Documentation and Validation of EveryCalc's Trajectory Tool

Thaddeus Hughes
hughes.thad@gmail.com
thaddeus-maximus.github.io

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Abstract

Hurling projectiles is something we humans really like doing. We've become exceedingly efficient at it. We make sport of it. We can guess at a trajectory pretty easily- but nailing it down, and tweaking it with an engineering mindset is harder. Lots of tools already exist to do this but I want to roll my own so I can add all the physics I want, along with the swept area of a projectile, and stacking the tolerance.

1 Basic Projectile Motion

Consider a ball of mass m, in a vacuum. It has only the force of gravity acting on it. That is to say,

$$\Sigma F_x = m\frac{dv_x}{dt} = 0 \tag{1}$$

$$\Sigma F_y = m\frac{dv_y}{dt} = -mg \tag{2}$$

To figure out the path, we'd also need to know the initial conidtions. Let's also say that the ball was launched at an angle θ from the horizontal at an initial velocity of $\bar{v}_0$, so that the x- and y- velocities would be

$$v_x(0) = \bar{v}_0\ cos(\theta) \tag{3}$$

$$v_y(0) = \bar{v}_0\ sin(\theta). \tag{4}$$

We'll start from a height of y_0 and at $x = -x_0$ from our target.

$$x(0) = -x_0 \tag{5}$$

$$y(0) = -y_0 \tag{6}$$

This is enough to get us a very simple simulation for projectile motion:

$$\frac{dv_x}{dt} = 0 \tag{7}$$
$$\frac{dv_y}{dt} = -g \tag{8}$$
$$\frac{dx}{dt} = v_x \tag{9}$$
$$\frac{dy}{dt} = v_y \tag{10}$$
$$v_x(0) = \bar{v}_0 \; cos(\theta) \tag{11}$$
$$v_y(0) = \bar{v}_0 \; sin(\theta) \tag{12}$$
$$x(0) = -x_0 \tag{13}$$
$$y(0) = y_0 \tag{14}$$
$$\text{terminate when } x \geq 0 \tag{15}$$

2 Swept Path with Parallel Curves

It's also worth knowing the swept zone that the target travels, since the object of firing a projectile may not be to hit a target per se, but to make it *through* a target. This may seem like a trivial task at first blush; just add on the radius of the ball to the y-direction, but then one realizes the projetile may not be striking the target dead-on. Creating an offset path, or parallel curve is necessary. The upper swept path (x_u, y_u) of a ball of radius r can be determined as

$$x_u = x + r(\hat{x} \cdot \hat{N}) \tag{16}$$
$$y_u = y + r(\hat{y} \cdot \hat{N}) \tag{17}$$

Where $\hat{x}$, $\hat{y}$, $\hat{N}$, $\hat{T}$ are unit vectors in the x-, y-, normal, and tangent directions.

$$\text{let } \bar{v} = \sqrt{v_x^2 + v_y^2} \tag{18}$$
$$\hat{x} \cdot \hat{N} = -v_x/\bar{v} \tag{19}$$
$$\hat{x} \cdot \hat{T} = +v_y/\bar{v} \tag{20}$$
$$\hat{y} \cdot \hat{N} = +v_x/\bar{v} \tag{21}$$
$$\hat{y} \cdot \hat{T} = +v_y/\bar{v} \tag{22}$$

The lower path would be found by reversing the direction of r, yielding

$$x_l = x - r(\hat{x} \cdot \hat{N}) \tag{23}$$
$$y_l = y - r(\hat{y} \cdot \hat{N}). \tag{24}$$

3 Aerodynamic Effects

There are multiple aerodynamic forces that can act on a ball.

$$F_{drag} = \frac{1}{2} \; C_{drag} \; \rho \; A \; \bar{v}^2 \tag{25}$$
$$F_{lift} = \frac{1}{2} \; C_{lift} \; \rho \; A \; \bar{v}^2 \tag{26}$$
$$F_{magnus} = \frac{\pi}{3} \; C_{magnus} \; \rho \; A \; \bar{v} \; \omega_{+\text{ccw}} \tag{27}$$

The lift and drag forces are standard equations, but the magnus force equation is derived from this NASA page. There are undoubtedly better models out there, but this is what I have currently.

