A REPORT

on

Application of Finite Element Galerkin Method to Solve Coupled Whitham-Broer-Kaup Equations

Prepared in partial fulfillment of

MATH F266: STUDY PROJECT

Submitted to

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Abstract

The approximate solution of the non linear system of equations known as the coupled Whitham–Broer–Kaup (WBK) to model shallow water waves is discussed here by the use of finite element method using Galerkin Approach. Before solving the WBK equations we solve the Linear and Non Linear Heat Equations, with both homogeneous and non-homogeneous boundary conditions using semidiscrete finite element method. We have used finite element method in x-direction and finite difference method in time direction. We obtain the rate of convergence for approximate solution of heat equation as 2 when used linear polynomial for approximation. The numerical result for heat equation validate the theoretical results.

Acknowledgement

I would like to thank the Department of Mathematics, Birla Institute of Technology and Sciences Pilani, Pilani Campus for providing me with this opportunity to work on such an interesting project which has great use in real industries. I would especially like to thank Dr. Sangita Yadav, Assistant Professor, Department of Mathematics to give me this wonderful opportunity to work on this project titled "Application of Finite Element Galerkin Method to solve Coupled Whitham-Broer-Kaup (WBK) equations". Being a student from the Department of Chemistry, but deeply interested in mathematics I requested her to take a project under her and she agreed. It would had been quite tough for me to understand the deeper mathematics required behind solving of the equations but it was due to her motivation and suggestions that I was able to move forward and understand the concepts. I would also like to thank her for sharing her knowledge with me and improving my understanding of concepts required behind the various steps of solving, telling the mathematical procedures and constructs which the project uses at various steps. During the course of the project I have learned a lot from her. This project would not have been possible without her support and helpful attitude.

Contents

1 Introduction

1.1 WBK Equations

The coupled Whitham–Broer–Kaup (WBK) equations were studied by Whitham[9], Broer[3] and Kaup [5]. The equations describe the propagation of shallow water waves, with different dispersion relations. The WBK equations are as follows,

$$u_t + uu_x + v_x + \beta u_{xx} = 0 \tag{1}$$
$$v_t - (uv)_x - \beta v_{xx} + \alpha u_{xxx} = 0 \tag{2}$$

where $u(x,t)$ is the horizontal velocity, $v(x,t)$ height of the liquid from equilibrium position of the liquid α, β are constants relating to the diffusion powers of the fluid. For background and model refer to [[9], [3], [5]] A variety of methods have been used for deriving the exact solution such as the Homotopy Analysis [7], Variational Iteration Method [6], Backlund transformation [4], sine–cosine method [2] etc. The system of equations is a very good model to describe dispersive waves. If α =0, $\beta \neq 0$ then the system represents the classical long wave equation that describes shallow water wave with diffusion [10]. If α=1 and $\beta = 0$ and, (1) is a modified form of Boussinesq equations[4].The solutions by inverse transformation were studied by Kaup[5] and Ablowitz[1] for the special case of (1), Kupershmidt discussed their symmetries and conservation laws. By using of Backlund transformation[4], three pairs of solutions of WBK equation were found by Fan[4]. Xie et al.[11] used hyperbolic function method and Wu elimination method obtained four pairs of solutions of WBK equation. Xu [12] presented an elliptic equation method for constructing new types of elliptic function solutions for the WBK equation. Sirendaoreji[8] suggested a new auxiliary ordinary differential equation method and constructed exact travelling wave solutions of the WBK equation in a unified way.In this paper we look the numerical approach to solve this problem by the use Finite Element Method and using the weak formulation of the equation.

1.2 Galerkin Method

Galerkin methods are a class of methods for converting a continuous operator problem (such as a differential equation) to a discrete problem. In principle, it is the equivalent of applying the method of variation of parameters to a function space, by converting the equation to a weak formulation. Galerkin method can be further used with the Finite Element Method to find an approximate solution.To understand Galerkin method we need an understanding of distribution theory and the concept of weak derivatives defined in a Sobolov Space.

1.2.1 Weak Formulation in term of weak derivates

Step 1: Consider the strong form of the differential equation as :

$$\begin{aligned}
&Lu(x) = f(x) \\
&\langle Lu, v\rangle - \langle f(x), v\rangle = 0 \\
&\int Lu(x)v(x)dx - \int f(x)v(x) = 0 \ \forall \ v \ \in V
\end{aligned}$$

Step 2: Reduce order of derivatives by integration by parts.

Step 3: Apply boundary conditions.

1.3 Finite Element Method

The finite element method (FEM) is a numerical method for solving problems of engineering and mathematical physics. This method is widely used in structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential. Applications include simulation of car crash, impact of force on bridges, wind thrush on structures etc. System of PDE's can be used to model a physical system occurring in nature and can be solved to get the desired state parameters of the system. But most of the modelling of the natural phenomena involves complex non – linear PDE's which can easily be solved by numerical methods and Finite Element Analysis is a technique which can help solving such system of equations by approximating the function by a finite number of basis functions, i.e. if $u(x)$ is a solution to our differential equation then $u^h(x) = \sum_{i=0}^{n} c_i \phi_i(x)$ where $\phi(x)$ is an element of the mesh. In our particular case u is a function of x and t. Thus finite element approximation is used as $u^h(x) = \sum_{i=0}^{n} \alpha_i(t)\phi_i(x)$.

1.4 Basis Functions

The basis are the approximation of elements of the mesh in form of mathematical functions. Many types of basis are used but most commonly used basis functions are the hat functions which are used in this paper for solving the 4 Heat Equations and the WBK Equations. The hat functions basis are defined as followed:

$$\phi_0(x) = \begin{cases} \\ -\frac{x-h}{h} & ih \le x \le (i+1)\,h \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_i(x) = \begin{cases} \frac{x-(i-1)h}{h} & (i-1)\,h \le x \le ih \\ -\frac{x-(i+1)h}{h} & ih \le x \le (i+1)\,h \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in \{1 \text{ ... N-1}\}$$

$$\phi_N(x) = \begin{cases} \frac{x-(N-1)h}{h} & (N-1)\,h \le x \le hN \\ 0 & \text{otherwise} \end{cases}$$

2 Linear Heat Equations

We are considering the heat equations as follows :
Equation:

$$u_t + \beta u_{xx} = f(x,t) \text{ for } x \in (0,1) \text{ and } t > 0 \quad (3)$$
$$\text{where } u = u(x,t)$$

