Cone product reformulation for global optimization

Dimitris Bertsimas
Operations Research Center, Massachusetts Institute of Technology, United States, d.bertsim@mit.edu

Danique de Moor
Amsterdam Business School, University of Amsterdam, The Netherlands, d.demoor@uva.nl

Dick den Hertog
Amsterdam Business School, University of Amsterdam, The Netherlands, d.denhertog@uva.nl

Thodoris Koukouvinos
Operations Research Center, Massachusetts Institute of Technology, United States, tkoukouv@mit.edu

Jianzhe Zhen
School of Economics and Management, University of Chinese Academy of Sciences, China, trevorzhen@gmail.com

In this paper, we study nonconvex optimization problems involving sum of linear times convex (SLC) functions as well as conic constraints belonging to one of the five basic cones, that is, linear cone, second order cone, power cone, exponential cone, and semidefinite cone. By using the Reformulation Perspectification Technique, we can obtain a convex relaxation by forming the perspective of each convex function and linearizing all product terms with newly introduced variables. To further tighten the approximation, we can pairwise multiply (parts of) the conic constraints. In this paper, we analyze all possibilities of multiplying conic constraints. Particularly noteworthy are the novel results involving the power cone and exponential cone. We delineate methods for deriving new, valid linear and second-order cone inequalities for pairwise constraint multiplications involving the power cone and exponential cone, thereby enhancing the strength of the approximation. Numerical experiments on a quadratic optimization problem over exponential cone constraints and on a robust palatable diet problem over power cone constraints, demonstrate that including additional inequalities generated from the proposed pairwise multiplications improves the approximation. Moreover, when incorporated in a branch and bound procedure the global optimal solution of the original nonconvex optimization problem can often be obtained faster than by BARON.

Key words: Reformulation-Linearization Technique, perspective function, conic optimization, nonconvex optimization, conjugate function, branch and bound

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1. Introduction

In this paper, we consider the following nonconvex optimization problem:

$$\min_{\boldsymbol{x}} \quad f_{00}(\boldsymbol{x}) + \sum_{i\in\mathcal{I}^0} \left(b_{i0} - \boldsymbol{a}_{i0}^\top \boldsymbol{x}\right) f_{i0}(\boldsymbol{x}) \tag{1a}$$

$$\text{s.t.} \quad f_{0k}(\boldsymbol{x}) + \sum_{i\in\mathcal{I}^k} \left(b_{ik} - \boldsymbol{a}_{ik}^\top \boldsymbol{x}\right) f_{ik}(\boldsymbol{x}) \le 0, \qquad k \in \mathcal{K}, \tag{1b}$$

$$c_j(\boldsymbol{x}) \leq 0, \qquad j \in \mathcal{J}, \tag{1c}$$

where $\boldsymbol{x}, \boldsymbol{a}_{ik} \in \mathbb{R}^{n_x}, b_{ik} \in \mathbb{R}$, for every $i \in \mathcal{I}_0^k = \mathcal{I}^k \cup \{0\}$, $k \in \mathcal{K}_0 = \mathcal{K} \cup \{0\}$, each function $f_{ik} : \mathbb{R}^{n_x} \to (-\infty, \infty]$ is proper, closed, and convex, and each inequality $c_j(\boldsymbol{x}) \leq 0$ is representable in one or more of the five basic cones, that is, linear cone, second-order cone, power cone, exponential cone, and semi-definite cone. Observe that (1) is nonconvex, since it contains products of linear and convex functions. A broad class of nonconvex problems can be written in the form of (1), such as concave minimization problems, which often occur due to economies of scale, and problems with a Difference of Convex (DC) objective and/or constraints, see (Bertsimas et al., 2023, Example 1).

Bertsimas et al. (2023) show how to obtain the following convex relaxation of problem (1), using the Reformulation Perspectification Technique with Branch and Bound (RPT-BB):

$$\min_{\boldsymbol{x}, \boldsymbol{U}} \quad f_{00}(\boldsymbol{x}) + \sum_{i \in \mathcal{I}^0} \left(b_{i0} - \boldsymbol{a}_{i0}^\top \boldsymbol{x}\right) f_{i0}\left(\frac{b_{i0}\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_{i0}}{b_{i0} - \boldsymbol{a}_{i0}^\top \boldsymbol{x}}\right) \tag{2a}$$

$$\text{s.t.} \quad f_{0k}(\boldsymbol{x}) + \sum_{i \in \mathcal{I}^k} \left(b_{ik} - \boldsymbol{a}_{ik}^\top \boldsymbol{x}\right) f_{ik}\left(\frac{b_{ik}\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_{ik}}{b_{ik} - \boldsymbol{a}_{ik}^\top \boldsymbol{x}}\right) \leq 0, \qquad k \in \mathcal{K}, \tag{2b}$$

$$c_j(\boldsymbol{x}) \leq 0, \qquad j \in \mathcal{J}, \tag{2c}$$

where $\boldsymbol{x}\boldsymbol{x}^\top$ is linearized by $\boldsymbol{U}$. In order to obtain bounds on $\boldsymbol{U}$, we can generate additional convex inequalities by pairwise multiplying (parts of) the cone inequalities and subsequently convexifying the resulting inequalities. We can further link $\boldsymbol{U}$ with $\boldsymbol{x}$, via the following Linear Matrix Inequality (LMI):

$$\begin{pmatrix} \boldsymbol{U} & \boldsymbol{x} \\ \boldsymbol{x}^\top & 1 \end{pmatrix} \succeq 0. \tag{3}$$

Obtaining convexifiable constraints from pairwise multiplication of linear or quadratic inequalities is well known from the Reformulation Linearization Technique (RLT), introduced by Sherali and Adams (1990). RLT consists of two steps, those are, a reformulation step and a linearization step. RLT generates redundant nonconvex quadratic constraints from pairwise multiplication of the existing linear inequalities in the reformulation step. In the linearization step, the nonconvex quadratic components are linearized by substituting each distinct product of variables by a newly introduced continuous variable. These additional generated constraints are not redundant anymore after linearization and serve as bounds on the newly introduced variables.

Linearizing the product of linear constraints is further explored in Sherali and Tuncbilek (1992) and Sherali and Tuncbilek (1995). Sturm and Zhang (2003) show how to multiply a linear inequality with a conic quadratic inequality and reformulate the resulting constraint as a conic quadratic inequality. Jiang and Li (2019) show how to obtain a conic quadratic inequality from pairwise multiplication of two conic quadratic inequalities. Jiang and Li (2019) and Anstreicher (2017) address the same multiplication by reformulating each conic quadratic inequality as an LMI and subsequently pairwise multiply them to finally obtain one additional LMI using either the Hadamard product or Kronecker product respectively. We also refer to Jiang and Li (2020) for an overview of RLT approximations for quadratic optimization problems.

Moreover, Bertsimas et al. (2023) show how to multiply a linear inequality with a general convex inequality and how to convexify the resulting inequality. However, they mention that the pairwise multiplication of two general convex inequalities does not necessarily yield a convexifiable inequality. Anstreicher (2017) shows how to obtain a convexifiable constraint from pairwise multiplication of two LMIs.

Note that the solution of Problem (2) provides a lower bound for Problem (1). Further, an upper bound can be obtained from local optimization algorithms. RPT-BB leverages both mechanisms for obtaining bounds in a systematic global optimization approach for solving nonconvex optimization problems. During BB, the gap of the RPT approximation is closed by cutting the feasible region through additionally generated hyperplanes.

In this paper, we analyze all 15 possibilities of pairwise multiplication of (parts of) the five basic cone inequalities and demonstrate how to convexify the resulting constraints. Notably, the outcomes involving a power cone or an exponential cone present new findings. Furthermore, we provide numerical examples that validate the effectiveness of the newly proposed constraints derived from the multiplication of the inequalities from two basic cone definitions. Finally, we report numerical examples indicating that the cone product reformulations introduced in this manuscript not only enhance the RPT approximation but also expedite the RPT-BB approach to solve the nonconvex optimization problem to global optimality, surpassing the existing methods described by Bertsimas et al. (2023).

Contributions. Our main contributions can be summarized as follows:

- In this paper, we show for all 15 possibilities of pairwise multiplications of the five basic cone constraints how to convexify the resulting constraints. Especially the results for the cases in which a power cone or an exponential cone is involved are new. In the case of a power cone inequality we generate additional valid inequalities by linearizing the left-hand side (LHS) of the power cone inequality, to further enhance the RPT approximation. Further, when multiplying a power cone inequality with other cone inequalities, we show how to find the best reformulation out of the infinitely many possible ones for a given $(\boldsymbol{x}, \boldsymbol{U})$, by using a robust optimization lens leveraging the adversarial approach. Moreover, in the case of an exponential cone inequality we generate additional linear and quadratic inequalities, utilizing the Taylor expansion of the exponential function to the first or second order, which we can then multiply with other constraints to further enhance the approximation. Finally, we report numerical examples indicating that the cone product reformulations introduced in this manuscript not only enhance the RPT approximation but also expedite the RPT-BB approach to solve the nonconvex optimization problem to global optimality, surpassing the existing methods described by Bertsimas et al. (2023).

- We show that there are two ways to multiply a conically representable convex constraint and a linear inequality that yield the same result. More precisely, we show that the additional inequalities generated from the pairwise multiplication of a conically representable convex constraint with a linear inequality lead to the same inequalities that one would obtain from first reformulating the conically representable convex constraint into cone constraints and then pairwise multiply them with the linear inequality.
- We demonstrate that each additional constraint derived from the pairwise multiplication of parts of two basic cone inequalities considered in this paper is valuable. Specifically, for each considered multiplication, we identify an example showing that the resulting additional constraint outperforms all other possible constraints derived from different pairwise multiplications.
- For constraints involving DC functions, that are conically representable, we derive additional cone constraints obtained from first order conditions. We illustrate the derived constraints on multiple small examples and also show that they improve the approximation of a nonconvex optimization problem.
- We demonstrate the effectiveness of the proposed pairwise multiplications involving a power cone and an exponential cone through numerical experiments on a nonconvex quadratic optimization problem with exponential cone constraints as well as a robust palatable diet problem, including power cone constraints. We demonstrate that the additional inequalities, which are generated from pairwise multiplications of cone inequalities outlined in this paper, enhance the approximation. Further, when incorporated in a branch and bound method, the computational time to find the global optimal solution is reduced, while frequently outperforming BARON.

Notation. The calligraphic letters $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$ and the corresponding capital Roman letters I, J, K, L are reserved for finite index sets and their respective cardinalities, *i.e.*, $\mathcal{I}=\{1,\ldots,I\}$ etc. The subscript 0 for an index set indicates that the set additionally includes 0, *i.e.*, $\mathcal{I}_0=\{0,\ldots,I\}$ etc. Let $\mathbb{R}^{m\times n}$ denote the set of real $m\times n$ matrices, and $\mathbb{S}^n$ the set of real $n\times n$ symmetric matrices. The *domain* of a function $f:\mathbb{R}^{n_{\boldsymbol{\nu}}}\to[-\infty,+\infty]$ is defined as $\mathrm{dom}(f)=\{\boldsymbol{\nu}\in\mathbb{R}^{n_{\boldsymbol{\nu}}} \mid f(\boldsymbol{\nu})<+\infty\}$. The function f is *proper* if $f(\boldsymbol{\nu})>-\infty$ for all $\boldsymbol{\nu}\in\mathbb{R}^{n_{\boldsymbol{\nu}}}$ and $f(\boldsymbol{\nu})<+\infty$ for at least one $\boldsymbol{\nu}\in\mathbb{R}^{n_{\boldsymbol{\nu}}}$, implying that $\mathrm{dom}(f)\neq\emptyset$. In addition, f is *closed* if f is lower semicontinuous and either $f(\boldsymbol{\nu})>-\infty$ for all $\boldsymbol{\nu}\in\mathbb{R}^{n_\nu}$ or $f(\boldsymbol{\nu})=-\infty$ for all $\boldsymbol{\nu}\in\mathbb{R}^{n_\nu}$. The *conjugate* of a function $f:\mathbb{R}^{n_\nu}\to[-\infty,+\infty]$ is the function $f^*:\mathbb{R}^{n_\nu}\to[-\infty,+\infty]$ defined through $f^*(\boldsymbol{w})=\sup_{\boldsymbol{\nu}}\left\{\boldsymbol{\nu}^\top\boldsymbol{w}-f(\boldsymbol{\nu})\right\}$. The conjugate $(f^*)^*$ of f^* is called the *biconjugate* of f and is abbreviated as f^{**}. The *perspective* $h:\mathbb{R}^{n_\nu}\times\mathbb{R}_+\to[-\infty,+\infty]$ of a proper, closed and convex function $f:\mathbb{R}^{n_\nu}\to(-\infty,+\infty)$ is defined for all $\boldsymbol{\nu}\in\mathbb{R}^{n_\nu}$ and $t\in\mathbb{R}_+$ as $h(\boldsymbol{\nu},t)=tf(\boldsymbol{\nu}/t)$ if $t>0$, and $h(\boldsymbol{\nu},0)=\delta^*_{\mathrm{dom}(f^*)}(\boldsymbol{\nu})$, where $\delta^*_{\mathrm{dom}(f^*)}$ denotes the recession function (Rockafellar, 1970, p. 67 and Theorem 13.3). For ease of exposition, we use $tf(\boldsymbol{\nu}/t)$ to denote the perspective function $h(\boldsymbol{\nu},t)$ in the rest of this paper.

2. Overview: five basic cone inequalities and their products

In this section, we give an overview of all 15 possibilities of pairwise multiplying (parts of) the five basic cone inequalities to obtain additional cone inequalities. The five basic cone inequalities are given by:

(L) Linear inequality:

$$b-\boldsymbol{a}^\top\boldsymbol{x}\geq 0,$$

where $\boldsymbol{x}\in\mathbb{R}^{n_x}$.

(Q) Conic quadratic inequality:

$$b-\boldsymbol{a}^\top\boldsymbol{x}\geq\left\|\boldsymbol{D}\boldsymbol{x}+\boldsymbol{p}\right\|,$$

where $\boldsymbol{x}\in\mathbb{R}^{n_x},\ \boldsymbol{D}\in\mathbb{R}^{L\times n_x},\ \boldsymbol{p}\in\mathbb{R}^L$.

(P) Power cone inequality:

$$\prod_{i=1}^m x_i^{\alpha_i}\geq\sqrt{\sum_{i=m+1}^{n_x}x_i^2},\quad x_1,\cdots,x_m\geq 0,$$

where $n_x>m$, $\alpha_1,\cdots,\alpha_m>0$ and $\sum_{i=1}^m\alpha_i=1$, or equivalently

$$((x_1,\cdots,x_m),(x_{m+1},\cdots,x_{n_x}))\in\mathcal{P}_{n_x}^{\boldsymbol{\alpha}}=\left\{\boldsymbol{x}\in\mathbb{R}^{n_x}:\prod_{i=1}^m x_i^{\alpha_i}\geq\sqrt{\sum_{i=m+1}^{n_x}x_i^2},\quad x_1,\cdots,x_m\geq 0\right\},$$

where $\mathcal{P}_{n_x}^{\boldsymbol{\alpha}}$ denotes the power cone, with $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_m)$.

(E) Exponential cone inequality:

$$x_1\geq x_2\exp\left(\frac{x_3}{x_2}\right),\quad x_2\geq 0,$$

or equivalently

$$(x_1,x_2,x_3)\in\mathcal{K}_{\exp}=\left\{(x_1,x_2,x_3):x_1\geq x_2\exp\left(\frac{x_3}{x_2}\right),\ x_2\geq 0\right\},$$

where $\mathcal{K}_{\exp}$ denotes the exponential cone.

(**S**) Semidefinite cone inequality/LMI:

$$\boldsymbol{A}(\boldsymbol{x}) \succeq 0,$$

where $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}_0 + \boldsymbol{A}_1 x_1 + \cdots + \boldsymbol{A}_{n_x} x_{n_x}$ and $\boldsymbol{A}, \boldsymbol{A}_i \in \mathbb{S}^{n_x}, i \in \{0, 1, \ldots, n_x\}$.

The results of the 15 possibilities of pairwise multiplying (parts of) the five basic cone inequalities to obtain additional cone inequalities are summarized in Table 1. In the remainder of this paper we focus on all cases involving a power cone inequality or an exponential cone inequality. We refer to Appendix A for the other cases, that have already been studied in the literature (i.e., Cases 1, 2, 5, 6, 9, 12, and 15 in Table 1). We note that Case 9 is not explicitly mentioned in the literature; however, the steps followed are from Anstreicher (2017). Therefore, we present the case in Appendix A and note in Table 1 that it is derived in this paper. Finally, we note that Cases 3(i), 4 and 14(i) are from the literature; however, we do include them in the main text since they are connected with other subcases involving a power cone inequality or an exponential cone inequality.

3. Product with a power cone inequality

In this section, we show how to obtain additional valid inequalities from a power cone inequality as outlined in Section 2. Moreover, we examine all cases in which we multiply (parts of) one of the five basic cone inequalities with the power cone inequality and show how to obtain the best reformulation for the resulting constraint. We relegate the discussion on multiplying parts of the power cone with parts of the exponential cone to Section 4.4.

3.1. Generating valid inequalities from a power cone inequality

We first show that we can generate valid power cone inequalities from one power cone inequality by linearizing the product terms in the LHS of the power cone inequality. First, observe that in the LHS of the power cone inequality we can decompose the powers of the different x_i such that we get powers of products of x_i, x_j and add a power of 1 to satisfy the power cone inequality. For example, $x_1^{0.4} x_2^{0.6} = x_1^{0.3} x_1^{0.1} x_2^{0.1} x_2^{0.5} = x_1^{0.3} u_{12}^{0.1} x_2^{0.5} 1^{0.1}$. In the general form we obtain the following:

$$\prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=1}^{m+1} x_i^2} \iff \prod_{i=1}^{m} x_i^{\varepsilon_i} \prod_{i \leq j}^{m} (x_i x_j)^{\beta_{ij}} 1^{\delta} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}, \tag{4}$$

$$\implies \prod_{i=1}^{m} x_i^{\varepsilon_i} \prod_{i \leq j}^{m} (u_{ij})^{\beta_{ij}} 1^{\delta} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}, \tag{5}$$

$$\iff ((x_1, \cdots, x_m, u_{11}, \cdots, u_{mm}, 1), (x_{m+1}, \cdots, x_{n_x})) \in \mathcal{P}_{n'_x}^{\boldsymbol{\alpha}'}, \tag{6}$$

where $n'_x = n_x + m(m+1)/2 + 1$, $\boldsymbol{\alpha}' = (\varepsilon_1, \cdots, \varepsilon_m, \beta_{11}, \cdots, \beta_{mm}, \delta)$, and $(\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta) \in \mathcal{U}$, where

$$\mathcal{U} = \left\{ (\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta) \in \mathbb{R}^{n'_x} : \sum_{i=1}^{m} \varepsilon_i + \sum_{i \leq j}^{m} \beta_{ij} + \delta = 1, \quad \varepsilon_i + \beta_{ii} + \sum_{j=1}^{m} \beta_{ij} = \alpha_i, \ \forall i, \quad \varepsilon_i, \beta_{ij}, \delta \geq 0, \ \forall i, j \right\}. \tag{7}$$

Case	Cone-1	Cone-2	Cone-1 × Cone-2	Reference	Discussed in	Remarks
1	L	L	L	Sherali and Adams (1990)	Appendix A.1	
2	L	Q	Q	Sturm and Zhang (2003)	Appendix A.2	
3	L	P	(i) mL + P (ii) mL + P	Bertsimas et al. (2023) This paper	Section 3.2 Section 3.2	 Best reformulation
4	L	E	L + E	Bertsimas et al. (2023)	Section 4.2	No decomposition
5	L	S	S	Anstreicher (2017)	Appendix A.3	
6	Q	Q	(i) 3Q (ii) S (iii) S	Jiang and Li (2019) Jiang and Li (2019) Anstreicher (2017)	Appendix A.4 Appendix A.4 Appendix A.4	 (i) and (ii) are at least as good as (iii)
7	Q	P	(i) mQ + 2P (ii) mQ + 2P	This paper This paper	Section 3.3 Section 3.3	 Best reformulation
8	Q	E	(i) 2Q + E (ii) 3Q + E (iii) 5Q + E	This paper This paper This paper	Section 4.3 Section 4.3 Section 4.3	No decomposition $x_3 < 0$ $x_3 \geq 0$
9	Q	S	S	This paper	Appendix A.5	
10	P	P	(i) $m_1 m_2$L + (m_1+m_2+1)P (ii) $m_1 m_2$L + (m_1+m_2+1)P	This paper This paper	Section 3.4 Section 3.4	 Best reformulation
11	P	E	(i) mL + 2P + mE (ii) mL + 3P + mE (iii) mL + 5P + mE	This paper This paper This paper	Section 4.4 Section 4.4 Section 4.4	No decomposition $x_3 < 0$ $x_3 \geq 0$
12	P	S	mS	Anstreicher (2017)	Appendix A.6	
13	E	E	(i) L + 5E (ii) L + 7E (iii) L + Q + 9E (iv) L + 8E	This paper This paper This paper This paper	Section 4.5 Section 4.5 Section 4.5 Section 4.5	No decomposition $x_3, x_6 < 0$ $x_3, x_6 \geq 0$ $\text{sign}(x_3) \neq \text{sign}(x_6)$
14	E	S	(i) 2S (ii) 3S (iii) 4S	Anstreicher (2017) This paper This paper	Section 4.6 Section 4.6 Section 4.6	No decomposition $x_3 < 0$ $x_3 \geq 0$
15	S	S	(i) S (ii) S	Jiang and Li (2019) Anstreicher (2017)	Appendix A.7 Appendix A.7	(ii) is at least as good as (i) only if the two cones are of the same size

Table 1 Results of multiplying (parts of) two cone inequalities as given in Section 2. Cone-1 × Cone-2 refers to the total additional cone inequalities resulting from all possible multiplications of the inequalities in cone 1 with the inequalities in cone 2 .

Note that there are infinite ways to add such a constraint, since there are infinite possibilities to choose $\boldsymbol{\varepsilon}$, $\boldsymbol{\beta}$ and δ. One could consider (5) as a robust constraint, where $(\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta)$ are the uncertain parameters, and enforce that the constraint should hold for all $(\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta)$ in $\mathcal{U}$. Hence, the inequality becomes

$$\prod_{i=1}^{m} x_i^{\varepsilon_i} \prod_{i \leq j}^{m} (u_{ij})^{\beta_{ij}} 1^{\delta} \geq \sqrt{\sum_{i=1}^{m+1} x_i^2}, \quad \forall (\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta) \in \mathcal{U}. \tag{8}$$

We can address the robust constraint (8) in multiple ways. One approach is to compute the robust counterpart, that is, reformulate the constraint in terms of $(\boldsymbol{x}, \boldsymbol{U})$ for the worst case values of $(\boldsymbol{\epsilon}, \boldsymbol{\beta}, \delta)$ in the uncertainty set $\mathcal{U}$. Another approach is to compute the dual of the outer optimization problem over $(\boldsymbol{x}, \boldsymbol{U})$, in which case the "$\forall$" quantifier in the primal becomes a "$\exists$" quantifier in the dual. In this case, the uncertain parameters become variables in the dual and their products with other dual variables are linearized with new variables, see Bertsimas and den Hertog (2022). The third approach, which we follow in this paper, is the adversarial approach, see Bertsimas and den Hertog (2022), which consists of the following steps: At each iteration, instead of considering semi-infinite inequality (8), we only consider a finite subset of scenarios for $(\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta) \in \mathcal{U}$ to determine an optimal $(\boldsymbol{x}, \boldsymbol{U})$ (master problem). Then, we find the worst-case scenario for $(\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta)$ by minimizing the LHS of (8), with $(\boldsymbol{x}, \boldsymbol{U})$ fixed at the value determined in the master problem (sub problem). Utilizing a log transformation, the sub problem to find the worst-case for $(\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta)$ is the following linear optimization problem:

$$\min_{\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta} \left\{ \sum_i \varepsilon_i \log x_i + \sum_{i \leq j} \beta_{ij} \log u_{ij} \;\middle|\; (\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta) \in \mathcal{U} \right\}. \tag{9}$$

If (8) is satisfied for this worst-case scenario, the adversarial approach terminates and we have found the optimal solution. Otherwise, this worst-case scenario is added to the finite subset of scenarios and we repeat the previous steps for this new subset. We refer to Appendix B for the pseudocode of the adversarial approach. We note that if the nominal feasible region over $(\boldsymbol{x}, \boldsymbol{U})$ is bounded, then the adversarial approach converges Mutapcic and Boyd (2009). Observe that the Lipschitz continuity assumption is satisfied since the logarithm function is Lipschitz continuous. The effectiveness of the proposed valid inequalities for the power cone is demonstrated through the following toy example.

