TorquePaper 2016.3: Sinusoidal Motion Profile (SMP)

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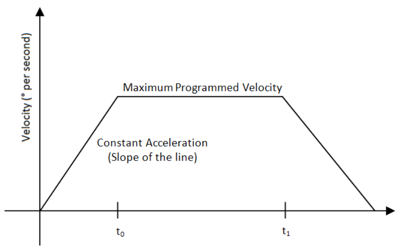
Contents

Chapter 1

A New Profile

1.1 Trapezoidal Motion Profile

The graph for TMP is represented by the graph of velocity vs. time in the following figure:



The problem with TMP is the initial (infinitesimal) amount of acceleration required over the first instant to start the profile. This creates inconsistencies in following the profile as time increases. To fix this, the line used to accelerate to maximum velocity must be replaced with a function that has, ideally, infinite derivatives. One such function is sin. The goal of this paper is to create a new motion profile that compensates for this inconsistency but is still efficient. The only constants defined by the user of SMP are v_{max}, a_{max}, and D, which are the maximum velocity, max acceleration, and distance of the profile of the robot at any time.

1.2 Sine Wave

A sin wave can be modified from it's parent function in order to fit a certain domain and range. In it's most general form, the modified sin wave is defined by:

$$sin(x) \rightarrow \frac{sin(kt - \frac{\pi}{2}) + 1}{c}, k = \frac{\pi}{x_{max}}, c = \frac{2}{y_{max}}$$

From the perspective of a motion profile, the maximum y value will be the maximum velocity ($y_{max} = t_{interval}$) and the maximum x value will be the time the profile takes to complete ($x_{max} = v_{max}$). Therefore, the new sin wave is defined as:

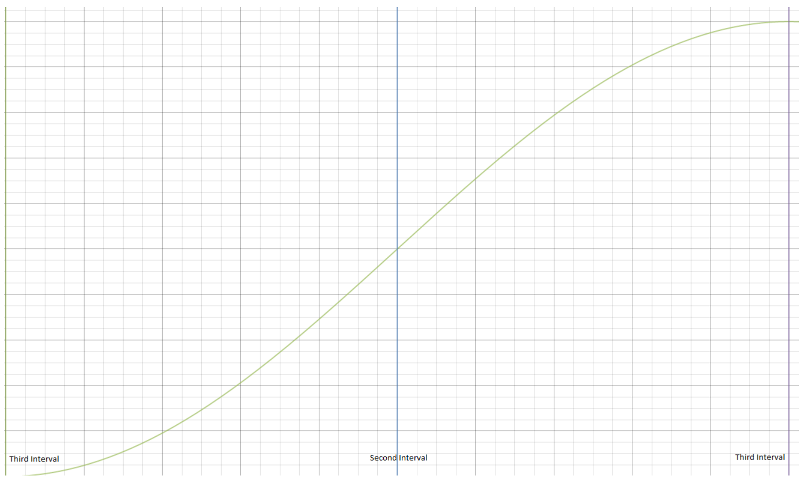
$$velocity = v_{mod}(t) = \frac{sin(kt - \frac{\pi}{2}) + 1}{c}, c = \frac{2}{v_{max}}, k = \frac{\pi}{t_{interval}}$$

The derivative of velocity is acceleration, and thus the derivative of this function is:

$$acceleration = a_{mod}(t) = \frac{k}{c} cos(kt - \frac{\pi}{2}), c = \frac{2}{v_{max}}, k = \frac{\pi}{t_{interval}}$$

1.3 Using the Modified Sin Wave

The acceleration portion of the SMP can be visualized in this image:



The equation for the velocity describes only the acceleration section of the SMP. The entire profile can be defined by the piecewise function defined below, which applies maximum velocity between the acceleration and deceleration intervals.

$$v(t) = \begin{cases} v_{mod}(t) & t < t_3 \\ v_{max} & t_3 \leq t \leq t_4 \\ -v_{mod}(t) & t > t_f \end{cases}$$

Using this piecewise function, the entire profile is shaped like the trapezoidal motion profile, but is much smoother and easier for a robot to follow.

Chapter 2

Why It Works

2.1 Problem with TMP

Chapter 3

Determining Time Intervals

3.1 SMP Time Invervals

The SMP graph is split into 6 important time intervals, referred to as $t_0 ... t_5$ and lastly t_f. The following table describes each interval.

t_1	the initial time of the SMP profile
t_2	the time at which the robot has the highest acceleration
t_3	the time at which the robot stops accelerating and is holding constant velocity
t_4	the time at which the robot starts decelerating
t_5	the time at which the robot has the highest deceleration
t_f	the final time of the profile (aka $t_profile$)

3.2 First Interval

The initial time (t_1) of the profile can be found by grabbing time from the robot's FGPA. However, for the sake of calculating the intervals, the initial time will be 0.

3.3 Second and Third Interval

The acceleration of the robot is highest at t_2, which can be proved by plugging in any numbers to the acceleration equation and determining the maximum of the function on the domain. $t_2 = \frac{t_3}{2}$ because the maximum acceleration occurs at the halfway point between 0 velocity and maximum velocity.

