# Probability 1

Daniel Yu

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# 1 Poisson Processes

**Definition 1.** A stochastic process $\{N(t)\}_{t\in[0,\infty)}$ is a counting process if

1. $N(t) = \{01, 2, 3, \ldots\} \forall t > 0$ and
2. $N(s) \leq N(t) \forall s \leq t$

$N(t)$ counts the number of events that occured between time 0 and time $t$.

**Note.** See the Appendix for the definition for asympotic notation for $O(x)$. We are using a definition that is no the traditional definition.

**Definition 2.** A counting process is **simple** if $\forall t \geq 0$,

$$P[N(t+h) - N(t) \geq 2] = O(h).$$

meaning the probability of more than 1 arrival in the interval $[t+h, t]$ goes to 0 fairly quickly (at rate $O(h)$ )

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We wish to study the generalization of **time-homogenous markov chains**. Let's make the assumptions that:

1. **the distribution of (N(t+s) - N(t)) the number of arrivals that occured between t to t+s does not depend on t!**
2. **for any $t \leq s$ the RVs N(t), N(s) - N(t) are independent** this is known as the independent increament property. This is saying that the number of arrivials in disjoint time intervals is independent.

**Definition 3.** Any simple counting process that satisfies (1), (2) is a time-homogenous poisson process on $\mathbb{R}$

To determine the process, we need only one parameter $\lambda > 0$. Consider for the interval, $[0, h]$:

$$P[N(h) - N(0) = 0] = 1 - \lambda h + O(h).$$

using the fact that $P[N(h) - N(0) = 1] = \lambda h$ and $P[N(h) - N(0) \geq 2] = O(h)$

**Theorem 1.** Let $N_\lambda(t)$ be the unique process which satisfies all these properties above:

$$P[N_\lambda(t) = k] = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

i.e. $N_\lambda(t) \sim Poisson(\lambda t)$

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*Proof.* Let $P_m(t) = P[N_\lambda(t) = m]$.

$$\begin{aligned}
P_m(t+h) &= \sum_{k=0}^{m} P[N_\lambda(1+h) = m \mid N_\lambda(t) = k] \cdot P[N_\lambda(t) = k] \\
&= \sum_{k=0}^{m} P[N_\lambda(1+h) - N_\lambda(t) + N_\lambda(t) = m \mid N_\lambda(t) = k] \cdot P[N_\lambda(t) = k] \\
&= \sum_{k=0}^{m} P[N_\lambda(1+h) - N_\lambda(t) = m-k \mid N_\lambda(t) = k] \cdot P[N_\lambda(t) = k] \\
&\text{Use the property that disjoint intervals are independent} \\
&= \sum_{k=0}^{m} P[N_\lambda(1+h) - N_\lambda(t) = m-k] \cdot P[N_\lambda(t) = k] \\
&= \text{using time homogenity} \\
&= \sum_{k=0}^{m} P[N_\lambda(h) - N_0(t) = m-k] \cdot P[N_\lambda(t) = k] \\
&= P[N_\lambda(h) - N_\lambda(0) = 0] \cdot P_m(t) + P[N_\lambda(h) - N_\lambda(0) = 0] \cdot P_{m-1}(t) + \\
&\sum_{k=0}^{m-2} P[N_\lambda(h) - N_0(t) = m-k] \cdot P_k(t) \\
&= (1 - \lambda h + O(h))P_m(t) + (\lambda h + O(h))P_{m-1}(t) + \sum_{k=0}^{m-2} O(h) \cdot P_k(t) \\
&\text{we can group and simplify by asymptotics since } cO(x) = O(x) \\
&= P_m(t) + \lambda h \left(-P_m(t) + P_m(t)\right) + O(h) \\
\frac{P_m(t+h) - P_m(t)}{h} &= \lambda(-P_m(t) + P_m(t)) + \frac{O(h)}{h}.
\end{aligned}$$

Let's take the limit as $h \to 0$:

$$\frac{dP_m(t)}{dt} = -\lambda P_m(t) + \lambda P_m(t), m \geq 1.$$

Repeating this for $m = 0$,m

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t).$$

What is $P_0(t)$?

$$\begin{aligned}
P_0' &= -\lambda P_0 \\
\Rightarrow \int_0^t \frac{P_0'(s)}{P_0(s)} ds & \\
&= \int_0^t -\lambda ds \\
&= \ln(P_0(t)) - \ln(P_0(0)) = -\lambda t \\
P_0(t) &= e^{-\lambda t}.
\end{aligned}$$

For $m = 1$: missing notes.

$e^{\lambda t} P_1(t) - P_1(0) = \lambda t \Rightarrow P_1(t) = \lambda t \cdot e^{-\lambda t}$
It is possible to solve the above by induction and get the closed form formula:

$$P_m(t) \frac{e^{-\lambda t} (\lambda t)^m}{m!}.$$

Method 2: Using Generating Functions

$$G(s,t) = \sum_{m=0}^{\infty} P_m(t) \cdot S^m.$$

Then, compute the derivative

$$\begin{aligned}
\frac{dG(s,t)}{dt} &= \sum_{m=0}^{\infty} P_m(t)' S^m \\
&= P_0'(t) + \sum_{m \geq 1} P_m'(t) S^m \\
&= -\lambda P_0(t) + \sum_{m \geq 1} [-\lambda P_m(t) + \lambda P_{m+1}(t)] S^m \\
&= \text{missing text} \\
&= \lambda G(s,t) + \lambda s G(s,t)
\end{aligned}$$

.

