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# Optismoothing: An optimization-driven approach to mesh smoothing

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**Abstract.** This paper presents a mesh smoothing technique that uses optimization principles to minimize a distortion metric throughout a mesh. A comparison is made with laplacian and isoparametric smoothing techniques.

## Introduction

The finite element method is a very popular and powerful means of analyzing complicated geometries. Quite often the most difficult and tedious part of the process is generating an acceptable mesh. An acceptable mesh is one which provides answers of sufficient accuracy at a reasonable cost. Since accuracy and error are functions of the geometry of the domain, the boundary conditions, the governing differential equations, as well as the geometry of the finite element mesh, there is no unique relationship between distortion and error. However, for most problems, distorted elements cause the convergence rate to decrease and the error to increase [1,2]. Creating well shaped elements of the desired type is a critical facet of mesh generation. This is especially true of automatic mesh generation where irregular mesh topologies are common. The process of improving the shape of existing elements in a mesh is known as *smoothing*. This paper presents a smoothing technique, referred to as *optismoothing*, which is based on optimization techniques.

## Prevalent Smoothing Techniques

Smoothing reduces element distortion in the mesh by moving the non-fixed nodes. Two prevalent types of mesh smoothing are the laplacian and isoparametric smoothing techniques.

### *Laplacian smoothing*

The laplacian method is the most common mesh smoothing algorithm employed by mesh generation codes. It determines each node's new location by averaging the coordinates of the nodes connected to it.

$$\vec{P} = \frac{1}{n} \sum_{i=1}^{n} \vec{P_i}, \tag{1}$$

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where $\vec{P}$ is the coordinate position vector of the node being considered, and $\vec{P_i}$ are the position vectors of the neighboring nodes directly connected to $\vec{P}$. Jones [3] developed several variations on this method by implementing various weighting schemes in the summation process. In Fig. 1, the new location for $\vec{P}$ is computed as the average of $\vec{P_1}$ through $\vec{P_6}$.

*Isoparametric smoothing*

In the laplacian smoothing technique, a node's smoothed position is a function of the nodes with which it shares an *edge*. By contrast, in the isoparametric method, a node's smoothed position is a function of the nodes with which it shares a *face*. For example, to compute the new position for a node $\vec{P_s}$ in Fig. 2, all of the nodes in the faces that are connected to $\vec{P_s}$ are used in the smoothing process.

The coordinates of all of the nodes in these faces that share an edge with $\vec{P_s}$ are summed while all other nodes (i.e., nodes on opposite corners from it) are subtracted. For example, in the upper face in Fig. 2, $\vec{P_2}$ and $\vec{P_4}$ are added, while $\vec{P_3}$ is subtracted.

## Optimization-driven smoothing

Optimization techniques locate the local minima (or maxima) of complicated functions and can allow for constraints to be applied to the solutions. Minimizing the distortion throughout a mesh is a natural application of these techniques. However, in order to use optimization techniques, a satisfactory means of measuring the distortion must be found. This distortion metric must be a *continuous* function of the coordinate positions of the nodes in order for the optimization algorithm to function properly.

Once an adequate measure of element distortion is found, it seems more natural to think of smoothing as a process that minimizes the distortion in a mesh (as opposed to a series of operations performed directly on the nodes). This section will briefly discuss a couple of common distortion metrics and then present a recently published metric that is well suited for optimization algorithms.

*Traditional distortion metrics*

Two of the most empirical rules to determine acceptable levels of distortion are to keep the aspect ratio less than 3 : 1 for stress computations and 7 : 1 for displacement computations [1]

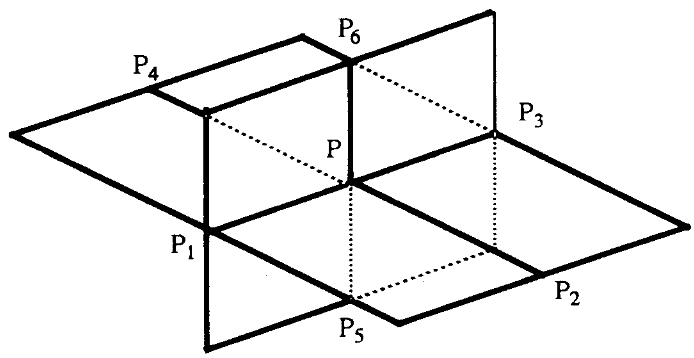



Fig. 1. Laplacian smoothing.

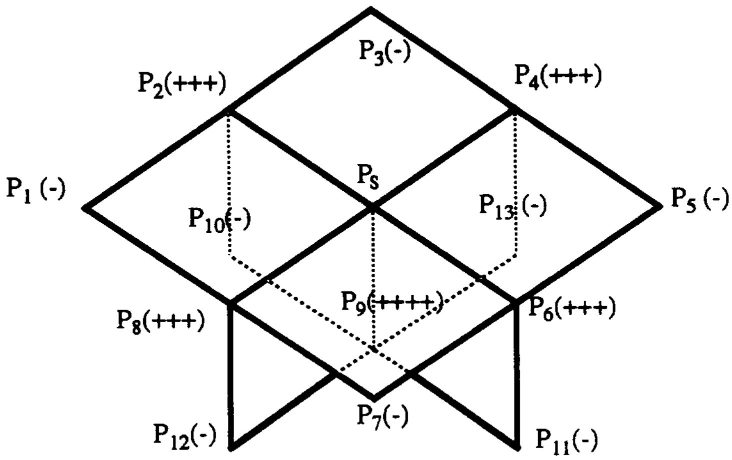



Fig. 2. Isoparametric smoothing.

and restrict the corner angles to range between 45 and 135 degrees [4]. (A number of other measures of element distortion have been developed and several of these are reviewed by Oddy [5].)

