NUS National University of Singapore | School of Computing

CS4277 / CS5477
3D Computer Vision

Lecture 1:
2D and 1D Projective Geometry

Assoc Prof. Lee Gim Hee

AY 2022/23

Semester 2

Course Schedule

Week	Date	Topic	Assignments
1	11 Jan	2D and 1D projective geometry	**Assignment 0:** Getting started with Python (Ungraded)
2	18 Jan	3D projective geometry, Circular points and Absolute conic	
3	25 Jan	Rigid body motion and Robust homography estimation	
4	01 Feb	Camera models and calibration	**Assignment 1:** Metric rectification and robust homography (10%) **Due:** 2359hrs, 07 Feb
5	08 Feb	Single view metrology	**Assignment 2**: Affine 3D measurement from vanishing line and point (10%) **Due:** 2359hrs, 14 Feb
6	15 Feb	The Fundamental and Essential matrices	
-	22 Feb	Semester Break	No lecture
7	**01 Mar**	**Mid-term Quiz (20%)**	**In-person Quiz (LT 15, 1900hrs – 2000hrs)**
8	08 Mar	Absolute pose estimation from points or lines	
9	15 Mar	Three-view geometry from points and/or lines	
10	22 Mar	Structure-from-Motion (SfM) and bundle adjustment	**Assignment 3:** SfM and Bundle adjustment (10%) **Due:** 2359hrs, 28 Mar
11	29 Mar	Two-view and multi-view stereo	**Assignment 4:** Dense 3D model from multi-view stereo (10%) **Due:** 2359hrs, 04 Apr
12	05 Apr	3D Point Cloud Processing	
13	12 Apr	Neural Field Representations	

Final Exam: 03 MAY 2023

Learning Outcomes

- Students should be able to:
  1. Explain the difference between Euclidean and Projective geometry.
  2. Use homogenous coordinates to represent points, lines and conics in the projective space.
  3. Describe the duality relation between lines and points, and conics and dual conics on a plane.
  4. Apply the hierarchy of transformations on points, lines and conics.

Acknowledgements

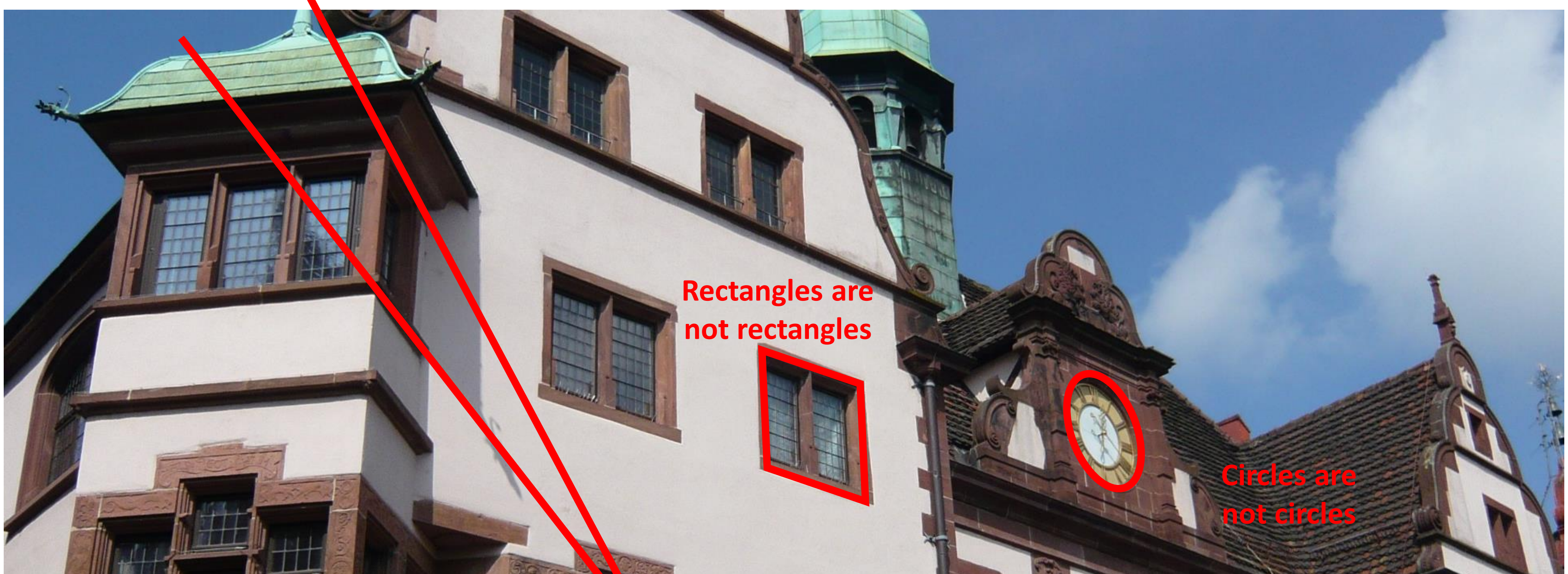
- A lot of slides and content of this lecture are adopted from:

1. R. Hartley, and Andrew Zisserman: “Multiple view geometry in computer vision”, Chapter 2.

2. Y. Ma, S. Soatto, J. Kosecka, S. S. Sastry, “ An invitation to 3-D vision”, Chapter 2.

Projective Transformation

- The mapping of scene objects onto an image is an example of a projective transformation.



Parallel lines meet at a finite point

G. H. Lee "A random building", Freiburg, Germany, 2013.

What is Projective Geometry?

- We saw that certain geometric properties are not preserved by projective transformation, e.g.

  1. A circle may appear as an ellipse
  2. Parallel lines may meet at a finite point
  3. A rectangle may appear as a parallelogram

- In fact, angles, distance, ratios of distances – none of these are preserved!

What is Projective Geometry?

- A property that is preserved is straightness, which is the most general requirement on the mapping.
- **A thought**: we may define a projective transformation as any mapping that preserves straight lines.
- More generally, we study geometric properties that are invariant with respect to projective transformations in projective geometry!

Euclidean vs Projective

- The familiar **Euclidean geometry** is an example of synthetic geometry.
- Use axiomatic method and its related tools, i.e. compass and straightedge to solve problems.
- **Projective geometry** uses coordinates and algebra – analytic geometry.
- We will see that one most important result is that geometry at infinity can now be nicely represented!

Raphel, “The School of Athens”, 1509-1511

Image source: https://en.wikipedia.org/wiki/Euclidean_geometry

The 2D Projective Plane

- We will look at the homogeneous notation for points $\mathbf{x}$ and lines $\mathbf{l}$ on a plane π, and the incidence relationship between points and lines.

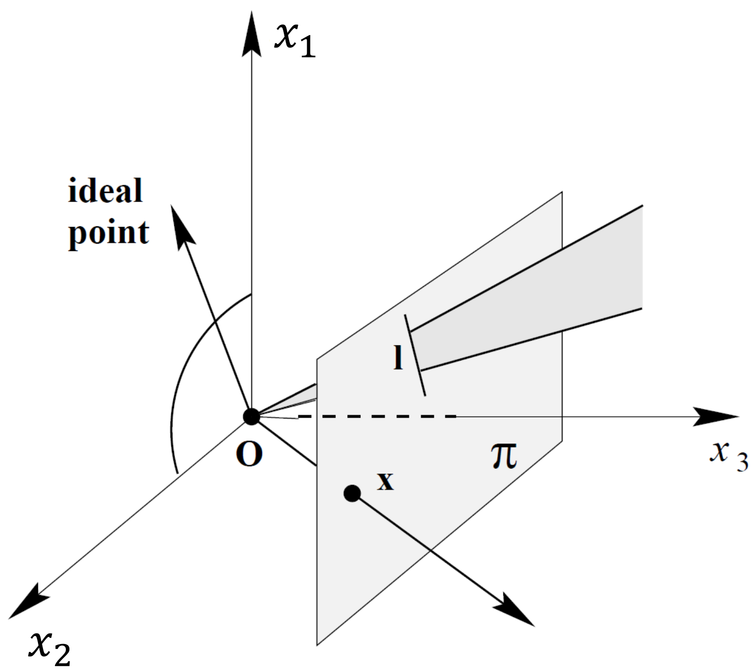


Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Homogeneous Representation of Lines and Points

- A line in the plane is represented by:

$$ax + by + c = 0$$

- Different choices of a, b and c giving rise to different lines.

