School of Computing
National University of Singapore
CS4277/CS5477: 3D Computer Vision
Semester 2, AY 2022/23

Exercise 1

Question 1

Show that the homogenous coordinates $[2.5 \quad 3.5 \quad 2]^\top$, $[5.0 \quad 7.0 \quad 4.0]^\top$ and $[3.90625 \quad 5.46875 \quad 3.125]^\top$ are the same point.

Solution:

Convert the given homogenous coordinates into Cartesian coordinates, i.e.

$[x \quad y \quad k]^\top \rightarrow [\frac{x}{k}, \quad \frac{y}{k}]$. We have:

$$[2.5 \quad 3.5 \quad 2]^\top \rightarrow \left[\frac{2.5}{2}, \quad \frac{3.5}{2}\right] = [1.25, \quad 1.75]$$

$$[5.0 \quad 7.0 \quad 4.0]^\top \rightarrow \left[\frac{5.0}{4.0}, \quad \frac{7.0}{4.0}\right] = [1.25, \quad 1.75]$$

$$[3.90625 \quad 5.46875 \quad 3.125]^\top \rightarrow \left[\frac{3.90625}{3.125}, \quad \frac{5.46875}{3.125}\right] = [1.25, \quad 1.75]$$

All the homogeneous coordinates convert to the same Cartesian coordinate. Therefore, they are the same point.

Question 2

Show that the following lines: $[6.2 \quad 2.3 \quad 1.2]^\top$, $[12.4 \quad 4.6 \quad 2.4]^\top$ and $[18.6 \quad 6.9 \quad 3.6]^\top$ in $\mathbb{P}^2$ represent the same line.

Solution:

A line in homogeneous coordinates can be converted to Cartesian coordinates as follow:

$$[a \quad b \quad c]^\top \rightarrow ax + by + c = 0.$$

We get:

$$[6.2 \quad 2.3 \quad 1.2]^\top \rightarrow 6.2x + 2.3y + 1.2 = 0$$

$[12.4 \quad 4.6 \quad 2.4]^\top \rightarrow 12.4x + 4.6y + 2.4 = 0$

$\Rightarrow 2.0(6.2x + 2.3y + 1.2) = 0$

$\Rightarrow 6.2x + 2.3y + 1.2 = 0$

$[18.6 \quad 6.9 \quad 3.6]^\top \rightarrow 18.6x + 6.9y + 3.6 = 0$

$\Rightarrow 3.0(6.2x + 2.3y + 1.2) = 0$

$\Rightarrow 6.2x + 2.3y + 1.2 = 0$

All three lines convert to the same Cartesian line equation. Therefore, they are the same line.

<u>Question 3</u>

a) Find the line in Cartesian space that passes the two points: $[5.1 \quad 8.4]^\top$ and $[6.3 \quad 10.4]^\top$ in $\mathbb{R}^2$.
b) Find the same line in projective space by first converting the two points into homogenous coordinates.

Solution:

a) Cartesian line equation is given by: $ax + by + c = 0 \Rightarrow \frac{a}{c}x + \frac{b}{c}y + 1 = 0.$

Let let $\alpha = \frac{a}{c}$ and $\beta = \frac{b}{c}$ and putting $[5.1 \quad 8.4]^\top$ into the line equation, we get:

$5.1\alpha \; + 8.4\beta + 1 = 0 - (1)$

Putting $[6.3 \quad 10.4]^\top$ into the line equation, we get: $6.3\alpha + 10.4\beta + 1 = 0 - (2)$

From (1) and (2), we get:

$$\begin{bmatrix} 5.1 & 8.4 \\ 6.3 & 10.4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 5.1 & 8.4 \\ 6.3 & 10.4 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -16.6667 \\ 10.0 \end{bmatrix}$$

Since there is only two degree-of-freedom in a line, it means the ratio is important, i.e. $\{a: b: c\}$. We can rewrite the ratio as $\left\{\frac{a}{c}: \frac{b}{c}: 1\right\} = \{\alpha: \beta: 1\}$.

Therefore, the line equation is given by: $\alpha x + \beta y + 1 = 0 \Rightarrow \mathbf{-16.6667x + 10.0y + 1 = 0}$

b) Convert the two given points into homogeneous coordinates, i.e., $[5.1 \quad 8.4 \quad 1.0]^\top$ and $[6.3 \quad 10.4 \quad 1.0]^\top$.

Taking the cross-product, we get the equation of the line in homogenous coordinates:

$[5.1 \quad 8.4 \quad 1.0]^\top \times [6.3 \quad 10.4 \quad 1.0]^\top = [-2.0 \quad 1.2 \quad 0.12]^\top = 0.12[-16.6667 \quad 10.0 \quad 1.0]^\top,$

which is obviously the same line as from part (a).

<u>Question 4</u>

How many degrees of freedom are there for a point on a line in $\mathbb{P}^2$? Find the family of points that lie on the line $\mathbf{l} = [2.6 \quad 8.9 \quad 1.2]^\top \in \mathbb{P}^2$.

Solution:

In general, a point in $\mathbb{P}^2$ has 2 degree-of-freedom. The point loses 1 degree-of-freedom on a line, i.e. it's remaining **1 degree-of-freedom**. This means that the family of points on a line is **parameterized by 1 parameter**.

Let $\mathbf{x} = [x, y, 1]^\top$be the homogenous coordinate of the point. Using the point-line incidence relation: $\mathbf{x}^\top \mathbf{l} = 0$, we get

$$[x, y, 1][2.6 \quad 8.9 \quad 1.2]^\top = 0 \Rightarrow 2.6x + 8.9y + 1.2 = 0.$$

Making x the subject, we get:

$$x = -3.426y - 0.462.$$

Putting x back into the point coordinates, we get:

$$\mathbf{x_{1D}} = [\boldsymbol{-3.426y - 0.462} \quad \boldsymbol{y} \quad \mathbf{1}]^\top,$$

which has 1 degree-of-freedom parameterized by y.

<u>Question 5</u>

Find the family of parallel lines that pass through the ideal point $\mathbf{x} = [2.0 \quad 3.0 \quad 0]^\top \in \mathbb{P}^2$.

Solution:

A line in $\mathbb{P}^2$ has 2 degree-of-freedom.

Intuition: Think of it this way. A line in $\mathbb{P}^2$ can be defined by 2 points, where each point gives 1 degree-of-freedom.

Fixing one point on the line, i.e. fixing this point to be the given ideal point $\mathbf{x} = [2.0 \quad 3.0 \quad 0]^\top \in \mathbb{P}^2$ reduces the line to 1 degree-of-freedom. Thus, the family of parallel lines that intersect at $\mathbf{x} = [2.0 \quad 3.0 \quad 0]^\top \in \mathbb{P}^2$ can be parameterized by 1 parameter. s

Two parallel lines have the same gradient and thus can be expressed as: $\mathbf{l} = [a \quad b \quad c]^\top$ and $\mathbf{l}' = [a \quad b \quad c']^\top$, respectively. Using the intersection relation of two lines, we have:

$\mathbf{l} \times \mathbf{l}' = \mathbf{x}$

$\Rightarrow \mathbf{x} = [a \quad b \quad c]^\top \times [a \quad b \quad c']^\top = (c' - c)[b \quad -a \quad 0]^\top$, where the scale $(c' - c)$ can be ignored. Thus, we get:

$\mathbf{x} = [b \quad -a \quad 0]^\top = [2.0 \quad 3.0 \quad 0]^\top \Rightarrow a = -3.0, b = 2.0.$

We can then obtain the family of parallel lines that intersect at the ideal point $\mathbf{x} = [2.0 \quad 3.0 \quad 0]^\top \in \mathbb{P}^2$ as:

$$\mathbf{l} = [-3.0 \quad 2.0 \quad c]^\top,$$

which is parameterized by c.