In vector form the conservation of momentum could be written as

$$m\frac{d\vec{v}}{dt} = -F_{drag}\hat{T} + F_{lift}\hat{N} + F_{magnus}\hat{N} - mg\hat{y} \tag{28}$$

This changes the model, when projected out into the x- and y- components to

$$\frac{dv_x}{dt} = \frac{-F_{drag}(\hat{x}\cdot\hat{T}) + F_{lift}(\hat{x}\cdot\hat{N}) + F_{magnus}(\hat{x}\cdot\hat{N})}{m} \tag{29}$$

$$\frac{dv_y}{dt} = \frac{-F_{drag}(\hat{y}\cdot\hat{T}) + F_{lift}(\hat{y}\cdot\hat{N}) + F_{magnus}(\hat{y}\cdot\hat{N})}{m} - g \tag{30}$$

$$\frac{dx}{dt} = v_x \tag{31}$$

$$\frac{dy}{dt} = v_y \tag{32}$$

$$v_x(0) = \bar{v}_0\ cos(\theta) \tag{33}$$

$$v_y(0) = \bar{v}_0\ sin(\theta) \tag{34}$$

$$x(0) = -x_0 \tag{35}$$

$$y(0) = y_0 \tag{36}$$

$$\text{terminate when } x \geq 0. \tag{37}$$

4 Tolerance Stacking

To determine accuracy, multiple iterations of the simulation can be ran with different permutations of input variables.

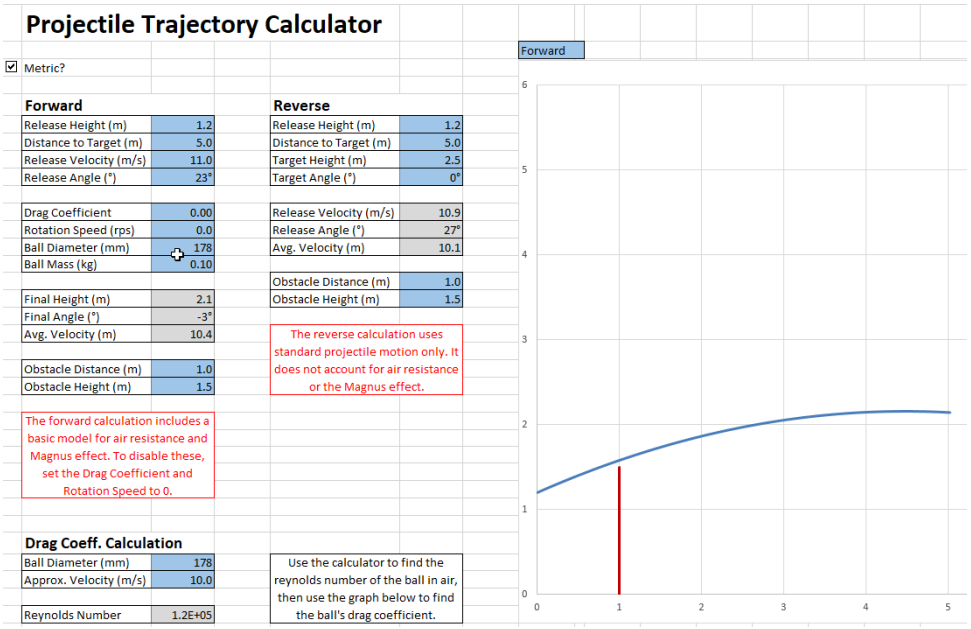
5 Reverse Computation

A bisection algorithm is used to solve for the appropriate distance/angle/velocity required to propel the projectile into the target. This has benefits over analytical solutions in that it can be used in conjunction with aerodynamic effects.

6 Validation Against Other Tools

I'll compare results to AMB's Design Spreadsheet.

