Some Class

Daniel Yu

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Contents

1 Riemann Stieltjes Integral

Take a partition of $[a, b]$ denoted as

$$P = \{a = x_0 \leq x_1 \leq \ldots \leq x_n = b\}.$$

and consider $f : [a, b] \to \mathbb{R}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ Then,

$$\overline{S}(P, f, \alpha) = \sum_{k=1}^{n} M_k(\alpha(x_k) - \alpha(x_{k-1})), M_k = \sup_{[x_{k-1}, x_k]} f.$$

$$\underline{S}(P, f, \alpha) = \sum_{k=1}^{n} m_k(\alpha(x_k) - \alpha(x_{k-1})), m_k = \inf_{[x_{k-1}, x_k]} f.$$

And,

$$m(\alpha(b) - \alpha(a)) \leq \underline{S}(P, f, \alpha) \leq \overline{S}(P, f, \alpha) \leq M(\alpha(b) - \alpha(a)).$$

One application is in probability when we find the expected value of R.V. X:

$$E[X] = \int_{-\infty}^{\infty} x dF(x).$$

where $= F(x)$ the cdf of X

Definition 1. Define the Riemann Stieltjes Integral as

$$\overline{\int_a^b} f d\alpha = \inf_P \overline{S}(P, f, \alpha).$$

,

$$\underline{\int_a^b} f d\alpha = \sup_P \underline{S}(P, f, \alpha).$$

Definition 2. $f : [a.b] \to \mathbb{R}$ satisfying the conditions above is called integrable (Riemann-Stieltjes Integrable with respect to alpha) if

$$\underline{\int_a^b} f d\alpha = \overline{\int_f^d} \alpha.$$

In this case we set

$$\int_a^b f d\alpha = \underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha.$$

and say that $f \in R(\alpha)$

Note. This doesn't always hold.

Consider the example of the dirchlet function. Take,

$$F(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}.$$

and consider $x \in [0, 1]$ with $F : [0, 1] \to \mathbb{R}$. Consider the partition $P = \{0 = x_0 \leq x_1 \leq \ldots \leq \ldots \leq x_{n-1} \leq x_n\}$, Then,

$$\overline{S}(P, F, \alpha) = \sum_{k=1}^{n} \sup_{[x_{k-1}, x_k]} f(\alpha(x_k) - \alpha(x_{k-1})) = \alpha(b) - \alpha(a) \Rightarrow \overline{\int_a^b} F d\alpha = \alpha(b) - \alpha(a) > 0.$$

But when considering the lower RS integral,

$$\underline{\int_a^b} F d\alpha = 0.$$

Hence, $F \in R(\alpha) \forall \alpha \neq$ constant.

Let $P = \{a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b\}$ be a parition over $[a, b]$.

Definition 3. A partition P_1 of $[a, b]$ is a refinement of P if $P_1 \supseteq P$.

Lemma 1. Let P_1 be a refinedment of P. Then,

$$\underline{\int} (P_1, F, \alpha) \leq \overline{\int} (P, F, \alpha).$$

and,

$$\underline{\int} (P_1, f, \alpha) \geq \underline{\int} (P, f, \alpha).$$

Proof. Take $P = \{a = x_0 \leq x_1 \leq \ldots \leq x_p \leq x_{p-1} \leq \ldots \leq x_{n-1} \leq x_n = b\}$ It is enough to prove the folloowing lemm when

$$P_1 = P \cup \{x\}, x \in [X_p, x_{p-1}].$$

.

$$\overline{S}(P_1, f, \alpha) - M_p(\alpha(x_p) + \alpha(_k)) + \sup_{x_p, x^k} f + \inf_{x_p, x^k} f.$$

□

Corollary 1. Let P_1, P_2 be two partitions of $[a, b]$ Then

$$\underline{S}(P_1, f, \alpha) \leq \overline{S}(P_2, f, \alpha).$$

Proof. Let P_1, P_2 be partitions of $[a, b]$. Then consider $P' = P_1 \cup P_2$ the common refinement. Then by Lemma 1,

$$\underline{S}(P_1, f, \alpha) \leq \underline{S}(P', f, \alpha) \leq \overline{S}(P, f, \alpha) \leq \overline{S}(P_2, f, \alpha).$$

by lemma 1 and we are done. □

Corollary 2.

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha.$$

Proof. Using the previous corollary,

$$\forall P_1, P_2, \underline{s}(P_1, f, \alpha) \leq \overline{S}(P_2, f, \alpha).$$

This implies that

$$\underline{\int_a^b} = \sup_{P_1} \underline{S}(P_1, f, \alpha) \leq \overline{S}(P_2, f, \alpha) \Rightarrow \underline{\int_a^b} f d\alpha \leq \inf_{P_2} \overline{S}(P_2, f, \alpha) = \overline{\int_a^b} f d\alpha.$$

□

Hence for any paritions, P_1, P_2 of $[a,b]$ we have the following inequality:

$$m(\alpha(b)-\alpha(a)) \leq \underline{S}(P_1,f,\alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq \overline{S}(P_2,f,\alpha) \leq M(\alpha(b)-\alpha(a)).$$

Proposition 1. A **criteria** for integrability is $f \in R(\alpha) \Leftrightarrow \forall \varepsilon > 0 \exists P$ a parition of $[a,b]$ such that

$$\overline{S}(P,f,\alpha) - \underline{S}(P,f,\alpha) \leq \varepsilon.$$

Proof. $\Rightarrow$ Assume that $f \in R(\alpha)$. Take $\varepsilon > 0$. Then, $\exists P_1$ a partition

$$\int_a^b \leq \overline{S}(P_1,f,\alpha) \leq \int_a^b f d\alpha + \frac{\varepsilon}{2}.$$

by definition. Similarly, $\exists P_2$

$$\int_a^b \geq \underline{S}(P_2,f,\alpha) \leq \int_a^b f d\alpha - \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$ then it follows from lemma 1 that,

$$\int_a^b \leq \overline{S}(P,f,\alpha) \leq \int_a^b f d\alpha + \frac{\varepsilon}{2}.$$

$$\int_a^b \geq \underline{S}(P,f,\alpha) \leq \int_a^b f d\alpha - \frac{\varepsilon}{2}.$$

Then,

$$\overline{S}(P) - \underline{S}(P) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$