37005 Fundamental of Derivative Security Pricing - Spring 2024
Group Assignment Part I

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Task 3

$\text{Payoffat}T : \max\left(S(0)e^{\mathrm{gT}}, S(0)^{\alpha}S(T)^{1-\alpha}\right)$
$= S(0)e^{\mathrm{gT}}\mathbb{I}_{\{S(0)e^{\mathrm{gT}}>S(0)^{\alpha}S(T)^{1-\alpha}\}} + S(0)^{\alpha}S(T)^{1-\alpha}\mathbb{I}_{\{S(0)e^{\mathrm{gT}}<S(0)^{\alpha}S(T)^{1-\alpha}\}}$
$= S(0)e^{\mathrm{gT}}\mathbb{I}_{\left\{S(0)e^{\frac{\mathrm{gT}}{1-\alpha}}>S(T)\right\}} + S(0)^{\alpha}S(T)^{1-\alpha}\mathbb{I}_{\left\{S(0)e^{\frac{\mathrm{gT}}{1-\alpha}}<S(T)\right\}}$
$V = B(0,T)\mathbb{E}_{\beta}\left[S(0)e^{\mathrm{gT}}\mathbb{I}_{\left\{S(0)e^{\frac{\mathrm{gT}}{1-\alpha}}>S(T)\right\}} + S(0)^{\alpha}S(T)^{1-\alpha}\mathbb{I}_{\left\{S(0)e^{\frac{\mathrm{gT}}{1-\alpha}}<S(T)\right\}}\right]$
$= B(0,T)\left[S(0)e^{\mathrm{gT}}P_{\beta}\left\{S(0)e^{\frac{\mathrm{gT}}{1-\alpha}} > S(T)\right\} + S(0)^{\alpha}\mathbb{E}_{\beta}\left[S(T)^{1-\alpha}\mathbb{I}_{\left\{S(0)e^{\frac{\mathrm{gT}}{1-\alpha}}<S(T)\right\}}\right]\right]$
$\text{Considerthedynamicsofthe}S(t)$
$dS(t) = S(t)\left((r(t)-q(t))dt + \sigma(t)dW_{\beta}(t)\right)$
$\Rightarrow \ln(S(T)) = \ln(S(0)) + \int_0^T r(t)dt - \int_0^T q(t)dt - \frac{1}{2}\int_0^T \sigma^2(t)dt + \int_0^T \sigma(t)dW_{\beta}(t)$
$= \ln(S(0)) - \ln(B(0,T)) + \ln(D(0,T)) - \frac{1}{2}\sum_{i=1}^{N}\int_{T_{i-1}}^{T_i}\sigma_i{}^2(t)dt + \sum_{i=1}^{N}\int_{T_{i-1}}^{T_i}\sigma_i(t)dW_{\beta}(t)\ (\text{where}T_N = T)$
$= \ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) - \frac{1}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right) + \sum_{i=1}^{N}\sigma_i\left(W_{\beta}\left(T_i\right) - W_{\beta}\left(T_{i-1}\right)\right)$
$\Rightarrow \ln(S(T)) \sim N_{\beta}\left(\ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) - \frac{1}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right), \sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right)\right)$
$\Rightarrow P_{\beta}\left\{S(0)e^{\frac{\mathrm{gT}}{1-\alpha}} > S(T)\right\} = \Phi\left(\frac{\ln\left(S(0)e^{\frac{\mathrm{gT}}{1-\alpha}}\right) - \left(\ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) - \frac{1}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)(T_i - T_{i-1})\right)}{\sqrt{\sum_{i=1}^{N}\sigma_i{}^2(t)(T_i - T_{i-1})}}\right)$
$\text{Let}Z(t,T) = \frac{S(T)^{1-\alpha}}{S(0)^{1-\alpha}}C\text{is}R/N\text{derivativetochangefrom}P_{\beta}\text{to}P_Q$
$\Rightarrow \frac{S(T)^{1-\alpha}}{S(0)^{1-\alpha}}C = \mathrm{Exp}\left[-\frac{(1-\alpha)^2}{2}\int_0^T \sigma^2(t)dt + (1-\alpha)\int_0^T \sigma(t)dW_{\beta}(t)\right]$
$\Rightarrow \ln\frac{1}{C} = (1-\alpha)\left[\ln\left(\frac{D(0,T)}{B(0,T)}\right)\right] - \frac{(1-\alpha)}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right) + \frac{(1-\alpha)^2}{2}\int_0^T \sigma^2(t)dt$