Example 1. Consider the following toy example

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & x_1 x_2 + x_1 + x_2 \\ \text{s.t.} \quad & x_1^{1/4} x_2^{3/4} \geq 1, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{10}$$

By applying RLT we obtain the following relaxation

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & u_{12} + x_1 + x_2 \\ \text{s.t.} \quad & x_1^{1/4} x_2^{3/4} \geq 1, \\ & x_1, x_2, u_{11}, u_{12}, u_{22} \geq 0. \end{aligned} \tag{11}$$

The solution of (11) appears to be

$$\boldsymbol{x}' = \begin{bmatrix} 0.44 \\ 1.32 \end{bmatrix} \quad \text{and} \quad \boldsymbol{U}' = \begin{bmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with objective value 1.7548, which constitutes a lower bound on (10). The obtained $\boldsymbol{x}'$ is a feasible solution to (10), and its corresponding value is 2.3320, which constitutes an upper bound on the optimal objective value of (10).

Next, we solve subproblem (9) in which we substitute $(\boldsymbol{x}', \boldsymbol{U}')$. We obtain the solution $(\boldsymbol{\varepsilon}^\top, \boldsymbol{\beta}^\top, \delta) = (0, 0, 0.125, 0, 0.375, 0.5)$ with objective value 0.5. Since $0.5 > 0$, we add the resulting additional inequality to (11) and obtain the optimal solution

$$\boldsymbol{x}' = \begin{bmatrix} 0.44 \\ 1.32 \end{bmatrix} \quad \text{and} \quad \boldsymbol{U}' = \begin{bmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{bmatrix} = \begin{bmatrix} 8.28 & 0 \\ 0 & 3.36 \end{bmatrix},$$

with objective value 1.7548. We again solve subproblem (9) in which we substitute $(\boldsymbol{x}', \boldsymbol{U}')$ and obtain the solution $(\boldsymbol{\varepsilon}^\top, \boldsymbol{\beta}^\top, \delta) = (0, 0.5, 0, 0.25, 0, 0.25)$ with objective value 0.25. Since $0.25 > 0$, we again add the resulting additional inequality to (11) and obtain the optimal solution

$$\boldsymbol{x}^* = \begin{bmatrix} 0.26 \\ 1.57 \end{bmatrix} \quad \text{and} \quad \boldsymbol{U}^* = \begin{bmatrix} u^*_{11} & u^*_{12} \\ u^*_{21} & u^*_{22} \end{bmatrix} = \begin{bmatrix} 11.16 & 0.40 \\ 0.40 & 4.39 \end{bmatrix},$$

with objective value 2.2341, which constitutes a tighter lower bound on (10). Solving (9) in which we substitute $(\boldsymbol{x}^*, \boldsymbol{U}^*)$, we obtain an objective value of 0. Hence, the adversarial approach terminates. The obtained $\boldsymbol{x}^*$ is a feasible solution to (10), and its corresponding value is 2.2341, which constitutes an upper bound on the optimal objective value of (10). Hence, $\boldsymbol{x}^*$ is an optimal solution to (10). □

3.2. Case 3 in Table 1: (L) × (P)

Consider one linear inequality and one power cone inequality

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad \begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_i \geq 0, \quad i = 1, \ldots, m, \end{cases}$$

where $\sum_{i=1}^m \alpha_i = 1, \boldsymbol{\alpha} \geq 0$.

Case 3(i) in Table 1. We multiply the linear inequality with the power cone inequality and obtain 1 additional power cone inequality:

$$\begin{aligned} & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \prod_{i=1}^{m} x_i^{\alpha_i} \geq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ \iff & \prod_{i=1}^{m} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{x} x_i)^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{x} x_i)^2} \\ \implies & \prod_{i=1}^{m} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^2} \,. \end{aligned} \tag{12}$$

Moreover, we multiply the linear inequality with the nonnegativity constraints of the power cone and obtain m additional linear inequalities, see Appendix A.1.

We note that in the approach that we describe here, we do not follow the same treatment as in Bertsimas et al. (2023), that is multiplying the argument of the convex function with the linear inequality and dividing the argument of the convex function by the linear inequality, although we obtain the same result. This follows from the homogeneity of the power cone. Namely, if $\boldsymbol{x}$ belongs to the power cone, then $(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{x}$ also belongs to the power cone.

Case 3(ii) in Table 1. Note that there are infinite possibilities for linearizing the LHS of (12). More precisely, we can write the LHS of (12) as follows:

$$\begin{aligned}
& (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \prod_{i=1}^{m} x_i^{\alpha_i} \geq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\
\iff & \left((b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})^2\right)^{\eta} \prod_{i\leq j}^{m} (x_i x_j)^{\beta_{ij}} \prod_{i=1}^{m} \left((b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) x_i\right)^{\epsilon_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{x} x_i)^2} \\
\implies & \left(b_1^2 - 2b_1 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_1\right)^{\eta} \prod_{i\leq j}^{m} (u_{ij})^{\beta_{ij}} \prod_{i=1}^{m} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^{\epsilon_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^2} \qquad (13) \\
\iff & ((b_1^2 - 2b_1 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_1, u_{11}, \cdots, u_{mm}, b_1 x_1 - \boldsymbol{a}_1^\top \boldsymbol{u}_1, \cdots, b_1 x_m - \boldsymbol{a}_1^\top \boldsymbol{u}_m), \\
& (b_1 x_{m+1} - \boldsymbol{a}_1^\top \boldsymbol{u}_{m+1}, \cdots, b_1 x_{n_x} - \boldsymbol{a}_1^\top \boldsymbol{u}_{n_x})) \in \mathcal{P}_{n_x'}^{\boldsymbol{\alpha}'},
\end{aligned}$$

where $n_x' = (1 + m(m+1)/2 + n_x)$, $\boldsymbol{\alpha}' = (\eta, \beta_{11}, \cdots, \beta_{mm}, \epsilon_1, \ldots, \epsilon_m)$,

$$\eta + \sum_{i\leq j}^{m} \beta_{ij} + \sum_{i=1}^{m} \epsilon_i = 1, \quad \beta_{ii} + \sum_{j=1}^{m} \beta_{ij} + \epsilon_i = \alpha_i, \ \forall i \in [m], \quad 2\eta + \sum_{i=1}^{m} \gamma_i = 1, \qquad (14)$$

and $(\eta, \boldsymbol{\beta}, \boldsymbol{\epsilon}) \geq \mathbf{0}$. We can consider (13) as a robust constraint, where $\eta, \boldsymbol{\beta}$ and $\boldsymbol{\epsilon}$ are the uncertain parameters and use the adversarial approach in a similar way as described in Section 3.1 to find the worst-case for $(\eta, \boldsymbol{\beta}, \boldsymbol{\epsilon})$, for a given $(\boldsymbol{x}, \boldsymbol{U})$ and repeat iteratively.

3.3. Case 7 in Table 1: (Q) $\times$ (P)

Consider one conic quadratic inequality and one power cone inequality

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq \|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\| \quad \text{and} \quad \begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0, \end{cases}$$

where $\boldsymbol{\alpha} \geq \mathbf{0}$, $\sum_{i=1}^{m} \alpha_i = 1$, and $\boldsymbol{D} \in \mathbb{R}^{L \times n_x}$, $\boldsymbol{p} \in \mathbb{R}^L$.

Case 7(i) in Table 1. We multiply the LHSs and RHSs of the conic quadratic inequality and the power cone inequality with each other and obtain 1 additional power cone inequality:

$$(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \prod_{i=1}^{m} x_i^{\alpha_i} \geq \|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\| \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \qquad (15)$$

$$\iff \prod_{i=1}^{m}(b_1x_i - \boldsymbol{a}_1^\top \boldsymbol{x}x_i)^{\alpha_i} \geq \left\|(\boldsymbol{D}\boldsymbol{x}+\boldsymbol{p})\boldsymbol{x}_{[m+1]}^\top\right\|_F$$
$$\Longrightarrow \prod_{i=1}^{m}(b_1x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^{\alpha_i} \geq \left\|\boldsymbol{D}\boldsymbol{U}_{[m+1]}+\boldsymbol{p}\boldsymbol{x}_{[m+1]}^\top\right\|_F,$$

where $\boldsymbol{x}_{[m+1]} = (x_{m+1} \ \cdots \ x_{n_x})$ and $\boldsymbol{U}_{[m+1]} = (\boldsymbol{u}_{m+1} \ \cdots \ \boldsymbol{u}_{n_x})$. Moreover, we multiply the conic quadratic inequality with the nonnegativity constraints and obtain m additional conic quadratic inequalities, see Appendix A.2. We further multiply the LHS of the conic quadratic inequality with both sides of the power cone inequality and obtain 1 additional power cone inequality, see Section 3.2.

Case 7(ii) in Table 1. Linearizing the LHS of (15), we obtain the LHS of (13). Hence we obtain the following power cone inequality

$$\begin{aligned}&((b_1 - \boldsymbol{a}_1^\top\boldsymbol{x}, b_1^2 - 2b_1\boldsymbol{a}_1^\top\boldsymbol{x} + \boldsymbol{a}_1^\top\boldsymbol{U}\boldsymbol{a}_1, x_1, \cdots, x_m, u_{11}, \cdots, u_{mm}, b_1x_1 - \boldsymbol{a}_1^\top\boldsymbol{u}_1, \cdots, b_1x_m - \boldsymbol{a}_1^\top\boldsymbol{u}_m),\\ &\ \left(\left(\boldsymbol{D}\boldsymbol{U}_{[m+1]} + \boldsymbol{p}\boldsymbol{x}_{[m+1]}^\top\right)_{11}, \cdots, \left(U_{[m+1]} + \boldsymbol{p}\boldsymbol{x}_{[m+1]}^\top\right)_{L,n_x-m-1}\right)) \in \mathcal{P}_{n_x'}^{\boldsymbol{\alpha}'},\end{aligned}$$

where $n_x' = (2+2m+m(m+1)/2+L(n_x-m-1)$ and $\boldsymbol{\alpha}' = (\delta, \eta, \varepsilon_1, \cdots, \varepsilon_m, \beta_{11}, \cdots, \beta_{mm}, \gamma_1, \cdots, \gamma_m)$. We can view the above inequality as a robust constraint, where $\delta, \boldsymbol{\varepsilon}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and η are the uncertain parameters and use the adversarial approach in a similar way as described in Section 3.1 to find the worst-case values for $(\delta, \boldsymbol{\varepsilon}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \eta)$ for a given $(\boldsymbol{x}, \boldsymbol{U})$, and repeat iteratively.

3.4. Case 10 in Table 1: (P) × (P)

Consider two power cone inequalities

$$\begin{cases}\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}\\ x_i \geq 0,\ i=1,\dots,m_1\end{cases} \quad \text{and} \quad \begin{cases}\prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2}\\ x_{\sigma(j)} \geq 0,\ j=1,\dots,m_2,\end{cases}$$

where σ is an arbitrary permutation, $n_x > m_1, m_2$, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \geq 0$ and $\sum_{i=1}^{m_1}\alpha_{1i} = \sum_{j=1}^{m_2}\alpha_{2j} = 1$.

Case 10(i) in Table 1. We multiply the left-hand sides and right-hand sides of the two power cone inequalities and obtain 1 additional power cone inequality:

$$\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}\sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \iff \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2 \sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \tag{16}$$

$$\Longrightarrow \prod_{i=1}^{m_1}\prod_{j=1}^{m_2} u_{i,\sigma(j)}^{\theta_{ij}} \geq \sqrt{\sum_{i=m_1+1}^{n_x}\sum_{j=m_2+1}^{n_x} u_{i,\sigma(j)}^2}, \tag{17}$$

where θ is such that

$$\sum_j \theta_{ij} = \alpha_{1i}, \quad \sum_i \theta_{ij} = \alpha_{2j}, \quad \theta_{ij} \geq 0, \quad i \in [m_1],\, j \in [m_2]. \tag{18}$$

Moreover, we multiply the nonnegativity constraints of one power cone with the nonnegativity constraints of the other power cone and obtain $m_1 m_2$ additional linear inequalities, see Appendix A.1. Finally, we multiply the nonnegativity constraints of each power cone with the power cone inequality of the other power cone and obtain $m_1 + m_2$ additional power cone inequalities, see Section 3.2.

Case 10(ii) in Table 1. Note that there are infinite number of possibilities for linearizing the LHS of (16). More precisely, we can write (17) as follows:

$$\begin{aligned}
&\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}\sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\
\Longleftrightarrow &\prod_{i=1}^{m_1}\prod_{j=i}^{m_1} (x_i x_j)^{\theta_{ij}} \prod_{i=1}^{m_2}\prod_{j=i}^{m_2} (x_{\sigma(i)} x_{\sigma(j)})^{\beta_{ij}} \prod_{i=1}^{m_1}\prod_{j=1}^{m_2} (x_i x_{\sigma(j)})^{\gamma_{ij}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}\sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\
\Longrightarrow &\prod_{i=1}^{m_1}\prod_{j=i}^{m_1} u_{ij}^{\theta_{ij}} \prod_{i=1}^{m_2}\prod_{j=i}^{m_2} u_{\sigma(i)\sigma(j)}^{\beta_{ij}} \prod_{i=1}^{m_1}\prod_{j=1}^{m_2} u_{i\sigma(j)}^{\gamma_{ij}} \geq \sqrt{\sum_{i=m_1+1}^{n_x}\sum_{j=m_2+1}^{n_x} u_{i,\sigma(j)}^2} \\
\Longrightarrow &((u_{11}, \cdots, u_{m_1 m_1}, u_{\sigma(1),\sigma(1)}, \cdots, u_{\sigma(m_2),\sigma(m_2)}, u_{1\sigma(1)}, \cdots, u_{m_1,\sigma(m_2)}), \\
&\quad (u_{m_1+1\sigma(m_2+1)}, \cdots, u_{n_x\sigma(n_x)})) \in \mathcal{P}_{n_x'}^{\boldsymbol{\alpha}'},
\end{aligned}$$

where $n_x' = m_1(m_1+1)/2 + m_2(m_2+1)/2 + m_1 m_2 + (n_x - m_1 - 1)(n_x - m_2 - 1)$ and $\boldsymbol{\alpha}' = (\theta_{11}, \cdots, \theta_{m_1 m_1}, \beta_{11}, \cdots, \beta_{m_2 m_2}, \gamma_{11}, \cdots, \gamma_{m_1 m_2})$. We can view the above inequality as a robust constraint, where $\boldsymbol{\theta}, \boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, are the uncertain parameters, which need to satisfy the following constraints:

$$\begin{aligned}
&\sum_{i=1}^{m_1}\sum_{j=i}^{m_1} \theta_{ij} + \sum_{i=1}^{m_2}\sum_{j=i}^{m_2} \beta_{ij} + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} \gamma_{ij} = 1, & \\
&\sum_{j=1}^{m_1} \theta_{ij} + \theta_{ii} + \sum_{j=1}^{m_2} \gamma_{ij} = \alpha_{1i}, & i \in [m_1], \\
&\sum_{i=1}^{m_2} \beta_{ij} + \beta_{jj} + \sum_{i=1}^{m_1} \gamma_{ij} = \alpha_{2j}, & j \in [m_2], \\
&\theta_{ij}, \beta_{ij}, \gamma_{ij} \geq 0, & i \in [m_1],\, j \in [m_2].
\end{aligned}$$

We can then use the adversarial approach in a similar way as described in Section 3.1 to find the worst-case values for $(\boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, for a given $(\boldsymbol{x}, \boldsymbol{U})$ and repeat iteratively.

4. Product with an exponential cone inequality

In this section, we derive valid inequalities from an exponential cone inequality, by leveraging the Taylor expansion of the exponential function, which we can then pairwise multiply with other existing inequalities to tighten the approximation of Problem (1). Moreover, we consider all cases in which we multiply (parts of) the inequalities belonging to one of the five basic cones with (parts of) the inequalities belonging to the exponential cone as given in Section 2.

4.1. Generating valid inequalities from an exponential cone inequality

Sometimes, the pairwise multiplication of exponential cone inequalities may not lead to constraints that involve products of the original variables, see Section 6.1, and hence do not tighten the approximation. For this reason we derive valid inequalities from the exponential cone inequality that we can use for pairwise multiplications. We further note that even if the pairwise multiplications of the original constraints involving exponential cone inequalities yield good bounds on the new variables, we can still improve them with the derived inequalities. Our main tool in deriving those inequalities, is the Taylor expansion of the exponential function, that is

$$\exp(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \ldots. \tag{19}$$

We can generate additional valid inequalities, depending on the sign of x_3. When $x_3 < 0$, there exists a $\xi \in \left[\frac{x_3}{x_2}, 0\right]$ such that

$$\exp\left(\frac{x_3}{x_2}\right) = 1 + \frac{x_3}{x_2} + \frac{\xi^2}{2} \geq 1 + \frac{x_3}{x_2}.$$

Therefore, we have $x_2 \exp\left(\frac{x_3}{x_2}\right) \geq x_2 + x_3$ and we derive the valid linear inequality

$$x_1 \geq x_2 + x_3. \tag{20}$$

When $x_3 \geq 0$, there exists a $\xi \in \left[0, \frac{x_3}{x_2}\right]$ such that

$$\exp\left(\frac{x_3}{x_2}\right) = 1 + \frac{x_3}{x_2} + \frac{x_3^2}{2x_2^2} + \frac{\xi^3}{6} \geq 1 + \frac{x_3}{x_2} + \frac{x_3^2}{2x_2^2}.$$

Therefore, we have $x_2 \exp\left(\frac{x_3}{x_2}\right) \geq x_2 + x_3 + \frac{x_3^2}{2x_2}$ and we derive the following valid inequalities

$$\begin{cases} x_1 \geq x_2 + x_3 + y, \\ \left\| \left(\sqrt{2}x_3, x_2 - y\right) \right\|_2 \leq x_2 + y. \end{cases} \tag{21}$$

We next show how we can obtain additional conic inequalities from pairwise multiplying the exponential cone inequality with one of the five basic cone inequalities as given in Section 2. We first provide a Lemma that will be useful in determining when some constraint multiplications are redundant.

Lemma 1. *Consider the constraint $f(\boldsymbol{x}) \leq 0$, where the function $f(\boldsymbol{x})$ is convex. Suppose that $f(\boldsymbol{x}) \geq g(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X}$, where $\mathcal{X}$ denotes the feasible set. Then, multiplying a linear inequality with $g(\boldsymbol{x}) \leq 0$ and perspectifying the result, yields a convex constraint that is redundant to the inequality that results from multiplying that linear inequality with $f(\boldsymbol{x}) \leq 0$.*

Proof. Let $a - \boldsymbol{b}^\top \boldsymbol{x} \geq 0$ denote the linear inequality. We obtain

$$(a - \boldsymbol{b}^\top \boldsymbol{x}) f(\boldsymbol{x}) \leq 0 \implies (a - \boldsymbol{b}^\top \boldsymbol{x}) f\left(\frac{(a - \boldsymbol{b}^\top \boldsymbol{x})\boldsymbol{x}}{a - \boldsymbol{b}^\top \boldsymbol{x}}\right) \leq 0 \implies (a - \boldsymbol{b}^\top \boldsymbol{x}) f\left(\frac{a\boldsymbol{x} - \boldsymbol{U}\boldsymbol{b}}{a - \boldsymbol{b}^\top \boldsymbol{x}}\right) \leq 0.$$

Since $f(\boldsymbol{x}) \geq g(\boldsymbol{x})$, we obtain that

$$(a - \boldsymbol{b}^\top \boldsymbol{x}) g\left(\frac{a\boldsymbol{x} - \boldsymbol{U}\boldsymbol{b}}{a - \boldsymbol{b}^\top \boldsymbol{x}}\right) \leq (a - \boldsymbol{b}^\top \boldsymbol{x}) f\left(\frac{a\boldsymbol{x} - \boldsymbol{U}\boldsymbol{b}}{a - \boldsymbol{b}^\top \boldsymbol{x}}\right) \leq 0.$$

Therefore, it follows that the constraint $(a - \boldsymbol{b}^\top \boldsymbol{x}) g\left(\frac{a\boldsymbol{x} - \boldsymbol{U}\boldsymbol{b}}{a - \boldsymbol{b}^\top \boldsymbol{x}}\right) \leq 0$ is redundant to the constraint $(a - \boldsymbol{b}^\top \boldsymbol{x}) f\left(\frac{a\boldsymbol{x} - \boldsymbol{U}\boldsymbol{b}}{a - \boldsymbol{b}^\top \boldsymbol{x}}\right) \leq 0$. □

4.2. Case 4 in Table 1: (L) × (E)

Consider one linear inequality and one exponential cone inequality

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad \begin{cases} x_1 \geq x_2 \exp\left(\dfrac{x_3}{x_2}\right) \\ x_2 \geq 0. \end{cases}$$

Bertsimas et al. (2023) show how to multiply a linear inequality with a convex inequality by first reformulating the resulting inequality in its perspective form, and subsequently linearizing all product terms. In the case of an exponential cone inequality this boils down to the following: We multiply the linear inequality with both sides of the exponential cone inequality and obtain 1 additional exponential cone inequality:

$$\begin{aligned} &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) x_1 \geq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) x_2 \exp\left(\frac{(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) x_3}{(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) x_2}\right) \\ \implies &b_1 x_1 - \boldsymbol{a}_1^\top \boldsymbol{u}_1 \geq (b_1 x_2 - \boldsymbol{a}_1^\top \boldsymbol{u}_2) \exp\left(\frac{b_1 x_3 - \boldsymbol{a}_1^\top \boldsymbol{u}_3}{b_1 x_2 - \boldsymbol{a}_1^\top \boldsymbol{u}_2}\right). \end{aligned} \tag{22}$$

Here, the first inequality follows from multiplying both the nominator and denominator in the exponential function by the left hand side (LHS) of the linear inequality, and the second inequality follows from linearizing the product terms $\boldsymbol{x}\boldsymbol{x}^\top$ by the matrix $\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$. Observe that the second inequality is jointly convex in $\boldsymbol{x}$ and $\boldsymbol{U}$, since the right hand side (RHS) is the perspective function of a convex function, which is convex, see Rockafellar (1970). Moreover, we multiply the linear inequality with the nonnegativity constraint of the exponential cone and obtain 1 additional linear inequality, see Appendix A.1.

