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Approximate traveling wave solutions for coupled Whitham–Broer–Kaup shallow water

D.D. Ganji [a,*], Houman B. Rokni [a], M.G. Sfahani [b], S.S. Ganji [c]

[a] Department of Mechanical Engineering, Babol University of Technology, P.O. Box 484, Babol, Iran
[b] Departments of Structural Engineering, Faculty of Civil Engineering, Babol University of Technology, P.O. Box 484, Babol, Iran
[c] Department of Civil and Transportation Engineering, Islamic Azad University, Science and Research Branch, Tehran, Iran

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ABSTRACT

Homotopy Perturbation Method (HPM) was used for computing the Coupled Whitham–Broer–Kaup Shallow Water. Then HPM solution verified against exact one and compared with another approximate solution, the Homotopy Analysis Method (HAM). The existent error of the methods computed and convergence of the HPM solution has presented. Results reveal that HPM is an effective and powerful in solving the non-linear systems in mechanic, analytically.

1. Introduction

The Shallow Water Equations (SWE) is derived from the depth-averaged Navier–Stokes equations. These equations are used to describe flow in vertically well-mixed water bodies where the horizontal length scales are much greater than the fluid depth (i.e., long wavelength phenomena) [1]. These Shallow Water Equations describe the motion of water bodies wherein the depth is small relative to the scale of the waves propagating on that body. When waves travel into areas of shallow water (SW), they begin to be affected by the ocean bottom. The free orbital motion of the water is disrupted, and water particles in orbital motion no longer return to their original position. As the water becomes shallower, the swell becomes higher and steeper, ultimately assuming the familiar sharp-crested wave shape. After the wave breaks, it becomes a wave of translation and erosion of the ocean bottom intensifies [2]. The speed of shallow water waves is independent of wavelength or wave period and is controlled by the depth of water: as deep water waves get to shallow areas their speed decrease and their amplitudes increase accordingly. Decreasing speed of waves as water becomes shallow has dramatic consequences on the beach. As the waves slow, their profile is laterally compressed and since each wave must carry the same energy it becomes higher. As the wave approaches shore this process continues until the height exceeds 1/7th the wave length and the wave becomes unstable. Then the wave breaks. Different type of motion in SW is shown in Fig. 1 [3].

The Shallow Water Equations (SWE) are using to study many physical phenomena such as storm surges, tidal fluctuations, tsunami waves, forces acting on off-shore structures and modeling the transport of chemical species. The equations also, are used to model the hydrodynamics of lakes, estuaries, tidal flats and coastal regions, as well as deep ocean tides. SWE preserves mass, energy, and mass weighted functions of the Potential Vorticity (PV) globally and locally, for smooth flows and appropriate boundary conditions [4]. Coupled Whitham–Broer–Kaup (CWBK) equations which have been studied by Whitham [5], Broer [6], and Kaup [7] describe the propagation of shallow water waves with different dispersion relations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\beta\frac{\partial^2 u}{\partial x^2}=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(uv)+\alpha\frac{\partial^3 u}{\partial x^3}-\beta\frac{\partial^2 v}{\partial x^2}=0, \tag{2}$$

when $\alpha = 0$ and $\beta = 0.5$, the WBK equations are reduced to the modified Boussinesq (MB) equations. When $\alpha = 1$ and $\beta = 0$, the WBK equations are reduced to the approximate long-wave (ALW) equations in shallow water [8]. Solution of these non-linear coupled Eqs. (1) and (2) can be obtained using classical numerical methods such as Runge–Kutta method or the forward Euler method (see Table 1).

* Corresponding author.
E-mail address: ddg_davood@yahoo.com (D.D. Ganji).

