Analysis

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Contents

Chapter 1

Fourier Analysis and Distributions on $\mathbb{R}^d$

1.1 Fourier Analysis on $\mathbb{T}$

Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. You can view $\mathbb{T}$ as $[0, 2\pi]$ or $[-\pi, \pi]$.

1. First, let's consider the Fourier series on $L^2([0, 2\pi])$. Considering the following function space,
$$C_{\text{per}}([0, 2\pi]) = \{f \in C([0, 2\pi])\colon f(0) = f(2\pi)\}$$

   Proposition 1.1.1. *For $1 \leq p < \infty$, $C_{\text{per}}([0, 2\pi])$ is dense in $L^p([0, 2\pi])$.*

   Proof. For any $f \in L^p([0, 2\pi])$, we have for any $\varepsilon > 0$, there is a $\delta > 0$ such that
$$\left\| f\chi_{[\delta, 2\pi-\delta]} - f \right\|_p < \varepsilon$$
   Moreover, there is a $g \in C_0(\mathbb{R})$ such that $\operatorname{supp} g \in (0, 2\pi)$ and
$$\left\| f\chi_{[\delta, 2\pi-\delta]} - g \right\|_p < \varepsilon$$
   Therefore, $g \to f$ and $g \in C_{\text{per}}([0, 2\pi])$. □

   Remark. In general, for a locally compact space X with a Radon measure μ, we have $C_c(X)$ is dense in $L^p(X)$ for $1 \leq p < \infty$.

   Proof. First since L^p is approximated by simple functions, it is necessary for approximating χ_E by $C_c(X)$ for any Borel set E. Because Radon measure μ is inner regular on any σ-finite set, for any $\varepsilon > 0$, there are compact $K \subset E$ and open $U \supset E$ such that $\mu(U \backslash K) < \varepsilon$. Then by the Urysohn's lemma, there is a $f \in C_c(X)$ such that $\chi_E \leq f \leq \chi_U$. Therefore,
$$\|\chi_E - f\|_p \leq \mu(U \backslash K)^{1/p} < \epsilon^{1/p}$$
   □

   Theorem 1.1.1. $\left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n \in \mathbb{Z}}$ *is an orthonormal basis of $L^2([0, 2\pi])$.*

Proof. By above, we only need to prove that $f \in C_{\text{per}}([0, 2\pi])$ can be expressed by $\left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n \in \mathbb{Z}}$. Let

$$a_n = \int_0^{2\pi} f(x) e^{-inx} dx$$

For any $0 < r < 1$, let

$$f_r(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} a_n r^{|n|} e^{inx}$$

and it is uniformly convex because of the boundedness of a_n (Cauchy's convergence principle). Next, we need to check

$$\|f - f_r\|_\infty \to 0, \quad r \to 1$$

which implies $\|f - f_r\|_2 \to 0$ because of their compact definition. Let

$$\begin{aligned}
P_r(x) &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{inx} \\
&= \frac{1}{2\pi} \left(\sum_{n=0}^{\infty} r^n e^{inx} + \sum_{n=1}^{\infty} r^n e^{-inx} \right) \\
&= \frac{1}{2\pi} \left(\frac{1}{1 - re^{ix}} + \frac{re^{-ix}}{1 - re^{-ix}} \right) \\
&= \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos x + r^2} > 0
\end{aligned}$$

And since it is uniformly convergent

$$\int_0^{2\pi} P_r(x) dx = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_0^{2\pi} r^{|n|} e^{inx} dx = 1$$

So $f(x) = \int_0^{2\pi} f(x) P_r(y) dy$. And

$$\begin{aligned}
f_r(x) &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_0^{2\pi} f(y) e^{-iny} dy \right) r^{|n|} e^{inx} \\
&= \int_0^{2\pi} f(y) \left(\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(x-y)} \right) dy \\
&= \int_0^{2\pi} f(x - y) P_r(y) dy
\end{aligned}$$

so we have

$$\begin{aligned}
|f_r(x) - f(x)| &\le \left(\int_\delta^{2\pi - \delta} + \int_0^\delta + \int_{2\pi - \delta}^{2\pi} \right) |f(x - y) - f(x)| \, P_r(y) dy \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned}$$

Since $f \in C_{\text{per}}([0, 2\pi])$, we can choose δ such that $\mathrm{II} + \mathrm{III} < \varepsilon$. Besides,

$$\mathrm{I} \le 2\pi \cdot 2 \|f\|_\infty \max_{\delta \le y \le 2\pi - \delta} P_r(y)$$

Therefore, we have

$$\|f - f_r\|_\infty \le 2 \|f\|_\infty \frac{1 - r^2}{1 - 2r \cos \delta + r^2} + \varepsilon$$

$\square$

Remark. On finite measure domain, clearly convergence in L^∞ (or uniform) implies convergence in L^2 but the converse is note true.

Remark. Note that the Fourier transform

$$\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$$

preserves the norm.

Corollary 1.1.1. *For $f \in L^2([0, 2\pi])$, let*

$$a_n = \int_0^{2\pi} f(x)e^{-inx}dx$$

called the Fourier coefficients, then

$$f(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} a_n e^{inx}$$

convergent in L^2.

Note that the above is about the L^2 convergence. For uniform convergence, we need more.

Proposition 1.1.2. *For $f \in C_{\text{per}}([0, 2\pi])$, if f is differentiable and $f' \in L^2([0, 2\pi])$, then*

$$f(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} a_n e^{inx}$$

convergence in uniformly.

Proof. Consider

$$b_n = \int_0^{2\pi} f'(x)e^{-inx}dx = ina_n \ \Rightarrow\ a_n = \frac{b_n}{in}$$

$f' \in L^2([0, 2\pi])$ implies $\sum_n |b_n|^2 = \|f'\|_2^2 < \infty$. Therefore, by the Cauchy-Schwartz inequality, $\sum_n |a_n| < \infty$. Thus, the above convergence is uniform. □

Theorem 1.1.2. $\left\{\frac{1}{(\sqrt{2\pi})^n} e^{i(k_1x_1+k_2x_2+\cdots+k_nx_n)}\right\}$ *is an orthonormal basis of $L^2([0, 2\pi]^n)$.*

Proof. Only to prove the case $n = 2$. For $f \in L^2([0, 2\pi]^2)$, assume

$$\int_{I^2} f(x, y)e^{-ikx}e^{-ily}dxdy = 0, \quad \forall\, k, l \in \mathbb{Z}$$

where $I = [0, 2\pi]$. Let

$$g(x) = \left(\int_0^{2\pi} |f(x, y)|^2\, dy\right)^{\frac{1}{2}} \ \Rightarrow\ g \in L^2$$

and $g < \infty$ *a.e.*. Let $f_k(x) = \int_0^{2\pi} f(x, y)e^{-iky}dy$, then by the Cauchy-Schwartz inequality,

$$|f_k(x)| \le \sqrt{2\pi}g(x) \ \Rightarrow\ f_k \in L^2$$

Because $\int_0^{2\pi} f_k(x)e^{-ilx}dx = 0$ for all l, by above theorem, $f_k = 0$ *a.e.*. Considering two sets

$$E = \{x \in [0, 2\pi] \colon g(x) = \infty\}$$
$$E_k = \{x \in [0, 2\pi] \colon f_k(x) \neq 0\}$$

then for any $x \in [0, 2\pi] \backslash (E \cup \bigcup_k E_k)$,

$$\int_0^{2\pi} |f(x,y)|^2 \, dy < \infty, \quad \int_0^{2\pi} f(x,y)e^{-iky}dy = 0$$

so $f(x,y) = 0$ for almost every y. Therefore, by the Fubini's theorem,

$$\int_{I^2} |f(x,y)| \, dxdy = \int_0^{2\pi} \left(\int_0^{2\pi} |f(x,y)| \, dy \right) dx = 0$$

which implies $f(x,y) = 0$ *a.e.*. □

2. Next, we consider the Fourier series on $L^1([0, 2\pi])$, which is because $L^2([0, 2\pi]) \subset L^1([0, 2\pi])$ (Hölder's inequality) but the converse is not true.

Remark. On a measurable space (X, μ) with $\mu(X) < \infty$, we have

$$L^q(X) \subset L^p(X), \quad 1 \leq p < q \leq \infty$$

Proof. First, for $1 \leq p < q < \infty$, since $\frac{p}{q} < 1$, by the Hölder's inequality,

$$\int_X |f|^p \, d\mu \leq \left(\int_X |f|^q \, d\mu \right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}}$$

Therefore, $L^q(X) \subset L^p(X)$. For $q = \infty$, if $f \in L^\infty(X)$, then $|f(x)| \leq M$ *a.e.*. So

$$\int_X |f|^p \, d\mu \leq M^p \mu(X)$$

$f \in L^p(X)$. □

Besides, note that by the Young's inequality, for $1 \leq p, q, r < \infty$ such that,

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

(1) $p = 1, q = p$ and $r = p$,

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p, \quad \forall \, f \in L^1(\mathbb{R}), g \in L^p(\mathbb{R})$$

(2) $p = q = r = 1$,

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad \forall \, f, g \in L^1(\mathbb{R})$$

Proposition 1.1.3. *For $f, g \in L^1([0, 2\pi])$, the Fourier coefficients of $f, g, f * g$, denoted by a_n, b_n, c_n, are well-defined and*

$$c_n = a_n b_n$$

Proof. The well-definition is clear.

$$\begin{aligned}
c_n &= \int_0^{2\pi} f * g(x) e^{-inx} dx \\
&= \int_0^{2\pi} \int_0^{2\pi} f(x-y) g(y) dy e^{-inx} dx \\
&= \int_0^{2\pi} \int_0^{2\pi} f(x-y) g(y) e^{-in(x-y)} e^{-iny} dy dx \\
&= \int_0^{2\pi} \left(\int_0^{2\pi} f(x-y) e^{-in(x-y)} dx \right) g(y) e^{-iny} dy \\
&= a_n b_n
\end{aligned}$$

□

Proposition 1.1.4. *For $f \in L^1([0, 2\pi])$ and $0 < r < 1$, let*

$$P_r(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{inx} = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos x + r^2}$$

then we can prove

$$\lim_{r \to 1^-} \|f - P_r * f\|_1 = 0$$

Proof. For any $\varepsilon > 0$, there is $g \in C_{\text{per}}([0, 2\pi])$ such that

$$\|f - g\|_1 < \varepsilon$$

Let $g_r = g * P_r$. Similarly as above, we can prove

$$\|g - g_r\|_\infty < \varepsilon', \Rightarrow \|g - g_r\|_1 < \varepsilon$$

because of g, g_r defined on $[0, 2\pi]$. Thus, there is r_0 such that for any $r > r_0$,

$$\begin{aligned}
\|f - P_r * f\|_1 &\leq \|f - g\|_1 + \|g - P_r * g\|_1 + \|P_r * g - P_r * f\|_1 \\
&\leq 2\varepsilon + \|P_r\|_1 \|f - g\|_1 \\
&\leq 3\varepsilon
\end{aligned}$$

because $\|P_r\|_1 = 1$. □

Theorem 1.1.3. *For $f \in L^1([0, 2\pi])$, let a_n be its Fourier coefficients. Then*

$$f(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

in L^1 sense.

Remark. If we further know $(a_n) \in \ell^1$, then this convergence is uniform by the Weierstrass test, that is, for a sequence $\{f_n\}$, if $\sup |f_n(x)| \leq M_n$ and $\sum_n M_n < \infty$, then $\{f_n\}$ is uniformly. So $\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} a_n e^{inx}$ converges to a $g(x)$ uniformly, and combining this with above we know, $g(x) = f(x)$ *a.e.*. So if $f \in C$, it converges to f uniformly. Moreover, if $f \in C^2$, then $(a_n) \in \ell^1$ by the property of Fourier transform $a_n = \frac{1}{n^2} \widehat{f''}$.

Theorem 1.1.4. *For a complex value function f, if f has the properties* *(i)* $f \in L^1(\mathbb{R})$,

(ii) $\sum_{n\in\mathbb{Z}} f(x+2\pi n)$ *converges absolutes on* $[0, 2\pi]$,

(iii) $\sum_{n\in\mathbb{Z}} \widehat{f}(n) < \infty$,

then

$$\sum_{n\in\mathbb{Z}} f(2\pi n) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \widehat{f}(n),$$

Proof. For any $x \in [0, 2\pi]$, let

$$\begin{aligned} F(x) &= \sum_{n\in\mathbb{Z}} f(x+2\pi n) \\ G(x) &= \sum_{n\in\mathbb{Z}} |f(x+2\pi n)| \end{aligned}$$

Note that

$$\begin{aligned} \int_0^{2\pi} G(x)dx &= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} |f(x+2\pi n)|\, dx \\ &= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} |f(x+2\pi n)|\, dx \\ &= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} |f(x)|\, dx \\ &= \int_{\mathbb{R}} |f(x)|\, dx < \infty \end{aligned}$$

because $f \in L^1(\mathbb{R})$ and also $\widehat{f}$ is well-defined. Besides, $|F(x)| \leq G(x)$ so $F \in L^1([0, 2\pi])$.

$$\begin{aligned} \int_0^{2\pi} F(x)e^{-ikx}dx &= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} f(x+2\pi n)e^{-ikx}dx \\ &= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} f(x+2\pi n)e^{-ikx}dx \\ &= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} f(x+2\pi n)e^{-ik(x+2\pi n)}dx \\ &= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} f(x)e^{-ikx}dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \\ &= \widehat{f}(k) \end{aligned}$$

which means $\widehat{F}(k) = \widehat{f}(k)$. Moreover, since $\sum_{n\in\mathbb{Z}} \widehat{f}(n) < \infty$,

$$F(x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx}, \ \textit{a.e.}$$

So $2\pi F(0) = 2\pi \sum_{n\in\mathbb{Z}} f(2\pi n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$. $\square$

3. Summability kernel: For $k_n \in L^1([0, 2\pi])$, if

(i) $\int_0^{2\pi} k_n(x)dx = 1$,

(ii) $\exists\, C > 0$ such that $\|k_n\|_1 \leq C$ for all n,

(iii) for $0 < \delta < \pi$,

$$\int_{\delta \leq |x| \leq \pi} |k_n(x)|\, dx \to 0, \quad \text{as } n \to \infty$$

then (k_n) is called a summability kernel over $\mathbb{T}$.

Theorem 1.1.5. *Let (k_n) be a summability kernel over $\mathbb{T}$. Then for any $f \in C(\mathbb{T}) = C_{per}([0, 2\pi])$,*

$$k_n * f \to f, \quad \textit{in uniform}$$

Remark. And also by the density of $C(\mathbb{T})$ in $L^p(\mathbb{T})$ $(1 \leq p < \infty)$, similarly as above, for $f \in L^p(\mathbb{T})$,

$$k_n * f \to f, \quad \text{in } L^p - \text{norm}$$

In fact,

$$P_r(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{inx} = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos x + r^2}$$

is a summability kernel over $\mathbb{T}$ with continuous index $r \in [0, 1)$. So the proof of this theorem is as same as the above proof.