<u>Question 6</u>

Find the angle between two lines $\mathbf{l}$ and $\mathbf{l}'$ which intersect at the point $\mathbf{x}_1 = [1.0 \quad 1.0 \quad 1.0]^\top \in \mathbb{P}^2$, and the points $\mathbf{x}_2 = [2.0 \quad 2.0 \quad 1.0]^\top \in \mathbb{P}^2$ and $\mathbf{x}_3 = [0.0 \quad 1.0 \quad 1.0]^\top \in \mathbb{P}^2$ lie on $\mathbf{l}$ and $\mathbf{l}'$, respectively.

Solution:

$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2 = [-1.0 \quad 1.0 \quad 0.0]^\top$; $\mathbf{l}' = \mathbf{x}_1 \times \mathbf{x}_3 = [0 \quad -1.0 \quad 1.0]^\top$;

The angle between the two lines is given by:

$$\cos\theta = \frac{\mathbf{l}^\top C_\infty^* \mathbf{l}'}{\sqrt{(\mathbf{l}^\top C_\infty^* \mathbf{l})(\mathbf{l}'^\top C_\infty^* \mathbf{l}')}} = -\frac{1}{\sqrt{2 * 1}} \Longrightarrow \boldsymbol{\theta = 0.785\mathbf{rad}}$$

where

$$\mathbf{l}^\top C_\infty^* \mathbf{l}' = [-1.0 \quad 1.0 \quad 0.0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} [0 \quad -1.0 \quad 1.0]^\top = -1,$$

$$\mathbf{l}^\top C_\infty^* \mathbf{l} = [-1.0 \quad 1.0 \quad 0.0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} [-1.0 \quad 1.0 \quad 0.0]^\top = 2,$$

$$\mathbf{l}^\top C_\infty^* \mathbf{l} = [0 \quad -1.0 \quad 1.0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} [0 \quad -1.0 \quad 1.0]^\top = 1.$$

Question 7

Find the equation of the conic in $\mathbb{P}^2$ that passes through the following five points: $(x, y) = (1.0, 5), (1.0, 11.0), (0.0, 5.4019), (0.0, 10.5981), (2.0, 5.4019), (2.0, 10.5981)$.

Solution

The equation of a conic in $\mathbb{R}^2$ is given by:

$$a{x_i}^2 + bx_iy_i + c{y_i}^2 + dx_i + ey_i + f = 0.$$

The equation can be rewritten into the following form given five points: $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ and (x_5, y_5)

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

, where $\mathbf{c} = [a \quad b \quad c \quad d \quad e \quad f]^\top$.

We get:

$$\begin{bmatrix} 1.0 & 5.0 & 25.0 & 1.0 & 5.0 & 1.0 \\ 1.0 & 11.0 & 121.0 & 1.0 & 11.0 & 1.0 \\ 0.0 & 0.0 & 29.1805 & 0.0 & 5.4019 & 1.0 \\ 0.0 & 0.0 & 112.3197 & 0.0 & 10.5981 & 1.0 \\ 4.0 & 10.8038 & 29.1805 & 2.0 & 5.4019 & 1.0 \end{bmatrix} \mathbf{c} = \mathbf{0},$$ which we write as $\mathrm{A}\mathbf{c} = \mathbf{0}.$

Taking the singular value decomposition of A, we get:

$\mathrm{A} = \mathrm{U\Sigma V^\top}$, where

$\mathrm{U} =$

-0.14811	-0.30433	-0.42362	0.42922	0.722336
-0.70411	-0.24157	0.507191	0.393973	-0.18281
-0.17049	0.094467	-0.68485	0.353342	-0.60675
-0.65021	0.510466	-0.21408	-0.46545	0.232767
-0.17451	-0.76127	-0.22013	-0.56486	-0.14997

$\Sigma =$

173.1923	0	0	0	0	0
0	13.34974	0	0	0	0
0	0	4.095998	0	0	0

0	0	0	1.576826	0	0
0	0	0	0	0.249137	0

V =

-0.00895	-0.26899	-0.19457	-0.91084	-0.24217	0.037709
-0.05988	-0.92912	0.264343	0.239204	-0.07784	1.23E-16
-0.99311	0.077897	0.079597	-0.03156	0.007531	0.01676
-0.00694	-0.15494	-0.08708	-0.19439	0.961709	-0.07542
-0.09954	-0.1776	-0.90246	0.25644	-0.08024	-0.26817
-0.01067	-0.0526	-0.25281	0.092737	0.062548	0.959529

The solution of c is the right orthogonal vector that corresponds to the least singular value, i.e.

$\mathrm{c} = [0.0377 \quad 0.0000 \quad 0.0168 \quad -0.0754 \quad -0.2682 \quad 0.9595]^\top$

The conics equation in $\mathbb{P}^2$ is given by:

$\mathbf{x}^\top \mathtt{C}\mathbf{x} = 0$, where

$$\mathtt{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}.$$

We get: $\mathrm{C} = \begin{bmatrix} 0.0377 & 0.0000 & -0.0377 \\ 0.0000 & 0.0168 & -0.1341 \\ -0.0377 & -0.1341 & 0.9595 \end{bmatrix}.$

<u>Question 8</u>

Find the two lines that $\mathbf{l}$ and $\mathbf{m}$ that forms the degenerate conics:

$$\mathrm{C} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

<u>Solution</u>

The degenerate conics formed by two lines $\mathbf{l}$ and $\mathbf{m}$ is given by:

$\mathrm{C} = \mathbf{l}\mathbf{m}^\top + \mathbf{m}\mathbf{l}^\top = [l_1 \quad l_2 \quad l_3]^\top [m_1 \quad m_2 \quad m_3] + [m_1 \quad m_2 \quad m_3]^\top [l_1 \quad l_2 \quad l_3]$

$$\Rightarrow \mathrm{C} = \begin{bmatrix} 2 * l_1 * m_1 & l_1 * m_2 + l_2 * m_1 & l_1 * m_3 + l_3 * m_1 \\ l_1 * m_2 + l_2 * m_1 & 2 * l_2 * m_2 & l_2 * m_3 + l_3 * m_2 \\ l_1 * m_3 + l_3 * m_1 & l_2 * m_3 + l_3 * m_2 & 2 * l_3 * m_3 \end{bmatrix}$$

$2 * l_1 * m_1 = 0$

$l_1 * m_2 + l_2 * m_1 = 1$

$l_1 * m_3 + l_3 * m_1 = -1$

$2 * l_2 * m_2 = -2$

$l_2 * m_3 + l_3 * m_2 = 1$

$2 * l_3 * m_3 = 0$

We have six equations and six unknowns $l_1, l_2, l_3, m_1, m_2, m_3$, we get:

$\mathbf{l} = [-1 \quad 1 \quad 0]^\top$,

$\mathbf{m} = [0 \quad -1 \quad 1]^\top$.

