Nelson/Siegel calibration

Discount factor

$$f(0,t) = f_0 + f_1 e^{-\frac{t}{\gamma}} + f_2 \frac{t}{\gamma} e^{-\frac{t}{\gamma}}$$

$$B(0,T) = \mathrm{Exp}\left[-\int_0^T f(0,t)\, dt\right]$$

$$= \mathrm{Exp}\left[-T f_0 - \left(\gamma - e^{-\frac{T}{\gamma}} \gamma\right) f_1 - \left(\gamma - e^{-\frac{T}{\gamma}} (T+\gamma)\right) f_2\right]$$

In[∘]:=

$$\int_0^T \left(f_0 + f_1 e^{-\frac{t}{\gamma}} + f_2 \frac{t}{\gamma} e^{-\frac{t}{\gamma}}\right) dt$$

Out[∘]=

$$T f_0 + \left(\gamma - e^{-\frac{T}{\gamma}} \gamma\right) f_1 + \left(\gamma - e^{-\frac{T}{\gamma}} (T+\gamma)\right) f_2$$

Bond valuation

$$V_j = 100 \left(\sum_{i=1}^{n_j} c_j \frac{\partial}{\partial \theta} B(0, T(j)_i) + \frac{\partial}{\partial \theta} B\left(0, T(j)_{n_j}\right)\right)$$

Objective function

$$\min_{f_0, f_1, f_2, \gamma} \left(\sum_{j=1}^{m} \left(\left(\frac{\max(0, \mathrm{Bid}_j - V_j)}{B_j}\right)^2 + \left(\frac{\max(0, V_j - \mathrm{Ask}_j)}{A_j}\right)^2\right)\right)$$

Gradient function

$$\text{Loss} = \sum_{j=1}^{n}\left(\left(\frac{\max(0, V_j - A_j)}{A_j}\right)^2 + \left(\frac{\max(0, B_j - V_j)}{V_j}\right)^2\right)$$

$$L_j = \left(\frac{\max(0, V_j - A_j)}{A_j}\right)^2 + \left(\frac{\max(0, B_j - V_j)}{B_j}\right)^2$$

$$\frac{\partial L}{\partial \theta} = \sum_{j=1}^{n}\frac{\partial L_j}{\partial \theta}$$

- $V_j > A_j$

$$\frac{\partial L_j}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{\max(0, V_j - A_j)}{A_j}\right)^2$$

$$= \frac{\partial}{\partial \theta}\left(\frac{V_j - A_j}{A_j}\right)^2$$

$$= 2\,\frac{V_j - A_j}{A_j^2}\,\frac{\partial}{\partial \theta}V_j$$

$$\Rightarrow \frac{\partial L_j}{\partial \theta} = \frac{V_j - A_j}{A_j^2}\,\frac{\partial}{\partial \theta}V_j$$

$$\Rightarrow \lim_{V_j \to A_j}\frac{\partial L_j}{\partial \theta} = 0$$

- $A_j > V_j > B_j$

$$\frac{\partial L_j}{\partial \theta} = 0$$

- $B_j > V_j$

$$\frac{\partial L_j}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{\max(0, B_j - V_j)}{B_j}\right)^2$$

$$= \frac{\partial}{\partial \theta} \left(\frac{B_j - V_j}{B_j} \right)^2$$

$$= 2 \frac{V_j - B_j}{{B_j}^2} \frac{\partial}{\partial \theta} V_j$$

$$\Rightarrow \lim_{V_j \to B_j} \frac{\partial L_j}{\partial \theta} = 0$$

$$\frac{\partial}{\partial \theta} V_j = 100 \left(\sum_{i=1}^{n_j} c_j \frac{\partial}{\partial \theta} B\,(0,\ T\,(j)_i) + \frac{\partial}{\partial \theta} B\left(0,\ T\,(j)_{n_j}\right) \right)$$

$$\frac{\partial}{\partial \theta} B\,(0,\ T) = \frac{\partial}{\partial \theta} \mathrm{Exp}\left[-T f_0 - \left(\gamma - e^{-\frac{T}{\gamma}} \gamma \right) f_1 - \left(\gamma - e^{-\frac{T}{\gamma}} (T + \gamma) \right) f_2 \right]$$

$$\theta = f_0 \Rightarrow \frac{\partial}{\partial \theta} B\,(0,\ T) = -B\,(0,\ T)\ T$$

$$\theta = f_1 \Rightarrow \frac{\partial}{\partial \theta} B\,(0,\ T) = B\,(0,\ T) \left(-\gamma + e^{-\frac{T}{\gamma}} \gamma \right)$$

$$\theta = f_2 \Rightarrow \frac{\partial}{\partial \theta} B\,(0,\ T) = B\,(0,\ T) \left(-\gamma + e^{-\frac{T}{\gamma}} (T + \gamma) \right)$$

$$\theta = \gamma \Rightarrow \frac{\partial}{\partial \theta} B\,(0,\ T) = B\,(0,\ T) \left(-\left(\left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}} T}{\gamma} \right) f_1 \right) - \left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}} T\,(T + \gamma)}{\gamma^2} \right) f_2 \right)$$

Hessian function

$$\frac{\partial^2 L}{\partial \theta_1\, \partial \theta_2} = \sum_{j=1}^{n} \frac{\partial^2 L_j}{\partial \theta_1\, \partial \theta_2}$$

$$- V_j > A_j$$

$$\frac{\partial^2 L_j}{\partial\theta_1\,\partial\theta_2} = \frac{\partial}{\partial\theta_2}\frac{\partial L_j}{\partial\theta_1}$$

$$= \frac{\partial}{\partial\theta_2}\left(2\,\frac{V_j - A_j}{A_j{}^2}\,\frac{\partial}{\partial\theta_1}V_j\right)$$

$$= 2\,\frac{V_j - A_j}{A_j{}^2}\,\frac{\partial^2 V_j}{\partial\theta_1\,\partial\theta_2} + 2\,\frac{1}{A_j{}^2}\left(\frac{\partial}{\partial\theta_2}V_j\right)\left(\frac{\partial}{\partial\theta_1}V_j\right)$$

