HOMEWORK 4 FOR MATH 7241, FALL 2024. DUE OCTOBER 10TH

1. Let X be an exponential random variable of parameter λ (i.e. its pdf is $\lambda e^{-\lambda x}$). One of the following statements is correct:

- $\mathbb{E}[X^2 \mid X > 1] = \mathbb{E}[(X+1)^2]$,
- $\mathbb{E}[X^2 \mid X > 1] = \mathbb{E}[X^2] + 1$,
- $\mathbb{E}[X^2 \mid X > 1] = (\mathbb{E}[X+1])^2$.

Without computing this directly, use the memorylessness property to explain which is correct.

2. Recall that X has the Poisson λ distribution if $\mathbb{P}[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}$. Given $\lambda, \mu > 0$, let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$, such that X and Y are independent. Show that $X + Y$ has the distribution $\text{Poisson}(\lambda + \mu)$.

Hint: You may want to use the Binomial identity: for any real x and integer k,

$$\sum_{r=0}^{k} \binom{k}{r} \cdot x^r = (1+x)^k.$$

3. Let a be a positive real number, and X be a random variable with $\mathbb{E}[X] = 0$, $\text{Var}(X) = a^2$.

- Show that, for any $b \geq a$, $\mathbb{P}[|X| > b] \leq (a/b)^2$.
- Show that, for any $b \geq a$, there exists at random variable Y with mean zero and variance a^2 such that $\mathbb{P}[|Y| > b] = (a/b)^2$.

4. We define a sequence of random variables inductively: $X_0 = 1$ with probability 1. Given X_n, we set X_{n+1} to be uniform on the interval $[0, X_n]$. Show that there exists a real number c such that $\frac{1}{n}\ln(X_n)$ converges to c in probability, and compute the value of c.

Hint: Write $\ln(X_n)$ as a sum of independent random variables, and use the weak Law of Large Numbers.

5. The last part of the balls and bins trilogy: we have r balls, to be distributed among n bins. Each of the n^r possible configurations is equally likely. Suppose that r and n are going to infinity so that $r/n \to c$ for some positive real number c. Let E_n be the number of empty bins. You may assume that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[E_n] = e^{-c}.$$

Show that E_n/n converges to e^{-c} in probability as n goes to infinity.

Hint: First, prove that the limit of the variance of E_n/n as n goes to infinity is zero, and then appeal to Chebyshev's inequality.