Probability 1

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1 Independence

Definition 1. Let Ω be the outcome space and P the probability measure. $A, B \subseteq \Omega$ are events. A, B are **independent** if $P[A \cap B] = P[A] \cdot P[B]$. We say X, Y random variables on Ω are independent if $\{X = a\}$ and $\{Y = b\}$ are independent events for every a,b in range of X, Y respectively.

Theorem 1. If $\{A_1, \ldots, A_n\}$ is a set of events then for $A \subseteq \{1, \ldots, n\}$:

$$P[\cap_{i \in I} A_i] = \prod_{i \in I} P[A_i].$$

if the set of events is independent.

Definition 2. Define P_B a new probability measure on Ω known as conditional probability.

$$P_B[A] = \frac{P[A \cap B]}{P[B]}.$$

We can check that this P_B is a probability measure:

1. $P_B[\Omega] = \frac{P[\Omega \cap B}{P[B]} = 1$
2. $P_B[\emptyset] = \frac{P[\emptyset \cap B]}{P[B]} = 0$

Then independence in terms of this probability measure is defined as:

$$P_B[A] = \frac{P[A \cap B]}{P[A]} = P[A].$$

You will also see the notation,

$$P[A|B] = \text{ probability of A given B.}$$

Note. Example
P is uniform on $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$. Let $X =$ outcome of first roll. $Y =$ outcome of second roll.

$$P[X = 1] = \{(1, 1), (1, 2), (1, 3), (1, 4)\} = \frac{4}{16} = \frac{1}{4}.$$

The same is true for $P[X = 2], \ldots, P[Y = 4]$. Thus, X, Y are also both uniform probability measures. And, $P[X = a, Y = b] = \frac{1}{16}\ \forall a, b \in \{1, 2, 3, 4\}$ is a uniform probability measure. Then, $P[X = a, Y = b] = P[x = a] \cdot P[y = b] = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ and the two rolls are independent!

Note. Example
Let M be the R.V. representing the maximum of two tosses. Are M and X independent?

$$P[M = 1] = \frac{1}{16}, P[X = 1] = \frac{1}{4}.$$

$$P[M = 1, X = 1] = \frac{1}{16} \neq P[M = 1] \cdot P[X = 1] = \frac{1}{16} \cdot \frac{1}{4}.$$

No! Which makes sense!

Note. Example
$R = \{$ first roll even$\}$ and $L = \{$ sum of two tolls is 6$\}$:

$$P[R] = \frac{1}{2}, P[L] = \frac{3}{16}.$$

$$P[L \cap R] = \frac{2}{16} = \frac{1}{8} \neq P[R] \cdot P[L] = \frac{1}{2} \cdot \frac{3}{16}.$$

They are not independent.

However, counterintuitively, when **L is the set of two rolls whose sum is 5** then the two varirables **are independent**, so this is why we need to prove events are independent.

Remark. Independence allows us now to consider **independent coupling** of distributions which allows us to solve the problem of marginals not giving enough information of joint distributions.

2 Independent Coupling of Distributions

I.e. joint distributions with independent variables

Definition 3. When $P[X = x], P[Y = y]$ where x, y are in the range and X, Y are independent, we can write the joint probabiltity as:

$$P[X = x, Y = y] = P[X = x] \cdot P[Y = y].$$

Definition 4. binomial distribution
Given $P \in [0, 1]$ and let $(X_1, X_2, \ldots, X_n)$ to be n independent **bernoulli**(p) random variables. Let $Y = X_1 + X_2 + X_n$, so the $range(Y) = \{0, 1, 2, \ldots, n\}$. We know that for each bernoulli $\Omega = \{0, 1\}$ and for all n independent bernoulli variables: $\Omega = \{0, 1\}^n$ and $\mid \Omega \mid= 2^n$. What is the distribution of Y?

$$P[Y = 0] = P[\{0, 0, 0, \ldots, 0\}] = P[X_1 = 0] + \ldots + P[X_n = 0] = (1 - p)^n .$$

$$P[Y = 1] = P[\{1, 0, \ldots, 0\} \cup \{0, 1, \ldots, 0\} \cup \ldots \cup \{0, 0, \ldots, 1\}] = n \cdot p \cdot (1 - p)^{n-1}.$$

$$P[Y = k] = \binom{n}{k} p^k (1 - p)^{n-k}.$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ In math terms, we describe $Y \sim binomial(n, p)$ the binomial distribution!

