The Fast Fourier Transform

Trigonometric Fourier series in $L_2(0, 2\pi)$

Complex exponential function

The Discrete Fourier Transform

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Numerical Analysis
The discrete and fast Fourier Transform

by Csaba Gáspár

Széchenyi István University

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Trigonometric Fourier series in $L_2(0, 2\pi)$

An arbitrary real function $f \in L_2(0, 2\pi)$ can be expressed as a trigonometric Fourier series which is convergent with respect to the $L_2(0, 2\pi)$-norm:

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

where the coefficients can be calculated as:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx,$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\, dx, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\, dx$$

Complex exponential function

For any $z \in \mathbf{C}$, define the **exponential series** as:

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

Euler's formula

For every $t \in \mathbf{R}$: $\quad e^{it} = \cos t + i \cdot \sin t$

Utilizing the well-known Taylor series of the sine and cosine functions:

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots, \qquad \sin t = \frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots$$

which implies that:

$$e^{it} = 1 + \frac{it}{1!} + \frac{i^2t^2}{2!} + \frac{i^3t^3}{3!} + \frac{i^4t^4}{4!} + \ldots = 1 + \frac{it}{1!} - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \ldots$$

Separating the real and imaginary parts, we have the theorem.

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The Discrete Fourier Transform (DFT)

If $f_0, f_1, ..., f_{N-1} \in \mathbf{C}$ is a finite sequence, then define its **discrete Fourier transform** as:

$$\hat{f}_k := \sum_{j=0}^{N-1} f_j \cdot e^{\frac{2\pi ikj}{N}} \qquad (k = 0, 1, ..., N-1)$$

Relationship with the Fourier series: Let f be a continuous function defined on the interval $[0, 2\pi)$, and denote by $f_j := f(\frac{2\pi j}{N})$. Then the sum $\frac{2\pi}{N} \cdot \sum_{j=0}^{N-1} f_j \cdot e^{\frac{2\pi ikj}{N}}$ is a Riemann sum of the integral $\int_0^{2\pi} f(x)e^{ikx}\,dx$. Utilizing Euler's formula, we have:

$$\frac{1}{N}\hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j \cdot e^{\frac{2\pi ikj}{N}} \approx \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{ikx}\,dx =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos kx\,dx + i \cdot \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin kx\,dx = a_k + ib_k,$$

where a_k, b_k are the trigonometric Fourier coefficients.

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The inverse Discrete Fourier Transform (iDFT)

Every finite sequence can uniquely be reconstructed from its DFT, namely:

$$f_k := \frac{1}{N} \cdot \sum_{j=0}^{N-1} \hat{f}_j \cdot e^{-\frac{2\pi ikj}{N}} \qquad (k = 0, 1, ..., N-1)$$

$$\frac{1}{N}\sum_{j=0}^{N-1} \hat{f}_j e^{-\frac{2\pi ikj}{N}} = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{r=0}^{N-1} f_r e^{\frac{2\pi irj}{N}} e^{-\frac{2\pi ikj}{N}} = \sum_{r=0}^{N-1} f_r \cdot \frac{1}{N}\sum_{j=0}^{N-1} e^{\frac{2\pi i(r-k)j}{N}}$$

$$= \sum_{r=0}^{N-1} f_r \cdot \frac{1}{N}\sum_{j=0}^{N-1} z^j \qquad (z := e^{\frac{2\pi i(r-k)}{N}})$$

If $r = k$, then $z = 1$, therefore the inner sum equals to 1. If $r \neq k$, then the inner sum is the sum of a finite geometric sequence, which equals to 0. This completes the proof.

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The Discrete Fourier Transform, numerical features

Total computational cost: $\mathcal{O}(N^2)$, which is too high!

The Fast Fourier Transform (FFT)

Denote by $F_N : \mathbf{C}^N \to \mathbf{C}^N$ the linear operator of the DFT:

$$(F_N f)_k = \sum_{j=0}^{N-1} f_j \cdot e^{\frac{2\pi i k j}{N}} \qquad (k = 0, 1, ..., N-1)$$

Assume that N is even: $N = 2N_1$. Let us separate the terms with even and odd indices in the expression of $(F_N f)_k$. First, let k be a 'small' index: $k = 0, 1, ..., N_1 - 1$:

$$(F_N f)_k = \sum_{\ell=0}^{N_1-1} f_{2\ell} \cdot e^{\frac{2\pi i k \cdot 2\ell}{N}} + \sum_{\ell=0}^{N_1-1} f_{2\ell+1} \cdot e^{\frac{2\pi i k \cdot (2\ell+1)}{N}} =$$

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The Fast Fourier Transform (FFT)

For both the 'small' and 'great' indices, both sums on the right-hand sides are discrete Fourier transforms with smaller vectors. The procedure can recursively be continued, if N is a power of two.

