

**Question 1**

Given the image coordinates of three vertices of a square poster with unit length:  $\mathbf{X}_1 = [0 \ 0 \ 0 \ 1]^T$ ,  $\mathbf{X}_2 = [1 \ 0 \ 0 \ 1]^T$  and  $\mathbf{X}_3 = [0 \ 1 \ 0 \ 1]^T$  are:  $\mathbf{x}_1 = [0.0 \ 0.0 \ 1.0]^T$ ,  $\mathbf{x}_2 = [99.50 \ 14.98 \ 1.0]^T$  and  $\mathbf{x}_3 = [-9.98 \ 149.25 \ 1.0]^T$ .

- a) Show that there exists a homography  $H$  that maps the vertices  $\mathbf{X}_i$  of the square poster in the 3D space to the image space  $\mathbf{x}_i$ , i.e.,  $H: \mathbf{X}_i \mapsto \mathbf{x}_i$ .
- b) We further know that the principal points  $p_x = p_y = 0$ , the skew parameter  $s = 0$ , and the extrinsic  $T_{cw} \in SE(3)$  (maps a point in the world to camera frame) consists of a rotation angle  $\theta$  around the z-axis and a translation  $t_z$  along the z-axis. Find the homography  $H$  that maps the vertices of the square poster in the 3D space to the image space.
- c) Using the homography  $H$ , find the focal lengths  $(f_x, f_y)$ , rotation angle  $\theta$  around the z-axis and the image of the absolute conic (IAC)  $\omega$ .
- d) Find the circular points on the image. Explain your answer.

**Show all your workings clearly.**

**(20 marks)**

## Solution

- a) The projection equation is:  $\mathbf{x} = K[R \quad \mathbf{t}]\mathbf{X}$ . Let us further write  $R = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3]$ . We assign  $xy$ -plane of the world frame to be on the plane of the poster and the origin to be on the first vertex. Thus, the 3D points can be written as:  $\mathbf{X} = [X \quad Y \quad 0 \quad 1]^\top$ . Putting back into the project equation, we get:

$$\mathbf{x} = K[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3 \quad \mathbf{t}][X \quad Y \quad 0 \quad 1]^\top \Rightarrow \mathbf{x} = K[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}][X \quad Y \quad 1]^\top, \text{ where}$$

$$H = K[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}] \text{ since we now have a } \mathbb{P}^2 \mapsto \mathbb{P}^2 \text{ mapping.}$$

(shown)

- b) Since  $p_x = p_y = s = 0$ , the intrinsic is given by:

$$K = \begin{bmatrix} f_x & s & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since there is only a rotation around the  $z$ -axis and a translation along the  $z$ -axis for  $T_{cw}$ , we have:

$$[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}] = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & t_z \end{bmatrix}.$$

Thus, the homography is given by:

$$H = K[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}] = \begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & t_z \end{bmatrix} = \begin{bmatrix} f_x r_{11} & f_x r_{12} & 0 \\ f_y r_{21} & f_y r_{22} & 0 \\ 0 & 0 & t_z \end{bmatrix}, \text{ which we}$$

represent as:

$$H = \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{bmatrix}. \text{ We now have 4 equations and 4 unknowns:}$$

$$\mathbf{x}_i = H\mathbf{X}_i$$

$$[x \quad y \quad 1.0]^\top = \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{bmatrix} [X \quad Y \quad 1]^\top$$

$$\Rightarrow x = Xh_{11} + Yh_{12} \text{ and } y = Xh_{21} + Yh_{22}.$$

Using  $\mathbf{x}_1 \leftrightarrow \mathbf{X}_1$ , we have:

$$h_{33} = 1.00$$

Using  $\mathbf{x}_2 \leftrightarrow \mathbf{X}_2$ , we have:

