Analysis I
Mean Value Theorem

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1 Local Max/Min

Definition 1. An open neighborhood of $x_0 \in X$ in a metric space (X, ρ) is an open set $U(x_0)$ of X such that $x_0 \in U(x_0)$.

Lemma 1. $f : (a, b) \to \mathbb{R}$ such that $f \in D(x_0)$ where $x_0 \in (a, b)$. Then,

1. Let $(a, b) \subseteq \mathbb{R}, a < b$. If $f'(x_0) > 0$ then $\exists$ an open neighborhood $U(x_0) \subseteq (a, b)$ of x_0 such that
   (a) $f(x) > f(x_0)$ for $x > x_0 \in U(x_0)$
   (b) $f(x) < f(x_0)$ for $x < x_0, x \in U(x_0)$

Proof. Assume that $f \in D(x_0), f'(x_0) > 0$. Then,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x)(x - x_0).$$

where $r \in C(x)$ and $r(x_0) = 0$ which implies that $r(x) \to 0 (x \to x_0)$. Then, we have the following

$$f(x) - f(x_0) = [f'(x_0) + r(x)](x - x_0).$$

Since $f'(x_0) > 0$ is positive and fixed, and since $r(x)$ is continuous and $r(x_0) = 0$, then we can find some open neighborhood around x_0 with $x \in U(x_0)$, such that $\mid r(x) \mid < \frac{f'(x_0)}{2}$. Then, $f'(x_0) + r(x) > \frac{f'(x_0)}{2} > 0 \forall x \in U(x_0)$. Then we can say the following:

- If $x > x_0$ for $x \in U(x_0)$, then $x - x_0 > 0$, so
$$f(x) - f(x_0) > 0 \Rightarrow f(x) > f(x_0).$$
- If $x < x_0$ for $x \in U(x_0)$, then $x - x_0 < 0$, so
$$f(x) - f(x_0) < 0 \Rightarrow f(x) < f(x_0).$$

A similar proof follows for (2) except $f'(x_0) < 0$ and the signs are flipped. □

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2. If $f'(x_0) < 0$ then $\exists$ an open neighborhood $U(x_0) \subseteq (a, b)$ of x_0 such that
   (a) $f(x) < f(x_0)$ for $x > x_0 \in U(x_0)$
   (b) $f(x) > f(x_0)$ for $x < x_0, x \in U(x_0)$.

Note. The above only shows this in one variable!

Definition 2. Assume that (X, ρ) metric space and we have $f : X \to \mathbb{R}$. Then we say that f has a local maximum or minimum at x_0 if $\exists U(x_0)$ open neighborhood in X such that:

1. maximum: $f(x) \leq f(x_0)$ for $x \in U(x_0)$
2. minimum: $f(x) \geq f(x_0)$ for $x \in U(x_0)$

Corollary 1. If $f : (a, b) \to \mathbb{R}$ that has a local maximum or minimum at $x_0 \in (a, b)$ and $f \in D(X_0)$. Then $f'(x_0) = 0$.

Proof. This is an immediate consequence of the lemma. Assume that there is a local max/min at x_0 and $f \in D(x_0)$.

Assume that $f'(x_0) > 0$, then we know that $\exists x_1 > x_0$ such that $f(x_1) > f(x_0)$ and there exists another $x_2 < x_0$, such that $f(x_2) < f(x_0)$ and x_0 can't be a local minimum or maximum. Similarly, if $f'(x_0) < 0$ then x_0 can't be a local minimum or maximum because $\exists x_1, x_2 \in U(x_0)$ such that $f(x_1) < f(x_0)$ and $f(x_2) > f(x_0)$.

