Analysis I
Compactness

Daniel Yu

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Contents

1 Compact Sets

Definition 1. A set of **open** sets $\{U_\alpha\}_{\alpha \in I}$ where I is a set of indicies is called an **open cover of E** if

$$E \subseteq \cup_{\alpha \in I} U_\alpha.$$

Definition 2. If $\{U_\alpha\}_{\alpha \in I}$ is an open cover in E and $I_1 \subseteq I$ such that

$$E \subseteq \cup_{\alpha \in I_1} U_\alpha.$$

then $\{U_\alpha\}_{\alpha \in I_1}$ is called a **subcover** of $\{U_\alpha\}_{\alpha \in I}$

Definition 3. The set $E \subseteq X$ is **compact** in X if any **open cover** of E has a **finite subcover**

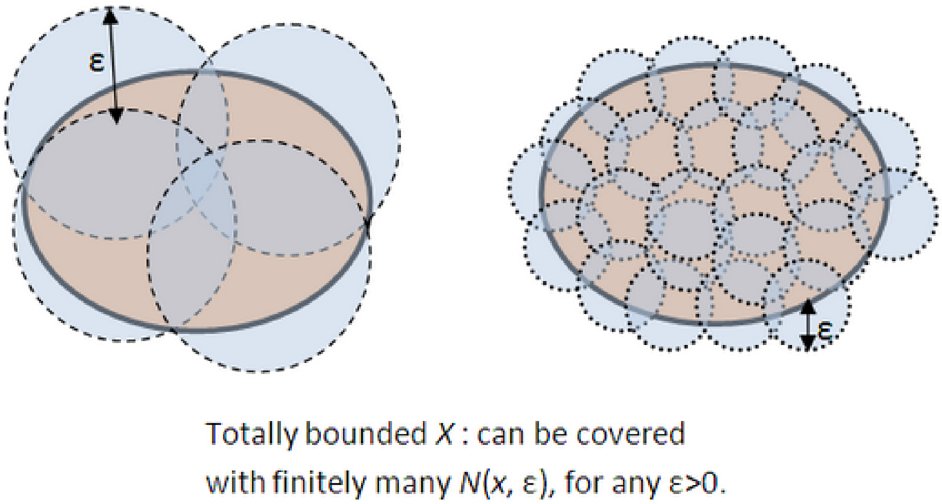

Figure 1: Compact Set

Definition 4. A subset $E \subseteq X$ is **bounded** if $\exists x_0 \in X, r > 0$, such that $E \subseteq B_r(x_0)$

Proposition 1. If $E \subseteq X$ is **compact** then E is **bounded**

Proof. By Contradiction
Assume that E is compact but not bounded. Take $x_0 \in X$ and consider the open cover of E, $U_n = \{B_n(x_0)\}$ for $n = 1, \ldots, n$. We know that each of these balls $B_n(x_0)$ is open by definition. Then $E \subseteq \cup_{n \geq 1} B_n(x_0)$. Since E is compact, then there exists finite subcover, so $\cup_{n \geq 1} B_n(x_0)$ is composed of finitely many balls. Then, $\exists 1 \leq n_1 < n_< \ldots < n_k$ such that $E \subseteq \cup_{j=1}^{k} B_{n_j}(x_0) = B_{n_k}(x_0)$ and E is contained in a ball with largest radius and this means E is bounded which is a contradiction! □

Note. In this proof, we are constructing a finite subcover of concentric balls starting from the origin with increasing radius, one possible out of many.

Definition 5. The **haussdorff property** is defined as follows. For any $x \neq y$ in (X, ρ) metric space. Then $\exists r_1, r_2 > 0$ such that $B_{r_1}(x) \cap B_{r_2}(y) = \emptyset$. This **Haussdorff space** is also known as the **T2 axiom**.

Note. This is always true for **metric spaces**

Proof. By contradiction
Take $0 < r < \frac{\rho(x,y)}{2}$. Assume $\exists z \in X$ such that $z \in B_r(x) \cap B_r(y)$. Then, by the triangle inequality:

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y) < r + r = 2r.$$

$$\frac{\rho(x,y)}{2} < r.$$

However, this contradicts the assumption $0 < r < \frac{\rho(x,y)}{2}$. Therefore, such a point z cannot exist. Any metric space must be haussdorff. $\square$

Proposition 2. If $E \subseteq X$ is compact then it is a closed set in X.

