# Chapter 5

# Polynomial Interpolation

Interpolation is often used to approximate a value of some function $f$ using two known values of that function at other points.

## 5.1 Direct Method

Using the similarities of triangles,

$$\frac{f(b)-f(a)}{b-a} = \frac{f(c)-f(a)}{c-a}$$
$$f(b)-f(a) = \frac{(b-a)}{(c-a)}[f(c)-f(a)]$$

thus

$$f(b) = f(a) + \frac{(b-a)}{(c-a)}\left[f(c)-f(a)\right] \tag{5.1}$$

and

$$b = a + \left[\frac{f(b)-f(a)}{f(c)-f(a)}\right](c-a) \tag{5.2}$$

**Example 5.1.1** Given the data below

| Time | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Distance | 0 | 6 | 39 | 67 | 100 |

1.) Find the distance traveled when $t = 2.3$ hrs.
Then $a = 2, c = 3, \& \ b = 2.3$
$f(a) = 39, f(c) = 67, \ \& \ f(b)?$

$$f(b) = f(a) + \frac{(b-a)}{(c-a)}\left[f(c)-f(a)\right] = 39 + \frac{(2.3-2)}{(3-2)}(67-39) = 47.4$$

2.) The time taken when the distance traveled is 80 miles.
Then $a = 3, c = 4, \& \ b = ?$
$f(a) = 67, f(c) = 100, \ \& \ f(b) = 80$

$$b = a + \left[\frac{f(b)-f(a)}{f(c)-f(a)}\right](c-a) = 3 + \frac{(80-67)}{(100-67)}(4-1) = 3.39394$$

**Example 5.1.2** The bus stages along Kampala-Jinja are 10 $km$ apart. An express bus traveling between the two towns only stops at these stages except in case of an emergency when its permitted to stop at a point between the two stages.
The fares (fee) between the first, second, third and fourth stages from Jinja are $Sh$ 110, $Sh$ 150, $Sh$ 185 and $Sh$ 200 respectively. On a certain day, a passenger paid to travel from Jinja in the bus upto the fourth stage, but he fell sick and had to be left on a health center 33 $km$ from Jinja.

1.) Given that he was refunded money for the distance he had not traveled, find the approximate amount of money he received.

| Distance ($x$) | 10 | 20 | 30 | 40 |
|---|---|---|---|---|
| Amount Paid | 110 | 150 | 185 | 200 |

Then $a = 30, c = 40, \& \ b = 33$
$f(a) = 185, f(c) = 200, \ \& \ f(b)?$

$$f(b) = f(a) + \frac{(b-a)}{(c-a)}\left[f(c) - f(a)\right] = 185 + \frac{(200-185)}{(40-30)}(33-30) = 189.5$$

The journey he had moved costed him $Sh$ 189.5, thus he was refunded
$200 - 189.5 = 10.5$ shillings

2.) Another person who had only $Sh.165$ was allowed to board a bus but would be left at a point worth his money, how far from Jinja would he be left.
Then $a = 20, c = 30, \& \ b =?$
$f(a) = 150, f(c) = 185, \ \& \ f(b) = 165$

$$b = a + \left[\frac{f(b) - f(a)}{f(c) - f(a)}\right](c-a) = 20 + \frac{(165-150)}{(185-150)}(30-20) = 24.286 \ km$$

**Example 5.1.3** The table below shows the values of $\cos x$.

| $80^\circ \ x'$ | 0 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $\cos \ 80^\circ \ x'$ | 0.1736 | 0.1708 | 0.1679 | 0.1650 | 0.1622 | .01593 |

(i) Find the value of $\cos 80^\circ 35'$.
Then $a = 30', c = 40', \& \ b = 35'$
$f(a) = 0.1650, f(c) = .01622, \ \& \ f(b)?$

$$f(b) = f(a) + \frac{(b-a)}{(c-a)}\left[f(c) - f(a)\right] = 0.1650 + \frac{(-0.0028)}{10'}5' = .01636$$

(ii) Find the $\cos^{-1} 0.1655$.
Then $a = 20', c = 30', \& \ b =?$
$f(a) = 0.1679, f(c) = 0.1650, \ \& \ f(b) = 0.1655$

$$b = a + \left[\frac{f(b) - f(a)}{f(c) - f(a)}\right](c-a) = 20' + \frac{(0.1655-0.1679)}{(0.1650-0.1679)}(30'-20') = 20' + 8.276' = 28.276'$$

$\cos^{-1} 0.1655 = 80^\circ 28.3'$

**Example 5.1.4** Use Linear interpolation to find the root of the equation $x^3 - x - 1 = 0$ which lies between $(1, 2)$

| $x$ | 1 | 2 |
|---|---|---|
| $f(x)$ | $-1$ | 5 |

Recall that a value $x$ is a root or a solution to an equation $f(x)$ if $x$ satisfy it, i.e if $f(x) = 0$. The question is to find the value of $x$ at $f(x) = 0$.
Then $a = 1, c = 2, \& \ b =?$
$f(a) = -1, f(c) = 5, \ \& \ f(b) = 0$

$$b = a + \left[\frac{f(b) - f(a)}{f(c) - f(a)}\right](c-a) = 1 + \frac{(0 -^- 1)}{(5 -^- 1)}(2-1) = 1.16667$$

**Example 5.1.5** Use linear interpolation to estimate the root of the equation

$$x^3 - 2x - 5 = 0$$

which lies in the interval $(2, 3)$. 2.0588

**Example 5.1.6** Use linear interpolation to estimate the root of the equation

$$x^2 - 2 = 0$$

which lies in the interval $(-2, -1)$. $-1.3333$

**Example 5.1.7** The following data gives the distance covered by a particle for a certain period of time.

| Time (s) | 0 | 1 | 2 |
|---|---|---|---|
| Distance(m) | 0 | 5 | 7 |

Estimate the time taken by a particle to cover a distance of 6 $m$. 1.5 s

### 5.1.1 Application

Linear interpolation is often used to fill the gaps in a table. Suppose you have a table listing the population of some country in 1970, 1980, 1990 and 2000, and that you want to estimate the population in 1994. Linear interpolation gives you an easy way to do this.

The basic operation of linear interpolation between two values is so commonly used in computer graphics that it is sometimes called a lerp in the jargon of computer graphics. The term can be used as a verb or noun for the operation. e.g. "Bresenham's algorithm lerps incrementally between the two endpoints of the line."

Lerp operations are built into the hardware of all modern computer graphics processors. They are often used as building blocks for more complex operations: for example, a bilinear interpolation can be accomplished in three lerps. Because this operation is cheap, it's also a good way to implement accurate lookup tables with quick lookup for smooth functions without having too many table entries.

### 5.1.2 History

Linear interpolation has been used since antiquity for filling the gaps in tables, often with astronomical data. It is believed that it was used in the Seleucid Empire (last three centuries BC) and by the Greek astronomer and mathematician Hipparchus (second century BC). A description of linear interpolation can be found in the Almagest (second century AD) of Ptolemy.

