MECH230 Section 2 - Fall 2024
Midterm 2

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Monday December 2, 2024 at 2:00-3:15 pm in Bechtel 202

Full Name ______________________________

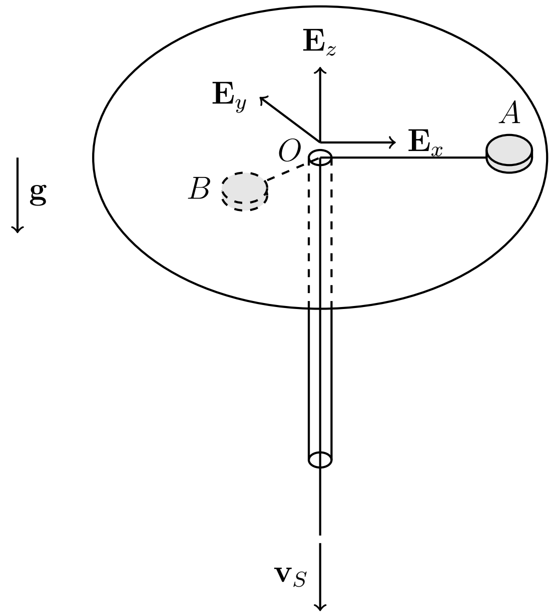
ID Number ______________________________

Instructions

- This is a closed book, closed notes exam. You are allowed to have your exam sheet, your solution booklet, your scratch booklet, and a blue pen on your desk only.
- This exam consists of 3 problems with several parts. Clearly indicate which part you are solving on your solution sheet.
- Write in a legible and neat way. Illegible solutions will not be graded.
- Write your name on all the papers you have.
- Write your answers in the answer booklet. We will only grade the solutions you submit in the answer booklet.
- When you enter the exam room, place your phone with the screen facing down on the floor under your seat and do not touch it until after you submit your exam.
- For your safety, you are not allowed to use any electronic devices, including calculators and smart watches, during the exam. If you have any of these devices, put them in your bag and close it.
- You are not allowed to leave the exam room during the exam. If you want to leave the room, you have to submit your exam.

Problem 1 [35 pts] Full Name ______________________

Consider a small coin of mass m lying on its flat side on the smooth horizontal surface of a table. The particle is attached by a chord that goes through the center O of the table and is being pulled with a constant velocity $\mathbf{v}_S = -c\mathbf{E}_z$. The puck is initially at location A where it is also given a horizontal velocity such that its velocity at A is $\mathbf{v}_A = -c\mathbf{E}_x + r_0\dot{\theta}_0\mathbf{E}_y$. After a certain time, the coin reaches location B. The chord remains taught during the motion.



(a) Draw the free body diagram of the coin during its motion. Write the expressions of the forces applied on the coin as applicable. Feel free to add basis vectors as you see fit.

The forces acting on the coin are

$$\mathbf{W} = -mg\mathbf{E}_z,$$
$$\mathbf{N} = N\mathbf{E}_z,$$
$$\mathbf{T} = -T\mathbf{e}_r.$$

(b) Write the work-energy theorem of the coin between locations A and B. Is the energy of the coin conserved between these two states? Explain.

The work-energy theorem between states A and B is

$$T_B - T_A = W_{\mathbf{W},AB} + W_{\mathbf{N},AB} + W_{\mathbf{T},AB}.$$

The weight force and the normal force are both workless since they act along $\mathbf{E}_z$ and there is no motion along $\mathbf{E}_z$. The tension force is not conservative and does work on the system. Hence, the energy of the system is not conserved between these two states.

(c) Prove that the angular momentum of the coin about point O, the center of the table, is conserved.

$$
\begin{aligned}
\mathbf{M}^O &= \mathbf{r} \times (-mg\mathbf{E}_z + N\mathbf{E}_z - T\mathbf{e}_r) \\
&= r\mathbf{e}_r \times (-T\mathbf{e}_r) \\
&= \mathbf{0}.
\end{aligned}
$$

So, $\mathbf{M}^O = \dot{\mathbf{H}}^O = \mathbf{0} \implies \mathbf{H}^O$ is conserved.

