \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| &= 0 \\ \begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} &= 0 \end{aligned}$$

The determinant can be given by

$$|A| = (1-\lambda)A_{11} + 2A_{12} + -1A_{13}$$

with cofactors

$$A_{11} = [-1]^2 \begin{vmatrix} -\lambda & 1 \\ -4 & 5-\lambda \end{vmatrix} \qquad A_{12} = [-1]^3 \begin{vmatrix} 1 & 1 \\ 4 & 5-\lambda \end{vmatrix} \qquad A_{13} = [-1]^4 \begin{vmatrix} 1 & -\lambda \\ 4 & -4 \end{vmatrix}$$

$$\begin{aligned} |A| = (1-\lambda) \begin{vmatrix} -\lambda & 1 \\ -4 & 5-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 4 & 5-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -\lambda \\ 4 & -4 \end{vmatrix} &= 0 \\ -(\lambda^3 - 6\lambda^2 + 11\lambda - 6) &= 0 \\ (\lambda-1)(\lambda-2)(\lambda-3) &= 0 \end{aligned}$$

To have

$$\lambda_1 = 1,\ \lambda_2 = 2,\quad \lambda_3 = 3$$

As the eigenvalues of A.

2.) For each eigenvalue we need to get the corresponding eigenvector by solving the homogeneous equation $(A - \lambda_i I_3)\boldsymbol{X} = 0$

For $\lambda_1 = 1$,

$$\begin{aligned} (A - \lambda I_n)\boldsymbol{X} &= 0 \\ \left(\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

whose system of equations is

$$\begin{array}{rcrcrcl} & & 2x_2 & - & x_3 & = & 0 \\ x_1 & - & x_2 & + & x_3 & = & 0 \\ 4x_1 & - & 4x_2 & + & 4x_3 & = & 0 \end{array}$$

By row reduction, the homogeneous equation generates infinetely many solutions given by

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{r}{2} \\ \\ \frac{r}{2} \\ \\ r \end{bmatrix} = r \begin{bmatrix} -\frac{1}{2} \\ \\ \frac{1}{2} \\ \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

A general eigenvectors with infinitely many solutions

For $\lambda_2 = 2$

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

For $\lambda_3 = 3$

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

Exercise 5.1.1 Compute the eigenvalues and associated eigenvectors of this matrix

$$\begin{bmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{bmatrix}$$

$$. \ \lambda_1 = 5; \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 4; \begin{bmatrix} -7 \\ -7 \\ 1 \end{bmatrix}, \quad \lambda_3 = 2; \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Example 5.1.3 Find the Eigen values of matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Solution : *To find the Eigen values it satisfy the condition,* $|A - \lambda I| = 0$

$$\begin{aligned} |A - \lambda I| &= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2\text{–}\lambda)(\lambda^2 - 4\lambda + 3)\text{–}(2 - \lambda) \end{aligned}$$

By solving the above equation, we get,

$$\lambda = 2 + \sqrt{2},\ 2 - \sqrt{2},\ 2.$$

■

Example 5.1.4 Find the eigenvalues of A, and for each eigenvalue, give one eigenvector, where

$$A = \begin{bmatrix} -7 & -2 & 10 \\ -3 & 2 & 3 \\ -6 & -2 & 9 \end{bmatrix}.$$

$$\begin{aligned}
\det A - \lambda I &= \begin{vmatrix} -7-\lambda & -2 & 10 \\ -3 & 2-\lambda & 3 \\ -6 & -2 & 9-\lambda \end{vmatrix} \\
&= (-7-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ -2 & 9-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -3 & 3 \\ -6 & 9-\lambda \end{vmatrix} + 10\begin{vmatrix} -3 & 2-\lambda \\ -6 & -2 \end{vmatrix} \\
&= (-7-\lambda)(\lambda^2 - 11\lambda + 24) + 2(3\lambda - 9) + 10(-6\lambda + 18) \\
&= -\lambda^3 + 4\lambda^2 - \lambda - 6 \\
&= -(\lambda + 1)(\lambda - 2)(\lambda - 3)
\end{aligned}$$

In the last step we factored the characteristic polynomial $-\lambda^3 + 4\lambda^2 - \lambda - 6$. Factoring polynomials of degree > 2 is not trivial; we'll assume the reader has access to methods for doing this accurately.[1]

Our eigenvalue s are $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. We now find corresponding eigenvectors.

For $\lambda_1 = -1$:

We need to solve the equation $(A - (-1)I)\boldsymbol{X} = \mathbf{0}$. To do this, we form the appropriate augmented matrix and put it into row reduced echelon form.

$$\left[\begin{array}{ccc|c} -6 & -2 & 10 & 0 \\ -3 & 3 & 3 & 0 \\ -6 & -2 & 10 & 0 \end{array}\right] \quad \overrightarrow{\text{rref}} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1.5 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Our solution, in vector form, is

$$\boldsymbol{X} = t\begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for t; a nice choice would get rid of the fractions. So we'll set $t = 2$ and choose $\boldsymbol{X}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ as our eigenvector.

For $\lambda_2 = 2$:

We need to solve the equation $(A - 2I)\boldsymbol{X} = \mathbf{0}$. To do this, we form the appropriate augmented matrix and put it into row reduced echelon form.

$$\left[\begin{array}{ccc|c} -9 & -2 & 10 & 0 \\ -3 & 0 & 3 & 0 \\ -6 & -2 & 7 & 0 \end{array}\right] \quad \overrightarrow{\text{rref}} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

[1] You probably learned how to do this in an algebra course. As a reminder, possible roots can be found by factoring the constant term (in this case, -6) of the polynomial. That is, the roots of this equation could be $\pm 1, \pm 2, \pm 3$ and ± 6. That's 12 things to check.

One could also graph this polynomial to find the roots. Graphing will show us that $\lambda = 3$ *looks* like a root, and a simple calculation will confirm that it is.

Our solution, in vector form, is

$$\boldsymbol{X} = r\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for r; again, a nice choice would get rid of the fractions. So we'll set $r = 2$ and choose $\boldsymbol{X}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ as our eigenvector.

For $\lambda_3 = 3$:

We need to solve the equation $(A - 3I)\boldsymbol{X} = \boldsymbol{0}$. To do this, we form the appropriate augmented matrix and put it into row reduced echelon form.

$$\left[\begin{array}{rrr|r} -10 & -2 & 10 & 0 \\ -3 & -1 & 3 & 0 \\ -6 & -2 & 6 & 0 \end{array}\right] \quad \overrightarrow{\text{rref}} \quad \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Our solution, in vector form, is (note that $x_2 = 0$):

$$\boldsymbol{X} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for s; an easy choice is $s = 1$, so $\boldsymbol{X}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as our eigenvector.

To summarize, we have the following eigenvalue/eigenvector pairs:

$$\begin{aligned} &\text{eigenvalue } \lambda_1 = -1, &&\text{with eigenvector } \boldsymbol{X}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \\ &\text{eigenvalue } \lambda_2 = 2, &&\text{with eigenvector } \boldsymbol{X}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &\text{eigenvalue } \lambda_3 = 3, &&\text{with eigenvector } \boldsymbol{X}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Exercise 5.1.2 Find the eigenvalues of A, and for each eigenvalue, give one eigenvector, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 3 & 4 \end{bmatrix}.$$

$$\left\{-2, \begin{bmatrix} -3 \\ -8 \\ 4 \end{bmatrix}\right\}; \left\{2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}; \left\{7, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Theorem 5.1.1 Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Theorem 5.1.2 If λ is an eigenvalue of an $n \times n$ matrix A and $\boldsymbol{X}$ is the corresponding eigenvector then λ^n is the eigenvalue of A^n, $n \geq 2$ and the eigenvector remains constant.

Proof : *λ is an eigenvalue of an $n \times n$ matrix A and $\boldsymbol{X}$ is the corresponding eigenvector implies that*

$$A\boldsymbol{X} = \lambda\boldsymbol{X} \tag{5.5}$$

Left multiply both sides of (5.5) *by A to get*

$$A(A\boldsymbol{X}) = A(\lambda\boldsymbol{X}) = \lambda A(\boldsymbol{X}) = \lambda(\lambda\boldsymbol{X}) = \lambda^2\boldsymbol{X} \tag{5.6}$$

From (5.6), *the eigenvalue of A^2 is λ^2 while the eigenvector remains the same* ($\boldsymbol{X}$).

Assume that the statement is true for $n = k$ that is

$$A^k\boldsymbol{X} = \lambda^k\boldsymbol{X} \tag{5.7}$$

We prove that it is true for $n = k+1$ thus $A^{k+1}(\boldsymbol{X}) = \lambda^{k+1}\boldsymbol{X}$. Consider

$$A^{k+1}\boldsymbol{X} = A^k(A\boldsymbol{X}) = A^k(\lambda\boldsymbol{X}) = \lambda(A^k(\boldsymbol{X})) = \lambda^{k+1}\boldsymbol{X} \tag{5.8}$$

Equation (5.8) *proves that λ^{k+1} is an eigenvalue for A^{k+1} while $\boldsymbol{X}$ is the same eigenvector.* ■

Theorem 5.1.3 Let A be an $n \times n$ matrix whose distinct n eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$,

1) The product of the eigenvalues of A is the equal to $\det A$, the determinant of A. i.e

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n = \det(A)$$

Proof: λ is an eigenvalue of an $n \times n$ matrix A and $\boldsymbol{X}$ is the corresponding eigenvector implies that

$$A\boldsymbol{X} = \lambda\boldsymbol{X} \tag{5.9}$$

For non trivial solutions to (5.9), the determinant of $(A - I_n\lambda)$ is zero.

Since the determinant of $(A - I_n\lambda)$ is the characteristic polynomial $P(\lambda)$ whose roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$ then

$$det(A) = (\lambda_1 - \lambda) \cdot (\lambda_2 - \lambda) \cdot (\lambda_3 - \lambda) \cdot \ldots \cdot (\lambda_n - \lambda) \tag{5.10}$$

Set $\lambda = 0$ in (5.10) to get $det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n$. ■

2) The sum of the eigenvalues of A is equal to $tr(A)$, the trace of A. i.e

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = tr(A)$$

Proof: The diagonal elements of the matrix $(A - I_n\lambda)$ are $\lambda_1 - \lambda, \lambda_2 - \lambda, \lambda_3 - \lambda, \ldots, \lambda_n - \lambda$ thus the trace of $(A - \lambda I_n)$ is

$$tr(A) = (\lambda_1 - \lambda) + (\lambda_2 - \lambda) + (\lambda_3 - \lambda) + \ldots + (\lambda_n - \lambda) \tag{5.11}$$

Set $\lambda = 0$ in (5.11) to get $tr(A) = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n$. ■

3) If A is triangular, then the diagonal elements of A are the eigenvalues of A.

4) If λ is an eigenvalue of A with eigenvector $\boldsymbol{X}$, then $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1} with eigenvector $\boldsymbol{X}$.

5) If λ is an eigenvalue of A , then λ is an eigenvalue of A^T.