### 5.1.3 Extensions

In demanding situations, linear interpolation is often not accurate enough (since not all points can be approximated to be on a straight line). In that case, it can be replaced by **polynomial interpolation or spline interpolation**.

Linear interpolation can also be extended to bilinear interpolation for interpolating functions of two variables. Bilinear interpolation is often used as a crude anti-aliasing filter. Similarly, trilinear interpolation is used to interpolate functions of three variables. Other extensions of linear interpolation can be applied to other kinds of mesh such as triangular and tetrahedral meshes.

## 5.2 Lagrange Interpolation

We may know the value of a function $f$ at a set of points $x_1, x_2, \ldots, x_N$. How do we estimate the value of the function at any point $x*$; how do we compute $f(x*)$. In general this is done by developing a smooth curve though the points $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_N, f(x_N))$.

Lagrange interpolation is a way to pass a polynomial of degree $N-1$ through $N$ points.

Lagrange polynomials are the interpolating polynomials that equal zero in all given points, save one. Say, given points $x_1, x_2, \ldots, x_N$, Lagrange's polynomial number $k$ is the product

$$P_k(x) = \frac{(x-x_1)}{(x_k-x_1)} \cdot \frac{(x-x_2)}{(x_k-x_2)} \cdots \frac{(x-x_{k-1})}{(x_k-x_{k-1})} \cdot \frac{(x-x_{k+1})}{(x_k-x_{k+1})} \cdots \frac{(x-x_N)}{(x_k-x_N)}$$

such that $P_k(x_k) = 1$ and $P_k(x_j) = 0$, for j different from k. In terms of Lagrange's polynomials the polynomial interpolation through the points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$ could be defined simply as

$$P(x) = y_1 P_1(x) + y_2 P_2(x) + ... + y_N P_N(x)$$

**Definition 5.2.1** Polynomial interpolation is a method of constructing a function and estimating values at non-tabular points between $x_0$ and $x_n$.

#### 5.2.0.1 Alternatively

Let $f(x)$ be a continuous function on $[a, b]$, such that

$$f(x_k) = f(x_k), \quad k = 0, 1, ..., n,$$

with $x_k \in [a, b]$. We call the set of points $\{x_k\}$ tabular points (or interpolating points), while the set of values $\{f(x_k)\}$ are called the tabular values of $f(x)$. We seek for a polynomial of degree $n$, such that

$$P_n(x_k) = f(x_k), \text{ for } k = 0, 1, 2, \ldots n. \tag{5.3}$$

Such a polynomial is called Lagrange's interpolation polynomial. However, the process of calculating $f(x), x \in [a, b], x \neq x_k \ \ \forall\, k$ from 0 to $n$, is called interpolation.
We now proceed to derive a formula for $P_n(x)$. However, there is need to introduce some key definitions.

**Definition 5.2.2**
Let $\{D_k : k = 0, 1, ..., n\}$ be any set of numbers. We define the products;

$$\prod_{k=0}^{n} D_k = D_0.D_1...D_n$$

$$\text{and} \quad \prod_{\substack{j=0 \\ j\neq k}}^{n} D_k = D_0.D_1...D_{j-1}.D_{j+1}...D_n \quad .$$

This definition is important in the following defining equation for Lagrange's interpolating polynomial.

**Theorem 5.2.1**
The Lagrange's interpolation polynomial $P_n(x)$ is given by,

$$P_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k) \tag{5.4}$$

with

$$L_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \left( \frac{x - x_j}{x_k - x_j} \right) \tag{5.5}$$

**Proof:** Because $P_n(x)$ is of degree $n$, we may write

$$L_k(x) = \alpha_k \prod_{\substack{j=0 \\ j\neq k}}^{n} ,$$

$\alpha_k$ constant.
For $P_n(x)$ to satisfy equation (5.3) we must have,

$$L_k(x_j) = \delta_{kj} (\quad \text{Kroneker} \quad \text{delta})$$

with

$$\delta_{kj} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{5.6}$$

With the condition (1.3) we have,

$$\alpha_k = \frac{1}{\prod\limits_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j)}$$

This completes the proof. ■

**Remark 5.2.1** Let us have some examples of interpolating polynomials

### 5.2.1 Linear Interpolating Polynomials

When $n = 1$, in equation (5.4) we have the polynomial as

$$\begin{aligned} P_1(x) &= \sum_{k=0}^{1} L_k(x) f(x_k) \\ &= L_0(x)f(x_0) + L_1(x)f(x_1) \\ \text{But } L_k(x) &= \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x-x_j)}{(x_k-x_j)} \\ \text{therefore } L_0(x) &= \left(\frac{x-x_1}{x_0-x_1}\right) \\ L_1(x) &= \left(\frac{x-x_0}{x_1-x_0}\right) \end{aligned}$$

Thus,

$$P_1(x) = \left(\frac{x-x_1}{x_0-x_1}\right) f(x_0) + \left(\frac{x-x_0}{x_1-x_0}\right) f(x_1) \tag{5.7}$$

Which is known as Lagrange's interpolating polynomial of degree one, popularly known as Linear interpolating polynomial.

**Example 5.2.1** Construct a linear interpolation polynomial for the data,

| $x$ | 0 | 1 |
|---|---|---|
| $f(x)$ | 1.0000 | 2.7183 |

Hence interpolate $f(0.5)$.

**Solution :** *We have that*

| | $x_0$ | $x_1$ |
|---|---|---|
| $x$ | *0* | *1* |
| $f(x)$ | 1.0000 | 2.7183 |
| | $f(x_0)$ | $f(x_1)$ |

*Substituting in equation (5.7) for linear interpolation we have,*

$$\begin{aligned} P_1(x) &= \left(\frac{x-x_1}{x_0-x_1}\right) f(x_0) + \left(\frac{x-x_0}{x_1-x_0}\right) f(x_1) \\ &= \left(\frac{x-1}{0-1}\right) f(0) + \left(\frac{x-0}{1-0}\right) f(1) \\ &= (1-x)(1.0000) + (x)(2.7183) \\ &= 1.0000 + 1.7183x \end{aligned}$$

*Hence*

$$\begin{aligned} P_1(0.5) &= 1.0000 + (1.7183)(0.5) \\ &= 1.0000 + 0.8592 \\ &= 1.8592 \end{aligned}$$

*In fact the data in this particular example describes the graph of $f(x) = e^x$ on $[0, 1]$.*

■

**Example 5.2.2** Find the linear Lagrange interpolating polynomial, $P_1(x)$, that passes through the points $(1,2)$ and $(3,4)$.

