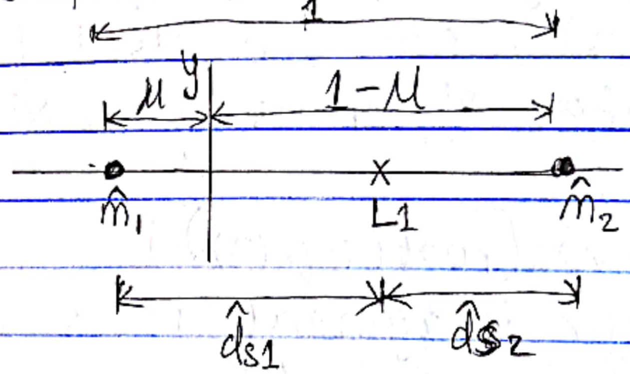
Assignment 3

1) To solve for L_1 Location in the CRTBP,



$$1 = \hat{d}_{s1} + \hat{d}_{s2}$$

where $\hat{d}_{si} = \|\hat{r}_{si}\|$, denotes the distance of L_1 from $\hat{m}_i$. Since L_1 falls within Region 1, which must satisfy the following conditions:-

$$\hat{d}_{s1} = x + \mu, \quad \hat{d}_{s2} = 1 - \mu - x = -\hat{d}_{s1} + 1 = 1 - \hat{d}_{s1}$$

$$x = \hat{d}_{s1} - \mu$$

We can substitute the above into our reduced equilibrium equation, which is given as

$$0 = -x + (1-\mu)\frac{(x+\mu)}{\hat{d}_{s1}^3} + \frac{\mu(x+\mu-1)}{\hat{d}_{s2}^3}$$

Since the components of $y = z = 0$,

$$0 = -(\hat{d}_{s1} - \mu) + (1-\mu)\frac{(\hat{d}_{s1})}{\hat{d}_{s1}^3} + \frac{\mu(1-\hat{d}_{s1})}{(1-\hat{d}_{s1})^3}$$

$$0 = \mu - \hat{d}_{s1} + \frac{(1-\mu)}{\hat{d}_{s1}^2} - \frac{\mu}{(1-\hat{d}_{s1})^2}$$

$$0 = \mu(\hat{d}_{s1}^2)(1-\hat{d}_{s1})^2 - \hat{d}_{s1}(\hat{d}_{s1}^2)(1-\hat{d}_{s1})^2 + (1-\mu)(1-\hat{d}_{s1})^2 - \mu\hat{d}_{s1}^2$$

$$0 = \mu\hat{d}_{s1}^2 + \mu\hat{d}_{s1}^2(-2\hat{d}_{s1}) + \mu\hat{d}_{s1}^2(\hat{d}_{s1}^2) - \hat{d}_{s1}^3 - \hat{d}_{s1}^3(-2\hat{d}_{s1}) - \hat{d}_{s1}^3(\hat{d}_{s1}^2) + (1-\mu) + (1-\mu)(-2\hat{d}_{s1}) + (1-\mu)(\hat{d}_{s1}^2) - \mu\hat{d}_{s1}^2$$

$$0 = -\hat{d}_{s1}^5 + \mu\hat{d}_{s1}^4 + 2\hat{d}_{s1}^4 - 2\mu\hat{d}_{s1}^3 - \hat{d}_{s1}^3 + (1-\mu)\hat{d}_{s1}^2 \cancel{+(1-\mu)d} - (2-2\mu)\hat{d}_{s1} + (1-\mu)$$

$$\therefore \; 0 = -\hat{d}_{s1}^5 + (2+\mu)\hat{d}_{s1}^4 + (-2\mu-1)\hat{d}_{s1}^3 + (1-\mu)\hat{d}_{s1}^2 + (2\mu-2)\hat{d}_{s1} + (1-\mu) \quad \text{—— (3)}$$

Therefore the polynomial expression for L_1 given in eq. (3), can be used to find the coordinates of L_1 using the MatLab "roots" function as:-

$$>> ds_1 = vpa(root(eq.(3)))$$

This gives us four imaginary roots and one real root of $\hat{ds}_1 = 0.849146$,

given than $\mu = 0.01213$, we can solve for

$$x = \hat{ds}_1 - \mu = 0.849146 - 0.01213$$

$$x = 0.837016$$

N/B:- If we substituting these values back into our our equilibrum condition, we will get zero, which satisfies, the condition. Hence our coordinate is verified as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0.837016 \\ 0 \\ 0 \end{bmatrix}$$

