HOMEWORK 8 FOR MATH 7241, FALL 2024. DUE NOVEMBER 7TH

1. Let $\{X_n\}$ be an irreducible, aperiodic finite-state Markov chain with transition matrix $P = (p_{i,j})_{i,j}$, and a stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$. Let Y_n keep track of the *two* previous states — that is, $Y_n = (X_{n-1}, X_n)$. Show that Y_n is a Markov chain, and compute its stationary distribution (in terms of P and π).

2. Let us consider a very simple version of monopoly: there are 4 squares, one marked 'GO,' the next 'Baltic', the third 'Free Parking', and the fourth 'Boardwalk,' arranged in a circle. At any turn, we toss two fair coins; we advance 1 square if both are tails, 2 if exactly one is heads, and 3 if both are heads. If we start at 'GO', what is the expected time when we first return to 'GO'? What is the expected number of visits to 'Boardwalk' before the first return to 'GO'?

3. Consider two Markov chains with transition matrices

$$P_1 = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix} \quad \text{and } P_2 = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

Check whether each of these chains is reversible.

4. We return to the setup of HW 6, problem 1. There are two jars in front of you, and a total of n balls divided among them. Every minute, you pick a ball uniformly at random, and move it to the other jar. Let X_n denote the number of balls in the first jar. Show that the Markov chain is reversible and find its stationary distribution.

5. Consider the two-state Markov chain given by the transition probabilities

$$P = \begin{pmatrix} 0.1 & 0.9 \\ 0.95 & 0.05 \end{pmatrix}.$$

- Compute the stationary distribution of this irreducible, aperiodic Markov chain.
- Find the eigenvalues and row eigenvectors of P.
- Show that the distance to mixing at time 13 is smaller than 1/16.