HOMEWORK 5 FOR MATH 7241, FALL 2024. DUE OCTOBER 17TH

1. Let $\vec{v} = (v_1, v_2, \ldots, v_n)$ be a random vector in $\mathbb{R}^n$, where each of the coordinates v_i is uniformly distributed in $[-1, 1]$, independently of each other. Show that $\|\vec{v}\|/\sqrt{n}$, the length of the vector divided by $\sqrt{n}$, converges in probability to $\sqrt{1/3}$.

2. You are playing a modified game of Roulette. There are 9 outcomes in total, each equally likely. four are red, four are blue, and one is green (numbered 0). You bet one dollar on red; if the ball lands on a red outcome, you will get two dollars back (giving you one extra dollar). If it lands on a non-red outcome, you get nothing (and thus lose one dollar). You play this game with the same strategy one hundred times. Use the Central Limit Theorem to estimate the probability of coming out ahead — i.e. winning more money than you lost. You may use a Normal distribution chart to evaluate the probability numerically, or leave your answer in terms of a Normal probability.

3. A battery has a random lifetime, which we model as an exponential random variable of mean 10 days. You must operate a lighthouse continuously, and can change a battery immediately as the previous one fails. You have a stockpile of 30 batteries. Use the Central Limit Theorem to estimate the probability that you can operate your lighthouse for a full year. You may use a Normal distribution chart to evaluate the probability numerically, or leave your answer in terms of a Normal probability.

4. Prove that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = 1/2.$$

Hint: The lefthand side is $\mathbb{P}[X_n \leq n]$ if X_n has the Poisson distribution of mean n. Use the Central Limit Theorem to show that, as n goes to infinity, this probability goes to $1/2$. Previous homework exercises may be used to show that one can apply the Central Limit Theorem to X_n.

5. Let X and Y be independent Normal random variables with mean 0 and variance 1.

- Show that $X + Y$ and $X - Y$ are both Normal random variables with mean 0 and variance 2.
- Show that $X + Y$ is independent of $X - Y$.

Hint: Moment generating functions may be useful in this problem!