Computers and Mathematics with Applications 62 (2011) 51–64

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Exponentially weighted Legendre–Gauss Tau methods for linear second-order differential equations☆

Mohamed K. El-Daou *

Applied Sciences Department, College of Technological Studies, P.O. Box 64287 Shuwaikh/B, 70453, Kuwait

ARTICLE INFO

Article history:
Received 27 July 2010
Received in revised form 18 April 2011
Accepted 18 April 2011

Keywords:
Tau Method
Legendre polynomials
Coefficient perturbation
Collocation

ABSTRACT

We develop a method to solve a class of second-order ordinary differential equations with highly oscillatory solutions. The method consists in combining three different techniques: Legendre–Gauss spectral Tau method, exponential fitting, and coefficient perturbation methods. With our approach, the resulting approximate solutions are expressed in terms of an exponentially weighted Legendre polynomial basis $\{e^{\omega_n x}L_n(x);\ n \geq 0\}$, where ω_n are appropriately chosen complex numbers. The accuracy and efficiency of the method are discussed and illustrated numerically.

1. Introduction

The numerical solution of ordinary differential equations (ODEs) with highly oscillatory solutions continues to represent a challenging problem in numerical analysis and computational physics. Many problems in practice exhibit an oscillatory behaviour. Among these are the pendulum-like systems and vibrations in classical mechanics, wave propagation and resonances in electromagnetism and the behaviour of the quantum particles. Another good illustration is the fundamental equation of the non-relativistic quantum mechanics known as the Schrödinger equation. In the past, the treatment of this class of problems relied on standard numerical methods such as Numerov, Runge–Kutta and de Vogelaere methods. However, those methods were found to be unsatisfactory and efforts were concentrated on developing numerical methods with an oscillatory character, for such techniques are more efficient in detecting the strong oscillations of the exact solution. The improved methods are based on Neumann series expansions [1], coefficient perturbation [2–5], and exponentially fitted versions of Runge–Kutta methods (see [6,7]).

Exponential fitting (EF) is a procedure for integrating functions whereby weighted sums with exponential weights are used. A vast body of research on this topic has accumulated since the works of Greenwood [8], Brock and Murray [9]. The appeal of this method lies in its space of exactness (i.e. the class of ODEs that can be integrated exactly by exponentially fitted techniques) being much wider than that of the traditional methods (see [10,6]). This special feature allows integration using a large step length without loss of accuracy throughout long time intervals.

The purpose of this paper is to utilize the EF technique to derive an exponentially weighted version for the Legendre–Gauss Tau method (LGT). The latter consists in constructing approximations written in terms of finite sums of Legendre polynomials with the unknown coefficients determined by collocation at Legendre–Gauss points. The basic LGT method was proposed by Lanczos in [11] to solve simple first-order linear ODEs and it was developed later by Ortiz [12] and by Gottlieb and Orszag [13] to treat problems with different degrees of complexity. A brief description of LGT is given in Section 2. In the same section, we try to get insight into the behaviour of LGT by solving a simple highly oscillatory equation.

☆ A preliminary version of this paper appeared in the conference proceedings (El-Daou, 2008 [22]).

* Tel.: +965 22623761.
E-mail address: mk.eldaou@paaet.edu.kw.

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.04.045

In Section 3, we review the main features of Coefficient Perturbation method (CPM) and analyse its error. This error analysis suggests how LGT should be modified in order to improve its performance. In Section 4, we present the modified version of LGT that we call *Exponentially weighted Legendre–Gauss Tau Method* (ELGT). Section 5 is concerned with analysing the error of ELGT, where we propose a reference-correction procedure that allows increasing the degree of accuracy. Numerical examples supporting our results will be given in Section 6. Finally, we point out that this work is a continuation of a recent paper [14] in which a special form of ELGT has been devised for approximating Sturm–Liouville eigenvalue problems.

2. The Legendre–Gauss Tau method

Let us consider the initial-value problem (IVP),

$$(Dy)(x) := \sum_{i=0}^{\nu} P_i(x) y^{(i)}(x) = f(x), \quad x \in I := [0, 1], \tag{1}$$

$$y^{(k)}(0) = \alpha_k \in \mathbf{R}, \quad k = 0, 1, \ldots, \nu - 1, \ (\nu \in \mathbf{N}) \tag{2}$$

where $P_i(x),\ i = 0, 1, \ldots, \nu$ are continuous functions in I with $P_\nu(x)$ not vanishing in I. If the interval $I = [a_\ell, a_r] \neq [0, 1]$ then it can be reduced to $[0, 1]$ by means of the affine map $x \to (x - a_\ell)/(a_r - a_\ell)$.

The Legendre–Gauss Tau method of order N, (LGT(N)), seeks an approximation $y_N(x)$ for $y(x)$ written as

$$y_N(x) = \sum_{i=0}^{N+\nu-1} c_i L_i(x), \tag{3}$$

where $L_i(x)$ stands for Legendre polynomial of degree i shifted to the interval I, and where the unknown coefficients $\{c_i,\ i = 0, 1, \ldots, N + \nu - 1\}$ are determined (i) by imposing the supplementary conditions (2) on $y_N(x)$,

$$y_N^{(k)}(0) = \alpha_k; \quad k = 0, 1, \ldots, \nu - 1,$$

and (ii) by forcing the residual $R_N(x) := Dy_N(x) - f(x)$ to vanish at the N zeros of $L_N(x)$, $\{z_i\}_{i=1}^N \subset I$, (known as Legendre–Gauss points),

$$R_N(z_i) = 0, \quad i = 1, 2, \ldots, N. \tag{4}$$

In the piecewise version of the LGT method, we consider a partition $0 = x_0 < x_1 < \cdots < x_M = 1$ of I and we use LGT(N) to solve the M IVPs,

$$(Dy_i)(x) = f(x), \quad x \in [x_{i-1}, x_i],\ i = 1, 2, \ldots, M,$$
$$y_i^{(k)}(x_{i-1}) = y_{i-1}^{(k)}(x_{i-1}), \qquad y_1^{(k)}(x_0) = \alpha_k, \quad k = 0, 1, \ldots, \nu - 1.$$

Throughout the paper, LGT(M, N) will refer to the piecewise LGT that uses N Legendre–Gauss points on each one of the M intervals $\{[x_{i-1}, x_i],\ i = 1, 2, \ldots, M\}$. When $M = 1$, LGT$(1, N) \equiv$ LGT(N).

It follows from (4) that LGT(N) requires N evaluations of the residual $R_N(x)$ at points $\{z_i\}_{i=1}^N$. This in turn implies that LGT(M, N) will require MN evaluations of R_N. It is appropriate then to measure the computational cost of LGT(M, N) by counting the number of evaluations of $R_N(x)$.

In order to see the performance of LGT(M, N), let us apply it to the following IVP,

$$y''(x) + 4x^2 y(x) = 2\cos x^2, \quad x \in [0, 40], \tag{5}$$
$$y(0) = 0, \qquad y'(0) = 0.$$

The exact solution, $y(x) = \sin x^2$, is highly oscillatory for large x (see Fig. 1).

We solved this problem with LGT($M = 800, N = 2$). That is, we used 800 intervals with step size $h = 1/20$ each, and two Legendre–Gauss points $\frac{1}{6}(3 \pm \sqrt{3})$ on each interval. In Table 1 we list the exact errors at some $\{x_i,\ i = 0, 1, \ldots, 800\}$ committed by LGT(M, N).

