ARTICLE NO. 79-1462R

J80-119

Direct Control of the Grid Point Distribution in Meshes Generated by Elliptic Equations

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A popular method of constructing computational meshes generates the physical grid numerically as the solution to an elliptic boundary value problem. The resulting grid point distribution in the interior of the flow region is difficult to control. An effective method of interior grid control is presented based on a modified elliptic system containing frer parameters. For a simply connected region, the free parameters are computed from the Dirichlet boundary values. The resulting interior grid point distribution is controlled entirely by a priori selection of the grid point distribution along the boundaries of the region. The method also enables one to control the local angle of intersection between transverse grid lines and the boundary. In particular, the transverse lines may be constrained to be locally orthogonal to the boundary. Multiply connected regions are treated by segmentation into simply connected subregions.

I. Introduction

THE generation of computational grids suitable for carrying out accurate numerical solutions to the three-dimensional Navier-Stokes equations is currently the subject of intensive research. The research reported herein on grid generation techniques forms part of a larger study aimed at the prediction of viscous, compressible flow in and about three-dimensional nozzles.[1,2] For a wide class of nozzle configurations, a three-dimensional grid can be constructed by a sequence of two-dimensional grids in successive cross-sectional planes. The present paper is concerned with numerical generation of two-dimensional grids.

It is important that the coordinate system and grid conform to the shape of the boundaries of the flow region because of the strong influence that the boundary conditions have on the accuracy of the computed flowfield. Thompson et al.,[3] have a general method for numerically generating boundary-conforming coordinate transformations. In the method, one computes the Cartesian coordinates (x,y) of the grid points in physical space as solutions to an elliptic system of partial-differential equations. The equations are formulated in a rectangular computation domain (ξ,η) and are solved numerically over a uniform, rectangular grid in that domain. The Thompson, Thames, Mastin (TTM) method employs the following inhomogeneous Laplace equations as the generating system:

$$\xi_{xx}+\xi_{yy}=P(\xi,\eta)$$
$$\eta_{xx}+\eta_{yy}=Q(\xi,\eta) \tag{1}$$

These equations are transformed to ξ,η coordinates by interchanging the roles of dependent and independent variables. This yields an elliptic system of quasilinear equations

$$\alpha x_{\xi\xi}-2\beta x_{\xi\eta}+\gamma x_{\eta\eta}=-J^2(Px_\xi+Qx_\eta)$$
$$\alpha y_{\xi\xi}-2\beta y_{\xi\eta}+\gamma y_{\eta\eta}=-J^2(Py_\xi+Qy_\eta)$$
$$\alpha=x_\eta^2+y_\eta^2$$
$$\beta=x_\xi x_\eta+y_\xi y_\eta$$
$$\gamma=x_\xi^2+y_\xi^2 \tag{2}$$

where J denotes the Jacobian of the transformation

$$J=\frac{\partial(x,y)}{\partial(\xi,\eta)}=x_\xi y_\eta-x_\eta y_\xi \tag{3}$$

Equations (2) are solved numerically on a uniform, rectangular grid $\Delta\xi$, $\Delta\eta$. For a simply connected physical domain such as a nozzle interior, Dirichlet boundary conditions may be specified over the entire closed boundary $A'B'C'O'$ of the computational domain (see Fig. 1). The boundary values are the (x,y) coordinates of the physical grid point that corresponds to each of the mesh points (ξ_k,η_ℓ) on the boundary of the rectangular computational domain.

The physical grid points may be spaced as desired along the boundaries of the flow region. However, in practice, it is difficut to control the spacing between grid points in the interior of the flow region. The interior grid spacing is governed primarily by the elliptic system Eqs. (2) itself. The boundary values have a strong influence on the grid spacing only in the immediate neighborhood of the boundaries, in spite of the strongly elliptic character of the equations. It is especially important to have good control over the interior grid spacing in high Reynolds number viscous flow problems, where accurate resolution of high gradient regions such as wall boundary layers demands a locally refined grid.

Various forms of the source terms P, Q in Eqs. (2) have been devised that contain adjustable parameters and provide some measure of control over the interior grid spacing. This ad hoc approach has been used with success in particular applications.[4-6] However, the forms of these source terms and the values of the adjustable parameters require artful selection and are problem dependent.