Case A: Metric units, no aero



Projectile Trajectory Calculator

Metric?

Forward	
Release Height (m)	1.2
Distance to Target (m)	5.0
Release Velocity (m/s)	11.0
Release Angle (°)	23°
Drag Coefficient	0.00
Rotation Speed (rps)	0.0
Ball Diameter (mm)	178
Ball Mass (kg)	0.10
Final Height (m)	2.1
Final Angle (°)	-3°
Avg. Velocity (m)	10.4
Obstacle Distance (m)	1.0
Obstacle Height (m)	1.5

The forward calculation includes a basic model for air resistance and Magnus effect. To disable these, set the Drag Coefficient and Rotation Speed to 0.

Reverse	
Release Height (m)	1.2
Distance to Target (m)	5.0
Target Height (m)	2.5
Target Angle (°)	0°
Release Velocity (m/s)	10.9
Release Angle (°)	27°
Avg. Velocity (m)	10.1
Obstacle Distance (m)	1.0
Obstacle Height (m)	1.5

The reverse calculation uses standard projectile motion only. It does not account for air resistance or the Magnus effect.

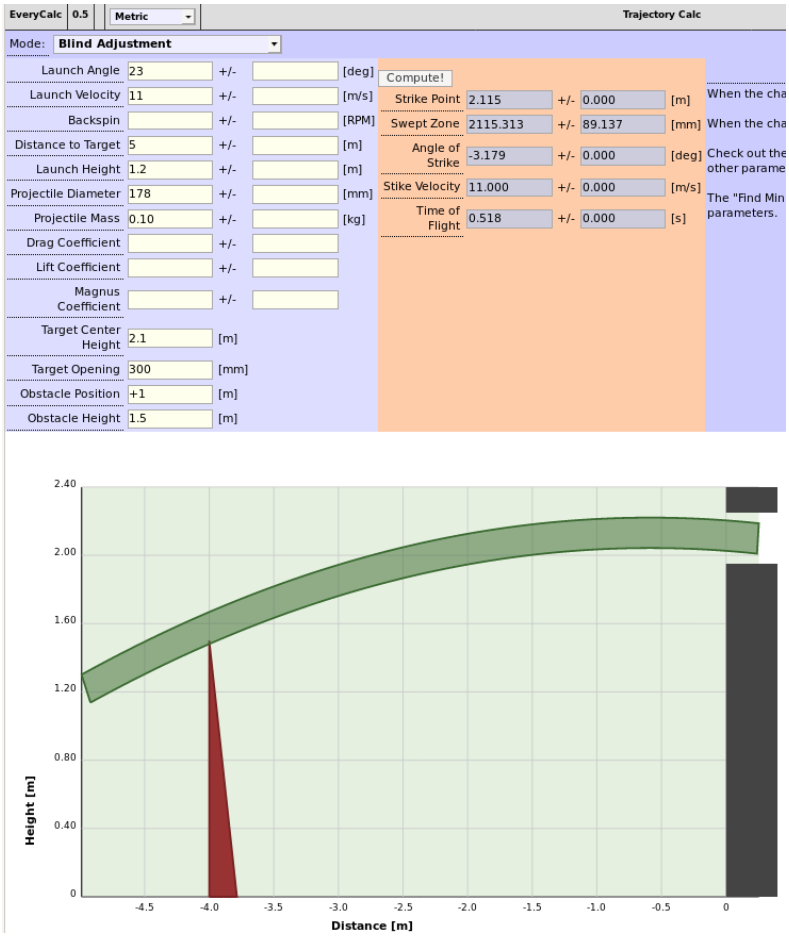
Forward

0 1 2 3 4 5 6

Drag Coeff. Calculation

Ball Diameter (mm)	178
Approx. Velocity (m/s)	10.0
Reynolds Number	1.2E+05

Use the calculator to find the reynolds number of the ball in air, then use the graph below to find the ball's drag coefficient.