It is clearly seen that the accuracy of LGT(M, N) deteriorates as we approach the end point near which the function oscillates very sharply. In order to understand this failure of LGT let us measure the residual $R_N(x)$ on the current interval $[X, X + h]$ for some $X := x_{i-1}$: Condition (4) implies that $R_N(x)$ is the Nth interpolant of $R(x) := Dy(x) - f(x)$ at $\{t_j = X + hz_j;\ j = 1, 2, \ldots, N\}$. Hence by the Cauchy remainder theorem (see [15]),

$$R(x) - R_N(x) = \frac{\prod_{j=1}^{N}(x - t_j)}{(N+1)!} R^{(N+1)}(\xi) \quad \text{for any } x \in [X, X + h]$$

where $\min\{x, t_1, t_2, \ldots, t_N\} < \xi < \max\{x, t_1, t_2, \ldots, t_N\}$. In this example, $f(x) = 2\cos x^2$ and $N = 2$. Therefore

$$R(x) - R_2(x) = \frac{(x - t_1)(x - t_2)}{3!}\{(Dy(\xi))''' + 24\xi\cos\xi^2 - 16\xi^3 \sin\xi^2\} \tag{6}$$

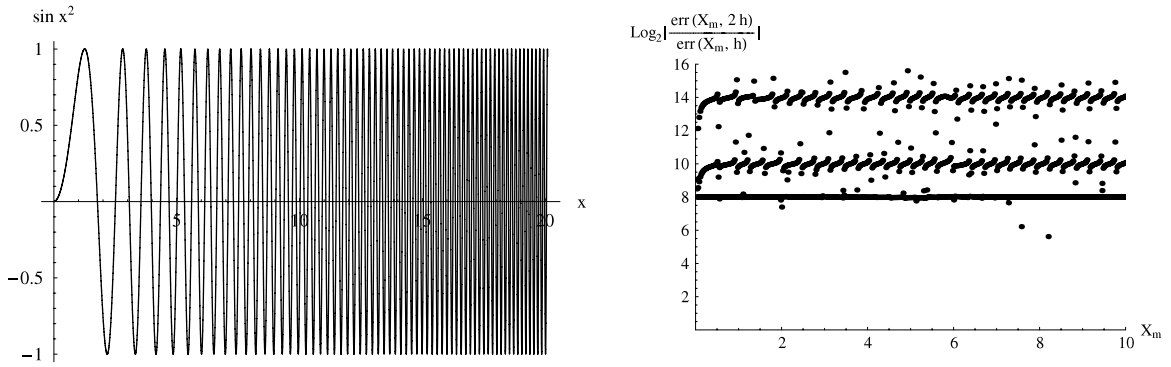

Fig. 1. Left (Eq. (5)): Plot of $\sin x^2$ in [0,20]. Right (Example 3): Plot of $\log_2 \left|\frac{\text{global err}[100,4,P]}{\text{global err}[200,4,P]}\right|$ at points x_m for ELGT(M, N, P) with $M = 100, 200$, $N = 4$ and $P = 0$ (lower line), 6 (middle line), 10 (upper line).

Table 1
(Example 1)—Comparison between ELGT(800,2) and LGT(800,2) errors for $0 \le x \le 40$.

LGT(800,2)			ELGT(800,2)		
x_k	$e_N(x_k)$	$e'_N(x_k)$	ω_k	$e_N(x_k)$	$e'_N(x_k)$
0.05	6.51E−10	−8.68E−9	0.05i	−1.26E−9	−7.71E−8
5.0	1.18E−4	5.42E−4	9.95i	1.70E−6	4.48E−5
10.0	6.07E−3	7.99E−2	19.95i	−3.01E−5	1.33E−3
15.0	2.67E−2	2.01E+0	29.95i	−3.43E−4	4.26E−3
20.0	−2.14E−1	9.48E+0	39.95i	−1.01E−3	−2.45E−2
25.0	−7.70E−1	−3.00E+1	49.95i	3.89E−4	−1.35E−1
30.0	−8.67E−1	1.17E+1	59.95i	5.55E−3	4.23E−2
35.0	−3.84E+5	9.81E+8	69.95i	−2.26E−3	4.54E−1
40.0	−2.91E+5	−7.19E+8	79.95i	−9.92E−3	−6.42E−1

for some $\xi \in \left[\min\left\{x, X + \frac{h}{6}(3 \pm \sqrt{3})\right\}, \max\left\{x, X + \frac{h}{6}(3 \pm \sqrt{3})\right\}\right]$ and for any $x \in [X, X + h]$. It is clear that when X is large then $16\xi^3 \sin \xi^2$ represents the deciding term on the right-hand side of (6). Indeed, this justifies the deterioration in the LGT error for $X \ge 30$ as shown in Table 1.

This observation suggests that it is more advantageous to construct, instead of (3), approximate solutions that involve exponential weights of the form $e^{\omega x}$, with suitably chosen frequencies ω. We will appreciate how such weights will contribute in improving the results and allow us to detect the strong oscillations throughout the domain of integration. In the next section, we will explain how the piecewise coefficient perturbation method will play a crucial role in achieving this goal.

3. Piecewise coefficient perturbation

The piecewise coefficient perturbation method (PPM) has been essentially devised to approximate second-order ODEs of the form

$$(Dy)(x) := y'' + b(x)y = 0, \quad x \in I, \qquad (7)$$
$$y(0) = \alpha_0, \qquad y'(0) = \alpha_1,$$

where $b(x)$ is a continuous function.

The basic idea of the PPM is to replace $b(x)$ by approximations $\tilde{b}_i(x)$ on each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, M$, and to approximate $y(x)$ on each $[x_{i-1}, x_i]$, by a finite sum of the form

$$Y_{mi}(x) := \sum_{k=0}^{m} y_{ki}(x), \quad (m \ge 0) \qquad (8)$$

where $y_{0i}(x)$, called *reference*, is the exact solution of

$$y''_{0i}(x) + \tilde{b}_i(x)y_{0i}(x) = 0, \quad x \in [x_{i-1}, x_i], \ i = 1, 2, \ldots, M, \qquad (9)$$
$$y^{(\ell)}_{0i}(x_{i-1}) = Y^{(\ell)}_{m,i-1}(x_{i-1}), \quad \ell = 0, 1, \ (Y^{(\ell)}_{m,0}(x_0) = \alpha_\ell)$$

and where $y_{ki}(x)$, $k = 1, 2, \ldots, m$, called *corrections*, are computed through solving the IVPs

$$y''_{ki}(x) + \tilde{b}_i(x)y_{ki}(x) = \delta b_i(x)y_{k-1,i}(x), \quad x \in [x_{i-1}, x_i] \qquad (10)$$
$$y^{(\ell)}_{ki}(x_{i-1}) = 0, \quad k \ge 1, \ \ell = 0, 1,$$

with $\delta b_i(x) := \tilde{b}_i(x) - b(x)$. In practice, $\tilde{b}_i(x),\ i = 1, 2, \ldots, M$ are chosen in a way that reference Eq. (9) can be solved analytically. The simplest way is through assuming that $\tilde{b}_i(x),\ i = 1, 2, \ldots, M$ are constants denoted by $\tilde{b}_i,\ i = 1, 2, \ldots, M$. Then, the method is called the CP method, (see [2]). Next we analyse the error of the CP method and the structure of its residual.

3.1. Structure of CP residual

Since there is no special restriction upon the manner of distributing the mesh points $0 = x_0 < x_1 < \cdots < x_M = 1$ of I we will focus on the interval $[x_{i-1}, x_i]$ of length $h_i = x_i - x_{i-1}$. For simplicity, we will omit the indices i and denote $X = x_{i-1}$, $h = h_i$, $Y_m(x) = Y_{mi}(x)$, $y_k(x) = y_{ki}(x)$ and $\delta b(x) = \delta b_i(x)$. The residual induced by the CP method is obtained by adding up Eqs. (9) and (10), $i = 1, 2, \ldots, m$. We find then that (8) satisfies

$$\begin{aligned} &Y_m''(x) + b(x)Y_m(x) = -\delta b(x)y_m(x), \quad x \in I_h := [X, X + h], \\ &Y_m(X) = \eta_0, \qquad Y_m'(X) = \eta_1, \end{aligned} \tag{11}$$

where $\{\eta_0, \eta_1\}$ are known initial conditions. Comparing (11) with (7), we observe that Y_m is the exact solution of a perturbed version of (7) where the perturbation occurs on the right-hand side as a residual of the form

$$R(x) = -\delta b(x)y_m(x). \tag{12}$$

The oscillatory phenomenon occurs when $\tilde{b}(x) \equiv \tilde{b}$ is a positive constant. Then setting $\omega = \sqrt{\tilde{b}}\mathrm{i} = \gamma\mathrm{i}$, we observe the following.