An example is given in Figs. 2 and 3 to illustrate the difficulty of controlling the grid in the nozzle interior. Figure 2 displays a quasielliptical grid generated by the TTM method

Presented as Paper 79-1462 at the AIAA 4th Computational Fluid Dynamics Conference, Williamsburg, Va., July 23-24, 1979; submitted Aug. 20, 1979; revision received Nov. 26, 1979.

Index categories: Computational Methods; Analytical and Numerical Methods.

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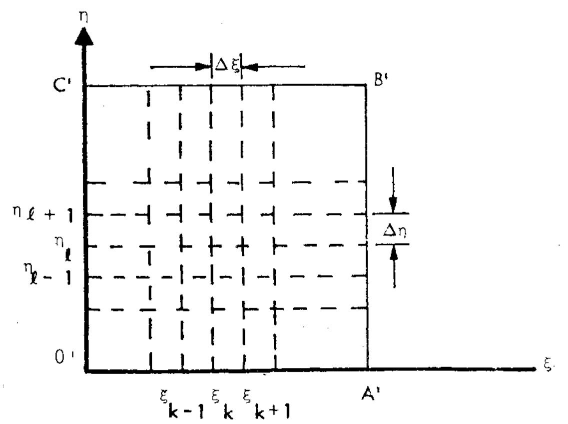

Fig. 1 Rectangular computational domain for the nozzle interior.

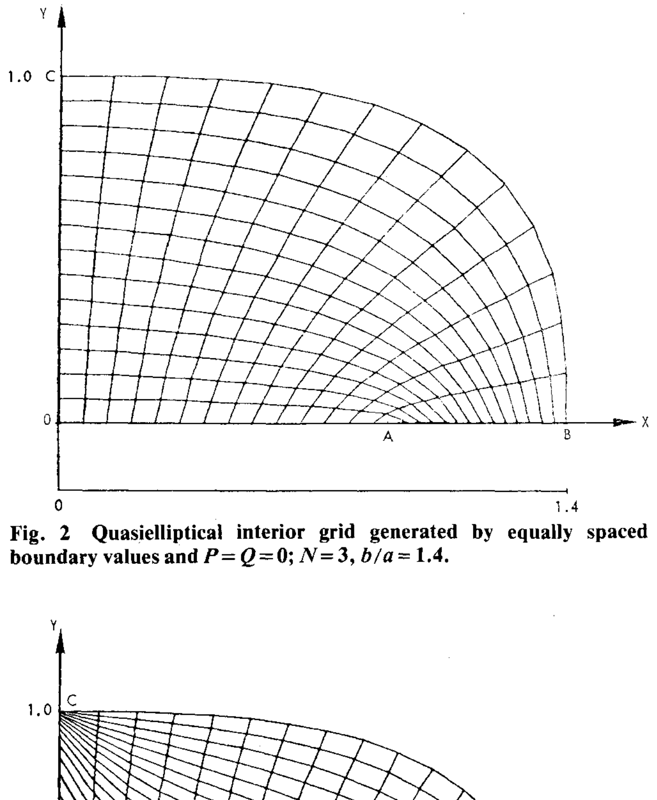

Fig. 2 Quasielliptical interior grid generated by equally spaced boundary values and $P=Q=0$; $N=3$, $b/a=1.4$.

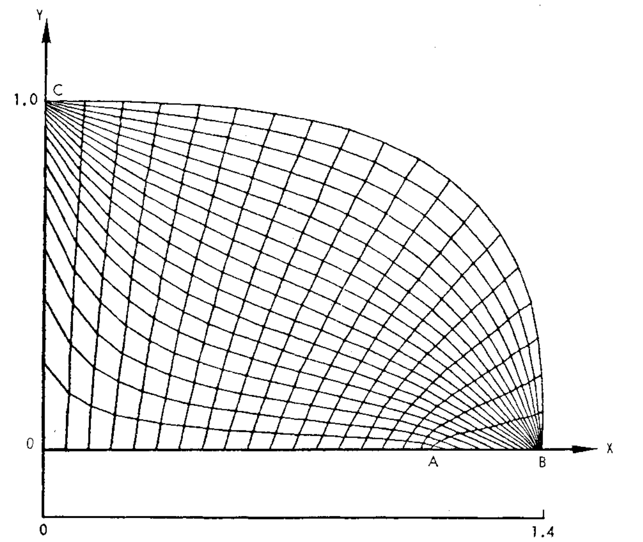

Fig. 3 Quasielliptical grid generated by exponentially spaced boundary values on AB and CO with $P=Q=0$.

