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Application of the variational iteration method to the Whitham–Broer–Kaup equations

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Abstract

Explicit traveling wave solutions including blow-up and periodic solutions of the Whitham–Broer–Kaup equations are obtained by the variational iteration method. Moreover, the results are compared with those obtained by the Adomian decomposition method, revealing that the variational iteration method is superior to the Adomian decomposition method.
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1. Introduction

Here, we consider the coupled Whitham–Broer–Kaup (WBK) equations which have been studied by Whitham [1], Broer [2] and Kaup [3]. The equations describe the propagation of shallow water waves, with different dispersion relations. The WBK equations are as follows,

$$
\begin{aligned}
u_t + uu_x + v_x + \beta u_{xx} &= 0, \\
v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} &= 0
\end{aligned}
\tag{1.1}
$$

where $u = u(x, t)$ is the horizontal velocity, $v = v(x, t)$ is the height that deviates from equilibrium position of the liquid, and α, β are constants which are represented in different diffusion powers [4].

Calculating the exact and numerical solutions of nonlinear equations in mathematical physics plays an important role in soliton theory [1–10]. Many explicit exact methods have been introduced in literature [1–10] including the Backlund transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine–cosine method, Painleve method, homogeneous balance method, and similarity reduction method. Recently, Xie et al. [11] applied the hyperbolic function method to the WBK equations and found some new solitary wave solutions. System (1.1) is a very good model to describe dispersive waves. If $\alpha = 0$, $\beta \neq 0$, then the system represents the classical long wave

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equation that describes shallow water wave with dispersion [4]. If $\alpha = 1$, $\beta = 0$, then the system represents the variant Boussinesq equation [11].

The variational iteration method (VIM) was first proposed by He [12–18]. In this method, the problems are initially approximated with possible unknowns. Then a correction functional is constructed by a general Lagrange multiplier which can be optimally identified via the variational theory. Being different from the other nonlinear analytical methods such as perturbation methods, this method is independent of small parameters so that it can be widely applied in nonlinear problems without linearization or small perturbations. This method was successfully applied to autonomous ordinary differential systems [17], the relaxation process [19], the nonlinear differential equations with convolution product nonlinearities [14], seepage flow with fractional derivatives in porous media [13], solitary solution [16], and the Helmholtz equation [20].

In this paper we find analytical approximate and exact traveling wave solutions of the system (1.1) using VIM. The accuracy of the solutions is demonstrated through some numerical examples. Three examples of special interest, namely, the WBK, modified Boussinesq (MB) and approximate long wave (ALW) equations are discussed in details and the results are compared with those found by Adomian decomposition method (ADM) [21].

2. The variational iteration method

To illustrate the basic concepts of the VIM, we consider the following differential equation [15],

$$Lu + Nu = g(x) \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term.

According to the VIM, one can construct a correction functional as follows,

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{Lu_n(s) + N\tilde{u}_n(s) - g(s)\} \mathrm{d}s, \tag{2.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation [12–18], i.e., $\delta\tilde{u}_n = 0$.

3. Applications

To solve Eq. (1.1) by means of the VIM, one can construct the following correction functional,

$$
\begin{aligned}
u_{n+1}(x, t) &= u_n(x, t) + \int_0^t \lambda_1 \{u_{n,s} + \tilde{u}_n u_{n,x} + v_{n,x} + \beta u_{n,xx}\} \mathrm{d}s, \\
v_{n+1}(x, t) &= v_n(x, t) + \int_0^t \lambda_2 \{v_{n,s} + (\tilde{u}_n v_n)_x + \alpha u_{n,xxx} - \beta v_{n,xx}\} \mathrm{d}s,
\end{aligned}
\tag{3.1}
$$

where λ_1 and λ_2 are general Lagrange multipliers, and $\tilde{u}_n u_{n,x}$ and $(\tilde{u}_n v_n)_x$ are considered as restricted variations, i.e. $\delta\tilde{u}_n u_{n,x} = 0$, $\delta(\tilde{u}_n v_n)_x = 0$.

