Solutions of linear systems of equations in the Sense of least squares,

Let $A \in M_{m \times n}$ be a Matrix $b \in \mathbb{R}^m$. the Vector

$x^+ := A^+ b \in \mathbb{R}^n$ → The generalized Solutions of the system $Ax = b$

There is examples in lct 6

The Jacobi iteration

Let $A \in M_{n \times n}$ be a regular Matrix, Let $b \in \mathbb{R}^n$ be a vector and Consider the system of equations

$$Ax = b$$

(2) Quadratic reg: Details.

let $x_1, \dots, x_M$ be given ~~values~~ Numbers and let $y_1 \dots y_M$ be some associated Values ($M \geq 3$)
Find polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ such that it fits to the given data "as exactly as possible"

The exact fitting: $a_0 + a_1 x_1 + a_2 x_1^2 = y_1$

$$\vdots$$

$$a_0 + a_1 x_M + a_2 x_M^2 = y_M$$

There is no exact sol in general. It is worth defining the 'best fitting' by minimizing the norm of residual:

$$F(a) := \| Aa - y \|^2 \to \min!$$

$$F(a_0, a_1, a_2) = \| A\underline{a} - \underline{y} \|^2 = \sum_{K=1}^{M} (a_0 + a_1 x_K + a_2 x_K^2 - y_K)^2 \to \min!$$

$$\frac{\partial F}{\partial a_0} = \sum_{K=1}^{M} 2 (a_0 + a_1 x_K + a_2 x_K^2 - y_K) 1 = 0$$

$$\frac{\partial F}{\partial a_1} = \sum_{K=1}^{M} 2 (a_0 + a_1 x_K + a_2 x_K^2 - y_K) x_K = 0$$

$$\frac{\partial F}{\partial a_2} = \sum_{K=1}^{M} 2 (a_0 + a_1 x_K + a_2 x_K^2 - y_K) \cdot x_K^2 = 0$$

whence a_0, a_1 can be computed. the coefficients a_q

$$a_0 \left(\sum_{K=1}^{M} 1\right) + a_1 \left(\sum_{K=1}^{M} x_K\right) + a_2 \sum_{K=1}^{M} (x_K^2) = \sum_{K=1}^{M} y_K$$

$$a_0 \left(\sum_{K=1}^{M} x_K\right) + a_1 \left(\sum_{K=1}^{M} x_K^2\right) + a_2 \left(\sum_{K=1}^{M} x_K^3\right) = \sum_{K=1}^{M} y_K x_K$$

$$a_0 \left(\sum_{K=1}^{M} x_K^2\right) + a_1 \left(\sum_{K=1}^{M} x_K^3\right) + a_2 \left(\sum_{K=2}^{M} x_K^4\right) = \sum_{K=1}^{M} y_K x_K^2$$

[1] Linear regression: Details

let $x_1, x_2 \dots x_m$ be given numbers and let $y_1 \dots y_m$ be some associated values.

- Find a polynomial $p(x) = a_0 + a_1 x$ such that it fits to the given data 'as exactly as possible'

$\Rightarrow$ the exact fitting $\rightarrow a_0 + a_1 x_k = y_k$ for all $(k = 1, 2 \dots M)$

$$\begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_m \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix} \Rightarrow \text{No exact solution.}$$

So best fitting is by minimizing the norm of the residual

$$F(a) := \|Aa - y\|^2 \rightarrow \min$$

$$\Rightarrow F(a_0, a_1) := \sum_{k=1}^{M} (a_0 + a_1 x_k - y_k)^2 \rightarrow \min$$

At the minimum, the partial derivates vanish:

$$\frac{\partial F}{\partial a_0} = \sum_{k=1}^{M} 2(a_0 + a_1 x_k - y_k) \cdot 1 = 0$$

$$\frac{\partial F}{\partial a_1} = \sum_{k=1}^{M} 2 \cdot (a_0 + a_1 x_k - y_k) \cdot x_k = 0$$

$\Rightarrow a_0$ & a_1 can be computed by solving the system of equations:

$$\sum_{k=1}^{M} y_k = a_0 \sum_{k=1}^{M} 1 + a_1 \sum_{k=1}^{M} x_k$$

$$\sum_{k=1}^{M} x_k y_k = a_0 \sum_{k=1}^{M} x_k + a_1 \sum_{k=1}^{M} x_k^2$$

By direct calculation $\Rightarrow A^* A a = A^* y$