Observe that the multiplications from the additional cases that can be obtained from the Taylor expansion are redundant. When $x_3 < 0$, we can apply Lemma 1 for $f(\boldsymbol{x}) = x_2 \exp\left(\frac{x_3}{x_2}\right)$ and $g(\boldsymbol{x}) = x_2 + x_3$. Further, when $x_3 \geq 0$, we can apply Lemma 1 for $f(\boldsymbol{x}) = x_2 \exp\left(\frac{x_3}{x_2}\right)$ and $g(\boldsymbol{x}) = x_2 + x_3 + \frac{x_3^2}{2x_2}$.

4.3. Case 8 in Table 1: (Q) × (E)

Consider one conic quadratic inequality and one exponential cone inequality

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\| \quad \text{and} \quad \begin{cases} x_1 \geq x_2 \exp\left(\dfrac{x_3}{x_2}\right) \\ x_2 \geq 0. \end{cases}$$

Case 8(i) in Table 1. We multiply the conic quadratic inequality with the nonnegativity constraint and the LHS of the exponential cone inequality and obtain 2 additional conic quadratic inequalities, see Appendix A.2. Moreover, we multiply the LHS of the conic quadratic inequality with the exponential cone inequality and obtain 1 additional exponential cone inequality, see Section 4.2.

Case 8(ii) in Table 1. When $x_3 < 0$, in addition to the inequalities in Case 8(i), we multiply linear inequality (20) with the conic quadratic inequality and obtain 1 additional conic quadratic inequality, see Appendix A.2.

Case 8(iii) in Table 1. When $x_3 \geq 0$, in addition to the inequalities in Case 8(i), we multiply the linear inequality in (21) with the conic quadratic inequality and obtain 1 additional conic quadratic inequality, see Appendix A.2. Moreover, we multiply the conic quadratic inequality in (21) with the initial conic quadratic inequality and obtain 2 additional conic quadratic inequalities, see Case 6(i) in Appendix A.4. Note that the multiplication of the RHS of the original conic quadratic inequality with the derived conic quadratic inequality is redundant by Lemma 1.

4.4. Case 11 in Table 1: (P) × (E)

Consider one power cone inequality and one exponential cone inequality

$$\begin{cases} \displaystyle\prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} x_1 \geq x_2 \exp\left(\dfrac{x_3}{x_2}\right) \\ x_2 \geq 0, \end{cases}$$

where $\boldsymbol{\alpha} \geq \boldsymbol{0}$ and $\sum_{i=1}^{m} \alpha_i = 1$.

Case 11(i) in Table 1. We multiply the nonnegativity constraints of the power cone with the nonnegativity constraint of the exponential cone and obtain m additional linear inequalities, see Appendix A.1. Moreover, we multiply the nonnegativity constraints of the power cone with the exponential cone inequality and obtain m additional exponential cone inequalities, see Case 4 in Section 4.2. Finally, we multiply the nonnegativity constraint of the exponential cone as well as the LHS of the exponential cone inequality with the power cone inequality and obtain 2 additional power cone inequalities, see Section 3.2. Hence we obtain the following set of additional inequalities:

$$\begin{cases} \prod_{i=1}^{m}(x_1x_i)^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x}(x_1x_i)^2} \\ \prod_{i=1}^{m}(x_2x_i)^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x}(x_2x_i)^2} \\ x_1x_1 \geq x_2x_1 \exp\left(\frac{x_3x_1}{x_2x_1}\right) \\ \qquad\vdots \\ x_1x_m \geq x_2x_m \exp\left(\frac{x_3x_m}{x_2x_m}\right) \end{cases} \implies \begin{cases} \prod_{i=1}^{m} u_{1i}^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} u_{1i}^2} \\ \prod_{i=1}^{m} u_{2i}^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} u_{2i}^2} \\ x_1, \cdots, x_m \geq 0 \\ u_{11} \geq u_{21} \exp\left(\frac{u_{31}}{u_{21}}\right) \\ \qquad\vdots \\ u_{1m} \geq u_{2m} \exp\left(\frac{u_{3m}}{u_{2m}}\right) \\ u_{21}, \cdots, u_{2m} \geq 0. \end{cases}$$

Case 11(ii) in Table 1. When $x_3 < 0$ (and hence is not part of the power cone), in addition to the inequalities in Case 11(i), we multiply the linear inequality (20) obtained from the Taylor expansion with the power cone and obtain 1 additional power cone inequality, see Section 3.2. Note that the multiplications of the derived linear inequality with the nonnegativities of the power cone are redundant by Lemma 1.

Case 11(iii) in Table 1. When $x_3 \geq 0$, we multiply the linear inequality and the quadratic inequality in (21), obtained from the Taylor expansion, with the power cone and obtain 3 additional power cone inequalities, see Sections 3.2 and 3.3, in addition to the inequalities in Case 11(i). Note that the multiplications of the derived linear inequality and the RHS of the derived conic quadratic inequality with the nonnegativity constraints of the power cone are redundant by Lemma 1.

Remark 1. Observe that also here, we can use the adversarial approach in a similar way as described in Section 3.1 to find the best power cone reformulation for a given $(\boldsymbol{x}, \boldsymbol{U})$.

4.5. Case 13 in Table 1: (E) $\times$ (E)

Consider two exponential cone inequalities

$$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0. \end{cases}$$

Case 13(i) in Table 1. We multiply the LHSs and RHSs of the exponential cone inequalities and obtain 1 additional exponential cone inequality:

$$x_1x_4 \geq x_2x_5 \exp\left(x_3x_5/x_2x_5 + x_6x_2/x_2x_5\right) \quad \Longrightarrow \quad u_{14} \geq u_{25} \exp\left((u_{35}+u_{26})/u_{25}\right).$$

Moreover, we multiply the nonnegativity constraints of the exponential cones with each other and obtain 1 additional linear inequality, see Appendix A.1, we multiply the nonnegativity constraints of each exponential cone with the exponential cone inequality of the other cone and obtain 2 additional exponential cone inequalities and finally we multiply the LHS of each exponential cone inequality with both sides of the other exponential cone inequality and obtain 2 additional exponential cone inequalities, see Section 4.2.

Case 13(ii) in Table 1. When $x_3 < 0$ and $x_6 < 0$, for each exponential cone we obtain an additional linear inequality from (19). For each exponential cone we multiply the derived linear inequality from (19) with the exponential cone inequality from the other cone and obtain 2 additional exponential cone inequalities, in addition to the inequalities in Case 13(i). Note that the multiplication of the two linear inequalities with each other and with the nonnegativity constraints of the other exponential cone are redundant by Lemma 1.

Case 13(iii) in Table 1. When $x_3 \geq 0$ and $x_6 \geq 0$, for each exponential cone, we obtain one additional linear and conic quadratic inequality from (19). We multiply each of those linear inequalities and the conic quadratic inequalities with the exponential cone inequalities belonging to the other exponential cone and obtain 4 additional exponential cone inequalities, see Section 4.2. We also multiply the conic quadratic inequalities with each other and obtain 1 additional conic quadratic inequality, see Appendix A.4. We note that the remaining constraint multiplications are redundant by Lemma 1.

Case 13(iv) in Table 1. When $x_3 < 0$ and $x_6 \geq 0$, we obtain the linear inequalities $x_1 \geq x_2 + x_3$ and $x_4 \geq x_5 + x_6 + \bar{y}$, and the quadratic inequality $x_2 + y \geq \left\|\sqrt{2}x_3, x_2 - y\right\|$ from the Taylor expansion of the exponential inequality. We multiply each of those linear inequalities with the exponential inequality belonging to the other exponential cone and obtain 2 additional exponential cone inequalities. Moreover, we multiply the LHS of the quadratic inequality with the exponential inequality belonging to the other exponential cone and obtain one more additional exponential cone inequality. Note that the remaining constraint multiplications are redundant by Lemma 1. Moreover, notice that we obtain the same number of additional constraints in case $x_3 \geq 0$ and $x_6 < 0$.

4.6. Case 14 in Table 1: (E) $\times$ (S)

Consider one exponential cone inequality and one LMI

$$\begin{cases} x_1 \geq x_2 \exp\left(\dfrac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0.$$

Case 14(i) in Table 1. We multiply the nonnegativity constraint and the LHS of the exponential cone inequality with the LMI and obtain 2 additional LMIs:

$$\begin{cases} x_1 \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\ x_2 \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} \boldsymbol{A}(\boldsymbol{u}_1) \succeq 0 \\ \boldsymbol{A}(\boldsymbol{u}_2) \succeq 0. \end{cases}$$

Note that the pairwise multiplication of a linear inequality with an LMI is already studied in Anstreicher (2017).

Case 14(ii) in Table 1. When $x_3 < 0$, in addition to the inequalities in Case 14(i), we multiply the linear inequality (20) with the LMI and obtain 1 additional LMI.

Case 14(iii) in Table 1. When $x_3 \geq 0$, we multiply the inequalities in (21) with the LMI and obtain 2 additional LMIs, see Appendix A.3 and Appendix A.5.

Remark 2. Note that for each case, we have only detailed how to pairwise multiply two generic inequalities from any of the five basic cones. However, one can also multiply each inequality by itself to derive additional inequalities. We refer to Appendix C for an overview of the number of additional inequalities one would obtain when applying full RPT, i.e., considering all possible constraint multiplications, including multiplications of the cone inequalities with themselves.

Remark 3. All considered additional constraint multiplications in this paper are valuable in the sense that for each case we have found an example demonstrating the dominance of the considered constraint over all other possible additional constraint multiplications. We refer to Section 5.2 for an overview of these dominance results.

5. Justification and enhancements

In this section, we investigate additional constraint multiplications and describe several ways to improve the approximation of nonconvex Problem (1). First, we have a result on the best order of the multiplication of a linear inequality with a conically representable constraint. Further, we identify the best linearization for quadratic inequalities and finally, for DC problems, we derive additional conic constraints, by leveraging first order conditions.

5.1. Justification for first reformulating into conic constraints

It might be the case that one of the constraints is not in conic form, but since it is conically representable, we can reformulate it such that it satisfies problem format (1). The question that then arises is which of the following options is better:

- Option 1: Multiply all linear constraints directly with this convex constraint that is not reformulated in conic form, following the methodology from Bertsimas et al. (2023).
- Option 2: Reformulate the conically representable constraints in conic form and then multiply this constraint with all linear constraints.

We will prove that both options lead to the same approximation. We use the definition of a conically representable constraint from Serrano (2015), that is, a constraint $f(\boldsymbol{x}) \leq 0$, where $f : \mathbb{R}^n \to \mathbb{R}$, is conically representable if its feasible set can be written as

$$\{\boldsymbol{x} \mid f(\boldsymbol{x}) \leq 0\} = \{\boldsymbol{x} \mid \exists \boldsymbol{u} \in \mathbb{R}^m,\ S(\boldsymbol{x},\boldsymbol{u}) = 0,\ T(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{K}\}, \tag{23}$$

where $S : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{k_1}$ and $T : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{k_2}$ are affine mappings and $\mathcal{K}$ is a cone. We have the following result.

Lemma 2. *Suppose the convex constraint $f(\boldsymbol{x}) \leq 0$ is conically representable, and suppose we multiply this constraint with a linear constraint $b - \boldsymbol{a}^\top \boldsymbol{x} \geq 0$. Then, the additional inequalities generated from the pairwise multiplication of the linear inequality with the convex constraint are equivalent for Options 1 and 2.*

Proof. Let S, T be the affine mappings that define the conic representation of the feasible set $\{\boldsymbol{x} \mid f(\boldsymbol{x}) \leq 0\}$ and let $\mathcal{K}$ be the corresponding cone. Let us denote the linear function $b - \boldsymbol{a}^\top \boldsymbol{x}$ by $\ell(\boldsymbol{x})$ and denote the linear function that results after linearizing $\boldsymbol{x}\ell(\boldsymbol{x})$ by $\tilde{\boldsymbol{\ell}}(\boldsymbol{x}, \boldsymbol{U})$, where $\boldsymbol{U} = \boldsymbol{x}\boldsymbol{x}^\top$, and $\tilde{\ell}_i(\boldsymbol{x}, \boldsymbol{U}) = bx_i - a^\top \boldsymbol{U}_i$. Then after RPT we obtain the constraint

$$\ell(\boldsymbol{x}) f\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})}\right) \leq 0. \tag{24}$$

On the other hand, we can first derive the affine mappings S and T that define the conic representation of the feasible set of the original constraint, and then multiply it by $\ell(\boldsymbol{x})$, and apply RPT. Then we obtain the set

$$\left\{ (\boldsymbol{x}, \boldsymbol{U}) \;\middle|\; \exists \boldsymbol{u} \in \mathbb{R}^m,\ S\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})}, \frac{\boldsymbol{u}}{\ell(\boldsymbol{x})}\right) = 0,\ T\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})}, \frac{\boldsymbol{u}}{\ell(\boldsymbol{x})}\right) \in \mathcal{K} \right\}. \tag{25}$$

Moreover we find

$$\left\{ (\boldsymbol{x}, \boldsymbol{U}) \;\middle|\; \ell(\boldsymbol{x}) f\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})}\right) \leq 0 \right\}$$

$$
\begin{aligned}
&= \left\{ (\boldsymbol{x}, \boldsymbol{U}) \;\middle|\; \exists \boldsymbol{u} \in \mathbb{R}^m,\ S\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x}, \boldsymbol{U})}{\ell(\boldsymbol{x})}, \boldsymbol{u} \right) = 0,\ T\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x}, \boldsymbol{U})}{\ell(\boldsymbol{x})}, \boldsymbol{u} \right) \in \mathcal{K} \right\} \\
&= \left\{ (\boldsymbol{x}, \boldsymbol{U}) \;\middle|\; \exists \boldsymbol{u} \in \mathbb{R}^m,\ S\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x}, \boldsymbol{U})}{\ell(\boldsymbol{x})}, \frac{\boldsymbol{u}}{\ell(\boldsymbol{x})} \right) = 0,\ T\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x}, \boldsymbol{U})}{\ell(\boldsymbol{x})}, \frac{\boldsymbol{u}}{\ell(\boldsymbol{x})} \right) \in \mathcal{K} \right\},
\end{aligned}
$$

which concludes the proof. Observe that if $\ell(\boldsymbol{x}) = 0$, then Bertsimas et al. (2023, Lemma 1) show that all constraints resulting from multiplying this equality constraint with any convex constraint are redundant to the constraints resulting from multiplying this equality constraint with the existing variables. □

5.2. Dominance results of the constraint multiplications considered in this paper

In this section, we demonstrate that each additional constraint derived from the pairwise multiplication of parts of two basic cone inequalities, as considered in this paper, is valuable (see Table 2). Specifically, for each multiplication analyzed, we provide an example demonstrating that the resulting additional constraint outperforms all other potential constraints derived from different pairwise multiplications. For further details on these examples, please refer to Appendix F.

Case	Cone-1	Cone-2	Constraints 1	Constraints 2	Example	Equation
3(i)	$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0$	$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0 \end{cases}$	$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0$	$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0 \end{cases}$	6	(51c), (51d)
3(ii)	$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0$	$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0 \end{cases}$	Best reformulation		1	
4	$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0$	$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases}$	$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0$	$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases}$	7	(53f)
7(i)	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \Vert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \Vert$	$\begin{cases} \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \\ x_i \geq 0,\ i = 1, \ldots, m_1 \end{cases}$	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \Vert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \Vert$	$\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}$	8	(55i)
			$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq 0$	$\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}$	9	(57i)
			$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \Vert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \Vert$	$x_i \geq 0,\ i = 1, \ldots, m$	10	(59d)

Case	Cone-1	Cone-2	Constraints 1	Constraints 2	Example	Equation
7(ii)	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$\begin{cases} \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \ge \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \\ x_i \ge 0,\ i = 1,\dots,m_1 \end{cases}$	Best reformulation		1	
8(i)	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$\begin{cases} x_1 \ge x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \ge 0 \end{cases}$	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$x_2 \ge 0$	11	(61h)
			$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$x_1 \ge 0$	12	(63h)
			$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge 0$	$x_1 \ge x_2 \exp\left(\frac{x_3}{x_2}\right)$	13	(65g)
8(ii)	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$\begin{cases} x_1 \ge x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \ge 0 \\ x_1 \ge x_2 + x_3 \end{cases}$	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$x_1 \ge x_2 + x_3$	14	(67i)
8(iii)	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$\begin{cases} x_1 \ge x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \ge 0 \\ x_1 \ge x_2 + x_3 + y \\ \lVert(\sqrt{2}x_3, x_2 - y)\rVert_2 \le x_2 + y \end{cases}$	$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \ge \lVert \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \rVert$	$\begin{cases} x_1 \ge x_2 + x_3 + y \\ \lVert(\sqrt{2}x_3, x_2 - y)\rVert_2 \le x_2 + y \end{cases}$	13	(65k)-(65n)
10(i)	$\begin{cases} \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \ge \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \\ x_i \ge 0,\ i = 1,\dots,m_1 \end{cases}$	$\begin{cases} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \ge \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\ x_{\sigma(j)} \ge 0,\ j = 1,\dots,m_2, \end{cases}$	$\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \ge \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2}$	$\prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \ge \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2}$	15	(69g)
			$x_i \ge 0,\ i = 1,\dots,m_1$	$\begin{cases} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \ge \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\ x_{\sigma(j)} \ge 0,\ j = 1,\dots,m_2, \end{cases}$	16 / 17	(71e), (71f) / (73c), (73d)
			$\begin{cases} \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \ge \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \\ x_i \ge 0,\ i = 1,\dots,m_1 \end{cases}$	$x_{\sigma(j)} \ge 0,\ j = 1,\dots,m_2$	16 / 17	(71e), (71f) / (73c), (73d)
10(ii)	$\begin{cases} \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \ge \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \\ x_i \ge 0,\ i = 1,\dots,m_1 \end{cases}$	$\begin{cases} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \ge \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\ x_{\sigma(j)} \ge 0,\ j = 1,\dots,m_2, \end{cases}$	Best reformulation		1	
11(i)	$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \ge \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \ge 0 \end{cases}$	$\begin{cases} x_1 \ge x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \ge 0 \end{cases}$	$x_1, \cdots, x_m \ge 0$	$\begin{cases} x_1 \ge x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \ge 0 \end{cases}$	16	(71j)
			$\prod_{i=1}^{m} x_i^{\alpha_i} \ge \sqrt{\sum_{i=m+1}^{n_x} x_i^2}$	$x_2 \ge 0$	18	(75j), (75k)
			$\prod_{i=1}^{m} x_i^{\alpha_i} \ge \sqrt{\sum_{i=m+1}^{n_x} x_i^2}$	$x_1 \ge 0$	16	(71k)
11(ii)	$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \ge \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \ge 0 \end{cases}$	$\begin{cases} x_1 \ge x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \ge 0 \\ x_1 \ge x_2 + x_3 \end{cases}$	$\prod_{i=1}^{m} x_i^{\alpha_i} \ge \sqrt{\sum_{i=m+1}^{n_x} x_i^2}$	$x_1 \ge x_2 + x_3$	19	(77l)
			$x_1, \cdots, x_m \ge 0$	$x_1 \ge x_2 + x_3$	Lemma 1	n/a
11(iii)	$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \ge \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \ge 0 \end{cases}$	$\begin{cases} x_1 \ge x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \ge 0 \\ x_1 \ge x_2 + x_3 + y \\ \lVert(\sqrt{2}x_3, x_2 - y)\rVert_2 \le x_2 + y \end{cases}$	$\prod_{i=1}^{m} x_i^{\alpha_i} \ge \sqrt{\sum_{i=m+1}^{n_x} x_i^2}$	$\begin{cases} x_1 \ge x_2 + x_3 + y \\ \lVert(\sqrt{2}x_3, x_2 - y)\rVert_2 \le x_2 + y \end{cases}$	17	(73l)-(73n)
			$x_1, \cdots, x_m \ge 0$	$\begin{cases} x_1 \ge x_2 + x_3 + y \\ \lVert(\sqrt{2}x_3, x_2 - y)\rVert_2 \le x_2 + y \end{cases}$	Lemma 1	n/a

Case	Cone-1	Cone-2	Constraints 1	Constraints 2	Example	Equation
13(i)	$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases}$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \end{cases}$	$x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	$x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right)$	18	(75h)
			$x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	$x_4 \geq 0$	20	(79f) - (79i)
			$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases}$	$x_5 \geq 0$	21	(81e), (81f)
			$x_1 \geq 0$	$x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right)$	20	(79f) - (79i)
			$x_2 \geq 0$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \end{cases}$	21	(81e), (81f)
13(ii)	$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \\ x_1 \geq x_2 + x_3 \end{cases}$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \\ x_4 \geq x_5 + x_6 \end{cases}$	$x_1 \geq x_2 + x_3$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \end{cases}$	22	-
			$x_1 \geq x_2 + x_3$	$x_4 \geq x_5 + x_6$	Lemma 1	n/a
			$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases}$	$x_4 \geq x_5 + x_6$	22	-
13(iii)	$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \\ x_1 \geq x_2 + x_3 + y_1 \\ \left\Vert(\sqrt{2}x_3, x_2 - y_1)\right\Vert_2 \leq x_2 + y_1 \end{cases}$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \\ x_4 \geq x_5 + x_6 + y_2 \\ \left\Vert(\sqrt{2}x_6, x_5 - y_2)\right\Vert_2 \leq x_5 + y_2 \end{cases}$	$\begin{cases} x_1 \geq x_2 + x_3 + y_1 \\ \left\Vert(\sqrt{2}x_3, x_2 - y_1)\right\Vert_2 \leq x_2 + y_1 \end{cases}$	$x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right)$	23	(84j), (84k)
			$\begin{cases} x_1 \geq x_2 + x_3 + y_1 \\ \left\Vert(\sqrt{2}x_3, x_2 - y_1)\right\Vert_2 \leq x_2 + y_1 \end{cases}$	$x_5 \geq 0$	Lemma 1	n/a
			$\begin{cases} x_1 \geq x_2 + x_3 + y_1 \\ \left\Vert(\sqrt{2}x_3, x_2 - y_1)\right\Vert_2 \leq x_2 + y_1 \end{cases}$	$\begin{cases} x_4 \geq x_5 + x_6 + y_2 \\ \left\Vert(\sqrt{2}x_6, x_5 - y_2)\right\Vert_2 \leq x_5 + y_2 \end{cases}$	Lemma 1	n/a
			$x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	$\begin{cases} x_4 \geq x_5 + x_6 + y_2 \\ \left\Vert(\sqrt{2}x_6, x_5 - y_2)\right\Vert_2 \leq x_5 + y_2 \end{cases}$	23	(84j), (84k)
			$x_2 \geq 0$	$\begin{cases} x_4 \geq x_5 + x_6 + y_2 \\ \left\Vert(\sqrt{2}x_6, x_5 - y_2)\right\Vert_2 \leq x_5 + y_2 \end{cases}$	Lemma 1	n/a
13(iv)	$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \\ x_1 \geq x_2 + x_3 \end{cases}$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \\ x_4 \geq x_5 + x_6 + y \\ \left\Vert(\sqrt{2}x_6, x_5 - y)\right\Vert_2 \leq x_5 + y \end{cases}$	$x_1 \geq x_2 + x_3$	$\begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0 \end{cases}$	22	-
			$x_1 \geq x_2 + x_3$	$\begin{cases} x_4 \geq x_5 + x_6 + y \\ \left\Vert(\sqrt{2}x_6, x_5 - y)\right\Vert_2 \leq x_5 + y \end{cases}$	Lemma 1	n/a
			$x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right)$	$\begin{cases} x_4 \geq x_5 + x_6 + y \\ \left\Vert(\sqrt{2}x_6, x_5 - y)\right\Vert_2 \leq x_5 + y \end{cases}$	23	(84j), (84k)
			$x_2 \geq 0$	$\begin{cases} x_4 \geq x_5 + x_6 + y \\ \left\Vert(\sqrt{2}x_6, x_5 - y)\right\Vert_2 \leq x_5 + y \end{cases}$	Lemma 1	n/a

Table 2 Dominance results of each additional inequality resulting from multiplying two cone inequalities as given in Section 2.