Making the above correction functional stationary, the following stationary conditions can be obtained,

$$
\begin{aligned}
&\lambda_1'(\tau) = 0, \\
&1 + \lambda_1(\tau)|_{\tau=t} = 0, \\
&\lambda_2'(\tau) = 0, \\
&1 + \lambda_2(\tau)|_{\tau=t} = 0.
\end{aligned}
\tag{3.2}
$$

The Lagrange multipliers, therefore, can be identified as,

$$\lambda_1 = \lambda_2 = -1. \tag{3.3}$$

Substituting the values of λ_1 and λ_2 from Eq. (3.3) into the correction functional of Eq. (3.1) leads to the following iteration formulae:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \{u_{n,s} + u_n u_{n,x} + v_{n,x} + \beta u_{n,xx}\} ds,$$

$$v_{n+1}(x, t) = v_n(x, t) - \int_0^t \{v_{n,s} + (u_n v_n)_x + \alpha u_{n,xxx} - \beta v_{n,xx}\} ds.$$
(3.4)

We first consider the application of the VIM to the WBK Eq. (1.1) with the initial conditions

$$u(x, 0) = \lambda - 2Bk \coth(k\xi),$$

$$v(x, 0) = -2B(B + \beta)k^2 \operatorname{csch}^2(k\xi),$$
(3.5)

where $B = \sqrt{\alpha + \beta^2}$ and $\xi = x + x_0$ and x_0, k, λ are arbitrary constants [11].

Using the initial guess given by Eq. (3.5) and by the iteration formula (3.4), one can obtain the following results,

$$u_0(x, t) = \lambda - 2Bk \coth(k\xi),$$
(3.6)

$$v_0(x, t) = -2B(B + \beta)k^2 \operatorname{csch}^2(k\xi),$$
(3.7)

$$u_1(x, t) = \frac{1}{4}\operatorname{csch}^3(k\xi)(2Bk \cosh(k\xi) - 2Bk \cosh(3k\xi) - \lambda(8Bk^2 t + 3) \sinh(k\xi) + \lambda \sinh(3k\xi))$$
(3.8)

$$v_1(x, t) = -Bk^2 \operatorname{csch}^4(k\xi)(-B - \beta - 8B^2 k^2 t + 8\alpha t k^2 + 2(Bkt\lambda + kt\lambda\beta) \sinh(2k\xi)$$
$$+ 8\beta^2 t k^2 (B + 4\beta^2 t k^2 - 4B^2 k^2 t + \beta + 4\alpha t k^2) \cosh(2k\xi))$$
(3.9)

$$u_2(x, t) = -\frac{1}{48}\operatorname{csch}^5(k\xi)((-72Bk^2 t\lambda - 30\lambda) \sinh(k\xi) + (24Bk^2 t\lambda + 15\lambda)$$
$$\times \sinh(3k\xi) - 3\lambda \sinh(5k\xi) + (-128B^2 k^5 \lambda^2 t^3 - 1056B^3 k^5 t^2 + 12Bk$$
$$+ 1056Bk^5 t^2 \alpha - 24Bk^3 t^2 \lambda^2 + 1056Bk^5 t^2 \beta^2) \cosh(k\xi) + 6Bk \cosh(5k\xi)$$
$$+ (-96B^3 k^5 t^2 - 18Bk + 24Bk^3 t^2 \lambda^2 + 96Bk^5 t^2 \beta^2 96Bk^5 t^2 \alpha) \cosh(3k\xi))$$
(3.10)

