Numerical Analysis
Singular Value Decomposition

by Csaba Gáspár

Széchenyi István University

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SVD for square, regular matrices

Let $A \in \mathbf{M}_{N\times N}$ be a regular matrix. Denote by $\lambda_j > 0$ the eigenvalues of the (self-adjoint, positive definite) matrix A^*A. Let v_j be the corresponding orthonormal eigenvectors $(j = 1, 2, ..., N)$.

Singular values

The numbers $\sigma_k := \sqrt{\lambda_k}$ $(k = 1, ..., N)$ are called the **singular values** of A.

Define the obviously orthogonal matrix $V := \left(v_1 \middle| v_2 \middle| \ldots \middle| v_N \right)$.

Introduce the (orthonormal!) vectors $u_k := \frac{Av_k}{\sigma_k}$, and the matrices

$$U := \left(u_1 \middle| u_2 \middle| \ldots \middle| u_N \right), \qquad S := \begin{pmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \sigma_N \end{pmatrix}.$$

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- The SVD requires computing a complete eigensystem of the matrix A^*A, which is expensive.
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The numbers $\sigma_k := \sqrt{\lambda_k}$ $(k = 1, ..., r)$ are called the **singular values** of A, where $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq ...$ and r is the last index for which $\sigma_r > 0$

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Now define the M-by-N 'diagonal' matrix:

$$S := \begin{pmatrix} \sigma_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & \sigma_r & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

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SVD, example

Let $\mathbf{0} \neq \mathbf{A} := \mathbf{a} \in \mathbf{M_{M\times 1}}$ be a column vector. Let $u_1 := \frac{a}{||a||}$ and complete u_1 by the vectors $u_2, ..., u_M$ such that they form an orthonormal system. Denote by

$$U := \left(\begin{array}{c|c|c|c} u_1 & u_2 & ... & u_M \end{array} \right).$$

The SVD of the matrix A is as follows:

$$A = U \cdot \begin{pmatrix} ||a|| \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} \cdot (1)$$

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$$A = U \cdot \begin{pmatrix} \sqrt{2}||a|| & 0 \\ 0 & 0 \\ 0 & 0 \\ ... & ... \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

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The generalized inverse

Let $A \in \mathbf{M}_{M\times N}$ be a matrix with the SVD $A = USV^*$, where $U \in \mathbf{M}_{M\times M}$, $V \in \mathbf{M}_{N\times N}$ are orthogonal matrices and S is the quasidiagonal matrix formed by the positive singular values:

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Denote by $$S^+ := \begin{pmatrix} \sigma_1^{-1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^{-1} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & \sigma_r^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbf{M}_{N\times M}$$

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The matrix $\boxed{A^+ := VS^+U^*}$ is called the **generalized inverse** of A (Moore-Penrose inverse or pseudoinverse)

If the matrix $A \in \mathbf{M}_{N\times N}$ is regular, then $A^+ = A^{-1}$.

The matrices AA^+ and A^+A are self-adjoint. Moreover, $AA^+A = A$ and $A^+AA^+ = A^+$.
The pseudoinverse is uniquely determined by the above properties.

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Generalized solution of linear systems

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The Singular Value Decomposition

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Image compression by SVD

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Let $A \in \mathbf{M}_{M \times N}$ be a matrix, $b \in \mathbf{R}^M$. The vector $\boxed{x^+ := A^+ b \in \mathbf{R}^N}$ is called the **generalized solution** of the system $Ax = b$.

The generalized solution $x^+ = A^+ b$ always uniquely exists, and this is the solution in the sense of least squares, i.e. it minimizes the functional $||Ax - b||^2$. If several minimizing vectors exist, then x^+ is the one that has the minimal Euclidean norm.

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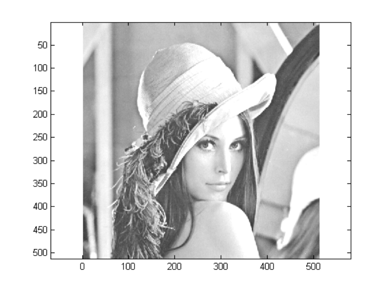
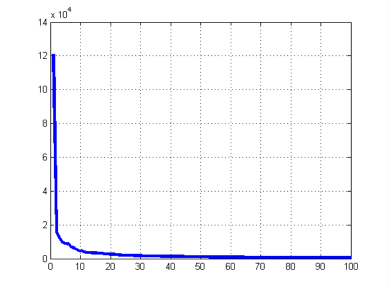
Image compression by SVD

An N-by-N grayscale image can be considered an N-by-N matrix. In general, *there are a lot of singular values that almost equal to* 0.





