Alexandria Engineering Journal (2018) 57, 1991–1998

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ORIGINAL ARTICLE

Numerical treatment for traveling wave solutions of fractional Whitham-Broer-Kaup equations

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Received 3 February 2017; revised 21 April 2017; accepted 23 April 2017
Available online 13 May 2017

KEYWORDS

Whitham-Broer-Kaup equations;
Non-linear partial fractional differential equation;
Laplace-Adomian decomposition method;
Numerical solutions

Abstract In the concerned article, we present the numerical solution of nonlinear coupled system of Whitham-Broer-Kaup equations (WBK) of fractional order. With the help of Laplace transform coupled with Adomian decomposition method, an iterative procedure is established to investigate approximate solution to the proposed coupled system of nonlinear partial fractional differential equations. The concerned techniques are demonstrated by some numerical examples. Also, we compared the results of our proposed method with the results of other well known numerical methods such as Variation iteration method (VIM), Adomian decomposition method (ADM) and Homotopy perturbation method (HPM). For computation, we use Maple 18 and Matlab.

1. Introduction

In applied sciences, engineering and technology, partial differential equations containing nonlinearities describe different phenomena, ranging from gravitation to dynamics [1–3]. In fact the nonlinear partial differential equations are important tools used to modeled nonlinear dynamical phenomena in different fields such as mathematical biology, plasma physics, solid state physics, and fluid dynamics, see [4]. The majority of dynamical systems can be represented by suitable set of partial differential equations. It is also well known that partial differential equations are used to solve mathematical problems, such as Calabi conjecture and the Poincare conjecture problems.

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Peer review under responsibility of Faculty of Engineering, Alexandria University.

It has been observed that in fluid dynamics, the nonlinear evolution of shallow water waves is represented by using coupled system of Whitham-Broer-Kaup (WBK) [14] equations. The coupled system of aforesaid equations was introduced by Whitham [10], Broer [11] and Kaup [12]. The aforesaid equations describe the shallow water wave propagation with different spreading relations, see [13]. The governing equations of the said phenomenon in the classical order are given by

$$\begin{cases} \phi_t + \phi\phi_x + \psi_x + q\phi_{xx} = 0, \\ \psi_t + \psi\phi_x + \phi\psi_x - q\psi_{xx} + p\phi_{xxx} = 0, \end{cases} \tag{1}$$

where $\phi = \phi(x,t), \psi = \psi(x,t)$ represents the horizontal velocity and height that deviates from equilibrium position of the liquid respectively and p, q are constant which are represented in different diffusion powers. To investigate solution to such like nonlinear partial differential equations is an important area of research for the last few decades. Various numerical techniques have been developed by many researchers to investigate approximate solutions to nonlinear partial differential equations. Recently, Mohyud-Din and Noor [15], used homo-

http://dx.doi.org/10.1016/j.aej.2017.04.012
1110-0168

topy perturbation techniques to investigate the solution of various classical order partial differential equations. Biazar and Aminikhah [16], used perturbation technique to solve the coupled system of Burgers and Brusselator equations. For approximate solutions to other classical order partial differential equations by using other methods, see [17–20]. In the same line the solution to the given nonlinear coupled system (1) of partial differential equation has been studied by various methods. Mohyud-Din et al. [21], studied the solution of classical order coupled system of (WBK) equation by perturbation techniques. So for, many efficient and powerful techniques were used by different researchers to investigate the solution of (WBK) coupled equation of classical order partial differential equations, such as Xie et al. in (2002), investigated the solution through hyperbolic method. Similarly, El-Sayed and Kaya, studied the system (1) by Adomian decomposition method (ADM). Also Ahmad et al. [4], investigated the solution of coupled system (1) by (ADM) coupled with He's polynomial.

Recently, partial differential equations of fractional order have got great attention, due to its numerous applications in various fields of applied sciences, such as control theory, pattern reorganization, signal processing and identification of systems, image processing, fluid mechanics, see [5,6,8,9]. Because, in many sanitation, fractional order differential equations describe many physical phenomenon more accurately as compared to classical order. Therefore strong motivation has been found for fine numerical solutions of fractional order differential equations. Like classical order partial order differential equations, fractional order partial differential equations have been investigated by the aforesaid methods for analytical and approximate solutions.

Since the investigation of the traveling wave solutions to nonlinear fractional order partial differential equations plays an important role in the study of nonlinear physical phenomena, various numbers of very efficient and powerful techniques are proposed for the approximate solutions to the said equations. Likewise, classical order (WBK) equations, the solution for fractional order (WBK) partial differential equations were also investigated by various techniques. Wang and Zheng [22], investigated the approximate solution of coupled system of (WBK) equations for fractional order provided as

$$\begin{cases} \frac{\partial^\delta \phi}{\partial t^\delta} + \phi \frac{\partial^\eta \phi}{\partial x^\eta} + \frac{\partial^\eta \psi}{\partial x^\eta} + q \frac{\partial^{2\eta} \phi}{\partial x^{2\eta}} = 0, \quad 0 < \delta,\ \eta \leq 1, \\ \frac{\partial^\delta \psi}{\partial t^\delta} + \phi \frac{\partial^\eta \psi}{\partial x^\eta} + \psi \frac{\partial^\eta \phi}{\partial x^\eta} + p \frac{\partial^{3\eta} \phi}{\partial x^{3\eta}} - q \frac{\partial^{2\eta} \psi}{\partial x^{2\eta}} = 0, \quad 0 < \delta,\ \eta \leq 1, \\ \phi(x,0) = \alpha(x), \quad \psi(x,0) = \beta(x), \end{cases} \tag{2}$$