Trigonometric Fourier series in $L_2(0, 2\pi)$

Complex exponential function

The Discrete Fourier Transform

The Fast Fourier Transform

Image compression by DFT

The Fast Fourier Transform, recursive algorithm

With recursive invocations:

- $N_1 := N/2$

$$f^{even} := (f_0, f_2, ..., f_{2N_1-2}), \quad f^{odd} := (f_1, f_3, ..., f_{2N_1-1})$$

- $\hat{f}^{even} := F_{N_1} f^{even}, \quad \hat{f}^{odd} := F_{N_1} f^{odd}$
-

$$(F_N f)_k \quad := \hat{f}_k^{even} + e^{\frac{2\pi i k}{N}} \cdot \hat{f}_k^{odd} \qquad (k = 0, 1, ..., N_1 - 1)$$

$$(F_N f)_{N_1+k} := \hat{f}_k^{even} - e^{\frac{2\pi i k}{N}} \cdot \hat{f}_k^{odd} \qquad (k = 0, 1, ..., N_1 - 1)$$

where for $N = 1$, $F_1 f := f$ (f has one component only)

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The Fast Fourier Transform, numerical features

Total computational cost: $\mathcal{O}(N \cdot \log N)$, which is much smaller than the computational cost of the original DFT.

Example:

$$f_0 \quad f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad f_7$$

$$f_0 \quad f_2 \quad f_4 \quad f_6 \;|\; f_1 \quad f_3 \quad f_5 \quad f_7$$

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The 2D Discrete Fourier Transform

The DFT of a matrix $f \in \mathbf{M}_{N\times N}$ is the matrix $\hat{f} \in \mathbf{M}_{N\times N}$ with the following entries:

$$\hat{f}_{k,j} := \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} f_{r,s} \cdot e^{\frac{2\pi ikr}{N}} e^{\frac{2\pi ijs}{N}}$$

The algorithm of the computation of the DFT of a matrix:

- For every row of the matrix f, substitute the 1D DFT of the corresponding row.
- For every column of this matrix, substitute the 1D DFT of the corresponding column.

This results in the 2D DFT of the original matrix f.

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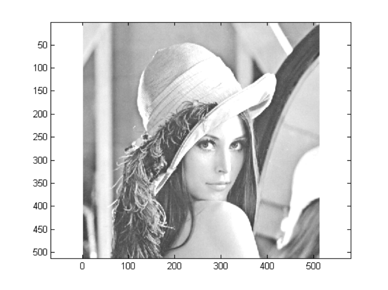
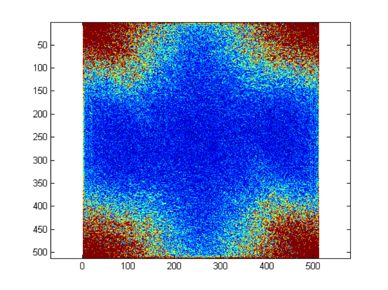
Image compression by DFT

Image compression by DFT

An N-by-N grayscale image can be considered an N-by-N matrix. In general, *there are a lot of Fourier coefficients that almost equal to* 0.

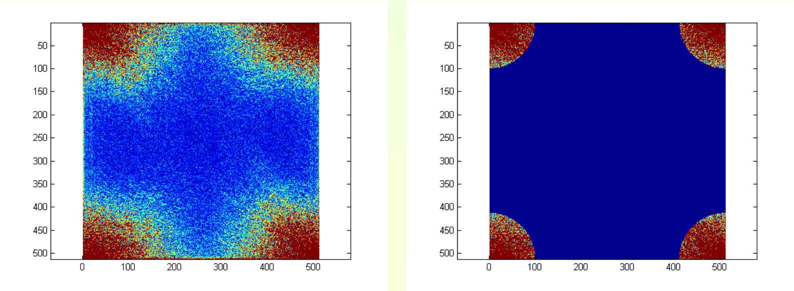




An 512-by-512 image and its Discrete Fourier Transform

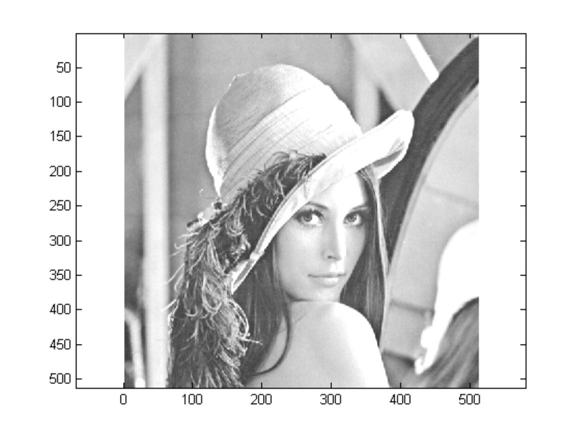
Image compression by DFT

The idea of the image compression: *to keep the Fourier coefficients which are closer to the corners than* r*, and to set the others to zero.* From the truncated DFT, the image can be approximately reconstructed.



