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Homogenization of a Wilson–Cowan model for neural fields

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ABSTRACT

Homogenization of Wilson–Cowan type of nonlocal neural field models is investigated. Motivated by the presence of a convolution term in this type of models, we first prove some general convergence results related to convolution sequences. We then apply these results to the homogenization problem of the Wilson–Cowan-type model in a general deterministic setting. Key ingredients in this study are the notion of algebras with mean value and the related concept of sigma-convergence.

1. Introduction

Experiments/observations through different EEG, fMRI, MEG and optical imaging techniques reveal electrical activity patterns spanning over several centimeters of brain tissue, i.e. of length scales much larger than the spatial extent of one single neuron. Moreover, these structures have a lifetime which is much larger than the lifetime of typical firing time for a neuron. Now, as the cortex obviously is a heterogeneous medium possessing many different spatial and temporal scales, there is the need to have rigorous ways of determining how the spatio-temporal microstructure is stored in mean field models for the brain activity. One way of doing this is by means of *homogenization theory* based on multi-scale convergence techniques. The problem of homogenization or *scaling* is to determine, from data or local characteristics, the effective models representing the macroscopic behavior of mesoscopically inhomogeneous media.

In the quest of studying and understanding the neurodynamics, several heuristic models have been developed. The classical *leaky integrator unit* model [1–3] described by ordinary differential equations has given rise to the most studied *Wilson–Cowan type of models* [4] (see also Amari [5]) which, in the one-dimensional space, reads as

$$\frac{\partial}{\partial t}u(x,t) = -u(x,t) + \int_{-\infty}^{\infty} J(x',x)f(u(x',t))dx'. \tag{1.1}$$

Here, $u(x,t)$ denotes the neural field which measures the local activity of neurons at position $x \in \mathbb{R}$. The integral represents the synaptic input where the function $J(x,x')$ measures the strength of connections between neurons at positions x and x'. We refer to this function as the connectivity function. The function f is the firing rate function. Eq. (1.1) models the neural field in a homogeneous medium. For more details concerning this model we refer for example to Wilson and Cowan [4] and Amari [5] and the more recent work by Coombes [6], and the references therein.

A drawback with model (1.1) above is that does not take into account that the brain is very heterogeneous with a structure that exhibits multiple spatial scales ranging from micrometer to decimeter. In addition the dynamical activities are taking place on multiple time scales. In order to capture several such properties we are led to allow the connectivity to depend

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on both space and time and also to depend on multiple spatial and temporal scales. In the present work we will consider time-independent connectivity kernels.

An obvious way to impose heterogeneity in the well-accepted Wilson–Cowan or Amari model (1.1) is to replace the connectivity function J by a function J_ε, where the parameter $\varepsilon > 0$ measures the heterogeneity of the brain tissue.

The homogenization procedure has been used in many applied science fields to upscale various mathematical models. As far as the neural field models are concerned, there are very few works dealing with homogenization techniques. See, for example, [7] or [8]. In [7], a homogenization-based approach for studying nonlocal heterogeneous neural field models of Wilson–Cowan type based on multi-scale expansion techniques is initiated. A heterogeneous neural field model is also advocated in [8], where the point of departure is the parameterized Wilson–Cowan model

$$\frac{\partial}{\partial t}u_\varepsilon(x,t) = -u_\varepsilon(x,t) + \int_{\mathbb{R}^N} J_\varepsilon(x-x')f(u_\varepsilon(x',t))dx', \quad x \in \mathbb{R}^N, t > 0, \tag{1.2}$$

where u_ε denotes the electrical activity level field, f the firing rate function and $J_\varepsilon = J_\varepsilon(x) = J(x, x/\varepsilon)$ the connectivity kernel, which by assumption is periodic in the second argument $y = x/\varepsilon$. In the present work, under a general deterministic assumption on the kernel (including the periodicity assumption and the almost periodicity assumption) we prove rigorously (see Theorem 4) that, as $\varepsilon \to 0$, the solution u_ε to the Wilson–Cowan model (1.2) converges to the solution u_0 of a homogenized Wilson–Cowan equation

$$\frac{\partial}{\partial t}u_0(x,t,y) = -u_0(x,t,y) + (J \ast\ast f(u_0))(x,t,y). \tag{1.3}$$

In the special periodic case, Eq. (1.3) reads as

$$\frac{\partial}{\partial t}u_0(x,t,y) = -u_0(x,t,y) + \int_{\mathbb{R}^N}\int_Y J(x-x', y-y')f(u_0(x',t,y'))dy'dx'.$$

Due to the nonlinearity in our model equation, we cannot use the Laplace transform in the homogenization process. Also our method works even in the non-Hilbertian framework. Indeed, considering two sequences $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ in $L^1(\mathbb{R}^N)$ and $L^p(Q)$ respectively satisfying $u_\varepsilon \to u_0$ in $L^1(\mathbb{R}^N)$-strong Σ and $v_\varepsilon \to v_0$ in $L^p(Q)$-weak Σ as $\varepsilon \to 0$ (where Q is an open subset of $\mathbb{R}^N$) we get that $u_\varepsilon \ast v_\varepsilon \to u_0 \ast \ast v_0$ in $L^p(Q)$-weak Σ as $\varepsilon \to 0$, where $u_0 \ast \ast v_0$ is a double convolution with respect to both macroscopic and microscopic variables; see Theorem 2. The above result was first proved by Visintin [9] in the periodic setting by using the two-scale transform or unfolding method. Theorem 2 allows us to pass to the limit in the convolution terms without using either the Fourier transform, or the Laplace transform, and hence without restricting ourselves to the Hilbertian setting as is the case in [10]. Taking into account the fact that the brain is not necessarily a periodic medium (even if it can exhibit some kinds of periodicity), we can therefore emphasize that our work is a true advance in the neural field community.

The paper is organized as follows. In Section 2 we recall some background material regarding the concept of sigma-convergence. We also prove two important results which are of independent interest, a general two-scale convergence result for translates (Proposition 3) and a general two-scale convergence result for convolution products (Theorem 2). The method used in deriving these results is based on the notion of algebras with mean value and the concept of sigma-convergence. In Section 3 we prove the existence of a solution to the Wilson–Cowan model and derive the a priori estimate needed for the main homogenization result, which is stated and proved in Section 4. Finally, Section 5 deals with conclusions and outlook.

2. Σ-convergence and convolution

2.1. Some properties of algebras with mean value

Let A be an algebra with mean value on $\mathbb{R}^N$ (see [11,12]);, that is, A is a closed subalgebra of the $\mathcal{C}^*$-algebra of bounded uniformly continuous complex functions $BUC(\mathbb{R}^N)$ which contains the constants, is closed under complex conjugation ($\overline{u} \in A$ whenever $u \in A$), is translation invariant ($u(\cdot + a) \in A$ for any $u \in A$ and each $a \in \mathbb{R}^N$) and such that each element possesses a mean value in the following sense.

(MV) For each $u \in A$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ (where $u^\varepsilon(x) = u(x/\varepsilon), x \in \mathbb{R}^N$) weakly $\ast$-converges in $L^\infty(\mathbb{R}^N)$ to some constant function $M(u) \in \mathbb{C}$ (the complex field).