Here is another example. First, the Dirichlet's kernel

$$D_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} = \frac{\sin\left(N + \frac{1}{2}\right)x}{\sin \frac{x}{2}} \Rightarrow (D_N * f)(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} \widehat{f}(n) e^{inx} =: S_N(f)$$

And let

$$\sigma_N(f) = \frac{1}{N+1} \sum_{n=0}^{N} S_n(f)$$

called the Cesàro summation. Define Fejér kernel as

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(f) = \frac{1}{2\pi} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{inx}$$

Note that

(i) $\sigma_N(f) = K_N * f$,

(ii) $K_N(x) = \frac{1}{2\pi} \frac{1}{N+1} \left(\frac{\sin \frac{N+1}{2} x}{\sin \frac{x}{2}} \right)^2$,

(iii) $\int_0^{2\pi} K_N(x)dx = 1$,

(iv) for $0 < \delta < \pi$,

$$\sup_{\delta \leq |x| \leq \pi} K_N(x) \to 0, \quad \text{as } N \to \infty$$

Therefore, (K_N) is also a summability kernel over $\mathbb{T}$. And so for $f \in L^1(\mathbb{T})$,

$$(K_N * f)(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) e^{inx} \to f, \quad \text{in } L^1$$

First, it can prove the injectivity of Fourier transform $\mathcal{F} \colon L^1(\mathbb{T}) \to \ell^\infty$.

Corollary 1.1.2. *Let $f \in L^1(\mathbb{T})$. If $\widehat{f}(n) = 0$, then $f = 0$* a.e..

Proof. It is because

$$(K_N * f)(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) e^{inx} = 0, \ \forall \ N$$

□

Another application of K_N is it can prove the Riemannian-Lebesgue lemma.

Corollary 1.1.3 (Riemannian-Lebesgue lemma)**.** *For $f \in L^1(\mathbb{T})$,*

$$\lim_{n\to\infty} \widehat{f}(n) \to 0$$

Proof. Let $P_N(x) = (K_N * f)(x)$. Because $K_N \in L^1(\mathbb{T})$ and

$$K_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{inx}$$

by the uniqueness of Fourier coefficients (since $\sum_{n\in\mathbb{Z}} a_n e^{inx} = 0$ implies $a_n = 0$ for all n),

$$\widehat{K_N}(n) = 0, \ \forall \ |n| > N$$

So

$$\widehat{P_N}(n) = \widehat{K_N}(n)\widehat{f}(n) = 0 \ \forall \ |n| > N$$

Besides, for any $\varepsilon > 0$ there exists a N_0 such that

$$\|f - P_{N_0}\|_1 < \varepsilon$$

Therefore,

$$\begin{aligned} \left|\widehat{f}(n)\right| &\leq \left|\widehat{f}(n) - \widehat{P_{N_0}}(n)\right| + \left|\widehat{P_{N_0}}(n)\right| \\ &\leq \varepsilon + \left|\widehat{P_{N_0}}(n)\right| \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left|\widehat{f}(n)\right| < \varepsilon$$

□

4. Different Convergence: Consider complex-valued function f on $[0, 2\pi]$ satisfying

   (1) f is differentiable on $[0, 2\pi]$ except for finitely many points,

   (2) f' is bounded when it is well-defined

   Then it can prove that $\forall \ x_0 \in [0, 2\pi]$, $f(x_0 + 0)$ and $f(x_0 - 0)$ exist and

   $$\forall \ \varepsilon, \varepsilon' > 0, \quad |f(x_0 + \varepsilon) - f(x_0 + \varepsilon')| \leq K \, |\varepsilon - \varepsilon'|$$

   In fact, this implies f satisfies $\alpha(\alpha \in (0, 1])$-order Lipschitz condition, that is there are $\delta, K > 0$ such that for any $\varepsilon \in (0, \delta]$,

   $$|f(x_0 + \varepsilon) - f(x_0 + 0)| \leq K\varepsilon^\alpha, \quad |f(x_0 - \varepsilon) - f(x_0 - 0)| \leq K\varepsilon^\alpha$$

This will imply that

$$\varphi_{x_0}(x) = (f(x_0+x) - f(x_0+0)) + (f(x_0-x) - f(x_0-0)), \quad \frac{\varphi_{x_0}(x)}{x} \in L^1([0,\delta]) \quad (1.1)$$

Under this condition, when $f \in L^1([0,2\pi])$ (which is extended to $\mathbb{R}$ by setting the period $T = 2\pi$), for any $x \in [0,2\pi]$ and the Fourier coefficients a_n,

$$\lim_{N\to\infty} \frac{1}{2\pi} \sum_{n=-N}^{N} a_n e^{inx} = \frac{f(x+0)+f(x-0)}{2}$$

Proof. By calculating,

$$\begin{aligned}
\frac{1}{2\pi} \sum_{n=-N}^{N} a_n e^{inx} &= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_0^{2\pi} f(y) e^{in(x-y)} dy \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(y) \sum_{n=-N}^{N} e^{in(x-y)} dy \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(y) \frac{\sin(N+\frac{1}{2})(x-y)}{\sin\frac{x-y}{2}} dy \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(y+x) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy \\
&= \frac{1}{2\pi} \left(\int_0^{\pi} + \int_{-\pi}^{0} \right) f(y+x) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy \\
&= \frac{1}{2\pi} \int_0^{\pi} (f(x+y) + f(x-y)) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy
\end{aligned}$$

If $f(x) = 1$, the $a_n = 0$ for $n \neq 0$ and $a_0 = 2\pi$. So above equality implies

$$1 = \frac{1}{2\pi} \int_0^{\pi} 2 \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy$$

Therefore, we have

$$\frac{f(x_0+0) - f(x_0-0)}{2} = \frac{1}{2\pi} \int_0^{\pi} (f(x_0+0) - f(x_0-0)) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy$$

And so

$$\frac{1}{2\pi} \sum_{n=-N}^{N} a_n e^{inx} - \frac{f(x+0)-f(x-0)}{2} = \frac{1}{2\pi} \int_0^{\pi} \varphi_x(y) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy$$

Let $G(y) = \frac{\varphi_x(y)}{\sin\frac{y}{2}}$. Then

$$\begin{aligned}
\int_0^{\pi} |G(y)|\, dy &= \int_0^{\pi} \left| \frac{\varphi_x(y)}{\frac{1}{2}y} \cdot \frac{\frac{1}{2}y}{\sin\frac{y}{2}} \right| dy \\
&\leq K \int_0^{\pi} \left| \frac{\varphi_x(y)}{\frac{1}{2}y} \right| dy \\
&\leq \infty
\end{aligned}$$

which is because $\frac{\varphi_x(y)}{\frac{1}{2}y} \in L^1([0,\pi])$ and $\left|\frac{\frac{1}{2}y}{\sin\frac{y}{2}}\right| \leq K$ on $[0,\pi]$. Therefore, $G \in L^1([0,\pi])$ so that by the Riemannian-Lebesgue lemma,

$$\int_0^\pi \varphi_x(y)\frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}}dy = \int_0^\pi G(y)\sin(N+\frac{1}{2})ydy \to 0, \quad \text{as } N \to \infty$$

$\square$

But if the condition (1.1) is not satisfied, what can we get?

Theorem 1.1.6. *For $f \in L^1(\mathbb{T})$, if $f(x_0+0)$ and $f(x_0-0)$ exist and let $\alpha = \frac{1}{2}(f(x_0+0)+f(x_0)-0)$, then*

$$\lim_{N\to\infty}\sigma_N(f)(x_0) = \alpha$$

Proof. Fix any $\varepsilon > 0$. Because $\sigma_N = K_N * f$ and $K_N(-x) = K_N(x)$ and $\int_\mathbb{T} K_N(x)dx = 1$,

$$\begin{aligned}
\sigma_N(f)(x_0) - \alpha &= \int_\mathbb{T} f(x_0-x)K_N(x)dx - \alpha \\
&= \int_{-\pi}^{\pi} f(x_0-x)K_N(x)dx - \alpha \\
&= \int_0^\pi (f(x_0-x)+f(x_0+x)-2\alpha)K_N(x)dx \\
&= \left(\int_0^\delta + \int_\delta^\pi\right)(f(x_0-x)+f(x_0+x)-2\alpha)K_N(x)dx
\end{aligned}$$

with $\delta > 0$ such that

$$|f(x_0-x)+f(x_0+x)-2\alpha| < \varepsilon, \quad x \in (0,\delta)$$

and

$$\sup_{\delta\leq|x|\leq\pi} K_N(x) < \varepsilon$$

Then, we have

$$|\sigma_N(f)(x_0)-\alpha| \leq \mathrm{I} + \mathrm{II}$$

where

$$\begin{aligned}
\mathrm{I} &= \int_0^\delta |(f(x_0-x)+f(x_0+x)-2\alpha)K_N(x)|\,dx \\
&\leq \varepsilon\int_0^\delta |K_N(x)|\,dx \\
&\leq \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{II} &= \int_\delta^\pi |(f(x_0-x)+f(x_0+x)-2\alpha)K_N(x)|\,dx \\
&\leq (\|f\|_1+\alpha)\varepsilon
\end{aligned}$$

$\square$

Note that if $S_N(f)(x_0)$ exists as $N\to\infty$, then it converges to α.

Now, let's consider more about the uniform convergence.

Theorem 1.1.7. *For f defined on $\mathbb{T}$, if there is a $\varphi \in L^1(\mathbb{T})$ with $\int_{\mathbb{T}} \varphi(x)dx = 0$ such that*

$$f(x) = \int_0^x \varphi(s)ds + f(0), \quad x \in \mathbb{T}$$

(which implies $f \in C(\mathbb{T})$), then $S_N(f) \to f$ uniformly as $N \to \infty$.

Remark. Note that if $f \in C^1$, then above condition is clearly satisfied.

Lemma 1.1.1. *The following lemmas are needed for proving above theorem.*

I. *For any continuous map $t \mapsto h_t$ from closed bounded interval $I \subset \mathbb{R}$ to $L^1(\mathbb{T})$, then*

$$\lim_{|n|\to\infty} \widehat{h_t}(n) = 0$$

uniformly on I.

II. *Let $I_N(t) = \int_0^t D_N(s)ds$ defined on $[-\pi, \pi]$. Then there is $C > 0$ such that for all $N \in \mathbb{N}_0$ and $t \in [-\pi, \pi]$,*

$$|I_N(t)| \leq C$$

III. *(Integral by Parts) For functions f, g defined on $[-\pi, \pi]$, if there are $\varphi, \psi \in L^1([-\pi, \pi])$ such that*

$$f(t) = \int_0^t \varphi(s)ds + f(0), \quad g(t) = \int_0^t \psi(s)ds + g(0)$$

then for any $[a, b] \subset [-\pi, \pi]$,

$$\int_a^b f(t)\psi(t)dt = f(b)g(b) - f(a)g(a) - \int_a^b \varphi(t)g(t)dt$$

Proof. I. For $\varepsilon > 0$, by the compactness of I, there is a partition $\{t_j\}$ of I such that

$$\left\|h_t - h_{t_j}\right\|_1 < \varepsilon, \quad \forall\, t \in [t_j, t_{j+1}]$$

For each h_{t_j}, let $P_j^{N_j}$ be the series such that $\left\|h_{t_j} - P_j\right\|_1 < \varepsilon$. Let $N = \max_j N_j$. Then for all $n > N$, $\forall\, t \in [t_j, t_{j+1}]$

$$\begin{aligned}\left|\widehat{h_t}(n)\right| &= \left|\widehat{h_t}(n) - \widehat{P_j^{N_j}}(n)\right| \\ &\leq \left\|h_t - P_j^{N_j}\right\|_1 \\ &\leq \left\|h_t - h_{t_j}\right\|_1 + \left\|h_{t_j} - P_j\right\|_1 \\ &< 2\varepsilon,\end{aligned}$$

II. It is because

$$\begin{aligned}I_N(t) &= \int_0^t \frac{\sin\left(N + \frac{1}{2}\right)s}{\sin\frac{s}{2}}ds \\ &= \int_0^t \left(\frac{1}{\frac{s}{2}} - \frac{1}{\sin\frac{s}{2}}\right)\sin\left(N + \frac{1}{2}\right)sds + 2\int_0^t \frac{\sin\left(N + \frac{1}{2}\right)s}{s}ds.\end{aligned}$$

The first term is bounded because of $s \sim \sin s$ as $s \to 0$. For the second term, it is $2\int_0^{\left(N+\frac{1}{2}\right)t} \frac{\sin x}{x}dx$, which converges to $\frac{\pi}{2}$ as $N \to \infty$.

III. It can be obtained by approximating L^1-functions by C-functions.

□

Proof of Theorem 1.1.7. By $S_N(f) = D_N * f$ and $\int_{-\pi}^{\pi} D_N(x)dx = 1$,

$$\begin{aligned} S_N(f)(t) - f(t) &= \int_{-\pi}^{\pi} (f(t-s) - f(t))D_N(s)ds \\ &= \int_{-\delta}^{\delta} + \int_{\delta \le |s| \le \pi} (f(t-s) - f(t))D_N(s)ds \end{aligned}$$

Let $\chi_\delta = \chi_{[-\pi,-\delta]\cup[\delta,\pi]}$. Then the second term is

$$\int_{-\pi}^{\pi} \underbrace{(f(t-s) - f(t))\frac{1}{\sin\frac{s}{2}}\chi_\delta(s)}_{=:h_t(s)} \sin\left(N + \frac{1}{2}\right) sds$$

For $t \mapsto h_t \in L^(\mathbb{T})$, by above lemma, this term converges uniformly to 0 as $N \to \infty$ on $\mathbb{T}$. For the first term, by the lemma,

$$\begin{aligned} &\int_{-\delta}^{\delta} (f(t-s) - f(t))D_N(s)ds \\ =& \left((f(t-\delta) - f(t))I_N(\delta) - (f(t+\delta) - f(t))I_N(-\delta) + \int_{-\delta}^{\delta} \varphi(t-s)I_N(s)ds\right) \\ =& \left((f(t+\delta) + f(t-\delta) - 2f(t))I_N(\delta) + \int_{-\delta}^{\delta} \varphi(t-s)I_N(s)ds\right). \end{aligned}$$

The first term can be arbitrarily small because f is uniformly continuous on $\mathbb{T}$ and I_N is bounded. For the second term, let $\varphi_0 \in C(\mathbb{T})$ such that $\|\varphi - \varphi_0\|_1 < \varepsilon$. Then

$$\begin{aligned} \left|\int_{-\delta}^{\delta} \varphi(t-s)I_N(s)ds\right| &\le C\int_{-\delta}^{\delta} |\varphi(t-s)|ds < C\int_{-\delta}^{\delta} |\varphi_0(t-s)|\,ds + 2\pi C\varepsilon \\ &\le 2C\delta \|\varphi_0\|_\infty + 2\pi C\varepsilon < 10C\varepsilon. \end{aligned}$$

which can be arbitrarily small. □

This technique can be applied to proving the Riemannian local principle.

Theorem 1.1.8. *For $f, g \in L^1(\mathbb{T})$, let $J \subset \mathbb{T}$ open interval such that $f = g$ on J. Then for closed interval $I \subset J$, $S_N(f) - S_N(g)$ converges to 0 uniformly as $N \to \infty$ on I.*

The proof is basically same by replacing $f(t-s) - f(t)$ with $f(t-s) - g(t-s)$.

