SZE, Numerical Linear Algebra, exam, 20.12.2022.

The solutions were evaluated as follows:

1 – 5. (description of algorithms)	3 points each
Problem 6 (LU deonposition)	5 points
Problem 7 (an application of Gershgorin's theorem)	4 points
Problem 8 (convergence of a Jacobi iteration)	6 points

The maximal number of points is 30. The connection between the number of points and the marks:

below 8 points:	fail (1)
8 - 13 points:	pass (2)
14 - 19 points:	satisfactory (3)
20 - 25 points:	good (4)
26 points and above:	excellent (5)

Some remarks:

- The algorithm descriptions should have been written in such a way that somebody – who knowns nothing about the algorithm – will be able to understand the method, moreover, will be able to reproduce it. In particular:
  - The Cholesky decomposition works for a **self-adjoint**, **positive definite** matrix A; if A has an LU-decomposition of the form $A = LU$ (where L is a normed, lower triangular matrix, U is an upper triangular matrix), then its Cholesky decomposition is as follows: $A = L_0 L_0^*$, where $L_0 \coloneqq L\sqrt{D}$ is a (not necessarily normed) lower triangular matrix, and D denotes the diagonal part of U. Here one should have briefly defined the square root of a diagonal matrix.
  - The fixed point iteration to solve the linear system of equations $x = Bx + f$ has the following recursive definition: $x^{(n+1)} \coloneqq Bx^{(n)} + f$ (starting from an arbitrary vector $x^{(0)}$). If $||B|| < 1$ with respect to an arbitrary matrix norm induced by a vector norm, the iteration is convergent with respect to the corresponding vector norm i.e. $x^{(n)} \to x^*$ (where x^* is (the unique) solution of the original problem $x = Bx + f$). Moreover, the convergence is achieved if the **spectral radius** of B (to be defined!) is less than 1.
  - The Richardson's method for the linear system of equations $Ax = b$ (where A is a **self-adjoint**, **positive definite** matrix) has the form $x^{(n+1)} \coloneqq (I - \omega A)x^{(n)} + \omega b$. The optimal value of the parameter is $\omega = \frac{2}{\lambda_{min} + \lambda_{max}}$, where λ_{min} (resp. λ_{max}) denotes the minimal (resp. maximal) eigenvalue of A.
  - The power iteration approximates the **dominant eigenvalue** (what does it mean?) of a **normal matrix** A (what does it mean?) as a limit of **Rayleigh quotients** (definition?). To compute them, one should consider the vector sequence $x^{(n+1)} \coloneqq Ax^{(n)}$; the starting vector $x^{(0)}$ has to satisfy a non-orthogonality condition (formulation?)
  - The Singular Value Decomposition ($A = USV^*$) algorithm starts with the calculation of the **eigensystem of the self-adjoint matrix** A^*A. The (positive) square roots of the eigenvalues are the singular values; the matrix V is constructed from the corresponding normed eigenvectors (how?). Then the matrix U is constructed from the vectors $u_k = \frac{1}{\sigma_k} \cdot Av_k$ (how?) Without outlining these items, nobody will be able to understand (and reproduce) the algorithm.

- Problem 7 (definiteness of a matrix): it is sufficient to realize that the Gershgorin intervals consist of negative numbers, so that all the eigenvalues are negative. Consequently, the matrix is negative definite.
- In Problem 8, the essential steps are:
    - to realize that the original matrix is **not** diagonally dominant, therefore the calculation of the transition matrix is needed;
    - to realize that both the row and the column norm of the transition matrix are equal to 1;
    - therefore one has to calculate the eigenvalues of the transition matrix;
    - since the absolute values of the eigenvalues of the transition matrix are less than 1, therefore the Jacobi iteration is convergent.

Results:

Neptun	Points	Mark
C63XG3	23	4
D4QPSX	10	2
H20R0Z	5	1

21.12.2022.

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