Analysis I

Daniel Yu

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Contents

1 Closure of a Set

Definition 1. Let (X, ρ) be a metric space where $E \subseteq X$. Denote $\overline{E}$ as the **closure**.

$$\overline{E} = \cap_{E \subseteq F} F \text{ and F is closed.}$$

And since, the intersection of any number of (including infinite) closed sets is closed.

1. $\overline{E}$ is closed
2. $\overline{E} \supseteq E$
3. If $E \subseteq B$ where B is closed, then $\overline{E} \subseteq B$

   Proof. Since closure of E is defined as the intersection of all closed sets that contain E, then for some B closed that contains E, $\overline{E}$ must be in the intersection of B and all other closed sets containing E, so $\overline{E}$ must be contained in B. □

4. $B \subseteq A \to \overline{B} \subseteq \overline{A}$

   Proof. It follows from the definition, that $\overline{B} \cap_{F' \supseteq B, F\text{closed}} F'$ and $\overline{A} = \cap_{F \subseteq A \subseteq B, F\text{closed}} F$. Any closed set F that contains A must also contain B. Hence, any set F that appears in $\cap_{F \supseteq A, \text{F closed}} F$ appears also in $\cap_{F' \supseteq B} F'$, so $\overline{A} \supseteq \overline{B}$ because $\cap_{F' \supseteq B} F'$ has all F and potentially more closed sets that it's intersecting, so their combined intersections will result in a set that is strictly smaller than the intersection of only Fs □

5. If E is closed $\to \overline{E} = E$

   Proof. If E is closed, then since $E \supseteq E$, $E \subseteq \overline{E}$ and by property (1) $E \subseteq \overline{E}$, so $\overline{E} = E$ □

6. $A \supseteq B \to A' \supseteq B'$ where A', B' represents the limit points of A, B respectively.

Remark. Intuitively, the closure of a set is turning an open set into a closed set, where the closed set is the minimum possible augmentation of the open set.

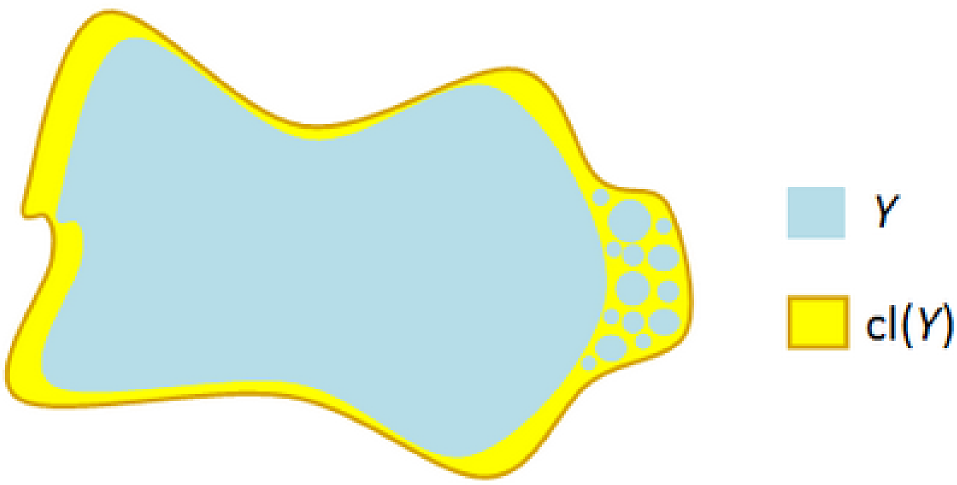

Figure 1: Closure of a Set

Proposition 1. $\overline{E} = E \cup E'$ (where E' is the set of limit points of E)

Proof. We will first prove $E \cup E'$ closed. Take $x \in X \cup E'$.Then, x is not an limit point of $E \rightarrow \exists r >$ such that $B_r^0(x) \cap E = \emptyset$. Since $x \notin E \rightarrow B_r(x) \cap E = \emptyset \rightarrow E \subseteq X \setminus B_r(x)$ and $X \setminus B_r(x)$ is closed since it is the complement of $B_r(x)$ which is open (all balls are open) in X. Then $E' \subseteq [X \setminus B_r(x)]' = X \setminus B_r(x)$ and $E' \cap B_r(x) = \emptyset$. Hence, $B_r(x) \cap (E \cup E') = \emptyset$ or equivalently,

$$B_r(x) \subseteq X \setminus (E \cup E').$$

We know that $X \setminus (E \cup E')$ must be open since an open set $B_r(x)$ is contained in it, so,

$$E \cup E'$$

is a closed set.

