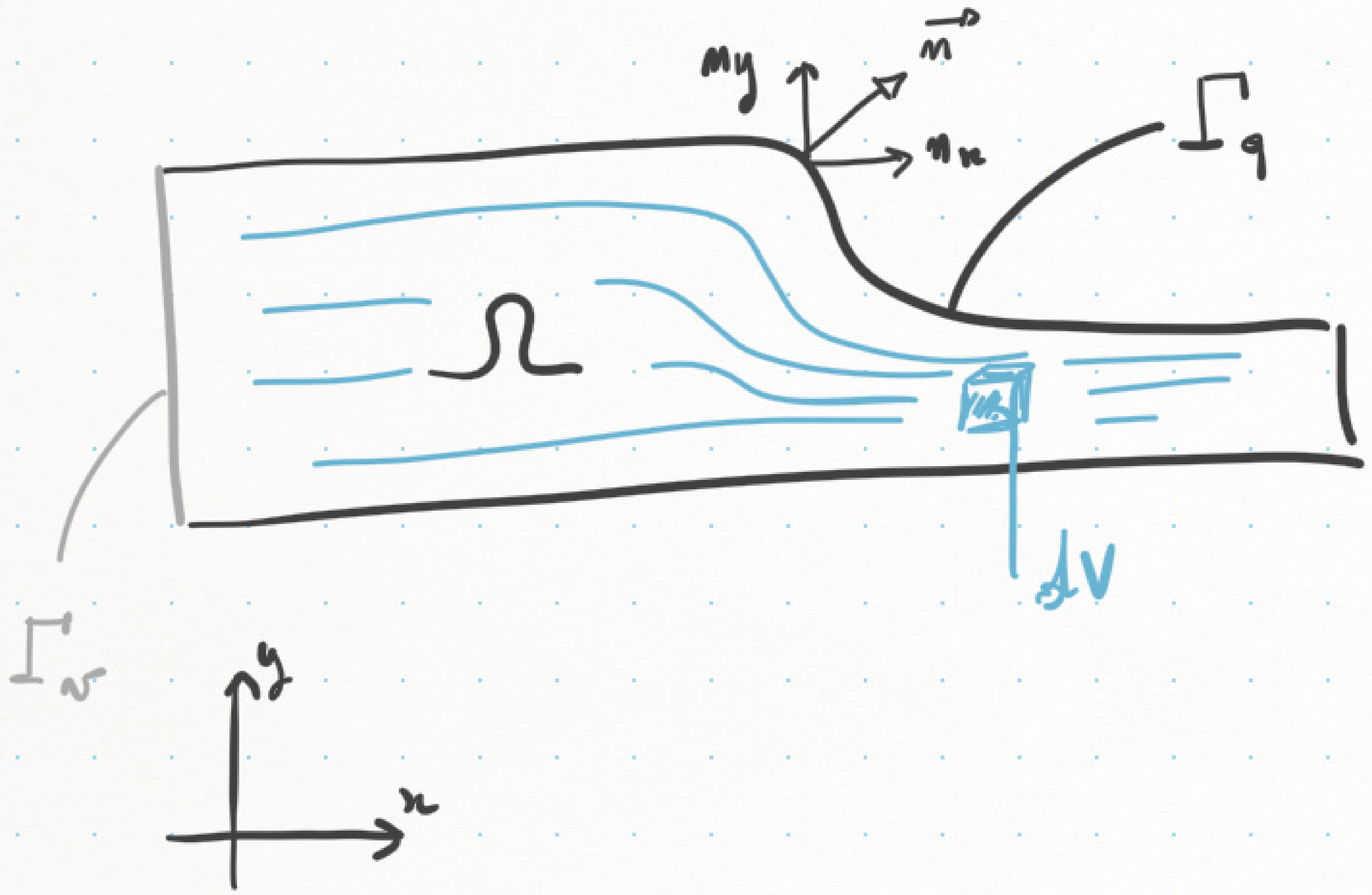
Fluid Mechanics



Momentum balance equation (on Ω)

velocity vector

$$\underbrace{\rho}_{\text{mass density}} \frac{D\vec{v}}{Dt} = \nabla \cdot \underbrace{\overline{\overline{t}}}_{\text{stress tensor}} + \underbrace{\vec{b}}_{\text{volumetric Forces}} \qquad (1)$$

