HOMEWORK 1 FOR MATH 7241, FALL 2024. DUE SEPTEMBER 19TH

1. Let Ω be a sample space, and $\mathbb{P}$ be a probability function on Ω. Suppose that $\{A_i\}_{i\in\{1,\ldots,n\}}$ is a finite collection of (not necessarily disjoint) subsets of Ω. Use the axioms of probability functions to show that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i).$$

Hint: If you are unsure where to start, try proving this for $n = 2$, and continuing from there.

2. Suppose that, for some constant two constants c_1 and c_2, the function

$$f_X(s) = \begin{cases} \frac{c_1}{\sqrt{s}} + c_2 s & s \in (0,4) \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function of some random variable X. Furthermore, $\mathbb{P}[X \leq 1] = 1/3$. What is the value of c_1 and c_2?.

3. Let N be a random variable that takes on non-negative integer values. Show that

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}[N \geq n].$$

4. Let X and Y be two random variables such that

$$\mathbb{P}[X = i] = \mathbb{P}[Y = i] = \begin{cases} 1/3, & i \in \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}.$$

Furthermore, assume that $\mathbb{P}[X = 1, Y = 1] = 1/12$, $\mathbb{P}[X = 2, Y = 3] = 1/12$, and $\mathbb{P}[X = 3, Y = 2] = 1/4$.

- What is the probability that $X = Y$?
- What is the *smallest* possible value of $\mathbb{P}[X = 2, Y = 1]$?

5. Consider a stick of length 1 unit. We pick a uniformly chosen point along the stick, and break it there. Let L be the length of the longer piece.

- What is the probability density function $F_L(s)$?
- What is the expected value of L?
- What is the variance of L?