Work & Energy for a particle

Define the mechanical power of a force $\underline{P}$ acting on a particle whose absolute velocity is $\underline{v}$ as

$$\mathcal{P} = \underline{P} \cdot \underline{v} . \qquad N . \frac{m}{s} = W \quad (\text{Watts})$$

$\underline{P}$ $\underline{v}$

The work done by a force $\underline{P}$ in an interval of time $[t_A, t_B]$ is the integral of its power with respect to time

$$W_{\underline{P},AB} = \int_{t_A}^{t_B} \underline{P} \cdot \underbrace{\underline{v}\, dt}_{d\underline{r}} \qquad \underline{v} = \frac{d\underline{r}}{dt}$$

$$= \int_{r_A}^{r_B} \underline{P} \cdot d\underline{r} \qquad N \cdot m = J \quad (\text{Joules}) .$$

$\underline{r_B} - \underline{r_A}$

$\underline{W}$

* $\mathcal{P} = \dot{W}$

$$\underline{r} = x\underline{E_x} + y\underline{E_y} + z\underline{E_z}$$

$$d\underline{r} = dx\,\underline{E_x} + dy\,\underline{E_y} + dz\,\underline{E_z}$$

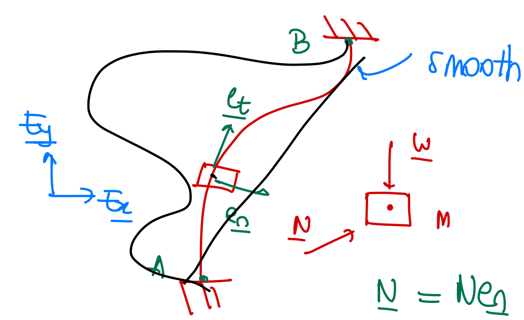
$$\underline{r} = r\underline{e_r}$$

$$d\underline{r} = d(r\underline{e_r}) = dr\,\underline{e_r} + r\,d\theta\,\underline{e_\theta}$$

$$d\underline{r} = \underline{v}\, dt = (\dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta})\, dt = \left(\frac{dr}{\cancel{dt}}\underline{e_r} + r\frac{d\theta}{\cancel{dt}}\underline{e_\theta}\right)\cancel{dt}$$

$$d\underline{r} = \underbrace{v\underline{e_t}\, dt}_{} = ds\,\underline{e_t} .$$

$v = \frac{ds}{dt}$



If $\underline{P} \perp \underline{v}$ then $\mathcal{P} = \underline{P} \cdot \underline{v} = 0$

$\underline{P} \perp d\underline{r}$ then $W_{\underline{P},AB} = 0 .$

$\underline{v}$

$$\underline{N} = N\underline{e_n}$$

$$\underline{W} = -mg\underline{E_y}$$

$$\underline{v} = v\underline{e_t}$$

$$W_{\underline{N},AB} = \int \underbrace{\underline{N} \cdot \underline{v}}_{0}\, dt$$

→ the normal force is workless.

→ what is the work of $\underline{W}$?

$$W_{\underline{W},AB} = \int_{t_A}^{t_B} \underline{W} \cdot \underline{v}\, dt$$

$$= \int_{r_A}^{r_B} \underline{W} \cdot d\underline{r}$$

$$= \int_{r_A}^{r_B} -mg\underline{E_y} \cdot (dx\,\underline{E_x} + dy\,\underline{E_y})$$

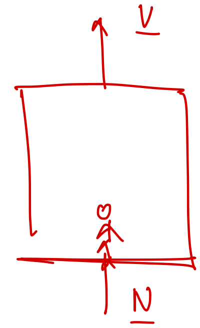
$$= \int_{y_A}^{y_B} - mg\, dy$$

$$= -mg \int_{y_A}^{y_B} dy$$

$$= (-)mg\underbrace{(y_B - y_A)}_{>0} < 0.$$

the particle is going up

$y_B > y_A$



$\underline{N} \cdot \underline{v} \neq 0$

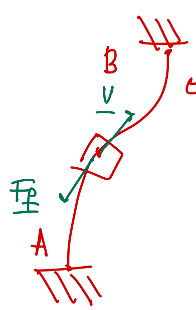
Normal force is not workless.

note. in general, the work a force between $[t_A, t_B]$ depends on the path of the particle between these two points.

