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The stability of a large gas bubble rising through liquid†

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The upper surface of a large gas bubble rising steadily through liquid under gravity is a statically unstable interface, and if the liquid were stationary small sinusoidal disturbances to the interface with wavelength exceeding the critical value λ_c determined by surface tension would grow exponentially. The existence of the deforming motion of the liquid adjoining the interface of a steadily rising bubble changes the nature of the problem of stability. It is shown that a small sinusoidal disturbance of the part of the interface that is approximately plane and horizontal remains sinusoidal, although with exponentially increasing wavelength. The amplitude of such a disturbance increases, from the instant at which $\lambda = \lambda_c$ until λ becomes comparable with the radius of curvature of the interface (R), and the largest amplification occurs for a disturbance whose initial wavelength is approximately equal to λ_c. With a plausible guess at the disturbance amplitude and wavelength at which bubble break-up due to nonlinear effects is inevitable, it is possible to obtain an approximate numerical relation between the initial magnitude of the disturbance and the maximum value of R for which a bubble remains intact. This relation applies both to a spherical-cap bubble in a large tank and a bubble rising in a vertical tube in which the liquid far ahead of the bubble is stationary. The few published observations of the maximum size of spherical-cap bubbles are not incompatible with the theory, but lack of information about the magnitude of the ambient disturbances in the liquid precludes any close comparison.

1. Introduction

The fact that gas bubbles with volumes exceeding about one centilitre take a spherical-cap shape when they rise steadily through liquid of small viscosity appears to have been noticed first by Taylor & Davies (1944) during World War II in laboratory studies of the rate of rise of large gas bubbles simulating those produced by submarine explosions. Later Davies & Taylor (1950) published their photographs of spherical-cap bubbles and their observations that show the steady speed of rise (U) to be approximately proportional to the square root of the radius of curvature of the top of the bubble (R), as would be expected if the bubble shape is independent of size and the drag on the bubble is independent of the viscosity of the liquid. This work and many later investigations have established that the top surface of a large gas bubble is normally quite steady and smooth and closely spherical, although ripples may be seen under some conditions. Figures 1(*a* and *b*) show two spherical-cap bubbles, and figure 1(*c*) a corresponding 'two-dimensional' circular-cap bubble, all

† With an appendix by Herbert E. Huppert.

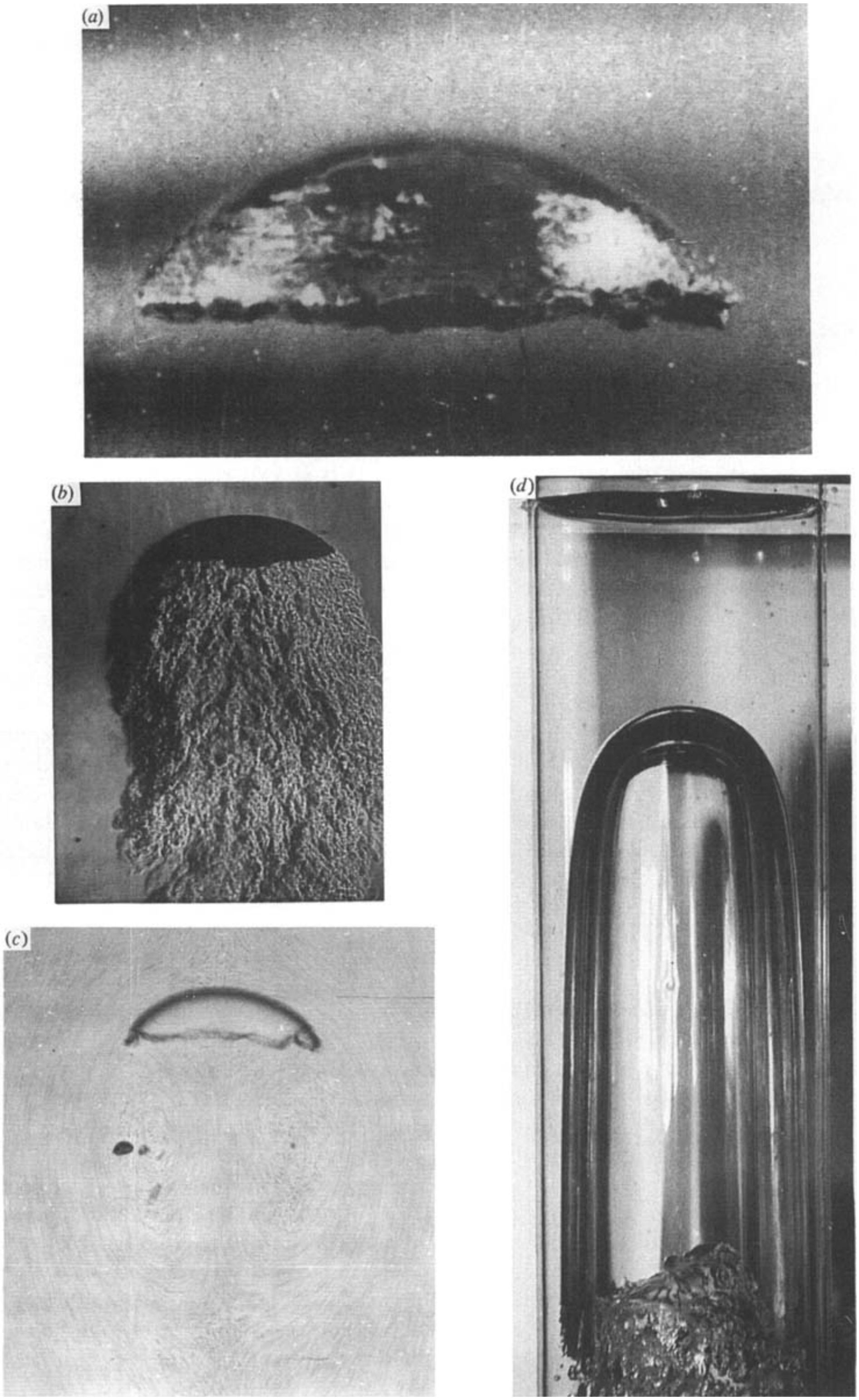

FIGURE 1. For caption see opposite.

large enough for the Reynolds number of the liquid flow and the Bond, or Eőtvős, number (defined as $g\rho d^2/\gamma$, where d is the diameter of a sphere of the same volume as the bubble and γ is the surface tension of the interface) both to be large compared with unity, indicating that effects of viscosity and surface tension on the steady bubble shape are negligible except possibly at singular places such as the sharp edge of the cap. For smaller bubble volumes there is a variety of bubble shapes, described in the book by Clift, Grace & Weber (1978).

Since the upper surface of a spherical-cap bubble separates liquid above from gas below it is at first sight remarkable that it is not gravitationally unstable. It is known that a horizontal plane interface between two stationary fluids with density difference $\Delta\rho$ is unstable to any small sinusoidal disturbance with wavelength greater than λ_c (which denotes $2\pi(\gamma/\Delta\rho g)^{\frac{1}{2}}$, and has the value 1.71 cm for an air–water interface). The existence of the streaming of the liquid around the upper surface of the bubble evidently is responsible for maintenance of the steady spherical shape. Intuitively one might argue that a growing small disturbance to the interface is swept round to the edge of the bubble by the liquid flow and that this limits the time available for growth of the disturbance by the familiar gravitational overturning. But since the speed of rise U is proportional to $(gR)^{\frac{1}{2}}$, the time spent by a material element of liquid in passing round the cap varies as $R^{\frac{1}{2}}$ and so is greater for larger bubbles. This is the theoretical basis for the opinion, reported by Clift *et al.* (1978), that the upper surfaces of spherical-cap bubbles above a certain size are indeed gravitationally unstable and that such bubbles break up into smaller bubbles with stable interfaces. It is of some practical value to know the maximum size of a spherical-cap bubble which remains intact, since that determines the speed with which a very large volume of gas would rise through liquid after breaking up into smaller stable units.

Similar considerations apply to large gas bubbles rising through stationary liquid in a vertical tube, one example of which is shown in figure 1(*d*). The shape of the upper part of the interface of such bubbles in a tube of circular cross-section is observed to be smooth and steady, and independent of bubble size provided the vertical extent of the bubble is large compared with the tube radius. Here too one would expect intuitively that the bubble is gravitationally unstable for sufficiently large radii of curvature of the top of the interface.

2. Previous work on the existence of a maximum bubble size

Experimental evidence concerning the existence of a maximum attainable size of spherical-cap gas bubbles rising steadily in unbounded liquid is surprisingly sparse. No mention of experimental data (nor of the possible existence of a maximum size) is made in the comprehensive review of research on bubble dynamics by Harper (1972) or in the article on spherical-cap bubbles by Wegener & Parlange (1973).

Grace, Wairegi & Brophy (1978) compiled a list of 66 different fluid–fluid disperse

FIGURE 1. Gas bubbles rising steadily through liquid. The Bond number and the Reynolds number of the liquid flow are large in each case. (*a*) Air bubble in water, from Davenport, Bradshaw & Richardson (1967); radius of curvature at top of bubble surface $R \approx 4.8$ cm. (*b*) Schlieren photograph of air bubble in water, from Wegener & Parlange (1973), showing the turbulent wake; $R \approx 4.5$ cm. (*c*) 'Two-dimensional' air bubble in water confined between two parallel plates 0.64 cm apart, from Collins 1965; $R \approx 7$ cm. The camera was moving with the bubble. (*d*) A long air bubble in a vertical circular tube containing water which is stationary far ahead of the bubble, from Davies & Taylor (1950); $R \approx 2.6$ cm.

systems for which observations of a maximum volume of a drop or bubble of the disperse phase had been made, either by themselves or by previous authors. For only five of these systems was the disperse phase gaseous and, as will be seen in §6, for only one of these five was the liquid-flow Reynolds number large enough for the bubbles to be likely to have the steady spherical-cap shape. In their discussion of the question Clift *et al.* (1978) report no observations of a maximum stable size of gas bubbles in water. However, there is some relevant war-time work by Temperley & Chambers (1945). These authors extended to larger bubble sizes the observations of rates of rise made by Taylor & Davies (1944) and found that they could not generate a single spherical-cap bubble of gas in water that remained intact with a radius of curvature of the cap exceeding about 15 cm. This corresponds to a maximum volume of about 1200 cm^3 if we assume a flat base of the bubble and a semi-angle of the cone subtended at the centre of curvature by the upper surface equal to the observed value of about 50°.