Then, we will show $E \cup E' \supseteq \overline{E}$

$$E \cup E' \supseteq E \rightarrow E \cup E' \supseteq \overline{E}.$$

by property (3) in the definition since $E \cup E'$ is closed.

For the other direction,

$$\overline{E} \supseteq E \rightarrow \overline{E}' \supseteq E'.$$

since $\overline{E}' \subseteq \overline{E}$,

$$\overline{E}' \supseteq E' \rightarrow \overline{E} \supseteq E'.$$

and by property (6). So, since $\overline{E} \supseteq E$ and $\overline{E} \supseteq E'$:

$$\overline{E} \supseteq E \cup E'.$$

and with both directions proven,

$$\overline{E} = E \cup E'.$$

□

2 Sequences

Definition 2. Take (X, ρ) a metric space. A **sequence** $\{x_n\}_{n \geq 1}$ is a map

$$\begin{aligned} \mathbb{N} &\longrightarrow X \\ n &\longmapsto x_n. \end{aligned}$$

The **image (range)** of a sequence,

$$im(\{x_n\}_{n \geq 1}) = \{x \in X | \exists k \geq 1 \text{such that} x_k = x\}.$$

Note. Example
Consider

$$x_n = \begin{cases} 1, & \text{n even} \\ 0, & \text{n odd} \end{cases}.$$

so $x_1 = 0, x_2 = 1, x_3 = 0, \ldots$. Then the range would be

$$im(\{x_n\}_{n \geq 1}) = \{0, 1\}.$$

Definition 3. Let $\{x_n\}_{n\geq 1}$ a sequence in X. We say that $x \in X$ is the **limit** of this sequence if $\forall \varepsilon > 0$, $\exists N \geq 1$ such that $\forall n \geq N$ we have that

$$x_n \in B_\varepsilon(x).$$

In this case we can say that $\{x_n\}_{n\geq 1}$ converges to x as $n \to \infty$. We write:

$$x_n \to x(n \to \infty).$$

or equivalently

$$\lim_{n\to\infty} x_n = x.$$

Note that we sometimes have to specify the metric space if x_n is in multiple metric spaces. The image must always be **countable** (even if the sequence converges to $x \in \mathbb{R}$ because the indices we map from are countable $\mathbb{N}$)

Note. Example
The previous example does not have a limit because x_n oscillates so the "tail" is never contained around 0 or 1 for any ε, because the next x_{i+1} flips the parity so it "jumps" and is not contained in that $B_\varepsilon(x)$ ball.

Proposition 2. If x', x'' are limits of $\{x_n\}_{n\geq 1}$ then $x' = x''$ (i.e. the limit is unique)

Proposition 3. If $\{x_n\}_{n\geq 1}$ converges, then $im(\{x_n\}_{n\geq 1}$ is a bounded set in X

2.1 Subsequence

Definition 4. Let $\{x_n\}_{n\geq 1}$ be a sequence and $1 \leq n_1 < n_2 < \ldots < n_k < \ldots$, i.e $n_j \in \mathbb{Z}$ where $j \geq 1$, then we get $\{x_{n_j}\}_{j\geq 1}$ is a **subsequence** of $\{x_n\}_{n\geq 1}$.

Proposition 4. If $x_n \to x$ as $n \to \infty$, then any subsequence $\{x_{n_j}\}_{j\geq 1}$ converges to x.

Theorem 1. Let (X, ρ) be a compact metric space and let $\{x_n\}_{n\geq 1}$ be a sequence in X. Then $\exists$ a subsequence $\{x_{n_j}\}_{j\geq 1}$ and $x \in X$ such that this $x_{n_j} \to x$ as $j \to \infty$ (In other words, every sequence in a compact metric space has a convergent subsequence.)

