Analysis I

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September 9, 2024

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1 Metric Spaces

Definition 1. Let X be a non-empty set and a function:

$$S : X \times X \to \mathbb{R}.$$

$$(x, y) \longmapsto S(x, y).$$

such that:

- $\forall x, y \in X$, $S(x, y) \geq 0$ and $S(x, y) = 0 \Leftrightarrow x = y$
- $S(y, x) = S(x, y)$
- **Triangle Inequality** $\forall x, y, z \in X$, $S(x, y) \leq S(x, z) + S(z, y)$ (imagine x,y,z as corners of a triangle)

Remark. Example

1. Distance on a number line:

$$S(x, y) = |x - y|, x, y \in \mathbb{R}.$$

2. Let $(X, || \cdot ||)$ be a normed space then

$$S(x, y) = || x - y ||.$$

Then S is a distance function on X. **This means all norm spaces are metric spaces**.

3. A metric space does not have to be a **Vector Space**, for example a metric space could simply be the collection of points defining a sphere and the distance would be the distance across the surface of the sphere of two points!

1.1 Balls in Metric Spaces

Definition 2. Denote metric space as (X, S). Then an (open) ball of radius r centered at x:

$$B_r(x) = \{y \in X | S(y, x) < r\}.$$

Remark. Let $X = \mathbb{R}$, $S(x, y) = |x - y|$:

$$B_{.5}(1) = \{.5 < y < 1.5 | y \in R\} = (.5, 1.5).$$

Definition 3. The punctured ball is a ball without the center and is defined as:

$$B_r^o(x) = B_r(x) \setminus \{x\}.$$

Note. Balls are a special type of **neighborhoods** (defined later)

1.2 Open and Closed Sets in Metric Space

Definition 4. A subset $U \subseteq X$ is **open** if $\forall x \in U \ \exists r > 0$ s.t.:

$$B_r(x) \subseteq U.$$

Remark. The analogy is that a closed interval $[0, 1) \subseteq \mathbb{R}$ would **not be an open set**. For example, then there is no ball centered at $B_r(0)$ because there exist $y < 0 \in \mathbb{R}$ but there are no points to the left of 0 in $[0, 1)$ and $r \not\geq 0$.

Remark. However, if X is restricted to $[0, 1)$ and we consider $U = [0, 1) \subseteq X$, this is an **open set** because there does not existed $y < 0 \in X = [0, 1)$ so a ball centered at 0 would be $B_{.5}(0) = [0, .5)$.

Definition 5. Let $E \subseteq X$. $x \in X$ is a **limit point** of E if $\forall r > 0$:

$$(B_r^o(x) \cap E) \neq \emptyset.$$

Note. We are working with points (and distances) that may be infinitesimially small. For example, if a point is a limit point, then any ball (neighborhood) centered at the limit point contains infinitely many limit points, because we could take $r' < r$.

Remark. Example
Consider $E = [0, 1) \subseteq X = \mathbb{R}$: All the interior points $0 < x < 1$ are limit points.
0 is a limit point
1 is a limit point
The set of all limit points in E would be $[0, 1]$. Note that this is different from open subset definition as 0 is included because the intersection $B_r^o(x) \cap E = (0, r)$, we don't have to restrict the ball (or neighborhood) to be inside E!

Definition 6. $E \subseteq X$ is **closed** if the set of limit points $E' \subseteq E$.

Note. Give a metric space (X, S), $X \subseteq X$ and X is open and closed by definition. Similarly, $\emptyset$ is both open and closed (vaccously).

Remark. Example

1. $X = \mathbb{R}, E = [0, 1)$, this set is not closed in in $(\mathbb{R}, S)$ as 1 is a limit point but is not in E.
2. However, for $X = [0, 1)$, $E = [.5, 1)$, E is closed in (X, S) since 1 is not a limit point (in fact it doesn't exist in X)

Proposition 1. Complement Rules – Metric Space is (X, S)

1. U is open $\Leftrightarrow X \setminus U$ is closed

   Proof. $\rightarrow$ Assume U open set, $U \subseteq X$. Take $x \in U$, by def. since U is open, $\exists r > 0$ such that $B_r(x) \subseteq U$ and so $B_r^0(x) \subseteq U$. Then, $B_r^0 \cap (X \setminus U) = \emptyset$, so x cannot be a limit point in $X \setminus U$.

