

Numerical Analysis

by Csab Gáspár

Matrix decom positions

decomposition
The Cholesky decomposition
The QR

Numerical Analysis Matrix decompositions

by Csaba Gáspár

Széchenyi István University

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LU decomposition by Gaussian elimination

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Matrix decom positions

The LU decomposition
The Cholesky decomposition
The QR decomposition

Let $A = [a_{kj}] \in \mathbf{M}_{N \times N}$ be a regular matrix, for which the Gaussian elimination can be performed without swapping rows.

$LU\ { m decomposition}$

The matrix A can be uniquely decomposed in the following form:

$$A = LU$$

where L is a normed lower triangular matrix (i.e. $L_{kk} = 1$, and $L_{kj} = 0$, if j > k), and U is an upper triangular matrix (i.e. $U_{kj} = 0$, whenever j < k).

If a linear system Ax=b should be solved with a lot of different right-hand sides b, then it is much cheaper to solve the pairs of equations instead: Ly=b, Ux=y, since only forward and backwar substitutions have to be performed.



LU decomposition by Gaussian elimination

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Matrix decompositions

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Matrix decompositions

- The product of (normed) lower triangular matrix is (normed) lower triangular
- The product of (normed) upper triangular matrix is (normed) upper triangular
- The inverse of a (normed) lower triangular matrix is (normed) lower triangular (if exists)
- The inverse of a (normed) upper triangular matrix is (normed) upper triangular (if exists)



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LU decomposition by Gaussian elimination

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Matrix decompositions
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The QR decomposition

Performing the decomposition by Gaussian elimination: At the elimination by the kth row, subtract the kth row multiplied by $l_{m,k} := a_{m,k}/a_{k,k}$ from the mth row (m = k + 1, ..., N).

From these numbers $l_{m,k}$, the matrix L can be assembled:

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{2,1} & 1 & 0 & \dots & 0 \\ l_{3,1} & l_{3,2} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ l_{N,1} & l_{N,2} & l_{N,3} & \dots & 1 \end{pmatrix}$$

After the elimination steps, from the original matrix A, we arrive at the matrix U.



LU decomposition by Gaussian elimination

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After the elimination steps, from the original matrix A, we arrive at the matrix U.



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Matrix decompositions

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Determine the LU decomposition of the matrix

$$A = \left(\begin{array}{ccc} 2 & -6 & 10\\ 2 & -5 & 3\\ 3 & -2 & 1 \end{array}\right)$$



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Matrix decon

The LU

decomposition The Cholesky decomposition

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$$\begin{pmatrix}
2 & -6 & 10 \\
2 & -5 & 3 \\
3 & -2 & 1
\end{pmatrix}$$

$$\begin{pmatrix}
2 & -6 & 10 \\
0 & 1 & -7 \\
3 & -2 & 1
\end{pmatrix}$$

$$\begin{pmatrix}
2 & -6 & 10 \\
0 & 1 & -7 \\
0 & 7 & -14
\end{pmatrix}$$

$$\left(\begin{array}{ccc}
2 & -6 & 10 \\
0 & 1 & -7 \\
0 & 0 & 35
\end{array}\right) = U$$

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
. & 1 & 0 \\
. & . & 1
\end{array}\right)$$

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
& & 1
\end{array}\right)$$

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
\frac{3}{2} & . & 1
\end{array}\right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & 7 & 1 \end{array}\right) = L$$



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Matrix decon positions

The *LU*decomposition
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 $\begin{pmatrix}
2 & -6 & 10 \\
2 & -5 & 3 \\
3 & -2 & 1
\end{pmatrix}$

$$\begin{pmatrix} 2 & -6 & 10 \\ 0 & 1 & -7 \\ 3 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix}
2 & -6 & 10 \\
0 & 1 & -7 \\
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\end{pmatrix}$$

$$\left(\begin{array}{ccc} 2 & -6 & 10\\ 0 & 1 & -7\\ 0 & 0 & 35 \end{array}\right) = U$$

$$\left(\begin{array}{cccc}
1 & 0 & 0 \\
. & 1 & 0 \\
. & . & 1
\end{array}\right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \cdot & \cdot & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
\frac{3}{2} & . & 1
\end{array}\right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & 7 & 1 \end{array}\right) = L$$



