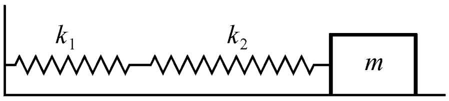
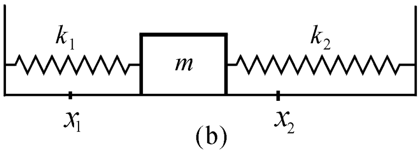
第七章 机械振动



(a)



(b)

7.1 解: (a) 弹簧 k_1 和 k_2 串联后等效为一个强度系数为 k 的弹簧.设 k_1 和 k_2 的形变量分别为 Δx_1 和 Δx_2 , k 的形变量为 Δx ,则有

$$\Delta x = \Delta x_1 + \Delta x_2$$

即 $$\frac{F}{k} = \frac{F}{k_1} + \frac{F}{k_2}$$

∴ $$k = \frac{k_1 k_2}{k_1 + k_2} \qquad \omega_a = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1 k_2}{m\left(k_1 + k_2\right)}}$$

(b) [法一] 设 m 偏离平衡位置的位移为 x (设向右为正方向),则 m 所受的合外力为

$$f = -\left(k_1 + k_1\right) x = -kx$$

这与两弹簧并联的情况是一样的.由此可得

$$\omega_b = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1 + k_2}{m}}$$

[法二] 设物体在平衡位置时,两弹簧伸长量分别为 x_1 和 x_2 ,则平衡条件给出 $k_1 x_1 = k_2 x_2$.此物体的平衡位置为原点.当物体向后移动 x 时,牛二定律给出

$$k_1\left(x_1 + x\right) - k_2\left(x_2 - x\right) = -m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\left(k_1 + k_2\right) x$$

∴ $$\omega_0 = \sqrt{\frac{k_1 + k_2}{m}}$$

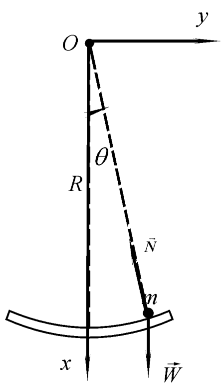


图7.2

7.2 解: 如图建立坐标系. θ 的正向与 Oz 轴的正向成右手螺旋.在以 Ox 为极轴的极坐标中,沿横向的运动微分方程可由牛二定律得

$$-mg\sin\theta = m\frac{\mathrm{d}\left(R\dot{\theta}\right)}{\mathrm{d}t}$$

当 θ 很小时 $\sin \approx \theta$

∴ $g\theta + R\ddot{\theta} = 0$

$$\ddot{\theta}+\frac{g}{R}\theta=0$$

$\therefore$ $\omega_0=\sqrt{\frac{g}{R}}$

$$T=\frac{2\pi}{\omega_0}=2\pi\sqrt{\frac{R}{g}}$$

7.3 解: (1) 振幅 $A=0.06\,\mathrm{m}$, $\omega_0=5$

周期 $T=\frac{2\pi}{\omega_0}=0.4\pi\,(\mathrm{s})\approx 1.26\,(\mathrm{s})$

(2) $t=0$ 时,初位移 $x_0=0.06\cos\pi=-0.06\ \mathrm{m}$

(3) $\ddot{x}=1.5\cos5t$ $t=0$ 时 $\ddot{x}_0=1.5\left(\mathrm{m/s^2}\right)$

$F_0=m\ddot{x}_0=2.5\times10^{-4}\times1.5=3.75\times10^{-4}\ (\mathrm{N})$

(4) $t=\pi\,(\mathrm{s})$ 时

$x=0.06\cos(5\pi+\pi)=0.06\,(\mathrm{m})$

$v_x=\dot{x}=-0.3\sin(5t+\pi)$

$=-0.3\sin(5\pi+\pi)$

$=0$

$a_x=\ddot{x}=-1.5\cos(5t+\pi)$

$=-1.5\cos(5\pi+\pi)$

$=-1.5\left(\mathrm{m/s^2}\right)$

7.4 解：设运动学方程为 $x=A\cos(\omega_0 t+\varphi)$

则 $v_x=-A\omega\sin(\omega_0 t+\varphi)$

$a_x=-A\omega^2\cos(\omega_0 t+\varphi)$

(1) $A=0.02\,\text{m}$ $v_m=A\omega_0=0.03\,\text{m}$

$\therefore\ \omega_0=\dfrac{0.03}{0.02}=1.5$

$\therefore\ T=\dfrac{2\pi}{\omega_0}=\dfrac{2\pi}{1.5}\doteq 4.19\quad(\text{s})$