While these empirical rules are intuitive and helpful in understanding the distortion in a mesh, they do not lend themselves to being easily programmed as continuous functions. More importantly, they neglect the coupling effects of stretching and shearing. While it may be more descriptive to know them separately, element performance depends on both together.

*Oddy's distortion metric*

Oddy [5] recently published a new distortion metric that lends itself well to optimization techniques. This section will present an abbreviated derivation of the equation (eqn. (8) below) used to compute the distortion metric for an element. (For a more complete derivation and description, see Oddy [5].)

The first step is to normalize the Jacobian matrix to remove the effect of element size:

$$J'_{ij} = \frac{J_{ij}}{|J|^{1/n}}, \tag{2}$$

where

$$n = 2 \quad \text{(for 2-D)}, \tag{3}$$

$$n = 3 \quad \text{(for 3-D)}. \tag{4}$$

Since an analogy can be drawn between element distortion and strain, Green's strain is used. Only one term from Green's strain, $C_{ij}$, is needed for the distortion metric computation. It is a direct function of the Jacobian:

$$C_{ij} = J'_{ki} J'_{kj}, \tag{5}$$

where

$$i, j = 1, 2 \quad \text{(for 2-D)}, \tag{6}$$

$$i, j = 1, 2, 3 \quad \text{(for 3-D)}. \tag{7}$$

The effects of local changes in area or volume are removed from the computation. Oddy's distortion metric, $D$, is given by:

$$D = C_{ij}C_{ij} - \frac{1}{n}(C_{kk})^2. \tag{8}$$

Oddy's distortion metric is capable of computing a *continuous* scalar evaluation of an element's distortion, and is sensitive to combined stretching and shearing of the element. Since it is a function of the Jacobian, it is not affected by rigid body displacements, is independent of element size and is general enough to work for many different types of elements.

*Optimizing the distortion metric*

Oddy's distortion metric very naturally lends itself to an optimization algorithm that computes the distortion of all of the elements in a mesh, sums them, and minimizes the sum. This allows the distortion of some elements to increase in order to reduce the distortion of the whole. The shortcomings of the traditional metrics are overcome with this metric in that it is easily programmed and is sensitive to combined stretching and shearing.

*Selection of an optimization algorithm*

The conjugate gradient method was chosen as the optimization technique. It is super-linear in convergence and does not require any matrix manipulations. How this algorithm performs for an extremely large number of degrees of freedom is a real concern, in that finite element meshes can easily exceed 10,000 degrees of freedom. Since the conjugate gradient method does not require the use of matrices (whose size would be determined by the number of degrees of freedom), it is superior to other optimization methods (such as Newton, quasi-Newton, GRG, and SQP). Gill [6] states the following:

> "Although conjugate-gradient-type algorithms are far from ideal, they are currently the *only* reasonable method available for a general problem in which the number of variables is extremely large"

Although the conjugate gradient method is an unconstrained optimization technique, it is adequate for many 2-D and 3-D meshes. Constrained methods *might* be required if it were necessary for the nodes to remain on a given surface.

The conjugate gradient method consist basically of the following:

$$X_0 = \text{arbitrary}, \tag{9}$$

$$G_0 = \nabla F(X_0), \tag{10}$$

$$S_0 = -G_0, \tag{11}$$

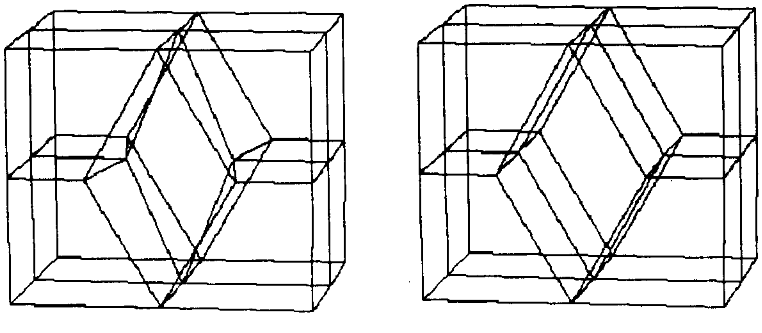
$$X_{i+1} = X_i + \alpha_i^* S_i, \tag{12}$$

$$G_{i+1} = \nabla F(X_{i+1}), \tag{13}$$

$$\beta_i = |G_{i+1}|^2/|G_i|^2, \tag{14}$$

$$S_{i+1} = -G_{i+1} + \beta_i S_i, \tag{15}$$

where: $X_i$ is the nodal coordinate array; $F$ is the sum total of element distortion values at a given $X$; $G_i$ is a gradient vector which stores the partial derivatives of total distortion with



(a) (b)

Fig. 3. (a) After length-weighted laplacian smoothing, the average value of Oddy's distortion metric is 0.34 and the largest value is 1.08. (b) After optismoothing, the average value of Oddy's distortion metric is 0.31 and the largest element distortion value is 0.735.

respect to nodal coordinate displacements; $S_i$ is the search direction; $\alpha_i^*$ is the distance to move in search direction (a quadratic line search was used); and $\beta_i$ is a correction factor to enhance convergence.

### *Comparison to other types of smoothing*

The optimization-driven smoothing has given very good results for the analyzed models. A five to twenty percent decrease in the total mesh distortion (as measured using Oddy's metric) over the length-weighted laplacian technique is common. An example is presented in Fig. 3.

## Conclusions

Optismoothing attempts to reduce distortion throughout the mesh, instead of concentrating its efforts directly on nodal operations. Optismoothing yields better results, especially around transition elements. This, of course, depends directly on the distortion metric used to drive the optimization. Experience has shown that one iteration is usually all that is needed for most applications. Optismoothing is also less likely to diverge.

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