where $\frac{\partial^\delta}{\partial t^\delta}$, $\frac{\partial^\eta}{\partial x^\eta}$ are partial fractional order derivatives in Caputo sense. They used an extended fractional Riccati sub-equation method to obtain the solutions for coupled system (2). Similarly, El-Borai et al. [23], solved coupled system (2) via exponential function method. In the same line, author [7], used the coupled fractional reduced differential transform method (CFRDTM) to obtain the analytical approximate solutions to the aforesaid model(2). The authors in [2], applied the residual power series method (RPSM) to investigate numerical solutions to the considered coupled system (2). Also, finite difference method [24], the finite element method [25], the exponential-function method [26], the Homotopy analysis method(HAM), Homotopy perturbation method (HPM), Variation iteration method (VIM), etc., were also used to find numerical solutions to the coupled system (2), for detail see [27–29]. In 1980, Adomian decomposition method (ADM) was introduced by Adomian, which is an effective method for finding numerical and explicit solution of a wide class of differential equations representing physical problems. This method works efficiently for both initial value problems and boundary value problem, for partial and ordinary differential equations including linear and nonlinear equations and also for stochastic system as well. Adomian decomposition method coupled with Laplace transform leads to a powerful method known as Laplace Adomian decomposition method (LADM). LADM has also been applied in many papers to obtain the numerical solution of fractional order nonlinear partial differential equations, see [22–24].

Inspired from the mentioned papers, in this article, we investigate the general as well as numerical solution of the coupled system of Whitham-Broer-Kaup equations of fractional order by using Laplace transform Adomian decomposition method (LADM). LADM is a simple and very efficient method in which no perturbation or liberalization is required. We compare the results of our proposed method with those of other well-known methods such as HPM, VIM, and ADM. We see that the proposed method is better than the mentioned method to find the solutions of nonlinear fractional order partial differential equations. For the computation purposes, we use Matlab and Maple 18. The convergence of the proposed method is also provided by extending the idea been discussed in [25,26].

2. Preliminaries

In the concerned section, we recall some basic definitions and results from fractional calculus. For the concerned materials, we refer [8–11,25,26].

Definition 2.1. The fractional integral of Riemann-Liouville type of order $\delta \in (0,1)$ of a function $\phi \in L^1(0,\infty), \mathbb{R})$ is defined as

$$I^\delta \phi(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \phi(s)\, ds,$$

provided that the integral on the right is converges point wise on $(0,\infty)$.

Definition 2.2. The Caputo fractional order derivative of a function ϕ on the interval $(0,\infty) \times (0,\infty)$ is defined by

$${}^cD^\delta \phi(x,t) = \frac{1}{\Gamma(n-\delta)} \int_0^t (t-s)^{n-\delta-1} \phi^{(n)}(x,s) ds,$$

where $n = [\delta] + 1$ and $[\delta]$ represents the integer part of δ. In particularly for $0 < \delta < 1$, one has

$${}^cD^\delta \phi(x,t) = \frac{1}{\Gamma(1-\delta)} \int_0^t \frac{1}{(t-s)^\delta} \frac{\partial}{\partial s} \phi(x,s) ds.$$

Lemma 2.1. *The following result holds for fractional differential equations*

$$I^\delta[{}^cD^\delta \phi](x,t) = \phi(x,t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, *where* $n = [\delta] + 1$ and $[\delta]$ *represents the integer part of* δ.

Definition 2.3. We recall the definition of Laplace transform of Caputo derivative as

$$\mathcal{L}\{{}^cD^\delta \phi(x,t)\} = s^\delta \phi(x,s) - \sum_{k=0}^{n-1} s^{\delta-k-1}\phi^{(k)}(x,0),$$
$$n-1 < \delta < n, \quad n \in N.$$

where $n = [\delta] + 1$ and $[\delta]$ represents the integer part of δ.

3. The Laplace Adomian decomposition method

This section is concerned with the general procedure for numerical treatment of (2), with given initial conditions. Taking Laplace transform of the system of Eq. (2) as

$$\begin{cases} \mathcal{L}\left[\frac{\partial^\delta \phi}{\partial t^\delta} + \phi\frac{\partial^\eta \phi}{\partial x^\eta} + \frac{\partial^\eta \psi}{\partial x^\eta} + q\frac{\partial^{2\eta}\phi}{\partial x^{2\eta}}\right] = 0, \\ \mathcal{L}\left[\frac{\partial^\delta \psi}{\partial t^\delta} + \phi\frac{\partial^\eta \psi}{\partial x^\eta} + \psi\frac{\partial^\eta \phi}{\partial x^\eta} + p\frac{\partial^{3\eta}\phi}{\partial x^{3\eta}} - q\frac{\partial^{2\eta}\psi}{\partial x^{2\eta}}\right] = 0, \end{cases} \tag{3}$$

Applying Laplace transform, we get

$$\begin{cases} s^\delta\mathcal{L}\{\phi(t,x)\} - s^{\delta-1}\phi(x,0) = -\mathcal{L}\left[\phi\frac{\partial^\eta \phi}{\partial x^\eta} + \frac{\partial^\eta \psi}{\partial x^\eta} + q\frac{\partial^{2\eta}\phi}{\partial x^{2\eta}}\right], \\ s^\delta\mathcal{L}\{\psi(x,t)\} - s^{\delta-1}\psi(x,0) = -\mathcal{L}\left[\phi\frac{\partial^\eta \psi}{\partial x^\eta} + \psi\frac{\partial^\eta \phi}{\partial x^\eta} + p\frac{\partial^{3\eta}\phi}{\partial x^{3\eta}} - q\frac{\partial^{2\eta}\psi}{\partial x^{2\eta}}\right], \end{cases} \tag{4}$$