by using simple Laplace equations ($P=Q=0$). The arc BC (the nozzle wall) was generated by the superellipse $(x/a)^n+(y/b)^n=1$ where $N=3$, $b/a=1.4$. The quasielliptical shape of the grid was achieved by specifying the boundary values so that the points $ABCO$ of the physical

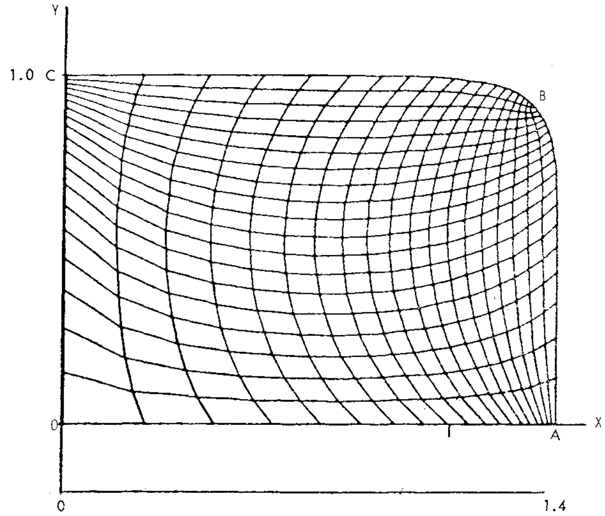

Fig. 4 Quasirectangular grid generated by exponentially spaced boundary values and $P=Q=0$; $N=10$, $b/a=1.4$.

domain map onto the points $A'B'C'O'$ of the rectangular computational domain of Fig. 1. The point A plays a role akin to that of the focus in the classical system of orthogonal elliptical coordinates. Equally spaced boundary values (x,y) were specified along each of the boundary segments AB, BC, CO, and OA. The resulting grid is smooth and regular in the interior and would be suitable for inviscid flow computations. However, this grid would be incapable of resolving the nozzle wall boundary layer in a viscous flow. In an attempt to obtain a locally refined grid near the nozzle wall, an exponential distribution of boundary values was specified along the boundary segments AB and CO. The resulting grid, displayed in Fig. 3, is wholly unsatisfactory. Evidently, the influence of the boundary values does not penetrate very deeply into the interior, in spite of the elliptic character of Eqs. (2). The interior grid is affected primarily by the generating equations, rather than by the boundary values.

Equally poor results are obtained whenever highly stretched boundary values are used with $P=Q=0$. A further example for a quasirectangular mapping of high-degree superellipse with exponentially stretched boundary values is displayed in Fig. 4.

II. Present Method

Simply Connected Regions

The described difficulty with interior grid control has been overcome by devising general source terms P and Q that are computed from the Dirichlet boundary values and that vary continuously throughout the computational domain. This results in an elliptic generating system whose solutions have the property that the grid point distribution in the interior is controlled entirely by a priori selection of the grid point distribution along the boundaries. This is accomplished by choosing source terms such that Eqs. (2) possess exponential solutions, although the terms themselves are not exponential functions. The terms contain arbitrary parameters that are evaluated locally at the boundaries of the computational domain by using a finite-difference representation of the limiting form of the system, Eqs. (2), that is valid at the boundaries. Interpolation of these parameters into the interior from the boundaries then defines the parameters at each mesh point of the computational domain. Numerical solution of Eqs. (2) by standard successive line over-relaxation (SLOR) then results in a grid point distribution throughout the physical domain that is controlled entirely by the distribution of grid points on the boundaries of that domain.

The source terms have the form

$$P=\phi(\xi,\eta)(\xi_x^2+\xi_y^2)$$

$$Q=\psi(\xi,\eta)(\eta_x^2+\eta_y^2) \tag{4}$$

where the parameters ϕ,ψ are yet to be specified. Upon introducing these terms, Eqs. (2) assume the form

$$\alpha(x_{\xi\xi}+\phi x_\xi)-2\beta x_{\xi\eta}+\gamma(x_{\eta\eta}+\psi x_\eta)=0$$

$$\alpha(y_{\xi\xi}+\phi y_\xi)-2\beta y_{\xi\eta}+\gamma(y_{\eta\eta}+\psi y_\eta)=0 \tag{5}$$

One can easily verify that Eqs. (5) possess exponential solutions if the parameters ϕ,ψ are locally constant. Given a set of boundary values (x,y) on the boundary of the computational domain, one can determine the parameters by requiring that the given boundary values satisfy appropriate limiting forms of Eqs. (5) along the boundary of the computational domain.‡ Equations that define the parameters φ and ψ in terms of the boundary values can be obtained from Eqs. (5) by imposing two a priori constraints on the local slope and curvature of the coordinate curves transverse to the boundary. In particular, one may impose the constraints that the transverse coordinate curves be locally straight and orthogonal to the boundary. This remarkable result can be proved as follows.