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Matrix decompositions

The *LU* decomposition
The Cholesky decomposition

 $\begin{pmatrix}
2 & -6 & 10 \\
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3 & -2 & 1
\end{pmatrix}$

$$\begin{pmatrix} 2 & -6 & 10 \\ 0 & 1 & -7 \\ 2 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -6 & 10 \\ 0 & 1 & -7 \\ 0 & 7 & -14 \end{pmatrix}$$

$$\left(\begin{array}{ccc} 2 & -6 & 10\\ 0 & 1 & -7\\ 0 & 0 & 35 \end{array}\right) = U$$

 $\left(\begin{array}{ccc}
1 & 0 & 0 \\
. & 1 & 0 \\
. & . & 1
\end{array}\right)$

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
. & . & 1
\end{array}\right)$$

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
\frac{3}{2} & . & 1
\end{array}\right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & 7 & 1 \end{array}\right) = L$$



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Matrix decon positions

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2 & -6 & 10 \\
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\end{pmatrix}$$

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2 & -6 & 10 \\
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\end{pmatrix}$$

$$\left(\begin{array}{ccc} 2 & -6 & 10\\ 0 & 1 & -7\\ 0 & 0 & 35 \end{array}\right) = U$$

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1 & 0 & 0 \\
. & 1 & 0 \\
. & . & 1
\end{array}\right)$$

$$\left(\begin{array}{cccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ & & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
\frac{3}{2} & . & 1
\end{array}\right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & 7 & 1 \end{array}\right) = L$$



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Matrix decom positions ___

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In the previous example, first, write down the entries of ${\cal L}$ and ${\cal U}$ which are a priori known:

$$U = \begin{pmatrix} 2 & -6 & 10 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -6 & 10 \\ 2 & 5 & 3 \end{pmatrix} - 1$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & \ell_{32} & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & -6 & 10 \\ 2 & -5 & 3 \\ 3 & -2 & 1 \end{pmatrix} = A$$



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Matrix decom positions

The LU decomposition The Cholesky decomposition The QR

Multiplying the 2nd row of L by the 2nd column of U, u_{22} can be computed: $u_{22}=1$.

$$U = \left(\begin{array}{ccc} 2 & -6 & 10\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33} \end{array}\right)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & \ell_{32} & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & -6 & 10 \\ 2 & -5 & 3 \\ 3 & -2 & 1 \end{pmatrix} = A$$



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Multiplying the 2nd row of L by the 2nd column of U, u_{22} can be computed: $u_{22}=1$.

$$U = \left(\begin{array}{ccc} 2 & -6 & 10\\ 0 & 1 & u_{23}\\ 0 & 0 & u_{33} \end{array}\right)$$

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Multiplying the 3rd row of L by the 2nd column of U, ℓ_{32} can be computed: $\ell_{32}=7$.

$$U = \left(\begin{array}{ccc} 2 & -6 & 10\\ 0 & 1 & u_{23}\\ 0 & 0 & u_{33} \end{array}\right)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & \ell_{32} & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & -6 & 10 \\ 2 & -5 & 3 \\ 3 & -2 & 1 \end{pmatrix} = A$$



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Multiplying the 3rd row of L by the 2nd column of U, ℓ_{32} can be computed: $\ell_{32}=7$.

$$U = \left(\begin{array}{ccc} 2 & -6 & 10\\ 0 & 1 & u_{23}\\ 0 & 0 & u_{33} \end{array}\right)$$

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positions

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Multiplying the 2nd row of L by the 3rd column of U, u_{23} can be computed: $u_{23}=-7$.

$$U = \begin{pmatrix} 2 & -6 & 10 \\ 0 & 1 & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{5} & 7 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & -6 & 10 \\ 2 & -5 & 3 \\ 3 & -2 & 1 \end{pmatrix} = A$$



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positions

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Multiplying the 2nd row of L by the 3rd column of U, u_{23} can be computed: $u_{23}=-7$.

$$U = \begin{pmatrix} 2 & -6 & 10 \\ 0 & 1 & -7 \\ 0 & 0 & u_{33} \end{pmatrix}$$

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The QR

Finally, multiplying the 3rd row of L by the 3rd column of U, u_{33} can be computed: $u_{33}=35$.

$$U = \begin{pmatrix} 2 & -6 & 10 \\ 0 & 1 & -7 \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{5} & 7 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & -6 & 10 \\ 2 & -5 & 3 \\ 3 & -2 & 1 \end{pmatrix} = A$$



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The LUdecomposition

Finally, multiplying the 3rd row of L by the 3rd column of U, u_{33} can be computed: $u_{33} = 35$.

