Hw 7

Daniel Yu

October 27, 2024

1. You enter a casino with 100 dollars, and continuously play a fair game: each minute, you either gain a dollar with probability 1/2, or lose a dollar with probability 1/2. You leave the casino when you are out of money, or after you have 400 dollars. What is the probability that you leave the casino with no money.

   Proof. We will denote $q_i = P[\text{leave the casino with no money } \mid \text{intial wealth is } i]$. Then,

   - $q_0 = 1$
   - $q_1 = \frac{1}{2}q_0 + \frac{1}{2}q_2$ (probability you lose your only dollar + probability you lose all from 2 dollars)
   - $q_2 = \frac{1}{2}q_1 + \frac{1}{2}q_3$ (probability you lose to go to q_1 + probability you win to go to go to 3 dollars and lose all)
   - $q_3 = \frac{1}{2}q_2 + \frac{1}{2}q_4$
   - ...
   - $q_{400} = 0$

   We want to solve for q_{100}.

$$\begin{aligned} q_{100} &= \frac{1}{2}q_{99} + \frac{1}{2}q_{101} \\ &= \frac{1}{2}\left(\frac{1}{2}q_{98} + \frac{1}{2}q_{100}\right) + \frac{1}{2}\left(\frac{1}{2}q_{100} + \frac{1}{2}q_{102}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}(\frac{1}{2}q_{97} + \frac{1}{2}q_{99}) + \frac{1}{2}(\frac{1}{2}q_{99} + \frac{1}{2}q_{101})\right) + \frac{1}{2}\left(\frac{1}{2}(\frac{1}{2}q_{99} + \frac{1}{2}q_{101}) + \frac{1}{2}(\frac{1}{2}q_{101} + \frac{1}{2}q_{103})\right) \\ &= \dots . \end{aligned}$$

   Notice the linear relationship:

$$q_{100} = \frac{1}{2}^k \sum_{i=1}^{2^{k-1}} q_{100-k+2i} + \frac{1}{2}^k \sum_{i=1}^{2^{k-1}} q_{100+k-2i}$$

   . This gets extremely complicated.

   We know that that the probability is symmetric for a random walk starting from q_{200}, i.e. $q_{200} = \frac{1}{2}$. Since q_i is a random walk, and only depends on q_{i+1}, we can see that the probability would continue to be symmetric. Notice that

$$q_i = \frac{1}{2}q_{i-1} + \frac{1}{2}q_{i+1} \Rightarrow 2q_i = q_{i-1} + q_{i+1} \Rightarrow q_i - q_{i-1} = q_{i+1} - q_i$$

   . This implies that the probability is an arthimetic series and thus the relationship between terms is linear as there is some constant $q_i = q_{i-1} + c, \forall i \in 1, \dots, 400$! Thus, $q_i = q_0 + c \cdot i = 1 + c \cdot i$. Substitute, $i = 400$

$$q_{400} = 0 = 1 - c \cdot 400 \Rightarrow c = \frac{1}{400}.$$

   and,

$$q_i = 1 - \frac{i}{400}.$$

   Thus, $q_{100} =$ the number of steps from the q_{400} over the total steps $= \frac{300}{400} = \frac{3}{4}$

   □

2. Let a, b be two numbers between 0 and 1. Consider the two-state Markov chain given by the transition probabilities

$$P_{a,b} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}.$$

   (a) For what values of a and b is this chain irreducible?

Proof. Transition matrices P are irreducible when $a = 1$ and $b = 1$ or when $0 < a < 1$ and $0 < b < 1$. When $a = 1, b = 1$ $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $P^2 = I$ identity matrix, so P is still irreducible since it is possible to get from any state to any other state in 2 steps. However, when $P = I$ and $a = 0, b = 0$, then there is no way to travel from 1 state or the other, so P is not irreducible. Finally, when $0 < a < 1, 0 < b < 1$, then it is possible to get to any state to any other state in 1 step, so P is irreducible. □

(b) For what values of a and b is it aperiodic?

Proof. The transition matrix P changes slightly for the markov chain to be aperiodic. P when $a = 1, b = 1$ is no longer a valid transition matrix for a aperiodic markov chain, in fact in this case there is a period of 2. The matrix P are of the forms $0 < a < 1, 0 < b < 1$ since P^N with $N = 1$ now satisfies the periodicity condition and clearly the matrix is also aperiodic when $a = 0, b = 0$ as the period is 1. □

(c) Assuming a and b satisfy the conditions required for the chain to be irreducible, find the stationary distribution of the Markov chain (as a function of a and b)

Proof. We would find the eigenvector associated with one.

$$\begin{aligned}
\vec{v} \cdot P_{a,b} &= 1 \cdot \vec{v} \\
\vec{v} \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} &= \vec{v} \\
\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} &= \begin{pmatrix} x & y \end{pmatrix} \\
\begin{pmatrix} (1-a)x + by & ax + (1-b)y \end{pmatrix} &= \begin{pmatrix} x & y \end{pmatrix}.
\end{aligned}$$

We know that:

$$\begin{aligned}
(1-a)x + by &= x \\
ax + (1-b)y &= y \\
x + y &= 1.
\end{aligned}$$

Solving,

$$\begin{align}
x &= 1 - y \tag{1} \\
-ax + by &= 0 \tag{2} \\
-a(1-y) + by &= 0 \tag{3} \\
-a + ay + by &= 0 \tag{4} \\
y(a+b) &= a \tag{5} \\
y &= \frac{a}{a+b} \tag{6} \\
x &= \frac{b}{a+b}. \tag{7}
\end{align}$$

Thus,

$$\vec{v}^T = \begin{pmatrix} \frac{b}{a+b} \\ \frac{a}{a+b} \end{pmatrix}.$$

□

3. We say a matrix is doubly stochastic if both the rows and columns of the matrix sum up to one. Let P be a doubly stochastic matrix that is the transition probability matrix for a Markov chain with N states. Find a stationary distribution for this Markov chain.