- It is a special case that the work of the weight only depends on the endpoints. $\Longrightarrow$ to be discussed.

- If the angle between $\underline{f}$ & $\underline{v}$ is acute, work > 0

  $\underline{f}$ & $\underline{v}$ is obtuse, work < 0.



$\underline{F}_f$

$\leftarrow$ rough

$$W_{\underline{F}_f, AB} = \int_{r_A}^{r_B} \underline{F}_f \cdot d\underline{r}$$

$$= \int_{s_A}^{s_B} -\mu_s \|\underline{N}\|\, v\, \underline{e}_t \cdot ds\, \underline{e}_t$$

$$= \int_{s_A}^{s_B} -\mu_s \|\underline{N}\|\, v\, ds \leftarrow \text{depends on the shape of the guide.}$$

< 0

arc-length

$\underline{v}_{rel} = \underline{v}_{collar} - \cancel{\underline{v}_{guide\ under\ it}}$

Kinetic energy

We define the kinetic energy of a particle to be $T = \frac{1}{2} m \underline{v} \cdot \underline{v}$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m(\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2) = \frac{1}{2} m v^2.$$

The Work Energy theorem

$$\rightarrow \boxed{\dot{T} = P}$$

the rate of change of the kinetic energy of a particle is equal to the mechanical power of the forces acting on it.

Proof

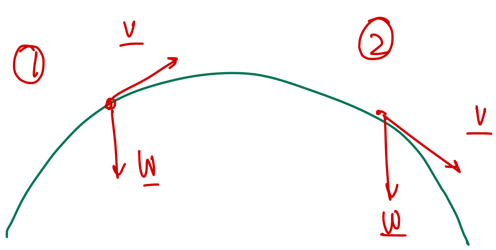
$$T = \frac{1}{2} m \underline{v} \cdot \underline{v}$$

$$\dot{T} = m\underline{a} \cdot \underline{v} = \underline{F} \cdot \underline{v} = P.$$

$$\int_{t_A}^{t_B} \dot{T}\, dt = \int_{t_A}^{t_B} P\, dt$$

$$\boxed{T_B - T_A = W_{\underline{F}, AB}}$$

* Projectile motion



$P = \underline{W} \cdot \underline{v} < 0$

$\dot{T} = P < 0.$

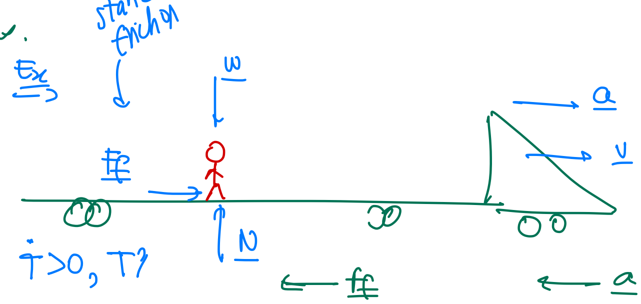
T is $\searrow$ decreasing.

$P = \underline{W} \cdot \underline{v} > 0$

$\dot{T} = P > 0$

T ?.

* A person on a bus.



$\underline{F_f} = m\underline{a}$

$W_{\underline{N}} = 0$

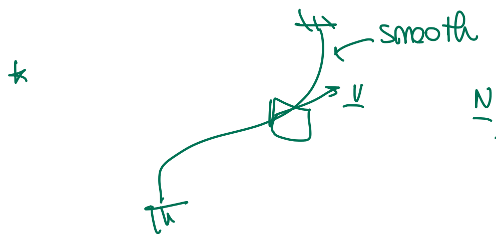
$W_{\underline{W}} = 0$

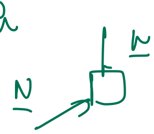
workless

$W_{\underline{fF}} = \underline{fF} \cdot \underline{v} > 0$ $\Rightarrow$ $\dot{T} > 0$, T ?

$$\underline{F} = m\underline{a}$$

$$\cancel{\underline{N}} + \cancel{\underline{w}} + \underline{F} = m\underline{a}$$





$W_{\underline{N}} = 0 \leftarrow$ workless

$W_{\underline{w}} < 0$

$\underline{w} \cdot \underline{v} < 0$, $\dot{T} < 0$, $T \searrow$.