(2) $a_m=A\omega_0^{\ 2}=0.02\times1.5^2=0.045\quad(\text{m/s}^2)$

(3) 初态时 $x_0=0$ $t=0$ $v_{0x}>0$

$$\begin{cases}\cos\varphi=0\\ \sin\varphi=-1\end{cases}$$

$\therefore\ \varphi=-\dfrac{\pi}{2}$

$\therefore\ x=0.02\cos\left(1.5t-\dfrac{\pi}{2}\right)$

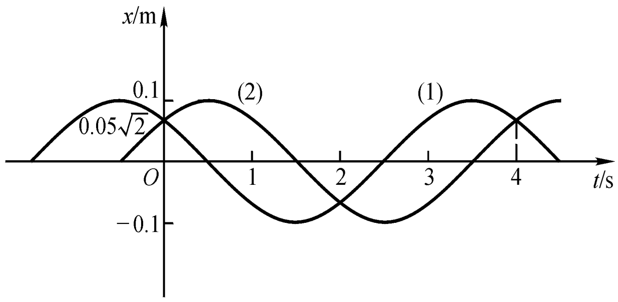


图7.5

7.5 解: $A=0.1\,\text{m}$ $T=4\,\text{s}$

$\omega_0=\dfrac{2\pi}{T}=0.5\pi$

对振动(1)而言 $x_1=0.1\cos\left(0.5\pi t+\varphi_1\right)$

当 $t=0$ 时

$$\begin{cases}x_1=0.05\sqrt{2}=0.1\cos\varphi_1\\ v_{x1}=-0.05\pi\sin\varphi_1<0\end{cases}$$

$\therefore\ \dot{\varphi}_1=\dfrac{\pi}{4}$

$\therefore\ x_1=0.1\cos\left(0.5\pi t+\dfrac{\pi}{4}\right)$

对振动(2)而言 $x_2=0.1\cos\left(0.5\pi t+\varphi_2\right)$

当 $t=0$ 时

$$\begin{cases} x_1 = 0.1\cos\varphi_2 = 0.05\sqrt{2} \\ v_{x1} = -0.05\pi\sin\varphi_2 > 0 \end{cases}$$

$\therefore$ $\varphi_2 = -\dfrac{\pi}{4}$

$\therefore$ $x_2 = 0.1\cos\left(0.5\pi t - \dfrac{\pi}{4}\right)$

$$\Delta\varphi = \varphi_1 - \varphi_2 = \frac{\pi}{2}$$

$\therefore$ φ_1 比 φ_2 超前 $\dfrac{\pi}{2}$

7.6 解: $A = 0.01\,\text{m}$ $a_m = \omega_0{}^2 A = 0.04\,\text{m/s}^2$

$\therefore$ $\omega_0 = \sqrt{\dfrac{0.04}{0.01}} = 2$ $\omega_0 = \sqrt{\dfrac{k}{m}}$

$\therefore$ $k = m\omega_0{}^2 = 0.1\times 4 = 0.4\,(\text{N/m})$

(1) $E = \dfrac{1}{2}kA^2 = \dfrac{1}{2}\times 0.4\times 0.01^2 = 2\times 10^{-5}$ J

(2) 通过平衡位置时 $x = 0$ $\therefore$ $E_p = 0$

$\therefore$ $E_k = E = 2\times 10^{-5}$ J

(3) $E_k = \dfrac{1}{2}kA^2\sin^2(\omega_0 t + \varphi)$

$$E_P = \frac{1}{2}kA^2\cos^2(\omega_0 t + \varphi)$$

$$x = 0.01\cos(2t + \varphi_1)$$

当 $t = 0$ 时 $x_0 = 0.01\,\text{m}$ $v_{x0} = 0$

$\therefore$ $\begin{cases} \cos\varphi_1 = 1 \\ \sin\varphi_1 = 0 \end{cases}$ $\varphi_1 = 0$

$\therefore$ $x = 0.01\cos 2t$

$\therefore\ E_k = 2\times10^{-5}\sin^2 2t$

$E_p = 2\times10^{-5}\cos^2 2t$

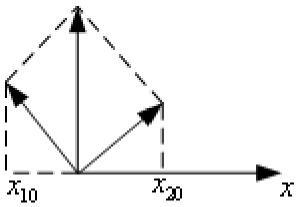
$E_k = E_p$ 时 $\sin^2 2t = \cos^2 2t$

$$\frac{1-\cos4t}{2} = \frac{2\cos4t+1}{2}$$

$\cos4t = 0$ $\quad 4t = \frac{\pi}{4}(2k+1)$

$\therefore\ t = \frac{\pi}{8}(2k+1)$ $\quad k = 0,1,2,3...$

7.7 解: $x = x_1 + x_2 = 0.05\cos\left(10t + \frac{3}{4}\pi\right) + 0.05\cos\left(10t + \frac{\pi}{4}\right)$