   $\leftarrow$ Assume $X \setminus U$ is a closed set, then use forward direction of (2) that the complement of the closed set is open. $\square$

2. E is closed $\Leftrightarrow X \setminus E$ is open

   Proof. $\rightarrow$ Assume that E is closed. Take a point in $x \in X \setminus E$. By definition, x is not a limit point of E because all the limit points of E are in E since E is closed. Thus, $\exists r > 0$ such that $B_r^0(x) \cap E = \emptyset$ and $B_r(x) \cap E = \emptyset$ (the center x also not in E). Then, the ball $B_r(x) \subseteq X \setminus E$. As this is true for any x, then by def. U is open.

   $\leftarrow$ Assume $X \setminus E$ is open, then use forward direction of (1) that the complement of the open set is closed! $\square$

Proposition 2. Union Rules – Metric Space (X, S)

1. If $U_1 \subseteq X$, $U_2 \subseteq X$ are open, then $U_1 \cup U_2$ and $U_1 \cap U_2$ are open.

   Proof. Assume $U_1, U_2 \subseteq X$ and are open. Consider $U_1 \cup U_2$, take $x \in U_1 \cup U_2$. Then $x \in U_1$. Since they are open, by definition, $\exists r > 0$ such that $B_r(x) \subseteq U_1$ for example and $B_r(x) \subseteq U_1 \cup U_2$. The same argument follows for $x \in U_2$. Since this is for any x, then $U_1 \cup U_2$ is open.

   Consider $U_1 \cap U_2$, take $x \in U_1 \cap U_2$ so $x \in U_1$ and $x \in U_2$, then by definition $\exists r_1, r_2 > 0$ such that $B_{r_1}(x) \subseteq U_1$ and $B_{r_2}(x) \subseteq U_2$. We can simply take the minimum $r = min(r_1, r_2)$ (since r_1, r_2 are non-negative),so $B_r(x) \subseteq U_1 \cap U_2$ and this is true for all x, so $U_1 \cap U_2$ is open. $\square$

2. If $U_\alpha, \alpha \in I$ set of indicies, then $\cup_{\alpha \in I} U_\alpha$ is open.

   Proof. Assume U_α is open for all $\alpha in I$. Take $x \in \cup_{alpha \in I} U_\alpha$ then there exists $\beta \in I$ such that $x \in U_\beta$. Since U_β is open $\exists r > 0$ such that $B_r(x) \subseteq U_\beta \subseteq \cup_{\alpha \in I} U_\alpha$ and this is true for any x, then $\cup_{\alpha \in I}$ is an open set in X. $\square$

Remark. Is infinite intersection $\cap_\alpha^\infty U_\alpha$ also open?

Proof. Counterexample
Take $X = \mathbb{R}$, $S(x, y) = \|x - y\|$. Then define $U_n = \left(-\frac{1}{n} - 1, 1 + \frac{1}{n}\right)$ which is clearly open. As we take $n = 1, 2, 3, \ldots$, $\cap_{n \geq 1} U_n = [-1, 1]$ which is closed, a counterexample. $\square$

This doesn't contradict the above (one could argue that you could repeatedly take $U_1 \cap U_2$ which we proved is open) because the above is for finitely many sets!

Note. normed spaces $\subseteq$ metric space $\subseteq$ topological space

Definition 7. Topological Space.
Let X be a set. A **topology on X** is a set of subsets that are called "open" (not all possible sets – Ex: $\tau = \{u_1, v_1, \ldots\}$) and satisfy the following properties.

1. $X, \emptyset$ are open
2. U_1, U_2 are open $\to U_1 \cup U_2$ and $U_1 \cap U_2$ are open.
3. If $U_\alpha, \alpha \in I$ are open $\to U_{\alpha \in I} U_\alpha$ is open.

Remark. A metric space is a a topological space with open sets following the definition given above.

Note. Example
$X = \{a, b\}$. Then let $\tau = \{\emptyset, \{a\}, X\}$. Property (1) is clearly satisfied. Property (2) is satisfied $\emptyset \cup \{a\} = \{a\} \in \tau$ so it is open. The same follows for all other unions. Then checking intersections: $X \cap \{a\} = \{a\} \in \tau$ so it is open. Property (3) follows quickly from (2).