   Proof. Let P be the doubly stochastic matrix that is the transition probability matrix for the markov chain with N states. This means that the rows and columns of P both sum to 1 and all entries are non-negative. From class we proved that $Null(P - I) = span\{\vec{1}\}$ i.e. $P \cdot \vec{1} = \vec{1}$. We can then consider P^T which must now be row-stochastic, so by the same theorem $Null(P^T - I) = \vec{1}$ and $P^T \cdot \vec{1} = \vec{1}$. By transpose law,
   $$P^T \cdot \vec{1} \Rightarrow \vec{1}^T P = \vec{1}^T.$$

Thus, the constant vector is a row-eigenvector and thus a stationary distribution for this markov chain. $\square$

4. Construct a Markov chain with exactly two distinct irreducible intercommunicating classes and at least three different stationary distributions.

Proof. Recall that intercommunicating classes are nodes that can be reached from each other (i.e strongly connected components). We can use this to construct a simple example of a markov chain with two distinct irreducible intercommunicating classes and at least 3 different stationary distributions. The markov chain below would have 2 intercommunicating classes: $C_0 = \{0\}, C_1 = \{1, 2, 3, 4\}$. The stationary distributions would be a combination of the stationary distribution for C_1, which has a period of 4, and is doubly stochastic with $\pi_1 = \begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$ and the stationary distribution for $C_0, \pi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Thus, we get:

$$\pi = \alpha \begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} + (1 - \alpha) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; \forall \alpha \in \mathbb{R}.$$

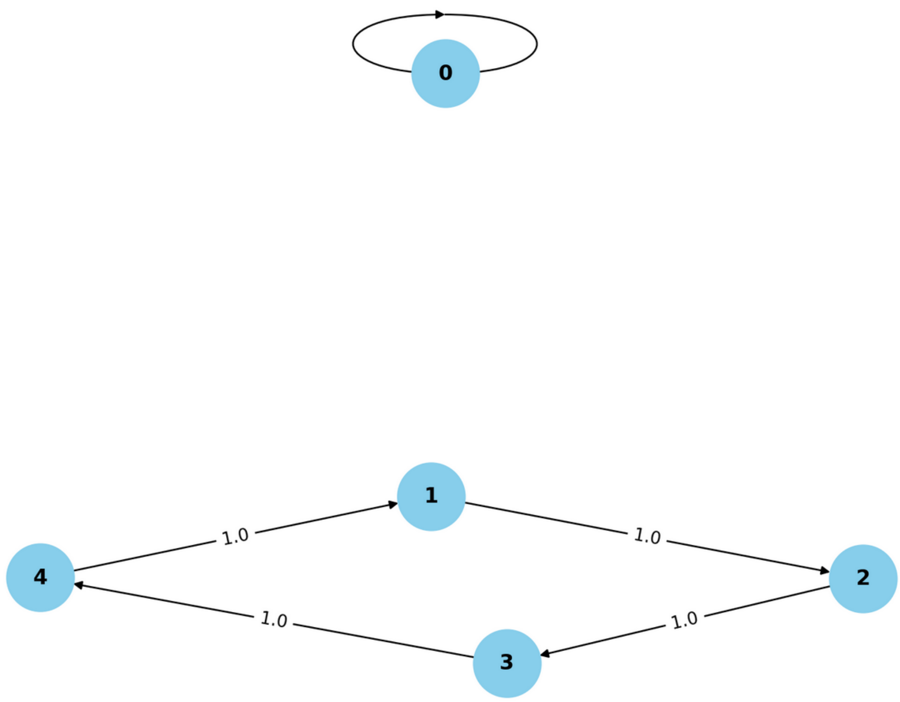

Figure 1: solution

$\square$

5. Assume that we toss a fair, four-sided die repeatedly. Let X_n be the outcome of the nth toss, and $S_n = \sum_{i=1}^{n} X_i$. Compute
$$\lim_{n \to \infty} P[S_n \text{is divisible by 5}].$$

Proof. The 5 states of the markov chain would be the states representing $S_n \mod 5 = \{0, 1, 2, 3, 4\}$. The transition matrix P would be

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}.$$

We know from question 3, that since P is a doubly stochastic matrix, then the stationary distribution vector is just the constant vector.

$$\vec{v} \cdot P = \vec{v}.$$

where $\vec{v} = c \cdot \vec{1}$ and since $\sum_{i=1}^{5} \vec{v}_i = 1$, then $c = \frac{1}{5}$, so

$$\vec{v}^T = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix}.$$

The probability that S_n is divisible by 5 as $n \to \infty$ is $S_n \mod 5 = 0$ so $\lim\limits_{n\to\infty} P[S_n \text{is divisible by } 5] = \vec{v}_1^T = \frac{1}{5}$ representing the probability that this markov chain lands in the state representing a modulus of 0. $\square$