HOMEWORK 7 FOR MATH 7241, FALL 2024. DUE OCTOBER 31ST

1. You enter a casino with 100 dollars, and continuously play a fair game: each minute, you either gain a dollar with probability 1/2, or lose a dollar with probability 1/2. You leave the casino when you are out of money, or after you have 400 dollars. What is the probability that you leave the casino with no money?

Hint: Do not try to compute NR directly (as this would require working with 399×399 matrix). Let $q_i = \mathbb{P}[\text{You leave the casino with no money} \mid \text{your initial wealth is i}]$. What is q_0? What is q_{400}? Use the Markov property to relate different values of q_i.

2. Let a, b be two numbers between 0 and 1. Consider the two-state Markov chain given by the transition probabilities

$$P_{a,b} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}.$$

- For what values of a and b is this chain irreducible?
- For what values of a and b is it aperiodic?
- Assuming a and b satisfy the conditions required for the chain to be irreducible, find the stationary distribution of the Markov chain (as a function of a and b).

3. We say a matrix is doubly stochastic if both the rows and columns of the matrix sum up to one. Let P be a doubly stochastic matrix that is the transition probability matrix for a Markov chain with N states. Find a stationary distribution for this Markov chain.

4. Construct a Markov chain with exactly two distinct irreducible intercommunicating classes and at least three different stationary distributions.

5. Assume that we toss a fair, four-sided die repeatedly. Let X_n be the outcome of the nth toss, and $S_n = \sum_{i=1}^n X_i$. Compute

$$\lim_{n\to\infty} \mathbb{P}[S_n \text{ is divisible by } 5].$$

Hint: The answer can be understood as an entry of the stationary vector of an appropriately formulated Markov chain with 5 states.