Corollary 1.1.4. *Let $f \in L^1(\mathbb{T})$ and $J \subset \mathbb{T}$ open interval. If $f \in C^1(J)$, then for closed interval $I \subset J$, then $S_N(f)$ converges to I uniformly.*

Proposition 1.1.5. *For $f \in L^1(\mathbb{T})$, there is a $g \in C^\infty(\mathbb{T})$ such that $f = g$* a.e. *if and only if for $\widehat{f} = (a_n)_{n\in\mathbb{Z}}$, $a_n = o(|n|^{-k})$ as $n \to \infty$.*

Proof. $\Rightarrow$: if $f \in C^\infty$, then

$$\widehat{f}(n) = \frac{\widehat{f^{(k)}}(n)}{(in)^k}$$

$\Leftarrow$: first, by $a_n = o(|n|^{-k})$, $(a_n) \in \ell^1$. So

$$S_N(f)(t) = \frac{1}{2\pi} \sum_{n=-N}^{N} a_n e^{int}$$

converse uniformly on $\mathbb{T}$, denoted by $g(t)$. And because $a_n = o(|n|^{-k})$, $g \in C^\infty$. Besides, by the uniqueness of Fourier series, $g = f$ *a.e.*. $\square$

5. More properties: First, let summarize the proved properties. Considering the Fourier transform

$$\mathcal{F}\colon L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$$

   (i) $\mathcal{F}$ is injective;

   (ii) $\operatorname{Im}\mathcal{F} \subset c_0(\mathbb{Z})$, where

$$c_0(\mathbb{Z}) = \{(a_n) \in \ell^\infty(\mathbb{Z})\colon a_n \to \infty, \text{ as } |n| \to \infty\}$$

   (iii) when restricting $\mathcal{F}$ on $L^2(\mathbb{T})$, $\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ preserves the norm.

1.2 Fourier Analysis on $\mathbb{R}$

1. Basic properties: First note that the Fourier transform can be well-defined on $L^1(\mathbb{R})$, because

$$\left|\int_{\mathbb{R}} f(x)e^{-ix\xi}dx\right| \le \int_{\mathbb{R}} \left|f(x)e^{-ix\xi}\right| dx = \int_{\mathbb{R}} |f(x)|\, dx < \infty$$

and the Fourier transform $\widehat{f}$ is uniformly continuous on $\mathbb{R}$, because

$$|\widehat{f}(\xi+\eta) - \widehat{f}(\xi)| = \left|\int_{\mathbb{R}} f(x)\left(e^{-i(\xi+\eta)x} - e^{-i\xi x}\right) dx\right| \le \int_{\mathbb{R}} |f(x)| \left|e^{-i\eta x} - 1\right| dx \le 2\,\|f\|_1$$

and by the DCT, the RHS is integrable and $\to 0$ as $\eta \to 0$ independent with ξ.

Examples of Fourier analysis:

   (1) Considering the characteristic function $\chi_{[-1,1]}(x)$ on $\mathbb{R}$, clearly $\chi_{[-1,1]} \in L^1 \cap L^2$. So

$$\begin{aligned}\widehat{\chi}_{[-1,1]}(\xi) &= \int_{\mathbb{R}} \chi_{[-1,1]}(x)e^{-ix\xi}dx \\ &= \int_{-1}^{1} \cos\xi x dx + i\int_{-1}^{1} \sin\xi x dx \\ &= 2\frac{\sin\xi}{\xi}\end{aligned}$$

   Note that $\widehat{\chi}_{[-1,1]}(\xi) \notin L^1$, because if it is, then $\chi_{[-1,1]} = \check{\widehat{\chi}}_{[-1,1]} \in C_0$.
   Moreover, based on these results and the inverse formula, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin\xi}{\xi}e^{ix\xi}d\xi = \chi_{[-1,1]}(x) \;\Rightarrow\; \int_{-\infty}^{\infty} \frac{\sin\xi}{\xi}d\xi = \pi\chi_{[-1,1]}(0) = \pi$$

Besides, we have

$$\int_{-\infty}^{\infty} \frac{\sin \alpha x}{x} dx = \begin{cases} \pi, & \alpha > 0 \\ -\pi, & \alpha < 0 \\ 0, & \alpha = 0 \end{cases}$$

(2) For $\alpha > 0$, considering

$$\frac{1}{\cosh \alpha x} = \frac{2}{e^{\alpha x} + e^{-\alpha x}} \in L^1 \cap L^2$$

by directly calculating. Then

$$\mathcal{F}\left(\frac{1}{\cosh \alpha x}\right)(\xi) = \int_{\mathbb{R}} \frac{2e^{-ix\xi}}{e^{\alpha x} + e^{-\alpha x}} dx$$

Then by the residues theorem, we have

$$\mathcal{F}\left(\frac{1}{\cosh \alpha x}\right)(\xi) = \frac{\pi}{\alpha} \frac{1}{\cosh \frac{\pi}{2\alpha}\xi}$$

2. Kernels: For $\lambda > 0$, define

$$D_\lambda(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} e^{i\xi x} d\xi, \quad K_\lambda(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{i\xi x} d\xi$$

It can easily obtain $D_\lambda(x) = \frac{\sin \lambda x}{\pi x} = \lambda D_1(\lambda x)$. And for $K_\lambda(x)$, from integral by parts,

$$K_\lambda(x) = \frac{\lambda}{2\pi} \left(\frac{\sin(\lambda x/2)}{\lambda x/2}\right)^2$$

and also $K_\lambda(x) = \lambda K_1(\lambda x)$.

In general, let's consider the summability kernel in continuous version. A family of $(k_\lambda)_{\lambda>0}$ in $L^1(\mathbb{R})$ is called a summability kernel on $\mathbb{R}$ if

(1) for all λ, $\int_{\mathbb{R}} k_\lambda(x) dx = 1$,

(2) there is a $C > 0$ such that $\|k_\lambda\|_1 \le C$ for all λ,

(3) for any $\delta > 0$,

$$\lim_{\lambda \to \infty} \int_{|x| \ge \delta} |k_\lambda(x)| \, dx = 0$$

Note that $(K_\lambda)_{\lambda>0}$ is a summability kernel but not $(D_\lambda)_{\lambda>0}$.

Similarly as the discrete case, the continuous summability kernel has the same properties.

Theorem 1.2.1. *Let $(k_\lambda)_{\lambda>0}$ be a summability kernel.*

(1) For uniform continuous $f \in L^1(\mathbb{R})$, then

$$\lim_{\lambda \to \infty} k_\lambda * f = f$$

uniformly.

(2) Let $1 \le p < \infty$. For any $f \in L^p(\mathbb{R})$,

$$\lim_{\lambda \to \infty} k_\lambda * f = f$$

in L^p-norm.

Proof. (1) Because f is uniformly continuous, for any $\varepsilon > 0$, there is a $\delta > 0$ *s.t.* for any $|y| < \delta$,

$$|f(x-y) - f(x)| < \varepsilon$$

Besides, $f \in L^1$,

$$\begin{aligned}
|(k_\lambda * f)(x) - f(x)| &\le \int_{\mathbb{R}} |f(x-y) - f(x)| \, |k_\lambda(y)| \, dy \\
&\le \left(\int_{|y|<\delta} + \int_{|y|\ge\delta} \right) |f(x-y) - f(x)| \, |k_\lambda(y)| \, dy \\
&\le \int_{|y|<\delta} \varepsilon \, |k_\lambda(y)| \, dy + \int_{|y|\ge\delta} 2 \, \|f\|_1 \, |k_\lambda(y)| \, dy
\end{aligned}$$

(2) For $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|y| < \delta \;\Rightarrow\; \int_{\mathbb{R}} |f(x-y) - f(x)|^p dx < \varepsilon$$

which is because $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.
For $p > 1$, let q be the conjugate of p. By the Hölder's inequality ($|f(x-y) - f(x)| \, |k_\lambda(y)| = (|f(x-y) - f(x)|^p \, |k_\lambda(y)|)^{\frac{1}{p}} \, |k_\lambda(y)|^{\frac{1}{q}}$),

$$\begin{aligned}
\|k_\lambda * f - f\|_p^p &\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-y) - f(x)| \, |k_\lambda(y)| \, dy \right)^p dx \\
&\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-y) - f(x)|^p \, |k_\lambda(y)| \, dy \right) \left(\int_{\mathbb{R}} |k_\lambda(y)| \, dy \right)^{p/q} dx \\
&\le C^{p/q} \int_{\mathbb{R}} |k_\lambda(y)| \left(\int_{\mathbb{R}} |f(x-y) - f(x)|^p dx \right) dy \\
&= 2^p C^{p/q} \left(\int_{|y|<\delta} \varepsilon \, |k_\lambda(y)| \, dy + \int_{|y|\ge\delta} \|f\|_p^p \, |k_\lambda(y)| \, dy \right) \to 0, \quad \text{as } \lambda \to \infty
\end{aligned}$$

For $p = 1$, it has the similar proof without the Hölder's inequality.

□

Corollary 1.2.1. *For $1 \le p < \infty$, $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.*

Proof. First, because $\|f - f\chi_{-M,M}\|_p \to 0$ as $M \to \infty$, we only need to consider f with $f(x) = 0$ for $|x| > M$. Choose $\varphi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi(x)dx = 1$. For any $\lambda > 0$, let $\varphi_\lambda(x) := \lambda\varphi(\lambda x)$. Then $(\varphi_\lambda)_{\lambda>0}$ is a summability kernel. So $\varphi_\lambda * f \to f$ in L^p. Besides, by the DCT, $\varphi_\lambda * f \in C^\infty$. Moreover, for any λ, $\operatorname{supp}\varphi_\lambda \in [-L, L]$ for some large L. Then if $|x| > M + L$, then because $|y| < m$ to guarantee $f(y) \ne 0$,

$$(\varphi_\lambda * f)(x) = \int_{-M}^{M} \varphi_\lambda(x-y) f(y) dy = 0$$

by $|x - y| > L$. So $\operatorname{supp}\varphi_\lambda * f \subset [-M - L, M + L]$. □

If $f \in C_c^\infty(\mathbb{R})$, then

$$\widehat{f'}(\xi) = i\xi\widehat{f}(\xi)$$

So by above corollary, we can prove the continuous version of Riemannian-Lebesgue lemma.

Lemma 1.2.1 (Riemannian-Lebesgue Lemma). *For any $f \in L^1(\mathbb{R})$,*

$$\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0$$

Proof. For $\varepsilon > 0$, there is a $g \in C_c^\infty(\mathbb{R})$ such that $\|f - g\|_1 < \varepsilon$. For g, we know $\widehat{g}(\xi) \to 0$ as $|\xi| \to \infty$. So

$$\left|\widehat{f}(\xi)\right| \leq |\widehat{f}(\xi) - \widehat{g}(\xi)| + |\widehat{g}(\xi)| < \|f - g\|_1 + \varepsilon < 2\varepsilon$$

□

3. Applications of Kernels:

Corollary 1.2.2. *(1) For $f \in L^1(\mathbb{R})$ and $\lambda > 0$, let $\sigma_\lambda(f) = K_\lambda * f$. Then for any $x \in \mathbb{R}$,*

$$\sigma_\lambda(f)(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) \widehat{f}(\xi) e^{i\xi x} d\xi$$

and $\sigma_\lambda(f) \to f$ in L^1 as $\lambda \to \infty$.

(2) Fourier transform

$$\mathcal{F} \colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$$

is an injection.

Corollary 1.2.3. *Assume $f, \widehat{f} \in L^(\mathbb{R})$. Then*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{i\xi x} d\xi (=: \check{\widehat{f}}), \ a.e.$$

Proof. By above, for any $x \in \mathbb{R}$,

$$\sigma_\lambda(f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \chi_{[-\lambda,\lambda]}(\xi) \left(1 - \frac{|\xi|}{\lambda}\right) \widehat{f}(\xi) e^{i\xi x} d\xi.$$

By the DCT, as $\lambda \to \infty$,

$$\sigma_\lambda(f)(x) \to \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{i\xi x} d\xi.$$

And combining this with $\sigma_\lambda(f) \to f$, we have the above result. □

Note that if $f \in C^2(\mathbb{R})$ and $f, f', f'' \in L^1(\mathbb{R})$, then the above formula always true for all $x \in \mathbb{R}$, because $\widehat{f}(\xi) = (i\xi)^{-2}\widehat{f''}(\xi) \in L^1(\mathbb{R})$.

Proposition 1.2.1. *If $F(\xi) \in L^1(\mathbb{R})$, then the Fourier inverse transform $\check{F}(x)$ exists and $\check{F}(x) \in C_0(\mathbb{R})$.*

Proof. First,

$$\check{F}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} F(\xi) e^{ix\xi} d\xi$$

The existence of $\check{F}(x)$ is similar as the above.

For the continuity of $\check{F}(x)$,

$$\begin{aligned}\left|\check{F}(x)-\check{F}(x_0)\right| &= \frac{1}{2\pi}\left|\int_{\mathbb{R}} F(\xi)e^{ix\xi} - F(\xi)e^{ix_0\xi}d\xi\right| \\ &\le \frac{1}{2\pi}\int_{\mathbb{R}} |F(\xi)|\left|e^{ix\xi}-e^{ix_0\xi}\right| d\xi \\ &\le \frac{1}{2\pi}\int_{\mathbb{R}} |F(\xi)|\left|e^{ix\xi}-e^{ix_0\xi}\right| d\xi \\ &\le \frac{1}{\pi}\int_{\mathbb{R}} |F(\xi)|\, d\xi\end{aligned}$$

and $e^{ix\xi} \to e^{ix_0\xi}$ as $x \to x_0$, so by the Dominant Convergence Theorem

$$\check{F}(x) \to \check{F}(x_0), \quad \text{as } x \to x_0$$

For the C_0,

$$\lim_{|x|\to\infty} \check{F}(x) = 0,$$

which is because of the Riemannian-Lebesgue lemma. □

Corollary 1.2.4. *For K_λ,*

$$\widehat{K_\lambda}(\xi) = \chi_{[-\lambda,\lambda]}(\xi)\left(1 - \frac{|\xi|}{\lambda}\right).$$

Proof. Let $\varphi_\lambda(\xi) = \chi_{[-\lambda,\lambda]}(\xi)\left(1 - \frac{|\xi|}{\lambda}\right)$. By definition,

$$K_\lambda(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_\lambda(\xi)e^{i\xi x}d\xi = \frac{1}{2\pi}\widehat{\varphi_\lambda}(-x)$$

Therefore, by the inverse formula,

$$\varphi_\lambda(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi_\lambda}(x)e^{i\xi x}dx = \int_{\mathbb{R}} K_\lambda(-x)e^{i\xi x}dx = \widehat{K_\lambda}(\xi)$$

□

1.3 Fourier Analysis on $\mathbb{R}^d$

On $\mathbb{R}^d$, the Fourier transform is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}dx, \quad \xi \in \mathbb{R}^d$$

1. Rapidly Decreasing Functions: For f defined on $\mathbb{R}^d$ and $k \in \mathbb{N}$, f is called a rapidly decreasing function if
$$\lim_{|x|\to\infty} |x|^k f(x) = 0$$
or $f(x) = o(|x|^{-k})$ as $|x| \to \infty$. The Schwartz space is defined as
$$\mathcal{S} = \mathcal{S}(\mathbb{R}^d) = \left\{f \in C^\infty(\mathbb{R}^d)\colon \partial^\alpha f \text{ is rapidly decreasing, } \forall\ \alpha\right\}$$
$f \in \mathcal{S}$ is called Schwartz rapidly decreasing function.