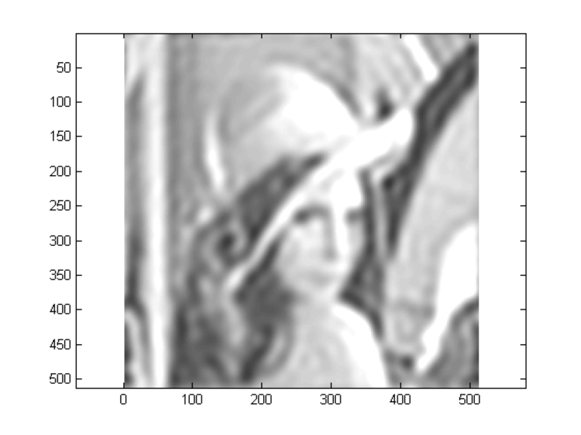
The original and truncated DFT of the image $(r = 100)$

Image compression by DFT



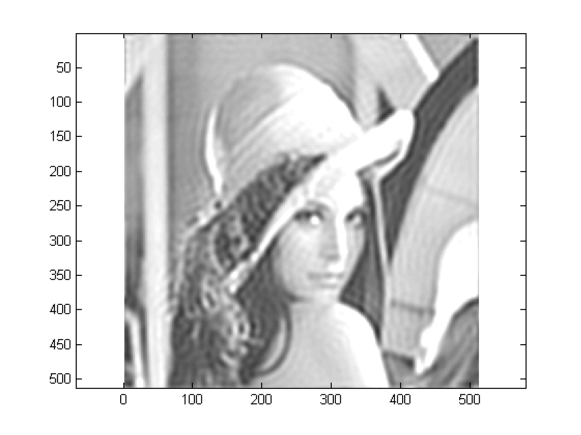
Original image

Image compression by DFT



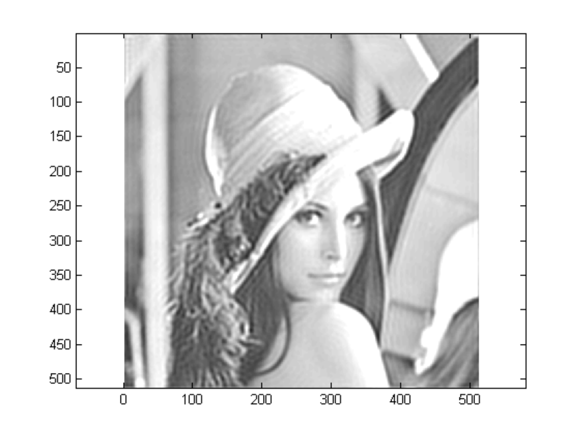
Reconstructed image, $r = 20$

Image compression by DFT



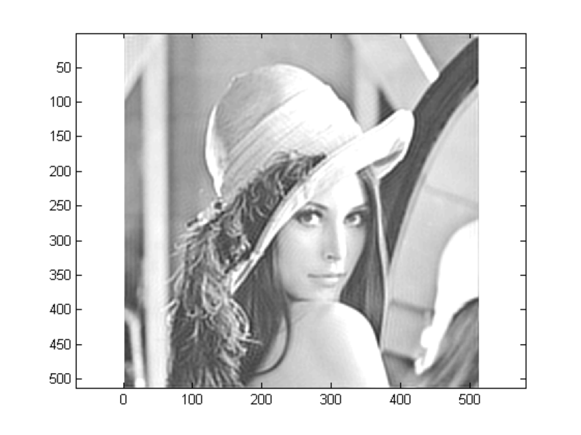
Reconstructed image, $r = 40$

Image compression by DFT



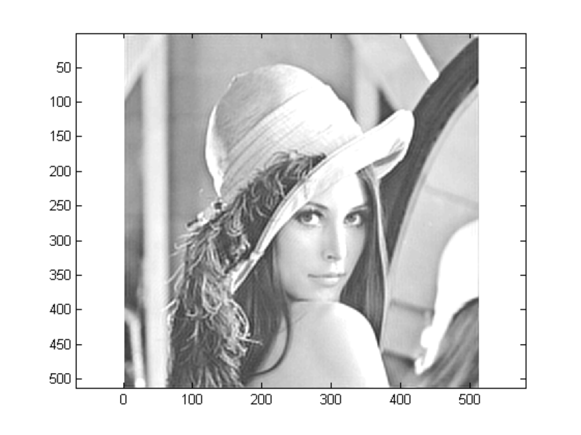
Reconstructed image, $r = 60$

Image compression by DFT



Reconstructed image, $r = 80$

Image compression by DFT



Reconstructed image, $r = 100$