(d) Use the result from part (c) to calculate the velocity of the coin when it reaches point B where $||\mathbf{r}_B - \mathbf{r}_O|| = \frac{r_0}{2}$.

We will use the conservation of angular momentum about O to calculate $\mathbf{v}_B$, the velocity of the particle at B.

$$
\begin{aligned}
\mathbf{H}^O &= \mathbf{r} \times m\mathbf{v}, \\
&= (r\mathbf{e}_r) \times m(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta), \\
&= mr^2\dot{\theta}\mathbf{E}_z.
\end{aligned}
$$

Between states A and B

$$
\begin{aligned}
mr_0^2\dot{\theta}_0 &= m\left(\frac{r_0}{2}\right)^2\dot{\theta}_B, \\
\dot{\theta}_B &= 4\dot{\theta}_0.
\end{aligned}
$$

Therefore,

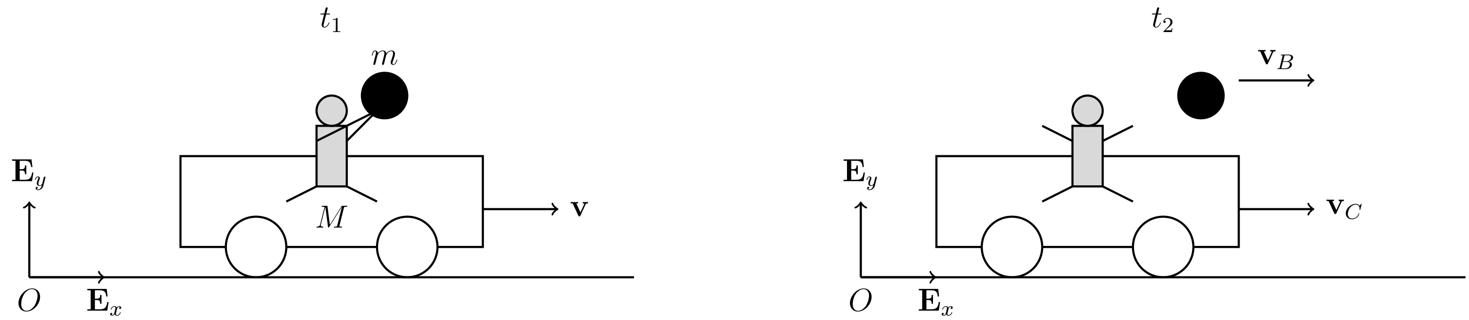
$$
\mathbf{v}_B = \dot{r}_B\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta = -c\mathbf{e}_r + \left(\frac{r_0}{2}\right)\left(4\dot{\theta}_0\right)\mathbf{e}_\theta = -c\mathbf{e}_r + 2r_0\dot{\theta}_0\mathbf{e}_\theta.
$$

(e) Calculate the work of the tension force between A and B.

$$
\begin{aligned}
W_{\mathbf{T},AB} &= T_B - T_A, \\
&= \frac{1}{2}m(c^2 + 4r_0^2\dot{\theta}_0^2) - \frac{1}{2}m(c^2 + r_0^2\dot{\theta}_0^2), \\
&= \frac{1}{2}m(3r_0^2\dot{\theta}_0^2).
\end{aligned}
$$

Problem 2 [30 pts] Full Name ______________________

Consider a child carrying a ball of mass m sitting inside a wagon. The wagon and the child have a combined mass M. At the initial time t_1 depicted, the child, the wagon, and the ball are all going at a velocity $\mathbf{v}(t_1) = v\mathbf{E}_x$. Just after t_1, the child throws the ball with a horizontal velocity such that at the instant $t_2 > t_1$, the ball has a velocity $\mathbf{v}_B(t_2) = c\mathbf{E}_x$. Neglect friction. The child does not move with respect to the wagon.



(a) Show that the linear momentum of the ball, child, and wagon in the $\mathbf{E}_x$ direction is conserved in the interval $[t_1, t_2]$.