Mass balance equation (on Ω)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad (2)$$

if $\rho = \text{const} \Rightarrow \nabla \cdot \vec{v} = 0$

Boundary conditions :

$$\vec{v} = \vec{v}_0 \quad \text{on } \Gamma_v \qquad (3)$$

$$\overline{\overline{t}} \cdot \vec{n} = -p_0 \vec{n} \quad \text{on } \Gamma_q \qquad (4)$$

$$\rho \underbrace{\frac{\partial \vec{v}}{\partial t} + \rho (\nabla \vec{v}) \cdot \vec{v}}_{\frac{D\vec{v}}{Dt}} = \nabla \cdot \bar{\bar{t}} + \vec{b} \qquad (1)$$

<u>For Newtonian fluids</u>

$$\bar{\bar{\tau}} = -p \bar{\bar{1}} + \mu \vec{\nabla} \vec{v} + \mu (\vec{\nabla} \vec{v})^T \qquad (5)$$

μ ← dynamic viscosity

<u>Pa · s</u>

$$= \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

(5) ⇒ (1)

N.S.

<u>Divergence form</u>

$$\begin{cases} \rho \frac{\partial \vec{v}}{\partial t} + \rho (\nabla \vec{v}) \cdot \vec{v} = \nabla \cdot \left[-p \bar{\bar{1}} + \mu \vec{\nabla} \vec{v} + \mu (\vec{\nabla} \vec{v})^T \right] + \vec{b} \\ \vec{\nabla} \cdot \vec{v} = 0 \end{cases}$$

In MOST SITUATIONS the viscosity μ is constant

$$\star = \underbrace{\nabla\cdot(-p\bar{\bar{1}})}_{(1)} + \underbrace{\nabla\cdot(\mu\nabla\vec{v})}_{(2)} + \underbrace{\nabla\cdot(\mu(\nabla\vec{v})^T)}_{(3)}$$

1) $\nabla\cdot(-p\bar{\bar{1}}) = \cancel{-p\underbrace{\nabla\cdot\bar{\bar{1}}}_{0}} - \bar{\bar{1}}\nabla p = -\nabla p$

2) $\nabla\cdot(\mu\nabla\vec{v}) = \mu\underbrace{\nabla\cdot\nabla}\vec{v} = \mu\nabla^2\vec{v} = \mu\Delta\vec{v}$

3) $\nabla\cdot(\mu(\nabla v)^T) = \mu\underbrace{\nabla\cdot[(\nabla v)^T]}_{=} = \mu\nabla\underbrace{(\nabla\cdot\vec{v})}_{=0} = 0$

So in this case the momentum equ.
can be expressed as:

Momentum equation Laplace Form :

$$\rho \cancel{\frac{\partial \vec{v}}{\partial t}} + \rho \cancel{\nabla \vec{v}} \cdot \vec{v} = -\nabla p + \mu \Delta \vec{v} + \vec{b}$$

when μ is constant in space.

temporal acceleration

Inertial term

We study here the stokes problem where we neglect inertial forces (low Reynolds)

$$\begin{cases} \nabla p - \mu \Delta \vec{v} = \vec{b} \\ \nabla \cdot \vec{v} = 0 \end{cases}$$

