# Analysis I
# Lecture 6

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# 1 Uniformly Continuous Maps

**Definition 1.** Let $(X, \rho_X), (Y, \rho_Y)$ metric spaces and $f : X \to Y$ is **uniformly continous** if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in X, \rho_X(x, y) < \delta$ then $\rho_Y(f(x), f(y)) < \varepsilon$.

1. imposing a stronger constraint than continuous because this is saying some $\delta$ works across the entire domain for $x, y \in X$
2. stating that $f$ behaves "similarly" across the whole domain
3. a function that is continuous at every point $\not\Leftrightarrow$ the function is uniformly continuous
4. consider behavior of functions with asymptotes, what can you say about them?

**Remark.** Recall that the definition of a continuous map is as follows. $f \in C(X, Y) \Leftrightarrow \forall x \in X, \forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall y \in X$, $\rho(y, x) < \delta$ we have $\rho(f(x), f(y)) < \varepsilon$

**Lemma 1.** If $f : X \to Y$ is uniformly continuous then $f \in C(X, Y)$ i.e. it is continuous.

**Note.** This is not true the other way around. A map that is continuous but not uniformly continuous. $f : [0, 1] \to \mathbb{R}$. $x_n = \frac{1}{n}, n \geq 1$. Consider

$$\rho(x_n, x_{n+1}) = \mid x_n + x_{n+1} \mid = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \to 0 \text{ as } n \to \infty.$$

and,

$$\rho(f(x_n), f(x_{n+1})) = \mid f(x_n) - f(x_{n+1}) \mid = \mid n - (n+1) \mid = 1.$$

Hence this map is continuous but not uniformly continuous (the same $\delta$ won't work across all $x_n, x_{n+1}$ pairs).

**Definition 2.** f is not uniformly continuous $\Leftrightarrow \exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x, y \in X \ \rho_X(x, y) < \delta$ such that $\delta_Y(f(x), f(y)) \geq \varepsilon$

**Theorem 1.** Assume that $f : X \to Y$ is continuous and $X$ is compact. Then, $f$ is uniformly continuous.

*Proof.* By contradition. Assume that $f : X \to Y$ is continuous and $X$ is comapct but $f$ is not uniformly continuous. Then, $\exists\, \varepsilon_0 > 0$ such that $\forall \delta > 0\, \exists x, y \in X$, $\rho_X(x, y) < \delta$ and $\rho_Y(f(x), f(y)) \geq \varepsilon_0$. We can choose a construction like so

1. $x_1, y_1 \in X$ such that $\rho_X(x_1, y_1) < 1$ and $\rho_Y(f(x_1), f(y_1)) \geq \varepsilon_0$
2. $x_2, y_2 \in X$ such that $\rho_X(x_2, y_2) < \frac{1}{2}$ and $\rho_Y(f(x_2), f(y_2)) \geq \varepsilon_0$.
3. $\ldots$
4. $x_n, y_n \in X$ such that $\rho_X(x_n, y_n) < \frac{1}{n}$ and $\rho_Y(f(x_n), f(y_n)) \geq \varepsilon_0$

Since $X$ is compact, then $\exists$ a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \to x^*$ as $j \to \infty$. Then $f \in C(x^*) \Rightarrow$ $\delta_0 > 0$ such that

$$f(B^X_{\delta_0}(x^*)) \subseteq B^Y_{\frac{\varepsilon_0}{2}}(f(x^*)).$$

Since $x_{n_j} \to x^*$ in $X$ as $j \to \infty$ then $\exists j_0 \geq 1$ such that $x_{n_j} \in B_{\frac{\delta_0}{2}}(x^*)\,, \forall j \geq j_0$. Now choose $j_0 \geq 1$ larger if necessary so that

$$\frac{1}{n_j} \leq \frac{\delta}{2}, \forall j \geq j_0.$$

Then by triangle inequality

$$\begin{aligned}
\rho_X(y_{n_j}, x^*) &\leq \rho_X(x^*, x_{n_j}) + \rho_X(x_{n_j}, y_{n_j}) \\
&< \frac{\delta_0}{2} + \frac{1}{n_j} \\
&< \frac{\delta_0}{2} + \frac{\delta_0}{2} \\
&= \delta_0.
\end{aligned}$$

$\Rightarrow$

$$x_{n_j}, y_{n_j} \in B^X_{\frac{\delta_0}{2}}(x^*), \forall j \geq j_0.$$

$\Rightarrow$

$$f(x_{n_j}), f(y_{n_j}) \in B^Y_{\frac{\varepsilon_0}{2}}(y^*).$$

$\Rightarrow$

$$\rho_Y(f(x_{n_j}), f(y_{n_j})) < \varepsilon_0 \forall j \geq j_0.$$

This is a contradiction to the construction of $x_n, y_n, n \geq 1$ since $\rho_Y(f(x_n), f(y_n)) \geq \varepsilon_0 \forall n \geq 1$. $\square$

**Remark.** What we essentially did in the proof above is that we took two sequences $x_i$'s and $y_i$'s that both converge to some point $x^*$ and as they converge the distance $\rho_X(x_n, y_n) \to 0$ as $n \to \infty$. So in $Y$, bothe sequences in $Y$ must converge to $f(x^*)$ and similarily the distances in $Y$ of $\rho_Y(f(x_n), f(y_n)) \to 0$.