Question 9

Given three points:

$\mathbf{x}_1 = [3.4072 \quad -2.4572 \quad 1.0000]^\top$, $\mathbf{x}_2 = [3.1428 \quad -2.5648 \quad 1.0000]^\top$,

$\mathbf{x}_3 = [4.2926 \quad -1.5002 \quad 1.0000]^\top$, and the

correspondences of the first two points after a similarity transformation:

$\mathbf{x}_1' = [13.6263 \quad -0.9310 \quad 1.0000]^\top$, $\mathbf{x}_2' = [12.9915 \quad -1.5055 \quad 1.0000]^\top$. Find $\mathbf{x}_4'$ after the transformation.

Solution

Similarity transformation is given by:

$$\mathbf{x}' = \mathrm{H}_s\mathbf{x} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ -s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x} \Rightarrow \quad \mathbf{x}' \times \mathrm{H}_s\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} x_2' - t_y - s * x_2 * \cos\theta - s * x_1 * \sin\theta \\ t_x - x_1' + s * x_1 * \cos\theta - s * x_2 * \sin\theta \\ x_1' * (t_y + s * x_2 * \cos\theta + s * x_1 * \sin\theta) - x_2' * (t_x + s * x_1 * \cos\theta - s * x_2 * \sin\theta) \end{bmatrix} = \mathbf{0}$$

where the third constraint is redundant. We get:

$-0.9310 - t_y - s * (-2.4572) * \cos\theta - s * (3.4072) * \sin\theta = 0$ -- (1)

$t_x - 13.6263 + s*(3.4072)*\cos\theta - s*(-2.4572)*\sin\theta = 0$ -- (2)

$-1.5055 - t_y - s*(-2.5648)*\cos\theta - s*(3.1428)*\sin\theta = 0$ -- (3)

$t_x - 12.9915 + s*(3.1428)*\cos\theta - s*(-2.5648)*\sin\theta = 0$ -- (4)

$\sin^2\theta + \cos^2\theta = 1$ -- (5)

Solving for the unknowns, we get:

$\theta = 0.3491, s = 3.0, t_x = 1.5, ty = 2.5.$

Thus, we get:

$$\mathbf{x}_3' = \mathrm{H}_s\mathbf{x}_3 = \begin{bmatrix} 3.0\cos(0.3491) & -3.0\sin(0.3491) & 1.5 \\ -3.0\sin(0.3491) & 3.0\cos(0.3491) & 2.5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4.2926 \\ -1.5002 \\ 1.0000 \end{bmatrix} = \begin{bmatrix} 15.1405 \\ 2.6754 \\ 1.0000 \end{bmatrix}.$$

Question 10

For $\bar{\mathbf{x}}' = \mathrm{H}_{2\times 2}\,\bar{\mathbf{x}} \in \mathbb{P}^1$, prove that:

$$\mathrm{Cross}(\bar{\mathbf{x}}_1', \bar{\mathbf{x}}_2', \bar{\mathbf{x}}_3', \bar{\mathbf{x}}_4') = \mathrm{Cross}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4).$$

Let

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \qquad \bar{\mathbf{x}}_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, \qquad \bar{\mathbf{x}}_i' = \mathbf{H}\bar{\mathbf{x}}_i = \begin{bmatrix} h_{11}x_{i1} + h_{12}x_{i2} \\ h_{21}x_{i1} + h_{22}x_{i2} \end{bmatrix},$$

$$Cross(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) = \frac{|\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2||\bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4|}{|\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_3||\bar{\mathbf{x}}_2, \bar{\mathbf{x}}_4|}$$

We get

$$\begin{aligned} |\bar{\mathbf{x}}_i', \bar{\mathbf{x}}_j'| &= (h_{11}x_{i1} + h_{12}x_{i2})(h_{21}x_{j1} + h_{22}x_{j2}) \\ &\quad - (h_{11}x_{j1} + h_{12}x_{j2})(h_{21}x_{i1} + h_{22}x_{i2}) \\ &= (x_{i1}x_{j2} - x_{i2}x_{j1})(h_{11}h_{22} - h_{12}h_{21}) \\ &= |\bar{\mathbf{x}}_i, \bar{\mathbf{x}}_j| det(\mathbf{H}) \end{aligned}$$

And thus

$$\begin{aligned} Cross(\bar{\mathbf{x}}_1', \bar{\mathbf{x}}_2', \bar{\mathbf{x}}_3', \bar{\mathbf{x}}_4') &= \frac{|\bar{\mathbf{x}}_1', \bar{\mathbf{x}}_2'||\bar{\mathbf{x}}_3', \bar{\mathbf{x}}_4'|}{|\bar{\mathbf{x}}_1', \bar{\mathbf{x}}_3'||\bar{\mathbf{x}}_2', \bar{\mathbf{x}}_4'|} \\ &= \frac{|\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2| det(\mathbf{H}) |\bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4| det(\mathbf{H})}{|\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_3| det(\mathbf{H}) |\bar{\mathbf{x}}_2, \bar{\mathbf{x}}_4| det(\mathbf{H})} \\ &= \frac{|\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2||\bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4|}{|\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_3||\bar{\mathbf{x}}_2, \bar{\mathbf{x}}_4|} \\ &= Cross(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) \end{aligned}$$

Question 11

Find the equation of the plane formed by the three points:

$\mathbf{X}_1 = [1.2000 \quad 2.2000 \quad 3.1000 \quad 1.0000]^\top$,

$\mathbf{X}_2 = [5.2000 \quad 2.6000 \quad 4.3000 \quad 1.0000]^\top$,

$\mathbf{X}_3 = [-3.2000 \quad 6.3000 \quad -3.1000 \quad 1.0000]^\top$.

Solution

The three points must be incident to the plane, i.e. $\mathbf{X}^\top \boldsymbol{\pi} = \mathbf{0}$. We get:

$$\begin{bmatrix} 1.2000 & 2.2000 & 3.1000 & 1.000 \\ 5.2000 & 2.6000 & 4.3000 & 1.000 \\ -3.2000 & 6.3000 & -3.1000 & 1.000 \end{bmatrix} \boldsymbol{\pi} = \mathbf{0}$$

Taking SVD of $\mathbf{X}^\top$, the solution of $\boldsymbol{\pi}$ is the right orthogonal vector that corresponds to the smallest eigenvalue, i.e. $\boldsymbol{\pi} = [0.0783 \quad -0.2066 \quad -0.1922 \quad 0.9562]^\top$.

Question 12

Find the family of points that lie on the plane $\boldsymbol{\pi} = [2.1 \quad 3.2 \quad 5.1 \quad 4.0]^\top$.

Solution

The points $\mathbf{X}$ on the plane $\boldsymbol{\pi} = [a, b, c, d]^\top$ may be written as: $\mathbf{X} = \mathrm{M}\mathbf{x}$. This means the following point-plane incidence relation must hold true, i.e. $\boldsymbol{\pi}^\top\mathbf{X} = \boldsymbol{\pi}^\top\mathrm{M}\mathbf{x} = \mathbf{0}$. Thus, we have $\boldsymbol{\pi}^\top\mathrm{M} = \mathbf{0}$. By inspection, we get:

$$\mathrm{M}^\top = [\mathbf{p} \mid I_{3\times3}], \qquad \text{where } \mathbf{p} = \left(-\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a}\right)^\top.$$

As a result, the family of points on the plane is given by:

$$\mathbf{X} = \begin{bmatrix} -\frac{3.2}{2.1} & 1 & 0 & 0 \\ -\frac{5.1}{2.1} & 0 & 1 & 0 \\ -\frac{4.0}{2.1} & 0 & 0 & 1 \end{bmatrix}^\top \mathbf{x}.$$

Question 13

Given three colinear points on the image: $\mathbf{x}_1 = [20 \quad 304 \quad 1]^\top$, $\mathbf{x}_2 = [40 \quad 508 \quad 1]^\top$ and $\mathbf{x}_2 = [80 \quad 916 \quad 1]^\top$, and the distance ratio of the corresponding 3D points is $d(\mathbf{X}_1, \mathbf{X}_2) : d(\mathbf{X}_2, \mathbf{X}_3) = 16.4838 : 32.1527$. Find the vanishing of the line formed by $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

Solution

The distance ratio of the image points is: $d(\mathbf{x}_1, \mathbf{x}_2) : d(\mathbf{x}_2, \mathbf{x}_3) = 204.9780 : 409.9561$.

