# Analysis I

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# Contents

# 1 Closure of a Set

**Definition 1.** Let $(X, \rho)$ be a metric space where $E \subseteq X$. Denote $\overline{E}$ as the **closure**.

$$\overline{E} = \cap_{E \subseteq F} F \text{ and F is closed.}$$

And since, the intersection of any number of (including infinite) closed sets is closed.

1. $\overline{E}$ is closed
2. $\overline{E} \supseteq E$
3. If $E \subseteq B$ where $B$ is closed, then $\overline{E} \subseteq B$

   *Proof.* Since closure of $E$ is defined as the intersection of all closed sets that contain $E$, then for some $B$ closed that contains $E$, $\overline{E}$ must be in the intersection of $B$ and all other closed sets containing $E$, so $\overline{E}$ must be contained in $B$. □

4. $B \subseteq A \rightarrow \overline{B} \subseteq \overline{A}$

   *Proof.* It follows from the definition, that $\overline{B} \cap_{F' \supseteq B, F\text{closed}} F'$ and $\overline{A} = \cap_{F \subseteq A \subseteq B, F\text{closed}} F$. Any closed set $F$ that contains $A$ must also contain $B$. Hence, any set $F$ that appears in $\cap_{F \supseteq A, \text{F closed}} F$ appears also in $\cap_{F' \supseteq B} F'$, so $\overline{A} \supseteq \overline{B}$ because $\cap_{F' \supseteq B} F'$ has all $F$ and potentially more closed sets that it's intersecting, so their combined intersections will result in a set that is strictly smaller than the intersection of only $F$s □

5. If $E$ is closed $\rightarrow \overline{E} = E$

   *Proof.* If $E$ is closed, then since $E \supseteq E$, $E \subseteq \overline{E}$ and by property (1) $E \subseteq \overline{E}$, so $\overline{E} = E$ □

6. $A \supseteq B \rightarrow A' \supseteq B'$ where $A', B'$ represents the limit points of $A, B$ respectively.

**Remark.** Intuitively, the closure of a set is turning an open set into a closed set, where the closed set is the minimum possible augmentation of the open set.

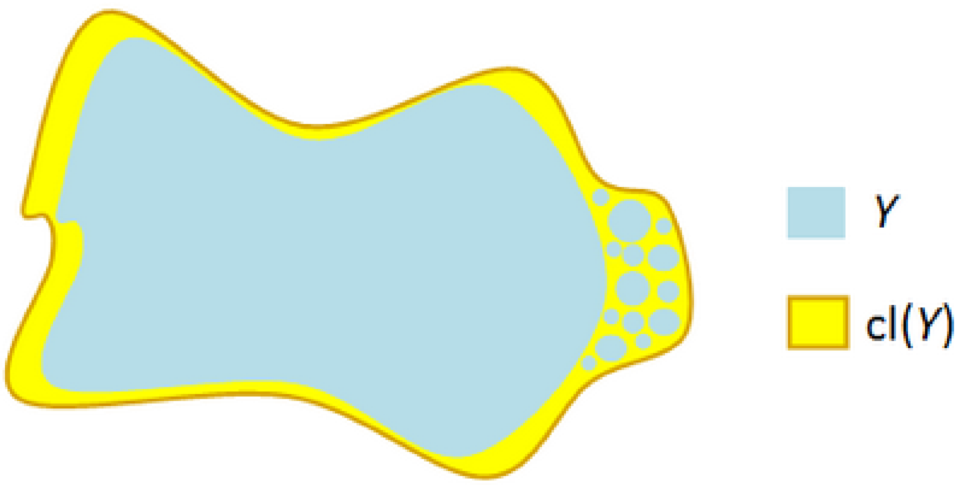


Figure 1: Closure of a Set

**Proposition 1.** $\overline{E} = E \cup E'$ (where $E'$ is the set of limit points of $E$)

*Proof.* We will first prove $E \cup E'$ closed. Take $x \in X \cup E'$.Then, $x$ is not an limit point of $E \to \exists r >$ such that $B_r^0(x) \cap E = \emptyset$. Since $x \notin E \to B_r(x) \cap E = \emptyset \to E \subseteq X \setminus B_r(x)$ and $X \setminus B_r(x)$ is closed since it is the complement of $B_r(x)$ which is open (all balls are open) in $X$. Then $E' \subseteq [X \setminus B_r(x)]' = X \setminus B_r(x)$ and $E' \cap B_r(x) = \emptyset$. Hence, $B_r(x) \cap (E \cup E') = \emptyset$ or equivalently,