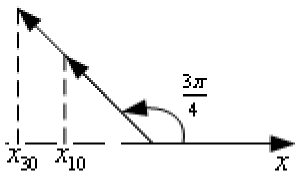


(1) 如图(旋转矢量图)

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos90^\circ}$$

$$= 0.05\sqrt{2} = 0.071(\mathrm{m})$$

$$\varphi = \frac{\pi}{2}$$



(2) 当 $\varphi_3 = \frac{3}{4}\pi$ 时

$$A_{max} = 0.13(\mathrm{m})$$

当 $\varphi_3 = -\frac{3}{4}\pi$ 时

$$A_{min} = 0.03(\mathrm{m})$$

7.8 解: 设两个分振动的振动方程分别为

$$x_1 = 0.1\sqrt{3}\cos(\omega t + \varphi_1)$$

$$x_2 = A_2\cos(\omega t + \varphi_2)$$

$$x = x_1 + x_2$$

图7.8

$$A_2=\sqrt{A^2+A_1^2-2AA_1\cos\theta}$$

$A=0.2\ \mathrm{m}\quad A_1=0.1\sqrt{3}\ \mathrm{m}\quad \theta=30^\circ$

$$A_2=\sqrt{0.04+0.03-2\times0.2\times0.1\sqrt{3}\times\cos30^\circ}$$

$$=0.1\,\mathrm{m}$$

$$\cos(\varphi_2-\varphi_1)=\frac{A^2-A_1^2-A_2^2}{2A_1A_2}$$

$$=\frac{0.04-0.03-0.01}{2\times0.1\sqrt{3}\times0.1}=0$$

$$\varphi_2-\varphi_1=\frac{\pi}{2}$$

7.9 解: 弱阻尼振动 $x=A\mathrm{e}^{-\beta t}\cos(\omega' t+\varphi)$

$$\omega'^2=\omega_0^2-\beta^2$$

由题意 $\dfrac{A\mathrm{e}^{-\beta t}}{A\mathrm{e}^{-\beta(t+T')}}=\dfrac{1}{3}$

取对数 $\ln\dfrac{A\mathrm{e}^{-\beta t}}{A\mathrm{e}^{-\beta(t+T')}}=\beta T'=\ln3$

$$\omega'=\frac{2\pi}{T'}=\frac{2\pi\beta}{\ln3}$$

$\therefore\ \omega_0=\sqrt{\omega'^2+\beta^2}=\beta\sqrt{\left(\dfrac{2\pi}{\ln3}\right)^2+1}$

$\therefore\ \dfrac{\omega'}{\omega_0}=\dfrac{\dfrac{2\pi\beta}{\ln3}}{\beta\sqrt{\left(\dfrac{2\pi}{\ln3}\right)^2+1}}$

$$=\frac{2\pi}{\sqrt{4\pi^2+(\ln3)^2}}$$

$$\frac{T'}{T}=\frac{2\pi/\omega'}{2\pi/\omega_0}=\frac{\omega_0}{\omega'}$$

$$=\sqrt{1+\left(\ln 3/2\pi\right)^2}$$

$$=1.015$$

7.10 解: 根据牛顿第二定律 $m\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=-kx-r\dfrac{\mathrm{d}x}{\mathrm{d}t}+F_0\cos 2t$

令 $\omega_0{}^2=\dfrac{k}{m}$ $\quad 2\beta=\dfrac{r}{m}$ $\quad f_0=\dfrac{F_0}{m}$

则方程变为 $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\beta\dfrac{\mathrm{d}x}{\mathrm{d}t}+\omega_0{}^2x=f_0\cos 2t$

当系统达到稳定状态时

$$x=A_0\cos\left(2t+\varphi\right)$$

其中 $A_0=\dfrac{f_0}{\sqrt{\left(\omega_0{}^2-2^2\right)^2+4\beta^2\times 2^2}}$

根据已知条件 $\omega_0{}^2=\dfrac{k}{m}=\dfrac{1.2\times 10^{-2}}{3\times 10^{-3}}=4$

$$f_0=\frac{F_0}{m}\qquad 2\beta=\frac{r}{m}$$

$\therefore$ $A_0=\dfrac{F_0}{2r}$

当 $r'=3r$， $A_0{}'=\dfrac{F_0}{6r}=\dfrac{1}{3}A_0$

$\therefore$ $\dfrac{A_0{}'}{A_0}=\dfrac{1}{3}$