To find an equation that determines the function φ along either of the horizontal boundaries $\eta=\eta_b=$const in Fig. 1, we first eliminate ψ between the two equations. This yields a single equation that can be cast in the form

$$\alpha[y_\eta(x_{\xi\xi}+\varphi x_\xi)-x_\eta(y_{\xi\xi}+\varphi y_\xi)]$$

$$=y_\eta^2[2\beta(x_\eta/y_\eta)_\xi+\gamma(x_\eta/y_\eta)_\eta] \tag{6}$$

Now, the ratio x_η/y_η is merely the slope dx/dy of the family of coordinate curves $\xi=$const that are transverse to the boundary $\eta=\eta_b$. We are at liberty to impose the constraint that these transverse coordinate curves $\xi=$const be locally straight (i.e., have zero curvature) in the neighborhood of the boundary. This constraint may be stated in the form

$$(x_\eta/y_\eta)_\eta=0 \text{ on } \eta=\eta_b \tag{7}$$

We now impose the further constraint that the transverse coordinate curves be locally orthogonal to the boundary $\eta=\eta_b$. The orthogonality condition may be found as follows. Let $r=(x,y)$ denote the radius vector. Then the vector that is locally tangent to a coordinate curve $\eta=$const is

$$r_\xi=(x_\xi,y_\xi) \tag{8a}$$

Similarly, the local tangent vector to a coordinate curve $\xi=$const is

$$r_\eta=(x_\eta,y_\eta) \tag{8b}$$

The two families of coordinate curves are then orthogonal if and only if

$$r_\xi\cdot r_\eta=0 \tag{9}$$

‡An earlier version of the method[1,7] employed limiting forms of Eqs. (5) that were derived under the assumption that all derivatives with respect to the curvilinear coordinate direction transverse to the boundary vanish locally at the boundary. The authors are indebted to a reviewer for pointing out the flaw in this assumption, which is valid only at a boundary that is both straight and parallel to one of the Cartesian coordinate axes x,y. The present, much more powerful, method[2] evolved from an effort to correct the flaw in the original technique as presented in Refs. 1 and 7.

This orthogonality condition may be expressed in the form

$$x_\xi x_\eta+y_\xi y_\eta=0 \text{ on } \eta=\eta_b \tag{10}$$

When Eq. (6) is evaluated at the boundary $\eta=\eta_b$, the second term in brackets on the right-hand side vanishes by virtue of Eq. (7); the first term in those brackets also vanishes because $\beta=0$ under the orthogonality condition, Eq. (10). The latter equation may be used to eliminate all η-differentiated terms from the left-hand side of Eq. (6). This yields a limiting form of the elliptic system that is valid at the boundary $\eta=\eta_b$ and that can be solved directly for the parameter φ.

$$\varphi=-(x_\xi x_{\xi\xi}+y_\xi y_{\xi\xi})/(x_\xi^2+y_\xi^2) \text{ on } \eta=\eta_b \tag{11}$$

The latter is a universally valid equation that can be used to compute the numerical value of φ at each grid point on the boundary in terms of the boundary values x,y once the differential operators are replaced by central-difference operators

$$(x_\xi)_{k,\ell}\sim(x_{k+1,\ell}-x_{k-1,\ell})/2\Delta\xi \tag{12a}$$

$$(x_{\xi\xi})_{k,\ell}\sim(x_{k+1,\ell}-2x_{k,\ell}+x_{k-1,\ell})/(\Delta\xi)^2 \tag{12b}$$

The corresponding expression that determines the parameter ψ along boundaries $\xi=$const can be obtained directly from Eq. (11) by the substitution $\varphi,\xi\rightarrow\psi,\eta$.

