Commutative Algebra

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Contents

Just Randomly dropped in lmao

1 Something

Note. Examples of Spec(A) and mSpec(A)

1. k field, $A = k[x]$, then $Spec(A) = \{(0)\} \cup \{(f)\,|f$ irreducible$\}$, $Spec(A) \backslash \{(0)\}$. When k algebraically closed, the irreducible polynomials in $k[x]$ are of the form $b(x-a)$, $a, b \in k \to spec(k[x]) = k$.

2. What if k not algebraically closed? $k = \mathbb{R}$. We set nonlinear irreducible $f \in k[x]$. Goal: determine a geometric interpretation of $(f) \in Spec(k[x])$ where $deg(f) > 1$ irreducible. Let $k[x, y]$, then $(0) \subseteq (X) \subseteq (x, y)$ where all ideals in this chain are prime, but only (x, y) maximal, $Spec(k[x, y]) \neq mSpec(k[x, y])$. Hw: prove $([x.y])$ is maximal. Generally, if $(a.b) \in k^2$, the ideal generated by $x - a$ and $y - b$:
$$M_{(a,b)} = (x - a, y - b)\,.$$
is maximal in $k[x, y]$. We can make a similar identitfication with at least some part of $mSpec(k[x, y])$ with k^2, the k-plane.

3. etc.

Highkey lost rn

Definition 1. Given a field k, then a field F containing k so that addition and multiplication agree, we call F a field extension of k and k the base field and write:
$$\frac{F}{k} = \text{ f over k.}$$

Note. Given a field k, there is no natural way to give the set k^2 the structure of a field:
$$(0, 1) \cdot (1, 0) = (0, 0) \text{ nonzero zero divisors and not a field.}$$
This is not to say that there are not definitions that turn k^2 into a ring: Consider $\mathbb{R}^2$ and define:
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$
$$(a + bi)(c + di) = (ac - bd) + i(ad + bc).$$

Remark. Holy!!!
$\mathbb{R}^2$ with the multiplication defined above is a field extension of $\mathbb{R}$ i.e. commonly known as $\mathbb{C}$:
$$\mathbb{C} \cong \frac{\mathbb{R}[x]}{x^2 + 1} = \mathbb{R}[i].$$
And this generalizes for other k, the underlying set!

2 Nilpotent elements and Nilradical

Definition 2.
1. $x \in A$ is nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$.
2. the set of all nilpotent elements of A is called the **nilradical** of A and denoted:
$$nilrad(A) = \{a \in A | a^n = 0, \text{some n} > 0\}.$$
3. A is reduced if nilrad(A) $=$ (0) :

Proposition 1. A integral domain $\to$ A reduced.

Proposition 2. Furthermore $\frac{A}{nilrad(A)}$ is always reduced.

Proposition 3. $nilrad(A) = \cap_{P \in Spec(A)} P$

Definition 3. The jacobson radical of ring A is the intersection of all maximal ideals of A.

$$Jrad(A) = \cap_{M \in mSpec(A)} M.$$

Proposition 4. $x \in Jrad(A) \Leftrightarrow 1 - xy \in A^x$ for all $y \in A$.

Proof. $\rightarrow$ Assume $x \in jrad(A)$ and $1 - xy \notin A^x$ some $y \in A$. Then we know that $1 - xy$ must be contained in some maximal ideal of A, denote it as $M \subsetneq A$ maximal ideal (any non-unit has to be in some maximal ideal). Since $x \in M$, then $xy \in M$ (since x is in an ideal). Then, $(1 - xy) + xy = 1 \in M$, so $M = A$ and hence is not a maximal ideal, a contradiction, so $1 - xy$ can't be contained in a maximal ideal of A.

$\leftarrow$ Suppose $x \notin M$ some maximal $M \subseteq A$ then the ideal generated by x and M must be all of A then $u + xy = 1$ for some $u \in M$ and $y \in A \Leftrightarrow 1 - xy \in M$ and hence $1 - xy \notin A^x$. $\square$

Definition 4. $I \subseteq A$ ideal, the **radical** of the ideal I is the set

$$\sqrt{I} = rad(I) = \{x \in A | x^n = I, \text{ some n} > 0\}.$$

Remark. In some sense, radical is a generalization of nilradical

$$nilrad(A) = rad((0)), (0) \subseteq A.$$

Proposition 5. $rad(I) = \pi^{-1}(nilrad(\frac{A}{I}))$ where $\phi : A \rightarrow \frac{A}{I}$

Proof. $x \in rad(I) \Leftrightarrow x^n \in I$ some $n \in N$, so $\pi(x^n) = \pi(x)^n = (x + I)^n = x^n + I = O_{\frac{A}{I}} = \pi(O_A)$.
I'm lost $\square$