SoFP

2. [MKB 03-080]

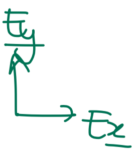
$x_B - x_A = 0.8$

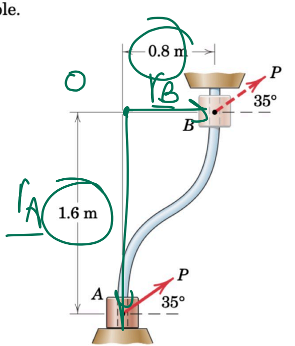
$y_B - y_A = 1.6$

3/80 The 2-kg collar is at rest in position A when the constant force P is applied as shown. Determine the speed of the collar as it passes position B if (*a*) $P = 25$ N and (*b*) $P = 40$ N. The curved rod lies in a vertical plane, and friction is negligible.

$\underline{r}_A = -1.6\,\underline{e}_y$

$\underline{r}_B = 0.8\,\underline{e}_x$





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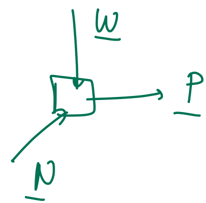


$$T_B - \cancel{T_A}^{\,0} = W_{\underline{F},AB}$$

$$T_B = \frac{1}{2} m \underline{v}_B \cdot \underline{v}_B = \frac{1}{2} m v_B^2 = \cancel{W_{\underline{N},AB}}^{\,0} + W_{\underline{W},AB} + W_{\underline{P},AB}$$

$$\frac{1}{2} m v_B^2 = W_{\underline{W}, AB} + W_{\underline{P}, AB}$$

$$\frac{1}{2} m v_B^2 = -mg(y_B - y_A) + \_\_\_\_\_\_ .$$

$$W_{\underline{P}, AB} = \int_{\underline{r}_A}^{\underline{r}_B} \underline{P} \cdot d\underline{r} = \int_{\underline{r}_A}^{\underline{r}_B} P(\cos 35^\circ \underline{E}_x + \sin 35^\circ \underline{E}_y) \cdot (dx\, \underline{E}_x + dy\, \underline{E}_y)$$

$$= \int_{x_A}^{x_B} P\cos 35^\circ dx + \int_{y_A}^{y_B} P\sin 35^\circ dy$$

$$= P\cos 35^\circ (x_B - x_A) + P\sin 35^\circ (y_B - y_A).$$

$$\boxed{\frac{1}{2} m v_B^2 = -mg(y_B - y_A) + P\cos 35^\circ (x_B - x_A) + P\sin 35^\circ (y_B - y_A).}$$

$v_B = \ldots$

- Power $= \underline{P} \cdot \underline{v}$
- $W = \int \dot{P}\, dt = \int_{t_A}^{t_B} \underline{P} \cdot \underline{v}\, dt = \int_{t_A}^{t_B} \underline{P} \cdot d\underline{r}$

  $W = 0$ if $\underline{P} \cdot \underline{v} = 0$ force is workless.

  $\underline{P} \perp \underline{v}$
- $T = \frac{1}{2} m \underline{v} \cdot \underline{v}$
- work-Energy theorem $\dot{T} = P$

  $T_B - T_A = W_{\underline{F}, AB}$

→ we derived this orig $\underline{F} = m\underline{a}$ BaLM

$$\int_{t_A}^{t_B} \underline{F} \cdot \underline{v}\, dt = \int_{t_A}^{t_B} m\, \underline{a} \cdot \underline{v}\, dt$$

Conservative forces

A force $\underline{F_c}$ is conservative if one can find a scalar function (called a potential energy function) $U = U(\underline{r})$ from which $\underline{f_c}$ is derivable:

$$\underline{F_c} = - \frac{\partial U}{\partial \underline{r}} = -\mathrm{grad}_{\underline{r}}\, U .$$

the minus sign is conventional.

$$W_{\underline{fc}, AB} = \int_{\underline{r_A}}^{\underline{r_B}} \underline{f_c} \cdot d\underline{r} = \int_{\underline{r_A}}^{\underline{r_B}} - \frac{\partial U}{\cancel{\partial \underline{r}}} \cdot \cancel{d\underline{r}} = -\int_{\underline{r_A}}^{\underline{r_B}} dU = -U_B + U_A .$$

$$\boxed{W_{\underline{fc}, AB} = -U_B + U_A}$$ where U is the potential energy function.

