Iterative Solvers of Ax = b
The Jacobi Method

CS 111: Introduction to Computational Science

Spring 2019 Lecture #7

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Administrative

- Homework #4 due next Monday
- Midterm #1 is NEXT week Thursday (May 2nd)
  - Will put up review questions this Thursday

Lecture Outline

- Iteration to solve Ax = b: **Jacobi Method**

- Intro to the **Conjugate Gradient Method**

$Ax = b$ Solvers We've Learned So Far

- Pivoting
- LU Factorization
- Cholesky
- QR Factorization

Jacobi Method

An **iterative** algorithm for determining the solutions of a ***diagonally dominant*** system of linear equations.

- **Diagonally dominant:**
  - for every row of the matrix,
    magnitude of the diagonal entry in a row
    ≥ the sum of the magnitudes of all
    the other (non-diagonal) entries in that row.

Diagonally Dominant Example

For example,

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

is ***strictly*** a diagonally dominant matrix

because:

$|a_{11}| > |a_{12}| + |a_{13}|$, that is $4 > 3$

$|a_{22}| > |a_{21}| + |a_{23}|$, that is $6 > 3$

$|a_{33}| > |a_{31}| + |a_{32}|$, that is $5 > 3$

Jacobi's Method

- Simple, but not the most efficient algorithm
- Given a system: $A\mathbf{x} = b$
  - Where we know A and b and want to find **x**

- We can start with a *guess* for **x**
  - Good starting point is the zero vector
  - We'll call it the 0th approximation, or $\mathbf{x}^{(0)}$

- We will use the first equation and the current values of $x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)}$ to find a new value $x_1^{(k+1)}$.
  - See examples on the blackboard and Jupyter

Jacobi Method

- Strict row diagonal dominance is a sufficient *but not necessary* condition for Jacobi's Method to converge
  - It sometimes converges even if this condition is not satisfied

- The standard convergence condition (for any iterative method) is when the ***spectral radius*** of the iteration matrix is less than 1
  - Written as: ρ **($D^{-1}.C$) < 1**
  - It means that the max. eigenvalue of **$D^{-1}.C$** should be < 1

- A nice write-up to read (optional) on Wikipedia: *https://en.wikipedia.org/wiki/Jacobi_method*

Conjugate Gradient Iteration

- Another (more efficient) iterative algorithm to solve $Ax = b$.
- Start with a guess $x^{(0)}$, then compute $x^{(1)}$, $x^{(2)}$, ...
- Stop when you think you're close enough.

- In theory, CG can be used to solve *any* system $Ax = b$, provided only that **A is SPD**.

- In practice, how well CG works depends on specifics of **A** in subtle ways, involving eigenvalues and condition number.

Your To-Dos

- Homework #4 – due **Monday, April 29th**

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