

Shape optimization for improved understanding of magmatic plumbing systems

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Abstract

In volcano geodesy, inverse problems caused by identifying the location and shape of magmatic bodies based on ground deformation data are common. Traditional approaches often rely on models with predefined shapes, which can limit their accuracy. To address this, we present a shape optimisation method using a level-set approach that flexibly determines the optimal shape of a magma chamber without prior shape assumptions. By minimising the discrepancy between observed and modelled surface displacements, our adapted algorithm becomes suitable for solving inverse volcano deformation problems. We explore the capabilities of this approach with synthetic data and apply it to InSAR observations of the Svartsengi volcanic system in Iceland, demonstrating its potential to improve volcanic hazard assessment after maturation through future work.

1 Introduction

1.1 Challenge

In volcano geodesy, inverse problems are central to estimating the position of magmatic bodies using ground motion as a proxy. Displacement is observed by geodetic measurements such as Global Navigation Satellite System (GNSS) point positioning, leveling campaigns, or Synthetic Aperture Radar (InSAR) interferometry within a volcanic field, and the subsurface processes causing the movement are inferred from these observations (Dzurisin, 2007). Magmatic sources are modeled as pressurized cavities that deform the surrounding host rocks and cause the surface to move. Various inversion methods based on parametric analytical or numerical models aim at finding the optimal values for the vector of d free parameters $\vec{m} \in \mathbb{R}^d$ of the model. Then an error function $J(\vec{m})$ is representative of the misfit between the observed displacements and the prediction of the model. \vec{m}_{opt} can then be found using various inversion techniques that minimize J : global optimization based on analytic (Cervelli et al., 2001) or numerical models (Hickey and Gottsmann, 2014, Charco and Galán del Sastre, 2014), Bayesian inference (Bagnardi and Hooper, 2018, Trasatti, 2022), or genetic algorithms (Velez et al., 2011) on analytic models. The choice of the method is constrained by the reasonable number of evaluations of $J(\vec{m})$: numerical models handle a complex description of the system, but are computationally expensive compared to analytic models, which on the other hand may lead to an oversimplification (Taylor et al., 2021).

However, each of these finite-dimensional optimization methods is limited by the intrinsic assumption of a definite parametric shape for the source. In fact, analytic expressions can be derived for only a few regular shapes such as point source (Mogi, 1958), finite sphere source (McTigue, 1987), or ellipsoidal source (Yang et al., 1988), and any numerically generated shape must be parameterized to be inverted. Even in the case where complex shapes are chosen, they would require additional describing parameters, and ultimately any of the above methods may face the curse of dimensionality. The goal of this paper is not to give a definitive answer to these limitations, but rather to lay the first stone for a new approach that overcomes these difficulties.

1.2 Shape optimization

Shape (and topology) optimization aims to find the shape that minimizes a given function defined on a given system, without the need for prior assumptions about shape and topology. It is actively developed by part of the

56 applied mathematics community and is widely used in engineering to find op-
57 timal designs for systems: In structural mechanics, to maximize the stiffness
58 of a solid structure such as a cantilever beam (Bendsøe and Sigmund, 2004),
59 in fluid-structure interaction on heat exchangers or flying obstacles (Fep-
60 pon et al., 2020), and even as a way to explore new architecture for buildings
61 (Beghini et al., 2014). Most finite element simulation and design software now
62 implements an embedded shape optimization module (Frei, 2015, Slavov and
63 Konsulova-Bakalova, 2019, Le Quilliec, 2014). However, its use has not yet
64 been reported in the context of inverse problems in volcano geodesy, where
65 it can overcome the shape hypothesis problem as long as an internal pressure
66 value is assumed.