Stokes problem
Laplace Form

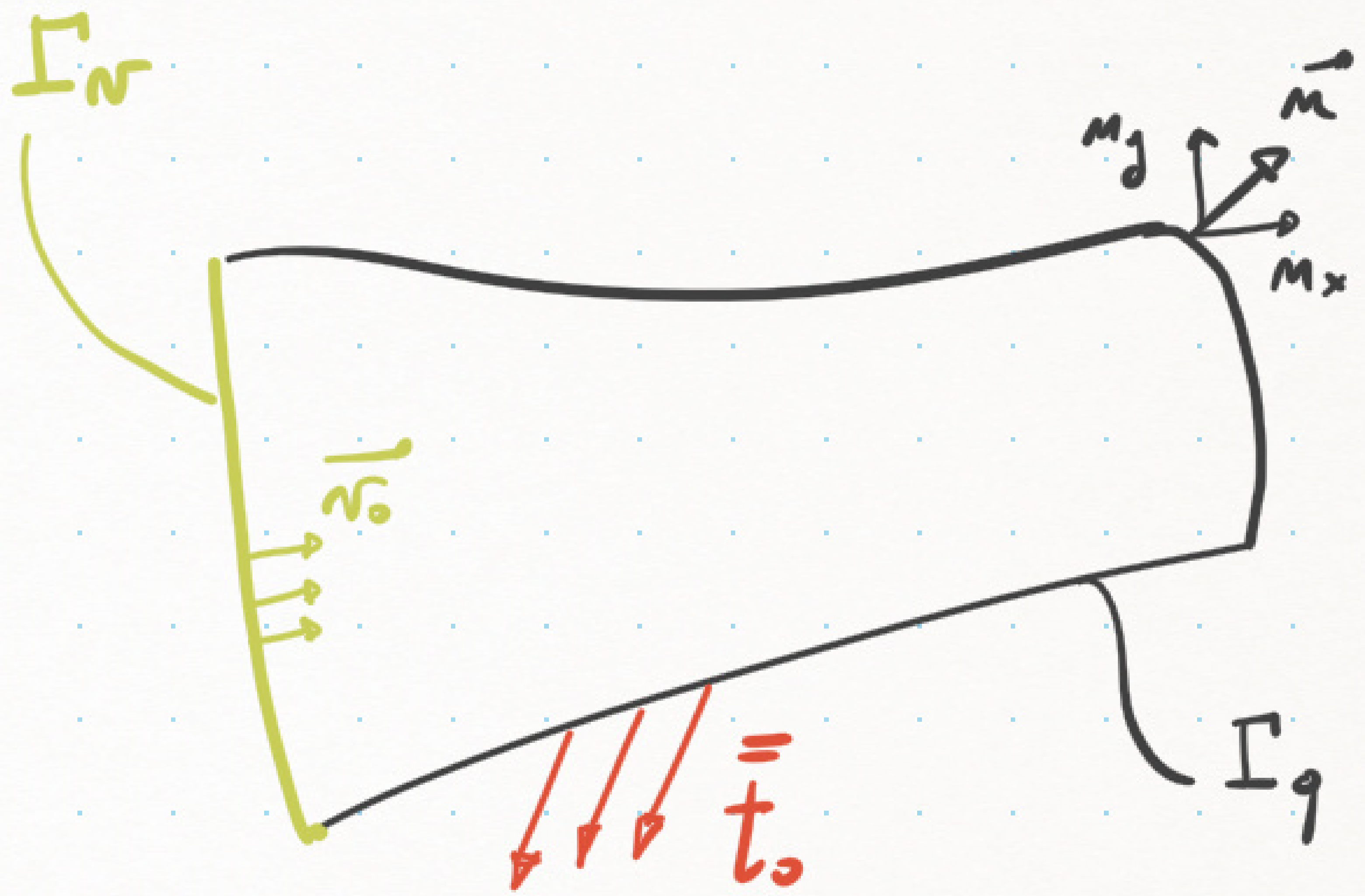
$$\boxed{\begin{cases} -\nabla \cdot \bar{\bar{t}} = \vec{b} \\ \nabla \cdot \vec{v} = 0 \end{cases}}$$

Stokes problem
Divergence Form.

We use this form since it is more general.

So the problem is defined as

$$
\begin{cases}
-\nabla \cdot \bar{\bar{t}} = \vec{b} & \text{in } \Omega \quad (a) \\
\nabla \cdot \vec{v} = 0 & \text{in } \Omega \quad (b) \\
\vec{v} = \vec{v}_0 & \text{in } \Gamma_v \quad (c) \\
\bar{\bar{t}} \cdot \vec{n} = -p_0 \vec{n} & \text{in } \Gamma_t \quad (d)^{*}
\end{cases}
$$



A more general b.c. is:

$$\bar{\bar{t}} \cdot \vec{n} = \bar{\bar{t}}_0$$

We define two weight functions:

$\vec{w}$	vectorial	associated	to eqn (a)
q	scalar	" "	to eqn (b)