**Solution :** *The tabulated data is given by*

| | | |
|---|---|---|
| $x$ | $x_0$ *1* | $x_1$ *3* |
| $f(x)$ | 2 | 4 |
| | $f(x_0)$ | $f(x_1)$ |

*The function* $P_1$ *can be obtained directly by substituting the the points* $(1,2)$ *and* $(3,4)$ *into the formula (5.7) for linear interpolation above to get:*

$$\begin{aligned}
P_1(x) &= \left(\frac{x-x_1}{x_0-x_1}\right) f(x_0) + \left(\frac{x-x_0}{x_1-x_0}\right) f(x_1) \\
&= \left(\frac{x-3}{3-1}\right) f(1) + \left(\frac{x-1}{3-1}\right) f(3) \\
&= \left(\frac{x-3}{3-1}\right) (2) + \left(\frac{x-1}{3-1}\right) (4) \\
&= \frac{2(3-x)+4(x-1)}{3-1} \\
&= \frac{6-2x+4x-4}{2} \\
&= \frac{2x+2}{2} \\
&= x+1
\end{aligned}$$

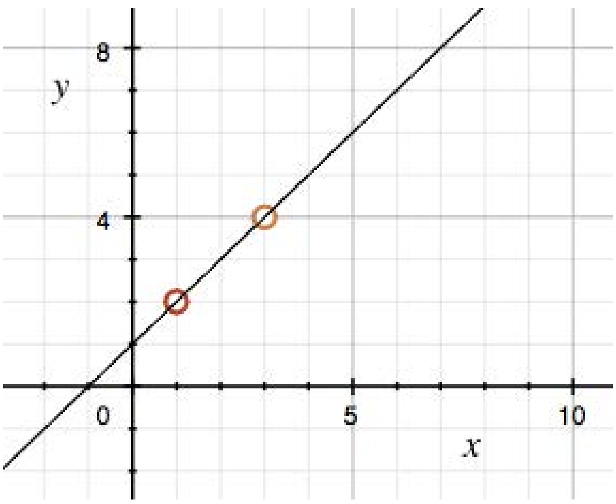



Figure 5.1: Graphical illustration of linear interpolation

■

**Example 5.2.3** Estimate the value of $\sqrt{5}$ using the linear Lagrange interpolating polynomial $P_1(x)$ that passes through the points $(1, 1)$ and $(9, 3)$ and evaluate the error of this approximation with the true value of $\sqrt{5} \approx 2.23606...$

**Solution :** *The tabulated data is given by*

| | $x_0$ | $x_1$ |
|---|---|---|
| $x$ | *1* | *9* |
| $f(x)$ | 1 | 3 |
| | $f(x_0)$ | $f(x_1)$ |

*The function $P_1$ can be obtained directly by substituting the the points $(1, 1)$ and $(9, 3)$ into the formula (5.7) for linear interpolation above to get:*

$$\begin{aligned}
P_1(x) &= \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1) \\
&= \left(\frac{x - 9}{9 - 1}\right) f(1) + \left(\frac{x - 1}{9 - 1}\right) f(9) \\
&= \left(\frac{x - 9}{9 - 1}\right) (1) + \left(\frac{x - 1}{9 - 1}\right) (3) \\
&= \frac{1(9 - x) + 3(x - 1)}{9 - 1} \\
&= \frac{9 - x + 3x - 3}{8} \\
&= \frac{2x + 6}{8} \\
&= \frac{x + 3}{4}
\end{aligned}$$

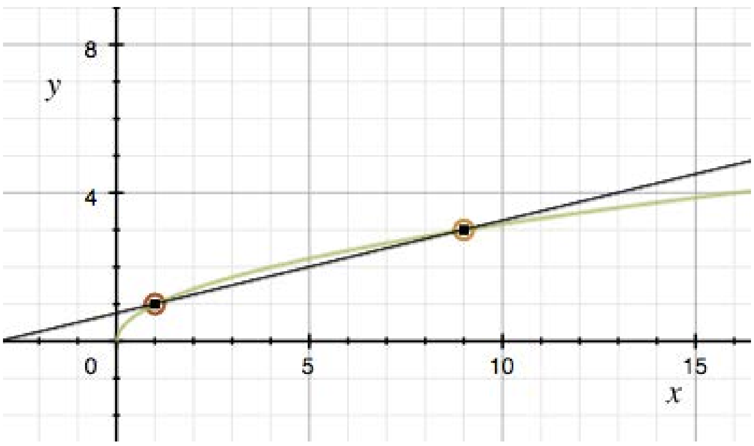



Figure 5.2: Graphical illustration of linear interpolation

*Now our approximation of $f(5) = \sqrt{5}$ is given by $P_1(5)$:*

$$P_1(5) = \frac{5 + 3}{4} = 2$$

*As we can see, our approximation is an underestimate of the true value of $\sqrt{5}$. We only obtained one significant digit of accuracy.* ∎

### 5.2.2 Quadrative Interpolating Polynomials

When $n = 2$ in equation (5.4) we get

$$\begin{aligned} P_2(x) &= \sum_{k=0}^{2} L_k(x) f(x_k) \\ &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \\ \text{with } L_k(x) &= \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)} \\ \text{then} \quad \mathrm{L_0(x)} &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\ L_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\ L_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}$$

Thus, we have,

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Which is called a quadratic interpolating polynomial. Generally they are better interpolating polynomials than the linear ones.

**Example 5.2.4** You are given that, $f(0) = -2$, $f(2) = 4$, and $f(3) = 10$. Find a Lagrange polynomial of degree 2 that fits the data.

Since $x_0 = 0$, $x_1 = 2$ and $x_2 = 3$ therefore $f(x_0) = -2$, $f(x_1) = 4$, $f(x_2) = 10$.
But

$$\begin{aligned} P_2(x) &= L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2) \\ L_0 &= \frac{(x - x_1).(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 2)(x - 3)}{(-2)(-3)} \\ &= \frac{1}{6}x^2 - \frac{5}{6}x + 1 \\ L_1 &= \frac{(x - x_0).(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 0)(x - 3)}{(2 - 0)(2 - 3)} \\ &= -\frac{1}{2}x^2 + \frac{3}{2}x \\ L_2 &= \frac{(x - x_0).(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 0)(x - 2)}{(3 - 0)(3 - 2)} \\ &= \frac{1}{3}x^2 - \frac{2}{3}x \end{aligned}$$

$$\begin{aligned} P_2(x) = f(x) &= L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2) \\ f(x) &= (-2)\left(\frac{1}{6}x^2 - \frac{5}{6}x + 1\right) + (4)\left(-\frac{1}{2}x^2 + \frac{3}{2}x\right) + (10)\left(\frac{1}{3}x^2 - \frac{2}{3}x\right) \\ &= x^2 + x - 2 \end{aligned}$$

Thus $P_2(x)$ can be used to interpolate $f(x)$ at any of the non tabular points.

**Example 5.2.5** Construct the quadratic Lagrange interpolating polynomial $P_2(x)$ that interpolates the points $(1,4),(2,1)$, and $(5,6)$.