67 Many paradigms coexist in shape optimization as reviewed by Sigmund
68 and Maute, 2013, one of the most popular being SIMP optimization, where a
69 density value is optimized for each element of the mesh with values between
70 0 (void) and 1 (material) before being black and white filtered to output
71 a design (Sigmund, 2001, Bendsøe and Sigmund, 2004), with several open
72 source implementations (Andreassen et al., 2011, Hunter et al., 2017). We
73 chose level-set shape optimization instead because it has the advantage of
74 providing an explicit representation of the boundary at each step of the op-
75 timization, which is crucial for us as explained later (section 2). For this, we
76 relied on the work of Dapogny and Feppon, 2023, who thoroughly described
77 and vulgarized the method, as well as providing a freely available open source
78 implementation of the method, `sotuto` (Dapogny and Feppon, 2022, Octo-
79 ber 4/2024), which we modified and extended to adapt it to inverse geodetic
80 problems.

81 2 Method

82 Here we briefly present the key ingredients of level set shape optimization
83 along with their implications for our problem. The full mathematical back-
84 ground on which it relies is not detailed, but see this chapter by Allaire et al.,
85 2021 for a comprehensive step-by-step description supported by proofs and
86 theorems. It is also worth noting that many aspects of secondary importance
87 to the method are not mentioned for the sake of brevity. For the unfamiliar
88 reader interested in understanding the method, the lecture (especially part
89 III) given by Dapogny and Bonnetier, n.d. at the Université Grenoble Alpes
90 is also recommended.

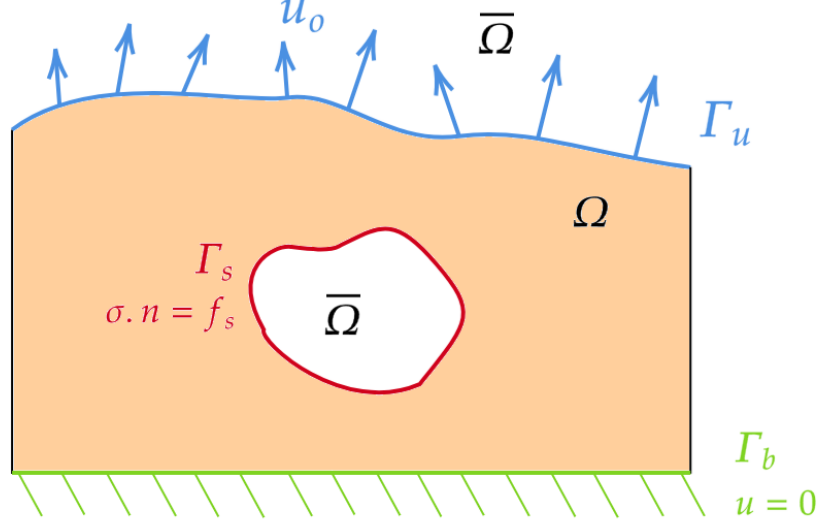


Figure 1: 2D sketch of the problem. The optimized boundary (where the level-set function is zero) is the magma chamber wall Γ_s subjected to a uniform normal load $\sigma(u).n = f_s$ on Γ_s , where $f_s = -\Delta P.n$, where n is the unit normal vector and ΔP is the pressure change between the magma source and the surrounding crust. The bottom surface Γ_b is fixed ($u = 0$). The other boundaries are free. The target displacement field u_o is known on the upper surface Γ_u .

91 2.1 Model

92 Let Ω be a bounded domain of \mathbb{R}^3 whose shape we want to optimize by
 93 modifying parts of its boundary $\partial\Omega$. As for classical analytical models of
 94 volcanic deformation induced by magmatic activity, Ω is a domain represent-
 95 ing a portion of the shallow Earth crust, including the volcano, assumed
 96 to be homogeneous, isotropic, and elastic. The governing equations are
 97 $-\text{div}(Ae(u)) = 0$ in Ω , where $e(u)$ is the strain tensor of the displacement
 98 field u and A is the constitutive law tensor, $Ae = 2\mu e + \lambda \text{tr}(e)Id$ for linear
 99 elasticity. Boundaries under different conditions, see Fig. 1 for all notations.