Proposition 1.3.1. *(1)* $C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}$,

(2) $f \in \mathcal{S}$ *implies* $\partial^\alpha f \in \mathcal{S}$ *for all* α,

(3) p *is a polynomial, then for* $f \in \mathcal{S}$, $pf \in \mathcal{S}$.

Proposition 1.3.2. *(1) For any* $f \in \mathcal{S}$ *and* α, $\widehat{\partial^\alpha f}(\xi)$ *exists and*

$$\widehat{\partial^\alpha f}(\xi) = i^{|\alpha|}\xi^{|\alpha|}\widehat{f}(\xi)$$

(2) For any $f \in \mathcal{S}$, $\widehat{f} \in C^\infty(\mathbb{R}^d)$ *and*

$$\widehat{x^\alpha f}(\xi) = i^{|\alpha|}(\partial \widehat{f})(\xi)$$

(3) For any $f \in \mathcal{S}$, $\widehat{f} \in \mathcal{S}$.

2. Inverse Formula: First, because

$$C_c^\infty(\mathbb{R}^d) \subset \mathcal{S} \subset L^p(\mathbb{R}^d)$$

$\mathcal{S}$ is dense in $L^p(\mathbb{R}^d)$ $(1 \le p < \infty)$.

Proposition 1.3.3 (Riemannian-Lebesgue Lemma)**.** *For any* $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \in C_0(\mathbb{R}^d)$.

Proof. Continuity of $\widehat{f}$ is obtained from the DCT. $\exists\, g \in C_c^\infty \subset \mathcal{S}$ such that $\|f-g\|_1 \le \varepsilon$. So

$$\left|\widehat{f}(\xi)\right| \le \left\|\widehat{f} - \widehat{g}\right\|_\infty + |\widehat{g}(\xi)| \le \|f-g\|_1 + |\widehat{g}(\xi)|$$

□

Theorem 1.3.1. *Assume* $f, \widehat{f} \in L^1(\mathbb{R}^d)$. *Then*

$$f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{f}(\xi)e^{i\xi\cdot x}d\xi,,\ \ a.e.$$

In particular,

$$\left(\mathcal{F}^2 f\right)(x) = (\mathcal{F}\widehat{f})(x) = (2\pi)^d f(-x).$$

Proof. Consider a function $\kappa \in L^1(\mathbb{R}^d)$ that is bounded and

(1) κ is continuous at 0 and $k(0) = 1$,

(2) let

$$k(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \kappa(\xi)e^{i\xi\cdot x}d\xi$$

and $k \in L^1$ with $\widehat{k}(0) = \int_{\mathbb{R}^d} k(x)dx = 1$.

Then for $\lambda > 0$, let $k_\lambda(x) = \lambda^d k(\lambda x)$ so $(k_\lambda)_{\lambda>0}$ is a summability kernel on $\mathbb{R}^d$. And therefore, $k_\lambda * f \to f$ in L^1. Besides,

$$\begin{aligned}
(k_\lambda * f)(x) &= \int_{\mathbb{R}^d} k_\lambda(x-y)f(y)dy \\
&= \frac{\lambda^d}{(2\pi)^d}\int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}^d}\kappa(\xi)e^{i\xi\cdot\lambda(x-y)}d\xi\right)dy \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\kappa(\eta/\lambda)e^{i\eta\cdot(x-y)}d\eta\right)f(y)dy \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\eta)\kappa(\eta/\lambda)e^{i\eta\cdot x}d\eta \\
&\to \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\eta)e^{i\eta\cdot x}d\eta \quad (\lambda \to \infty).
\end{aligned}$$

by the DCT. □

Remark. An example of κ is

$$\kappa(x) = \prod_{i=1}^{d} \chi_{[-1,1]}(x_i)(1 - |x_i|)$$

Corollary 1.3.1. *(1) For $f \in L^1(\mathbb{R}^d)$, $\widehat{f} = 0$ implies $f = 0$.*

(2) The Fourier transform $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$ is a bijection.

Proof. (1) is by the inverse formula, which implies that $\mathcal{F}$ is an injective. For (2), for any $f \in \mathcal{S}$,

$$\left(\mathcal{F}^4 f\right)(x) = (2\pi)^d \left(\mathcal{F}^2 f\right)(-x) = (2\pi)^{2d} f(x)$$

let $g = \mathcal{F}^3\left((2\pi)^{-2d} f\right)$, then $\mathcal{F}(g) = f$. So $\mathcal{F}$ is a surjection. $\square$

Proposition 1.3.4. *For any $f, g \in \mathcal{S}$, $f * g \in \mathcal{S}$.*

Proof. First, for $\phi, \psi \in \mathcal{S}$, $\phi\psi \in \mathcal{S}$. First, $f * g \in L^1(\mathbb{R}^d)$ and $\widehat{f}, \widehat{g} \in \mathcal{S}$. Therefore,

$$\widehat{f * g} = \widehat{f}\widehat{g} \in \mathcal{S}$$

and thus by the inverse, $f * g = \mathcal{F}^{-1}(\widehat{f * g}) \in \mathcal{S}$. $\square$

3. Plancherel Theorem:

Lemma 1.3.1. *For any $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, there is a $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $f_n \to f$ both in L^1 and L^2 norms.*

Theorem 1.3.2. *There is a unique linear map*

$$\mathcal{F}\colon L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$$

such that

(1) for any $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\widehat{f} = \mathcal{F}(f)$,

(2) for any $f \in L^2(\mathbb{R}^d)$, $\|\mathcal{F}f\|_2 = (2\pi)^{d/2}\|f\|_2$.

Proof. I. Isometry on $\mathcal{S}$: For $f, g \in \mathcal{S}$, let $h(x) = \bar{\widehat{g}}(x)$. So

$$\widehat{h}(\xi) = \int_{\mathbb{R}^d} \overline{\widehat{g}(x)} e^{-i\xi\cdot x} dx = \overline{\int_{\mathbb{R}^d} \widehat{g}(x) e^{i\xi\cdot x} dx} = (2\pi)^d \overline{g(\xi)}$$

and because

$$\int_{\mathbb{R}^d} f(x)\widehat{h}(x)dx = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)h(y)e^{-ix\cdot y} dxdy = \int_{\mathbb{R}^d} \widehat{f}(y)h(y)dy$$

we have

$$(2\pi)^d\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$$

II. Constructing $\mathcal{F}$: For $f \in L^2(\mathbb{R}^d)$, let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $f_n \to f$ in L^2. Because

$$\left\| \widehat{f_n} - \widehat{f_m} \right\|_2 = (2\pi)^{d/2} \|f_n - f_m\|_2$$

$(\widehat{f_n})_{n\in\mathbb{N}}$ is a Cauchy in $L^2(\mathbb{R}^d)$. By the completeness of $L^2(\mathbb{R}^d)$, there is a $\widehat{f} \in L^2(\mathbb{R}^d)$ such that $\widehat{f_n}) \to \widehat{f})$ in L^2. Therefore,

$$\mathcal{F}(f) := \widehat{f}$$

And this definition is independent with the choice of the f_n, because if there are such g_n and $\widehat{g}$, then

$$\left\| \widehat{f} - \widehat{g} \right\|_2 \leq \left\| \widehat{f} - \widehat{f_n} \right\|_2 + \|\widehat{g_n} - \widehat{g}\|_2 + \left\| \widehat{f_n} - \widehat{g_n} \right\|_2,$$

where these terms are all arbitrarily small. Besides, we have $\|\mathcal{F}(f)\|_2 = \|f\|_2$, which implies $\mathcal{F}$ is an injective. And the linearity of $\mathcal{F}$ can be induced by the linearity of the Fourier transform on $\mathcal{S}$.

III. Proof of (1): For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, there is a $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $f_n \to f$ both in L^1 and L^2 norms. So by above $\widehat{f_n} = \mathcal{F}f_n \to \mathcal{F}f$ in L^2. By the definition of Fourier transform on $L^1 \cap L^2$,

$$\left\| \widehat{f_n} - \widehat{f} \right\|_\infty \leq \|f_n - f\|_1$$

So $\widehat{f_n} \to \widehat{f}$ uniformly. Then on any ball $B \subset \mathbb{R}^d$,

$$\int_B \left| \mathcal{F}(f)(\xi) - \widehat{f}(\xi) \right| d\xi \leq \int_B |\mathcal{F}(f)(\xi) - \mathcal{F}(f_n)(\xi)| \, d\xi + \int_B \left| \widehat{f_n}(\xi) - \widehat{f}(\xi) \right| d\xi$$

The the second term can be arbitrarily small because of $\widehat{f_n} \to \widehat{f}$ uniformly. For the first term, by

$$m(B)^{1/2} \left(\int_B |\mathcal{F}f(\xi) - \mathcal{F}f_n(\xi)|^2 \, d\xi \right)^{1/2} \leq m(B)^{1/2} \|\mathcal{F}f - \mathcal{F}f_n\|_2$$

it can also be arbitrarily small. Therefore, on any ball B, $\mathcal{F}(f) = \widehat{f}$. So $\mathcal{F}(f) = \widehat{f}$.

IV. Surjectivity: First, we have known

$$\mathcal{S} \subset \operatorname{Im} \mathcal{F} \subset L^2(\mathbb{R}^d)$$

So only need to check the closedness of $\operatorname{Im} \mathcal{F}$. Assume

$$\mathcal{F}f_n \to g \in L^2(\mathbb{R}^d), \quad n \to \infty$$

in L^2. Then by the Cauchy of $(\mathcal{F}f_n)_{n\in\mathbb{N}}$ and isometry of $\mathcal{F}$, $(f_n)_{n\in\mathbb{N}}$ is also Cauchy. So there is a $f \in L^2$ such that $f_n \to f$ in L^2. Then

$$\|\mathcal{F}f_n - \mathcal{F}f\|_2 = (2\pi)^{d/2} \|f_n - f\|_2 \to 0$$

implies $\mathcal{F}f = g \in \operatorname{Im} \mathcal{F}$.

V. The uniqueness of $\mathcal{F}$ is by the density of $L^1 \cap L^2$ in L^2.

□

1.4 Distributions

1. Definitions:

   Definition 1.4.1 (Test Function Space)**.** Consider the set

   $$\mathcal{D} = \mathcal{D}\left(\mathbb{R}^d\right) = C_c^\infty\left(\mathbb{R}^d\right)$$

   with the topology defined as for $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}$, $\varphi_n \to \varphi$ in $\mathcal{D}$ if and only if there is a compact $K \subset \mathbb{R}^d$ such that $\operatorname{supp}\varphi \cup \bigcup_n \operatorname{supp}\varphi_n \subset K$ and for any α $\partial^\alpha\varphi_n \to \partial^\alpha\varphi$ uniformly on $\mathbb{R}^n$.

   Remark. In fact, $\mathcal{D}$ equipped with this topology is a locally convex topological space. It is induced by a family of seminorms indexed by any compact set $K \subset \mathbb{R}^d$ and $n \in \mathbb{N}_0$,

   $$p_{K,n}(f) := \sup_{x\in K}\sup_{|\alpha|\le n} |\partial^\alpha f(x)|$$

   Definition 1.4.2. The set of distributions on $\mathbb{R}^d$ is

   $$\mathcal{D}' = \mathcal{D}'(\mathbb{R}^d) = \{T\colon \mathcal{D} \to \mathbb{C}\colon T \text{ is linear and continuous}\}$$

   where $\mathcal{D}$ is equipped with the above topology.

   Example 1.4.1 (Locally Integrable Functions)**.**

   Definition 1.4.3. The locally integrable function space is

   $$L^1_{loc}(\mathbb{R}^d) := \left\{u\colon \mathbb{R}^d \to \mathbb{C}\colon \int_K |u(x)|\,dx < \infty,\ \forall\ K \subset \mathbb{R}^d \text{ compact}\right\}$$

   Note that

   - for any $p \in [1,\infty]$, $L^p(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$ by the Hölder's Inequality

     $$\int_K |u(x)|dx \le \|u\|_p\|1\|_{p'} \le \|u\|_p|K|^{\frac{1}{p'}} < \infty$$

   - $C(\Omega) \subset L^1_{loc}(\Omega)$.

   Proposition 1.4.1. *$f \in L^1_{loc}$ if and only if*

   $$\int_{\mathbb{R}^d} |f(x)\varphi(x)|\,dx < \infty, \quad \forall\ \varphi \in \mathcal{D}$$

   Proof. $\Rightarrow$ is clearly by definition.
   $\Leftarrow$: For any compact $K \subset \mathbb{R}^d$, there is a $\varphi \in \mathcal{D}$ with $K \subset \operatorname{supp}\varphi \subset K'$ and $\varphi(x) = 1$ on K. Then

   $$\int_K |f(x)|\,dx \le \int_{\mathbb{R}^d} |f(x)\varphi(x)|\,dx < \infty$$

   □

Let $f \in L^1_{loc}(\mathbb{R}^d)$. Define $T_f \colon \mathcal{D} \to \mathbb{C}$ by

$$\langle T_f, \varphi \rangle := \int_{\mathbb{R}^d} f(x)\varphi(x)dx$$

Then for $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}$, $\varphi_n \to \varphi$ in $\mathcal{D}$, then $\exists$ compact K containing all supports and so

$$\langle T_f, \varphi_n \rangle := \int_K f(x)\varphi_n(x)dx \to \int_K f(x)\varphi(x)dx = \langle T_f, \varphi \rangle$$

by the DCT. Therefore, $T_f \in \mathcal{D}'$. And in the following we do not differ T_f and f. Moreover,

$$T_f = 0 \;\Rightarrow\; f = 0$$

Proof. For any open set U, considering an open ball $B \subset \mathbb{R}^d$ such that $U \subset B$. Let $\varphi_n \in \mathcal{D}$ and $0 \leq \varphi_n \leq 1$ such that $\varphi_n \to \chi_U$. Then

$$0 = \int_B f(x)\varphi_n(x)dx \to \int_B f(x)\chi_U(x)dx = \int_U f(x)dx = 0$$

□

which also means $f = g$ in L^1_{loc} if and only if

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} g(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}$$

Example 1.4.2 (Dirichlet's Delta Function)**.** Define $\delta \colon \mathcal{D} \to \mathbb{C}$ as

$$\langle \delta, \varphi \rangle = \varphi(0)$$

Clearly, $\delta \in \mathcal{D}'$. Note that $\delta \notin L^1_{loc}$. More generally, for any μ on $\mathbb{R}^d$, define $T_\mu \in \mathcal{D}'$ as

$$\langle T_\mu, \varphi \rangle = \int_{\mathbb{R}^d} \varphi d\mu$$

2. Derivatives: For $T \in \mathcal{D}'$, $\partial^\alpha T$ defined as

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|}\langle T, \partial\varphi \rangle, \quad \varphi \in \mathcal{D}$$

Remark. (1) This definition is a generalization. Because if $f \in C^\infty$, then by the integral by parts we have

$$\langle \partial^\alpha f, \varphi \rangle = \int_{\mathbb{R}^d} (\partial^\alpha f)(x)\varphi(x)dx = (-1)^{|\alpha|}\int_{\mathbb{R}^d} f(x)(\partial^\alpha \varphi)(x)dx = (-1)^{|\alpha|}\langle f, \partial^\alpha \varphi \rangle$$

Therefore, we can see if $f \in C^\infty(\mathbb{R}^d)$,

$$\partial^\alpha T_f = T_{\partial^\alpha f}$$

(2) For $f \in L^1_{loc}$, if we define

$$\langle \partial^\alpha f, \varphi \rangle = \int_{\mathbb{R}^d} f(x)\partial^\alpha \varphi(x)dx, \quad \varphi \in \mathcal{D}$$

then $\partial^\alpha f \in L^1_{loc}$ and it is uniquely determined by above proposition and example. Then we can see

$$\partial^\alpha T_f = T_{\partial^\alpha f}$$

Moreover, for $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ with $\varphi_n \to \varphi$ in $\mathcal{D}$,

$$\langle \partial^\alpha T, \varphi_n \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi_n \rangle \to (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle = \langle \partial^\alpha T, \varphi \rangle$$

which means $\partial^\alpha T \in \mathcal{D}'$.