Definition 5. Geometric Distirbution
Consider the experiment of tossing a coin until H shows up. Let $P[X = H] = p$ where X is a coin toss.

$$\Omega = \{H, TH, TTH, \ldots\}.$$

Note that omega is countably infinite, even if you toss a coin a million times you can't be sure there will be a heads.

$$P[H] = p.$$

$$P[TH] = (1 - p)p.$$

$$P[TTH] = (1 - p)^2 p.$$

$$P[T^n H] = (1 - p)^n p.$$

Let $Z : \Omega \to \mathbb{R}$, the position of H i.e. how many T's. Then $Z \sim Geo(p)$

2.1 Counterintuitive stuff

Note. Example

Let X, Y be two Ber($\frac{1}{2}$) random variables. Let $S = \begin{cases} 1 \text{ if } X = Y \\ 0 \text{ otherwise} \end{cases}$

Is S independent of X? Yes! (verify by hand) The same holds for S and Y! S is pairwise independent with X, Y. However, $\{X, Y, S\}$ are not jointly independently!!!

$$P[X = 1, Y = 1, S = 0] = 0.$$

but,

$$P[X=1]\cdot P[Y=1]\cdot P[S=0]=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\neq 0.$$

pairwise independence does not imply joint independence

Note. Example
Setup an experiment as follows: $X=\{0,1\}$ where $P[X=1]=\frac{1}{2}$.

- If $X=1$, then I toss unfair coin with $P[H]=\frac{3}{4}$ twice, independently
- If $X=0$, then I toss unfair coin with $P[H]=\frac{1}{4}$ twice, independently

$$\Omega=\{0,1\}\cdot\{H,T\}\cdot\{H,T\}.$$

Let's analyze the probabilities

$$P[\{1,H,H\}]=P[X=1,\{H,H\}]=P[\{H,H\}|X=1]\cdot P[X=1]=\frac{9}{32}.$$

$$P[\{0,H,H\}]=P[X=0,\{H,H\}]=P[\{H,H\}|X=0]\cdot P[X=0]=\frac{1}{32}.$$

Then, since the above are disjoint:

$$P[\{H,H\}]=\frac{5}{16}.$$

Does $P[\{H,H\}]=P[\text{ first toss H}]\cdot P[\text{ second toss H}]$?

$$P[\text{ first toss H}]=\frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{2}.$$

$$P[\text{ second toss H}]=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}=\frac{1}{2}.$$

$$P[\{H,H\}]=\frac{5}{16}\neq\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}.$$

Thus, the "first coin being heads" is NOT independent of the "second coin being heads"!!! Despite the experiment being clearly defined with the first coin and second coin being tossed independently.

The intuition is that **information is being passed from the two statements** where if we know that the first toss is heads, because the probabilities are weighted to a certain path of events to lead to that outcome, this informs us about the probability of the second coin toss being heads!!!!

Proposition 1. Independent R.V. factor w.r.t to expectations: Let $f(x), g(y)$ be arbitrary functions and X, Y are R.V.

$$E[f(x)\cdot g(y)]=E[f(x)]\cdot E[g\,(y)].$$

Proof.

$$\begin{aligned}
E[f(x)\cdot g(y)] &= \sum_{a\in range(x)}\ \sum_{b\in range(y)} f(a)g(b)P[X=a,Y=b]\\
&= \sum_{a\in range(x)}\ \sum_{b\in range(y)} f(a)g(b)P[X=a]P[Y=b]\\
&= \sum_{a\in range(x)} f(a)P[X=a] \sum_{b\in range(y)} g(b)P[Y=b]\\
&= E[f(x)]\cdot E[g(y)].
\end{aligned}$$

□

Corollary 1. If X, Y are independent random variables, they are uncorrelated. Recall that $Cov(X,Y)=E[XY]-E[X]E[Y]$ and when $Cov(X,Y)=0$ then they are uncorrelated.