Proof. Let us first assume that $im(\{x_n\}_{n\geq 1})$ has finitely many elements: $y_1, y_2, \ldots, y_n (y_k \neq y_j$ if $k \neq j)$. Then, $\exists \{x_{n_j}\}_{j\geq 1}$ and $1 \leq l \leq N$ such that $x_{n_j} = y_l$ $(\forall j \geq 1)$ (this is true by pigeonhole principle, finite y_l, infinite x_{n_j}, so at least one distinct value for y_l must appear infinitely in the sequence $\{x_n\}$, then this guarantees there is some subsequence $\{x_{n_j}\}_{j\geq 1}$ whose terms are equal to a fixed y_l). Then, clearly $x_{n_j} \to y_l$ as $(j \to \infty)$.

Now assume that the range of $\{x_n\}_{n\geq 1}$ is not finite. Let $S = im(\{x_n\}_{n\geq 1})$. Since X is compact and $S \subseteq X$ has infinitely many points. We can use the proposition from the previous lecture, which tells us that $\exists x \in X$ that is a limit point of this set. Define a ball centered at x.

$$B_{\frac{1}{n}}(x) n \geq 1.$$

For $n = 1$, $\exists x_{n_1}, n_1 \geq 1$ such that $x_{n_1} \in B_1(x)$. For $n = 2$, $\exists x_{n_2}$, $n_2 > n_1$ such that $x_{n_2} B_{\frac{1}{2}}(x)$. …For $n = k \exists x_{n_k}, n_k > n_{k-1}$ such that $x_{n_k} \in B_{\frac{1}{k}}(x)$ (by definition of limit point). Then, $x_{n_j} \to x$ as $j \to \infty$ $\square$

2.2 Cauchy Sequence

Definition 5. Let (X, ρ) a metric space. $E \neq \emptyset \subseteq X$ and E bounded. The we can defined

$$diam(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

Can be thought of as the max distance between two points in the set E i.e. the diameter.

Lemma 1. Let $E \subseteq X$ be a bounded set. Then,

$$diam\overline{E} = diam E.$$

Proof. Since $\overline{E} \supseteq E \rightarrow diam\overline{E} \geq diam E$.

Take $x, y \in \overline{E} = E \cup E' \rightarrow \forall \varepsilon > 0$, $\exists x', y' \in E$ such that $\rho(x, x') < \varepsilon$ and $\rho(y, y') < \varepsilon$. Here we say that x and y are either an element of E or a limit point of E so they are ε close to x, y respectively. Then by the triangle inequality (applied twice),

$$\begin{aligned} \rho(x, y) &\leq \rho(x, x') + \rho(x', y') + \rho(y', y) \\ &< \varepsilon + diam(E) + \varepsilon \\ &< diam(E) + 2\varepsilon. \end{aligned}$$

Hence for any $x, y \in \overline{E}$, $\forall \varepsilon > 0$ we have that:

$$\rho(x, y) < diam(E) + 2\varepsilon.$$

For fixed ε and any x,y

$$diam(\overline{E}) \leq diam(E) + 2\varepsilon.$$

From here follows (a bit counterintuitive but consider if $a \leq b + \varepsilon$ was not true),

$$diam\overline{E} \leq diam(E).$$

As $diam\overline{E} \geq diam E$ and $diam\overline{E} \leq diam E$, then $diam\overline{E} = diam E$ $\square$

Note. For $diam(\overline{E})$ to even be a valid question, $\overline{E}$ must be bounded so we use the fact that the closure of a bounded set is always bounded.