5.3. The best linearization for the quadratic case

In this section we describe that as for power cone inequalities, also for quadratic inequalities there are multiple choices for linearization. We first give an example that shows that different choices may lead to different solutions.

EXAMPLE 2. Consider the following toy example

$$
\begin{aligned}
\max_{\boldsymbol{x}} \quad & x_1^2 + x_2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 \leq 1, \\
& x_1, x_2 \geq 0.
\end{aligned}
\tag{26}
$$

The optimal solution of this problem is $(\frac{1}{2}\sqrt{3}, \frac{1}{2})$ and the optimal value is $\frac{5}{4}$. By applying RLT we obtain the following relaxation

$$
\begin{aligned}
\max_{\boldsymbol{x}} \quad & u_{11} + x_2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 \leq 1, \\
& u_{11} + u_{22} \leq 1, \\
& x_1, x_2, u_{11}, u_{22} \geq 0.
\end{aligned}
\tag{27}
$$

The solution of (27) appears to be $u_{11} = x_2 = 1$, $u_{22} = x_1 = 0$, with optimal value 2. This solution is suboptimal for the original Problem (26). However, if we add the inequality that occurs when we partially linearize, i.e., the inequality $u_{11} + x_2^2 \leq 1$, then we do obtain the optimal solution of (26). It can easily be verified that if we add the LMI (3) to (27), then we also get the optimal solution to (26). □

The question hence arises whether we should linearize all quadratic terms, or only a part of these terms, such that the remaining part is convex. The following lemma shows that linearizing all quadratic terms in combination with adding the LMI (3) yields the tightest approximation. Hence, when LMI (3) is included, then the full linearization always yields the best approximation.

LEMMA 3. *Consider the quadratic inequality*

$$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x} + c \leq 0,$$

where $\boldsymbol{A}$ is not necessarily positive semidefinite. Then the linearization that gives the tightest approximation of the above quadratic inequality is obtained by linearizing all quadratic terms if LMI (3) is included in the constraints.

Proof. We search for the best value of a semidefinite matrix $\boldsymbol{B}$ such that linearizing $\boldsymbol{x}^\top(\boldsymbol{A} - \boldsymbol{B})\boldsymbol{x}$ and keeping $\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x}$ yields the tightest approximation. In other words we consider the following inequality

$$\max_{\boldsymbol{B} \succeq \boldsymbol{0}} \left\{ \boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} + \operatorname{Tr}\left((\boldsymbol{A} - \boldsymbol{B})\boldsymbol{U}\right) + \boldsymbol{b}^\top \boldsymbol{x} + c \right\} \leq 0,$$

which is equivalent to

$$\max_{\boldsymbol{B}\succeq \boldsymbol{0}} \left\{\operatorname{Tr}\left(\boldsymbol{B}(\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{U})\right)\right\} + \operatorname{Tr}(\boldsymbol{A}\boldsymbol{U}) + \boldsymbol{b}^\top \boldsymbol{x} + c \le 0.$$

Taking the dual of the maximization problem we obtain that this is equivalent to

$$\operatorname{Tr}(\boldsymbol{A}\boldsymbol{U}) + \boldsymbol{b}^\top \boldsymbol{x} + c \le 0, \quad \begin{pmatrix} \boldsymbol{U} & \boldsymbol{x} \\ \boldsymbol{x}^\top & 1 \end{pmatrix} \succeq 0,$$

where the LMI is the same as (3). Therefore, the LMI can be interpreted (from its dual) as obtaining the best $\boldsymbol{B}$, and we do not need to add different LMIs. □

5.4. First-order conditions for DC problems

Derivation. In this section, we consider the following DC constraint

$$c_0(\boldsymbol{x}) - c_1(\boldsymbol{x}) \le 0,$$

where $c_0, c_1 : \mathbb{R}^{n_x} \to (-\infty, +\infty]$ are proper, closed, and convex functions. Rockafellar (1970) shows that, using the biconjugate reformulation, the above inequality can be equivalently written as

$$\begin{aligned} c_0(\boldsymbol{x}) - \sup_{\boldsymbol{y}\in \operatorname{dom}(c_1^*)} \{\boldsymbol{x}^\top \boldsymbol{y} - c_1^*(\boldsymbol{y})\} \le 0 &\iff c_0(\boldsymbol{x}) + \inf_{\boldsymbol{y}\in \operatorname{dom}(c_1^*)} \{-\boldsymbol{x}^\top \boldsymbol{y} + c_1^*(\boldsymbol{y})\} \le 0 \\ &\iff \begin{cases} c_0(\boldsymbol{x}) - \boldsymbol{x}^\top \boldsymbol{y} + c_1^*(\boldsymbol{y}) \le 0, \\ \boldsymbol{y} \in \operatorname{dom}(c_1^*), \end{cases} \end{aligned}$$

as long as the infimum is attained, see also (Bertsimas et al., 2023, Example 1). Note that the obtained problem in this case is in the format of Problem (1). Now suppose $c_1(\boldsymbol{x})$ is differentiable, then we have

$$\boldsymbol{y} = \nabla_{\boldsymbol{x}} c_1(\boldsymbol{x}). \tag{28}$$

We can leverage this extra equation to get a better approximation, as illustrated in the following examples.

Examples.

EXAMPLE 3. Suppose $c_1(x) = -\log(x)$. Then (28) becomes $y = -1/x$, or $xy = -1$. We introduce the variable v to linearize the product xy and obtain the equality $v = -1$. □

EXAMPLE 4. Suppose $c_1(x) = x\log(x)$. Then (28) becomes $y = 1 + \log(x)$. Hence, we can add the following convex inequalities to Problem (1): $y \le 1 + \log(x)$, $v \ge x + x\log(x)$, where $v = xy$. □

EXAMPLE 5. Suppose $c_1(\boldsymbol{x}) = \log\left(\sum_j \exp(x_j)\right)$. Then (28) becomes

$$y_i = \frac{\exp(x_i)}{\sum_j \exp(x_j)} \iff \log(y_i) + \log\left(\sum_j \exp(x_j)\right) = x_i \iff y_i \log(y_i) + y_i \log\left(\sum_j \exp(x_j)\right) = x_i y_i.$$

We linearize $y_i x_j$ with v_{ij} and obtain the following convex inequality

$$y_i \log(y_i) + y_i \log\left(\sum_j \exp\left(\frac{v_{ij}}{y_i}\right)\right) \leq v_{ii}, \quad \forall i \tag{29}$$

which we can include in Problem (1). Note that $y_i \log\left(\sum_j \exp\left(\frac{v_{ij}}{y_i}\right)\right)$ is a perspective function of the convex function $\log\left(\sum_j \exp(v_{ij})\right)$. $\square$

Showing the benefit in a nonconvex optimization problem. We consider the following problem:

$$\begin{aligned}
\max_{\boldsymbol{x}} \quad & \log\left(\sum_{i=1}^{n_x} \exp(x_i)\right) && && \text{(30a)}\\
\text{s.t.} \quad & x_1 \exp(x_i) \leq \rho, && i \in \{1, \ldots, n_x\}, && \text{(30b)}\\
& x_1 \geq 0. && && \text{(30c)}
\end{aligned}$$

Using the biconjugate of the convex objective, we obtain the following equivalent problem:

$$\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{y}} \quad & \boldsymbol{x}^\top \boldsymbol{y} + \sum_{i=1}^{n_x} w_i && && \text{(31a)}\\
\text{s.t.} \quad & x_1 \exp(x_i) \leq \rho, && i \in \{1, \ldots, n_x\}, && \text{(31b)}\\
& y_i \exp\left(\frac{w_i}{y_i}\right) \leq 1, && i \in \{1, \ldots, n_x\}, && \text{(31c)}\\
& \sum_{i=1}^{n_x} y_i = 1, && && \text{(31d)}\\
& x_1, \boldsymbol{y} \geq \boldsymbol{0}. && && \text{(31e)}
\end{aligned}$$

We can consider the valid inequalities (29) derived from the first order conditions, see Example 5. We next compare the obtained upper bounds for Problem (31), when we use the decomposition of the exponential cone as well as the first order conditions. The results are illustrated in Table 3. In all instances we fix $\rho = 1$. We refer to Appendix D.1 for the formulations of the three different approximations. From Table 3 we observe that when we include the inequalities obtained from the decomposition of the exponential cone, the upper bound improves. We further notice a more significant improvement in the upper bound when including the inequalities obtained from the first order conditions.

n_x	w/o dec-foc	w dec w/o foc	w/o dec w foc	w dec-foc
10	243,922	196,411	29,335	20,314
20	435,374	429,614	26,559	10,820
30	452,940	243,679	28,396	9,909
40	542,497	487,828	20,532	11,032
50	317,332	255,552	30,662	15,259

Table 3 Comparison of the obtained upper bounds for Problem (31), including the LMI, with and without the decomposition of the exponential cone (dec) and the first order conditions (foc). n_x is the dimension of $\boldsymbol{x}$.

6. Numerical experiments

In this section, we demonstrate empirically the benefit of the cone product reformulations introduced in this paper. More precisely, we consider a quadratic optimization problem over exponential cone constraints, demonstrating the value of the proposed methodology for the exponential cone as well as a robust palatable diet problem showing the benefit of the proposed methodology for the power cone.

All numerical experiments are performed on an Intel i9 2.3GHz CPU core with 16 GB RAM. All computations for RPT-BB and SCIP are conducted with MOSEK version 9.2.45 (MOSEK ApS, 2020), Gurobi version 9.0.2 (Gurobi Optimization, 2019), and implemented using Julia 1.5.3 and the Julia package `JuMP.jl` version 0.21.6, and all computations for BARON are conducted with BARON version 20.10.16 (Sahinidis, 1996) implemented using the Python package `pyomo` version 6.4.1.

6.1. Quadratic optimization with exponential cone constraints

In this section we consider the following problem

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \boldsymbol{x}^\top \boldsymbol{A}_0 \boldsymbol{x} + \boldsymbol{b}_0^\top \boldsymbol{x} + c_0 && \text{(32a)}\\
\text{s.t.} \quad & \log\left(\sum_{i=1}^{n_x} \exp(-x_i)\right) \le \alpha, && \text{(32b)}\\
& \sum_{i=1}^{n_x} \exp(x_i) \le \beta. && \text{(32c)}
\end{aligned}
$$

Using the conic representation of constraints (32b), (32c), Problem (32) is equivalent to the following problem:

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \boldsymbol{x}^\top \boldsymbol{A}_0 \boldsymbol{x} + \boldsymbol{b}_0^\top \boldsymbol{x} + c_0 && && \text{(33a)}\\
\text{s.t.} \quad & \sum_{i=1}^{n_x} z_i \le 1, && && \text{(33b)}\\
& \exp(-x_i - \alpha) \le z_i, && i \in \{1, \dots, n_x\}, && \text{(33c)}
\end{aligned}
$$

$$\sum_{i=1}^{n_x} t_i \le \beta, \tag{33d}$$

$$\exp(x_i) \le t_i, \qquad i \in \{1, \ldots, n_x\}. \tag{33e}$$

Using the decomposition of the exponential cone as explained in Section 4.1, we generate the following additional valid linear inequalities from (33c) and (33e)

$$z_i \ge -x_i - \alpha + 1, \quad t_i \ge x_i + 1, \quad i \in \{1, \ldots, n_x\},$$

respectively. We note that without the decomposition of the exponential cone, the relaxation obtained from multiplying constraints and linearizing products with new variables is unbounded, since $\boldsymbol{x}\boldsymbol{x}^\top$ does not appear in the constraints. However, we can link them by considering the decomposition of the exponential cone and as a result obtain tighter bounds. In Table 4 we solve Problem (33) to optimality using RPT-BB, with and without LMI (3), on six instances which reflect the average of 10 randomly generated instances. We also compare the results with BARON. The formulation after multiplying all constraints and the data generation for each instance are summarized in Appendix D.2 and Appendix E respectively. The maximum time limit is equal to 3600 seconds, hence if the computation time equals 3600*, the optimum cannot be found within 3600 seconds and all approaches return the best value they can obtain within 3600 seconds. A ”-” indicates that no solution was found after one hour.

Instance	n_x	w/o LMI			w LMI			BARON	
		Opt	Time(s)	Hyp	Opt	Time(s)	Hyp	Opt	Time(s)
1	5	-102	0.1	0	-102	0.1	0	-102	1
2	10	-175.8	0.1	0	-175.8	0.1	0	-175.8	0.5
3	10	-1885.4	0.2	0	-1885.4	0.7	0	-1885.4	220.2
4	20	-8172.8	4.4	1	-8172.8	25.1	1	-8172.8	3600*
5	50	-37306.6	101.4	3.2	-37306.6	2100.2	3.1	-37306.6	3600*
6	100	-326577.3	75.4	1	-	3600*	0	-326577.3	3600*

Table 4 Optimal value (Opt) and computation time (Time) comparisons for Problem (33). Hyp represents the total number of hyperplanes generated during branch and n_x represents the problem dimension.

From Table 4, we observe that for all instances we were able to solve the problem to optimality with branch and bound in less computational time than BARON, when decomposing the exponential cone. Moreover, we observe that in instances 4, 5, and 6 corresponding to 20, 50, and 100 variables respectively, when using the proposed valid inequalities the problem could be solved to optimality in seconds, while BARON located the global optimal solution but could not prove optimality within one hour. Finally, we note that in instance 6 which involves 100 variables, the problem could not be solved at the root node after one hour when including the LMI.

6.2. Robust palatable diet problem

In this section, we consider the palatable diet problem where there is uncertainty in the coefficients of one nutrient. The palatable diet problem is an important part of the World Food Programme's (WFP) food supply chain. The problem is to maximize palatability, while satisfying diet requirements. The main variables are the ration variables r_k, i.e. the amount of ingredient k in the ration. Further, the palatability is defined as a function $\hat{h}(\boldsymbol{r})$, which we assume is quadratic, that is $\hat{h}(\boldsymbol{r}) = \boldsymbol{r}^\top \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}^\top \boldsymbol{r} + d$. Utilizing the dataset from Maragno et al. (2023), consisting of observations $(\boldsymbol{r}_i, \hat{h}(\boldsymbol{r}_i))$, we find the values of $\boldsymbol{A}, \boldsymbol{b}$ and d that fit them best by regression. Moreover, we include diet constraints, ensuring that the total nutritional value of a certain nutrient l is not below the required nutritional value η_l for that nutrient, that is $\sum_{k\in\mathcal{K}} \beta_{kl} r_k \geq \eta_l$. Finally, we also have a budget constraint, that is $\sum_{k\in\mathcal{K}} c_k r_k \leq W$. The problem formulation is as follows:

$$
\begin{aligned}
\max_{\boldsymbol{r}} \quad & \boldsymbol{r}^\top \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}^\top \boldsymbol{r} + d && && \text{(34a)}\\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}} c_k r_k \leq W, && && \text{(34b)}\\
& \sum_{k\in\mathcal{K}} \beta_{kl} r_k \geq \eta_l && l \in \mathcal{L}, && \text{(34c)}\\
& \boldsymbol{r} \geq \boldsymbol{0}. && && \text{(34d)}
\end{aligned}
$$

It is often the case that the nutrient coefficients are uncertain. Assuming uncertainty in the coefficients of nutrient m, we obtain the following robust constraint:

$$
(\boldsymbol{\beta}_m + \boldsymbol{z})^\top \boldsymbol{r} \geq \eta_m, \ \forall \boldsymbol{z} \in \mathcal{U},
$$

where $\mathcal{U} = \{\boldsymbol{z} : \|\boldsymbol{z}\|_p \leq \rho\}$, for $p \geq 1$. In this case, the robust counterpart is as follows (Bertsimas and den Hertog (2022))

$$
\boldsymbol{\beta}_m^\top \boldsymbol{r} - \rho \|\boldsymbol{r}\|_q \geq \eta_m,
$$

where $1/p + 1/q = 1$. The constraint can be written as $\|\boldsymbol{r}\|_q \leq \frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m)$ and, by using auxiliary variables $\boldsymbol{t}$, can be reformulated as the following set of linear and power cone inequalities:

$$
\begin{cases}
\displaystyle\sum_k t_k = \frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m), \\
t_k^{1/q} \left(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m)\right)^{1-1/q} \geq |r_k|, \quad k \in \mathcal{K}.
\end{cases}
$$

Hence, the final problem formulation is as follows:

$$
\begin{aligned}
\max_{\boldsymbol{r},\boldsymbol{t}} \quad & \boldsymbol{r}^\top \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}^\top \boldsymbol{r} + d && \text{(35a)}\\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}} c_k r_k \leq W, && \text{(35b)}
\end{aligned}
$$

$$\sum_k \beta_{kl} r_k \geq \eta_l, \qquad l \in \mathcal{L} \setminus \{m\}, \tag{35c}$$

$$\sum_k t_k = \frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m), \tag{35d}$$

$$t_k^{1/q}(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m))^{1-1/q} \geq |r_k|, \qquad k \in \mathcal{K}, \tag{35e}$$

$$\boldsymbol{r} \geq \mathbf{0}. \tag{35f}$$

We compare the optimal value and computational time for RPT-BB applied to Problem (35) with and without the multiplication of power cone inequalities with each other, while including LMI (3). We also compare the results with the results obtained by BARON. The results for ThiamineB1 and NicacinB3 as the robust nutrients, are illustrated in Table 5. The nutrient coefficients β_{kl} are from Peters et al. (2022) and the costs c_k are from de Moor et al. (2024). The problem formulation after multiplying all constraints is provided in Appendix D.3. We set the maximum time limit equal to 3600 seconds, hence if the computation time equals 3600^*, the optimum cannot be found within 3600 seconds and all approaches return the best value they can obtain within 3600 seconds. We fix $p = 3$, $\rho = 0.1$, $W = 5$.

Rob Nutr	w/o additions			w additions			BARON	
	Opt	Time(s)	Hyp	Opt	Time(s)	Hyp	Opt	Time(s)
ThiamineB1	269.1	111.2	9	269.1	16	0	269.1	1.9
NicacinB3	213	44.5	4	213	39.6	1	212.2	3600^*

Table 5 Optimal value (Opt) and computation time in seconds (Time) comparisons for Problem (35) with and without the proposed additions. Hyp represents the total number of hyperplanes generated during branch and bound.

From Table 5 we observe that for ThiamineB1 as the robust nutrient, all methods find the global optimal solution, with BARON achieving the best computational time. We also notice that including the multiplication of power cone inequalities improves the approximation and as a result the computational time decreases from 111.20 to 16.01 seconds. In case the robust nutrient is NicacinB3, we observe that adding the power cone multiplications improves the computational time, while also finding the global optimal solution. In this case BARON could not solve the problem within one hour and returned a solution with slightly smaller objective value.

7. Discussion and conclusion

In this paper, we studied in detail the pairwise multiplications of cone inequalities. In particular, we showed how we can pairwise multiply one of the five basic cone constraints with exponential and power cone inequalities and obtain additional convex constraints. Moreover, we derived valid inequalities from exponential and power cone inequalities, which can further strengthen the approximation. Further, we provided examples showing that each of the obtained inequalities can tighten the approximation. In addition, for DC problems we derived valid inequalities from first order conditions. In the numerical experiments, we provided empirical evidence, suggesting that the cone product reformulations introduced in this paper improve the approximation, while often leading to smaller computational times than BARON. In future work, it would be interesting to investigate adaptations of the proposed methodology, including partial constraint multiplications as well as partial generation of product variables.

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Appendix

A. Multiplication of cone inequalities from the literature

In this appendix, we provide all multiplications of cone inequalities from Table 1 that are from the literature.

A.1. Case 1 in Table 1: (L) $\times$ (L)

Consider two linear inequalities

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq 0.$$

Multiplying the two linear inequalities yields 1 additional linear inequality (Sherali and Alameddine, 1992):

$$
\begin{aligned}
(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \geq 0 &\iff b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{a}_2 \geq 0 \\
&\implies b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_2 \geq 0.
\end{aligned}
$$

A.2. Case 2 in Table 1: (L) $\times$ (Q)

Consider one linear inequality and one conic quadratic inequality

$$
b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\| \boldsymbol{D}\boldsymbol{x} + \boldsymbol{d} \right\|.
$$

Multiplying the linear inequality with both sides of the conic quadratic inequality yields 1 additional conic quadratic inequality (Sturm and Zhang (2003)) :

$$
\begin{aligned}
&(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \left\| \boldsymbol{D}\boldsymbol{x} + \boldsymbol{d} \right\| \leq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \\
\iff &\left\| (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(\boldsymbol{D}\boldsymbol{x} + \boldsymbol{d}) \right\| \leq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \\
\implies &\left\| b_1 \boldsymbol{D}\boldsymbol{x} + b_1 \boldsymbol{d} - \boldsymbol{D}\boldsymbol{U}\boldsymbol{a}_1 - \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{d} \right\| \leq b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_2.
\end{aligned}
$$

A.3. Case 5 in Table 1: (L) $\times$ (S)

Consider one linear inequality and one LMI respectively

$$
b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0.
$$

We apply RPT to the multiplication of these inequalities, and obtain one additional LMI:

$$
\begin{aligned}
&(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\
\iff &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \boldsymbol{A}_0 + (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \boldsymbol{A}_1 x_1 + \cdots + (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \boldsymbol{A}_{n_x} x_{n_x} \succeq 0 \\
\implies &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \boldsymbol{A}_0 + (b_1 x_1 - \boldsymbol{a}_1^\top \boldsymbol{u}_1) \boldsymbol{A}_1 + \cdots + (b_1 x_{n_x} - \boldsymbol{a}_1^\top \boldsymbol{u}_{n_x}) \boldsymbol{A}_{n_x} \succeq 0.
\end{aligned}
$$

A.4. Case 6 in Table 1 (Q) $\times$ (Q)

Consider two conic quadratic inequalities

$$
b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq \left\| \boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1 \right\| \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\| \boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2 \right\|. \tag{36}
$$

We multiply the left-hand side of the first conic quadratic inequality with both sides of the second conic quadratic inequality and the left-hand side of the second conic quadratic inequality with both sides of the first conic quadratic inequality to obtain 2 additional conic quadratic inequalities, see

Appendix A.2. Moreover, we multiply the left-hand sides and right-hand sides of the conic quadratic inequalities with each other and obtain 1 additional conic quadratic inequality:

$$
\begin{aligned}
& (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \geq \|\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1\| \|\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2\| && (37) \\
\iff \quad & b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{a}_2 \geq \|(\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1)(\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2)^\top\|_F && (38) \\
\implies \quad & b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_2 \geq \|\boldsymbol{D}_1 \boldsymbol{U} \boldsymbol{D}_2^\top + \boldsymbol{p}_1 \boldsymbol{x}^\top \boldsymbol{D}^\top + \boldsymbol{D}\boldsymbol{x}\boldsymbol{p}_2^\top + \boldsymbol{p}_1 \boldsymbol{p}_2^\top\|_F. && (39)
\end{aligned}
$$

This is Case 6(i) in Table 1.

