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Nodal solutions of boundary value problems with boundary conditions involving Riemann–Stieltjes integrals

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ABSTRACT

We study the nonlinear boundary value problem consisting of the equation $-y'' = \sum_{i=1}^{m} w_i(t)f_i(y)$ and a boundary condition involving a Riemann–Stieltjes integral. By relating it to the eigenvalues of the corresponding linear Sturm–Liouville problem with a two-point separated boundary condition, we obtain results on the existence and nonexistence of nodal solutions of this problem. The shooting method and an energy function are used to prove the main results.

1. Introduction

We are concerned with the boundary value problem (BVP) consisting of the equation

$$-y'' = \sum_{i=1}^{m} w_i(t)f_i(y), \quad t \in (a, b), \tag{1.1}$$

and the boundary condition (BC)

$$\begin{aligned} &\cos\alpha\, y(a) - \sin\alpha\, y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) - \int_a^b y(s)\mathrm{d}\xi(s) = 0, \end{aligned} \tag{1.2}$$

where $m \geq 1$ is an integer, $a, b \in \mathbb{R}$ with $a < b$ and the integral in BC (1.2) is the Riemann–Stieltjes integral with respect to $\xi(s)$ with $\xi(s)$ a function of bounded variation. In the case where $\xi(s) = s$, the Riemann–Stieltjes integral in the second condition of (1.2) reduces to the Riemann integral. In the case that $\xi(s) = \sum_{j=1}^{d} k_j\chi(s - \eta_j)$, where $d \geq 1$, $k_j \in \mathbb{R}, j = 1, \ldots, m$, $\{\eta_j\}_{j=1}^{d}$ is a strictly increasing sequence of distinct points in (a, b), and $\chi(s)$ is the characteristic function on $[0, \infty)$, i.e.,

$$\chi(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0, \end{cases}$$

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the second equation in (1.2) reduces to the multi-point BC

$$y(b) - \sum_{j=1}^{d} k_j y(\eta_j) = 0. \tag{1.3}$$

We assume throughout, and without further mention, that the following conditions hold:

(H1) For each $i \in \{1, \ldots, m\}$, $w_i \in C^1[a, b]$ and $w_i(t) > 0$;
(H2) for each $i \in \{1, \ldots, m\}$, $f_i \in C(\mathbb{R})$, f_i is locally Lipschitz on $(-\infty, 0) \cup (0, \infty)$, and $f_i(y) > 0$ and $f_i(-y) = -f_i(y)$ for all $y > 0$;
(H3) for each $i \in \{1, \ldots, m\}$, there exist extended real numbers $(f_i)_0, (f_i)_\infty \in [0, \infty]$ such that

$$(f_i)_0 = \lim_{y\to 0} f_i(y)/y \quad \text{and} \quad (f_i)_\infty = \lim_{|y|\to\infty} f_i(y)/y.$$

Remark 1.1. The oddness assumptions for f_i in (H2) is only for convenience. Our results can be extended to the case when $yf_i(y) > 0$ for $y \neq 0$ and $i \in \{1, \ldots, m\}$, without the oddness assumption.

The existence of positive solutions of BVPs with multi-point BCs have been studied extensively; see, for example, [1–8] and the references therein. Riemann–Stieltjes integral formulations of the multi-point BCs have also been considered by many authors, notably by Infante and Webb [9–14]. In these nice papers, the existence of positive solutions are studied by using the fixed-point index theory. In recent years, the existence of nodal solutions, i.e., solutions with a specific zero-counting property in (a, b), has also attracted much attention in the research of BVPs. Great progress has been made in the study of such solutions for BVPs consisting of Eq. (1.1) with two-point separated BCs and certain types of nonlocal BCs; see [15–23]. Regarding problems with nonlocal BC (1.2), results on the existence of nodal solutions have been obtained only for special cases. In fact, Ma [24], Ma and O'Regan [25], Rynne [26], Xu [27], and Xu et al. [28] studied the special BVP consisting of the equation