3) Similarly we can use the above resuts to obtain our Jacobi constant for L_1,

Jacobi constant;- $C = 2\left[\frac{x^2+y^2}{2} + \frac{1-\mu}{\hat{ds}_1} + \frac{\mu}{\hat{ds}_2}\right] - (x'^2 + y'^2 + z'^2)$

where $x' = y' = z' = 0$

and we know that $\hat{ds}_2 = 1 - \hat{ds}_1$

$$C = \frac{(0.837016)^2}{1} + \frac{2(1-0.01213)}{0.849146} + \frac{2(0.01213)}{(1-0.849146)}$$

Therefore, we obtain our Jacobi constant for L_1 as, $C = 3.18815$

4) In order to Linearise for L_1, let consider following state equation. where

$$X_1 = x \quad X_1' = x' = X_2$$
$$Y_1 = y \quad Y_1' = y' = Y_2$$
$$X_2' = x'' = 2y' + x - (1-\mu)\frac{(x+\mu)}{\hat{d}_{s1}^3} - \mu\frac{(x+\mu-1)}{\hat{d}_{s2}^3}$$

$$Y_2' = y'' = -2x' + y - y\frac{(1-\mu)}{\hat{d}_{s1}^3} - y\frac{\mu}{\hat{d}_{s2}^3}$$

Since $z = z' = z'' = 0$, we can write the above in matrix form as,

$$f(X,Y) = \begin{bmatrix} f(X,Y)_1 \\ f(X,Y)_2 \\ f(X,Y)_3 \\ f(X,Y)_4 \end{bmatrix} = \begin{bmatrix} X_1' \\ Y_1' \\ X_2' \\ Y_2' \end{bmatrix} = \begin{bmatrix} X_2 \\ Y_2 \\ 2Y_2 + X_1 - (1-\mu)\frac{(X_1+\mu)}{\hat{d}_{s1}^3} - \mu\frac{(X_1+\mu-1)}{\hat{d}_{s2}^3} \\ -2X_2 + Y_1 - Y_1\frac{(1-\mu)}{\hat{d}_{s1}^3} - Y_1\frac{\mu}{\hat{d}_{s2}^3} \end{bmatrix}$$

$f(X,Y) = 0$ yields our equilibrum condition. So we can linearise about this equilibrium point as

$$\delta(x',y') = \delta(\dot{x},\dot{y}) = \frac{\partial(f(X,Y)_i)}{\partial(X_i,Y_i)}\bigg|_{(X_i,Y_i) = (X_{ir},Y_{ir}) = (x,y)} \delta(X,Y)$$

equilibrium points

so that

$$\delta(x',y') = \begin{bmatrix} \delta x' \\ \delta y' \\ \delta x'' \\ \delta y'' \end{bmatrix} = \begin{bmatrix} \frac{\partial f(X,Y)_1}{\partial X_1} & \frac{\partial f(X,Y)_1}{\partial Y_1} & \frac{\partial f(X,Y)_1}{\partial X_2} & \frac{\partial f(X,Y)_1}{\partial Y_2} \\ \frac{\partial f(X,Y)_2}{\partial X_1} & \frac{\partial f(X,Y)_2}{\partial Y_1} & \frac{\partial f(X,Y)_2}{\partial X_2} & \frac{\partial f(X,Y)_2}{\partial Y_2} \\ \frac{\partial f(X,Y)_3}{\partial X_1} & \frac{\partial f(X,Y)_3}{\partial Y_1} & \frac{\partial f(X,Y)_3}{\partial X_2} & \frac{\partial f(X,Y)_3}{\partial Y_2} \\ \frac{\partial f(X,Y)_4}{\partial X_1} & \frac{\partial f(X,Y)_4}{\partial Y_1} & \frac{\partial f(X,Y)_4}{\partial X_2} & \frac{\partial f(X,Y)_4}{\partial Y_2} \end{bmatrix} \delta(X,Y)$$