Convert into $\mathbb{P}^1$, we get:

Image points: $[0 \quad 1]^\top$, $[204.9780 \quad 1]^\top$ and $[614.9341 \quad 1]^\top$;

3D points: $[0 \quad 1]^\top$, $[16.4838 \quad 1]^\top$ and $[48.6365 \quad 1]^\top$.

From $\mathbf{x}' = \mathrm{H}\mathbf{x}$**,** we take the "cross-product", i.e. the determinant of $[\mathbf{x}' \quad \mathrm{H}\mathbf{x}]$ since $\mathbf{x}'$ is a 2x2 vector. Thus, for each point we get one equation, i.e.

$$h_{21} * x * x' - h_{12} - h_{11} * x + h_{22} * x' = 0$$

$$\Rightarrow [-x \quad -1 \quad xx' \quad x'] \begin{bmatrix} h_{11} \\ h_{12} \\ h_{21} \\ h_{22} \end{bmatrix} = 0$$

Putting the 3-point correspondences into the equation, we get:

$$\begin{bmatrix} -204.9780 & 1 & (204.9780)(16.4838) & 16.4838 \\ -614.9341 & 1 & (614.9341)(48.6365) & 48.6365 \\ -0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{21} \\ h_{22} \end{bmatrix} = 0.$$

Taking SVD of the 3x4 matrix, the null space is the right singular vector that corresponds to the least singular value, i.e.

$$\begin{bmatrix} h_{11} \\ h_{12} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0.0808 \\ 0 \\ 0 \\ 0.9967 \end{bmatrix} \Rightarrow \mathrm{H}_{2\times 2} = \begin{bmatrix} 0.0808 & 0.0000 \\ 0.0000 & 0.9967 \end{bmatrix}.$$

The 3D vanishing point in $\mathbb{P}^1$ is given by $[1 \quad 0]^\top$, and it is imaged to the point:

$$\mathbf{x} = \mathrm{H}^{-1}\mathbf{x}' = [0.0808 \quad 0.00]^\top.$$

Question 14

It was mid-autumn festival. Your best friend Xiaoming, who is a whimsical computer vision scientist and photographer, sent you a post card. The front of the postcard is a photo of the full moon taken with his latest DSLR camera. On the back of the postcard, Xiaoming wrote a lantern riddle for you to solve.

a) Assuming that the full moon is *infinitely far away*, the camera has zero skew and the pixels are square. Let us denote the homogenous coordinates of an edge pixel $[u, v, 1]^\top$ from the image of the moon, prove that: $f^2 + (p_x - u)^2 + (p_y - v)^2 = 0$, where f and (p_x, p_y) are the focal length and principal point of the camera. **Show all your workings clearly**.

b) Suppose that the image of the full moon is given by the Cartesian equation: $0.0001x^2 + 0.0001y^2 - 0.05x - 0.05y + 13.5 = 0$, and we further know that the principal point of the camera is $(p_x, p_y) = (250,250)$. Find the focal length f of the camera. **Show all your workings clearly**.

Solution

a)

Zero skew and square pixels means the camera intrinsic matrix is given by:

$$\mathrm{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the moon is infinitely far away, it can be taken to be the absolute conic, i.e. $\mathrm{C} = \mathrm{I}_{3\times 3}$ and

$$[X \quad Y \quad Z]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0, \text{ or equivalently } X^2 + Y^2 + Z^2 = 0.$$

Since $\mathbf{x} = \mathrm{P}\mathbf{X} \Longrightarrow [u, v, 1]^\top = \mathrm{K}[\mathrm{R} \mid \mathbf{t}][X, Y, Z, 0]^\top = \mathrm{KR}[X, Y, Z]^\top$, using $[X, Y, Z]\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0$, we write:

$((\mathrm{KR})^{-1}\mathbf{x})^\top((\mathrm{KR})^{-1}\mathbf{x}) = 0.$

This gives us:

$$\mathbf{x}^\top(\mathrm{KR})^{-\top}(\mathrm{KR})^{-1}\mathbf{x} = 0$$

$$\Longrightarrow \mathbf{x}^\top\mathrm{K}^{-\top}\,\mathrm{R}^{-\top}\mathrm{R}^{-1}\mathrm{K}^{-1}\mathbf{x} = 0$$

$$\Longrightarrow \mathbf{x}^\top\mathrm{K}^{-\top}\,\mathrm{K}^{-1}\mathbf{x} = 0.$$

From $\mathrm{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$, we get

$$\mathrm{K}^{-1} = \begin{bmatrix} 1/f & 0 & -p_x/f \\ 0 & 1/f & -p_y/f \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathrm{K}^{-\top} = \begin{bmatrix} 1/f & 0 & 0 \\ 0 & 1/f & 0 \\ -p_x/f & -p_y/f & 1 \end{bmatrix}.$$

$$\text{Thus, } \mathrm{K}^{-\top}\mathrm{K}^{-1} = \begin{bmatrix} 1/f^2 & 0 & -p_x/f^2 \\ 0 & 1/f^2 & -p_y/f^2 \\ -p_x/f^2 & -p_y/f^2 & \frac{p_x^2}{f^2} + \frac{p_y^2}{f^2} + 1 \end{bmatrix} = \frac{1}{f^2}\begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ -p_x & -p_y & p_x^2 + p_y^2 + f^2 \end{bmatrix}.$$

Putting $\mathbf{x} = [u, v, 1]^\top$ and $\mathrm{K}^{-\top}\mathrm{K}^{-1}$ into $\mathbf{x}^\top\mathrm{K}^{-\top}\,\mathrm{K}^{-1}\mathbf{x} = 0$, we get:

$$\frac{1}{f^2}[u,v,1]\begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ -p_x & -p_y & p_x^2+p_y^2+f^2 \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$= \frac{1}{f^2}[u-p_x, v-p_y, -up_x - vp_y + p_x^2 + p_y^2 + f^2]\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$= \frac{1}{f^2}[u^2 - up_x + v^2 - vp_y - up_x - vp_y + p_x^2 + p_y^2 + f^2]$$

$$= \frac{1}{f^2}[u^2 - up_x + v^2 - vp_y - up_x - vp_y + p_x^2 + p_y^2 + f^2]$$

$$= \frac{1}{f^2}[u^2 - 2up_x + p_x^2 + v^2 - 2vp_y + p_y^2 + f^2]$$

$$= \frac{1}{f^2}[(p_x - u)^2 + (p_y - v)^2 + f^2].$$

$\Longrightarrow (p_x - u)^2 + (p_y - v)^2 + f^2 = 0$ **(QED)**.

b)

Convert the cartesian equation into homogenous equation. Let $x = \frac{u}{w}$ and $y = \frac{v}{w}$, we get :

$$0.0001\left(\frac{u}{w}\right)^2 + 0.0001\left(\frac{v}{w}\right)^2 - 0.05\left(\frac{u}{w}\right) - 0.05\left(\frac{v}{w}\right) + 13.5 = 0.$$

We need 1 point (u, v) to find f from $(p_x - u)^2 + (p_y - v)^2 + f^2 = 0$.