In the literature also two LMIs are proposed. First observe that the two conic quadratic inequalities (36) can be written as

$$
b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq \|\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1\| \iff \begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \boldsymbol{I} \end{bmatrix} \succeq 0
$$

and

$$
b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \|\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2\| \iff \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \boldsymbol{I} \end{bmatrix} \succeq 0.
$$

We now assume that, without loss of generality, the matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are of the same size. Indeed, suppose that $\boldsymbol{D}_1$ has less rows than $\boldsymbol{D}_2$, then we can extend matrix $\boldsymbol{D}_1$ by zero rows or by copying scaled versions of some of the original rows. Using the fact that the Kronecker product of two positive semidefinite matrices is also positive semidefinite (Horn and Johnson, 1991, Theorem 4.2.12) and linearizing each element of the product, we obtain

$$
\begin{aligned}
& \begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \boldsymbol{I} \end{bmatrix} \otimes \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \boldsymbol{I} \end{bmatrix} \succeq 0 \\
\implies & \left[\begin{array}{cc|cc|c|cc}
\alpha & \boldsymbol{\gamma}^\top & \delta_1 & \boldsymbol{\eta}_1^\top & \cdots & \delta_r & \boldsymbol{\eta}_r^\top \\
\boldsymbol{\gamma} & \alpha I & \boldsymbol{\eta}_1 & \delta_1 I & & \boldsymbol{\eta}_r & \delta_r I \\
\hline
\delta_1 & \boldsymbol{\eta}_1^\top & \alpha & \boldsymbol{\gamma}^\top & & & \\
\boldsymbol{\eta}_1 & \delta_1 I & \boldsymbol{\gamma} & \alpha I & & & \\
\hline
\vdots & & & & \ddots & & \\
\hline
\delta_r & \boldsymbol{\eta}_r^\top & & & & \alpha & \boldsymbol{\gamma}^\top \\
\boldsymbol{\eta}_r & \delta_r I & & & & \boldsymbol{\gamma} & \alpha I
\end{array}\right] \succeq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= b_1 b_2 - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} + a_1^\top U a_2 \\
\boldsymbol{\gamma} &= b_1 (\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2) - (a_1^\top \boldsymbol{x}) \boldsymbol{p}_2 - \boldsymbol{D}_2 \boldsymbol{U} \boldsymbol{a}_1 \\
\delta_i &= b_2 (\boldsymbol{d}_{1i}^\top \boldsymbol{x}) + p_{1i} (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) - \boldsymbol{d}_{1i}^\top \boldsymbol{U} \boldsymbol{a}_1, \quad i = 1, \ldots, r \\
\boldsymbol{\eta}_i &= (\boldsymbol{d}_{1i}^\top \boldsymbol{x} + p_{1i}) \boldsymbol{p}_2 + p_{1i} \boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{D}_2 \boldsymbol{U} \boldsymbol{d}_{1i}, \quad i = 1, \ldots, r,
\end{aligned}
$$

and $\boldsymbol{d}_{1i}$ and $\boldsymbol{d}_{2i}$ denote the i-th row of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. This is Case 6(ii) in Table 1. Another LMI is proposed by Jiang and Li (2019), using the Hadamard product instead of the Kronecker product. It follows for the Hadamard product that

$$\begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1\boldsymbol{x}+\boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1\boldsymbol{x}+\boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \circ \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2\boldsymbol{x}+\boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2\boldsymbol{x}+\boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0,$$

which implies

$$\begin{bmatrix} \alpha & \boldsymbol{\beta}^\top \\ \boldsymbol{\beta} & \alpha \boldsymbol{I} \end{bmatrix} \succeq 0, \tag{40}$$

where

$$\alpha = b_1 b_2 - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} + a_1^\top \boldsymbol{U} a_2 \tag{41}$$

$$\beta_i = \boldsymbol{d}_{1i} U \boldsymbol{d}_{1i} + p_{1i} p_{2i} + p_{2i} \boldsymbol{d}_{1i} \boldsymbol{x} + p_{1i} \boldsymbol{d}_{2i} \boldsymbol{x}, \quad i = 1, \ldots, r, \tag{42}$$

and $\boldsymbol{d}_{1i}$ and $\boldsymbol{d}_{2i}$ is the i-th row of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. Notice that the matrix in the left-hand side of (40) has an arrow structure, and hence LMI (40) is equivalent with the following conic quadratic inequality:

$$\|\beta\|_2 \leq \alpha. \tag{43}$$

It can easily be verified that (43) is a weaker inequality than (39). This is Case 6(iii) in Table 1.

A.5. Case 9 in Table 1: (Q) $\times$ (S)

Consider one conic quadratic inequality and one LMI

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x}+\boldsymbol{p}\right\| \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0. \tag{44}$$

First observe that the conic quadratic inequality can be formulated as an LMI (Anstreicher, 2017):

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x}+\boldsymbol{p}\right\| \iff \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}\boldsymbol{x}+\boldsymbol{p})^\top \\ \boldsymbol{D}\boldsymbol{x}+\boldsymbol{p} & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0.$$

We now multiply these inequalities. Using the fact that the Kronecker product of two positive semidefinite matrices is also positive semidefinite (Horn and Johnson, 1991, Theorem 4.2.12), we obtain 1 additional LMI:

$$\begin{aligned}
&\left(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} - \left\|\boldsymbol{D}\boldsymbol{x}+\boldsymbol{p}\right\|\right) \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \implies \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}\boldsymbol{x}+\boldsymbol{p})^\top \\ \boldsymbol{D}\boldsymbol{x}+\boldsymbol{p} & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \otimes \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\
&\iff \begin{bmatrix} (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) & (\boldsymbol{d}_1^\top \boldsymbol{x} + \boldsymbol{p}_1)\boldsymbol{A}(\boldsymbol{x}) & \cdots & (\boldsymbol{d}_r^\top \boldsymbol{x} + \boldsymbol{p}_r)\boldsymbol{A}(\boldsymbol{x}) \\ (d_1^\top \boldsymbol{x} + \boldsymbol{p}_1)\boldsymbol{A}(\boldsymbol{x}) & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) & & \\ \vdots & & \ddots & \\ (d_r^\top \boldsymbol{x} + \boldsymbol{p}_r)\boldsymbol{A}(\boldsymbol{x}) & & & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) \end{bmatrix} \succeq 0 \\
&\implies \begin{bmatrix} \boldsymbol{A}(b_2\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) & \boldsymbol{A}(p_1\boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_1) & \cdots & \boldsymbol{A}(p_r\boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_r) \\ \boldsymbol{A}(p_1\boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_1) & \boldsymbol{A}(b_2\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) & & \\ \vdots & & \ddots & \\ \boldsymbol{A}(p_r\boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_r) & & & \boldsymbol{A}(b_2\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) \end{bmatrix} \succeq 0,
\end{aligned} \tag{45}$$

where $\boldsymbol{d}_i$ is the i-th row of $\boldsymbol{D}$. We could also directly multiply the left-hand side of the conic quadratic inequality with the LMI and obtain 1 additional LMI:

$$A(b_2\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) \succeq 0,$$

which is also implied by (45).

A.6. Case 12 in Table 1: (P) × (S)

Consider one power cone inequality and one LMI

$$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0 \end{cases} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0.$$

We multiply the nonnegativity constraints of the power cone with the LMI and obtain m additional LMIs:

$$x_i\boldsymbol{A}(\boldsymbol{x}) \succeq 0 \quad \Longrightarrow \quad \boldsymbol{A}(\boldsymbol{u}_i) \succeq 0, \qquad i = 1, \cdots, m.$$

A.7. Case 15 in Table 1: (S) × (S)

Consider two LMIs

$$\begin{cases} \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\ \boldsymbol{B}(\boldsymbol{x}) \succeq 0. \end{cases}$$

If $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are of different sizes, it follows from (Horn and Johnson, 1991, Theorem 4.2.12) that the Kronecker product of $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ is positive semidefinite, that is $\boldsymbol{A}(\boldsymbol{x}) \otimes \boldsymbol{B}(\boldsymbol{x}) \succeq 0$. Notice that each element in the Kronecker product is the multiplication of two affine functions of $\boldsymbol{x}$. After linearizing the quadratic terms in $\boldsymbol{A}(\boldsymbol{x}) \otimes \boldsymbol{B}(\boldsymbol{x})$ with the matrix $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{U})$, which is linear in both $\boldsymbol{x}$ and $\boldsymbol{U}$, we obtain Case 15(i) of Table 1. If $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are of the same size, it follows from the Schur Product Theorem (Schur, 1911; Horn and Johnson, 1991, Theorem 5.2.1) that the Hadamard product of $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ is positive semidefinite, that is $\boldsymbol{A}(\boldsymbol{x}) \circ \boldsymbol{B}(\boldsymbol{x}) \succeq 0$. Notice that each element in the Hadamard product is the multiplication of two affine functions of $\boldsymbol{x}$. After linearizing the quadratic terms in $\boldsymbol{A}(\boldsymbol{x}) \circ \boldsymbol{B}(\boldsymbol{x})$ with the matrix $\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{U})$, which is linear in both $\boldsymbol{x}$ and $\boldsymbol{U}$, we obtain Case 15(ii) of Table 1.

B. Adversarial approach for best power cone reformulation

In Algorithm 1 we include generic pseudocode for the adversarial approach, utilized for finding the best reformulation when multiplying a cone inequality with the power cone. The function $g(\boldsymbol{x}, \boldsymbol{U})$ refers to the right-hand side of the constraint obtained after the multiplication of a cone inequality with a power cone inequality.

Algorithm 1 Adversarial approach for best reformulation

Input: $\boldsymbol{\theta}^0$: Initial guess for the uncertain parameters.

Output: $(\boldsymbol{x}^*, \boldsymbol{U}^*)$: Optimal solutions of the best reformulated problem.

1: Initialize $\mathcal{V} = \{\boldsymbol{\theta}^0\}$.
2: Solve the master problem with input $\mathcal{V}$ and obtain optimal solutions $(\boldsymbol{x}^*, \boldsymbol{U}^*)$.
3: Solve the sub-problem with input $(\boldsymbol{x}^*, \boldsymbol{U}^*)$ and obtain optimal solution $\boldsymbol{\theta}^*$ with cost c^*.
4: **if** $c^* < \log(g(\boldsymbol{x}^*, \boldsymbol{U}^*))$ **then**
5: $\quad \mathcal{V} = \mathcal{V} \cup \{\boldsymbol{\theta}^*\}$.
6: $\quad$ Go to Step 2.
7: **else**
8: $\quad$ Return the optimal solutions $(\boldsymbol{x}^*, \boldsymbol{U}^*)$.
9: **end if**

C. Full RPT

In this appendix, we describe the number of additional inequalities one would obtain when applying full RPT, i.e., the total additional conic inequalities resulting from the multiplication of all pairwise multiplications of the constraints in the two cones, including multiplications of all inequalities in the same cone.

Case 1 in Table 6. Multiplying each linear inequality with itself yields 2 additional linear inequalities. Hence, with full RPT we would obtain in total *3 additional linear inequalities.*

Case 2 in Table 6. Multiplying the linear inequality with itself yields 1 additional linear inequality, see Case 1 in Appendix A.1. Multiplying the conic quadratic inequality with itself yields 2 additional conic quadratic inequalities, see Case 6 in Appendix A.4. Hence, with full RPT we would obtain *1 additional linear inequality and 3 additional conic quadratic inequalities.*

Case 3(i) and 3(ii) in Table 6. Multiplying the linear inequality with itself and the nonnegativity constraints of the power cone with themselves yields $1 + m(m+1)/2$ additional linear inequalities, see Case 1 in Appendix A.1. Multiplying the power cone inequality with the nonnegativity constraints yields m additional power cone inequalities, see Case 3 in Section 3.2. Multiplying the power cone inequality with itself yields 1 additional power cone inequality, see Case 10 in Section 3.4. Hence, with full RPT we would obtain in total $m + 1 + m(m+1)/2$ *additional linear inequalities and* $m + 2$ *additional power cone inequalities.*

Case 4 in Table 6. Multiplying the linear inequality with itself and the nonnegativity constraint of the exponential cone with itself yields 2 additional linear inequalities, see Case 1 in Appendix A.1.

Multiplying the nonnegativity constraint of the exponential cone with the exponential cone inequality and the exponential cone inequality with itself yields 3 additional exponential cone inequalities, see Case 13 in Section 4.5. Hence, with full RPT we would obtain in total *3 additional linear inequalities and 4 additional exponential cone inequalities.*

Case 5 in Table 6. Multiplying the linear inequality with itself yields 1 additional linear inequality, see Case 1 in Appendix A.1. Multiplying the LMI with itself yields 1 additional LMI, see Case 15 in Appendix A.7. Hence, with full RPT we would obtain *1 additional linear inequality and 2 additional LMIs.*

Case 6(i) in Table 6. We can further multiply both quadratic inequalities with themselves to obtain 4 additional quadratic inequalities, see Case 6(i) in Appendix A.4. Hence, with full RPT we would obtain in total *7 additional quadratic inequalities.*

Case 6(ii) and 6(iii) in Table 6. We can multiply both quadratic inequalities with themselves as explained in Appendix A.4 for Case 6(ii) and 6(iii) to obtain 2 additional LMIs. Hence, with full RPT we would obtain in total *3 additional LMIs.*

Case 7(i) and 7(ii) in Table 6. Multiplying the conic quadratic inequality with itself yields 2 additional conic quadratic inequalities, see Case 6 in Appendix A.4. Multiplying the nonnegativity constraints with each other yields $m(m+1)/2$ additional linear inequalities, see Case 1 in Appendix A.1. Further, multiplying the nonnegativity constraints with the power cone inequality yields m additional power cone inequalities, see Case 3 in Section 3.2. Multiplying the power cone inequality with itself yields 1 additional power cone inequality, see Case 10 in Section 3.4. Hence, with full RPT, we would obtain $m(m+1)/2$ *additional linear inequalities,* $m+2$ *additional conic quadratic inequalities, and* $m+3$ *additional power cone inequalities.*

Case 8(i) in Table 6. Multiplying the conic quadratic inequality with itself yields 2 additional conic quadratic inequalities, see Case 6 in Appendix A.4. Multiplying the nonnegativity constraint with itself and the exponential cone inequality yields 1 additional linear inequality, see Case 1 in Appendix 4.2, and 1 additional exponential cone inequality, see Case 4 in Section 4.2. Multiplying the exponential cone inequality with itself yields 2 additional exponential cone inequalities, see Case 13 in Section 4.5. Hence, with full RPT we would obtain *1 additional linear inequality, 4 additional conic quadratic inequalities, and 4 additional exponential cone inequalities.*

Case 8(ii) in Table 6. We obtain 1 additional conic quadratic inequality and 1 additional exponential cone inequality from the multiplication of the derived linear inequality with the conic quadratic inequality and the exponential cone inequality, respectively. The remaining possible constraint multiplications are redundant from Lemma 1. Hence, together with the inequalities in 8(i) in Table 6, with full RPT we obtain in total *1 additional linear inequality, 5 additional conic quadratic inequalities and 5 additional exponential cone inequalities.*

Case 8(iii) in Table 6. In addition to the inequalities from Case 8(ii), we also obtain 2 additional conic quadratic inequalities and 1 additional exponential cone inequality from the multiplication of the derived conic quadratic inequality with the conic quadartic inequality and the exponential cone inequality, respectively. The remaining possible constraint multiplications are redundant from Lemma 1. Hence, in this case, with full RPT we would obtain in total *1 additional linear inequality, 7 additional conic quadratic inequalities, and 6 additional exponential cone inequalities.*

Case 9 in Table 6. Multiplying the conic quadratic inequality with itself yields 2 additional conic quadratic inequalities, see Case 6 in Appendix A.4. Further, multiplying the LMI with itself yields 1 additional LMI, see Case 15 in Appendix A.7. Hence, with full RPT we would obtain in total *2 additional conic quadratic inequalities and 2 additional LMIs.*

Case 10(i) and 10(ii) in Table 6. Multiplying the nonnegativity constraints of each cone with themselves and the power cone inequality of the same cone, results in $m_1(m_1+1)/2 + m_2(m_2+1)/2$ additional linear inequalities, see Case 1 in Appendix A.1, and $m_1 + m_2$ additional power cone inequalities, see Case 3 in Section 3.2. Moreover, multiplying each power cone inequality with itself yields 2 additional power cone inequalities, see Case 10 in Section 3.4. Hence, with full RPT we would obtain in total $m_1(m_1+1)/2 + m_2(m_2+1)/2 + m_1m_2$ *additional linear inequalities and* $2m_1 + 2m_2 + 3$ *additional power cone inequalities.*

Case 11(i) in Table 6. Multiplying the nonnegativity constraints of the power cone with themselves and the power cone inequality yields $m(m+1)/2$ additional linear inequalities and m additional power cone inequalities, see Case 1 in Appendix A.1 and Case 3 in Section 3.2 respectively. Multiplying the power cone inequality with itself yields one additional power cone inequality, see Case 10 in Section 3.4. Also, multiplying the nonnegativity constraint of the exponential cone with itself and the exponential cone inequality and multiplying the exponential cone inequality with itself yields 1 linear and 3 additional new exponential cone inequalities, see Case 13 in Section 4.5. Hence, with full RPT we would obtain in total $(m+1)(m+2)/2$ *additional linear inequalities,* $m+3$ *additional exponential cone inequalities and* $m+3$ *additional power cone inequalities.*

Case 11(ii) in Table 6. We obtain 1 additional exponential cone inequality and 1 additional power cone inequality from the multiplication of the derived linear inequality with the exponential cone inequality and power cone inequality, respectively. The remaining possible constraint multiplications are redundant from Lemma 1. Hence, together with the inequalities in 11(i) in Table 6, with full RPT we obtain in total $(m+1)(m+2)/2$ *additional linear inequalities,* $m+4$ *additional exponential cone inequalities and* $m+4$ *additional power cone inequalities.*

Case 11(iii) in Table 6. In addition to the inequalities from Case 11(ii), we obtain 1 additional exponential cone inequality and 2 additional power cone inequalities from the multiplication of the derived conic quadratic inequality with the exponential cone inequality and the power cone inequality, respectively. The remaining possible constraint multiplications are redundant from Lemma 1. Hence, together with the inequalities in 11(ii) in Table 6, with full RPT we obtain in total $(m+1)(m+2)/2$ *additional linear inequalities,* $m+5$ *additional exponential cone inequalities, and* $m+6$ *additional power cone inequalities.*

Case 12 in Table 6. Multiplying the nonnegativity constraints with themselves and the power cone inequality yields $m(m+1)/2$ additional linear inequalities, see Case 1 in Appendix A.1 and m additional power cone inequalities, see Case 3 in Section 3.2 Multiplying the power cone inequality with itself yields 1 additional power cone inequality, see Case 10 in Section 3.4. Moreover, multiplying the LMI with itself yields 1 additional LMI, see Case 15 in Appendix A.7. Hence, with full RPT we obtain in total $m(m+1)/2$ *additional linear inequalities,* $m+1$ *additional power cone inequalities, and* $m+1$ *additional LMIs.*

Case 13(i) in Table 6. Multiplying the nonnegativity constraint of each exponential cone with itself and the exponential cone inequality of the same exponential cone yields 2 additional linear inequalities, see Case 1 in Appendix A.1, and 2 additional exponential cone inequalities, see Case 4 in Section 4.2. Multiplying the left-hand side of each exponential cone inequality with both sides of the same exponential cone inequality yields 2 additional exponential cone inequalities, see Case 4 in Section 4.2. Moreover, multiplying the left-hand sides and right-hand sides of both inequalities yields 2 additional exponential cone inequalities:

$$\begin{cases} x_1^2 \geq x_2^2 \exp\left(2x_3x_2/x_2^2\right) \\ x_4^2 \geq x_5^2 \exp\left(2x_6x_5/x_5^2\right) \end{cases} \implies \begin{cases} u_{11} \geq u_{22} \exp\left(2u_{23}/u_{22}\right) \\ u_{44} \geq u_{55} \exp\left(2u_{56}/u_{55}\right). \end{cases}$$

Hence, with full RPT we would obtain in total *3 additional linear inequalities and 11 additional exponential cone inequalities.*

Case 13(ii) in Table 6. We multiply each of the derived linear inequalities $x_1 \geq x_2 + x_3$ and $x_4 \geq x_5 + x_6$ with the exponential cone inequalities and obtain 4 additional exponential cone inequalities. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, with full RPT we would obtain in total *3 additional linear inequalities and 15 additional exponential cone inequalities.*

Case 13(iii) in Table 6. We multiply each of the derived linear and conic quadratic inequalities with the exponential cone inequalities and obtain 8 additional exponential cone inequalities. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, with full RPT we would obtain in total *3 additional linear inequalities and 19 additional exponential cone inequalities.*

Case 13(iv) in Table 6. We multiply the derived linear and conic quadratic inequalities with the exponential cone inequalities and obtain 6 additional exponential cone inequalities. The remaining possible constraint multiplications are redundant by Lemma 1. Hence, with full RPT we would obtain in total *3 additional linear inequalities and 17 additional exponential cone inequalities.*

Case 14(i) in Table 6. Multiplying the exponential cone with itself, yields 1 additional linear inequality and 3 additional exponential cone inequalities. Moreover, multiplying the LMI with itself yields 1 additional LMI. Hence, with full RPT we would obtain *1 additional linear inequality, 3 additional exponential cone inequalities, and 3 additional LMIs.*

Case 14(ii) in Table 6. Multiplying the linear inequality resulting from the decomposition of the exponential cone with itself, the nonnegativity constraint of the exponential cone, and the exponential cone inequality gives 2 additional linear inequalities and 1 additional exponential cone inequality. Hence, together with the inequalities in 14(i) in Table 6 and the inequalities resulting from multiplying this linear inequality with the LMI as explained in Section A.3, with full RPT we obtain in total *3 additional linear inequalities, 4 additional exponential cone inequalities, and 4 additional LMIs.*

Case 14(iii) in Table 6. Multiplying the quadratic inequality resulting from the decomposition of the exponential cone with itself, the nonnegativity constraint of the exponential cone, the linear inequality resulting from the decomposition of the exponential cone, and the exponential cone inequality gives 5 additional quadratic inequalities and one additional exponential cone inequality. Hence, together with the inequalities in 14(ii) in Table 6 and the inequalities resulting from multiplying this quadratic inequality with the LMI as explained in Section A.5, with full RPT we obtain in total *3 additional linear inequalities, 5 additional quadratic inequalities, 5 additional exponential cone inequalities and 5 LMIs*

Case 15(i) and 15(ii) in Table 6. Multiplying the LMIs with itself, yields 2 additional LMIs, hence with full RPT we would obtain in total *3 additional LMIs.*

D. RPT formulations of numerical experiments

In this section, we include the formulations obtained when multiplying all constraints in the problems encountered in the numerical experiments.