Example 1.4.3. (1) Let

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Then $H \in L^1_{xloc}$. For any $\varphi \in \mathcal{D}$,

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_{\mathbb{R}} H(x)\varphi'(x)dx = -\int_0^\infty \varphi'(x)dx = \varphi(0)$$

So $H' = \delta$.

(2) For any α and any $\varphi \in \mathcal{D}$,

$$\langle \partial^\alpha \delta, \varphi \rangle = (-1)^{|\alpha|}\langle \delta, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|}\partial^\alpha \varphi(0)$$

(3) For $g \in L^1_{loc}(\mathbb{R})$, if

$$f(x) = \int_0^x g(t)dt + f(0)$$

then $(T_f)' = T_g$ by

$$\left\langle (T_f)', \varphi \right\rangle = -\left\langle T_f, \varphi' \right\rangle = -\int_{\mathbb{R}} f(x)\varphi'(x)dx = -\left([f(x)\varphi(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} g(x)\varphi(x)dx \right) = \langle g, \varphi \rangle$$

(4) Consider $f(x) = \log|x| \in L^1_{loc}(\mathbb{R})$.

$$\begin{aligned} -\int_{|x|\geq\varepsilon} \varphi'(x)\log|x|dx - \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}dx &= -[\varphi(x)\log|x|]_\varepsilon^\infty - [\varphi(x)\log|x|]_{-\infty}^{-\varepsilon} \\ &= (\varphi(\varepsilon) - \varphi(-\varepsilon))\log\varepsilon \end{aligned}$$

The RHS converges to 0 as $\varepsilon \to 0+$. Therefore,

$$\left\langle (T_f)', \varphi \right\rangle = -\int_{\mathbb{R}} \varphi'(x)\log|x|dx = \lim_{\varepsilon \searrow 0} \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}dx =: \left\langle \text{p. v.}\left(\frac{1}{x}\right), \varphi \right\rangle$$

So $(T_f)' = \text{p. v.}\left(\frac{1}{x}\right)$.

Proposition 1.4.2. *For $T \in \mathcal{D}'(\mathbb{R})$, $T' = 0$ implies T is constant distribution,* i.e. *$\langle T, \varphi \rangle = \langle c, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.*

Proof. Fix $\varphi_0 \in \mathcal{D}$ with $\int_{\mathbb{R}} \varphi_0(x)dx = 1$. The for any $\varphi \in \mathcal{D}$, there is a $\psi \in \mathcal{D}$ such that

$$\varphi = \psi' + \alpha\varphi_0, \quad \alpha = \langle 1, \varphi \rangle = \int_{\mathbb{R}} \varphi(x)dx$$

In fact,

$$\psi(x) = \int_{-\infty}^x (\varphi(t) - \alpha\varphi_0(t))dt$$

Let $c := \langle T, \varphi_0 \rangle$. Then by $T' = 0$,

$$\langle T, \varphi \rangle = \langle T, \psi' \rangle + \alpha \langle T, \varphi_0 \rangle = -\langle T', \psi \rangle + \alpha c = c\langle 1, \varphi \rangle$$

□

3. Convolution: For $\varphi \in \mathcal{D}$ and $x, y \in \mathbb{R}^d$,

$$\varphi^{\sim}(x) = \varphi(-x), \quad (\tau_x \varphi)(y) = \varphi(y - x)$$

Remark. For $\varphi_n \to \varphi$ in $\mathcal{D}$, it is clearly $\tau_x \varphi_n \to \tau_x \varphi$ and $\varphi_n^{\sim} \to \varphi^{\sim}$ in $\mathcal{D}$.

For $f \in L^1_{loc}$ and $\varphi \in \mathcal{D}$,

$$(f * \varphi)(x) = \int_{\mathbb{R}^d} f(y)\psi(x - y)dy = \int_{\mathbb{R}^d} f(y) (\tau_x \psi^{\sim})(y)dy = \langle f, \tau_x \psi^{\sim} \rangle$$

Based on this, for $T \in \mathcal{D}'$ and $\psi \in \mathcal{D}$,

$$(T * \psi)(x) = \langle T, \tau_x \psi^{\sim} \rangle, \quad x \in \mathbb{R}^d$$

So by this definition, we have for any $f \in L^1_{loc}$ and $\varphi \in \mathcal{D}$,

$$T_f * \varphi = f * \varphi$$

Example 1.4.4. For any $\psi \in C^\infty(\mathbb{R}^d)$,

$$(\delta * \psi)(x) = \langle \delta, \tau_x \psi^{\sim} \rangle = (\tau_x \psi^{\sim})(0) = \psi^{\sim}(-x) = \psi(x)$$

Proposition 1.4.3. *For any $T \in \mathcal{D}'$ and $\psi \in \mathcal{D}$,*

(1) $\tau_t(T * \psi) = T * (\tau_t \psi)$ for all t.
(2) $T * \psi \in C^\infty$ and for any α,

$$\partial^\alpha (T * \psi) = (\partial^\alpha T) * \psi = T * (\partial^\alpha \psi)$$

Proof. (1) By the definition, $(T * (\tau_t \psi))(x) = \langle T, \tau_x (\tau_t \psi)^{\sim} \rangle$.

$$(\tau_x (\tau_t \psi)^{\sim})(y) = (\tau_t \psi)^{\sim}(y - x) = (\tau_t \psi)(x - y) = \psi(x - y - t) = (\tau_{x-t} \psi^{\sim})(y)$$

So we have

$$\langle T, \tau_x (\tau_t \psi)^{\sim} \rangle = \langle T, \tau_{x-t} \psi^{\sim} \rangle = (T * \psi)(x - t) = (\tau_t (T * \psi))(x)$$

(2) First, for the second equality,

$$((\partial^\alpha T) * \psi)(x) = \langle \partial^\alpha T, \tau_x \psi^{\sim} \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha (\tau_x \psi^{\sim}) \rangle$$

and for $\partial^\alpha (\tau_x \psi^{\sim})$,

$$\partial^\alpha (\tau_x \psi^{\sim})(y) = \partial^\alpha ([y \mapsto \psi(x - y)])(y) = (-1)^{|\alpha|} (\partial^\alpha \psi)(x - y) = (-1)^{|\alpha|} \tau_x (\partial^\alpha \psi)^{\sim}(y)$$

So

$$((\partial^\alpha T) * \psi)(x) = \langle T, \tau_x (\partial^\alpha \psi)^{\sim} \rangle = (T * (\partial^\alpha \psi))(x)$$

Next, for the first equality, firstly note that it can prove $T * \psi \in C^\infty$. To prove the the first equality, it only needs to prove that for any directional derivative D_e for a unit vector e. For $r \neq 0$, let $\eta_r = (\tau_{-re} - \tau_0)/r$. Then as $r \to 0$,

$$(\eta_r \psi)(y) = \frac{\psi(y + re) - \psi(y)}{r} \to (D_e \psi)(y)$$

and for any β,

$$\partial^\beta (\eta_r \psi)(y) = \frac{\left(\partial^\beta \psi\right)(y + re) - \left(\partial^\beta \psi\right)(y)}{r} \to \left(D_e \partial^\beta \psi\right)(y) = \partial^\beta (D_e \psi)(y)$$

Moreover, compactness of the support means that the convergence is uniform. Therefore, by the definition, $\eta_r \psi \to D_e \psi$ in $\mathcal{D}$. And so $\tau_x (\eta_r \psi)^\sim \to \tau_x (D_e \psi)^\sim$. By the continuity of T,

$$\langle T, \tau_x (\eta_r \psi)^\sim \rangle \to \langle T, \tau_x (D_e \psi)^\sim \rangle = (T * (D_e \psi))(x)$$

where the left hand side is

$$(T * (\eta_r \psi))(x) = (\eta_r (T * \psi))(x) = \frac{(T * \psi)(x + re) - (T * \psi)(x)}{r}.$$

Therefore, as $r \to 0$, we have

$$D_e (T * \psi) = T * (D_e \psi)$$

□

For any $f \in L^1_{loc}(\mathbb{R}^d)$ and $\varphi, \psi \in \mathcal{D}$,

$$\begin{aligned} \langle f * \psi, \varphi \rangle &= \int_{\mathbb{R}^d} (f * \psi)(x) \varphi(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) \psi(x - y) \varphi(x) dy dx \\ &= \int_{\mathbb{P}_d} f(y) \left(\psi^\sim * \varphi\right)(y) dy = \langle f, \psi^\sim * \varphi \rangle \end{aligned}$$

This result can be extended to $\mathcal{D}'$.

Proposition 1.4.4. *For any $T \in \mathcal{D}'$ and $\varphi, \psi \in \mathcal{D}$,*

$$\langle T * \psi, \varphi \rangle = \langle T, \psi^\sim * \varphi \rangle$$

Note that because $T * \psi \in C^\infty$, this $\langle \cdot, \cdot \rangle$ is the integral.

Proof. Let

$$S_\varepsilon(x) = \varepsilon^d \sum_{v \in \mathbb{Z}^d} \psi^\sim (x - \varepsilon v) \varphi(\varepsilon v)$$

Then we have

$$\left(\psi^\sim * \varphi\right)(x) = \int_{\mathbb{R}^d} \psi^\sim (x - y) \varphi(y) dy = \lim_{\varepsilon \to 0} S_\varepsilon(x)$$

and for any α,

$$\partial^\alpha S_\varepsilon(x) = \varepsilon^d \sum_{v \in \mathbb{Z}^d} \partial^\alpha \left(\psi^\sim\right)(x - \varepsilon v) \varphi(\varepsilon v) \to \int_{\mathbb{R}^d} \partial^\alpha \left(\psi^\sim\right)(x - y) \varphi(y) dy = \left(\left(\partial^\alpha \left(\psi^\sim\right)\right) * \varphi\right)(x)$$

so $S_\varepsilon \to \psi^\sim * \varphi$ in $\mathcal{D}$. And thus

$$\langle T, S_\varepsilon \rangle \to \langle T, \psi^\sim * \varphi \rangle$$

On the other hand,

$$\begin{aligned} \langle T, S_\varepsilon \rangle &= \varepsilon^d \sum_{v \in \mathbb{Z}^d} \langle T, \tau_{\varepsilon v} \psi^\sim \rangle \varphi(\varepsilon v) = \varepsilon^d \sum_{v \in \mathbb{Z}^d} (T * \psi)(\varepsilon v) \varphi(\varepsilon v) \\ &\to \int_{\mathbb{R}^d} (T * \psi)(y) \varphi(y) dy = \langle T * \psi, \varphi \rangle \end{aligned}$$

□

4. Support of Distribution: Generally, the support of f on $\mathbb{R}^d$ is the complementary of the maximal open set U such that $f|_U = 0$, *i.e.*

$$\operatorname{supp} f := \mathbb{R}^d \backslash \bigcup_{U \text{ open}} \{f|_U = 0\}$$

For any open $U \subset \mathbb{R}^d$, let

$$\mathcal{D}(U) = \{\varphi \in \mathcal{D} \mid \operatorname{supp} \varphi \subset U\}$$

Let $T \in \mathcal{D}'$.

$$\mathcal{U}(T) := \big\{U \subset_{\text{open}} \mathbb{R}^d \colon \forall\, \varphi \in \mathcal{D}(U),\ \langle T, \varphi\rangle = 0\big\}$$

Then the support of T is defined as

$$\operatorname{supp} T := \mathbb{R}^d \backslash \bigcup \mathcal{U}(T)$$

Remark. For $f \in L^1_{loc}(\mathbb{R}^d)$,

$$\operatorname{supp} T_f = \operatorname{supp} f$$

which is because, first for U with $f|_U = 0$, clearly we have $\langle T_f, \varphi\rangle = 0$ for any $\varphi \in \mathcal{D}(U)$, and conversely, for U with $\langle T_f, \varphi\rangle = 0$ for any $\varphi \in \mathcal{D}(U)$, then $f|_U = 0$.

Proposition 1.4.5. *Let $T \in \mathcal{D}'$. If for any $\lambda \in \Lambda$, $U_\lambda \in \mathcal{U}(T)$, then $\bigcup_{\lambda\in\Lambda} U_\lambda \in \mathcal{U}(T)$.*

Proof. Let $U = \bigcup_{\lambda\in\Lambda} U_\lambda$. For $(U_\lambda)_{\lambda\in\Lambda}$, choose a partition of unity $(\psi_\lambda)_{\lambda\in\Lambda}$. Then for $\varphi \in \mathcal{U}$, because $\operatorname{supp}\varphi$ is compact, there is $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that

$$\varphi = \sum_{i=1}^{n} \varphi\psi_{\lambda_i}$$

Then we have

$$\langle T, \varphi\rangle = \sum_{i=1}^{n} \langle T, \varphi\psi_{\lambda_i}\rangle = 0$$

by $\operatorname{supp}\psi_{\lambda_i} \subset U_{\lambda_i} \in \mathcal{U}(T)$. □

For example, $\operatorname{supp}\delta = \{0\}$.

Proposition 1.4.6. *Let $T \in \mathcal{D}'$.*

(1) If $\varphi \in \mathcal{D}$ with $\varphi = 0$ near around $\operatorname{supp} T$, then $\langle T, \varphi\rangle = 0$.

(2) For any α, $\operatorname{supp}\partial^\alpha T \subset \operatorname{supp} T$.

(3) For any $\psi \in \mathcal{D}$,

$$\operatorname{supp}(T * \psi) \subset \operatorname{supp} T + \operatorname{supp}\psi.$$

Proof. Let $U = \bigcup \mathcal{U}(T) = \mathbb{R}^d \backslash \operatorname{supp} T$.

(1) Assume open V contains $\operatorname{supp} T$ such that $\operatorname{supp}\varphi \subset \mathbb{R}^d \backslash V$. So for any $x \in \mathbb{R}^d \backslash V$, choose open $U_x \cap \operatorname{supp} T = \varnothing$. Let

$$U_1 := \bigcup_{x\in\mathbb{R}^d\backslash V} U_x \subset U$$

Then $\operatorname{supp}\varphi \subset U_1 \subset U$. So $\langle T, \varphi\rangle = 0$.

(2) First, for any $\varphi \subset \mathcal{D}(U)$, $\partial^\alpha \varphi \in \mathcal{D}(U)$, then

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle = 0$$

So $U \in \mathcal{U}(\partial^\alpha T)$.