$$U = \left(\begin{array}{ccc} 2 & -6 & 10\\ 0 & 1 & -7\\ 0 & 0 & 35 \end{array}\right)$$

$$L = \left(\begin{array}{rrr} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & 7 & 1 \end{array}\right)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & 7 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & -6 & 10 \\ 2 & -5 & 3 \\ 3 & -2 & 1 \end{pmatrix} = A$$



LU decomposition, numerical features

Numerical Analysis

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Matrix decom positions The LU

 $\begin{array}{c} {\rm decomposition} \\ {\rm The~Cholesky} \\ {\rm decomposition} \\ {\rm The~} QR \\ \\ \end{array}$

Computational cost

The necessary number of arithmetic operations of the LU decomposition of an $N \times N$ matrix is $\mathcal{O}(N^3)$.

Calculation of the determinant

Once the LU decomposition A = LU has been performed, the determinant of A can be calculated as:

$$\det(A) = \det(L) \cdot \det(U) = u_{11} \cdot u_{22} \cdot \dots \cdot u_{NN}$$



LU decomposition, numerical features

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Matrix decom

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positions

The LU

The Cholesky decomposition The QR

Let $A \in \mathbf{M}_{N \times N}$ be a **self-adjoint, positive definite** matrix (which assures that the Gaussian elimination can be performed without swapping rows). Consider the LU decomposition of A:

$$A = LU$$

Denote by D the diagonal part of U, then U=DU', where U' is a normed upper trianglular matrix. Since A is self-adjoint, the uniqueness of the LU decomposition implies that

$$U' = L^*$$

therefore

$$A = LDL^*$$



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Matrix decompositions

The LUdecomposition

The LU decomposition
The Cholesky decomposition

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Matrix decompositions

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LDL^* decomposition

Every self-adjoint, positive definite matrix \boldsymbol{A} can be uniquely decomposed in the following form:

$$A = LDL^*$$

where L is a normed lower triangular matrix and D is a diagonal matrix, the diagonal entries of which are positive.



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Matrix dec

The LU decompositi

The Cholesky decomposition

Let $A \in \mathbf{M}_{N \times N}$ be a self-adjoint, positive definite matrix with the LDL^* decomposition:

$$A = L_0 D L_0^*$$

Ther

$$A = L_0 \sqrt{D} \sqrt{D} L_0^* = (L_0 \sqrt{D}) \cdot (L_0 \sqrt{D})^* =: LL^*$$

Cholesky decomposition

Every self-adjoint, positive definite matrix A can be uniquely decomposed in the following form:

$$A = LL^*$$

where L is a not necessarily normed lower triangular matrix, with positive diagonal elements.



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Let $A \in \mathbf{M}_{N \times N}$ be a self-adjoint, positive definite matrix with the LDL^* decomposition:

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Then

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Matrix decor

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The QRdecomposition

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Cholesky decomposition

Every self-adjoint, positive definite matrix A can be uniquely decomposed in the following form:

$$A = LL^*$$

where ${\cal L}$ is a not necessarily normed lower triangular matrix, with positive diagonal elements.



The Cholesky decomposition, numerical features

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Computational cost

The number of necessary arithmetic operations of the Cholesky decomposition of an $N\times N$ self-adjoint, positive definite matrix is $\mathcal{O}(N^3)$.

The Cholesky decomposition is less sensitive to the roundoff errors than the LU decomposition.

If a linear system Ax = b should be solved with a lot of different right-hand sides b, then it is cheaper to solve the pairs of equations instead: Ly = b, $L^*x = y$, since only forward and backward substitutions have to be performed.



The Cholesky decomposition, numerical features

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The Cholesky decomposition, numerical features

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The Cholesky decomposition

Computational cost

The number of necessary arithmetic operations of the Cholesky decomposition of an $N\times N$ self-adjoint, positive definite matrix is $\mathcal{O}(N^3)$.

The Cholesky decomposition is less sensitive to the roundoff errors than the ${\cal L}{\cal U}$ decomposition.

If a linear system Ax=b should be solved with a lot of different right-hand sides b, then it is cheaper to solve the pairs of equations instead: Ly=b, $L^*x=y$, since only forward and backward substitutions have to be performed.



