Haskelleene

a very Haskell implmentation of automata, regular expressions, and Kleene's algorithm

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Introduction

"Automata are pretty cool."

\- Liam Chung

What is an automaton?

An basic version of a state machine. It takes inputs from some *alphabet*, moving between *states* that may or may not *accept*.

```
data DetAut l s = DA { states :: [s]
                     , accept :: [s]
                     , delta  :: l -> s -> s }

data NDetAut l s = NA { nstates :: [s]
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How do we encode `delta`**?**

Automaton data: `AutData`

A simpler representation of our data:

```
type TDict l s = [(s, [(Maybe l, s)])]
data AutData l s = AD { stateData :: [s]
                      , acceptData :: [s]
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```

Then, encode this data into our nicer types.

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Need to check if data is safe for DAs! **How?**

The Alphabet

Implementing semantics

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Implementing semantics

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Implementing semantics

Seq and Star cases

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Deterministic Automata

Non-Deterministic Automata

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For instance, the following represents a NA:

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\begin{tikzcd}
1 \arrow[r, shift left, "\epsilon"] \arrow[d, "b"] & 2 \arrow[l, shift left, "c"] \arrow[dl, "a"] \arrow[d, "\epsilon"] \arrow[loop above, "b"] \\
3 & 4
\end{tikzcd}
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Our Haskell implementation of the type of NA:

```
data NDetAut l s = NA { nstates :: [s]
                      , naccept :: [s]
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```

Semantics of Non-Deterministic Automata

Definition

A NA accepts an input string u if there is a *possible* path that terminates at an accepting state.

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Suppose the initial state is 1, and the only accepting state is 4:

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1 \arrow[r, shift left, "\epsilon"] \arrow[d, "b"'] & 2 \arrow[l, shift left, "c"] \arrow[dl, "a"] \arrow[d, "\epsilon"] \arrow[loop above, "b"] \\
3 & \color{red}{4}
\end{tikzcd}
$$

Then this NA accepts

$$\epsilon, b, c, bc, \cdots$$

Algorithm for Running NA

General Idea

Searching paths in a finite graph in general requires a lot of computational resources. However, we do not need to output the whole path, thus we only keep track of how states transform.

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Active states: $([ba], 1)$.

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Say we run the input *ba* on the previous example:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\epsilon"] \arrow[d, "b"] & \color{green}{2} \arrow[l, shift left, "c"] \arrow[dl, "a"] \arrow[d, "\epsilon"] \arrow[loop above, "b"] \\ \color{green}{3} & 4 \end{tikzcd}$$

Active states: $([a], 3), ([ba], 2)$.

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Active states: $([a], 3), ([a], 2), ([ba], 4)$.

Algorithm for Running NA

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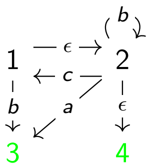
Searching paths in a finite graph in general requires a lot of computational resources. However, we do not need to output the whole path, thus we only keep track of how states transform.

Say we run the input *bca* on the previous example:



Active states: $([a], 3), ([], 3), ([a], 4), ([ba], 4)$.

Haskell Implementation of Semantics for NA

The function `runNA` is defined as follows:

```
runNA :: (Alphabet l, Ord s) =>
         NDetAut l s  -> s -> [l] -> [([l], s)]
runNA na st input =
  case input of
    [] -> ([],) <$> epReachable (ndelta na) st
    (w:ws) -> concatMap (\s -> runNA na s input) nsucc ++
              case wsucc of
                [] -> [(input,st)]
                ls -> concatMap (\s -> runNA na s ws) ls
      where wsucc = ndelta na (Just w) st
    where   nsucc = ndelta na Nothing  st
```

Here the function `epReachable` calculates all the states that is reachable from the current state via ϵ-transitions.

Equivalence between DA and NA

Evidently, any DA is a NA. On the other hand, we can simulate running NA deterministically, basically via the same idea as `runNA`:

- States are subsets of states of a NA.
- A subset is accepting iff it contains some accepting state.
- Under an input I, a subset transforms to those states reachable from some state via I (with ϵ-transitions).

Haskell Implementation

```
fromNA :: (Alphabet l, Ord s) =>
          NDetAut l s -> DetAut l (Set.Set s)
fromNA nda = DA { states = Set.toList dasts
                , accept = Set.toList $ Set.filter
                    acchelp dasts
                , delta = fromTransNA ntrans
                }
  where ndasts = nstates nda
        dasts  = Set.powerSet $ Set.fromList ndasts
        ndaacc = naccept nda
        acchelp set = not $ Set.disjoint set
                          $ Set.fromList ndaacc
        ntrans = ndelta nda

fromTransNA :: (Alphabet l, Ord s) =>
               (Maybe l -> s -> [s]) -> l -> Set.Set s ->
                    Set.Set s
fromTransNA ntrans sym set = result
  where starts = listUnions (epReachable ntrans) set
        step = listUnions (ntrans $ Just sym) starts
        result = listUnions (epReachable ntrans) step
        listUnions f input = Set.unions $ Set.map Set.
            fromList $ Set.map f input
```

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Kleene's Theorem

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- *it is represented by a state in a finite DA*

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- *it is represented by a state in a finite DA*
- *it is represented by a state in a finite NA*
- *it is represented by a regular expression*

in such a case, the language l is called ***regular****.*