- The CP reference, being the exact solution of (9), is given as

$$y_0(x) = p_{1,0}\mathrm{e}^{\omega x} + p_{2,0}\mathrm{e}^{-\omega x} \quad \text{where } p_{1,0} = \frac{\eta_0}{2} - \mathrm{i}\frac{\eta_1}{2\gamma} \text{ and } p_{2,0} = \overline{p_{1,0}}. \tag{13}$$

- The first correction, implicitly defined by (10) with $k = 1$, is given by

$$y_1(x) = \int_X^x G(x, t)\delta b(t)y_0(t)\mathrm{d}t$$

where $G(x, t)$ is the Green function associated with $\tilde{D}u = u''(x) + \tilde{b}u(x)$

$$G(x, t) = \frac{1}{2r}(\mathrm{e}^{\omega(x-t)} - \mathrm{e}^{-\omega(x-t)}).$$

The latter and (13) allow us to write $y_1(x)$ as

$$y_1(x) = p_{1,1}(x)\mathrm{e}^{\omega x} + p_{2,1}(x)\mathrm{e}^{-\omega x},$$

where

$$p_{1,1} = \frac{1}{2r}\int_X^x \delta b(t)(p_{1,0} + p_{2,0}\mathrm{e}^{-2\omega t})\mathrm{d}t \quad \text{and} \quad p_{2,1} = \overline{p_{1,1}}.$$

- Proceeding this way we can prove that the kth CP correction takes the form $y_k(x) = p_{1,k}(x)\mathrm{e}^{\omega x} + p_{2,k}(x)\mathrm{e}^{-\omega x}$, for some functions $\{p_{1,k}(x),\ p_{2,k}(x)\}$ that involve two constants fixed by the initial conditions $y_k(X) = y_k'(X) = 0$,
- The CP residual (12) takes the form

$$R(x) = -(\delta b p_{1,m})\mathrm{e}^{\omega x} - (\delta b p_{2m})\mathrm{e}^{-\omega x}. \tag{14}$$

This structure of the CP residual will be very constructive in ensuring the close dependence between the error function and the quality of perturbation measured by $\delta b(x)$. Later, this will allow us to propose a technique that reduces the error substantially.

3.2. Analysing the CP error

Let $\varepsilon_m(x) := y(x) - Y_m(x)$. The difference between (7) and (11), taking into account (14), gives

$$\begin{aligned} &\varepsilon_m'' + b(x)\varepsilon_m = (\delta b p_{1m})\mathrm{e}^{\omega x} + (\delta b p_{2m})\mathrm{e}^{-\omega x}, \quad x \in I_h \\ &\varepsilon_m(X) = y(X) - Y_m(X), \qquad \varepsilon_m'(X) = y'(X) - Y_m'(X). \end{aligned}$$

The error $\varepsilon_m(x)$ is formally represented as

$$\varepsilon_m(x) = \int_X^x G^*(x, t)\delta b(t)\mathrm{d}t + G(x, X)\varepsilon_m'(X) + G_x(x, X)\varepsilon_m(X), \tag{15}$$

$G(x, t)$ being the Green function associated with D and

$$G^*(x, t) := G(x, t)[p_{1,m}(t)e^{\omega t} + p_{2,m}(t)e^{-\omega t}].$$

For the local truncation error, let $\varepsilon_m(X) = \varepsilon'_m(X) = 0$ and take norms in (15), to get

$$\|\varepsilon_m\| \le \kappa \|\delta b\| \quad \text{where } \|G^*\| \le \kappa$$

for some constant $\kappa = \kappa(\omega)$. As far as the CP method is concerned the smallest $\|\delta b\|$ is achieved if $\tilde{b}$ is the best zeroth approximation of $b(x)$ in $L^2[I_h]$,

$$\tilde{b} = \frac{1}{h}\int_X^{X+h} b(t)\mathrm{d}t. \tag{16}$$

which, due to the Cauchy remainder theorem (see [15]), implies that

$$\delta b(x) = L_{1,h}(x)\left[\frac{1}{2}b'(X + \xi h)\right] \quad \text{for some } \xi \in (0, 1).$$

We conclude that the residual (14) can be written as

$$R(x) = L_{1,h}(x)\tau_1(x)e^{\omega x} + L_{1,h}(x)\tau_2(x)e^{-\omega x}. \tag{17}$$

This result provides us with a powerful opportunity to improve on the accuracy of the CP method: the residual (17), that determines the CP error, is a multiple of the first degree Legendre polynomial $L_{1,h}(x)$. Then, we can undoubtedly improve the CP method much better if we construct an alternative method of which the residual is a multiple of a higher-order Legendre polynomial, say $L_{n,h}(x)$, $n \ge 1$. In other words, we seek a method that could lead to a residual of the form

$$R_N(x) = L_{N,h}(x)\tau_1(x)e^{\omega x} + L_{N,h}(x)\tau_2(x)e^{-\omega x}. \tag{18}$$

This is indeed possible and can be achieved by the LGT as explained next.

But before proceeding, there are two computational aspects that need be treated.

1. Although the CP procedure can be implemented with any form of $\delta b(x) := \tilde{b}(x) - b(x)$, practical difficulties arise in computing the corrections when Eq. (10) cannot be integrated analytically, a case that may happen when $\delta b(x)$ has a complicated form. To cure this drawback we can, as suggested in [2] page 203, replace $b(x)$ by a pilot reference function, that is a polynomial with degree enough high to ensure that $b(x)$ is well approximated, and (10) is solved accurately.
2. Similarly, if the integration in (16) cannot be performed analytically, then we can replace $b(t)$ by a polynomial, $b_n(t)$ say, and then approximate $\tilde{b}$ by

$$\tilde{b} \approx \frac{1}{h}\int_X^{X+h} b_n(t)\mathrm{d}t.$$

Actually, we can prove that the degree of $b_n(t)$ does not contribute significantly in the computation of $\tilde{b}$ and ω. Let us replace $b(x)$ in (16) by its Taylor's series expansion near $\bar{X} := X + \frac{h}{2}$. Then we have

$$\begin{aligned}\tilde{b} &= \frac{1}{h}\int_X^{X+h}\left(\sum_{k\ge0}\frac{b^{(k)}(\bar{X})}{k!}(t-\bar{X})^k\right)\mathrm{d}t = \sum_{k\ge0}\frac{b^{(2k)}(\bar{X})}{(2k+1)!}\left(\frac{h}{2}\right)^{2k}\\ &= b(\bar{X}) + \left(\frac{h}{2}\right)^2\left[\sum_{k\ge1}\frac{b^{(2k)}(\bar{X})}{(2k+1)!}\left(\frac{h}{2}\right)^{2(k-2)}\right] = b(\bar{X}) + O(h^2).\end{aligned} \tag{19}$$

Since $\omega = \sqrt{\tilde{b}}\mathrm{i}$, we can write

$$\omega = \mathrm{i}\sqrt{b(\bar{X}) + O(h^2)} = \mathrm{i}(\gamma + O(h)) \quad \text{with } \gamma := \sqrt{b(\bar{X})}.$$

Therefore, the oscillatory weight becomes

$$\begin{aligned}e^{\omega x} &= \cos(\gamma + O(h))x + \mathrm{i}\sin(\gamma + O(h))x\\ &= \cos\gamma x\cos O(h)x - \sin\gamma x\sin O(h)x + \mathrm{i}\cos\gamma x\sin O(h)x + \mathrm{i}\sin\gamma x\cos O(h)x.\end{aligned}$$

Noting that $\cos O(h)x = 1 + O(h^2x^2)$ and $\sin O(h)x = O(hx)$, we obtain

$$e^{\omega x} = \cos\gamma x + \mathrm{i}\sin\gamma x + O(h) = e^{\mathrm{i}\gamma x} + O(h).$$

This shows that the oscillation of $e^{\omega x}$ is driven by $e^{\mathrm{i}\gamma x}$ and thus we can safely drop $O(h^2)$ in (19) and consider

$$\tilde{b} \approx \tilde{b}_0 := b(\bar{X})$$

which is the zeroth interpolant of $b(x)$ at the midpoint $\bar{X}$ of I_h.

4. Exponentially weighted LGT

Let us consider the second-order linear IVP

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0, \quad x \in I, \tag{20}$$

$$y(0) = \alpha_0, \qquad y'(0) = \alpha_1, \tag{21}$$

where $a(x)$ and $b(x)$ are continuous functions on I. For each $\omega \in \mathbf{C}$, associate to (20) the auxiliary linear differential equation

$$(D_\omega u)(x) := u''(x) + [2\omega + a(x)]u'(x) + [\omega^2 + a(x)\omega + b(x)]u(x) = 0. \tag{22}$$