Once the parameter ψ is defined at each mesh point of the vertical boundaries $\xi=$const in the computational domain, its value at interior mesh points can be computed by linear interpolation along horizontal mesh lines $\eta=$const. Similarly, ϕ is computed by interpolation along the vertical mesh lines $\xi=$const between the boundaries at which it is defined by Eq. (11). Equations (5) are then solved by SLOR iteration in the computational domain to generate the grid in the physical domain. The procedure for evaluating the parameters φ and ψ insures that the grid throughout the interior of the computational domain will be governed by the grid point distribution that is assigned on the boundaries, and that the transverse grid lines will be locally orthogonal to the boundaries.

Figures 5 and 6 illustrate grids generated from the same boundary conditions as the grids shown in Figs. 3 and 4, respectively. The grids shown in Figs. 5 and 6 were generated by the described technique for calculating P and Q. Note the smooth, regular character of the grids, the near-orthogonality of the grid lines, and the preservation throughout the domain of the grid point distribution specified on the boundaries.

Although the grids illustrated in Figs. 5 and 6 were generated using an exponential distribution of grid points along the physical boundaries of the flow region, the present technique is valid, in general, for any arbitrary boundary point distribution. An example of the ability of the present method of automatically constructing grids that conform to nonexponential distributions of boundary grid points is given in Fig. 7, which illustrates a grid generated for a plug nozzle. This grid has double clustering of grid points at the top and bottom of the grid as well as clustering at the right-hand side.

It is instructive to explore the geometric interpretation of Eq. (11), which is used to evaluate the parameter φ along a boundary $\eta=\eta_b=$const. That equation can be recast in the form

$$s_{\xi\xi}+\varphi s_\xi=0 \text{ on } \eta=\eta_b \tag{13a}$$

where s denotes the arc length along the boundary curve

$$ds=\sqrt{dx^2+dy^2} \qquad s_\xi=(x_\xi^2+y_\xi^2)^{1/2} \tag{13b}$$

Clearly, Eq. (13a) possesses exponential solutions if φ is constant. Thus, the use of Eq. (11) to evaluate the parameter

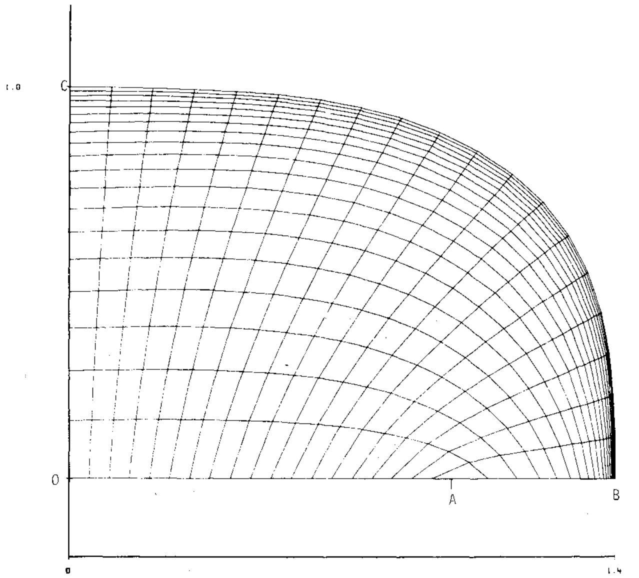

Fig. 5 Quasielliptical grid, exponentially spaced boundary values on *AB* and *CO*, *P,Q* computed from the boundary values; $N=3$, $b/a=1.4$.

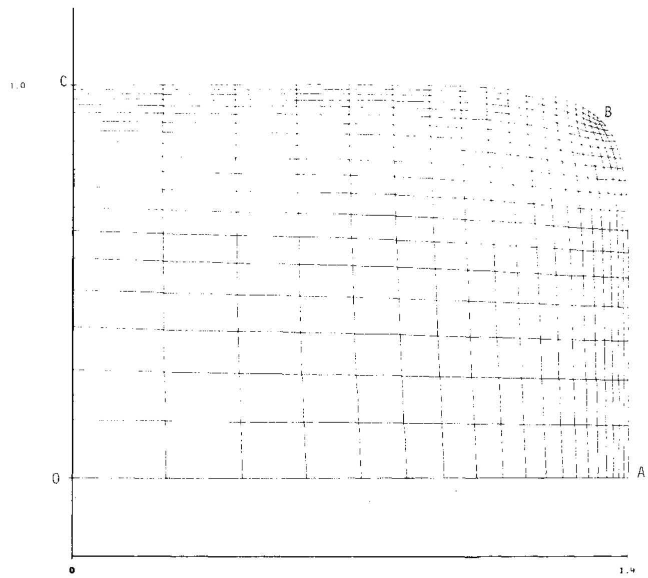

Fig. 6 Quasirectangular grid, exponentially spaced boundary values, *P,Q* computed from the boundary values; $N=10$, $b/a=1.4$.

