AE/ME8112 Problem Set PS1 **Due Wednesday May 26th at 4:00PM**

Instructions:
For each problem, state all your assumptions, explain your approach and solution method, include a good copy of any preparation work that needs to be done by hand (derivations, discretizations… etc.) in your report.

If the problem requires writing a computer program, include your code and its output in your report.

In addition, email your codes to Prof. Dworkin by the due date and time. (i.e., your codes should appear in your report AND in separate files.) Your codes may be run to verify output, or put through a plagiarism checker. All codes that you email should be as separate attachments, titled
PSnum_Qnum_Lastname_Firstname_Studentnum.dat (or .f90 or .f95 or .c etc.)
Example: PS1_Q2_Dworkin_Seth_0000001.f90. Make sure that they can be opened and read in a text editor such as Notepad or equivalent.

Email your codes and report in the same email. Your email should be timestamped by Dr. Dworkin's email provider by 3:59:59 PM on the due date or else your assignment will be considered late. All of your report must be contained in a single PDF file. (I.e., your email should have one PDF report, and each code that you wrote as a separate attachment.)

For all programs that you write, include a header with your full name as registered at the university, student number, date, assignment number, question number, description of the program, and programming language used.

Before you hand in your solutions, refer to the document on assignment preparation instructions posted on D2L in the "Problem Sets" folder.

Questions
1. There is no programming required for this question.
a) Derive the finite difference discretization,

$$\left.\frac{\partial \phi}{\partial x}\right|_i \approx \frac{(\phi_{i+1}-\phi_i)\frac{\Delta x_-}{\Delta x_+} - (\phi_{i-1}-\phi_i)\frac{\Delta x_+}{\Delta x_-}}{(\Delta x_- + \Delta x_+)} + R_i$$

where $\left.\frac{\partial \phi}{\partial x}\right|_i$ is $\frac{\partial \phi}{\partial x}$ at *i*, R_i is the sum of all the truncated terms (the residual), $\Delta x_+ = (x_{i+1} - x_i)$ and $\Delta x_- = (x_i - x_{i-1})$ by combining two Taylor Series, one for ϕ_{i-1} centered at *i,* and one for ϕ_{i+1} centered at *i.* Assume that $\Delta x_+ \neq \Delta x_-$. Maintain up to and including third order derivative terms (terms containing $\frac{\partial^3 \phi}{\partial x^3}$) throughout the derivation. What is the largest truncated term in R_i?

b) What is $\left.\frac{\partial \phi}{\partial x}\right|_i$ if $\Delta x_+ = \Delta x_-$? Does this make sense? Explain in three sentences or less.

c) Derive the finite difference discretization,

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i \approx \frac{(\phi_{i-1}-\phi_i)\frac{2}{\Delta x_-} + (\phi_{i+1}-\phi_i)\frac{2}{\Delta x_+}}{(\Delta x_- + \Delta x_+)} + R_i$$

by combining the same two Taylor Series. Again assume that $\Delta x_+ \neq \Delta x_-$. What is the largest truncated term in R_i for this discretization?

d) What is $\left.\dfrac{\partial^2 \phi}{\partial x^2}\right|_i$ if $\Delta x_+ = \Delta x_-$? Does this make sense? Explain in three sentences or less.

2. Consider the problem:

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \cos(x) = 0 \qquad \lambda = 1$$

with boundary conditions: $T(0) = T(2\pi) = 1$

a) Solve for $T(x)$ analytically.

b) Write a program to solve for $T(x)$ on an equispaced grid using *finite volumes*. Run your program with $N = 8$, 16, 32, and 64 control volumes. For each of the four runs calculate the following three error norms:

$$L_1 = \sum_{i=1}^{N} \left|T_i - T_{exact}\right|, \quad L_2 = \sum_{i=1}^{N} \left(T_i - T_{exact}\right)^2, \quad L_\infty = \max\left|T_i - T_{exact}\right|$$

Have your code output the three norms and $T(x)$ for each run.

c) Write a program to solve for $T(x)$ on an equispaced grid using *finite differences*. Run your program with $N = 9$, 17, 33, and 65 points. For each of the four runs calculate the same three error norms as in part b). Have your code output the three norms and $T(x)$ for each run.

d) nth order accuracy implies that the error should vary as $(\Delta x)^n$. Plot your error as a function of Δx for parts b) and c) (on separate graphs). What is n? Explain in two sentences or less if it is what you expected and why. In five sentences or less, which would you recommend for this problem, finite differences or finite volumes? Why?

3. Consider the problem:

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + 100x^2 = 0 \qquad \lambda = 1$$

with boundary conditions:

$$\frac{\partial T}{\partial x}(0) = 0, \qquad T(1) = 0$$

a) Solve the equation analytically. Plot or sketch *T(x)* from 0 to 1. You should discover that T varies slowly near $x = 0$, and rapidly near $x = 1$.

b) Write a computer program to solve this problem on a *non-uniform grid* using finite differences and 81 points. Use a first spacing of Δx_1, a second of $\Delta x_2 = \alpha \Delta x_1$, a third of $\Delta x_3 = \alpha \Delta x_2$ and so on for $0.7 < \alpha < 1.3$. For a domain of length L divided into N segments, the first control volume must be of size

$$\Delta x_1 = L \frac{(1-\alpha)}{(1-\alpha^N)}.$$

Note that the above expression for Δx_1 cannot be used when $\alpha = 1$.

c) What range would you expect α to be confined to for this particular problem and why? Find the value of α to four or more decimal places that leads to a minimum value of L_∞. Compare that value of L_∞ to L_∞ for an equispaced grid.

d) Plot L_∞ as a function of α for $0.7 < \alpha < 1.3$. In four sentences or less, comment on the results.

****