Clearly if $\phi(x)$ is a solution of (22), then

$$D[\phi(x)e^{\omega x}] = (D_\omega \phi(x))e^{\omega x} = 0.$$

Therefore, any exact solution $y(x)$ of (20) can be expressed as

$$y(x) = c_1\phi_1(x)e^{\omega_1 x} + c_2\phi_2(x)e^{\omega_2 x}, \tag{23}$$

whenever $\phi_1(x)$ and $\phi_2(x)$ satisfy the differential equations

$$(D_{\omega_j}\phi_j)(x) = 0, \quad j = 1, 2. \tag{24}$$

Expression (23) holds true for any set of constants $\{\omega_1, \omega_2, c_1, c_2\}$. But we are interested in identifying the values of $\{\omega_1, \omega_2\}$ that make D_ω sufficiently smooth so that Eq. (22) becomes more amenable by the classical LGT method. To this end, let us consider the Volterra integral operator associated to D_ω

$$(Ku)(x) = u(x) - \int_X^x k(x, t)u(t)dt, \quad x \in [X, X + h]$$

where $k(x, t) := -2\omega - a(t) - (\omega^2 + a(t)\omega + b(t) - a'(t))(x - t)$. Taking norms, we find that

$$\begin{aligned} \|K(u)\|_\infty &\le \|u\|_\infty + \int_X^x |-2\omega - a(t) - (\omega^2 + a(t)\omega + b(t) - a'(t))(x - t)|\|u\|_\infty dt \\ &\le \|u\|_\infty + \{|2\omega| + \|a\|_\infty + (|\omega^2| + \|a\|_\infty|\omega| + \|b\|_\infty + \|a'\|_\infty)h\}h\|u\|_\infty \end{aligned} \tag{25}$$

yielding

$$\|K\|_{\mathrm{op}} = O(|\omega|^2)$$

which is valid for any ω. Now if ω_1 and ω_2 are chosen to be the roots of

$$\omega^2 + a(\bar{X})\omega + b(\bar{X}) = 0, \tag{26}$$

then the coefficient of $u(x)$ in (22) can be written as

$$\begin{aligned} \omega^2 + a(x)\omega + b(x) &= \omega^2 + (a(\bar{X}) + \delta a)\omega + b(\bar{X}) + \delta b \\ &= [\omega^2 + \omega a(\bar{X}) + b(\bar{X})] + \omega\delta a + \delta b = \omega\delta a + \delta b \end{aligned}$$

where $\delta a(x) := a(x) - a(\bar{X})$ and $\delta b(x) := b(x) - b(\bar{X})$. Therefore, (22) becomes

$$(D_{\omega_j}\phi_j)(x) := \phi_j''(x) + (2\omega_j + a(x))\phi_j'(x) + (\omega_j\delta a + \delta b)\phi(x) = 0, \quad j = 1, 2 \tag{27}$$

and consequently (25) reduces to

$$\|K(z)\|_\infty \le \{|2\omega_j| + \|a\|_\infty + (\|\delta a\|_\infty|\omega_j| + \|\delta b\|_\infty)h\}h\|z\|_\infty$$

which yields

$$\|K\|_{\mathrm{op}} = O(|\omega_j|).$$

Thus the smoothness of K is achieved when

$$\omega_k = \frac{-a(\bar{X}) + (-1)^k\sqrt{a(\bar{X})^2 - 4b(\bar{X})}}{2}, \quad k = 1, 2. \tag{28}$$

The exact solution of any second-order linear ODE (20) can be represented then in terms of variable amplitudes $\{\phi_1(x), \phi_2(x)\}$ defined by (27) and oscillatory (or hyperbolic) weights $\{e^{\omega_1 x}, e^{\omega_2 x}\}$ where $\{\omega_1, \omega_2\}$ are given by (28). It turns out that in order to obtain the exact solution $y(x)$, we need to find the amplitudes $\{\phi_1(x), \phi_2(x)\}$, and then determine the constants $\{c_1, c_2\}$ for $y(x) = c_1\phi_1(x)e^{\omega_1 x} + c_2\phi_2(x)e^{\omega_2 x}$ in a way that the given initial conditions are fulfilled.

Analytically, solving (27) is not easier, however, than solving the original problem (20). But, computationally, since the oscillatory factor is taken out, numerical methods that approximate the smooth solutions of (27) could be more successful

than approximating (20) directly, specially when $y(x)$ exhibits sharp variations. Next we will propose a procedure for the LGT that can effectively generate approximations $\{\phi_{N,1}(x),\ \phi_{N,2}(x)\}$ for the functions $\{\phi_1(x),\ \phi_2(x)\}$ defined by (27) and then construct an approximation $Y_N(x) = c_1\phi_{N,1}(x)e^{\omega_1 x} + c_2\phi_{N,2}(x)e^{\omega_2 x}$ for $y(x)$. We will refer to this procedure by ELGT(M, N) where M indicates the number of steps and N is the number of Legendre–Gauss points used in each subinterval $I_h := [X, X + h]$.

ELGT procedure. The following algorithm lists the steps of the ELGT(M, N) procedure for approximating the IVP (20)–(21) on the current interval I_h.

1. Provide $\{z_k\}_{k=1}^N$, the N Legendre–Gauss points in [0,1].
2. Set $\bar{X} = X + h/2$ and $t_k = X + hz_k$, $k = 1, 2, \ldots, N$.
3. Compute $\{\omega_1, \omega_2\}$ from (28).
4. Obtain two LGT(M, N) approximations $\phi_{N,\ell}(x) = \sum_{j=0}^N d_{\ell j}L_{j,h}(x)$, $\ell = 1, 2$, where the coefficients $\{d_{\ell j}\}$ satisfy the linear systems,
$$(D_{\omega_\ell}\phi_{N,\ell})(t_k) = 0, \quad k = 1, 2, \ldots, N,$$
$$\phi_{N,\ell}(X) = (-1)^\ell, \quad \ell = 1, 2.$$
5. Construct ELGT(M, N) approximation $Y_N(x) = c_1\phi_{N,1}e^{\omega_1 x} + c_2\phi_{N,2}e^{\omega_2 x}$ where $\{c_1, c_2\}$ are fixed by left-end conditions
$$Y_N(X) = \tilde{y}_0 \quad \text{and} \quad Y_N'(X) = \tilde{y}_1.$$

We identify now the residual of ELGT(M, N). That is the defect resulting from the substitution of $Y_N(x)$ in (20).

Theorem 1. *The ELGT(M,N) approximant $Y_N(x)$ produces the residual*

$$R_N(x) = L_{N,h}(x)\tau_1(x)e^{\omega_1 x} + L_{N,h}(x)\tau_2(x)e^{\omega_2 x}. \tag{29}$$

Note that this $R_N(x)$ is identical to (18) when $a(x) \equiv 0$.

Proof. Since $\{z_k\}_{k=1}^N$ are the zeros of $L_N(x)$, it follows from step 4. In the ELGT procedure that $\{(D_{\omega_\ell}\phi_{N,\ell})(x),\ \ell = 1, 2\}$ are multiples of $L_{N,h}(x)$; that is, there exist some functions $\{\mu_\ell(x),\ \ell = 1, 2\}$ such that

$$(D_{\omega_\ell}\phi_{N,\ell})(x) = L_{N,h}(x)\mu_\ell(x), \quad \ell = 1, 2.$$

Therefore, if D is operated on $Y_N(x)$ we get, as required,

$$\begin{aligned} DY_N(x) &= D[c_1\phi_{N,1}e^{\omega_1 x} + c_2\phi_{N,2}e^{\omega_2 x}], \\ &= c_1L_{N,h}(x)\mu_1(x)e^{\omega_1 x} + c_2L_{N,h}(x)\mu_2(x)e^{\omega_2 x}. \end{aligned}$$

The ELGT procedure can be extended to approximate non-homogeneous second-order ODEs of the form

$$\begin{aligned} &Dy = y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in I, \\ &y(0) = \alpha_0, \qquad y'(0) = \alpha_1. \end{aligned} \tag{30}$$

The general solution of the latter is written as

$$y(x) = \text{const}_1u_1(x) + \text{const}_2u_2(x) + U(x),$$

where $\{u_1(x), u_2(x)\}$ are two particular solutions of $Dy = 0$ and $U(x)$ is a particular solution of $DU = f(x)$. Then the ELGT approximation for (30) in the subinterval $[X, X + h]$ takes the form,

$$Y_N(x) = c_1\phi_{N,1}(x)e^{\omega_1 x} + c_2\phi_{N,2}(x)e^{\omega_2 x} + U_N(x)$$

where

- $\{\phi_{N,1}(x), \phi_{N,2}(x)\}$ are computed by means of the ELGT procedure,
- $U_N(x)$ is obtained by replacing step 5. in the ELGT procedure, by the following steps:

5. Compute the coefficients $\{C_j,\ j = 0, 1, \ldots, \bar{N} + 1\}$ of
$$U_{N,i} = \sum_{j=0}^{\bar{N}/2} C_jL_{j,h}(x)e^{\omega_1 x} + \sum_{j=0}^{\bar{N}/2} C_{\bar{N}/2+1+j}L_{j,h}(x)e^{\omega_2 x}$$
by solving
$$(DU_N)(t_k) = f(t_k), \quad k = 1, 2, \ldots, \bar{N},$$
$$U_N(X) = 0, \qquad U_N'(X) = 1,$$
where $\bar{N} = N$ when N is even (resp. $\bar{N} = N + 1$ when N is odd).

