Analysis I
Continuous Maps

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1 Continuous Maps

Definition 1. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. Define the following function

$$\begin{aligned} f : X &\longrightarrow Y \\ x_0 &\longmapsto f(x_0) = y_0. \end{aligned}$$

The map f is **continuous at the point** $x_0 \in X \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0,$

$$f\left(B^X_\delta(x_0)\right) \subseteq B^Y_\varepsilon(f(x_0)).$$

So the image of some ball around x_0 in X is always a subset of a ball around $f(x_0)$ in Y

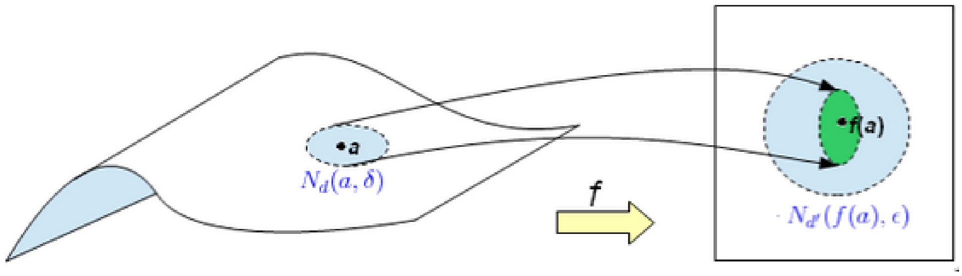

Figure 1: Continuous Map Diagram

Note. when we say a map is continous, we mean either the map is continuous over the domain $(\forall x_0 \in X)$ or that it is continuous for some specific x_0. For whichever case it is, it will be specified.

1. Non-continuous function:

$$D(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, \text{ else} \end{cases}.$$

and $D : \mathbb{R} \to \mathbb{R}$ known as the dirichlet function

2. Consider $f(x) = x \cdot D(x)$, this function is only only continuous at 0, $f \in C(0)$ or consider the Riemann function

$$R(x) = \begin{cases} \frac{1}{q}, x = \frac{p}{q} \\ 0, x \in \frac{\mathbb{R}}{\mathbb{Q}} \end{cases}.$$

Note. Example

1. Take $f(x) = x$, $f : [0, 1] \to \mathbb{R}$. Without loss of generality, let $x_0 = \frac{1}{2}$ and $f(x_0) = y_0 = \frac{1}{2}$. Then, $\forall \varepsilon > 0, B^Y_\varepsilon(y_0) = (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ and we can take $\delta = \varepsilon$ so that $f(B^X_\delta(x_0)) = f((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)) = (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon) \subseteq B^Y_\varepsilon(y_0)$.
2. Consider

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto f(x) = (x, x). \end{aligned}$$

this function takes the line in $\mathbb{R}$ to a "diagonal" line in $\mathbb{R}^2$.

3. Consider

$$\begin{aligned} g : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, x) &\longmapsto g((x, x)) = x^2. \end{aligned}$$

Remark. Exercise
Consider the maps and show they are continuous

1. $(x,y) \longmapsto xy$, $f : \mathbb{R}^2 \to \mathbb{R}$
2. $(x,y) \longmapsto x+y$, $f : \mathbb{R}^2 \to \mathbb{R}$
3. $(x,y) \longmapsto \frac{x}{y}$, $f : \mathbb{R}^2 \setminus \{y=0\} \to \mathbb{R}$

Theorem 1. The composition of continuous maps is continous. The map $(g \cdot f)(x) = g(f(x))$, $x \in X$ is called the composed map where if $f : X \to Y$ and $g : Y \to Z$, then $g \cdot f : X \to Z$ (Kind of just follows out from set theory)

Definition 2. If $f : X \to Y$ is a continuous map at the point x_0, then $f \in C(x_0)$ where $C(x_0)$ is the class of all maps from $X \to Y$ that are continuous at x_0

Theorem 2. If $f \in C(x_0)$ and $g \in C(f(x_0))$ then $g \cdot f \in C(x_0)$.

Proof. Take $\varepsilon > 0$ and consider the ball $B_\varepsilon^Z(z_0)$. Since $g \in C(f(x_0)), \exists \tilde{\delta} > 0$ such that

$$g\left(B_{\tilde{\delta}}^Y(y_0)\right) \subseteq B_\varepsilon^Z(z_0).$$

Similarly, since $f \in C(x_0)$ then $\exists \delta > 0$ such that

$$f\left(B_\delta^X(x_0)\right) \subseteq B_{\tilde{\delta}}^Y(y_0).$$

If then follows from (1) and (2) that

$$(g \cdot f)\left(B_\delta^X(x_0)\right) = g(f\left(B_\delta^X(x_0)\right)) \subseteq g\left(B_{\tilde{\delta}}^Y(y_0)\right) \subseteq B_\varepsilon^Z(z_0).$$

Hence this implies $g \cdot f \in C(x_0)$ ☐

Theorem 3. $f : X \to Y$ is continous on X (i.e. continous at all points $x_0 \in X$) $\Leftrightarrow \forall U_{\text{open}} \subseteq Y$, then $f^{-1}(U)$ is also open in X. Note f^{-1} not necessarily a function.