In the interval $[t_1, t_2]$, for the combined system of the ball, child, and wagon, there are no external forces in the $\mathbf{E}_x$ direction acting on the system. The force from the child on the ball and its opposite cancel out. Hence, the linear momentum of this combined system in the $\mathbf{E}_x$ direction is conserved.

(b) Find the velocity of the wagon and child at time t_2, $\mathbf{v}_C(t_2)$, resulting from the child throwing the ball off the wagon.

Writing the conservation of the angular momentum of the combined system along $\mathbf{E}_x$ between instances t_1 and t_2,

$$\begin{aligned}
(M+m)\mathbf{v}(t_1) &= M\mathbf{v}_C(t_2) + m\mathbf{v}_B(t_2), \\
(M+m)v\mathbf{E}_x &= M\mathbf{v}_C + mc\mathbf{E}_x, \\
\mathbf{v}_C &= \frac{((M+m)v - mc)\mathbf{E}_x}{M}.
\end{aligned}$$

(c) Find the average of the force $\mathbf{F}_a$ applied by the child on the ball during the interval $[t_1, t_2]$.

To calculate the average of force $\mathbf{F}_a$ applied by the child on the ball, we write the linear-impulse linear-momentum equation of the system of the child and wagon. The applied force on the ball is horizontal. For this system, the weight and normal force sum to zero.

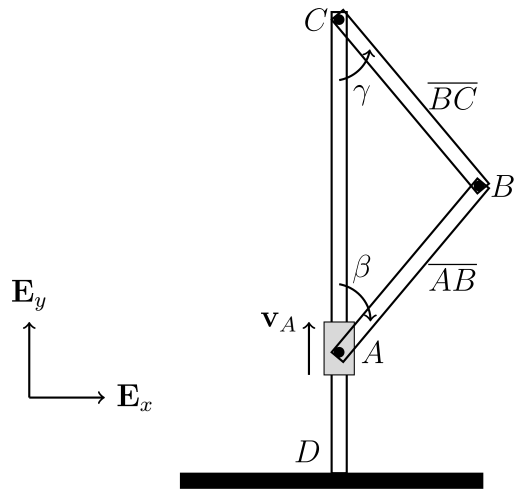
$$\int_{t_1}^{t_2} \mathbf{F}_a dt = \mathbf{F}_{a,avg}\Delta t = M(\mathbf{v}_C(t_2) - \mathbf{v}(t_1)),$$

$$\mathbf{F}_{a,avg} = \frac{((M+m)v - mc)\mathbf{E}_x - M\mathbf{v}}{t_2 - t_1} = \frac{m(v-c)}{t_2 - t_1}\mathbf{E}_x.$$

Alternatively, you can apply the linear impulse - linear momentum equation on the system of the ball alone.

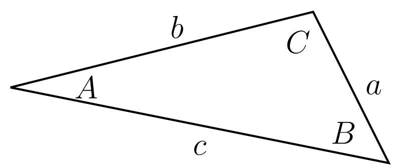
Problem 3 [35 pts]

Consider the following setup consisting of three rigid links connected by pins at B and C and by a collar at A which is pinned to link AB and can slide freely along the fixed link CD. Links CB a AB have given lengths $\overline{CB}$ and $\overline{AB}$ respectively. The collar at A has a velocity $\mathbf{v}_A = v_A \mathbf{E}_y$.



(a) Recall the sine and cosine laws and find a relationship between the angles γ and β.

Referring to the below triangle with interior angles A, B, and C and with side lengths a, b, and c, the sine law and cosine law are respectively



$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}, \tag{1}$$

$$c^2 = a^2 + b^2 - 2ab\cos(C). \tag{2}$$

Using the sine law, we have

$$\frac{\sin(\gamma)}{\overline{AB}} = \frac{\sin(\beta)}{\overline{BC}}.$$

(b) Calculate the angular velocities $\boldsymbol{\omega}_{AB}$ of link AB and $\boldsymbol{\omega}_{BC}$ of link BC in terms to v_A, γ and β.

