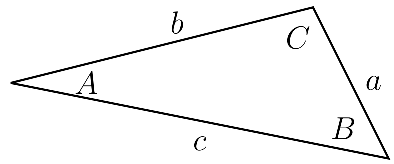
Sine Law and Cosine Law



Referring to the above triangle with interior angles A, B, and C and with side lengths a, b, and c, the sine law and cosine law are respectively

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}, \tag{1}$$
$$c^2 = a^2 + b^2 - 2ab\cos(C). \tag{2}$$

Chain Rule Let θ be some function of time such that $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \ddot{\theta}\mathbf{E}_z$, then

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\theta}{dt}. \tag{3}$$

Spring Force A spring of stiffness K with unstretched length ℓ_0 whose base is at point A and whose free end is attached to a mass m with position vector $\mathbf{r}$ applies a force on m that is

$$\mathbf{F}_s = -K(||\mathbf{r} - \mathbf{r}_A|| - \ell_0)\frac{\mathbf{r} - \mathbf{r}_A}{||\mathbf{r} - \mathbf{r}_A||}. \tag{4}$$

Friction Force

- Static friction is unknown but satisfies that static friction criterion $||\mathbf{F}_f|| \leq \mu_s||\mathbf{N}||$.
- Kinetic friction is prescribed according to Coulomb's friction model to be $\mathbf{F}_f = -\mu_k||\mathbf{N}||\frac{\mathbf{v}_{rel}}{||\mathbf{v}_{rel}||}$.

Rigid Body (RB) Kinematics If A is fixed to the RB and B moving with respect to it, then

$$\begin{aligned}
\mathbf{r}_B - \mathbf{r}_A &= x\mathbf{e}_x + y\mathbf{e}_y, \\
\mathbf{v}_B - \mathbf{v}_A &= \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A) + \mathbf{v}_{rel}, \\
\mathbf{a}_B - \mathbf{a}_A &= \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)) + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel},
\end{aligned} \tag{5}$$

where the angular velocity and acceleration of the RB are resp. $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ and $\boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z$ and

$$\mathbf{v}_{rel} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y, \quad \text{and} \quad \mathbf{a}_{rel} = \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y. \tag{6}$$

The corotation basis is taken such that

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \quad \text{and} \quad \mathbf{e}_y = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y. \tag{7}$$

Then, $\dot{\mathbf{e}}_x = \boldsymbol{\omega} \times \mathbf{e}_x$ and $\dot{\mathbf{e}}_y = \boldsymbol{\omega} \times \mathbf{e}_y$.

Sum of Forces and Moments Consider a rigid body with K applied forces $\mathbf{F}_i$ at points with position vectors $\mathbf{r}_i$ and an applied couple $\mathbf{M}^e$, then the sum of forces and the moments about an arbitrary point P on the body are respectively

$$\mathbf{F} = \sum_{i=1}^{K} \mathbf{F}_i, \quad \text{and} \quad \mathbf{M}^P = \sum_{i=1}^{K} (\mathbf{r}_i - \mathbf{r}_P) \times \mathbf{F}_i + \mathbf{M}_e. \tag{8}$$

Balance of Linear Momentum (BoLM) The BoLM of a RB is written as

$$\mathbf{F} = \dot{\mathbf{G}} = m\mathbf{a}_C, \tag{9}$$

where $\mathbf{G} = \int_{\mathcal{B}} \mathbf{v} dm = m\mathbf{v}_C$ is the linear momentum of the body $\mathcal{B}$ and C is its center of mass, $\mathbf{r}_C = \frac{\int_{\mathcal{B}} \mathbf{r} dm}{\int_{\mathcal{B}} dm}$. C behaves like a material point of the rigid body.

The **linear impulse - linear momentum equation** is

$$\int_{t_A}^{t^B} \mathbf{F} dt = \mathbf{G}(t_B) - \mathbf{G}(t_A). \tag{10}$$

Balance of Angular Momenum (BoAM) The BoAM of a RB has three equivalent forms

$$\begin{aligned}
&\mathbf{M}^O = \dot{\mathbf{H}}^O \quad \text{about a fixed point } O, \\
&\mathbf{M}^C = \dot{\mathbf{H}}^C \quad \text{about the center of mass } C, \\
&\mathbf{M}^P = \dot{\mathbf{H}}^P + (\mathbf{v}_P - \mathbf{v}_C) \times \mathbf{G} = \dot{\mathbf{H}}^C + (\mathbf{r}_C - \mathbf{r}_P) \times m\mathbf{a}_C \quad \text{about a material point } P \text{ on } \mathcal{B}.
\end{aligned} \tag{11}$$

where the angular momentum of a rigid body $\mathcal{B}$ relative to any material point P on the body is

$$\mathbf{H}^P = \int_{\mathcal{B}} (\mathbf{r} - \mathbf{r}_P) \times \mathbf{v} dm = \mathbf{H}^C + (\mathbf{r}_C - \mathbf{r}_P) \times \mathbf{G}, \quad \text{where} \quad \mathbf{H}^C = \int_B (\mathbf{r} - \mathbf{r}_C) \times \mathbf{v} dm. \tag{12}$$