$$B_r(x) \subseteq X \setminus (E \cup E').$$

We know that $X \setminus (E \cup E')$ must be open since an open set $B_r(x)$ is contained in it, so,

$$E \cup E'$$

is a closed set.

Then, we will show $E \cup E' \supseteq \overline{E}$

$$E \cup E' \supseteq E \to E \cup E' \supseteq \overline{E}.$$

by property (3) in the definition since $E \cup E'$ is closed.

For the other direction,

$$\overline{E} \supseteq E \to \overline{E}' \supseteq E'.$$

since $\overline{E}' \subseteq \overline{E}$,

$$\overline{E}' \supseteq E' \to \overline{E} \supseteq E'.$$

and by property (6). So, since $\overline{E} \supseteq E$ and $\overline{E} \supseteq E'$ :

$$\overline{E} \supseteq E \cup E'.$$

and with both directions proven,

$$\overline{E} = E \cup E'.$$

□

# 2 Sequences

**Definition 2.** Take $(X, \rho)$ a metric space. A **sequence** $\{x_n\}_{n \geq 1}$ is a map

$$\mathbb{N} \longrightarrow X$$
$$n \longmapsto x_n.$$

The **image (range)** of a sequence,

$$im(\{x_n\}_{n \geq 1}) = \{x \in X | \exists k \geq 1 \text{such that} x_k = x\}.$$

**Note.** Example
Consider

$$x_n = \begin{cases} 1, & \text{n even} \\ 0, & \text{n odd} \end{cases}.$$

so $x_1 = 0, x_2 = 1, x_3 = 0, \ldots$. Then the range would be

$$im\left(\{x_n\}_{n \geq 1}\right) = \{0, 1\}.$$

**Definition 3.** Let $\{x_n\}_{n\geq 1}$ a sequence in $X$. We say that $x \in X$ is the **limit** of this sequence if $\forall \varepsilon > 0$, $\exists N \geq 1$ such that $\forall n \geq N$ we have that

$$x_n \in B_\varepsilon(x).$$

In this case we can say that $\{x_n\}_{n\geq 1}$ converges to $x$ as $n \to \infty$. We write:

$$x_n \to x(n \to \infty).$$

or equivalently

$$\lim_{n\to\infty} x_n = x.$$

Note that we sometimes have to specify the metric space if $x_n$ is in multiple metric spaces. The image must always be **countable** (even if the sequence converges to $x \in \mathbb{R}$ because the indices we map from are countable $\mathbb{N}$)

**Note.** Example
The previous example does not have a limit because $x_n$ oscillates so the "tail" is never contained around 0 or 1 for any $\varepsilon$, because the next $x_{i+1}$ flips the parity so it "jumps" and is not contained in that $B_\varepsilon(x)$ ball.

**Proposition 2.** If $x', x''$ are limits of $\{x_n\}_{n\geq 1}$ then $x' = x''$ (i.e. the limit is unique)

**Proposition 3.** If $\{x_n\}_{n\geq 1}$ converges, then $im(\{x_n\}_{n\geq 1}$ is a bounded set in $X$

## 2.1 Subsequence

**Definition 4.** Let $\{x_n\}_{n\geq 1}$ be a sequence and $1 \leq n_1 < n_2 < \ldots < n_k < \ldots$, i.e $n_j \in \mathbb{Z}$ where $j \geq 1$, then we get $\{x_{n_j}\}_{j\geq 1}$ is a **subsequence** of $\{x_n\}_{n\geq 1}$.