- Thus, a line may naturally be represented by the vector $(a, b, c)^\top$.

Homogeneous Representation of Lines and Points

- The correspondence between lines and vectors $(a, b, c)^\top$ is not one-to-one.
- Since the lines $ax + by + c = 0$ and $(ka)x + (kb)y + (kc) = 0$ are the same, $\forall k \neq 0$.
- Thus $(a, b, c)^\top$ and $k(a, b, c)^\top$ represent the same line, for any non-zero k, i.e., equivalence class.
- **Note:** the vector $(0, 0, 0)^\top$ does not correspond to any line.

Homogeneous Representation of Lines and Points

- A point $\mathbf{x} = (x, y)^\top$ lies on the line $\mathbf{l} = (a, b, c)^\top$ if and only if $ax + by + c = 0$, i.e.

$$(x, y, 1)(a, b, c)^\top = (x, y, 1)\mathbf{l} = 0;$$

- Similarly, for any constant non-zero k,

$$(kx, ky, k)(a, b, c)^\top = k(x, y, 1)\mathbf{l} = (x, y, 1)\mathbf{l} = 0.$$

Homogeneous Representation of Lines and Points

- Hence, $(kx, ky, k)^\top \in \mathbb{P}^2$ for varying values of k to be a representation of the point $(x, y)^\top \in \mathbb{R}^2$ in the Cartesian space, i.e.

$$\mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{P}^2 \equiv \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)^\top \in \mathbb{R}^2.$$

- Now we can use (x, y, k), where $k = 0$ to represent the point at infinity, i.e. $\left(\frac{x}{0}, \frac{y}{0}\right)$ which is infinite.

Homogeneous Representation of Lines and Points

- More formally: The point $\mathbf{x}$ lies on the line $\mathbf{l}$ if and only if $\mathbf{x}^\top \mathbf{l} = 0$.

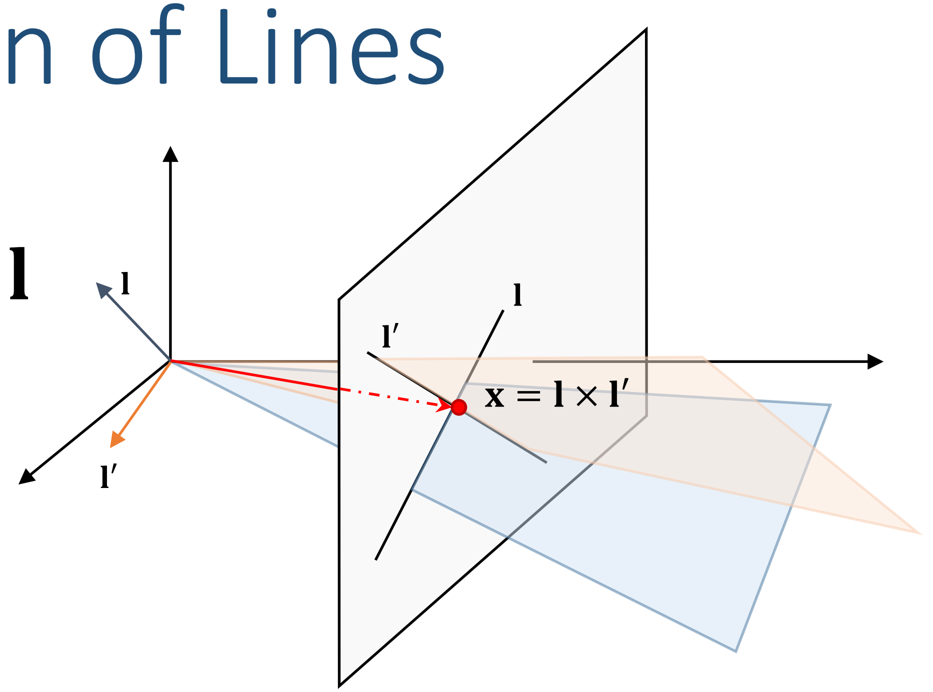
- Note that the expression $\mathbf{x}^\top \mathbf{l}$ is just the inner or scalar product of the two vectors $\mathbf{l}$ and $\mathbf{x}$; the scalar product:

$$\mathbf{x}^\top \mathbf{l} = \mathbf{l}^\top \mathbf{x} = \mathbf{x}.\mathbf{l}$$

- Degree of freedom (dof): a point has 2 dof – x and y coordinates; a line also has 2 dof – two independent ratios $\{a : b : c\}$.

Intersection of Lines

- The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$.



Proof:

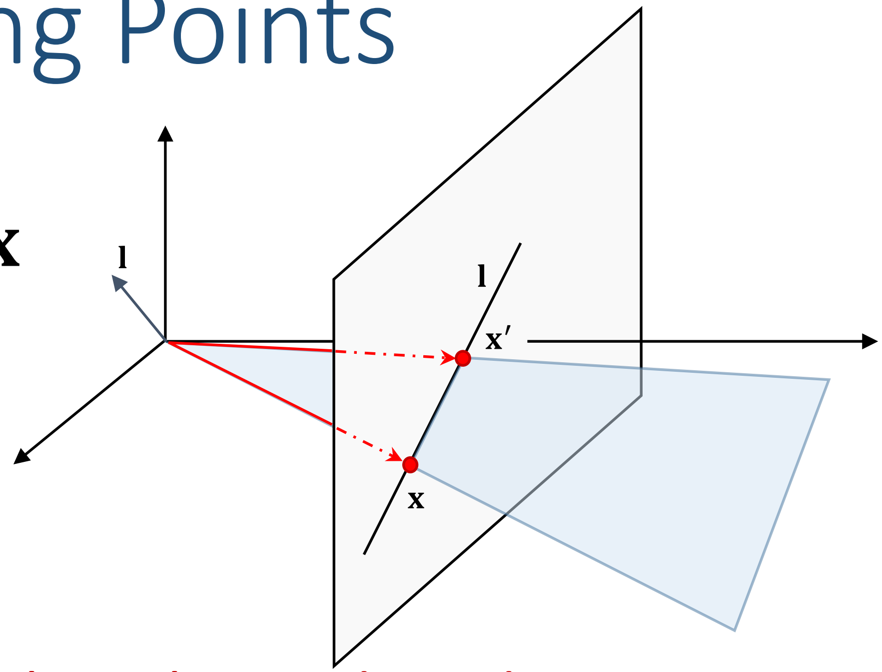
Given two lines $\mathbf{l} = (a, b, c)^\mathsf{T}$ and $\mathbf{l}' = (a', b', c')^\mathsf{T}$, the triple scalar product identity gives $\mathbf{l}.(\mathbf{l} \times \mathbf{l}') = \mathbf{l}'.(\mathbf{l} \times \mathbf{l}') = 0$, which we rewrite as:

$$\mathbf{l}^\mathsf{T}\mathbf{x} = \mathbf{l}'^\mathsf{T}\mathbf{x} = 0.$$

If $\mathbf{x}$ is thought of as representing a point, then $\mathbf{x}$ lies on both lines $\mathbf{l}$ and $\mathbf{l}'$, and hence is the intersection of the two lines. $\square$

Line Joining Points

- The line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.



