Analysis I

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Contents

1 Connected Sets

Define connected sets as a set that is not a not connected sets

Definition 1. (X, ρ) is not connected if $\exists U_{\text{open}} \neq \emptyset, V_{\text{open}} \neq \emptyset$ such that

$$X = U \cup V, U \cap V = \emptyset.$$

This is also written as the disjoint union:

$$X = U \amalg V.$$

Not-connectedness is an intrinsic propery of the metric space.

Definition 2. let $E \subseteq X$. E is not connected in X if (E, ρ) is not connected.

Proposition 1. $(\rightarrow)$ $E \subseteq X$ is not connected w.r.t to $X \Leftrightarrow \exists A \neq \emptyset, B \neq \emptyset$ such that $E = A \cup B$ and $\overline{A} \cap B = \emptyset, A \cap \overline{B}$ (closures are in X)

Proof. Assume that $E \subseteq X$ is not connected in $X \Rightarrow (E, \rho)$ is not connected $\Rightarrow \exists U_{\text{open}} \neq \emptyset \subseteq E, V_{\text{open}} \neq \emptyset \subseteq E$ and $E = U \cup V$ where $U \cap V = \emptyset$.

By homework 1, $\exists \tilde{U} \subseteq X$, $\tilde{U} \cap E = U$. Then,

$$\tilde{U} \cap V = \left(\tilde{U} \cap E\right) \cap V = U \cap V = \emptyset.$$

Then,

$$V \subseteq \left(X \setminus \tilde{U}\right) \Rightarrow \overline{V}^X \subseteq \left(X \setminus \tilde{U}\right) \Rightarrow \overline{V} \cap \tilde{U} \Rightarrow \overline{V} \cap U = \emptyset.$$

We can apply the same argument to V and obtain that $\overline{U}^X \cap V = \emptyset$. In addition, U, V are nonempty and $E = U \cap V$.

$(\leftarrow)$ Assume that $\exists A \neq \emptyset, B \neq \emptyset$ such that $E = A \cup B$ and $\overline{A}^X \cap B = \emptyset, A \cap \overline{B}^X$. Since

$$\overline{A} \cap B = \emptyset \Rightarrow B \subseteq X \setminus \overline{A}.$$

Note that $X \setminus \overline{A}$ must then be open since the complement of a closed set is open.

$$\left(X \setminus \overline{A}\right) \cap E = (X \cap E) \setminus \left(\overline{A} \cap E\right) = (A \cup B) \setminus \overline{A} = B.$$

Since $B = \left(X \setminus \overline{A}\right) \cap E$, the intersection of an open set with an open set (a metric space is open and closed), then B is an open set in (E, ρ). We can apply the same arugment to A to show that A is open in (E, ρ).

We also know that $A \neq \emptyset, B \neq \emptyset, E = A \cup B$ where A, B are open, and $A \cap B = \emptyset$ because $\overline{A} \cap B = \emptyset, A \cap \overline{B} = \emptyset$. Thus, E is not connected in X as (E, ρ) is not connected □

Note. Example
Consider the set in $\mathbb{R}^2$. Define $E = \{(0, y) \mid -1 \leq y \leq 1\} \coprod \{\sin \frac{1}{x} \mid 0 < x \leq \frac{2}{\pi}\}$ where $\coprod$ is the disjoint union. This is an example of a **connected set**