Theorem 5.1.4 Let A be an $n \times n$ matrix, if A is invertible then A does not have an eigenvalue of 0.

5.1.1 Algebraic and Geometric Multiplicity of an Eigenvalue

Definition 5.1.3 Suppose that A is a square matrix and λ is an eigenvalue of A. Then the ***algebraic multiplicity*** of λ, denoted by $\alpha_A(\lambda)$, is the highest power of $(x-\lambda)$ that divides the characteristic polynomial, $\mathscr{P}_A(x)$.

Definition 5.1.4 Suppose that A is a square matrix and λ is an eigenvalue of A. Then the ***eigenspace*** of A for λ, denoted as $\xi_A(\lambda)$, is the set of all the eigenvectors of A for λ, together with the inclusion of the zero vector.

Definition 5.1.5 Suppose that A is a square matrix and λ is an eigenvalue of A. Then the ***geometric multiplicity*** of λ, denoted by $\gamma_A(\lambda)$, is the dimension of the eigenspace $\xi_A(\lambda)$.

Example 5.1.5 Compute the eigenvalues and their corresponding eigenvectors

$$A=\begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

Then the characteristic polynomial is given by

$$\begin{aligned}
\mathscr{P}_A(\lambda) &= \det(A-\lambda I_3) \\
&= \begin{bmatrix} -13-\lambda & -8 & -4 \\ 12 & 7-\lambda & 4 \\ 24 & 16 & 7-\lambda \end{bmatrix} \\
&= (-13-\lambda)\begin{bmatrix} 7-\lambda & 4 \\ 16 & 7-\lambda \end{bmatrix} + (-8)(-1)\begin{bmatrix} 12 & 4 \\ 24 & 7-\lambda \end{bmatrix} + (-4)\begin{bmatrix} 12 & 7-\lambda \\ 24 & 16 \end{bmatrix} \\
&= (-13-\lambda)((7-\lambda)(7-\lambda)-4(16)) \\
&\quad + (-8)(-1)(12(7-\lambda)-4(24)) \\
&\quad + (-4)(12(16)-(7-\lambda)(24)) \\
&= 3+5\lambda+\lambda^2-\lambda^3 \\
\mathscr{P}_A(\lambda) &= -(\lambda-3)(\lambda+1)^2
\end{aligned}$$

The eigenvalues are $\lambda_1=3, \lambda_2=-1$

$$\text{For } \lambda_1=3, \Rightarrow \alpha_A(3)=1 \qquad \xi_A(3)=\left\{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right\} \quad \Rightarrow \quad \gamma_A(3)=1$$

$$\text{For } \lambda_2=-1, \Rightarrow \alpha_A(-1)=2 \qquad \xi_A(-1)=\left\{\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}\right\} \quad \Rightarrow \quad \gamma_A(-1)=2$$

For each eigenvalue, it has the same number of algebraic and geometric multiplicity.

Example 5.1.6 Find the complex eigenvalues for $A=\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

The characteristic polynomial is

$$\mathscr{P}_A(\lambda)=(2-\lambda)(1-\lambda)-(1)(-1)=\lambda^2-3\lambda+3=\left(\lambda-\frac{3+i\sqrt{3}}{2}\right)\left(\lambda-\frac{3-i\sqrt{3}}{2}\right)$$

The complex value eigenvalues are

$$\lambda_1=\frac{3+i\sqrt{3}}{2}, \quad \lambda_2=\frac{3-i\sqrt{3}}{2}$$

with $\alpha_A(\lambda_1)=1,\ \alpha_A(\lambda_2)=1$ algebraic multiplicities.

Example 5.1.7 Consider the matrix

$$B = \begin{bmatrix} -2 & 1 & -2 & -4 \\ 12 & 1 & 4 & 9 \\ 6 & 5 & -2 & -4 \\ 3 & -4 & 5 & 10 \end{bmatrix}$$

then, the characteristic polynomial for matrix B is given by

$$\mathscr{P}_B(\lambda) = 8 - 20\lambda + 18\lambda^2 - 7\lambda^3 + \lambda^4 = (\lambda - 1)(\lambda - 2)^3$$

So the eigenvalues are $\lambda = 1,\ 2$ with algebraic multiplicities $\alpha_B(1) = 1$ and $\alpha_B(2) = 3$.

Computing eigenvectors,

$$\lambda = 1 \quad : \quad \xi_B(1) = \left\{ \begin{bmatrix} -1 \\ 3 \\ 3 \\ 0 \end{bmatrix} \right\} \quad \Rightarrow \quad \gamma_B(1) = 1$$

$$\lambda = 2 \quad : \quad \xi_B(2) = \left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix} \right\} \quad \Rightarrow \quad \gamma_B(2) = 1$$

This example is of interest because of the discrepancy between the two multiplicities for $\lambda = 2$. In many of our examples the algebraic and geometric multiplicities will be equal for all of the eigenvalues (as it was for $\lambda = 1$ in this example), so keep this example in mind.

Example 5.1.8 Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

then, the characteristic polynomial for matrix C is given by

$$\mathscr{P}_C(\lambda) = -3 + 4\lambda + 2\lambda^2 - 4\lambda^3 + \lambda^4 = (\lambda - 3)(\lambda - 1)^2(\lambda + 1)$$

Computing their corresponding eigenvectors,

$$\lambda = 3 \quad \Rightarrow \quad \alpha_C(3) = 1 \quad \& \quad \xi_C(3) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \Rightarrow \quad \gamma_C(3) = 1$$

$$\lambda = 1 \quad \Rightarrow \quad \alpha_C(1) = 2 \quad \& \quad \xi_C(1) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \quad \Rightarrow \quad \gamma_C(1) = 2$$

$$\lambda = -1 \quad \Rightarrow \quad \alpha_C(-1) = 1 \quad \& \quad \xi_C(-1) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \Rightarrow \quad \gamma_C(-1) = 1$$

This example is of interest, where the geometric multiplicities is same as for the algebraic multiplicities, because A is a symmetric matrix.

5.2 Similar Matrices

The notion of matrices being *similar* is a lot like saying two matrices are row-equivalent. Two similar matrices are not equal, but they share many important properties.

This section 5.2 will be devoted in part to discovering just what these common properties are.

Definition 5.2.1 The matrix B is said to be similar to another matrix A if $\exists$ a non singular matrix P such that

$$B = P^{-1}AP. \tag{5.12}$$

Thus given any matrix A, we can find a matrix B similar to A using $B = P^{-1}AP$.

Example 5.2.1 Given

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix},$$

find a matrix B similar to A using an invertible matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

It is easy to show that P is a singular matrix.

$$B = P^{-1}AP$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow B = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Example 5.2.2 Define

$$A = \begin{bmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{bmatrix}$$

Check that P is nonsingular and then compute

$$\begin{aligned} B &= P^{-1}AP \\ &= \begin{bmatrix} 10 & 1 & 0 & 2 & -5 \\ -1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & 0 & 1 \\ -4 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -10 & -27 & -29 & -80 & -25 \\ -2 & 6 & 6 & 10 & -2 \\ -3 & 11 & -9 & -14 & -9 \\ -1 & -13 & 0 & -10 & -1 \\ 11 & 35 & 6 & 49 & 19 \end{bmatrix} \end{aligned}$$

So by this construction, we know that B and A are similar.

Example 5.2.3 Define

$$A = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

Check that P is nonsingular and then compute

$$\begin{aligned} B &= P^{-1}AP \\ &= \begin{bmatrix} -6 & -4 & -1 \\ -3 & -2 & -1 \\ 5 & 3 & 1 \end{bmatrix} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

So by this construction, we know that A and B are similar. But before we move on, look at how pleasing the form of A is. Not convinced? Then consider that several computations related to A are especially easy. For example, $\det A = (-1)(3)(-1) = 3$. Similarly, the characteristic polynomial is straightforward to compute by hand,

$$P_A(\lambda) = (-1-\lambda)(3-\lambda)(-1-\lambda) = -(\lambda-3)(\lambda+1)^2$$

and since the result is already factored, the eigenvalues are transparently $\lambda = 3, -1$. Finally, the eigenvectors of A are just the standard unit vectors.

Example 5.2.4 Square matrices A and B are similar to each other through the invertible matrix P:

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & -3 \\ 6 & 5 \end{bmatrix} \qquad P = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Solution : *To prove that they are similar matrices, we must first find the inverse of matrix P:*

$$P^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

And now we check that they are similar by calculating the matrix product that defines the similarity of two matrices:

$$B = P^{-1}AP?$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 6 & 5 \end{bmatrix} = B$$

The similarity relation is fulfilled, so matrix B is similar to matrix A, they are similar matrices. ■

5.2.1 Properties of Similar Matrices

Theorem 5.2.1 If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.

> **Proof :** *If A is similar to B then there exists an invertible matrix P such that $A = P^{-1}BP$, then the characteristic equation of A is given by*
>
> $$\begin{aligned} P_A(\lambda) &= \det(A - \lambda I_n) \\ &= \det\left(S^{-1}BS - \lambda I_n\right) \\ &= \det\left(S^{-1}BS - \lambda S^{-1}I_nS\right) \\ &= \det\left(S^{-1}BS - S^{-1}\lambda I_nS\right) \\ &= \det\left(S^{-1}(B - \lambda I_n)S\right) \\ &= \det\left(S^{-1}\right)\det(B - \lambda I_n)\det(S) \\ &= \det\left(S^{-1}\right)\det(S)\det(B - \lambda I_n) \\ &= \det\left(S^{-1}S\right)\det(B - \lambda I_n) \\ &= \det(I_n)\det(B - \lambda I_n) \\ &= 1\det(B - \lambda I_n) \\ &= \det(B - \lambda I_n) \\ &= P_B(\lambda) \end{aligned}$$
>
> ■

Note 5.2.1 So similar matrices not only have the same set of eigenvalues, the algebraic multiplicities of these eigenvalues will also be the same. However, be careful with this theorem. It is tempting to think the converse is true, and argue that if two matrices have the same eigenvalues, then they are similar. Not so, as the following example illustrates.

Example 5.2.5 Define

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and check that

$$P_A(\lambda) = P_B(\lambda) = 1 - 2\lambda + \lambda^2 = (\lambda - 1)^2$$

and so A and B have equal characteristic polynomials. If the converse of Theorem 5.2.1 were true, then A and B would be similar. Suppose this is the case. More precisely, suppose there is a nonsingular matrix S so that $A = S^{-1}BS$. Then

$$A = S^{-1}BS = S^{-1}I_2S = S^{-1}S = I_2$$

Clearly $A \neq I_2$ and this contradiction tells us that the converse of Theorem 5.2.1 is false.