Notice The work of a conservative force only depends on the potential energy at the endpoints.

$\Rightarrow$ path independent work.

Conservative forces
- any constant force (including weight)
- the spring force
- gravitational force.

* Any constant force

$$\boxed{\begin{array}{l} \underline{C} \\ U = -\underline{C} \cdot \underline{r} \end{array}}$$

$$-\frac{\partial U}{\partial \underline{r}} = \underline{C}$$

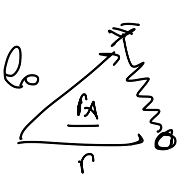
eg. $\underline{w} = -mg\underline{E_z}$

$\underline{E_z} \uparrow$ $\quad U = -\underline{w} \cdot \underline{r} = -mg\underline{E_z} \cdot (x\underline{E_x} + y\underline{E_y} + z\underline{E_z}) = mgz$

$U_{\underline{w}} = -U_B + U_A = -mgz_B + mgz_A = mg(z_A - z_B)$ (if we are going from A to B)

* Spring force

$$\boxed{\begin{array}{l} \underline{F_s} = -k\varepsilon \dfrac{\underline{r} - \underline{r_A}}{\|\underline{r} - \underline{r_A}\|} \\ U = \frac{1}{2} k \varepsilon^2 \end{array}}, \qquad \varepsilon = s = \|\underline{r} - \underline{r_A}\| - \ell_0$$



$$W_{\underline{F_s},AB} = -\frac{1}{2}k\varepsilon_B^2 + \frac{1}{2}k\varepsilon_A^2 .$$

* Gravitational force

$$\boxed{\begin{aligned} \underline{F_G} &= G\frac{M_e m}{(R_e+h)^2}(-\underline{e_r}) \\ U &= -\frac{GM_e m}{r} \end{aligned}}$$

If the only forces doing on the system are conservative, then the total energy of the system is conserved.

$$T_B - T_A = W_{\underline{F},AB}$$

$$\underline{F} = \underset{\text{conservative}}{\underline{F_c}} + \underset{\text{nonconservative}}{\underline{F_{nc}}}$$

$$T_B - T_A = \underbrace{W_{\underline{F_c},AB}}_{-U_B+U_A} + W_{\underline{F_{nc}},AB}$$

$$T_B - T_A = -U_B + U_A + W_{\underline{F_{nc}},AB}$$

$$T_B + U_B - (T_A + U_A) = W_{\underline{F_{nc}},AB}$$

Define the total mechanical energy $E = T + U$ (T: kinetic, U: potential.)

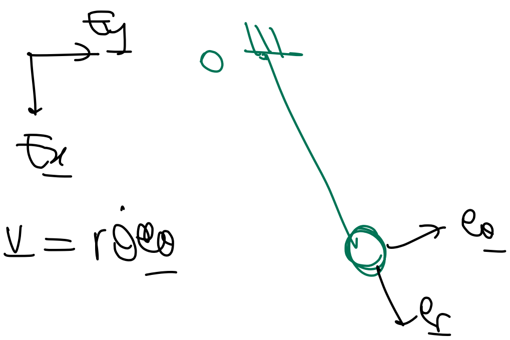
$$\boxed{E_B - E_A = W_{\underline{F_{nc}},AB}}$$

The change in total mechanical energy equals the work done by $\underline{F_{nc}}$

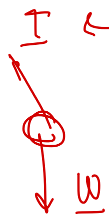
If $W_{\underline{F_{nc}},AB} = 0$ (meaning that the nonconservative forces do not do any work)

$$\boxed{E_A = E_B}$$ ← Energy of the system is conserved.

eg. The simple pendulum Is the energy of this system conserved or not? why?



