Documentation and Validation of EveryCalc's Trajectory Tool

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Abstract

Hurling projectiles is something we humans really like doing. We've become exceedingly efficient at it. We make sport of it. We can guess at a trajectory pretty easily- but nailing it down, and tweaking it with an engineering mindset is harder. Lots of tools already exist to do this but I want to roll my own so I can add all the physics I want, along with the swept area of a projectile, and stacking the tolerance.

Basic Projectile Motion

Consider a ball of mass m, in a vacuum. It has only the force of gravity acting on it. That is to say,

$$\Sigma F_x = m\frac{dv_x}{dt} = 0 \tag{1}$$

$$\Sigma F_y = m\frac{dv_y}{dt} = -mg \tag{2}$$

To figure out the path, we'd also need to know the initial conidtions. Let's also say that the ball was launched at an angle θ from the horizontal at an initial velocity of $\bar{v}_0$, so that the x- and y- velocities would be

$$v_x(0) = \bar{v}_0 \ cos(\theta) \tag{3}$$

$$v_y(0) = \bar{v}_0 \ sin(\theta). \tag{4}$$

We'll start from a height of y_0 and at $x = -x_0$ from our target.

$$x(0) = -x_0 \tag{5}$$

$$y(0) = -y_0 \tag{6}$$

This is enough to get us a very simple simulation for projectile motion:

$$\frac{dv_x}{dt} = 0 \tag{7}$$
$$\frac{dv_y}{dt} = -g \tag{8}$$
$$\frac{dx}{dt} = v_x \tag{9}$$
$$\frac{dy}{dt} = v_y \tag{10}$$
$$v_x(0) = \bar{v}_0 \ cos(\theta) \tag{11}$$
$$v_y(0) = \bar{v}_0 \ sin(\theta) \tag{12}$$
$$x(0) = -x_0 \tag{13}$$
$$y(0) = y_0 \tag{14}$$
$$\text{terminate when } x \geq 0 \tag{15}$$

Parallel Curves

It's also worth knowing the swept zone that the target travels, since the object of firing a projectile may not be to hit a target per se, but to make it *through* a target. This may seem like a trivial task at first blush; just add on the radius of the ball to the y-direction, but then one realizes the projetile may not be striking the target dead-on. Creating an offset path, or parallel curve is necessary. The upper swept path (x_u, y_u of a ball of radius r can be determined as

$$x_u = x + r\hat{x} \cdot \hat{N} \tag{16}$$
$$y_u = y + r\hat{y} \cdot \hat{N} \tag{17}$$

Where $\hat{x}$, $\hat{y}$, $\hat{N}$, $\hat{T}$ are unit vectors in the x-, y-, normal, and tangent directions.

$$\text{let } \bar{v} = \sqrt{v_x^2 + v_y^2} \tag{18}$$
$$\hat{x} \cdot \hat{N} = -v_x/\bar{v} \tag{19}$$
$$\hat{x} \cdot \hat{T} = +v_y/\bar{v} \tag{20}$$
$$\hat{y} \cdot \hat{N} = +v_x/\bar{v} \tag{21}$$
$$\hat{y} \cdot \hat{T} = +v_y/\bar{v} \tag{22}$$

The lower path would be found by reversing the direction of r, yielding

$$x_l = x - r\hat{x} \cdot \hat{N} \tag{23}$$
$$y_l = y - r\hat{y} \cdot \hat{N}. \tag{24}$$

Aerodynamic Effects

There are multiple aerodynamic forces that can act on a ball.

$$F_{drag} = \frac{1}{2}\rho C_{drag} A\bar{v}^2(-\hat{T}) \tag{25}$$
$$F_{lift} = \frac{1}{2}\rho C_{drag} A\bar{v}^2(+\hat{N}) \tag{26}$$
$$F_{magnus} = \frac{\pi}{3}\rho C_{drag} A\bar{v}\omega_{+\text{ccw}}(+\hat{N}) \tag{27}$$

The lift and drag forces are standard equations, but the magnus force equation is derived from this NASA page. There are undoubtedly better models out there, but this is what I have currently.

This changes the model to

$$
\begin{aligned}
\frac{dv_x}{dt} &= \frac{-F_{drag}\hat{x}\cdot\hat{T} + F_{lift}\hat{x}\cdot\hat{N} + F_{magnus}\hat{x}\cdot\hat{N}}{m} && (28)\\
\frac{dv_y}{dt} &= \frac{-g - F_{drag}\hat{y}\cdot\hat{T} + F_{lift}\hat{y}\cdot\hat{N} + F_{magnus}\hat{y}\cdot\hat{N}}{m} && (29)\\
\frac{dx}{dt} &= v_x && (30)\\
\frac{dy}{dt} &= v_y && (31)\\
v_x(0) &= \bar{v}_0\ cos(\theta) && (32)\\
v_y(0) &= \bar{v}_0\ sin(\theta) && (33)\\
x(0) &= -x_0 && (34)\\
y(0) &= y_0 && (35)\\
\text{terminate when } x &\geq 0. && (36)
\end{aligned}
$$

Tolerance Stacking

To determine accuracy, multiple iterations of the simulation can be ran with different permutations of input variables.