$- A_j > V_j > B_j$

$$\frac{\partial^2 L_j}{\partial\theta_1\,\partial\theta_2} = 0$$

$- B_j > V_j$

$$\frac{\partial^2 L_j}{\partial\theta_1\,\partial\theta_2} = \frac{\partial}{\partial\theta_2}\frac{\partial L_j}{\partial\theta_1}$$

$$= \frac{\partial}{\partial\theta_2}\left(2\,\frac{V_j - B_j}{B_j{}^2}\,\frac{\partial}{\partial\theta}V_j\right)$$

$$= 2\,\frac{V_j - B_j}{B_j{}^2}\,\frac{\partial^2 V_j}{\partial\theta_1\,\partial\theta_2} + 2\,\frac{1}{B_j{}^2}\left(\frac{\partial}{\partial\theta_2}V_j\right)\left(\frac{\partial}{\partial\theta_1}V_j\right)$$

$$\frac{\partial^2 V_j}{\partial\theta_1\,\partial\theta_2} = 100\left(\sum_{i=1}^{n_j} c_j\,\frac{\partial^2}{\partial\theta_1\,\partial\theta_2}B\,(0,\ T\,(j)_i) + \frac{\partial^2}{\partial\theta_1\,\partial\theta_2}B\left(0,\ T\,(j)_{n_j}\right)\right)$$

$$\frac{\partial^2}{\partial\theta_1\,\partial\theta_2}B\,(0,\ T\,(j)_i) = \frac{\partial^2}{\partial\theta_1\,\partial\theta_2}\mathrm{Exp}\left[-T\,f_0 - \left(\gamma - e^{-\frac{T}{\gamma}}\,\gamma\right)f_1 - \left(\gamma - e^{-\frac{T}{\gamma}}\,(T+\gamma)\right)f_2\right]$$

$$\theta_1 = f_0,\ \theta_2 = f_0 \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2}B\,(0,\ T) = B\,(0,\ T)\ T^2$$

$$\theta_1 = f_0,\ \theta_2 = f_1 \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) = -B(0, T)\ T \left(-\gamma + e^{-\frac{T}{\gamma}}\gamma\right)$$

$$\theta_1 = f_0,\ \theta_2 = f_2 \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) = -B(0, T)\ T \left(-\gamma + e^{-\frac{T}{\gamma}}(T+\gamma)\right)$$

$$\theta_1 = f_0,\ \theta_2 = \gamma \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) = -B(0, T)\ T \left(-\left(\left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}}T}{\gamma}\right) f_1\right) - \left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}}T(T+\gamma)}{\gamma^2}\right) f_2\right)$$

$$\theta_1 = f_1,\ \theta_2 = f_1 \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) = B(0, T) \left(-\gamma + e^{-\frac{T}{\gamma}}\gamma\right)^2$$

$$\theta_1 = f_1,\ \theta_2 = f_2 \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) = B(0, T) \left(-\gamma + e^{-\frac{T}{\gamma}}\gamma\right)\left(-\gamma + e^{-\frac{T}{\gamma}}(T+\gamma)\right)$$

$$\theta_1 = f_1,\ \theta_2 = \gamma \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) = B(0, T) \left(\left(-1 + e^{-\frac{T}{\gamma}} + \frac{e^{-\frac{T}{\gamma}}T}{\gamma}\right) + \left(-\left(\left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}}T}{\gamma}\right) f_1\right) - \left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}}T(T+\gamma)}{\gamma^2}\right) f_2\right)\right)$$

$$\theta_1 = f_2,\ \theta_2 = f_2 \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) = B(0, T) \left(-\gamma + e^{-\frac{T}{\gamma}}(T+\gamma)\right)^2$$

$$\theta_1 = f_2,\ \theta_2 = \gamma \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) =$$

$$B(0, T) \left(\left(-1 + e^{-\frac{T}{\gamma}} + \frac{e^{-\frac{T}{\gamma}}T(T+\gamma)}{\gamma^2}\right) + \left(-\gamma + e^{-\frac{T}{\gamma}}(T+\gamma)\right)\left(-\left(\left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}}T}{\gamma}\right) f_1\right) - \left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}}T(T+\gamma)}{\gamma^2}\right) f_2\right)\right)$$

$$\theta_1 = \gamma,\ \theta_2 = \gamma \Rightarrow \frac{\partial^2}{\partial\theta_1\,\partial\theta_2} B(0, T) = B(0, T)$$

$$\left(\left(\frac{e^{-\frac{T}{\gamma}}T^2 f_1}{\gamma^3} - \left(-\frac{2\,e^{-\frac{T}{\gamma}}T}{\gamma^2} - \frac{e^{-\frac{T}{\gamma}}T^2(T+\gamma)}{\gamma^4} + \frac{2\,e^{-\frac{T}{\gamma}}T(T+\gamma)}{\gamma^3}\right) f_2\right) + \left(-\left(\left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}}T}{\gamma}\right) f_1\right) - \left(1 - e^{-\frac{T}{\gamma}} - \frac{e^{-\frac{T}{\gamma}}T(T+\gamma)}{\gamma^2}\right) f_2\right)^2\right)$$