$\underline{v} = r\dot{\theta}\underline{e}_\theta$



$\underline{T}$ ← workless

$\underline{T}\cdot\underline{v} = T\underline{e}_r \cdot r\dot{\theta}\underline{e}_\theta = 0$

$\underline{W}$ ← conservative

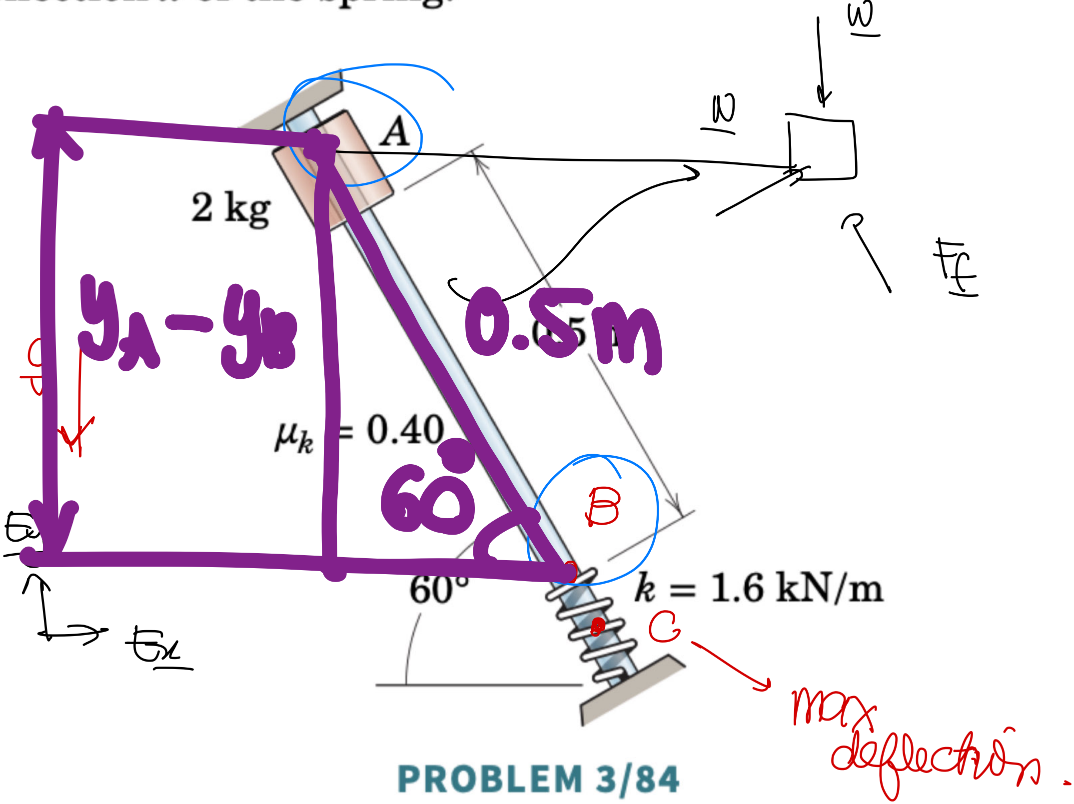
the only force doing work on the system is conservative, so the energy of the system is conserved.

Whenever you want to use energy conservation, you have to justify your usage by confirming that all the forces doing work on the system are conservative.

$$E = T + V = \frac{1}{2}m\underline{v}\cdot\underline{v} + U_{\underline{W}}$$

$$\boxed{E = \frac{1}{2}m(r\dot{\theta})^2 + mgx}$$

3/84 The 2-kg collar is released from rest at A and slides down the inclined fixed rod in the vertical plane. The coefficient of kinetic friction is 0.40. Calculate (a) the velocity v of the collar as it strikes the spring and (b) the maximum deflection x of the spring.



PROBLEM 3/84

V_B ? Work energy theorem between A and B

$-y_B + y_A$

$$T_B - \cancel{T_A}^{\,0} = U_{W,AB} + U_{N,AB} + U_{F_f,AB}$$

$$\frac{1}{2} m v_B^2 = + mg y_B - mg y_A + \int \cancel{N \cdot v}^{\,0}\, dt + \int_{t_A}^{t_B} F_f \cdot v\, dt$$

find from $F = ma$ (BoLM)

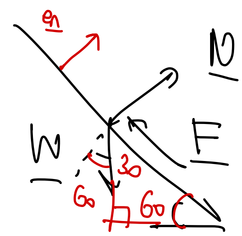
$$\int_{t_A}^{t_B} F_f \cdot v\, dt = -\int_{t_A}^{t_B} \mu_k \|N\| \frac{v\, e_t}{v} \cdot v\, e_t\, dt$$

$$= -\underbrace{\mu_k \|N\|}_{\text{constant.}} \int_{t_A}^{t_B} v\, dt$$

$$= -\mu_k \|N\| \int_{s_A}^{s_B} ds$$

$$= -\mu_k \|N\| (s_B - s_A)$$

$$= -(0.4)\|N\|(0.5)$$



$$N = mg\cos 60$$

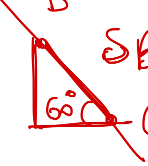
$V_B = \ldots$

* Write the W-E theorem between B and C

$$\cancel{T_C}^{\,0} - T_B = U_{W,BC} + \cancel{U_{N,BC}}^{\,0} + U_{F_f,BC} + U_{F_s,BC}$$