Theorem 5.2.2 Properties of similarity. Suppose A, B and C are square matrices of size n,

1) A is similar to A, that is $A = P^{-1}AP$. (Reflexive)

> **Proof :** *Need only demonstrate a nonsingular matrix that effects a similarity transformation of A to A. I_n is nonsingular matrix.*
>
> $$I_n^{-1}AI_n = I_nAI_n = A$$
>
> ■

2) If A is similar to B, then B is similar to A. (Symmetric)

Proof : *If*

$$A = S^{-1}BS \tag{5.13}$$

To show that

$$B = R^{-1}AR \tag{5.14}$$

Let $R = S^{-1}$, since is also nonsingular (invertible) if S is invertible. Idea is to use any invertible matrix.

$$\begin{aligned}
R^{-1}AR &= \left(S^{-1}\right)^{-1} A \left(S^{-1}\right) \\
&= SAS^{-1} \\
&= SS^{-1}BSS^{-1} && \text{by (5.13)} \\
&= \left(SS^{-1}\right) B \left(SS^{-1}\right) \\
&= I_n B I_n \\
&= B
\end{aligned}$$

■

3) If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

Proof : *If*

$$A = S^{-1}BS \tag{5.15}$$
$$B = R^{-1}CR \tag{5.16}$$

To show that

$$A = Q^{-1}CQ \tag{5.17}$$

Notice how we have to assume $S \neq R$, as will usually be the case. Since S and R are invertible, so too RS is invertible - A good candidate for Q.

$$\begin{aligned}
Q^{-1}CQ &= (RS)^{-1} C (RS) \\
&= S^{-1}R^{-1}CRS \\
&= S^{-1}\left(R^{-1}CR\right) S \\
&= S^{-1}BS && \text{by (5.16)} \\
&= A && \text{by (5.15)}
\end{aligned}$$

■

4) If B is similar to A, then B^k is similar to A^k, that is

$$B^k = P^{-1}A^kP \tag{5.18}$$

for any positive integer k.

Proof : *If B is similar to A, then we have*

$$B = P^{-1}AP$$

for some nonsingular matrix P.

Then we have for a positive integer k

$$\begin{aligned} B^k &= (P^{-1}AP)^k \\ &= \underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{k \text{ times}} \\ &= P^{-1}A^kP \end{aligned}$$

since we can cancel P and P^{-1} in between.

Hence B^k is similar to A^k. ■

From Eqn (5.18)

$$A^k = PB^kP^{-1} \tag{5.19}$$

Used to compute the powers of a matrix.

5) If A is similar to the identity matrix I, then $A = I$.

Proof : *Since A is similar to I, there exists a nonsingular matrix P such that*

$$A = P^{-1}IP.$$

Since $P^{-1}IP = I$, we have $A = I$. ■

6) If A or B is nonsingular, then AB is similar to BA.

Proof : *Suppose first that A is nonsingular. Then A is invertible, hence the inverse matrix A^{-1} exists.*

Then we have

$$A^{-1}(AB)A = A^{-1}ABA = IBA = BA,$$

hence AB and BA are similar.

Analogously, if B is nonsingular, then the inverse matrix B^{-1} exists.

We have

$$B^{-1}(BA)B = B^{-1}BAB = IAB = AB,$$

hence AB and BA are similar. ■

Remark 5.2.1 Property 1), 2), 3) show that similarity is an equivalence relation.

Example 5.2.6 The only matrix similar to the zero matrix is itself:

$$P^{-1}ZP = ZP = Z.$$

The identity matrix has the same property:

$$P^{-1}IP = P^{-1}P = I.$$

Example 5.2.7 Given

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

1.) Find a matrix B similar to A using an invertible matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution :

$$B = P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

as given in Example 5.2.1. ■

2.) Compute A^8

Solution : *Applying Eqn (5.19) to compute powers of a matrix A using it's similar matrx B.*

$$\begin{aligned} A^k &= PB^kP^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^8 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^8 & 0 \\ 0 & 3^8 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 6561 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -6049 & 6305 \\ -12610 & 12866 \end{bmatrix} \end{aligned}$$

Which is a power of a matrix by using a similar matrix B, other than using the matrix A directly as below:

$$A^2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -11 & 19 \\ -38 & 46 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -11 & 19 \\ -38 & 46 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -49 & 65 \\ -130 & 146 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} -49 & 65 \\ -130 & 146 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -179 & 211 \\ -422 & 454 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} -179 & 211 \\ -422 & 454 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -601 & 665 \\ -1330 & 1394 \end{bmatrix}$$

$$A^7 = \begin{bmatrix} -601 & 665 \\ -1330 & 1394 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1931 & 2059 \\ -4118 & 4246 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} -1931 & 2059 \\ -4118 & 4246 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -6049 & 6305 \\ -12610 & 12866 \end{bmatrix}$$

■

Remark 5.2.2 For higher powers, the computation would be more costly.

Example 5.2.8 Compute A^{100}, where

$$A = \begin{bmatrix} 5 & 13 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}^{-1}.$$

Solution : *By the fact, we have*

$$A = \begin{bmatrix} 5 & 13 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}^{-1} \tag{5.20}$$

would imply that matrix $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *is similar to matrix* $A = \begin{bmatrix} 5 & 13 \\ -2 & -5 \end{bmatrix}$ *using an invertible matrix* $P = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$. *That is because, the system 5.20 is in form of*

$$A = PBP^{-1}$$

implying that

$$B = P^{-1}AP$$

Therefore, for powers on matrices, we apply the equations (5.18) or (5.19)

$$\begin{aligned} B^k &= P^{-1}A^kP \\ A^k &= PB^kP^{-1} \end{aligned}$$

$$A^{100} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{100} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}^{-1}.$$

such that

$$A^{100} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

■

Example 5.2.9 Let B be similar to matrix A, that is $B = P^{-1}AP$. If

$$B^k = \begin{bmatrix} 0 & 2^k \\ 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}.$$

What is A^k?

A. $\begin{bmatrix} 3 \cdot 2^k & 2^{k+2} \\ 9 \cdot 2^k & 2^{k+2} \end{bmatrix}$ B. $\begin{bmatrix} 0 & 2^k \\ 0 & 0 \end{bmatrix}$ C. $\begin{bmatrix} 2^k & -2^{k+2} \\ 2^k & -2^{k+2} \end{bmatrix}$ D. $\begin{bmatrix} 3 \cdot 2^{k+1} & -2^{k+2} \\ 9 \cdot 2^k & -3 \cdot 2^{k+1} \end{bmatrix}$

Solution : *Given* $P = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, *we have that* $P^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$ *Applying the powers of a matrix Eqn (5.19)*

$$\begin{aligned} A^k &= PB^kP^{-1} \\ &= \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2^k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \cdot 2^k \\ 0 & 3 \cdot 2^k \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \cdot 2^k & -4 \cdot 2^k \\ 9 \cdot 2^k & -6 \cdot 2^k \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 2^1 \cdot 2^k & -2^2 \cdot 2^k \\ 9 \cdot 2^k & -3 \cdot 2^1 \cdot 2^k \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 2^{k+1} & -2^{k+2} \\ 9 \cdot 2^k & -3 \cdot 2^{k+1} \end{bmatrix} \end{aligned}$$

Therefore, the correct solution is D). ■

Exercise 5.2.1 Let $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$. A has two eigen vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. What is A^8?

A. $\begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$ C. $\begin{bmatrix} 766 & 765 \\ 510 & -509 \end{bmatrix}$

B. $\begin{bmatrix} -766 & -765 \\ 510 & -509 \end{bmatrix}$ D. $\begin{bmatrix} 766 & -765 \\ -510 & -509 \end{bmatrix}$

5.3 Diagonisable Matrices

Good things happen when a matrix is similar to a diagonal matrix. For example, the eigenvalues of the matrix are the entries on the diagonal of the diagonal matrix. And it can be a much simpler matter to compute high powers of the matrix. Diagonalizable matrices are also of interest in more abstract settings.

But the problem is that, are the similar matrices always diagonal? NO, some matrices can have similar matrices that are always diagonal (diagonisable matrices), some not.

Definition 5.3.1 Diagonalizable Matrix
An $n \times n$ matrix A is said to be diagonalizable, if it is similar to diagonal matrix, that is "If there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix"

Remark 5.3.1 A matrix A is said to be diagonisable, if a matrix B similar to A is a diagonal matrix D, that is

$$D = P^{-1}AP \tag{5.21}$$

Example 5.3.1 Let

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \quad \text{with} \quad P = \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and is similar to a diagonal matrix, as can be seen by the following computation with the nonsingular matrix S,

$$\begin{aligned} B = P^{-1}AP &= \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & -1 \\ 2 & 3 & 1 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ B &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D \end{aligned}$$

1.) The matrix B similar to A is a diagonal matrix.

2.) The matrix D is similar to A.

3.) Thus A is diagonisable.

But how to think of a magic matrix P that can be used in a similarity transformation to produce a diagonal matrix? Next theorems.

Remark 5.3.2 Diagonizable matrices are very special similar matrices (Just more simplified, non-zero entries of similar matrices only at the diagonal). All properties of similar matrices also apply to Diaginisable matrices.

Example 5.3.2 Define

$$C = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \quad \text{with} \qquad P = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

Check that P is nonsingular and then compute

$$\begin{aligned} B &= P^{-1}CP \\ &= \begin{bmatrix} -6 & -4 & -1 \\ -3 & -2 & -1 \\ 5 & 3 & 1 \end{bmatrix} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D \end{aligned}$$

So by this construction, we know that D is similar to C. But before we move on, look at how pleasing the form of D is. Not convinced? Then consider that several computations related to D are especially easy.

Theorem 5.3.1 Diagonisable Matrices and Linear Independence of eigenvectors. An $n \times n$ matrix A is diagonalizable *iff it has n linearly independent eigenvectors.* In this case, A is similar to diagonal matrix D with $D = P^{-1}AP$, where D is the matrix formed with eigenvalues of A while P is made up of the columns of linearly independent eigenvectors of A.

Example 5.3.3 $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ is diagonalizable by , since from Example 5.1.1 on page (p. 194), the *eigen vectors* of A are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Infact with

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad D = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

a diagonal matrix.

Example 5.3.4 Consider the matrix

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

The eigenvalues and their corresponding eigenvectors are

$$\lambda_1 = 3 \qquad \left\{ \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_2 = \lambda_3 = -1 \qquad \left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Define the matrix P to be the 3×3 matrix whose columns are the three basis vectors in the eigenspaces for F,

$$P = \begin{bmatrix} -1/2 & -2/3 & -1/3 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Check that P is nonsingular. Then the three columns of S are a linearly independent set. By Theorem 5.3.1 we now know that F is diagonalizable. If we apply the matrix P to F in a similarity transformation, the result will be a diagonal matrix with the eigenvalues of F on the diagonal. The eigenvalues appear on the diagonal of the matrix in the same order as the eigenvectors appear in P. So,

$$\begin{aligned}
P^{-1}FP &= \begin{bmatrix} -1/2 & -2/3 & -1/3 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} -1/2 & -2/3 & -1/3 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 4 & 2 \\ -3 & -1 & -1 \\ -6 & -4 & -1 \end{bmatrix} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} -1/2 & -2/3 & -1/3 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{aligned}$$

A diagonal matrix. Thus D simlar to F. That is $D = P^{-1}FP$

Corollary 5.3.1 $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by $L(x) = Ax \ \ \forall \in \mathbb{R}^n$. Then A is diagonalizable with n-linearly independent vectors (eigenvectors) and the matrix of L with respect to S is diagonal.

