Numerical Analysis
Gaussian elimination

by Csaba Gáspár

Széchenyi István University

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Linear systems of equations

Let $A = [a_{kj}] \in \mathbf{M}_{N \times N}$ be a given matrix, and let $b \in \mathbf{R}^N$ be a given vector. Consider the equation

$$Ax = b$$

This is equivalent to the following system of linear equations with N unknowns:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N
\end{aligned}$$

The system is **homogeneous**, if $b = \mathbf{0}$. In this case, $x = \mathbf{0}$ is always a solution (**trivial solution**). The solution x is said to be a **nontrivial solution**, if at least one component of x differs from zero.

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Solvability of linear systems

The matrix $A \in \mathbf{M}_{N\times N}$ is regular if and only if the equation $Ax = b$ has a solution for every right-hand side. In this case, the solution is unique, namely: $x = A^{-1}b$.

The matrix $A \in \mathbf{M}_{N\times N}$ is regular if and only if the only the trivial solution solves the corresponding homogeneous equation $Ax = \mathbf{0}$, i.e. the matrix A is singular if and only if the corresponding homogeneous equation has a nontrivial solution (in this case, an infinite number of nontrivial solutions exist).

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The Gaussian elimination

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\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{NN}x_N &= b_N
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Divide the 1st equation by the coefficient a_{11} (**pivot element**):

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Now let us subtract the 1st row multiplied by a_{k1}, from the kth row: $(k = 2, 3, ..., N)$:

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The procedure is repeated for the 2nd, 3rd, ..., Nth equations:

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x_1 + \tilde{a}_{12}x_2 + \tilde{a}_{13}x_3 + \ldots + \tilde{a}_{1N}x_N &= \tilde{b}_1 \\
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The components $x_{N-1}, x_{N-2}..., x_1$ can be computed by back-substitutions.

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Elimination, example

$$\begin{array}{rcrcrcr} 2x_1 & - & 6x_2 & + & 10x_3 & = & -12 \\ 2x_1 & - & 5x_2 & + & 3x_3 & = & -4 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 3 \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 5x_3 & = & -6 \\ 2x_1 & - & 5x_2 & + & 3x_3 & = & -4 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 3 \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 5x_3 & = & -6 \\ & & x_2 & - & 7x_3 & = & 8 \\ & & 7x_2 & - & 14x_3 & = & 21 \end{array}$$

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Pivoting

Computational cost: $\mathcal{O}(N^3)$.

If a pivot element is zero, then the algorithm stops.

Partial pivoting:
Swap the kth row with the rth row, where $r \geq k$ is the index for which $|a'_{rk}|$ is maximal.

Complete pivoting:
Swap the kth equation with the rth one, and swap the kth unknown with the pth one where $r \geq k$ and $p \geq k$ are the indices for which $|a'_{rp}|$ is maximal.

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The Gauss-Jordan elimination

Idea: we eliminate not only from the next equations but also from the previous ones.

Example:

$$\left(\begin{array}{rrr|r} 2 & -6 & 10 & -12 \\ 2 & -5 & 3 & -4 \\ 3 & -2 & 1 & 3 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -3 & 5 & -6 \\ 2 & -5 & 3 & -4 \\ 3 & -2 & 1 & 3 \end{array}\right) \to$$

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Elimination in case of singular matrix

Example:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 1 \\ -2x_1 & + & x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & - & 2x_3 & = & 1 \end{array}$$

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There is no solution.

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Some unknown e.g. x_3 can be defined arbitrarily: $x_3 := t$.

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An infinite number of nontrivial solutions exist: $x_1 = t$, $x_2 = t$, $x_3 = t$.

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Matrix inversion by Gaussian elimination

Let $A \in \mathbf{M}_{N\times N}$ be a regular matrix. Then $AA^{-1} = I$. Denote by a_1, a_2, ... , a_N the columns of the inverse matrix A^{-1}. Similarly, let e_1, e_2, ... ,e_N be the column vectors of the unit matrix I, then

$$A \cdot \left(\begin{array}{c|c|c|c} a_1 & a_2 & \ldots & a_N \end{array} \right) = \left(\begin{array}{c|c|c|c} e_1 & e_2 & \ldots & e_N \end{array} \right)$$

i.e.

$$Aa_k = e_k \qquad (k = 1, 2, ..., N)$$

That is, N different systems of equations have to be solved (with different right-hand sides but with a common matrix).

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Matrix inversion by Gaussian elimination

Let $A \in \mathbf{M}_{N\times N}$ be a regular matrix. Then $AA^{-1} = I$. Denote by a_1, a_2, ... , a_N the columns of the inverse matrix A^{-1}. Similarly, let e_1, e_2, ... ,e_N be the column vectors of the unit matrix I, then

$$A \cdot \left(\begin{array}{c|c|c|c} a_1 & a_2 & \dots & a_N \end{array} \right) = \left(\begin{array}{c|c|c|c} e_1 & e_2 & \dots & e_N \end{array} \right)$$

i.e.

$$Aa_k = e_k \qquad (k = 1, 2, ..., N)$$

That is, N different systems of equations have to be solved (with different right-hand sides but with a common matrix).

Matrix inversion by Gaussian elimination

Example:

$$\left(\begin{array}{ccc|ccc} -3 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 0 & \frac{4}{3} & 1 & -\frac{2}{3} & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{4}{3} & 1 & -\frac{2}{3} & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{9} & -\frac{2}{3} & -\frac{4}{9} & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -6 & -4 & 9 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 4 & 3 & -6 \\ 0 & 0 & 1 & -6 & -4 & 9 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -2 & 4 \\ 0 & 1 & 0 & 4 & 3 & -6 \\ 0 & 0 & 1 & -6 & -4 & 9 \end{array}\right)$$

Matrix inversion by Gaussian elimination

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Matrix inversion by Gaussian elimination

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Matrix inversion by Gaussian elimination

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Matrix inversion by Gaussian elimination

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