$$y'' + f(y) = 0, \qquad t \in (0, 1), \tag{1.4}$$

and the multi-point BC

$$y(0) = 0, \qquad y(1) - \sum_{j=1}^{d} k_j y(\eta_j) = 0. \tag{1.5}$$

Among them, [25,26] used a standard global bifurcation method to establish the existence of nodal solutions of BVP (1.4), (1.5) by relating it to the eigenvalues of the corresponding linear Sturm–Liouville problem (SLP) with the multi-point BC (1.5). However, the establishment of these results relies heavily on the direct computation of the eigenvalues and eigenfunctions of the SLP associated with BVP (1.4), (1.5), and hence cannot be extended to a general BVP with variable coefficient functions.

Motivated by the above work, Kong et al. [18] established the existence of nodal solutions of the BVP consisting of the equation

$$y'' + w(t)f(y) = 0, \quad t \in (a, b), \tag{1.6}$$

and the multi-point BC

$$\begin{aligned} &\cos\alpha\, y(a) - \sin\alpha\, y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) - \sum_{j=1}^{d} k_j y(\eta_j) = 0, \end{aligned} \tag{1.7}$$

by relating it to the eigenvalues of the corresponding linear SLP with a two-point separated boundary condition. This provides a new direction for the research of nonlocal BVPs and the results are significant since eigenvalues are easy to calculate for two-point linear self-adjoint SLPs using standard software packages such as those in [29].

In this paper, we apply the ideas in [18] to study the existence and nonexistence of nodal solutions of the more general BVP (1.1), (1.2). The shooting method and a generalized energy function play key roles in the proofs. Note that our results are for Eq. (1.1) with multiple terms on the right hand side, and one equation in BC (1.2) given by a Riemann–Stieltjes integral. To the best of the authors' knowledge, no work has been done on the existence of nodal solutions for this problem.

Our results for BVP (1.1), (1.2) are established using the eigenvalues, $\{\lambda_n\}_{n=0}^{\infty}$, of the linear SLP consisting of the equation

$$-y'' = \lambda \sum_{i=1}^{m} w_i(t) y, \quad t \in (a, b), \tag{1.8}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha\, y(a) - \sin\alpha\, y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y'(b) = 0. \end{aligned} \tag{1.9}$$

It is well known that any eigenfunction associated with λ_n has n simple zeros in (a, b); see [30, Theorem 4.3.2].

Note that the function $\xi(s)$ given in BC (1.2) is of bounded variation on $[a, b]$. Thus, there are two nondecreasing functions $\xi_1(s)$ and $\xi_2(s)$ such that

$$\xi(s) = \xi_1(s) - \xi_2(s), \quad s \in [a, b]. \tag{1.10}$$

We study the solutions of BVP (1.1), (1.2) in the following function class T_k^γ, which is a variation of the definition in [26].

Definition 1.1. A solution y of BVP (1.1), (1.2) is said to belong to class $\mathcal{T}_k^\gamma$ for $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\gamma \in \{+, -\}$ if

(i) y and y' have only simple zeros in $[a, b]$,
(ii) y' has exactly $k + 1$ zeros in (a, b) if $\alpha \in [0, \pi/2)$ and has exactly k zeros in (a, b) if $\alpha \in [\pi/2, \pi)$,
(iii) there is exactly one zero of y strictly between any two consecutive zeros of y',
(iv) $\gamma y(t) > 0$ in a right-neighborhood of a.

Remark 1.2. It is easy to see that for $y \in \mathcal{T}_k^\gamma$ with $k \in \mathbb{N}_0$ and $\gamma \in \{+, -\}$, y may have k or $k + 1$ zeros in (a, b).