The Boundary Conditions of (3)

$$u(0,t) = g_0(t) \quad (4)$$
$$u(1,t) = g_1(t) \quad (5)$$
$$\text{for } t > 0$$

The Weak Form of the equation (3)

$$\langle u_t \ + \beta u_{xx}, w\rangle = \langle f(x,t), w\rangle$$
$$\text{or}$$
$$\int_0^1 w(u_t + \beta u_{xx})dx = \int_0^1 wf(x,t)dx \quad (6)$$

Applying integration by parts to u_{xx} in 6 we get :

$$\int_0^1 wu_t dx - \beta \int_0^1 w_x u_x dx + \beta(w(1)u_x(1) - w(0)u_x(0)) = \int_0^1 wf(x,t)dx \quad \forall\ w \in V \quad (7)$$

We consider a finite dimensional subspace $V_h \subset V$. We approximate the solution u by $u^h \in V_h$ which can be written as

$$u \approx u^h = \sum_{i=1}^{n-1} \phi_i(x)\gamma_i(t) + \phi_0(x)g_0(t) + \phi_n(x)g_1(t) \quad (8)$$

where $\{\phi_i\}$ are nodal basis functions of V_h. Substituting (8) in (7):

$$\int_0^1 w(\sum_{i=1}^{n-1} \phi_i(x)\gamma_i'(t) + \phi_0(x)g_0'(t) + \phi_n(x)g_1'(t))dx - \beta \int_0^1 w_x(\sum_{i=1}^{n-1} \phi_i'(x)\gamma_i(t) + \phi_0'(x)g_0(t)$$
$$+ \phi_n'(x)g_1(t))dx + \beta\ (w(1)\phi_n'(1)g_1(t) - w(0)\phi_0'(0)g_0(t)) = \int_0^1 wf(x,t)dx$$
$$\text{for } w = \phi_i, i = 1, 2, \cdots, n-1.$$

Since w represents basis function and is only a function of x , we can take the summation out and integrate.

$$\sum_{i=1}^{n-1} \gamma_i'(t) \int_0^1 w\phi_i(x)dx + g_0'(t) \int_0^1 w\phi_0(x)dx + g_1'(t) \int_0^1 w\phi_n(x)dx - \beta \sum_{i=1}^{n-1} \gamma_i(t) \int_0^1 w_x\phi_i'(x)dx$$
$$- \beta g_0(t) \int_0^1 w_x\phi_0'(x)dx - \beta g_1(t) \int_0^1 w_x\phi_n'(x)dx + \beta w(1)\phi_n'(1)g_1(t) - \beta w(0)\phi_0'(0)g_0(t) = \int_0^1 wf(x,t)dx$$

Collecting the summations and constants, we get

$$\sum_{i=1}^{n-1} \gamma_i'(t) \int_0^1 w\phi_i(x)dx - \beta \sum_{i=1}^{n-1} \gamma_i(t) \int_0^1 w_x\phi_i'(x)dx + g_0'(t) \int_0^1 w\phi_0(x)dx + g_1'(t) \int_0^1 w\phi_n(x)dx$$
$$- \beta g_0(t) \int_0^1 w_x\phi_0'(x)dx - \beta g_1(t) \int_0^1 w_x\phi_n'(x)dx + \beta w(1)\phi_n'(1)g_1(t) - \beta w(0)\phi_0'(0)g_0(t) = \int_0^1 wf(x,t)dx$$

Writing equations in matrix form, we get

$$\boxed{\mathbf{A}\boldsymbol{\gamma}' - \beta\mathbf{D}\boldsymbol{\gamma} + \mathbf{E_1} = F}$$

where

A is a n-1 x n-1 square matrix with

$A_{ji} = \int_0^1 \phi_j(x)\phi_i(x)dx$

D is a n-1 x n-1 square matrix with

$D_{ji} = \int_0^1 \phi_j'(x)\phi_i'(x)\ dx$

$\mathbf{E_1}$ is n-1 x 1 column matrices containing the sum of the constants

F is a n-1 x 1 square matrix with

$F_j = \int_0^1 \phi_j(x)f(x,t)\ dx,$

$i, j \in$ {1, 2, .. ,n -1 }

Since $w = \phi_j \ \forall j \in 1, 2, 3, 4, \ldots.\ n-1$ where n is the number of elements of the mesh.
To solve the following system of differential equations we use the Scipy's inbuilt Runge-Kutta 45 Ode Solver.

2.1 Numerical Solution

2.1.1 Parameters:

$$\beta = -1$$

2.1.2 Homogeneous Boundary Conditions:

2.1.3 Initial Conditions:

$$u(x,0) = \sin(\pi x)$$
$$f(x,t) = \left(1 - \pi^2\beta\right) e^t \sin \pi x$$

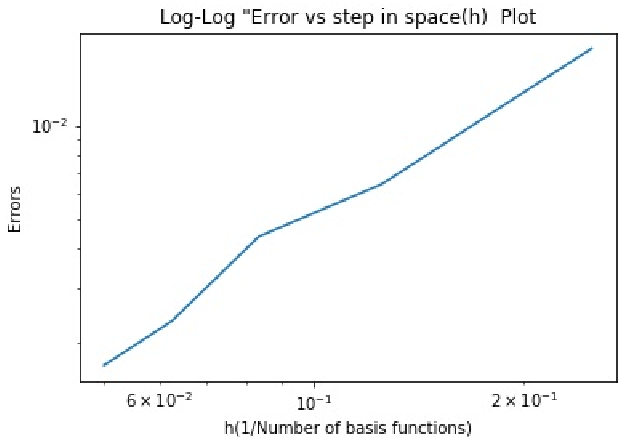
2.1.4 Exact Solution:

$$u(x,t) = e^t \sin(\pi x)$$

2.1.5 Boundary Conditions:

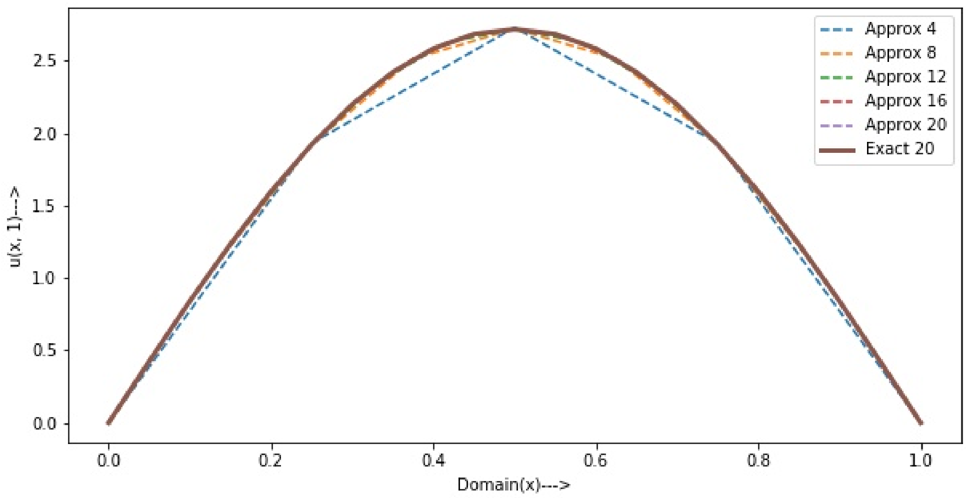
$$g_0(t) = 0$$
$$g_1(t) = 0$$
$$\text{for } t > 0$$