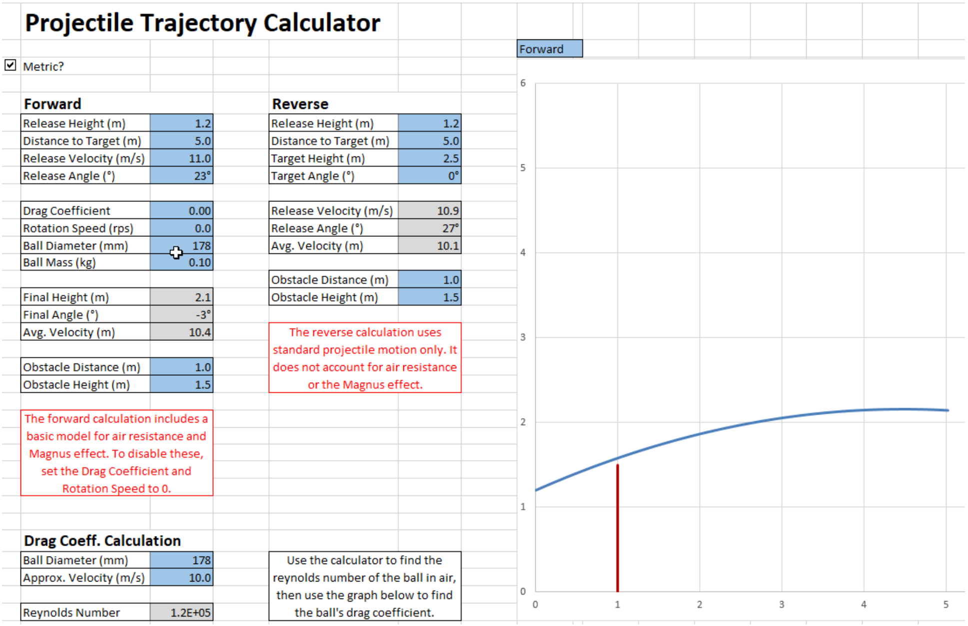
Reverse Computation

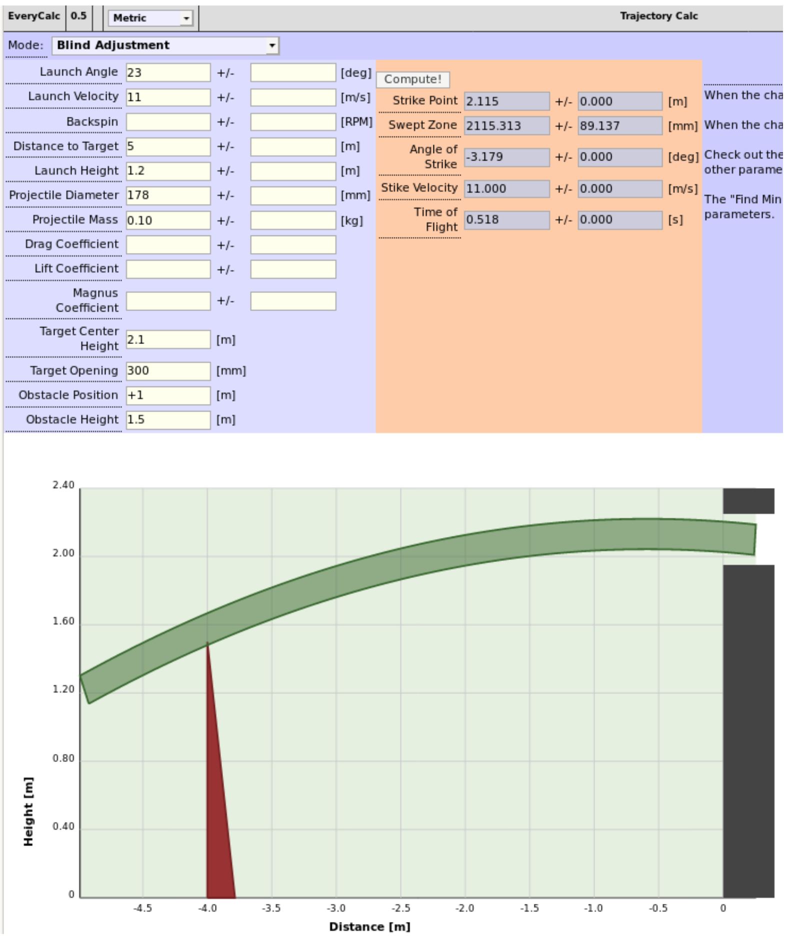
A bisection algorithm is used to solve for the appropriate distance/angle/velocity required to propel the projectile into the target. This has benefits over analytical solutions in that it can be used in conjunction with aerodynamic effects.