2. Main results

In this paper, we will use the notation $h_\pm(t) := \max\{0, \pm h(t)\}$ for any function h. For each $i \in \{1, \ldots, m\}$, let $F_i(y) = \int_0^y f_i(\xi)\, d\xi$ for $y \in \mathbb{R}$ and denote

$$H(t, y) := \sum_{i=1}^m w_i(t) F_i(y) \quad \text{and} \quad k_0 = \int_a^b l(t) dt, \tag{2.1}$$

where

$$l(t) := \max\left\{ \frac{(w_1')_-(t)}{w_1(t)}, \ldots, \frac{(w_m')_-(t)}{w_m(t)} \right\}.$$

By (H2), each F_i is strictly increasing on $[0, \infty)$. Thus, for any fixed $t \in [a, b]$, $H(t, y)$ is strictly increasing in y on $[0, \infty)$, and hence, is invertible in y on $[0, \infty)$. We denote by $H_+^{-1}(t, y)$ its inverse. Similarly, $H(t, y)$ has an inverse $H_-^{-1}(t, y)$ in y on $(-\infty, 0]$.

Note that assumption (H2) implies that F_i is even. Therefore, for $t \in [a, b]$,

$$H_-^{-1}(t, y) = -H_+^{-1}(t, y), \quad y \in [0, \infty). \tag{2.2}$$

It is useful to note that for the special case with $m = 1$, we have $H(t, y) = w(t) F(y)$. Then, for $t \in [a, b]$,

$$H_\pm^{-1}(t, y) = F_\pm^{-1}\left(\frac{y}{w(t)} \right), \quad y \in [0, \infty). \tag{2.3}$$

We now present our main results on the existence and nonexistence of nodal solutions of BVP (1.1), (1.2) with the proofs given in a later section. In what follows, we let λ_n be the nth eigenvalue of the SLP (1.8), (1.9) for $n \in \mathbb{N}_0$. The first theorem concerns the existence of certain types of nodal solutions.

Theorem 2.1. *Assume that for some $n \in \mathbb{N}_0$ and all $t \in [a, b]$, either*

$$\sum_{i=1}^m w_i(t)(f_i)_0 < \lambda_n \sum_{i=1}^m w_i(t) \quad \textit{and} \quad \lambda_{n+1} \sum_{i=1}^m w_i(t) < \sum_{i=1}^m w_i(t)(f_i)_\infty, \tag{2.4}$$

or

$$\sum_{i=1}^m w_i(t)(f_i)_\infty < \lambda_n \sum_{i=1}^m w_i(t) \quad \textit{and} \quad \lambda_{n+1} \sum_{i=1}^m w_i(t) < \sum_{i=1}^m w_i(t)(f_i)_0. \tag{2.5}$$

Suppose further that, for any $c > 0$,

$$\int_a^b H_+^{-1}\left(s, ce^{k_0}\right) d\left(\xi_1(s) + \xi_2(s)\right) < H_+^{-1}(b, c), \tag{2.6}$$

where $\xi_1(s)$ and $\xi_2(s)$ are given by (1.10). Then BVP (1.1), (1.2) has a solution $y_n^\gamma \in \mathcal{T}_n^\gamma$ for $\gamma \in \{+, -\}$.

As consequences of Theorem 2.1, we have the following corollaries. The first one is for the special case of Eq. (1.1) with $m = 1$.

Corollary 2.1. *Consider the equation*

$$-y'' = w(t)f(y). \tag{2.7}$$

Assume that for some $n \in \mathbb{N}_0$,

$$\text{either } f_0 < \lambda_n \quad \text{and} \quad \lambda_{n+1} < f_\infty \quad \text{or} \quad f_\infty < \lambda_n \quad \text{and} \quad \lambda_{n+1} < f_0. \tag{2.8}$$

If, for any $c > 0$,

$$\int_a^b F_+^{-1}\Big(\frac{ce^{k_0}}{w(s)}\Big)\mathrm{d}(\xi_1(s) + \xi_2(s)) < F_+^{-1}\Big(\frac{c}{w(b)}\Big), \tag{2.9}$$

then BVP (2.7), (1.2) has a solution $y_n^\gamma \in \mathcal{T}_n^\gamma$ *for* $\gamma \in \{+, -\}$.

In particular, when $w(t) \equiv w_0 > 0$, *if (2.8) holds and*

$$\int_a^b \mathrm{d}(\xi_1(s) + \xi_2(s)) < 1, \tag{2.10}$$

then BVP (2.7), (1.2) has a solution $y_n^\gamma \in \mathcal{T}_n^\gamma$ *for* $\gamma \in \{+, -\}$.