6. Compute $Y_N(x) = c_1\phi_{N,1}(x)e^{\omega_1 x} + c_2\phi_{N,2}(x)e^{\omega_2 x} + U_N(x)$ where $\{c_1, c_2\}$ are fixed by left-end conditions $Y_N(X) = \tilde{y}_0$ and $Y'_N(X) = \tilde{y}_1$.

When the frequencies ω_1 and ω_2 are complex conjugates, the ELGT procedure can be simplified considerably. □

Corollary 2. *Suppose that for some interval I_h, $\omega_1 = \overline{\omega_2}$. Then*

$$Y_N(x) = 2\mathrm{Re}[c\phi_N(x)e^{\omega(x-X)}] \quad \textit{and} \quad (DY_N)(x) = 2\mathrm{Re}[L_{N,h}(x)\mu(x)e^{\omega(x-X)}],$$

where $\omega := \omega_1 = \overline{\omega_2}$,

$$\phi_N(x) := \phi_{N,1}(x) = \overline{\phi_{N,2}(x)},$$

$$\mu(x) := \mu_1(x) = \overline{\mu_2(x)} \quad \textit{and} \quad c = \frac{\omega\tilde{y}_0 - \tilde{y}_1}{\omega - \overline{\omega}}.$$

Proof. Since $\omega_1 = \overline{\omega_2} \equiv \omega$, and the initial conditions in step 4 are opposite (i.e. ± 1), we have

$$d_{2j} = -\overline{d_{1j}}, \qquad c_2 = -\overline{c_1}, \qquad \phi_N = -\sum_{j=0}^{N} \overline{d_{1j}} L_{j,h}(x).$$

Therefore, letting $c_1 = c$, and $\phi_N = \phi_{N,1}$ we arrive at

$$\begin{aligned} Y_N(x) &= c_1\phi_{N,1}e^{\omega_1(x-X)} + c_2\phi_{N,2}e^{\omega_2(x-X)} \\ &= c_1\phi_{N,1}e^{\omega(x-X)} + (-\overline{c_1})(-\overline{\phi_{N,1}})e^{-\omega(x-X)} \\ &= c\phi_N e^{\omega(x-X)} + \overline{c}\,\overline{\phi_N}e^{-\omega(x-X)} = 2\mathrm{Re}[c\phi_N(x)e^{\omega(x-X)}] \end{aligned}$$

as required. The second assertion follows from (29). □

5. Error analysis

In this section, we establish a correction procedure that permits to increase the accuracy of ELGT(M, N) approximation. But first of all let us investigate the exactness of the ELGT.

5.1. Exactness of ELGT(M, N)

We have seen that the exact solution of any second-order ODE of the form (20) can be represented as (23),

$$y(x) = c_1\phi_1(x)e^{\omega_1 x} + c_2\phi_2(x)e^{\omega_2 x},$$

where $\{\phi_i(x),\ i = 1, 2\}$ satisfy the auxiliary ODEs (24). In general, there is no specific method that can integrate the auxiliary ODEs exactly. We can prove, however, that when $\{\phi_i(x),\ i = 1, 2\}$ are exact polynomials, then ELGT(M, N) will reproduce the analytic solution of Eq. (20).

Theorem 3. *If the amplitudes $\{\phi_1(x), \phi_2(x)\}$ of the analytic solution defined by (23) are polynomial, then ELGT(M,N) can reproduce $y(x)$ exactly provided that $N > \max\{\deg[\phi_1], \deg[\phi_2]\}$, $a(\bar{X}) = \omega_1 + \omega_2$ and $b(\bar{X}) = \omega_1\omega_2$.*

Proof. This follows from the fact that the classical LGT itself is of degree $N + 1$. That is, if the LGT is applied to an ODE whose exact solution is a polynomial of degree $N + 1$ then it can reproduce this solution exactly. □

Remark. By construction, ELGT(M, N) is exact when it is applied to non-homogeneous differential equations with constant coefficients like

$$y''(x) - \omega^2 y(x) = k(k-1)x^{k-2}e^{\omega x} + 2\omega x^{k-1}e^{\omega x}. \tag{31}$$

whose exact solution is $y(x) = x^k e^{\omega x}$.

5.2. Error estimation of ELGT(M, N)

Let $e_N(x) := y(x) - Y_N(x)$ be the error function in I_h. In [16], we gave an infinite series representation for $e_N(x)$, recalled in Theorem 4. In order to formulate this theorem, we introduce the following recursions: for all $k \geq 0$, let

$$A_{k+1}(x) := A'_k(x) + B_k(x) - a(x)A_k(x),$$
$$B_{k+1}(x) := B'_k(x) - b(x)A_k(x),$$

with $A_0(x) := 0$ and $B_0(x) := 1$.

Theorem 4. *If $a(x)$ and $b(x)$ belong to $\mathcal{C}^\infty[I_h]$ then, for $\ell = 0, 1$*

$$e_N^{(\ell)}(x) = \sum_{k\geq 0} \frac{1}{k!}\{\mathcal{F}_{k\ell}(x) + (x-X)^k \Delta_{k+\ell}(X)\}$$

for all $x \in I_h$, where

$$\Delta_k(X) = [A_k(X)e_N'(X) + B_k(X)e_N(X)],$$
$$\mathcal{F}_{k\ell}(x) := \int_X^x A_{k+\ell}(t)(x-t)^k F(t)\mathrm{d}t, \quad \mathcal{F}_k := \mathcal{F}_{k0}$$

with $F(x) := -R_N(x)$ where $R_N(x)$ is given by (29).

Consequently, the exact solution of (30) has the expansion

$$y^{(\ell)}(x) = Y_N^{(\ell)}(x) + \delta_1^{(\ell)}(x) + \delta_2^{(\ell)}(x) + \delta_3^{(\ell)}(x) + \cdots \tag{32}$$

where $\{\delta_k^{(\ell)};\ k = 1, 2, \ldots\}$ are called corrections *and given by*

$$\delta_{k+1}^{(\ell)}(x) = \frac{1}{k!}\{\mathcal{F}_{k\ell}(x) + (x-X)^k \Delta_{k+\ell}(X)\}, \quad k = 0, 1, 2, \ldots. \tag{33}$$

Thus, in order to increase the accuracy of the reference solution Y_N we can accumulate as many corrections δ_i's as required. The K-stage corrected solution will be then

$$Y_{N,K}^{(\ell)} = Y_N^{(\ell)} + \delta_1^{(\ell)} + \delta_2^{(\ell)} + \cdots + \delta_K^{(\ell)},$$

and notation ELGT(M, N, P) will stand for the ELGT(M, N) with P corrections $\{\delta_1, \delta_2, \ldots, \delta_K\}$.

The contribution of each correction δ_k to the accuracy of ELGT(M, N) is summarized next.

Theorem 5. *Let $e_{N,K} := y(x) - Y_{N,K}(x)$. Under the above assumptions and notations, we have*

$$e_{N,K}(X+h) = \begin{cases} O(h^{2N+1}) & \text{if } k \leq N, \\ O(h^{2N+d+1}) & \text{if } k > N \text{ with } d = N-K. \end{cases}$$

Proof. The accuracy of $Y_{N,K}$ is measured by the order of δ_{K+1} in terms of h because $e_{N,K} = \delta_{K+1} + \delta_{K+2} + \cdots$.