Lemma 2. Let $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_j \supseteq \ldots$ where $K_j \neq \emptyset$ are compact sets in X. Then, $\cap_{j \geq 1} K_j \neq \emptyset$

Proof. Consider

$$G_j = X \setminus K_j, j \geq 1.$$

Since K_j is compact, then it is closed, so G_j is open $\forall j \geq 1$. Assume that all conditions in the lemma hold but $\cap_{j \geq 1} K_j = \emptyset$. Then,

$$\cup_{j \geq 1} G_j = \cup_{j \geq 1} (X \setminus K_j) = X \setminus [\cap_{j \geq 1} (X \setminus (X \setminus K))] = X \setminus [\cap_{j \geq 1} K_j] = .$$

TODO

$\square$

Let (X, ρ) a metric space, $\{x_n\}_{n \geq 1}$ is a sequence in X

Definition 6. $\{x_n\}_{n\geq 1}$ is a **Cauchy sequence** if and only if $\forall \varepsilon > 0\ \exists N \geq 1$ such that

$$\forall n, m \geq N, \rho(x_n, x_m) < \varepsilon.$$

An example would be an asymptote that doesn't go to infinity. A cauchy sequence must be a **bounded set**

Note. Notice that the difference in defintion between cauchy and limit is that cauchy doesn't assume the existence of x such that $x_n \to x$. Usually, a cauchy is used to find x and show it exists. So cauchy sequence $\supseteq$ convergent sequence

Definition 7. A metric space (X, ρ) is **complete** if any cauchy sequence in X converges to some point $x \in X$.

Proposition 5. If $x_n \to x$ as $n \to \infty$ then $\{x_n\}_{n\geq 1}$ is a cauchy sequence.

Proof. □

Proposition 6. If $\{x_n\}_{n\geq 1}$ be a cauchy sequence then $im(\{x_n\}_{n\geq 1})$ is bounded (the same as saying the sequence is bounded)

Proof. □

Proposition 7. $\{x_n\}_{n\geq 1}$ is a cauchy sequence $\Leftrightarrow$ $diam\left(\{x_n\}_{n\geq N}\right) \to 0$ as $N \to \infty$ (by the previous proposition we know that $\{x_n\}_{n\geq N}$ is bounded, so the diam is well-defined)

Proof. $\to$ Assume that $\{x_n\}_{n\geq 1}$ is a cauchy sequence then $\forall \varepsilon\ \exists N \geq 1$ such that $\forall m, n \geq N, \rho(x_n, x_m) < \varepsilon$. Notice that this means $diam(\{x_n\}_{n\geq N}) \leq \varepsilon$ (the tail of the sequence is contained in a ball of radius ε). Hence, $\forall \varepsilon > 0$, $\exists N \geq 1$ such that $k \geq N$,

$$diam\left(\{x_n\}_{n\geq k}\right) \leq \varepsilon \Leftrightarrow diam\left(\{x_n\}_{n\geq k}\right) \to 0 (k \to \infty).$$

$\leftarrow$ Assume that $diam\left(\{x_n\}_{n\geq N}\right) \to 0$ as $N \to \infty$ then $\forall \varepsilon > 0$, $\exists N_0 \geq 1$ such that $\forall N \geq N_0$, we have $diam(\{x_n\}_{n\geq N}) < \varepsilon$. (We are now looking at $B_\varepsilon^{\mathbb{R}}(0) = (-\varepsilon, \varepsilon)$ and saying the diameter is a element in this ball). This implies that $\forall m, n \geq N$,

$$\rho(x_n, x_m) < \varepsilon.$$

Then $\forall \varepsilon > 0$, $\exists N_0 \geq 1$ such that $\forall m, n \geq N_0$. Hence, $\{x_n\}_{n\geq 1}$ is a cauchy sequence. □

Note. CounterExample
A cauchy sequence in a metric space that does not converge (i.e. the metric space is not complete).

Consider $X = [0, 1) \subseteq \mathbb{R}$ with standard ρ. Take $x_n = 1 - \frac{1}{n}, n \geq 1$. Then this sequence is a cauchy sequence in $\mathbb{R}$ and since distance is inherited by X, then the sequence is also cauchy in X. However, while the $\{x_n\}_{n\geq 1}$ converges in $\mathbb{R}$, it does not converge in X.

Theorem 2. If X is compact $\to$ X is complete. (but not the other way around)

Theorem 3. $\mathbb{R}^n$ is complete (observe that it is not compact so we can't use the theorem above)