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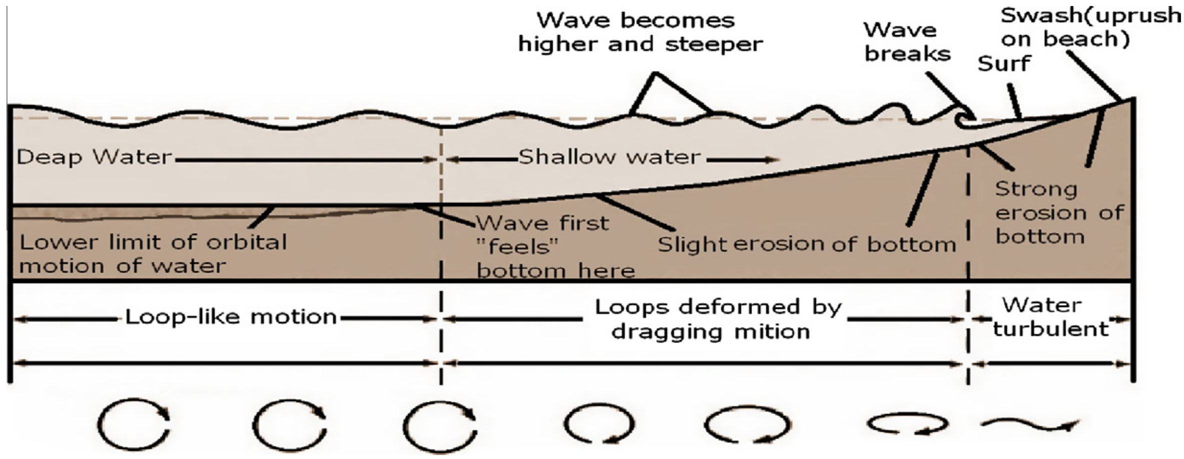

Fig. 1. Changing the motion of water particles.

Recently, advances were reported on producing and developments of analytical approaches for solving non-linear engineering problems. Some of these well-known methods includes Homotopy Perturbation [9–21], Adomian Decomposition [22,23], Amplitude–Frequency Formulation [24], Parameter–Expansion [25,26], Energy Balance [27–30], Variational Approach [28,31–33], Variational Iteration [34–36], Max–Min [37,38] and Homotopy Analysis Method [39].

This lead to researchers considers these methods for finding analytical wave solution for Coupled Whitham–Broer–Kaup (CWBK) equations. Using Adomian Decomposition Method, Salah and Kaya [40], had presented explicit and numerical traveling wave solutions of Whitham–Broer–Kaup equations in the form of a convergent power series with easily computable components, which contain blow-up and periodic solutions. Rashidi et al. [39], consider Homotopy Analysis Method to obtain the approximate traveling wave solutions of the coupled Whitham–Broer–Kaup (WBK) equations in shallow water. Rafei and Daniali [41], have obtained explicit traveling wave solutions including blow-up and periodic solutions of the Whitham–Broer–Kaup equations by the aid of variational iteration method. Trying to find exact traveling wave solutions for coupled WBK equations, Guiqiong and Zhibin [42], had found that the auxiliary equation method is also applicable to a coupled system of two different equations involving both even-order and odd-order partial derivative terms. Furthermore, singular traveling wave solutions can also be obtained by considering other types of exact solutions of auxiliary equation. Also some other exact solution was reported for coupled WBK equations [43,44].

Homotopy Perturbation Method (HPM) is known as a powerful and congruous method in applications to the linear and non-linear engineering problems and has been developed by scientists. In this paper we consider HPM to finding analytical traveling wave solutions for non-linear coupled WBK equation.

2. Basic idea of HPM

The Homotopy Perturbation Method (HPM) was proposed by He [10–12] and was further developed and improved by He [12].

To illustrate the Homotopy Perturbation Method (HPM) [11–13], Ji-Huan He considered the following non-linear differential equation:

$$A(u) - f(r) = 0, \ \ r \in \Omega, \tag{3}$$

With boundary conditions of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \ \ r \in \Gamma, \tag{4}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.

