

# Homework 6

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1. There are two jars in front of you and a total  $n$  balls divided among them. Every minute, you pick a ball uniformly at random, and move it to another jar. Let  $X_n$  denote the number of balls in the first jar. Argue that  $\{X_n\}$  is a markov chain and calculate its transition probabilities.

*Proof.* Consider the markov chain with states  $\{S_0, S_1, \dots, S_n\}$ . The different states represent the different numbers of balls that could belong in the first jar at each timestep  $X_n$ . Thus, for state  $S_i$ , the transition probability to  $S_{i+1}$  would be  $\frac{n-i}{n}$  the probability a ball is chosen from the second jar and moved to the first jar. Similarly, the transition probability from  $S_i$  to  $S_{i-1}$  would be  $\frac{i}{n}$  the probability a ball is moved from the first jar to the second jar. This would be for all states  $S_1, \dots, S_n$  and then for  $S_0$ , the transition probability would be 1 to  $S_1$  and  $S_n$  the transition probability would be 1 to  $S_{n-1}$ .

$\{X_n\}$  is a markov chain because at every time  $X_n$ ,

$$\begin{aligned} P[X_n = S_j \mid X_{n-1} = S_{j_{n-1}}, \dots, X_0 = S_{j_0}] &= P[X_n = S_j \mid X_{n-1} = S_{j_{n-1}}] \\ &= \begin{cases} \frac{n-j_{n-1}}{n}, S_j = S_{j_{n-1}} + 1 \\ \frac{j_{n-1}}{n}, S_j = S_{j_{n-1}} - 1 \\ 0, \text{ otherwise} \end{cases} \end{aligned}$$

By construction, the probability of the next state is only dependent on the previous state. Thus, since it's a stochastic process (since  $X_k$  are timesteps) with the above property, then  $\{X_n\}$  is a markov chain.  $\square$

2. Let  $Y_n$  be a time homogenous markov chain with states  $\{1, 2, 3\}$ . The transition probabilities for  $Y_n$  are summarized in the matrix below where  $P_{i,j} = P[Y_1 = j \mid Y_0 = i]$ .

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

If  $P[Y_0 = 1] = P[Y_0 = 3] = \frac{1}{2}$ , compute  $E[Y_3]$ .

*Proof.*

$$P^3 = \begin{pmatrix} 0.4583 & 0.3009 & 0.2407 \\ 0.4259 & 0.1944 & 0.3796 \\ 0.4722 & 0.3472 & 0.1806 \end{pmatrix}, V_0 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

We know that:

$$\begin{aligned} E[Y_3] &= \sum_{k=1}^3 k \cdot P[Y_3 = k] \\ &= \sum_{k=1}^3 k \left( \sum_{i=1}^3 P[Y_0 = i] \cdot P[Y_3 = k \mid Y_0 = i] \right) \\ &= \sum_{k=1}^3 k (v_0 \cdot P^3)_k \\ &= \sum_{k=1}^3 k (0.46525 \quad 0.32405 \quad 0.21065)_k \\ &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot (0.46525 \quad 0.32405 \quad 0.21065) \\ &= 1.7453. \end{aligned}$$

$\square$

3. Suppose that  $P$  is transition matrix for some finite-state Markov chain. Suppose that there exists an integer  $r \geq 1$  such that every entry in  $P^r$  is strictly positive. Argue that, for any  $n \geq r$ , every entry of  $P^n$  is also strictly positive.

*Proof.* Since  $P$  is a transition matrix, then we know that every entry is non-negative. Furthermore,  $\forall i, \sum_j P_{i,j} = 1$ , the matrix must be row stochastic. If there exists an integer  $r \geq 1$  such that every matrix entry in  $P^r$  is also strictly positive, then this means that:

$$P^r = P \times P \times P \dots \times P.$$

is strictly positive as well. Suppose that  $P$  had a column of all zeros. Let this column be the  $i$ th column. Then, since whatever row multiplied by that column results in 0,  $\forall j, P_{i,j}^2 = 0$ .

$$\begin{pmatrix} \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \end{pmatrix} \times \begin{pmatrix} \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \end{pmatrix}.$$

Thus,  $\forall j, P_{i,j}^r = P \times P \times P \dots \times P = 0$ . This violates our assumption that  $P^r$  has strictly positive entries. So there, can't be a column of all zeros. Now let's use induction, we assume that  $P^r$  is strictly positive, then  $P^{r+1} = P^r \cdot P$  must also be strictly positive, because every row of  $P^r$  has every value as non-zero and  $P$  must have every column containing at least one non-zero value. So,  $P_{i,j}^{r+1} > 0$ . By induction, this hold for  $n \geq r$ !  $\square$

4. We have a Markov chain on five states, numbered 1, 2, 3, 4, 5. At each state, the Markov chain is equally likely to go to any state whose number is greater than or equal to itself. The state 5 is an absorbing state. Let  $T$  be the first time of absorption — that is,  $T = \min\{t | X_t = 5\}$ . Find the expected value of  $T$ , if  $X_0 = 1$ .

*Proof.* We first must calculate  $N = (I - Q)^{-1}$ . Since 5 is the only absorbing state, then  $Q$  consists of the states 1, 2, 3, 4 and the connections between them and  $I$  is just 1.