$$v_2(x, t) = -\frac{2}{96}Bk^2 \operatorname{csch}^8(k\xi)(384k^4 t^2 B\beta^2 - 960k^4 t^2 \alpha\beta + 960k^4 t^2 \beta B^2 - 30B$$
$$+ 48k^2 t^2 \lambda^2 B + 128B^2 k^4 \lambda^2 t^3 + 384k^4 t^2 B\alpha - 30\beta - 384k^4 t^2 B^3 + 96\alpha t k^2$$
$$- 960k^4 t^2 \beta^3 - 96B^2 k^2 t + 96\beta^2 t k^2 + 48k^2 t^2 \lambda^2 \beta + 128Bk^4 \lambda^2 t^3 \beta$$
$$+ (-108\alpha t k^2 - 54k^2 t^2 \lambda^2 B - 360k^4 t^2 \beta B^2 + 360k^4 t^2 \beta\alpha + 360k^4 t^2 \beta^3$$
$$- 108\beta^2 k^2 t + 108B^2 k^2 t + 45B - 64Bk^4 \lambda^2 t^3 \beta + 72k^4 t^2 B^3 - 72k^4 t^2 B\beta^2$$
$$- 64B^2 k^4 \lambda^2 t^3 - 72k^4 t^2 B\alpha + 45\beta - 54k^2 t^2 \lambda^2 \beta) \cosh(2k\xi) + (-18B - 18\beta$$
$$+ 576k^4 t^2 \beta^3 - 64Bk^4 \lambda^2 t^3 \beta - 288k^4 t^2 B\alpha - 576k^4 t^2 \beta B^2 - 288k^4 t^2 B\beta^2$$
$$+ 288k^4 t^2 B^3 - 64B^2 k^4 \lambda^2 t^3 + 576k^4 t^2 \beta\alpha) \cosh(4k\xi) + (3\beta - 24k^4 t^2 \beta B^2$$
$$- 24k^4 t^2 B\beta^2 - 12B^2 k^2 t + 6k^2 t^2 \lambda^2 B + 24k^4 t^2 B^3 + 12\beta^2 t k^2 + 6k^2 t^2 \lambda^2 \beta$$
$$+ 12\alpha t k^2 - 24k^4 t^2 B\alpha + 24k^4 t^2 \beta^3 + 3B + 24k^4 t^2 \beta\alpha) \cosh(6k\xi)$$
$$+ (456k^3 t^2 \lambda B^2 - 456k^3 t^2 \beta^2 \lambda - 896Bk^5 \lambda t^3 \alpha + 30kt\lambda\beta + 896B^3 k^5 \lambda t^3$$
$$+ 30Bkt\lambda - 456k^3 t^2 \lambda\alpha - 896Bk^5 \lambda t^3 \beta^2) \sinh(2k\xi) + (192k^3 t^2 \lambda\alpha$$
$$+ 192k^3 t^2 \beta^2 \lambda - 24kt\lambda\beta - 128Bk^5 \lambda t^3 \beta^2 - 128Bk^5 \lambda t^3 \alpha - 192k^3 t^2 \lambda B^2$$
$$+ 128B^3 k^5 \lambda t^3 - 24Bkt\lambda) \sinh(4k\xi) + (6Bkt\lambda - 24k^3 t^2 \lambda B^2 + 24k^3 t^2 \beta^2 \lambda$$
$$+ 6kt\lambda\beta + 24k^3 t^2 \lambda\alpha) \sinh(6k\xi)).$$
(3.11)

In a similar way, one can obtain the other components, and then the two functions $u(x, t)$ and $v(x, t)$ in closed form can be readily found:

$$u(x, t) = \lambda - 2Bk \coth(k\xi - \lambda t),$$
$$v(x, t) = -2B(B + \beta)k^2 \operatorname{csch}^2(k\xi - \lambda t). \tag{3.12}$$

As a special case, if $\alpha = 1$ and $\beta = 0$, the WBK equations are reduced to the Modified Boussinesq (MB) equations [11]. We consider the initial conditions of the MB equations as,

$$u(x, 0) = \lambda - 2k \coth(k\xi),$$
$$v(x, 0) = -2k^2 \operatorname{csch}^2(k\xi), \tag{3.13}$$

where k, λ are constants still to be determined, and $\xi = x + x_0$ is an arbitrary constant [11].

One can apply the initial guess given by Eq. (3.13) and the iteration formula (3.4), which leads to the following:

$$u_0(x, t) = \lambda - 2k \coth(k\xi),$$
$$u_1(x, t) = u_0(x, t) - \int_0^t \{u_{0,s} + u_0 u_{0,x} + v_{0,x}\} ds,$$
$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \{u_{n,s} + u_n u_{n,x} + v_{n,x}\} ds, \quad n \geq 1, \tag{3.14}$$

$$v_0(x, t) = -2k^2 \operatorname{csch}^2(k\xi),$$
$$v_1(x, t) = v_0(x, t) - \int_0^t \{v_{0,s} + (u_0 v_0)_x + u_{0,xxx}\} ds,$$
$$v_{n+1}(x, t) = v_n(x, t) - \int_0^t \{v_{n,s} + (u_n v_n)_x + u_{n,xxx}\} ds, \quad n \geq 1. \tag{3.15}$$

Performing the calculations in (3.14) and (3.15), using *Maple* gives the exact solutions as,

$$u(x, t) = \lambda - 2k \coth(k\xi - \lambda t),$$
$$v(x, t) = -2k^2 \operatorname{csch}^2(k\xi - \lambda t). \tag{3.16}$$

Moreover, if $\alpha = 0$ and $\beta = 1/2$, the WBK equations are reduced to the approximate long wave (ALW) equation in shallow water [11]. One can compute the ALW equation with the initial conditions,

$$u(x, 0) = \lambda - k \coth(k\xi),$$
$$v(x, 0) = -k^2 \operatorname{csch}^2(k\xi), \tag{3.17}$$

where k, λ are constants still to be determined, and $\xi = x + x_0$, [11]. The VIM leads to the following:

$$u_0(x, t) = \lambda - k \coth(k\xi),$$
$$u_1(x, t) = u_0(x, t) - \int_0^t \left\{u_{0,s} + u_0 u_{0,x} + v_{0,x} + \frac{1}{2}u_{0,xx}\right\} ds,$$
$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left\{u_{n,s} + u_n u_{n,x} + v_{n,x} + \frac{1}{2}u_{n,xx}\right\} ds, \quad n \geq 1, \tag{3.18}$$