Assuming that the solutions, $\phi(x,t), \psi(x,t)$ in the form of infinite series are given by

$$\phi(x,t) = \sum_{i=0}^{\infty}\phi_i(x,t), \quad \psi(x,t) = \sum_{i=0}^{\infty}\psi_i(x,t). \tag{5}$$

and the nonlinear terms $\phi(x,t)\frac{\partial^\eta \phi(x,t)}{\partial x^\eta}$, $\psi(x,t)\frac{\partial^\eta \phi(x,t)}{\partial x^\eta}$, $\phi(x,t)\frac{\partial^\eta \psi(x,t)}{\partial x^\eta}$ involved in the system are decompose as

$$\phi(x,t)\frac{\partial^\eta \phi(x,t)}{\partial x^\eta} = \sum_{i=0}^{\infty} Q_i, \quad \psi(x,t)\frac{\partial^\eta \phi(x,t)}{\partial x^\eta} = \sum_{i=0}^{\infty} R_i, \quad \phi(x,t)\frac{\partial^\eta \psi(x,t)}{\partial x^\eta} = \sum_{i=0}^{\infty} S_i. \tag{6}$$

where Q_i, R_i and S_i are Adomian polynomials defined as

$$\begin{aligned} Q_i &= \frac{1}{\Gamma(i+1)}\frac{d^i}{d\lambda^i}\left[\sum_{i=0}^{n}\lambda^i\phi_i\sum_{i=0}^{n}\lambda^i\frac{\partial^\eta\phi_i}{\partial x^\eta}\right]\Big|_{\lambda=0}, \\ R_i &= \frac{1}{\Gamma(i+1)}\frac{d^i}{d\lambda^i}\left[\sum_{i=0}^{n}\lambda^i\psi_i\sum_{i=0}^{n}\lambda^i\frac{\partial^\eta\phi_i}{\partial x^\eta}\right]\Big|_{\lambda=0}, \\ S_i &= \frac{1}{\Gamma(i+1)}\frac{d^i}{d\lambda^i}\left[\sum_{i=0}^{n}\lambda^i\phi_i\sum_{i=0}^{n}\lambda^i\frac{\partial^\eta\psi_i}{\partial x^\eta}\right]\Big|_{\lambda=0}. \end{aligned} \tag{7}$$

Now, applying inverse Laplace transform and initial conditions to system (4), we get

$$\begin{cases} \phi(x,t) = \alpha(x) - \mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{\phi\frac{\partial^\eta \phi}{\partial x^\eta} + \frac{\partial^\eta \psi}{\partial x^\eta} + q\frac{\partial^{2\eta}\phi}{\partial x^{2\eta}}\right\}\right], \\ \psi(x,t) = \beta(x) + \mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{\phi\frac{\partial^\eta \psi}{\partial x^\eta} + \psi\frac{\partial^\eta \phi}{\partial x^\eta} + p\frac{\partial^{3\eta}\phi}{\partial x^{3\eta}} - q\frac{\partial^{2\eta}\psi}{\partial x^{2\eta}}\right\}\right], \end{cases} \tag{8}$$

using (5), (6) in system (8), and comparing both sides of coupled equations, we get

$$\begin{aligned} &\phi_0(x,t) = \phi_0(x,0) = \alpha(x), \quad \psi_0(x,t) = \psi_0(x,0) = \beta(x), \\ &\phi_1(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{Q_0 + \frac{\partial^\eta\psi_0(x,t)}{\partial x^\eta} + q\frac{\partial^{2\eta}\phi_0(x,t)}{\partial x^{2\eta}}\right\}\right], \\ &\psi_1(x,t)) = -\mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{R_0 + S_0 + p\frac{\partial^{3\eta}\phi_0(x,t)}{\partial x^{3\eta}} - q\frac{\partial^{2\eta}\psi_0(x,t)}{\partial x^{2\eta}}\right\}\right], \\ &\phi_2(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{Q_1 + \frac{\partial^\eta\psi_1(x,t)}{\partial x^\eta} + q\frac{\partial^{2\eta}\phi_1(x,t)}{\partial x^{2\eta}}\right\}\right], \\ &\psi_2(x,t)) = -\mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{R_1 + S_1 + p\frac{\partial^{3\eta}\phi_1(x,t)}{\partial x^{3\eta}} - q\frac{\partial^{2\eta}\psi_1(x,t)}{\partial x^{2\eta}}\right\}\right], \\ &\phi_3(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{Q_2 + \frac{\partial^\eta\psi_2(x,t)}{\partial x^\eta} + q\frac{\partial^{2\eta}\phi_2(x,t)}{\partial x^{2\eta}}\right\}\right], \\ &\psi_3(x,t)) = -\mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{R_2 + S_2 + p\frac{\partial^{3\eta}\phi_2(x,t)}{\partial x^{3\eta}} - q\frac{\partial^{2\eta}\psi_2(x,t)}{\partial x^{2\eta}}\right\}\right], \\ &\vdots \\ &\phi_n(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{Q_{n-1} + \frac{\partial^\eta\psi_{n-1}(x,t)}{\partial x^\eta} + q\frac{\partial^{2\eta}\phi_{n-1}(x,t)}{\partial x^{2\eta}}\right\}\right], \\ &\qquad n \geqslant 1, \\ &\psi_n(x,t)) = -\mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{R_{n-1} + S_{n-1} + p\frac{\partial^{3\eta}\phi_{n-1}(x,t)}{\partial x^{3\eta}} \right.\right. \\ &\qquad \left.\left. - q\frac{\partial^{2\eta}\psi_{n-1}(x,t)}{\partial x^{2\eta}}\right\}\right], \quad n \geqslant 1. \end{aligned} \tag{9}$$