100 The part of $\partial\Omega$ to be optimized is Γ_s , the boundary magma chamber,
 101 which is modeled as an empty, uniformly pressurized cavity. Therefore, a
 102 value for the internal pressure ΔP must be assumed (see Discussion for devel-
 103 opment). In the following text we talk about optimizing $\partial\Omega$, but in practice

only $\Gamma_s \subset \partial\Omega$ is of interest and will be modified, any other boundary will be fixed during the iterations.

We want to find $\partial\Omega$ such that the displacement of the model $u(\Omega)$ is as close as possible to the observed displacement u_o on the surface Γ_u . Thus, the unconstrained shape optimization problem we want to solve is the minimization of a squared RMS discrepancy

$$\min_{\Omega} J(\Omega) = \int_{\Gamma_u} (u(\Omega) - u_o)^2 dS \quad (1)$$

2.2 Hadamard Boundary Variation

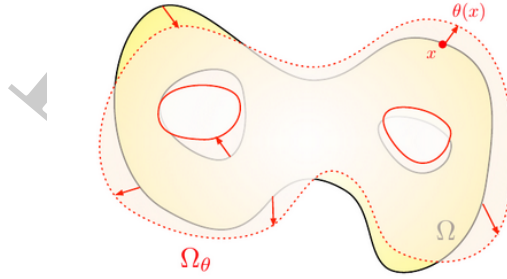


Figure 2: Reproduced from Allaire et al., 2021

Overall, this method can be considered a classical iterative gradient descent algorithm. J is first initialized at J_0 with an instructed first guess for Ω_0 and then iteratively decreased by moving $\partial\Omega$ of a given step in a given descent direction $\theta : \mathbb{R}^3 \mapsto \mathbb{R}^3 \in W^{1,\text{inf}}$ (the Sobolev space of uniformly bounded functions, Allaire et al., 2021) chosen using the shape derivative $J'(\Omega)(\theta)$.

The boundary variation method of Hadamard, 1908 introduces the notion of shape differentiation $F'(\Omega)(\theta)$ of a functional F defined on Ω in the direction θ . In short, such a derivative is based on the variation of a bounded domain $\Omega \mapsto \Omega_\theta := (Id + \theta)(\Omega)$: the surface $\partial\Omega$ is slightly moved according to a small vector field $\theta(x)$, as shown in Fig. 2. Once such a derivative exists, one can compute a descending direction at the n^{th} step θ_n , such as $J'(\Omega)(\theta_n) \leq 0$, so $J_{n+1} \leq J_n$, to decrease the value of J at each iteration.

In our case, after derivation based on the Cea, 1986 formal method, we found under the variational form :

$$J'(\Omega)(\theta) = \int_{\Gamma_s} \left(Ae(u) : e(p) + \frac{\partial f_s}{\partial n} p + \frac{\partial p}{\partial n} f_s + \kappa f_s p \right) \cdot \theta \cdot ndS \quad (2)$$

where $\kappa = \text{div}(n)$ is the mean curvature at the boundary, and p is the adjoint solution of

$$\forall v \in H^1(\mathbb{R}^3), \int_{\Gamma_u} 2(u_\Omega - u_o)vdS + \int_{\Omega} Ae(v) : e(p)dV = 0 \quad (3)$$

and $p = 0$ on Γ_b

From there, we can trivially move Ω in the direction $\theta = -A$ (where A is the integrand term in parentheses) to ensure that $J'(\Omega)(\theta) \leq 0$. This guarantees that $J(\Omega_{n+1}) \leq J(\Omega_n)$: the series $J(\Omega_n)$ converges to a minimum.

2.3 Level-set representation

A key issue is the representation of the surface to be optimized. The level set method allows to track dramatic changes as well as topology variations (creation of new holes). A certain function $\phi : D \mapsto \mathbb{R}$ is defined over the domain $D \in \mathbb{R}^3$ in such a way that the shape boundary is the level set 0, i.e. reads $\partial\Omega = \phi(x = 0)$. Basically, ϕ can be taken as the signed distance between any point x and $\partial\Omega$, as shown in the example fig. 3. In this way, $\partial\Omega$ is implicitly manipulated when transforming ϕ .