The operator A can be divided into two parts L and N, where L is linear and N is non-linear. Therefore (3) can be rewritten as follows:

$$A(u) = L(u) + N(u). \tag{5}$$

So,

$$L(u) + N(u) = f(r). \tag{6}$$

By the homotopy technique [16,17]. He constructed a Homotopy $v(r,p):\Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)(L(v) - L(y_0)) + p[A(v) - f(r)] = 0, \tag{7}$$

where $r \in \Omega_i$ and $p \in [0,1]$ is an imbedding parameter, and y_0 is an initial approximation of (3).

Hence, it is easy to see

$$H(v,0) = L(v) - L(y_0) = 0, H(v,1) = A(v) - f(r), \tag{8}$$

And changing the variation of p from 0 to 1 is the same as changing $H(v,p)$ from $L(v) - L(y_0)$ to $A(v) - f(r)$.

In topology, this is called deformation, $L(v) - L(y_0)$ and $A(v) - f(r)$ are called homotopic.

Owing to the fact that $0 \leqslant p \leqslant 1$ can be considered as a small parameter, by applying the perturbation technique, we can assume that the solution of (7) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots \tag{9}$$

And the best approximation for solution is:

$$u(x) = \lim v \overset{p\to 1}{\to} u(x) = v_0 + v_1 + v_2 + v_3 + \cdots \tag{10}$$

The essential idea of this method is to introduce an implicit small Parameter, p which takes the values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to one, the problem goes through a sequence of deformation, the solution of each of which is "close" to that at the previous stage of the deformation. Eventually at $p = 1$, the problem takes the original form and the final stage of deformation gives the desired solution.

The series (10) is convergent for most of the cases, and also the rate of convergence depends on how we choose $A(v)$, furthermore He made the following proposals:

1. The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e., $p \to 1$.
2. The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

Recently, application of the HPM to linear and non-linear problems has been developed by scientists and engineers [15–18]. In the next section, we are trying to apply HPM as a powerful method for solving Eqs. (1) and (2).

3. Application of HPM

First, we consider the coupled Whitham–Broer–Kaup (WBK) Eqs. (1) and (2), with initial condition [39]

$$u(x,0) = \lambda - 2k(\alpha+\beta^2)^{.5}\coth[k(x+x_0)], \tag{11}$$

$$v(x,0) = -2k^2(\alpha+\beta(\alpha+\beta^2)^{.5}+\beta^2)\mathrm{csch}^2[k(x+x_0)], \tag{12}$$

It can continuously trace an implicitly defined curve from a starting point $H(v,0)$ to a solution function $H(v,1)$.

$$H_1(u,p) = (1-p)\left(\frac{\partial u}{\partial t}-0\right)+p\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\beta\frac{\partial^2 u}{\partial x^2}\right]=0, \tag{13}$$

$$H_2(v,p) = (1-p)\left(\frac{\partial v}{\partial t}-0\right)+p\left[\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(uv)+\alpha\frac{\partial^3 u}{\partial x^3}-\beta\frac{\partial^2 v}{\partial x^2}\right]=0, \tag{14}$$

Substituting (9) into (13) and (14), and equating the coefficients of like terms with the identical powers of p, we obtain for first equation:

$$p^0 : \frac{\partial}{\partial t}u_0(x,t) = 0, \tag{15}$$

$$p^1 : u_0(x,t)\left(\frac{\partial}{\partial x}u_0(x,t)\right)+\frac{\partial}{\partial x}v_0(x,t)+\frac{\partial}{\partial t}u_1(x,t)=0, \tag{16}$$

$$p^2 : \frac{\partial}{\partial x}v_1(x,t)+u_0(x,t)\left(\frac{\partial}{\partial x}u_1(x,t)\right)+u_1(x,t)\left(\frac{\partial}{\partial x}u_0(x,t)\right)+\frac{\partial}{\partial t}u_2(x,t)=0, \tag{17}$$