**Proposition 4.** If $x_n \to x$ as $n \to \infty$, then any subsequence $\{x_{n_j}\}_{j\geq 1}$ converges to $x$.

**Theorem 1.** Let $(X, \rho)$ be a compact metric space and let $\{x_n\}_{n\geq 1}$ be a sequence in $X$. Then $\exists$ a subsequence $\{x_{n_j}\}_{j\geq 1}$ and $x \in X$ such that this $x_{n_j} \to x$ as $j \to \infty$ (In other words, every sequence in a compact metric space has a convergent subsequence.)

*Proof.* Let us first assume that $im(\{x_n\}_{n\geq 1})$ has finitely many elements: $y_1, y_2, \ldots, y_n (y_k \neq y_j$ if $k \neq j)$. Then, $\exists \{x_{n_j}\}_{j\geq 1}$ and $1 \leq l \leq N$ such that $x_{n_j} = y_l$ $(\forall j \geq 1)$ (this is true by pigeonhole principle, finite $y_l$, infinite $x_{n_j}$, so at least one distinct value for $y_l$ must appear infinitely in the sequence $\{x_n\}$, then this guarantees there is some subsequence $\{x_{n_j}\}_{j\geq 1}$ whose terms are equal to a fixed $y_l$). Then, clearly $x_{n_j} \to y_l$ as $(j \to \infty)$.

Now assume that the range of $\{x_n\}_{n\geq 1}$ is not finite. Let $S = im(\{x_n\}_{n\geq 1})$. Since $X$ is compact and $S \subseteq X$ has infinitely many points. We can use the proposition from the previous lecture, which tells us that $\exists x \in X$ that is a limit point of this set. Define a ball centered at $x$.

$$B_{\frac{1}{n}}(x) n \geq 1.$$

For $n = 1$, $\exists x_{n_1}, n_1 \geq 1$ such that $x_{n_1} \in B_1(x)$. For $n = 2$, $\exists x_{n_2}$, $n_2 > n_1$ such that $x_{n2} B_{\frac{1}{2}}(x)$. $\ldots$ For $n = k \exists x_{n_k}, n_k > n_{k-1}$ such that $x_{n_k} \in B_{\frac{1}{k}}(x)$ (by definition of limit point). Then, $x_{n_j} \to x$ as $j \to \infty$ $\square$

## 2.2 Diameter

**Definition 5.** Let $(X, \rho)$ a metric space. $E \neq \emptyset \subseteq X$ and $E$ bounded. The we can defined

$$diam(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

Can be thought of as the max distance between two points in the set $E$ i.e. the diameter.

**Lemma 1.** Let $E \subseteq X$ be a bounded set. Then,

$$diam\overline{E} = diam E.$$

*Proof.* Since $\overline{E} \supseteq E \to diam\overline{E} \geq diam E$.

Take $x, y \in \overline{E} = E \cup E' \to \forall\varepsilon > 0$, $\exists x', y' \in E$ such that $\rho(x, x') < \varepsilon$ and $\rho(y, y') < \varepsilon$. Here we say that x and y are either an element of $E$ or a limit point of $E$ so they are $\varepsilon$ close to $x, y$ respectively. Then by the triangle inequality (applied twice),

$$\begin{aligned}\rho(x, y) &\leq \rho(x, x') + \rho(x', y') + \rho(y', y)\\ &< \varepsilon + diam(E) + \varepsilon\\ &< diam(E) + 2\varepsilon.\end{aligned}$$

Hence for any $x, y \in \overline{E}$, $\forall\varepsilon > 0$ we have that:

$$\rho(x, y) < diam(E) + 2\varepsilon.$$

For fixed $\varepsilon$ and any x,y

$$diam(\overline{E}) \leq diam(E) + 2\varepsilon.$$

From here follows (a bit counterintuitive but consider if $a \leq b + \varepsilon$ was not true),

$$diam\overline{E} \leq diam(E).$$

As $diam\overline{E} \geq diam E$ and $diam\overline{E} \leq diam E$, then $diam\overline{E} = diam E$ □

**Note.** For $diam(\overline{E})$ to even be a valid question, $\overline{E}$ must be bounded so we use the fact that the closure of a bounded set is always bounded.

