HOMEWORK 3 FOR MATH 7241, FALL 2024. DUE OCTOBER 4RD

1. Similarly to the last week's problem, we have r balls, to be distributed among n bins. Each of the n^r possible configurations is equally likely. Suppose that r and n are going to infinity so that $r/n \to c$ for some positive real number c. Let E_n be the number of empty bins. Show that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[E_n]$$

exists, and compute it as a function of c.

Hint: write E_n as a sum of random variables that take on the values 0 and 1.

2. A fair, six-sided die is rolled until a number shows up twice in a row. Let N be the amount of rolls required for that to occur (so, if the sequence of rolls is $(5,4,3,4,3,5,1,1)$, $N = 8$. Find the expected value of N.

3. Consider the following variant on the Monty Hall problem: there are five screens in front of a contestant. Two of the screens hide prizes, while the other three hide goats. The contestant picks a screen. After making that choice, the host reveals what is behind a different screen — namely, that there was a prize behind it. He then allows the contestant a choice: to keep their original screen, or switch (choosing one of the remaining three screens uniformly at random). Should the contestant switch screens?

4. Let p and q be two real numbers in $(0,1)$. Let V be a geometric random variable of parameter p — that is, $\mathbb{P}[V = k] = (1-p)^{k-1}\cdot p$. We have V customers visit a candy store in a given day. Each customer buys a chocolate bar with probability q; otherwise, they leave without purchasing anything. The behavior of each customer is independent of all others.

- What is the expected number of chocolate bars sold in a day?
- What is the probability that the number of chocolate bars sold is equal to the number of customers that visited the store on a particular day?

5. Let X and Y be two independent exponential random variables of parameter 1 – that is, both of their pdf are e^{-x} when $x \geq 0$, and zero otherwise. Conditional on X and Y, let Z be a uniform random variable on $[-X, Y]$. What is the mean and variance of Z?