# Hw 8

Daniel Yu

November 6, 2024

1. Let $X_n$ be an irreducible, aperiodic finite-state Markov chain with transition matrix $P = (p_{i,j})_{i,j}$, and a stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$. Let $Y_n$ keep track of the two previous states — that is, $Y_n = (X_{n-1}, X_n)$. Show that $Y_n$ is a Markov chain, and compute its stationary distribution (in terms of P and $\pi$).

---

*Proof.* Since $X_n, X_{n-1}$ are previous states in the Markov Chain, then they will follow the transition matrix $P$ and have the same markov chain dynamics.

Notice that $Y_n$ is a markov chain because $P[Y_n \mid Y_{n-1}, Y_{n-2}, \ldots, Y_1] = P[Y_n \mid Y_{n-1}]$ because:

$$\begin{aligned} P[Y_n \mid Y_{n-1}, Y_{n-2}, \ldots, Y_1] &= P[(X_{n-1}, X_n) \mid (X_{n-2}, X_{n-1}), (X_{n-3}, X_{n-2}) \ldots, (X_2, X_1)] \\ &= P[(X_{n-1}, X_n) \mid (X_{n-2}, X_{n-1})] \\ &= P[Y_n \mid Y_{n-1}]. \end{aligned}$$

Now consider, the distribution of $Y_n$.

$$\begin{aligned} P(Y_n = (c,d) \mid Y_{n-1} = (a,b)) &= P[(X_{n-1}, X_n) = (c,d) \mid (X_{n-2}, X_{n-1}) = (a,b)] \\ &= P[(X_{n-1}, X_n) = (c,d) \mid (X_{n-2}, X_{n-1}) = (a,b), X_{n-1} = X_{n-1}] + \\ &\quad P[(X_{n-1}, X_n) = (c,d) \mid (X_{n-2}, X_{n-1}) = (a,b), X_{n-1} \neq X_{n-1}] \\ &= P[(X_{n-1}, X_n) = (b,d) \mid (X_{n-2}, X_{n-1}) = (a,b), X_{n-1} = X_{n-1}] + 0 \\ &= P[X_n = d \cap X_{n-1} = b \cap X_{n-2} = a] \\ &= P[X_n = d \mid X_{n-1} = b \cap X_{n-2} = a] \cdot P[X_{n-1} = b \mid X_{n-2} = a] \cdot P[X_{n-2} = a] \\ &= P[X_n = d \mid X_{n-1} = b] \cdot P[X_{n-1} = b \mid X_{n-2} = a] \cdot P[X_{n-2} = a] \\ &= P[X_n = d \mid X_{n-1} = b] \cdot P[X_{n-1} = b \mid X_{n-2} = a] \left( \sum_{i \in \Omega} P[X_{n-2} = a \mid X_0 = i] \right) \\ &= P_{b,d} P_{a,b} \left( v \cdot P^{n-2} \right)_a \\ &\text{as } n \to \infty \\ &= P_{b,d} P_{a,b} \pi_a. \end{aligned}$$

The probability matrix $P_{Y_n}((b,c),(a,b)) = P_{b,c} P_{a,b}$ and the stationary distribution is $\pi = P_{b,d} P_{a,b} \pi_a$. ☐

2. Let us consider a very simple version of monopoly: there are 4 squares, one marked 'GO,' the next 'Baltic', the third 'Free Parking', and the fourth 'Boardwalk,' arranged in a circle. At any turn, we toss two fair coins; we advance 1 square if both are tails, 2 if exactly one is heads, and 3 if both are heads. If we start at 'GO', what is the expected time when we first return to 'GO' ? What is the expected number of visits to 'Boardwalk' before the first return to 'GO' ?

---

*Proof.* The transition matrix would be

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}.$$

The markov chain would be of the form below.

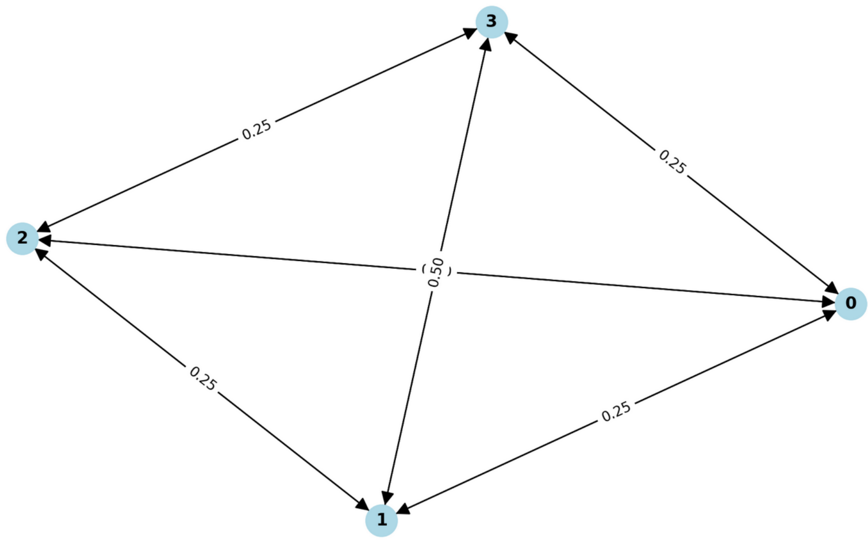


Figure 1: Markov Chain

Notice that $E[\text{ GO} \mid \text{GO}]$ is just the sojurn time: $E[S_{\text{GO}} \mid X_0 = \text{GO}] = \frac{1}{\pi_{\text{GO}}}$.

Since $P$ is doubly stochastic, then $\pi$ is a constant vector, and $\pi = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$. Thus,

$$E[S_{\text{GO}} \mid X_0 = \text{GO}] = \frac{1}{.25} = 4.$$

$\square$