**Solution :** *The tabulated data is given by*

| | $x_0$ | $x_1$ | $x_2$ |
|---|---|---|---|
| $x$ | 1 | 2 | 5 |
| $f(x)$ | 4 | 1 | 6 |
| | $f(x_0)$ | $f(x_1)$ | $f(x_2)$ |

*such that*

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-5)}{(1-2)(1-5)} \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-5)}{(2-1)(2-5)} \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-2)}{(5-1)(5-2)} \end{aligned}$$

*with*

$$f(x_0)=4,\ f(x_1)=1,\ f(x_2)=6$$

*we have*

$$\begin{aligned} P_2(x) &= L_0f(x_0)+L_1f(x_1)+L_2f(x_2) \\ P_2(x) &= 4\frac{(x-2)(x-5)}{(1-2)(1-5)}+1\frac{(x-1)(x-5)}{(2-1)(2-5)}+6\frac{(x-1)(x-2)}{(5-1)(5-2)} \\ P_2(x) &= (x-2)(x-5)-\frac{1}{3}(x-1)(x-5)+\frac{1}{2}(x-1)(x-2) \\ &= x^2-5x-2x+10-\frac{1}{3}x^2+\frac{5}{3}x+\frac{1}{3}x-\frac{5}{3}+\frac{1}{2}x^2-\frac{1}{2}x-x+1 \\ &= x^2-5x-2x+10-\frac{1}{3}x^2+2x-\frac{5}{3}+\frac{1}{2}x^2-\frac{1}{2}x-x+1 \\ &= \left(x^2-\frac{1}{3}x^2+\frac{1}{2}x^2\right)\left(-5x-2x+2x-\frac{1}{2}x-x\right)+\left(10-\frac{5}{3}+1\right) \\ &= \frac{7}{6}x^2-\frac{13}{2}x+\frac{28}{3} \end{aligned}$$

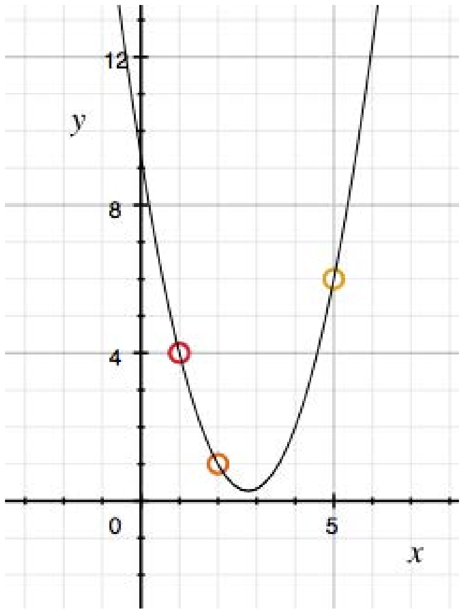



Figure 5.3: Graphical illustration of quadratic interpolation, the graph of $y = P_2(x)$

■

**Example 5.2.6** Find the Lagrange interpolation expansion for $f(x) = x^2 - 2x + 1$ relative to $x_0 = -1$ , $x_1 = 0$, and $x_2 = 1$ .

**Solution :** *The Lagrange polynomials are*

$$L_0 = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}(x^2 - x)$$
$$L_1 = \frac{(x+1)(x-1)}{(0+1)(0-1)} = -(x^2 - 1)$$
$$L_2 = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{1}{2}(x^2 + x)$$

*Because* $f(-1) = 4$ , $f(0) = 1$ , *and* $f(1) = 0$ , *the expansion is*

$$\begin{aligned} P_2 &= L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2) \\ &= 4\frac{1}{2}(x^2 - x) + 1\left(-(x^2 - 1)\right) + 0\left(\frac{1}{2}(x^2 + x)\right) \\ &= 2(x^2 - x) - (x^2 - 1) \\ &= x^2 - 2x + 1 \end{aligned}$$

*Therefore*

$$f(x) = x^2 - 2x + 1$$

*We note that, both equations are the same.* ■

**Example 5.2.7** Construct the quadratic Lagrange interpolating polynomial $P_2(x)$ that interpolates the points $(1,2), (3,4)$, and $(5,6)$.

**Solution :** *The tabulated data is given by*

| | $x_0$ | $x_1$ | $x_2$ |
|---|---|---|---|
| $x$ | 1 | 3 | 5 |
| $f(x)$ | 2 | 4 | 6 |
| | $f(x_0)$ | $f(x_1)$ | $f(x_2)$ |

*such that*

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-3)(x-5)}{(1-3)(1-5)} \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-5)}{(3-1)(3-5)} \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-3)}{(5-1)(5-3)} \end{aligned}$$

*with*

$$f(x_0) = 2, \; f(x_1) = 4, \; f(x_2) = 6$$

*we have*

$$\begin{aligned} P_2(x) &= L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2) \\ &= 2\frac{(x-3)(x-5)}{(1-3)(1-5)} + 4\frac{(x-1)(x-5)}{(3-1)(3-5)} + 6\frac{(x-1)(x-3)}{(5-1)(5-3)} \\ &= \frac{1}{4}(x-3)(x-5) - (x-1)(x-5) + \frac{3}{4}(x-1)(x-3) \\ &= \frac{1}{4}x^2 - \frac{3}{4}x - \frac{5}{4}x + \frac{15}{4} - x^2 + 5x + x - 5 + \frac{3}{4}x^2 - \frac{3}{4}x - \frac{9}{4}x + \frac{9}{4} \\ &= x + 1 \end{aligned}$$

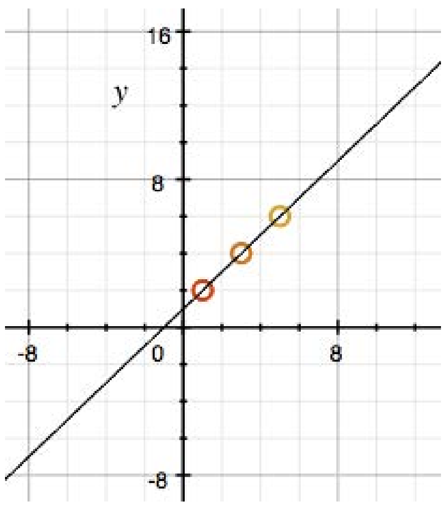



Figure 5.4: Graphical illustration of quadratic interpolation, the graph of $y = P_2(x)$

*Note that this example shows that $P_2$ need not be quadratic and may be a polynomial of lesser degree.* ■

### 5.2.3 The Cubic Interpolating Polynomial

When $n = 3$ in equation (5.4) we get,

$$\begin{aligned}
P_3(x) &= \sum_{k=0}^{3} L_k(x) f(x_k) \\
&= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3) \\
\text{and with } L_k(x) &= \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)} \\
\text{then,} \quad \mathrm{L_0(x)} &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\
L_1(x) &= \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
L_2(x) &= \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\
L_3(x) &= \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}
\end{aligned}$$

Thus,

$$P_3 = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

Construction of the cubic polynomials from available data will be tested in the text questions at the end of this lecture. Note that the higher degree Lagrange polynomials can be constructed with ease.

