Elementary Toposes

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1 Introduction

An *elementary topos* is a Cartesian closed category with finite limits and a subobject classifier. The goal of this project is to formalise this definition in a suitable way. Namely, in mathlib Cartesian closed categories are defined via *chosen* finite products, which realise finite products as an algebraic operation in a category; for each finite product diagram we *choose* a specific object to serve as the product, rather than only asking for the existence of such a product. Classically, we can still, using the axiom of choice, get chosen limits from existence.

The first part of the project will be extending this to define finite chosen limits in general – this will allow us to work more naturally in a topos, and keep with the existing definition of Cartesian closedness by getting an instance for chosen finite products from limits.

The second part will be defining a subobject classifier, which we will again see as a chosen subobject classifier – an object of truth values Ω in the category with a monomorphism $\mathrm{true} : 1 \to \Omega$ and a unique characteristic morphism $\chi_U : X \to \Omega$ for any monomorphism $U \to X$.

If time permits, we will also define a chosen natural numbers object, allowing us to also consider NNO-topoi.

After the definition, the goal will be to apply it and give instances for some common examples, like finite sets, Type, presheaves, and sheaves.

2 Chosen Finite Limits

The first step is to generalise chosen finite products to general finite limit cones.

Definition 1 (Chosen Limit)**.** Let $F : J \to C$ be a functor. A *chosen limit* for F is a specified limit cone for F.

3 Subobject Classifiers

4 Elementary Toposes

5 Natural Numbers Objects (Optional)

6 Examples

6.1 Finite Sets

6.2 The Category of Types

6.3 Presheaves

6.4 Sheaves

7 Further Developments