Instructions:
For each problem, state all your assumptions, explain your approach and solution method, include a good copy of any preparation work that needs to be done by hand (derivations, discretizations… etc.) in your report.

If the problem requires writing a computer program, include your code and its output in your report.

In addition, email your codes to Prof. Dworkin by the due date and time. Your codes may be run to verify output, or put through a plagiarism checker. All codes that you email should be as separate attachments, titled PSnum_Qnum_Lastname_Firstname_Studentnum.dat (or .f90 or .f95 or .c etc.)
Example: PS3_Q1_Dworkin_Seth_0000001.f90. Make sure that they can be opened and read in a text editor such as Notepad or equivalent.

Email your codes and report in the same email. Your email should be timestamped by Dr. Dworkin's email provider by 3:59:59 PM on the due date or else your assignment will be considered late. All of your report must be contained in a single PDF file. (I.e., your email should have one PDF report which contains your codes, and in addition it should have each code that you wrote as a separate attachment.)

For all programs that you write, include a header with your full name as registered at the university, student number, date, assignment number, question number, description of the program, and programming language used.

Before you hand in your assignment, refer to the document on assignment preparation instructions posted on D2L under Content – Assignments.

Questions
1. Consider the 1D convection/diffusion problem governed by:

$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right)$$

in the domain $0 \le x \le L$, with $\phi(0) = 1$ and $\phi(L) = 0$.

a) Solve the problem analytically. Write a program to solve this equation using finite volumes and a Peclet number $\left(\frac{\rho u L}{\Gamma}\right)$ of $P_e = 50$ on an equispaced grid. Assume constant diffusivity and density. Use ghost points to discretize your boundary conditions. Discretize the diffusion term as per usual. Generate a discretization for the convective term as a function of an input variable, α where $\alpha = 1$ corresponds to an upwinded scheme, and $\alpha = 0$ corresponds to a central difference scheme. Calculate the error L_2 defined as

$$L_2 = \frac{\sqrt{\sum_{i=1}^{N} (\phi(x_i) - \phi_{exact})^2}}{N}$$

Run the program with 32, 64, and 128 control volumes for each of $\alpha = 0$, 1, and $P_e^2/(P_e^2+5)$. Note that contrary to what was in the lecture notes, here for simplicity, we are using P_e instead of P_Δ for the calculation of α. Generate a table with these nine values of L_2. Plot and compare the three solutions on the grid with 64 c.v.s. Discuss your results in three sentences or less.

b) Plot L_2 vs Δx for each of $\alpha = 0$, 1, and $P_e^2/(P_e^2+5)$ and determine the expected order of accuracy for the three cases. Extrapolate to determine the value of P_e for which L_2 of $\alpha = 0$ and L_2 of $\alpha = 1$ would be roughly the same. Discuss your results in three sentences or less.

c) Reuse your program to determine the value of P_e for which the solution generated with $\alpha = 0$ no longer oscillates.

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2. Consider the problem of transient heat conduction in a heat sink plate that cools a hot object such as a CPU (see lecture 6, page 5) governed by:

$$\rho c_p \frac{dT}{dt} = \nabla \cdot (\lambda \nabla T) + S$$

The plate is initially at temperature T = 25°C. Via slow and steady air circulation, the ambient air surrounding the plate is maintained at T_∞ = 25°C. The dimensions of the plate are 5 cm × 5 cm × 3.5 mm. The edge (y = 0) that comes into contact with the heat source is exposed to a temperature of 150°C. At the three other sides, cooling occurs via convection according to:

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = h\left(T_\infty - T(0,y,t)\right), \quad \lambda \left.\frac{\partial T}{\partial x}\right|_{x=5cm} = h\left(T_\infty - T(5,y,t)\right), \quad \lambda \left.\frac{\partial T}{\partial y}\right|_{y=5cm} = h\left(T_\infty - T(x,5,t)\right),$$

The source term relates to convective cooling of the larger flat surfaces and is characterized by

$$S = \frac{2h_{side} A\left(T_\infty - T(x,y,t)\right)}{V}$$

The material properties are:

$\rho = 1716 \ \ kg/m^3$

$c_p = 4817 \ \ J/kg \cdot K$

$\lambda = 14.6 \ \ W/m \cdot K$

$h = 472 \ \ W/m^2 \cdot K$

$h_{side} = 36.4 \, W/m^2 \cdot K$

Using a timestep of 0.01 s, a fully implicit scheme, and 80 x 80 equispaced control volumes, write a program to determine the final temperature distribution in the plate, and the time, t_{SS}, at which that occurs, characterized by when

$$\frac{\sqrt{\sum_{ind=1}^{N} \left(T(x,y,t^n) - T(x,y,t^{n+1})\right)^2}}{N} < Tol_{ss}$$

It is recommended that you convert all temperatures to K before writing your program. Use ghost points to discretize your boundary conditions.

Use your preconditioned Bi-CGSTAB subroutine to solve your linear system at each timestep. Note that you will likely need to modify it to handle the two new vectors associated with the pentadiagonal system that were not present in the tridiagonal system. Use a linear system residual tolerance of 10^{-9}.

Plot the final temperature distribution in the plate. Plot the T distribution in the plate with TOL_{SS} = 5e-3, 5e-5, 5e-8, and 5e-10, Comment on the time it takes to reach steady state in three sentences or less.

Comment on the program performance and the evolution of temperature within the plate in three sentences or less.