Probability I Hw 2

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September 23, 2024

1. Are the random variables T_n, P_n, S_n independent?

   Proof. Proof by contradiction
   Let $\Omega = \{T, P, S\}^n$ the possible combinations of tops, pants, and shoes where $P(T) = \frac{1}{2}, P(P) = \frac{1}{3}, P(S) = \frac{1}{6}$ for each week. Then the random variables $T_n, P_n, S_n : \Omega \to \{1, 2, 3, \dots, n\} \subseteq \mathbb{R}$ represent the number of tops, pants, and shoes respectively that were bought after n weeks where each week is independent of the previous weeks. Thus, $T_n, P_n, S_n \sim Binomial(n, p)$.

   For T_n, P_n, S_n to be independent random variables, they must be pairwise independent and jointly independent i.e. $P(T_n = a \cap P_n = b \cap S_n = c) = P(T_n = a) \cdot P(P_n = b) \cdot P(S_n = c)$. We know that $P(T_n = a) = \binom{n}{a} \left(\frac{1}{2}\right)^a \left(\frac{1}{2}\right)^{n-a} = \binom{n}{a} \left(\frac{1}{2}\right)^n$, $P(P_n = b) = \binom{n}{b} \left(\frac{1}{3}\right)^b \left(\frac{2}{3}\right)^{n-b}$, $P(S_n = c) = \binom{n}{c} \left(\frac{1}{6}\right)^c \left(\frac{5}{6}\right)^{n-c}$. However, $P(T_n = a \cap P_n = b \cap S_n = c) = \binom{n}{a} (\frac{1}{2})^a \cdot \binom{n-a}{b} (\frac{1}{3})^b \cdot \binom{n-b-a}{c} \frac{1}{6}^c$ with the given constraint that $c = n - a - b$ else the probability is 0. Clearly, the left hand side and right hand side are not the same.

   For example, consider the $P(T_n = n)$ i.e. when all the items chosen after n weeks are tops, $P(T_n = n) =$ no. ways$\cdot$ probability $= 1 \cdot \frac{1}{2}^n$. Then $P(T_n = n \cap P_n = b \cap S_n = c) = 0$ when $b, c > 0$. However, suppose $b, c = 1$, $P(T_n = n) \cdot P(P_n = 1) \cdot P(S_n = 1) = \frac{1}{2}^n \cdot \binom{n}{1} \frac{1}{3} \frac{1}{3}^{n-1} \cdot \binom{n}{1} \frac{1}{6} \frac{5}{6}^{n-1} \neq 0$!
   Thus, the three random variables are not independent! □

2. In the same setup as problem 1, compute $E[T_n - P_n]$ and $Var(T_n - P_n)$.

   Proof. The expected value is additive, so $E[T_n - P_n] = E[T_n] - E[P_n]$. We know $E[T_n] = \frac{n}{2}$ and $E[P_n] = \frac{n}{3}$, so $E[T_n - P_n] = \frac{n}{2} - \frac{n}{3} = \frac{n}{6}$.

   To compute $Var(T_n - P_n)$, let

   $$Var(T_n - P_n) = E[(T_n - P_n)^2] - E[T_n - P_n]^2.$$

   Consider the random variable $D_n = T_n - P_n$. We could attempt to map out the probabilities for each outcome for the random variable D_n and D_n^2 but that gets extremely complicated. Instead, recognize that each week's outcome is independent from each other weeks outcome by definition, so we can break down the cumulative difference of $T_i - P_i$ for $i = 1, 2, \dots, n$ weeks as the sum of the individual differences each week, i.e. $D_n = d_1 + d_2 + \ldots + d_n$ where each d_i is the random variable representing the difference between tops and pants for that week (also recall that selecting one item is mutually exclusive with selecting all other items), so $d_i \in \{-1, 0, 1\}$ where $d_i = 1$ means that there was a top selected and $d_i = 0$ means that there was a hat selected and $d_i = -1$ menas that there was a pants selected. Although variance is nonlinear in general , since each d_i is independent, $Var(D_n) = Var(d_1 + d_2 + \ldots + d_n) = Var(d_1) + Var(d_2) + \ldots + Var(d_n)$ as independent variables X, Y have $Cov(X, Y) = 0$ (independence $>$ correlation). This gives:

   $$\begin{aligned} Var(T_n - P_n) = Var(D_n) &= Var(d_1) + Var(d_2) + \ldots + Var(d_n) \\ &= \sum_{i=1}^{n} Var(d_i) \\ &= nVar(d_1) \text{ since each } d_i \text{ has the same distribution} \\ &= n(E[d_1^2] + E[d_1]^2). \end{aligned}$$