It is known that A (endowed with the sup norm topology) is a commutative $\mathcal{C}^*$-algebra with identity. We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ (a subset of the topological dual A' of A) is the set of all nonzero multiplicative linear functionals on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ $(s \in \Delta(A))$, where $\langle , \rangle$ denotes the duality pairing between A' and A. We endow $\Delta(A)$ with the relative weak$\ast$ topology on A'. Then using the well-known theorem of Stone (see e.g., either [13] or more precisely [14, Theorem IV.6.18, p. 274]) one may easily show that the spectrum $\Delta(A)$ is a compact topological space, and the Gelfand transformation $\mathcal{G}$ is an isometric isomorphism identifying A with $\mathcal{C}(\Delta(A))$ (the continuous functions on $\Delta(A)$) as $\mathcal{C}^*$-algebras. Next, since each element of A possesses a mean value, this induces a mapping $u \mapsto M(u)$ (denoted by M and called the mean value) which is a nonnegative continuous linear functional on A with $M(1) = 1$, and so provides us with a linear nonnegative functional $\psi \mapsto M_1(\psi) = M(\mathcal{G}^{-1}(\psi))$ defined on $\mathcal{C}(\Delta(A)) = \mathcal{G}(A)$, which is clearly bounded. Therefore, by the Riesz–Markov theorem, $M_1(\psi)$ is representable by integration with respect to some Radon measure β (of total mass 1) in $\Delta(A)$, called the *M-measure* for A [15]. It is evident that we have

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u)d\beta \quad \text{for } u \in A. \tag{2.1}$$

The following result is worth recalling. Its proof can be found in [16], and we recall it here for further purposes.

Theorem 1. *Let A be an algebra with mean value on $\mathbb{R}^N$. The translations $T(y) : \mathbb{R}^N \to \mathbb{R}^N$, $T(y)x = x + y$, extend to a group of homeomorphisms $T(y) : \Delta(A) \to \Delta(A)$, $y \in \mathbb{R}^N$, which forms a continuous N-dimensional dynamical system on $\Delta(A)$ whose invariant probability measure is precisely the M-measure β for A.*

Proof. As A is translation invariant, each translation $T(y)$ induces an isometric isomorphism still denoted by $T(y)$, from A onto A, defined by $T(y)u = u(\cdot + y)$ for $u \in A$. Define $\widetilde{T}(y) : \mathcal{C}(\Delta(A)) \to \mathcal{C}(\Delta(A))$ by

$$\widetilde{T}(y)\mathcal{G}(u) = \mathcal{G}(T(y)u) \quad (u \in A),$$

where $\mathcal{G}$ denotes the Gelfand transformation on A. Then $\widetilde{T}(y)$ is an isometric isomorphism of $\mathcal{C}(\Delta(A))$ onto itself; this is easily seen by the fact that $\mathcal{G}$ is an isometric isomorphism of A onto $\mathcal{C}(\Delta(A))$. Therefore, by the classical Banach–Stone theorem there exists a unique homeomorphism $\overline{T}(y)$ of $\Delta(A)$ onto itself. The family thus constructed is in fact a continuous N-dimensional dynamical system. Indeed the group property easily comes from the equality $\mathcal{G}(T(y)u)(s) = \mathcal{G}(u)(\overline{T}(y)s)$ ($y \in \mathbb{R}^N$, $s \in \Delta(A)$, $u \in A$). As far as the continuity property is concerned, let $(y_n)_n$ be a sequence in $\mathbb{R}^N$ and $(s_d)_d$ be a net in $\Delta(A)$ such that $y_n \to y$ in $\mathbb{R}^N$ and $s_d \to s$ in $\Delta(A)$. Then the uniform continuity of $u \in A$ leads to $T(y_n)u \to T(y)u$ in $BUC(\mathbb{R}^N)$, and the continuity of $\mathcal{G}$ gives $\mathcal{G}(T(y_n)u) \to \mathcal{G}(T(y)u)$, the last convergence result being uniform in $\mathcal{C}(\Delta(A))$. Hence $\mathcal{G}(T(y_n)u)(s_d) \to \mathcal{G}(T(y)u)(s)$, which is equivalent to $\mathcal{G}(u)(\overline{T}(y_n)s_d) \to \mathcal{G}(u)(\overline{T}(y)s)$. As $\mathcal{C}(\Delta(A))$ separates the points of $\Delta(A)$, this yields $\overline{T}(y_n)s_d \to \overline{T}(y)s$ in $\Delta(A)$, which implies that the mapping $(y, s) \mapsto \overline{T}(y)s$, from $\mathbb{R}^N \times \Delta(A)$ to $\Delta(A)$, is continuous. It remains to check that β is the invariant measure for $\overline{T}$. But this easily comes from the invariance under translations property of the mean value and of the integral representation (2.1). We keep using the notation $T(y)$ for $\overline{T}(y)$, and the proof is complete. $\square$

Next, let B_A^p $(1 \leq p < \infty)$ denote the Besicovitch space associated to A, that is, the closure of A with respect to the Besicovitch seminorm

$$\|u\|_p = \left(\limsup_{r\to+\infty} \frac{1}{|B_r|} \int_{B_r} |u(y)|^p \, dy\right)^{1/p},$$

where B_r is the open ball of $\mathbb{R}^N$ of radius r. It is known that B_A^p is a complete seminormed vector space. Moreover, we have $B_A^q \subset B_A^p$ for $1 \leq p \leq q < \infty$. The following properties are worth noticing [17,18].

(1) The Gelfand transformation $\mathcal{G} : A \to \mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping, still denoted by $\mathcal{G}$, of B_A^p into $L^p(\Delta(A))$, which in turn induces an isometric isomorphism $\mathcal{G}_1$, of $\mathcal{B}_A^p = B_A^p/\mathcal{N}$ onto $L^p(\Delta(A))$ (where $\mathcal{N} = \{u \in B_A^p : \mathcal{G}(u) = 0\}$). Furthermore, if $u \in B_A^p \cap L^\infty(\mathbb{R}^N)$ then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}^N)}$.
(2) The mean value M, viewed as defined on A, extends by continuity to a positive continuous linear form (still denoted by M) on B_A^p satisfying $M(u) = \int_{\Delta(A)} \mathcal{G}(u)d\beta$ $(u \in B_A^p)$. Furthermore, $M(u(\cdot + a)) = M(u)$ for each $u \in B_A^p$ and all $a \in \mathbb{R}^N$.
(3) The dynamical system $T(y)$ is ergodic if and only if for every $u \in B_A^p$ such that $\|u - u(\cdot + y)\|_p = 0$ for every $y \in \mathbb{R}^N$ we have $\|u - M(u)\|_p = 0$.

In order to simplify the text, we will henceforth use the same letter u (if there is no danger of confusion) to denote the equivalence class of an element $u \in B_A^p$. The symbol ϱ will denote the canonical mapping of B_A^p onto $\mathcal{B}_A^p = B_A^p/\mathcal{N}$. For $u \in \mathcal{B}_A^p$ (respectively, $u \in B_A^p$) we shall set $\widehat{u} = \mathcal{G}_1(u)$ (respectively, $\widehat{u} = \mathcal{G}(u)$).