Remark 2.1. Let the second condition of BC (1.2) be replaced by the multi-point BC (1.3), then we may choose $\xi(s) = \xi_1(s) - \xi_2(s)$ with

$$\xi_1(s) = \sum_{j=1}^{d}(k_j)_+\chi(s - \eta_j) \quad \text{and} \quad \xi_2(s) = \sum_{j=1}^{m}(k_j)_-\chi(s - \eta_j),$$

where $(k_j)_\pm = \max\{\pm k_j, 0\}$. Hence, $\xi_1(s) + \xi_2(s) = \sum_{j=1}^{d} |k_j|\chi(s - \eta_j)$. Then, it is easy to see that (2.9) reduces to

$$\sum_{j=1}^{d} |k_j|F_+^{-1}\Big(\frac{ce^{k_0}}{w(\eta_j)}\Big) < F_+^{-1}\Big(\frac{c}{w(b)}\Big). \tag{2.11}$$

In particular, when $f(y) = |y|^{r-1}y$ for $r > 0$, then (2.11) reduces to

$$\sum_{j=1}^{d} |k_j|\Big(\frac{w(b)e^{k_0}}{w(\eta_j)}\Big)^{1/(r+1)} < 1;$$

and when $w(t) \equiv w_0 > 0$ and $m = 1$, then (2.11) reduces to

$$\sum_{j=1}^{d} |k_j| < 1.$$

Therefore, it is easy to see that for the case when $m = 1$, Theorem 2.1 covers the main results in [18] for BVPs with multi-point BCs.

Corollary 2.2. *Assume that (2.6) holds and either (2.4) or (2.5) holds with* $n = 0$. *Then*

(i) *BVP (1.1), (1.2) has positive and negative solutions in* $\mathcal{T}_0^\gamma$ *for* $\gamma \in \{+, -\}$ *if* $\xi(s)$ *is increasing on* $[a, b]$;
(ii) *BVP (1.1), (1.2) has solutions in* $\mathcal{T}_0^\gamma$ *for* $\gamma \in \{+, -\}$ *with exactly one zero in* (a, b) *if* $\xi(s)$ *is decreasing on* $[a, b]$ *and such that* $\int_a^{b-\epsilon} \mathrm{d}\xi(s) < 0$ *for some small* $\epsilon > 0$.

Corollary 2.3. *Let* $\alpha \in [0, \pi/2)$. *Assume that (2.6) holds, and either*

$$\sum_{i=1}^{m}(f_i)_0 = 0 \quad \text{and} \quad \sum_{i=1}^{m}(f_i)_\infty = \infty, \tag{2.12}$$

or

$$\sum_{i=1}^{m}(f_i)_\infty = 0 \quad \text{and} \quad \sum_{i=1}^{m}(f_i)_0 = \infty. \tag{2.13}$$

Then for all $n \geq 0$ *and* $\gamma \in \{+, -\}$, *BVP (1.1), (1.2) has a solution* $y_n^\gamma \in \mathcal{T}_n^\gamma$.

Here we observe that (2.12) and (2.13) mean that

$$(f_i)_0 = 0 \quad \text{for all } i \in \{1, 2, \ldots, m\}, \quad \text{and} \quad (f_j)_\infty = \infty \quad \text{for some } j \in \{1, 2, \ldots, m\}$$

and

$$(f_i)_\infty = 0 \quad \text{for all } i \in \{1, 2, \ldots, m\}, \quad \text{and} \quad (f_j)_0 = \infty \quad \text{for some } j \in \{1, 2, \ldots, m\},$$

respectively.

The next theorem is about the nonexistence of certain types of nodal solutions.

Theorem 2.2. (i) *Assume that for some $n \in \mathbb{N}_0$,*

$$\sum_{i=1}^{m} w_i(t) f_i(y)/y < \lambda_n \sum_{i=1}^{m} w_i(t)$$

for all $t \in [a, b]$ and $y \neq 0$. Then BVP (1.1), (1.2) has no solution in $\mathcal{T}_j^\gamma$ for all $j \geq n$ and $\gamma \in \{+, -\}$.

