Numerical Analysis
Scattered data interpolation problems

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The problem of scattered data interpolation

Let $x_1, x_2, ..., x_N \in \mathbf{R}^2$ be given locations on the place (**interpolation points**), and let $f_1, f_2, ..., f_N \in \mathbf{R}$ be some predefined values associated to the interpolation points.

The basic problem of interpolation

Find a function $f : \mathbf{R}^2 \to \mathbf{R}$ (*as smooth as possible*) **interpolation function**, which satisfies the **interpolation conditions**:

$$f(x_k) = f_k \qquad (k = 1, 2, ..., N)$$

No special structure (grid of mesh) of the interpolation points is assumed.

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The Shepard method

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Shepard's method

If x is not an interpolation points, then:

$$f(x) := \frac{\sum_{j=1}^{N} f_j \cdot \frac{1}{||x - x_j||^p}}{\sum_{j=1}^{N} \frac{1}{||x - x_j||^p}} \qquad (p > 0; \quad \text{usually } p = 2 \quad \text{or } p = 4)$$

For arbitrary interpolation point x_k: $f(x) \to f_k$ whenever $x \to x_k$; moreover, the partial derivatives tend to zero: $\partial_1 f(x), \partial_2 f(x) \to 0$.

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The Shepard method, numerical features

- Low accuracy
- The computational cost is moderate: at each point of evaluation, $\mathcal{O}(N)$ arithmetic operations are necessary
- Numerically stable
- If $||x|| \to +\infty$, then $f(x) \to \frac{1}{N} \cdot \sum_{j=1}^{N} f_j$

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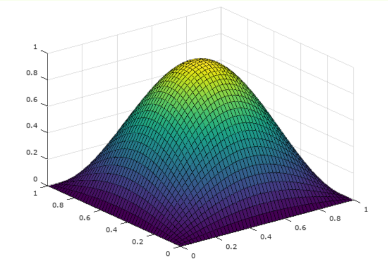
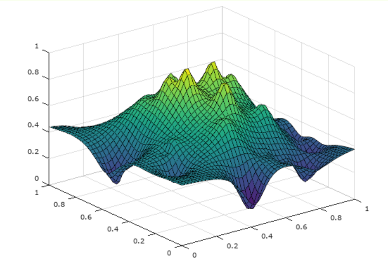
The Shepard method, example

Test function:

$$f(x,y) := \sin \pi x \cdot \sin \pi y$$

defined on the unit square $\Omega := \{(x,y) \in \mathbf{R}^2 : \; 0 \le x, y \le 1\}$.
30 interpolation points has been defined in the unit square in a random way.





Test function and Shepard interpolant (with $p = 2$).

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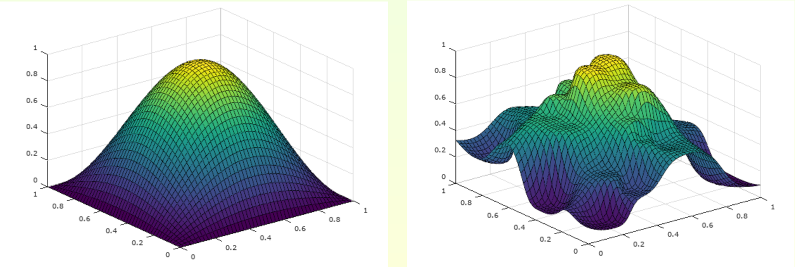

Test function and Shepard interpolant (with $p = 4$).

The method of radial basis functions

Let $x_1, x_2, ..., x_N \in \mathbf{R}^2$ be given **interpolation points** on the plane, and let $f_1, f_2, ..., f_N \in \mathbf{R}$ be given values associated to the interpolation points. Let Φ be a predefined radial (i.e. circularly symmetric) function, i.e. suppose that $\Phi(x)$ depends only on $||x||$.

The method of radial basis functions

Seek the interpolation function is the following form:

$$f(x) := \sum_{j=1}^{N} \alpha_j \cdot \Phi(x - x_j)$$

The a priori unknown coefficients α_j can be computed by solving the system of interpolation equations:

$$\sum_{j=1}^{N} \alpha_j \cdot \Phi(x_k - x_j) = f_k \qquad (k = 1, 2, ..., N)$$

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Some usual choices of Φ:

- $\Phi(x) := \sqrt{||x||^2 + c^2}$ (Method of multiquadrics, MQ)
- $\Phi(x) := \frac{1}{\sqrt{||x||^2+c^2}}$ (Inverse multiquadrics, iMQ)
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- Very good accuracy
- At each point of evaluation, the number of necessary arithmetic operations is $\mathcal{O}(N)$, but the computational cost of the calculation of the coefficients is $\mathcal{O}(N^3)$
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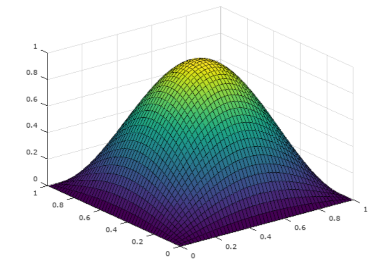

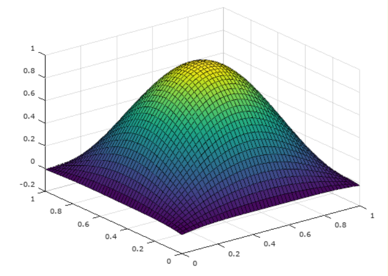

Test function and MQ-interpolant (with parameter $c = 1$).