For second equation:

$$p^0 : \frac{\partial}{\partial t}v_0(x,t) = 0, \tag{18}$$

$$p^1 : \frac{\partial^3}{\partial x^3}u_0(x,t)+\left(\frac{\partial}{\partial x}u_0(x,t)\right)v_0(x,t)+u_0(x,t)\left(\frac{\partial}{\partial x}v_1(x,t)\right)+\frac{\partial}{\partial t}v_1(x,t)=0, \tag{19}$$

$$p^2 : \frac{\partial^3}{\partial x^3}u_1(x,t)+\left(\frac{\partial}{\partial x}u_0(x,t)\right)v_1(x,t)+v_0(x,t)\left(\frac{\partial}{\partial x}u_1(x,t)\right)+u_0(x,t)\left(\frac{\partial}{\partial x}v_1(x,t)\right)+u_1(x,t)\left(\frac{\partial}{\partial x}v_0(x,t)\right)+\frac{\partial}{\partial t}v_2(x,t)=0, \tag{20}$$

Then we have,

$$u_0(x,t) = \lambda - 2k(\alpha+\beta^2)^{.5}\coth[k(x+x_0)], \tag{21}$$

$$u_1(x,t) = -\frac{4k^2\sqrt{\alpha+\beta^2}\lambda t}{\cosh(2kx+2kx_0)-1}, \tag{22}$$

$$u_2(x,t) = \frac{40t^2\cosh\left(\frac{4}{5}x+8\right)-120t^2+80t^2\cosh\left(\frac{2}{5}x+4\right)-\frac{201}{2}t^2\sinh\left(\frac{4}{5}x+8\right)-201t^2\sinh\left(\frac{2}{5}x+4\right)}{62,500\cosh\left(\frac{6}{5}x+12\right)+93,75,000\cosh\left(\frac{2}{5}x+4\right)-37,50,000\cosh\left(\frac{4}{5}x+8\right)-62,50,000}, \tag{23}$$

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t), \tag{24}$$

And,

$$v_0(x,t) = -2k^2(\alpha+\beta(\alpha+\beta^2)^{.5}+\beta^2)\csc h^2[k(x+x_0)], \tag{25}$$

$$v_1(x,t) = -\frac{16k^3(\beta\sqrt{\alpha+\beta^2}+\beta^2+\alpha)\lambda t(\sinh(2k(x+x_0))}{-\cosh(4k(x+x_0))-3+4\cosh(2k(x+x_0))}, \tag{26}$$

$$v_2(x,t) = \frac{-67t^2\cosh\left(\frac{3}{5}x+6\right)+33t^2\cosh\left(\frac{1}{5}x+2\right)+\frac{1}{2}t^2\cosh(x+10)}{-62,500\sinh\left(\frac{7}{5}x+14\right)+218,75,000\sinh\left(\frac{1}{5}x+2\right)-43,75,000\cosh(x+10)-131,25,000\sinh\left(\frac{3}{5}x+6\right)}, \tag{27}$$

$$v(x,t) = v_0(x,t)+v_1(x,t)+v_2(x,t). \tag{28}$$

4. Numerical experiments

According to the Xie et al. [8], the closed form solutions for coupled WBK is as follows:

$$u(x,t) = \lambda - 2k(\alpha+\beta^2)^{.5}\coth[k(x+x_0)-\lambda t], \tag{29}$$

$$v(x,t) = -2k^2(\alpha+\beta(\alpha+\beta^2)^{.5}+\beta^2)\csc h^2[k(x+x_0)-\lambda t]. \tag{30}$$

Table 2 presents numerical illustration for HPM solution of coupled WBK obtained in this paper and comparison of these results with exact one for $\alpha = 0.5$ and $\beta = 1$ when $k = 0.2$, $\lambda = 0.005$ and $x_0 = 10$. As time goes and in different position of x, HPM show a good congruity with the exact solution. Also, comparing the accuracy of HPM with HAM solution which was given in the tables declares the benefits of HPM over HAM for the analytical solution of the non-linear problem, according to simplicity, shorter procedure

Table 1
Values of dimensionless parameters α and β.