Proof:

Given two points $\mathbf{x}$ and $\mathbf{x}'$, the triple scalar product identity gives $\mathbf{x}.(\mathbf{x} \times \mathbf{x}') = \mathbf{x}'.(\mathbf{x} \times \mathbf{x}') = 0$, which we rewrite as:

$$\mathbf{x}^\top \mathbf{l} = \mathbf{x}'^\top \mathbf{l} = 0.$$

If $\mathbf{l}$ is thought of as representing a line, then $\mathbf{l}$ contains both points $\mathbf{x}$ and $\mathbf{x}'$, and hence is the line joining the two points.

Ideal Points and the Line at Infinity

Intersection of parallel lines

- Consider two parallel lines $ax + by + c = 0$ and $ax + by + c' = 0$, i.e., $\mathbf{l} = (a, b, c)^\top$ and $\mathbf{l}' = (a, b, c')^\top$.

- The intersection is $\mathbf{l} \times \mathbf{l}' = (c' - c)(b, -a, 0)^\top$, i.e $(b, -a, 0)^\top$ ignoring the scale factor $(c' - c)$.

- $(b, -a, 0)^\top$ is an infinite point and this implies that parallel lines meet at infinity.

Ideal Points and the Line at Infinity

Example:

Consider the two lines $x = 1$ and $x = 2$. Here the two lines are parallel, and consequently intersect "at infinity".

In homogeneous notation the lines are $\mathbf{l} = (-1, 0, 1)^\top$, $\mathbf{l} = (-1, 0, 2)^\top$, and their intersection point is:

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which is the point at infinity in the direction of the *y*-axis.

Ideal Points and the Line at Infinity

- The points $\mathbf{x} = (x_1, x_2, x_3)^\top$ with last coordinate $x_3 = 0$ are known as ideal points, or points at infinity.
- The set of all ideal points may be written $(x_1, x_2, 0)^\top$, with a particular point specified by the ratio $x_1 : x_2$.
- Note that this set lies on a single line, the line at infinity, denoted by the vector $\mathbf{l}_\infty = (0, 0, 1)^\top$.

Proof:

$$(0, 0, 1)(x_1, x_2, 0)^\top = 0.$$

Ideal Points and the Line at Infinity

- The parallel lines $\mathbf{l} = (a, b, c)^\top$ and $\mathbf{l}' = (a, b, c')^\top$ intersects $\mathbf{l}_\infty$ at the ideal point $(b, -a, 0)^\top$ for all c's.
- In inhomogeneous notation $(b, -a)^\top$ is a vector tangent to the line, and orthogonal to the line normal (a, b), and so represents the line's *direction*.
- As the line's direction varies, the ideal point $(b, -a, 0)^\top$ varies over $\mathbf{l}_\infty$.
- Hence, the line at infinity can be thought of as the set of directions of lines in the plane.

Duality principle

- Notice how the role of points and lines may be interchanged in:

1. Incidence equations, i.e. $\mathbf{l}^\top \mathbf{x} = 0$ and $\mathbf{x}^\top \mathbf{l} = 0$.
2. Intersection of two lines and the line through two points, i.e. $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$ and $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.

- These observations lead to the duality principle.

Duality principle

- **Duality principle.** To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem.
- Consequently, it not necessary to prove the dual of a given theorem once the original theorem has been proven.
- The proof of the dual theorem will be the dual of the proof of the original theorem.

Conics and Dual Conics

- A conic is a curve described by a second-degree equation in the plane.
- In Euclidean geometry, conics are of three main types: hyperbola, ellipse, and parabola.
- These three types of conic arise as conic sections generated by planes of differing orientation.
- **Note**: there are also degenerate conics, which we will define later.

Conics and Dual Conics

Types of conics:

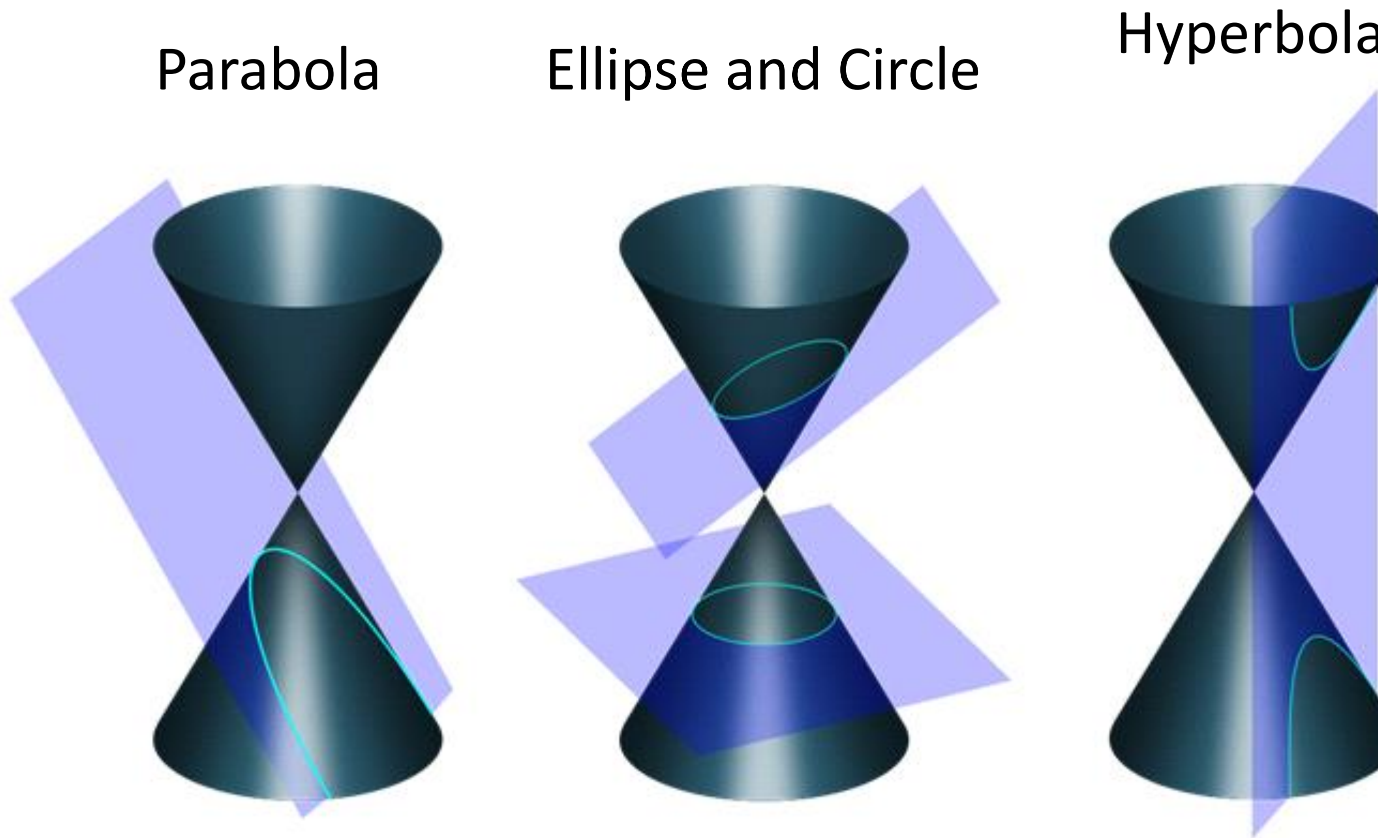


Image source: https://en.wikipedia.org/wiki/Conic_section

Conics and Dual Conics

- The equation of a conic in inhomogeneous coordinates is a polynomial of degree 2, i.e.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

- "Homogenizing" this by the replacements: $x \rightarrow \frac{x_1}{x_3}, y \rightarrow \frac{x_2}{x_3}$ gives

$$a{x_1}^2 + bx_1x_2 + c{x_2}^2 + dx_1x_3 + ex_2x_3 + f{x_3}^2 = 0$$

Conics and Dual Conics

- Or in matrix form: $\mathbf{x}^{\mathsf{T}} \mathtt{C} \mathbf{x} = 0$, where C is symmetric and given by:

$$\mathtt{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}.$$

- C is a homogeneous representation of a conic.