Mathematically, this can be represented by setting $t = t_2 = \frac{t_3}{2}$. Note that because the velocity equation is only defined for the interval between t_1 and t_3, $t_{end} = t_3$, and thus $c = \frac{2}{t_3}$. Plugging these values into $a_mod(t)$, we get the following derivation.

$$a_{max} = \frac{k}{c}cos(\frac{t_3}{2}k - \frac{\pi}{2}), c = \frac{2}{v_{max}}, k = \frac{\pi}{t_3}$$

$$a_{max} = \frac{\frac{\pi}{t_3}}{\frac{2}{v_{max}}}cos(\frac{\pi}{t_3}\frac{t_3}{2} - \frac{\pi}{2})$$

$$a_{max} = \frac{v_{max}\pi}{2t_3}cos(\frac{\pi}{2} - \frac{\pi}{2})$$

$$a_{max} = \frac{v_{max}\pi}{2t_3}$$

$$t_3 = \frac{v_{max}\pi}{2a_{max}}$$

$$t_2 = \frac{t_3}{2} = \frac{v_{max}\pi}{4a_{max}}$$

3.4 Fourth Interval

We start with the assumption that the addition of the integral of the acceleration and deceleration portions plus the integral of the constant velocity portion results in the total distance that the profile travels. This is represented by:

$$D = 2\int_{t_1}^{t_3} v_{mod}(t)dt + \int_{t_3}^{t_4} v_{max}dt$$

This can be simplified through the following derivation, to an equation for the third and fourth intervals.

$$D = 2\int_{t_1}^{t_3} v_{mod}(t)dt + v_{max}(t_4 - t_3)$$

$$D = 2\int_{t_1}^{t_3} \frac{sin(kt - \frac{\pi}{2}) + 1}{c}dt + v_{max}(t_4 - t_3)$$

$$D = 2\left[\int_{t_1}^{t_3} \frac{sin(kt - \frac{\pi}{2})}{c}dt + \int_{t_1}^{t_3} \frac{1}{c}dt\right] + v_{max}(t_4 - t_3)$$

$$D = 2\left[\int_{t_1}^{t_3} \frac{sin(kt - \frac{\pi}{2})}{c}dt + \frac{t_3 - t_1}{c}\right] + v_{max}(t_4 - t_3)$$

$$substitution \rightarrow u = kt - \frac{\pi}{2}, du = kdt, \frac{1}{k}du = dt$$

$$D = 2\left[-\frac{1}{ck}\left[cos(kt - \frac{\pi}{2})\right]_{t_1}^{t_3} + \frac{t_3 - t_1}{c}\right] + v_{max}(t_4 - t_3)$$

$$D = 2\left[-\frac{1}{ck}\left[cos(kt_3 - \frac{\pi}{2}) - cos(kt_1 - \frac{\pi}{2})\right] + \frac{t_3 - t_1}{c}\right] + v_{max}(t_4 - t_3)$$

From here, t_1 is the beginning of the profile and can be set to 0 for the rest of this derivation.

$$D = 2\left[-\frac{1}{ck}cos(kt_3 - \frac{\pi}{2}) + \frac{t_3}{c}\right] + v_{max}(t_4 - t_3)$$

Finally, we must plug in $k = \frac{\pi}{t_3}$ and $c = \frac{2}{v_{max}}$.

$$D = 2\left[-\frac{t_3 v_{max}}{2\pi}cos(\frac{\pi}{2}) + \frac{t_3 v_{max}}{2}\right] + v_{max}(t_4 - t_3)$$

$$D = 2(\frac{t_3 v_{max}}{2}) + v_{max}(t_4 - t_3)$$

$$D = t_3 v_{max} + v_{max}(t_4 - t_3)$$

$$D = v_{max}(t_3 + t_4 - t_3)$$

$$t_4 = \frac{D}{v_{max}}$$

3.5 Final Interval

The total time of the profile, t_f, can be easily defined using symmetry. The acceleration and deceleration portions are exactly equal, which means $t_f = 2t_3 + (t_4 - t_3) = t_4 + t_3$.

$$t_f = t_4 + t_3$$

$$t_f = \frac{D}{v_{max}} + \frac{v_{max}\pi}{2a_{max}}$$

$$t_f = \frac{2a_{max}D}{2a_{max}v_{max}} + \frac{(v_{max})^2\pi}{2v_{max}a_{max}}$$

$$t_f = \frac{2a_{max}D + (v_{max})^2\pi}{2a_{max}v_{max}}$$

3.6 Fifth Interval

The fifth interval is easily defined by the equation $t_f - t_5 = t_2$, or $t_5 = t_f - t_2$. This requires that t_f is defined.

$$t_5 = t_f - t_2$$

$$t_5 = \frac{2a_{max}D + (v_{max})^2\pi}{2a_{max}v_{max}} - \frac{v_{max}\pi}{4a_{max}}$$

$$t_5 = \frac{4a_{max}D + (v_{max})^2\pi}{4a_{max}v_{max}}$$

Chapter 4

Time Interval Calculation Alternatives

4.1 Second Interval Alt

4.2 Fifth Interval Alt