Mode	α	β
1	0.5	1.0
2	0.0	0.5
3	1.0	0.0

Table 2
Comparing absolute errors (%) for $u(x,t)$ and $v(x,t)$ given by HPM and HAM when $k = 0.2, \lambda = 0.005, x_0 = 10$, mode 1.

x	$t = 1$				$t = 2$				$t = 3$			
	$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$	
	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM
1	0.81	0.81	0.01	0.01	1.62	1.62	0.04	0.04	2.43	2.66	0.06	0.05
3	0.80	0.80	0.00	0.01	1.60	1.60	0.01	0.01	2.39	2.66	0.02	0.02
5	0.80	0.80	0.00	0.00	1.59	1.59	0.00	0.00	2.38	2.66	0.01	0.01
7	0.79	0.79	0.00	0.00	1.59	1.59	0.00	0.00	2.37	2.66	0.00	0.00

Table 3
Comparing absolute errors (%) for $u(x,t)$ and $v(x,t)$ given by HPM and HAM when $k = 0.2, \lambda = 0.005, x_0 = 10$, mode 2.

x	$t = 1$				$t = 2$				$t = 3$			
	$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$	
	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM
1	0.01	0.02	0.52	0.81	0.03	0.04	1.04	1.62	0.05	0.06	1.57	2.43
3	0.00	0.00	0.51	0.80	0.01	0.01	1.02	1.60	0.02	0.02	1.53	2.39
5	0.00	0.00	0.50	0.80	0.00	0.00	1.00	1.59	0.01	0.01	1.51	2.38
7	0.00	0.00	0.50	0.79	0.00	0.00	1.00	1.59	0.00	0.00	1.50	2.37

Table 4
Comparing absolute errors (%) for $u(x,t)$ and $v(x,t)$ given by HPM and HAM when $k = 0.2, \lambda = 0.005, x_0 = 10$, mode 3.

x	$t = 1$				$t = 2$				$t = 3$			
	$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$	
	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM	HPM	HAM
1	0.02	0.02	0.81	0.81	0.04	0.04	1.62	1.62	0.06	0.06	2.43	2.43
3	0.00	0.00	0.80	0.80	0.01	0.01	1.60	1.60	0.02	0.02	2.39	2.39
5	0.00	0.00	0.80	0.80	0.00	0.00	1.59	1.59	0.01	0.01	2.38	2.38
7	0.00	0.00	0.79	0.79	0.00	0.00	1.59	1.59	0.00	0.00	2.37	2.37

and better results in some instants. By the mode 2 as $\alpha = 0.0$ and $\beta = 0.5$ when the coupled WBK equation reduces to modified Boussinesq equations, no special changes observed in HPM results in Table 3. This happened due to mode 3 (Table 4) when $\alpha = 1.0$ and $\beta = 0.0$ and problem reduces to approximate long-wave (ALW) equations in shallow water, similarly.

Moreover, for further illustration, comparison of HPM and exact solution were done via plotting the wave's deformation at different instances in Figs. 2 and 3. A good compatibility is observed between HPM and the closed form solution in these figures.

5. Conclusions

In this paper, we consider non-linear coupled Whitham–Broer–Kaup (WBK) Shallow Water Equation for finding an analytical solution via Homotopy Perturbation Method (HPM). In addition, modified Boussinesq equation and the approximate equation for long water waves, as the special cases of WBK equation, also considered for the corresponding solitary wave solutions. Three coupled non-linear equations with initial conditions are discussed as demonstrations. A very good agreement between the