To solve the above we know that for our L_1

$$\hat{d}_{s1} = x_1 + \mu \quad \& \quad \hat{d}_{s2} = 1 - \mu - x_1$$

we will substitute this into x_2' when differentiating with respect to x_1

we get

$$\begin{bmatrix} \delta x' \\ \delta y' \\ \delta x'' \\ \delta y'' \end{bmatrix} = \begin{bmatrix} 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ 1 - \frac{2(\mu-1)}{(\mu+x)^3} - \frac{2\mu}{(\mu+x-1)^3}, & 0, & 0, & 2 \\ 0, & 1 - \frac{(1-\mu)}{\hat{d}_{s1}^3} - \frac{\mu}{\hat{d}_{s2}^3}, & -2, & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta x' \\ \delta y' \end{bmatrix}$$

putting back $x + \mu = \hat{d}_{s1}$ & $1 - x - \mu = \hat{d}_{s2}$ for the 3rd row

we get

$$\begin{bmatrix} \delta x' \\ \delta y' \\ \delta x'' \\ \delta y'' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 + 2\left[\frac{(1-\mu)}{\hat{d}_{s1}^3} + \frac{\mu}{\hat{d}_{s2}^3}\right] & 0 & 0 & 2 \\ 0 & 1 - \left[\frac{1-\mu}{\hat{d}_{s1}^3} + \frac{\mu}{\hat{d}_{s2}^3}\right] & -2 & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta x' \\ \delta y' \end{bmatrix}$$

Lets have that $A = \frac{1-\mu}{\hat{d}_{s1}^3} + \frac{\mu}{\hat{d}_{s2}^3}$, the above becomes

$$\begin{bmatrix} \delta x' \\ \delta y' \\ \delta x'' \\ \delta y'' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1+2A & 0 & 0 & 2 \\ 0 & 1-A & -2 & 0 \end{bmatrix}}_{F} \begin{bmatrix} \delta x \\ \delta y \\ \delta x' \\ \delta y' \end{bmatrix} \quad (4)$$

$F \to$ used for obtaining characteristic equation

Therefore the equations of motion for L_1 is Linearised as

5) Having obtained $x_{L1} = 0.837016$, $y_{L1} = 0$, The eigenvalues of F from eqn (4) are found to be $\lambda_s = -2.9318$, $\lambda_u = 2.9318$, $\lambda_c = 2.3342i$, $\bar{\lambda}_c = -2.3342i$

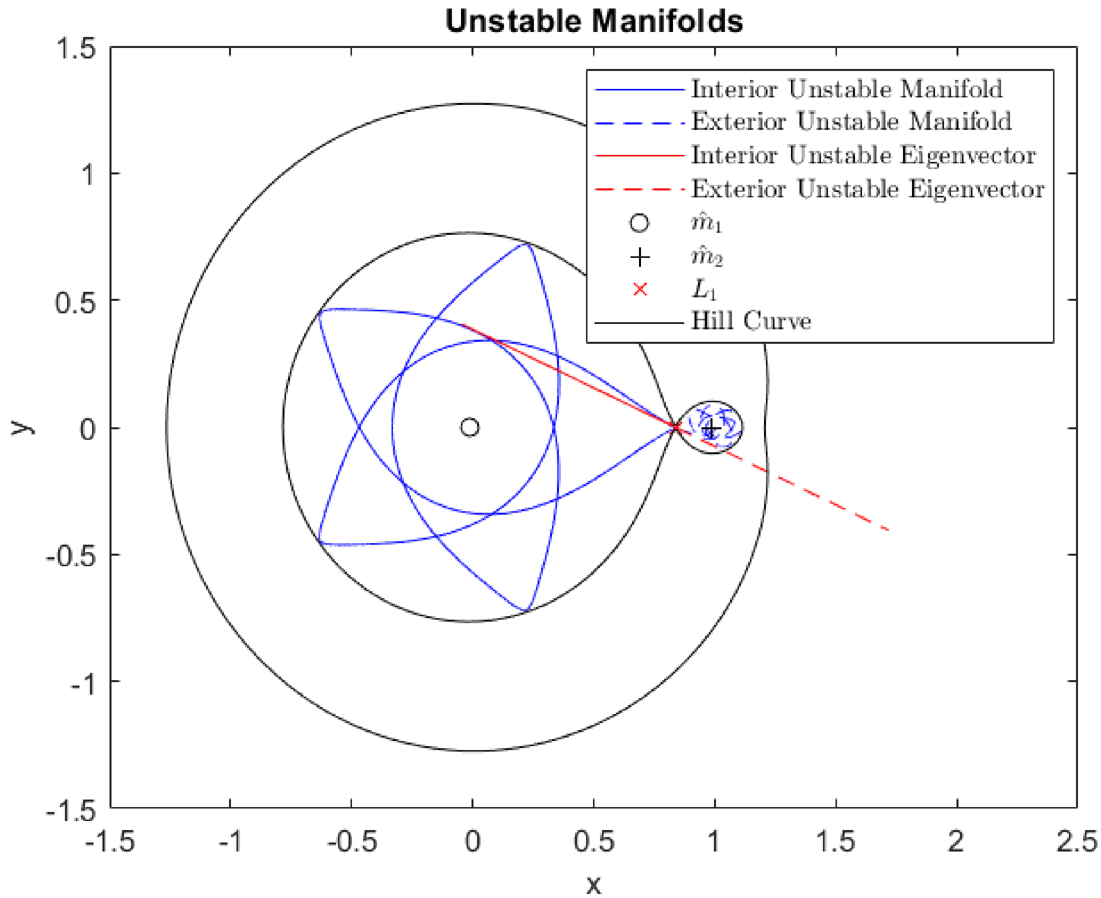
we know th λ_s will lead to a stable manifold since it is negative and λ_u will lead to unstable manifold And the pair $\lambda_c, \bar{\lambda}_c$ leads to periodic motion for linearized system, which are called centre manifold. The corresponding eigenvectors are;