(3) For $x \in \mathbb{R}^d$ with $(T * \psi)(x) = \langle T, \tau_x \psi^\sim \rangle \neq 0$, by (1)

$$\operatorname{supp} T \cap \operatorname{supp}(\tau_x \psi^\sim) \neq \varnothing$$

So there is a $y \in \operatorname{supp} T \cap \operatorname{supp}(\tau_x \psi^\sim)$. And $y \in \operatorname{supp}(\tau_x \psi^\sim)$ implies $x - y \in \operatorname{supp} \psi$. Therefore,

$$x \in y + \operatorname{supp} \psi \subset \operatorname{supp} T + \operatorname{supp} \psi$$

And the closedness of $\operatorname{supp} T + \operatorname{supp} \psi$ is by the compactness of $\operatorname{supp} T$ and $\operatorname{supp} \psi$.

Note that if $A, B \subset \mathbb{R}^d$ are compact, then $A + B$ is closed. □

5. Distributions with Compact Support:

Proposition 1.4.7. *Linear map $T \colon \mathcal{D} \to \mathbb{C}$ is continuous if and only if for any compact $K \subset \mathbb{R}^d$, there is a $C > 0$ and $N \in \mathbb{N}$ such that for any $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \subset K$ we have*

$$|\langle T, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_\infty$$

Remark. This result can be obtained from the general result of the theory of locally convex topological vector space.

Lemma 1.4.1. *Assume $\psi_n \to \psi$ in $\mathcal{D}$.*

(1) For any $R \in \mathcal{D}'$ with compact support, $R * \psi_n \to R * \psi$ in $\mathcal{D}$.

(2) For any $R \in \mathcal{D}'$ and $\eta \in \mathcal{D}$, $\eta(R * \psi_n) \to \eta(R * \psi)$ in $\mathcal{D}$.

Proof. $\psi_n \to \psi$ in $\mathcal{D}$ implies that there is a compact K such that $\operatorname{supp} \psi_n, \operatorname{supp} \psi \in K$.

(1) Because $\operatorname{supp} R$ is compact,

$$\operatorname{supp} R * \psi_n, \operatorname{supp} R * \psi \in K + \operatorname{supp} R$$

Besides, for any $x \in K + \operatorname{supp} R$ and α,

$$\begin{aligned}
&|\partial^\alpha (R * \psi_n)(x) - \partial^\alpha (R * \psi)(x)| = |(R * (\partial^\alpha \psi_n))(x) - (R * (\partial^\alpha \psi))(x)| \\
&= |\langle R, \tau_x (\partial^\alpha \psi_n - \partial^\alpha \psi)^\sim \rangle| \leq C \sum_{|\beta| \leq N} \left\| \partial^\beta (\tau_x (\partial^\alpha \psi_n - \partial^\alpha \psi)^\sim) \right\|_\infty \\
&= C \sum_{|\beta| \leq N} \left\| \partial^{\alpha+\beta} \psi_n - \partial^{\alpha+\beta} \psi \right\|_\infty \to 0.
\end{aligned}$$

where C is independent with x, so it is uniform.

(2) For any $x \in \operatorname{supp} \eta$ and α,

$$\begin{aligned}
&|\partial^\alpha (\eta (R * \psi_n))(x) - \partial^\alpha (\eta (R * \psi))(x)| \\
&= \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1 \alpha_2} |(\partial^{\alpha_1} \eta)(x)| \, |\partial^{\alpha_2} (R * \psi_n)(x) - \partial^{\alpha_2} (R * \psi)(x)| \\
&\leq \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1 \alpha_2} \|\partial^{\alpha_1} \eta\|_\infty C \sum_{|\beta| \leq N} \left\| \partial^{\alpha_2 + \beta} \psi_n - \partial^{\alpha_2 + \beta} \psi \right\|_\infty \to 0.
\end{aligned}$$

□

For $T \in \mathcal{D}'$, define $T^{\sim} \in \mathcal{D}'$ as

$$\langle T^{\sim}, \varphi \rangle = \langle T, \varphi^{\sim} \rangle, \quad \varphi \in \mathcal{D}$$

By this definition, we have $T_f^{\sim} = T_{f^{\sim}}$ for any $f \in L^1_{loc}$. And we have $\operatorname{supp} T^{\sim} = -\operatorname{supp} T$, like the usual function.

Proposition 1.4.8. *For any $\varphi, \psi \in \mathcal{D}$ and $T \in \mathcal{D}'$, we have*

(1) $T^{\sim} * \psi^{\sim} = (T * \psi)^{\sim}$.
(2) $(T * \psi) * \varphi = T * (\psi * \varphi)$.

Proof. (1) For $x \in \mathbb{R}^d$,

$$(\tau_x \psi)^{\sim}(y) = (\tau_x \psi)(-y) = \psi(-y-x) = \psi^{\sim}(x+y) = (\tau_{-x}\psi^{\sim})(y)$$

so

$$(T^{\sim} * \psi^{\sim})(x) = \langle T^{\sim}, \tau_x \psi \rangle = \langle T, \tau_{-x}\psi^{\sim} \rangle = (T * \psi)(-x) = (T * \psi)^{\sim}(x)$$

(2) For $x \in \mathbb{R}^d$,

$$((T * \psi) * \varphi)(x) = \langle T * \psi, \tau_x \varphi^{\sim} \rangle = \langle T, \psi^{\sim} * (\tau_x \varphi^{\sim}) \rangle$$

On the other hand,

$$\begin{aligned}\psi^{\sim} * (\tau_x \varphi^{\sim}) &= \int_{\mathbb{R}^d} \psi^{\sim}(\cdot - y)\varphi^{\sim}(y-x)dy = \int_{\mathbb{R}^d} \psi^{\sim}(\cdot - x - y)\varphi^{\sim}(y)dy \\ &= \tau_x(\psi^{\sim} * \varphi^{\sim}) = \tau_x((\psi * \varphi)^{\sim})\end{aligned}$$

So we have the desired equality.

□

Proposition 1.4.9. *Let $T \in \mathcal{D}'$ with compact support.*

(1) Extending T to a linear map $\bar{T} \colon C^\infty(\mathbb{R}^d) \to \mathbb{C}$ by

$$\langle \bar{T}, \varphi \rangle = \langle T, \eta\varphi \rangle, \quad \varphi \in C^\infty(\mathbb{R}^d)$$

where $\eta \in \mathcal{D}$ such that $\eta = 1$ near around $\operatorname{supp} T$. Note that this extension is independent with the choice of η.

(2) If $\varphi, \psi \in C^\infty(\mathbb{R}^d)$ with $\varphi = \psi$ near around $\operatorname{supp} T$, then $\langle \bar{T}, \varphi \rangle = \langle \bar{T}, \psi \rangle$.

Proof. (1) First, for $\varphi \in \mathcal{D}$, $\varphi - \eta\varphi = 0$ near around $\operatorname{supp} T$. So $\langle T, \varphi \rangle = \langle T, \eta\varphi \rangle$, and it is an extension.
If $\eta, \zeta = 1$ near around $\operatorname{supp} T$, then $\eta\varphi - \zeta\varphi = 0$ near around $\operatorname{supp} T$. So $\langle T, \eta\varphi \rangle = \langle T, \zeta\varphi \rangle$.

(2) Similarly, $\eta\varphi - \eta\psi = 0$ near around $\operatorname{supp} T$.

□

For $T, S \in \mathcal{D}'$, define $T * S \in \mathcal{D}'$ as

(1) when $\operatorname{supp} S$ is compact,

$$\langle T * S, \varphi \rangle = \langle T, S^{\sim} * \varphi \rangle, \quad \varphi \in \mathcal{D}$$

where the compactness of $\operatorname{supp} S$ implies $S^{\sim} * \varphi \in \mathcal{D}$.

(2) when $\operatorname{supp} T$ is compact, consider $\bar{T}$ on $C^{\infty}(\mathbb{R}^d)$,

$$\langle T * S, \varphi \rangle = \left\langle \bar{T}, S^{\sim} * \varphi \right\rangle, \quad \varphi \in \mathcal{D}$$

By this definition, we can see

(a) For $f \in L^1_{loc}$ and $\varphi \in \mathcal{D}$,then $\operatorname{supp} T_\varphi = \operatorname{supp} \varphi$ is compact and

$$T_f * T_\varphi = T_{f*\varphi}$$

(b) More generally, for $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, we have

$$T * T_\varphi = T_{T*\varphi}$$

(c) For any $T \in \mathcal{D}'$,

$$\langle T * \delta, \varphi \rangle = \langle T, \delta^{\sim} * \varphi \rangle = \langle T, \delta * \varphi \rangle = \langle T, \varphi \rangle, \quad \varphi \in \mathcal{D}$$

Proposition 1.4.10. *For $T, S \in \mathcal{D}'$, if S or T has the compact support, then*

$$T * S = S * T$$

Proof. First, for $\varphi, \psi \in \mathcal{D}$, check

$$\langle T * S, \varphi * \psi \rangle = \langle S * T, \varphi * \psi \rangle$$

Assume S has the compact support and $\operatorname{supp} S \subset U \subset K$ for open U and compact K.

$$\begin{aligned} \langle T * S, \varphi * \psi \rangle &= \langle T, S^{\sim} * (\varphi * \psi) \rangle = \langle T, (S^{\sim} * \varphi) * \psi \rangle = \langle T, \psi * (S^{\sim} * \varphi) \rangle \\ &= \langle T * \psi^{\sim}, S^{\sim} * \varphi \rangle = \langle T^{\sim} * \psi, S * \varphi^{\sim} \rangle \end{aligned}$$

Let $\eta \in \mathcal{D}$ such that $\eta = 1$ near around $K - \operatorname{supp} \varphi$. Because

$$\operatorname{supp} (S * \varphi^{\sim}) \subset K + \operatorname{supp} \varphi^{\sim} = K - \operatorname{supp} \varphi$$

we have

$$\langle S * \varphi^{\sim}, \eta \, (T^{\sim} * \psi) \rangle = \langle S, \varphi * (\eta \, (T^{\sim} * \psi)) \rangle = \langle S, (\eta \, (T^{\sim}{}_{*} * \psi)) * \varphi \rangle$$

For any $x \in K$,

$$((\eta \, (T^{\sim} * \psi)) * \varphi) \, (x) = \int_{\operatorname{supp} \varphi} \underbrace{\eta(x-y)}_{=1} \, (T^{\sim} * \psi) \, (x-y) \varphi(y) dy = ((T^{\sim} * \psi) * \varphi) \, (x)$$

Therefore, On U, $(\eta \, (T^{\sim} * \psi)) * \varphi = (T^{\sim} * \psi) * \varphi$. Then consider the extension of S,

$$\begin{aligned} \langle S, (\eta \, (T^{\sim} * \psi)) * \varphi \rangle &= \left\langle \bar{S}, (T^{\sim} * \psi) * \varphi \right\rangle = \left\langle \bar{S}, T^{\sim} * (\psi * \varphi) \right\rangle = \langle S * T, \psi * \varphi \rangle \\ &= \langle S * T, \varphi * \psi \rangle. \end{aligned}$$

Next, $\{\varphi * \psi \colon \varphi, \psi \in \mathcal{D}\}$ is dense in $\mathcal{D}$. For $\varphi \in \mathcal{D}$ with $\int_{\mathbb{R}^d} \varphi(x) dx = 1$, let

$$\varphi_n(x) = n^d \varphi(nx)$$

Then $(\varphi_n)_{n \in \mathbb{N}}$ is a summability kernel. So for any $\psi \in \mathcal{D}$, $\varphi_n * \psi \to \psi$ in $\mathcal{D}$. By the continuity of $T * S$ and $S * T$,

$$\langle T * S, \psi \rangle = \langle S * T, \psi \rangle$$

Therefore $T * S = S * T$. ☐

By this, we get $T * \delta = T = \delta * T$ for all $T \in \mathcal{D}'$.

Proposition 1.4.11. *For $T, S \in \mathcal{D}'$, assume T or S has the compact support. Then for any α,*

$$\partial^\alpha (T * S) = (\partial^\alpha T) * S = T * (\partial^\alpha S)$$

Proof. Assume S has the compact support. For any $\varphi \in \mathcal{D}$,

$$\langle \partial^\alpha (T * S), \varphi \rangle = (-1)^{|\alpha|} \langle T * S, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle T, S^\sim * (\partial^\alpha \varphi) \rangle$$

Because $S^\sim * (\partial^\alpha \varphi) = \partial^\alpha (S^\sim * \varphi)$, the right hand side is

$$(-1)^{|\alpha|} \langle T, \partial^\alpha (S^\sim * \varphi) \rangle = \langle \partial^\alpha T, S^\sim * \varphi \rangle = \langle (\partial^\alpha T) * S, \varphi \rangle$$

So the first equality is obtained. Besides, for $\psi \in \mathcal{D}$,

$$\begin{aligned} \langle \partial^\alpha (S^\sim), \psi \rangle &= (-1)^{|\alpha|} \langle S^\sim, \partial^\alpha \psi \rangle = (-1)^{|\alpha|} \langle S, (\partial^\alpha \psi)^\sim \rangle = \langle S, \partial^\alpha (\psi^\sim) \rangle \\ &= (-1)^{|\alpha|} \langle \partial^\alpha S, \psi^\sim \rangle = (-1)^{|\alpha|} \langle (\partial^\alpha S)^\sim, \psi \rangle \end{aligned}$$

so $\partial^\alpha (S^\sim) = (-1)^{|\alpha|} (\partial^\alpha S)^\sim$. And by $S^\sim * (\partial^\alpha \varphi) = (\partial^\alpha (S^\sim)) * \varphi$,

$$(-1)^{|\alpha|} \langle T, S^\sim * (\partial^\alpha \varphi) \rangle = (-1)^{|\alpha|} \langle T, (\partial^\alpha (S^\sim)) * \varphi \rangle = \langle T, (\partial^\alpha S)^\sim * \varphi \rangle = \langle T * (\partial^\alpha S), \varphi \rangle .$$

□

Proposition 1.4.12. *For $T, S \in \mathcal{D}'$, assume T or S has the compact support. Then*

$$\operatorname{supp}(T * S) \subset \operatorname{supp} T + \operatorname{supp} S$$

Proof. Only need to prove for $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \cap (\operatorname{supp} T + \operatorname{supp} S) = \varnothing$, which implies $\operatorname{supp} T \cap (\operatorname{supp} \varphi - \operatorname{supp} S) = \varnothing$, we have $\langle T * S, \varphi \rangle = 0$. First, we have

$$\operatorname{supp} (S^\sim * \varphi) \subset \operatorname{supp} S^\sim + \operatorname{supp} \varphi = \operatorname{supp} \varphi - \operatorname{supp} S$$

Therefore,

$$\langle T * S, \varphi \rangle = \left\langle \bar{T}, S^\sim * \varphi \right\rangle = 0$$

□

6. Order of Distribution: Let $T \in \mathcal{D}$. We say T has the order $N \in \mathbb{N}$ if there is a $C > 0$ such that for any $\varphi \in \mathcal{D}$

$$|\langle T, \varphi \rangle| \le C \sum_{|\alpha| \le N} \|\partial^\alpha \varphi\|_\infty$$

Example 1.4.5. Let $f \in L^1(\mathbb{R}^d)$. For any $\varphi \in \mathbb{D}$,

$$|\langle \partial^\alpha f, \varphi \rangle| \le \|f\|_1 \|\partial^\alpha \varphi\|_\infty$$

Therefore, $\partial^\alpha f$ has the order $|\alpha|$.