### 5.2.4 Higher Order Lagrange Interpolating Polynomials

**Definition 5.2.3** The $n$th Order Lagrange Interpolating Polynomial that passes through the $n+1$ points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$ where $x_0, x_1, \ldots, x_n$ are distinct numbers is

$$P_n(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots + f(x_n)L_n(x) \tag{5.8}$$

with

$$L_k(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)} = \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)} \tag{5.9}$$

**Example 5.2.8** Find the quartic Lagrange interpolating polynomial, $P4(x)$, that interpolates the points $(-1,4), (1,1), (2,0), (6,4)$, and $(7,-1)$.

**Solution :** *Applying the formula above directly and we get that:*

$$P_4(x) = y_0L_0(x) + y_1L_1(x) + y_2L_2(x) + y_3L_3(x) + y_4L_4(x)$$

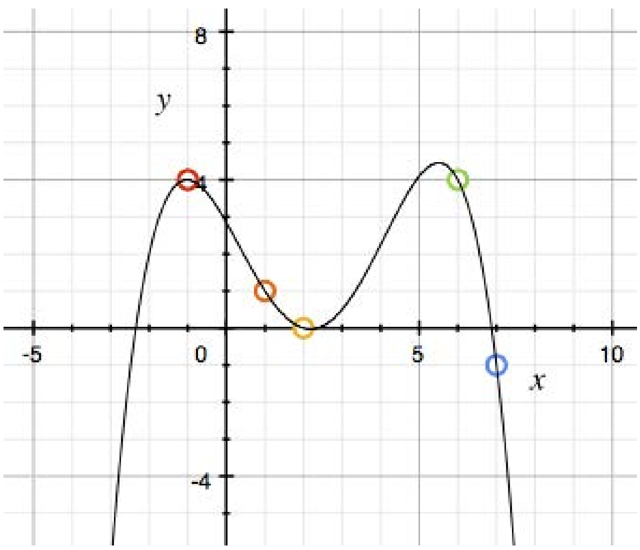
$$P_4(x) = y_0\frac{\prod l^n_{j=0,j\neq 0}(x-x_j)}{\prod^n_{j=0,j\neq 0}(x_0-x_j)} + y_1\frac{\prod^n_{j=0,j\neq 1}(x-x_j)}{\prod^n_{j=0,j\neq 1}(x_1-x_j)} + y_2\frac{\prod^n_{j=0,j\neq 2}(x-x_j)}{\prod^n_{j=0,j\neq 2}(x_2-x_j)} + y_3\frac{\prod^n_{j=0,j\neq 3}(x-x_j)}{\prod^n_{j=0,j\neq 3}(x_3-x_j)}$$

$$P_4(x) = 4\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} + 1\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$
$$+ 4\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} - 1\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$$P_4(x) = 4\frac{(x-1)(x-2)(x-6)(x-7)}{(-1-1)(-1-2)(-1-6)(-1-7)} + 1\frac{(x+1)(x-2)(x-6)(x-7)}{(1+1)(1-2)(1-6)(1-7)}$$
$$+ 4\frac{(x+1)(x-1)(x-2)(x-7)}{(6+1)(6-1)(6-2)(6-7)} - 1\frac{(x+1)(x-1)(x-2)(x-6)}{(7+1)(7-1)(7-2)(7-6)}$$

$$P_4(x) = \frac{1}{84}(x-1)(x-2)(x-6)(x-7) - \frac{1}{60}(x+1)(x-2)(x-6)(x-7)$$
$$- \frac{1}{35}(x+1)(x-1)(x-2)(x-7) - \frac{1}{240}(x+1)(x-1)(x-2)(x-6)$$
$$= -\frac{3}{80}x^4 + \frac{11}{30}x^3 - \frac{13}{16}x^2 + \frac{13}{30}x + \frac{1}{20}$$

*The graph of $y = P_4(x)$ is given below*




■

### 5.2.5 Uniqueness of Lagrange Interpolating Polynomials

**Theorem 5.2.2** Lagrange's interpolation polynomial $P_n(x)$ is unique.

**Proof:** *The proof proceeds by contradiction. Let $P_n(x)$ and $Q_n(x)$ be two different polynomials which interpolate $f(x)$ over the set of points $\{x_k : k = 0, 1, \ldots, n\}$ which belong to the interval $[a, b]$, then*

$$P_n(x_k) = Q(x_k) = f(x_k), \text{ for } k = 0, 1, 2, \ldots, n.$$

*Lets define,*

$$r(x) = P(x) - Q(x) \quad \forall \quad x \in [a, b],$$

*then $r(x)$ has at most degree $n$. Since $r(x_k) = 0$ for $k = 0, 1, 2, \ldots, n$, it has $n + 1$ distinct zeros in $[a, b]$. This contradicts the fundamental law of algebra which states that a non-zero polynomial of degree $n$ cannot have more than $n$ zeros and so $P_n(x)$ and $Q_n(x)$ are the same polynomials.* ■

**Note 5.2.1** The uniqueness means that you cannot find any other polynomial of the same degree which can interpolate the data.

**Exercise 5.2.1**

1.) Construct a linear interpolating polynomial $P_1(x)$ for the function $f(x) = \dfrac{1}{x}$ on the interval $[1, 2]$. Use your polynomial to interpolate $f(x)$ at $x = 1.2$ .

2.) Given that $x_0 = 0, \quad x_1 = \dfrac{1}{2}$ and $x_2 = 1$ for $f(x) = e^x$. Construct a Lagrange polynomial that agrees with f(x) at the interpolating points.

3.) Find a third degree Lagrange polynomial that goes through the points $(0, 0), \quad (1, 1), \quad (8, 2)$ and $(27, 3)$.
Use the polynomial to find $q$ for $(20, q)$. Also construct a linear interpolating polynomial using only $(8, 2)$ and $(27, 3)$, then use the linear polynomial to estimate $q$. Compare the estimated $q^s$ and comment on your results given that the data is of the function $y = x^{\frac{1}{3}}$.

4.) Find the interpolating polynomials going through

(a) $(0, 1)$ and $(2, 3)$

(b) $(-1, a), \quad (0, b)$ and $(1, c)$

(c) $(0, 1), \quad (1, 0), \quad (2, 1)$ and $(3, 0)$

5.) Given the table below, use Lagrange's interpolation polynomials of degree one, two and three to find $f(2.5)$

| $x$ | 2.0 | 2.2 | 2.4 | 2.6 |
|---|---|---|---|---|
| $f(x)$ | 0.5102 | 0.5208 | 0.5104 | 0.4813 |

Solution

(a) For $n = 1$, since need to predict $f(2.5)$ then $x_0 = 2.4, x_2 = 2.6$

$$\begin{aligned}
P_1(x) &= (\frac{x - x_1}{x_0 - x_1})f(x_0) + (\frac{x - x_0}{x_1 - x_0})f(x_1) \\
&= (\frac{x - 2.6}{2.4 - 2.6})0.5104 + (\frac{x - 2.4}{2.6 - 2.4})0.4813 \\
&= 0.49585
\end{aligned}$$

so for $x = 2.5, f(2.5) = 0.49585$

(b) For $n = 2, x_0 = 2.2, x_1 = 2.4, x_2 = 2.6$

$$\begin{aligned}
&= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}f(x_2) \\
&= 0.53814
\end{aligned}$$

6.) Repeat questions in linear-interpolation (review-section) using the Lagrange linear interpolating polynomial. comment on the accuracy of the two techniques.