- Only the ratios of the matrix elements are important, multiplying C by a non-zero scalar has no effect.

Conics and Dual Conics

$$\mathbf{x}^\mathsf{T} \mathtt{C} \mathbf{x} = 0, \qquad \mathtt{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}.$$

- The conic has five degrees of freedom which can be thought of as the ratios $\{a : b : c : d : e : f\}$.
- Or equivalently the six elements of a symmetric matrix less one for scale.

Five Points Define a Conic

- Each point $\mathbf{x}_i = (x_i, y_i)$ places one constraint on the conic coefficients:

$$a{x_i}^2 + bx_iy_i + c{y_i}^2 + dx_i + ey_i + f = 0.$$

- This constraint can be written as:

$$\begin{pmatrix} x_i^2 & x_iy_i & y_i^2 & x_i & y_i & 1 \end{pmatrix} \mathbf{c} = 0$$

- where $\mathbf{c} = (a, b, c, d, e, f)^\top$ is the conic C represented as a 6-vector.

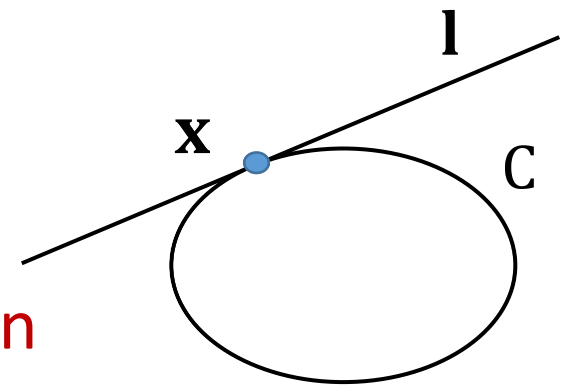
Five Points Define a Conic

- Stacking the constraints from five points we obtain

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

- The conic is the null vector of this 5×6 matrix.
- This shows that a conic is determined uniquely (up to scale) by five points in general position.

Conics and Dual Conics



Tangent lines to conics:

The line $\mathbf{l}$ tangent to C at a point $\mathbf{x}$ on C is given by $\mathbf{l} = \mathrm{C}\mathbf{x}$.

Proof:

The line $\mathbf{l} = \mathrm{C}\mathbf{x}$ passes through $\mathbf{x}$, since $\mathbf{l}^\top \mathbf{x} = \mathbf{x}^\top \mathrm{C}\mathbf{x} = 0$. If $\mathbf{l}$ has one-point contact with the conic, then it is a tangent, and we are done.

$\square$

Conics and Dual Conics

- The conic C defined as far is more properly termed a point conic, as it defines an equation on points.
- There is also a dual (line) conic which defines an equation on lines denoted as C^*(3x3 matrix).
- A line $\mathbf{l}$ *tangent* to the conic C satisfies $\mathbf{l}^\top C^* \mathbf{l} = 0$.
- A dual conic has five degrees of freedom and can be computed from five lines.

Conics and Dual Conics

- For a non-singular symmetric matrix $\mathrm{C}^* = \mathrm{C}^{-1}$ (up to scale).

Proof:

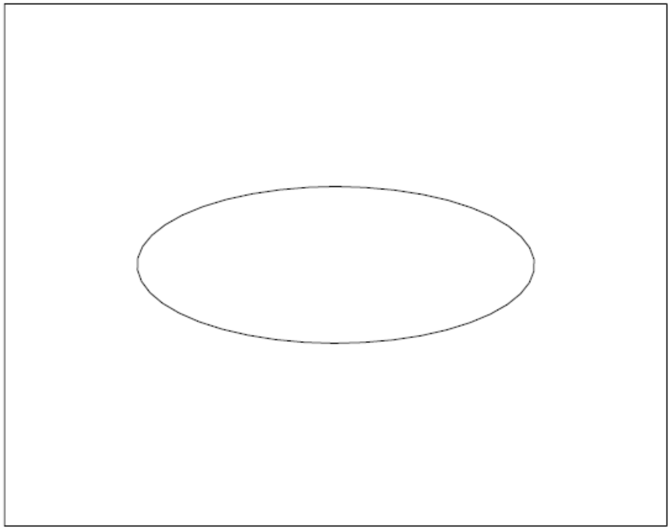
A point $\mathbf{x}$ on C, the tangent is $\mathbf{l} = \mathrm{C}\mathbf{x}$ and this implies $\mathbf{x} = \mathrm{C}^{-1}\mathbf{l}$, i.e. $\mathrm{C}^* = \mathrm{C}^{-1}$ and $\mathbf{x} = \mathrm{C}^*\mathbf{l}$.

Furthermore, since $\mathbf{x}$ satisfies $\mathbf{x}^\top \mathrm{C}\mathbf{x} = 0$, we obtain $(\mathrm{C}^{-1}\mathbf{l})^\top \mathrm{C}(\mathrm{C}^{-1}\mathbf{l}) = \mathbf{l}^\top \mathrm{C}^{-1}\mathbf{l} = 0$, where $\mathrm{C}^{-\top} = \mathrm{C}^{-1}$; we can write as $\mathbf{l}^\top \mathrm{C}^*\mathbf{l} = 0$.

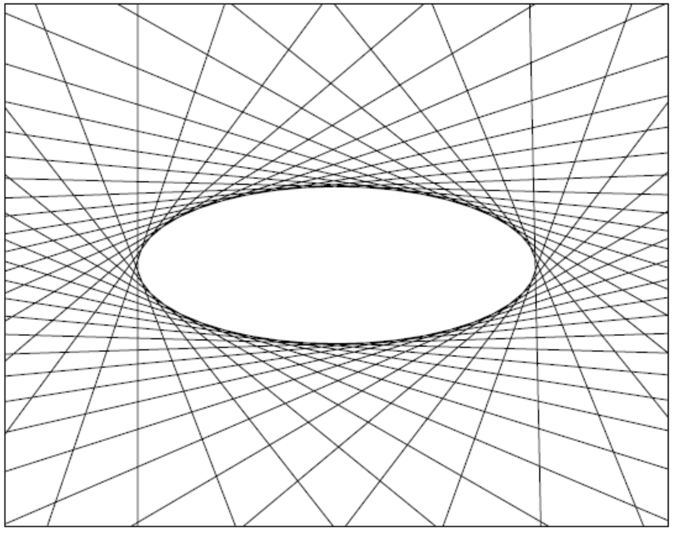
$\square$

Conics and Dual Conics

- Dual conics are also known as conic envelopes:



Points $\mathbf{x}$ satisfying $\mathbf{x}^\top \mathrm{C}\mathbf{x} = 0$ lie on a point conic.



Lines $\mathbf{l}$ satisfying $\mathbf{l}^\top \mathrm{C}^*\mathbf{l} = 0$ are tangent to the point conic C. The conic C is the envelope of the lines $\mathbf{l}$.

Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Degenerate Conics

- Suppose that $\mathbf{l}$ meets the conic at another point $\mathbf{y}$, then $\mathbf{y}^\top \mathrm{C}\mathbf{y} = 0$ and $\mathbf{x}^\top \mathrm{C}\mathbf{y} = \mathbf{l}^\top \mathbf{y} = 0$.

- From this it follows:

$$(\mathbf{x} + \alpha\mathbf{y})^\top \mathrm{C}(\mathbf{x} + \alpha\mathbf{y}) = 0 \text{, for all } \alpha \in \mathbb{R}$$

$$\mathbf{l} = \mathrm{C}\mathbf{x} = \mathrm{C}\mathbf{y} \;\text{ and }\; \mathbf{l}^\top \mathrm{y} = \mathbf{l}^\top \mathbf{x} = \mathbf{0};\;\; \mathrm{rank}(\mathrm{C}) < 3$$

- This means that the whole line $\mathbf{l} = \mathrm{C}\mathbf{x}$ joining $\mathbf{x}$ and $\mathbf{y}$ lies on the conic C, which is therefore degenerate.