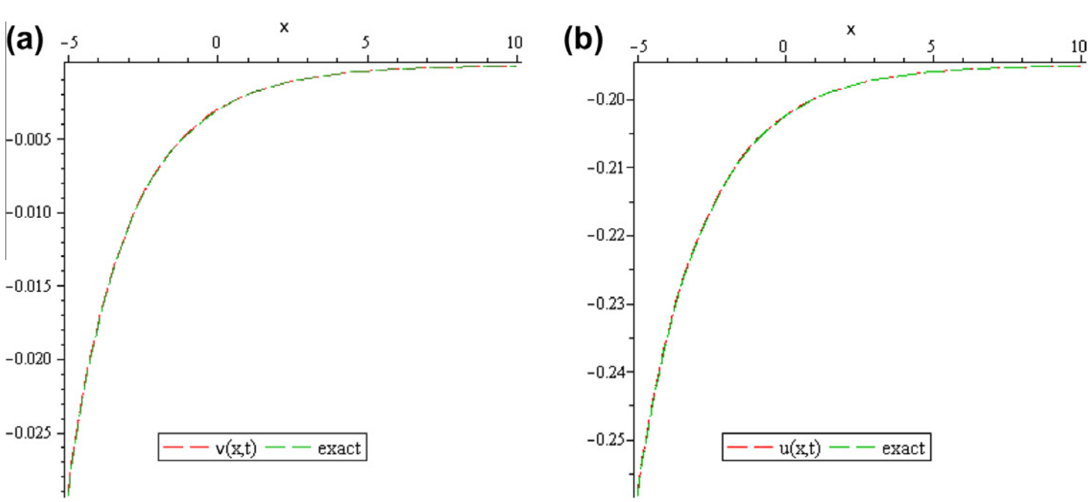

Fig. 2. Comparing the exact solution with HPM: (a) $v(x,t)$ and (b) $u(x,t)$ at $t = 1$.

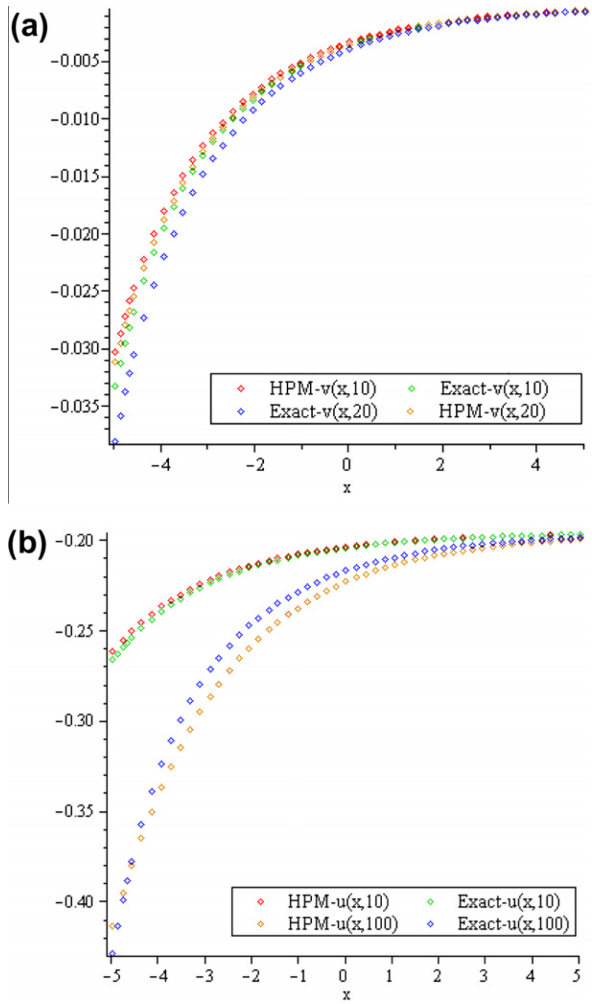

Fig. 3. Propagation of: (a) $v(x, t)$ and (b) $u(x, t)$ versus time.

results of the HPM and the exact solutions was observed, which confirms the validity of the HPM. It may be concluded that the HPM methodology is very powerful and efficient technique in finding analytical solutions for wide classes of problems and can be also easy to be extended to other non-linear evaluation equations, with the aid of Mathematica, Matlab or Maple. Furthermore, as the HPM does not require discretization, it is not affected by computation round off errors, and large computer memory as well as consumed time which are issues in the calculation procedure. Also it is noteworthy to point out that the advantage of the HPM is the fast convergence of the solutions.

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