for λ_s, $u_s = \begin{bmatrix} -0.2933 \\ -0.1350 \\ 0.8598 \\ 0.3956 \end{bmatrix}$

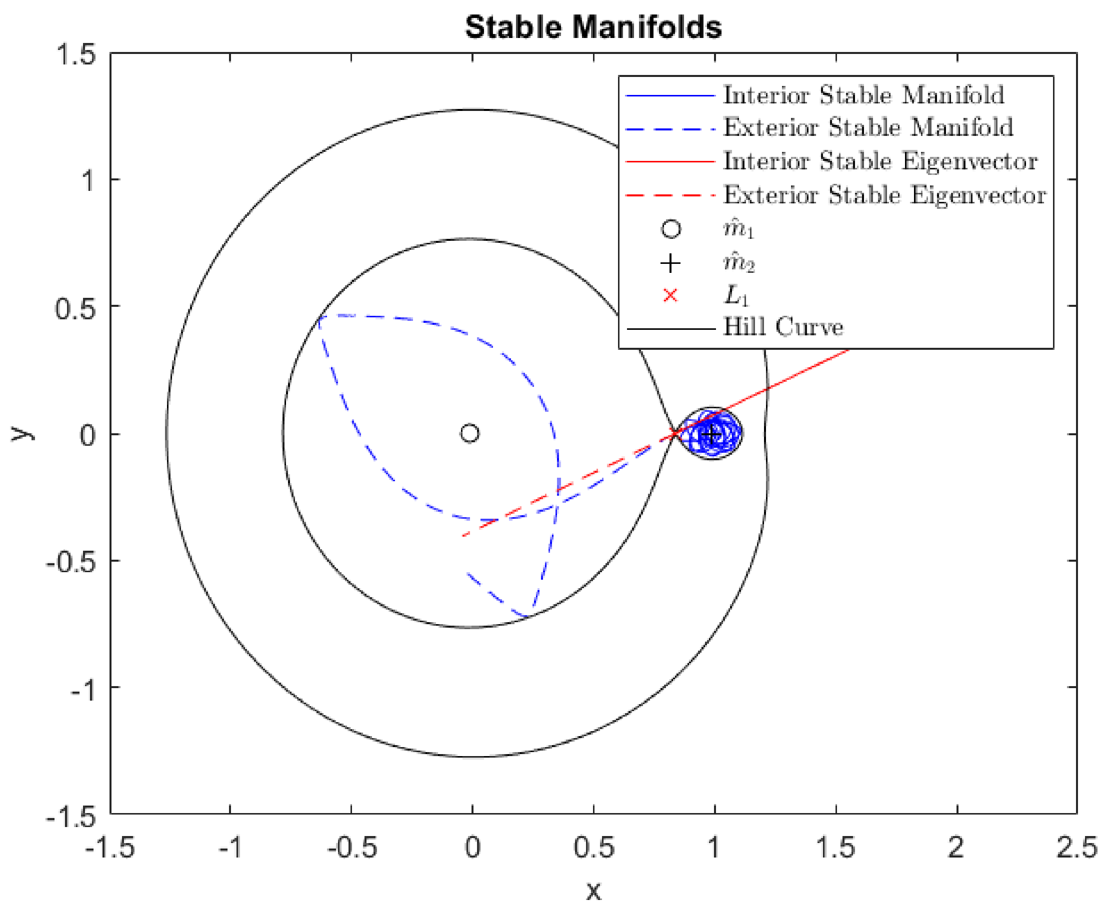
for λ_u, $u_u = \begin{bmatrix} -0.2933 \\ 0.1350 \\ -0.8598 \\ 0.3956 \end{bmatrix}$