$$= (1-\alpha)\left[\ln\left(\frac{D(0,T)}{B(0,T)}\right)\right] + \frac{(1-\alpha)^2-(1-\alpha)}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right)$$
$$\Rightarrow \frac{1}{C} = \left(\frac{D(0,T)}{B(0,T)}\right)^{(1-\alpha)}\text{Exp}\left[\frac{\alpha^2-\alpha}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right)\right]$$

$$\mathbb{E}_\beta\left[S(T)^{1-\alpha}\mathbb{I}_{\left\{S(0)e^{\frac{\text{gT}}{1-\alpha}}<S(T)\right\}}\right]$$
$$= \mathbb{E}_Q\left[S(T)^{1-\alpha}\mathbb{I}_{\left\{S(0)e^{\frac{\text{gT}}{1-\alpha}}<S(T)\right\}}\frac{1}{\frac{S(T)^{1-\alpha}}{S(0)^{1-\alpha}}C}\right]$$
$$= \frac{S(0)^{1-\alpha}}{C}P_Q\left\{S(0)e^{\frac{\text{gT}}{1-\alpha}} < S(T)\right\}$$
$$= \left(S(0)\frac{D(0,T)}{B(0,T)}\right)^{(1-\alpha)}\text{Exp}\left[\frac{\alpha^2-\alpha}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right)\right]P_Q\left\{S(0)e^{\frac{\text{gT}}{1-\alpha}} < S(T)\right\}$$

UnderGirsanovTheorem

$$dW_Q(t) = dW_\beta(t) - (1-\alpha)\sigma(t)dt$$
$$\Rightarrow \ln(S(T)) = \ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) - \frac{1}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right) + \sum_{i=1}^{N}\int_{T_{i-1}}^{T_i}\sigma_i(t)dW_\beta(t)$$
$$= \ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) - \frac{1}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right) + \sum_{i=1}^{N}\int_{T_{i-1}}^{T_i}\sigma_i(t)\left[dW_Q(t) + (1-\alpha)\sigma_i(t)dt\right]$$
$$= \ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) - \frac{1}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right) + (1-\alpha)\sum_{i=1}^{N}\int_{T_{i-1}}^{T_i}\sigma_i{}^2(t)dt + \sum_{i=1}^{N}\int_{T_{i-1}}^{T_i}\sigma_i(t)dW_Q(t)$$
$$= \ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) + \left(\frac{1}{2}-\alpha\right)\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right) + \sum_{i=1}^{N}\sigma_i(t)\left(W_Q\left(T_i\right) - W_Q\left(T_{i-1}\right)\right)$$
$$\Rightarrow \ln(S(T)) \sim N_Q\left(\ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) + \left(\frac{1}{2}-\alpha\right)\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right), \sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right)\right)$$
$$P_Q\left\{S(0)e^{\frac{\text{gT}}{1-\alpha}} < S(T)\right\} = \Phi\left(\frac{\ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) + \left(\frac{1}{2}-\alpha\right)\sum_{i=1}^{N}\sigma_i{}^2(t)(T_i - T_{i-1}) - \ln\left(S(0)e^{\frac{\text{gT}}{1-\alpha}}\right)}{\sqrt{\sum_{i=1}^{N}\sigma_i{}^2(t)(T_i - T_{i-1})}}\right)$$

Then

$$V = B(0,T)\left[S(0)e^{\text{gT}}\Phi\left(h_1\right) + S(0)\left(\frac{D(0,T)}{B(0,T)}\right)^{(1-\alpha)}\text{Exp}\left[\frac{\alpha^2-\alpha}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)\left(T_i - T_{i-1}\right)\right]\Phi\left(h_2\right)\right]$$
$$h_1 = \frac{\ln\left(S(0)e^{\frac{\text{gT}}{1-\alpha}}\right) - \left(\ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) - \frac{1}{2}\sum_{i=1}^{N}\sigma_i{}^2(t)(T_i - T_{i-1})\right)}{\sqrt{\sum_{i=1}^{N}\sigma_i{}^2(t)(T_i - T_{i-1})}}$$
$$h_2 = \frac{\ln\left(S(0)\frac{D(0,T)}{B(0,T)}\right) + \left(\frac{1}{2}-\alpha\right)\sum_{i=1}^{N}\sigma_i{}^2(t)(T_i - T_{i-1}) - \ln\left(S(0)e^{\frac{\text{gT}}{1-\alpha}}\right)}{\sqrt{\sum_{i=1}^{N}\sigma_i{}^2(t)(T_i - T_{i-1})}}$$