Setting $w = 1$ and $v = 0$, we get

$0.0001u^2 - 0.05u + 13.5 = 0.$

Solving the roots, we get:

$$u = \frac{0.05 \pm \sqrt{0.05^2 - 4(0.0001)(13.5)}}{2(0.0001)} \Longrightarrow u = 250 \pm 269.2582i.$$

Putting into $(p_x - u)^2 + (p_y - v)^2 + f^2 = 0$, we get:

$$(250 - 250 + 269.2582\mathrm{i})^2 + (250)^2 + f^2 = 0$$

$$f^2 = 269.2582^2 - 250^2 \Longrightarrow f = 100.$$

Question 15

Your other best friend Kumar is an avid night sky photographer. He took two photographs of the brightest star in the night sky that is *infinitely far away* with an *in-plane rotation* between the second and first image planes, i.e. as shown in Fig. 1, there is only a rotation θ around the principal axis $\boldsymbol{Z}$ of the camera between the views.

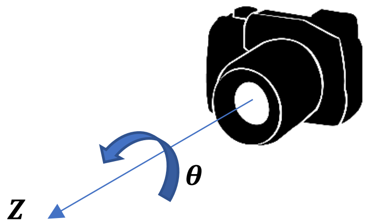

Figure 1. Camera in-plane rotation.

Given that the image homogenous coordinates of the star in the first and second views are $[200.1262, 150.2062, 1.00]^{\mathrm{T}}$ and $[-128.1647, -261.866, 1.00]^{\mathrm{T}}$, respectively. Find the in-plane rotation θ between the views. We further assume that the camera intrinsic matrix is given by: $\mathrm{K} = \begin{bmatrix} 270 & 0 & 512 \\ 0 & 270 & 256 \\ 0 & 0 & 1 \end{bmatrix}$. **Show all your workings clearly**.

Solution

The directions of the rays in the two views are given by:

$\boldsymbol{d} = \frac{\mathrm{K}^{-1}\mathbf{x}}{||\mathrm{K}^{-1}\mathbf{x}||} = \begin{bmatrix} -0.7323 \\ -0.2484 \\ 0.6340 \end{bmatrix}$, since we have

$\mathrm{K}^{-1}\mathbf{x} = \begin{bmatrix} 270 & 0 & 512 \\ 0 & 270 & 256 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 200.1262 \\ 150.2062 \\ 1.00 \end{bmatrix} = \begin{bmatrix} -1.1550 \\ -0.3917 \\ 1.00 \end{bmatrix}$ and $||\mathrm{K}^{-1}\mathbf{x}|| = 1.5772.$

$\boldsymbol{d}' = \frac{\mathrm{K}^{-1}\mathbf{x}'}{||\mathrm{K}^{-1}\mathbf{x}'||} = \begin{bmatrix} -0.6781 \\ -0.3718 \\ 0.6340 \end{bmatrix}$, since we have

$\mathrm{K}^{-1}\mathbf{x}' = \begin{bmatrix} 270 & 0 & 512 \\ 0 & 270 & 256 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 223.2545 \\ 97.6847 \\ 1.00 \end{bmatrix} = \begin{bmatrix} -1.0694 \\ -0.5864 \\ 0.6340 \end{bmatrix}$ and $||\mathrm{K}^{-1}\mathbf{x}|| = 1.5772.$

Let $\mathrm{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the rotation matrix which represents the in-plane rotation that transforms a point in the second view into the first view. We thus have:

$$\boldsymbol{d}' = \mathrm{R}\boldsymbol{d} \implies \begin{bmatrix} -0.6781 \\ -0.3718 \\ 0.6340 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.7323 \\ -0.2484 \\ 0.6340 \end{bmatrix}.$$

We get two equations with two unknowns $\cos\theta$ and $\sin\theta$, i.e.

$-0.6781 = -0.7323\cos\theta + 0.2484\sin\theta$ – (1)

$-0.3718 = -0.2484\cos\theta - 0.7323\sin\theta$ – (2)

Solving for the unknowns, we get:

$$\begin{bmatrix} -0.7323 & 0.2484 \\ -0.2484 & -0.7323 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} -0.6781 \\ -0.3718 \end{bmatrix}, \text{ where}$$

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} -0.7323 & 0.2484 \\ -0.2484 & -0.7323 \end{bmatrix}^{-1} \begin{bmatrix} -0.6781 \\ -0.3718 \end{bmatrix} = \begin{bmatrix} 0.9849 \\ 0.1736 \end{bmatrix}$$

$$\implies \tan\theta = \frac{0.1736}{0.9849} \text{ and } \theta = \tan^{-1}\frac{0.1736}{0.9849} = 0.1745 \text{ rad.}$$

Question 16

Given the homogeneous coordinates of a 2D (image)-3D (scene) point correspondence $\mathbf{x} \leftrightarrow \mathbf{Y}$, where

$\mathbf{x} = [432.0351,\ 424.6289,\ \ 1.0000]^\top; \mathbf{Y} = [62.2385,\ \ 47.3612,\ \ 31.8379,\ \ 1.0000]^\top;$

We further know that the 3D scene points are expressed in a fixed world frame, and the camera has an intrinsics given by:

$K = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$, where f is the unknown focal length, and $c_x = 270, c_y = 240$ is the principal point. The extrinsics of the camera is given by:

$$[\,\mathrm{R} \mid \mathbf{t}\,] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $[\,\mathrm{R} \mid \mathbf{t}\,]$ transforms a 3D point in the world frame into the camera frame. Find the unknown focal length f and angle θ. **Show all your workings clearly.**

Useful equation: $\cos^2\alpha + \sin^2\alpha = 1.$

Solution

The camera projection equation is given by: $\lambda\mathbf{x} = \mathrm{P}\mathbf{Y}$**.** Taking cross product to eliminate the unknown scale λ, we get:

$\mathbf{x} \times \mathrm{P}\mathbf{Y} = \mathbf{0}$

$$\begin{bmatrix} x_x \\ x_y \\ 1 \end{bmatrix} \times \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Y_x \\ Y_y \\ Y_z \\ 1 \end{bmatrix} = \mathbf{0}$$

Collecting the unknowns into the form of Ax=0, we get:

$$\begin{bmatrix} -Y_y & -Y_x & Y_z * x_y - Y_z * c_y \\ Y_x & -Y_y & Y_z * c_x - Y_z * x_x \\ (Y_y * x_x - Y_x * x_y) * f * \cos\theta & (Y_x * x_x + Y_y * x_y) * f * \sin\theta & Y_z * c_y * x_x - Y_z * c_x * x_y \end{bmatrix}$$
$$\begin{bmatrix} f\cos\theta \\ f\sin\theta \\ 1 \end{bmatrix} = \mathbf{0}.$$

Remarks: It is also correct to solve the unknown scale λ directly from the third element. Alternatively, the scale can be eliminated by normalizing (i.e. dividing) over the third element.

Only the first two equations are independent. Thus, we get 2 equations to solve for the 2 unknowns $a = f\cos\theta$ and $b = f\sin\theta$, i.e.