Validation Against Other Tools

I'll compare results to AMB's Design Spreadsheet.

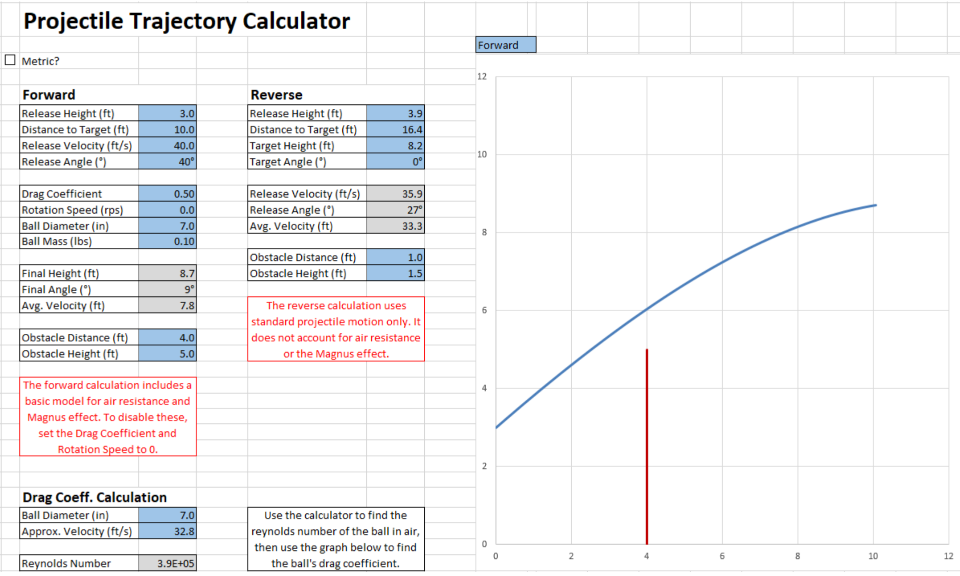
Case A: Metric units, no aero

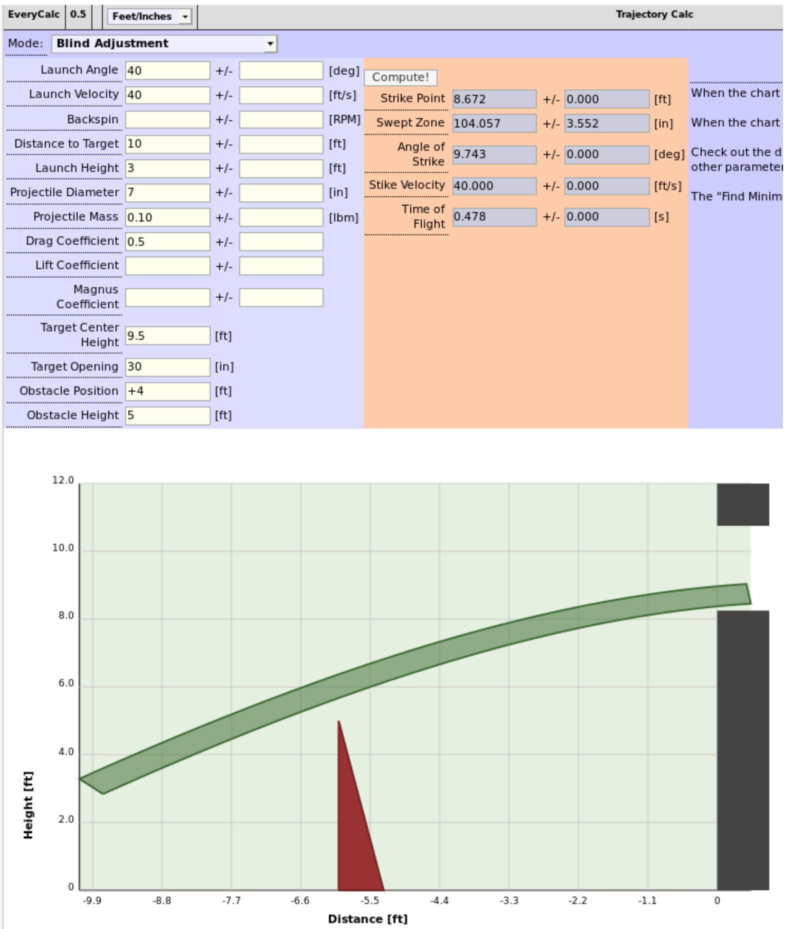




Look only at the "Forward" portions of AMB's sheet. Effectively the same result.

Case B: English units, drag included





Look only at the "Forward" portions of AMB's sheet. Effectively the same result.