Take $X = 0$ and let us find the order of each correction at the left-end point h. For $K \geq 0$ and $x = h$, $\delta_{K+1}(h)$ reduces to

$$\delta_{K+1}(h) = \frac{1}{K!}\mathcal{F}_K(h).$$

Analyse $\mathcal{F}_K(h)$:

$$\begin{aligned}\mathcal{F}_K(h) &= \int_0^h A_K(t)(h-t)^K F(t)\mathrm{d}t \\ &= \int_0^h A_K(t)(h-t)^K[\tau_1(t)\mathrm{e}^{\omega_1 t}L_{N,h}(t) + \tau_2(t)\mathrm{e}^{\omega_2 t}L_{N,h}(t)]\mathrm{d}t = \Im_1 + \Im_2.\end{aligned}$$

For $\Im \in \{\Im_1, \Im_2\}$, $\omega \in \{\omega_1, \omega_2\}$ and $\tau(t) := \sum_{j=0}^r \tau_j t^j \in \{\tau_1(t), \tau_2(t)\}$ we have

$$\begin{aligned}\Im &= \sum_{j=0}^r \tau_j \int_0^h A_K(t)(h-t)^K t^j \mathrm{e}^{\omega t}L_{N,h}(t)\mathrm{d}t \\ &= \sum_{j=0}^r \tau_j \langle A_K(t)(h-t)^K t^j \mathrm{e}^{\omega t}|L_{N,h}\rangle \\ &= \tau_0\langle A_K(t)(h-t)^K \mathrm{e}^{\omega t}|L_{N,h}\rangle + \tau_1\langle A_K(t)(h-t)^K t\mathrm{e}^{\omega t}|L_{N,h}\rangle + \cdots \\ &= \begin{cases} \tau_0 O(h^N) = O(h^{2N+1}) & \text{if } K \leq N, \\ \tau_0 O(h^K) = O(h^{N+K+1}) & \text{if } K > N. \end{cases}\end{aligned}$$

The last assertion follows from the fact that $\tau_0 = O(h^{N+1})$ (see [17]) and from Lemma 6. Thus,

$$\Im_1 \text{ and } \Im_2 = \begin{cases} O(h^{2N+1}) & \text{if } K \leq N, \\ O(h^{2N+d+1}) & \text{if } K > N \text{ with } d = N-K. \end{cases} \quad \square$$

Lemma 6. 1. *If* $f(t) = \sum_{m=0}^{\infty} f_m L_{m,h}(t)$, $t \in [0, h]$, *then* $f_m = O(h^m)$.
2. *If, further,* $f(t) = (h - t)^k g(t)$, *then*

$$f_m = \begin{cases} O(h^m) \text{ if } k \leq m, \\ O(h^k) \text{ if } k > m. \end{cases}$$

5.3. Step size control

The automatic control of the step size is an important part of any numerical method for solving initial-value problems. In addition to obtaining the approximate function value at a specific point by a numerical technique, there are in each step three things to be done: (i) estimate the local truncation error, (ii) decide whether the computed values can be accepted or whether a shorter step from the previous point should be taken, and (iii) determine the step to be used next. We shall describe here an approach to control the step size for the ELGT.

Suppose that $(X, \tilde{y}(X), \tilde{y}'(X))$ approximates $(X, y(X), y'(X))$ at a point X. Applying the ELGT method in the interval $[X, X + h]$, we obtain an approximation $(X + h, \tilde{y}(X + h), \tilde{y}'(X + h))$ of $(X + h, y(X + h), y'(X + h))$. Let $l_i := y^{(i)}(X + h) - \tilde{y}^{(i)}(X + h)$, $i = 0, 1$ and $l := \max\{|l_0|, |l_1|\}$. Our policy is to accept the computed value if l is kept below a given tolerance $\epsilon > 0$. If such a condition is not met, then, owing to the control step length strategy given in [18], we can adapt a smaller step size h' in the following way:

$$h' = h \min(\kappa^*, \max(\kappa_*, \kappa(\epsilon/l)^{1/(2N+1)})) \tag{34}$$

where $\kappa \leq 1$ represents a preset safety factor that is necessary in order to have an acceptable error in the following step, κ^* and κ_* stand for the maximum and minimum acceptable increasing and decreasing factors respectively.

It is clear that controlling the step size using (34) depends strongly on a sufficiently good estimate of the local truncation error l. Such estimates are often obtained by means of the Richardson extrapolation process. In this work, we explain how to derive estimates for l by means of formulae (32)–(33). Starting at X we first take one step of length h and apply the ELGT in the interval $[X, X + h]$ which results in an approximation $(X + h, y^*(X + h), y^{*\prime}(X + h))$. It follows from (32) that

$$l_i = y^{(i)}(X + h) - y^{*(i)}(X + h) = \delta_1^{(i)}(X + h) + \delta_2^{(i)}(X + h) + \cdots.$$

Taking only a finite number of corrections δ_i's, say n_l, we gain the estimate

$$l_i \approx \delta_1^{(i)}(X + h) + \delta_2^{(i)}(X + h) + \cdots + \delta_{n_l}^{(i)}(X + h). \tag{35}$$

Therefore, the local truncation error at $X + h$ is approximately given by

$$l \approx \max\{|l_0| + |l_1|\}, \tag{36}$$

with l_i, $i = 0, 1$, being computed by (35).

Formula (35) is not only useful in adjusting the step size according to (34) but also helps in improving the level of precision in the computed solutions at the right end of interval $[X, X+h']$ before advancing the calculations to the subsequent interval. That is, we apply ELGT on $[X+h', X+h'+h]$ supplied with the corrected initial conditions $y^{(i)}(X+h') = Y^{(i)}(X+h') + l_i$, $i = 0, 1$. This will be illustrated numerically in Section 6.

5.4. Computational costs

Since, as we mentioned earlier, LGT(M, N) performs MN evaluations of $R_N(x)$, ELGT(M, N) will require, in turn, $2MN$ evaluations, because LGT(M, N) is called twice by the ELGT procedure when the ODE is homogeneous. If $\{\omega_1, \omega_2\}$ are complex conjugates, then the LGT is called only once and therefore both methods have the same number of evaluations. When the ODE is non-homogeneous, the LGT is executed one more time and therefore the ELGT performs MN evaluations more. Since ELGT(M, N) can reach a certain prescribed accuracy using a relatively wider step size (that is, small M), its computational cost will be reduced considerably compared to that of the LGT. This will be illustrated later in Example 2.

6. Numerical examples

In order to illustrate the theoretical results of the previous sections, we have considered the numerical solution of some linear and nonlinear second-order differential equations.

Example 1. Let us solve the linear IVP (5) by ELGT(800,2). The errors committed by ELGT(800,2) compared to those of LGT(800,2) are listed in Table 1. One can easily appreciate the significant improvement throughout the interval of integration and in particular at the right-end point. The frequencies $\{\omega_k\}_{k\geq 1}$ used on subintervals $\{[x_{k-1}, x_k], k \geq 1\}$ are also displayed in the same table. We observe that the sizes of those ω_k increase as the exact solution becomes strongly oscillatory.

We solved the IVP (5) with the ELGT and the LGT using variable step size adapted according to formulae (34)–(36). A comparison between the two methods is given in Table 2 and in Fig. 2. In our computations, we have taken $\epsilon = 10^{-14}$ and the safety factors that were used in [6]: $\kappa = 0.9$, $\kappa^* = 2$ and $\kappa_* = 0.5$.

Table 2
(Example 1)—Comparison of step sizes for ELGT and LGT with $N = 4$ near $x = 40$. Algorithm (34)–(36) is used.

Method	No. steps	Max step	Min step	$e_N(40.0)$	$e'_N(40.0)$	l
ELGT	1156	0.25917	0.02159	3.67E−14	1.22E−12	4.78E−10
LGT	3909	0.2637	0.00480	−1.71E−13	−4.13E−12	4.79E−10

Table 3
(Example 2)—Comparison of computational costs of ELGT and LGT.

Method	$e_N(30)$	M	N	Freq.(ω_M)	Time (s)	Evaluations $R(x) = MN$
ELGT	5.213E−06	300	2	0.0164+60.9i	0.672	600
LGT	5.007E−06	20 000	2		25.203	40 000
ELGT	1.714E−06	100	4	0.0164+60.7i	0.438	400
LGT	3.419E−06	1 500	4		3.625	6 000

Table 4
(Example 2)—Comparison of step sizes for ELGT and LGT with $N = 4$ near $x = 30$.

Method	No. steps	Max step	Min step	$e_N(30.0)$	$e'_N(30.0)$	ℓ
ELGT	413	0.2154	0.055417	5.67E−13	3.89E−11	4.30E−11
LGT	2194	0.1846	0.006555	2.84E−13	4.93E−12	6.05E−10

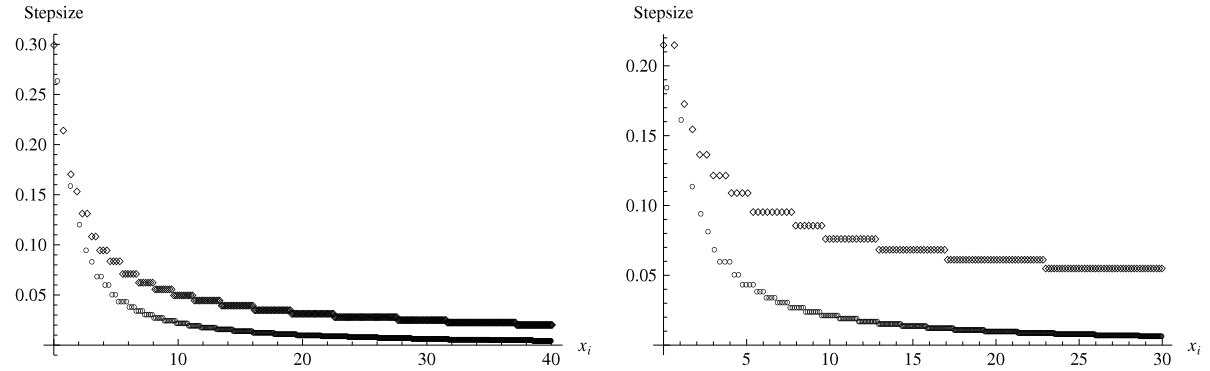

Fig. 2. (Example 1, left)—(Example 2, right): The step size decays for LGT (○) and ELGT (◇).