Thus, $f'(x_0) = 0$ when there is a local min/max at x_0 $\square$

2 Mean Value Theorem

Theorem 1. Rolle's Theorem
Assume that $f : [a, b] \to \mathbb{R}$ such that $f \in C([a, b]), f \in D((a, b))$ and $f(a) = f(b)$. Then, $\exists x_* \in (a, b)$ such that

$$f'(x_*) = 0.$$

Proof. Since the interval $[a, b]$ is compact in $\mathbb{R}$ and $f \in C([a, b])$, then $\exists x_m, x_M \in [a, b]$ s.t. $f(x_m) = inf_{[a,b]}f$ and $f(x_M) = sup_{[a,b]}f$. Consider the case when $x_m, x_M \in \{a, b\}$ the min/max are equivalent and $f(x_m) = f(x_M)$ (as $f(a) = f(b)$ by assumption). Then it's clear that the function f must be a map from $[a, b] \to c$, a constant. The derivative of f along any point along (a, b) is 0. Otherwise, one of the points $x_m \in (a, b)$ or $x_M \in (a, b)$. Without loss of generality, assume that $x_m \in (a, b)$ so take $f'(x_m)$, by the corollary we can say that $f'(x_m) = 0$ since $f(x_m)$ is a global minima and thus must have a derivative equal to 0. Then, $x_* = x_m$. The same logic follows if $x_M \in (a, b)$. $\square$

Theorem 2. Mean Value Theorem
Assume that $f : [a, b] \to \mathbb{R}$ such that $f \in C([a, b]) \cap D((a, b))$ then $\exists x_* \in (a, b)$ such that

$$f(b) - f(a) = f'(x_*)(b - a).$$

also written as

$$f'(x_*) = \frac{f(b) - f(a)}{b - a}.$$

Proof. If $f(b) \neq f(a)$ then

$$F(x) = \frac{f(x) - f(a)}{f(b) - f(a)} - \frac{x - a}{b - a}.$$

Then, $F(a) = 0, F(b) = 0$. Note that F is an affine function so differentiable and continuous by construction (continuous functions are additive). Also $F \in C([a, b]) \cap D((a, b))/$ By **Rolle's Theorem**, $\exists x_* \in (a, b)$ such that $F'(x_*) = 0$ and the derivative is $F'(x) = \frac{f'(x)}{f(b)-f(a)} \Rightarrow F'(x_0) = \frac{f'(x_*)}{f(b)-f(a)} - \frac{1}{b-a} = F(x_*) = 0$, and

$$\frac{f'(x_*)}{f(b) - f(a)} = \frac{1}{b - a}$$

$$f'(x_*) = \frac{f(b) - f(a)}{b - a}.$$

When $f(b) = f(a)$, then the mean value theorem follows from Rolles Theorem.

$$f(b) - f(a) = f'(x_*)(b - a) \Rightarrow 0 = f'(x^*)(b - a) \Rightarrow 0 = 0.$$

□

Proposition 1. Assume that $f : (a, b) \to \mathbb{R}$, $f \in D((a, b))$. Then,

1. If $f'(x) \geq 0 \ \forall x \in (a, b)$ then we have that $f(x) \geq f(y)$ for $x \geq y, x, y \in (a, b)$. This is knon as **monotonically increasing**.
2. If $f'(x) > 0 \ \forall x \in (a, b)$ then $f(x) > f(y)$ for $x > y, x, y \in (a, b)$. This is known as **strictly monotonically increasing**
3. If $f'(x) \leq 0 \ \forall x \in (a, b)$ then we have that $f(x) \leq f(y)$ for $x \leq y, x, y \in (a, b)$. This is knon as **monotonically decreasing**.
4. If $f'(x) < 0 \ \forall x \in (a, b)$ then $f(x) < f(y)$ for $x < y, x, y \in (a, b)$. This is known as **strictly monotonically decreasing**

Proof. The proof follows from the mean value theorem. Consider $x_1, x_2 \in (a, b)$ where $x_1 < x_2$, then by the mean value theorem,

$$f(x_2) - f(x_1) = f'(x_*)(x_2 - x_1).$$

and we know that if $f'(x) \geq 0$ then, we can see that $f(x_2) \geq f(x_1)$. If $f'(x) > 0$, then $f(x_2) > f(x_1)$. The same follows for $f'(X) \leq 0, f'(x) < 0$. □

Remark. It is not the case that if a function is strictly monotonically increasing (or descreasing) then it is not the case that $f'(x) > 0($ or $f'(x) < 0)$. Consider $f(x) = x^3$, it is strictly increasing but $f'(0) = 0$.