Probability I Hw 2

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September 23, 2024

1. Are the random variables T_n, P_n, S_n independent?

   Proof. Proof by contradiction
   Let $\Omega = \{T, P, S\}^n$ the possible combinations of tops, pants, and shoes where $P(T) = \frac{1}{2}, P(P) = \frac{1}{3}, P(S) = \frac{1}{6}$ for each week. Then the random variables $T_n, P_n, S_n : \Omega \to \{1, 2, 3, \ldots, n\} \subseteq \mathbb{R}$ represent the number of tops, pants, and shoes respectively that were bought after n weeks where each week is independent of the previous weeks. Thus, $T_n, P_n, S_n \sim \mathit{Binomial}(n, p)$.

   For T_n, P_n, S_n to be independent random variables, they must be pairwise independent and jointly independent i.e. $P(T_n = a \cap P_n = b \cap S_n = c) = P(T_n = a) \cdot P(P_n = b) \cdot P(S_n = c)$. We know that $P(T_n = a) = \binom{n}{a}\left(\frac{1}{2}\right)^a\left(\frac{1}{2}\right)^{n-a} = \binom{n}{a}\left(\frac{1}{2}\right)^n$, $P(P_n = b) = \binom{n}{b}\left(\frac{1}{3}\right)^b\left(\frac{2}{3}\right)^{n-b}$, $P(S_n = c) = \binom{n}{c}\left(\frac{1}{6}\right)^c\left(\frac{5}{6}\right)^{n-c}$. However, $P(T_n = a \cap P_n = b \cap S_n = c) = \binom{n}{a}\left(\frac{1}{2}\right)^a \cdot \binom{n-a}{b}\left(\frac{1}{3}\right)^b \cdot \binom{n-b-a}{c}\frac{1}{6}^c$ with the given constraint that $c = n - a - b$ else the probability is 0. Clearly, the left hand side and right hand side are not the same.

   For example, consider the $P(T_n = n)$ i.e. when all the items chosen after n weeks are tops, $P(T_n = n) =$ no. ways$\cdot$ probability $= 1 \cdot \frac{1}{2}^n$. Then $P(T_n = n \cap P_n = b \cap S_n = c) = 0$ when $b, c > 0$. However, suppose $b, c = 1$, $P(T_n = n) \cdot P(P_n = 1) \cdot P(S_n = 1) = \frac{1}{2}^n \cdot \binom{n}{1}\frac{1}{3}\frac{1}{3}^{n-1} \cdot \binom{n}{1}\frac{1}{6}\frac{5}{6}^{n-1} \neq 0$!
   Thus, the three random variables are not independent! $\square$

2. In the same setup as problem 1, compute $E[T_n - P_n]$ and $Var(T_n - P_n)$.

   Proof. The expected value is additive, so $E[T_n - P_n] = E[T_n] - E[P_n]$. We know $E[T_n] = \frac{n}{2}$ and $E[P_n] = \frac{n}{3}$, so $E[T_n - P_n] = \frac{n}{2} - \frac{n}{3} = \frac{n}{6}$.

   Variance is not additive, so we have to consider the random variable $D_n = T_n - P_n$. We know that $P(D_1 = 1) = P(T_n = a \cap P_n = a - 1) = \binom{n}{a}\left(\frac{1}{2}\right)^a \cdot \binom{n-a}{a-1}\left(\frac{1}{3}\right)^{a-1} \cdot \frac{1}{6}^{n-2a+1}$

   $$\begin{aligned} Var(T_n - P_n) &= E[(T_n - P_n)^2] - E[T_n - P_n]^2 \\ &= . \end{aligned}$$

   $\square$

3. Show that $P[R \geq 11] \leq \frac{1}{2}$.

   Proof. We toss a fair coin 1000 times, so $\Omega = \{H, T\}^{1000}$ where each toss is independent of all others. The probability that the largest consecutive run is exactly r is bounded by $P[R = r] =$

   Why isn't this true??? $P[R = r] = (1000 - r + 1)! \cdot \frac{1}{2}^r \cdot \frac{1}{2}^{1000-r} = (1000 - r + 1)! \cdot \frac{1}{2}^{1000}$. $\square$

4. Party that fits 20 people, 24 people are invited

   (a) What is the expected number of people who will attend and the probability that all attendees fit inside the venue if the probability is $\frac{5}{6}$ that any one person shows up?

   Note. Since, each person is indpendent of each other and they each have $\frac{5}{6}$ probability to show up. The number of people who show up $X \sim \mathit{Binomial}(24, \frac{5}{6})$, so $E[X] = \frac{5}{6} \cdot 24 = 20$.

   The probability that all attendes will fit inside the venue is $P[X \leq 20] = 1 - P[X \geq 21] = 1 - (\binom{24}{21}\frac{5}{6}^{21}\frac{1}{6}^3 + \binom{24}{22}\frac{5}{6}^{22}\frac{1}{6}^2 + \binom{24}{23}\frac{5}{6}^{23}\frac{1}{6}^1 + \frac{5}{6}^{24}) \approx 0.584$

   (b) Suppose now that 24 people come in groups of 6 with probability $\frac{5}{6}$, what is the expected number of people that will arrive and the probability that all attendees fit in the venue.

   Note. There are now 4 groups of people with 6 people each and each group of people has $\frac{5}{6}$ chance to come. Now, X represents the number of groups and $E[X] = 0 \cdot \frac{5}{6}^4 + 6 \cdot \frac{5}{6}^1\frac{1}{6}^3 + 12 \cdot$

$\frac{5}{6}^2 \frac{1}{6}^2 + 18 \cdot \frac{5}{6}^3 \frac{1}{6}^1 + 24 \cdot \frac{5}{6}^4 = 13.56$

The probability that all the attendees fit inside the venue is now $P[X \leq 3] = \sum_{k=1}^{3} \frac{5}{6}^k \cdot \frac{1}{6}^{4-k} = 0.1196$

5. Suppose we have r balls to be distributed among n bins. Each of n^r configurations are equally likely. For any $k \in \{1, 2, \ldots, n\}$, calculate the probability first k bins are empty.

   Proof. Let K be the random variable denotating the first k bins that are empty. If the configurations are equally likely, then $P[K = k] = \frac{\text{no configurations with first k bins empty}}{\text{total configurations}} P[K = k] = \frac{(n-k)^r}{n^r} = \frac{n-k}{n}^r$

   Why is this wrong??? Let K be the random variable denotating the first k bins that are empty. If the configurations are equally likely, then $P[K = k] = \frac{\text{no configurations with first k bins empty}}{\text{total configurations}} = \frac{\binom{n+r-k-1}{r}}{\binom{n+r-1}{r}}$ □