Example 2. This example compares the computational costs of ELGT and LGT that are necessary to attain a prescribed degree of accuracy.

The highly oscillatory function $y(x) = \sin(x^2 + x)$ satisfies the following homogeneous linear IVP

$$y'' - \frac{2}{2x+1}y' + (2x+1)^2 y = 0, \quad x \in [0, 30] \tag{37}$$
$$y(0) = 0, \qquad y'(0) = 1.$$

We wish to approximate the value of $y(x)$ at $x = 30$. So we computed two approximations $\tilde{y}(30)$ having the same order of accuracy by means of ELGT and LGT. Table 3 displays the error committed by both methods, and the necessary costs required by each one of them. Using the Mathematica package we have found that with 2 Legendre–Gauss points (i.e. $N = 2$), and 300 steps of uniform length (i.e. $M = 300$), ELGT(M, N) yields an approximate value with error $\approx 5.E - 06$. LGT(M, 2), on the other hand, needs $M = 20\,000$ steps to reach the same level of accuracy. We repeated the calculations with $N = 4$ and listed the results in the same table.

Note the following ratios:

$$\frac{\text{time(LGT},N=2)}{\text{time(LGT},N=4)} = \frac{25.203}{3.625} = 6.950 \text{ and } \frac{\text{eval(LGT},N=2)}{\text{eval(LGT},N=4)} = \frac{40\,000}{6000} = 6.667.$$
$$\frac{\text{time(ELGT},N=2)}{\text{time(ELGT},N=4)} = \frac{0.672}{0.438} = 1.53 \text{ and } \frac{\text{eval(ELGT},N=2)}{\text{eval(ELGT},N=4)} = \frac{600}{400} = 1.50.$$

Problem (37) was solved with variable step size adapted according to formulae (34)–(36). A comparison between the two methods is given in Table 4 and in Fig. 2 (note that $\epsilon = 10^{-12}$). Again the safety factors were taken in the previous example.

Table 5
(Example 3) Global errors at x_k's obtained by ELGT(M, N, P) for $M = 100, 200$, $N = 4$ and $P = 0, 6, 10$.

	Method		Order		
x_k	ELGT(100,4, 0)	ELGT(200, 4, 0)	h^8		
	ELGT(100, 4, 6)	ELGT(200, 4, 6)		h^{10}	
	ELGT(100, 4, 10)	ELGT(200, 4, 10)			h^{14}
3.0	1.77E−09	5.83E−12	8		
	7.39E−11	6.97E−14		10	
	7.53E−15	4.88E−19			14
4.0	2.41E−09	9.81E−12	8		
	1.83E−10	1.52E−13		10	
	5.16E−15	4.90E−19			13
5.0	1.28E−09	5.66E−12	8		
	1.60E−10	7.89E−14		11	
	1.12E−14	1.52E−18			13
8.0	1.03E−06	3.69E−09	8		
	1.35E−07	1.63E−10		10	
	2.94E−10	2.52E−14			14
9.0	3.59E−06	1.25E−08	8		
	7.59E−07	8.20E−10		10	
	3.12E−09	2.21E−13			14
10.0	6.25E−06	2.12E −08	8		
	2.17E−06	2.11E−09		10	
	1.46E−08	9.10E−13			14

Table 6
(Example 2) Exact error $e_N(x_m)$ at some points x_m for $h = 1$ for $N = 10, 12, \ldots, 20$.

x_m	$e_N(x_m)$					
	$N = 10$	$N = 12$	$N = 14$	$N = 16$	$N = 18$	$N = 20$
1	7.34E−9	−9.62E−13	−8.55E−16	−4.45E−20	1.03E−23	4.45E−28
2	−9.81E−8	−1.70E−11	1.43E−14	5.34E−18	3.83E−22	−4.76E−26
3	3.19E−7	2.01E−10	1.45E−14	−3.03E−17	−1.14E−20	−1.57E−24
4	−3.05E−7	−4.79E−10	−2.90E−13	−5.93E−17	−9.77E−22	8.48E−26
5	−8.71E−7	7.85E−10	3.56E−12	1.86E−15	−1.79E−18	−5.04E−22
6	2.54E−6	1.40E−8	−2.58E−11	−1.42E−13	1.18E−17	1.69E−19
7	−1.87E−4	4.89E−7	−3.51E−9	1.09E−11	8.77E−15	4.90E−18
8	2.34E−2	−2.70E−4	1.72E−6	−5.94E−9	1.33E−11	−1.95E−14
9	1.30E−1	1.03E−4	−2.30E−5	2.89E−7	−1.74E−9	6.02E−12
10	−4.98E−1	−4.31E−1	9.78E−3	−1.26E−4	9.98E−7	−5.03E−9

Example 3. Consider the non-homogeneous linear IVP

$$y'' + 4x^2 y = (4x^2 - \omega^2)\sin(\omega x) - 2\sin(x^2), \quad 0 \le x \le 10, \tag{38}$$
$$y(0) = 1, \qquad y'(0) = \omega,$$

with exact solution $y(x) = \sin(\omega x) + \cos(x^2)$. Let $\{0 = x_0 < x_1 \ldots < x_M = 10\}$ be a uniform partition of [0,10]. We applied ELGT(M, N, P) with $M = 100, 200$, $N = 4$ and $P = 0, 6, 10$. The global errors at x_m are displayed Table 5 and plotted in Fig. 1. Note that the last 3 columns ensure that for $P > 2N$, the order of ELGT(M, N, P) is $N + P$.

Variations of ELGT error in terms N. To see the variations of the ELGT error at points $\{x_m; m = 1, 2, \ldots, M\}$ with respect to N, the order of $L_N(x)$, we report in Table 6 the exact errors when (38) is solved by ELGT(M, N) with $M = 10$ (i.e. $h = 1$), and $N = 10, 12, \ldots, 20$. In order to explain the dependence between the global error and N, we plot in Fig. 3 the pairs $\{(N, \ln|e_N(x_m)|); N = 10, 12, \ldots, 20\}$, for each $m = 1, 2, \ldots, M$, ($M = 10$ and 20). It is observed that the error decays exponentially with respect to N, in accordance with the following asymptotic relation

$$\text{error of ELGT}(M, N) \sim \frac{1}{N! c_N^N}; \quad (c_N^N := \text{leading coefficient of } L_N(x))$$

that can be derived from an earlier result given in [17].

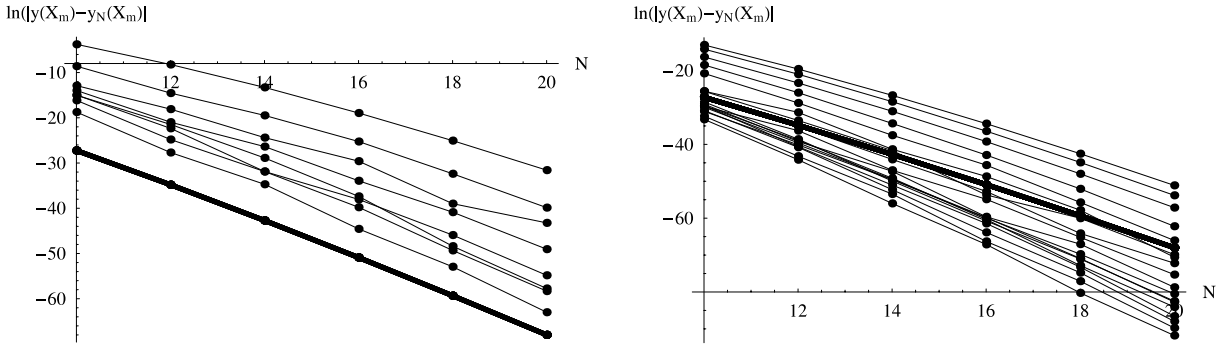

Fig. 3. (Example 3) The order of ELGT(M, N) method versus N: variations with N of $\ln|y(x_m)-y_N(x_m)|$ compared to the (heavy dark) plot of $\ln\left|\frac{1}{N!c_N^N}\right|$, $N = 10, 12, \ldots, 20$, $m = 1, 2, \ldots, 8$ and $M = 10$ (left), $M = 20$ (right).