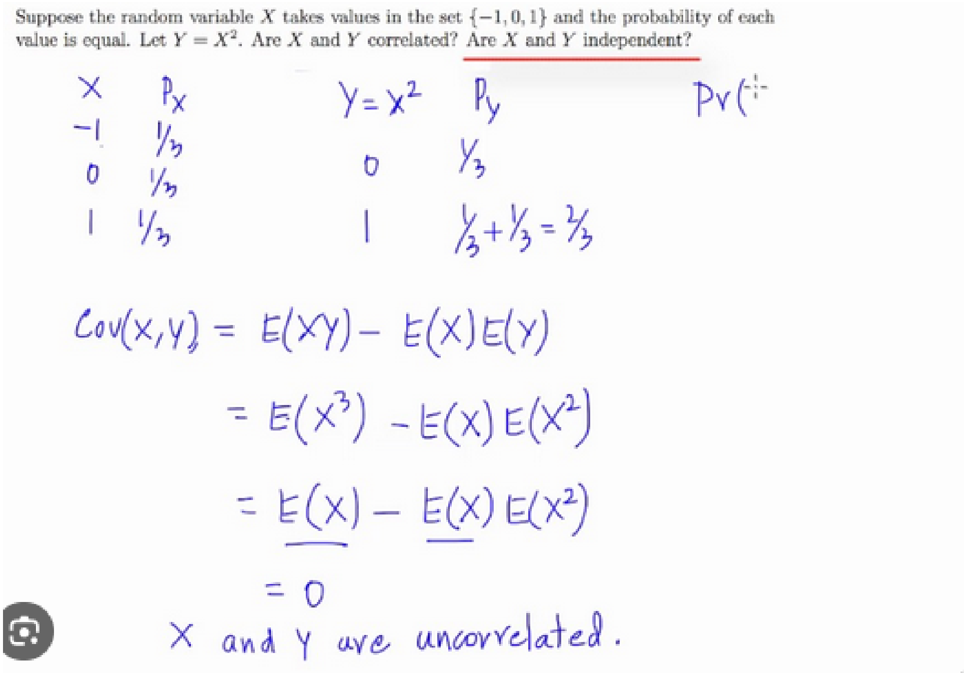

Figure 1: Uncorrelated but Not Independent

Note. Correlation x= Independence

Finish the analysis at home!

2.2 Independence for Continuous R.V.

Definition 6. X, Y are independent if

$$P[X \leq s, Y \leq t] = P[X \leq s] \cdot P[Y \leq t] \forall s, t.$$

For cdfs:

$$F_{x,y}(s,t) = F_x(s) - F_x(t).$$

Taking partial derivatives, we see the pdfs multiply through:

$$f_{X,Y}(s,t) = f_x(s) \cdot f_Y(t).$$

Definition 7. continuous probability on a continuous space we use pdfs. If X, Y have a joint distribution with joint pdf $f_{X,Y}(s,t)$:

$$P[Y = t, X = s] = \frac{P[Y = t, X = s]}{P[X = s]}.$$

As the marginal pdf of X:

$$\int_{-\infty}^{\infty} f_{X,Y}(s,t) = \text{ marginal pdf of X} f_X(s).$$

pdf of Y given $X = s$:

$$f_{Y|X=s}(t) = \frac{f_{X,Y}(s,t)}{\int_{-\infty}^{\infty} f_{X,Y}(s,t)dt}.$$