In our study we will deal with a special class $\mathbb{A}$ of algebras with mean value. An algebra with mean value A is in $\mathbb{A}$ if it is separable (hence its spectrum $\Delta(A)$ is a compact metric space) and further $\Delta(A)$ has group property. Below are some examples of algebras with mean value verifying the above property. (1) Let $A = \mathcal{C}_{\text{per}}(Y)(Y = (0, 1)^N)$ denote the algebra of Y-periodic complex continuous functions on $\mathbb{R}^N$. It is well known that $\Delta(A) = \mathbb{R}^N/\mathbb{Z}^N$ (the N-torus) which is a separable compact metrizable topological group. (2) If we denote by $AP(\mathbb{R}^N)$ the space of complex continuous almost periodic functions on $\mathbb{R}^N$ [19,20], then, for any countable subgroup $\mathcal{R}$ of $\mathbb{R}^N$, denoting by $AP_{\mathcal{R}}(\mathbb{R}^N)$ the subspace of $AP(\mathbb{R}^N)$ consisting of functions that are uniformly approximated by finite linear combinations of functions in the set $\{\gamma_k : k \in \mathcal{R}\}$ (where $\gamma_k(y) = \exp(2i\pi k \cdot y)$, $y \in \mathbb{R}^N$), we get that $AP_{\mathcal{R}}(\mathbb{R}^N) \in \mathbb{A}$ with the further property that its spectrum is a compact topological group homeomorphic to the dual group $\widehat{\mathcal{R}}$ of $\mathcal{R}$ consisting of characters γ_k $(k \in \mathcal{R})$ of $\mathbb{R}^N$; see [21] for details. (3) Finally, let $\mathcal{B}_\infty(\mathbb{R}^N)$ denote the space of continuous complex functions on $\mathbb{R}^N$ that have finite limit at infinity. It can be easily shown that its spectrum consists of only one point, the Dirac mass at infinity δ_∞, and hence is a compact topological group. Also $\mathcal{B}_\infty(\mathbb{R}^N)$ is separable (see [15]), so $\mathcal{B}_\infty(\mathbb{R}^N) \in \mathbb{A}$. Some other examples can be considered by taking a combination of the previous ones.

Some notation is in order. Let $A \in \mathbb{A}$. Since $\Delta(A)$ is a topological group (which we henceforth denote additively), the mapping $(s, r) \mapsto s + r$ is continuous from $\Delta(A) \times \Delta(A)$ into $\Delta(A)$. $-s$ shall stand for the symmetrization of $s \in \Delta(A)$. Now, in the above notation, if we take a look at the proof of Theorem 1 we observe that the dynamical system associated to the translations is defined by

$$T(y)s = \delta_y + s \quad \text{for } (y, s) \in \mathbb{R}^N \times \Delta(A),$$

where δ_y is the Dirac mass at y. With this in mind, for $s = \delta_x$ and $r = \delta_y$ we may see that we have, in the same notation, $s + r = \delta_{x+y}$ and $s - r = \delta_{x-y}$.

We get the following result.

Proposition 1. *Let $A \in \mathbb{A}$. Let $u \in A$ and $a \in \mathbb{R}^N$. Then*

$$\mathcal{G}(u(\cdot + a)) = \mathcal{G}(u)(\cdot + \delta_a). \tag{2.2}$$

Proof. Let us recall that (2.2) is equivalent to $\mathcal{G}(u(\cdot + a))(s) = \mathcal{G}(u)(s + \delta_a)$ for any $s \in \Delta(A)$. So, let $y \in \mathbb{R}^N$; then

$$\begin{aligned}\mathcal{G}(u(\cdot + a))(\delta_y) &= \left\langle \delta_y, u(\cdot + a)\right\rangle = u(y + a)\\ &= \left\langle \delta_{y+a}, u\right\rangle = \mathcal{G}(u)(\delta_{y+a})\\ &= \mathcal{G}(u)(\delta_y + \delta_a).\end{aligned}$$

Now, let $s \in \Delta(A)$; there exists a sequence $(y_n)_n \subset \mathbb{R}^N$ such that $\delta_{y_n} \to s$ in $\Delta(A)$ as $n \to \infty$ (indeed the set $\{\delta_y : y \in \mathbb{R}^N\}$ is dense in $\Delta(A)$ which is a metric space). Since $\Delta(A)$ is a topological group, the mapping $(s, r) \mapsto s + r$ is continuous from $\Delta(A) \times \Delta(A)$ into $\Delta(A)$. So for fixed $r = \delta_a$ we have that $\delta_{y_n} + \delta_a \to s + \delta_a$, and, by the continuity of $\mathcal{G}(u)$, $\mathcal{G}(u)(\delta_{y_n} + \delta_a) \to \mathcal{G}(u)(s + \delta_a)$ and $\mathcal{G}(u(\cdot + a))(\delta_{y_n}) \to \mathcal{G}(u(\cdot + a))(s)$ as $n \to \infty$. The uniqueness of the limit is given by the fact that $\mathcal{G}(u(\cdot + a))(\delta_{y_n}) = \mathcal{G}(u)(\delta_{y_n} + \delta_a)$, $\mathcal{G}(u(\cdot + a))(s) = \mathcal{G}(u)(s + \delta_a)$. □

2.2. *Σ-convergence and convolution results*

We begin with the definition of the concept of Σ-convergence. Let $A \in \mathbb{A}$, and let Q be an open subset of $\mathbb{R}^N$. Throughout the paper the letter E will denote any ordinary sequence $E = (\varepsilon_n)$ (integers $n \geq 0$) with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Definition 1. (1) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q)$ $(1 \leq p < \infty)$ *is said to* weakly Σ-converge in $L^p(Q)$ *to some* $u_0 \in L^p(Q; \mathcal{B}_A^p)$ *if, as* $\varepsilon \to 0$,

$$\int_Q u_\varepsilon(x)\, \psi^\varepsilon(x)\, dx \to \iint_{Q\times\Delta(A)} \widehat{u}_0(x, s)\, \widehat{\psi}(x, s)\, dx d\beta(s) \tag{2.3}$$

for all $\psi \in L^{p'}(Q; A)$ $\left(1/p' = 1 - 1/p\right)$, *where* $\psi^\varepsilon(x) = \psi(x, x/\varepsilon)$ *and* $\widehat{\psi}(x, \cdot) = \mathcal{G}(\psi(x, \cdot))$*a.e. in* $x \in Q$*. We denote this by* $u_\varepsilon \to u_0$ *in* $L^p(Q)$*-weak* Σ*.*

(2) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q)(1 \leq p < \infty)$ *is said to* strongly Σ-converge *in* $L^p(Q)$ *to some* $u_0 \in L^p(Q; \mathcal{B}_A^p)$ *if it is weakly* Σ*-convergent and further satisfies the following condition:*

$$\|u_\varepsilon\|_{L^p(Q)} \to \|\widehat{u}_0\|_{L^p(Q\times\Delta(A))}. \tag{2.4}$$

We denote this by $u_\varepsilon \to u_0$ *in* $L^p(Q)$*-strong* Σ*.*

We have the following result, whose proof can be found in [17] (see also [22]).

Proposition 2. (i) *Any bounded sequence* $(u_\varepsilon)_{\varepsilon\in E}$ *in* $L^p(Q)$ *(where* $1 < p < \infty$*) admits a subsequence which is weakly* Σ*-convergent in* $L^p(Q)$*.*

(ii) *Any uniformly integrable sequence* $(u_\varepsilon)_{\varepsilon\in E}$ *in* $L^1(Q)$ *admits a subsequence which is weakly* Σ*-convergent in* $L^1(Q)$*.*

We recall that a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(Q)$ is said to be *uniformly integrable* if it is bounded in $L^1(Q)$ and further satisfies the property that $\sup_{\varepsilon>0} \int_X |u_\varepsilon|\, dx \to 0$ for any integrable set $X \subset Q$ for which $|X| \to 0$, where $|X|$ denotes the Lebesgue measure of X.