The very few published investigations of the existence of a maximum size of spherical-cap bubbles thus support the intuitive notion that increase of size ultimately leads to instability and break-up of the upper surface. The fact that the instability is delayed until the cap radius is as large as 15 cm in the case of gas bubbles in water calls for some explanation.

For prediction purposes Grace *et al.* (1978) have put forward a semi-empirical model of the growth of small disturbances to the upper surface of a rising bubble or drop in a general fluid–fluid disperse system. They make use of the algebraic equation for the proportional growth rate σ of a small sinusoidal deformation of a plane horizontal interface between two stationary fluids given by Bellman & Pennington (1954). This equation includes allowance for the effects of (1) gravity, which causes growth for all values of the disturbance wavelength λ when the upper fluid is the denser, (2) surface tension, which restricts positive growth to values of λ exceeding the critical value λ_c and determines a value of λ for which σ has a maximum, and (3) the viscosity of the two fluids, which reduces σ for all values of λ, especially the smaller values, and so increases the value of λ for which σ has a maximum (σ_m say) but does not change the range of wavelengths for which σ is positive. Grace *et al.* argue that a drop or bubble will break up if the time available for growth of a disturbance at the interface, τ_a say, is sufficiently large by comparison with the minimum e-folding time of a disturbance given by the Bellman & Pennington expression (i.e. σ_m^{-1}). τ_a is estimated as the time for a fluid element to pass nearly round the bubble surface in the external flow ('nearly', because a small region near the forward stagnation point has to be excluded to avoid logarithmic divergence of τ_a), and the value of τ_a/σ_m^{-1} above which break-up of gas bubbles occurs was then found empirically from their observations of the maximum bubble size in five air–liquid systems to be 3.8. For air bubbles in water this 'correlation' gives a maximum bubble volume of 62 cm^3, which is much smaller than the maximum volume 1200 cm^3 observed by Temperley & Chambers.

The model proposed by Grace *et al.* is based on the idea that a limited time only is available for growth of a disturbance to the upper surface of a bubble, and seems to incorporate the essential means by which an upper limit to the size of a stable bubble is determined. However, aside from the roughness with which the time available for amplification of a disturbance is estimated, from a theoretical viewpoint the model needs improvement (for high-Reynolds-number bubbles) in the following respects:

(*a*) In reality the velocity of a material element of liquid at the bubble surface increases with distance from the forward stagnation point, and so the liquid is subjected to a pure straining motion with contraction in the direction normal to the interface and extension in orthogonal directions. One consequence of this contractional motion is that it tends to decrease the amplitude of a disturbance.

(*b*) The growth rate of a disturbance is not constant throughout its lifetime, because the wavelength of the disturbance is being increased continually by the extensional motion; and so the most 'dangerous' disturbance does not grow at the maximum rate σ_m calculated by Bellman & Pennington. This use of the maximum growth rate obscures the way in which surface tension exerts a controlling influence, as we shall see.

(*c*) There is no quantitative dependence of the maximum bubble size on the initial magnitude of the disturbance to the interface.

In the case of gas bubbles rising in a vertical tube, there appear not to be any published observations that show the existence of a maximum value of the radius of curvature of the top of the bubble. The model of instability proposed by Grace *et al.* (1978) is applicable in principle, although the empirical relations needed for quantitative predictions from their model are lacking in this case.

3. Equations governing a small disturbance to the top of a large gas bubble

We now investigate mathematically the behaviour of a small disturbance to the upper surface of a gas bubble rising steadily through liquid, with allowance for the deforming motion of the liquid streaming around the bubble.† The pressure in the gas is assumed to be uniform. The bubble is taken to be sufficiently large for the Reynolds number of the liquid flow to be large, implying that the liquid viscosity has negligible effect on the undisturbed bubble shape. Note that the specificity of this fluid–fluid disperse system is in contrast to the wide range of systems to which Grace *et al.*'s instability model is intended to apply, and that the prospects for definite results are thereby improved. On the other hand, the analysis to be given here is more general in that it applies to bubbles rising in vertical tubes as well as to spherical-cap bubbles in unbounded liquid.

The effect of viscosity on the disturbed motion of the liquid will also be neglected, with the consequential simplification that the motion is irrotational. The justification for this is that only the fairly large lengthscales of the disturbance at which growth occurs are relevant. Bellman & Pennington's equation for the rate of growth of a sinusoidal disturbance to a stationary horizontal interface between a gas (below) and a viscous liquid shows that the effect of viscosity is quite negligible in the case of water for all the growing disturbances with wavelengths above λ_c; and the same qualitative conclusion seems likely to hold for disturbances on an extending interface.

It is evident from figure 1 (*a*) and other similar photographs that the whole of the upper surface of a large gas bubble rising steadily through unbounded liquid of small viscosity is spherical to a good approximation. That is not equally true of a gas bubble spanning a vertical tube of circular cross-section, but there is here a central part of the bubble surface which is approximately spherical (see figure 1 *d*). We shall assume the relevant central part of the upper surface in the undisturbed state of the

† An interesting attempt to do just this for a liquid–gas interface in general and for the top of a spherical-cap bubble in particular was made by Dagan (1975), but was flawed by his assumption that the wavelength of a disturbance remains constant.

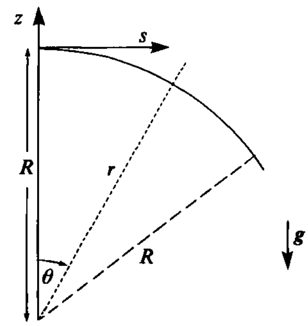

FIGURE 2. Sketch showing the coordinate system, which moves with the bubble.

bubble to be exactly spherical, with radius of curvature R. The origin of a spherical coordinate system which moves with a steadily rising bubble is located at the centre of curvature (see figure 2). The upper surface of the bubble in the disturbed state is given by

$$r = R+\eta(\theta, \phi, t),$$

where $|\eta| \ll R$. The motion in the liquid above the interface will be supposed to be irrotational, with velocity potential Φ and velocity $\boldsymbol{u} = \nabla\Phi$. In the undisturbed axisymmetric state $\Phi = \Phi_0$, and in the disturbed state $\Phi = \Phi_0+\Phi_1$ say.

The disturbed flow must satisfy two matching conditions at the bubble surface. The first is the kinematical condition

$$\frac{\partial\eta}{\partial t}+R\left(\frac{\boldsymbol{u}}{r}\right)_{r=R+\eta}\cdot\nabla\eta = \left(\frac{\partial\Phi}{\partial r}\right)_{r=R+\eta}. \tag{3.1}$$

To the first order in the small disturbance quantities this becomes

$$\frac{\partial\eta}{\partial t}+\frac{q_0}{R}\frac{\partial\eta}{\partial\theta} = \left(\frac{\partial\Phi_0}{\partial r}\right)_{r=R+\eta}+\left(\frac{\partial\Phi_1}{\partial r}\right)_{r=R}, \tag{3.2}$$

where q_0 is the magnitude of the (poloidal) liquid velocity at the bubble surface in the undistured flow. Now since Φ_0 satisfies Laplace's equation, we have

$$\left(\frac{\partial\Phi_0}{\partial r}\right)_{r=R+\eta} = \eta\left(\frac{\partial^2\Phi_0}{\partial r^2}\right)_{r=R} = -\frac{\eta}{R\sin\theta}\frac{\partial(q_0\sin\theta)}{\partial\theta}$$

to first order, whence (3.2) becomes

$$\frac{\partial\eta}{\partial t}+\frac{q_0}{R}\frac{\partial\eta}{\partial\theta}+\frac{\eta}{R\sin\theta}\frac{\partial(q_0\sin\theta)}{\partial\theta} = \left(\frac{\partial\Phi}{\partial r}\right)_{r=R}. \tag{3.3}$$

The other condition is that the pressure be continuous across the bubble surface, and hence constant. For the disturbed state this is expressed as

$$\left(\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_{r=R+\eta}+g(R+\eta)\cos\theta+\frac{\gamma K}{\rho} = \text{const.}, \tag{3.4}$$

where γ is the surface tension at the interface (assumed to be uniform) and K is the sum of the two local principal curvatures of the interface. In the absence of the disturbance this reduces to

$$\tfrac{1}{2}q_0^2+gR(\cos\theta-1) = 0, \tag{3.5}$$

and on subtracting (3.5) from (3.4) and retaining only first-order small quantities we find

$$\left(\frac{\partial \Phi_1}{\partial t}\right)_{r=R}+\frac{q_0}{R}\left(\frac{\partial \Phi_1}{\partial \theta}\right)_{r=R}-\frac{q_0^2\,\eta}{R}+g\eta\cos\theta+\frac{\gamma}{\rho}\nabla^2\eta=0. \tag{3.6}$$

The relations (3.3) and (3.6) together with the equation

$$\nabla^2\Phi_1=0$$

and the expression (3.5) for q_0 as a function of θ presumably determine the behaviour of the disturbance. I have been able to make progress with these equations only by confining attention to the central region of the upper surface of the bubble, where $\theta \ll 1$, which we expect to be the most unstable part of the interface. In this region q_0 is necessarily given approximately by the linear expression

$$q_0 = kR\theta, \tag{3.7}$$

and the constant of proportionality k must take the value

$$k=(g/R)^{\frac{1}{2}} \tag{3.8}$$

for consistency with the condition of uniform pressure at the boundary expressed by (3.5). (The constant k in (3.7) is also a parameter of the global irrotational flow field, and is related to the speed of rise of the bubble in a way that depends on the bubble shape and on the presence of boundaries, but this relation plays no part in the stability analysis.) Note that the only aspect of the undisturbed flow field that appears in (3.3) and (3.6) is the poloidal velocity q_0 at the interface, and that when q_0 has the linear form (3.7) the interface motion in the neighbourhood of $\theta=0$ is a uniform isotropic pure straining such that all material line elements in the interface are being extended at the rate k.