Proposition 1.4.13. *Let $T \in \mathcal{D}'$ with the compact support. Then there is an $N \in \mathbb{N}$ such that T has the order N.*

Proof. Let $\operatorname{supp} T \subset U \subset K$ for open U and compact K. Then there is an $\eta \in \mathcal{D}$ such that $\eta = 1$ on U and $\eta = 0$ on $\mathbb{R}^d \backslash K$. First, for $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \subset K$, by the continuity of T, there is a $C > 0$ and $N \in \mathbb{N}$ such that

$$|\langle T, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha \varphi\|_\infty .$$

For genera $\varphi \in \mathcal{D}$, because $\operatorname{supp} \eta\varphi \subset K$,

$$|\langle T, \varphi \rangle| = |\langle T, \eta\varphi \rangle| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha (\eta\varphi)\|_\infty \leq C \sum_{|\alpha| \leq N} \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1 \alpha_2} \|\partial^{\alpha_1} \eta\|_\infty \|\partial^{\alpha_2} \varphi\|_\infty .$$

□

Proposition 1.4.14. *Let $T \in \mathcal{D}'$ with compact support and let its order be N. Then for any $f \in C^N(\mathbb{R}^d)$, $T * f \in \mathcal{D}'$ is a continuous function on $\mathbb{R}^d$.*

Proof. Let $\psi \in \mathcal{D}$ with $\int_{\mathbb{R}^d} \psi(x) dx = 1$ and define $\psi_n(x) = n^d \psi(nx)$. Then $(\psi_n)_{n \in \mathbb{N}}$ is a summability kernel. For $x \in \mathbb{R}^d$, define

$$F_n(x) = \left\langle \bar{T}, \tau_x (\psi_n * f)^\sim \right\rangle$$

Note that $\tau_x (\psi_n * f)^\sim \in C^\infty$ and there is a $\eta \in \mathcal{D}$ with $\eta = 1$ near $\operatorname{supp} T$ such that

$$\langle \bar{T}, \varphi \rangle = \langle T, \eta\varphi \rangle, \quad \varphi \in C^\infty \left(\mathbb{R}^d\right)$$

(1) Limits and continuity: First, since T has the order N, there is a $C > 0$ such that

$$|F_n(x) - F_m(x)| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha (\eta \tau_x (\psi_n * f)^\sim) - \partial^\alpha (\eta \tau_x (\psi_m * f)^\sim)\|_\infty = \mathrm{I}$$

By $(\psi_n * f)^\sim = \psi_n^\sim * f^\sim$, we have

$$\partial^\alpha (\eta \tau_x (\psi_n * f)^\sim) = \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1 \alpha_2} (\partial^{\alpha_1} \eta) \partial^{\alpha_2} [\tau_x (\psi_n^\sim * f^\sim)]$$

And because ,$\partial^{\alpha_2} [\tau_x (\psi_n^\sim * f^\sim)] = \tau_x \partial^{\alpha_2} (\psi_n^\sim * f^\sim) = \tau_x (\psi_n^\sim * \partial^{\alpha_2} (f^\sim))$,

$$\mathrm{I} \leq C \sum_{|\alpha| \leq N} \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1 \alpha_2} \|(\partial^{\alpha_1} \eta) \tau_x (\psi_n^\sim * \partial^{\alpha_2} (f^\sim) - \psi_m^\sim * \partial^{\alpha_2} (f^\sim))\|_\infty$$

Because $\psi_n^\sim * g \to g$ uniformly on $\mathbb{R}^d$, as $n, m \to \infty$, the right hand side converges to 0 uniformly. Therefore,

$$F(x) = \lim_{n \to \infty} F_n(x)$$

uniformly. And the continuity of $F_n(x)$ implies the continuity of $F(x)$.

(2) Check: $T * f = F$, *i.e.* for any $\varphi \in \mathcal{D}$,

$$\langle T, \eta (f^\sim * \varphi) \rangle = \langle F, \varphi \rangle$$

For $\varepsilon > 0$,

$$S_{n,\varepsilon}(x) = \varepsilon^d \sum_{v \in \mathbb{Z}^d} \eta(x) (\psi_n * f)^\sim (x - \varepsilon v) \varphi(\varepsilon v)$$

Then

$$\begin{aligned}\langle T, S_{n,\varepsilon}\rangle &= \varepsilon^d \sum_{v\in\mathbb{Z}^d} \langle T, \eta\tau_{\varepsilon v}\left(\psi_n * f\right)^{\sim}\rangle\, \varphi(\varepsilon v)\\ &= \varepsilon^d \sum_{v\in\mathbb{Z}^d} F_n(\varepsilon v)\varphi(\varepsilon v) \underset{n\to\infty}{\to} \varepsilon^d \sum_{v\in\mathbb{Z}^d} F(\varepsilon v)\varphi(\varepsilon v) \underset{\varepsilon\to 0}{\to} \langle F, \varphi\rangle.\end{aligned}$$

On the other hand, for any $x \in \mathbb{R}^d$,

$$S_{n,\varepsilon}(x) \underset{\varepsilon\to 0}{\to} \eta(x)\left(\left(\psi_n * f\right)^{\sim} * \varphi\right)(x)$$

And for $x \in \operatorname{supp}\eta$, above convergence is uniform and for all β with $|\beta| \le N$,

$$\partial^\beta S_{n,\varepsilon} \to \partial^\beta\left(\eta\left(\left(\psi_n * f\right)^{\sim} * \varphi\right)\right)$$

converges uniformly. And because T has order N,

$$\langle T, S_{n,\varepsilon}\rangle \underset{\varepsilon\to 0}{\to} \langle T, \eta\left(\left(\psi_n * f\right)^{\sim} * \varphi\right)\rangle$$

uniformly. And by $\left(\psi_n * f\right)^{\sim} \to f^{\sim}$ uniformly,

$$\langle T, \eta\left(\left(\psi_n * f\right)^{\sim} * \varphi\right)\rangle \underset{n\to\infty}{\to} \langle T, \eta\left(f^{\sim} * \varphi\right)\rangle.$$

□

Theorem 1.4.1. *Let $T \in \mathcal{D}'$ with compact support and order N. Then there is a continuous function g such that*

$$\partial_1^{N+2}\partial_2^{N+2}\cdots\partial_d^{N+2} g = T$$

Proof. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, define $E(x)$ as

$$E(x) = \begin{cases} x_1^{N+1}\cdots x_d^{N+1}/((N+1)!)^d & x_i > 0,\ \forall\ i,\\ 0, & \text{others}\end{cases}$$

Then $E \in C^N$ and

$$\partial_1^{N+2}\cdots\partial_d^{N+2} E = \delta$$

Let $g = T * E$. Then by above, g is continuous and

$$\partial_1^{N+2}\cdots\partial_d^{N+2} g = T * \left(\partial_1^{N+2}\cdots\partial_d^{N+2} E\right) = T * \delta = T.$$

□

Corollary 1.4.1. *(1) Let $T \in \mathcal{D}'$ that has the form as the derivatives of continuous function. Then for any open $U \subset \mathbb{R}^d$, there are a continuous function f and α such that for any $\varphi \in \mathcal{D}(U)$,*

$$\langle T, \varphi\rangle = \langle \partial^\alpha f, \varphi\rangle$$

(2) Let $T \in \mathcal{D}'$ with compact support. If $\operatorname{supp} T \subset U$ open, then there are continuous $f_1, \cdots, f_n$ and $\alpha_1, \cdots, \alpha_n$ such that $\operatorname{supp} f_k \subset U$ and

$$T = \sum_{k=1}^n \partial^{\alpha_k} f_k$$

Proof. (1) Choose $\eta \in \mathcal{D}$ with $\eta = 1$ on U and $\langle S, \varphi \rangle = \langle T, \eta\varphi \rangle$. Then it can be obtained by above theorem.

(2) Generally, for $\eta \in C^\infty$ and $T \in \mathcal{D}'$, we have

$$\eta\left(\partial^\beta T\right) = \sum_{\alpha \leq \beta} (-1)^{|\alpha|} \frac{\beta!}{(\beta-\alpha)!\alpha!} \partial^{\beta-\alpha}\left(\left(\partial^\alpha \eta\right) T\right)$$

Let $\eta \in \mathcal{D}$ with $\eta = 1$ on $\operatorname{supp} T$. Because $T = \partial^\beta g$,

$$T = \eta\left(\partial^\beta g\right) = \sum_{\alpha < \beta} c_{\alpha\beta} \partial^{\beta-\alpha}\left(\left(\partial^\alpha \eta\right) g\right)$$

□

Theorem 1.4.2. *Let $T \in \mathcal{D}'$ with $\operatorname{supp} T = \{0\}$. Then there is a $N \in \mathbb{N}$ such that*

$$T = \sum_{|\alpha| \leq N} a_\alpha \partial^\alpha \delta$$

1.5 Fourier Analysis on Distributions

1. Tempered Distributions: For any $f \in L^1(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}$, by the Fubini's theorem,

$$\int_{\mathbb{R}^d} \widehat{f}(y)\varphi(y)dy = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)\varphi(y)e^{-iy\cdot x}dxdy = \int_{\mathbb{R}^d} f(x)\widehat{\varphi}(x)dx$$

So we have $\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle$. Motivated by this, for $T \in \mathcal{D}$, its Fourier transform $\widehat{T}$ is defined

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle, \quad \varphi \in \mathcal{S}$$

But it cannot be extended to $\mathcal{D}$. To solve this problem, we need more about distributions.

For $N \in \mathbb{N}$, defined $\|\cdot\|_N$ on $\mathcal{S}$ as

$$\|\varphi\|_N = \sum_{|\alpha|, |\beta| \leq N} \sup_{x \in \mathbb{R}^d} \left| x^\alpha \left(\partial^\beta \varphi\right)(x) \right|, \quad \varphi \in \mathcal{S}.$$

Let $\mathcal{S}$ be equipped with the topology induced by the family of seminorms $\{\|\cdot\|_N\}_{N \in \mathbb{N}_0}$.

Remark. When considering $\mathcal{D} \subset \mathcal{S}$, the topology defined on $\mathcal{D}$ as above is equivalent to this new topology on $\mathcal{D}$ induced by these seminorms.

Proposition 1.5.1. *$\mathcal{D}$ is dense in $\mathcal{S}$.*

Proof. Let $\varphi \in \mathcal{S}$. Choose $\psi \in \mathcal{D}$ such that $\psi = 1$ on $[-1, 1]^d$. Let $\psi_n(x) = \psi(x/n)$ and $\varphi_n = \psi_n \varphi$. Then for any β,

$$\left(\partial^\beta \varphi_n\right)(x) - \left(\partial^\beta \varphi\right)(x) = \partial^\beta\left(\left(\psi_n - 1\right)\varphi\right)(x) = \sum_{\beta_1 + \beta_2 = \beta} c_{\beta_1\beta_2} \partial^{\beta_1}\left(\psi_n - 1\right)(x)\left(\partial^{\beta_2}\varphi\right)(x)$$

and thus for any α,

$$\left| x^\alpha \partial^\beta \left(\varphi_n - \varphi\right)(x) \right| = \sum_{\beta_1 + \beta_2 = \beta} c_{\beta_1\beta_2} \left| \partial^{\beta_1}\left(\psi_n - 1\right)(x) \right| \left| x^\alpha \left(\partial^{\beta_2}\varphi\right)(x) \right|.$$

First, by $\left\|\partial^{\beta_1}\left(\psi_n - 1\right)\right\|_\infty = n^{-|\beta_1|}\left\|\partial^{\beta_1}\psi\right\|_\infty$, $\left\|\partial^{\beta_1}\left(\psi_n - 1\right)\right\|_\infty$ is bounded with respect to n. Then for $|x| > n$, we have $\left| x^\alpha \left(\partial^{\beta_2}\varphi\right)(x) \right| \to 0$ as $n \to \infty$. And for $|x| \leq n$, $\partial^{\beta_1}\left(\psi_n - 1\right)(x) = 0$. Therefore, $\varphi_n \to \varphi$ in $\mathcal{S}$. □

Considering distributions on $\mathcal{S}$, defined as

$$\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d) = \{T \colon \mathcal{S} \to \mathbb{C} \colon T \text{ is linear and continuous.}\}$$

and the element in $\mathcal{S}'$ is called a tempered distribution.

Because $\mathcal{S}$ is a locally convex topological space and $\|\varphi\|_{N_1} \leq \|\varphi\|_{N_2}$ with $N_1 \leq N_2$, $T \in \mathcal{D}'$ if and only if there are $N \in \mathbb{N}_0$ and $C > 0$,

$$|\langle T, \varphi \rangle| \leq C\|\varphi\|_N, \quad \varphi \in \mathcal{S}$$

By this definition, because $\mathcal{D}$'s topology can be also induced by the same topology on $\mathcal{S}$ and $\mathcal{D}$ is dense in $\mathcal{S}$, we have the following corollary.

Corollary 1.5.1. *$T \mapsto T|_{\mathcal{D}}$ is a map from $\mathcal{S}' \to \mathcal{D}'$ and this map is injective. Moreover, $\mathcal{S}' \subset \mathcal{D}'$ under this restriction.*

Example 1.5.1. (1) For $f(x) = e^x$, clearly it is in $\mathcal{D}'$. But $f \notin \mathcal{S}'$.

(2) Let $f \in L^1_{loc}(\mathbb{R}^d)$ satisfy that there is a $N \in \mathbb{N}$ such that

$$\int_{|x|\geq 1} \frac{|f(x)|}{|x|^N} dx < \infty$$

Then for any $\varphi \in \mathcal{S}$,

$$|\langle f, \varphi \rangle| \leq \int_{|x|<1} |f(x)||\varphi(x)|dx + \int_{|x|\geq 1} \frac{|f(x)|}{|x|^N}|x|^N|\varphi(x)|dx \leq C\|\varphi\|_N$$

Therefore, $f \in \mathcal{S}'$.

(3) For any measure μ defined on $(\mathbb{R}^d, \mathcal{R}^d)$, if there is a $N \in \mathbb{N}$ such that

$$\int_{|x|\geq 1} |x|^{-N} d\mu(x) < \infty$$

then by defining $\langle \mu, \varphi \rangle := \int \varphi(x) d\mu(x)$ for $\varphi \in \mathcal{S}$, $\mu \in \mathcal{S}'$ because

$$|\langle \mu, \varphi \rangle| \leq \int_{|x|<1} |\varphi| d\mu + \int_{|x|\geq 1} \frac{1}{|x|^N}|x|^N|\varphi(x)|d\mu(x) \leq C\|\varphi\|_N$$

For example, the Dirac delta function δ can be viewed as the measure with $\mu(\{0\}) = \mu(\mathbb{R}^d) = 1$, called the Dirac measure.

Remark. A function on $\mathbb{R}^d$ is called polynomial growth if there is $N \in \mathbb{N}$ and $C > 0$ such that for all $x \in \mathbb{R}^d$,

$$|f(x)| \leq C\left(1 + |x|^N\right)$$

Then measurable $f \in \mathcal{S}'$. Moreover, $L^p(\mathbb{R}^d) \subset \mathcal{S}'$.

