Analysis
Taylor's Theorem

Daniel Yu

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1 Taylor's Theorem

Lemma 1. Let
$$Q(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0.$$
a polynomial with real coefficients. Then $Q \in D(\mathbb{R})$ and $Q'(x) = a_m m x^{m-1} + \ldots + a_1$ ($\Rightarrow Q \in C(\mathbb{R})$).

Proof. . Take $x_0 \in \mathbb{R}$ and $f(x) = x^m$. Then,

$$\begin{aligned}(x_0+h)^m &= x_0^m + \binom{m}{1} x_0^{m-1}h + \{\binom{m}{2} x_0^{m-2}h^2 + \ldots + h^m\}\\ &= x_0^m + \binom{m}{1} x_0^{m-1}h + h\{\binom{m}{2} x_0^{m-2}h^1 + \ldots + h^{m-1}\}\\ &= x_0^m + m x_0^{m-1}h + h \cdot r(h).\end{aligned}$$

Notice that $f'(x_0) = m x_0^{m-1}$, so $x^m \in D(x_0)$ and $(x^m)\,|_{x=x_0} = m x_0^{m-1}$.

□

Definition 1. Define the Taylor's polynomial of degree n at point x_0 is a polynomial approximation of the function $f(x)$ at the point x_0 which matches the functions value and first n derivatives at that point. It describes the *neighborhood around* $f(x_0)$

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^n(x_0)}{(n)!}(x-x_0)^n.$$

Definition 2.
$$f \in D^m((a,b)) \text{ if } f, f', \ldots, f^{m-1} \in D((a,b)).$$
$$f \in C^m((a,b)) \text{ if } f, f', \ldots, f^{m=1}, f^m \in C((a,b)).$$

Note. $f \in D((a,b)) \Rightarrow f \in C^{m-1}((a,b))$

Theorem 1. Taylor's Theorem (Lagrange)
Assume that $f : (a,b) \to \mathbb{R}$ and $f \in D^n((a,b))$ where $n \geq 1$, $x \in (a,b)$. Then, $\forall x \in (a,b)$ $\exists x_* \in (x_0, x)$ [or $x_* \in (x, x_0)$ if $x \leq x_0$] such that:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(x_*)}{(n+1)!}(x-x_0)^{n+1}.$$

Note that $\frac{f^{(n+1)}(x_*)}{(n+1)!}(x-x_0)^{n+1}$ is known as the *Langrangian Error Term* which represents the difference between the taylor polynomial T_n and $f(x)$

Proof. Consider the taylor polynomial,

$$T_{n-1}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{n-1}(x_0)}{(n-1)!}(x-x_0)^{n-1}.$$

where $T_{n-1}(x_0) = f(x_0), T^1_{n-1}(x_0) = f'(x_0), \ldots, T^{n-1}_{n-1}(x_0) = f^{n-1}(x_0)$ We will prove the taylors theorem with x replaced by x_1. Then, take $x_0, x_1 \in (a,b)$, $x_1 \neq x_0$. First, we find $M \in \mathbb{R}$ such that:

$$f(x_1) = T_{n-1}(x_1) + \frac{M}{n!}(x_1 - x_0)^n.$$

Since M obviously exists since we can solve th elinear equation. Our task now is to show that $M = f^n(x_*)$ for some $x_* \in (x_0, x_1)$. Consider the function,

$$g(x) = f(x) - T_{n-1}(x) - \frac{M}{n!}(x-x_0)^n.$$

Then,

1. Take $g(x_0) = 0$, $g'(x_0) = 0, \ldots, g^{n-1}(x_0) = 0$.
2. Take $g(x_1) = 0$.

Then, $g \in D((x_0, x_1)) \cap C([x_0, x_1])$. By the mean-value theorem, $\exists x_1 \in (a,b)$ such that

$$g'(x_2) = 0.$$

Since by (1), $g'(x_0) = 0$, and $g' \in D((x_0, x_1)) \cap C([x_0, x_1])$ we find $x_3 \in (x_0, x_2)$ such that $g''(x_3) = 0$. (apply the mean value theorem again, this time with endpoints $\{x_0, x_2\}$).

Continuing this process, we find: $\exists x_{n+1} \in (x_0, x_n)$ such that,

$$g^n(x_{n+1}) = 0.$$

Then we set $x_* = x_{n+1}$. Then $g^n(x_*) = 0$. Clearly, $x_* \in (x_0, x_1)$. But $g^n(x) = f^n(x) - M \Rightarrow 0 = g^n(x_*) = f^n(x_*) - M \Rightarrow M = f^n(x_*)$

□

Lemma 2. For any $a > 0$ we have that

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0.$$

Proof. Take $a > 0$. Then,

$$\frac{a^n}{n!} = \frac{a}{1} \cdot \frac{a}{2} \cdot \frac{a}{3} \cdot \ldots \cdot \frac{a}{n}.$$

choose $n_0 \geq 1$ such that $\frac{a}{n_0} < \frac{1}{2}$. Then, for $n \geq n_0$,

$$\begin{aligned} \frac{a}{n!} &= [\frac{a}{1}\frac{a}{2}\frac{a}{3} \cdots \frac{a}{n_0}][\frac{a}{n_0+1} \cdots \frac{a}{n}] \\ &\leq [\frac{a}{1}\frac{a}{2}\frac{a}{3} \cdots \frac{a}{n_0}](\frac{1}{2})^{n-n_0}, n \geq n_0. \end{aligned}$$

This decays to 0 as $n \to \infty$, so $\frac{a^n}{n!} \to 0$ $\square$

Lemma 3. Stirling Approximation
$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^e$ (rate of growth)

2 Series

Definition 3. $(a_n)_{n\geq 1}, a_n \in \mathbb{R}$ (or $\mathbb{C}$), $n \geq 1$. We write

$$\sum_{n=1}^{\infty} a_n = A.$$

and say that that the series converges to A if the partial sums $S_N = \sum_{n=1}^{N} a_n$ converges to A, i.e. $S_N \to A$ ($N \to \infty$).

Equivalently, $\mid \sum_{n=1}^{N} a_n - A \mid \to 0$ as $(N \to \infty)$.

Definition 4. For series, we will often consider the "upper limit",

$$\overline{\lim_{n\to\infty}} x_n = \lim_{n\to\infty}(\sup\{x_n \mid n \geq N\}).$$

1. $\overline{\lim_{n\to\infty}} \mid a_{\frac{n}{a_{n+1}}} \mid < 1$ converges
2. $\overline{\lim_{n\to\infty}} \sqrt{\mid a_n \mid}^n < 1$ converges