Analysis I - Hw 1

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1 Problem 1

Let (X, ρ) be a metric space and E, a non-empty subset in X. Consider the new metric space, (E, ρ). Prove that a set $U \subseteq E$ is open in $E \Leftrightarrow \exists$ an open set $\tilde{U}$ in X such that $U = \tilde{U} \cap E$. Similarly, prove this for closed sets.

Proof. $\rightarrow$ Assume that a set $U \subseteq E$ is open in $E \subseteq X$ with $(X, \rho), (E, \rho)$ being the respective metric spaces. This means that $\forall x \in U\ \exists r > 0$ such that

$$B_r(x) = \{y \in E \mid \rho(x, y) < r\} \subseteq U.$$

If U is also open in X i.e. $U = \tilde{U}$ then we are done. Assume that U is not also open in X. This means now that, $\exists x \in U$, $\forall r_i > 0$,

$$B'_{r_i}(x) = \{y \in X \mid \rho(x, y) < r\} \not\subseteq U.$$

This is only possible because for any $B_{r_i}(x)' \setminus B_r(x) \subseteq X$ and $B_{r_i}(x)' \setminus B_r(x) \not\subseteq U$.

$$B_{r_i}(x)' \setminus B_r(x) \subseteq X \setminus U.$$

and,

$$B_{r_i}(x)' \setminus B_r(x) \subseteq X \setminus E.$$

so,

$$B_{r_i}(x)' \cap E = B_r(x).$$

Now, choose $r \in \{r_1, r_2, \ldots\}$ where $r_i > 0$. Then,

$$\tilde{U} = B_r(x)' \cup U = B_r(x)' \cup (U \setminus B_r(x)).$$

is open. This can be seen because for any point $y \in B_r(x)'$, we can generate:

$$B_{r'}(y) = \{z \in \tilde{U} \mid \rho(y, z) > r' = \rho(x, z) - r\}.$$

We know by the triangle inequality that

$$\begin{aligned}
\rho(x, z) &\leq \rho(x, y) + \rho(y, z) \\
&< r + \rho(y, z) \\
\rho(y, z) &> p(x, z) - r.
\end{aligned}$$

So, for any point in $y \in B_r(x)'$ we can generate a new ball $B_{r'}(y)$ so that:

$$B_{r'}(y) \subseteq B_r(x)' \subseteq \tilde{U}.$$

Since we can do this for any $x_j \in X$ such that $\forall r_i > 0$, $B'_{r_i}(x_j) = \{y \in X \mid \rho(x, y) < r\} \not\subseteq U$. We will refine our construction to select $\hat{r}_j \in \{r_1, \ldots\}$ for each of the balls above. Denote this set as $B' = \{B_{\hat{r}_j}(x_j)'\}$

$$\tilde{U} = \cup_{\alpha \in B'} B_{\hat{r}_j}(x_j)' \cup U = \cup_{\alpha \in B'} B_{\hat{r}_j}(x_j)' \cup (U \setminus \cup_{x_j \in X} B_r(x_j)).$$

Since each of the balls $B_{\hat{r}_j}(x_j)$ is open and the same argument follows that for any point we can take a ball of a small enough radius and since we know $U \setminus B_r(x)$ contains only points with open balls, then the whole set $\tilde{U}$ must be open. We know that each of the $B_r(x)' \setminus B_r(x) \in X \setminus E$, so:

$$\tilde{U} \cap E = B_r(x) \cup U \setminus B_r(x) = U.$$

$\leftarrow$ Assume that $\exists$ an open set $\tilde{U} \subseteq X$ such that $U = \tilde{U} \cap E$. By set theory, $U \subseteq E$. We know that $\tilde{U}$ is open, so

$$\forall x \in \tilde{U}, \exists r > 0, \text{ such that } B_r(x) \subseteq \tilde{U}.$$