### 5.2.6 Error analysis in Lagrange's interpolation

## Truncation errors in Lagrange interpolation

We assume that $f(x_k) \equiv f(x_k), \quad k = 0, 1, 2, \ldots, n$ are exact and we consider the truncation error $e(x) = f(x) - P_n(x)$ for $x \in [a, b]$ and $P_n(x)$ the Lagrange polynomial of degree $n$ as defined in earlier.

Apart from the fact that $e(x_k) = 0,$ for $k = 0, 1, 2, \ldots, n$ with $x_k \in [a, b]$, we can say nothing more about $e(x)$ for any $x \neq x_k$.

In addition we have that $f(x)$ has at least $n+1$ continuous derivatives on $[a, b]$, then it is possible to express $e(x)$ in terms of $f^{n+1}(x)$. We now state without proof two necessary Lemmas.

**Lemma 5.2.1**
Given

$$q_{n+1}(x) = \prod_{k=0}^{n}(x - x_k), \qquad x \in [a, b]$$

of degree $(n + 1)$,
$q'_{n+1}(x)$ is of degree $n$ such that

$$q'_{n+1}(x_j) = \sum_{\substack{k=0 \\ k \neq j}}^{n}(x_j - x_k)$$

**Lemma 5.2.2**
Let a function $g(x)$ be defined on $[a, b]$. Let $\{S_k : k = 0, 1, \ldots, n\}$ be a set of distinct points each belonging to $[a, b]$, with $S_0 < S_1 < S_2 < \ldots < S_n$. Suppose that:

(i) $g^{(n)}(x)(n \geq \text{integer})$ is continuous on $[S_0, S_n]$.

(ii) $g(S_k) = 0, k = 0, 1, \ldots, n$.

Then there is at least a number $\xi \in (S_0, S_n)$ such that , $g^{(n)}(\xi) = 0$

**Lemma 5.2.3**
The expression in the truncation error $e(x)$ is given by:

$$e(x) = \prod_{k=0}^{n}(x - x_k)\frac{f^{(n+1)}(\xi)}{(n + 1)!}$$

with $\xi = \xi(x)$ and $\min_k(x, x_k) < \xi < \max_k(x, x_k)$

**Proof**

For points $x = x_k, \quad k = 0, 1, 2, \ldots, n$, the theorem is trivially satisfied.
Suppose $x \neq x_k$, we define

$$q(t) = \prod_{k=0}^{n}(t - x_k)$$

and

$$g(t) = f(t) - P_n(t) - \frac{q(t)}{q(x)}(f(x) - P_n(x)) \text{ for } t, x \in [a, b].$$

Now $g(x_k) = 0, \quad k = 0, 1, 2, \ldots n$ Also $g(x) = 0 \quad (x \neq x_k)$. So for each fixed $x \neq x_k, \; g(t)$ has $n + 2$ distinct zeros.
Since $g(t)$ satisfies all the conditions of Lemma 1.2, we deduce that there is a number $\xi$ such that $g^{(n+1)}(\xi) = 0$ where

$$\min_k(x, xk) < \xi < \max_k(x, x_k).$$

Now

$$\begin{aligned} g^{(n+1)}(t) &= f^{(n+1)}(t) - P_n^{(n+1)}(t) - \frac{(f(x) - P_n)q^{(n+1)}(t)}{q(x)} \\ &= f^{(n+1)}(t) - (n+1)! \frac{(f(x) - P_n(x))}{q(x)} \end{aligned}$$

Since

$$P^{n+1}(t) = 0$$

and

$$q_{n+1}(t) = (n+1)!$$

The result of the theorem follows when $t = \xi$.

**Example 5.2.9** Using two point linear interpolation of $e^x$ on $[0, 1]$ therefore

$$e^x = \frac{x(x-1)}{2} e^{\xi}, \quad \xi \in (0, 1).$$

But $x(x - 1)$ is maximum or minimum at $x = \dfrac{1}{2}$ with maximum absolute value equal to 1. So $e(x)$ has a maximum value equal to $\dfrac{e}{8} = 0.3398$ (4dp)

### 5.2.7 Rounding errors in Lagrange polynomials

Errors known as rounding errors are usually introduced in the functional evaluation of $f(x_k)$. Suppose rounding errors $E_k$ occur in the data $f(x_k) = f(x_k), \quad k = 0, 1, 2, \ldots, n$ respectively. Let $P(x)$ and $P^{\star}(x)$ denote, respectively, the interpolating polynomials using exact and inexact data.
Thus,

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} (f(x_k) + E_k) L_k(x) \\ P_n^{\star}(x) &= \sum_{k=0}^{n} f(x_k) L_k(x) \\ \text{so} \quad |P_n(x) - P_n^{\star}(x)| &\leq \sum_{k=0}^{n} |E_k||L_k(x)| \end{aligned}$$

Which is the rounding error bound. If data was rounded to $m$ decimal places, then the absolute maximum error is $|E_k| \leq \dfrac{1}{2} \times 10^{-m}$ for each $k = 0, 1, \ldots, n$.

**Example 5.2.10** Find the rounding error bound in linear interpolation, of $e^x$ for $x \in [0, 1]$, when data were rounded to four digits.

**Solution**

$$|P_1(x) - P_1*(x)| \leq \frac{1}{2}10^{-4}(|L_0(x)| + |L_1(x)|)$$

$$\text{Now,} \quad \mathrm{L}_0(\mathrm{x}) = 1 - x$$

$$\text{and} \quad \mathrm{L}_1(\mathrm{x}) = x$$

$$\text{and so on} \quad [0,1], \; |\mathrm{L}_0(\mathrm{x})| + |\mathrm{L}_1(\mathrm{x})| = 1 - x + x = 1.$$

$$\text{Thus,} \quad |P_1(x) - P_1^{\star}(x)| \leq \frac{1}{2}10^{-4},$$

which says that the effect of rounding errors in the data, on $P_1(x)$ maintains the same maximum magnitude.