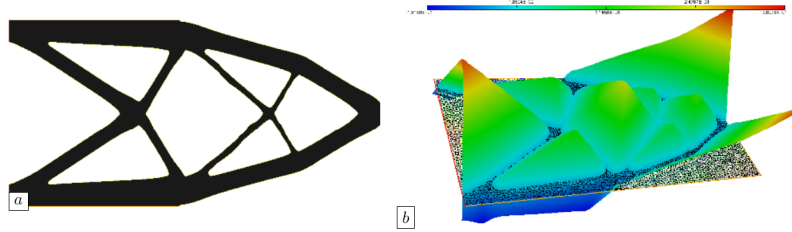


Figure 3: Reproduced from Dapogny and Feppon, 2023

Ω_n is then deformed by advecting the corresponding ϕ_n with a velocity field $V(x) = \tau_n \theta_n$, where τ_n is the additional step size. The advection equation usually appears in fluid mechanics to describe the evolution of a quantity transported by a given velocity field, but here there is a smooth and flexible way to modify ϕ which ensures smoothness of Ω_{n+1} and change of topology (see Allaire et al., 2021).

2.4 Numerical implementation

In practice, the D domain is discretized into a mesh T_n on which each variational form is solved at each iteration n . This includes the solution of the

148 elasticity to get u_n , the adjoint state p_n , the computation of the shape gradi-
 149 ent J'_n , the descent direction θ_n , the advection of ϕ_n . In **sotuto** it is achieved
 150 by calling scripts written in FreeFem++, a finite element software that allows
 151 solving any integral form of elliptic PDE (Hecht, 2012).

152 Once the new form Ω_{n+1} is computed and discretized thanks to a local
 153 remeshing phase, a new evaluation of J^{n+1} is performed. Since τ is arbitrarily
 154 fixed and initialized to 1, it can happen that Ω_n is shifted by too large a step
 155 and so $J_{n+1} \geq J_n$. To adjust the step size, a line search procedure is
 156 implemented and adjusts the step size by decreasing it if the new iteration
 157 is the worst to ensure an improvement of J by computing a new Ω_{n+1} being
 158 a less deformed version of Ω_n . On the contrary, if Ω_{n+1} is accepted, τ is
 159 increased to speed up convergence. A tolerance is set to accept iterations if
 160 the increase in J is reasonable.

161 The global optimization loop has no termination criterion. Thus, it is up
 162 to the user to stop it when no significant improvement in J can be achieved,
 163 or when the shape is not realistic.

164 The loop and the line search are implemented in Python in **sotuto**. Then
 165 the FreeFem scripts are called by the Python script core and data is ex-
 166 changed via temporary files.

167 The above aspects are implemented in **sotuto**. However, we extended its
 168 functionality to handle our geophysical problem, in a fork we called **magmaOpt**.
 169 This included: scripts to create the domain and initial source with a flexible
 170 mesher GMSH which handle complex geometries such as the one generated
 171 by topography Geuzaine et al., 2009, adapting FreeFem scripts to different
 172 error functions, allowing optimization of the loaded boundary Γ_s and so on.

173 3 Validation with synthetic data

174 To test the method, the idea is to do a kind of cross-validation. On the
 175 one hand, we form synthetic observation data from a known source. On
 176 the other hand, we initialized the algorithm with a first guess for the source
 177 shape and location. We expect the algorithm to iteratively modify the shape
 178 of the source and converge to the correct shape and location. In fact, the
 179 3D location of the source (e.g., its center of gravity for a random shape) is
 180 not directly optimized as a vector of discrete parameters, but is modified by
 181 the simple fact that the boundary is free to move in any direction, and thus
 182 can take on a kind of "average rigid body motion" as it gradually moves the
 183 center in a given direction.

184 In practice, the synthetic observed surface displacement field is derived
 185 from the McTigue, 1987 solution, an analytical approximation of the dis-

186 placement caused by a uniformly pressurized spherical cavity (the magma
 187 domain) embedded in an isotropic, homogeneous, and planar elastic medium
 188 (the host crust) with elastic constants $E = 10\text{GPa}$ and $\mu = 0.25$.

