Elementary Toposes

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1 Introduction

An *elementary topos* is a Cartesian closed category with finite limits and a subobject classifier. The goal of this project is to formalise this definition in a suitable way. Namely, in mathlib Cartesian closed categories are defined via *chosen* finite products, which realise finite products as an algebraic operation in a category; for each finite product diagram we *choose* a specific object to serve as the product, rather than only asking for the existence of such a product. Classically, we can still, using the axiom of choice, get chosen limits from mere existence.

The first part of the project will be extending this to define finite chosen limits in general – this will allow us to work more naturally in a topos, and keep with the existing definition of Cartesian closedness by getting an instance for chosen finite products from limits.

The second part will be defining a subobject classifier, which we will again see as a chosen subobject classifier – an object of truth values Ω in the category with a monomorphism true : $1 \to \Omega$ and a unique characteristic morphism $\chi_U : X \to \Omega$ for any monomorphism $U \to X$.

If time permits, we will also define a chosen natural numbers object, allowing us to also consider NNO-topoi.

After the definition, the goal will be to apply it and give instances for some common examples, like finite sets, Type, presheaves, and sheaves.

2 Chosen Finite Limits

The first step is to generalise chosen finite products to general finite limit cones. Note that chosen limits are also limits in the sense of there existing a limit cone. Being a chosen limit is a strictly stronger property.

Definition 1 (Chosen Limit)**.** Let $F : J \to C$ be a functor. A *chosen limit* for F is a specified limit cone for F.

Proposition 2 (Chosen Limit implies Limit)**.** *If we have a chosen limit for a functor F, then F has a limit.*

Proof. An inhabited set is nonempty. □

Under choice, the reverse also holds.

Proposition 3 (Existence of limits implies chosen limit)**.**

Proof. Classically, a nonempty set is inhabited. □

Definition 4 (Chosen Limits of Shape)**.** A category C has *chosen limits of shape J* if every functor $F : J \to C$ has a chosen limit.

Definition 5 (Chosen Finite Limits)**.** A category C has *chosen finite limits* if it has chosen limits of shape J for every finite category J, i. e. for every finite limit cone.

Note that Mathlib already provides a concept of chosen finite product it uses internally for Cartesian closed categories.

Definition 6 (Chosen Finite Products)**.** A category C has *chosen finite products* if it is equipped with:

- A choice of a limit binary fan (product) for any two objects of the category.
- A choice of a terminal object.

More precisely, this consists of chosen limit cones for the pair functor $X \leftarrow \cdot \to Y$ and for the empty functor from the empty category.

Note that our definition is compatible, so that a category with chosen finite limits in particular has an instance of chosen finite products.

Lemma 7 (Chosen Finite Products from Limits)**.** *If a category C has chosen finite limits, then it has chosen finite products.*

Proof. We construct the limit cones for the pair and the terminal object. □

3 Cartesian Closedness

The notion of a Cartesian closed category is already defined in Mathlib, so we just state the definitions.

Definition 8 (Monoidal category)**.** In a *monoidal category $\mathcal{C}$*, we can take the *tensor product* of objects, $X \otimes Y$ and of morphisms $f \otimes g$. The tensor product does not need to be strictly associative on objects, but there is a specified *associator*, $\alpha_{XYZ} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$. There is a tensor unit $\mathbf{1}_{\mathcal{C}}$, with specified left and right *unitor isomorphisms* $\lambda_X : \mathbf{1}_{\mathcal{C}} \otimes X \cong X$ and $\rho_X : X \otimes \mathbf{1}_{\mathcal{C}} \cong X$.

These associators and unitors satisfy the pentagon and triangle identities.

Definition 9 (Monoidal closed category)**.**

Definition 10 (Cartesian closed category)**.** A category $\mathcal{C}$ is *cartesian closed* if it has finite products and every object is exponentiable.

We define this as a monoidal closed category with respect to the cartesian monoidal structure.

4 Subobject Classifiers

Definition 11 (Subobject Classifier)**.**

In a category C with chosen finite limits, a chosen *subobject classifier* consists of:

1. An object Ω
2. A monomorphism $\mathrm{true} : 1 \to \Omega$

3. For every monomorphism $m : U \to X$, a unique assigned morphism $\chi_m : X \to \Omega$ (the *characteristic morphism*) such that the square

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\mathrm{true}"] \\
X \arrow[r, "\chi_m"'] & \Omega
\end{tikzcd}$$

is a pullback.

5 Elementary Toposes

Definition 12 (Elementary Topos)**.** An *elementary topos* is a category $\mathcal{C}$ that

1. has chosen finite limits,
2. is cartesian closed for the inherited chosen finite products,
3. has a subobject classifier.

Definition 13 (Natural Numbers Object)**.** In a topos $\mathcal{C}$, a *natural numbers object* is an object N with morphisms $0 : 1 \to N$ and $s : N \to N$ satisfying the appropriate universal property.

Definition 14 (Topos with NNO)**.** A topos equipped with a natural numbers object.

6 Examples

6.1 Finite Sets

6.2 The Category of Types

6.3 Presheaves

6.4 Sheaves