$$v_0(x, t) = -k^2 \operatorname{csch}^2(k\xi),$$
$$v_1(x, t) = v_0(x, t) - \int_0^t \left\{v_{0,s} + (u_0 v_0)_x - \frac{1}{2}v_{0,xx}\right\} ds,$$
$$v_{n+1}(x, t) = v_n(x, t) - \int_0^t \left\{v_{n,s} + (u_n v_n)_x - \frac{1}{2}v_{n,xx}\right\} ds, \quad n \geq 1. \tag{3.19}$$

Performing the calculations in (3.18) and (3.19), using *Maple*, the components of the VIM can be easily obtained, of which $u(x, t)$ and $v(x, t)$ are evaluated in series form. Consequently, using Taylor series, one can obtain the closed

Table 1
The VIM and ADM results for $u(x, t)$ and $v(x, t)$

(x, t)	$|u_{\text{Exact}} - u_{\text{VIM}}|$	$|u_{\text{Exact}} - u_{\text{ADM}}|$	$|v_{\text{Exact}} - v_{\text{VIM}}|$	$|v_{\text{Exact}} - v_{\text{ADM}}|$
(0.1, 0.1)	1.23033E−04	1.04892E−04	1.10430E−04	6.41419E−03
(0.1, 0.3)	3.69597E−04	9.64474E−05	3.31865E−04	5.99783E−03
(0.1, 0.5)	6.16873E−04	8.88312E−05	5.54071E−04	5.61507E−03
(0.2, 0.1)	1.19869E−04	4.25408E−04	1.07016E−04	1.33181E−02
(0.2, 0.3)	3.60098E−04	3.91098E−04	3.21601E−04	1.24441E−02
(0.2, 0.5)	6.01006E−04	3.60161E−04	5.36927E−04	1.16416E−02
(0.3, 0.1)	1.16789E−04	9.71922E−04	1.03737E−04	2.07641E−02
(0.3, 0.3)	3.50866E−04	8.93309E−04	3.11737E−04	1.93852E−02
(0.3, 0.5)	5.85610E−04	8.22452E−04	5.20447E−04	1.81209E−02
(0.4, 0.1)	1.13829E−04	1.75596E−03	1.00579E−04	2.88100E−02
(0.4, 0.3)	3.41948E−04	1.61430E−03	3.02245E−04	2.68724E−02
(0.4, 0.5)	5.70710E−04	1.48578E−03	5.04593E−04	2.50985E−02
(0.5, 0.1)	1.10936E−04	2.79519E−03	9.75385E−05	3.75193E−02
(0.5, 0.3)	3.33274E−04	2.56714E−03	2.93107E−04	3.49617E−02
(0.5, 0.5)	5.56235E−04	2.36184E−03	4.89335E−04	3.26239E−02

$k = 0.1, \lambda = 0.005, \alpha = 1.5, \beta = 1.5$ and $x_0 = 10$, for the WKB equation (1.1).

Table 2
The VIM and ADM results for $u(x, t)$ and $v(x, t)$

(x, t)	$|u_{\text{Exact}} - u_{\text{VIM}}|$	$|u_{\text{Exact}} - u_{\text{ADM}}|$	$|v_{\text{Exact}} - v_{\text{VIM}}|$	$|v_{\text{Exact}} - v_{\text{ADM}}|$
(0.1, 0.1)	6.35269E−05	8.16297E−07	1.65942E−05	5.88676E−05
(0.1, 0.3)	1.90854E−04	7.64245E−07	4.98691E−05	5.56914E−05
(0.1, 0.5)	3.18549E−04	7.16083E−07	8.32598E−05	5.27169E−05
(0.2, 0.1)	6.18930E−05	3.26243E−06	1.60813E−05	1.18213E−04
(0.2, 0.3)	1.85945E−04	3.05458E−06	4.83269E−05	1.11833E−04
(0.2, 0.5)	3.10352E−04	2.86226E−06	8.06837E−05	1.05858E−04
(0.3, 0.1)	6.03095E−05	7.33445E−06	1.55880E−05	1.78041E−04
(0.3, 0.3)	1.81187E−04	6.86758E−06	4.68440E−05	1.68429E−04
(0.3, 0.5)	3.02408E−04	6.43557E−06	7.82068E−05	1.59428E−04
(0.4, 0.1)	5.87746E−05	1.30286E−05	1.51135E−05	2.38356E−04
(0.4, 0.3)	1.76574E−04	1.22000E−05	4.54174E−05	2.25483E−04
(0.4, 0.5)	2.94707E−04	1.14333E−05	7.58243E−05	2.13430E−04
(0.5, 0.1)	5.72867E−05	2.03415E−05	1.46569E−05	2.99162E−04
(0.5, 0.3)	1.72102E−04	1.90489E−05	4.40448E−05	2.83001E−04
(0.5, 0.5)	2.87241E−04	1.78528E−05	7.35317E−05	2.67868E−04