In this way, we get the solution in the form of infinite series as provided by

$$\begin{aligned} \phi(x,t) &= \phi_0(x,t) + \phi_1(x,t) + \phi_2(x,t) + \cdots = \sum_{n=0}^{\infty}\phi_n(x,t), \\ \psi(x,t) &= \psi_0(x,t) + \psi_1(x,t) + \psi_2(x,t) + \cdots = \sum_{n=0}^{\infty}\psi_n(x,t). \end{aligned} \tag{10}$$

To check the convergence of the series (10), one can use classical techniques, to give sufficient conditions of convergence of the proposed method.

Convergence of the proposed method: The solution obtained via using the proposed method is in the form of infinite series. The concerned series is rapidly convergent. For further detail about the convergence of the proposed method, we refer [30,31].

4. Numerical experiments

In concerned section, we studied the coupled system of Whitham-Broer-Kaup equations with different initial conditions (see Figs. 1 and 2).

Example 4.1. Consider the coupled system of Whitham-Broer-Kaup equations with $p = 3$, $q = 1$ and $\eta = 1$, we obtain the proposed equation like considered in [4] in fractional order as given by

$$\begin{cases} \frac{\partial^\delta\phi(x,t)}{\partial t^\delta} = -\phi(x,t)\frac{\partial\phi(x,t)}{\partial x} - \frac{\partial\psi(x,t)}{\partial x} - \frac{\partial\phi(x,t)}{\partial x}, \\ \frac{\partial^\delta\psi(x,t)}{\partial t^\delta} = -\phi(x,t)\frac{\partial\psi(x,t)}{\partial x} - \psi(x,t)\frac{\partial\phi(x,t)}{\partial x} + \frac{\partial^2\psi(x,t)}{\partial x^2} - 3\frac{\partial^3\phi(x,t)}{\partial x^3}, \end{cases} \tag{11}$$

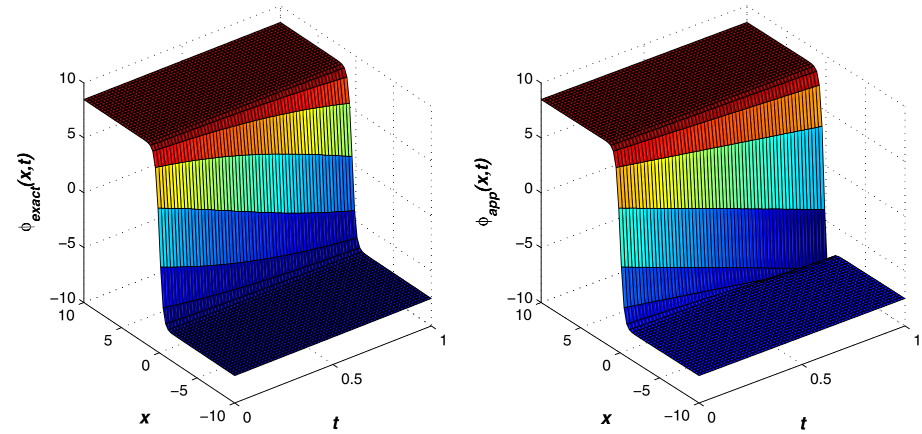

Figure 1 Comparison of approximate and exact plot of $\phi(x,t)$ at $\delta = 1$.

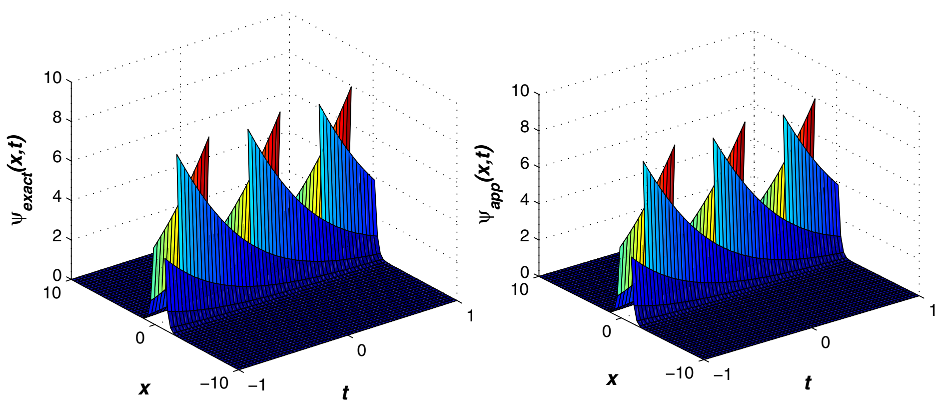

Figure 2 Comparison of approximate and exact plots of $\psi(t,x)$ at $\delta = 1$ for Example 4.1.

subject to the initial conditions

$$\begin{aligned}\phi(x,0) &= \frac{1}{2} - 8\ \tanh(-2x),\\ \psi(x,0) &= 16 - 16\ \tanh^2(-2x).\end{aligned} \tag{12}$$