(a) · $\vec{w}$ and $\int_\Omega$

$$-\int_\Omega (\vec{\nabla} \cdot \bar{\bar{t}}) \cdot \vec{w} \, d\Omega = \int_\Omega \vec{b} \cdot \vec{w} \, d\Omega$$

<u>GREEN FORMULA</u> For tensor quantity

$$-\int_\Omega (\nabla \cdot \bar{\bar{t}}) \cdot \vec{w} \, d\Omega = \int_\Omega \bar{\bar{t}} : \vec{\nabla} \vec{w} \, d\Omega - \underbrace{\int_\Gamma (\bar{\bar{t}} \cdot \vec{n}) \cdot \vec{w} \, d\Gamma}_{\text{we split this term}}$$

$$\int_\Omega \bar{\bar{t}} : \vec{\nabla} \vec{w} - \underbrace{\cancel{\int_{\Gamma_v} (\bar{\bar{t}} \cdot \vec{n}) \cdot \vec{w} \, d\Gamma}}_{\substack{=0 \\ \text{For the F.E.M.}}} - \int_{\Gamma_t} \underbrace{(\bar{\bar{t}} \cdot \vec{n})}_{= -p_0 \vec{n}} \cdot \vec{w} \, d\Gamma = \int_\Omega \vec{b} \cdot \vec{w} \, d\Omega$$

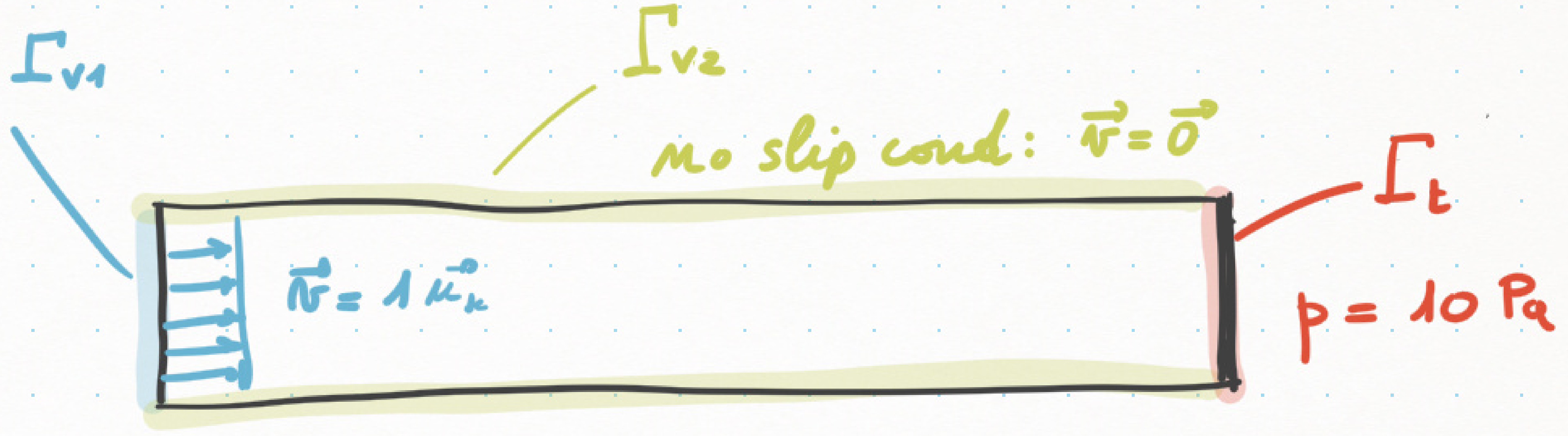
$$\int_\Omega \bar{\bar{t}} : \nabla\vec{w}\, d\Omega - \int_{\Gamma_t} \underbrace{(\bar{\bar{t}}\cdot\vec{n})}_{\text{Eq. (d)}}\cdot\vec{w}\, d\Gamma = \int_\Omega \vec{b}\cdot\vec{w}\, d\Omega$$

$$\begin{cases} \int_\Omega \left[-p\,\bar{\bar{1}} + \mu\nabla\vec{v} + \mu(\nabla\vec{v})^T\right] : \nabla\vec{w}\, d\Omega + \int_{\Gamma_t} p_0\,\vec{n}\cdot\vec{w}\, d\Gamma = \int_\Omega \vec{b}\cdot\vec{w}\, d\Omega \\ \\ \int_\Omega (\nabla\cdot\vec{v})\cdot q\, d\Omega = 0 \end{cases}$$