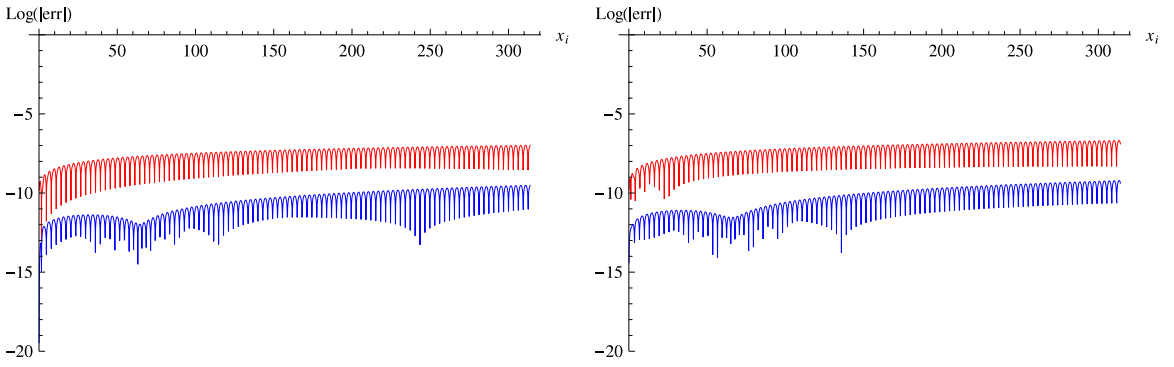

Fig. 4. Example 4 (left: case 1, right: case 2). Plot of $\log|e_N(x_i)|$ for ELGT(3000, 2, P) at $x_i = \mathrm{i}\pi/30$; $i = 1, 2, \ldots, 3000$ for $P = 0$ (upper line) and $P = 4$ (lower line).

Example 4 (*Mathieu Equation*)**.** Here we consider Mathieu's equation

$$y'' + (\delta + \epsilon\cos(2x))y = 0, \quad x \geq 0, \ (\delta, \epsilon \in \mathbf{R}) \tag{39}$$

$$y(0) = y_0; \qquad y'(0) = y_0', \tag{40}$$

where δ and ϵ are real constants. For given values of δ and ϵ, either all the solutions of Eq. (39) are bounded (stable) or at least an unbounded solution exists. The $\delta - \epsilon$ parameter plane is thus divided into stable and unstable regions. The boundaries between these two regions are demarcated by one-dimensional transition curves. We shall consider two cases:

Case 1: $\epsilon = 0.001$, $\delta = 1.000499968748047$, $y_0 = 0$, $y_0' = 1$, which corresponds to one of the transition curves of (39).
Case 2: $\epsilon = 0.001$, $\delta = 0.999791843656178$, $y_0 = -1.557212993975872$, $y_0' = 1$ which corresponds to a solution of (39) near to the transition curve.

The two versions of this problem have been solved by Chebyshev methods proposed by Richardson and Panovsky [19], and later on by Vigo-Aguiar and Ferrandiz [10] using an adapted multistep technique. We have solved the same problems by ELGT(M, N, P) with $M = 3000$, step length $\pi/30$ and $N = 2$. A plot of the decimal logarithm for the error at points $\{x_i = \mathrm{i}\pi/30\}_{i=1}^{M}$ is displayed in Fig. 4. We notice that for $N = 2$, the ELGT produces results very close to those obtained by Vigo-Aguiar and Ferrandiz [10]. Experiments with $N = 4$ have shown that the ELGT has advantages on both techniques, in both cases, transition curve and near transition curve.

7. Conclusion

In this paper we proposed an exponentially weighted version of the Legendre–Gauss Tau method for integrating linear and nonlinear second-order ODEs with highly oscillatory solutions. This method, denoted by ELGT, produces approximations expressed in an exponentially weighted Legendre polynomial basis $\{e^{\omega x}L_n(x); n \geq 0\}$, with complex frequencies ω. Such weights play an instrumental role in detecting the sharp variations in the solution, and permit to employ large step size for integration while keeping the level of accuracy throughout long intervals.

The accuracy of the ELGT can be measured either in terms of the step size h or in terms of the degree of Legendre polynomial L_N. In the former we obtain errors of order $O(h^{2N})$ and the latter results in errors of order $O\left(\frac{1}{N!}\right)$. A preliminary

version of this method was tested on a class of Sturm–Liouville problems where the accuracy was discussed in terms of ω; significant results were reported in [14]. A similar discussion can be carried out for other types of ill-posed problems and for mixed integro-differential equations (see [20]) and for systems of ODEs.

Finally, it is worth mentioning that the ELGT becomes more appropriate for automation if it is simulated by a finite difference algorithm. It is quite likely that such a task can be achieved with the help of the equivalence between the Tau method and the implicit Runge–Kutta methods (see [21]).

References

[1] A. Iserles, On the method of Neumann series for highly oscillatory equations, BIT 44 (2004) 473–488.
[2] L.Gr. Ixaru, Numerical Methods for Differential Equations and Applications, D. Reidel Publishing Co., Dordrecht, 1984.
[3] L.Gr. Ixaru, H. De Meyer, G. Vaned Berghe, CP methods for the Schrödinger equation, revisited, J. Comput. Appl. Math. 88 (1997) 289–314.
[4] V. Ledoux, M. Van Daele, G. Vanden Berghe, CP methods of higher order for Sturm–Liouville and Schrödinger equations, Comput. Phys. Commun. 162 (2004) 151–165.
[5] V. Ledoux, M. Rizea, L.Gr. Ixaru, G. Vanden Berghe, M. Van Daele, Solution of the Schrödinger equation by a high order perturbation method based on linear reference potential, Comput. Phys. Commun. 175 (2006) 424–439.
[6] L.Gr. Ixaru, G. Vanden Berghe, Exponential Fitting, Kluwer Academic Publishers, Dordrecht, 2004.
[7] G. Vanden Berghe, M. Van Daele, H. Van de Vyver, Exponential fitted Runge–Kutta methods of collocation type: fixed or variable knot points? J. Comput. Appl. Math. 159 (2003) 217–239.
[8] R.E. Greenwood, Numerical integration of linear sums of exponential functions, Ann. Math. Stat. 20 (1949) 608–611.
[9] P. Brock, F.J. Murray, The use of exponential sums in step by step integration, MTAC 6 (1952) 63–78.
[10] J. Vigo-Aguiar, J.M. Ferrandiz, A general procedure for the adaptation of multistep algorithms to the integration of oscillatory problems, SIAM J. Numer. Anal. 35 (1998) 1684–1708.
[11] C. Lanczos, Applied Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1956.
[12] E.L. Ortiz, The Tau method, SIAM J. Numer. Anal. 6 (1969) 480–492.
[13] D. Gottlieb, S.A. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, in: Series in Appl. Math., SIAM, Philadelphia, 1977.
[14] M.K. El-Daou, N.R. Al-Matar, An improved Tau method for a class of Sturm–Liouville problems, Appl. Math. Comput. 216 (2010) 1923–1937.
[15] P.J. Davis, Interpolation and Approximation, Dover Pub., NY, 1975.
[16] M.K. El-Daou, Computable error bounds for coefficients perturbation methods, Computing 69 (2002) 305–317.
[17] M.K. El-Daou, E.L. Ortiz, Error analysis of the Tau method: dependence of the error on the degree and on the length of the interval of approximation, Comput. Math. Appl. 25 (1993) 33–45.
[18] E. Hairer, S.P. Nörsett, G. Wanner, Solving Ordinary Differential Equations I, Nonstiff Problems, Springer-Verlag, Berlin, 1993.
[19] D.L. Richardson, J. Panovsky, A family of implicit Chebyshev methods for the numerical integration of first-order differential equations, in: Proc. AAS/AIAA Spaceflight Mechanics, Victoria, BC, Canada, 1993.
[20] S.M. Hosseini, S. Shahmorad, Numerical piecewise approximate solution of Fredholm integro-differential equations by the Tau method, Appl. Math. Modelling 29 (2005) 1005–1021.
[21] M.K. El-Daou, E.L. Ortiz, The Tau method as an analytic tool in the discussion of equivalence results across numerical methods, Computing 60 (1998) 365–376.
[22] M.K. El-Daou, Algorithms for differential equations with oscillatory solutions, in: World Congress on Engineering, 2008, pp. 388–395.