Theorem 5.3.2 Diagonisable Matrices and Multiplicity of the eigenvalues.

1) A matrix A is diagonalizable if all its eigenvalues <u>are real & distinct</u> (different). However, if eigenvalues of A are real but not distinct (polynomial with repeated roots). A may or may not be diagonalizable and

2) If the eigenvalues of A are real but not distinct, A is diagonalizable if and only if for each eigenvalue of a multiplicity k we can find k linearly independent **non-zero** eigenvectors.

Example 5.3.5 If $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, is it diagonalizable ?
$\lambda_1 = 1 \ \lambda_2 = \lambda_3 = 5$ repeated thus eigenvectors are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus its diagonalizable since:

1.) The eigenvalues and eigenvectors have the same multiplicity $= 3$ since real eigenvalues and repeated (Theorem 5.3.2, 2)) or

2.) We can find a matrix $P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ such that $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ a diagonal matrix.

Example 5.3.6 Refer to Example 5.1.1 on page (p. 194), is diagonisable since $\lambda_1 = 2, \ \lambda_2 = 3$ are real and distinct.

Example 5.3.7 $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is it diagonalizable ?

The characteristic equation will give

$$P(\lambda) = \lambda(\lambda - 1)^2 = 0 \quad \Rightarrow \quad \lambda_1 = 0 \ \lambda_2 = \lambda_3 = 1$$

Repeated eigen values, thus computing eigenvectors

$$(A - \lambda I_n)\boldsymbol{X} = 0 \ \Rightarrow \ \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_2 = \lambda_3 = 1$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda_1 = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus diagonalizable by Theorem 5.3.2, 2). Equal multiplicity.

Example 5.3.8 Given $A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$. Is it diagonalizable ?
$\lambda_1 = -1 \ \lambda_2 = -1$ real and repeated eigen values. Thus the eigen vectors are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus matrix A is not diagonalizable since

1.) the eigenvalues and eigenvectors do not have the same multiplicity $2 \neq 1$

2.) we cant find a matrix P such that $P^{-1}AP$ a diagonal matrix.

Example 5.3.9 If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ is it diagonalizable ?
$\lambda_1 = 4$, $\lambda_2 = 2+\sqrt{3}$, $\lambda_3 = 2-\sqrt{3}$ Since the eigenvalues are real and distinct, its diagonalizable.

1.) True by Theorem 5.3.2, 1) for distinct eigen values and not multiplicity since no repeated eigenvalues.

2.) But we can find a matrix P such that $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$ a diagonal matrix.

This method is sort of lengthy.

Note 5.3.1

1) A matrix $A(n \times n)$ may fail to be diagonalizable if
   1.) Its eigenvalues are not all real
   2.) It does not have n linearly independent eigenvectors.
   3.) it does not have the same multiplicity of eigenvalues and vectors for real and repeated eigenvalues.
2) Not all diagonalizable matrices have n distinct eigenvalues.
3) If A is triangular, its eigenvalues are the elements on its main diagonal.
4) Let λ be a fixed real eigenvalue of A. The set S consisting of all eigenvectors associated with λ together with $\mathbf{0}$ vector form a space of $\mathbb{R}^n$ called the eigenspace associated with λ.
5) If a matrix is diagonalizable, there is a very fast way to compute its powers A^n, $n > 2$. If A is diagonalizable, then
$$A^n = PD^nP^{-1} \tag{5.22}$$
but not $A^n = P^{-1}D^nP$

Proof : *From the definition of diagonisable matrices equation (5.21), matrix D is similar to A, that is*

$$D = P^{-1}AP$$

From Similarity of matrix property 4 by equation (5.18)

$$\begin{aligned} B^k &= P^{-1}A^kP \\ D^k &= P^{-1}A^kP \\ A^k &= PD^kP^{-1} \qquad Eqn(5.19) \\ A^n &= PD^nP^{-1} \end{aligned}$$

by proof by induction. Since for

$$\begin{aligned} \text{For } n = 1 \quad A &= PDP^{-1} \qquad Eqn(5.21) \\ A^2 &= PDP^{-1}PDP^{-1} = PD^2P^{-1} \\ \text{For } n = k \quad A^k &= PD^kP^{-1} \qquad (Assume) \\ \text{For } n = k+1 \quad A^{k+1} &= A \cdot A^k = PDP^{-1}PD^kP^{-1} \\ &= PDD^kP^{-1} \\ &= PD^{k+1}P^{-1} \end{aligned}$$

■

Remark 5.3.3 When matrix P is not given, and need a similar matrix, we apply Theorem 5.3.1 on page (p. 212) to generate columns for P since diagonal matrices are similar matrices too.

Remark 5.3.4 Eqn (5.22) is a unique way of writing the general Eqn (5.19), where $B = D$, where the matrix B similar to A is a diagonal matrix.

Example 5.3.10 Given $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

1.) Find the matrix B similar to A?

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \quad \Rightarrow \quad B = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Also note that 1 and 4 are the eigenvalues of matrix A. Note that $B = D$, so A is Diagonisable.

2.) Compute A^4

$$A^4 = PB^4P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 171 & 85 \\ 170 & 86 \end{bmatrix}$$

Exercise 5.3.1 Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, compute A^8.

Example 5.3.11 For a matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$$\Rightarrow \; D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{bmatrix}$$

$$\begin{aligned}
A^4 &= \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}^4 \begin{bmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6^4 & 0 \\ 0 & 1^4 \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{bmatrix} \\
&= \begin{bmatrix} 1037 & 1036 \\ 259 & 260 \end{bmatrix} \\
A^{18} &= \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}^{18} \begin{bmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6^{18} & 0 \\ 0 & 1^{18} \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{bmatrix} \\
&= \begin{bmatrix} 203262 & 193262 \\ 48316 & 58316 \end{bmatrix}
\end{aligned}$$

5.4 Eigen Values and Eigen Vectors Chapter Examples

Example 5.4.1 The eigenvalues of

$$\begin{bmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 0 & 0 & 37 \end{bmatrix}$$

are

A. $-19,\ 5,\ 37$ B. $19,\ -5,\ -37$ C. $2,\ -3,\ 7$ D. $3,\ -5,\ 37$

> **Solution :** *The correct answer is (A). The eigenvalues of an upper triangular matrix are simply the diagonal entries of the matrix. Hence* $5, -19,$ *and* 37 *are the eigenvalues of the matrix.*
>
> *Alternately, look at*
>
> $$\begin{aligned} |A - \lambda I| &= 0 \\ \left| \begin{bmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 0 & 0 & 37 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| &= 0 \\ \begin{vmatrix} 5-\lambda & 6 & 17 \\ 0 & -19-\lambda & 23 \\ 0 & 0 & 37-\lambda \end{vmatrix} &= 0 \\ (5-\lambda)(-19-\lambda)(37-\lambda) &= 0 \end{aligned}$$
>
> *Then* $-5, -19, 37$ *are the roots of the equation; and hence, the eigenvalues of* A. ∎

Example 5.4.2 If $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix}$, the eigenvalue corresponding to the eigenvector is

A. 1 B. 4 C. -4.5 D. 6

> **Solution :** *If* A *is a* $n \times n$ *matrix and* λ *is one of the eigenvalues and* X *is a corresponding eigenvector, then* $AX = \lambda X$
>
> $$\begin{aligned} \begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} &= \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -18 \\ -16 \\ 4 \end{bmatrix} &= \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} \\ 4 \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} &= \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} \end{aligned}$$
>
> *such that* $\lambda = 4$
>
> *The solution is B.* ∎

Example 5.4.3 The eigenvalues of the following matrix

$$\begin{bmatrix} 3 & 2 & 9 \\ 7 & 5 & 13 \\ 6 & 17 & 19 \end{bmatrix}$$

are given by solving the cubic equation

A. $\lambda^3 - 27\lambda^2 + 167\lambda - 285$

B. $\lambda^3 - 27\lambda^2 - 122\lambda - 313$

C. $\lambda^3 + 27\lambda^2 + 167\lambda + 285$

D. $\lambda^3 + 23.23\lambda^2 - 158.3\lambda + 313$

Solution : *To find the equations of* $A = \begin{bmatrix} 3 & 2 & 9 \\ 7 & 5 & 13 \\ 6 & 17 & 19 \end{bmatrix}$ *we solve*

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 3 & 2 & 9 \\ 7 & 5 & 13 \\ 6 & 17 & 19 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) &= 0 \\ \det\left(\begin{bmatrix} 3-\lambda & 2 & 9 \\ 7 & 5-\lambda & 13 \\ 6 & 17 & 19-\lambda \end{bmatrix}\right) &= 0 \end{aligned}$$

Using the cofactor method with Row1

$$(3-\lambda)\begin{vmatrix} 5-\lambda & 13 \\ 17 & 19-\lambda \end{vmatrix} - 2\begin{vmatrix} 7 & 13 \\ 6 & 19-\lambda \end{vmatrix} + 9\begin{vmatrix} 7 & 5-\lambda \\ 6 & 17 \end{vmatrix} = 0$$

we have

$$\begin{aligned} (3-\lambda)((5-\lambda)(19-\lambda) - 13x17) - 2(7(19-\lambda) - 13x6) + 9(7x17 - 6(5-\lambda)) &= 0 \\ \lambda^3 - 27\lambda^2 - 122\lambda - 313 &= 0 \end{aligned}$$

such that, the solution is B. ■

Example 5.4.4 The eigenvalues of a 4×4 matrix $[A]$ are given as 2, -3, 13, and 7. The $|\det(A)|$ then is

A. 546

B. 19

C. 25

D. cannot be determined

Solution : *If* $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n$ *are the eigenvalues of a* $n \times n$ *matrix* A*, then*

$$\begin{aligned} |\det(A)| &= |\lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n| \\ &= |\lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4| \\ &= |2 \times (-3) \times 13 \times 7| \\ &= 546 \end{aligned}$$

Therefore, the solution is A. ■

Example 5.4.5 If one of the eigenvalues of $A_{n\times n}$ is zero, it implies

A. The solution to $AX = C$ system of equations is unique

B. The determinant of A is zero

C. The solution to $AX = 0$ system of equations is trivial

D. The determinant of A is nonzero

Solution : *For a $n \times n$ matrix A with $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n$ as the eigenvectors*

$$|det(A)| = |\lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n|$$

Since one of the eigenvalues is zero,

$$|det(A)| = 0$$

$$|det(A)| = 0$$

such that the solution is B ■

Example 5.4.6 Given that matrix $[A] = \begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -3 \end{bmatrix}$ has an eigenvalue value of 4 with the corresponding eigenvectors of $[X] = \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$,then $[A]^5[X]$ is

A $\begin{bmatrix} -18 \\ -16 \\ 4 \end{bmatrix}$ B $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$ C $\begin{bmatrix} -4608 \\ -4096 \\ 1024 \end{bmatrix}$ D $\begin{bmatrix} -0.004395 \\ -0.003906 \\ 0.0009766 \end{bmatrix}$

Solution : *If for a $n \times n$ matrix A, such that λ is an eigenvalue and X is the corresponding eigenvector, then*

$$\begin{aligned} A^m X &= \lambda^m X \\ A^5 X &= \lambda^5 X = 4^5 \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4608 \\ -4096 \\ 1024 \end{bmatrix} \end{aligned}$$

such that the solution is C. ■

Example 5.4.7 If $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix}$, the eigenvalue corresponding to the eigen vector is

A. 1 B. 4 C. -4.5 D. 6

Example 5.4.8 Which of the following matrices is diagonalizable?

A. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ C. $\begin{bmatrix} 2 & -3 \\ 3 & 8 \end{bmatrix}$ D. $\begin{bmatrix} -1 & -2 \\ 2 & -5 \end{bmatrix}$

Example 5.4.9 Let A and B be similar. Let $B = P^{-1}AP$. If $B^k = \begin{bmatrix} 0 & 2^k \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. What is A^k?

A. $\begin{bmatrix} 0 & 2^k \\ 0 & 0 \end{bmatrix}$ B. $\begin{bmatrix} 3\cdot 2^k & -2^{k+2} \\ 9\cdot 2^k & -3\cdot 2^{k+1} \end{bmatrix}$ C. $\begin{bmatrix} 2^k & -2^{k+2} \\ 2^k & -2^{k+2} \end{bmatrix}$ D. $\begin{bmatrix} 3\cdot 2^k & 2^{k+2} \\ 9\cdot 2^k & 2^{k+2} \end{bmatrix}$

Example 5.4.10 The eigenvalues of $\begin{bmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 0 & 0 & 37 \end{bmatrix}$ are

A. $-19,\ 5,\ 37$ B. $19, -5, -37$ C. $2, -3, 7$ D. $3, -5, 37$

Example 5.4.11 Give an example of a similar matrix to B of order 5 given

$$B = \begin{bmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{bmatrix}$$

Solution : *Check that P is **nonsingular** and then compute*

$$A = P^{-1}BP$$
$$= \begin{bmatrix} 10 & 1 & 0 & 2 & -5 \\ -1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & 0 & 1 \\ -4 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -10 & -27 & -29 & -80 & -25 \\ -2 & 6 & 6 & 10 & -2 \\ -3 & 11 & -9 & -14 & -9 \\ -1 & -13 & 0 & -10 & -1 \\ 11 & 35 & 6 & 49 & 19 \end{bmatrix}$$

So by this construction, we know that A and B are similar. ■

Example 5.4.12 Compute the eigenvalues and associated eigenvectors of this matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 2;\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\lambda_3 = 3;\ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

Example 5.4.13 Compute the eigenvalues and associated eigenvectors of this matrix

$$A = \begin{bmatrix} -7 & -2 & 10 \\ -3 & 2 & 3 \\ -6 & -2 & 9 \end{bmatrix}$$

$$\lambda_1 = -1; \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \lambda_2 = 2; \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \lambda_3 = 3; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Example 5.4.14 Let $A \in 3 \times 3$ matrix have characteristic polynomial

$$(\lambda + 1)^2(\lambda - 3)$$

One of the eigenvalues has two eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. The other eigenvalue has corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Determine A.

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

To have

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 5.4.15 Let A be the following matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

Determine the eigenvalues and eigenvectors over $\mathbb{C}$.

The eigenvalues of A over $\mathbb{C}$ are the roots, in $\mathbb{C}$, of its characteristic polynomial

$$P(\lambda) = \det(A - \lambda I) = (1 - \lambda)(\lambda^2 - 2\lambda + 3)$$

We thus obtain the three roots $\lambda_1 = 1$, $\lambda_2 = 1 + i\sqrt{2}$ and $\lambda_3 = \overline{\lambda_2} = 1 - i\sqrt{2}$. An eigenvector corresponding to λ_1 will be a basis of the null space of $A - I$ and can be taken as

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

An eigenvector corresponding to λ_2 satisfies $\left(A - \left(1 + i\sqrt{2}\,\right) I\right) \boldsymbol{v}_2 = \mathbf{0}$ and can be taken to be

$$\boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 + i\sqrt{2} \end{bmatrix}$$

An eigenvector corresponding to λ_3 *satisfies* $\left(A - \left(1 - i\sqrt{2}\,\right) I\right) \boldsymbol{v}_3 = \mathbf{0}$ *and can be taken as*

$$\boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 - i\sqrt{2} \end{bmatrix}$$

We note that in this case, where the matrix A *is a real-valued matrix, that* $\boldsymbol{v}_3 = \overline{\boldsymbol{v}_2}$*. In fact, if* A *is a real matrix and* $\boldsymbol{v}$ *is the eigenvector associated with* $\lambda \in \mathbb{C}$*, then*

$$\overline{A\boldsymbol{v}} = A\overline{\boldsymbol{v}} = \overline{\lambda \boldsymbol{v}} = \overline{\lambda}\overline{\boldsymbol{v}}$$

which shows that the eigenvector $\boldsymbol{w}$ *corresponding to* $\overline{\lambda}$ *is* $\boldsymbol{w} = \boldsymbol{v}$*.*

Example 5.4.16

1.) Is $\lambda = 4$ an eigenvalue of $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If it is, find an associated eigenvector.

The eigenvalues of A *are the solutions to the characteristic equation* $\det(A - \lambda I) = 0$*, which gives*

$$-\lambda^3 + 11\lambda^2 - 32\lambda + 16 = 0.$$

Since $\lambda = 4$ *is a solution of this equation, it is indeed an eigenvalue of* A*. Its eigenvector is a nontrivial solution of* $A\boldsymbol{x} = 4\boldsymbol{x}$*, or* $(A - 4I)\boldsymbol{x} = \mathbf{0}$*, which gives a nonzero multiple of*

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

2.) Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$?

For $\boldsymbol{x}$ *to be an eigenvector of* A*, it must satisfy the equation* $A\boldsymbol{x} = \lambda\boldsymbol{x}$ *for some* λ*. Here, the product* $A\boldsymbol{x}$ *is*

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, $\boldsymbol{x}$ *is an eigenvector corresponding to the eigenvalue* $\lambda = 0$

Example 5.4.17 Let us consider the following matrices

$$A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Determine the characteristic polynomials of A and B, their eigenvalues and their associated eigenvectors.

- *The eigenvalues of A are the roots of its characteristic polynomial*

$$P(\lambda) = \det(A - \lambda I) = \lambda^2 - 6\lambda + 8.$$

We find its two roots $\lambda_1 = 2$ and $\lambda_2 = 4$. The eigenvector corresponding to $\lambda = 2$ satisfies $A\boldsymbol{v}_1 = 2\boldsymbol{v}_1$ and is given by

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The eigenvector corresponding to $\lambda = 4$ satisfies $A\boldsymbol{v}_2 = 4\boldsymbol{v}_2$ and is given by

$$\boldsymbol{v}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

- *We compute* $\det(B - \lambda I)$ *by a cofactor expansion along the third row of $B - \lambda I$ giving*

$$P(\lambda) = \det(B - \lambda I) = (2 - \lambda)(-1 - \lambda)(4 - \lambda).$$

Therefore, the eigenvalues are $\lambda = 2, \lambda = -1$ and $\lambda = 4$. The eigenvector corresponding to $\lambda = 2$ satisfies $B\boldsymbol{v}_1 = 2\boldsymbol{v}_1$ and is given by

$$\boldsymbol{v}_1 = \begin{bmatrix} 1/3 \\ 1/2 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to $\lambda = -1$ satisfies $B\boldsymbol{v}_2 = -\boldsymbol{v}_2$ and is given by

$$\boldsymbol{v}_2 = \begin{bmatrix} 5/3 \\ 1 \\ 0 \end{bmatrix}.$$

The eigenvector corresponding to $\lambda = 4$ satisfies $B\boldsymbol{v}_3 = 4\boldsymbol{v}_3$ and is given by

$$\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Example 5.4.18 Consider

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix},\ B = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix},\ C = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix},\ D = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \text{ and } E = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}.$$

- For each matrix find its eigenvalues and the corresponding eigenvectors.
- Which of these matrices are diagonalizable?

Solutions to Example 5.4.18

- *The characteristic equation*
$$\det(A - \lambda I)x = \lambda^2 - 8\lambda + 16 = 0$$
has solutions $\lambda_1 = \lambda_2 = 4$. *Therefore* A *has only* 4 *as eigenvalue with algebraic multiplicity* 2. *The associated eigenspace,* $Nul\ (A - 4I)$, *is generated by*
$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
and has dimension 1. *According to theorem,* A *is not diagonalizable.*

- B *has two eigenvalues:* 4, *with algebraic multiplicity* 2, *whose eigenspace has dimension* 1 *generated by*
$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
and 5 *with eigenvector*
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$
This shows, according to theorem, that B *is not diagonalizable.*

- $\mathcal{C}$ *has three different (distinct) eigenvalues,* $1, 2$ *and* 3. *This shows, according to theorem, that* $\mathcal{C}$ *is in fact diagonalizable. The eigenvectors are*
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \textit{for } \lambda = 1,$$
$$\begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix} \quad \textit{for } \lambda = 2,$$
$$\begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix} \quad \textit{for } \lambda = 3.$$

- D *has eigenvalues* $8, 2, 2$ *and three linearly independent eigenvectors given as the columns of* $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, *therefore* D *is diagonalizable.*

- E *has* 5 *as an eigenvalue with algebraic multiplicity* 2 *and only one eigenvector* $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, *hence* E *is not diagonalizable.*

Example 5.4.19 Show that $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$ is not diagonalizable.

[$\lambda_1 = \lambda_2 = \lambda_3 = 5$ and eigenvector of only one multiplicity i.e $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Example 5.4.20 Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

1.) Find the eigenvalues and their corresponding vectors.

$\lambda_1 = \lambda_2 = 1, \lambda_3 = 3.$

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} s \\ -2r \\ r \end{bmatrix} = r\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \lambda_1 = \lambda_2 = 1 \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{for } \lambda_3 = 3
\end{aligned}$$

Thus $(0, -2, 1), (1, 0, 0), (1, 0, 1)$ are the eigenvectors.

2.) Determine the diagonal matrix. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

3.) Determine the invertible matrix P which diagonalize A. $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

4.) Use the result to compute A^2. $A^2 = PD^2P^{-1} = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 1 & 0 \\ 0 & 4 & 9 \end{bmatrix}$

Example 5.4.21 Diagonalise the following matrices

1.) $\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ 2.) $\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ 3.) $\begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{aligned}
D_{(1)} &= \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \\
D_{(2)} &= \begin{bmatrix} -1 & -1/13 & -1 \\ -3/2 & -6/13 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} -16/21 & -2/7 & -4/21 \\ 13/12 & 0 & 13/12 \\ -9/28 & 2/7 & 3/28 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} \\
D_{(3)} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}
\end{aligned}$$

Example 5.4.22 The matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not diagonalisable over $\mathbb{F}$.