The exact solution of (11) at classical order $\delta = 1$ and taking $p = 3$, $q = 1$, $-10 \leqslant x \leq 10$ and $-1 \leqslant t \leq 1$ in [4], is given by

$$\begin{aligned}\phi(x,t) &= \frac{1}{2} - 8\ \tanh\left[-2\left(x - \frac{t}{2}\right)\right], \quad \psi(x,t)\\ &= 16 - 16\ \tanh^2\left[-2\left(x - \frac{t}{2}\right)\right].\end{aligned} \tag{13}$$

Thus using the procedure (9), we have

$$\begin{aligned}\phi(x,t) &= \frac{1}{2} - 8\ \tanh(-2x)\\ &\quad - \mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{\sum_{i=0}^{\infty} Q_i + \frac{\partial}{\partial x}\sum_{i=0}^{\infty}\psi_i(x,t) + \frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\phi_i(x,t)\right\}\right],\\ \psi(x,t)) &= 16 - 16\ \tanh^2(-2x) - \mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{\sum_{i=0}^{\infty} R_i + \sum_{i=0}^{\infty} S_i\right.\right.\\ &\quad \left.\left. - \frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\psi_i(x,t) + 3\frac{\partial^3}{\partial x^3}\sum_{i=0}^{\infty}\phi_i(x,t)\right\}\right],\end{aligned}$$

where the nonlinear terms of (11) are represented by Q_i, R_i, S_i. Comparing both sides of (14), we have

$$\begin{aligned}\phi_0(x,0) &= \frac{1}{2} - 8\ \tanh(-2x),\\ \psi_0(x,0)) &= 16 - 16\ \tanh^2(-2x),\\ \phi_1(x,t) &= -8sech^2(-2x)\frac{t^\delta}{\Gamma(\delta+1)},\\ \psi_1(x,t) &= -32sech^2(-2x)\ \tanh(-2x)\frac{t^\delta}{\Gamma(\delta+1)},\\ \phi_2(x,t) &= 16sech^2(-2x)\{3\ \tanh(-2x) - 8\ \tanh^2(-2x)\\ &\quad + 4sech^2(-2x)\}\frac{t^{2\delta}}{\Gamma(2\delta+1)},\\ \psi_2(x,t) &= -32sech^2(-2x)\{96\ \tanh(-2x) - 32\ \tanh^3(-2x)\\ &\quad + 40sech^2(-2x)\ \tanh(-2x) - 2\ \tanh^2(-2x)\\ &\quad - 25sech^2(-2x)\}\frac{t^{2\delta}}{\Gamma(2\delta+1)},\end{aligned} \tag{15}$$

On the similar way, one can obtain the remaining terms and finally, we get the solution in the form of infinite series as given by

$$\begin{aligned}\phi(x,t) &= \phi_0(x,t) + \phi_1(x,t) + \phi_2(x,t) + \phi_3(x,t) + \cdots,\\ \psi(x,t) &= \psi_0(x,t) + \psi_1(x,t) + \psi_2(x,t) + \psi_3(x,t) + \cdots.\end{aligned} \tag{16}$$

The solution to the coupled system (11), after three terms is given by

$$
\begin{aligned}
\phi(x,t) &= \frac{1}{2} - 8\ \tanh(-2x) - 8sech^2(-2x)\frac{t^\delta}{\Gamma(\delta+1)} \\
&\quad + 16sech^2(-2x)\{3\ \tanh(-2x) - 8\ \tanh^2(-2x) \\
&\quad + 4sech^2(-2x)\}\frac{t^{2\delta}}{\Gamma(2\delta+1)}, \\
\psi(x,t) &= 16 - 16\ \tanh^2(-2x) - 32sech^2(-2x) \\
&\quad \times \tanh(-2x)\frac{t^\delta}{\Gamma(\delta+1)} - 32sech^2(-2x)\{96\ \tanh(-2x) \\
&\quad - 32\ \tanh^3(-2x) + 40sech^2(-2x)\ \tanh(-2x) \\
&\quad - 2\ \tanh^2(-2x) - 25sech^2(-2x)\}\frac{t^{2\delta}}{\Gamma(2\delta+1)}.
\end{aligned} \tag{17}
$$

We obtain the following series at classical order $\delta = 1$, using (17) is given by

$$
\begin{aligned}
\phi(x,t) &= \frac{1}{2} - 8\ \tanh(-2x) - 8sech(-2x)^2 t + 8sech^2(-2x) \\
&\quad \times \left\{3tanh(-2x) + 8tanh(-2x)^2 + 4sech^2(-2x)\right\}t^2, \\
\psi(x,t) &= 16 - 16\ \tanh^2(-2x) - 32sech^2(-2x)tanh(-2x)t \\
&\quad - 16sech^2(-2x)\{96tanh(-2x) - 32tanh^3(-2x) \\
&\quad + 40sech^2(-2x)tanh(-2x) - 2tanh^2(-2x) \\
&\quad - 25sech^2(-2x)\}t^2.
\end{aligned} \tag{18}
$$