When $\theta \ll 1$ and terms of the second degree in θ can be neglected, (3.3) and (3.6) become, with the aid of (3.7),

$$\frac{\partial \eta}{\partial t}+ks\frac{\partial \eta}{\partial s}+2k\eta=\left(\frac{\partial \Phi_1}{\partial z}\right)_{z=0}, \tag{3.9}$$

$$\left(\frac{\partial \Phi_1}{\partial t}\right)_{z=0}+ks\left(\frac{\partial \Phi_1}{\partial s}\right)_{z=0}+g\eta+\frac{\gamma}{\rho}\nabla^2\eta=0, \tag{3.10}$$

where $s=R\theta$ and $z=r-R$. z, s and the azimuthal angle ϕ are effectively cylindrical coordinates and the plane $z=0$ is horizontal and tangential to the interface. In this approximation these two equations describe a disturbance to a horizontal plane interface which is being stretched isotropically with rate of extension k.

The 'convective' terms in (3.9) and (3.10) indicate that the lengthscale of the disturbance in the horizontal plane is being extended, like the interface itself. Consequently eigenmodes of disturbance that preserve their dependence on the streamwise distance s and whose amplitudes vary exponentially with respect to time do not exist. It follows that it will not be possible to find a clear-cut criterion for instability in the form of a critical value of some parameter of the problem above or below which some eigenmodes grow exponentially. However, since the liquid flow in the neighbourhood of the upper-most point on the bubble surface is irrotational with uniform rate of strain, the change of form of a disturbance is a simple stretch (or linear transformation) and this allows us to analyse the evolution of a disturbance of arbitrary initial form. We shall see that certain modes of disturbance are amplified although for a limited time only, and that the total amplification can be calculated. Thus a criterion for the occurrence of a specified total amplification of a small disturbance can be given, but not (from the linear analysis alone) for the occurrence of bubble break-up. This is effectively also the position with the model proposed by Grace *et al.* (1978).

We now represent an arbitrary small disturbance to the plane interface at the initial instant $t = 0$ in terms of a complete set of orthogonal functions of the position vector $\boldsymbol{s}$ (with polar coordinates s, ϕ) in the horizontal plane. One possibility is a Fourier expansion (in two dimensions), of which a typical term is

$$\eta(\boldsymbol{s}) = A\exp(in\boldsymbol{j}\cdot\boldsymbol{s}), \quad \Phi_1(\boldsymbol{s},z) = -B\exp\{n(i\boldsymbol{j}\cdot\boldsymbol{s}-z)\}, \tag{3.11}$$

where $n\boldsymbol{j}$ is the vector wavenumber, $\boldsymbol{j}$ being a unit vector in the horizontal plane. Another possibility is a Fourier–Bessel expansion, of which a typical term is

$$\eta(\boldsymbol{s}) = AJ_p(ns)\cos p\phi, \quad \Phi_1(\boldsymbol{s},z) = -BJ_p(ns)\cos p\phi\, \mathrm{e}^{-nz}, \tag{3.12}$$

where p may take any one of the values 0, 1, 2, ... The form of the dependence of Φ_1 on z in (3.11) and (3.12) is a consequence of it being a harmonic function. If now we allow n, as well as A and B, to be a function of t alone and substitute either of these representations of η and Φ_1 in (3.9) and (3.10) we obtain two equations, the real and imaginary parts of which add up to zero if

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -kn, \quad \text{i.e. } n = n_0\,\mathrm{e}^{-kt}, \tag{3.13}$$

where n_0 is the value of n at $t = 0$, and

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -2kA + nB, \tag{3.14}$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \left(g - \frac{\gamma n^2}{\rho}\right)A. \tag{3.15}$$

Thus a disturbance that initially has a sinusoidal or a Fourier–Bessel form continues to have that form, but the lengthscale of the dependence on s remains constant only with respect to material coordinates, as anticipated. The equations (3.14) and (3.15) determining the time dependence of the two amplitudes A and B are independent of the form of representation of the disturbance, so there is no need to choose between the Fourier and the Fourier–Bessel representations until the nature of the real initial disturbance in an actual flow system is under consideration. For convenience we shall refer to n and $\lambda(=2\pi/n)$ as the 'wavenumber' and 'wavelength' of the disturbance, regardless of whether (3.11) or (3.12) is the form of the disturbance.

The interpretation of the terms in (3.14) and (3.15) is straightforward. The term $-2kA$ in (3.14) represents the continual reduction in amplitude of the disturbance to the interface resulting from the contractional motion of the liquid in the vertical direction. The other terms have the same form as those that would be present when $k = 0$, and when n is constant they describe the growth of a disturbance to a plane horizontal interface between two stationary immiscible fluids the lower of which has negligible density. Equations (3.14) and (3.15) thus represent a generalization of the familiar Taylor-instability analysis to a case in which the interface is being extended isotropically. The effect of this imposed extensional motion is to introduce the stabilizing effect of the associated contraction of the fluid in the vertical direction and to vary the disturbance wavenumber according to (3.13); and the latter effect renders impossible an exponential dependence of the disturbance amplitude on time. These features of the behaviour of a small disturbance were also found by Tomotika (1936) in a study of the capillary instability of a thread of one viscous liquid immersed in a second liquid in axisymmetric pure straining motion, by Moore & Griffith-Jones (1974) in their analysis of the stability of a vortex sheet which is being extended in the direction normal to the vorticity, and by Frankel & Weihs (1985) in their analysis of the stability of a shaped-charge liquid jet in which the velocity increases with axial distance parallel to the jet.

On eliminating B from (3.14) and (3.15) we obtain finally

$$\frac{\mathrm{d}^2A}{\mathrm{d}t^2}+3k\frac{\mathrm{d}A}{\mathrm{d}t}+A\left\{2k^2-gn\left(1-\frac{n^2}{n_c^2}\right)\right\}=0, \tag{3.16}$$

where $n_c = 2\pi/\lambda_c = (\rho g/\gamma)^{\frac{1}{2}}$ is the critical wavenumber for an interface between two stationary fluids.

4. Solution of the disturbance equation

For the purposes of numerical integration of (3.16) it is desirable to introduce the dimensionless quantities

$$\mathscr{A}=\frac{A}{L},\quad \tau=kt,\quad \alpha=\frac{gn_c}{k^2}=n_c R,\quad \beta=\frac{n_c}{n_0}=\frac{\lambda_0}{\lambda_c}, \tag{4.1}$$

where L is any conveniently chosen length, whence (3.16) becomes

$$\frac{\mathrm{d}^2\mathscr{A}}{\mathrm{d}\tau^2}+3\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\tau}+\mathscr{A}\{2-\alpha\beta^{-1}\,\mathrm{e}^{-\tau}(1-\beta^{-2}\,\mathrm{e}^{-2\tau})\}=0. \tag{4.2}$$

The evolution of the disturbance is seen to be controlled by the two parameters α and β. For a given gas–liquid system α is proportional to the bubble size and β is proportional to the initial wavelength of the disturbance.

The initial conditions satisfied by A have now to be specified. In stability problems for which unstable eigenmodes exist it is enough to know that a disturbance of arbitrary form and non-zero magnitude is generated initially by some external means, because an eigenmode grows indefinitely regardless of its initial amplitude. However, in the present problem the initial form and magnitude of the disturbance affect the final magnitude and must be considered explicitly. The choice of the initial conditions, which is not straightforward, should presumably reflect the assumed origin of the disturbance to the bubble surface.

It seems likely that the disturbance originates in slowly decaying motions in patches of the liquid through which the bubble is passing. Turbulent motions in a large body of liquid, perhaps arising from the previous passage of bubbles through it, may persist for some time. For instance a turbulent motion with a root-mean-square velocity fluctuation of 1 mm/s and lengthscale of 50 mm has a half-life of the order of a minute, and decays by generating 'eddies' on smaller scales. Such ambient motions in the liquid are inevitably rotational and persist (although not without change) as they are convected through the irrotational velocity field surrounding the bubble. A velocity fluctuation of certain magnitude and with a lengthscale $2\pi/n_0$ may be generated at the bubble surface in this way, and the corresponding initial conditions for the disturbance of the interface appear to be

$$A = 0 \quad \text{and} \quad \frac{\mathrm{d}A}{\mathrm{d}t} = n_0 B_0, \quad = u_0 \text{ say}, \quad \text{at } t = 0.$$

In terms of the non-dimensional variables

$$\tau = kt, \quad \mathscr{A} = \frac{kA}{u_0}, \tag{4.3}$$

these initial conditions are

$$\mathscr{A} = 0, \quad \frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\tau} = 1 \quad \text{at } \tau = 0. \tag{4.4}$$

Note that u_0 is the amplitude of the sinusoidal variation, with wavenumber n_0, of the normal component of the velocity of the liquid at the interface at $t = 0$.

Another possible origin of the disturbance to the upper surface of a spherical-cap bubble is the shaking or wobbling of the whole bubble that accompanies the oscillation of the sharp edge. The back-and-forth motion of the sharp edge, with occasional detachment of a small gas bubble from the parent bubble, is evident in some photographs, although it appears not to have any visible influence on the smooth spherical upper surface of the bubble. It is difficult to see any direct connection between the oscillatory motion of the sharp edge and the existence of disturbances to the upper surface of the bubble with relatively small lengthscales comparable with λ_c. And in the case of a gas bubble of large vertical extent in a tube, like that in figure 1(*d*), the rear edge of the bubble is probably too far away to produce any disturbance to the nose of the bubble. For these reasons we shall adopt the initial conditions (4.4) (with (4.3)) with the understanding that the initial interface velocity u_0 can be determined, in principle, from a consideration of the random velocity distribution that exists in the liquid ahead of the rising bubble.