φ locally at each point along the boundary $\eta=\eta_b$ is equivalent to constructing a local exponential curve-fit to the arc length between the preassigned boundary grid points. The interpolation of the parameters φ,ψ into the interior of the ξ,η domain simply extends the range of the curve-fit. Then the elliptic system, Eqs. (5), merely provides a reliable, automatic means for translating the parameters into a local exponential curve-fit at each interior point that reflects the boundary grid point distribution and has the properties of regularity and monotonicity that is required of nonsingular coordinate transformations. The resulting grid has the further desirable property that the two families of grid lines are locally orthogonal at the boundaries.

We now show that the general method has the flexibility to allow one to control at will the angle of intersection between the two families of grid lines at boundaries. This can be accomplished as follows. In place of the orthogonality condition, Eq. (9), we use the corresponding condition

$$r_\xi \cdot r_\eta = |r_\xi|\,|r_\eta|\cos\theta \tag{14}$$

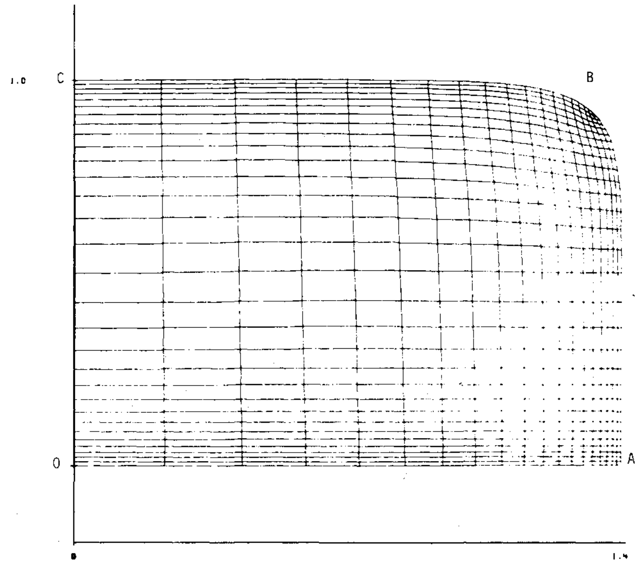

Fig. 7 Grid for plug nozzle; $N=10$, $b/a=1.4$.

where θ denotes the local angle of intersection between the boundary curve $\eta=\eta_b$ and the transverse coordinate curves $\xi=$const. A more convenient representation of this condition is:

$$x_\xi x_\eta + y_\xi y_\eta = (x_\xi y_\eta - y_\xi x_\eta)\cot\theta \tag{15}$$

which satisfies Eq. (14) identically. Upon inserting the zero-curvature constraint, Eq. (7), into Eq. (6), all η-differentiated terms in the resulting equation can be eliminated with the aid of Eq. (15). This yields the following equation for the parameter φ.

$$\varphi = -2(\sin\theta)_\xi/\sin\theta - [(x_\xi - y_\xi\cot\theta)x_{\xi\xi} + (y_\xi + x_\xi\cot\theta)y_{\xi\xi}]/(x_\xi^2+y_\xi^2) \tag{16}$$

Equation (16) can be used in the same fashion as Eq. (11) to compute numerically the parameter φ in the terms of the preassigned boundary values x,y and of any preassigned distribution of θ as a function of position along the boundary curve $\eta=\eta_b$.

Multiply Connected Regions

The examples shown thus far illustrate the application of the method only to simply connected regions for which the Dirichlet boundary value problem is well posed. However, multiply connected regions can be treated by segmentation into simply connected subregions for which the Dirichlet problem remains well posed. Because the present method allows one to control the angle at which each transverse grid line intersects a boundary, the geometric shape of the boundary curve that separates any two contiguous subregions can be quite arbitrary. Both the slope and curvature of the transverse grid lines will automatically remain continuous across the boundary as long at the same boundary values of x,y, and θ are used in solving Eqs. (5) for both subregions flanking the boundary.

