

Haskelleene

a very Haskell implementation of automata,
regular expressions, and Kleene's algorithm

Liam Chung, Brendan Dufty, Lingyuan Ye

31st May, 2024

Table of Contents

Introduction

Finite Automata

Regular Expressions

Kleene's Theorem

Table of Contents

Introduction

Finite Automata

Regular Expressions

Kleene's Theorem

Introduction

"Automata are pretty cool."

- Liam Chung

Table of Contents

Introduction

Finite Automata

Regular Expressions

Kleene's Theorem

What is an automaton?

An basic version of a state machine. It takes inputs from some *alphabet*, moving between *states* that may or may not *accept*.

```
data DetAut l s = DA { states :: [s]
                      , accept :: [s]
                      , delta  :: l -> s -> s }

data NDetAut l s = NA { nstates :: [s]
                       , naccept :: [s]
                       , ndelta  :: Maybe l -> s -> [s] }
```

What is an automaton?

An basic version of a state machine. It takes inputs from some *alphabet*, moving between *states* that may or may not *accept*.

```
data DetAut l s = DA { states :: [s]
                      , accept :: [s]
                      , delta  :: l -> s -> s }

data NDetAut l s = NA { nstates :: [s]
                       , naccept :: [s]
                       , ndelta  :: Maybe l -> s -> [s] }
```

- ▶ a state in a deterministic automaton accepts a word if that words leads to an accepting state.

What is an automaton?

An basic version of a state machine. It takes inputs from some *alphabet*, moving between *states* that may or may not *accept*.

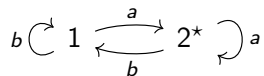
```
data DetAut l s = DA { states :: [s]
                      , accept :: [s]
                      , delta  :: l -> s -> s }

data NDetAut l s = NA { nstates :: [s]
                       , naccept :: [s]
                       , ndelta  :: Maybe l -> s -> [s] }
```

- ▶ a state in a deterministic automaton accepts a word if that words leads to an accepting state.
- ▶ a state in a non-deterministic automaton accepts a word if *there exists a path* to an accepting state.

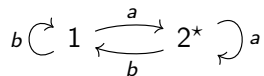
Examples

A deterministic automaton:

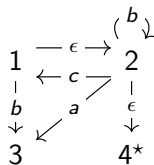


Examples

A deterministic automaton:



A non-deterministic one:



Algorithm for Running DAs

Given an automaton, a starting state, and an input word...

```
run :: DetAut l s -> s -> [l] -> s
run _ s0 [] = s0
run da s0 (w:ws) = run da (delta da w s0) ws

acceptDA :: (Eq s) => DetAut l s -> s -> [l] -> Bool
acceptDA da s0 w = run da s0 w 'elem' accept da
```

Algorithm for Running DAs

Given an automaton, a starting state, and an input word...

```
run :: DetAut l s -> s -> [l] -> s
run _ s0 [] = s0
run da s0 (w:ws) = run da (delta da w s0) ws

acceptDA :: (Eq s) => DetAut l s -> s -> [l] -> Bool
acceptDA da s0 w = run da s0 w 'elem' accept da
```

...it's that simple! Mostly due to how much work we put into encoding automata.

Algorithm for Running DAs

Given an automaton, a starting state, and an input word...

```
run :: DetAut l s -> s -> [l] -> s
run _ s0 [] = s0
run da s0 (w:ws) = run da (delta da w s0) ws

acceptDA :: (Eq s) => DetAut l s -> s -> [l] -> Bool
acceptDA da s0 w = run da s0 w 'elem' accept da
```

...it's that simple! Mostly due to how much work we put into encoding automata.

Unfortunately, running NAs is less simple.

Algorithm for Running NAs

General Idea

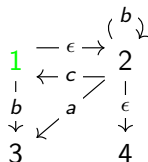
Searching paths in a finite graph in general requires a lot of computational resources. However, we do not need to output the whole path, thus we only keep track of how states transform.

Algorithm for Running NAs

General Idea

Searching paths in a finite graph in general requires a lot of computational resources. However, we do not need to output the whole path, thus we only keep track of how states transform.

Say we run the input ba on the previous example:



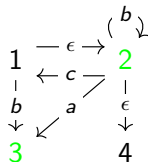
Active states: $([ba], 1)$.

Algorithm for Running NAs

General Idea

Searching paths in a finite graph in general requires a lot of computational resources. However, we do not need to output the whole path, thus we only keep track of how states transform.

Say we run the input ba on the previous example:



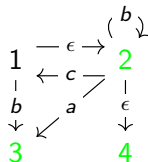
Active states: $([a], 3)$, $([ba], 2)$.

Algorithm for Running NAs

General Idea

Searching paths in a finite graph in general requires a lot of computational resources. However, we do not need to output the whole path, thus we only keep track of how states transform.

Say we run the input ba on the previous example:



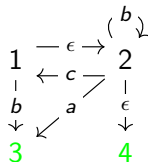
Active states: $([a], 3)$, $([a], 2)$, $([ba], 4)$.