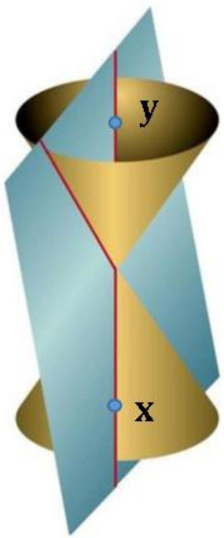
Degenerate Conics

Case 1: Two intersecting lines

The conic $\mathtt{C} = \mathbf{l}\,\mathbf{m}^\top + \mathbf{m}\,\mathbf{l}^\top$ is composed of two lines $\mathbf{l}$ and $\mathbf{m}$. Points on $\mathbf{l}$ satisfy $\mathbf{l}^\top\mathbf{x} = 0$, and are on the conic since

$$\mathbf{x}^\top\mathrm{C}\mathbf{x} = (\mathbf{x}^\top\mathbf{l})(\mathbf{m}^\top\mathbf{x}) + (\mathbf{x}^\top\mathbf{m})(\mathbf{l}^\top\mathbf{x}) = 0.$$

$$\mathrm{rank}(\mathrm{C}) = 2,$$



We can see geometrically that this is two straight lines:

$$\mathbf{x}^\top\mathbf{l} = \mathbf{l}^\top\mathbf{x} = l_1x_1 + l_2x_2 + l_3$$
$$\mathbf{m}^\top\mathbf{x} = \mathbf{x}^\top\mathbf{m} = m_1x_1 + m_2x_2 + m_3$$

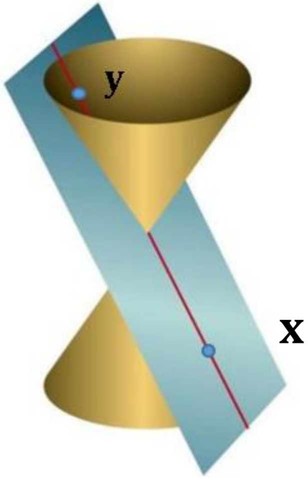
Image source: https://slideplayer.com/slide/12844330/

Degenerate Conics

Case 2: Repeated Lines

Similar method can be used to show that $\mathrm{C} = \mathbf{l}\mathbf{l}^\top + \mathbf{l}\mathbf{l}^\top$ contains a repeated line. We have

$$\mathbf{x}^\top \mathrm{C}\mathbf{x} = (\mathbf{x}^\top \mathbf{l})(\mathbf{l}^\top \mathbf{x}) + (\mathbf{x}^\top \mathbf{l})(\mathbf{l}^\top \mathbf{x}) = 0,$$
$$\mathrm{rank}(\mathrm{C}) = 1,$$



which consists of

$$\mathbf{x}^\top \mathbf{l} = \mathbf{l}^\top \mathbf{x} = l_1 x_1 + l_2 x_2 + l_3.$$

Image source: https://slideplayer.com/slide/12844330/

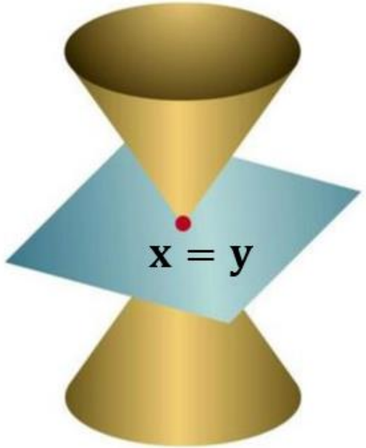
Degenerate Dual Conics

Case 3: Repeated Points

The line conic $\mathrm{C}^* = \mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top$, where $\mathbf{x} = \mathbf{y}$ has rank 1 and consists of lines passing through either of the two points.

Note that for matrices that are not invertible $(\mathrm{C}^*)^* \neq \mathrm{C}$.

- Degenerate dual (line) conics include two points (rank 2), and a repeated point (rank 1).



Planar Projective Transformations

- 2D projective geometry is the study of properties of the projective plane $\mathbb{P}^2$ that are invariant under a group of transformations known as projectivities.

- A projectivity is an invertible mapping h from $\mathbb{P}^2$ to itself such that three points $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ lie on the same line if and only if $h(\mathbf{x}_1)$, $h(\mathbf{x}_2)$ and $h(\mathbf{x}_3)$ do.

- A projectivity is also called a collineation, a projective transformation or a homography.

Planar Projective Transformations

Theorem:

A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a projectivity if and only if there exists a non-singular 3×3 matrix H such that for any point in $\mathbb{P}^2$ represented by a vector $\mathbf{x}$ it is true that $h(\mathbf{x}) = \mathrm{H}\mathbf{x}$.

Note: We show partial proof of collinearity is preserved (i.e., projectivity) by the transformation and skip the converse which is harder to prove, i.e., each projectivity arises in this way.

Planar Projective Transformations

Partial Proof:

- Let $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ lie on a line $\mathbf{l}$. Thus, $\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$ and $\mathbf{l}^\top \mathbf{x}_3 = 0 \Rightarrow (\mathbf{x}_1 \times \mathbf{x}_2)^\top \mathbf{x}_3 = 0$.
- Let H be a non-singular 3×3 matrix.
- We have $\mathbf{l}' = \mathbf{x}_1' \times \mathbf{x}_2' = \mathrm{H}\mathbf{x}_1 \times \mathrm{H}\mathbf{x}_2$ and $\mathbf{l}'^\top \mathbf{x}_3' = (\mathrm{H}\mathbf{x}_1 \times \mathrm{H}\mathbf{x}_2)^\top \mathrm{H}\mathbf{x}_3 = 0$.

Planar Projective Transformations

Partial Proof:

- Since $(\mathrm{H}\mathbf{x}_1 \times \mathrm{H}\mathbf{x}_2) = (\det \mathrm{H})(\mathrm{H}^{-1})^\top(\mathbf{x}_1 \times \mathbf{x}_2)$, we get:

$$\left((\det \mathrm{H})(\mathrm{H}^{-1})^\top(\mathbf{x}_1 \times \mathbf{x}_2)\right)^\top \mathrm{H}\mathbf{x}_3 = 0$$

$$\Rightarrow \left((\det \mathrm{H})(\mathrm{H}^{-1})^\top(\mathbf{x}_1 \times \mathbf{x}_2)\right)^\top \mathrm{H}\mathbf{x}_3 = 0$$

$$\Rightarrow (\det \mathrm{H})(\mathbf{x}_1 \times \mathbf{x}_2)^\top(\mathrm{H}^{-1})\mathrm{H}\mathbf{x}_3 = 0$$

$$\Rightarrow (\mathbf{x}_1 \times \mathbf{x}_2)^\top \mathbf{x}_3 = 0$$

- Thus, the points $\mathbf{x}_i'$ all lie on the line $\mathbf{l}'$, and hence collinearity is preserved by the transformation.

Planar Projective Transformations

- We now define planar projective transformation as:

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \qquad \mathbf{x}' = \mathrm{H}\mathbf{x}.$$

- Properties of H:

1. Non-singular 3×3 matrix;
2. Homogeneous matrix since only the ratio of the matrix elements is significant;
3. Eight degrees of freedom, i.e., eight independent ratios amongst the nine elements of H.

Planar Projective Transformations

- Central projection maps points on one plane to points on another plane.
- And represented by a linear mapping of homogeneous coordinates $\mathbf{x}' = \mathrm{H}\mathbf{x}$.