## 2.3 Cauchy Sequence

**Lemma 2.** Let $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_j \supseteq \ldots$ where $K_j \neq \emptyset$ are compact sets in $X$. Then, $\cap_{j\geq 1} K_j \neq \emptyset$

*Proof.* Consider

$$G_j = X \setminus K_j, j \geq 1.$$

Since $K_j$ is compact, then it is closed, so $G_j$ is open $\forall j \geq 1$. Assume that all conditions in the lemma hold but $\cap_{j\geq 1} K_j = \emptyset$. Then,

$$\cup_{j\geq 1} G_j = \cup_{j\geq 1} (X \setminus K_j) = X \setminus [\cap_{j\geq 1} (X \setminus (X \setminus K))] = X \setminus [\cap_{j\geq 1} K_j] = .$$

TODO

□

Let $(X, \rho)$ a metric space, $\{x_n\}_{n\geq 1}$ is a sequence in $X$

**Definition 6.** $\{x_n\}_{n\geq 1}$ is a **Cauchy sequence** if and only if $\forall \varepsilon > 0\ \exists N \geq 1$ such that

$$\forall n, m \geq N, \rho(x_n, x_m) < \varepsilon.$$

An example would be an asymptote that doesn't go to infinity. A cauchy sequence must be a **bounded set**, or more precisely the range of the cauchy sequence must be **bounded**

**Note.** Notice that the difference in defintion between cauchy and limit is that cauchy doesn't assume the existence of $x$ such that $x_n \to x$. Usually, a cauchy is used to find $x$ and show it exists. So cauchy sequence $\supseteq$ convergent sequence

**Note.** CounterExample
A cauchy sequence in a metric space that does not converge (i.e. the metric space is not complete).

Consider $X = [0, 1) \subseteq \mathbb{R}$ with standard $\rho$. Take $x_n = 1 - \frac{1}{n}, n \geq 1$. Then this sequence is a cauchy sequence in $\mathbb{R}$ and since distance is inherited by $X$, then the sequence is also cauchy in $X$. However, while the $\{x_n\}_{n\geq 1}$ converges in $\mathbb{R}$, it does not converge in $X$.

**Definition 7.** A metric space $(X, \rho)$ is **complete** $\Leftrightarrow$ any cauchy sequence in $X$ converges to some point $x \in X$.

**Proposition 5.** If $x_n \to x$ as $n \to \infty$ then $\{x_n\}_{n\geq 1}$ is a cauchy sequence.

*Proof.* □

**Proposition 6.** If $\{x_n\}_{n\geq 1}$ be a cauchy sequence then $im(\{x_n\}_{n\geq 1})$ is bounded (the same as saying the sequence is bounded)

*Proof.* □

**Proposition 7.** $\{x_n\}_{n\geq 1}$ is a cauchy sequence $\Leftrightarrow$ $diam\left(\{x_n\}_{n\geq N}\right) \to 0$ as $N \to \infty$ (by the previous proposition we know that $\{x_n\}_{n\geq N}$ is bounded, so the diam is well-defined)

*Proof.* $\to$ Assume that $\{x_n\}_{n\geq 1}$ is a cauchy sequence then $\forall \varepsilon\ \exists N \geq 1$ such that $\forall m, n \geq N, \rho(x_n, x_m) < \varepsilon$. Notice that this means $diam(\{x_n\}_{n\geq N}) \leq \varepsilon$ (the tail of the sequence is contained in a ball of radius $\varepsilon$). Hence, $\forall \varepsilon > 0$, $\exists N \geq 1$ such that $k \geq N$,

$$diam\left(\{x_n\}_{n\geq k}\right) \leq \varepsilon \Leftrightarrow diam\left(\{x_n\}_{n\geq k}\right) \to 0 (k \to \infty).$$