$-Y_y * a - X_x * b + Y_z * x_y - Y_z * c_y = 0$

$-47.3612 * a - 62.2385 * b + 31.8379 * \ 424.6289 - 31.8379 * 240 = 0$

$-47.3612 * a - 62.2385 * b + 5.8782e + 03 = 0$ -- (1)

$Y_x * \ a - Y_y * b + Y_z * c_x - Y_z * x_x = 0$

$62.2385 * \ a - 47.3612 * b + 31.8379 * 270 - 31.8379 * 432.0351 = 0$

$62.2385 * \ a - 47.3612 * b - 5.1589e + 03 = 0$ -- (2)

Solving for a and b gives us: a=98.0065 and b=19.8670.

This implies $f\cos\theta = 98.0065$ and $f\sin\theta = 19.8670$. Taking the sum of the squares of both equations, we get:

$$f^2(\cos^2\theta + \sin^2\theta) = 98.0065^2 + 19.8670^2 \ \Rightarrow \boldsymbol{f} = \mathbf{100}.$$

Substituting back into $f\cos\theta = 98.0065$, we get:

$100\cos\theta = 98.0065 \Longrightarrow \boldsymbol{\theta} = \mathbf{0.2}$.

Question 17

The z-coordinate of a 3D point lying on a plane in the first camera reference frame F_{C1} is $z = 20$. We further note that this plane is parallel to the image plane of the first camera, where the z-axis of F_{C1} is perpendicular to the plane. A second camera view of the 3D point is taken. The second camera frame F_{C1} is related to the first camera frame F_{C2} by a rigid transformation $\mathrm{T}_{C2C1} \in \mathrm{SE}(3)$, which consists of only a rotation around the z-axis and a translation along the x-axis.

Given the camera normalized coordinates (i.e. $\hat{\mathbf{x}} = \mathrm{inv}(\mathrm{K})\mathbf{x}$, where K is the camera intrinsic) in the first and second camera images:

$\hat{\mathbf{x}}^{C1} = [0.00 \quad 0.50 \quad 1.00]^\top$ and $\hat{\mathbf{x}}^{C2} = [-0.0493 \quad 0.49 \quad 1.00]^\top$.

Find $\mathrm{T}_{C2C1} \in \mathrm{SE}(3)$ that relates the reference frames of the two cameras.

Solution

Since the plane is parallel to the first camera image plane where the z-axis of F_{C1} is perpendicular to the plane, its normal vector is given by:

$\mathbf{N} = [0 \quad 0 \quad 1]^\top$ and the distance of the plane to F_{C1} is given by the Z coordinate of the 3D points, i.e. $d = 20$.

The image points of camera 2 and camera 1 are related by a homography given by:

$\hat{\mathbf{x}}^{C2} = \left(\mathrm{R} + \frac{\mathbf{tN}^\top}{d}\right)\hat{\mathbf{x}}^{C1}$, where $\mathrm{R} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $\mathbf{t} = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix}$.

Thus, we get:

$\frac{t_x}{20} + \cos\gamma * x_1^{C1} - \sin\gamma * x_2^{C1} - x_1^{C2} = 0$ -- (1)

$\cos\gamma * x_2^{C1} + \sin\gamma * x_1^{C1} - x_2^{C2} = 0$ -- (2)

Putting the camera normalized point into Equation (2), we get:

$\cos\gamma * (0.5) + \sin\gamma * (0) - 0.49 = 0 \Rightarrow \boldsymbol{\gamma} = \mathbf{0.2\ rad}$

Putting the point into Equation (1), we get:

$\frac{t_x}{20} + \cos(0.2) * (0) - \sin(0.2) * (0.5) - (-0.0493) = 0 \Rightarrow \boldsymbol{t_x} = \mathbf{1}$.

Alternatively

Since the 3D point lies on a plane parallel to the first camera image, its coordinate is given by: $\mathbf{X} = [0.00 \quad 10.0 \quad 20.0 \quad 1.00]^\top$.

The projection into the second camera image is then given by:

$$\hat{\mathbf{x}}^{C2} = [\mathrm{R} \quad \mathrm{t}]\,\mathbf{X},\ \text{ where } \mathrm{T}_{C2C1} = \begin{bmatrix} \mathrm{R} & \mathrm{t} \\ 0 & 1 \end{bmatrix} \text{ with } \mathrm{R} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{t} = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow [xw, yw, w]^\top = [\mathrm{R} \quad \mathrm{t}]\,[X, Y, Z, 1]^\top$$

We get: $w = Z = 20.0$. Putting back into the equation, we get:

$$\frac{X\cos\gamma - Y\sin\gamma + t_x}{w} = x \Rightarrow (0.00)\cos\gamma - (10.0)\sin\gamma + t_x = -0.0493 * 20.0$$

$\Rightarrow -(10.0)\sin\gamma + t_x = -0.9860$ – (1)

$$\frac{X\sin\gamma + Y\cos\gamma}{w} = y \Rightarrow (0.00)\sin\gamma + (10.0)\cos\gamma = 0.49 * 20.0$$

$\Rightarrow \cos\gamma = 0.98 \Rightarrow \boldsymbol{\gamma} = \mathbf{0.2\ rad}$ – (2)

Putting back into equation (1), we get:

$$-(10.0)\sin(0.2) + t_x = -0.9860 \Rightarrow \boldsymbol{t_x} = \mathbf{1.00}$$

Question 18

a) Given the projection matrices of the two views:

$$\mathrm{P} = [\,\mathrm{I}_{3\times3} \mid \mathbf{0}_{3\times1}\,] \text{ and } \mathrm{P}' = [\,\mathrm{A} \mid \mathbf{a}\,],$$

where A is a 3×3 matrix and $\mathbf{a}$ is a 3×1 vector, and a plane defined by $\boldsymbol{\pi}^\top\mathbf{X} = 0$ with $\boldsymbol{\pi} = [\mathbf{v}^\top \quad 1]^\top$, prove that the homography induced by the plane is $\mathbf{x}' = \mathrm{H}\mathbf{x}$ with:

$$\mathrm{H} = \mathrm{A} - \mathbf{a}\mathbf{v}^\top.$$

$\mathbf{X} \in \mathbb{P}^3$ is the point in the 3D space, and $\mathbf{x} \in \mathbb{P}^2$ and $\mathbf{x}' \in \mathbb{P}^2$ are the projections of $\mathbf{X}$ into the first and second views, respectively.

b) Further prove that the homography can be written as $\mathrm{H} = \mathrm{A} - \mathbf{e}'\mathbf{v}^\top$, where $\mathbf{e}'$ is the epipole in the second view.

c) Any three points in the 3D space form a plane, and therefore it is intuitive that there is a homography that relates any three image point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ for $i = 1,2,3$. Show that the homography induced by the plane of the three 3D points is given by:

$$\mathrm{H} = \mathrm{A} - \mathbf{e}'(\mathrm{M}^{-1}\mathbf{b})^\top,$$

where $\mathbf{b}$ is a 3-vector with components:

$$b_i = \left(\mathbf{x}'_i \times (\mathrm{A}\mathbf{x}_i)\right)^\top (\mathbf{x}'_i \times \mathbf{e}')/\|\mathbf{x}'_i \times \mathbf{e}'\|^2,$$

and M is a 3×3 matrix with rows $\mathbf{x}_i^\top$.

d) Give two degenerate cases of $\mathrm{H} = \mathrm{A} - \mathbf{e}'(\mathrm{M}^{-1}\mathbf{b})^\top$. Be specific on what constitutes the degeneracies.

Solution

a)

We first back-project the point $\mathbf{x}$ in the first view and determine the intersection point $\mathbf{X}$ of this ray with the plane $\boldsymbol{\pi}$. The 3D point $\mathbf{X}$ is then projected into the second view to compute H.