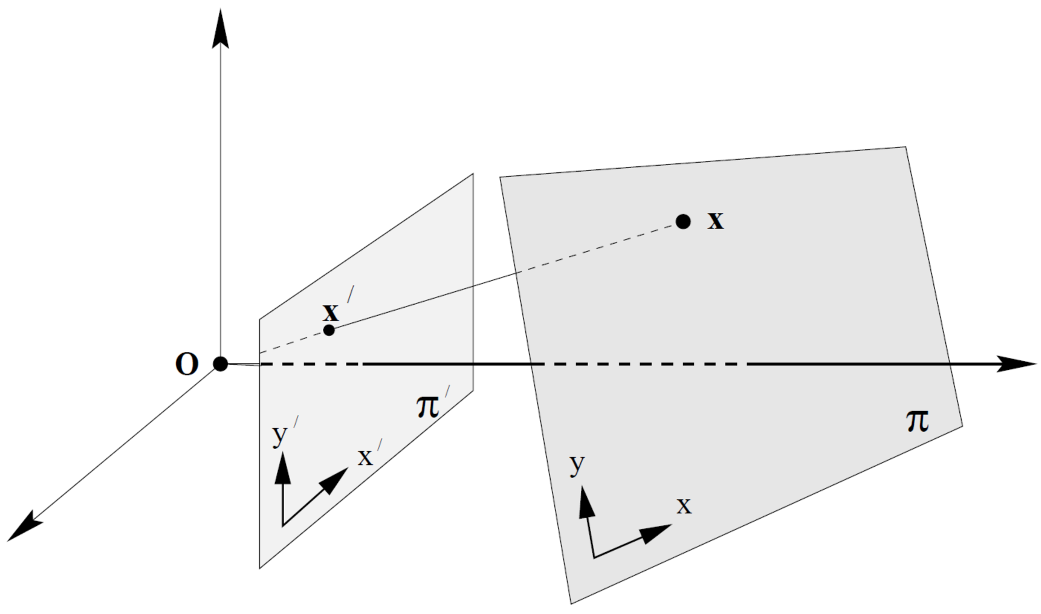
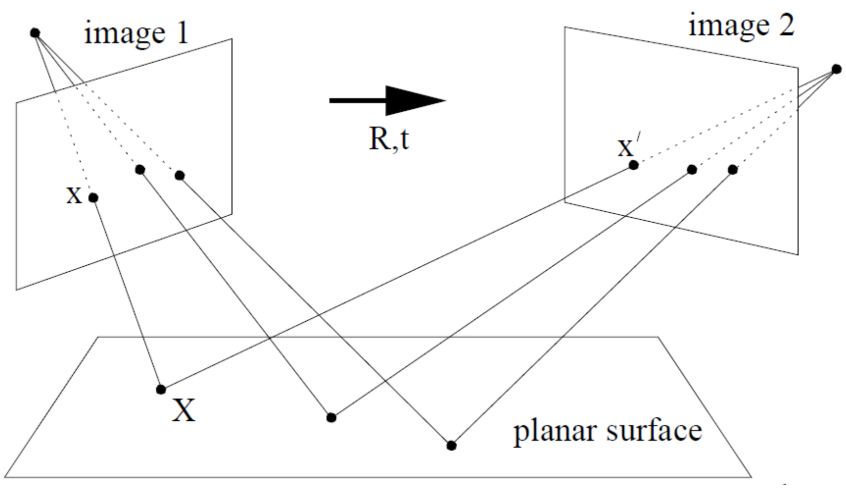
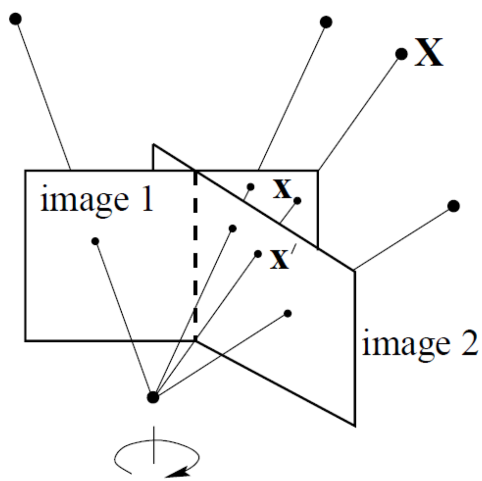


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Planar Projective Transformations

- Examples of a projective transformation $\mathrm{x}' = \mathrm{H}\mathrm{x}$, arising in perspective images.





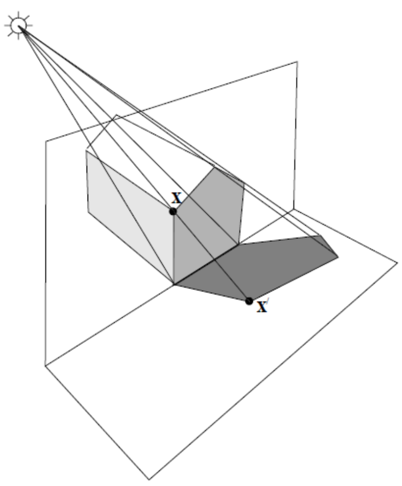


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Transformations of Lines and Conics

- We have seen earlier that if points $\mathbf{x}_i$ lie on a line $\mathbf{l}$, then the transformed points $\mathbf{x}_\mathbf{i}' = \mathrm{H}\mathbf{x}_\mathbf{i}$ under a projective transformation lie on the line $\mathbf{l}'$.

- In this way, incidence of points on lines is preserved, and thus $\mathbf{l}'^\top\mathbf{x}_i' = \mathbf{l}^\top\mathrm{H}^{-1}\mathrm{H}\mathbf{x}_i = 0$ must be satisfied.

- This means that under the point transformation $\mathbf{x}' = \mathrm{H}\mathbf{x}$, a line transforms as:

$$\mathbf{l}' = \mathrm{H}^{-\top}\mathbf{l}\text{, or}\qquad \mathbf{l}'^\top = \mathbf{l}^\top\mathrm{H}^{-1}.$$

Transformations of Lines and Conics

- Under a point transformation $\mathbf{x}' = \mathrm{H}\mathbf{x}$, a conic C transforms to $\mathrm{C}' = \mathrm{H}^{-\top}\mathrm{C}\mathrm{H}^{-1}$.
- Under a point transformation $\mathbf{x}' = \mathrm{H}\mathbf{x}$, a dual conic C^* transforms to $\mathrm{C}^{*\prime} = \mathrm{H}\mathrm{C}^*\mathrm{H}^\top$.

Proof:

Under a point transformation $\mathbf{x}' = \mathrm{H}\mathbf{x}$,

$$\begin{aligned} \mathbf{x}^\top \mathrm{C}\mathbf{x} &= \mathbf{x}'^\top [\mathrm{H}^{-1}]^\top \mathrm{C}\mathrm{H}^{-1}\mathbf{x}' \\ &= \mathbf{x}'^\top \mathrm{H}^{-\top}\mathrm{C}\mathrm{H}^{-1}\mathbf{x}' \end{aligned}$$

which is a quadratic form $\mathbf{x}'^\top\mathrm{C}'\mathbf{x}'$ with $\mathrm{C}' = \mathrm{H}^{-\top}\mathrm{C}\mathrm{H}^{-1}$. $\square$

Hierarchy of Transformations: Isometries

- Isometries are transformations of the plane $\mathbb{R}^2$ that preserve Euclidean distance, and represented as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$, or

$$\mathbf{x}' = \mathrm{H}_{\mathrm{E}}\mathbf{x} = \begin{bmatrix} \mathrm{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x}$$

- where $\epsilon = \pm 1$.

Hierarchy of Transformations: Isometries

- If ϵ =1, then the isometry is orientation-preserving and is a Euclidean transformation (rotation matrix R and translation t).

- If $\epsilon = -1$, then the isometry reverses orientation, e.g., reflection.

- **Invariants:** Length, angle and area.

Hierarchy of Transformations: Similarity

- Similarity transformation is an isometry composed with an isotropic scaling, and represented as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$, or

$$\mathbf{x}' = \mathrm{H}_\mathrm{S}\mathbf{x} = \begin{bmatrix} s\mathrm{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \mathbf{x}$$

- where the scalar s represents the isotropic scaling.