Then,

$$\frac{dG(s,t)}{dt} = \lambda G(s,t)(S-1).$$

Divide by $G(s,t)$ and integrat w.r.t. $t$,

$$\ln[G(s,t)] - \ln[G(s,t)] = \lambda s - 1) \cdot t.$$

missing text

$$\begin{aligned}
\sum_{m=0}^{\infty} P_m(t) \cdot S^m &= e^{-\lambda t} \cdot e^{\lambda s t} \\
&= e^{-\lambda t} \sum_{m=0}^{\infty} \frac{(\lambda s t)^m}{m!} \\
&\text{canceling both sides and removing the summations} \\
P_m(t) &= \frac{e^{-\lambda t} (\lambda t)^m}{m!}.
\end{aligned}$$

□

**Note.** The continuous version of this is the Brownian motion

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Let's go through an example. Let $N(t)$ be a poisson point process of increment 1. What is

$$P[N(1) = 1, N(2) = 1, N(3) = 2, N(4) = 2, N(5) = 3].$$

?

**Method 1**
We can use the new arrivals per interval:

$$= P[\text{1 arrival [0,1], 0 arrival in [1,2], one arrival in [2,3], zero arrivasl [3,4], 1 arrova; [4,5]}].$$

By tyhe time independent increment property:

$$\begin{aligned}
P[\text{ events above}] &= P[\text{1 arrival [0,1]}]P[\text{0 arrival [1,2]}]P[\text{1 arrival [2,3]}P[\text{0 arrival [3,4]}]P[\text{1 arrival [4,5]}] \\
&= P[Poisson(1) = 1]^3 P[Poisson(1) = 0]^2 \\
&= \frac{1}{e^5}
\end{aligned}$$

.

**Method 2**
Using joints intelligently

$$\begin{aligned} P[N(1)=1,N(2)=1,N(3)=2,N(4)=2,N(5)=3] &= P[N(1)-N(0)=1]\cdot P[N(2)-N(1)=0]\cdot P[N(3)-N(2)=1] \\ &= \cdot P[N(4)-N(3)=0]\cdot P[N(5)-N(4)=1]. \end{aligned}$$

.

$$P[X=k] = \frac{\mu^k e^{-\mu}}{k!}.$$

For each interval $[a,b]$, the mean is $\mu = \lambda(b-a)$. Since $\lambda = 1$ and each interval is of length 1, we have $\mu = 1$ for each increment:

$$\begin{aligned} P[N(1)-N(0)=1] &= P[\text{Poisson}(1)=1] = \frac{1^1 e^{-1}}{1!} = \frac{1}{e}, \\ P[N(2)-N(1)=0] &= P[\text{Poisson}(1)=0] = \frac{1^0 e^{-1}}{0!} = \frac{1}{e}, \\ P[N(3)-N(2)=1] &= P[\text{Poisson}(1)=1] = \frac{1}{e}, \\ P[N(4)-N(3)=0] &= P[\text{Poisson}(1)=0] = \frac{1}{e}, \\ P[N(5)-N(4)=1] &= P[\text{Poisson}(1)=1] = \frac{1}{e}. \end{aligned}$$

**Step 4: Combine Probabilities**
The total joint probability is:

$$P[N(1)=1,N(2)=1,N(3)=2,N(4)=2,N(5)=3] = \frac{1}{e}\cdot\frac{1}{e}\cdot\frac{1}{e}\cdot\frac{1}{e}\cdot\frac{1}{e} = \frac{1}{e^5}.$$

**Final Result:**
Both methods yield the same result:

$$P[N(1)=1,N(2)=1,N(3)=2,N(4)=2,N(5)=3] = \frac{1}{e^5}.$$

# 2 Appendix: Asysmpotics

**Definition 4.** $f(x) = O(x)$ if $\lim\limits_{x\to 0} \frac{f(x)}{x} = 0$

**Note.** This not equivalent to the usual definition of big-O notation i.e. that $f(x)$ is $O(g(x))$ if $f(x) \le cg(x)$.

1. consider $x^2 \Rightarrow \lim\limits_{x\to 0} \frac{x^2}{x} = \lim\limits_{x\to 0} x = 0$ but is not $O(x)$ in the traditional definition.
2. Consider $x \Rightarrow \lim\limits_{x\to 0} \frac{x}{x} = 1 \neq 0$ but is $O(x)$ in the traditional definion.

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In this notation if $f(x)$ is twice differentiable at $x = 0$, then we can use the taylor series expansion with the error term $r(x) = O(x^2)$:

$$f(x) = f(0) + f'(0) + \frac{f''(0)}{2}x^2 + O(x^2).$$

$$(\frac{1}{x} + 1 + O(1)) \cdot (1 + O(x)) = \left(\frac{2}{x} + 2\right) + O(1).$$

some missing text here $\frac{f(x)}{x} + f(x) \to 0$