189 Usually, the quantities to be determined with parametric inversion based
 190 on a McTigue model are the location and the radius. The pressure change
 191 can also be left as a free parameter, but is interchangeable with the radius,
 192 so one must be fixed to determine the other, see (Greiner, 2021) for more
 193 details. For the synthetic source, we fixed these free parameters to $z = -5\text{km}$,
 194 $\Delta P = 10\text{MPa}$, $R = 1.5\text{km}$, which are typical values for inverted magmatic
 195 domains.

196 `magmaOpt` is then allowed to run freely, without any termination condition,
 197 to see whether or not it succeeds in converging from the ellipsoid to the
 198 McTigue sphere we used to generate the synthetic displacement.

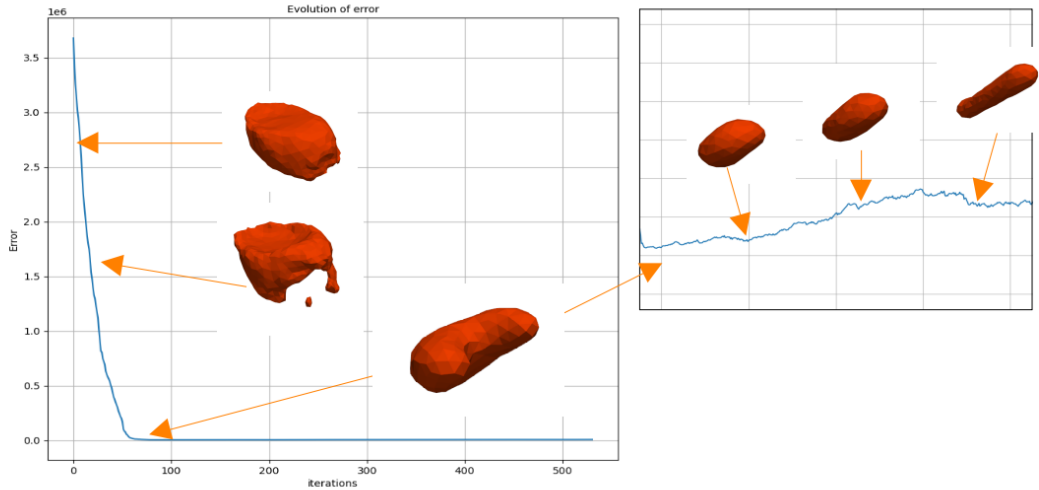


Figure 4: Evolution of error and successive shapes taken by the magma source during an optimization loop. The initial guess is a flat ellipsoid of semi-axes $r_x = 2\text{km}$, $r_y = 3\text{km}$, $r_z = 1\text{km}$ centered on the true spherical source. The minimum is reached at iteration 82.

199 As shown in the figure 4, the algorithm seems to converge to a minimum.
 200 After that, the slope of the cost function is positive because a small increase
 201 in J is allowed. It is obvious that no other minima are found, as the shape
 202 evolves towards a stick-shaped feature, far from the expected solution. We
 203 can also discuss the minima found. The surface reached is obviously not a
 204 sphere, but it is closer than any shape found before. We expect the shape to

205 be closer to a sphere with a finer mesh defined. Many improvements could be
 206 realized: for example, once it is obvious that the algorithm will not converge
 207 to a better solution, we could restart the algorithm on the best solution
 208 found, set new evolution parameters, and allow a finer mesh. By repeating
 209 this process automatically, it may be possible to arrive at a more likely shape
 210 for the magma reservoir.

211 4 Real test case : Svartsengi 2022 inflation

212 We now apply the method to infer the shape of a magma domain in a recent
 213 period of volcanic unrest and eruption in SW Iceland by evaluating the shape
 214 of a magma body responsible for the ground inflation observed from 21 April
 215 to 14 June 2022 at Svartsengi on the Reykjanes peninsula. This is one of
 216 5 inflation episodes that preceded catastrophic dike breaches and eruptions
 217 at the Sundhnúkur crater row, which caused the destruction of the city of
 218 Grindavík (Sigmundsson et al., 2024).