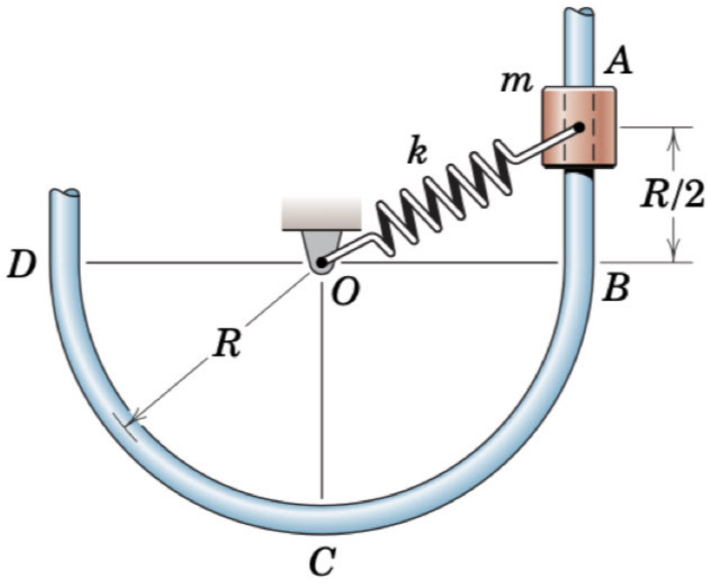
$$-\frac{1}{2} m v_B^2 = mg\underbrace{(y_B + y_C)}_{?} - \mu_k mg\cos 60\underbrace{(s_C - s_B)}_{?} - \frac{1}{2}k\varepsilon_C^2 + \cancel{\frac{1}{2}k\varepsilon_B^2}^{\,0}$$

$(s_B - s_C)^2$



$$\boxed{y_B - y_C = (s_B - s_C)\sin 60^\circ}$$

3/94 The collar of mass m is released from rest while in position A and subsequently travels with negligible friction along the vertical-plane circular guide. Determine the normal force (magnitude and direction) exerted by the guide on the collar (a) just before the collar passes point B, (b) just after the collar passes point B (i.e., the collar is now on the curved portion of the guide), (c) as the collar passes point C, and (d) just before the collar passes point D. Use the values $m = 0.4$ kg, $R = 1.2$ m, and $k = 200$ N/m. The unstretched length of the spring is $0.8R$.



PROBLEM 3/94

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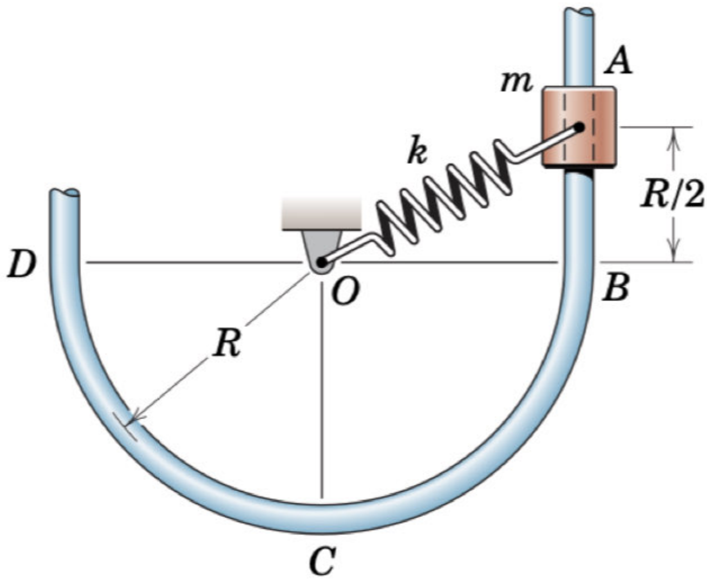

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