For the first view, we have:

$$\mathbf{x} = \mathrm{P}\mathbf{X} = [\,\mathrm{I}_{3\times 3} \mid \mathbf{0}_{3\times 1}\,]\mathbf{X}.$$

This implies that $\mathbf{X} = [\mathbf{x}^\top \quad \rho]^\top$ is a valid 3D point, where ρ is a scalar that parameterized the 3D point.

Since the 3D point lies on the plane, it satisfies $\boldsymbol{\pi}^\top\mathbf{X} = 0$. As a result, we have:

$[\mathbf{v}^\top \quad 1][\mathbf{x}^\top \quad \rho]^\top = 0 \Rightarrow \mathbf{v}^\top\mathbf{x} + \rho = 0 \Rightarrow \rho = -\mathbf{v}^\top\mathbf{x}\,.$

Thus, we get $\mathbf{X} = [\mathbf{x}^\top \quad -\mathbf{x}^\top\mathbf{v}]^\top$. The 3D point $\mathbf{X}$ projects into the second view as:

$$\mathbf{x}' = \mathrm{P}'\mathbf{X} = [\,\mathrm{A} \mid \mathbf{a}\,][\mathbf{x}^\top \quad -\mathbf{x}^\top\mathbf{v}]^\top = \mathbf{A}\mathbf{x} - \mathbf{a}\mathbf{v}^\top\mathbf{x} = (\mathbf{A} - \mathbf{a}\mathbf{v}^\top)\mathbf{x},$$

where $\mathrm{H} = (\mathbf{A} - \mathbf{a}\mathbf{v}^\top)$.

(shown)

b)

The null-space of P gives the camera center of the first view $\mathbf{C}$, i.e. $\mathrm{P}\mathbf{C} = 0$. By inspection, we see that $\mathbf{C} = [0 \quad 0 \quad 0 \quad 1]^\top$. Thus, the epipole $\mathbf{e}'$ in the second view is given by:

$$\mathbf{e}' = \mathrm{P}'\mathbf{C} = [\,\mathrm{A} \mid \mathbf{a}\,][0 \quad 0 \quad 0 \quad 1]^\top = \mathbf{a}.$$

Putting $\mathbf{a} = \mathbf{e}'$ into $\mathrm{H} = (\mathbf{A} - \mathbf{a}\mathbf{v}^\top)$, we get $\mathrm{H} = (\mathbf{A} - \mathbf{e}'\mathbf{v}^\top)$.

(shown)

c)

Since $\mathbf{x}' = \mathrm{H}\mathbf{x}$, we have $\mathbf{x}' = \mathrm{H}\mathbf{x} = (\mathbf{A} - \mathbf{e}'\mathbf{v}^\top)\mathbf{x}$. Taking the cross-product of both sides with $\mathbf{x}'$, we get:

$$\mathbf{x}' \times (\mathbf{A} - \mathbf{e}'\mathbf{v}^\top)\mathbf{x} = (\mathbf{x}' \times \mathbf{A}\mathbf{x}) \; - (\mathbf{x}' \times \mathbf{e}')(\mathbf{v}^\top\mathbf{x}) = 0.$$

As a result, we have:

$$(\mathbf{x}' \times \mathbf{e}')(\mathbf{v}^\top\mathbf{x}) = (\mathbf{x}' \times \mathbf{A}\mathbf{x}).$$

Taking the transpose of both sides give:

$$(\mathbf{x}^\top\mathbf{v})(\mathbf{x}' \times \mathbf{e}')^\top = (\mathbf{x}' \times \mathbf{A}\mathbf{x})^\top.$$

We make $(\mathbf{x}^\top\mathbf{v})$ the subject by multiplying both sides with $(\mathbf{x}' \times \mathbf{e}')$, i.e.

$$(\mathbf{x}^\top\mathbf{v})(\mathbf{x}' \times \mathbf{e}')^\top(\mathbf{x}' \times \mathbf{e}') = (\mathbf{x}' \times \mathbf{A}\mathbf{x})^\top(\mathbf{x}' \times \mathbf{e}'),$$

which we can rewrite as:

$$\mathbf{x}^\top\mathbf{v} = \frac{(\mathbf{x}' \times \mathbf{A}\mathbf{x})^\top(\mathbf{x}' \times \mathbf{e}')}{(\mathbf{x}' \times \mathbf{e}')^\top(\mathbf{x}' \times \mathbf{e}')}$$

since the denominator $(\mathbf{x}' \times \mathbf{e}')^\top(\mathbf{x}' \times \mathbf{e}')$ is now a scalar value. $\mathbf{x}_i^\top\mathbf{v} = b_i$ since $\mathrm{M}^{-1} = \mathbf{x}^{-\top}$ **.**

d)

Case 1: $\mathrm{M}^\top = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$ is not full rank. This causes M^{-1} to be undefined. It happens when $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are not linearly independent. This is true when the **three 3D points are collinear**.

Case 2: $\mathbf{x}' \times \mathbf{e}' = 0$. This occurs when $\mathbf{x}' = \alpha\mathbf{e}'$, where α is a scalar value, i.e. $\mathbf{x}'$ **is the same point as the epipole**. $\mathbf{x}' \times \mathbf{e}' = 0$ causes $\mathbf{x}^\top\mathbf{v}$ to be undefined.

Question 19

Given a *projective* reconstruction $\{\mathrm{P}_i, \mathbf{X}_j\}$, where P_i is the i^{th} camera projection matrix and $\mathbf{X}_j \in \mathbb{P}^3$ is the j^{th} 3D point. We wish to determine a rectifying homography H such that $\{\mathrm{P}_i\mathrm{H}, \mathrm{H}^{-1}\mathbf{X}_j\}$ is a *metric* reconstruction.

a) Let $\mathrm{H} = \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$, where $\mathrm{A} \in \mathbb{R}^{3\times3}$, $\mathbf{t} \in \mathbb{R}^{3\times1}$, $\mathbf{v} \in \mathbb{R}^{3\times1}$, and $v \in \mathbb{R}$ can be set to 1 since H only has 15 degree-of-freedom. We further let the first camera projection matrix in the projective reconstruction be $\mathrm{P}_1 = [\mathrm{I}_{3\times3} \quad 0_{3\times1}]$, show that H takes the form:

$$\mathrm{H} = \begin{bmatrix} \mathrm{K}_1 & \mathbf{0} \\ -\mathbf{p}^\top\mathrm{K}_1 & 1 \end{bmatrix},$$

where K_1 is the 3×3 intrinsic matrix of the first camera and $\mathbf{p} = -\mathrm{K}_1^{-\top}\mathbf{v}$.

b) Find the plane at infinity in the projective reconstruction in terms of K_1 and $\mathbf{v}$.

c) Denoting the other camera projection matrices as $\mathrm{P}_i = [\mathrm{A}_i \quad \mathbf{a}_i]$, show that the following expression:

$$\omega_i^* = (\mathrm{A}_i - \mathbf{a}_i\mathbf{p}^\top)\omega_1^*(\mathrm{A}_i - \mathbf{a}_i\mathbf{p}^\top)^\top$$

is true, where ω^* is the dual image of the absolute conic (DIAC), A_i is the first 3×3 entries and $\mathbf{a}_i$ is the last column of P_i.

d) Given $\mathrm{K}_1 = \mathrm{K}_2 = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = [0 \quad 0 \quad 0]^\top$ and the camera matrix:

$$\mathrm{P}_2 = \begin{bmatrix} 0.00 & 0.00 & 100.00 & 100.00 \\ 0.00 & 1.00 & 0.00 & 100.00 \\ -0.01 & 0.00 & 0.00 & 1.00 \end{bmatrix}$$

under projective reconstruction, find the camera matrix P_{M2} under metric reconstruction.