2.1.6 Convergence Rate



Hom ROC.jpg

2.1.7 Convergence Graph



Hom Conv.jpg

2.1.8 Error Table

Index	N	Errors
0	4	0.017742
1	8	0.006470
2	12	0.004384
3	16	0.002335
4	20	0.001683

2.1.9 Slope

Experimental = 1.44578324
Theoretical = 2

2.1.10 Non Homogeneous Boundary Conditions:

2.1.11 Initial Conditions:

$$u(x,0) = \sin(x)$$
$$f(x,t) = (1-\beta)\, e^t \sin x$$

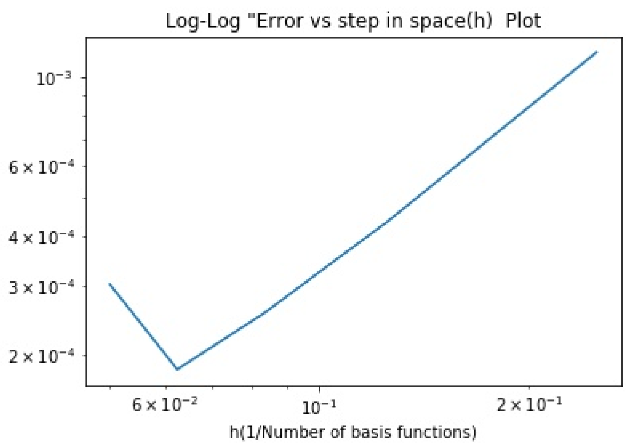
2.1.12 Exact Solution:

$$u(x,t) = e^t \sin x$$

2.1.13 Boundary Conditions:

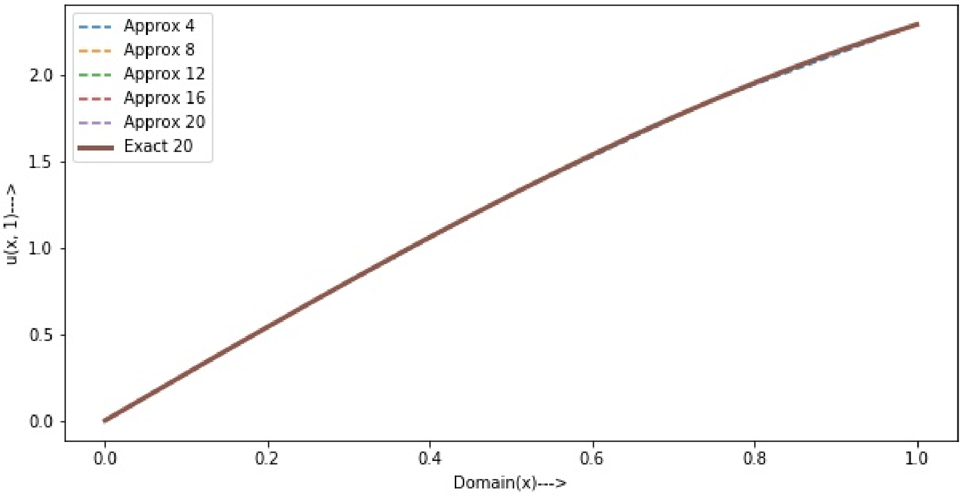
$$g_0(t) = 0$$
$$g_1(t) = e^t \sin(1)$$
$$\text{for } t > 0$$

2.1.14 Convergence Rate



NonHom ROC.jpg

2.1.15 Convergence Graph



NonHom Conv.jpg

2.1.16 Error Table

Index	N	Errors
0	4	0.001156
1	8	0.000433
2	12	0.000256
3	16	0.000185
4	20	0.000302

2.1.17 Slope

Experimental = 0.99928785
Theoretical = 2

3 Non Linear Heat Equations

We are considering the heat equations as follows :
Equation:

$$u_t + uu_x + \beta u_{xx} = f(x,t) \text{ for } x \in (0,1) \text{ and } t > 0 \tag{9}$$
$$\text{where } u = u(x,t)$$

The Boundary Conditions of (3)

$$u(0,t) = g_0(t) \tag{10}$$
$$u(1,t) = g_1(t) \tag{11}$$
$$\text{for } t > 0$$

The Weak Form of the equation (9)

$$\langle u_t + uu_x + \beta u_{xx}, w\rangle = \langle f(x,t), w\rangle$$
$$\text{or}$$
$$\int_0^1 w(u_t + uu_x + \beta u_{xx})dx = \int_0^1 wf(x,t)dx \tag{12}$$

Applying integration by parts to u_{xx} in 12 we get :

$$\int_0^1 wu_t dx + \int_0^1 wuu_x dx - \beta\int_0^1 w_x u_x dx + \beta(w(1)u_x(1) - w(0)u_x(0)) = \int_0^1 wf(x,t)dx \quad \forall\ w \in V \tag{13}$$

We consider a finite dimensional subspace $V_h \subset V$. We approximate the solution u by $u^h \in V_h$ which can be written as

$$u \approx u^h = \sum_{i=1}^{n-1} \phi_i(x)\gamma_i(t) + \phi_0(x)g_0(t) + \phi_n(x)g_1(t) \tag{14}$$

where $\{\phi_i\}$ are nodal basis functions of V_h. Substituting (14) in (13):

$$\begin{aligned}
&\int_0^1 w(\sum_{i=1}^{n-1} \phi_i(x)\gamma_i'(t) + \phi_0(x)g_0'(t) + \phi_n(x)g_1'(t))dx \\
&+ \int_0^1 w(\sum_{i=1}^{n-1} \phi_i(x)\gamma_i(t) + \phi_0(x)g_0(t) + \phi_n(x)g_1(t))(\sum_{k=1}^{n-1} \phi_k'(x)\gamma_k(t) + \phi_0'(x)g_0(t) + \phi_n'(x)g_1(t))dx \\
&- \beta\int_0^1 w_x(\sum_{i=1}^{n-1} \phi_i'(x)\gamma_i(t) + \phi_0'(x)g_0(t) + \phi_n'(x)g_1(t))dx \\
&+ \beta\ (w(1)\phi_n'(1)g_1(t) - w(0)\phi_0'(0)g_0(t)) = \int_0^1 wf(x,t)dx
\end{aligned}$$