for $\lambda_c, \bar{\lambda}_c$, $u_c = u_R + j u_I$, where $u_R = \begin{bmatrix} 0.1058 \\ 0 \\ 0 \\ 0.8854 \end{bmatrix}$, $u_I = \begin{bmatrix} 0 \\ 0.3793 \\ 0.2469 \\ 0 \end{bmatrix}$

Since this is a planar motion we can remove z & z' from ~~oure~~ our parameter q, so numerically integrating our equation of motion, backwards in time, with initial condition $q(0) = q_{L1} \pm \epsilon u_s$ for stable manifold, $q(0) = q_{L1} \pm u_u$ for unstable manifold, where $q_{L1} = [x_{L1}; y_{L1}, 0, 0]^T$. Lets ~~the lebelt~~ Lebel the two regions of the eigen vectors u_s as 'interior' and 'exterior' manifolds which denotes whethe they go into the Earth region or are in the moon region since our Hill curve for a Jacobi constant of $C = 3.1882$, seals moving out of the Earth-moon Region, but only allows transfers between the Earth and the moon region. Similary for the unstable manifold.



Unstable Manifolds
Interior Unstable Manifold
Exterior Unstable Manifold
Interior Unstable Eigenvector
Exterior Unstable Eigenvector
$\hat{m}_1$
$\hat{m}_2$
L_1
Hill Curve
x
y



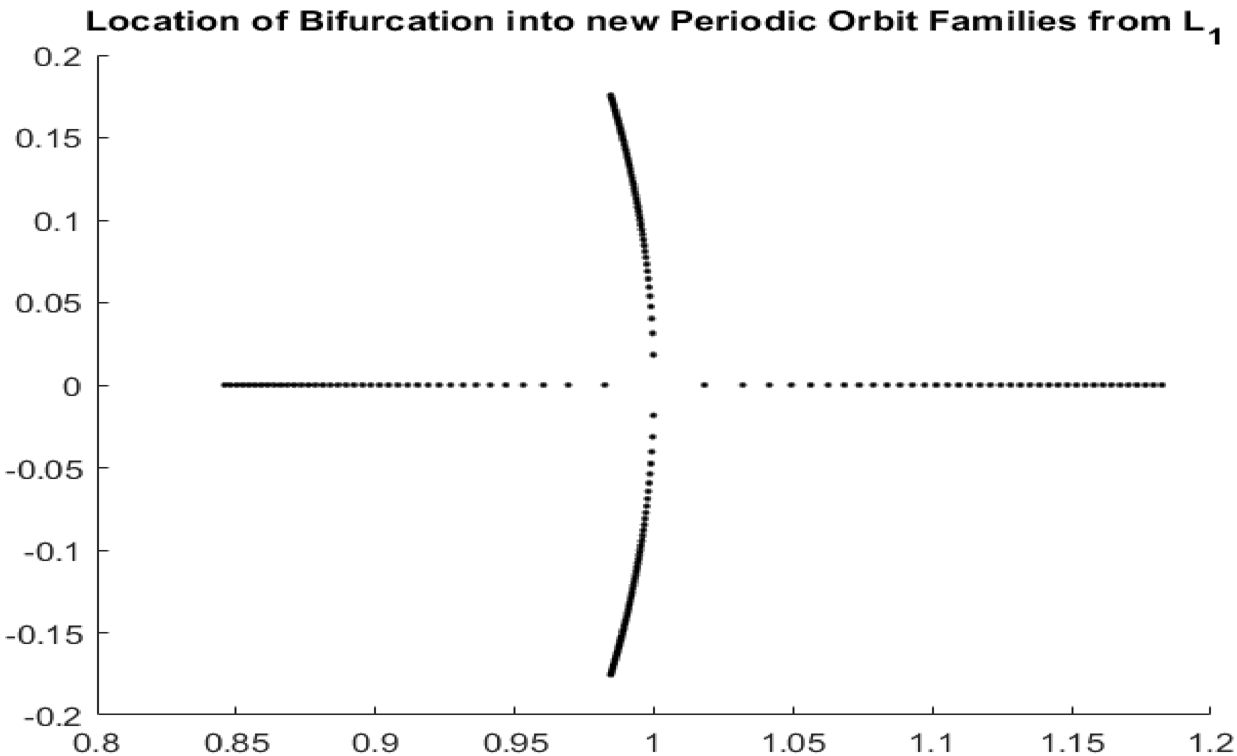
Stable Manifolds
Interior Stable Manifold
Exterior Stable Manifold
Interior Stable Eigenvector
Exterior Stable Eigenvector
$\hat{m}_1$
$\hat{m}_2$
L_1
Hill Curve
x
y