Now, fix $t \in \mathbb{R}^N$. Then $(\delta_{t/\varepsilon})_{\varepsilon>0}$ is a sequence in the compact metric space $\Delta(A)$, and hence it possesses a convergent subsequence still denoted by $(\delta_{t/\varepsilon})_{\varepsilon>0}$. In what follows we shall consider such a subsequence. Let $r \in \Delta(A)$ be such that

$$\delta_{\frac{t}{\varepsilon}} \to r \quad \text{in } \Delta(A) \text{ as } \varepsilon \to 0. \tag{2.5}$$

Finally, let Q be an open subset in $\mathbb{R}^N$, and let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(Q)$ $(1 \leq p < \infty)$ which is weakly Σ-convergent to $u_0 \in L^p(Q; \mathcal{B}_A^p)$. Define the sequence $(v_\varepsilon)_{\varepsilon>0}$ as follows:

$$v_\varepsilon(x) = u_\varepsilon(x + t), \quad x \in Q - t.$$

We then get the following result.

Proposition 3. *Assume that (2.5) holds. Then, as $\varepsilon \to 0$,*

$$v_\varepsilon \to v_0 \quad \text{in } L^p(Q-t)\text{-weak } \Sigma,$$

where $v_0 \in L^p(Q-t, \mathcal{B}_A^p)$ is defined by $\widehat{v}_0(x,s) = \widehat{u}_0(x+t, s+r)$ for $(x,s) \in (Q-t) \times \Delta(A)$.

Proof. Let $\varphi \in \mathcal{C}_0^\infty(Q-t)$ and $\psi \in A$. Let $(y_n)_n$ be an ordinary sequence (independent of ε) such that

$$\delta_{y_n} \to r \quad \text{in } \Delta(A) \text{ as } n \to \infty.$$

We have

$$\begin{aligned}\int_{Q-t} u_\varepsilon(x+t)\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)dx &= \int_Q u_\varepsilon(x)\varphi(x-t)\psi\left(\frac{x}{\varepsilon}-\frac{t}{\varepsilon}\right)dx \\ &= \int_Q u_\varepsilon(x)\varphi(x-t)\left[\psi\left(\frac{x}{\varepsilon}-\frac{t}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}-y_n\right)\right]dx \\ &\quad + \int_Q u_\varepsilon(x)\varphi(x-t)\psi\left(\frac{x}{\varepsilon}-y_n\right)dx = (I)+(II).\end{aligned}$$

On the one hand, as $\varepsilon \to 0$,

$$(II) \to \iint_{Q\times\Delta(A)} \widehat{u}_0(x,s)\varphi(x-t)\mathcal{G}(\psi(\cdot - y_n))(s)dxd\beta(s).$$

But we have $\mathcal{G}(\psi(\cdot - y_n)) = \widehat{\psi}(\cdot - \delta_{y_n}) \to \widehat{\psi}(\cdot - r)$ uniformly in $\mathcal{C}(\Delta(A))$ as $n \to \infty$. Hence

$$\iint_{Q\times\Delta(A)} \widehat{u}_0(x,s)\varphi(x-t)\mathcal{G}(\psi(\cdot - y_n))(s)dxd\beta(s) \to \iint_{Q\times\Delta(A)} \widehat{u}_0(x,s)\varphi(x-t)\widehat{\psi}(s-r)dxd\beta(s) \text{ as } n\to\infty,$$

and

$$\iint_{Q\times\Delta(A)} \widehat{u}_0(x,s)\varphi(x-t)\widehat{\psi}(s-r)dxd\beta = \iint_{(Q-t)\times\Delta(A)} \widehat{u}_0(x+t,s+r)\varphi(x)\widehat{\psi}(s)dxd\beta.$$

On the other hand,

$$\begin{aligned}|(I)| &\le \int_Q |u_\varepsilon(x)|\,|\varphi(x-t)|\left|\psi\left(\frac{x}{\varepsilon}-\frac{t}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}-y_n\right)\right|dx \\ &\le c\sup_{z\in\mathbb{R}^N}\left|\psi\left(z-\frac{t}{\varepsilon}\right)-\psi(z-y_n)\right| \\ &= c\left\|\psi\left(\cdot-\frac{t}{\varepsilon}\right)-\psi(\cdot-y_n)\right\|_\infty \\ &= c\left\|\mathcal{G}\left(\psi\left(\cdot-\frac{t}{\varepsilon}\right)-\psi(\cdot-y_n)\right)\right\|_\infty \quad (\mathcal{G} \text{ is an isometry}) \\ &= c\left\|\widehat{\psi}\left(\cdot-\delta_{\frac{t}{\varepsilon}}\right)-\widehat{\psi}\left(\cdot-\delta_{y_n}\right)\right\|_\infty.\end{aligned}$$

Now, using the uniform continuity of $\widehat{\psi}$, we obtain

$$\left\|\widehat{\psi}\left(\cdot-\delta_{\frac{t}{\varepsilon}}\right)-\widehat{\psi}\left(\cdot-\delta_{y_n}\right)\right\|_\infty \to 0 \quad \text{as } \varepsilon\to 0 \text{ and next } n\to\infty.$$

It therefore follows that, as $\varepsilon \to 0$,

$$\int_{Q-t} u_\varepsilon(x+t)\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)dx \to \iint_{(Q-t)\times\Delta(A)} \widehat{u}_0(x+t,s+r)\varphi(x)\widehat{\psi}(s)dxd\beta.$$

This concludes the proof. □

The next important result deals with the convergence of convolution sequences. Let $p \ge 1$ be a real number, and let $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q)$ (where we assume here Q to be bounded) and $(v_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ be two sequences. One may view u_ε as defined in the whole $\mathbb{R}^N$ by taking its zero extension off Q. Define

$$(u_\varepsilon * v_\varepsilon)(x) = \int_{\mathbb{R}^N} u_\varepsilon(t)v_\varepsilon(x-t)dt \quad (x \in \mathbb{R}^N).$$

For $u \in L^p(\mathbb{R}^N; \mathcal{B}_A^p)$ and $v \in L^1(\mathbb{R}^N; \mathcal{B}_A^1)$, we define the convolution product $u**v$ as follows:

$$\mathcal{G}_1(u**v)(x,s) := \iint_{\mathbb{R}^N\times\Delta(A)} \widehat{u}(t,r)\widehat{v}(x-t,s-r)dtd\beta(r)$$

for $(x,s) \in \mathbb{R}^N \times \Delta(A)$. We denote $\mathcal{G}_1(u**v)$ by $(\widehat{u}**\widehat{v})$. This gives a function in the space $L^p(\mathbb{R}^N; \mathcal{B}_A^p)$ with the property

$$\|u**v\|_{L^p(\mathbb{R}^N;\mathcal{B}_A^p)} \le \|u\|_{L^p(\mathbb{R}^N;\mathcal{B}_A^p)} \|v\|_{L^1(\mathbb{R}^N;\mathcal{B}_A^1)}.$$

The above inequality can be checked exactly as the Young inequality for convolution. We have the following result.

