Analysis I
Lecture 8

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1 Limits

Definition 1. Let $f : X \to Y$, $(X, \rho_x), (Y, \rho_y)$ metric spaces $x_0 \in X$. Then,

$$f(x) \to y_0 \text{ in } Y \text{ as } x \to x_0..$$

"$f(x)$ converges to y_0 as x converges to x_0". Alternatively, this is written as:

$$\lim_{x \to x_0} f(x) = y_0.$$

if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$f(B_\delta^{oX}(x_0)) \subseteq B_\varepsilon^Y(y_0).$$

Note. This does not necessarily mean continuity because we are considering the punctured ball B_δ^{oX} so it is not necessarily true that $f(x_0) = y_0$., Think of a discontinuous jump, this converges to y_0 but is not continuous

Remark. We are interested of the behavior of the function around the point not the point itself.

Proposition 1. $f : X \to Y$ is continuous at $x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$.

Proof. $(\Rightarrow)$ Assume that $f \in C(x_0)$ then $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$f(B_\delta^X(x_0)) \subseteq B_\varepsilon^Y(f(x_0)).$$

by definition of continuity. Then trivially, this implies that the image of the punctured ball must also be a subset of $B_\varepsilon^Y(f(x_0))$. Then the limit of $\lim_{x \to x_0} f(x) = f(x_0)$.

$(\Leftarrow)$. Assume that $\lim_{x \to x_0} = f(x_0) \Rightarrow \forall \varepsilon > 0 \exists \delta > 0$ such that

$$f(B_\delta^{oX}, (x_0) \subseteq B_{\varepsilon \varepsilon}^{\ Y}(f(x_0)) \Rightarrow f(B_\delta^X(x_0)) \subseteq B_\varepsilon^Y(f(x_0)).$$

since $f(x_0) \in B_\varepsilon^Y(f(x_0))$. Thus, $f \in C(x_0)$. □

Corollary 1.] We can extend the definition of $\lim_{x \to x_0} f(x)$ also in the case when $f = X \setminus \{x_0\} \to Y$ is not defined at x_0

Proposition 2. $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ $x_0 \in X$. Then

1. If there exist $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ then $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$.

   Proof. Assume that $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$. Take $\varepsilon > 0$, by definition $\exists B_{\delta_1}^X(x_0)$ such that

   $$f\left(B_{\delta_1}^{oX}(x_0)\right) \subseteq \left(A - \frac{\varepsilon}{2}, A + \frac{\varepsilon}{2}\right) \Rightarrow \|f(x) - A\| < \frac{\varepsilon}{2} \text{ since } \|x - x_0\| < \delta_1, x \neq x_0.$$

   Similary, since $\lim_{x \to x_0} g(x) = B \Rightarrow \delta_2 > 0$ such that

   $$\|g(x) - B\| < \frac{\varepsilon}{2} \text{ since } \|x - x_0\| < \delta_2, x \neq x_0.$$

   Then take $\delta = \min(\delta_1, \delta_2)$. Then, for $\|x - x_0\| < \delta, x \neq x_0$ we have

   $$\begin{aligned} \|(f(x) - A) + (g(x) - B)\| &= \|\,(f(x) + g(x)) - (A + B)\| \\ &\leq \|f(x) - A\| + \|g(x) - B\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

   Thus, the limit of $f(x) + g(x)$ is $A + B$. □

2. If $\exists \lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ then $\lim_{x \to x_0} f(x)g(x) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$.

3. If $\exists \lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ and $\lim_{x \to x_0} g(x) \neq 0 (g(x) \neq 0 \forall x \in X)$. Then, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$.

2 Differentiable functions

Definition 2. Let $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. f is differentiable at a point x_0 if $\exists \alpha \in \mathbb{R}, r : (a, b) \setminus \{x_0\} \to \mathbb{R}$ such that $r(x) \to 0$ as $x \to x_0$ such that $\forall x \in (a, b), x \neq x_0$, we have:

$$f(x) = f(x_0) + \alpha(x - x_0) + f(x)(x - x_0).$$

If f is differentiable at x_0. then we set $f'(x_0) = \alpha$ and call it the derivative of f at x_0.

Note. α is the derivative and $\alpha(x - x_0)$ represents the tangent line at x_0. $r(x)$ is the error term (goes to 0 as $x \to x_0$).

Note. If f is differentiable at x_0 then we write that $f \in D(x_0)$

Remark. The above definition is equivalent to the "usual" (calculus) definition of the limit is:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

We can rewrite $r(x)$ as:

$$r(x) = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \to 0.$$

and,

$$r(x)(x - x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0).$$

which after some rearrangment gives us the definition of $f(x)$

Remark. If $f \in D(x_0)$ then we can set $r(x_0) = 0$. Then, $f : (a, b) \to \mathbb{R}$ and $r \in C(x_0)$ and $f(x) = f(x_0) + \alpha(x - x_0) + f(x)(x - x_0)$ holds for all $x \in (a, b)$

Remark. $f \in D(x_0) \Leftrightarrow f(x) = f(x_0) + \alpha(x - x_0) + f(x)(x - x_0)$ holds for $x \in (x_0 - \varepsilon, x_0 + \varepsilon) \subseteq (a, b)$ for some $\varepsilon > 0$

Proposition 3. If $f \in D(x_0) \Rightarrow f \in C(x_0)$. (differentialbility is stronger than continuity)