$$99.50 = (1)h_{11} + (0)h_{12} \Rightarrow h_{11} = 99.50$$

$$14.98 = (1)h_{21} + (0)h_{22} \Rightarrow h_{21} = 14.98$$

Using  $\mathbf{x}_3 \leftrightarrow \mathbf{X}_3$ , we have:

$$-9.98 = (0)h_{11} + (1)h_{12} \Rightarrow h_{12} = -9.98$$

$$149.25 = (0)h_{21} + (1)h_{22} \Rightarrow h_{22} = 149.25$$

Therefore, we get:

$$H = \begin{bmatrix} f_x c\theta & -f_x s\theta & 0 \\ f_y c\theta & f_y s\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 99.50 & -9.98 & 0 \\ 14.98 & 149.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

c)  $H = \begin{bmatrix} f_x c\theta & -f_x s\theta & 0 \\ f_y c\theta & f_y s\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 99.50 & -9.98 & 0 \\ 14.98 & 149.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , we get:

$$f_x c\theta = 99.50 \text{ and } f_x s\theta = 9.98 \Rightarrow \tan \theta = \frac{99.50}{9.98} \Rightarrow \theta = \mathbf{0.1 \text{ rad.}}$$

$$f_x = \frac{99.50}{\cos 0.1} = \mathbf{100} \text{ and } f_y = \frac{14.98}{\sin 0.1} = \mathbf{150}.$$

$$\text{Image of absolute conic } \omega = (KK^T)^{-1} = \begin{bmatrix} \frac{1}{f_x^2} & 0 & 0 \\ 0 & \frac{1}{f_y^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{100^2} & 0 & 0 \\ 0 & \frac{1}{150^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

d) Circular points are:  $[1 \quad \pm i \quad 0]^T$ . Since any plane intersects the plane at infinity at the circular points, the images of the circular points are given by:

$$H[1 \quad \pm i \quad 0]^T \Rightarrow \mathbf{h}_1 \pm i\mathbf{h}_2 = \begin{bmatrix} 99.50 \\ 14.98 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} -9.98 \\ 149.25 \\ 0 \end{bmatrix}.$$

## **Question 2**

- a) Given four collinear points in  $\mathbb{P}^3$ :

$$\mathbf{X}_1 = [2.00 \ 1.00 \ 2.00 \ 1.00]^\top, \quad \mathbf{X}_2 = [3.00 \ 4.00 \ 4.0000 \ 1.00]^\top, \\ \mathbf{X}_3 = [5.00 \ 10.00 \ 8.00 \ 1.00]^\top, \quad \mathbf{X}_4 = [2.60 \ 2.80 \ 3.20 \ 1.00]^\top,$$

and the projections of  $\mathbf{X}_1$ ,  $\mathbf{X}_2$  and  $\mathbf{X}_3$  on an image in  $\mathbb{P}^2$ :

$$\mathbf{x}_1 = [133.89 \ 124.25 \ 1.00]^\top, \quad \mathbf{x}_2 = [104.41 \ 136.56 \ 1.00]^\top, \\ \mathbf{x}_3 = [86.63 \ 143.99 \ 1.00]^\top,$$

find the distance between the projection of  $\mathbf{X}_4$  and  $\mathbf{x}_1$ .

- b) A camera that has undergone a translation along the x-axis, and observes a point correspondence in two image frames:

$$\mathbf{x}_1 = [50 \ 100 \ 1]^\top \leftrightarrow \mathbf{x}'_1 = [100 \ 150 \ 1]^\top, \\ \mathbf{x}_2 = [450 \ 150 \ 1]^\top \leftrightarrow \mathbf{x}'_2 = [600 \ 300 \ 1]^\top.$$

We further note that the camera intrinsic has principal points  $p_x = p_y = 0$  and skew parameter  $s = 0$ . Find the fundamental matrix  $F$  that relates that two image frames.