Example 5.4.23 If we work over $\mathbb{R}$ the the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has no eigenvectors and so, in particular we can not find an invertible P such that $P^{-1}AP$ is diagonal.

Example 5.4.24 Let A denote the matrix

$$A = \begin{bmatrix} -\dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \\ \\ \dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \end{bmatrix}$$

Find the characteristic polynomial of A, and use it to find all eigenvalues of A or to show that none exist.

The characteristic polynomial of A is $t^2 + \sqrt{3}t + 1$, so this polynomial has no real roots, and hence A has no eigenvalues.

Example 5.4.25

1.) Let A denote the matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

Find the eigenvalues of A.

The eigenvalues are 0 (with multiplicity 1) and 5 (with multiplicity 2). (Note: Even if I don't ask explicitly, you should always give the algebraic multiplicities of eigenvalues.)

2.) Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for A.

For $\lambda = 0$, the kernel is spanned by $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

We normalize to get a unit vector: $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Simillary for $\lambda = 5$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(It happens that these vectors are already orthogonal to each other. If they weren't, we would have to use Gram-Schmidt to find an orthonormal basis for E_5.) Combining the bases for E_0 and E_5, we get an orthonormal basis for $\mathbb{R}^3$.

3.) Find a 3×3 orthogonal matrix S and a 3×3 diagonal matrix D such that $A = SDS^T$.

4.) For any integer t, write an explicit formula for A^t.

$$A^t = \left(SDS^{-1}\right)^t$$

Since S is orthogonal, $S^{-1} = S^T$.

$$\begin{aligned} A^t &= \left(SDS^{-1}\right)^t = SD^tS^{-1} = SD^tS^T \\ &= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5^t & 0 \\ 0 & 0 & 5^t \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5^{t-1} & 0 & -2 \cdot 5^{t-1} \\ 0 & 5^t & 0 \\ -2 \cdot 5^{t-1} & 0 & 4 \cdot 5^{t-1} \end{bmatrix} \end{aligned}$$

As a sanity check, you should verify that plugging in $t = 1$ gives A.

Example 5.4.26 *True or False?* If A is an orthogonal matrix, there exists an orthonormal basis of eigenvectors for A.

False. The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *is orthogonal, but it doesn't have any eigenvectors at all, let alone an orthonormal basis of them.*

Example 5.4.27 *True or False?* If A and B are matrices whose eigenvalues, counted with their algebraic multiplicities, are the same, then A and B are similar.

False. The matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

has 1 as an eigenvalue with eigenvalue 2, as does $B = I_2$, but A is not similar to B. (If it were, we would have $A = SBS^{-1} = SI_2S^{-1} = SS^{-1} = I_2$, which isn't true.)

Example 5.4.28 This problem is about the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

Find the eigenvalues of A^TA and also of AA^T . For both matrices find a complete set of orthonormal eigenvectors.

$$\lambda_1 = 70, \lambda_2 = 0 \;:\; x_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \textit{and} \quad x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 70, \lambda_2 = 0, \lambda_3 = 0 \;:\; x_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \textit{and} \quad x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad x_3 = \frac{1}{\sqrt{70}} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$$

Exercise 5.4.1 Let the matrix

$$A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

act on $\mathbb{C}^2$.

1.) Find the eigenvalues of A as well as a basis of each eigenspace in $\mathbb{C}^2$.

The eigenvalues are the roots of $\det(A - \lambda I) = 0$, *which in this case are found by*

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \lambda_2 - 4\lambda + 13 = 0 \quad \Rightarrow \quad \lambda_\pm = 2 \pm 3i.$$

A basis for the eigenspace corresponding to the eigenvalue $\lambda_\pm$ *is obtained by solving* $(A - \lambda_\pm I)\boldsymbol{x} = \mathbf{0}$, *which gives*

$$\boldsymbol{v}_\pm = \begin{bmatrix} 1 \pm 3i \\ 2 \end{bmatrix}$$

2.) Find an invertible matrix P and a matrix C of the form

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

such that $A = PCP^{-1}$.

if A *is a real* 2×2 *matrix with complex eigenvalue* $\lambda = a - bi$ *(with* $b \neq 0$*) and associated eigenvector* $\boldsymbol{v}$, *then*

$$A = PCP^{-1} \qquad \textit{where} \;\; P = [\textit{Re}\; \boldsymbol{v} \;\; \textit{Im}\; \boldsymbol{v}] \;\; \textit{and} \;\; C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

In this case, we have

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \qquad (\textit{for} \;\; \lambda = 2 - 3i),$$
$$P = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \qquad (\textit{for} \;\; \lambda = 2 + 3i).$$

Remark 5.4.1 Recall that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

All real numbers $\mathbb{R}$ are complex numbers $\mathbb{C}$ but a complex number is not a real number.

Example 5.4.29 Let

$$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

1.) Verify that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are eigenvectors of A.

We have $A\boldsymbol{v}_1 = 10\boldsymbol{v}_1$ *and* $A\boldsymbol{v}_2 = \boldsymbol{v}_2$, *therefore* $\boldsymbol{v}_1$ *and* $\boldsymbol{v}_2$ *are eigenvectors of* A.

2.) Find the orthogonal diagonalization of A.

As A is symmetric, A is diagonalizable, and moreover there exists an orthogonal matrix P, so that $A = PDP^T$. We remark that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are two eigenvectors which are orthogonal. The third eigenvector is orthogonal to the two preceding ones. Since we are in $\mathbb{R}^3$, the orthogonal complement of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is of dimension 1 and it suffices to find a vector

$$\boldsymbol{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

so that $\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 0$ and $\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0$. Therefore we solve :

$$\begin{bmatrix} -2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The system has a free variable, and one can write the solution as :

$$\boldsymbol{v}_3 = z \begin{bmatrix} 1/4 \\ -1/4 \\ 1 \end{bmatrix}$$

In the following one verifies that $A\boldsymbol{v}_3 = \lambda_3 \boldsymbol{v}_3$, with $\lambda_3 = 1$. (One could also determine $\boldsymbol{v}_3$ using the vector product $\boldsymbol{v}_1 \times \boldsymbol{v}_2$.)

3.) Deduce the spectral decomposition of A.

Example 5.4.30 Let A, B and C be the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- For each of them determine the characteristic polynomial, eigenvalues, and the associated eigenvectors in $\mathbb{R}$ and/or $\mathbb{C}$.
- Generalize the notion of diagonalization for a set different from $\mathbb{R}$ (for example $\mathbb{C}$): a matrix A is diagonalizable if $A = PDP^{-1}$ where D is a diagonal matrix, and the entries in all these matrices lie in the considered set ($\mathbb{C}$ in the example). For each matrix A, B and C, indicate whether they are diagonalizable over $\mathbb{R}$ and/or $\mathbb{C}$.

Solutions to Example 5.4.30

- *The characteristic polynomial of A is*

$$P(\lambda) = \det(A - \lambda I) = (\lambda - 1)[-\lambda^2 + \lambda + 1].$$

with the three roots $\lambda_1 = 1, \quad \lambda_2 = (1+\sqrt{5})/2, \quad \lambda_3 = (1-\sqrt{5})/2, \quad \lambda_i \in \mathbb{R}, \quad \lambda_i \in \mathbb{C}$. The eigenvector corresponding to λ_1 is given by

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The eigenvector corresponding to λ_2 *is given by*

$$\boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -2 \\ 1-\sqrt{5} \end{bmatrix}.$$

The eigenvector corresponding to λ_3 *is given by*

$$\boldsymbol{v}_3 = \begin{bmatrix} -2 \\ -2 \\ 1+\sqrt{5} \end{bmatrix}.$$

The matrix A *is therefore diagonalizable in* $\mathbb{R}$ *and in* $\mathbb{C}$.

- *The characteristic polynomial of* B *is*

$$P(\lambda) = \det(A - \lambda I) = (1-\lambda)[\lambda^2 + 1],$$

with the three roots $\lambda_1 = 1,\quad \lambda_2 = i,\quad \lambda_3 = -i,\ \lambda_1 \in \mathbb{R},\ \lambda_1 \in \mathbb{C},\ \lambda_2 \in \mathbb{C},\ \lambda_3 \in \mathbb{C}$. *The eigenvector corresponding to* λ_1 *is given by*

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to λ_2 *is given by*

$$\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}.$$

The eigenvector corresponding to λ_3 *is given by*

$$\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}.$$

The matrix B *is therefore diagonalizable in* $\mathbb{C}$ *but not in* $\mathbb{R}$.

- *The characteristic polynomial of* C *is*

$$P(\lambda) = \det(A - \lambda I) = (1-\lambda)^2,$$

with the roots $\lambda = 1, \lambda \in \mathbb{R}, \lambda \in \mathbb{C}$. *Therefore* C *has only one eigenvalue with algebraic multiplicity* 2. *The eigenvector corresponding to* λ *is given by*

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

thus the geometric multiplicity is 1. *The matrix* C *is not diagonalizable in* $\mathbb{R}$ *nor in* $\mathbb{C}$ *according to theorem.*

Example 5.4.31 *True* or *False*? All matrices are assumed to be of dimension $n \times n$ unless specified otherwise.

1) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some vector $\boldsymbol{x}$, then λ is an eigenvalue of A.
2) A matrix A is not invertible if and only if 0 is an eigenvalue of A.
3) A number c is an eigenvalue of A if and only if the equation $(A - cI)\boldsymbol{x} = 0$ has a nontrivial solution.
4) Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.
5) To find the eigenvalues of A, reduce A to row echelon form.
6) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some scalar λ, then $\boldsymbol{x}$ is an eigenvector of A.
7) If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
8) A steady-state vector for a stochastic matrix is actually an eigenvector.
9) The eigenvalues of a matrix are on its main diagonal.
10) An eigenspace of A is a null space of a certain matrix.
11) If A is invertible and 1 is an eigenvalue for A, then 1 is also an eigenvalue of A^{-1}.
12) If A is row equivalent to the identity matrix I, then A is diagonalizable.
13) If A contains a row or column of zeros, then 0 is an eigenvalue of A.
14) Each eigenvalue of A is also an eigenvalue of A^2.
15) Each eigenvector of A is also an eigenvector of A^2.
16) Each eigenvector of an invertible matrix A is also an eigenvector of A^{-1}.
17) Eigenvalues must be nonzero scalars.
18) Eigenvectors must be nonzero vectors.
19) Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.
20) Similar matrices always have exactly the same eigenvalues.
21) Similar matrices always have exactly the same eigenvectors.
22) The sum of two eigenvectors of a matrix A is also an eigenvector of A.
23) The eigenvalues of an upper triangular matrix A are exactly the nonzero entries on the diagonal of A.
24) The matrices A and A^T have the same eigenvalues, counting multiplicities.
25) If a 5×5 matrix A has fewer than 5 distinct eigenvalues, then A is not diagonalizable.
26) There exists a 2×2 matrix that has no eigenvectors in $\mathbb{R}^2$.
27) If A is diagonalizable, then the columns of A are linearly independent.
28) A nonzero vector cannot correspond to two different eigenvalues of A.