Since w represents basis function and is only a function of x , we can take the summation out and integrate.

$$\sum_{i=1}^{n-1}\gamma_i'(t)\int_0^1 w\phi_i(x)dx + g_0'(t)\int_0^1 w\phi_0(x)dx + g_1'(t)\int_0^1 w\phi_n(x)dx$$
$$+\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\gamma_i(t)\gamma_k(t)\int_0^1 w\phi_i(x)\phi_k'(x)dx + g_0(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_0'(x)dx$$
$$+g_1(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_n'(x)dx + g_0(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_0(x)\phi_k'(x)dx + {g_0}^2(t)\int_0^1 w\phi_0(x)\phi_0'(x)dx$$
$$+g_0(t)g_1(t)\int_0^1 w\phi_0(x)\phi_n'(x)dx + g_1(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_n(x)\phi_k'(x)dx + g_1(t)g_0(t)\int_0^1 w\phi_n(x)\phi_0'(x)dx$$
$$+g_1^2(t)\int_0^1 w\phi_n(x)\phi_n'(x)dx - \beta\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w_x\phi_i'(x)dx - \beta g_0(t)\int_0^1 w_x\phi_0'(x)dx - \beta g_1(t)\int_0^1 w_x\phi_n'(x)dx$$
$$+\beta w(1)\phi_n'(1)g_1(t) - \beta w(0)\phi_0'(0)g_0(t) = \int_0^1 wf(x,t)dx$$

Collecting the summations and constants, we get

$$\sum_{i=1}^{n-1}\gamma_i'(t)\int_0^1 w\phi_i(x)dx + \sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\gamma_i(t)\gamma_k(t)\int_0^1 w\phi_i(x)\phi_k'(x)dx + g_0(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_0'(x)dx$$
$$+g_1(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_n'(x)dx + g_0(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_0(x)\phi_k'(x)dx$$
$$+g_1(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_n(x)\phi_k'(x)dx - \beta\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w_x\phi_i'(x)dx + g_0'(t)\int_0^1 w\phi_0(x)dx$$
$$+g_1'(t)\int_0^1 w\phi_n(x)dx + {g_0}^2(t)\int_0^1 w\phi_0(x)\phi_0'(x)dx + g_0(t)g_1(t)\int_0^1 w\phi_0(x)\phi_n'(x)dx$$
$$+g_1(t)g_0(t)\int_0^1 w\phi_n(x)\phi_0'(x)dx + g_1^2(t)\int_0^1 w\phi_n(x)\phi_n'(x)dx - \beta g_0(t)\int_0^1 w_x\phi_0'(x)dx$$
$$-\beta g_1(t)\int_0^1 w_x\phi_n'(x)dx + \beta w(1)\phi_n'(1)g_1(t) - \beta w(0)\phi_0'(0)g_0(t) = \int_0^1 wf(x,t)dx$$

Writing equations in matrix form, we get

$$
\mathbf{A}\boldsymbol{\gamma}' + \mathbf{B}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) + g_0(t)\mathbf{B_{top}}\boldsymbol{\gamma} + g_1(t)\mathbf{B_{bottom}}\boldsymbol{\gamma} + g_0(t)\mathbf{B_{left}}\gamma \\
+ g_1(t)\mathbf{B_{right}}\gamma - \beta\mathbf{D}\boldsymbol{\gamma} + \mathbf{E_1} = F
$$

where

A is a n-1 x n-1 square matrix with

$$A_{ji} = \int_0^1 \phi_j(x)\phi_i(x)dx$$

$\mathbf{B}(\boldsymbol{\xi}, \boldsymbol{\eta})$ is a n-1 x 1 column matrix with

$B_j(\xi, \eta) = \xi^T B^j \eta$ and B^j is a square matrix with

$$B^j_{ki} = \int_0^1 \phi_j(x)\phi_i(x)\phi_k'(x)dx$$

$\mathbf{B_{top}}$ is a n-1 x n-1 square matrix with

$$B_{top_{ji}} = \int_0^1 \phi_j\phi_i(x)\phi_0'(x)dx$$

$\mathbf{B_{bottom}}$ is a n-1 x n-1 square matrix with

$$B_{bottom_{ji}} = \int_0^1 \phi_j\phi_i(x)\phi_n'(x)dx$$

$\mathbf{B_{left}}$ is a n-1 x n-1 square matrix with

$$B_{left_{jk}} = \int_0^1 \phi_j\phi_0(x)\phi_k'(x)dx$$

$\mathbf{B_{right}}$ is a n-1 x n-1 square matrix with

$$B_{right_{jk}} = \int_0^1 \phi_j\phi_n(x)\phi_k'(x)dx$$

D is a n-1 x n-1 square matrix with

$$D_{ji} = \int_0^1 \phi_j'(x)\phi_i'(x)dx$$

$\mathbf{E_1}$ is n-1 x 1 column matrices containing the sum of the constants

F is a n-1 x 1 square matrix with

$$F_j = \int_0^1 \phi_j(x)f(x,t)\ dx,$$

$i, j \in$ {1, 2, .. ,n -1 }

Since $w = \phi_j \ \forall j \in$ 1, 2, 3, 4, ... $n-1$ where n is the number of elements of the mesh.
To solve the following system of differential equations we use the Scipy's inbuilt Runge-Kutta 45 Ode Solver.