Equation (4.2) has been integrated numerically over a range of values of τ, beginning at $\tau = 0$ where the initial conditions (4.4) are imposed, for a number of different combinations of values of α and β. Figure 3 shows the solution for the case $\alpha = n_c R = 50$ and a number of different values of $\beta(= \lambda_0/\lambda_c)$, some smaller than unity and some larger, indicating the effect of varying λ_0 alone. The abscissa variable for all these curves has been chosen as

$$\tau' = \tau + \ln\beta = \ln\frac{n_c}{n} = \ln\frac{\lambda}{\lambda_c} \tag{4.5}$$

in order to show the relation between the growth of the disturbance and the instantaneous value of λ/λ_c. A disturbance to a stationary plane interface is purely oscillatory (in the absence of viscosity) if $\lambda < \lambda_c$, and grows exponentially if $\lambda > \lambda_c$. Here the behaviour is more complicated because the disturbance wavelength is changing continually and there is the additional decrease of the disturbance amplitude due to the vertical contraction in the liquid. Nevertheless one can see from the enlargement of the region near the origin in figure 4 that in cases where $\lambda_0 < \lambda_c$ the amplitude initially tends to oscillate and that as λ increases through λ_c the amplitude begins to grow, at an increasing rate as λ increases through the value $\surd 3\lambda_c$ (i.e. $\tau' = 0.55$) where the growth rate of an inviscid disturbance to a stationary horizontal interface has its maximum value. As λ becomes still larger (figure 3) the growth rate due to the effect of gravity alone tends to zero, and since the squashing

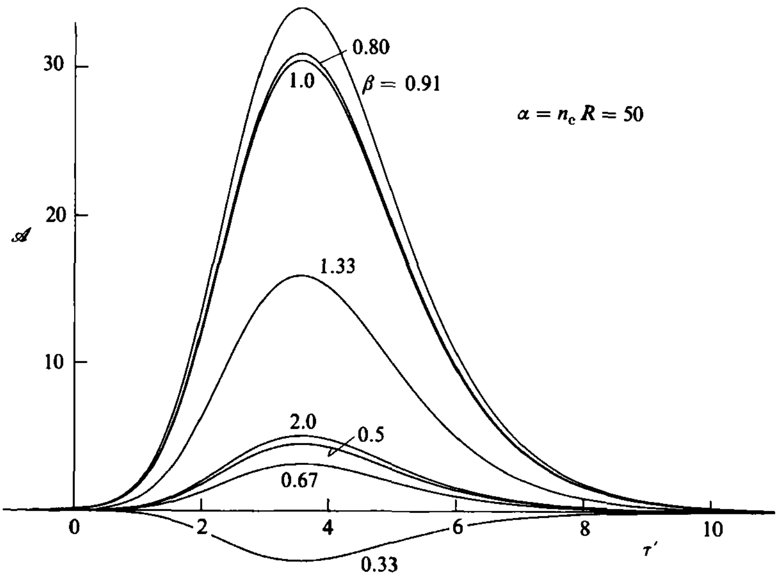

FIGURE 3. Calculated values of the non-dimensional amplitude of the disturbance ($\mathscr{A}$) as a function of $\tau'(=\tau+\ln\beta = \ln\lambda/\lambda_c)$ for various values of $\beta(=\lambda_0/\lambda_c)$, with $\alpha(=gn_c/k^2 = n_c R) = 50$.

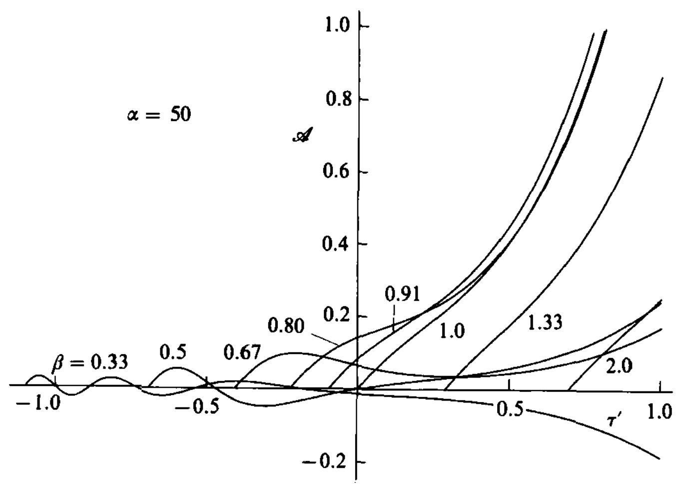

FIGURE 4. Enlarged view of the curves of figure 3 near the origin.

effect of the vertical contraction in the liquid is independent of λ the amplitude ultimately decreases to zero. Figures 5 and 6 show the solutions for the same family of values of β in the case $\alpha = 80$.

As would be expected, the value of $\beta(=\lambda_0/\lambda_c)$ for which the amplitude is largest when $\lambda > \lambda_c$ is approximately 1.0. A disturbance whose initial wavelength is less than λ_c experiences an initial period of damped oscillation (the 'damping' being due to the ambient vertical contraction) and any disturbance for which $\lambda_0 > \lambda_c$ is amplified for a period that is smaller than the period of amplification when $\lambda_0 = \lambda_c$. It will be seen from figures 3 and 5 that the maximum of $\mathscr{A}$ (for given $\tau' > 0$) actually occurs at a value of β that is a little less than unity and that varies slightly with α. The reason

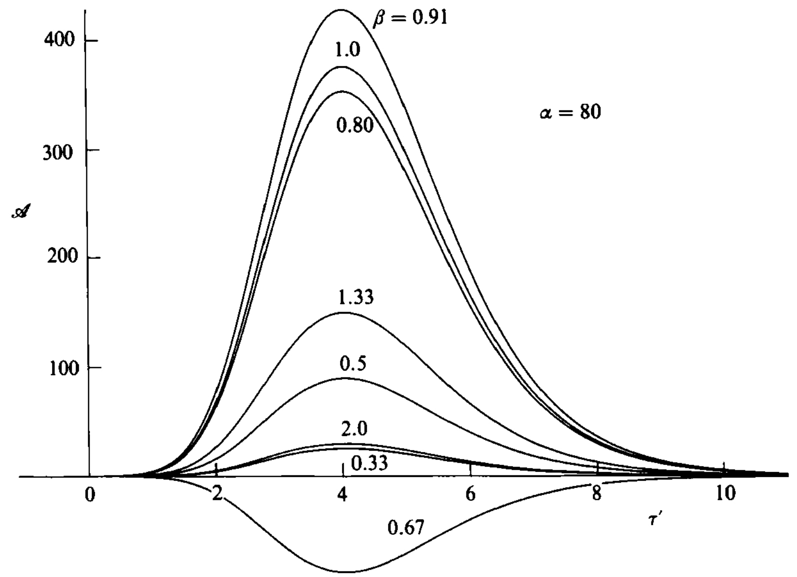

FIGURE 5. Calculated values of $\mathscr{A}$ as a function of τ' for various values of β, with $\alpha = 80$.

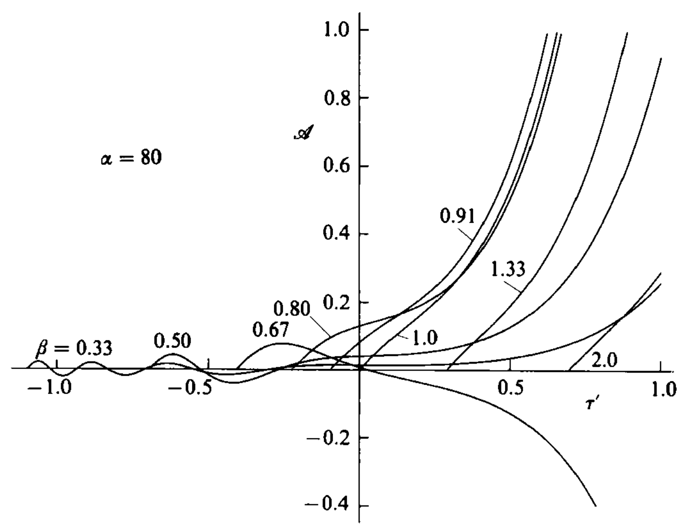

FIGURE 6. Enlarged view of the curves of figure 5 near the origin.

appears from figures 4 and 6 to be that some combinations of values of $\mathscr{A}$ and $\mathrm{d}\mathscr{A}/\mathrm{d}t$ at the instant at which λ reaches the value λ_c (i.e. $\tau' = 0$) give the disturbance a better start to the growth phase than others. Bearing in mind the uncertainty about the initial conditions in an actual flow system, the difference between the curve in figure 3 (or 5) for which $\beta = 1$ and that for which $\mathscr{A}$ is greatest is probably not significant. For the later purpose of estimation of the largest possible size of a stable bubble, we shall suppose that the initial wavelength of the disturbance whose (non-dimensional) amplitude reaches the largest value at a given τ' (> 0) is λ_c, corresponding to $\beta = 1$, regardless of the value of α. (It is of course the magnitude of the *dimensional* amplitude A that is relevant to break-up of a bubble, and if there are large variations in the initial velocity disturbance u_0 as a function of the wavelength λ_0 that must also be taken into account in an identification of the most 'dangerous' disturbance.)