Case	Cone-1	Cone-2	Full RPT
1	L	L	3L
2	L	Q	L + 3Q
3	L	P	(i) $(m+1+m(m+1)/2)$L + $(m+2)$P (ii) $(m+1+m(m+1)/2)$L + $(m+2)$P
4	L	E	3L + 4E
5	L	S	L + 2S
6	Q	Q	(i) 7Q (ii) 3S (iii) 3S
7	Q	P	(i) $m(m+1)/2$L + $(m+2)$Q + $(m+3)$P (ii) $m(m+1)/2$L + $(m+2)$Q + $(m+3)$P
8	Q	E	(i) L + 4Q + 4E (ii) L + 5Q + 5E (iii) L + 7Q + 6E
9	Q	S	2Q + 2S
10	P	P	(i) $(m_1(m_1+1)/2+m_2(m_2+1)/2+m_1m_2)$L + $(2m_1+2m_2+3)$P (ii) $(m_1(m_1+1)/2+m_2(m_2+1)/2+m_1m_2)$L + $(2m_1+2m_2+3)$P
11	P	E	(i) $((m+1)(m+2)/2)$L + $(m+3)$P + $(m+3)$E (ii) $((m+1)(m+2)/2+1)$L + $(m+4)$P + $(m+4)$E (iii) $((m+1)(m+2)/2+2)$L + 2Q + $(m+6)$P + $(m+5)$E
12	P	S	$(m(m+1)/2)$L + $(m+1)$P + $(m+1)$S
13	E	E	(i) 3L + 11E (ii) 3L + 15E (iii) 3L + 19E (iv) 3L + 17E
14	E	S	(i) L + 3E + 3S (ii) 3L + 4E + 4S (iii) 3L + 5Q + 5E + 5S
15	S	S	(i) 3S (ii) 3S

Table 6 Results of multiplying the inequalities in two of the five basic cones as given in Section 2 when applying full RPT.

D.1. RPT formulation of Section 5.3

We linearize $\boldsymbol{x}\boldsymbol{x}^\top$ with $\boldsymbol{X}$, $\boldsymbol{y}\boldsymbol{y}^\top$ with $\boldsymbol{Y}$, $\boldsymbol{x}\boldsymbol{y}^\top$ with $\boldsymbol{U}$, $\boldsymbol{x}\boldsymbol{w}^\top$ with $\boldsymbol{Q}$, $\boldsymbol{y}\boldsymbol{w}^\top$ with $\boldsymbol{P}$, and $\boldsymbol{w}\boldsymbol{w}^\top$ with $\boldsymbol{W}$. After pairwise multiplying all constraints, we obtain the following problem:

$$\max_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{X},\\ \boldsymbol{Y},\boldsymbol{U},\boldsymbol{Q}\\ \boldsymbol{P},\boldsymbol{W}}} \quad \mathrm{Tr}(\boldsymbol{U}) + \sum_{i=1}^{n} w_i \tag{46a}$$

$$
\begin{aligned}
\text{s.t.} \quad & x_1 \exp\left(\frac{X_{1i}}{x_1}\right) \le \rho, && i \in \{1,\ldots,n\}, && \text{(46b)}\\
& y_i \exp\left(\frac{w_i}{y_i}\right) \le 1, && i \in \{1,\ldots,n\}, && \text{(46c)}\\
& U_{1i} \exp\left(\frac{Q_{1i}}{U_{1i}}\right) \le x_1, && i \in \{1,\ldots,n\}, && \text{(46d)}\\
& Y_{ij} \exp\left(\frac{P_{ji}}{Y_{ij}}\right) \le y_j, && i,j \in \{1,\ldots,n\}, && \text{(46e)}\\
& Y_{ij} \exp\left(\frac{P_{ij}+P_{ji}}{Y_{ij}}\right) \le 1, && i \le j \in \{1,\ldots,n\}, && \text{(46f)}\\
& \sum_{i=1}^{n} y_i = 1, && && \text{(46g)}\\
& \sum_{i=1}^{n} \boldsymbol{U}_i = \boldsymbol{x}, && && \text{(46h)}\\
& \sum_{i=1}^{n} \boldsymbol{Y}_i = \boldsymbol{y}, && && \text{(46i)}\\
& \sum_{i=1}^{n} \boldsymbol{P}_i^\top = \boldsymbol{w}, && && \text{(46j)}\\
& x_1, \boldsymbol{U}_1, \boldsymbol{Y}, \boldsymbol{y} \ge \boldsymbol{0}, && && \text{(46k)}\\
& \begin{pmatrix} \boldsymbol{X} & \boldsymbol{U} & \boldsymbol{Q} & \boldsymbol{x} \\ \boldsymbol{U}^\top & \boldsymbol{Y} & \boldsymbol{P} & \boldsymbol{y} \\ \boldsymbol{Q}^\top & \boldsymbol{P}^\top & \boldsymbol{W} & \boldsymbol{w} \\ \boldsymbol{x}^\top & \boldsymbol{y}^\top & \boldsymbol{w}^\top & 1 \end{pmatrix} \succeq \boldsymbol{0}. && && \text{(46l)}
\end{aligned}
$$

Observe that as (31b) is nonconvex, we do not include this constraint in (46), but we include the convexified constraint (46b). We can further decompose the exponential cone inequalities for $\boldsymbol{y}$ and multiply them with $x_1, \boldsymbol{y}$ and each other to obtain the following additional inequalities

$$
\begin{aligned}
& 1 \ge y_i + w_i, && i \in \{1,\ldots,n\},\\
& x_1 \ge U_{1i} + Q_{1i}, && i \in \{1,\ldots,n\},\\
& \boldsymbol{y} \ge \boldsymbol{Y}_i + \boldsymbol{P}_i, && i \in \{1,\ldots,n\},\\
& Y_{ij} + W_{ij} + P_{ij} + P_{ji} - y_i - y_j - w_i - w_j + 1 \ge 0, && i,j \in \{1,\ldots,n\}.
\end{aligned}
$$

We can reformulate the first order conditions into cone inequalities and then multiply them with the rest to obtain more cone inequalities. More precisely, from Example 5, we have the convex inequality

$$
y_i \log(y_i) + y_i \log\left(\sum_j \exp\left(\frac{U_{ji}}{y_i}\right)\right) \le U_{ii}.
$$

Using epigraphical variables t_i for the terms $y_i \log(y_i)$ as well as epigraphical variables r_i for the terms $y_i \log\left(\sum_j \exp\left(\frac{U_{ji}}{y_i}\right)\right)$ we can reformulate it as conic constraints. We linearize $\boldsymbol{x}\boldsymbol{t}^\top$, $\boldsymbol{y}\boldsymbol{t}^\top$, $\boldsymbol{w}\boldsymbol{t}^\top$ with $\boldsymbol{H}, \boldsymbol{G}$ and $\boldsymbol{R}$, respectively, and $q_{ij}y_i$ with V_{ij}. We have the following:

$$
\begin{aligned}
& t_i + r_i \leq U_{ii}, && i \in \{1,\ldots,n\},\\
& y_i \exp\left(\frac{-t_i}{y_i}\right) \leq 1, && i \in \{1,\ldots,n\},\\
& \sum_j q_{ij} \leq 1, && i \in \{1,\ldots,n\},\\
& y_i \exp\left(\frac{U_{ji} - r_i}{y_i}\right) \leq V_{ij}, && i,j \in \{1,\ldots,n\},\\
& \sum_j V_{ij} \leq y_i, && i \in \{1,\ldots,n\},\\
& U_{1i} \exp\left(\frac{-H_{1i}}{U_{1i}}\right) \leq 1, && i \in \{1,\ldots,n\},\\
& Y_{ij} \exp\left(\frac{-G_{ji}}{Y_{ij}}\right) \leq 1, && i,j \in \{1,\ldots,n\},\\
& Y_{ij} \exp\left(\frac{P_{ji} - G_{ij}}{Y_{ij}}\right) \leq 1, && i,j \in \{1,\ldots,n\},\\
& Y_{ij} \exp\left(\frac{-G_{ji} - G_{ij}}{Y_{ij}}\right) \leq 1, && i,j \in \{1,\ldots,n\},\\
& \sum_i \boldsymbol{G}_i^\top = \boldsymbol{t},\\
& \begin{pmatrix} \boldsymbol{X} & \boldsymbol{U} & \boldsymbol{Q} & \boldsymbol{H} & \boldsymbol{x}\\ \boldsymbol{U}^\top & \boldsymbol{Y} & \boldsymbol{P} & \boldsymbol{G} & \boldsymbol{y}\\ \boldsymbol{Q}^\top & \boldsymbol{P}^\top & \boldsymbol{W} & \boldsymbol{R} & \boldsymbol{w}\\ \boldsymbol{H}^\top & \boldsymbol{G}^\top & \boldsymbol{R}^\top & \boldsymbol{T} & \boldsymbol{t}\\ \boldsymbol{x}^\top & \boldsymbol{y}^\top & \boldsymbol{w}^\top & \boldsymbol{t}^\top & 1 \end{pmatrix} \succeq \boldsymbol{0}.
\end{aligned}
$$

We can further decompose the exponential cone inequalities obtained from first order conditions and obtain the following:

$$
\begin{aligned}
& 1 \geq y_i - t_i, && i \in \{1,\ldots,n\},\\
& x_1 \geq U_{1i} - H_{1i}, && i \in \{1,\ldots,n\},\\
& \boldsymbol{y} \geq \boldsymbol{Y}_i - \boldsymbol{G}_i, && i \in \{1,\ldots,n\},\\
& (y_i - Y_{ij} + G_{ij}) \exp\left(\frac{w_i - P_{ji} + R_{ij}}{y_i - Y_{ij} + G_{ij}}\right) \leq 1 - y_j + t_j, && i,j \in \{1,\ldots,n\},\\
& Y_{ij} + P_{ji} - G_{ij} - R_{ij} - y_i - y_j - w_i + t_j + 1 \geq 0, && i,j \in \{1,\ldots,n\},\\
& Y_{ij} + T_{ij} - G_{ij} - G_{ji} - y_i - y_j + t_i + t_j + 1 \geq 0, && i,j \in \{1,\ldots,n\}.
\end{aligned}
$$

D.2. RPT formulation of Section 6.1

We linearize $\boldsymbol{x}\boldsymbol{x}^\top$ with $\boldsymbol{X}$, $\boldsymbol{z}\boldsymbol{z}^\top$ with $\boldsymbol{Z}$, $\boldsymbol{x}\boldsymbol{z}^\top$ with $\boldsymbol{V}$, $\boldsymbol{x}\boldsymbol{t}^\top$ with $\boldsymbol{W}$ and $\boldsymbol{z}\boldsymbol{t}^\top$ with $\boldsymbol{Q}$. When multiplying the constraints in Problem (33), without any additions, we obtain the following problem:

$$
\begin{aligned}
\min_{\substack{\boldsymbol{x},\boldsymbol{z},\boldsymbol{X},\\ \boldsymbol{V},\boldsymbol{Z}}} \quad & \operatorname{Tr}(\boldsymbol{A}_0\boldsymbol{X}) + \boldsymbol{b}_0^\top \boldsymbol{x} + c_0 && && (47a)\\
\text{s.t.} \quad & \sum_{i=1}^{n_x} z_i \le 1, && && (47b)\\
& \exp(-x_i - a) \le z_i, && i \in \{1,\dots,n_x\}, && (47c)\\
& \sum_{i=1}^{n_x} t_i \le \beta, && && (47d)\\
& \exp(x_i) \le t_i, && i \in \{1,\dots,n_x\}, && (47e)\\
& \left(1-\sum_{j=1}^{n_x} z_j\right)\exp\left(\frac{-x_i-\alpha+\sum_{j=1}^{n_x}V_{ij}+\alpha\sum_{j=1}^{n_x}z_j}{1-\sum_{j=1}^{n_x}z_j}\right) \le z_i - \sum_{j=1}^{n_x} Z_{ij} && i \in \{1,\dots,n_x\}, && (47f)\\
& \beta - \sum_{j=1}^{n_x} t_j - \beta\sum_{j=1}^{n_x} z_j + \sum_{i,j=1}^{n_x} Q_{ij} \ge 0, && && (47g)\\
& \left(1-\sum_{j=1}^{n_x} z_j\right)\exp\left(\frac{x_i-\sum_{j=1}^{n_x}V_{ij}}{1-\sum_{j=1}^{n_x}z_j}\right) \le t_i - \sum_{j=1}^{n_x} Q_{ji}, && i \in \{1,\dots,n_x\}, && (47h)\\
& \sum_{i,j=1}^{n_x} Z_{ij} - 2\sum_{i=1}^{n_x} z_i + 1 \ge 0, && && (47i)\\
& \left(\beta-\sum_{j=1}^{n_x} t_j\right)\exp\left(\frac{-\beta x_i-\alpha\beta+\sum_{j=1}^{n_x}W_{ij}+\alpha\sum_{j=1}^{n_x}t_j}{\beta-\sum_{j=1}^{n_x}t_j}\right) \le z_i\beta - \sum_{j=1}^{n_x} Q_{ij}, && i \in \{1,\dots,n_x\}, && (47j)\\
& \left(\beta-\sum_{j=1}^{n_x} t_j\right)\exp\left(\frac{\beta x_i-\sum_{j=1}^{n_x}W_{ij}}{\beta-\sum_{j=1}^{n_x}t_j}\right) \le \beta t_i - \sum_{j=1}^{n_x} T_{ij}, && i \in \{1,\dots,n_x\}, && (47k)\\
& \sum_{i,j=1}^{n_x} T_{ij} - 2\sum_{j=1}^{n_x} t_j + \beta^2 \ge 0, && && (47l)\\
& z_j \exp\left(\frac{V_{ij}}{z_j}\right) \le Q_{ji}, && i,j \in \{1,\dots,n_x\}, && (47m)\\
& \exp(-x_i - x_j - 2\alpha) \le Z_{ij}, && i \le j \in \{1,\dots,n_x\}, && (47n)\\
& \exp(-x_i - a + x_j) \le Q_{ij}, && i,j \in \{1,\dots,n_x\}, && (47o)\\
& z_j \exp\left(\frac{-V_{ij} - a z_j}{z_j}\right) \le Z_{ij}, && i,j \in \{1,\dots,n_x\}, && (47p)\\
& t_j \exp\left(\frac{-W_{ij} - a t_j}{t_j}\right) \le Q_{ij}, && i,j \in \{1,\dots,n_x\}, && (47q)\\
& t_j \exp\left(\frac{W_{ij}}{t_j}\right) \le T_{ij}, && i,j \in \{1,\dots,n_x\}, && (47r)\\
& \exp(x_i + x_j) \le T_{ij}, && i \le j \in \{1,\dots,n_x\}. && (47s)
\end{aligned}
$$

Further, from the decomposition of the exponential cone we obtain the following additional constraints:

$$
-x_i - \alpha + 1 \le z_i, \qquad i \in \{1,\dots,n_x\}, \tag{48a}
$$

$$
x_i + 1 \le t_i, \qquad i \in \{1,\dots,n_x\}, \tag{48b}
$$

$$
z_i - \sum_{j=1}^{n_x} Z_{ij} + x_i + \alpha - 1 - \sum_{j=1}^{n_x} V_{ij} - \alpha\sum_{j=1}^{n_x} z_j + \sum_{j=1}^{n_x} z_j \ge 0, \qquad i \in \{1,\dots,n_x\}, \tag{48c}
$$

$$(z_j + x_j + \alpha - 1)\exp\left(\frac{-V_{ij} - X_{ij} - \alpha x_i + x_i - \alpha(z_j + x_j + \alpha - 1)}{z_j + x_j + \alpha - 1}\right) \le Z_{ij} + V_{ji} + \alpha z_i - z_i, \quad i, j \in \{1, \ldots, n_x\}, \tag{48d}$$

$$(z_j + x_j + \alpha - 1)\exp\left(\frac{V_{ij} + X_{ij} + \alpha x_i - x_i}{z_j + x_j + \alpha - 1}\right) \le Q_{ji} + W_{ji} + \alpha t_i - t_i, \quad i, j \in \{1, \ldots, n_x\}, \tag{48e}$$

$$(\alpha - 1)(z_j + x_j + \alpha - 1) + Z_{ij} + V_{ji} + (\alpha - 1)z_i + V_{ij} + X_{ij} + (\alpha - 1)x_i \ge 0, \quad i, j \in \{1, \ldots, n_x\}, \tag{48f}$$

$$t_i - \sum_{j=1}^{n_x} Q_{ji} - x_i + \sum_{j=1}^{n_x} V_{ij} - 1 + \sum_{j=1}^{n_x} z_j \ge 0, \quad i \in \{1, \ldots, n_x\}, \tag{48g}$$

$$(t_j - x_j - 1)\exp\left(\frac{-W_{ij} - \alpha t_j + X_{ij} + \alpha(x_j + 1) + x_i}{t_j - x_j - 1}\right) \le Q_{ij} - V_{ji} - z_i, \quad i, j \in \{1, \ldots, n_x\}, \tag{48h}$$

$$(t_j - x_j - 1)\exp\left(\frac{W_{ij} - X_{ij} - x_i}{t_j - x_j - 1}\right) \le T_{ij} - Q_{ji} - t_i, \quad i, j \in \{1, \ldots, n_x\}, \tag{48i}$$

$$T_{ij} - W_{ij} - t_j - W_{ji} + X_{ij} + x_j - t_i + x_i + 1 \ge 0, \quad i, j \in \{1, \ldots, n_x\}, \tag{48j}$$

$$Q_{ji} + W_{ji} + \alpha t_i - t_i - V_{ij} - X_{ij} - \alpha x_i + x_i - z_j - x_j - \alpha + 1 \ge 0, \quad i, j \in \{1, \ldots, n_x\}, \tag{48k}$$

$$\beta z_j - \sum_i Q_{ji} + \beta x_j - \sum_i W_{ji} - \alpha \sum_i t_i + \alpha\beta - \beta + \sum_i t_i \ge 0, \quad j \in \{1, \ldots, n_x\}, \tag{48l}$$

$$\beta t_j - \sum_i T_{ij} - \beta x_j + \sum_i W_{ji} - \beta + \sum_i t_i \ge 0, \quad j \in \{1, \ldots, n_x\}. \tag{48m}$$

D.3. RPT formulation of Section 6.2

We linearize $\boldsymbol{r}\boldsymbol{r}^\top$, $\boldsymbol{t}\boldsymbol{t}^\top$, $\boldsymbol{r}\boldsymbol{t}^\top$ with $\boldsymbol{R}, \boldsymbol{T}$ and $\boldsymbol{V}$ respectively. We multiply all constraints in problem (35) to obtain the following problem:

$$\begin{aligned}
\max_{\substack{\boldsymbol{r},\boldsymbol{t},\boldsymbol{R},\\ \boldsymbol{T},\boldsymbol{V}}} \quad & \boldsymbol{r}^\top \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}^\top \boldsymbol{r} + d && && \text{(49a)}\\
\text{s.t.} \quad & \boldsymbol{c}^\top \boldsymbol{r} \le W, && && \text{(49b)}\\
& \boldsymbol{\beta}_l^\top \boldsymbol{r} \ge \eta_l, && l \in \mathcal{L} \setminus \{m\}, && \text{(49c)}\\
& \sum_k t_k = \frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m), && && \text{(49d)}\\
& t_k^{1/q}\left(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m)\right)^{1-1/q} \ge r_k, && k \in \mathcal{K}, && \text{(49e)}\\
& \boldsymbol{\beta}_l^\top \boldsymbol{R}_k \ge \eta_l r_k, && l \in \mathcal{L} \setminus \{m\}, k \in \mathcal{K}, && \text{(49f)}\\
& \left(\boldsymbol{\beta}_l^\top \boldsymbol{V}_k - \eta_l t_k\right)^{\frac{1}{q}} \left(\frac{1}{\rho}\left(\boldsymbol{\beta}_1^\top \boldsymbol{R}\boldsymbol{\beta}_l - \eta_l \boldsymbol{\beta}_1^\top \boldsymbol{r} - \eta_1 \boldsymbol{\beta}_l^\top \boldsymbol{r} + \eta_1\eta_l\right)\right)^{1-1/q} && && \\
& \quad \ge \boldsymbol{\beta}_l^\top \boldsymbol{R}_k - \eta_l r_k, && l \in \mathcal{L} \setminus \{m\}, k \in \mathcal{K}, && \text{(49g)}\\
& \boldsymbol{c}^\top \boldsymbol{R}_k \le W r_k, && k \in \mathcal{K}, && \text{(49h)}\\
& W^2 - 2W\boldsymbol{c}^\top \boldsymbol{r} + \boldsymbol{c}^\top \boldsymbol{R}\boldsymbol{c} \ge 0, && && \text{(49i)}
\end{aligned}$$

$$W\boldsymbol{\beta}_l^\top \boldsymbol{r} + \eta_l \boldsymbol{c}^\top \boldsymbol{r} - \boldsymbol{c}^\top \boldsymbol{R}\boldsymbol{\beta}_l - W\eta_l \geq 0, \qquad l \in \mathcal{L} \setminus \{m\}, \quad \text{(49j)}$$

$$\left(Wt_k - \boldsymbol{c}^\top \boldsymbol{V}_k\right)^{\frac{1}{q}} \left(\frac{1}{\rho}\left(W\boldsymbol{\beta}_m^\top \boldsymbol{r} - W\eta_m + \eta_m \boldsymbol{c}^\top \boldsymbol{r} - \boldsymbol{c}^\top \boldsymbol{R}\boldsymbol{\beta}_m\right)\right)^{1-1/q} \geq Wr_k - \boldsymbol{c}^\top \boldsymbol{R}_k \qquad k \in \mathcal{K}, \quad \text{(49k)}$$

$$\sum_k \boldsymbol{V}_k = \frac{1}{\rho}(\boldsymbol{R}\boldsymbol{\beta}_m - \eta_m \boldsymbol{r}), \quad \text{(49l)}$$

$$\sum_k \boldsymbol{T}_k = \frac{1}{\rho}(\boldsymbol{V}^\top \boldsymbol{\beta}_m - \eta_m \boldsymbol{t}), \quad \text{(49m)}$$

$$V_{k'k}^{1/q}\left(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{R}_{k'} - \eta_m r_{k'})\right)^{1-1/q} \geq R_{kk'}, \qquad k, k' \in \mathcal{K}, \quad \text{(49n)}$$

$$T_{kk'}^{\theta_{11}} \left(\frac{1}{\rho}\left(\boldsymbol{\beta}_m^\top \boldsymbol{V}_k - \eta_m t_k\right)\right)^{\theta_{12}} \left(\frac{1}{\rho}\left(\boldsymbol{\beta}_m^\top \boldsymbol{V}_{k'} - \eta_m t_{k'}\right)\right)^{\theta_{21}} \cdot \left(\frac{1}{\rho^2}\left(\boldsymbol{\beta}_m^\top \boldsymbol{R}\boldsymbol{\beta}_m - 2\eta_m \boldsymbol{\beta}_m^\top \boldsymbol{r} + \eta_m^2\right)\right)^{\theta_{22}} \geq R_{kk'}, \qquad k, k' \in \mathcal{K}, \quad \text{(49o)}$$

$$\boldsymbol{r}, \boldsymbol{R} \geq \boldsymbol{0}, \quad \text{(49p)}$$

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{V} & \boldsymbol{r} \\ \boldsymbol{V}^\top & \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{r}^\top & \boldsymbol{t}^\top & 1 \end{pmatrix} \succeq \boldsymbol{0}. \quad \text{(49q)}$$

For the multiplication of the power cone constraints we generate a feasible $\boldsymbol{\theta}$ that satisfies the following:

$$\theta_{11} + \theta_{21} = 1/q, \;\; \theta_{12} + \theta_{22} = 1 - 1/q,$$

$$\theta_{11} + \theta_{12} = 1/q, \;\; \theta_{21} + \theta_{22} = 1 - 1/q.$$

E. Data generation of Section 6.1

In problem instance 1 the objective is defined as $f(\boldsymbol{x}) = -\frac{1}{2}\sum_{i=1}^{20}(x_i + 5)^2$ and in problem instance 2 it is defined as $f(\boldsymbol{x}) = -\frac{1}{2}\sum_{i=1}^{20}(x_i + 7)^2$. In problem instances 3, 4, 5, and 6, the matrix $\boldsymbol{A}_0$ is generated as $\boldsymbol{L}^\top \boldsymbol{L}$, where $\boldsymbol{L} \in \mathbb{R}^{n \times n}$, with $L_{ij} \sim [0, 1]$, and further $\boldsymbol{b}_0 = \boldsymbol{0}$, $c_0 = 0$. We summarize all parameters describing each instance in Table 7.