Proof. $\to$ Assume $f : X \to Y$ is continous. Take $U_{\text{open}} \subseteq Y$ and consider $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$, called the pre-image. Let us prove that $f^{-1}(U)$ is open in X. Take $x_0 \in f^{-1}(U)$. Then, $f(x_0) \in U$. Since U is open in $Y, \exists \varepsilon > 0$ such that $B_\varepsilon^Y(f(x_0)) \subseteq U$. It now follows that since $f \in C(x_0)$, $\exists \delta > 0$ such that $f(B_\delta^X(x_0)) \subseteq B_\varepsilon^Y(f(x_0)) \subseteq U \Rightarrow f^{-1}(U) \supseteq B_\delta^X(x_0) \Rightarrow f^{-1}(U)$ is open since this is true for any $x_0 \in f^{-1}(U)$.

$\leftarrow$ Assume that $\forall U_{\text{open}} \in Y$, we have $f^{-1}(U)$ is open in X. Take $x_0 \in X$ and consider the corresponding point $y_0 = f(x_0)$. Take $\varepsilon > 0$ and consider the ball $B_\varepsilon^Y(y_0)$. Since a ball is an open set, we can apply our assumption,

$$x_0 \in f^{-1}(B_\varepsilon^Y(y_0))_{\text{open}} \subseteq X.$$

Since $f^{-1}(B_\varepsilon^Y(y_0))$ is open, $\exists \delta > 0$ such that

$$B_\delta^X(x_0) \subseteq f^{-1}(B_\varepsilon^Y(y_0)) \Rightarrow f(B_\delta^X(x_0)) \subseteq B_\varepsilon^Y(f(x_0)).$$

Hence, $f \in C(x_0)$ and since this is true from any $x_0 \in X$, $f \in C(X,Y)$. ☐

Theorem 4. Let $f : X \to Y$ be a continous map and $K_{\text{compact}} \subseteq X$. Then $f(K) \subseteq Y$ is compact in Y.

Proof. Assume $f : X \to Y$ is continuous and $K_{\text{compact}} \subseteq X$. Let $\{U_\alpha\}_{\alpha \in T}$ be an open cover of $f(K)$. Consider the set of open sets in X (by theorem 2):

$$\{f^{-1}(U_\alpha)\}_{\alpha \in I}.$$

Then this is an open cover of K, $\cup_{\alpha \in I} f^{-1}(U_\alpha) \supseteq K$ (since for any U_α we can take $x_0 \in f^{-1}(U_\alpha) \subseteq X$ and this will be true for all x_0). Take $x_0 \in K$, $f(x_0) \in f(K)$ and since $\{U_\alpha\}_{\alpha \in I}$ is a cover of $f(K) \exists \beta \in I$ such that

$$f(x_0) \in U_\beta \Rightarrow x_0 \in f^{-1}(U_\beta).$$

Since $K \subseteq X$ compact, $\exists \alpha_1, \ldots, \alpha_N \in I$ such that

$$\cup_{j=1}^N f^{-1}(U_{\alpha_j}) \supseteq K.$$

Then $\{U_{\alpha_j}\}_{j=1}^N$ is a finite subcover of $f(K)$,

$$\cup_{j=1}^N U_{\alpha_j} \supseteq f(K).$$

Take $y_0 \in f(K)$ then $y_0 = f(x_0), x_0 \in K$ then $\exists x_0 \in f^{-1}(U_{\alpha_{j_0}}) \Rightarrow f(x_0) \in U_{\alpha_{j_0}}$ □

Remark. (However, the preimage of a compact set is not necessarily compact). If $f : X \to Y$ is continuous and K is compact in Y then $f^{-1}(K)$ is not necessarily compact in X. Consider

$$f(x) = \sin x, f : \mathbb{R} \to \mathbb{R}, K = [-1, 1], f^{-1}(K) = \mathbb{R} \text{ not compact.}$$

Theorem 5. Let $f : X \to \mathbb{R}$ be a continuous map and X is compact then $\exists x_m, x_M \in X$ such that $f(x_m) = \inf_X f = \inf(\{f(x) \mid x \in X\})$ and $f(x_M) = \sup_X f$.

Proof. Since X is compact and $f : X \to \mathbb{R}$ is continuous $\Rightarrow im(f(X)) \subseteq \mathbb{R}$ is compact from the previous theorem (theorem 4), so $f(X)$ is bounded and closed. Let $\alpha = \sup_X f$ (this $\alpha \in \mathbb{R}$ exists since $\{f(x) | x \in X\}$ is bounded in $\mathbb{R}$). If $\alpha \in f(X)$ then $\alpha = f(x_M)$ for some $x_M \in X$. If $\alpha \notin f(X)$, then α must be a limit point of $f(X)$. Since $f(X)$ is closed, $\Rightarrow d \in f(X)$, a contradiction. Hence, $\alpha \in f(X)$ and there must exist some $f(x_m) = \inf_X f$. A similar argument can be made for the $\sup_X f$.

□