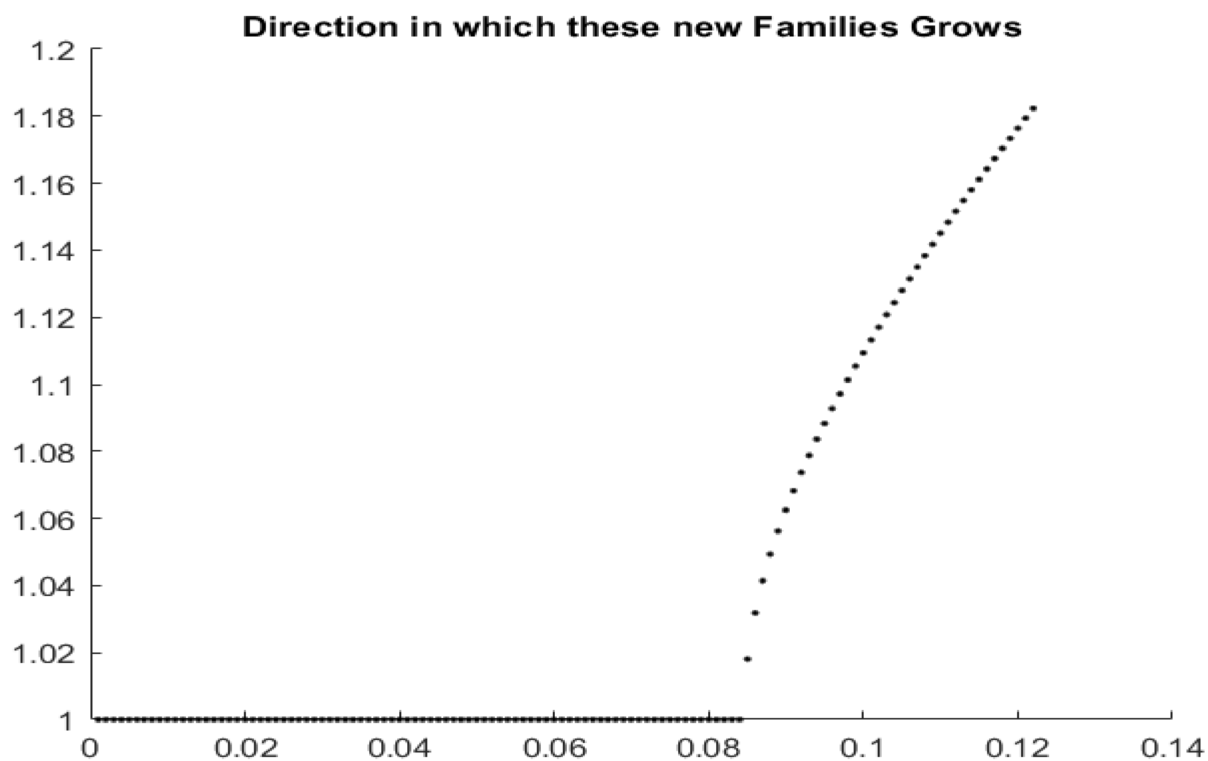
From the figure above we can see that for the stable manifold, the interior ~~regi~~manifold is in the moon region and a transfer to the Exterior ~~reg~~manifold is in the Earth region. We also observe that the stable manifold does not cross the Hills curve. Also for the unstable manifold, the interior is around the Earth region, unlike the stable manifold. And the Exterior ~~m~~manifold is around the moon region, ~~This~~ manifold also stays within the Hills curve.