We observe in passing that the curves in each of figures 3 and 5 have a similar shape, suggesting that the amplitude can be represented approximately in the form

$$\mathscr{A}(\tau,\beta) = C(\beta)f(\tau'), \tag{4.6}$$

whereas there is no such similarity among the curves in figures 4 and 6. This similarity at large values of λ/λ_c presumably is connected with the fact that when τ is replaced by τ' (defined in (4.5)) as independent variable the differential equation (4.2) becomes

$$\frac{d^2\mathscr{A}}{d\tau'^2}+3\frac{d\mathscr{A}}{d\tau'}+\mathscr{A}\{2-\alpha\, e^{-\tau'}(1-e^{-2\tau'})\} = 0 \tag{4.7}$$

and does not contain β, although β now appears explicitly in the initial conditions

$$\mathscr{A} = 0, \quad \frac{d\mathscr{A}}{d\tau'} = 1 \quad \text{at } \tau' = \ln\beta. \tag{4.8}$$

It seems that the influence of the initial conditions is confined to control of a multiplying constant C when τ' is larger than about 1.0.†

Now that the effect of varying the initial wavelength has been examined, we may consider the influence of the important parameter α that measures the size of the bubble (see (4.1)). Figure 7 shows computed solutions of (4.2) with the initial conditions (4.4) for several different values of α and with β chosen as 1.0 in accordance with the previous discussion. The range of values of the amplitude $\mathscr{A}$ is very wide, so the ordinate has a logarithmic scale, the quantity plotted being $\ln(\mathscr{A}+1)$ since this remains finite at $\tau = 0$ and differs imperceptibly from $\ln\mathscr{A}$ when $\mathscr{A} \gg 1$. The abscissa variable in figure 7 has been chosen as

$$\tau'' = \tau - \ln\frac{\alpha}{\beta} = \tau - \ln n_0 R = \ln\frac{\lambda}{2\pi R}, \tag{4.9}$$

because the curves for different values of α are then found to differ approximately by an additive constant only (again possibly because (4.2) with τ'' as the independent variable and neglect of the surface-tension term $\beta^{-2}\, e^{-2\tau}$, which is small at the larger values of τ, does not contain α or β, although these parameters now appear in the initial conditions). The solutions are shown only to values of τ'' near their maxima; at larger values of τ'' all the curves tend to zero like those in figures 3 and 5.

The plot in figure 7 also has the merit of revealing the limitations on the validity of the analysis at the larger values of τ or τ''. The disturbance equations (3.9) and (3.10) are valid in the neighbourhood of the top of the bubble where the interface resembles a horizontal plane, and the disturbance forms (3.11) and (3.12) are applicable when the wavelength λ is small compared with the distance over which these forms hold; in particular, the z-dependence of the forms (3.11) and (3.12) can be justified only if the assumed dependence on s applies over several wavelengths.

† My colleague Dr Herbert Huppert has been able to show mathematically that this similarity property of solutions of (4.7) and (4.8), and likewise that represented by (5.2), hold provided $\alpha \gg 1$, and he has kindly agreed to his proof being reproduced in the Appendix.

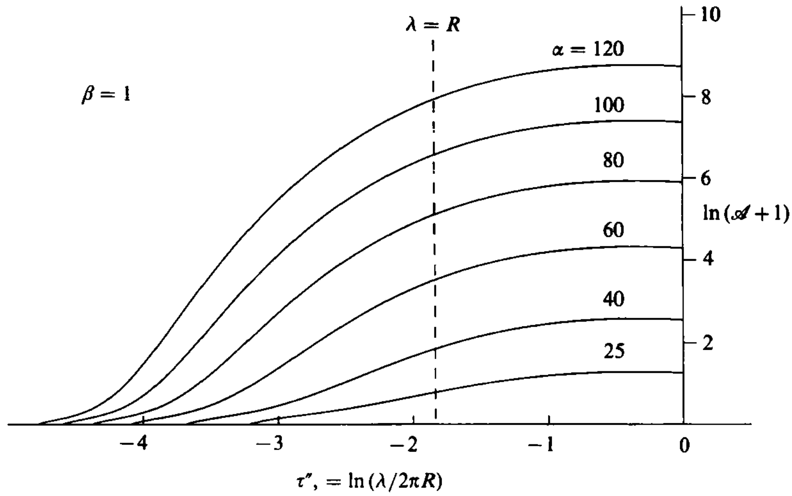

FIGURE 7. Calculated values of $\ln(\mathscr{A}+1)$ as a function of $\tau''(=\tau - \ln \alpha\beta^{-1} = \ln \lambda/2\pi R)$ for various values of $\alpha(= gn_c/k^2 = n_c R)$, with $\beta(= \lambda_0/\lambda_c) = 1$.

The numerical solutions are consequently beginning to be of uncertain accuracy when $\lambda = R$, that is, when

$$\tau'' = -\ln 2\pi = -1.84,$$

which is indicated by a broken line in the figure. The disturbance comes to the end of its lifetime, not because it is swept over the edge of the bubble, but because the wavelength becomes too large for a periodic structure to be contained on the interface. We note that the maxima of the curves lie outside the range of validity. Caution is thus needed in any attempt to base a criterion for break-up of a bubble on the calculated maximum amplification of a given initial disturbance.

Even though the curves in figure 7 are unreliable at $\tau'' > -1.84$, it seems clear that the value of $\ln(\mathscr{A}+1)$ for a *small* disturbance satisfying linearized equations valid over the whole of the bubble surface does reach a maximum and subsequently falls to zero. The rate of extension of the interface in the flow direction decreases with distance from the top point of the bubble, so that the wavelength becomes non-uniform, but this rate of extension and the associated rate of contraction in the normal direction remain positive over the whole of the interface thereby ensuring that the disturbance amplitude ultimately diminishes to zero everywhere.

5. The criterion for break-up of a bubble

If a steadily rising bubble with a steady smooth surface experiences an infinitesimal disturbance at an initial instant, the disturbance amplitude grows in the manner calculated above and may become large enough to cause finite-amplitude effects leading to break-up of the bubble. In a problem like this in which exponentially growing eigenmodes of disturbance do not exist, finite-amplitude effects do not necessarily occur; whether they do depends on both the initial disturbance amplitude and the degree of amplification. The objective of the present study is an estimate of the minimum level of the initial disturbance required to activate finite-amplitude effects of the type that will lead inevitably to break-up of a bubble of given size. A simple and precise criterion is unlikely to exist, because a bubble can doubtless break up in many different ways. Analysis of the process of disruption of a bubble using the full equations of motion is desirable but would be a massive computational problem beyond my scope. Instead I shall make an expedient hypothesis about the absolute amplitude and wavelength of a disturbance which by nonlinear processes will lead inevitably to break-up of a bubble.

It is natural to think in terms of the disturbance amplitude alone when searching for a criterion for subsequent break-up, but I think the disturbance wavelength is also relevant. If we think, for instance, of the extreme case in which the wavelength of a disturbance to the surface of a spherical-cap bubble is appreciably larger than R, the disturbed interface is obtained approximately by displacing the undisturbed interface as a whole and is unlikely to generate finite-amplitude effects however large the amplitude. The break-up of the bubble is likely to result from gravitational instability of the upper surface proceeding at a rate made faster by nonlinear effects and fast enough to overcome the smoothing action of the extensional motion of the interface. The strength of these nonlinear effects is measured, not by the disturbance amplitude alone, but by the change in the general shape of the bubble since this determines the change in the global irrotational flow about the bubble. Putting it concretely, my picture of the break-up process is that, if a shallow depression forms on the upper surface, the subsequent deepening of the depression by gravitational instability leads to less liquid flowing round the bubble and so to a smaller rate of extension of the interface, which in turn diminishes the smoothing action of the extensional motion and hastens the deepening of the depression.

The change in the global irrotational flow resulting from an amplified disturbance will be greatest when the disturbance wavelength is comparable with R, say when $\lambda = \lambda_1$. And the importance of nonlinear effects associated with a disturbance of this wavelength will be determined by the amplitude of the disturbance relative to R. This leads us then to the working hypothesis that bubble break-up is inevitable, as a consequence of nonlinear effects, if the amplitude A of a disturbance, as given by the preceding linear equations, is equal to or greater than a critical value ξR at the instant at which $\lambda + \lambda_1$, where ξ is an absolute constant. In terms of non-dimensional quantities this criterion for break-up is

$$\mathscr{A}_{\lambda=\lambda_1} \geqslant \frac{\xi(gR)^{\frac{1}{2}}}{u_0}, \tag{5.1}$$

in which use has been made of (3.8) and (4.3).

It will be recalled that the numerical integration of (4.2) (with $\beta = 1$) revealed an approximate similarity of shape of the solutions for different values of α (see figure 7 and the relation (4.9)). This similarity implies that $\mathscr{A}$ is approximately of the form

$$\mathscr{A}(\tau, \alpha) = K(\alpha)\, F(\tau'') \tag{5.2}$$

except at the smaller values of τ, and hence that the ratio of the values of $\mathscr{A}$ at two different values of τ'' (neither being small) is independent of α. Choosing different values of the wavelength λ_1, about which we know only that it should be comparable with R, is thus equivalent to choosing different values of ξ in the criterion (5.1). We are free to make a special choice of λ_1 on grounds of convenience, say $\lambda_1 = R$, thereby transferring all the arbitrariness in the criterion (5.1) into the constant ξ.

The above hypothesis thus says that the maximum or break-up value of the top radius of curvature ($R = R_b$, say) of a gas bubble in a given liquid which remains

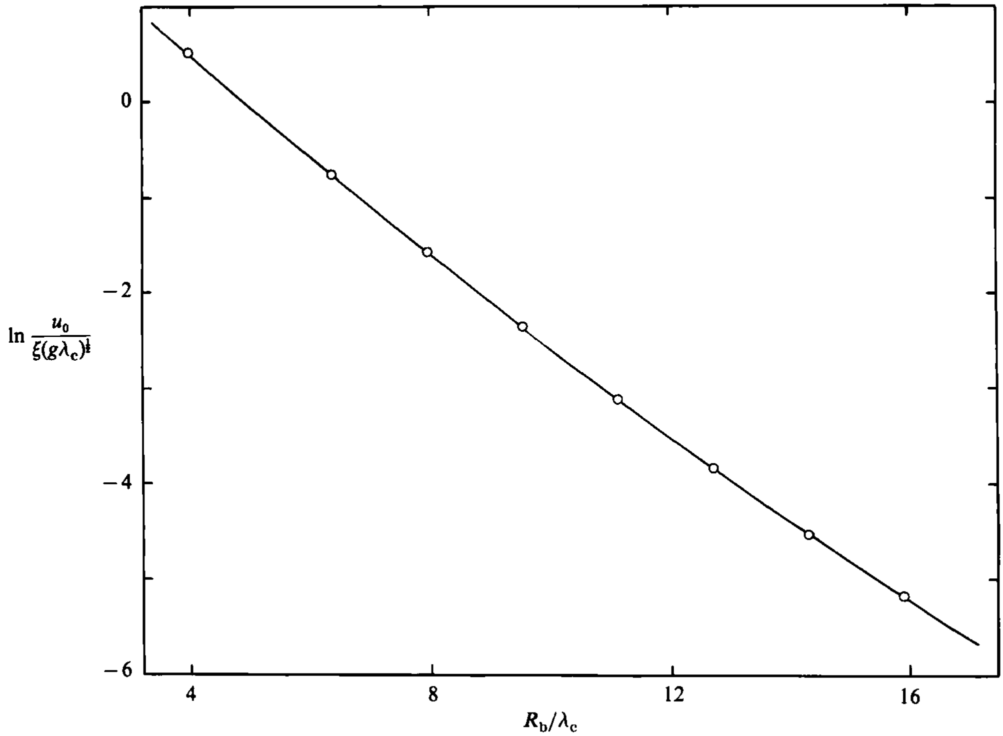

FIGURE 8. The calculated relation between the initial disturbance velocity u_0 and the radius of curvature of the largest bubble which does not break up (R_b), according to the criterion (5.3).