Example 4.2. Consider the system of coupled equations(2) with $p = 0$, $q = \frac{1}{2}$ and $\delta = 1$, we obtain (see Figs. 3 and 4)

$$
\begin{cases}
\frac{\partial^\delta \phi(x,t)}{\partial t^\delta} = -\phi(x,t)\frac{\partial \phi(x,t)}{\partial x} - \frac{\partial \psi(x,t)}{\partial x} - \frac{1}{2}\frac{\partial \phi(x,t)}{\partial x}, \\
\frac{\partial^\delta \psi(x,t)}{\partial t^\delta} = -\phi(x,t)\frac{\partial \psi(x,t)}{\partial x} - \psi(x,t)\frac{\partial \phi(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 \psi(x,t)}{\partial x^2},
\end{cases} \tag{19}
$$

subject to the initial conditions

$$
\begin{aligned}
\phi(x,0) &= \omega - k\coth[k(x+h)], \\
\psi(x,0) &= -k^2 cosech^2[k(x+h)].
\end{aligned} \tag{20}
$$

The exact solution of (19) at classical order $\delta = 1$ and taking $k = 0.1$, $\omega = 0.005$, $h = 10, p = 0$, $q = 0.5$ in [4], is given by

$$
\begin{aligned}
\phi(x,t) &= \omega - k\coth[k(x+h-\omega t)], \quad \psi(x,t) \\
&= -k^2 \csc h^2[k(x+h-\omega t)].
\end{aligned} \tag{21}
$$

We find the solution at fractional order by using the procedure (9). Thus applying the mentioned procedure, we have

$$
\begin{aligned}
\phi(x,t) &= \omega - k\coth[k(x+h)] \\
&\quad - \mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{\sum_{i=0}^{\infty} Q_i + \frac{\partial}{\partial x}\sum_{i=0}^{\infty}\psi_i(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\phi_i(x,t)\right\}\right], \\
\psi(x,t)) &= -k^2 csch^2[k(x+h)] \\
&\quad - \mathcal{L}^{-1}\left[\frac{1}{s^\delta}\mathcal{L}\left\{\sum_{i=0}^{\infty} R_i + \sum_{i=0}^{\infty} S_i - \frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\psi_i(x,t)\right\}\right],
\end{aligned} \tag{22}
$$

where the nonlinear terms of (19) are expresses by Q_i, R_i, S_i. Comparing both sides of (22), we have

$$
\begin{aligned}
\phi_0(x,0) &= \omega - k\coth[k(x+h)] \\
\psi_0(x,0)) &= -k^2 csch^2[k(x+h)] \\
\phi_1(x,t) &= -\frac{\omega k^2 t^\delta csch^2[k(x+h)]}{\Gamma(\delta+1)} \\
\psi_1(x,t) &= -\frac{2\omega k^3 csch^2[k(x+h)]\coth[k(x+h)]t^\delta}{\Gamma(\delta+1)} \\
\phi_2(x,t) &= \omega k^4 cseh^2[k(x+x_0)]\left\{\frac{2\omega k\Gamma(2\delta+1)t^{3\delta}}{(\Gamma(\delta+1))^2\Gamma(3\delta+1)}\right. \\
&\quad \left. - \frac{(3\coth^2[k(x+h)]-1)t^{2\delta}}{\Gamma(2\delta+1)}\right\} \\
\psi_2(x,t) &= \frac{1}{\Gamma(\delta+1)}\left[2\omega k^5 cseh^2[k(x+x_0)]\right. \\
&\quad \times \left\{\frac{\omega k cseh^2[k(x+h)](3\coth^2[k(x+h)]-1)t^{3\delta}}{\Gamma(\delta+1)\Gamma(3\delta+1)}\right. \\
&\quad - \frac{2\coth[k(x+h)](3cseh^2[k(x+h)]-1)t^{2\delta}}{\Gamma(2\delta+1)} \\
&\quad \left.\left. + \frac{2\omega k cseh^2[k(x+h)]\coth^2[k(x+h)]t^{3\delta}}{\Gamma(\delta+1)\Gamma(3\delta+1)}\right\}\right]
\end{aligned} \tag{23}
$$

At this way we can generate a series solution to the coupled system (19) given by

$$
\begin{aligned}
\phi(x,t) &= \phi_0(x,t) + \phi_1(x,t) + \phi_2(x,t) + \phi_3(x,t) + \cdots, \\
\psi(x,t) &= \psi_0(x,t) + \psi_1(x,t) + \psi_2(x,t) + \psi_3(x,t) + \cdots.
\end{aligned} \tag{24}
$$

The solution to the coupled system (19), after three terms is given by

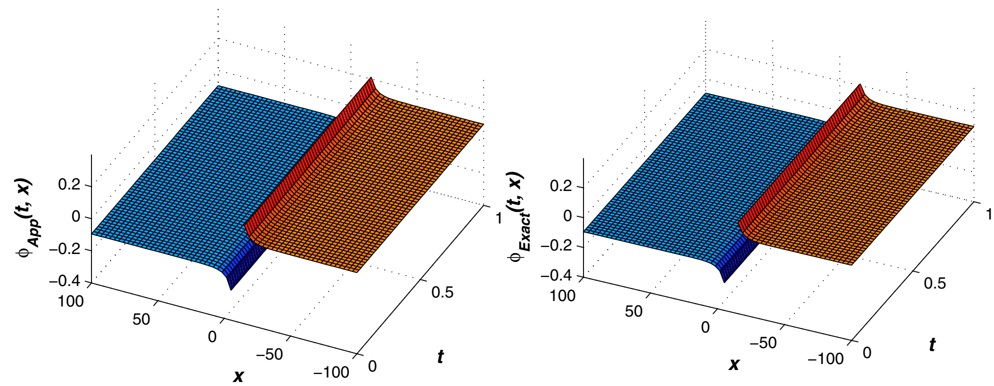

Figure 3 Comparison of approximate and exact plot of $\phi(t,x)$ at $\delta = 1$.