Letting $\mathbf{r} - \mathbf{r}_C = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_Z$ can calculate

$$\begin{bmatrix} \mathbf{H}^C \cdot \mathbf{e}_x \\ \mathbf{H}^C \cdot \mathbf{e}_y \\ \mathbf{H}^C \cdot \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} I^C_{xx} & I^C_{xy} & I^C_{xz} \\ I^C_{xy} & I^C_{yy} & I^C_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \cdot \mathbf{e}_x \\ \boldsymbol{\omega} \cdot \mathbf{e}_y \\ \boldsymbol{\omega} \cdot \mathbf{e}_z \end{bmatrix} \tag{13}$$

where

$$\begin{matrix} I^C_{xx} = \int_B (y^2 + z^2) dm, & I^C_{yy} = \int_B (x^2 + z^2) dm, & I^C_{zz} = \int_B (x^2 + y^2) dm, \\ I^C_{xy} = -\int_B xy dm, & I^C_{yz} = -\int_B yz dm, & I^C_{xz} = -\int_B xz dm \end{matrix} \tag{14}$$

Thus, for $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$, we have

$$\begin{aligned}
\dot{\mathbf{H}}^O &= (I^O_{xz}\dot{\omega} - I^O_{yz}\omega^2)\mathbf{e}_x + (I^O_{yz}\dot{\omega} + I^O_{xz}\omega^2)\mathbf{e}_y + I^O_{zz}\dot{\omega}\mathbf{E}_z, \\
\dot{\mathbf{H}}^C &= (I^C_{xz}\dot{\omega} - I^C_{yz}\omega^2)\mathbf{e}_x + (I^C_{yz}\dot{\omega} + I^C_{xz}\omega^2)\mathbf{e}_y + I^C_{zz}\dot{\omega}\mathbf{E}_z, \\
\dot{\mathbf{H}}^P &= (I^P_{xz}\dot{\omega} - I^P_{yz}\omega^2)\mathbf{e}_x + (I^P_{yz}\dot{\omega} + I^P_{xz}\omega^2)\mathbf{e}_y + I^P_{zz}\dot{\omega}\mathbf{E}_z + \frac{d}{dt}\left((\mathbf{r}_C - \mathbf{r}_P) \times m\mathbf{v}_P\right).
\end{aligned} \tag{15}$$

The **angular impulse - angular momentum equations** are equivalently

$$\begin{aligned}\int_{t_A}^{t_B} \mathbf{M}^C dt &= \mathbf{H}^C(t_B) - \mathbf{H}^C(t_A) \quad \text{where } C \text{ is the center of mass,}\\ \int_{t_A}^{t_B} \mathbf{M}^O dt &= \mathbf{H}^O(t_B) - \mathbf{H}^C(t_A) \quad \text{if there is a fixed point } O.\end{aligned} \tag{16}$$

Parallel axis theorem Consider a material point A on the rigid body such that $\mathbf{r}_A - \mathbf{r}_C = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z$, then according to the parallel axis theorem

$$I_{xx}^A = I_{xx}^C + m(A_y^2 + A_z^2), \quad I_{xy}^A = I_{xy}^C - mA_xA_y, \quad etc. \tag{17}$$

The Koenig decomposition for the **kinetic energy of a RB** is

$$T = \frac{1}{2}m\mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2}\mathbf{H}^C \cdot \boldsymbol{\omega}. \tag{18}$$

If the body has a point O with zero velocity, $\mathbf{v}_O = \mathbf{0}$, then

$$T = \frac{1}{2}m\mathbf{v}_O \cdot \mathbf{v}_O. \tag{19}$$

The work-energy theorem for a RB between $[t_A, t_B]$ is

$$T_B - T_A = \sum_{i=1}^{K} W_{\mathbf{F}_i,AB} + W_{\mathbf{M}_e,AB}, \tag{20}$$

where the work of a force $\mathbf{F}_i$ applied at a point with position vector $\mathbf{r}_i$ over the interval $[t_A, t_B]$ is

$$W_{\mathbf{F}_i,AB} = \int_{t_A}^{t_B} \mathbf{F}_i \cdot \mathbf{v}_i dt = \int_{\mathbf{r}_i(t_A)}^{\mathbf{r}_i(t_B)} \mathbf{F} \cdot d\mathbf{r}_i \tag{21}$$

and the work of a moment is

$$W_{\mathbf{M},AB} = \int_{t_A}^{t_B} \mathbf{M} \cdot \boldsymbol{\omega} dt. \tag{22}$$

Conservative Force A conservative $\mathbf{F}_c$ is such that $W_{\mathbf{F}_c,AB} = -(U_B - U_A)$. Examples include

- any constant force $\mathbf{C}$ with $U = -\mathbf{C} \cdot \mathbf{r}$,
- the gravitational force $\mathbf{F}_G = G\frac{M_e m}{(R_e+h)^2}(-\mathbf{e}_r)$ with $U = -\frac{GM_e m}{r}$, and
- the spring force $\mathbf{F}_s = -K\varepsilon\frac{\mathbf{r}-\mathbf{r}_A}{||\mathbf{r}-\mathbf{r}_A||}$ with $U = \frac{1}{2}K\varepsilon^2$. The spring stretch $\varepsilon = ||\mathbf{r} - \mathbf{r}_A|| - \ell_0$.