intact when subjected to a disturbance with initial normal velocity magnitude u_0 and wavelength λ_c (this being the value of λ_0 that maximizes $\mathscr{A}$) is given by

$$\mathscr{A}_{\lambda=R=R_b}\left(\frac{\lambda_c}{R_b}\right)^{\frac{1}{2}}\frac{u_0}{\xi(g\lambda_c)^{\frac{1}{2}}}=1\,; \tag{5.3}$$

equivalently, (5.3) gives the minimum value of u_0 that will cause disruption of a bubble of radius of curvature R_b in a given liquid. The value of $\mathscr{A}$ at $\tau'' = -\ln 2\pi = -1.838$ is a known function of $\alpha(=2\pi R/\lambda_c)$, as indicated in figure 7, from which we find the relation between R_b/λ_c and $u_0/\xi(g\lambda_c)^{\frac{1}{2}}$ shown in figure 8.

The disturbance amplitude $A = \xi R$ indicates the threshold of significant nonlinear effects. The constant ξ clearly is small compared with unity, but there is little that can be said about it on theoretical grounds. The change in the curvature of the interface due to the disturbance may have some relevance to the incidence of nonlinear effects associated with the change in the bubble shape. The disturbance curvature $\nabla^2\eta$ is equal to $-n^2\eta$ for either of the two disturbance forms (3.11) and (3.12), and it can be shown from the numerical solutions that the amplitude of this variable curvature reaches a maximum at $\lambda = 0.60R$ (i.e. $\tau'' = -2.35$), which is possibly just small enough to lie within the range of validity of the calculations. If we supposed that bubble break-up is inevitable when this maximum amplitude of the disturbance curvature is equal to or exceeds $2/R$, meaning that there is then at least one point on the interface where the total curvature is zero, that may be shown to correspond to the choice $\xi = 0.041$. And if we took as our criterion for break-up the

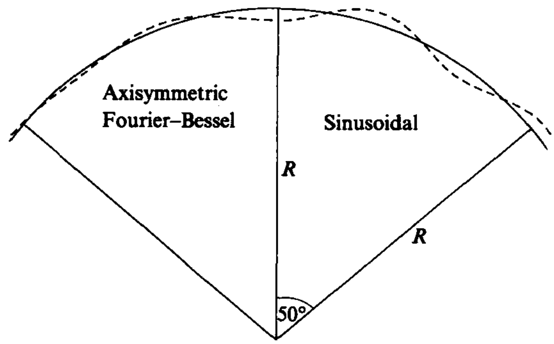

FIGURE 9. Sketch showing the intercept of the disturbed bubble surface by a vertical plane through the origin at the instant at which $\lambda = 0.60R$. The disturbance amplitude is $A = 0.036R$. The broken curve on the right-hand side shows the sinusoidal disturbance (3.11) with the wavenumber vector $n\boldsymbol{j}$ in the plane of the figure, and that on the left shows the Fourier–Bessel disturbance (3.12) with $p = 0$. For both forms of disturbance the total curvature of the interface at $\theta = 0$ is $2/R$. Observations show that $\theta \approx 50°$ at the edge of a spherical-cap bubble.

occurrence of a positive (concave upwards) total curvature equal to $2/R$, as would seem to be more reasonable, that would correspond to $\xi = 0.082$.

The threshold disturbance in this latter case ($\xi = 0.082$) is sketched in figure 9, which shows two alternative forms of disturbed interface, both with $\lambda = 0.60R$ (the instant at which the disturbance curvature reaches its maximum) and

$$n^2 A = \frac{4}{R}, \quad \text{i.e. } A = 0.036R.$$

On the right-hand side of the central axis is a sinusoidal disturbance with wavenumber vector in the plane of the figure and phase chosen for convenience so that the maximum total curvature occurs at the top point of the interface ($\theta = 0$), and on the left is an axisymmetric ($p = 0$) Fourier–Bessel disturbance with the centre of symmetry chosen for convenience of drawing to be at $\theta = 0$. For both these disturbed interfaces the total curvature is $2/R$ at $\theta = 0$. Choosing $\xi = 0.082$ in the break-up criterion (5.3) is equivalent to supposing that the disturbances shown in figure 9 are critical in the sense that nonlinear processes are just strong enough to cause disruption of the bubble at a later time by deepening the central depression at a rate greater than that given by the linear theory. The appearance of the interface in figure 9 seems to me to make plausible a value of ξ not greatly different from 0.082, but in the absence of nonlinear analysis of the bubble shape this can only be a very rough estimate.

The mode of break-up of a bubble on which our hypothesis is based would presumably lead to a spherical-cap bubble breaking up into a small number of bubbles of comparable size.

6. Concluding remarks on the maximum size of bubbles

The strong influence of surface tension, as represented by the parameter λ_c, on the amplitude of a small disturbance at the instant at which λ/R reaches any given value of order unity is evident from figure 7. Surface tension has some influence on the growth rate at all wavelengths, but the influence on the smallest wavelength for which there is positive growth is more important. The disturbance amplitude grows only during the 'window' of time in which λ increases from λ_c to a value of order R, and it is this duration of growth that primarily determines the magnification of the amplitude. This control of the period of growth by surface tension is the key to understanding why gas bubbles in liquid do not break up unless R is a good deal larger than λ_c.

As anticipated, there is no unique value of the break-up radius R_b for a given gas–liquid system; it depends on the initial magnitude of the disturbance generated by external causes. It will be recalled that u_0 is the magnitude of the variation of the normal velocity of the interface at the initial instant, when $A = 0$ and $\lambda = \lambda_0$. (And λ_0 was later chosen to be equal to λ_c, corresponding to $\beta = 1$, on the grounds that this would yield approximately the largest amplification of the disturbance at a given later time.) The origin of the disturbance was supposed to be a patch of motion with non-zero vorticity in the liquid ahead of the rising bubble. The value of u_0 is likely to vary considerably according to the circumstances of the observations of real bubbles, and there does not appear to be a particular value with special significance. Even when the velocity fluctuations in the liquid ahead of the bubble can be observed, there is the question of the relation between these fluctuations and the disturbance velocity at the bubble surface, which is a sizeable problem in itself and will not be considered here.

For practical purposes the numerical relationship between the dimensional quantities u_0 and R_b for particular gas–liquid systems is more informative than figure 8. Figure 10 shows the relationship for three values of $\lambda_c(=2\pi(\gamma/\rho g)^{\frac{1}{2}})$, one of which, viz. $\lambda_c = 1.71$ cm, is appropriate for air bubbles in water. The numbers on the abscissa scale give the value of u_0 in cm/s when ξ is chosen as 0.082.

The values of λ_c for some other common liquids given in table 1 are less than that for water, and lie within the range 0.80 cm $\leqslant \lambda_c \leqslant$ 1.71 cm covered by the three curves in figure 10. The difference between the values of R_b for air bubbles in water and for air bubbles in, say, carbon tetrachloride ($\lambda_c = 0.82$ cm) is greater at the smaller levels of the initial disturbance.

As remarked earlier, these estimates of the upper bound to values of the radius of curvature of the top of the bubble surface for which the bubble remains intact apply both to spherical-cap bubbles in unbounded liquid and to bubbles confined in a vertical tube of circular cross-section. The part of the interface near its highest point is the seat of the instability, according to the views put forward here, and the extent and shape of the remainder of the interface do not have a major influence on the instability.

For spherical-cap bubbles, we have the observation by Temperley & Chambers (1945) that gas bubbles in a large tank of water (depth 33 ft.), some generated by a small explosion and some by turning over a bucket of air, remained intact for values of R up to about 15 cm. (Temperley & Chambers photographed the bubbles and measured the cap radius on the photographs.) The surface tension at the interfaces of these bubbles is likely to be about the same as for air and water, in which case according to figure 10 this break-up size corresponds to an initial disturbance of magnitude $u_0 = 0.22$ cm/s if we take ξ as 0.082. This is not an unreasonable figure for a tank of water through which many large gas bubbles have passed during the same day. Temperley & Chambers reported that it was rather difficult to generate single bubbles for which R was close to the maximum.