Proof. Assume that E is compact. We will use the fact that we already proved if a set U is open, then it's complement U^c is closed. So we will show $X \setminus E$ is an open set. If $E = X$ then E is closed since it is the whole metric space and metric spaces are closed sets. Assume $E \neq X$. Then, take $x \in X \setminus E$. For any $y \in E$, $\exists r_y > 0$ such that $B_{r_y}(x) \cap B_{r_y}(y) = \emptyset$. Then, $\{B_{r_y}(y)\}_{y \in E}$ is an open cover of E (in fact, even the centers would cover E). Since E is compact $\exists y_1, \ldots, y_N \in E$ such that $E \subseteq \cup_{j=1}^{N} B_{r_{y_j}}(y_j)$, a finite subcover. Take $r = min\{r_{y_i}, r_{y_2}, \ldots, r_{y_N}\} > 0$. This means $\forall 1 \leq j \leq N$:

$$B_r(x) \cap B_{y_j}(y_j) \subseteq B_{r_j}(x) \cap B_{r_j}(y_j) = \emptyset.$$

So,

$$B_r(x) \cap (\cup_{j=1}^{N} B_{r_j}(y_j)) = \emptyset.$$

Then, $B_r(x) \cap E = \emptyset$ and $B_r(x) \subseteq X \setminus E$. Since we can make this argument for any $x \in X \setminus E$, then $X \setminus E$ is open. And E must be closed set in X. $\square$

Proposition 3. Let $E \subseteq X$ compact and $A \subseteq E$ and A closed in X. Then, A is compact in X.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of A. We will prove that it has a finite subcover. Since A is closed, then $X \setminus A$ is open and we get $\{\{U_\alpha\}_{\alpha \in I}, X \setminus A\}$ which is an open cover of E. Since E is compact, $\exists$ a finite subcover of E, $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_N}, X \setminus A\}$. Then, $\{U_{\alpha_1}, \ldots, U_{\alpha_N}\} \subseteq \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_N}, X \setminus A\}$ and is a finite subcover of A. $\square$

Proposition 4. Let E be a compact set and let $S \subseteq E$ compact that has infinitely many element. Then $\exists$ a limit point of S inside E. (will be used later on when talking about convergence).

Proof. By Contradiction
Assume E is compact and $S \subseteq E$ and S has infinitely many points and S does not have a limit point in E. This means that since $S \subseteq E$, if there is no limit point in E, then S does not have a limit point in general. Then for $\forall y \in E$ we have that y is not a limit point of S meaning,

$$\forall y \in E, \exists r_y > 0 \text{ such that } B_{r_y}(y) \text{ contains only finnitely many points from S.}$$

Then,

$$\{B_{r_y}(y)\}_{y \in E}.$$

is an open cover of E. Since E is compact, $\exists$ a finite subcover of E:

$$\{B_{r_{y_1}}(y_1), \ldots, B_{r_{y_N}}(y_N)\} \text{ where } y_1, \ldots, y_N \in E.$$

Then, $E \subseteq \cup_{i=1}^{N} B_{r_{y_j}}(y_j)$ and since N is a finite number and for each $B_{r_{y_j}}(y_j)$ there only exist finitely many elements from S, then this means that $|E|$ is finite and since $S \subseteq E$, S has only finitely many elements, but this is a contradiction! $\square$

2 Characterization of Compact Sets in $\mathbb{R}^n$

2.1 Suprenum and Infinitum in $\mathbb{R}$

Definition 6. M is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E$, $x \leq M$

Definition 7. Let $E \subseteq \mathbb{R}$ and let E be bounded above (i.e. $\exists$ an upper bound of E), then $alpha = sup(E)$ is known as the **suprenum** of E if it satisfies:

1. α is an upper bound
2. $\forall \varepsilon > 0$, the interval $(\alpha - \varepsilon, \alpha] \cap E \neq \emptyset$

Remark. Maximum vs Suprenum
Maximum can be thought of as the suprenum that belongs to the set. However, **the suprenum may not necessarily belong to the set** (think limits). The same holds for the minimum and infinitum.