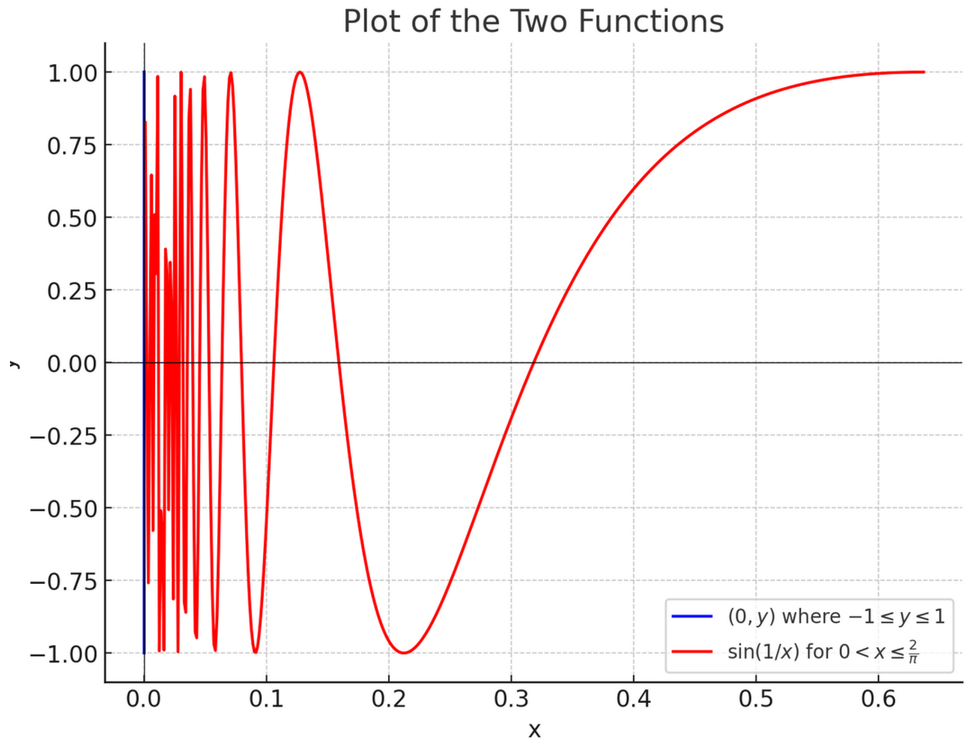

Figure 1: E is connected!

Note that the red line never reaches the blue, so they don't intersect.

Theorem 1. Let $f : X \to Y$ a continous map and $E \subseteq X$ (connected in X) $\Rightarrow f(E)$ is connected in Y.

Proof. Since $f : X \to Y$ is continuous, then $f : E \to f(E)$ is also continuous. We are given that (E, ρ) is connected. Consider the strongest case where $E = X$, so (X, ρ_X) is connected and $Y = f(X)$. Since by subspace topology, we can always restrict to $E \subseteq X, f : E \to f(E)$, then the above case is enough to prove the statement for all cases.

Thus, we will prove that if $f : X \to Y$ is continuous and X is not connected and $f(X) = Y$ then Y is connected. Prove this by contradiction, assume that the above holds but Y is not connected. Then,

$$Y = U \amalg V, U_{\text{open}} \neq \emptyset, V_{\text{open}} \neq \emptyset.$$

Consider the pre-images which we know are open due to continuous map theorems.

$$f^{-1}(U) \subseteq X, f^{-1}(V) \subseteq X.$$

We also know that both pre-images are not empty because the image is not empty and the map is onto (we construct $f(X) = Y$).

$$f^{-1}(U) \neq \emptyset, f^{-1}(V) \neq \emptyset.$$

We can use the following result from set theory:

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V).$$

And we know that $U \cap V = \emptyset$ because $U \amalg V$ (disjoint union), so

$$\begin{aligned} f^{-1}(U \cap V) &= f^{-1}(\emptyset) \\ &= \emptyset \\ &= f^{-1}(U) \cap f^{-1}(V). \end{aligned}$$

This means that open set $f^{-1}(U), f^{-1}(V)$ in X and which $f^{-1}(U) \cup f^{-1}(V) = X$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, so $f^{-1}(U) \amalg f^{-1}(V) = X$ and X is not connected. This is a contradiction. Thus, if $f : X \to Y$ is continuous and X is not connected and $f(X) = Y$ then Y is connected which is enough to prove the more general statement $f : X \to Y$ continuous map and $E \subseteq X \Rightarrow f(E)$ connected in Y.