Hierarchy of Transformations: Similarity

- A similarity transformation is also known as an *equi-form* transformation, because it preserves “shape” (form).

- H_s has four degrees of freedom (3 isometry + 1 scale) and can be computed from two-point correspondences.

- **Invariants:** Angles, ratio of two lengths and ratio of areas.

Hierarchy of Transformations: Affinity

- Affine transformation is a non-singular linear transformation followed by a translation, and represented as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \text{, or} \quad \mathbf{x}' = \mathrm{H}_{\mathrm{A}} \mathbf{x} = \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x}$$

- A is a 2×2 non-singular matrix.

- H_A has six degrees of freedom and can be computed from three-point correspondences.

- **Invariants:** parallel lines, ratio of lengths of parallel line segments and ratio of areas.

Hierarchy of Transformations: Affinity

- The affine matrix A can always be decomposed as:

$$\mathtt{A} \;=\; \mathrm{R}(\theta)\,\mathrm{R}(-\phi)\,\mathrm{D}\,\mathrm{R}(\phi)$$

- $\mathrm{R}(\theta)$ and $\mathrm{R}(\phi)$ are rotations by θ and ϕ respectively, and D is a diagonal matrix:

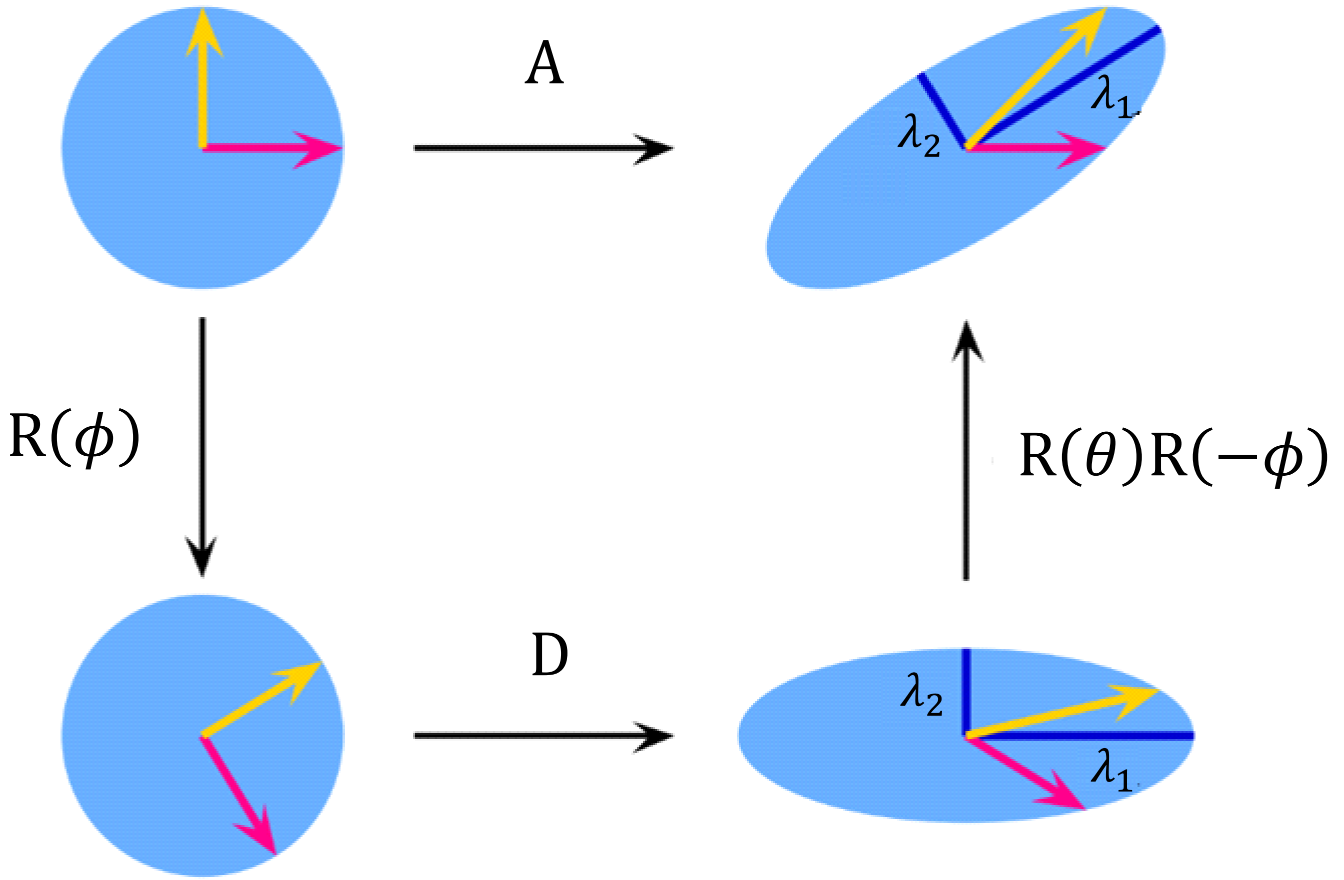
$$\mathtt{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

- This decomposition follows directly from the Singular Value Decomposition (SVD):

$$\mathrm{A} = \mathrm{UDV}^\top = (\mathrm{UV}^\top)(\mathrm{VDV}^\top) = \mathrm{R}(\theta)\big(\mathrm{R}(-\phi)\mathrm{DR}(\phi)\big).$$

- Since U and V are orthogonal matrices.

Hierarchy of Transformations: Affinity



$$A = R(\theta)\big(R(-\phi)DR(\phi)\big)$$

Image modified from: https://en.wikipedia.org/wiki/Singular_value_decomposition

Hierarchy of Transformations: Projective

- Projective transformation is a general non-singular linear transformation of homogeneous coordinates, and represented as:

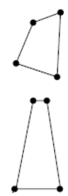
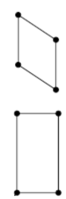
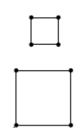
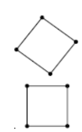
$$\mathbf{x}' = \mathrm{H}_\mathrm{P}\mathbf{x} = \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \mathbf{x} \ ,$$

- where the vector $\mathbf{v} = (v_1, v_2)^\mathsf{T}$ and v can be 0.
- H_p has nine elements with only their ratio significant, so the transformation has eight degrees of freedom.

Hierarchy of Transformations: Projective

- Note, it is not always possible to scale the matrix such that v is unity since v might be zero.
- A projective transformation between two planes can be computed from four-point correspondences, with no three collinear on either plane.
- Not possible to distinguish between orientation preserving and orientation reversing projectivities in $\mathbb{P}^2$.
- **Invariants:** order of contact, tangency (2-pt contact) and cross ratio (details later).

Hierarchy of Transformations

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$ (more later).
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (more later).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Source: Page 44, Table 2.1, "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Decomposition of a Projective Transformation

- A projective transformation can be decomposed into a chain of transformations:

$$\mathtt{H} = \mathtt{H}_\mathrm{S}\,\mathtt{H}_\mathrm{A}\,\mathtt{H}_\mathrm{P} = \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathtt{K} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix}$$

- A a non-singular matrix given by $\mathrm{A} = s\mathrm{RK} + \mathbf{tv}^\mathsf{T}$.
- K an upper-triangular matrix normalized as $\det(\mathrm{K}) = 1$.
- Decomposition is valid provided $v \neq 0$, and is unique if s is chosen positive.
- We will see that this decomposition preserves geometric properties of $\mathbf{l}_\infty$ and the circular points (Lecture 2).

Projective Geometry of 1D (Line), $\mathbb{P}^1$

- We denote a point on the line as the homogeneous coordinates $\bar{\mathbf{x}}' = (x_1, x_2)^\top$.

- $x_2 = 0$ is an ideal point of the line.

