HOMEWORK 2 FOR MATH 7241, FALL 2024. DUE SEPTEMBER 26TH

1. Every week, you go out shopping for one item of clothes. Each week you make a decision, independent of all other weeks: with probability 1/2, you buy a top; with probability 1/3, you buy a pair of pants; with probability 1/6, you buy a pair of shoes. Let T_n, P_n, and S_n be the number of tops, pants, and shoes (respectively) that you buy after n weeks. Are the random variables T_n, P_n and S_n independent random variables? Prove it or provide a counterexample

2. In the same setup as Problem 1, compute $\mathbb{E}[T_n - P_n]$ and $\mathrm{Var}(T_n - P_n)$.

3. Suppose you toss a fair coin 1000 times, where each toss is independent of all others. Let R be the random variable that counts the longest consecutive run of 'Heads' (so, for example, if the sequence of tosses was $(H, H, T, T, H, T, H, H, H, H, T, T, \dots T)$, $R = 4$). Show that $\mathbb{P}[R \geq 11] \leq 1/2$.

Hint: Do not compute this probability exactly! Instead, write the event $\{R \geq 11\}$ as a union of simple events, and use the union bound (problem 1 from the previous homework).

4. You are planning a party, and you know the venue fits 20 people.
   - Suppose you invite 24 people, knowing that each person has a probability of 5/6 of arriving, and that the decisions are taken independently for each person. What is the expected number of people who attend, and what is the probability that all the attendees fit inside the venue? You may round this to the nearest 1000th.
   - Next, suppose that you invite 24 people, but they are made up of six friend-groups, with four people in each. Either the entire friend group shows up (with probability 5/6) or no member of the friend group shows up. What is the expected number of people who attend, and what is the probability that all the attendees fit inside the venue? You may round this to the nearest 1000th.

5. Suppose we have r balls, to be distributed among n bins. Each of the n^r possible configurations is equally likely. For any $k \in \{1, 2, \dots n\}$, calculate the probability that the first k bins are empty.