Solutions to Exercise 5.4.31

1) *False*	8) *True*	15) *True*	22) *False*
2) *True*	9) *False*	16) *True*	23) *False*
3) *True*	10) *True*	17) *False*	24) *True*
4) *True*	11) *True*	18) *True*	25) *False*
5) *False*	12) *False*	19) *False*	26) *True*
6) *False*	13) *True*	20) *True*	27) *False*
7) *False*	14) *False*	21) *False*	28) *True*

Exercise 5.4.2 Which of the following matrices are diagonalizable? In case of diagonalizable matrices, compute its eigenvalues and their corresponding eigenvectors.

1.) $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$ 2.) $\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Exercise 5.4.3 Find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$
$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

Example 5.4.32 Compute a matrix T similar to $S = \begin{bmatrix} 0 & -1 & -2 \\ 2 & 3 & 2 \\ 4 & 5 & 2 \end{bmatrix}$ using a non-singular matrix $P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$T = P^{-1}SP = \begin{bmatrix} 1 & -3 & -6 \\ 0 & 2 & 0 \\ -1/3 & 3 & 2 \end{bmatrix}$

Example 5.4.33 Given $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$. Find the matrix B similar to A?

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 3/7 & 1/7 \\ -1/7 & 2/7 \end{bmatrix} \Rightarrow B = P^{-1}AP = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

Example 5.4.34 Given the matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$, find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix?

$P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

Example 5.4.35 Find the product of Eigen values of a matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 6 & 0 \\ 3 & 1 & 2 \end{bmatrix}$.

Solution : *According to the property of Eigen values, the product of the Eigen values of a given matrix is equal to the determinant of the matrix*

$$|A| = 1(12 - 0)–2(0) + 4(8) = -60.$$

■

Example 5.4.36 If one of the eigenvalues of $A_{n\times n}$ is zero, it implies

A. The solution to $AX = b$ system of equations is unique

B. The determinant of A is zero

C. The solution to $AX = 0$ system of equations is trivial

D. The determinant of A is nonzero

Solution : *For an $n \times n$ matrix A with $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n$ as the eigenvectors*

$$|\det(A)| = |\lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n|$$

Since one of the eigenvalues is zero,

$$\det(A) = 0$$

Therefore, the solution is B). ■

Example 5.4.37 Given that matrix $A = \begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -3 \end{bmatrix}$ has an eigenvalue value of 4 with the corresponding eigenvectors of $X = \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$,then A^5X is

A. $\begin{bmatrix} -18 \\ -16 \\ 4 \end{bmatrix}$ B. $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$ C. $\begin{bmatrix} -4608 \\ -4096 \\ 1024 \end{bmatrix}$ D. $\begin{bmatrix} -0.004395 \\ -0.003906 \\ 0.0009766 \end{bmatrix}$

Solution : *If for an $n\times n$ matrix A, λ is an eigenvalue and X is the corresponding eigenvector, then*

$$A^mX = \lambda^mX$$

Therefore

$$A^5X = \lambda^5X = 4^5 \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4608 \\ -4096 \\ 1024 \end{bmatrix}$$

Such that, the solution is C.) ■

5.4.1 Proofs

Example 5.4.38 Let A and B be two real $n \times n$ matrices. Show that AB and BA have the same eigenvalues.

First we note that it suffices to show that if λ is an eigenvalue of AB then λ is an eigenvalue of BA, because if this implication is true, the inverse implication is then true by symmetry (renaming $B = A$ and $A = B$). Let us consider two different cases.

- $\lambda = 0$.
  $\lambda = 0$ is an eigenvalue of AB if and only if $\det(AB) = 0 = \det(A)\det(B)$, *which implies that either* $\det(A) = 0$ *or* $\det(B) = 0$. *But then we get* $\det(BA) = 0$ *and this is true if and only if $\lambda = 0$ is an eigenvalue of BA.*

- $\lambda \neq 0$.
  If $\lambda \neq 0$ is an eigenvalue of AB, then

$$\exists\ \boldsymbol{v} \neq 0 \quad \textit{such that} \quad AB\boldsymbol{v} = \lambda\boldsymbol{v} \neq 0. \tag{5.23}$$

  Calling $\boldsymbol{w} = B\boldsymbol{v}$, then $\boldsymbol{w} \neq 0$ because $\boldsymbol{v} \neq 0$ and $\boldsymbol{v} \notin$ Null (B) by hypothesis (if $\boldsymbol{v} \in$ Null (B) then $AB\boldsymbol{v} = 0$ which contradict the hypothesis that $\boldsymbol{v}$ is an eigenvector associated to a non-zero eigenvalue λ). But then

$$BA\boldsymbol{w} = B(AB\boldsymbol{v}) = B(\lambda\boldsymbol{v}) = \lambda Bv = \lambda\boldsymbol{w} \tag{5.24}$$

  by linearity of B, which proves the statement.

Example 5.4.39 1.) Show that if λ is an eigenvalue of an invertible matrix A then λ^{-1} is an eigenvalue of A^{-1}. (Hint: take a nonzero vector $\boldsymbol{x}$ verifying $A\boldsymbol{x} = \lambda\boldsymbol{x}$.)

If λ is an eigenvalue of A, there exists a non-zero $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$: We deduce

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} \Rightarrow \boldsymbol{x} = \lambda A^{-1}\boldsymbol{x} \Rightarrow A^{-1}\boldsymbol{x} = \frac{1}{\lambda}\boldsymbol{x}.$$

Which means that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

2.) Show that A and A^T have the same eigenvalues. (Hint: find a relation between $\det(A-\lambda I)$ and $\det(A^T - \lambda I)$.) Does there exist a simple relation between the eigenvectors of A and those of A^T?

Since $\det(A - \lambda I) = \det(A - \lambda I)^T$ *and* $I = I^T$, *we get*

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I).$$

Then the characteristic polynomials of A and A^T are the same and these two matrices have the same eigenvalues.

3.) Let A be an $n \times n$ matrix where the sum of the entries in each row is the same number r. Show that r is an eigenvalue of A. (Hint: find an eigenvector.) Show that if the entries in A are positive, the absolute value of any other eigenvalue is less than r.

Let $\boldsymbol{v}$ be the vector in $\mathbb{R}^n$ with all its entries 1. We have $A\boldsymbol{v} = r\boldsymbol{v}$ and r is an eigenvalue of A. Let λ be an eigenvalue of A; there is a non-zero vector v such that: $A\boldsymbol{v} = \lambda\boldsymbol{v}$:

$$\begin{aligned} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n &= \lambda v_1 \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n &= \lambda v_2 \\ \cdots &= \cdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n &= \lambda v_n \end{aligned}$$

Let M, where $1 \leq M \leq n$, be such that $v_M = max(v_i)$. The M^{th} row from the system above can be written as

$$\lambda = a_{M1}\frac{v_1}{v_M} + a_{M2}\frac{v_2}{v_M} + \cdots + a_{Mn}\frac{v_n}{v_M}.$$

As $\frac{v_i}{v_M} \leq 1$ for every i, we get, since $a_{Mi} \geq 0, \forall i$:

$$\begin{aligned} |\lambda| &= \left| a_{M1}\frac{v_1}{v_M} + a_{M2}\frac{v_2}{v_M} + \cdots + a_{Mn}\frac{v_n}{v_M} \right| \\ &\leq a_{M1}\left|\frac{v_1}{v_M}\right| + a_{M2}\left|\frac{v_2}{v_M}\right| + \cdots + a_{Mn}\left|\frac{v_n}{v_M}\right| \\ &\leq a_{M1} + a_{M2} + \cdots + a_{Mn} = r, \end{aligned}$$

as we wanted to show.

4.) Let A be an $n \times n$ matrix where the sum of the entries in each column is the same number c. Show that c is an eigenvalue of A. (Hint: use the results from the two previous questions.)

The matrix A^T satisfies the condition that the sum of the entries in each of its rows is c. By last item, c is an eigenvalue of A^T and by the second one, c is an eigenvalue of A.

Example 5.4.40 Let $A = PDP^{-1}$ with $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Show that

$$A^2 = PD^2P^{-1},$$

then compute A^5 and A^{10}.

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}.$$

Since D is diagonal, its square is

$$D^2 = \begin{bmatrix} 2^2 & 0 \\ 0 & 1^2 \end{bmatrix}$$

More generally, the power of a diagonalizable matrix A is

$$A^n = PD^nP^{-1}$$

for every $n = 1, 2, 3, \cdots$. This is easy to show by induction:

By hypothesis, the property is satisfied for $n = 1$.

The following line shows that if the property is true for n it is also true for $n + 1$:

$$A^{n+1} = PD^nP^{-1}PDP^{-1} = PD^nDP^{-1} = PD^{n+1}P^{-1}.$$

Based on $n = 1$ we see that the property is true for $n = 2, n = 3$, etc.

Example 5.4.41 Let A be an $n \times n$ matrix with n different real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n$ (repeating eigenvalues according to their algebraic multiplicities). Recall that $\lambda \mapsto \det(A - \lambda I)$ is a polynomial function of degree n.

1.) Show that $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$.

When A has n different eigenvalues, it is diagonalizable and $A = PDP^{-1}$ with D diagonal with diagonal entries $\lambda_1, \cdots, \lambda_n$. For such a matrix A, one gets

$$\det(A - \lambda I) = \det(PDP^{-1} - \lambda I) = \det(P(D - \lambda I)P^{-1}) = \det(D - \lambda I),$$

which is easily seen to be $(\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$.

This is still true when the matrix has repeated eigenvalues and their are counted with multiplicities (the statement of the exercise is not clear, sorry!) but you are not supposed to solve that case.

2.) Explain why $\det A$ is the product of the n eigenvalues of A.

The identity

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

holds for any value of λ. In particular, for $\lambda = 0$, we get

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Example 5.4.42 Find the eigenvalues and their corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

Solution : *Find Eigenvalues*

We first find the matrix $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & -\lambda & 0 \\ -2 & 2 & 1-\lambda \end{bmatrix}$$

Write the characteristic equation.

$$Det(A - \lambda I) = (1-\lambda)(-\lambda(1-\lambda)) - 1(2-2\lambda) = 0$$

factor and rewrite the equation as

$$(1-\lambda)(\lambda-2)(\lambda+1) = 0$$

which gives 3 solutions

$$\lambda = -1, \lambda = 1, \lambda = 2$$

Find Eigenvectors

- *Eigenvectors for $\lambda = -1$ Substitute λ by - 1 in the matrix equation $(A - \lambda I)X = 0$ with*

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

  Row reduce to echelon form gives

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

  The solutions to the above system and are given by

$$x_3 = t, x_2 = -t/2, x_1 = t/2, t \in \mathbb{R}$$

  Hence the eigenvector corresponding to the eigenvalue $\lambda = -2$ is given by

$$X = t \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

- *Eigenvectors for $\lambda = 1$*

  Substitute λ by 1 in the matrix equation $(A - \lambda I)X = 0$.