An 512-by-512 image and the first 100 singular values

Image compression by SVD

The idea of the image compression: *to keep the first m singular values and to set the others to zero*.

$$\left(\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right) \cdot \left(\begin{array}{c|c} S_{11} & 0 \\ \hline 0 & S_{22} \end{array}\right) \cdot \left(\begin{array}{c|c} V_{11}^* & V_{12}^* \\ \hline V_{21}^* & V_{22}^* \end{array}\right) \approx$$

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To calculate this matrix product, it is sufficient to keep *the first m columns from U and the first m rows from V^**.

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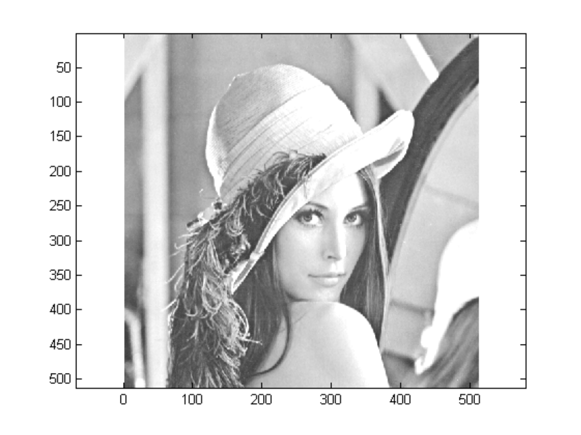
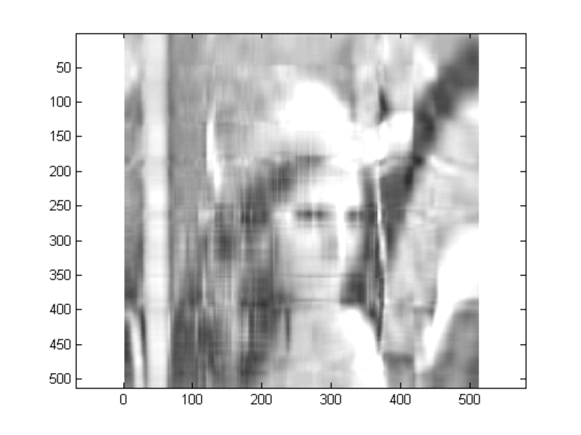
To calculate this matrix product, it is sufficient to keep *the first m columns from U and the first m rows from V^*.*

Image compression by SVD



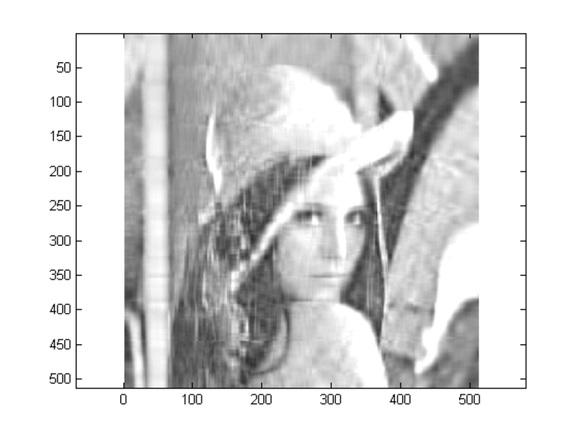
Original image

Image compression by SVD



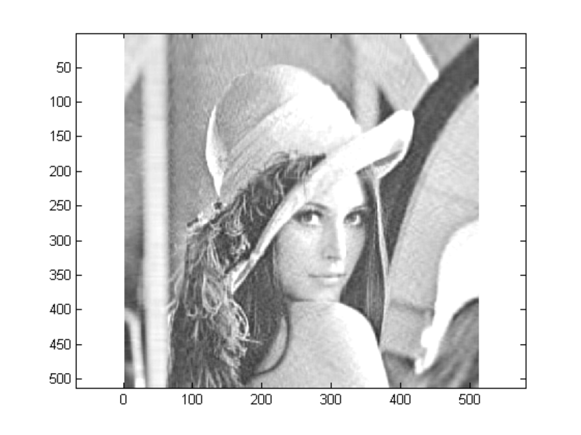
Reconstructed image, $m = 10$

Image compression by SVD



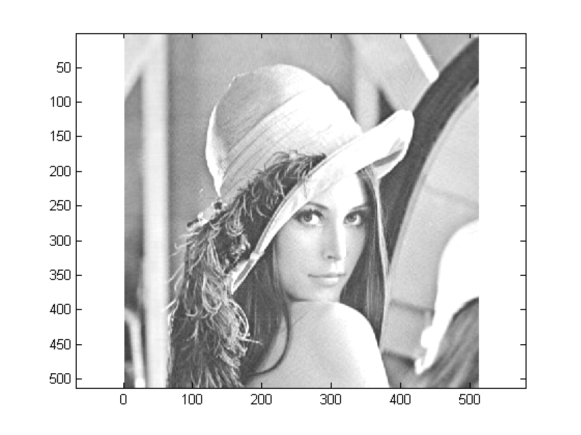
Reconstructed image, $m = 20$

Image compression by SVD



Reconstructed image, $m = 50$

Image compression by SVD



Reconstructed image, $m = 100$