Definition 8. Then the infinitum $\beta = inf(E)$is defined as:

1. β is a lower bound
2. $\forall \varepsilon > 0$, the interval $[\beta, \beta + \varepsilon) \cap E \neq \emptyset$

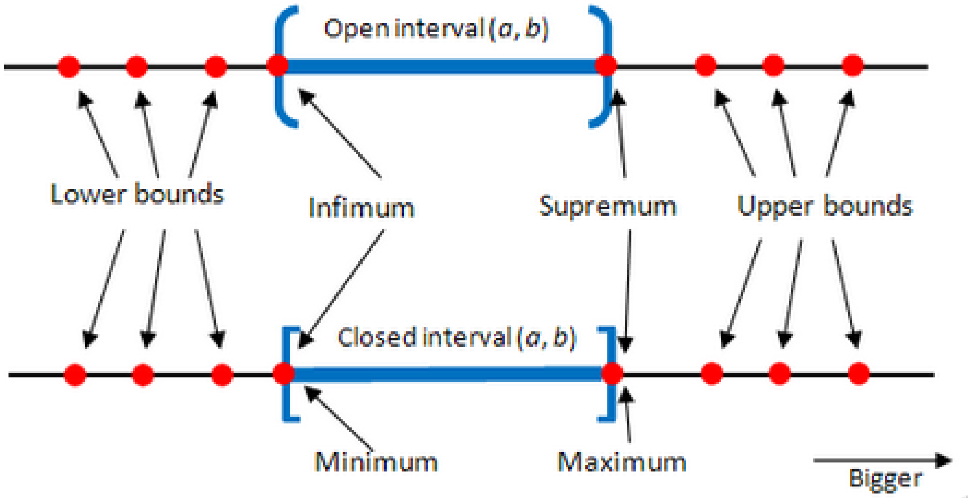

Figure 2: Suprenum and Infinitum vs Maximum and Minimum

Theorem 1. If $E \subseteq \mathbb{R}$ is bounded above then $\exists$ $sup(E)$. A similar statement holds for $inf(E)$ if E is bounded below.

Proposition 5. The $sup(E)$, E is bounded above, is unique.

Proof. By Contradiction
Assume that there exists two suprenum α_1, α_2, $\alpha_1 \neq \alpha_2$. Then WLOG, let $\alpha_1 < \alpha_2$. This means that for α_1 in E, by definition, $\exists \varepsilon_1$ such that the interval $(\alpha_1 - \varepsilon_1, \alpha_1] \cap E \neq \theta$ and for α_2 in E, $\exists \varepsilon_2$ such that $(\alpha_2 - \varepsilon_2, \alpha_2] \neq \theta$. However, because $\alpha_1 < \alpha_2$ then $\alpha_1 \in (\alpha_2 - \varepsilon_2, \alpha_2]$ which is non-empty with intersection with E, so there are elements $\alpha_1 < \alpha_1 + \varepsilon_1 \in E$. This means that α_1 is not an upper bound and can't be a suprenum! $\square$

Proposition 6. If M is an upper bound of $E \subseteq \mathbb{R}$ then $sup(E) \leq M$.

Proof. $\square$

Lemma 1. Let $I_1 \supseteq I_2 \supseteq \ldots \supset I_k \supseteq \ldots$ be a sequence of closed intervals

$$I_k = [a_k, b_k] \subseteq \mathbb{R} \text{ for all } k = 1, 2, \ldots.$$

Then,

$$\cap_{k=1}^{\infty} I_k \neq \emptyset.$$

Proof. Assume $I_1 \supseteq I_2 \supseteq \ldots \supset I_k \supseteq \ldots$ are closed intervals. Then for any given $k \geq 1$, $a_l \leq b_k$, $\forall l \geq 1$ (i.e. all the "lower bound"'s are smaller than any "upper bound"). Then b_k is an upper bound of $\{a_1, a_2, \ldots\}$ so through *Theorem 1*: $\alpha = sup_{l \geq 1}\{a_l\}$ exists and

$$\alpha \leq b_k \forall k \in 1, 2, \ldots.$$

. Then, α is a lower bound for $\{b_k | k \in \mathbb{N}\}$. And by *theorem 1* again,

$$\exists \beta = inf\{b_k | k \in \mathbb{N}\}.$$

and, $\alpha \leq \beta$. Then, $\forall k \in \mathbb{N}$,

$$a_k \leq \alpha \leq \beta \leq b_k \Leftrightarrow [\alpha, \beta] \subseteq I_k.$$