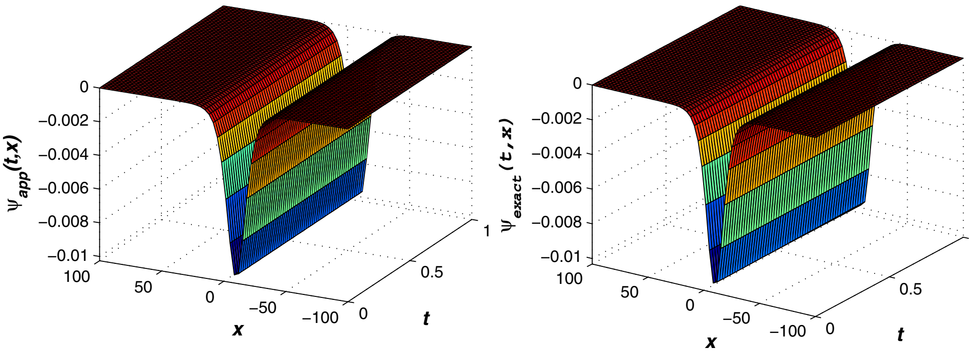

Figure 4 Comparison of approximate and exact plots of $\psi(t,x)$ at $\delta = 1$ for Example 4.2.

Table 1 The comparison between (HAM), (ADM), (VIM) and (LADM) results for $\phi(t,x)$ for Example 4.2.

(t,x)	$\lvert\phi_{Exact} - \phi_{ADM}\rvert$	$\lvert\phi_{Exact} - \phi_{HAM}\rvert$	$\lvert\phi_{Exact} - \phi_{VIM}\rvert$	$\left\lvert\phi_{Exact} - \phi^{(3)}_{LADM}\right\rvert$	$\left\lvert\phi_{Exact} - \phi^{(4)}_{LADM}\right\rvert$
(0.1, 0.1)	8.02989×10^{-6}	3.17634×10^{-5}	1.23033×10^{-4}	7.1000×10^{-9}	2.1009×10^{-18}
(0.1, 0.3)	7.38281×10^{-6}	2.69597×10^{-4}	3.69597×10^{-4}	6.5000×10^{-9}	2.9001×10^{-18}
(0.1, 0.5)	6.79923×10^{-6}	1.59274×10^{-4}	4.9278×10^{-4}	5.9000×10^{-9}	6.1023×10^{-18}
(0.2, 0.1)	2.97172×-5	9.29725×-5	1.69274×10^{-5}	2.8200×10^{-8}	1.6201×10^{-18}
(0.2, 0.3)	2.73673×-5	1.55176×-4	1.8921×10^{-4}	2.5900×10^{-8}	4.5129×10^{-17}
(0.2, 0.5)	2.73673×-5	1.55176×-4	1.55176×-4	2.4100×10^{-8}	1.9100×10^{-17}
(0.3, 0.1)	7.32051×-5	3.01549×-5	1.12345×-5	6.3367×10^{-8}	1.4367×10^{-16}
(0.3, 0.3)	6.73006×-5	9.05935×-5	6.55176×-5	5.8500×10^{-8}	5.7501×10^{-16}
(0.3, 0.5)	6.19760×-5	1.51204×-4	2.12346×-5	5.4000×10^{-8}	5.4000×10^{-16}
(0.4, 0.1)	1.31032×-4	2.93874×-5	7.36153×-5	1.124×10^{-7}	7.3124×10^{-16}
(0.4, 0.3)	1.20455×-4	8.82871×-5	9.5016×-5	1.0390×10^{-7}	2.1390×10^{-14}
(0.4, 0.5)	1.10919×-4	1.47354×-4	8.23160×-4	9.6100×10^{-8}	1.6100×10^{-14}
(0.5, 0.1)	2.06186×-4	2.86433×-5	5.551760×-5	1.7550×10^{-7}	1.7550×10^{-14}
(0.5, 0.3)	1.89528×-4	8.60509×-5	3.21715×-6	1.6220×10^{-7}	8.3220×10^{-14}
(0.5, 0.5)	1.74510×-4	1.43620×-4	2.00176×-5	1.5010×10^{-7}	4.1235×10^{-14}

Table 2 The comparison between (HAM), (ADM), (VIM) and our proposed (LADM) results for $\psi(t,x)$ for Example 4.2.