Performing velocity analysis on links AB and BC and noting that $\mathbf{v}_C = 0$, we get

$$\begin{aligned}
\mathbf{v}_B - \mathbf{v}_A = \omega_{AB}\mathbf{E}_Z \times (\mathbf{r}_B - \mathbf{r}_A) &= \omega_{AB}\mathbf{E}_Z \times \overline{AB}(\cos(\beta)\mathbf{E}_y + \sin(\beta)\mathbf{E}_x), \\
&= \overline{AB}\omega_{AB}\left(-\cos(\beta)\mathbf{E}_x + \sin(\beta)\mathbf{E}_y\right), \\
\mathbf{v}_B - \mathbf{v}_C = \omega_{BC}\mathbf{E}_Z \times (\mathbf{r}_B - \mathbf{r}_C) &= \omega_{BC}\mathbf{E}_Z \times (\overline{BC}(-\cos(\gamma)\mathbf{E}_y + \sin(\gamma)\mathbf{E}_x), \\
&= \overline{BC}\omega_{BC}\left(\cos(\gamma)\mathbf{E}_x + \sin(\gamma)\mathbf{E}_y\right).
\end{aligned}$$

Eliminating $\mathbf{v}_B$, we get

$$v_A\mathbf{E}_y + \overline{AB}\omega_{AB}\left(-\cos(\beta)\mathbf{E}_x + \sin(\beta)\mathbf{E}_y\right) = \overline{BC}\omega_{BC}\left(\cos(\gamma)\mathbf{E}_x + \sin(\gamma)\mathbf{E}_y\right).$$

Projecting along $\mathbf{E}_x$ and $\mathbf{E}_y$, we get respectively

$$\begin{aligned}
-\omega_{AB}\overline{AB}\cos(\beta) &= \omega_{BC}\overline{BC}\cos(\gamma), \\
v_A + \omega_{AB}\overline{AB}\sin(\beta) &= \omega_{BC}\overline{BC}\sin(\gamma),
\end{aligned}$$

and solve the system to get

$$\begin{aligned}
\boldsymbol{\omega}_{BC} &= \frac{v_A}{\overline{BC}(\sin(\gamma) + \tan(\beta)\cos(\gamma))}\mathbf{E}_z, \\
\boldsymbol{\omega}_{AB} &= -\frac{v_A}{\overline{AB}(\sin(\beta) + \tan(\gamma)\cos(\beta))}\mathbf{E}_z,
\end{aligned}$$

(c) Indicate on a sketch the instantaneous center of rotation of link AB. Explain your reasoning.

The IC of line AB lies along BC that is perpendicular to $\mathbf{v}_B$ and also lies on the perpendicular to CD at A.

Kinematics in Cartesian Coordinates

$$\begin{aligned}\mathbf{r} &= x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z,\\ \mathbf{v} &= v_x\mathbf{E}_x + v_y\mathbf{E}_y + v_z\mathbf{E}_z = \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y + \dot{z}\mathbf{E}_z,\\ \mathbf{a} &= a_x\mathbf{E}_x + a_y\mathbf{E}_y + a_z\mathbf{E}_z = \ddot{x}\mathbf{E}_x + \ddot{y}\mathbf{E}_y + \ddot{z}\mathbf{E}_z.\end{aligned} \tag{3}$$

Rectilinear Motion Consider a rectilinear motion of a particle in the direction of $\mathbf{E}_x$.

$$\begin{aligned}\mathbf{r} &= x\mathbf{E}_x,\\ \mathbf{v} &= v\mathbf{E}_x = \dot{x}\mathbf{E}_x,\\ \mathbf{a} &= a\mathbf{E}_x = \ddot{x}\mathbf{E}_x.\end{aligned} \tag{4}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}. \tag{5}$$

Kinematics in Cylindrical Polar Coordinates

$$\begin{aligned}\mathbf{r} &= r\mathbf{e}_r + z\mathbf{E}_z,\\ \mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{E}_z,\\ \mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{E}_z,\end{aligned} \tag{6}$$

where

$$\mathbf{e}_r = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_\theta = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y. \tag{7}$$