Numerical Analysis

The Cholesky decomposition

$$A := \left(\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{4}{3}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{2}{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$



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Matrix decompositions

The LU

The Cholesky decomposition

 $A := \left(\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right)$

The LU decomposition (please check):

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

Thus, the Cholesky decomposition: $A = L \cdot L^*$, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{4}{3}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{2}{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$



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Matrix decor positions

The LU decomposition

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The LU decomposition (please check):

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

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The LU decomposition (please check):

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

Thus, the Cholesky decomposition: $A = L \cdot L^*$, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{4}{3}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{2}{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$



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positions

decomposition The Cholesky decomposition In the previous example, first, write down the entries of ${\cal L}$ which are a priori known:

$$L^* = \begin{pmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ 0 & \ell_{22} & \ell_{32} \\ 0 & 0 & \ell_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



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Matrix decor positions

The LU decompositio

The Cholesky decomposition The QR

Multiplying the first row of L by the first column of L^* , ℓ_{11} can be computed: $\ell_{11} = \sqrt{2}$.

$$L^* = \left(\begin{array}{ccc} \ell_{11} & \ell_{21} & \ell_{31} \\ 0 & \ell_{22} & \ell_{32} \\ 0 & 0 & \ell_{33} \end{array}\right)$$

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The Cholesky decomposition

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Numerical Analysis

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Multiplying the 2nd and 3rd rows of L by the first column of L^* , ℓ_{21} , ℓ_{31} can be computed: $\ell_{21} = \sqrt{\frac{1}{2}}$, $\ell_{31} = 0$.

$$L^* = \begin{pmatrix} \sqrt{2} & \ell_{21} & \ell_{31} \\ 0 & \ell_{22} & \ell_{32} \\ 0 & 0 & \ell_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



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Matrix decompositions

The Cholesky decomposition

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$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{\frac{1}{2}} & \ell_{22} & 0 \\ 0 & \ell_{22} & \ell_{22} \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



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Multiplying the 2nd row of L by the 2nd column of L^* , ℓ_{22} can be computed: $\ell_{22} = \sqrt{\frac{3}{2}}$.

$$L^* = \left(\begin{array}{ccc} \sqrt{2} & \sqrt{\frac{1}{2}} & 0\\ 0 & \ell_{22} & \ell_{32}\\ 0 & 0 & \ell_{33} \end{array}\right)$$

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{\frac{1}{2}} & \ell_{22} & 0 \\ 0 & \ell_{32} & \ell_{33} \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



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Multiplying the 2nd row of L by the 2nd column of L^* , ℓ_{22} can be computed: $\ell_{22} = \sqrt{\frac{3}{2}}$.

$$L^* = \begin{pmatrix} \sqrt{2} & \sqrt{\frac{1}{2}} & 0\\ 0 & \sqrt{\frac{3}{2}} & \ell_{32}\\ 0 & 0 & \ell_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & \ell_{22} & \ell_{23} \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



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Multiplying the 3rd row of L by the 2nd column of L^* , ℓ_{32} can be computed: $\ell_{32} = \sqrt{\frac{2}{3}}$.

$$L^* = \begin{pmatrix} \sqrt{2} & \sqrt{\frac{1}{2}} & 0\\ 0 & \sqrt{\frac{3}{2}} & \ell_{32}\\ 0 & 0 & \ell_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & \ell_{22} & \ell_{23} \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



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Multiplying the 3rd row of L by the 2nd column of L^* , ℓ_{32} can be computed: $\ell_{32} = \sqrt{\frac{2}{3}}$.

$$L^* = \begin{pmatrix} \sqrt{2} & \sqrt{\frac{1}{2}} & 0\\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{2}{3}}\\ 0 & 0 & \ell_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{2}{2}} & \ell_{33} \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



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Finally, multiplying the 3rd row of L by the 3rd column of L^* , ℓ_{33} can be computed: $\ell_{33}=\sqrt{\frac{4}{3}}$.

$$L^* = \begin{pmatrix} \sqrt{2} & \sqrt{\frac{1}{2}} & 0\\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{2}{3}}\\ 0 & 0 & \ell_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{2}{3}} & \ell_{33} \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



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Finally, multiplying the 3rd row of L by the 3rd column of L^* , ℓ_{33} can be computed: $\ell_{33} = \sqrt{\frac{4}{3}}$.

$$L^* = \begin{pmatrix} \sqrt{2} & \sqrt{\frac{1}{2}} & 0\\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{2}{3}}\\ 0 & 0 & \sqrt{\frac{4}{3}} \end{pmatrix}$$

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{2}{2}} & \sqrt{\frac{4}{2}} \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$$



