

37005 Fundamental of Derivative Security Pricing  
- Spring 2024  
Group Assignment Part I

Group ID: 1  
Group Members:  
- Hayoung Lee  
- Quoc Thai Tran  
- Alexis Cullet  
- Ziqi Zhou

### Task 3

$$\begin{aligned} \text{Payoff at } T &: \max(S(0)e^{gT}, S(0)^\alpha S(T)^{1-\alpha}) \\ &= S(0)e^{gT} \mathbb{I}_{\{S(0)e^{gT} > S(0)^\alpha S(T)^{1-\alpha}\}} + S(0)^\alpha S(T)^{1-\alpha} \mathbb{I}_{\{S(0)e^{gT} < S(0)^\alpha S(T)^{1-\alpha}\}} \\ &= S(0)e^{gT} \mathbb{I}_{\left\{S(0)e^{\frac{gT}{1-\alpha}} > S(T)\right\}} + S(0)^\alpha S(T)^{1-\alpha} \mathbb{I}_{\left\{S(0)e^{\frac{gT}{1-\alpha}} < S(T)\right\}} \end{aligned}$$

$$\begin{aligned} V &= B(0, T) \mathbb{E}_\beta \left[ S(0)e^{gT} \mathbb{I}_{\left\{S(0)e^{\frac{gT}{1-\alpha}} > S(T)\right\}} + S(0)^\alpha S(T)^{1-\alpha} \mathbb{I}_{\left\{S(0)e^{\frac{gT}{1-\alpha}} < S(T)\right\}} \right] \\ &= B(0, T) \left[ S(0)e^{gT} P_\beta \left\{ S(0)e^{\frac{gT}{1-\alpha}} > S(T) \right\} + S(0)^\alpha \mathbb{E}_\beta \left[ S(T)^{1-\alpha} \mathbb{I}_{\left\{S(0)e^{\frac{gT}{1-\alpha}} < S(T)\right\}} \right] \right] \end{aligned}$$

Consider the dynamics of the  $S(t)$

$$\begin{aligned} dS(t) &= S(t) ((r(t) - q(t))dt + \sigma(t)dW_\beta(t)) \\ \Rightarrow \ln(S(T)) &= \ln(S(0)) + \int_0^T r(t)dt - \int_0^T q(t)dt - \frac{1}{2} \int_0^T \sigma^2(t)dt + \int_0^T \sigma(t)dW_\beta(t) \\ &= \ln(S(0)) - \ln(B(0, T)) + \ln(D(0, T)) - \frac{1}{2} \sum_{i=1}^N \int_{T_{i-1}}^{T_i} \sigma_i^2(t)dt + \sum_{i=1}^N \int_{T_{i-1}}^{T_i} \sigma_i(t)dW_\beta(t) \quad (\text{where } T_N = T) \\ &= \ln \left( S(0) \frac{D(0, T)}{B(0, T)} \right) - \frac{1}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) + \sum_{i=1}^N \sigma_i (W_\beta(T_i) - W_\beta(T_{i-1})) \\ \Rightarrow \ln(S(T)) &\sim N_\beta \left( \ln \left( S(0) \frac{D(0, T)}{B(0, T)} \right) - \frac{1}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}), \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) \right) \\ \Rightarrow P_\beta \left\{ S(0)e^{\frac{gT}{1-\alpha}} > S(T) \right\} &= \Phi \left( \frac{\ln \left( S(0)e^{\frac{gT}{1-\alpha}} \right) - \left( \ln \left( S(0) \frac{D(0, T)}{B(0, T)} \right) - \frac{1}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) \right)}{\sqrt{\sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1})}} \right) \end{aligned}$$

Let  $Z(t, T) = \frac{S(T)^{1-\alpha}}{S(0)^{1-\alpha}}$  is  $R/N$  derivative to change from  $P_\beta$  to  $P_Q$

$$\begin{aligned} \Rightarrow \frac{S(T)^{1-\alpha}}{S(0)^{1-\alpha}} C &= \text{Exp} \left[ -\frac{(1-\alpha)^2}{2} \int_0^T \sigma^2(t)dt + (1-\alpha) \int_0^T \sigma(t)dW_\beta(t) \right] \\ \Rightarrow \ln \frac{1}{C} &= (1-\alpha) \left[ \ln \left( \frac{D(0, T)}{B(0, T)} \right) \right] - \frac{(1-\alpha)}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) + \frac{(1-\alpha)^2}{2} \int_0^T \sigma^2(t)dt \end{aligned}$$