Solution

a)

Let P_M be the first camera projective matrix in the metric reconstruction. It is given by:

$$\mathrm{P}_M = \mathrm{P}_1\mathrm{H}$$

$$\Rightarrow \mathrm{K}_1[\mathrm{I}_{3\times3} \quad 0_{3\times1}] = [\mathrm{I}_{3\times3} \quad 0_{3\times1}]\begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$$

$\Rightarrow [\mathrm{K}_1 \quad 0_{3\times1}] = [\mathrm{I}_{3\times3} \quad 0_{3\times1}] \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} = [\mathrm{A} \quad \mathbf{t}]$.

Therefore, $\mathrm{A} = \mathrm{K}_1$ and $\mathbf{t} = 0$.

Since $\mathbf{p} = -\mathrm{K}_1^{-\top}\mathbf{v}$, we can see $-\mathbf{p}^\top\mathrm{K}_1 = -\mathbf{v}^\top\mathrm{K}_1\mathrm{K}_1 = \mathbf{v}$. **(shown)**

b)

$$\boldsymbol{\pi} = \mathrm{H}^{-\top}\boldsymbol{\pi}_\infty \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{K}_1 & \mathbf{0} \\ -\mathbf{p}^\top\mathrm{K}_1 & 1 \end{bmatrix}^\top \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{K}_1^\top & -\mathrm{K}_1^\top\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$$

We get:

$$\mathrm{K}_1^\top \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} - \mathrm{K}_1^\top\mathbf{p}\pi_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{-- (1)}$$

$$\pi_4 = 1 \qquad \text{-- (2)}$$

Putting (2) into (1), we get:

$\mathrm{K}_1^\top \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = -\mathrm{K}_1^\top\mathbf{p} \Rightarrow \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = -\mathbf{p}$. This implies the $\boldsymbol{\pi} = \begin{bmatrix} -\mathbf{p} \\ 1 \end{bmatrix}$.

c)

We denote the projection matrices under metric reconstruction as $\mathrm{P}_{Mi} = \mathrm{K}_i[\mathrm{R}_i \quad \mathbf{t}_i]$, we get:

$$\mathrm{P}_{Mi} = \mathrm{P}_i\mathrm{H} \Rightarrow \mathrm{K}_i[\mathrm{R}_i \quad \mathbf{t}_i] = [\mathrm{A}_i \quad a_i] \begin{bmatrix} \mathrm{K}_1 & \mathbf{0} \\ -\mathbf{p}^\top\mathrm{K}_1 & 1 \end{bmatrix}$$

$\Rightarrow \mathrm{K}_i\mathrm{R}_i = \mathrm{A}_i\mathrm{K}_1 - a_i\mathbf{p}^\top\mathrm{K}_1$, and getting rid of R_i by $\mathrm{K}_i\mathrm{R}_i(\mathrm{K}_i\mathrm{R}_i)^\top = \mathrm{K}_i\mathrm{K}_i^\top$, we have

$$\mathrm{K}_i\mathrm{K}_i^\top = (\mathrm{A}_i\mathrm{K}_1 - a_i\mathbf{p}^\top\mathrm{K}_1)(\mathrm{A}_i\mathrm{K}_1 - a_i\mathbf{p}^\top\mathrm{K}_1)^\top = (\mathrm{A}_i - a_i\mathbf{p}^\top)\mathrm{K}_1\mathrm{K}_1^\top(\mathrm{A}_i - a_i\mathbf{p}^\top)^\top$$

$\Rightarrow \omega_i^* = (\mathrm{A}_i - a_i\mathbf{p}^\top)\omega_1^*(\mathrm{A}_i - a_i\mathbf{p}^\top)^\top$. **(shown)**

d)

Given $\mathrm{P}_2 = \begin{bmatrix} 0.00 & 0.00 & 100.00 & 100.00 \\ 0.00 & 1.00 & 0.00 & 100.00 \\ -0.01 & 0.00 & 0.00 & 1.00 \end{bmatrix}$

$$\Rightarrow \mathrm{A}_2 = \begin{bmatrix} 0.00 & 0.00 & 100.00 \\ 0.00 & 1.00 & 0.00 \\ -0.01 & 0.00 & 0.00 \end{bmatrix} \text{ and } \mathrm{a}_2 = \begin{bmatrix} 100.00 \\ 100.00 \\ 1.00 \end{bmatrix}$$

Since $\mathrm{K}_1 = \mathrm{K}_2$, we have:

$\omega_i^* = (\mathrm{A}_i - a_i\mathbf{p}^\top)\omega_1^*(\mathrm{A}_i - a_i\mathbf{p}^\top)^\top \Rightarrow \mathrm{KK}^\top = (\mathrm{A}_i - a_i\mathbf{p}^\top)\mathrm{KK}^\top(\mathrm{A}_i - a_i\mathbf{p}^\top)^\top$.

Furthermore, $s = 0$ and $p_x = p_y = 0$, and thus we get:

$$\mathrm{KK}^\top = \begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} f_x^2 & 0 & 0 \\ 0 & f_y^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{p} = -\mathrm{K}^{-\top}\,\mathbf{v} = -\begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_i\mathbf{p}^\top = \begin{bmatrix} 100 \\ 100 \\ 1 \end{bmatrix}[p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(\mathrm{A}_i - a_i\mathbf{p}^\top)$

$$= \begin{bmatrix} 0.00 & 0.00 & 100.00 \\ 0.00 & 1.00 & 0.00 \\ -0.01 & 0.00 & 0.00 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.00 & 0.00 & 100.00 \\ 0.00 & 1.00 & 0.00 \\ -0.01 & 0.00 & 0.00 \end{bmatrix}.$$

$(\mathrm{A}_i - a_i\mathbf{p}^\top)\mathrm{KK}^\top$

$$= \begin{bmatrix} 0.00 & 0.00 & 100.00 \\ 0.00 & 1.00 & 0.00 \\ -0.01 & 0.00 & 0.00 \end{bmatrix}\begin{bmatrix} f_x^2 & 0 & 0 \\ 0 & f_y^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.00 & 0.00 & 100 \\ 0.00 & f_y^2 & 0.00 \\ -0.01f_x^2 & 0.00 & 0.00 \end{bmatrix}.$$

$$(\mathrm{A}_i - a_i\mathbf{p}^\top)\mathrm{K}\mathrm{K}^\top(\mathrm{A}_i - a_i\mathbf{p}^\top)$$

$$= \begin{bmatrix} 0.00 & 0.00 & 100 \\ 0.00 & f_y^2 & 0.00 \\ -0.01f_x^2 & 0.00 & 0.00 \end{bmatrix} \begin{bmatrix} 0.00 & 0.00 & -0.01 \\ 0.00 & 1.00 & 0.00 \\ 100.00 & 0.00 & 0.00 \end{bmatrix}$$

$$= \begin{bmatrix} 100^2 & 0.00 & 0.00 \\ 0.00 & f_y^2 & 0.00 \\ 0 & 0.00 & 0.01^2 f_x^2 \end{bmatrix}.$$

Since $f_x = f_y = f$, we get $100^2 = f^2 \Rightarrow \boldsymbol{f} = \mathbf{100}$.