**Show all your workings clearly.**

**(20 marks)**

### Solution

a)

Turn the collinear  $\mathbb{P}^3$  points into  $\mathbb{P}^1$ :

$$\begin{aligned}\mathbf{P}_1 &= [\text{dist}(\mathbf{X}_1, \mathbf{X}_1), 1]^\top = [0, 1]^\top, & \mathbf{P}_2 &= [\text{dist}(\mathbf{X}_2, \mathbf{X}_1), 1]^\top = [3.74, 1]^\top, \\ \mathbf{P}_3 &= [\text{dist}(\mathbf{X}_3, \mathbf{X}_1), 1]^\top = [11.23, 1]^\top, & \mathbf{P}_4 &= [\text{dist}(\mathbf{X}_4, \mathbf{X}_1), 1]^\top = [2.25, 1]^\top.\end{aligned}$$

Turn the collinear  $\mathbb{P}^2$  points into  $\mathbb{P}^1$ :

$$\begin{aligned}\mathbf{p}_1 &= [\text{dist}(\mathbf{x}_1, \mathbf{x}_1), 1]^\top = [0, 1]^\top, & \mathbf{p}_2 &= [\text{dist}(\mathbf{x}_2, \mathbf{x}_1), 1]^\top = [31.94, 1]^\top, \\ \mathbf{p}_3 &= [\text{dist}(\mathbf{x}_3, \mathbf{x}_1), 1]^\top = [51.21, 1]^\top.\end{aligned}$$

$$\text{CrossRatio} = \frac{\det[\mathbf{P}_1 \ \mathbf{P}_2] \det[\mathbf{P}_3 \ \mathbf{P}_4]}{\det[\mathbf{P}_1 \ \mathbf{P}_3] \det[\mathbf{P}_2 \ \mathbf{P}_4]} = \frac{(-3.7417)(8.9800)}{(-11.2250)(1.4967)} = 2.00$$

$$\det[\mathbf{p}_3 \ \mathbf{p}_4] = p_3 - p_4 = 51.21 - p_4$$

$$\det[\mathbf{p}_2 \ \mathbf{p}_4] = p_2 - p_4 = 31.94 - p_4$$

$$\text{CrossRatio} = \frac{\det[\mathbf{p}_1 \ \mathbf{p}_2] \det[\mathbf{p}_3 \ \mathbf{p}_4]}{\det[\mathbf{p}_1 \ \mathbf{p}_3] \det[\mathbf{p}_2 \ \mathbf{p}_4]} = \frac{(-31.94)(51.21 - p_4)}{(-51.21)(31.94 - p_4)} = 2.00$$

$$\Rightarrow (31.94)(51.21 - p_4) = (102.4200)(31.94 - p_4)$$

$$\Rightarrow 1635.6 - 31.94p_4 = 3271.3 - 102.42p_4$$

$$\Rightarrow 70.4800p_4 = 1635.7$$

$$\Rightarrow p_4 = 23.21.$$

The distance between the projection of  $\mathbf{X}_4$  and  $\mathbf{x}_1$  is  $p_4 = 23.21$ .

b)

$$\mathbf{K}^{-1} = \begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix},$$

$$\Rightarrow [\mathbf{t}]_{\times} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix}$$

$$\mathbf{F} = \mathbf{K}^{-\top} [\mathbf{t}]_{\times} \mathbf{R} \mathbf{K}^{-1} = \begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x/f_y \\ 0 & t_x & 0 \end{bmatrix} \begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x/f_y \\ 0 & t_x/f_y & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{bmatrix}.$$

Using the epipolar geometry, we have:

$$\mathbf{x}'^{\top} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{bmatrix} \mathbf{x} = 0$$

$$[100 \quad 150 \quad 1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \\ 1 \end{bmatrix} = [0 \quad b \quad -150b] \begin{bmatrix} 50 \\ 100 \\ 1 \end{bmatrix}$$

$$= 100b - 150b = -50b = 0$$

$$\Rightarrow b = 0$$

$$\Rightarrow \mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ i.e. a degenerate case where } \mathbf{F} \text{ does not exist.}$$