Elementary Toposes

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1 Introduction

An *elementary topos* is a Cartesian closed category with finite limits and a subobject classifier. The goal of this project is to formalise this definition in a suitable way. Namely, in mathlib Cartesian closed categories are defined via *chosen* finite products, which realise finite products as an algebraic operation in a category; for each finite product diagram we *choose* a specific object to serve as the product, rather than only asking for the existence of such a product. Classically, we can still, using the axiom of choice, get chosen limits from existence.

The first part of the project will be extending this to define finite chosen limits in general – this will allow us to work more naturally in a topos, and keep with the existing definition of Cartesian closedness by getting an instance for chosen finite products from limits.

The second part will be defining a subobject classifier, which we will again see as a chosen subobject classifier – an object of truth values Ω in the category with a monomorphism $\mathrm{true} : 1 \to \Omega$ and a unique characteristic morphism $\chi_U : X \to \Omega$ for any monomorphism $U \to X$.

If time permits, we will also define a chosen natural numbers object, allowing us to also consider NNO-topoi.

After the definition, the goal will be to apply it and give instances for some common examples, like finite sets, Type, presheaves, and sheaves.

2 Chosen Finite Limits

The first step is to generalise chosen finite products to general finite limit cones. Note that chosen limits are also limits in the sense of there existing a limit cone. Being a chosen limit is a strictly stronger property.

Definition 1 (Chosen Limit)**.** Let $F : J \to C$ be a functor. A *chosen limit* for F is a specified limit cone for F.

Proposition 2 (Chosen Limit implies Limit)**.** *If we have a chosen limit for a functor F, then F has a limit.*

Definition 3 (Chosen Limits of Shape)**.** A category C has *chosen limits of shape J* if every functor $F : J \to C$ has a chosen limit.

Definition 4 (Chosen Finite Limits)**.** A category C has *chosen finite limits* if it has chosen limits of shape J for every finite category J, i. e. for every finite limit cone.

Note that Mathlib already provides a concept of chosen finite product it uses internally for Cartesian closed categories.

Definition 5 (Chosen Finite Products)**.** A category C has *chosen finite products* if it is equipped with:

- A choice of a limit binary fan (product) for any two objects of the category.
- A choice of a terminal object.

More precisely, this consists of chosen limit cones for the pair functor $X \leftarrow \cdot \rightarrow Y$ and for the empty functor from the empty category.

Note that our definition is compatible, so that a category with chosen finite limits in particular has an instance of chosen finite products.

Lemma 6 (Chosen Finite Products from Limits)**.** *If a category C has chosen finite limits, then it has chosen finite products.*

Proof. We construct the limit cones for the pair and the terminal object. $\square$

3 Subobject Classifiers

4 Elementary Toposes

5 Examples

5.1 Finite Sets

5.2 The Category of Types

5.3 Presheaves

5.4 Sheaves