3.1 Numerical Solution

3.1.1 Parameters:

$$\beta = -1$$

3.1.2 Homogeneous Boundary Conditions:

3.1.3 Initial Conditions:

$$u(x,0) = \sin \pi x$$
$$f(x,t) = (1 - \beta\pi^2)e^t \sin \pi x + \pi e^{2t} \sin \pi x \cos \pi x$$

3.1.4 Exact Solution:

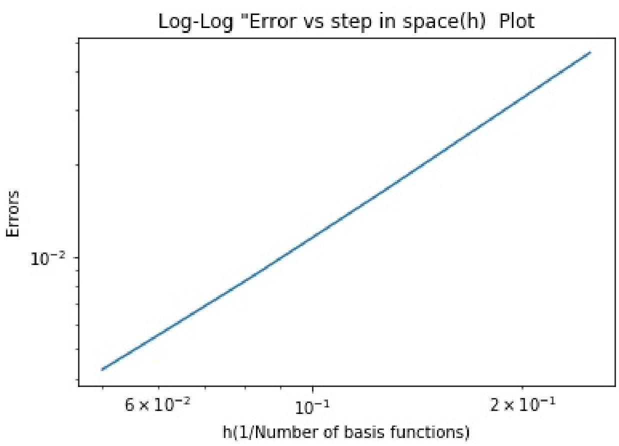
$$u(x,t) = e^t \sin(\pi x)$$

3.1.5 Boundary Conditions:

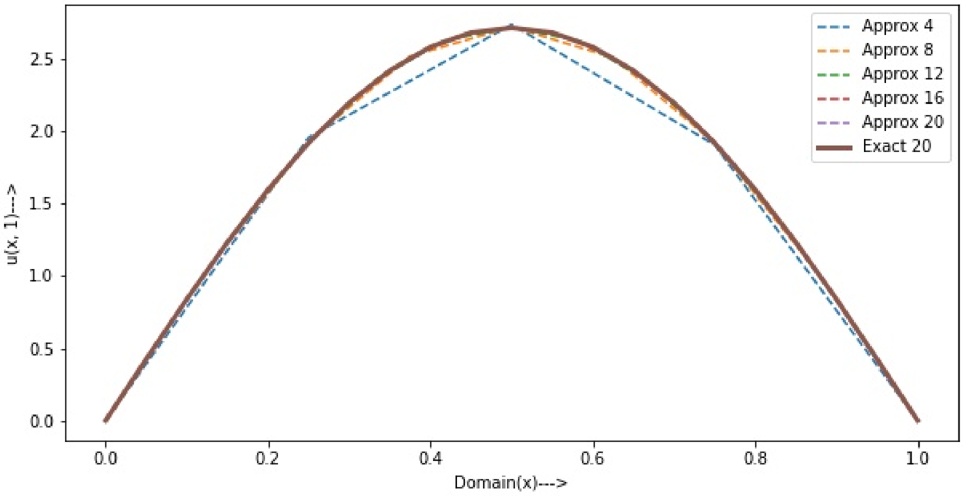
$$g_0(t) = 0$$
$$g_1(t) = 0$$
$$\text{for } t > 0$$

3.1.6 Convergence Rate

Hom ROC.jpg



3.1.7 Convergence Graph



Hom Conv.jpg

3.1.8 Error Table

Index	N	Errors
0	4	0.046143
1	8	0.016039
2	12	0.008799
3	16	0.005853
4	20	0.004283

3.1.9 Slope

Experimental = 1.47920829
Theoretical = 2

3.1.10 Non Homogeneous Boundary Conditions:

3.1.11 Initial Conditions:

$$u(x,0) = \sin(x)$$
$$f(x,t) = (1-\beta)e^t \sin x + e^{2t} \sin x \cos x$$

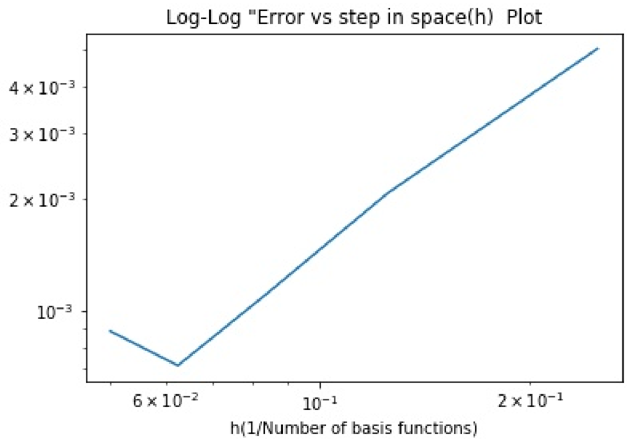
3.1.12 Exact Solution:

$$u(x,t) = e^t \sin x$$

3.1.13 Boundary Conditions:

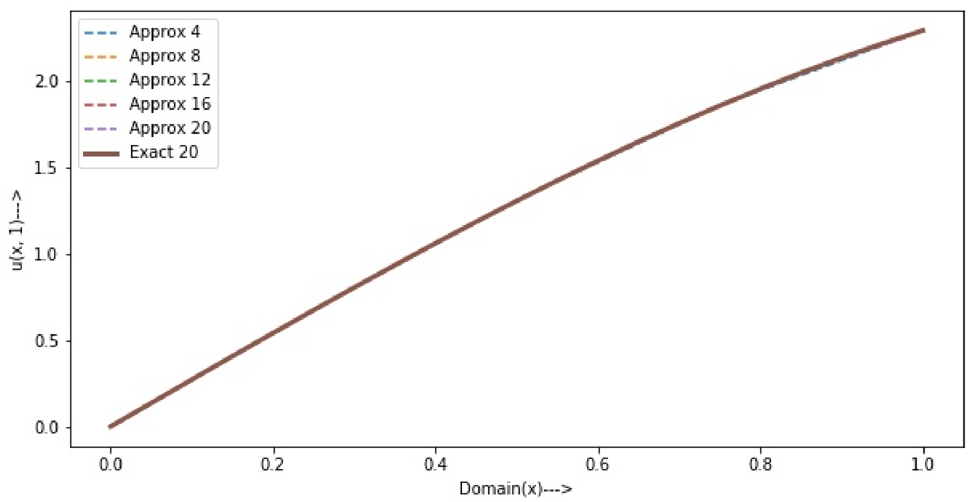
$$g_0(t) = 0$$
$$g_1(t) = e^t \sin(1)$$
$$\text{for } t > 0$$

3.1.14 Convergence Rate



NHom ROC.jpg

3.1.15 Convergence Graph



NHom Conv.jpg

3.1.16 Error Table

Index	N	Errors
0	4	0.005043
1	8	0.002068
2	12	0.001101
3	16	0.000714
4	20	0.000883

