37005 Fundamentals of Derivative Security Pricing

Group Assignment

Due by: 11:59PM on 3 November 2024

Assignments should be submitted electronically via the Canvas site for this subject. Please submit a typeset document (Microsoft Word, or if you're using LaTeX, PDF) as your assignment report (handwritten, scanned documents will not be accepted). All required programming should be done in Python. Please submit all Python code which you wrote to produce your report.

The file `spx_quotedata20220308_all.xlsx` contains end–of–day bid and ask prices of options on the S&P 500 index, traded on the CBOE on 8 March 2022. These options are European. Note that the level of the index on that day was 4170.7002.

Assume the following:

- Risk–free interest rates are deterministic, but time–dependent. Thus the continuously compounded short rate $r(t)$ is a deterministic function of t.
- The underlying S&P 500 index pays dividends, which can be adequately approximated by a continuous dividend rate $q(t)$, which is a deterministic function of t.

Tasks:

1. For option pricing we need a term structure of zero coupon bond prices

$$B(0,T)=\exp\left\{-\int_0^T r(s)ds\right\} \tag{1}$$

and "dividend discount factors"

$$D(0,T)=\exp\left\{-\int_0^T q(s)ds\right\} \tag{2}$$

Put/call parity asserts that for puts $P(K,T)$ and calls $C(K,T)$ of strike K and maturity T on the underlying S, it holds that

$$C(K,T)-P(K,T)=D(0,T)S(0)-B(0,T)K \tag{3}$$

Using "mid" prices in the middle of the bid/ask spread for calls and puts, for each maturity in the data set, determine $B(0,T)$ and $D(0,T)$ such that the sum of squared deviations from the identity (3) is minimised. *(1 mark)*

2. For each maturity in the data set, choose strikes closest to the forward price of the index for that maturity. Using option bid prices for these strikes, determine an

implied term structure of volatility, i.e., determine a piecewise constant function $\sigma_{\text{bid}}(t)$ such that the Black/Scholes prices resulting from this volatility equal the observed bid prices. Do the same to determine $\sigma_{\text{ask}}(t)$ from the ask prices. *(4 marks)*

Note: In the present context "piecewise constant function" means that for the maturities T_i in the data set (with $T_0 = 0$ corresponding to "today", 8 March 2022), there are corresponding constants c_i such that

$$\sigma(t) = c_i \text{ when } T_i \leq t < T_{i+1}$$

Your task is to determine these c_i.

3. Denoting the index level at time t by $S(t)$, define the ex-dividend return between t and T by

$$h(t,T) = \frac{1}{T-t} \ln \frac{S(T)}{S(t)}$$

An investment bank wishes to offer a client a contract where it guarantees an ex-dividend return of g between today ($t = 0$) and some maturity T, in exchange for a share α in any realised ex-dividend return above g. Thus, if the client invests $S(0)$ today, the client's payoff at time T will be

$$\begin{aligned} & \max(S(0)e^{gT}, S(0)e^{(1-\alpha)h(0,T)T}) \\ = \quad & \max\left(S(0)e^{gT}, S(t)\left(\frac{S(T)}{S(0)}\right)^{1-\alpha}\right) \\ = \quad & \max(S(0)e^{gT}, S(0)^{\alpha}S(T)^{1-\alpha}) \end{aligned} \tag{4}$$

Assuming Black/Scholes dynamics with deterministic time–dependent volatility $\sigma(t)$, $0 \leq t \leq T$, derive the formula for the time 0 price of this payoff. *(5 marks)*

Note: Don't forget that the index pays dividends at the continuous rate $q(t)$!

4. Suppose that the investment bank enters this contract on 8 March 2022, with maturity T on 18 December 2026. Suppose further that the only payment by the client is the initial investment amount $S(0)$. If $\alpha = 0.25$, what would be the "fair" guarantee level g consistent with the data considered in Task 2 and the formula derived in Task 3? What if $\alpha = 0.25$ or $\alpha = 0.75$? *(3 marks)*

5. Suppose that instead of using the formula derived in Task 3, the investment bank considers a static hedge of the payoff using all market–traded standard options in the data set expiring on 18 December 2026. For each of the three (α, g) combinations obtained in Task 4, what is the minimum cost of a static hedge eliminating all market risk exposure of the investment bank? *(4 marks)*

Note: A "static hedge" in the present context means buying and/or selling standard options with various strikes at time 0 and holding this position unchanged until 18 December 2026, in such a way that the combined payoff of the standard options and a short position in (4) is non-negative no matter what the outcome $S(T)$.

6. Discuss and interpret briefly your results from Tasks 4 and 5. *(3 marks)*