**Example 5.2.11** The table below gives the tabulated values of the probability integral,

$$I = \sqrt{\frac{2}{\pi}} \int_0^x e^{\frac{-t^2}{2}} dt$$

Use linear interpolation to find the value of $I$ when $x = 1.125$. Estimate also the error bound on the truncation error over $[1, 1.25]$

| | | | | | | |
|---|---|---|---|---|---|---|
| x | 1.00 | 1.05 | 1.10 | 1.15 | 1.20 | 1.25 |
| I | 0.683 | 0.705 | 0.729 | 0.750 | 0.770 | 0.789 |

**Solution**

**Note 5.2.2**
At the tabular points $x_0, x_1, ..., x_n$, the truncation error is zero, hence we can write

$$f(x) = P_n(x) + R\prod_{i=0}^{n}(x - x_i) \text{ (R constant depending on x)}$$

Let,

$$F(x) = f(t) - P_n(t) - \{f(x) - P_n(x)\}\frac{\prod_{i=0}^{n}(t - x_i)}{\prod_{i=0}^{n}(x - x_i)} \quad (*)$$

then

$$F(x_i) = f(x_i) - P_n(x_i) = 0, \quad i = 0, 1, 2, \ldots n$$

Further

$$F(x) = f(x) - P_n(x) - f(x) + P_n(x) = 0, \quad \forall \; x \in (a, b), \quad x \neq x_i,$$

necessarily and $i = 0, 1, 2, \ldots n$.
Thus, $F(t)$ vanishes in $(a, b)$ at $n + 2$ distinct points. On applying Rolle's theorem repeatedly we conclude that there exists $c \in (a, b)$ such that, $F^{(n+1)}(c) = 0$.
Differentiating equation $(*)$ $(n + 1)$ times and putting $t = c$, gives,

$$f(x) = P_n(x) + \prod_{i=0}^{n}(x - x_i)\frac{f^{(n+1)}(c)}{(n + 1)!}$$

For linear interpolation, $n = 1$, so,

$$P_1(x) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1)$$

with

$$x_0 = 1.10, \quad x_1 = 1.15 \quad \Rightarrow f(x_0) = .729, \quad f(x_1) = 0.750$$

thus ,

$$\begin{aligned} P_1(1.125) &= \frac{(1.125 - 1.15)(0.729)}{-0.05} + \frac{(1.125 - 1.10)(0.75)}{0.05} \\ &= 0.5(0.729 + 0.750) = 0.7395 \end{aligned}$$

Truncation error

$$\prod_{i=0}^{1}(x - x_i)\frac{f^{(2)}(c)}{2!}$$

Now $(x - x_0)(x - x_1)$ is minimum at $x = \dfrac{x_0 + x_1}{2}$ and is equal to $\dfrac{(x_1 - x_0)^2}{4}$ in magnitude.

$$f'(x) = \sqrt{\frac{2}{\pi}}e^{\frac{-x^2}{2}}, \qquad f''(x) = -\frac{\sqrt{2}}{\pi}xe^{-\frac{x^2}{2}}$$

$$f'''(x) = -\frac{\sqrt{2}}{\pi}e^{\frac{-x^2}{2}} + \frac{\sqrt{2}}{\pi}x^2e^{\frac{-x^2}{2}}$$

and is 0 if $x = \pm 1$. Thus ,

$$|f''(x)| \leq \frac{\sqrt{2}}{\pi\sqrt{e}}$$

A bound on the truncation error is given by,

$$\frac{(0.05)^2}{8\pi}\sqrt{\frac{2}{e}} \simeq 0.00009$$

**Exercise 5.2.2**

1.) Compute a bound on the truncation error for $e^x$ on $[1, 1.4]$ when fourth degree polynomial is used to interpolate $e^x$.

2.) Obtain error bounds for both linear and quadratic interpolation for $\sin hx$ over the interval $[1.90, 2.10]$

3.) A table of values for $\dfrac{x^4 - x}{12}$ is constructed for $0 \leq x \leq 1$ in such a way that the error in linear interpolation does not exceed $\epsilon$ if rounding errors were neglected. Show that for uniform spacing $h$, then $h$ does not exceed the value $2\sqrt{2\epsilon}$.

4.) Find the rounding error bound when quadratic interpolation, of $e^x$ for $x \in [0, 1]$, when data were rounded to four digits.

**Note 5.2.3** Use Lagrange interpolation to find an appropriate function passing through the given points. Sketch a graph of this function based only on the given points and what you think the curve must be. Compare your sketch with the graph created by graphing technology.

1.) A linear function passing through the points $(-1, 3)$ and $(2, 1)$.

2.) A quadratic function passing through the points $(-1, 3)$, $(0, 2)$, and $(2, 1)$.

3.) A cubic function passing through the points $(-1,3),(0,2),(1,5)$, and $(2,1)$.

4.) A quartic (fourth degree) polynomial function passing through $(-2,4),(-1,3),(0,2),(1,5)$, and $(2,1)$.

**Note 5.2.4** Finding a quadratic function that resembles other functions: By choosing three noncolinear points on any curve we can use Lagrange's interpolation to find a parabola that passes through those points. For each of the following functions find a parabola that passes through the graphs of the functions when at points with the indicated first coordinates. Use graphing technology to draw a sketch of the function and the quadratic function you find. Discuss how you might use the function you find to estimate the value of the given function.

(a) $f(x) = x^5 - 4x^3 + 2; x = 0, 1, 2.$

(b) $f(x) = \sqrt{x}; x = 0, 1, 4.$

(c) $f(x) = 2^x; x = -1, 0, 1.$

(d) $f(x) = 2^x; x = 0, 1, 2.$

(e) $f(x) = \sin\left(\frac{\pi}{2}x\right); x = 0, 1, 2.$

(f) $f(x) = \cos\left(\frac{\pi}{2}x\right); x = -1, 0, 1$ .

Describe a general procedure for finding a polynomial function of degree $n$ that passes through $n+1$ given points with distinct first coordinates.

**Note 5.2.5** Lagrange's interpolation formula has the disadvantage that the degree of the approximating polynomial must be chosen at the outset; an alternative approach is discussed in the next Step. Thus, Lagrange's formula is mainly of theoretical interest for us here; in passing, we mention that there are some important applications of this formula beyond the scope of this book - for example, the construction of basis functions to solve differential equations using a spectral (discrete ordinate) method.

**Note 5.2.6** Given the data below,

| $x$ | -3 | 1 | 4 | 5 | 7 |
|---|---|---|---|---|---|
| $f(x)$ | -28 | 4 | 28 | 36 | 52 |

Use linear interpolation to approximate

(a) $f(3)$.

(b) $f(x)$ if $x = 5$.

(c) $x$ if $f(x) = 12$.

(d) the root/solution of $f(x)$. $f(x) = 8x - 4$

**Example 5.2.12** Use Lagrange's polynomial to approximate $f(2)$ for

| | $x_0$ | $x_1$ |
|---|---|---|
| $x$ | 1 | 3 |
| $f(x)$ | 5 | 21 |

$$f(x) = 8x - 3$$

$$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} = \frac{1}{2}(3 - x)$$
$$L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)} = \frac{1}{2}(x - 1)$$

$$\begin{aligned} P_1(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) \\ &= \frac{1}{2}(3 - x)(5) + \frac{1}{2}(x - 1)(21) \\ &= \frac{1}{2}[15 - 5x + 21x - 21] \\ &= \frac{1}{2}[16x - 6] \\ &\approx 8x - 3 \end{aligned}$$

$$f(2) = 8(2) - 3 = 13$$

**Example 5.2.13** The Lagrange basis polynomials **(n=N-1)** on the data to approximate $f(3.5)$

| | $x_0$ | $x_1$ | $x_2$ |
|---|---|---|---|
| $x$ | 1 | 2 | 4 |
| $f(x)$ | 3 | 2 | 1 |