Instance	n_x	α	β
1	5	2	20
2	5	2	20
3	10	2	3
4	20	3	4
5	50	3	4
6	100	13	20

Table 7 Problem (32) parameters for each instance. n_x refers to the number of variables and α, β to the constraint parameters.

F. Dominance results of the constraint multiplications considered in this paper

In this appendix we show that all considered additional constraint multiplications in this paper are valuable in the sense that for each case we have found an example demonstrating the dominance of the considered constraint over all other possible additional constraint multiplications.

REMARK 4. In all convex relaxations discussed in the examples in this appendix we consider the additional inequalities $u_{ii} \geq 0$, where u_{ii} linearizes the product term x_i^2 for $i \in \{1, \ldots, n_x\}$.

EXAMPLE 6. We consider the following toy example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \tag{50a}$$

$$\text{s.t.} \quad x_3 \geq 5, \tag{50b}$$

$$x_1^{0.5} x_2^{0.5} \geq x_1 + x_2, \tag{50c}$$

$$x_1, x_2 \geq 0. \tag{50d}$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x}, \boldsymbol{U}} \quad u_{13} \tag{51a}$$

$$(50b) - (50d),$$

$$u_{33} - 10x_3 + 25 \geq 0, \tag{51b}$$

$$(u_{13} - 5x_1)^{0.5}(u_{23} - 5x_2)^{0.5} \geq u_{13} + u_{23} - 5x_1 - 5x_2, \tag{51c}$$

$$u_{i3} - 5x_i \geq 0, \qquad i = 1, 2, \tag{51d}$$

$$u_{i1}^{0.5} u_{i2}^{0.5} \geq u_{i1} + u_{i2}, \qquad i = 1, 2, \tag{51e}$$

$$u_{11}^{0.5} u_{22}^{0.5} \geq u_{11} + 2u_{12} + u_{22}, \tag{51f}$$

$$u_{11}, u_{12}, u_{22} \geq 0. \tag{51g}$$

We list the values obtained when including/excluding each one of the additional individual valid constraints in Table 8. From Table 8 we observe that the most valuable constraint multiplications

Con	w/o	w
(51b)	0.00	0.00
(51c), (51d)	$-\infty$	0.00
(51e), (51g)	0.00	0.00
(51f)	0.00	0.00

Table 8 Comparison of the optimal values for Problem (51), with and without each of the proposed multiplications.

in this case are (51c), (51d), which result from the multiplication of the linear inequality (50b) with the power cone inequalities (50c) and (50d). ☐

EXAMPLE 7. Consider the following toy example

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_1 x_3 && \text{(52a)}\\
\text{s.t.} \quad & x_2 \exp(x_1) \leq 5, && \text{(52b)}\\
& x_1 \geq 4, && \text{(52c)}\\
& x_2 \geq \exp(-x_3). && \text{(52d)}
\end{aligned}
$$

We obtain the following RPT relaxation

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && \text{(53a)}\\
\text{s.t.} \quad & x_2 \exp\left(\frac{u_{12}}{x_2}\right) \leq 5, && \text{(53b)}\\
& (52c) - (52d), && \\
& u_{11} - 8x_1 + 16 \geq 0, && \text{(53c)}\\
& u_{22} \geq \exp\left(-2x_3\right), && \text{(53d)}\\
& u_{22} \geq x_2 \exp\left(\frac{-u_{23}}{x_2}\right), && \text{(53e)}\\
& u_{12} - 4x_2 \geq (x_1 - 4) \exp\left(\frac{-u_{13} + 4x_3}{x_1 - 4}\right). && \text{(53f)}
\end{aligned}
$$

We list the values obtained when including/excluding each one of the additional individual valid constraints in Table 9. □

Con	w/o	w
(53c)	9.56	9.56
(53d)	9.56	9.56
(53e)	9.56	9.56
(53f)	$-\infty$	9.56

Table 9 Comparison of the optimal values for Problem (53), with and without each of the proposed multiplications.

EXAMPLE 8. Consider the following toyexample

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_3 x_4 && \text{(54a)}\\
\text{s.t.} \quad & 3x_1 \geq \|\mathrm{diag}(\boldsymbol{a})\boldsymbol{x}\|, && \text{(54b)}\\
& x_1^{0.5} x_2^{0.5} \geq |x_3|, && \text{(54c)}\\
& x_1, x_2 \geq 0, && \text{(54d)}\\
& x_1 \exp(x_1) \leq 5, && \text{(54e)}\\
& x_1 \exp(x_2) \leq 5, && \text{(54f)}
\end{aligned}
$$

where $\boldsymbol{a} = (1,0,0,1)^\top$. We obtain the following RPT relaxation

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{34} && \text{(55a)}\\
\text{s.t.} \quad & (54b)-(54d), \\
& x_1 \exp\left(\frac{u_{11}}{x_1}\right) \le 5, && \text{(55b)}\\
& x_1 \exp\left(\frac{u_{12}}{x_1}\right) \le 5, && \text{(55c)}\\
& 9u_{11} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{U}\operatorname{diag}(\boldsymbol{a})^\top\right\|_F, && \text{(55d)}\\
& u_{11}^{0.5}u_{12}^{0.5} \ge |u_{13}|, && \text{(55e)}\\
& u_{12}^{0.5}u_{22}^{0.5} \ge |u_{23}|, && \text{(55f)}\\
& u_{11}^{0.5}u_{22}^{0.5} \ge |u_{33}|, && \text{(55g)}\\
& 3u_{1i} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_i\right\|, \qquad i=1,2, && \text{(55h)}\\
& (3u_{11})^{0.5}(3u_{12})^{0.5} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_3\right\|, && \text{(55i)}\\
& u_{11}, u_{12}, u_{22} \ge 0. && \text{(55j)}
\end{aligned}
$$

We list the values obtained when including/excluding each one of the additional individual valid constraints in Table 10. □

Con	w/o	w
(55d)	-5.52	-5.52
(55e), (55f), (55j)	-5.52	-5.52
(55g)	-5.52	-5.52
(55h)	-5.52	-5.52
(55i)	$-\infty$	-5.52

Table 10 Comparison of the optimal values for Problem (55), with and without each of the proposed multiplications.

EXAMPLE 9. Consider the following toyexample

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_3x_4 && \text{(56a)}\\
\text{s.t.} \quad & 3x_3 \ge \|\operatorname{diag}(\boldsymbol{a})\boldsymbol{x}\|, && \text{(56b)}\\
& x_1^{0.5}x_2^{0.5} \ge |x_4|, && \text{(56c)}\\
& x_1\exp(x_3) \le 5, && \text{(56d)}\\
& x_2\exp(x_3) \le 5, && \text{(56e)}\\
& x_1, x_2 \ge 0, && \text{(56f)}
\end{aligned}
$$

where $\boldsymbol{a} = (1,1,0,0)^\top$. We obtain the following RPT relaxation

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{34} && && (57a)\\
\text{s.t.} \quad & (56b) - (56c), \\
& x_1 \exp\left(\frac{u_{13}}{x_1}\right) \le 5, && && (57b)\\
& x_2 \exp\left(\frac{u_{23}}{x_2}\right) \le 5, && && (57c)\\
& 3u_{33} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_3\right\|, && && (57d)\\
& 9u_{33} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{U}\operatorname{diag}(\boldsymbol{a})^\top\right\|_F, && && (57e)\\
& u_{1i}^{0.5}u_{2i}^{0.5} \ge |u_{i4}|, && i = 1,2, && (57f)\\
& u_{11}^{0.5}u_{22}^{0.5} \ge |u_{44}|, && && (57g)\\
& \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_i\right\| \le 3u_{i3}, && i = 1,2, && (57h)\\
& (3u_{13})^{0.5}(3u_{23})^{0.5} \ge |3u_{34}|, && && (57i)\\
& (3u_{13})^{0.5}(3u_{23})^{0.5} \ge \left\|\operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_4\right\|, && && (57j)\\
& x_1, x_2, u_{11}, u_{12}, u_{22} \ge 0. && && (57k)
\end{aligned}
$$

We list the values obtained when including/excluding each one of the additional individual valid constraints in Table 11. From Table 11 we observe that the most valuable constraint multiplication in

Con	w/o	w
(57d)	-1.84	-1.84
(57e)	-1.84	-1.84
(57f), (57k)	-1.84	-1.84
(57g)	-1.84	-1.84
(57h)	-1.84	-1.84
(57i)	$-\infty$	-1.84
(57j)	-1.84	-1.84

Table 11 Comparison of the optimal values for Problem (57), with and without each of the proposed multiplications.

this case is (57i), which results from the multiplication of the LHS of the conic quadratic inequality (56b) with the power cone inequality (56c). □

EXAMPLE 10. Consider the following toyexample

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_1 x_3 && (58a)\\
\text{s.t.} \quad & x_1 \exp(x_1) \le 5, && (58b)\\
& x_2 \exp(x_2) \le 5, && (58c)
\end{aligned}
$$

$$3x_2 \geq \left\|\mathrm{diag}(\boldsymbol{a})\boldsymbol{x}\right\|, \tag{58d}$$

$$x_1^{0.5}x_2^{0.5} \geq 2x_2, \tag{58e}$$

$$x_1, x_2 \geq 0, \tag{58f}$$

where $\boldsymbol{a} = (0,1,1)^\top$. We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && && (59a)\\
\text{s.t.} \quad & (58d) - (58f), \\
& x_1 \exp\left(\frac{u_{11}}{x_1}\right) \leq 5, && && (59b)\\
& x_2 \exp\left(\frac{u_{22}}{x_2}\right) \leq 5, && && (59c)\\
& 3u_{i2} \geq \left\|\mathrm{diag}(\boldsymbol{a})\boldsymbol{u}_i\right\|, && i = 1,2, && (59d)\\
& \left\|\mathrm{diag}(\boldsymbol{a})\boldsymbol{U}\mathrm{diag}(\boldsymbol{a})^\top\right\|_F \leq 9u_{22}, && && (59e)\\
& u_{i1}^{0.5}u_{i2}^{0.5} \geq 2u_{i2}, && i = 1,2, && (59f)\\
& u_{11}, u_{12}, u_{22} \geq 0, && && (59g)\\
& u_{11}^{0.5}u_{22}^{0.5} \geq 4u_{22}, && && (59h)\\
& (3u_{12})^{0.5}(3u_{22})^{0.5} \geq \left\|2\mathrm{diag}(\boldsymbol{a})\boldsymbol{u}_2\right\|. && && (59i)
\end{aligned}$$

We list the values obtained when including/excluding each one of the individual constraints in Table 12. From Table (12) we observe that the most valuable constraint multiplications in this case

Con	w/o	w
(59d)	-∞	-1.30
(59e)	-1.30	-1.30
(59f), (59g)	-∞	-1.30
(59h)	-1.30	-1.30
(59i)	-1.30	-1.30

Table 12 Comparison of the optimal values for Problem (59), with and without each of the proposed multiplications.

are (59d) and (59f), (59g). (59d) results from the multiplication of the conic quadratic inequality (58d) with the nonnegativity constraints of the power cone (58f). (59f), (59g) result from the multiplications of the nonnegativity constraints (58f) with the power cone inequalities (58d) and (58f). □

EXAMPLE 11. Consider the following toy example

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_1 x_3 && \text{(60a)}\\
\text{s.t.} \quad & 3x_2 \geq \|\mathrm{Diag}(\boldsymbol{a})\boldsymbol{x}\|, && \text{(60b)}\\
& x_3 \exp\left(\frac{-x_2}{x_3}\right) \leq 5, && \text{(60c)}\\
& x_3 \geq 0, && \text{(60d)}\\
& x_3 \exp(x_2) \leq 10, && \text{(60e)}
\end{aligned}
$$

where $\boldsymbol{a} = (1,1,0)^\top$. We obtain the following RPT relaxation

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && \text{(61a)}\\
\text{s.t.} \quad & \text{(60b)} - \text{(60d)}, && \text{(61b)}\\
& x_3 \exp\left(\frac{u_{23}}{x_3}\right) \leq 10, && \text{(61c)}\\
& 9u_{22} \geq \|3\mathrm{Diag}(\boldsymbol{a})\boldsymbol{u}_2\|, && \text{(61d)}\\
& 9u_{22} \geq \left\|\mathrm{Diag}(\boldsymbol{a})\boldsymbol{U}\mathrm{Diag}(\boldsymbol{a})^\top\right\|_F, && \text{(61e)}\\
& u_{33} \exp\left(\frac{-2u_{23}}{u_{33}}\right) \leq 25, && \text{(61f)}\\
& u_{33} \exp\left(\frac{-u_{23}}{u_{33}}\right) \leq 5x_3, && \text{(61g)}\\
& 3u_{23} \geq \|\mathrm{Diag}(\boldsymbol{a})\boldsymbol{u}_3\|, && \text{(61h)}\\
& u_{23} \exp\left(\frac{-u_{22}}{u_{23}}\right) \leq 5x_2, && \text{(61i)}\\
& u_{33} \geq 0. && \text{(61j)}
\end{aligned}
$$

We list the values obtained when including/excluding each one of the additional individual valid constraints in Table 13. From Table (13) we observe that the most valuable constraint multiplication

Con	w/o	w
(61d)	-10.41	-10.41
(61e)	-10.41	-10.41
(61f), (61j)	-10.41	-10.41
(61g)	-10.41	-10.41
(61h)	$-\infty$	-10.41
(61i)	-10.41	-10.41

Table 13 Comparison of the optimal values for Problem (61), with and without each of the proposed multiplications.

in this case is (61h), which results from the multiplication of the LHS of the conic quadratic inequality (60b) with the exponential cone inequality (60e). We observe that when constraint (61f) is not included, the problem is unbounded, whereas when it is included a lower bound of -14.72 is obtained. □

EXAMPLE 12. Consider the following toy example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \tag{62a}$$
$$\text{s.t.} \quad 3x_1 \geq \|\text{Diag}(\boldsymbol{a})\boldsymbol{x}\|, \tag{62b}$$
$$x_3 \geq \exp(x_1), \tag{62c}$$
$$x_2 x_3 \geq 3, \tag{62d}$$

where $\boldsymbol{a} = (1,1,0)^\top$. We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{13} \tag{63a}$$
$$\text{s.t.} \quad (62b) - (62c) \tag{63b}$$
$$u_{23} \geq 3, \tag{63c}$$
$$9u_{11} \geq \|3\text{Diag}(\boldsymbol{a})\boldsymbol{u}_1\|, \tag{63d}$$
$$9u_{11} \geq \left\|\text{Diag}(\boldsymbol{a})\boldsymbol{U}\text{Diag}(\boldsymbol{a})^\top\right\|_F, \tag{63e}$$
$$u_{33} \geq \exp(2x_1), \tag{63f}$$
$$u_{33} \geq x_3 \exp\left(\frac{u_{13}}{x_3}\right), \tag{63g}$$
$$3u_{13} \geq \|\text{Diag}(\boldsymbol{a})\boldsymbol{u}_3\|, \tag{63h}$$
$$u_{13} \geq x_1 \exp\left(\frac{u_{11}}{x_1}\right). \tag{63i}$$

We list the values obtained when including/excluding each one of the additional individual valid constraints in Table 14. From Table 14 we observe that the most valuable constraint multiplication

Con	w/o	w
(63d)	1.06	1.06
(63e)	1.06	1.06
(63f)	1.06	1.06
(63g)	1.06	1.06
(63h)	0.00	1.06
(63i)	1.06	1.06

Table 14 Comparison of the optimal values for Problem (63a), with and without each of the proposed multiplications.

in this case is (63h), which results from the multiplication of the conic quadratic inequality (62b) with the LHS of the exponential cone inequality (62c). □

EXAMPLE 13. Consider the following toyexample

$$\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_3x_4 && \text{(64a)}\\
\text{s.t.} \quad & x_1+x_2 \geq \|\mathrm{diag}(\boldsymbol{a})\boldsymbol{x}\|, && \text{(64b)}\\
& x_2+x_3 \geq \exp(x_1+x_2), && \text{(64c)}\\
& (x_1+x_2)\exp(x_i) \leq 5, \quad i\in[4]. && \text{(64d)}
\end{aligned}$$

where $\boldsymbol{a}=(0,1,0,1)^\top$. We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{34} && \text{(65a)}\\
\text{s.t.} \quad & (64b)-(64c), &&\\
& (x_1+x_2)\exp\left(\frac{u_{1i}+u_{2i}}{x_1+x_2}\right) \leq 5, \quad i\in[4], && \text{(65b)}\\
& \left\|\mathrm{diag}(\boldsymbol{a})(\boldsymbol{u}_1+\boldsymbol{u}_2)\right\| \leq u_{11}+2u_{12}+u_{22}, && \text{(65c)}\\
& \left\|\mathrm{diag}(\boldsymbol{a})\boldsymbol{U}\mathrm{diag}(\boldsymbol{a})^\top\right\|_F \leq u_{11}+2u_{12}+u_{22}, && \text{(65d)}\\
& \left\|\mathrm{diag}(\boldsymbol{a})(\boldsymbol{u}_2+\boldsymbol{u}_3)\right\| \leq u_{12}+u_{22}+u_{13}+u_{23}, && \text{(65e)}\\
& (x_2+x_3)\exp\left(\frac{u_{12}+u_{22}+u_{13}+u_{23}}{x_2+x_3}\right) \leq u_{22}+2u_{23}+u_{33}, && \text{(65f)}\\
& \exp(2x_1+2x_2) \leq u_{22}+2u_{23}+u_{33}, && \text{(65g)}\\
& (x_1+x_2)\exp\left(\frac{u_{11}+2u_{12}+u_{22}}{x_1+x_2}\right) \leq u_{12}+u_{22}+u_{13}+u_{23}, && \text{(65h)}\\
& x_3 \geq x_1+1+y, && \text{(65i)}\\
& \left\|\left(\sqrt{2}(x_1+x_2),\ 1-y\right)\right\| \leq 1+y, && \text{(65j)}\\
& \left\|\mathrm{diag}(\boldsymbol{a})\left(\boldsymbol{u}_3-\boldsymbol{u}_1-\boldsymbol{x}-\boldsymbol{z}\right)\right\| \leq u_{13}+u_{23}-u_{11}-u_{12}-x_1-x_2-z_1-z_2, && \text{(65k)}\\
& \left\|\left(\sqrt{2}(u_{11}+2u_{12}+u_{22}),\ x_1+x_2-z_1-z_2\right)\right\| \leq x_1+x_2+z_1+z_2, && \text{(65l)}\\
& \left\|\mathrm{diag}(\boldsymbol{a})\left(\boldsymbol{x}+\boldsymbol{z}\right)\right\| \leq x_1+x_2+z_1+z_2, && \text{(65m)}\\
& \left\|\begin{pmatrix}\sqrt{2}(u_{12}+u_{22}) & \sqrt{2}(u_{14}+u_{24})\\ x_2-z_2 & x_4-z_4\end{pmatrix}\right\| \leq x_1+x_2+z_1+z_2, && \text{(65n)}\\
& (x_3-x_1-1-y)\exp\left(\frac{u_{13}+u_{23}-u_{11}-u_{12}-x_1-x_2-z_1-z_2}{x_3-x_1-1-y}\right) &&\\
& \qquad \leq u_{23}+u_{33}-u_{12}-u_{13}-x_2-x_3-z_2-z_3, && \text{(65o)}\\
& (1+y)\exp\left(\frac{x_1+x_2+z_1+z_2}{1+y}\right) \leq x_2+x_3+z_2+z_3, && \text{(65p)}\\
& \left\|\left(\sqrt{2}(u_{12}+u_{13}+u_{22}+u_{23}),\ x_2+x_3-z_2-z_3\right)\right\| \leq x_2+x_3+z_2+z_3, && \text{(65q)}\\
& u_{11}+u_{33}-2u_{13}+2x_1-2x_3+2z_1-2z_3+2y+q+1 \geq 0, && \text{(65r)}\\
& \left\|\left(\sqrt{2}(u_{13}+u_{23}-u_{11}-u_{12}-x_1-x_2-z_1-z_2),\ x_3-x_1-1-y-z_3+z_1+y+q\right)\right\| &&\\
& \qquad \leq x_3-x_1-1-y+z_3-z_1-y-q, && \text{(65s)}
\end{aligned}$$

$$\left\| \left(\sqrt{2}(x_1+x_2+z_1+z_2),\ 1+q\right) \right\| \le 1+2y+q, \tag{65t}$$

$$\left\| \begin{pmatrix} 2(u_{11}+2u_{12}+u_{22}) & \sqrt{2}(x_1+x_2-z_1-z_2) \\ \sqrt{2}(x_1+x_2-z_1-z_2) & 1-2y+q \end{pmatrix} \right\| \le 1+2y+q. \tag{65u}$$

We list the values obtained when including/excluding each one of the additional individual valid constraints in Table 15. From Table 15 we observe that some of the most valuable constraint

Con	w/o	w
(65c)	-2.88	-2.52
(65d)	$-\infty$	-2.52
(65e)	-2.52	-2.52
(65f)	-2.52	-2.52
(65g)	-2.52	-2.52
(65h)	-2.58	-2.52
(65k)-(65n)	-2.90	-2.52
(65o)-(65p)	-2.52	-2.52
(65q)-(65u)	-2.52	-2.52

Table 15 Comparison of the optimal values for Problem (65), with and without each of the proposed multiplications.

multiplications in this case are (65h) and (65k)-(65n). (65h) results from the multiplication of the exponential cone inequality (64c) with the LHS of the quadratic inequality (64b). (65k)-(65n) result from the multiplication of the inequalities (77i) and (77j) result from the multiplication of the decomposition of the exponential cone with the conic quadratic inequality (64b). □

Example 14. Consider the following toy example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \tag{66a}$$

$$\text{s.t.} \quad x_2 \exp(x_i) \le 5,\ i=1,2,3, \tag{66b}$$

$$2x_2 \ge \left\| \operatorname{diag}(\boldsymbol{a})\boldsymbol{x} \right\|, \tag{66c}$$

$$x_2 \ge \exp(-x_3). \tag{66d}$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{13} \tag{67a}$$

$$\text{s.t.} \quad x_2 \exp\left(\frac{u_{i2}}{x_2}\right) \le 5,\ i=1,2,3, \tag{67b}$$

$$(66c) - (66d),$$

$$2u_{22} \ge \left\| \operatorname{diag}(\boldsymbol{a})\boldsymbol{u}_2 \right\|, \tag{67c}$$

$$4u_{22} \ge \left\| \operatorname{diag}(\boldsymbol{a})\boldsymbol{U}\operatorname{diag}(\boldsymbol{a})^\top \right\|_F, \tag{67d}$$

$$x_2 \exp\left(\frac{-u_{23}}{x_2}\right) \leq u_{22}, \tag{67e}$$

$$\exp(-2x_3) \leq u_{22}, \tag{67f}$$

$$x_2 \geq 1 - x_3, \tag{67g}$$

$$u_{22} + u_{33} + 2u_{23} - 2x_2 - 2x_3 + 1 \geq 0, \tag{67h}$$

$$2u_{22} - 2x_2 + 2u_{23} \geq \left\|\mathrm{diag}(\boldsymbol{a})\,(\boldsymbol{u}_2 - \boldsymbol{x} + \boldsymbol{u}_3)\,\right\|, \tag{67i}$$

$$(x_2 - 1 + x_3)\exp\left(\frac{-u_{23} + x_3 - u_{33}}{x_2 - 1 + x_3}\right) \leq u_{22} - x_2 + u_{23}. \tag{67j}$$

