HOMEWORK 6 FOR MATH 7241, FALL 2024. DUE OCTOBER 24TH

1. There are two jars in front of you, and a total of n balls divided among them. Every minute, you pick a ball uniformly at random, and move it to the other jar. Let X_n denote the number of balls in the first jar. Argue that $\{X_n\}$ is a Markov chain, and calculate its transition probabilities.

2. Let Y_n be a (time-homogeneous) Markov chain with states $\{1, 2, 3\}$. The transition probabilities for Y_n are summarized in the matrix below, where $P_{i,j} = \mathbb{P}[Y_1 = j \mid Y_0 = i]$:

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

If $\mathbb{P}[Y_0 = 1] = \mathbb{P}[Y_0 = 3] = 1/2$, compute $\mathbb{E}[Y_3]$.

3. Suppose that P is transition matrix for some finite-state Markov chain. Suppose that there exists an integer $r \geq 1$ such that every entry in P^r is strictly positive. Argue that, for any $n \geq r$, every entry of P^n is also strictly positive.

4. We have a Markov chain on five states, numbered $1, 2, 3, 4, 5$. At each state, the Markov chain is equally likely to go to any state whose number is greater than or equal to itself. The state 5 is an absorbing state. Let T be the first time of absorption — that is, $T = \min\{t : X_t = 5\}$. Find the expected value of T, if $X_0 = 1$.

5. In tennis, a player must win by at least two points. If the players are tied (and both have one at least three points), deuce is reached. If one player is leading by one, they have advantage. Assume that player one has advantage, but has a probability of $2/5$ of winning any given point; further assume that the outcomes of different points are independent. What is the probability that player one wins the game?