$k = 0.1, \lambda = 0.005, \alpha = 1, \beta = 0$ and $x_0 = 10$, for the MB equation (1.1).

form solutions as,

$$u(x, t) = \lambda - k \coth(k\xi - \lambda t),$$
$$v(x, t) = -2k^2 \text{csch}^2(k\xi - \lambda t). \tag{3.20}$$

Tables 1–3 show the absolute errors by VIM and ADM. It reveals from that we obtain a very good approximation from a few iterations. However, some extra terms can be calculated in order to achieve a better accuracy of VIM with the help of *Maple*. The absolute errors of ADM in Tables 1–3 were derived from Ref. [21].

Our numerical approximations show a very good accuracy, and reveal the advantages of VIM over the Adomian method. It is obvious that the overall errors can be made smaller by considering some extra iterations by the variational iteration formulae.

Furthermore, as VIM does not require discretization of the variables, namely, time and space, it is not affected by computation roundoff errors and large computer memory and consumed time are issues in the calculation procedure.

Table 3
The VIM and ADM results for $u(x, t)$ and $v(x, t)$

(x, t)	$|u_{\text{Exact}} - u_{\text{VIM}}|$	$|u_{\text{Exact}} - u_{\text{ADM}}|$	$|v_{\text{Exact}} - v_{\text{VIM}}|$	$|v_{\text{Exact}} - v_{\text{ADM}}|$
(0.1, 0.1)	3.17634E−05	8.02989E−06	8.29712E−06	4.81902E−04
(0.1, 0.3)	9.54273E−05	7.38281E−06	2.49346E−05	4.50818E−04
(0.1, 0.5)	1.59274E−04	6.79923E−06	4.16299E−05	4.22221E−04
(0.2, 0.1)	3.09466E−05	3.23228E−05	8.04063E−06	9.76644E−04
(0.2, 0.3)	9.29725E−05	2.97172E−05	2.41634E−05	9.13502E−04
(0.2, 0.5)	1.55176E−04	2.73673E−05	4.03419E−05	8.55426E−04
(0.3, 0.1)	3.01549E−05	7.32051E−05	7.79401E−06	1.48482E−03
(0.3, 0.3)	9.05935E−05	6.73006E−05	2.34220E−05	1.38858E−03
(0.3, 0.5)	1.51204E−04	6.19760E−05	3.91034E−05	1.30009E−03
(0.4, 0.1)	2.93874E−05	1.31032E−04	7.55675E−06	2.00705E−03
(0.4, 0.3)	8.82871E−05	1.20455E−04	2.27087E−05	1.87661E−03
(0.4, 0.5)	1.47354E−04	1.10919E−04	3.79121E−05	1.75670E−03
(0.5, 0.1)	2.86433E−05	2.06186E−04	7.32847E−06	2.54396E−03
(0.5, 0.3)	8.60509E−05	1.89528E−04	2.20224E−05	2.37815E−03
(0.5, 0.5)	1.43620E−04	1.74510E−04	3.67658E−05	2.22578E−03

$k = 0.1$, $\lambda = 0.005$, $\alpha = 0$, $\beta = 0.5$ and $x_0 = 10$, for the ALW equation (1.1).

4. Conclusion

In this study, the variational iteration method (VIM) was used for finding the exact and approximate traveling wave solutions of the Whitham–Broer–Kaup (WBK) equations in shallow water. The method can be easily extended to the other nonlinear evaluation equations with the aid of *Maple* (or *Matlab*, *Mathematica*, etc.). We demonstrated the robustness of the method with three coupled nonlinear equations with the initial conditions.

It may be concluded that the variational iteration method is a very powerful and efficient technique to find exact or approximate solutions for a wide classes of problems. It is also worth pointing out that the advantage of VIM is the fast convergence of the solutions. Moreover, VIM does not require linearization or perturbation for obtaining closed form solutions. Furthermore, it does not need any discretization to get numerical solutions. The comparison of the variational iteration method with the Adomian decomposition method reveals that the approximate solutions obtained by VIM converge to its exact solution faster than those obtained by ADM.

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