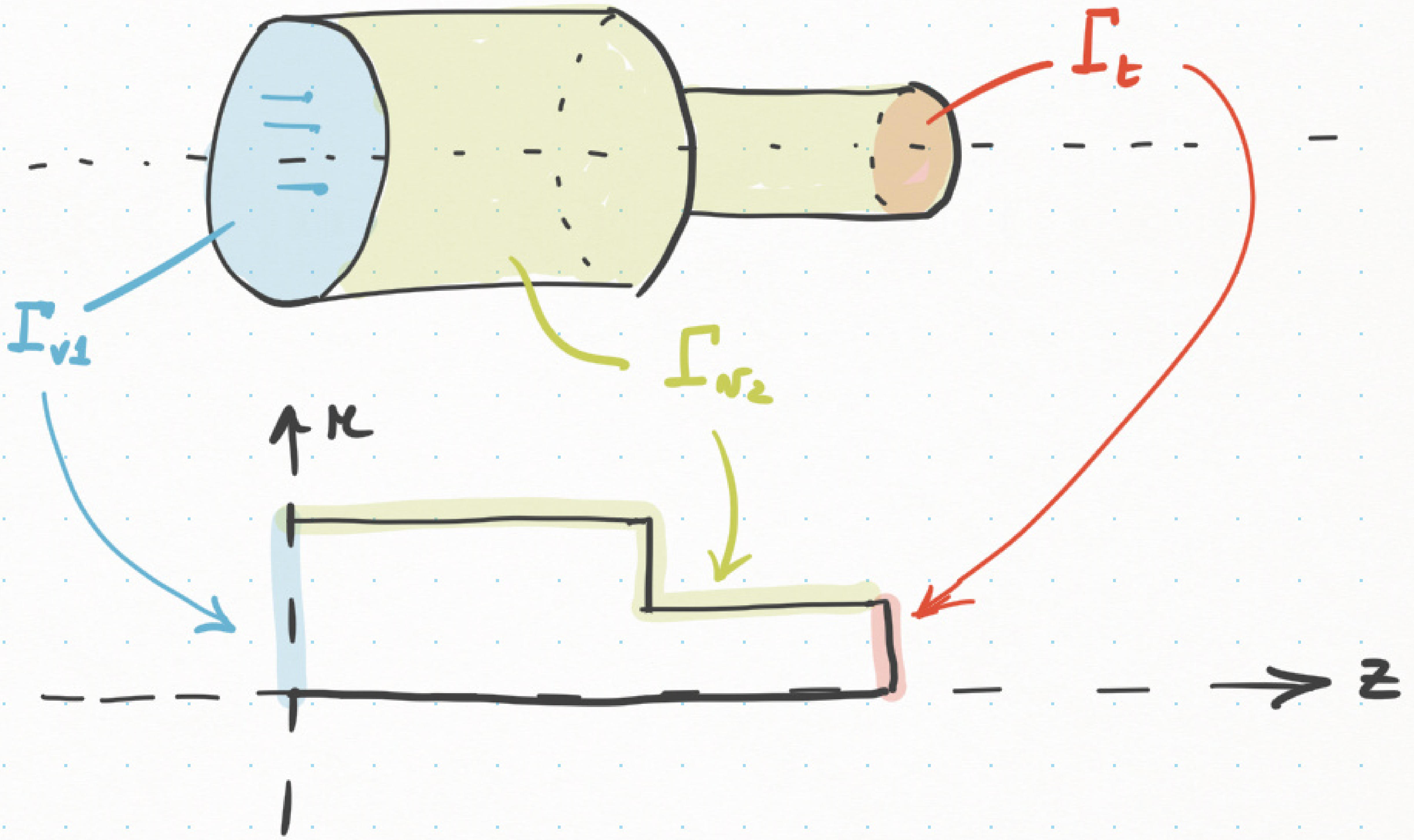
these two equations are implemented in FEniCS.

Applications :

1. Stokes_poiseuille.py



2. Stokes_3D.py



3. Stokes.axialsymetrie.py