$\leftarrow$ Assume that $diam\left(\{x_n\}_{n\geq N}\right) \to 0$ as $N \to \infty$ then $\forall \varepsilon > 0$, $\exists N_0 \geq 1$ such that $\forall N \geq N_0$, we have $diam(\{x_n\}_{n\geq N}) < \varepsilon$. (We are now looking at $B^{\mathbb{R}}_{\varepsilon}(0) = (-\varepsilon, \varepsilon)$ and saying the diameter is a element in this ball). This implies that $\forall m, n \geq N$,

$$\rho(x_n, x_m) < \varepsilon.$$

Then $\forall \varepsilon > 0$, $\exists N_0 \geq 1$ such that $\forall m, n \geq N_0$. Hence, $\{x_n\}_{n\geq 1}$ is a cauchy sequence. □

**Theorem 2.** If $X$ is compact $\to$ $X$ is complete. (but not the other way around)

*Proof.* Assume that $X$ is compact. Take a cauchy sequence $\{x_n\}_{n\geq 1}$ in $X$. Consider the tail of this sequence,

$$E_N = \{x_n\}_{n\geq N}, N = 1, 2, 3, \ldots.$$

Then,

1. $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_N \supseteq \ldots$
2. For any $N \geq 1$, $E_N$ is a bounded set and $diam(E_N) \to 0$ as $N \to \infty$ since $E_N$ is a subsequence (relative order maintained) of $\{x_n\}_{n\geq 1}$ then it is also cauchy and we can apply the proposition.

Hence, by taking the closure,

$$X \supseteq \overline{E_1} \supseteq \overline{E_2} \supseteq \ldots \supseteq \overline{E_N} \supseteq \ldots.$$

and each $\overline{E_i}$ is closed and we proved that closed subsets of compact sets are also compact, so $\overline{E_i}$ are also compact in $X$. On the other side, we have

$$diam(\overline{E_N}) \leftarrow 0 \text{ as } N \leftarrow \infty.$$

by lemma 1 (since we know $diam(\overline{E}) = diam(E)$). It follows from before that $\cap_{j\geq 1} K_j \neq \emptyset \to \exists x \in X$, such that $x \in \overline{E_N}$, $\forall N \geq 1$ (it is in the intersection). This means $\forall \varepsilon > 0$, $\exists N_0 \geq 1$ such that $\forall N \geq N_0$,

$$diam(\overline{E_N}) < \varepsilon \to diam(\overline{E_{N_0}}) < \varepsilon \to \rho(x, x_n) < \varepsilon, \forall n \geq N_0.$$

Hence, $\forall \varepsilon > 0, \exists N_0 \geq 1$ such taht $\forall n > N_0$,

$$\rho(x, x_n) < \varepsilon \Leftrightarrow x_n \in B_\varepsilon(x), \forall n \geq N_0.$$

Thus,

$$x_n \to x \text{ as } n \to \infty.$$

□

**Theorem 3.** $\mathbb{R}^n$ is complete (observe that $\mathbb{R}^n$ is not compact so we can't use the theorem above). Note that this proof does not hold for infinite dimensional spaces.

*Proof.* Take a cauchy sequence in $\mathbb{R}^n$, $\{x_n\}_{n\geq 1}$. Let $S = range(\{x_n\}_{n\geq 1})$. Thus, $S \neq$ and $S$ is bounded. Consider $\overline{S}$. Then, $\overline{S}$ is closed and bounded subset of $\mathbb{R}^n$ so $\overline{S}$ is compact in $\mathbb{R}^n$ (Note we only proved this for $\mathbb{R}^n$ ). Then, $\{x_n\}_{n\geq 1}$ is a cauchy sequence in $(\overline{S}, \rho)$ (as $\overline{S}$ contains all of $\{x_n\}_{n\geq 1}$). Now we have a compact metric space and a cauchy sequence, so by theorem 1, $\exists x \in \overline{S}$ such that in $\overline{S}$

$$x_n \to x \text{ as } n \to \infty.$$

this implies

$$x_n \to x \text{ as } n \to \infty.$$

in $X$! □