Proof. If $f \in D(x_0)$ then $\forall x \in (a, b)$,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x)(x - x_0).$$

where $r(x_0) = 0$ and $r \in C(x_0) \Rightarrow r(x) \to r(x_0) = 0$ as $x \to x_0$. Let's consider the limit of f:

$$\begin{aligned}
\lim_{x\to x_0} f(x) &= \lim_{x\to x_0} [f(x_0) + f'(x_0)(x - x_0) + r(x)(x - x_0)] \\
&= \lim_{x\to x_0} f(x_0) + \lim_{x\to x_0} f'(x_0)(x - x_0) + \lim_{x\to x_0} r(x)(x - x_0) \\
&= \lim_{x\to x_0} f(x_0) + \lim_{x\to x_0} f'(x_0) \lim_{x\to x_0} (x - x_0) + \lim_{x\to x_0} \lim_{x\to x_0} r(x)(x - x_0) \\
&= f(x_0) + f'(x_0) \cdot 0 + 0 \cdot 0 \\
&= f(x_0).
\end{aligned}$$

By the proposition since

$$\lim_{x\to x_0} f(x) = f(x_0).$$

then f is continuous at x_0 □

Proposition 4. If $f, g : (a, b) \to \mathbb{R}$ and $f, g \in D(x_0)$, then

1. $(\alpha f + \beta g) \in D(x_0)$ and $(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0)$
2. $f \times g \in D(x_0)$ and $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$

   Proof. We need to consider for $x \neq x_0$, the equality $\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0} + \frac{f(x)g(x)-f(x)g(x_0)}{x-x_0}$

   $$\begin{aligned}
   \lim_{x\to x_0} (f \times g)(x_0) &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
   &= \lim_{x\to x_0} \frac{f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{f(x)g(x) - f(x)g(x_0)}{x - x_0} \\
   &= \lim_{x\to x_0} g(x_0) \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} f(x) \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
   &= g(x_0)f'(x_0) + f(x_0)g'(x_0).
   \end{aligned}$$

   and we are done! □
3. If $g(x_0) \neq 0$ then $\frac{f}{g} \in D(x_0)$ and $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$

   Note. Since $f \in D(x_0) \Rightarrow f \in C(x_0)$. On the other side, $g(x_0) \neq 0$ and hence by continuity $g(x) \neq 0$ for $x \in (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$. This is so that $\frac{f}{g}$ is defined over some small interval $(x - \varepsilon, x + \varepsilon)$ which is needed for definition of differentiability at a point.

2.1 Chain Rule

Consider the composed functoin: $(g \cdot f)(x) = g(f(x)), x \in (a, b)$ where $f : (a, b) \to \mathbb{R}$ such that $f((a, b)) \subseteq (c, d), g : (c, d) \to \mathbb{R}$

Theorem 1. In the situation above, if $f \in D(x_0)$ and $g \in D(f(x_0))$ then $g \cdot f \in D(x_0)$ and

$$(g \cdot f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

Proof. Assume that $f \in D(x_0)$ and $g \in D(f(x_0))$ then $\forall x \in (a, b)$,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r_x(x)(x - x_0).$$

where $r_x \in C(x_0)$ and $r_x(x_0) = 0$. Similarly, $\forall y \in (c, d)$

$$g(y) = g(y_0) + g'(y_0)(y - y_0) + r_y(y)(y - y_0).$$

where $r_y \in C(y_0)$ and $r_y(y_0) = 0$. Then from here follows:

$$\begin{aligned}
g(y) = g(f(x)) &= g(f(x_0)) + g'(f(x_0))(f(x) - f(x_0)) + r_y(f(x))(f(x) - f(x_0)) \\
&= g(f(x_0)) + g'(f(x_0))(f(x_0) + f'(x_0)(x - x_0) + r_x(x)(x - x_0) - f(x_0)) \\
&+ r_y(f(x))(f(x_0) + f'(x_0)(x - x_0) + r_x(x)(x - x_0) - f(x_0)) \\
&= g(f(x_0)) + g'(f(x_0))(f'(x_0)(x - x_0) + r_x(x)(x - x_0)) + r_y(f(x))(f'(x_0)(x - x_0) + r_x(x)(x - x_0)) \\
&= g(f(x_0)) + g'(f(x_0))f'(x_0)(x - x_0) \\
&+ [g'(f(x_0))r_x(x)(x - x_0) + r_y(f(x))f'(x_0)(x - x_0) + r_y(f(x))r_x\,(x)\,(x - x_0)] \\
&= g(f(x_0)) + [g'(f(x_0))f'(x_0)](x - x_0) + [g'(f(x_0))r_x(x) + r_y(f(x))f'(x_0) + r_y(f(x))r_x\,(x)](x - x_0) \\
&= g(f(x_0)) + \alpha(x - x_0) + r(x)(x - x_0).
\end{aligned}$$

Thus, $g'(f(x_0))r_x(x) + r_y(f(x))f'(x_0) + r_y(f(x))r_x\,(x) = r(x)$. All we need to do is show that $r(x) \to 0$ as $x \to x_0$, then it will follow that $g \cdot f \in D(x_0)$. A quick observation reveals that $r_x(x) \to 0$ as $x \to x_0$ and since $r_2 \in C(f(x_0))$ and $f \in C(x_0) \Rightarrow r_2 \cdot f \in C(x_0) \Rightarrow r_y(f(x)) \to r_y(f(x_0)) = 0$ since $g(x) \in D(x_0)$ as $x \to x_0$. Thus, $r(x) \to 0$ as $x \to x_0$. $\square$