Kinematics in the Serret-Frenet Basis

$$v = ||\mathbf{v}|| = \frac{ds}{dt}, \quad \mathbf{e}_t = \frac{\mathbf{v}}{v}, \quad \frac{d\mathbf{e}_t}{ds} = \kappa\mathbf{e}_n, \quad \mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n, \quad \frac{d\mathbf{e}_b}{ds} = -\tau\mathbf{e}_n, \quad \rho = \frac{1}{\kappa}. \tag{8}$$

$$\begin{aligned}\mathbf{v} &= v\mathbf{e}_t.\\ \mathbf{a} &= \dot{v}\mathbf{e}_t + \kappa v^2\mathbf{e}_n.\end{aligned} \tag{9}$$

The Balance of Linear Momentum for a particle $\mathbf{F} = \dot{\mathbf{G}}$ where $\mathbf{G} = m\mathbf{v}$.

Spring Forces A spring of stiffness K with unstretched length ℓ_0 whose base is at point A and whose free end is attached to a mass m with position vector $\mathbf{r}$ applies a force on m that is

$$\mathbf{F}_s = -K(||\mathbf{r} - \mathbf{r}_A|| - \ell_0)\frac{\mathbf{r} - \mathbf{r}_A}{||\mathbf{r} - \mathbf{r}_A||}. \tag{10}$$

Friction Forces

- Static friction is unknown but satisfies that static friction criterion $||\mathbf{F}_f|| \leq \mu_s||\mathbf{N}||$.
- Kinetic friction is prescribed according to Coulomb's friction model to be $\mathbf{F}_f = -\mu_k||\mathbf{N}||\frac{\mathbf{v}_{rel}}{||\mathbf{v}_{rel}||}$.

Power The power of a force $\mathbf{F}$ acting on a particle with velocity $\mathbf{v}$ is defined to be $\mathcal{P} = \mathbf{F}\cdot\mathbf{v}$.

Work The work of a force $\mathbf{F}$ over the interval $[t_A, t_B]$ is $W_{\mathbf{F},AB} = \int_{t_A}^{t_B} \mathbf{F}\cdot\mathbf{v}dt = \int_{\mathbf{r}(t_A)}^{\mathbf{r}(t_B)} \mathbf{F}\cdot d\mathbf{r}$.

Kinetic Energy The kinetic energy of a particle is $T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$.

The work energy theorem for a particle $T_B - T_A = W_{\mathbf{F},AB}$.

A conservative force $\mathbf{F}_c$ is such that $W_{\mathbf{F}_c,AB} = -(U_B - U_A)$. Examples include

- any constant force $\mathbf{C}$ with $U = -\mathbf{C}\cdot\mathbf{r}$,
- the gravitational force $\mathbf{F}_G = G\frac{M_e m}{(R_e+h)^2}(-\mathbf{e}_r)$ with $U = -\frac{GM_e m}{r}$, and
- the spring force $\mathbf{F}_s = -K\varepsilon\frac{\mathbf{r}-\mathbf{r}_A}{||\mathbf{r}-\mathbf{r}_A||}$ with $U = \frac{1}{2}K\varepsilon^2$. The spring stretch $\varepsilon = ||\mathbf{r}-\mathbf{r}_A|| - \ell_0$.

Linear impulse - linear momentum equation $\int_{t_A}^{t_B} \mathbf{F}dt = \mathbf{G}_B - \mathbf{G}_A$.

Angular momentum of a particle about the fixed origin O is $\mathbf{H}^O = \mathbf{r}\times m\mathbf{v}$.

Moment The moment of a force $\mathbf{F}$ applied at point A about point P is $\mathbf{M}^P = (\mathbf{r}_A - \mathbf{r}_P)\times\mathbf{F}$.

Balance of angular momentum of a particle $\mathbf{M}^O = \dot{\mathbf{H}}^O$.