- A projective transformation of a line is represented by a 2×2 homogeneous matrix,

$$\bar{\mathbf{x}}' = \mathrm{H}_{2\times 2}\bar{\mathbf{x}}$$

- $\mathrm{H}_{2\times 2}$ has 3 dof corresponding to 4 elements less one for over scaling and can be computed from 3 points.

Projective Geometry of 1D (Line), $\mathbb{P}^1$

The Cross Ratio

- The cross ratio is the basic projective invariant of $\mathbb{P}^1$. Given 4 points $\bar{\mathbf{x}}_i$ the *cross ratio* is defined as:

$$\mathrm{Cross}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) = \frac{|\bar{\mathbf{x}}_1 \bar{\mathbf{x}}_2||\bar{\mathbf{x}}_3 \bar{\mathbf{x}}_4|}{|\bar{\mathbf{x}}_1 \bar{\mathbf{x}}_3||\bar{\mathbf{x}}_2 \bar{\mathbf{x}}_4|} ,$$

where

$$|\bar{\mathbf{x}}_i \bar{\mathbf{x}}_j| = \det \begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix} .$$

- If each point $\bar{\mathbf{x}}_i$ is a finite point and $x_{i2} = 1$, then $|\bar{\mathbf{x}}_i \bar{\mathbf{x}}_j|$ represents the signed distance from $\bar{\mathbf{x}}_i$ to $\bar{\mathbf{x}}_j$.

- Definition of the cross ratio is also valid if one of the points $\bar{\mathbf{x}}_i$ is an ideal point.

Projective Geometry of 1D (Line), $\mathbb{P}^1$

- The value of the cross ratio is invariant under any projective transformation of the line: if $\bar{\mathrm{x}}' = \mathrm{H}_{2\times 2}\bar{\mathrm{x}}$ then

$$\mathrm{Cross}(\bar{\mathbf{x}}'_1, \bar{\mathbf{x}}'_2, \bar{\mathbf{x}}'_3, \bar{\mathbf{x}}'_4) = \mathrm{Cross}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4).$$

Exercise: Prove it!

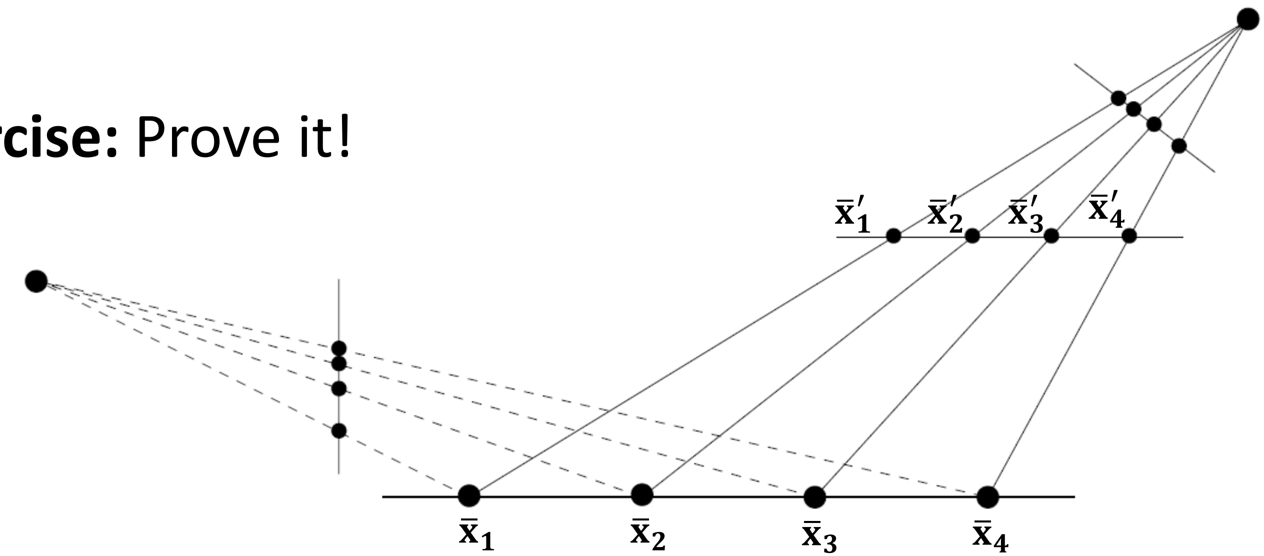


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Projective Geometry of 1D (Line), $\mathbb{P}^1$

Concurrent Lines

- A configuration of concurrent lines is dual to collinear points on a line, i.e. concurrent lines on a plane are also in $\mathbb{P}^1$.

- Four concurrent lines $\mathbf{l}_i$ intersect the line $\mathbf{l}$ in the four points $\bar{\mathbf{x}}_i$.

- The cross ratio of these lines is an invariant to projective transformations of the plane.

- Its value is given by the cross ratio of the points, $\mathrm{Cross}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)$.

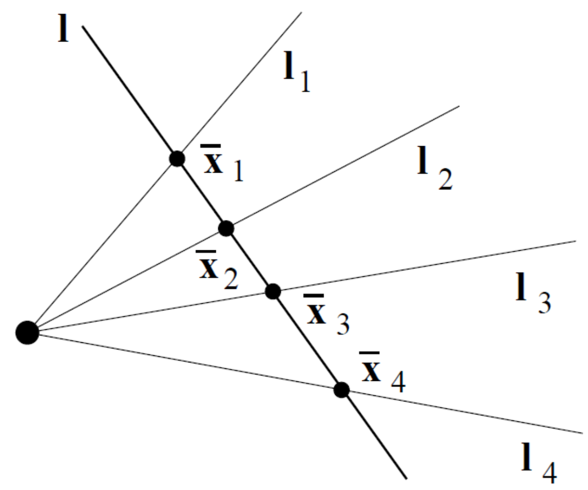


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Projective Geometry of 1D (Line), $\mathbb{P}^1$

Concurrent Lines

- Coplanar points $\mathbf{x}_i$ are imaged onto a line $\mathbf{l}$ (also in the plane) by a projection with centre $\mathbf{C}$.

- May be thought of as representing projection of points in $\mathbb{P}^2$ into a 1-dimensional image.

- In particular, the line $\mathbf{l}$ represents an 1D analogue of the image plane.

- The cross ratio of the image points $\bar{\mathbf{x}}_\mathrm{i}$ is invariant to the position of the image line $\mathbf{l}$.

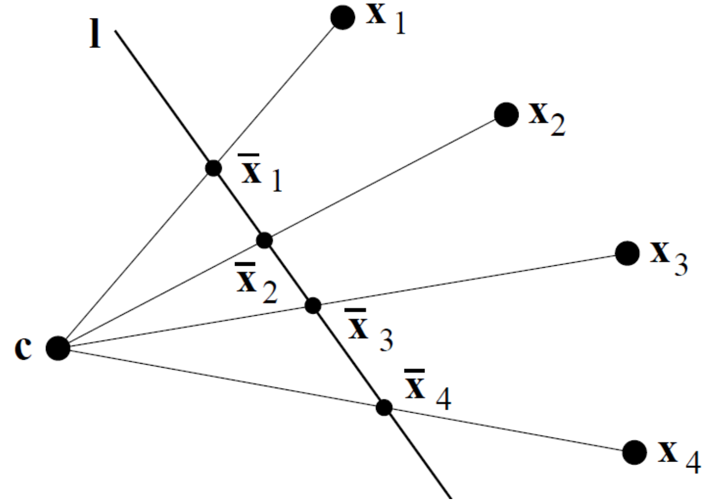


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Summary

- We have looked at how to:

  1. Explain the difference between Euclidean and Projective geometry.

  2. Use homogenous coordinates to represent points, lines and conics in the projective space.

  3. Describe the duality relation between lines and points, and conics and dual conics on a plane.

  4. Apply the hierarchy of transformations on points, lines and conics.