EveryCalc 0.5 | Metric | Trajectory Calc

Mode: Blind Adjustment

Launch Angle	23	+/-		[deg]
Launch Velocity	11	+/-		[m/s]
Backspin		+/-		[RPM]
Distance to Target	5	+/-		[m]
Launch Height	1.2	+/-		[m]
Projectile Diameter	178	+/-		[mm]
Projectile Mass	0.10	+/-		[kg]
Drag Coefficient		+/-		
Lift Coefficient		+/-		
Magnus Coefficient		+/-		
Target Center Height	2.1	[m]		
Target Opening	300	[mm]		
Obstacle Position	+1	[m]		
Obstacle Height	1.5	[m]		

Compute!

Strike Point	2.115	+/-	0.000	[m]
Swept Zone	2115.313	+/-	89.137	[mm]
Angle of Strike	-3.179	+/-	0.000	[deg]
Stike Velocity	11.000	+/-	0.000	[m/s]
Time of Flight	0.518	+/-	0.000	[s]

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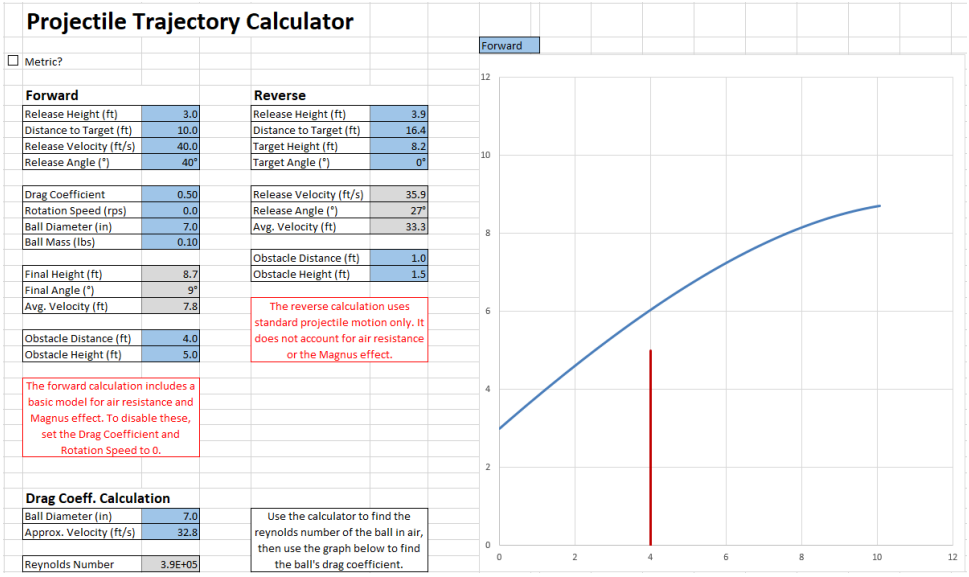
Height [m]: 0 0.40 0.80 1.20 1.60 2.00 2.40

Distance [m]: -4.5 -4.0 -3.5 -3.0 -2.5 -2.0 -1.5 -1.0 -0.5 0

Look only at the "Forward" portions of AMB's sheet. Effectively the same result.

Case B: English units, drag included



Projectile Trajectory Calculator

Metric? ☐

Forward	
Release Height (ft)	3.0
Distance to Target (ft)	10.0
Release Velocity (ft/s)	40.0
Release Angle (°)	40°
Drag Coefficient	0.50
Rotation Speed (rps)	0.0
Ball Diameter (in)	7.0
Ball Mass (lbs)	0.10
Final Height (ft)	8.7
Final Angle (°)	9°
Avg. Velocity (ft)	7.8
Obstacle Distance (ft)	4.0
Obstacle Height (ft)	5.0

The forward calculation includes a basic model for air resistance and Magnus effect. To disable these, set the Drag Coefficient and Rotation Speed to 0.