Consider the ball $B_r(x) \cap E$, the intersection of the ball $B_r(x) \subseteq X$ with E. We know that this intersection must be open in E because we are taking an open set in X and intersecting it with E which is open with respect to itself. Since $B_r(x) \subseteq \tilde{U}$, then $B_r(x) \cap E \subseteq \tilde{U} \cap E = U$. So for every $x \in U$, $\exists r > 0$ such that an open set is formed in E:

$$B_r(x) \cap E \subseteq U.$$

$U \subseteq E$ is open $\square$

Note. Intuition
Consider $U \subseteq E$ an open set. This means that $\forall x \in U\ \exists r > 0$ such that

$$B_r(x) \subseteq U.$$

If we consider U in $X \supseteq E$, then U may not be necessarily be open in X because there may $x' \in X$ but $x' \notin E$ such that $\exists B_r(x)\ \forall r > 0$ such that $x' \in B_r(x)$. Since $B_r(x) \subseteq U \subseteq E$, if $x' \in B_r(x)$, then:

$$B_r(x) \nsubseteq U.$$

For example, let $X = \mathbb{R}$, the closed interval $[0, 1)$ would not be open because there is no ball centered at $B_r(0)$ of any radius greater than 0 that is a subset of $[0, 1)$. However, if we restrict $X = [0, 1)$, then $[0, 1)$ becomes open since $B_r(0) = [0, r) \subseteq [0, 1)$ when $r < 1$ (now the $x < 0$ don't exist). The idea is that we can find an open set $\tilde{U}$ in X that is an analogue of U in E.

2 Problem 2

Given $K \subseteq E$, then prove K is compact in $E \Leftrightarrow K$ is compact in X.

Proof. $\rightarrow$ If K is compact in $E \subseteq X$, then for any open cover $\{U_\alpha\}_{\alpha \in I}$ in E such that $K \subseteq \cup_{\alpha \in I} U_\alpha$ that covers K and there is some finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\} \subseteq E$ that covers K. Since $E \subseteq X$, we can use the statement from problem 1. For each open subset $U_{\alpha_i} \subseteq E$, there exists open subset $U_{\beta_i} \subseteq X$ such that $U_{\alpha_i} = U_{\beta_i} \cap E \rightarrow U_{\alpha_i} \subseteq U_{\beta_i}$. So,

$$K \subseteq \cup_{\alpha_i \in I_1} U_{\alpha_i} \subseteq \cup_{\beta_i \in I_1} U_{\beta_i}.$$

Thus, we can construct a finite subcover $\{U_{\beta_1}, U_{\beta_2}, \ldots, U_{\beta_n}\}$ of K in X. As any open cover of K in X can be restricted $\{U_\beta \cap E\}_{\beta \in I}$ to be an open cover of K in E and any open cover of K in E can be augmented to be an open cover of K in X, $\{\{U_\alpha\}_{\alpha \in I}, X \setminus E\}$, then any open cover in X can be mapped to some open cover in E such that we can follow the construction above to create a finite subcover of K in X. So, K is compact in X.

$\leftarrow$ Assume that K is compact in X and $K \subseteq E$. This means for any open cover of K in X:

$$\{U_\alpha\}_{\alpha \in I}.$$

there exists a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ such that

$$K \subseteq \cup_{\alpha \in I_1} U_\alpha \subseteq X.$$

Then since $E \subseteq X$, we can use the statement from problem 1. For each open subset $U_{\alpha_i} \subseteq X$, there exists open subset $U_{\beta_i} \subseteq E$ such that $U_{\beta_i} = U_{\alpha_i} \cap E \rightarrow U_{\beta_i} \subseteq E$. And since, $K \subseteq E$,

$$U_{\alpha_i} \cap K \subseteq U_{\alpha_i} \cap E = U_{\beta_i}.$$

so,

$$(\cup_{\alpha \in I_1} U_{\alpha_i}) \cap K = K \subseteq (\cup_{\alpha \in I_1} U_{\beta_i}) \cap E = \cup_{\beta \in I_1} U_{\beta_i}.$$

and $\{U_{\beta_1}, \ldots, U_{\beta_n}\}$ is an finite subcover of K in E. Then the open cover of K in E would just be $\{U_\beta\}_{\beta \in I}$. As any open cover of K in X can be restricted $\{U_\alpha \cap E\}_{\alpha \in I}$ to be an open cover of K in E and any open cover of K in E can be augmented to be an open cover of K in X, $\{\{U_\beta\}_{\beta \in I}, X \setminus E\}$, then for any open cover in E which can be mapped to some open cover in X, we can follow the construction above to create a finite subcover of K in E and K is compact. $\square$

Due October 2nd