3.1.17 Slope

Experimental = 1.20899177
Theoretical = 2

4 WBK Equations

We are considering the heat equations as follows :
Equation:

$$u_t + uu_x + v_x + \beta u_{xx} = 0 \text{ for } x \in (0,1) \text{ and } t > 0 \tag{15}$$
$$v_t + (vu)_x - \beta v_{xx} = 0 \text{ for } x \in (0,1) \text{ and } t > 0 \tag{16}$$
$$\text{where } u = u(x,t) \text{ and } v = v(x,t)$$

The Boundary Conditions of (15) and (16)

$$u(0,t) = g_0(t) \tag{17}$$
$$u(1,t) = g_1(t) \tag{18}$$
$$v(0,t) = h_0(t) \tag{19}$$
$$v(1,t) = h_1(t) \tag{20}$$
$$\text{for } t > 0$$

The Weak Form of the equations (15) and (16)

$$\langle u_t + uu_x + v_x + \beta u_{xx}, w\rangle = 0$$
$$\langle v_t + (uv)_x - \beta v_{xx}, w\rangle = 0$$
$$\text{or}$$
$$\int_0^1 w(u_t + uu_x + v_x + \beta u_{xx})dx = 0 \tag{21}$$
$$\int_0^1 w(v_t + (uv)_x - \beta v_{xx})dx = 0 \tag{22}$$

Applying integration by parts to u_{xx} and v_{xx} in (21) and (22) we get :

$$\int_0^1 wu_t dx + \int_0^1 wuu_x dx + \int_0^1 wv_x dx - \beta \int_0^1 w_x u_x dx$$
$$+ \beta(w(1)u_x(1) - w(0)u_x(0)) = 0 \quad \forall\, w \in V \tag{23}$$

$$\int_0^1 wv_t dx + \int_0^1 wuv_x dx + \int_0^1 wvu_x dx + \beta \int_0^1 w_x u_x dx$$
$$- \beta(w(1)v_x(1) - w(0)v_x(0))dx = 0 \quad \forall\, w \in V \tag{24}$$

We consider a finite dimensional subspace $V_h \subset V$. We approximate the solution u by $u^h \in V_h$ and v by $v^h \in V_h$ which can be written as

$$u \approx u^h = \sum_{i=1}^{n-1} \phi_i(x)\gamma_i(t) + \phi_0(x)g_0(t) + \phi_n(x)g_1(t) \tag{25}$$

$$v \approx v^h = \sum_{i=1}^{n-1} \phi_i(x)\delta_i(t) + \phi_0(x)h_0(t) + \phi_n(x)h_1(t) \tag{26}$$

where $\{\phi_i\}$ are nodal basis functions of V_h. Substituting (25) in (23) and (26) in (24) :

$$\begin{aligned}
&\int_0^1 w(\sum_{i=1}^{n-1} \phi_i(x)\gamma_i'(t) + \phi_0(x)g_0'(t) + \phi_n(x)g_1'(t))dx \\
&+ \int_0^1 w(\sum_{i=1}^{n-1} \phi_i(x)\gamma_i(t) + \phi_0(x)g_0(t) + \phi_n(x)g_1(t))(\sum_{k=1}^{n-1} \phi_k'(x)\gamma_k(t) + \phi_0'(x)g_0(t) + \phi_n'(x)g_1(t))dx \\
&+ \int_0^1 w(\sum_{i=1}^{n-1} \phi_i'(x)\delta_i(t) + \phi_0'(x)h_0(t) + \phi_n'(x)h_1(t)) \\
&- \beta \int_0^1 w_x(\sum_{i=1}^{n-1} \phi_i'(x)\gamma_i(t) + \phi_0'(x)g_0(t) + \phi_n'(x)g_1(t))dx \\
&+ \beta\ (w(1)\phi_n'(1)g_1(t) - w(0)\phi_0'(0)g_0(t)) = 0
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 w(\sum_{i=1}^{n-1} \phi_i(x)\delta_i'(t) + \phi_0(x)h_0'(t) + \phi_n(x)h_1'(t))dx \\
&+ \int_0^1 w(\sum_{i=1}^{n-1} \phi_i(x)\gamma_i(t) + \phi_0(x)g_0(t) + \phi_n(x)g_1(t))(\sum_{k=1}^{n-1} \phi_k'(x)\delta_k(t) + \phi_0'(x)h_0(t) + \phi_n'(x)h_1(t))dx \\
&+ \int_0^1 w(\sum_{i=1}^{n-1} \phi_i(x)\delta_i(t) + \phi_0(x)h_0(t) + \phi_n(x)h_1(t))(\sum_{k=1}^{n-1} \phi_k'(x)\gamma_k(t) + \phi_0'(x)g_0(t) + \phi_n'(x)g_1(t))dx \\
&+ \beta \int_0^1 w_x(\sum_{i=1}^{n-1} \phi_i'(x)\delta_i(t) + \phi_0'(x)h_0(t) + \phi_n'(x)h_1(t))dx \\
&- \beta\ (w(1)\phi_n'(1)h_1(t) - w(0)\phi_0'(0)h_0(t)) = 0
\end{aligned}$$

Since w represents basis function and is only a function of x , we can take the summation out and integrate.