Reverse	
Release Height (ft)	3.9
Distance to Target (ft)	16.4
Target Height (ft)	8.2
Target Angle (°)	0°
Release Velocity (ft/s)	35.9
Release Angle (°)	27°
Avg. Velocity (ft)	33.3
Obstacle Distance (ft)	1.0
Obstacle Height (ft)	1.5

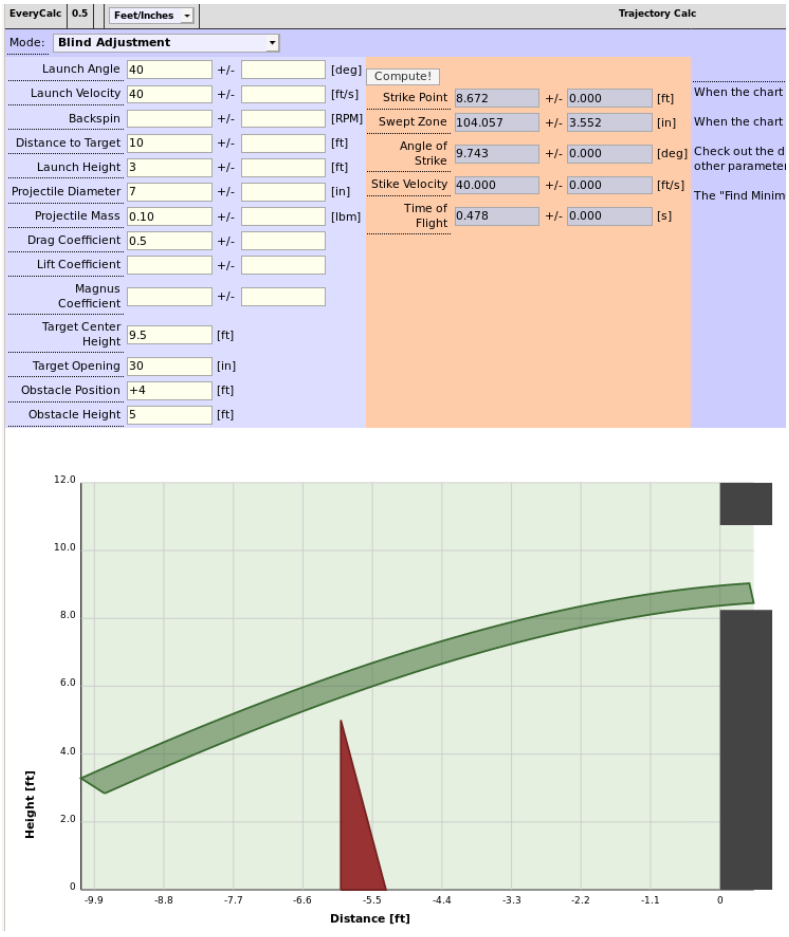
The reverse calculation uses standard projectile motion only. It does not account for air resistance or the Magnus effect.

Drag Coeff. Calculation

Ball Diameter (in)	7.0
Approx. Velocity (ft/s)	32.8
Reynolds Number	3.9E+05

Use the calculator to find the reynolds number of the ball in air, then use the graph below to find the ball's drag coefficient.

Forward



EveryCalc | 0.5 | Feet/Inches | Trajectory Calc

Mode: **Blind Adjustment**

Launch Angle	40	+/-		[deg]
Launch Velocity	40	+/-		[ft/s]
Backspin		+/-		[RPM]
Distance to Target	10	+/-		[ft]
Launch Height	3	+/-		[ft]
Projectile Diameter	7	+/-		[in]
Projectile Mass	0.10	+/-		[lbm]
Drag Coefficient	0.5	+/-		
Lift Coefficient		+/-		
Magnus Coefficient		+/-		
Target Center Height	9.5			[ft]
Target Opening	30			[in]
Obstacle Position	+4			[ft]
Obstacle Height	5			[ft]

Compute!

Strike Point	8.672	+/-	0.000	[ft]
Swept Zone	104.057	+/-	3.552	[in]
Angle of Strike	9.743	+/-	0.000	[deg]
Stike Velocity	40.000	+/-	0.000	[ft/s]
Time of Flight	0.478	+/-	0.000	[s]

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Look only at the "Forward" portions of AMB's sheet. Effectively the same result.