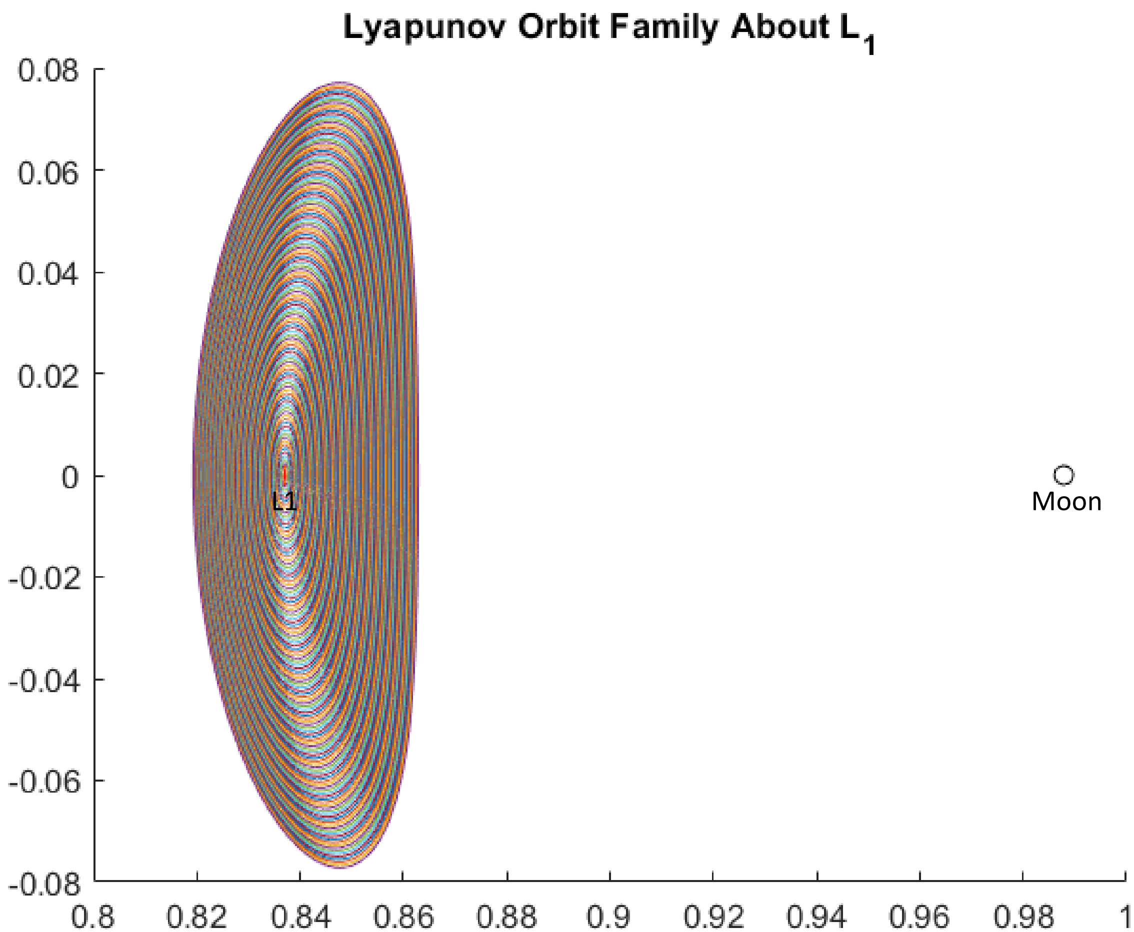
b) using the center manifold eigen values and vectors as starting point, we can grow a family of planar lyapunov orbits about L_1. This is done by examining the associate monodromy matrices of L_1, we get the following eigen values

$$\lambda_{mc} = 0.9845 + 0.1755i$$
$$\bar{\lambda}_{mc} = 0.9845 - 0.1755i$$

we can use this to obtain the bifurcation locations of the new periodic orbit families as shown below. And its associated eigenstructure can be used to determine its directions as ~~the gr~~ the families grows. This is also shown in the figure below. The new families about L_1 is also shown below:


Location of Bifurcation into new Periodic Orbit Families from L_1
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2


Direction in which these new Families Grows
1.2
1.18
1.16
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14



Lyapunov Orbit Family About L_1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
0.8
0.82
0.84
0.86
0.88
0.9
0.92
0.94
0.96
0.98
1
L1
Moon