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduce to echelon form gives

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to the above system and are given by

$$x_3 = 0, x_2 = t, x_1 = t, t \in \mathbb{R}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = 1$ *is given by*

$$X = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- *Eigenvectors for* $\lambda = 2$

  Substitute λ *by* 1 *in the matrix equation* $(A - \lambda I)X = 0$.

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduce to echelon form gives

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to the above system and are given by

$$x_3 = t, x_2 = -t/2, x_1 = -t, t \in \mathbb{R}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = 2$ *is given by*

$$X = t \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

■

Example 5.4.43

1.) What is meant by a matrix B being similar to A.

Solution : *The matrix B is said to be similar to another matrix A if $\exists$ a non singular matrix P such that*

$$B = P^{-1}AP. \tag{5.25}$$

$\blacksquare$

2.) Find a matrix B similar to A given

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Solution :

$$\begin{aligned} P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \Rightarrow \quad P^{-1} &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ \Rightarrow \quad B \quad = \quad P^{-1}AP &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \end{aligned}$$

$\blacksquare$

3.) Find the diagonal matrix of A.

Solution :

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$\blacksquare$

4.) Determine A^5.

$$\begin{aligned} A^5 &= PD^5P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^5 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^5 & 0 \\ 0 & 3^5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -179 & 211 \\ -422 & 454 \end{bmatrix} \end{aligned}$$

5.5 Eigen Values and Eigen Vectors Chapter Exercises

Exercise 5.5.1 Given matrix $A = \begin{bmatrix} 0 & 2 & 1 \\ 16 & 4 & -6 \\ -16 & 4 & 10 \end{bmatrix}$, find

1.) the eigenvalues of A

2.) their corresponding eigenvectors

3.) an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Exercise 5.5.2 Given that $A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$

1.) Find the eigenvalues of the matrices A, B and $A + B$.

2.) Are the eigenvalues of $A + B$ equal to eigenvalues of A plus eigenvalues of B?

3.) Are there any special matrices in which the eigenvalues of $A+B$ are sums of the eigenvalues of independent matrices?

Exercise 5.5.3 Find the eigenvalues of A and B and AB and BA for:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

1.) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

2.) Are the eigenvalues of AB equal to the eigenvalues of BA?

Exercise 5.5.4 Suppose λ is an eigenvalue of A. Prove that the eigenspace

$$E_\lambda = \{x \in \mathbb{R}^n : Ax = \lambda x\}$$

together with the zero vector is a subspace of $\mathbb{R}^n$.

Exercise 5.5.5 Suppose that A is a real diagonalizable matrix and that all the eigenvalues of A are non-negative. Prove that there is a matrix B such that $B^2 = A$. [Choose $D = diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \ldots, \sqrt{\lambda_n})$.]

Exercise 5.5.6 Show that the following matrices are not diagonalizable

(a) $\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

Exercise 5.5.7 If x is an eigenvector of matrix A. Prove that any other scalar multiple of x is also an eigenvector of A.

Exercise 5.5.8 Show that if A is both diagonalizable and invertible, then so is A^{-1}.

Exercise 5.5.9 Let A be a 5×5 matrix with two eigenvalues. If one eigenspace is three dimensional, find the dimension of the other eigenvalue for A to be diagonalizable.

Exercise 5.5.10 Find which of the matrices are diagonalizable. For those which are find P, and D such that $P^{-1}AP = D$ (D is a diagonal matrix).

1.) $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ 2.) $\begin{bmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \end{bmatrix}$ 3.) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ 4.) $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Exercise 5.5.11 Find basis for eigenspaces associated with each eigenvalue of the following matrices.

1.) $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 2.) $\begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 3.) $\begin{bmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}$

Exercise 5.5.12 Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$ find A^9 [Hint P such that $P^{-1}AP = D$) $A^9 = PD^9P^{-1}$]

Exercise 5.5.13 Given

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

Show that if $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then $B = P^{-1}AP$

Exercise 5.5.14 Let A be an $n \times n$ matrix. One of the the following statements in not equivalent to the statement:

The number 0 is not an eigenvalue of A.

Which one?

A. The homogeneous system $A\vec{x} = \vec{0}$ has a non-trivial solution.

B. A is invertible.

C. The determinant of A is not zero.

D. The columns of A span $\mathbb{R}^n$.

E. The rows of A are linearly independent.

F. The rank of A is n.

A

$$\begin{aligned} \det(A - 0I) &= 0 \\ \det(A) &= 0 \Rightarrow \therefore\ A \text{ is invertible.} \Rightarrow B, C, D, E, F. \end{aligned}$$

Exercise 5.5.15 The matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ is not diagonalizable over the reals. Why?

A. Because A does not have any real eigenvectors

B. Because A does not have any real eigenvalues..

C. Because A does not have three distinct eigenvalues.

D. Because A does not have three independent eigenvectors.

E. Because A is lower triangular.

F. Because 'diagonalizable' is too difficult to say very quickly.

D

Note that

A. False: Not true: $A\boldsymbol{e}_1 = 4\boldsymbol{e}_1$

B. False: Not true: $A\boldsymbol{e}_1 = 4\boldsymbol{e}_1, \quad \lambda = 4$ *is real eigen value*

C. False: This is not necessary

D. True: Compute the eigen values together with their corresponding eigen vectors. $3 \neq 2$

E. False: Many lower triangular matrices are diagonizable, I_3

F. False: Wrong

Exercise 5.5.16 State whether the following are *true* or *false*. If true, explain why, if false, give a numerical example to illustrate.

1.) $\begin{bmatrix} 6 & 0 \\ 1 & 5 \end{bmatrix}$ is diagonalizable. *True*

True because the 2×2 *matrix has two distinct eigen values namely 6 and 5.*

2.) If 0 is an eigenvalue of 4×4 matrix A, then A is invertible. *False*

For zero an eigen value,

$$\begin{aligned} \det(A - 0I) &= 0 \\ \det A &= 0 \end{aligned}$$

Thus A is not invertible.

Exercise 5.5.17 Let $B = C^{-1}AC$ for some matrix C. Let $\vec{v}$ be an eigenvector of A with eigenvalue λ. Show that $C^{-1}\vec{v}$ is an eigenvector of B with eigenvalue λ.

Exercise 5.5.18 Let $A = \begin{bmatrix} -15 & 28 \\ -8 & 15 \end{bmatrix}$.

1.) Find the eigenvalues and compute an eigenvector for each eigenvalue.

2.) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

3.) Compute A^{37}.

Exercise 5.5.19 Let

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \\ \dfrac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be a linear system of differential equation where

$$A = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}.$$

1.) Find the eigenvalues and find an eigenvector for each eigenvalue for A. Note: The eigenvalues are complex-valued.

2.) Find the general real solution to the linear system of differential equations.

Exercise 5.5.20 For what pair of real numbers (c_1, c_2) is the matrix $\begin{bmatrix} 2 & 0 \\ c_1 & c_2 \end{bmatrix}$ diagonalizable?

A. $(1, 2)$

B. $(2, -2)$

C. $(-2, 2)$

D. $(-1, 2)$

E. none of the preceding.

Exercise 5.5.21 One of the eigenvalues of the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 0 & -2 \\ -6 & 6 & -6 \end{bmatrix}$ is -1. The other eigenvalues of A are:

A. 2 and -1

B. 1 and 3

C. 0 and 1

D. 2 and 3

E. -3 and 0

F. -1 and 3

Exercise 5.5.22 Suppose $A = \begin{bmatrix} 3 & 0 & 4 \\ 0 & -1 & 0 \\ 4 & 0 & 3 \end{bmatrix}$.

1.) Compute the cubic polynomial $\det(A - \lambda I_3)$, then factorize it, and hence find the eigenvalues of A.

2.) Find a basis for $\ker(A + I_3)$.

3.) Find a basis for $\ker(A - 7I_3)$.

4.) Collect all the basis vectors you found in (2) and (3), and construct a matrix P with these vectors as columns. Is your matrix P invertible?

Exercise 5.5.23 Determine whether the matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is diagonalizable.

Exercise 5.5.24

1.) Let A be a square matrix. Explain what it means to say that matrix

   1.) A is similar to the matrix B.

   2.) A is diagonalizable

2.) Given that A, B and C are $n \times n$ matrices and that A is similar to B, and B is similar to C. Show that A is similar to C.

3.) Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

   1.) Compute the eigenvalues of A.

   2.) Determine whether A is diagonalizable and if so find the diagonalizing matrix P.

Exercise 5.5.25 Define an eigenvector and eigenvalue for matrix A. Show that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

Remark 5.5.1 Two similar matrices are not row-equivalent, but they share many important properties.

Exercise 5.5.26 A matrix A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

Exercise 5.5.27

1.) Define the following terms

   (a) A characteristic polynomial of a matrix A.

   (b) A matrix B being similar to a matrix A.

   (c) An eigenvalue and eigenvector of a matrix A.

2.) (a) Show that similar matrices have the same characteristic polynomial. What can you say about their eigenvectors?

   (b) Find the eigenvalues and the bases for the eigenspace of

$$A = \begin{bmatrix} 5 & -3 & -6 \\ 0 & 2 & 0 \\ 3 & -3 & -4 \end{bmatrix}$$

   (c) Is A similar to a diagonal matrix? If so find the diagonal matrix.

   (d) Determine the invertible matrix P which diagonalizes A.

Exercise 5.5.28

1.) Define an eigenvector of a matrix A.

2.) Show that $r\boldsymbol{x}$ is an eigenvector of A if $\boldsymbol{x}$ is an eigenvector of A where $r \in \mathbb{R}$.

Exercise 5.5.29 If λ is an eigenvalue for an invertible $n \times n$ matrix A. Prove that $\dfrac{1}{\lambda}$ is the eigenvalue for the matrix A^{-1}.

Exercise 5.5.30 Let $A = [a_{ij}]$ be an $n \times n$ matrix and λ is an eigenvalue of A. Which of the following is true?

A. $\det(A - \lambda I) = 0$;

B. $\det(A - \lambda I) \neq 0$;

C. $Ax = \lambda x$ for all x in R^n;

D. The linear system $(A - \lambda I)x = 0$ has only trivial solutions.

Exercise 5.5.31 Let A be a 3×3 matrix with the characteristic polynomial $p(\lambda) = \lambda(\lambda - 1)(\lambda - 3)$. Which of the following statements is wrong?

A. A is not invertible;

B. There are three eigenvectors v_1, v_2, v_3 which form an eigenbasis of R^3.

C. Each eigenspace of A is one-dimensional;

D. The linear system $(A - 3I)x = b$ has unique solutions for each b in R^3;

Exercise 5.5.32 Let A and B be two $n \times n$ matrices such that A and B are similar. Let λ be an eigenvalue of A. Which of the following statements is wrong?

A. λ is an eigenvalue of A^T ;

B. λ is an eigenvalue of A^{-1};

C. λ is an eigenvalue of B;

D. λ is an eigenvalue of B^T .

Chapter 6

Linear Algebra By Python