Observations of the rate of rise of spherical-cap bubbles larger than those observed by Taylor & Davies were also made by Allred & Blount (1953). They generated bubbles of different sizes by rotating rapidly a hemisphere partially filled with air

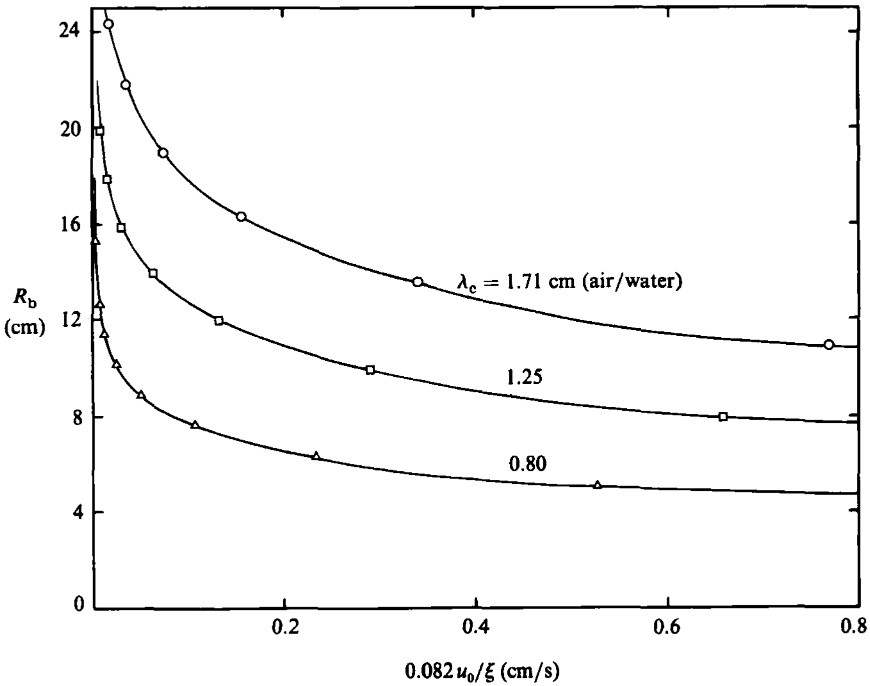

FIGURE 10. The calculated relation between R_b and u_0, for three different values of the wavelength $\lambda_c\,(= 2\pi(\gamma/\rho g)^{\frac{1}{2}})$ that is critical for a disturbance to a stationary interface.

	γ dyn/cm	ρ gm/cm^3	λ_c cm
Water	72.8	1.00	1.712
Glycerine	63	1.26	1.42
Ethylene glycol	42	1.10	1.24
Mercury	487	13.61	1.200
Ethyl alcohol	22	0.79	1.06
Aq. soap solution	25	1.0	1.00
Carbon tetrachloride	27	1.60	0.82

TABLE 1. Values of the wavelength λ_c that is critical for a sinusoidal disturbance to a stationary horizontal interface, for some common liquids above air; $\lambda_c = 2\pi(\gamma/\rho g)^{\frac{1}{2}}$.

near the bottom of a tank 3 ft. deep and 18 in. square in horizontal dimensions, and the bubble radii were measured from photographs of the bubble silhouette. The cap radius of the bubble corresponding to the last point in their figure showing their observations of speed as a function of size is 15.9 cm. The authors do not say whether they attempted to generate even larger bubbles (whereas Temperley & Chambers do report that they tried and were unsuccessful). Allred & Blount's report also contains an interesting sequence of photographs of a large bubble (with volume equivalent to R = 8–10 cm) as it leaves the upturned hemisphere. They found that a spray of drops usually broke off as the bubble interface left the lip of the hemisphere and flew upwards inside the bubble striking the upper surface and generating a disturbance there. It is unfortunately not easy in the photographs to distinguish between growth of the interface disturbance due to gravitational instability and that due to the

Liquid	ρ gm/cm³	μ gm/cm s	γ dyn/cm	λ_c cm	d_{max} cm	U cm/s	$\frac{\rho U d_{max}}{\mu}$	$\frac{\rho g d_{max}^2}{\gamma}$
Ethylene glycol	1.100	0.135	42.0	1.24	6.1	53.3	2650	24.2
Paraffin oil	0.883	2.00	37.5	1.30	6.3	55.0	153	23.5
Aq. sugar solution	1.310	0.54	73.1	1.51	4.6	49.2	549	9.3
Aq. sugar solution	1.366	3.40	76.2	1.51	6.3	54.1	137	17.4
Aq. sugar solution	1.392	13.40	79.1	1.51	8.9	64.2	59	34.7

TABLE 2. Properties of the five liquids in which Grace *et al.* (1978) observed the maximum size of a rising air bubble. d_{max} is the diameter of the sphere of the same volume as the largest bubble, and U is its observed speed of rise.

momentum of the impinging drops, but the enlargement of the horizontal lengthscale of the disturbances due to the flow of the liquid round the bubble is clearly visible.

Observations of the maximum volume of air bubbles that remain intact in five different liquids in a wide tank were made by Grace *et al.* (1978), and table 2 shows the relevant physical properties of these five liquids. In the last two columns values of the Reynolds number and Bond number for the bubbles of maximum size are shown. Unfortunately for the present purpose it appears from the shape-regime diagram compiled by Clift *et al.* (1978, figure 2.5) that only for the first of these liquids, ethylene glycol, were the values of the Reynolds and Bond numbers large enough for the bubbles to be likely to have a clear spherical-cap shape. For this liquid Grace *et al.* give 119 cm³ as the observed maximum volume of bubbles remaining intact, and for a bubble of spherical-cap shape with a semi-cone angle of 50° this corresponds to $R_b = 7.0$ cm. This observation is compatible with the curve for $\lambda_c = 1.25$ (which is appropriate for ethylene glycol) in figure 10 if $\xi = 0.082$ and if the magnitude of the initial disturbance was rather large. Grace *et al.* say that they allowed 'at least one minute...between injections' of bubbles of known volume in a tank 46 cm in diameter and 2.8 m deep.

No observation of the break-up of gas bubbles rising in a blocked vertical tube appear to have been reported, and in view of the values of R_b shown in figure 10 this is hardly surprising. In the case of a bubble whose vertical extent is large compared with the tube radius, like that in figure 1 (*d*) and like the interface in a draining tube, the tube radius a is approximately $1.5R$ (see figure 9 in Davies & Taylor 1950). A long vertical tube full of liquid might be expected to be 'quieter' than a wide tank since the tube wall has a damping effect on velocity fluctuations in the liquid, but if for definiteness we suppose the initial disturbance magnitude to be the same as in the experiments by Temperley & Chambers so that $R_b = 15$ cm in an air–water system, then the interface of a long bubble rising through water in a vertical tube will be disrupted only if $a > 23$ cm. Tubes of this size will not often be used in the laboratory. It would be interesting to know what the outcome of break-up is in this case of long bubbles in a tube. It may be noted in this connection that a gas bubble hugging the wall of a vertical tube, with the interface intersecting the wall at an angle of 60° in the vertical plane of symmetry, is known to be a possible steady configuration, and that such an interface is likely to be less gravitationally unstable as a consequence of its inclination to the horizontal.

Similar considerations of the stability of the interface no doubt apply to 'two-dimensional' bubbles in the liquid-filled space between two close vertical plates, like that shown in figure 1 (*c*). Provided the liquid motion is irrotational over most of the gap between the plates, the analysis in §3 of the behaviour of a small disturbance to the interface needs modification in only one minor respect, viz. to allow for the fact that the rate of contraction in the liquid normal to the interface is here equal to the rate of extension of the interface, not twice it. Thus the numerical coefficient of kA in the first term on the right-hand side of (3.14) becomes -1 instead of -2, with consequential changes in some of the numbers in (3.16), and the numerical integrations need to be done again. It is unlikely that the results will be radically different since the dominating gravity term is unchanged. No observations of the maximum size of such two-dimensional bubbles appear to have been made, although Clift & Grace (1972) reported that an air bubble of cap radius about 10 cm rising through sugar solution split into two smaller bubbles as a consequence of a disturbance to the upper surface of the bubble. Two-dimensional bubbles might be more convenient experimentally for an investigation of the instability of the interface since the quantity of liquid required is much smaller; on the other hand there may be uncertainties about the influence of the sidewalls.

In a thesis which was drawn to my attention after this paper had been submitted, Pelce (1986) has considered theoretically the stability of the interface of a long gas bubble rising in a liquid-filled tube. He supposes that the wavelength of a small disturbance is stretched like the material of the interface and that at any instant the rate of growth of the amplitude is the same as that for an (inclined) interface between stationary fluids. The growth continues as the disturbance is carried down the side of the bubble, and the bubble is regarded as unstable if the disturbance amplitude attains a value comparable with the tube radius. The resulting criterion for instability appears to be different in form from that given here.

I am glad to acknowledge the expert help of Dr Joyce Wheeler with the computing.

Appendix. Asymptotic solutions of (4.7) and (4.8)

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The aim of this Appendix is to prove that the solutions of (4.7) subject to the initial conditions (4.8) can be represented in the forms (4.6) and (5.2) provided τ' is sufficiently large and $\alpha \gg 1$.

The proof commences with the definition

$$\Psi(\tau') = \mathrm{e}^{3\tau'/2}\mathscr{A}(\tau'), \tag{A 1}$$

in terms of which the equations become

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\tau'^2} - p(\tau')\,\Psi = 0, \tag{A 2}$$

$$\Psi = 0, \quad \frac{\mathrm{d}\Psi}{\mathrm{d}\tau'} = \beta^{\frac{3}{2}} \quad \text{at } \tau' = \ln\beta, \tag{A 3}$$

where

$$p(\tau') = \alpha\,\mathrm{e}^{-\tau'}(1-\mathrm{e}^{-2\tau'}) + \tfrac{1}{4} \tag{A 4}$$

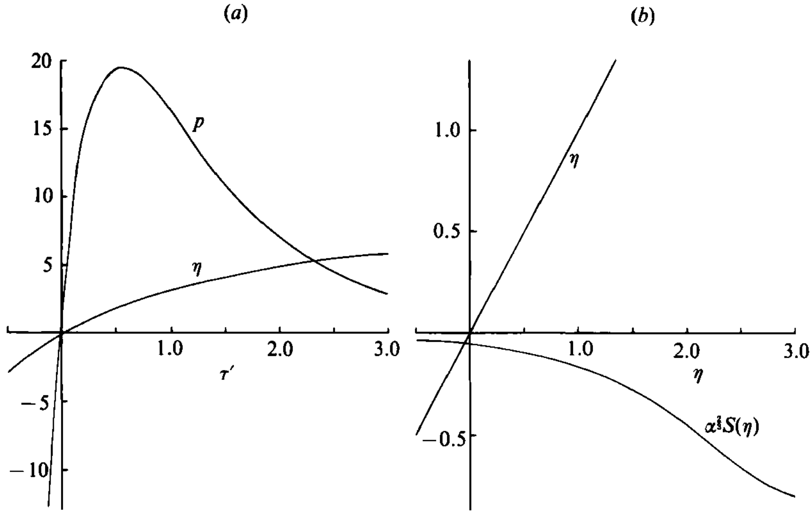

FIGURE 11. (a) Graphs of $p(\tau')$ and $\eta(\tau')$ for $\alpha = 50$. $p(\tau')$ has a maximum of $(8\sqrt{3\alpha}+9)/36$ at $\tau' = \frac{1}{2}\ln 3$. (b) Graph of $\alpha^{\frac{2}{3}}S(\eta)$ for comparison with η itself, for $\alpha = 50$.