The Gram-Schmidt orthogonalization

Numerical Analysis

decomposition

Let $a_1, a_2, ..., a_N \in \mathbf{R}^N$ be linearly independent vectors. Define $ilde{e}_1:=a_1$, $e_1:=rac{ ilde{e}_1}{|| ilde{e}_1||}$, and for k=2,3,...,N:

$$\tilde{e}_k := a_k - \sum_{j=1}^{k-1} \langle a_k, e_j \rangle e_j, \qquad e_k := \frac{\tilde{e}_k}{||\tilde{e}_k||}$$



The Gram-Schmidt orthogonalization

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The Cholesky decomposition

The QRdecomposition Let $a_1, a_2, ..., a_N \in \mathbf{R}^N$ be linearly independent vectors. Define $\tilde{e}_1 := a_1$, $e_1 := \frac{\tilde{e}_1}{||\tilde{e}_1||}$, and for k = 2, 3, ..., N:

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Gram-Schmidt orthogonalization

The vectors $e_1,e_2,...,e_N$ form an **orthonormal vector system** (i.e. $\langle e_k,e_j\rangle=0$, if $k\neq j$, and $||e_k||=1$), and the vector systems $a_1,a_2,...,a_k$ and $e_1,e_2,...,e_k$ generate the same subspace of \mathbf{R}^N for all k=1,2,...,N.



Orthogonal matrices

Numerical Analysis

The matrix $A \in \mathbf{M}_{N \times N}$ is called **orthogonal**, if its column vectors form an orthonormal system in \mathbf{R}^N .

decomposition

The QR

Example: The 2-by-2 matrix
$$A:=\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$
 is an orthogonal matrix.



Orthogonal matrices

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Orthogonal matrices

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Example: The 2-by-2 matrix
$$A:=\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$
 is an orthogonal matrix.

The inverse of orthogonal matrices can be computed in an extremely simple way:

If the matrix $A \in \mathbf{M}_{N \times N}$ is orthogonal, then $A^{-1} = A^*$



The QR decomposition

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The LU decomposition

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The OR

decomposition

Every regular matrix $A \in \mathbf{M}_{N \times N}$ can be uniquely decomposed in the form A = QR, where Q is an orthogonal matrix, R is an upper triangular matrix with positive diagonal entries.

Denote by $a_1,a_2,...,a_N$ its column vectors. Consider the orthonormal basis $e_1,e_2,...,e_N$ obtained from $a_1,a_2,...,a_N$ by a Gram-Schmidt orthogonalization. Then

$$Q = \left(\begin{array}{c|ccc} e_1 & e_2 & \dots & e_N \end{array} \right), \qquad R = \left(\begin{array}{cccc} r_{11} & r_{12} & r_{13} & \dots \\ 0 & r_{22} & r_{23} & \dots \\ 0 & 0 & r_{33} & \dots \\ \dots & \dots & \dots & \dots \end{array} \right)$$

where $r_{kj} := \langle e_k, a_j \rangle$

After performing the QR decomposition, the solution of the linear system Ax = b is extremely simple with low computational cost:

$$Rx = Q^{-1}b = Q^*b$$



The QR decomposition

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The QRdecomposition

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The QR decomposition

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Gaspar

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The QR decomposition

Compute the QR decomposition of the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$.

First,
$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
, $a_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$.

Next, perform a Gram-Schmidt-ortogonalization for these vectors:

$$\tilde{e}_1 = a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\tilde{e}_2 = a_2 - \langle a_2, e_1 \rangle \cdot e_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\tilde{e}_3 = a_3 - \langle a_3, e_1 \rangle \cdot e_1 - \langle a_3, e_2 \rangle \cdot e_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$



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decomposition

The QR decomposition

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latrix decom

 $\begin{array}{c} \textbf{positions} \\ \textbf{The } LU \\ \textbf{decomposition} \end{array}$

The Cholesky decomposition

The $\overline{Q}R$ decomposition

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Numerical Analysis

The QRdecomposition

Thus we have obtained:
$$Q = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$r_{11} = ||\tilde{e}_1|| = \sqrt{2},$$
 $r_{12} = \langle e_1, a_2 \rangle = 0,$ $r_{13} = \langle e_1, a_3 \rangle = 2\sqrt{2}$
 $r_{22} = ||\tilde{e}_2|| = 2,$ $r_{23} = \langle e_2, a_3 \rangle = 0$

$$r_{33} = ||\tilde{e}_3|| = \sqrt{|\tilde{e}_3|}$$

$$R = \left(\begin{array}{ccc} \sqrt{2} & 0 & 2\sqrt{2} \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{2} \end{array}\right)$$



Numerical Analysis

The QRdecomposition

Thus we have obtained:
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Moreover:

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Numerical Analysis

The QRdecomposition

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