219 The observational data used are the line-of-sight (LOS) displacement
 220 maps of the area from Cosmo SkyMed available in Parks et al., 2024, the
 221 data used in Sigmundsson et al., 2024. After uniform downsampling and
 222 mesh reprojection (the data points must be aligned on the mesh nodes), the
 223 ascending A32 and descending D132 tracks were both used in the RMS error
 224 function we adapted to the LOS geometry.

$$J(\Omega) = \sum_{i \in tck} \alpha_i \int_{\Gamma_u} (L_i(u(x)) - l_o^i(x))^2 dS \quad (4)$$

225 Where $tck = \{A125, D132\}$. For each track i , α_i is the weight of the track
 226 ($\forall i, \alpha_i = 1$ here), $L_i : \mathbb{R}^3 \mapsto \mathbb{R}$ is the function that projects the 3D sur-
 227 face displacement given by the model into the LOS geometry, and l_o^i is the
 228 observed LOS displacement.

229 We then used the framework developed above, only projecting the InSAR
 230 data onto the mesh of D and modifying the expression of the error function
 231 in `magmaOpt`.

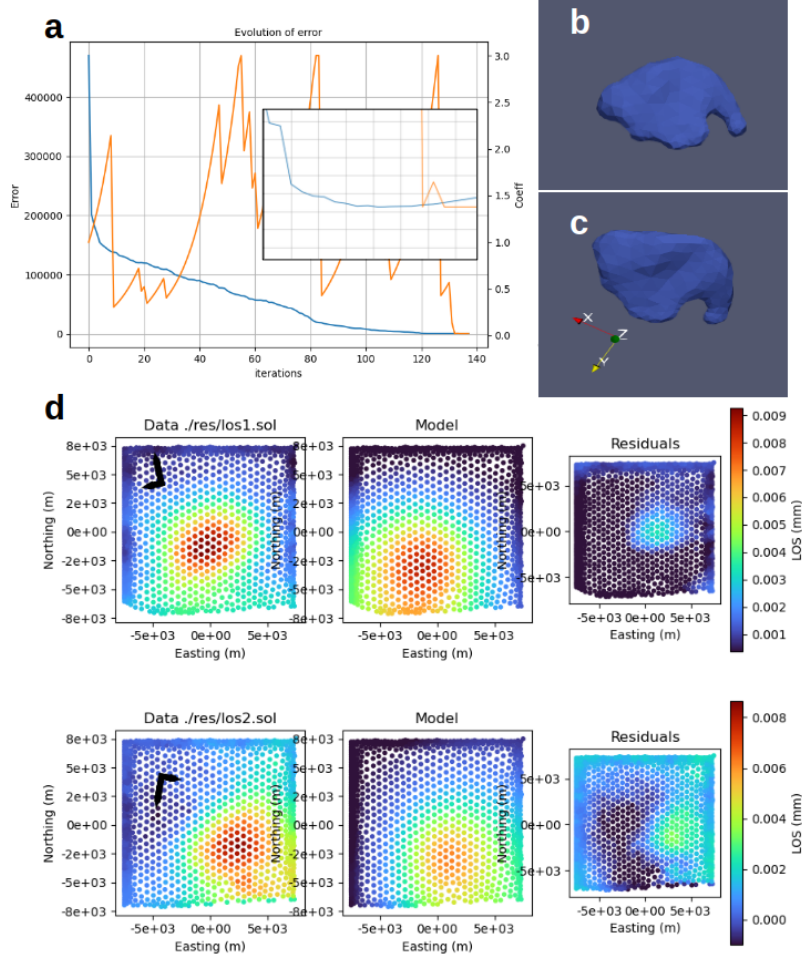


Figure 5: a) Convergence plot with embedded zoom. The blue line is the error and the orange line is the evolution of τ . Minima are reached at iteration 128. b,c) Side and top view of the source Γ_s minimizing J . d) Data, model and residuals of the LOS displacements at iteration 128 for the two InSAR tracks A32 (top) and D132 (bottom). Black arrows are heading and looking directions, coordinates are ISN16 islands shifted to a local origin (2529373E, 179745N).