   The distribution of d_1 is $P(d_1 = 1) = \frac{1}{2}$, $P(d_1 = 0) = \frac{1}{6}$, $P(d_1 = -1) = \frac{1}{3}$. So,

   $$E[d_1]^2 = \left(1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{6} + -1 \cdot \frac{1}{3}\right)^2 = \frac{1}{6}^2 = \frac{1}{36}.$$

   and the distribution of d_1^2 is $P(d_1^2 = 1) = P(d_1 = 1) + P(d_1 = -1) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $P(d_1^2 = 0) = P(d_1 = 0) = \frac{1}{6}$

   $$E[d_1^2] = \frac{5}{6} + 0 = \frac{5}{6}.$$

The answer is then:

$$n(E[d_1^2] + E[d_1]^2) = n(\frac{5}{6} - \frac{1}{36})$$
$$= n(\frac{29}{36}).$$

□

3. Show that $P[R \geq 11] \leq \frac{1}{2}$.

Proof. We toss a fair coin 1000 times, so $\Omega = \{H, T\}^{1000}$ where each toss is independent of all others. R is the random variable representing the longest consecutive run of 'Heads'. The $P(R \geq 11) = P(R = 11 \cup R = 12 \cup \ldots \cup R = 1000) = P(R = 11) + P(R = 12) + \ldots + P(R = 1000) = \sum_{i=11}^{1000} P(R = i)$. However, it is really hard to compute the probability of $P(R = i)$ explicitly. Instead, what we can do is break $P[R \geq 11]$ into a union of not necessarily disjoint events and compute their sums to provide an upper bound on the probability of $P[R \geq 11]$. Begin our decomposition as follows:

$$P[R \geq 11] = P[\{\text{coins 1 to 11 are heads} \cup \text{coins 2 to 12 are heads} \cup \ldots \cup \text{coins 990 to 1000 are heads}\}$$
$$\leq P[\text{coins 1 to 11 are heads}] + P[\text{coins 2 to 12 are heads}] + \ldots + P[\text{coins 990 to 1000 are heads}].$$

Notice that each of the events are not disjoint. For example, consider the event when coins 1 to 13 are heads, this would be counted for coins 1 to 11 are heads and coins 2 to 12 are heads. However, the probabilities of these events are now much easier to compute. Each $\forall i = 1, \ldots, 990$:

$$P(C_i) = \frac{1}{2}^{11}.$$

Then, we know :

$$P[R \geq 11] \leq \sum_{i=1}^{990} \frac{1}{2}^{11}$$
$$\leq 990 * \frac{1}{2}^{11}$$
$$\leq .483\ldots$$
$$\leq .5.$$

□

4. Party that fits 20 people, 24 people are invited

   (a) What is the expected number of people who will attend and the probability that all attendees fit inside the venue if the probability is $\frac{5}{6}$ that any one person shows up?

   Note. Since, each person is indpendent of each other and they each have $\frac{5}{6}$ probability to show up. The number of people who show up $X \sim Binomial(24, \frac{5}{6})$, so $E[X] = \frac{5}{6} \cdot 24 = 20$.

   The probability that all attendes will fit inside the venue is $P[X \leq 20] = 1 - P[X \geq 21] = 1 - (\binom{24}{21} \frac{5}{6}^{21} \frac{1}{6}^{3} + \binom{24}{22} \frac{5}{6}^{22} \frac{1}{6}^{2} + \binom{24}{23} \frac{5}{6}^{23} \frac{1}{6}^{1} + \frac{5}{6}^{24}) \approx 0.584$

   (b) Suppose now that 24 people come in groups of 4 with probability $\frac{5}{6}$, what is the expected number of people that will arrive and the probability that all attendees fit in the venue.

   Note. There are now 6 groups of people with 4 people each and each group of people has $\frac{5}{6}$ chance to come. Now, X represents the number of people and not groups. We know that $\Omega = \{(0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), \ldots, (1, 1, 1, 1, 1, 1)\}$, the attendance pattern of groups coming, and X maps from Ω to $\mathbb{R}$, so $X = \{0, 4, 8, 12, 16, 20, 24\}$. If we let Y be the random variabl e denoting the number of groups that came. Then $E[X] = E[4Y] = 4E[Y] = 4\left(\frac{5}{6} \cdot 6\right) = 20$ The probability that all the attendees fit inside the venue is now $P[X \leq 20] = P[Y \leq 5] = \sum_{k=1}^{5} \binom{6}{k} \frac{5}{6}^{k} \cdot \frac{1}{6}^{6-k} \approx 0.665$

5. Suppose we have r balls to be distributed among n bins. Each of n^r configurations are equally likely. For any $k \in \{1, 2, \ldots, n\}$, calculate the probability first k bins are empty.

Proof. Let K be the random variable denotating the first k bins that are empty. If the configurations are equally likely, then $P[K = k] = \frac{\text{no configurations with first k bins empty}}{\text{total configurations}} P[K = k] = \frac{(n-k)^r}{n^r} = \frac{n-k}{n}^r$ $\square$

Question: Doesn't stars and bars apply? Let K be the random variable denotating the first k bins that are empty. If the configurations are equally likely, then $P[K = k] = \frac{\text{no configurations with first k bins empty}}{\text{total configurations}} = \frac{\binom{n+r-k-1}{r}}{\binom{n+r-1}{r}}$