(ii) *Assume that for some $n \in \mathbb{N}_0$,*

$$\sum_{i=1}^{m} w_i(t) f_i(y)/y > \lambda_{n+1} \sum_{i=1}^{m} w_i(t)$$

for all $t \in [a, b]$ and $y \neq 0$. Then BVP (1.1), (1.2) has no solution in $\mathcal{T}_j^\gamma$ for all $j \leq n$ and $\gamma \in \{+, -\}$.

3. Proofs of the main results

To prove Theorem 2.1, we need some preliminaries. Lemmas 3.1–3.3 have been proved in [19] for the case when $m = 1$ and can be extended to the general case with essentially the same proofs. The first lemma is a generalization of [19, Propositions 3.1 and 3.2 and Corollary 3.1].

Lemma 3.1. *Any initial value problem associated with Eq. (1.1) has a unique solution which exists on the whole interval $[a, b]$. Consequently, the solution depends continuously on the initial condition.*

As a direct result from Lemma 3.1, we have the following:

Corollary 3.1. *For any nontrivial solution y of Eq. (1.1), y and y' have only simple zeros in $[a, b]$.*

Proof. Assume that y has a non-simple zero $t_0 \in [a, b]$, i.e., $y(t_0) = y'(t_0) = 0$. Then, from Lemma 3.1, $y(t) \equiv 0$ on $[a, b]$. Assume that y' has a non-simple zero $t_0 \in [a, b]$, i.e., $y'(t_0) = y''(t_0) = 0$. From Eq. (1.1) and noting that $w_i(t) > 0$ for $i = 1, \ldots, m$, we have $f_i(y(t_0)) = 0,\ i = 1, \ldots, m$. Hence, $y(t_0) = 0$ and as above, $y(t) \equiv 0$ on $[a, b]$. In either case, we have reached a contradiction. □

For $\gamma \in \{+, -\}$, let $y(t, \rho)$ be the solution of Eq. (1.1) satisfying

$$y(a) = \gamma \rho \sin\alpha \quad \text{and} \quad y'(a) = \gamma \rho \cos\alpha, \tag{3.1}$$

where $\rho > 0$ is a parameter. Let $\theta(t, \rho)$ be the Prüfer angle of $y(t, \rho)$, i.e., $\theta(\cdot, \rho)$ is a continuous function on $[a, b]$ such that

$$\tan\theta(t, \rho) = y(t, \rho)/y'(t, \rho) \quad \text{and} \quad \theta(a, \rho) = \alpha.$$

By Lemma 3.1, $\theta(t, \rho)$ is continuous in ρ on $(0, \infty)$ for any $t \in [a, b]$. The following results are generalizations of [19, Lemmas 4.1, 4.2, 4.4, and 4.5].

Lemma 3.2. (i) *Assume that for some $n \in \mathbb{N}_0$,*

$$\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n \sum_{i=1}^{m} w_i(t)$$

for all $t \in [a, b]$. Then there exists $\rho_* > 0$ such that $\theta(b, \rho) < n\pi + \pi/2$ for all $\rho \in (0, \rho_*)$.

(ii) *Assume that for some $n \in \mathbb{N}_0$,*

$$\lambda_n \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$$

for all $t \in [a, b]$. Then there exists $\rho^* > 0$ such that $\theta(b, \rho) > n\pi + \pi/2$ for all $\rho \in (\rho^*, \infty)$.

Lemma 3.3. (i) *Assume that for some* $n \in \mathbb{N}_0$,

$$\sum_{i=1}^{m} w_i(t)(f_i)_\infty < \lambda_n \sum_{i=1}^{m} w_i(t)$$

for all $t \in [a, b]$. *Then there exists* $\rho^* > 0$ *such that* $\theta(b, \rho) < n\pi + \pi/2$ *for all* $\rho \in (\rho^*, \infty)$.