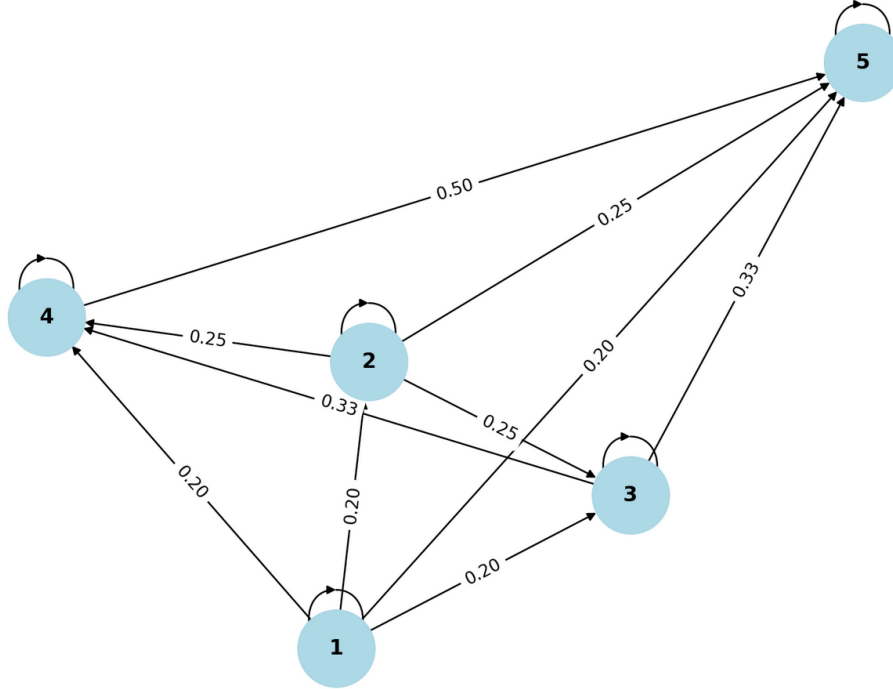


Figure 1: Problem 4 Markov Chain

Thus, we have the following matrix

$$\begin{pmatrix} 0.20 & 0.20 & 0.20 & 0.20 & 0.20 \\ 0.00 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.00 & 0.00 & 0.33 & 0.33 & 0.33 \\ 0.00 & 0.00 & 0.00 & 0.50 & 0.50 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{pmatrix}.$$

Then,

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.20 & 0.20 & 0.20 & 0.20 \\ 0.00 & 0.25 & 0.25 & 0.25 \\ 0.00 & 0.00 & 0.33 & 0.33 \\ 0.00 & 0.00 & 0.00 & 0.50 \end{pmatrix}^{-1} = \begin{pmatrix} 1.25 & 0.3333 & 0.4975 & 0.9950 \\ 0.00 & 1.3333 & 0.4975 & 0.9950 \\ 0.00 & 0.00 & 1.4925 & 0.9851 \\ 0.00 & 0.00 & 0.00 & 2.00 \end{pmatrix}.$$

We know that

$$E[T \mid X_0 = 1] = \sum_{j=1}^n (N)_{1,j} = (N \cdot \vec{1})_1 = 3.076.$$

□

5. In tennis, a player must win by at least two points. If the players are tied (and both have won at least three points), deuce is reached. If one player is leading by one, they have advantage. Assume that player one has advantage, but has a probability of  $2/5$  of winning any given point; further assume that the outcomes of different points are independent. What is the probability that player one wins the game?

*Proof.* We are given a game of tennis where Player 1 has the advantage, i.e. player 1 and 2 have score both at least 3 points and Player 1 is one point ahead. The probability of Player 1 winning a point is  $\frac{2}{5}$  and the probability of Player losing a point is  $\frac{3}{5}$ . Naturally, if either player has a lead of 2 points, then they win the game. Thus, we can construct the following markov chain which represents the point differential between player 1 and 2. Since we are given that Player 1 has advantage, we don't have to consider the previous points, only that the initial state is player 1 has 1 point ahead. Denote the probability that Player 1 then wins from advantage as  $W_A$  and wins

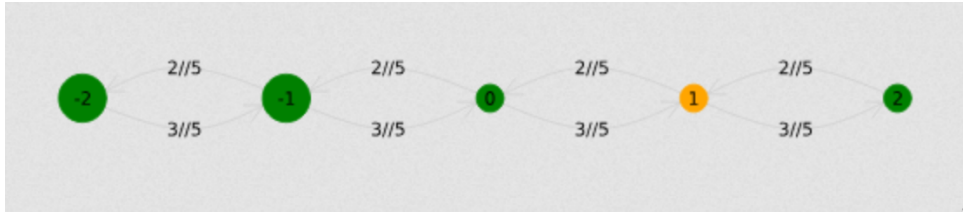


Figure 2: Problem 5 Graph

from Deuce as  $W_D$  and wins from down 1 as  $W_L$

$$W_A = \frac{2}{5} + \frac{3}{5}W_D \tag{1}$$

$$W_D = \frac{2}{5}W_A + \frac{3}{5}W_L \tag{2}$$

$$W_L = \frac{2}{5}W_D + \frac{3}{5} \cdot 0. \tag{3}$$

Substitute  $W_L$  into  $W_D$

$$\begin{aligned} W_D &= \frac{2}{5}W_A + \frac{3}{5}W_L \\ &= \frac{2}{5}W_A + \frac{3}{5}\left(\frac{2}{5}W_D\right) \\ \frac{19}{25}W_D &= \frac{2}{5}W_A \\ W_D &= \frac{10}{19}W_A. \end{aligned}$$

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Substitute into  $W_A$ :

$$\begin{aligned}W_A &= \frac{2}{5} + \frac{3}{5}W_D \\&= \frac{2}{5} + \frac{3}{5}\left(\frac{10}{19}W_A\right) \\&= \frac{2}{5} + \frac{6}{19}W_A \\&= \frac{38}{95} + \frac{30}{95}W_A \\ \frac{65}{95}W_A &= \frac{38}{95} \\&= \frac{38}{65}.\end{aligned}$$

□