(t,x)	$\lvert\phi_{Exact} - \phi_{ADM}\rvert$	$\lvert\phi_{Exact} - \phi_{HAM}\rvert$	$\lvert\phi_{Exact} - \phi_{VIM}\rvert$	$\left\lvert\phi_{Exact} - \phi^{(3)}_{LADM}\right\rvert$	$\left\lvert\phi_{Exact} - \phi^{(4)}_{LADM}\right\rvert$
(0.1, 0.1)	4.81902×-4	8.29712×-6	1.23033×10^{-4}	9.5512×10^{-10}	1.43534×10^{-18}
(0.1, 0.3)	4.50818×-4	2.49346×-5	1.7600×10^{-4}	8.0600×10^{-10}	1.8500×10^{-18}
(0.1, 0.5)	4.22221×-4	4.16299×-5	2.69597×10^{-4}	6.7700×10^{-10}	5.1234×10^{-17}
(0.2, 0.1)	9.76644×-4	8.04063×-6	2.69597×10^{-4}	3.8210×10^{-9}	1.2210×10^{-17}
(0.2, 0.3)	9.13502×-4	2.41634×-4	2.69597×10^{-4}	3.224×10^{-9}	8.224×10^{-17}
(0.2, 0.5)	8.55426×-4	4.03419×-5	2.69597×10^{-4}	2.7060×10^{-9}	4.8706×10^{-16}
(0.3, 0.1)	1.48482×-3	7.79401×-6	2.69597×10^{-4}	8.597×10^{-9}	1.0597×10^{-17}
(0.3, 0.3)	1.38858×-3	2.34220×-5	2.69597×10^{-4}	7.252×10^{-9}	9.252×10^{-17}
(0.3, 0.5)	1.30009×-3	3.91034×-5	2.69597×10^{-4}	6.0910×10^{-9}	3.6791×10^{-16}
(0.4, 0.1)	2.00705×-3	7.55675×-6	2.69597×10^{-4}	1.5284×10^{-8}	7.1284×10^{-16}
(0.4, 0.3)	1.87661×-3	2.27087×-5	2.69597×10^{-4}	1.2893×10^{-8}	5.3893×10^{-16}
(0.4, 0.5)	1.75670×-3	3.79121×-5	2.69597×10^{-4}	1.0827×10^{-8}	1.1316×10^{-14}
(0.5, 0.1)	2.54396×-3	7.32847×-6	2.69597×10^{-4}	2.3880×10^{-8}	7.3880×10^{-15}
(0.5, 0.3)	2.37815×-3	2.20224×-5	2.69597×10^{-4}	2.0144×10^{-8}	5.1144×10^{-15}
(0.5, 0.5)	2.22578×-3	3.67658×-5	2.69597×10^{-4}	1.6916×10^{-8}	2.0916×10^{-14}

$$\phi(x,t) = \omega - k\coth[k(x+h)] - \frac{\omega k^2 t^\delta csch^2[k(x+h)]}{\Gamma(\delta+1)} + \omega k^4 cseh^2[k(x+h)]\left\{\frac{2\omega k\Gamma(2\delta+1)t^{3\delta}}{(\Gamma(\delta+1))^2\Gamma(3\delta+1)} - \frac{(3\coth^2[k(x+h)]-1)t^{2\delta}}{\Gamma(2\delta+1)}\right\},$$

$$\psi(x,t) = -k^2 csch^2[k(x+h)] - \frac{2\omega k^3 csch^2[k(x+h)]\coth[k(x+h)]t^\delta}{\Gamma(\delta+1)} + \frac{1}{\Gamma(\delta+1)}\left[2\omega k^5 cseh^2[k(x+h)] \times \left\{\frac{\omega k cseh^2[k(x+h)](3\coth^2[k(x+h)]-1)t^{3\delta}}{\Gamma(\delta+1)\Gamma(3\delta+1)} - \frac{2\coth[k(x+h)](3cseh^2[k(x+h)]-1)t^{2\delta}}{\Gamma(2\delta+1)} + \frac{2\omega k cseh^2[k(x+h)]\coth^2[k(x+h)]t^{3\delta}}{\Gamma(\delta+1)\Gamma(3\delta+1)}\right\}\right]. \tag{25}$$

We obtain the following series at classical order $\delta = 1$, $k = 0.1$, $\omega = 0.005$, $h = 10$, using (25)

$$\phi(x,t) = 0.005 - 0.1\coth(0.1x+10) - 0.0005csch^2(0.1x+10)t + 5\times10^{-7}csch^2(0.1x+10)0.003t^3 - 0.5(3\coth^2(0.1x+10) - 1.))t^2,$$

$$\psi(x,t) = -0.01\operatorname{csc}h^2(0.1x+10) - 0.000010csch^2(0.1x+10) \times \coth(0.1x+10)t + 1.0\times10^{-7}\operatorname{csc}h^2(0.1x+10) \times [8.3\times10^{-5}t^3\operatorname{csc}h^2(0.1x+10)(3\coth(0.1x+10)-1) - t^2\coth(0.1x+10)(3\operatorname{csc}h^2(0.1x+10)-1) + 1.6\times10^{-4}t^3\operatorname{csc}h^2(0.1x+10)\coth(0.1x+10)]. \tag{26}$$

In Tables 1 and 2, we provide comparison between Homotopy perturbation method (HPM) [4], Adomian decomposition method (ADM) and Variation iteration method (VIM) [7] with the proposed method. We see that the proposed method (LADM) provides excellent solutions as compared to the aforesaid methods, which demonstrate the efficiency and reliability of the considered method.

5. Conclusion

This manuscript deals with the application of Laplace Adomian decomposition technique to a nonlinear coupled system of Whitham-Broer-Kaup equations of fractional order. The considered method has been well studied for coupled system of fractional order linear and nonlinear ordinary differential equations. Numerical results confirm that the concerned method is quite efficient and reliable to obtain the approximate solutions for such like nonlinear fractional partial differential equations. In comparison with other analytical techniques, the proposed technique is an efficient and simple tool to investigate numerical solution of nonlinear coupled systems of fractional partial differential equations. The results, received by this method ensure the applicability and reliability of the proposed method for nonlinear fractional partial differential equations. It is obvious that time fractional differential equations can be easily treated by using this method for their numerical solutions. Less amount of computation gives very accurate approximate solution which is the main advantage of this method.

Acknowledgments

We are really thankful to the anonymous referees for their useful comments which improved the quality of the paper.

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