(ii) *Assume that for some* $n \in \mathbb{N}_0$,

$$\lambda_n \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_0$$

for all $t \in [a, b]$. *Then there exists* $\rho_* > 0$ *such that* $\theta(b, \rho) > n\pi + \pi/2$ *for all* $\rho \in (0, \rho_*)$.

Proof of Theorem 2.1. We first prove it under the assumption that (2.4) holds. Without loss of generality, we assume that $\gamma = +$. The case with $\gamma = -$ can be proved in the same way. Let $y(t, \rho)$ be the solution of Eq. (1.1) satisfying (3.1) with $\gamma = +$, and $\theta(t, \rho)$ its Prüfer angle. By Lemma 3.2, there exist $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b, \rho) < n\pi + \pi/2 \quad \text{for all } \rho \in (0, \rho_*)$$

and

$$\theta(b, \rho) > (n + 1)\pi + \pi/2 \quad \text{for all } \rho \in (\rho^*, \infty).$$

By the continuity of $\theta(t, \rho)$ in ρ, there exist $\rho_* \le \rho_n < \rho_{n+1} \le \rho^*$ such that

$$\theta(b, \rho_n) = n\pi + \pi/2 \quad \text{and} \quad \theta(b, \rho_{n+1}) = (n + 1)\pi + \pi/2, \tag{3.2}$$

and

$$n\pi + \pi/2 < \theta(b, \rho) < (n + 1)\pi + \pi/2 \quad \text{for } \rho_n < \rho < \rho_{n+1}. \tag{3.3}$$

Then, for all $t \in [a, b]$ and all $\rho > 0$, we define an energy function $E(t, \rho)$ for $y(t, \rho)$ by

$$E(t, \rho) = \frac{1}{2}\big(y'(t, \rho)\big)^2 + H\big(t, y(t, \rho)\big), \tag{3.4}$$

where $H(t, y)$ is defined in (2.1). By (H1) and (H2), $F_i(y) \ge 0$ on $\mathbb{R}$ for each $i \in \{1, \dots, m\}$. Then, $E(t, \rho) \ge 0$ on $[a, b]$. By (1.1) and the definition of $l(t)$, we have

$$\begin{aligned} E'(t, \rho) &= \sum_{i=1}^{m} w_i'(t)F_i(y(t, \rho)) \ge \sum_{i=1}^{m} \frac{w_i'(t)}{w_i(t)} w_i(t)F_i(y(t, \rho)) \\ &\ge -l(t)\Big(\frac{1}{2}\big(y'(t, \rho)\big)^2 + H\big(t, y(t, \rho)\big)\Big) = -l(t)E(t, \rho). \end{aligned}$$

Thus, $E'(t, \rho) + l(t)E(t, \rho) \ge 0$ for all $t \in [a, b]$ and $\rho > 0$. By solving this inequality, we obtain

$$E(s, \rho) \le E(b, \rho)\mathrm{e}^{\int_s^b l(\tau)\mathrm{d}\tau} \le E(b, \rho)\mathrm{e}^{k_0}, \quad s \in [a, b]. \tag{3.5}$$

We observe from (3.4) that, for $\rho = \rho_n$ and $\rho = \rho_{n+1}$,

$$E(s, \rho) \ge H\big(s, y(s, \rho)\big) \quad \text{and} \quad E(b, \rho) = H\big(b, y(b, \rho)\big). \tag{3.6}$$

Recall that, for fixed t, $H_+^{-1}(t, y)$ is increasing in y on $[0, \infty)$. Thus, from (3.6) and (2.2), we see that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$ and $s \in [a, b]$,

$$|y(s, \rho)| \le H_+^{-1}(s, E(s, \rho)) \quad \text{and} \quad |y(b, \rho)| = H_+^{-1}(b, E(b, \rho)). \tag{3.7}$$

