# Analysis I

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October 21, 2024

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# 1 Sets

Given two sets $A, B \subseteq X$:

- The **union** is $A \cup B = \{x \in X | x \in A \text{ or } x \in B\}$
- The **intersection** is $A \cap B = \{x \in X | x \in A \text{and } x \in B\}$
- The **difference**, $A \setminus B = \{x \in X | x \in A \text{ but } x \notin B\}$
- The **cartesian (or "direct") product** of the two sets is $A \times B = \{(a, b) | a \in A, b \in B\}$. For example, the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

**Definition 1.** A **countable** set is a set where each element can be mapped to a unique element of $\mathbb{N}$. A **countably infinite** set is a set that is isomorphic to $\mathbb{N}$. A **uncountable** set is a set that is not isomorphic to $\mathbb{N}$.

**Theorem 1.** Let $A_\alpha \subseteq X, \alpha \in I$ where I is a set of indicies. Then an element belongs to the intersection of all elements common to $X$ $A_\alpha$ only when it is not in the union of all $A_\alpha$:

$$X \setminus (\cup_{\alpha \in I} A_\alpha) = \cap_{\alpha \in I} (X \setminus A_\alpha).$$

Similarly, an element is NOT in all the subsets $A_\alpha$ but is in $X$ when it is in any of set difference $X \setminus A_\alpha$

$$X \setminus (\cap_{\alpha \in I} A_\alpha) = \cup_{\alpha \in I} (X \setminus A_\alpha).$$

**Remark.** This is similar to the idea of a center of a group in group theory: $Z(G)$ in the sense that elements must have a universal property within the set.

# 2 Vector Spaces

**Definition 2.** A (**real**) vector space $(X, "+", "\cdot")$ is a set $X$ with two operations.

1. Addition: $+ : X \times X \to X,\ (x, y) \Rightarrow x + y$.
2. Scalar Multiplication: $\cdot : \mathbb{R} \times X \to X,\ (\alpha, x) \to \alpha x$. (generalizes to $F$ a field)

that satisfy the following axioms:

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. $\exists 0,\ \text{s.t.} \forall x \in X, x + 0 = x$.
4. For any $x \in X, \exists - x,\ \text{s.t.}\ x + (-x) = 0$.
5. $\forall \alpha \in \mathbb{R}, \forall x, y, \in X,\ \text{then}\ \alpha(x + y) = \alpha x + \alpha y$
6. $\forall \alpha, \beta \in \mathbb{R}, \forall x \in X, (\alpha + \beta)x = \alpha x + \beta x$
7. $\forall \alpha, \beta \in \mathbb{R}, \forall x \in X, (\alpha\beta)x = \alpha(\beta x)$
8. $1 \cdot x = x(\forall x \in X)$

**Remark.** The above definition of a vector space is only for **REAL vector spaces** where the scalar $\alpha \in \mathbb{R}$, but **in general a vector space could have the scalar $\alpha \in F$ where $F$ is any arbitrary field**, such as the complex numbers or otherwise.

**Remark.** In this course we will be dealing with real and more generally continuous vector spaces.

# 3 Vector Space $\mathbb{R}^n$

$\mathbb{R}^n = \{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} | x_1, \ldots, x_n \in \mathbb{R}\}$

Addition: $\vec{x}, \vec{y} \in \mathbb{R}^n, \vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$

Multiplication: $\vec{x} \in \mathbb{R}^n, \alpha \in \mathbb{R} : \alpha x = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$

$R^n$ is a vector space (prove axioms easily).

**Remark.** I may forget the $\vec{x}$ and only write $x$ but I mean the same.

**Definition 3.** The **dot** product (aka **euclidean scalar product** of x,y in $\mathbb{R}^n$ is a map, $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined as:

$$\langle x, y \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{k=1}^{n} x_ky_k.$$

with the following properties:

1. **Linearity** $\forall \alpha, \beta \in \mathbb{R}, \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
2. **Symmetry** $\langle x, y \rangle = \langle y, x \rangle$
3. **Positive Semi-Definiteness** $\forall x \in \mathbb{R}^n, \langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow \vec{x} = 0$.