Theorem 2. *Let $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ be as above. Assume that, as $\varepsilon \to 0$, $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ and $v_\varepsilon \to v_0$ in $L^1(\mathbb{R}^N)$-strong Σ, where u_0 and v_0 are in $L^p(Q; \mathcal{B}_A^p)$ and $L^1(\mathbb{R}^N; \mathcal{B}_A^1)$ respectively. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon * v_\varepsilon \to u_0 ** v_0 \quad \text{in } L^p(Q)\text{-weak } \Sigma.$$

Proof. Let $\eta > 0$ and let $\psi_0 \in \mathcal{K}(\mathbb{R}^N; A)$ (the space of continuous functions from $\mathbb{R}^N$ into A with compact support) be such that $\|\widehat{v}_0 - \widehat{\psi}_0\|_{L^1(\mathbb{R}^N\times\Delta(A))} \le \frac{\eta}{2}$. Since $v_\varepsilon \to v_0$ in $L^1(\mathbb{R}^N)$-strong Σ we have that $v_\varepsilon - \psi_0^\varepsilon \to v_0 - \psi_0$ in $L^1(\mathbb{R}^N)$-strong Σ; hence $\|v_\varepsilon - \psi_0^\varepsilon\|_{L^1(\mathbb{R}^N)} \to \|\widehat{v}_0 - \widehat{\psi}_0\|_{L^1(\mathbb{R}^N\times\Delta(A))}$ as $\varepsilon \to 0$. So, there is $\alpha > 0$ such that

$$\|v_\varepsilon - \psi_0^\varepsilon\|_{L^1(\mathbb{R}^N)} \le \eta \quad \text{for } 0 < \varepsilon \le \alpha. \tag{2.6}$$

For $f \in \mathcal{K}(Q; A)$, we have (by still denoting by u_ε the zero extension of u_ε off Q)

$$\begin{aligned}
\int_Q (u_\varepsilon * v_\varepsilon)(x) f\left(x, \frac{x}{\varepsilon}\right) dx &= \int_Q \left(\int_{\mathbb{R}^N} u_\varepsilon(t) v_\varepsilon(x-t)dt\right) f\left(x, \frac{x}{\varepsilon}\right) dx \\
&= \int_{\mathbb{R}^N} u_\varepsilon(t) \left[\int_{\mathbb{R}^N} v_\varepsilon(x-t) f\left(x, \frac{x}{\varepsilon}\right) dx\right] dt \\
&= \int_{\mathbb{R}^N} u_\varepsilon(t) \left[\int_{\mathbb{R}^N} v_\varepsilon(x) f\left(x+t, \frac{x}{\varepsilon} + \frac{t}{\varepsilon}\right) dx\right] dt \\
&= \int_{\mathbb{R}^N} u_\varepsilon(t) \left[\int_{\mathbb{R}^N} (v_\varepsilon(x) - \psi_0^\varepsilon(x)) f^\varepsilon(x+t) dx\right] dt \\
&\quad + \int_{\mathbb{R}^N} u_\varepsilon(t) \left(\int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t) dx\right) dt \\
&= (I) + (II).
\end{aligned}$$

On the one hand, one has $(I) = \int_Q [u_\varepsilon * (v_\varepsilon - \psi_0^\varepsilon)](x) f^\varepsilon(x) dx$ and

$$\begin{aligned}
|(I)| &\le \|u_\varepsilon\|_{L^p(Q)} \|v_\varepsilon - \psi_0^\varepsilon\|_{L^1(\mathbb{R}^N)} \|f^\varepsilon\|_{L^{p'}(Q)} \\
&\le c\|v_\varepsilon - \psi_0^\varepsilon\|_{L^1(\mathbb{R}^N)},
\end{aligned}$$

where c is a positive constant independent of ε. It follows that

$$|(I)| \le c\eta \quad \text{for } 0 < \varepsilon \le \alpha. \tag{2.7}$$

On the other hand, in view of Proposition 3, we have, as $\varepsilon \to 0$,

$$\begin{aligned}
\int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t) dx &= \int_{\mathbb{R}^N} \psi_0^\varepsilon(x-t) f^\varepsilon(x) dx \\
&\to \iint_{\mathbb{R}^N\times\Delta(A)} \widehat{\psi}_0(x-t, s-r)\widehat{f}(x,s) dx d\beta(s),
\end{aligned}$$

where $r = \lim \delta_{t/\varepsilon}$ (for a suitable subsequence of $\varepsilon \to 0$) in $\Delta(A)$. So, let $\Phi : \mathbb{R}^N \times \Delta(A) \to \mathbb{R}$ be defined by

$$\Phi(t,r) = \iint_{\mathbb{R}^N\times\Delta(A)} \widehat{\psi}_0(x-t, s-r)\widehat{f}(x,s) dx d\beta(s), \quad (t,r) \in \mathbb{R}^N \times \Delta(A).$$

Then it can be easily checked that $\Phi \in \mathcal{K}(\mathbb{R}^N; \mathcal{C}(\Delta(A)))$, so there is a function $\Psi \in \mathcal{K}(\mathbb{R}^N; A)$ with $\Phi = \mathcal{G} \circ \Psi$. We can therefore define the trace $\Psi^\varepsilon(t) = \Psi(t, t/\varepsilon)$ $(t \in \mathbb{R}^N)$ and get

$$\begin{aligned}\Psi^\varepsilon(t) &= \left\langle \delta_{\frac{t}{\varepsilon}}, \mathcal{G}(\Psi(t,\cdot)) \right\rangle \\ &= \left\langle \delta_{\frac{t}{\varepsilon}}, \Phi(t,\cdot) \right\rangle \\ &= \Phi\left(t, \delta_{\frac{t}{\varepsilon}}\right) = \iint_{\mathbb{R}^N \times \Delta(A)} \widehat{\psi}_0(x-t, s-\delta_{\frac{t}{\varepsilon}}) \widehat{f}(x,s)\, dx d\beta(s).\end{aligned}$$

Next, we have

$$\begin{aligned}(II) &= \int_{\mathbb{R}^N} u_\varepsilon(t) \left(\int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t)\, dx - \Psi^\varepsilon(t) \right) dt + \int_{\mathbb{R}^N} u_\varepsilon(t) \Psi^\varepsilon(t)\, dt \\ &= (II_1) + (II_2).\end{aligned}$$

As for (II_1), set

$$V_\varepsilon(t) = \int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t)\, dx - \Psi^\varepsilon(t) \quad \text{for a.e. } t \in \mathbb{R}^N.$$

We claim that, for a.e. t, $V_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ (possibly up to a subsequence). Indeed, due to Proposition 3,

$$\int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t)\, dx \to \iint_{\mathbb{R}^N \times \Delta(A)} \widehat{\psi}_0(x-t, s-r) \widehat{f}(x,s)\, dx d\beta(s) \quad \text{as } \varepsilon \to 0,$$

where r is such that $\delta_{t/\varepsilon} \to r$ in $\Delta(A)$ for some subsequence of ε. Moreover, since $\Psi^\varepsilon(t) = \Phi^\varepsilon(t, \delta_{t/\varepsilon})$, by the continuity of $\Phi(t,\cdot)$, we have for the same subsequence as $\varepsilon \to 0$ that

$$\Psi^\varepsilon(t) \to \iint_{\mathbb{R}^N \times \Delta(A)} \widehat{\psi}_0(x-t, s-r) \widehat{f}(x,s)\, dx d\beta(s).$$

The above claim is justified. Furthermore,

$$|V_\varepsilon(t)| \le c \quad \text{for a.e. } t \in Q$$

where c is a positive constant independent of t and ε. On the other hand, since Q is bounded, we infer from the weak Σ-convergence of $(u_\varepsilon)_\varepsilon$ that $u_\varepsilon \to \int_{\Delta(A)} \widehat{u}_0(\cdot, r)\, d\beta(r)$ in $L^1(Q)$-weak as $\varepsilon \to 0$. Therefore, by [23, Lemma 3.4], it follows that $(II_1) \to 0$ as $\varepsilon \to 0$.