and is graphed in figure 11 (a) for $\alpha = 50$. The shape of the curve is similar for all large values of α. We now introduce new variables (Erdelyi 1956, p. 93) by setting

$$Z(\eta) = \left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau'}\right)^{\frac{1}{2}} \Psi(\tau'), \quad \eta(\tau') = \left[\frac{3}{2}\int_{\tau'_0}^{\tau'} p^{\frac{1}{2}}(x)\,\mathrm{d}x\right]^{\frac{2}{3}}, \tag{A 5a, b}$$

where $p(\tau'_0) = 0$, and $\eta(\tau')$ also is graphed in figure 11(a). Note that $\eta(\tau') \gtrless 0$ according to whether $\tau' \gtrless \tau'_0$. The differential equation can now be written as

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\eta^2} - \eta Z = S(\eta)\,Z, \tag{A 6}$$

where
$$S(\eta) = \frac{1}{2}\frac{\eta'''}{\eta'^3} - \frac{3}{4}\frac{\eta''^2}{\eta'^4} \tag{A 7}$$

and a prime to η denotes differentiation with respect to τ'. $S(\eta)$ is plotted in figure 11(b) along with η. We now denote two linearly independent solutions of (A 6) as $H_1(\eta)$ and $H_2(\eta)$ such that $H_1(\eta)$ is exponentially large while $H_2(\eta)$ is exponentially small for large (positive) η. In terms of these and the relationships (A 1), (A 3) and (A 5), the solution of (4.7) and (4.8) becomes

$$\mathscr{A}(\tau') = \beta^{\frac{3}{2}} W^{-1} \left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau'}\right)_*^{-\frac{1}{2}} \left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau'}\right)^{-\frac{1}{2}} \mathrm{e}^{-3\tau'/2} \Big(H_1(\eta_*)\,H_2(\eta) - H_2(\eta_*)\,H_1(\eta)\Big), \tag{A 8}$$

where W is the constant Wronskian of $H_1(\eta)$ and $H_2(\eta)$ and the subscript $_*$ indicates evaluation at $\tau' = \ln\beta = \tau'_*$.

For large α, $S(\eta)$ is of order $\alpha^{-\frac{2}{3}}$. Thus (A 6) is dominated by the left-hand side and the solution to lowest order can be expressed in terms of the Airy functions as

$$H_1(\eta) = \mathrm{Bi}(\eta), \quad H_2(\eta) = \mathrm{Ai}(\eta).$$

Further, because τ_0' is approximately $-1/8\alpha$, it can, to lowest order, be equated to zero in (A 5*b*). Then for positive τ', because η is large (of order $\alpha^{\frac{1}{3}}$), $H_1(\eta) \gg H_2(\eta)$. As long as β is not too large on the other hand, neither $H_1(\eta_*)$ nor $H_2(\eta_*)$ is very much larger than the other. Inserting these relationships into the last bracket of (A 8), we obtain

$$\mathscr{A}(\tau') \approx -\beta^{\frac{3}{2}} W^{-1} \left(\frac{d\eta}{d\tau'}\right)_*^{-\frac{1}{2}} H_2(\eta_*) \left(\frac{d\eta}{d\tau'}\right)^{-\frac{1}{2}} e^{-3\tau'/2} H_1(\eta). \tag{A 9}$$

None of the first four terms of (A 9) is a function of τ' and (minus) their product can be identified with the $C(\beta)$ of (4.6). Additionally, none of the last three terms of (A 9) is a function of β and their product can be identified with the $f(\tau')$ of (4.6). This completes the first part of the proof.

It is now of interest to go further and evaluate some of the terms explicitly, at least to lowest order. Using the result $W = -\pi^{-1}$ (Abramowitz & Stegun 1965) and the fact that

$$\frac{d\eta}{d\tau'} = \left(\frac{p}{\eta}\right)^{\frac{1}{2}}, \tag{A 10}$$

we can rewrite (A 9) so that

$$C(\beta) = \pi \beta^{\frac{3}{2}} \left(\frac{\eta_*}{p_*}\right)^{\frac{1}{4}} \mathrm{Ai}(\eta_*), \tag{A 11}$$

$$f(\tau') = \left(\frac{\eta}{p}\right)^{\frac{1}{4}} e^{-3\tau'/2}\, \mathrm{Bi}(\eta) \tag{A 12}$$

$$\approx \pi^{-\frac{1}{2}} p^{-\frac{1}{4}} \exp\left\{\int_0^{\tau'} p^{\frac{1}{2}}(x)\, dx - \tfrac{3}{2}\tau'\right\}, \tag{A 13}$$

where (A 13) is obtained from (A 12) by using the asymptotic representation of $\mathrm{Bi}(\eta)$. Following the spirit of the asymptotic analysis, we evaluate the maximum of $f(\tau')$ by determining the stationary point of the rapidly varying exponential term in (A 13). This occurs when $p^{\frac{1}{2}} = \frac{3}{2}$ at $\tau' = s$, where

$$\alpha\, e^{-s}(1 - e^{-2s}) = 2. \tag{A 14}$$

For large α therefore, $s \approx \ln \frac{1}{2}\alpha$.

We can now prove the validity of the decomposition (5.2). For $\beta = 1$, from (4.5) and (4.9)

$$\tau'' = \tau' - \ln \alpha. \tag{A 15}$$

Thus with respect to τ'' coordinates, $\mathscr{A}$ has a maximum, dependent on α, say $K(\alpha)$, at $\tau'' = -\ln 2 = -0.693\ldots$. Considered as a function of τ'',

$$p = e^{-\tau''} + \tfrac{1}{4} - \alpha^{-2} e^{-3\tau''}, \tag{A 16}$$

in which α appears only to order α^{-2}. Thus integration with respect to τ'' of the linear differential equation for $\mathscr{A}$ commencing at $\tau'' = -\ln 2$ (independent of α) with $\mathscr{A} = K(\alpha)$ and $d\mathscr{A}/d\tau'' = 0$ will lead to solution of the form

$$\mathscr{A} = K(\alpha) F(\tau''), \tag{A 17}$$

where F is approximately independent of α until τ'' is sufficiently negative that the last term of (A 16) becomes important.

Numerical evaluation of $C(\beta)f(\tau')$ from (A 11) and (A 12) was conducted to investigate the accuracy of the asymptotic formulae for various values of α. The results agreed well with direct numerical solution of the differential equation as graphed in figures 3, 5 and 7. For example, for $\alpha = 50$ the error was 3% for $\tau = 0.4$ ($\tau'' = -3.4$) and less for larger values of τ.

REFERENCES

Abramowitz, M. & Stegun, I. A. 1965 *Handbook of Mathematical Functions*. U.S. Government Printing Office.

Allred, J. & Blount, G. 1953 Experimental studies of Taylor instability. *Los Alamos Scientific Laboratory Rep.* LA 1600.

Bellman, R. & Pennington, R. H. 1954 Effects of surface tension and viscosity on Taylor instability. *Q. Appl. Maths* **12**, 151–162.

Clift, R. & Grace, J. R. 1972 The mechanism of bubble break-up in fluidised beds. *Chem. Engng Sci.* **27**, 2309–2310.

Clift, R., Grace, J. R. & Weber, M. E. 1978 *Bubbles, Drops and Particles*. Academic.

Collins, R. 1965 Structure and behaviour of wakes behind two-dimensional air bubbles in water. *Chem. Engng Sci.* **20**, 851–853.

Dagan, G. 1975 Taylor instability of a non-uniform free-surface flow. *J. Fluid Mech.* **67**,113–123.

Davenport, W. G., Bradshaw, A. V. & Richardson, F. D. 1967 Behaviour of spherical-cap bubbles in liquid metals. *J. Iron Steel Inst.* **205**, 1034–1042.

Davies, R. M. & Taylor, G. I. 1950 The mechanics of large bubbles rising through extended liquids and through liquids in tubes. *Proc. R. Soc. Lond.* A **200**, 375–390; and *Scientific Papers of G. I. Taylor*, vol. 3, p. 479–492. Cambridge University Press.

Erdelyi, A. 1956 *Asymptotic Expansions*. Dover.

Frankel, I. & Weihs, D. 1985 Stability of a capillary jet with linearly increasing axial velocity (with application to shaped charges). *J. Fluid Mech.* **155**, 289–307.

Grace, J. R., Wairegi, T. & Brophy, J. 1978 Break-up of drops and bubbles in stagnant media. *Can. J. Chem. Engng* **65**, 3–8.

Harper, J. F. 1972 The motion of bubbles and drops through liquids. *Adv. Appl. Mech.* **12**, 59–129.

Moore, D. W. & Griffith-Jones, R. 1974 The stability of an expanding circular vortex sheet. *Mathematika* **21**, 128.

Pelce, P. 1986 Dynamique des front courbes. Doctoral thesis submitted to the University of Provence.

Taylor, G. I. & Davies, R. M. 1944 The rate of rise of large volumes of gas in water. Reproduced in *Underwater Explosions Research*, vol. 2. Office of Naval Research, Washington, 1950.

Temperley, H. N. V. & Chambers, L. G. 1945 The rate of rise of large volumes of gas in water. Reproduced in *Underwater Explosions Research*, vol. 2. Office of Naval Research, Washington, 1950.

Tomotika, S. 1936 Breaking up of a drop of viscous liquid immersed in another viscous fluid which is extending at a uniform rate. *Proc. R. Soc. Lond.* A **153**, 302–318.

Wegener, P. P. & Parlange, J.-Y. 1973 Spherical-cap bubbles. *Ann. Rev. Fluid Mech.* **5**, 79–100.