Algorithm for Running NAs

General Idea

Searching paths in a finite graph in general requires a lot of computational resources. However, we do not need to output the whole path, thus we only keep track of how states transform.

Say we run the input bca on the previous example:



Active states: $([a], 3)$, $([], 3)$, $([a], 4)$, $([ba], 4)$.

Haskell Implementation of Semantics for NA

The function `runNA` is defined as follows:

```
runNA :: (Alphabet l, Ord s) =>
        NDetAut l s  -> s -> [l] -> [[l], s]
runNA na st input =
  case input of
    [] -> ([],) <$> epReachable (ndelta na) st
    (w:ws) -> concatMap (\s -> runNA na s input) nsucc ++
      case wsucc of
        [] -> [(input,st)]
        ls -> concatMap (\s -> runNA na s ws) ls
    where wsucc = ndelta na (Just w) st
    where nsucc = ndelta na Nothing st
```

Here the function `epReachable` calculates states reachable from the current one via ϵ -transitions.

Equivalence between DA and NA

Evidently, any DA is a NA. On the other hand, we can simulate running NA deterministically, basically via the same idea as `runNA`:

- ▶ States are subsets of states of a NA.
- ▶ A subset is accepting iff it contains some accepting state.
- ▶ Under an input I , a subset transforms to those states reachable from some state via I (with ϵ -transitions).

Haskell Implementation

```
fromNA :: (Alphabet l, Ord s) =>
    NDetAut l s -> DetAut l (Set.Set s)
fromNA nda = DA { states = Set.toList dasts
                 , accept = Set.toList $ Set.filter
                     acchelp dasts
                 , delta = fromTransNA ntrans
                 }

    where ndasts = nstates nda
          dasts  = Set.powerSet $ Set.fromList ndasts
          ndaacc = naccept nda
          acchelp set = not $ Set.disjoint set
                      $ Set.fromList ndaacc
          ntrans = ndelta nda

fromTransNA :: (Alphabet l, Ord s) =>
    (Maybe l -> s -> [s]) -> l -> Set.Set s ->
    Set.Set s
fromTransNA ntrans sym set = result
    where starts = listUnions (epReachable ntrans) set
          step = listUnions (ntrans $ Just sym) starts
          result = listUnions (epReachable ntrans) step
          listUnions f input = Set.unions $ Set.map Set.
              fromList $ Set.map f input
```

Table of Contents

Introduction

Finite Automata

Regular Expressions

Kleene's Theorem

What is a regular expression?

A “finite representation” of a potentially infinite language:

```
data Regex l = Empty |  
              Epsilon |  
              L l |  
              Alt (Regex l) (Regex l) |  
              Seq (Regex l) (Regex l) |  
              Star (Regex l)  
deriving (Eq, Show)
```

N.B. not the same as the commonly known, “programmer’s” regular expression!

Implementing semantics

How to check if a word is in the language described by a Regex?

```
regexAccept :: Eq l => Regex l -> [l] -> Bool
-- the empty language accepts no words
regexAccept Empty _      = False
-- if down to empty string, only accept empty word
regexAccept Epsilon []   = True
regexAccept Epsilon _    = False
-- if down to a single letter, only accept that letter
regexAccept (L _) []     = False
regexAccept (L l) [c]    = l == c
regexAccept (L _) _      = False
regexAccept (Alt r r') cs =
    regexAccept r cs || regexAccept r' cs
...
```

These cases are not so bad!

Seq and Star cases

```
regexAccept (Seq r r') cs = any (regexAccept r' . snd)
                               (initCheck r cs)
regexAccept (Star _) [] = True
regexAccept (Star r) cs =
  any (regexAccept (Star r) . snd) $ ignoreEmpty $
    initCheck r cs
  where ignoreEmpty = if regexAccept r [] then init else
    id
```

Table of Contents

Introduction

Finite Automata

Regular Expressions

Kleene's Theorem

Kleene's Theorem

Theorem

The following are equivalent, for a language $l \in \mathcal{P}(X)$:

- ▶ *it is represented by a state in a finite DA*

Kleene's Theorem

Theorem

The following are equivalent, for a language $L \in \mathcal{P}(X)$:

- ▶ *it is represented by a state in a finite DA*
- ▶ *it is represented by a state in a finite NA*

Kleene's Theorem

Theorem

The following are equivalent, for a language $L \in \mathcal{P}(X)$:

- ▶ *it is represented by a state in a finite DA*
- ▶ *it is represented by a state in a finite NA*
- ▶ *it is represented by a regular expression*

Kleene's Theorem

Theorem

The following are equivalent, for a language $l \in \mathcal{P}(X)$:

- ▶ *it is represented by a state in a finite DA*
- ▶ *it is represented by a state in a finite NA*
- ▶ *it is represented by a regular expression*

*in such a case, the language l is called **regular**.*