□

1.1 connected sets in $\mathbb{R}$

Theorem 2. $E \subseteq \mathbb{R}$ is not connected $\Leftrightarrow \exists x, y \in E$, s.t. $x < y$ and $x < z < y$, where $z \notin E$

Proof. ($\Rightarrow$) Assume that E is not connected. Then, $\exists U \neq \emptyset, V \neq \emptyset$, U, V open in E, $E = U \amalg V$ ($U = X \setminus V$ but V is open, so U is open and closed and by the same arugment V is also open and closed). Take $x \in U, y \in V$. Without loss of generality, assume that $x < y$. Consider

$$\alpha = sup\left([x, y] \cap U\right).$$

which must exist as $[x, y]$ is bounded and U is bounded. Consider the following cases:

1. If $\alpha \notin E$ then $x < \alpha < y$ (since $x, y \in E$ and $\alpha \notin E$). The theorem thus holds for all $z = \alpha$
2. If $\alpha \in E$ then either:
   (a) $\alpha \in U$ then since U is open in E, $\exists \varepsilon > 0$ such that

   $$B_\varepsilon(\alpha)^E = ((\alpha - \varepsilon, \alpha + \varepsilon) \cap E) \subseteq U.$$

   and $\alpha + \varepsilon < y$. Take $z \in (\alpha, \alpha + \varepsilon)$,
      i. If $z \notin E$ then $x \leq \alpha < z < \alpha + \varepsilon < y$. Hence, the theorem follows in this case
      ii. If $z \in E$, then $z > \alpha$, $e \in [x, y] \cap U \Rightarrow \alpha$ is not the $sup([x, y] \cap U)$ so this arugment is not possible.

   (b) If $\alpha \in V$ then this means that $\alpha \notin ([x, y] \cap U)$. Then α is a limit point of $[x, y] \cap U \Rightarrow$ $\alpha \in E$ is a limit point of U. Note that U is both open and closed in E which implies that $\alpha \in U$, but we know $V \cap U = \emptyset$, so this is a contradiction!

$\square$

Definition 3. A subset E is convex if $\forall x, y \in E$ we have that $[x, y] = \{tx + (1-t)y \mid 0 \leq t \leq 1\} \subseteq E$

Theorem 3. $E \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ E is convex.

Proof. $\square$

Corollary 1. A subset $E \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ E is one of the following sets:
$\emptyset, \mathbb{R}, (a, b), [a, b], [a, b), (a, b], (-\infty, a], (a, \infty), [a, \infty)$

1.2 Intermediate Value Theorem

Corollary 2. If $f : [a, b] \to \mathbb{R}$ be continuous and $f(a) < f(b)$. Then if $f(a) < c < f(b)$ then $\exists x_* \in [a, b]$ such that $f(x_*) = c$.

Proof. The interval $[a, b]$ is connected and compact (in $[a, b]$, we restrict to $[a, b]$ because the function only defined as $[a, b]$) then since $f : [a, b] \to \mathbb{R}$ is continuous, $f([a, b])$ must also be connected and compact in $\mathbb{R} \Rightarrow f([a, b]) = [A, B]$. Since the interval exists then $f(x_*) \in [A, B]$ exists and must be the image of $x_* \in [a, b]$.

Note. We don't necessarily need compactness just to prove the above theorem, we use compactness for the stronger conclusion that the image of a closed interval under a continuous map is also a closed interval (where closed interval in $\mathbb{R}$ is connected and compact).

$\square$

2 Path Connected

Definition 4. X is path connected if $\forall x, y \in X$ $\exists$ a continuous map $\gamma[0,1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$.

Proposition 2. If (X, ρ) is path connected $\Rightarrow$ X is connected i.e. $\{\text{path connected sets}\} \subseteq \{\text{connected sets}\}$