So,

$$\cap_{k=1}^{\infty} I_k \supset [\alpha, \beta].$$

and the intersection is non-empty as $[\alpha, \beta]$ contains at least 1 element (when $\alpha = \beta$) $\square$

Note. If the intervals were not closed. For example: $I_n = \left(0, \frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$, then $\cap_{n \geq 0} I_n = \emptyset$

Theorem 2. If $a \leq b$, then $I = [a, b] \subseteq \mathbb{R}$ is compact.

Proof. By contradiction
Assume that I is not compact, then $\exists$ an open cover $\{U_\alpha\}_{\alpha \in A}$ of the interval I that does not have a finite subcover. Take the midpoint of the interval $\frac{a+b}{2}$ and split $I_1 = I$ into

$$I_1' = [a, \frac{a+b}{2}]$$

and

$$I_1'' = [\frac{a+b}{2}, b]$$

. One of these intervals cannot be covered by finitely many sets from the collection $\{U_\alpha\}_{\alpha \in A}$, otherwise if both could be covered by finitely many, then I would have a finite subcover. Let us call this non-finitely covered interval $I_2 \subseteq I_1$. Continuing with this process for I_n, we construct $I_1 \supseteq I_2 \ldots \supseteq I_k \supseteq \ldots$, where:

1. $I_k \supset I_{k+1} \forall k \geq 1$
2. $\forall k \geq 1, I_k$ can not be covered by finitely many sets from $\{U_\alpha\}_{\alpha \in A}$
3. $\mid I_k \mid = \frac{b-a}{2^{k-1}}, k = 1, 2, \ldots$

By *Lemma 1*, $\exists x \in \cap_{k=1}^{\infty} I_k$. Then $\exists \beta \in A$ such that $x \in U_\beta$. Since U_β is open, $\exists \varepsilon > 0$ such that $U_\beta \supseteq (x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$. It now follows from property (3), that $\exists k \geq 1$ such that $I_k \subseteq (x - \varepsilon, x + \varepsilon) \subseteq U_\beta$. This is a contradiction with property (2) because I_k can be covered by $\{U_\beta\}$ which is a finite subcover of size 1! Thus, $I = [a, b]$ is compact $\square$

3 Compact Sets in $\mathbb{R}^n$

Consider the rectangular box I_n:

$$[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | a_k \leq x_k \leq b_k \forall k = 1, 2, \ldots\}.$$

Theorem 3. *$n \geq 1, I^n$ is bounded in $\mathbb{R}^n$ The proof follows the argument in theorem (2)*

Theorem 4. *$E \supseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ E is closed and bounded.*

Proof. $\rightarrow$: Since E is compact $\rightarrow$ E is closed and bounded by prop 1 and 2.

$\leftarrow$: Assume that E is closed and bounded in $\mathbb{R}^n$. Then $\exists$ a rectangular box B inside $\mathbb{R}^n$ such that $E \subseteq B$ (we can consider the ball centered at E, then draw a box around it). But now realize that E is closed and B is compact (by theorem 3) and by Proposition 3, this means that E is compact. $\square$

Note. The above is not true in general, i.e. $E \subseteq X \not\Leftrightarrow$ E is closed and bounded. For example, consider $X = [0, 1)$ and $\rho(x, y) =| x - y |$, so (X, ρ) is the metric space. Define $E = [\frac{1}{2}, 1)$. Then E is closed and bounded in X. But E is not compact. Consider the open cover of E,

$$U_k = \left(0, 1 - \frac{1}{k+2}\right) k = 1, 2, 3, \ldots.$$

Then, $\cup_{k \geq 1} U_k \supseteq E$. But this open cover does not have a finite subcover that covers E!

$$\lim_{k \to \infty} U_k = (0, 1)\,.$$

So there is no way to contain $(1 - \varepsilon, 1) \subseteq E$ as $\varepsilon \to 0$ with finite subsets, it gets closer and closer but never reaches.