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{1}{3}(x - 2)(x - 4)$$
$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = -\frac{1}{2}(x - 1)(x - 4)$$
$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{1}{6}(x - 1)(x - 2)$$

$$\begin{aligned} P_2(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \\ &= \frac{1}{3}(x - 2)(x - 4)(3) - \frac{1}{2}(x - 1)(x - 4)(2) + \frac{1}{6}(x - 1)(x - 2) \\ &= (x - 2)(x - 4) - (x - 1)(x - 4) + \frac{1}{6}(x - 1)(x - 2) \\ &= \frac{1}{6}x^2 - \frac{3}{2}x + \frac{13}{3} \\ &\approx 0.1667x^2 - 1.5x + 4.3333 \end{aligned}$$

$$f(3.5) \approx 0.1667(3.5)^2 - 1.5(3.5) + 4.3333 \approx 1.125075$$

**Note 5.2.7** Lagrange error for $n = 2$,

$$f(x) = P_2(x) + \frac{f^{(3)}(\xi)}{3!}(x - x_0)(x - x_1)(x - x_2),\ \xi \in [a, b],\ \text{where } \{x_0, x_1, x_2\} \subseteq [a, b]$$

**Note 5.2.8** The identity

$$\sum_{k=0}^{n} L_k(x) = 1$$

(established by setting $f(x) = 1$) may be used as a check. Note also that with $n = 1$ we recover the linear interpolation formula:

$$\begin{aligned} P_1(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) \\ &= \frac{(x - x_1)}{(x_0 - x_1)}f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)}f(x_1) \\ &= f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)}[f(x_1) - f(x_0)] \end{aligned}$$

**Example 5.2.14** Use Lagrange's interpolation formula to find the interpolating polynomial $P_3(x)$ through the points $(0, 3), (1, 2), (2, 7)$, and $(4, 59)$ and then find the approximate value of $P_3(3)$.

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 4 |
| $f(x)$ | 3 | 2 | 7 | 59 |

Here $n = 3 = 4 - 1$
The Lagrange coefficients are:

$$\begin{aligned} L_0(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{(x - 1)(x - 2)(x - 4)}{(0 - 1)(0 - 2)(0 - 4)} = -\frac{1}{8}(x^3 - 7x^2 + 14x - 8) \\ L_1(x) &= \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{(x - 0)(x - 2)(x - 4)}{(1 - 0)(1 - 2)(1 - 4)} = \frac{1}{3}(x^3 - 6x^2 + 8x) \\ L_2(x) &= \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = \frac{(x - 0)(x - 1)(x - 4)}{(2 - 0)(2 - 1)(2 - 4)} = -\frac{1}{4}(x^3 - 5x^2 + 4x) \\ L_3(x) &= \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{(x - 0)(x - 1)(x - 2)}{(4 - 0)(4 - 1)(4 - 2)} = \frac{1}{24}(x^3 - 3x^2 + 2x) \end{aligned}$$

(The student should verify that $[L_0(x) + L_1(x) + L_2(x) + L_3(x) = 1]$

Hence, the required polynomial is

$$\begin{aligned} P_3(x) = &-\frac{3}{8}(x^3 - 7x^2 + 14x - 8) + \frac{2}{3}(x^3 - 6x^2 + 8x) \\ &-\frac{7}{4}(x^3 - 5x^2 + 4x) + \frac{59}{24}(x^3 - 3x^2 + 2x) \end{aligned}$$

$= \frac{1}{24}\left[-9x^3 + 63x^2 - 126x + 72 + 16x^3 - 96x^2 + 128x - 42x^3 + 210x^2 - 168x + 59x^3 - 177x^2 + 118x\right]$

$$\begin{aligned} &= \frac{1}{24}\left[-9x^3 + 63x^2 - 126x + 72 + 16x^3 - 96x^2 + 128x - 42x^3 + 210x^2 - 168x + 59x^3 - 177x^2 + 118x\right] \\ &= \frac{1}{24}\left[24x^3 + 0x^2 - 48x + 72\right] \\ &= x^3 - 2x + 3 \end{aligned}$$

Consequently $f(3) \approx P(3) = 3^3 - 2(3) + 3 = 24$, However, note that, if the explicit form of the interpolating polynomial were not required, one would proceed to evaluate $P_3(x)$ for some value of $x$ directly from the factored forms of $L_k(x)$. Thus, in order to evaluate $P_3(3)$, one has

$$L_0(3) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)} = \frac{1}{4}. \text{ etc.}$$

**Exercise 5.2.3** Given that $f(-2) = 46, f(-1) = 4, f(1) = 4, f(3) = 156$, and $f(4) = 484$, use Lagrange's interpolation formula to estimate the value of $f(0)$.

**Example 5.2.15** Use Lagrange interpolation polynomial for the data below

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|
| $x$ | 1 | 2 | 4 | 5 |
| $f(x)$ | 3 | 8 | 54 | 107 |

to show that $P(x) = x^3 - x^2 + x + 2$ The Lagrange coefficients are:

$$\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-2)(x-4)(x-5)}{(1-3)(1-4)(1-5)} = -\frac{1}{12}(x^3 - 11x^2 + 38x - 40)\\
L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-1)(x-4)(x-5)}{(2-1)(2-4)(2-5)} = \frac{1}{6}(x^3 - 10x^2 + 29x - 20)\\
L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-1)(x-2)(x-5)}{(4-1)(4-2)(4-5)} = -\frac{1}{6}(x^3 - 8x^2 + 17x - 10)\\
L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-1)(x-2)(x-4)}{(5-1)(5-2)(5-4)} = \frac{1}{12}(x^3 - 7x^2 + 14x - 8)
\end{aligned}$$

(The student should verify that $[L_0(x) + L_1(x) + L_2(x) + L_3(x) = 1]$ Hence, the required polynomial is

$$\begin{aligned}
P_3(x) = &-\frac{1}{12}(x^3 - 11x^2 + 38x - 40)[3] + \frac{1}{6}(x^3 - 10x^2 + 29x - 20)[8]\\
&+\frac{1}{6}(x^3 - 8x^2 + 17x - 10)[54] + \frac{1}{12}(x^3 - 7x^2 + 14x - 8)[107]
\end{aligned}$$

$= \frac{1}{12}\left[-3x^3 + 33x^2 - 114x + 120 + 16x^3 - 160x^2 + 464x - 320 - 108x^3 + 864x^2 - 1836x + 1080 + 107x^3 - 749x^2 + 1498x -\right.$

$$\begin{aligned}
&= \frac{1}{12}\left[-3x^3 + 33x^2 - 114x + 120 + 16x^3 - 160x^2 + 464x - 320 - 108x^3 + 864x^2 - 1836x + 1080 + 107x^3\right.\\
&= \frac{1}{12}\left[12x^3 - 12x^2 + 12x + 24\right]\\
&= x^3 - x^2 + x + 2
\end{aligned}$$

Consequently $f(3.5) \approx P(3.5) = (3.5)^3 - (3.5)^2 + (3.5) + 2 = 36.125$,