Define

$$\Gamma(\rho) = y(b, \rho) - \int_a^b y(s, \rho)\mathrm{d}\xi(s). \tag{3.8}$$

Let $n = 2k$ with $k \in \mathbb{N}_0$. Since $y(b, \rho_{2k}) > 0$ and $y(b, \rho_{2k+1}) < 0$, then using a similar argument as in [18] (see (2.11) and (2.12) there), from (3.5), (3.7) and (2.6), we can see that

$$\begin{aligned} \Gamma(\rho_{2k}) &= y(b, \rho_{2k}) - \int_a^b y(s, \rho_{2k})\mathrm{d}\xi(s) \\ &\ge y(b, \rho_{2k}) - \int_a^b |y(s, \rho_{2k})|\mathrm{d}(\xi_1(s) + \xi_2(s)) > 0 \end{aligned} \tag{3.9}$$

and

$$\Gamma(\rho_{2k+1}) = y(b, \rho_{2k+1}) - \int_a^b y(s, \rho_{2k+1}) \mathrm{d}\xi(s)$$
$$\leq -|y(b, \rho_{2k+1})| + \int_a^b |y(s, \rho_{2k+1})| \mathrm{d}(\xi_1(s) + \xi_2(s)) < 0. \quad (3.10)$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k+1$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1}, \rho_{2k+2})$ such that $\Gamma(\bar{\rho}) = 0$. In both cases, from (3.3),

$$n\pi + \pi/2 < \theta(b, \bar{\rho}) < (n+1)\pi + \pi/2.$$

Note that for $t \in (a, b)$ with $y(t) \neq 0$, $\theta(t)$ satisfies the equation

$$\theta'(t, \rho) = \cos^2\theta(t, \rho) + \sum_{i=1}^m w_i(t) \frac{f_i(y(t, \rho))}{y(t, \rho)} \sin^2\theta(t, \rho). \quad (3.11)$$

By (H1) and (H2), $\theta(\cdot, \rho)$ is strictly increasing on $[a, b]$. We note that $y(t) = 0$ if and only if $\theta(t, \rho) = 0 \pmod \pi$ and $y'(t) = 0$ if and only if $\theta(t, \rho) = \pi/2 \pmod \pi$. Thus, y' has exactly $n+1$ zeros in (a, b) if $\alpha \in [0, \pi/2)$ and n zeros in (a, b) if $\alpha \in [\pi/2, \pi)$, and y has exactly one zero strictly between any two consecutive zeros of y'. Initial condition (3.1) implies that $y(t, \bar{\rho}) > 0$ in a right-neighborhood of a. Therefore, $y(t, \bar{\rho}) \in \mathcal{T}_n^+$.

The proof when (2.5) holds is essentially the same as above except that the discussion is based on Lemma 3.3 instead of Lemma 3.2. □

Proof of Corollary 2.1. Since $m = 1$, by (2.3), $H_\pm^{-1}(t, ce^{k_0}) = F_\pm^{-1}(ce^{k_0}/w(t))$. In this case, conditions (2.4)–(2.6) reduce to (2.8)–(2.9). Then the conclusion follows from Theorem 2.1.

In addition, when $w(t) \equiv w_0 > 0$, from the definition of $l(t)$, $l(t) \equiv 0$. Hence, $k_0 = \int_a^b l(t)\mathrm{d}t = 0$. By (2.3), $H_\pm^{-1}(t, ce^{k_0}) = F_\pm^{-1}(c/w_0)$. In this case, (2.9) reduces to (2.10). Then the conclusion follows from Theorem 2.1. □

Proof of Corollary 2.2. Without loss of generality, let $\gamma = +$. The case for $\gamma = -$ can be proved in the same way. Let $y(t, \rho)$ be the solution of Eq. (1.1) satisfying (3.1) with $\gamma = +$, and $\theta(t, \rho)$ its Prüfer angle. Let ρ_0 and ρ_1 be given in (3.2) with $n = 0$. Then for the function $\Gamma(\rho)$ defined by (3.8), from (3.9) and (3.10), we have $\Gamma(\rho_0) > 0$ and $\Gamma(\rho_1) < 0$.