232 The results shown in figure 5 are encouraging: after providing an initial
 233 guess located at the center of inflation at depth for a sphere of radius RR ,
 234 the algorithm is able to iteratively change the shape and depth of the magma
 235 domain to finally result in a sill-like flattened spheroid whose centroid is lo-
 236 cated at DD depth. This is consistent with the presumed depth found in
 237 the supporting information of Sigmundsson et al., 2024, which performs an

analytical model-based inversion. Although the pressure must be fixed, as explained in 1.2, the result can be used to compare the final shape of the magmatic intrusion and give a richer insight into it. Here we see interesting features, such as an increasing thickness on the north side, that can't be traced by any other method. The algorithm produces features that we consider to be artifacts, probably due to mesh refinement problems, such as small holes or horn-shaped features.

5 Discussion

This work paves the way for a new class of methods that tackle an unknown geometry of the magmatic domains, thus giving the possibility to explore irregular shapes that are more likely to exist compared to any other usually assumed regular shapes. However, even if the first results presented are promising, many questions remain to be answered. First of all, the internal pressure of the chamber must be specified, which is a strong hypothesis. In this context, the precise shape of the source should be determined as a second step. The traditional analytical model-based inversion would be run first, giving a pressure and a first educated guess for the position and shape of Ω_0 . Then a more realistic shape could be sought with a shape optimization taking the output of the inversion as an initial guess.

Adding constraints may also be an interesting way to explore. For example, the volume of the source could be constrained to be within bounds or even to match a certain value. The implemented shape optimization is certainly able to handle constraints as described by [allaire2002](#). The physical meaning of the best shape might benefit from a more constrained problem, and the less influencing deeper part of the source might be less random.

To better understand the influence of data partitioning and variability, additional tests could be run with synthetic data. We can think of tests such as masking part of the surface displacement field, introducing noise and parasitic signals, reducing the number of data points, as is often the case in reality with areas of volcanic systems lacking data coverage (glacier, river, lava, forest) and subjected to perturbations (atmospheric distortion, weather).

It is also important to mention that the behavior of the algorithm is influenced by numerous parameters of varying importance, starting from the discretization length (element size) or the domain extent, to the limits of the step size τ , the regularization length, or the number of iterations allowed by the line search. A systematic study of each of these parameters would be beneficial in assessing the quality of the shape inferred.

276 Exploring a way to quantify the uncertainty of the answer is also crucial.
277 For example, a sensitivity analysis approach could be considered, as well as
278 the inclusion of a probabilistic quantity.

279 6 Conclusion

280 The present study has successfully demonstrated the application of inverse
281 problems and computational methods to infer the shape of a magma domain
282 beneath a volcano using ground inflation data from satellite observations.
283 The shape optimization technique used in this research showed a new way to
284 identify the most likely shape of the magma chamber. It was intended as an
285 opening to new methods rather than a complete solution.

286 We have shown that modifying a shape optimization algorithm to handle
287 geophysical problems is feasible and of interest. Tests on synthetic data
288 showed to some extent the relevance of the approach, although the best
289 sources found exposed the limitations faced by these first attempts of shape
290 optimization for volcano geodesy. The test on real data showed a concrete
291 case of how the method could be used after being more mature.

292 The perspectives are numerous. The numerical nature of the models
293 allows to easily add complexities to the modeling, such as complex mechanical
294 behavior of the crust (plasticity, viscoelasticity, poroelasticity), additional
295 loads (tectonic stress, tidal loads, glacier weights), or inhomogeneities. We
296 hope that the open source code `magmaOpt` developed by us will be modified
297 and extended by future work.

298 By addressing these limitations and extending this approach, researchers
299 can further improve the accuracy and reliability of magma domain shape
300 inference. Ultimately, the development of more sophisticated models will
301 enable geophysicists to better monitor volcanic activity, predict eruptions,
302 and provide critical support for hazard mitigation strategies.

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