$$
\begin{aligned}
&\sum_{i=1}^{n-1}\gamma_i'(t)\int_0^1 w\phi_i(x)dx + g_0'(t)\int_0^1 w\phi_0(x)dx + g_1'(t)\int_0^1 w\phi_n(x)dx\\
&+\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\gamma_i(t)\gamma_k(t)\int_0^1 w\phi_i(x)\phi_k'(x)dx + g_0(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_0'(x)dx\\
&+g_1(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_n'(x)dx + g_0(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_0(x)\phi_k'(x)dx + {g_0}^2(t)\int_0^1 w\phi_0(x)\phi_0'(x)dx\\
&+g_0(t)g_1(t)\int_0^1 w\phi_0(x)\phi_n'(x)dx + g_1(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_n(x)\phi_k'(x)dx + g_1(t)g_0(t)\int_0^1 w\phi_n(x)\phi_0'(x)dx\\
&+g_1^2(t)\int_0^1 w\phi_n(x)\phi_n'(x)dx + \sum_{i=1}^{n-1}\delta_i(t)\int_0^1 w\phi_i'(x)dx + h_0(t)\int_0^1 w\phi_0'(x)dx + h_1(t)\int_0^1 w\phi_n'(x)dx\\
&-\beta\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w_x\phi_i'(x)dx - \beta g_0(t)\int_0^1 w_x\phi_0'(x)dx - \beta g_1(t)\int_0^1 w_x\phi_n'(x)dx\\
&+\beta w(1)\phi_n'(1)g_1(t) - \beta w(0)\phi_0'(0)g_0(t) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n-1}\delta_i'(t)\int_0^1 w\phi_i(x)dx + h_0'(t)\int_0^1 w\phi_0(x)dx + h_1'(t)\int_0^1 w\phi_n(x)dx\\
&+\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\gamma_i(t)\delta_k(t)\int_0^1 w\phi_i(x)\phi_k'(x)dx + h_0(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_0'(x)dx\\
&+h_1(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_n'(x)dx + g_0(t)\sum_{k=1}^{n-1}\delta_k(t)\int_0^1 w\phi_0(x)\phi_k'(x)dx + g_0(t)h_0(t)\int_0^1 w\phi_0(x)\phi_0'(x)dx\\
&+g_0(t)h_1(t)\int_0^1 w\phi_0(x)\phi_n'(x)dx + g_1(t)\sum_{k=1}^{n-1}\delta_k(t)\int_0^1 w\phi_n(x)\phi_k'(x)dx + g_1(t)h_0(t)\int_0^1 w\phi_n(x)\phi_0'(x)dx\\
&+g_1(t)h_1(t)\int_0^1 w\phi_n(x)\phi_n'(x)dx + \sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\delta_i(t)\gamma_k(t)\int_0^1 w\phi_i(x)\phi_k'(x) + g_0(t)\sum_{i=1}^{n-1}\delta_i(t)\int_0^1 w\phi_i(x)\phi_0'(x)dx\\
&+g_1(t)\sum_{i=1}^{n-1}\delta_i(t)\int_0^1 w\phi_i(x)\phi_n'(x)dx + h_0(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_0(x)\phi_k'(x)dx + h_0(t)g_0(t)\int_0^1 w\phi_0(x)\phi_0'(x)\\
&+h_0(t)g_1(t)\int_0^1 w\phi_0(x)\phi_n'(x)dx + h_1(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_n(x)\phi_k'(x)dx + h_1(t)g_0(t)\int_0^1 w\phi_n(x)\phi_0'(x)dx\\
&+h_1(t)g_1(t)\int_0^1 w\phi_n(x)\phi_n'(x)dx + \beta\sum_{i=1}^{n-1}\delta_i(t)\int_0^1 w_x\phi_i'(x)dx + \beta h_0(t)\int_0^1 w_x\phi_0'(x)dx\\
&+\beta h_1(t)\int_0^1 w_x\phi_n'(x)dx - \beta w(1)\phi_n'(1)h_1(t) + \beta w(0)\phi_0'(0)h_0(t) = 0
\end{aligned}
$$

Collecting the summations and constants, we get

$$
\begin{aligned}
&\sum_{i=1}^{n-1}\gamma_i'(t)\int_0^1 w\phi_i(x)dx+\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\gamma_i(t)\gamma_k(t)\int_0^1 w\phi_i(x)\phi_k'(x)dx+g_0(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_0'(x)dx\\
&+g_1(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_n'(x)dx+g_0(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_0(x)\phi_k'(x)dx\\
&+g_1(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_n(x)\phi_k'(x)dx+\sum_{i=1}^{n-1}\delta_i(t)\int_0^1 w\phi_i'(x)dx-\beta\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w_x\phi_i'(x)dx\\
&+g_0'(t)\int_0^1 w\phi_0(x)dx+g_1'(t)\int_0^1 w\phi_n(x)dx+{g_0}^2(t)\int_0^1 w\phi_0(x)\phi_0'(x)dx+g_0(t)g_1(t)\int_0^1 w\phi_0(x)\phi_n'(x)dx\\
&+g_1(t)g_0(t)\int_0^1 w\phi_n(x)\phi_0'(x)dx+g_1^2(t)\int_0^1 w\phi_n(x)\phi_n'(x)dx+h_0(t)\int_0^1 w\phi_0'(x)dx+h_1(t)\int_0^1 w\phi_n'(x)dx\\
&-\beta g_0(t)\int_0^1 w_x\phi_0'(x)dx-\beta g_1(t)\int_0^1 w_x\phi_n'(x)dx+\beta w(1)\phi_n'(1)g_1(t)-\beta w(0)\phi_0'(0)g_0(t)=0
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n-1}\delta_i'(t)\int_0^1 w\phi_i(x)dx+\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\gamma_i(t)\delta_k(t)\int_0^1 w\phi_i(x)\phi_k'(x)dx+h_0(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_0'(x)dx\\
&+h_1(t)\sum_{i=1}^{n-1}\gamma_i(t)\int_0^1 w\phi_i(x)\phi_n'(x)dx+g_0(t)\sum_{k=1}^{n-1}\delta_k(t)\int_0^1 w\phi_0(x)\phi_k'(x)dx\\
&+g_1(t)\sum_{k=1}^{n-1}\delta_k(t)\int_0^1 w\phi_n(x)\phi_k'(x)dx+\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\delta_i(t)\gamma_k(t)\int_0^1 w\phi_i(x)\phi_k'(x)dx\\
&+g_0(t)\sum_{i=1}^{n-1}\delta_i(t)\int_0^1 w\phi_i(x)\phi_0'(x)dx+g_1(t)\sum_{i=1}^{n-1}\delta_i(t)\int_0^1 w\phi_i(x)\phi_n'(x)dx\\
&+h_0(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_0(x)\phi_k'(x)dx+h_1(t)\sum_{k=1}^{n-1}\gamma_k(t)\int_0^1 w\phi_n(x)\phi_k'(x)dx+\beta\sum_{i=1}^{n-1}\delta_i(t)\int_0^1 w_x\phi_i'(x)dx\\
&+h_0'(t)\int_0^1 w\phi_0(x)dx+h_1'(t)\int_0^1 w\phi_n(x)dx+g_0(t)h_0(t)\int_0^1 w\phi_0(x)\phi_0'(x)dx\\
&+g_0(t)h_1(t)\int_0^1 w\phi_0(x)\phi_n'(x)dx+g_1(t)h_0(t)\int_0^1 w\phi_n(x)\phi_0'(x)dx+g_1(t)h_1(t)\int_0^1 w\phi_n(x)\phi_n'(x)dx\\
&+h_0(t)g_0(t)\int_0^1 w\phi_0(x)\phi_0'(x)dx+h_0(t)g_1(t)\int_0^1 w\phi_0(x)\phi_n'(x)dx+h_1(t)g_0(t)\int_0^1 w\phi_n(x)\phi_0'(x)dx\\
&+h_1(t)g_1(t)\int_0^1 w\phi_n(x)\phi_n'(x)dx+\beta h_0(t)\int_0^1 w_x\phi_0'(x)dx+\beta h_1(t)\int_0^1 w_x\phi_n'(x)dx\\
&-\beta w(1)\phi_n'(1)h_1(t)+\beta w(0)\phi_0'(0)h_0(t)=0
\end{aligned}
$$