Figure 8 shows the physical grid generated by the described technique (using $\theta=\pi/2$) for a doubly connected region consisting of the interior and exterior flow regions of a superelliptical nozzle. The grid, highly refined near the nozzle walls to resolve the wall boundary layers, was generated simply by inputting an exponential grid point distribution along each boundary. Observe the smooth, regular character of the grid, the near-orthogonality of the grid lines to the nozzle wall, and the nearly uniform spacing between the wall

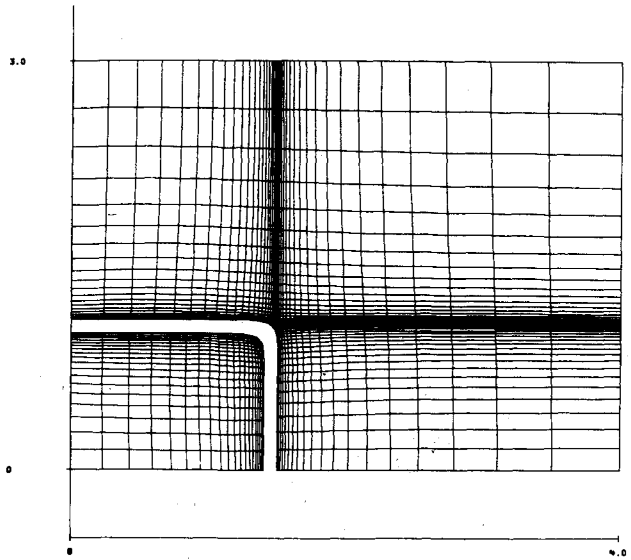

Fig. 8a Grid for the interior and exterior of a nozzle.

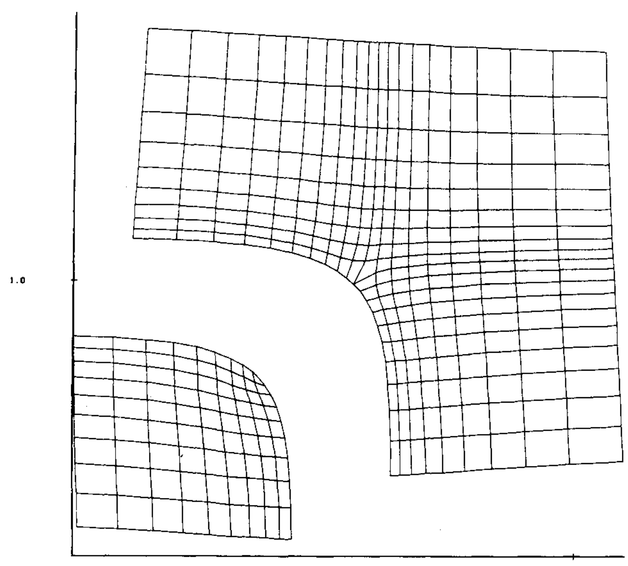

Fig. 8b Magnified view of nozzle wall corner region in Fig. 8a.

and nearest wall-like grid line over the whole length of the wall.

III. Concluding Remarks

To summarize, a simple, easily usable method has been presented for controlling grids generated by inhomogeneous Laplace equations. The method allows the grid point distribution in the interior of a region to be controlled directly by the selection of the grid point distribution along the boundaries. This is accomplished by introducing simple source terms whose mathematical form is independent of the boundary shape and of the boundary grid point distribution.

The source terms generate local exponential solutions, but contain free parameters that can conform to an arbitrary nonexponential boundary point distribution. The parameters are evaluated locally at the boundaries using limiting forms of the elliptic equations. In essence, the numerical evaluation of the free parameters at the boundaries simply is a convenient way to construct a curve-fit to the boundary values with local embedded exponential functions. The interpolation of these parameters into the interior of the computational domain simply extends the range of the curve-fit. Then the elliptic equation system merely provides a reliable, automatic means for translating the parameters into a local exponential curve-fit at each interior point that reflects the boundary value distribution, and has the properties of regularity and monotonicity required of nonsingular coordinate transformations. The method allows one to control directly the angle of intersection between the two families of coordinate curves at boundaries. In particular, one can impose the constraint that the two families be locally orthogonal at any or all boundaries.

The primary feature of the method is its fundamental simplicity, because it enables one to control the distribution of grid points throughout an entire domain by manipulating only the small subset of points situated on the boundary of the domain.

Acknowledgment

This research was sponsored by the NASA Langley Research Center under contract NAS1-15084.

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