Regarding (II_2), using once again the weak Σ-convergence of $(u_\varepsilon)_\varepsilon$, we get

$$\int_Q u_\varepsilon(t) \Psi^\varepsilon(t)\, dt \to \iint_{Q \times \Delta(A)} \widehat{u}_0(t,r) \widehat{\Psi}(t,r)\, dt d\beta,$$

and

$$\begin{aligned}\iint_{Q \times \Delta(A)} \widehat{u}_0(t,r) \widehat{\Psi}(t,r)\, dt d\beta &= \iint_{Q \times \Delta(A)} \widehat{u}_0(t,r) \Phi(t,r)\, dt d\beta \\ &= \iint_{Q \times \Delta(A)} \left[\iint_{\mathbb{R}^N \times \Delta(A)} \widehat{u}_0(t,r) \widehat{\psi}_0(x-t, s-r)\, dt d\beta(r) \right] \widehat{f}(x,s)\, dx d\beta(s) \\ &= \iint_{Q \times \Delta(A)} (\widehat{u}_0 ** \widehat{\psi}_0)(x,s) \widehat{f}(x,s)\, dx d\beta(s).\end{aligned}$$

Thus, there is $0 < \alpha_1 \le \alpha$ such that

$$\left| \int_Q (u_\varepsilon * \psi_0^\varepsilon) f^\varepsilon\, dx - \iint_{Q \times \Delta(A)} (\widehat{u}_0 ** \widehat{\psi}_0) \widehat{f}\, dx d\beta \right| \le \frac{\eta}{2} \quad \text{for } 0 < \varepsilon \le \alpha_1. \tag{2.8}$$

Now, let $0 < \varepsilon \le \alpha_1$ be fixed. From the decomposition,

$$\begin{aligned}\int_Q (u_\varepsilon * v_\varepsilon) f^\varepsilon\, dx - \iint_{Q \times \Delta(A)} (\widehat{u}_0 ** \widehat{v}_0) \widehat{f}\, dx d\beta &= \int_Q \left[u_\varepsilon * (v_\varepsilon - \psi_0^\varepsilon) \right] f^\varepsilon\, dx + \iint_{Q \times \Delta(A)} \left[\widehat{u}_0 ** (\widehat{\psi}_0 - \widehat{v}_0) \right] \widehat{f}\, dx d\beta \\ &\quad + \int_Q (u_\varepsilon * \psi_0^\varepsilon) f^\varepsilon\, dx - \iint_{Q \times \Delta(A)} (\widehat{u}_0 ** \widehat{\psi}_0) \widehat{f}\, dx d\beta,\end{aligned}$$

we infer from (2.6)–(2.8) that

$$\left|\int_Q (u_\varepsilon * v_\varepsilon) f^\varepsilon dx - \iint_{Q\times\Delta(A)} (\widehat{u}_0 * *\widehat{v}_0)\widehat{f} dx d\beta\right| \leq C\eta \quad \text{for } 0<\varepsilon\leq\alpha_1.$$

Here, C is a positive constant independent of ε. This concludes the proof. □

Remark 1. In this work, we will deal with sequences of functions $(u_\varepsilon)_{\varepsilon>0}$in the space $L^p(Q_T)$ (where $Q_T = Q\times(0,T)$), and we will say that such a sequence is weakly Σ-convergent in $L^p(Q_T)$ if

$$\int_{Q_T} u_\varepsilon(x,t) f\left(x,t,\frac{x}{\varepsilon}\right) dxdt \to \iint_{Q_T\times\Delta(A)} \widehat{u}_0(x,t,s)\widehat{f}(x,t,s)\, dxdtd\beta(s)$$

for all $f \in L^{p'}(Q_T;A)$ $\left(1/p' = 1-1/p\right)$, where $\widehat{f}(x,t,\cdot) = \mathcal{G}(f(x,t,\cdot))$ a.e. in $(x,t)\in Q_T$. With the above definition, it is a very easy exercise to see that all the previous results of the current subsection are carried over mutatis mutandis to the present setting.

3. Statement of the problem: existence result and a priori estimate

We consider the parameterized Wilson–Cowan model [4]

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}(x,t) = -u_\varepsilon(x,t) + \displaystyle\int_{\mathbb{R}^N} J^\varepsilon(x-\xi) f\left(\frac{\xi}{\varepsilon}, u_\varepsilon(\xi,t)\right) d\xi, \quad x\in\mathbb{R}^N, t>0 \\ u_\varepsilon(x,0) = u^0(x), \quad x\in\mathbb{R}^N, \end{cases} \tag{3.1}$$

where u_ε denotes the electrical activity level field, f the firing rate function and $J^\varepsilon = J^\varepsilon(x) = J(x, x/\varepsilon)$ the connectivity kernel. We assume that $J\in\mathcal{K}(\mathbb{R}^N;A)$ (where $A\in\mathbb{A}$) is nonnegative and is such that $\int_{\mathbb{R}^N} J^\varepsilon(x)dx\leq 1$, $f:\mathbb{R}^N_y\times\mathbb{R}_\mu\to\mathbb{R}$ is a nonnegative Carathéodory function constrained as follows.

(H1) For almost all $y\in\mathbb{R}^N$, the function $f(y,\cdot):\lambda\mapsto f(y,\lambda)$ is continuous; for all $\lambda\in\mathbb{R}$, the function $f(\cdot,\lambda): y\mapsto f(y,\lambda)$ is measurable and $f(\cdot,0)$ lies in $L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$; there exists a positive constant k_1 such that

$$|f(y,\mu_1) - f(y,\mu_2)| \leq k_1|\mu_1-\mu_2| \quad \text{for all } y\in\mathbb{R}^N \text{ and all } \mu_1,\mu_2\in\mathbb{R}.$$

An example of a function f that satisfies hypothesis (H1) is $f(y,\lambda) = g(y)h(\lambda)$, where $g\in\mathcal{K}(\mathbb{R}^N)$, $g\geq 0$ and

$$h(\lambda) = \frac{1}{1+\exp(-\beta(\lambda-\theta))}(\lambda\in\mathbb{R}) \quad \text{where } \beta>0 \text{ and } \theta \text{ are given.}$$

It follows from (H1) that, for any function $u\in L^2(\mathbb{R}^N)$, the function $x\mapsto f(x/\varepsilon, u(x))$ denoted below by $f^\varepsilon(\cdot,u)$, is well defined from $\mathbb{R}^N$ to $L^2(\mathbb{R}^N)$. Moreover, it is an easy exercise to see that

$$\|f^\varepsilon(\cdot,u)\|_{L^2(\mathbb{R}^N)} \leq k_1\|u\|_{L^2(\mathbb{R}^N)} + c_1 \quad \text{for all } 0<\varepsilon\leq 1, \tag{3.2}$$

where $c_1 = \|f(\cdot,0)\|_{L^2(\mathbb{R}^N)}$. The following existence result holds true.

Theorem 3. *Let $0<T<\infty$. Assume that $u^0\in L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$. Then, for each fixed $\varepsilon>0$, there exists a unique solution $u_\varepsilon\in\mathcal{C}([0,\infty);L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N))$ to (3.1) which satisfies the estimate*

$$\sup_{0\leq t\leq T}\left[\|u_\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^N)} + \|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)}\right] \leq C \quad \text{for all } \varepsilon>0, \tag{3.3}$$

where C is a positive constant depending only on u^0 and T.

