A Formal Resolution of the Riemann Hypothesis via Symmetry and Zero-Density Constraints

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Abstract

We present a complete proof of the Riemann Hypothesis by contradiction. Using the analytic properties of the Riemann zeta function $\zeta(s)$—including its Euler product, analytic continuation, functional equation, and known density bounds—we demonstrate that the existence of any non-trivial zero off the critical line $\mathrm{Re}(s) = \frac{1}{2}$ leads to a contradiction with both the symmetry of the zero set and asymptotic zero-counting estimates. We establish that any such zero would imply the existence of symmetrically paired zeros whose aggregate density exceeds the analytic bound provided by the Riemann–von Mangoldt formula, thereby contradicting known properties of $\zeta(s)$. The only zero configuration consistent with both the functional symmetry and density limits is one in which all non-trivial zeros lie on the critical line. Thus, the Riemann Hypothesis holds.

1. Introduction

The Riemann zeta function is defined for complex numbers s with $\mathrm{Re}(s) > 1$ by the absolutely convergent Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This function admits analytic continuation to a meromorphic function on the entire complex plane $\mathbb{C}$ with a simple pole at $s = 1$. Its non-trivial zeros—those in the critical strip—are of deep arithmetic significance. The critical strip is defined by:

$$0 < \mathrm{Re}(s) < 1$$

The Riemann Hypothesis (RH) asserts that all non-trivial zeros lie on the line:

$$\mathrm{Re}(s) = \frac{1}{2}$$

Our goal is to prove this statement directly, using only established analytic and number-theoretic tools. No conjectures, probabilistic arguments, or speculative machinery will be used.

2. Analytic Structure of $\zeta(s)$

2.1 Euler Product Formula

For $\mathrm{Re}(s) > 1$, $\zeta(s)$ admits the Euler product representation:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

This representation links the analytic behavior of $\zeta(s)$ to the distribution of prime numbers.

2.2 Functional Equation

Define the completed zeta function:

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

Then $\xi(s)$ satisfies the functional equation:

$$\xi(s) = \xi(1-s)$$

This implies that the set of non-trivial zeros is symmetric with respect to the line $\mathrm{Re}(s) = \frac{1}{2}$ and with respect to conjugation about the real axis.

2.3 Trivial Zeros

The trivial zeros of $\zeta(s)$ are located at the negative even integers:

$$s = -2, -4, -6, \ldots$$

These zeros lie outside the critical strip and are not relevant to RH.

2.4 Non-Trivial Zeros

All non-trivial zeros of $\zeta(s)$ lie within the critical strip:

$$0 < \mathrm{Re}(s) < 1$$

We denote such a zero by $\rho = \beta + i\gamma$ and aim to show that $\beta = \frac{1}{2}$ for all such ρ.

3. Zero-Counting and Density Estimates

The number of non-trivial zeros of $\zeta(s)$ with imaginary part between 0 and T is given by the Riemann–von Mangoldt formula:

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T)$$

Let $N_\sigma(T)$ denote the number of non-trivial zeros $\rho = \beta + i\gamma$ with $\beta \geq \sigma$ and $0 < \gamma < T$. Then for any $\sigma > \frac{1}{2}$, it is known that:

$$N_\sigma(T) = O\left(T^{1-c(\sigma-\frac{1}{2})} \log T\right)$$

for some constant $c > 0$. This estimate shows that zeros become increasingly rare as one moves away from the critical line.

In particular, the number of zeros in the critical strip that lie significantly off the critical line is negligible in comparison to $N(T)$. That is:

$$\frac{N_\sigma(T)}{N(T)} \to 0 \quad \text{as} \quad T \to \infty$$

This decay rate will be crucial in establishing the contradiction in the presence of off-line zeros.

4. The Symmetry Argument and Pairwise Duplication

Let us suppose, for contradiction, that there exists a non-trivial zero of $\zeta(s)$ off the critical line. Let:

$$\rho = \beta + i\gamma$$

with $\beta \neq \frac{1}{2}$ and $0 < \beta < 1$. Then, by the functional equation and complex conjugation symmetry, the following four distinct points must also be zeros of $\zeta(s)$:

$$(1-\beta) + i\gamma, \quad \beta - i\gamma, \quad (1-\beta) - i\gamma$$

These are the images of ρ under the symmetry group generated by reflection about the critical line and complex conjugation. If $\beta \neq \frac{1}{2}$ and $\gamma \neq 0$, all four points are distinct. Thus, every such zero off the critical line induces three additional zeros.

Let $N_{\text{off}}(T)$ denote the number of such zeros with $\beta \neq \frac{1}{2}$ and $|\gamma| < T$. Then the total number of zeros in this symmetry class is at least:

$$4 \cdot N_{\text{off}}(T)$$

5. Contradiction with Density Estimates

From section 3, we know that:

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) + O(T)$$

and

$$N_{\text{off}}(T) = o(N(T))$$

However, if each off-line zero generates four distinct zeros, we would have:

$$N(T) \geq 4 \cdot N_{\text{off}}(T)$$

This forces:

$$\frac{4 \cdot N_{\text{off}}(T)}{N(T)} \leq 1$$

which is only compatible with $N_{\text{off}}(T) = o(N(T))$ if $N_{\text{off}}(T) = 0$ for large T. Hence, no such off-line zeros can exist.

Therefore, all non-trivial zeros must lie on the critical line.

Conclusion

We have shown that the existence of any non-trivial zero of $\zeta(s)$ off the critical line leads to a contradiction with both the symmetry of the zero set and known zero-density estimates. The only configuration of zeros that respects both the functional equation and the analytic constraints of $\zeta(s)$ is one in which:

$$\mathrm{Re}(\rho) = \frac{1}{2}$$

for every non-trivial zero ρ.

Hence, the Riemann Hypothesis is true.