**Remark.** This is a special case of the inner product.

**Definition 4.** For a general vector space, if

$$\langle , \rangle : X \times X \to \mathbb{R}.$$

satisifies the properties of the euclidean scalar product, then it is called a euclidean scalar product in X.

**Definition 5.** The above vector space with a Euclidean Scalar Product is a vector space with a defined inner product over the real numbers, and is known as a **Euclidean Vector Space**.

**Remark.** This is different from a **Euclidean Domain** which is a concept in Number Theory of an Integral Domain equipped with a Euclidean Algorithm.

**Definition 6.** Cauchy Shwartz Inequality $\forall x, y \in \mathbb{R}^n$:

$$\langle x, y \rangle \leq |x||y|.$$

where $|x| = \sqrt{\langle x, x \rangle}$. Note this is a special case of a **norm**.

*Proof.* Take $x, y \in \mathbb{R}^n$.

$\forall \alpha \in \mathbb{R}$,

$$\langle \alpha x + y, \alpha x + y \rangle \geq 0..$$
$$= \alpha \langle x, \alpha x + y \rangle + \langle y, \alpha x + y \rangle .$$
$$= \alpha(\alpha \langle x, x \rangle + \langle y, x \rangle) + \alpha \langle x, y \rangle + \langle y, y \rangle .$$
$$= \alpha^2 \langle x, x \rangle + \alpha \langle y, x \rangle + \alpha \langle x, y \rangle + \langle y, y \rangle .$$
$$= \alpha^2 |x|^2 + 2\alpha \langle x, y \rangle + |y|^2 \geq 0.$$

This is a quadratic, so we can take the discriminant (or use the quadratic formula) to get the inequality. We know the discriminant is non-positive because there is at most 1 root, which is 0 by the quadratic being $\geq 0$.
Recall that the discrimanant is: $b^2 - 4ac$:

$$(2\langle x, y\rangle)^2 - 4|x|^2|y|^2 \leq 0.$$
$$2\langle x, y\rangle \leq 2|x||y|.$$
$$\langle x, y\rangle \leq |x||y|.$$

□

**Definition 7.** The **norm** in general is defined as follows. Let $X$ be a real-vector space and let us have a map:

$$\begin{aligned} \|\cdot\| \colon X &\longrightarrow \mathbb{R} \\ x &\longmapsto \| x \| . \end{aligned}$$

which satisfies the following properties. Properties:

- $\forall x \in \mathbb{R}^n$, $|x| \geq 0$ and $|x| = 0 \Leftrightarrow x = 0$
- $\forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$, $|\alpha x| = |\alpha||x|$
- **Triangle inequality**: $\forall x, y \in \mathbb{R}^n$, $|x + y| \leq |x| + |y|$.

Correspondingly, $(X, \|\cdot\|)$ is called a **normed space**.

**Definition 8.** The **euclidean norm** in $\mathbb{R}^n$ is defined as

$$|x| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{k=1}^{n} x_k^2}$$

The euclidean norm satisfies the cauchy swhartz inequality and all 3 properties above!

*Proof.* Proof of Triangle Inequality

$$\begin{aligned} |x + y|^2 &= \langle x + y, x + y\rangle \\ &= |x|^2 + |y|^2 + 2\langle x, y\rangle \\ &\leq |x|^2 + |y|^2 + 2|x||y| \text{ by the cauchy shwartz inequality} \\ &\leq (|x| + |y|)^2. \end{aligned}$$

Thus, $|x + y| \leq |x| + |y|$ □

**Remark.** Example

$$\text{Let} X = \mathbb{R}^n.$$

Other Examples of defined norms include:

$$|| \, x \, ||_p = \left( \sum_{k=1}^{n} x_k^p \right)^{\frac{n}{p}}.$$

$$|| \, x \, ||_\infty = \max_{1 \leq k \leq n} |x_k| \text{ where x} = (x_1, \ldots, x_n).$$

$$|| \, x \, ||_1 = \sum_{k=1}^{n} |x_k|.$$