Collisions The pre-impact and post-impact velocities of two particles A and B are related by $e = -\frac{\mathbf{v}'_B\cdot\mathbf{n}-\mathbf{v}'_A\cdot\mathbf{n}}{\mathbf{v}_B\cdot\mathbf{n}-\mathbf{v}_A\cdot\mathbf{n}}$ where e is restitution coefficient and $\mathbf{n}$ is perpendicular to the plane of collision.

Kinematics of a system of particles For a system of n particles each with mass m_i and position vector $\mathbf{r}_i$ from the origin, its center of mass C has

$$\mathbf{r}_C = \frac{\sum_{i=1}^n m_i\mathbf{r}_i}{m = \sum_{i=1}^n m_i}, \quad \mathbf{v}_C = \frac{\sum_{i=1}^n m_i\mathbf{v}_i}{m}, \quad \text{and} \quad \mathbf{a}_C = \frac{\sum_{i=1}^n m_i\mathbf{a}_i}{m}. \tag{11}$$

Its linear momentum is $\mathbf{G} = \sum_{i=1}^n \mathbf{G}_i = m\mathbf{v}_C$.

Its angular momentum about point P is $\mathbf{H}^P = \sum_{i=1}^n (\mathbf{r}_i - \mathbf{r}_P)\times m_i\mathbf{v}_i = \mathbf{H}^C + (\mathbf{r}-\mathbf{r}_P)\times\mathbf{G}$. where $\mathbf{H} = \sum_{i=1}^n(\mathbf{r}_i - \mathbf{r}_C)\times m_k\mathbf{v}_i = \sum_{i=1}^n(\mathbf{r}_i - \mathbf{r}_C)\times m_i(\mathbf{v}_i - \mathbf{v}_C)$.

Its kinetic energy is $T = \sum_{i=1}^n T_i = \sum_{i=1}^n \frac{1}{2}m_i\mathbf{v}_i\cdot\mathbf{v}_i = \frac{1}{2}m\mathbf{v}_C\cdot\mathbf{v}_C + \frac{1}{2}\sum_{i=1}^n m_k(\mathbf{v}_i - \mathbf{v}_C)\cdot(\mathbf{v}_i - \mathbf{v}_C)$.

Kinetics of a system of particles

$$\mathbf{F} = \sum_{i=1}^n \mathbf{F}_i = \sum_{i=1}^n m_i\mathbf{a}_i = m\mathbf{a}_c, \quad \text{and} \quad \dot{\mathbf{H}}^P = \mathbf{M}^P - \mathbf{v}^P\times\mathbf{G}. \tag{12}$$

for any point P. If P is a fixed point O, then $\mathbf{M}^O = \dot{\mathbf{H}}^O$ and if P is C, then $\mathbf{M}^C = \dot{\mathbf{H}}^C$.

Its work-energy theorem $\sum_{i=1}^n \dot{T}_i = \sum_{i=1}^n \mathbf{F}_i\cdot\mathbf{v}_i$; $\mathbf{v}_i$ is the velocity of the point where $\mathbf{F}_i$ acts.

Rigid Body (RB) Kinematics If A is fixed to the RB and B moving with respect to it, then

$$\mathbf{r}_B - \mathbf{r}_A = x\mathbf{e}_x + y\mathbf{e}_y, \tag{13}$$
$$\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega}\times(\mathbf{r}_B - \mathbf{r}_A) + \mathbf{v}_{rel}, \tag{14}$$
$$\mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha}\times(\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times(\mathbf{r}_B - \mathbf{r}_A)) + 2\boldsymbol{\omega}\times\mathbf{v}_{rel} + \mathbf{a}_{rel}, \tag{15}$$

where the angular velocity and acceleration of the RB are resp. $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ and $\boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z$ and

$$\mathbf{v}_{rel} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y, \quad \mathbf{a}_{rel} = \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y, \quad \dot{\mathbf{e}}_x = \boldsymbol{\omega}\times\mathbf{e}_x \quad \text{and } \dot{\mathbf{e}}_y = \boldsymbol{\omega}\times\mathbf{e}_y. \tag{16}$$