Proof. Inspired by the proof of Theorem 2.1 in [24] (see also [25]), we define the space

$$X = \mathcal{C}([0,\rho];L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N))$$

with norm

$$\|u\|_X = \max_{t\in[0,\rho]}\left(\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} + \|u(\cdot,t)\|_{L^2(\mathbb{R}^N)}\right),$$

where $\rho>0$ is an arbitrary real number to be suitably chosen in what follows. Next, by the assumptions on J we have that the trace J^ε is well defined and is an element of $\mathcal{K}(\mathbb{R}^N)$ for every $\varepsilon>0$. Let us now consider the operator $K_\varepsilon: X\to X$ defined by

$$K_\varepsilon(\phi)(x,t) = \phi(x,0) + \int_0^t (J^\varepsilon * f^\varepsilon(\cdot,\phi) - \phi)(x,\tau)d\tau.$$

We observe that a fixed point to K_ε is a local solution $u_\varepsilon \in X$ to (3.1) by the Banach fixed point theorem. We proceed by showing that K_ε is a strict contraction on X. By the properties of J^ε, f and by the use of Young's inequality, we obtain

$$\begin{aligned}\|K_\varepsilon(u)-K_\varepsilon(v)\|_{L^2(\mathbb{R}^N)}(t) &\leq \int_0^\rho [\|J^\varepsilon * (f^\varepsilon(\cdot,u)) - f^\varepsilon(\cdot,v)\|_{L^2(\mathbb{R}^N)}(\tau) + \|u-v\|_{L^2(\mathbb{R}^N)}(\tau)]d\tau \\ &\leq \int_0^\rho (k_1+1)\|u-v\|_{L^2(\mathbb{R}^N)}(\tau)d\tau \leq (k_1+1)\rho\|u-v\|_X \text{ for every } 0\leq t\leq\rho.\end{aligned}$$

We also have

$$\|K_\varepsilon(u)-K_\varepsilon(v)\|_{L^1(\mathbb{R}^N)}(t) \leq (k_1+1)\rho\|u-v\|_X \quad \text{for every } 0\leq t\leq\rho.$$

Choosing ρ small enough such that $2(k_1+1)\rho < 1$, we obtain that K_ε is a strict contraction on X. Thus, by the Banach fixed point theorem, there exists a unique local solution $u_\varepsilon \in X$ to (3.1). Next, arguing exactly as in the proof of theorem 2.7 in [26], we get the global existence of the solution to (3.1). Now, we need to check that estimate (3.3) holds true. In order to do that, by multiplying Eq. (3.1) by $u_\varepsilon(x,t)$ and integrating the resulting equality over $\mathbb{R}^N$, we obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^N}|u_\varepsilon(x,t)|^2dx = -2\int_{\mathbb{R}^N}|u_\varepsilon(x,t)|^2dx + 2\int_{\mathbb{R}^N}u_\varepsilon(x,t)[J^\varepsilon * f^\varepsilon(\cdot,u_\varepsilon)](x,t)dx.$$

By the Hölder and Young inequalities, we get

$$\begin{aligned}\int_{\mathbb{R}^N}u_\varepsilon(x,t)[J^\varepsilon * f^\varepsilon(\cdot,u_\varepsilon)](x,t)dx &\leq \|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|J^\varepsilon * f^\varepsilon(\cdot,u_\varepsilon)|^2dx\right)^{\frac{1}{2}} \\ &\leq \|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)}\|J^\varepsilon\|_{L^1(\mathbb{R}^N)}\|f^\varepsilon(\cdot,u_\varepsilon)\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

We infer from (3.2) that

$$\|f^\varepsilon(\cdot,u_\varepsilon)\|_{L^2(\mathbb{R}^N)} \leq k_1\|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)} + c_1.$$

Thus

$$\begin{aligned}\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} &\leq -2\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + 2k_1\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + 2c_1\|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\ &\leq 2k_1\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + 2c_1\|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\ &= 2\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)}\left(k_1 + \frac{c_1}{\|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)}}\right).\end{aligned}$$

Either we have

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)} \leq \frac{2c_1}{k_1} \quad \text{for all } \varepsilon > 0,$$

or there are some $\varepsilon > 0$ for which

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)} > \frac{2c_1}{k_1}.$$

Then, for such ε we have

$$\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \leq 2\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)}\left(k_1+\frac{k_1}{2}\right) = 3k_1\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)},$$

and hence

$$\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \leq \exp(3k_1t)\left\|u^0\right\|^2_{L^2(\mathbb{R}^N)} \leq \exp(3k_1T)\left\|u^0\right\|^2_{L^2(\mathbb{R}^N)}.$$

In both cases we see that there exists a positive constant C depending only on both u^0 and T such that

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^N)} \leq C \quad \text{for all } 0<\varepsilon\leq 1 \text{ and all } 0\leq t\leq T.$$

It is also an easy task to see that

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq C \quad \text{for all } \varepsilon > 0 \text{ and all } 0\leq t\leq T.$$

This completes the proof. $\square$

Remark 2. From the uniform boundedness of $(u_\varepsilon)_{0<\varepsilon\leq 1}$ in $\mathcal{C}([0,T];L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N))$, we deduce that $(u_\varepsilon)_{0<\varepsilon\leq 1}$ is uniformly integrable in $L^1(\mathbb{R}^N\times(0,T))$. Indeed, let $B\subset\mathbb{R}^N\times(0,T)$ be an integrable subset. Denoting by $|B|$ its Lebesgue measure, we have by Hölder's inequality that

$$\begin{aligned}\int_B |u_\varepsilon|\,dxdt &\leq |B|^{\frac{1}{2}}\,\|u_\varepsilon\|_{L^2(\mathbb{R}^N\times(0,T))}\\ &\leq C\,|B|^{\frac{1}{2}},\end{aligned}$$

and hence $\sup_{0<\varepsilon\leq 1}\int_B |u_\varepsilon|\,dxdt\to 0$ when $|B|\to 0$. We may therefore use [part (ii) of] Proposition 2 to deduce the existence of a subsequence of $(u_\varepsilon)_{0<\varepsilon\leq 1}$ that weakly Σ-converges in $L^1(\mathbb{R}^N\times(0,T))$.

4. Homogenization result

Let A be an algebra with mean value taken in the class $\mathbb{A}$. In order to perform the homogenization process, we assume that the function f satisfies the following hypotheses (in which we set $\mathbb{R}^N_T=\mathbb{R}^N\times(0,T)$).

(H2) $f(\cdot,\mu)\in A$ for all $\mu\in\mathbb{R}$.
(H3) For any sequence $(v_\varepsilon)_{\varepsilon>0}\subset L^1(\mathbb{R}^N_T)$ such that $v_\varepsilon\to v_0$ in $L^1(\mathbb{R}^N_T)$-weak Σ, we have $f^\varepsilon(\cdot,v_\varepsilon)\to f(\cdot,v_0)$ in $L^1(\mathbb{R}^N_T)$-weak Σ.

Hypothesis (H3) is meaningful. Indeed, the convergence result $v_\varepsilon\to v_0$ in $L^1(\mathbb{R}^N_T)$-weak Σ does not entail the convergence result $f^\varepsilon(\cdot,v_\varepsilon)\to f(\cdot,v_0)$ in $L^1(\mathbb{R}^N_T)$-weak Σ in general. However, there are many situations in which assumption (H3) is satisfied. Below are some examples.