Writing equations in matrix form, we get

$$\boxed{\begin{aligned}&\mathbf{A}\boldsymbol{\gamma}' + \mathbf{B}(\boldsymbol{\gamma},\boldsymbol{\gamma}) + g_0(t)\mathbf{B_{top}}\boldsymbol{\gamma} + g_1(t)\mathbf{B_{bottom}}\boldsymbol{\gamma} + g_0(t)\mathbf{B_{left}}\gamma \\ &+ g_1(t)\mathbf{B_{right}}\gamma + \mathbf{C}\boldsymbol{\delta} - \beta\mathbf{D}\boldsymbol{\gamma} + \mathbf{E_1} = 0 \\[2ex] &\mathbf{A}\gamma' + \mathbf{B}(\boldsymbol{\delta},\gamma) + h_0(t)\mathbf{B_{top}}\boldsymbol{\gamma} + h_1(t)\mathbf{B_{bottom}}\boldsymbol{\gamma} + g_0(t)\mathbf{B_{left}}\boldsymbol{\delta} \\ &+ g_1(t)\mathbf{B_{right}}\boldsymbol{\delta} + \mathbf{B}(\gamma,\boldsymbol{\delta}) + g_0(t)\mathbf{B_{top}}\boldsymbol{\delta} + g_1(t)\mathbf{B_{bottom}}\boldsymbol{\delta} \\ &+ h_0(t)\mathbf{B_{left}}\boldsymbol{\gamma} + h_1(t)\mathbf{B_{right}}\boldsymbol{\gamma} + \beta\mathbf{D}\boldsymbol{\delta} + \mathbf{E_2} = 0\end{aligned}}$$

where

A is a n-1 x n-1 square matrix with

$$A_{ji} = \int_0^1 \phi_j(x)\phi_i(x)dx$$

$\mathbf{B}(\ \boldsymbol{\xi},\ \ \boldsymbol{\eta})$ is a n-1 x 1 column matrix with

$B_j(\xi,\eta) = \xi^T B^j \eta$ and B^j is a square matrix with

$$B^j_{ki} = \int_0^1 \phi_j(x)\phi_i(x)\phi'_k(x)dx$$

$\mathbf{B_{top}}$ is a n-1 x n-1 square matrix with

$$B_{top_{ji}} = \int_0^1 \phi_j\phi_i(x)\phi'_0(x)dx$$

$\mathbf{B_{bottom}}$ is a n-1 x n-1 square matrix with

$$B_{bottom_{ji}} = \int_0^1 \phi_j\phi_i(x)\phi'_n(x)dx$$

$\mathbf{B_{left}}$ is a n-1 x n-1 square matrix with

$$B_{left_{jk}} = \int_0^1 \phi_j\phi_0(x)\phi'_k(x)dx$$

$\mathbf{B_{right}}$ is a n-1 x n-1 square matrix with

$$B_{right_{jk}} = \int_0^1 \phi_j\phi_n(x)\phi'_k(x)dx$$

C is a n-1 x n-1 square matrix with

$$C_{ji} = \int_0^1 \phi_j(x)\phi'_i(x)dx$$

D is a n-1 x n-1 square matrix with

$$D_{ji} = \int_0^1 \phi'_j(x)\phi'_i(x)dx$$

$\mathbf{E_1}$ and $\mathbf{E_2}$ are n-1 x 1 column matrices containing the sum of the constants,

$i, j \in 1, 2, .., n-1$

Since $w = \ \phi_j \ \ \forall j \ \in$ 1, 2, 3, 4, ... $n-1$ where n is the number of elements of the mesh.
To solve the following system of differential equations we use the Scipy's inbuilt Runge-Kutta 45 Ode Solver.

4.1 Numerical Solution

4.1.1 Parameters:

$$k = 0.2, \lambda = 0.005, \alpha = 0, \beta = 0.5 \text{ and } x_0 = 10$$
$$b = \sqrt{\alpha + \beta^2}$$

4.1.2 Boundary Conditions:

$$\begin{aligned}
g_0(t) &= \lambda - 2kb\coth[kx_0 - \lambda t] \\
g_1(t) &= \lambda - 2kb\coth[(1+x_0)k - \lambda t] \\
g_0'(t) &= -2k\lambda b\text{csch}^2(kx_0 - \lambda t) \\
g_1'(t) &= -2k\lambda b\text{csch}^2((1+x_0)k - \lambda t)
\end{aligned}$$

$$\begin{aligned}
h_0(t) &= -2k^2 b(b+\beta)\text{csch}^2[kx_0 - \lambda t] \\
h_1(t) &= -2k^2 b(b+\beta)\text{csch}^2[(1+x_0)k - \lambda t] \\
h_0'(t) &= -4k^2\lambda b(b+\beta)\text{csch}^2[kx_0 - \lambda t]\text{coth}[kx_0 - \lambda t] \\
h_1'(t) &= -4k^2\lambda b(b+\beta)\text{csch}^2[(1+x_0)k - \lambda t]\text{coth}[(1+x_0)k - \lambda t] \\
&\text{for } t > 0
\end{aligned}$$

4.1.3 Initial Conditions:

$$\begin{aligned}
u(x,0) &= \lambda - 2kb\coth[(x+x_0)k] \\
v(x,0) &= -2k^2 b(b+\beta)\text{csch}^2[(x+x_0)k]
\end{aligned}$$

4.1.4 Exact Solution:

$$\begin{aligned}
u(x,t) &= \lambda - 2kb\coth[(x+x_0)k - \lambda t] \\
v(x,t) &= -2k^2 b(b+\beta)\text{csch}^2[(x+x_0)k - \lambda t]
\end{aligned}$$

5 Conclusion

In this paper we have discussed and designed a solver for solution of mentioned Linear and Non Linear Heat Equations. Still the main objective of this project, i.e the WBK equations are not solved yet in this paper. The solver for WBK equations is made but due to some error in calculation the WBK equations are giving blown up results. The Python(Jupyter Notebook) code is available on Github at the following link Maths SOP Final in the master branch. The project was initially stored at Maths SOP for temporary files created during the Project.

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