We list the values obtained when including/excluding each one of the individual constraints in Table 16. From Table (16) we observe that the most valuable constraint multiplication in this

Con	w/o	w
(67c)	-8.21	-8.21
(67d)	-8.21	-8.21
(67e)	-8.21	-8.21
(67f)	-8.21	-8.21
(67h)	-8.21	-8.21
(67i)	$-\infty$	-8.21
(67j)	-8.21	-8.21

Table 16 Comparison of the optimal values for Problem (67), with and without each of the proposed multiplications.

case is (67i), which results from the multiplication of the conic quadratic inequality (66c) with the additional valid linear inequality (67g), resulting from the decomposition of (66d). We observe that the multiplication of the linear with the exponential cone inequality is necessary in this case. □

Example 15. Consider the following toy example

$$\min_{\boldsymbol{x}} \quad x_1 \exp(x_1) \tag{68a}$$

$$\text{s.t.} \quad x_1 x_1 \geq 10, \tag{68b}$$

$$x_2 x_2 \geq 5, \tag{68c}$$

$$x_3 x_3 \geq 10, \tag{68d}$$

$$x_2 \exp(x_1) \leq 25, \tag{68e}$$

$$x_1^{0.5} x_2^{0.5} \geq x_3, \tag{68f}$$

$$x_1, x_2 \geq 0, \tag{68g}$$

$$x_3 \geq 0. \tag{68h}$$

We obtain the following RPT relaxation

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & x_1 \exp\left(\frac{u_{11}}{x_1}\right) && && (69a)\\
\text{s.t.} \quad & u_{11} \geq 10, && && (69b)\\
& u_{22} \geq 5, && && (69c)\\
& u_{33} \geq 10, && && (69d)\\
& x_2 \exp\left(\frac{u_{22}}{x_2}\right) \leq 25, && && (69e)\\
& (68f) - (68h) && && \\
& u_{i1}^{0.5} u_{i2}^{0.5} \geq u_{i3}, && i = 1, 2, && (69f)\\
& u_{11}^{0.5} u_{22}^{0.5} \geq u_{33}, && && (69g)\\
& u_{13}^{0.5} u_{23}^{0.5} \geq u_{33}, && && (69h)\\
& u_{11}, u_{12}, u_{22} \geq 0, && && (69i)\\
& u_{13}, u_{23} \geq 0, && && (69j)\\
& u_{33} \geq 0. && && (69k)
\end{aligned}
$$

We list the values obtained when including/excluding each one of the individual constraints in Table 17. From Table 17 we observe that the most valuable constraint multiplication is (69g), which

Con	w/o	w
(69f), (69i)	29.56	29.56
(69g)	27.18	29.56
(69h), (69j)	29.56	29.56
(69k)	29.56	29.56

Table 17 Comparison of the optimal values for Problem (69a), with and without each of the proposed multiplications.

results from the multiplication of the power cone inequality (68f) with itself. □

Example 16. Consider the following toy example

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & x_1 x_3 && (70a)\\
\text{s.t.} \quad & x_1 x_2 \geq 1, && (70b)\\
& x_2 x_3 \geq 1, && (70c)\\
& x_3 x_3 \geq 5, && (70d)\\
& x_1^{0.5} x_2^{0.5} \geq x_3, && (70e)\\
& x_1, x_2 \geq 0, && (70f)\\
& x_3 \geq \exp(x_2). && (70g)
\end{aligned}
$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{13} \tag{71a}$$
$$\text{s.t.} \quad u_{12} \geq 1, \tag{71b}$$
$$u_{23} \geq 1, \tag{71c}$$
$$u_{33} \geq 5, \tag{71d}$$
$$(70e) - (70g),$$
$$u_{11}, u_{22}, u_{12} \geq 0, \tag{71e}$$
$$u_{i1}^{0.5} u_{i2}^{0.5} \geq u_{i3}, \qquad i = 1, 2, \tag{71f}$$
$$u_{11}^{0.5} u_{22}^{0.5} \geq u_{33}, \tag{71g}$$
$$u_{33} \geq x_3 \exp\left(\frac{u_{23}}{x_3}\right), \tag{71h}$$
$$u_{33} \geq \exp(2x_2), \tag{71i}$$
$$x_i \exp\left(\frac{u_{i2}}{x_i}\right) \leq u_{i3}, \qquad i = 1, 2, \tag{71j}$$
$$u_{13}^{0.5} u_{23}^{0.5} \geq u_{33}. \tag{71k}$$

We list the values obtained when including/excluding each one of the individual constraints in Table 18. From Table 18 we observe that the most valuable constraint multiplications in this case are (71e)

Con	w/o	w
(71e), (71f)	13.59	13.65
(71g)	13.65	13.65
(71h)	13.65	13.65
(71i)	13.65	13.65
(71j)	13.59	13.65
(71k)	9.07	13.65

Table 18 Comparison of the optimal values for Problem (71a), with and without each of the proposed multiplications.

- (71f), (71j), and (71k). The first two inequalities result from the multiplication of the nonnegativity constraints (70f) of the power cone with the power cone inequalities (70e) and (70f). The third inequality results from the multiplication of the nonnegativity constraints (70f) of the power cone with the exponential cone inequality (70g). The last inequality results from the multiplication of the LHS of the exponential inequality (70g) with the power cone inequality (70e). □

Example 17. Consider the following toyexample

$$\min_{\boldsymbol{x}} \quad x_3 x_4 \tag{72a}$$

$$\text{s.t.} \quad x_1^{0.5} x_2^{0.5} \geq \|\mathrm{diag}(\boldsymbol{a})\boldsymbol{x}\|, \tag{72b}$$

$$x_1, x_2 \geq 0, \tag{72c}$$

$$x_2 + x_3 \geq \exp(x_1 + x_2), \tag{72d}$$

$$(x_1 + x_2)\exp(x_i) \leq 5, \quad i \in [4]. \tag{72e}$$

where $\boldsymbol{a} = (0,1,0,1)^\top$. We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{34} \tag{73a}$$

$$\text{s.t.} \quad (72b) - (72d),$$

$$(x_1 + x_2)\exp\left(\frac{u_{1i} + u_{2i}}{x_1 + x_2}\right) \leq 5, \quad i \in [4], \tag{73b}$$

$$u_{1i}^{0.5} u_{2i}^{0.5} \geq \left\|\mathrm{diag}(\boldsymbol{a})\boldsymbol{u}_i\right\|, \quad i = 1, 2 \tag{73c}$$

$$u_{11}, u_{12}, u_{22} \geq 0, \tag{73d}$$

$$u_{11}^{0.5} u_{22}^{0.5} \geq \left\|\mathrm{diag}(\boldsymbol{a})\boldsymbol{U}\mathrm{diag}(\boldsymbol{a})^\top\right\|_F, \tag{73e}$$

$$u_{22} + 2u_{23} + u_{33} \geq (x_2 + x_3)\exp\left(\frac{u_{12} + u_{22} + u_{13} + u_{23}}{x_2 + x_3}\right), \tag{73f}$$

$$u_{22} + 2u_{23} + u_{33} \geq \exp(2x_1 + 2x_2), \tag{73g}$$

$$u_{i2} + u_{i3} \geq x_i \exp\left(\frac{u_{1i} + u_{2i}}{x_i}\right), \quad i = 1, 2, \tag{73h}$$

$$(u_{12} + u_{13})^{0.5}(u_{22} + u_{23})^{0.5} \geq \left\|\mathrm{diag}(\boldsymbol{a})(\boldsymbol{u}_2 + \boldsymbol{u}_3)\right\|, \tag{73i}$$

$$x_3 \geq x_1 + 1 + y, \tag{73j}$$

$$1 + y \geq \left\|\left(\sqrt{2}(x_1 + x_2),\ 1 - y\right)\right\|, \tag{73k}$$

$$(u_{13} - u_{11} - x_1 - z_1)^{0.5}(u_{23} - u_{12} - x_2 - z_2)^{0.5} \geq \left\|\mathrm{diag}(\boldsymbol{a})(\boldsymbol{u}_3 - \boldsymbol{u}_1 - \boldsymbol{x} - \boldsymbol{z})\right\|, \tag{73l}$$

$$(x_1 + z_1)^{0.5}(x_2 + z_2)^{0.5} \geq \left\|\mathrm{diag}(\boldsymbol{a})\,(\boldsymbol{x} + \boldsymbol{z})\right\|, \tag{73m}$$

$$(x_1 + z_1)^{0.5}(x_2 + z_2)^{0.5} \geq \left\|\begin{pmatrix} \sqrt{2}(u_{12} + u_{22}) & \sqrt{2}(u_{14} + u_{24}) \\ x_2 - z_2 & x_4 - z_4 \end{pmatrix}\right\|, \tag{73n}$$

$$u_{23} + u_{33} - u_{12} - u_{13} - x_2 - x_3 - z_2 - z_3 \geq \\ (x_3 - x_1 - 1 - y)\exp\left(\frac{u_{13} + u_{23} - u_{11} - u_{12} - x_1 - x_2 - z_1 - z_2}{x_3 - x_1 - 1 - y}\right), \tag{73o}$$

$$x_2 + x_3 + z_2 + z_3 \geq (1 + y)\exp\left(\frac{x_1 + x_2 + z_1 + z_2}{1 + y}\right), \tag{73p}$$

$$x_2 + x_3 + z_2 + z_3 \geq \left\|\left(\sqrt{2}(u_{12} + u_{13} + u_{22} + u_{23}),\ x_2 + x_3 - z_2 - z_3\right)\right\|, \tag{73q}$$

$$x_3 - x_1 - 1 - y + z_3 - z_1 - y - q \geq \\ \left\|\left(\sqrt{2}(u_{13} + u_{23} - u_{11} - u_{12} - x_1 - x_2 - z_1 - z_2),\ x_3 - x_1 - 1 - y - z_3 + z_1 + y + q\right)\right\|, \tag{73r}$$

$$1 + 2y + q \geq \left\|\left(\sqrt{2}(x_1 + x_2 + z_1 + z_2),\ 1 + q\right)\right\|, \tag{73s}$$

$$1+2y+q \geq \left\| \begin{pmatrix} 2(u_{11}+2u_{12}+u_{22}) & \sqrt{2}(x_1+x_2-z_1-z_2) \\ \sqrt{2}(x_1+x_2-z_1-z_2) & 1-2y+q \end{pmatrix} \right\|_F. \tag{73t}$$

We list the values obtained when including/excluding each one of the additional individual valid constraints in Table 19. From Table (19) we observe that the most valuable constraint multiplications

Con	w/o	w
(73c)-(73d)	-0.79	-0.65
(73e)	-0.65	-0.65
(73f)	-0.65	-0.65
(73g)	-0.65	-0.65
(73h)	-0.65	-0.65
(73i)	-0.65	-0.65
(73l)-(73n)	-0.76	-0.65
(73o)-(73p)	-0.65	-0.65
(73q)-(73t)	-0.65	-0.65

Table 19 Comparison of the optimal values for Problem (73), with and without each of the proposed multiplications.

in this case are (73c)-(73d) and (73l)-(73n). (73c)-(73d) result from the multiplication of the nonnegativity constraints (72c) of the power cone with the power cone inequalities (72b) and (72c). (73l)-(73n) result from the multiplication of the additional inequalities (73j) and (73k) with the power cone inequality (72b). □

EXAMPLE 18. Consider the following toy example

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & x_1x_3 && (74a)\\ \text{s.t.} \quad & x_1x_2 \geq 1, && (74b)\\ & x_2x_3 \geq 1, && (74c)\\ & x_1^{0.5}x_2^{0.5} \geq x_3, && (74d)\\ & 5 \geq x_3 \exp\left(\frac{x_2}{x_3}\right), && (74e)\\ & x_1, x_2 \geq 0, && (74f)\\ & x_3 \geq 0. && (74g) \end{aligned}$$

We obtain the following RPT relaxation

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{13} && (75a)\\ \text{s.t.} \quad & u_{12} \geq 1, && (75b)\\ & u_{23} \geq 1, && (75c)\\ & (74d)-(74g), \end{aligned}$$

$$u_{i1}^{0.5}u_{i2}^{0.5} \geq u_{i3}, \qquad i=1,2, \tag{75d}$$

$$u_{11}, u_{12}, u_{22} \geq 0, \tag{75e}$$

$$u_{11}^{0.5}u_{22}^{0.5} \geq u_{33}, \tag{75f}$$

$$5x_3 \geq u_{33}\exp\left(\frac{u_{23}}{u_{33}}\right), \tag{75g}$$

$$25 \geq u_{33}\exp\left(\frac{2u_{23}}{u_{33}}\right), \tag{75h}$$

$$5x_i \geq u_{i3}\exp\left(\frac{u_{i2}}{u_{i3}}\right), \qquad i=1,2, \tag{75i}$$

$$u_{13}, u_{23} \geq 0, \tag{75j}$$

$$u_{13}^{0.5}u_{23}^{0.5} \geq u_{33}, \tag{75k}$$

$$u_{33} \geq 0. \tag{75l}$$

We list the values obtained when including/excluding each one of the individual constraints in Table 20. From Table 20 we observe that the most valuable constraint multiplications in this case

Con	w/o	w
(75d), (75e)	0.27	0.27
(75f)	0.27	0.27
(75g)	0.27	0.27
(75h)	0.05	0.27
(75i), (75j)	0.27	0.27
(75j), (75k)	0.05	0.27
(75l)	0.27	0.27

Table 20 Comparison of the optimal values for Problem (75), with and without each of the proposed multiplications.

are (75h) and (75j), (75j). The first inequality results from the multiplication of exponential cone inequality (74e) with itself, while the last two inequalities result from the multiplication of the nonnegativity constraint (74g) of the exponential cone inequality with the power cone inequalities (74f) and (74d), respectively. □

Example 19. Consider the following toy example

$$\min_{\boldsymbol{x}} \quad x_3 x_4 \tag{76a}$$

$$\text{s.t.} \quad x_1^{0.5}x_2^{0.5} \geq |x_3|, \tag{76b}$$

$$x_1, x_2 \geq 0. \tag{76c}$$

$$x_2 \geq \exp(-x_4), \tag{76d}$$

$$x_2\exp(x_1) \leq 5, \tag{76e}$$

$$x_2 \exp(x_2) \le 10, \tag{76f}$$

$$x_1 \exp(x_4) \le 10, \tag{76g}$$

$$x_2 \exp(x_4) \le 10, \tag{76h}$$

We obtain the following RPT relaxation

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{U}} \quad & u_{34} && && \text{(77a)}\\
\text{s.t.} \quad & (76b) - (76d) \\
& x_2 \exp\left(\frac{u_{12}}{x_2}\right) \le 5, && && \text{(77b)}\\
& x_2 \exp\left(\frac{u_{22}}{x_2}\right) \le 10, && && \text{(77c)}\\
& x_1 \exp\left(\frac{u_{14}}{x_1}\right) \le 10, && && \text{(77d)}\\
& x_2 \exp\left(\frac{u_{24}}{x_2}\right) \le 10, && && \text{(77e)}\\
& u_{1i}^{0.5} u_{2i}^{0.5} \ge |u_{i3}|, && i = 1,2, && \text{(77f)}\\
& u_{11}, u_{12}, u_{12} \ge 0, && && \text{(77g)}\\
& u_{11}^{0.5} u_{22}^{0.5} \ge |u_{33}|, && && \text{(77h)}\\
& u_{22} \ge \exp(-2x_4), && && \text{(77i)}\\
& u_{i2} \ge x_i \exp\left(\frac{-u_{i4}}{x_i}\right), && i = 1,2, && \text{(77j)}\\
& x_2 \ge -x_4 + 1, && && \text{(77k)}\\
& (u_{12} + u_{14} - x_1)^{0.5} (u_{22} + u_{24} - x_2)^{0.5} \ge |u_{23} + u_{34} - x_3|, && && \text{(77l)}\\
& u_{22} + u_{24} - x_2 \ge (x_2 + x_4 - 1) \exp\left(\frac{-u_{24} - u_{44} + x_4}{x_2 + x_4 - 1}\right). && && \text{(77m)}
\end{aligned}$$

We list the values obtained when including/excluding each one of the additional constraints in Table 21. From Table 21 we observe that the most valuable constraint multiplications in this case are (77f)

Con	w/o	w
(77f), (77g)	-∞	-8.43
(77h)	-8.43	-8.43
(77i)	-8.43	-8.43
(77j)	-8.43	-8.43
(77l)	-∞	-8.43
(77m)	-8.43	-8.43

Table 21 Comparison of the optimal values for Problem (77a), with and without each of the proposed multiplications.

- (77g), and (77l). (77f) - (77g) result from the multiplication of the nonnegativity constraints (76c) with the power cone inequalities (76b) and (76c). (77l) results from the multiplication of inequality (77k), resulting from the decomposition of the exponential cone, with the power cone inequality (76b). □

EXAMPLE 20. Consider the following toy example

$$\min_{\boldsymbol{x}} \quad x_1 x_2 \tag{78a}$$
$$\text{s.t.} \quad x_1 x_3 \geq 5, \tag{78b}$$
$$x_1 \geq \exp(x_3), \tag{78c}$$
$$x_2 \geq \exp(x_3). \tag{78d}$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{12} \tag{79a}$$
$$\text{s.t.} \quad u_{13} \geq 5, \tag{79b}$$
$$(78c) - (78d),$$
$$u_{11} \geq x_1 \exp\left(\frac{u_{13}}{x_1}\right), \tag{79c}$$
$$u_{12} \geq x_2 \exp\left(\frac{u_{23}}{x_2}\right), \tag{79d}$$
$$u_{12} \geq x_1 \exp\left(\frac{u_{13}}{x_1}\right), \tag{79e}$$
$$u_{22} \geq x_2 \exp\left(\frac{u_{23}}{x_2}\right), \tag{79f}$$
$$u_{11} \geq \exp\left(2x_3\right), \tag{79g}$$
$$u_{22} \geq \exp\left(2x_3\right), \tag{79h}$$
$$u_{12} \geq \exp\left(2x_3\right). \tag{79i}$$

We list the values obtained when including/excluding each one of the individual constraints in Table 22. From Table 22, we observe that the most valuable constraint multiplications in this case are

Con	w/o	w
(79c) - (79f)	0.00	13.59
(79g) - (79i)	13.59	13.59

Table 22 Comparison of the optimal values for Problem (78), with and without each of the proposed multiplications.

(79c) - (79f), which result from the multiplication of the LHS of the exponential cone inequalities (78c) and (78d) with both sides of the exponential cone inequalities (78c) and (78d). □

Example 21. Consider the following toy example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \tag{80a}$$

$$\text{s.t.} \quad x_1 x_2 \geq 1, \tag{80b}$$

$$x_3 \geq x_1 \exp\left(\frac{x_2}{x_1}\right), \tag{80c}$$

$$x_1 \geq 0. \tag{80d}$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{13} \tag{81a}$$

$$\text{s.t.} \quad u_{12} \geq 1, \tag{81b}$$

$$(80c) - (80d),$$

$$u_{33} \geq u_{13} \exp\left(\frac{u_{23}}{u_{13}}\right), \tag{81c}$$

$$u_{33} \geq u_{11} \exp\left(\frac{2u_{12}}{u_{11}}\right), \tag{81d}$$

$$u_{13} \geq u_{11} \exp\left(\frac{u_{12}}{u_{11}}\right), \tag{81e}$$

$$u_{11} \geq 0. \tag{81f}$$

We list the values obtained when including/excluding each one of the individual constraints in Table 23. From Table 23, we observe that the most valuable constraint multiplications are in this

Con	w/o	w
(81c)	2.72	2.72
(81d)	2.72	2.72
(81e), (81f)	0.0	2.72

Table 23 Comparison of the optimal values for Problem (81), with and without each of the proposed multiplications.

case (81e) and (81f), which result from the multiplication of the nonnegativity constraint (80d) with the exponential cone inequalities (80c) and (80d). □

Example 22. We consider the group of constraints resulting from the following multiplications of constraints from numerical experiment 6.1:

$$\left\{ \begin{array}{l} (z_j + x_j + a - 1)\ \exp(-x_i - a) \ \leq\ z_i\,(z_j + x_j + a - 1)\,, \\ (z_j + x_j + a - 1)\ \exp(x_i) \ \leq\ t_i\,(z_j + x_j + a - 1)\,, \\ (t_j - x_j - 1)\ \exp(-x_i - a) \ \leq\ z_i\,(t_j - x_j - 1)\,, \\ (t_j - x_j - 1)\ \exp(x_i) \ \leq\ t_i\,(t_j - x_j - 1) \end{array} \right\}, \tag{82}$$

Cons	w/o	w
Instance 1	-119.98	-102
Instance 2	-191.27	-175.82

Table 24 Comparison of the optimal values for Problem (33), with and without each of the proposed groups of constraint multiplications, for Instance 1.

which consists of all multiplications of a linear inequality derived from the decomposition of one of the exponential cones with all other other exponential cones. We list the values obtained when including/excluding this group of constraints in Table 24 for Instances 1 and 2.

□

EXAMPLE 23. Consider the following toy example

$$\min_{\boldsymbol{x}} \quad x_1 x_3 \tag{83a}$$
$$x_1 \geq x_3, \tag{83b}$$
$$x_2 + x_3 \geq \exp(x_2), \tag{83c}$$
$$x_2 \geq 0. \tag{83d}$$

We obtain the following RPT relaxation

$$\min_{\boldsymbol{x},\boldsymbol{U}} \quad u_{13} \tag{84a}$$
$$\text{s.t.} \quad (83b) - (83d)$$
$$u_{12} \geq u_{23}, \tag{84b}$$
$$u_{11} - 2u_{13} + u_{33} \geq 0, \tag{84c}$$
$$u_{22} + u_{23} \geq x_2 \exp\left(\frac{u_{22}}{x_2}\right), \tag{84d}$$
$$u_{22} + 2u_{23} + u_{33} \geq \exp(2x_2), \tag{84e}$$
$$u_{22} + 2u_{23} + u_{33} \geq (x_2 + x_3)\exp\left(\frac{u_{22} + u_{23}}{x_2 + x_3}\right), \tag{84f}$$
$$u_{12} - u_{23} + u_{13} - u_{33} \geq (x_1 - x_3)\exp\left(\frac{u_{12} - u_{23}}{x_1 - x_3}\right), \tag{84g}$$
$$x_3 \geq 1 + y, \tag{84h}$$
$$\left\| \left(\sqrt{2}x_2, 1 - y\right) \right\| \leq 1 + y, \tag{84i}$$
$$u_{23} + u_{33} - x_2 - x_3 - z_2 - z_3 \geq (x_3 - 1 - y)\exp\left(\frac{u_{23} - x_2 - z_2}{x_3 - 1 - y}\right), \tag{84j}$$
$$x_2 + x_3 + z_2 + z_3 \geq (1 + y)\exp\left(\frac{x_2 + z_2}{1 + y}\right), \tag{84k}$$

We list the values obtained when including/excluding each of the additional constraints in Table 25. From Table 25 we observe that the most valuable constraint multiplications in this case are (84j)

Con	w/o	w
(84b)	1.00	1.00
(84c)	1.00	1.00
(84d)	1.00	1.00
(84e)	1.00	1.00
(84f)	1.00	1.00
(84g)	1.00	1.00
(84j), (84k)	0.61	1.00

Table 25 Comparison of the optimal values for Problem (84), with and without each of the proposed multiplications.

and (84k), which result from the multiplication of the inequalities (84h) and (84i) resulting from the decomposition of the exponential cone with the exponential cone inequality (83c). □