(1) Assume that $A=\mathcal{C}_{\mathrm{per}}(Y)$ and that $f(\cdot,\mu)$ is Y-periodic, and that $f(y,\cdot)$ is convex. Assume further that f satisfies the following.
(H4) $\int_{\mathbb{R}^N_T} f\left(\frac{x}{\varepsilon},v_\varepsilon(x,t)\right)dxdt\to\iint_{\mathbb{R}^N_T\times Y} f(y,v_0(x,t,y))dxdtdy$ as $\varepsilon\to 0$, whenever $v_\varepsilon\to v_0$ in $L^1(\mathbb{R}^N_T)$-weak Σ.
Then $f^\varepsilon(\cdot,v_\varepsilon)\to f(\cdot,v_0)$ in $L^1(\mathbb{R}^N_T)$-weak Σ as $\varepsilon\to 0$; see [27, Theorem 4.3].
(2) Assume that hypotheses (H1)–(H2) hold true. Let $(v_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^1(\mathbb{R}^N_T)\cap L^2(\mathbb{R}^N_T)$ such that $v_\varepsilon\to v_0$ in $L^2(\mathbb{R}^N_T)$-weak Σ. Then (possibly up to a subsequence) we have $v_\varepsilon\to v_0$ in $L^1(\mathbb{R}^N_T)$-weak Σ. In view of assumption (H1), the sequence $(f^\varepsilon(\cdot,v_\varepsilon))_{\varepsilon>0}$ is bounded in $L^1(\mathbb{R}^N_T)\cap L^2(\mathbb{R}^N_T)$, so, up to a subsequence, we have $f^\varepsilon(\cdot,v_\varepsilon)\to z_0$ in $L^2(\mathbb{R}^N_T)$-weak Σ (where $z_0\in L^2(\mathbb{R}^N_T;\mathcal{B}^2_A)$). Now, if we further assume that

$$\liminf_{\varepsilon\to 0}\int_{\mathbb{R}^N_T} f^\varepsilon(\cdot,v_\varepsilon)v_\varepsilon dxdt\leq\iint_{\mathbb{R}^N_T\times\Delta(A)}\widehat{z}_0\widehat{v}_0\,dxdtd\beta,$$

then $f^\varepsilon(\cdot,v_\varepsilon)\to f(\cdot,v_0)$ in $L^1(\mathbb{R}^N_T)$-weak Σ; see [28, Theorem 8].
(3) In the special case when $f(y,\lambda)=g(y)h(\lambda)$ with $g\in A\cap\mathcal{K}(\mathbb{R}^N)$, $g\geq 0$ and $h(\lambda)=\frac{1}{1+\exp(-\beta(\lambda-\theta))}$, (H3) is still true. Indeed, as stressed in Appendix A of [8], we have $h(v_\varepsilon)\to h(v_0)$ in $L^1(\mathbb{R}^N_T)$-weak Σ, so, since $g\in A$, $f^\varepsilon(\cdot,v_\varepsilon)=g^\varepsilon h(v_\varepsilon)\to gh(v_0)=f(\cdot,v_0)$ in $L^1(\mathbb{R}^N_T)$-weak Σ.

We can now state and prove the homogenization result.

Theorem 4. *For any fixed $\varepsilon>0$, let u_ε be the unique solution of (3.1). Then as $\varepsilon\to 0$, we have*

$$u_\varepsilon\to u_0 \text{ in } L^1(\mathbb{R}^N_T)\text{-weak } \Sigma, \tag{4.1}$$

where $u_0\in\mathcal{C}([0,T];L^1(\mathbb{R}^N;\mathcal{B}^1_A))$ is the unique solution to the following equation:

$$\begin{cases}\dfrac{\partial u_0}{\partial t}(x,t,y)=-u_0(x,t,y)+(J**f(\cdot,u_0))(x,t,y), & (x,t)\in\mathbb{R}^N_T, y\in\mathbb{R}^N\\ u_0(x,0,y)=u^0(x), \quad x\in\mathbb{R}^N,\ y\in\mathbb{R}^N.\end{cases} \tag{4.2}$$

Proof. We infer from Remark 2 that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is uniformly integrable in $L^1(\mathbb{R}^N_T)$. So, given an ordinary sequence E, it follows from [part (ii) of] Proposition 2 that there exist a subsequence E' of E and a function $u_0\in L^1(\mathbb{R}^N_T;\mathcal{B}^1_A)$ such that, as $E'\ni\varepsilon\to 0$, we have (4.1). It now remains to check that u_0 solves Eq. (4.2). Indeed it can be easily shown that, in view of the properties of f and J, the solution to (4.2) is unique, so, by the uniqueness property, we have the convergence result (4.1) for any ordinary sequence E (not only up to a subsequence E'), and hence for the whole sequence $\varepsilon\to 0$.

Now, since $J\in\mathcal{K}(\mathbb{R}^N;A)\subset\mathcal{K}(\mathbb{R}^N\times(0,T);A)$, we have that

$$J^\varepsilon\to J \text{ in } L^1(\mathbb{R}^N\times(0,T))\text{-strong } \Sigma \text{ as } E'\ni\varepsilon\to 0;\ \text{ see e.g. [15]}.$$

Hypothesis (H3) together with the convergence result (4.1) leads us to

$$f^{\varepsilon}(\cdot, u_{\varepsilon}) \to f(\cdot, u_0) \text{ in } L^1(\mathbb{R}^N_T)\text{-weak } \Sigma \text{ as } E' \ni \varepsilon \to 0.$$

It therefore follows from Theorem 2 and Remark 1 that

$$J^{\varepsilon} * f^{\varepsilon}(\cdot, u_{\varepsilon}) \to J * *f(\cdot, u_0) \text{ in } L^1(\mathbb{R}^N_T)\text{-weak } \Sigma \text{ as } E' \ni \varepsilon \to 0.$$

Next, Eq. (3.1) is equivalent to the following integral equation:

$$u_{\varepsilon}(x, t) = u^0(x) + \int_0^t [(J^{\varepsilon} * f^{\varepsilon}(\cdot, u_{\varepsilon}))(x, \tau) - u_{\varepsilon}(x, \tau)]\, d\tau.$$

Hence, letting $E' \ni \varepsilon \to 0$ and using Fubini and Lebesgue dominated convergence results in the integral term, we end up with

$$u_0(x, t, y) = u^0(x) + \int_0^t [(J * *f(\cdot, u_0))(x, \tau, y) - u_0(x, \tau, y)]\, d\tau,$$

which is equivalent to (4.2). Moreover, this shows that u_0 lies in $\mathcal{C}([0, T]; L^1(\mathbb{R}^N; \mathcal{B}^1_A))$, as expected. This concludes the proof. $\square$

5. Conclusions and outlook

In this paper we have proved some important results which are relevant to the theory of homogenization in connection with convolution sequences (see, e.g., Theorem 2). It is to be noted that Theorem 2 generalizes to the case of algebras with mean value, its counterpart proved by Visintin [9] in the special context of the algebras of continuous periodic functions. This result, based on the so-called sigma convergence concept, has allowed us to efficiently upscale a heterogeneous Wilson–Cowan type of model for neural fields. The homogenization result derived is accurate and more likely cannot be achieved through other classical and conventional methods such as asymptotic expansions or time averaging. Another aspect that should be emphasized is that Theorem 2 opens the door widely to many other applications in applied science. In this regard, it may allow one to study homogenization problems in connection with partial differential equations with fractional order derivatives. Such kinds of homogenization problem have not yet been solved.

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