(i) Assume that $\xi(s)$ is increasing on $[a, b]$. Since $\xi(s) = \xi_1(s) - \xi_2(s)$, we may take $\xi(s) = \xi_1(s)$. From (3.2) and (3.3) with $n = 0$ and by the continuity of $\theta(t, \rho)$ in ρ, there exists $\tilde{\rho} \in (\rho_0, \rho_1)$ such that $\theta(b, \tilde{\rho}) = \pi$ and $\pi/2 < \theta(b, \rho) < \pi$ for $\rho \in (\rho_0, \tilde{\rho})$. Note that $\theta(s, \tilde{\rho}) < \pi$ for $s \in [a, b)$. Thus, $y(b, \tilde{\rho}) = 0$ and $y(s, \tilde{\rho}) > 0$ for $s \in [a, b)$. By the mean value theorem for the Riemann–Stieltjes integral and the continuity of solution y, there exists $t_0 \in [a, b]$ such that

$$\Gamma(\tilde{\rho}) = -\int_a^b y(s, \tilde{\rho}) \mathrm{d}\xi_1(s) = -y(t_0, \tilde{\rho}) \int_a^b \mathrm{d}\xi_1(s) \leq 0.$$

Therefore, there exists $\bar{\rho} \in (\rho_0, \tilde{\rho}]$ such that $\Gamma(\bar{\rho}) = 0$. This means that $y(t, \bar{\rho})$ is a solution of BVP (1.1), (1.2). Note that $\theta(b, \bar{\rho}) \in (\pi/2, \pi)$, we have $y(t, \bar{\rho}) \in \mathcal{T}_0^+$ and is a positive solution.

(ii) Assume that $\xi(s)$ is decreasing on $[a, b]$ such that $\int_a^{b-\epsilon} \mathrm{d}\xi(s) < 0$ for some small $\epsilon > 0$. Since $\xi(s)$ is decreasing, we may take $\xi(s) = -\xi_2(s)$. As in (i), there exists $\tilde{\rho} \in (\rho_0, \rho_1)$ such that $\theta(b, \tilde{\rho}) = \pi$ and $3\pi/2 > \theta(b, \rho) > \pi$ for $\rho \in (\tilde{\rho}, \rho_1)$. We note that $\theta(s, \tilde{\rho}) < \pi$ for $s \in [a, b)$. Thus, $y(b, \tilde{\rho}) = 0$ and $y(s, \tilde{\rho}) > 0$ for $s \in [a, b)$. Thus, for some $t_0 \in [a, b-\epsilon]$,

$$\Gamma(\tilde{\rho}) = -\int_a^b y(s, \tilde{\rho}) \mathrm{d}\xi_2(s) \geq -\int_a^{b-\epsilon} y(s, \tilde{\rho}) \mathrm{d}\xi_2(s) = -y(t_0, \tilde{\rho}) \int_a^{b-\epsilon} \mathrm{d}\xi_2(s) > 0.$$

Therefore, there exists $\bar{\rho} \in (\tilde{\rho}, \rho_1)$ such that $\Gamma(\bar{\rho}) = 0$. This means that $y(t, \bar{\rho})$ is a solution of BVP (1.1), (1.2). Note that $\theta(b, \bar{\rho}) \in (\pi, 3\pi/2)$, we have $y(t, \bar{\rho}) \in \mathcal{T}_0^+$ and has exactly one zero in (a, b). □

Proof of Corollary 2.3. It is easy to see that $\lambda_0 = 0$ is the first eigenvalue of the BVP consisting of Eq. (1.8) and the BC

$$y'(a) = y'(b) = 0,$$

i.e., the BC (1.9) with $\alpha = \pi/2$. In fact, $y_0(t) \equiv 1$ is an associated eigenfunction. From [31, Theorem 4.2], we see that λ_0, as a function of α, is strictly decreasing. This shows that $\lambda_0 > 0$ for $\alpha \in [0, \pi/2)$. Then, by Theorem 2.1, BVP (1.1), (1.2) has a solution $y_n^\gamma \in \mathcal{T}_n^\gamma$ for all $n \geq 0$ and $\gamma \in \{+, -\}$. □

Finally, the proof of Theorem 2.2 is essentially the same as that of [18, Theorem 2.2] and hence is omitted.

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