$$\begin{aligned}
&= (1 - \alpha) \left[ \ln \left( \frac{D(0,T)}{B(0,T)} \right) \right] + \frac{(1-\alpha)^2 - (1-\alpha)}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) \\
&\Rightarrow \frac{1}{C} = \left( \frac{D(0,T)}{B(0,T)} \right)^{(1-\alpha)} \text{Exp} \left[ \frac{\alpha^2 - \alpha}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) \right]
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}_\beta \left[ S(T)^{1-\alpha} \mathbb{I}_{\left\{ S(0)e^{\frac{gT}{1-\alpha}} < S(T) \right\}} \right] \\
&= \mathbb{E}_Q \left[ S(T)^{1-\alpha} \mathbb{I}_{\left\{ S(0)e^{\frac{gT}{1-\alpha}} < S(T) \right\}} \frac{1}{\frac{S(T)^{1-\alpha}}{S(0)^{1-\alpha}} C} \right] \\
&= \frac{S(0)^{1-\alpha}}{C} P_Q \left\{ S(0)e^{\frac{gT}{1-\alpha}} < S(T) \right\} \\
&= \left( S(0) \frac{D(0,T)}{B(0,T)} \right)^{(1-\alpha)} \text{Exp} \left[ \frac{\alpha^2 - \alpha}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) \right] P_Q \left\{ S(0)e^{\frac{gT}{1-\alpha}} < S(T) \right\}
\end{aligned}$$

Under Girsanov Theorem

$$\begin{aligned}
dW_Q(t) &= dW_\beta(t) - (1 - \alpha)\sigma(t)dt \\
&\Rightarrow \ln(S(T)) = \ln \left( S(0) \frac{D(0,T)}{B(0,T)} \right) - \frac{1}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) + \sum_{i=1}^N \int_{T_{i-1}}^{T_i} \sigma_i(t) dW_\beta(t) \\
&= \ln \left( S(0) \frac{D(0,T)}{B(0,T)} \right) - \frac{1}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) + \sum_{i=1}^N \int_{T_{i-1}}^{T_i} \sigma_i(t) [dW_Q(t) + (1 - \alpha)\sigma_i(t)dt] \\
&= \ln \left( S(0) \frac{D(0,T)}{B(0,T)} \right) - \frac{1}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) + (1 - \alpha) \sum_{i=1}^N \int_{T_{i-1}}^{T_i} \sigma_i^2(t) dt + \\
&\quad \sum_{i=1}^N \int_{T_{i-1}}^{T_i} \sigma_i(t) dW_Q(t) \\
&= \ln \left( S(0) \frac{D(0,T)}{B(0,T)} \right) + \left( \frac{1}{2} - \alpha \right) \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) + \sum_{i=1}^N \sigma_i(t) (W_Q(T_i) - W_Q(T_{i-1})) \\
&\Rightarrow \ln(S(T)) \sim N_Q \left( \ln \left( S(0) \frac{D(0,T)}{B(0,T)} \right) + \left( \frac{1}{2} - \alpha \right) \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}), \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) \right) \\
P_Q \left\{ S(0)e^{\frac{gT}{1-\alpha}} < S(T) \right\} &= \Phi \left( \frac{\ln \left( S(0) \frac{D(0,T)}{B(0,T)} \right) + \left( \frac{1}{2} - \alpha \right) \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) - \ln \left( S(0)e^{\frac{gT}{1-\alpha}} \right)}{\sqrt{\sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1})}} \right)
\end{aligned}$$

Then

$$\begin{aligned}
V &= B(0, T) \left[ S(0)e^{gT} \Phi(h_1) + S(0) \left( \frac{D(0,T)}{B(0,T)} \right)^{(1-\alpha)} \text{Exp} \left[ \frac{\alpha^2 - \alpha}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) \right] \Phi(h_2) \right] \\
h_1 &= \frac{\ln \left( S(0)e^{\frac{gT}{1-\alpha}} \right) - \left( \ln \left( S(0) \frac{D(0,T)}{B(0,T)} \right) - \frac{1}{2} \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) \right)}{\sqrt{\sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1})}} \\
h_2 &= \frac{\ln \left( S(0) \frac{D(0,T)}{B(0,T)} \right) + \left( \frac{1}{2} - \alpha \right) \sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1}) - \ln \left( S(0)e^{\frac{gT}{1-\alpha}} \right)}{\sqrt{\sum_{i=1}^N \sigma_i^2(t) (T_i - T_{i-1})}}
\end{aligned}$$