Proposition 1.5.2. *If $T \in \mathcal{D}'$ with compact support, then for its extension $\bar{T}$ on C^∞, $\bar{T}\big|_{\mathcal{S}} \in \mathcal{S}'$.*

Proof. Choose $\varphi_n \to \varphi$ in $\mathcal{S}$. Let $\eta \in \mathcal{D}$ such that $\eta = 1$ near around $\operatorname{supp} T$. Then we have $\eta\varphi_n \to \eta\varphi$ in $\mathcal{D}$. So

$$\langle \bar{T}, \varphi_n \rangle = \langle T, \eta\varphi_n \rangle \to \langle T, \eta\varphi \rangle = \langle \bar{T}, \varphi \rangle$$

□

For $T \in \mathcal{S}'$ and α, $x^\alpha T \colon \mathcal{S} \to \mathbb{C}$ is defined as

$$\langle x^\alpha T, \varphi \rangle = \langle T, x^\alpha \varphi \rangle$$

where $x^\alpha \varphi$ is $x \mapsto x^\alpha \varphi(x)$. Note that for $\varphi \in \mathcal{S}$, $x^\alpha \varphi \in \mathcal{S}$.

Proposition 1.5.3. *For $T \in \mathcal{S}'$ and α, $x^\alpha T, \partial^\alpha T \in \mathcal{S}'$.*

Proof. For $\varphi \in \mathcal{S}$,

$$\begin{aligned} |\langle x^\alpha T, \varphi \rangle| &= |\langle T, x^\alpha \varphi \rangle| \le C \, \|x^\alpha \varphi\|_N \le C \|\varphi\|_{N+|\alpha|}, \\ |\langle \partial^\alpha T, \varphi \rangle| &= |\langle T, \partial^\alpha \varphi \rangle| \le C \, \|\partial^\alpha \varphi\|_N \le C \|\varphi\|_{N+|\alpha|} \end{aligned}$$

So $x^\alpha T, \partial^\alpha T \in \mathcal{S}'$. $\square$

2. Fourier Transform on $\mathcal{S}'$: For $T \in \mathcal{S}'$, define $\widehat{T} \colon \mathcal{S} \to \mathbb{C}$ as

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle, \quad \varphi \in \mathcal{S}$$

Remark. By this definition, for $f \in L^1_{loc}$, by $\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle$, we have $\widehat{T_f} = T_{\widehat{f}}$. So as $f \in L^1$.

Proposition 1.5.4. *For $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\widehat{\varphi}_n \to \widehat{\varphi}$ in $\mathcal{S}$. So $T \in \mathcal{S}'$ implies $\widehat{T} \in \mathcal{S}'$.*

Proof. For $\varphi \in \mathcal{S}$,

$$|\widehat{\varphi}(\xi)| \le \int_{\mathbb{R}^d} |\varphi(x)| dx = \int_{\mathbb{R}^d} \frac{1}{1+|x|^{d+1}} \underbrace{\left(1+|x|^{d+1}\right) |\varphi(x)|}_{\le \|\varphi\|_{d+1}} dx.$$

So $\|\widehat{\varphi}\|_\infty \le C \|\varphi\|_{d+1}$ for some $C > 0$. Moreover, for α,

$$\left(\partial^\alpha \varphi\right)^\wedge(\xi) = i^{|\alpha|} \xi^\alpha \widehat{\varphi}(\xi), \quad \left(x^\alpha \varphi\right)^\wedge(\xi) = i^{|\alpha|} \left(\partial^\alpha \widehat{\varphi}\right)(\xi)$$

So we have

$$\left|\xi^\alpha \left(\partial^\beta \widehat{\varphi}\right)(\xi)\right| = \left|\xi^\alpha \left(x^\beta \varphi\right)^\wedge(\xi)\right| = \left|\left(\partial^\alpha \left(x^\beta \varphi\right)\right)^\wedge(\xi)\right| \le C \left\|\partial^\alpha \left(x^\beta \varphi\right)\right\|_{d+1} \le C_1 \|\varphi\|_{|\alpha|+|\beta|+d+1}.$$

and thus $\|\widehat{\varphi}\|_N \le C_2 \|\varphi\|_{2N+d+1}$. $\square$

Recall $\mathcal{F} \colon \mathcal{S} \to \mathcal{S}$ is a bijection. For $S \in \mathcal{S}'$, define $\check{S} \colon \mathcal{S} \to \mathbb{C}$ as

$$\langle \check{S}, \varphi \rangle = \langle S, \mathcal{F}^{-1} \varphi \rangle, \quad \varphi \in \mathcal{S}$$

Then $\check{S}$ is called the inverse transform of S. Moreover, by $\mathcal{F}^2 \varphi = (2\pi)^d \varphi^\sim$, $\mathcal{F}^{-1} \varphi = (2\pi)^{-2d} \mathcal{F}^3 \varphi$. So by above proposition,

$$\varphi_n \to \varphi \quad \Rightarrow \quad \mathcal{F}^{-1} \varphi_n \to \mathcal{F}^{-1} \varphi$$

in $\mathcal{S}$. So $S \in \mathcal{S}'$ implies $\check{S} \in \mathcal{S}'$.

Proposition 1.5.5. *Fourier transform $\mathcal{F} \colon \mathcal{S}' \to \mathcal{S}'$ is a bijection and $\mathcal{F}^{-1} S = \check{S}$.*

Proof. For $\varphi \in \mathcal{S}$,

$$\langle \check{\widehat{T}}, \varphi \rangle = \langle \widehat{T}, \mathcal{F}^{-1}\varphi \rangle = \langle T, \mathcal{F}\mathcal{F}^{-1}\varphi \rangle = \langle T, \varphi \rangle$$

Therefore, $\check{\widehat{T}} = T$ and similarly $\widehat{\check{T}} = T$. □

Proposition 1.5.6. *For $T \in \mathcal{S}'$ and α,*

$$\widehat{\partial^\alpha T} = i^{|\alpha|} x^\alpha \widehat{T}, \quad \widehat{x^\alpha T} = i^{|\alpha|} \partial^\alpha \widehat{T}$$

Proof. For any $\varphi \in \mathcal{S}$,

$$\begin{aligned} \left\langle \widehat{\partial^\alpha T}, \varphi \right\rangle &= \langle \partial^\alpha T, \widehat{\varphi} \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \widehat{\varphi} \rangle = i^{|\alpha|} \left\langle T, \widehat{x^\alpha \varphi} \right\rangle = i^{|\alpha|} \left\langle x^\alpha \widehat{T}, \varphi \right\rangle, \\ \left\langle \widehat{x^\alpha T}, \varphi \right\rangle &= \langle T, x^\alpha \widehat{\varphi} \rangle = i^{-|\alpha|} \left\langle T, \widehat{\partial^\alpha \varphi} \right\rangle = i^{-|\alpha|} \left\langle \widehat{T}, \partial^\alpha \varphi \right\rangle = i^{|\alpha|} \left\langle \partial^\alpha \widehat{T}, \varphi \right\rangle. \end{aligned}$$

□

Proposition 1.5.7. *For $T \in \mathcal{S}'$,*

$$\mathcal{F}^2 T = (2\pi)^d T^\sim, \quad \mathcal{F}^4 T = (2\pi)^{2d} T$$

Proof. For any $\varphi \in \mathcal{S}$,

$$\left\langle \mathcal{F}^2 T, \varphi \right\rangle = \left\langle T, \mathcal{F}^2 \varphi \right\rangle = \left\langle T, (2\pi)^d \varphi^\sim \right\rangle = (2\pi)^d \left\langle T^\sim, \varphi \right\rangle.$$

□

Proposition 1.5.8. *Equipping $\mathcal{S}'$ with the wk*-topology, then*

$$T_n \to T \quad \Rightarrow \quad \widehat{T_n} \to \widehat{T}$$

Proof. For any $\varphi \in \mathcal{S}$,

$$\left\langle \widehat{T_n}, \varphi \right\rangle = \langle T_n, \widehat{\varphi} \rangle \to \langle T, \widehat{\varphi} \rangle = \langle \widehat{T}, \varphi \rangle$$

□

Example 1.5.2. (1) For the $\delta \in \mathcal{D}'$, because $\operatorname{supp} \delta = 0$, $\delta \in \mathcal{S}'$. And by

$$\langle \widehat{\delta}, \varphi \rangle = \langle \delta, \widehat{\varphi} \rangle = \widehat{\varphi}(0) = \int_{\mathbb{R}^d} \varphi(x) dx = \langle 1, \varphi \rangle$$

we have $\widehat{\delta} = 1$. And by the inverse formula,

$$\widehat{1} = \mathcal{F}^2 \delta = (2\pi)^d \delta$$

And based on this, for any monomial $x^\alpha = x^\alpha \cdot 1$,

$$\widehat{x^\alpha} = i^{|\alpha|} \partial^\alpha \widehat{1} = (2\pi)^d i^{|\alpha|} \partial^\alpha \delta.$$

(2) For $a \in \mathbb{R}^d$, let $\delta_a \in \mathcal{S}'$ defined as

$$\langle \delta_a, \varphi \rangle = \varphi(a)$$

Then

$$\widehat{\delta_a} = \left(x \mapsto e^{-ia \cdot x} \right)$$

(3) On L^2, $\mathcal{F}\colon L^2 \to L^2$ bijective isometry. For any $f \in L^2$, let $f_n = f\chi_{[-n,n]^d} \in L^1$. $f_n \to f$ in L^2 implies $\widehat{f_n} \to \widehat{f}$ in L^2. Besides, because $\mathcal{S} \subset L^2$, for any $\varphi\mathcal{S}$ and $g_n \to g$ in L^2 implies $\langle g_n, \varphi\rangle \to \langle g, \varphi\rangle$. So

$$\left\langle \widehat{T_f}, \varphi \right\rangle = \langle f, \widehat{\varphi}\rangle = \lim_{n\to\infty} \langle f_n, \widehat{\varphi}\rangle = \lim_{n\to\infty} \left\langle \widehat{f_n}, \varphi \right\rangle = \langle \widehat{f}, \varphi\rangle.$$

Therefore, $\widehat{T_f} = T_{\widehat{f}}$.

(4) By the Poisson Summation Formula, for any $\varphi \in \mathcal{S}$,

$$\sum_{n\in\mathbb{Z}} \widehat{\varphi}(n) = 2\pi \sum_{k\in\mathbb{Z}} \varphi(2\pi k)$$

we have

$$\mathcal{F}\left(\sum_{n\in\mathbb{Z}} \delta_n\right) = 2\pi \sum_{k\in\mathbb{Z}} \delta_{2\pi k}, \quad \mathcal{F}\left(\sum_{k\in\mathbb{Z}} \delta_{2\pi k}\right) = \sum_{n\in\mathbb{Z}} \delta_n$$

3. Polynomial Growth Functions:

Theorem 1.5.1. *For any $T \in \mathcal{S}'$, there is a polynomial growth continuous function f such that $T = \partial^\alpha f$*

For $a \in \mathbb{R}^d$, let $\Sigma_a = \prod_{i=1}^d [a_i, \infty)$.

Proposition 1.5.9. *(1) For any $a, b \in \mathbb{R}^d$, $\Sigma_a \cap (-\Sigma_b)$ is either empty or a closed rectangular so compact.*

(2) For any $a \in \mathbb{R}^d$ and compact K, there is a $b \in \mathbb{R}^d$ such that $\Sigma_a + K \subset \Sigma_b$

Proposition 1.5.10. *Let $T \in \mathcal{D}'$ and $\varphi \in C^\infty$ with $\operatorname{supp} T \subset \Sigma_a$ and $\operatorname{supp} \varphi \subset -\Sigma_b$. Let $\eta \in \mathcal{D}$ such that $\eta = 1$ near around $\Sigma_a \cap (-\Sigma_b)$. Define*

$$\langle T, \varphi\rangle = \langle T, \eta\varphi\rangle$$

which is independent with the choice of η. Then for any α,

$$\langle \partial^\alpha T, \varphi\rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi\rangle .$$

Proposition 1.5.11. *Let $T, S \in \mathcal{D}'$ with $\operatorname{supp} T \subset \Sigma_a$ and $\operatorname{supp} S \subset \Sigma_b$. Define*

$$\langle T * S, \varphi\rangle = \langle T, S^\sim * \varphi\rangle , \quad \varphi \in \mathcal{D}$$

Then $T * S \in \mathcal{D}'$.

Remark. For the definition, $S^\sim * \varphi \in C^\infty$ with the support contained in $\operatorname{supp} \varphi - \Sigma_b$ that is contained in $-\Sigma_c$ by the above proposition. So $\langle T, S^\sim * \varphi\rangle$ can be defined.

Similarly as above, we have

$$\partial^\alpha (T * S) = (\partial^\alpha T) * S = T * (\partial^\alpha S)$$

Proposition 1.5.12. *Let $T \in \mathcal{S}'$ with the order N and $f \in C^N$. And assume $\operatorname{supp} T \subset \Sigma_a$ and $\operatorname{supp} f \subset \Sigma_b$.*

(1) Then $T * f \in \mathcal{D}'$ is a continuous function on $\mathbb{R}^d$.

(2) If for any $|\alpha| < N$, $\partial^\alpha f$ is polynomial growth, then $T * f$ is polynomial growth.

4. Convolution with Rapidly Decreasing Functions: For $T \in \mathcal{S}'$ and $\psi \in \mathcal{S}$,

$$(T * \psi)(x) = \langle T, \tau_x \psi^\sim \rangle, \quad x \in \mathbb{R}^d$$

Lemma 1.5.1. *Let f be polynomial growth and continue. Let $\psi \in \mathcal{S}$. Then $f * \psi \in C^\infty$ and polynomial growth. Moreover, for any α,*

$$\partial^\alpha (f * \psi) = f * (\partial^\alpha \psi)$$

Proof. For any $x, y \in \mathbb{R}^d$, we have

$$1 + |x| \leq (1 + |x - y|)(1 + |y|)$$

Because f is polynomial growth, there is a $C > 0$ and $M \in \mathbb{N}$ such that $|f(x)| \leq C(1 + |x|)^M$. Therefore, for any $x \in \mathbb{R}^d$,

$$\begin{aligned}
|(f * \psi)(x)| &\leq \int_{\mathbb{R}^d} |f(y)||\psi(x - y)|dy \\
&\leq \int_{\mathbb{R}^d} \frac{C}{(1 + |y|)^{d+1}} \frac{(1 + |y|)^{M+d+1}}{(1 + |x - y|)^{M+d+1}} (1 + |x - y|)^{M+d+1} |\psi(x - y)|dy \\
&\leq \int_{\mathbb{R}^d} \frac{Cdy}{(1 + |y|)^{d+1}} (1 + |x|)^{M+d+1} C_1 \|\psi\|_{M+d+1}
\end{aligned}$$

So $f * \psi$ is polynomial growth.

To show C^∞, let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$, then for any $h \neq 0$,

$$\frac{(f * \psi)(x + he_1) - (f * \psi)(x)}{h} = \int_{\mathbb{R}^d} f(y) \frac{\psi(x + he_1 - y) - \psi(x - y)}{h} dy.$$

By the mean value theorem, there is a $\xi \in (0, 1)$ such that if let $z = x + \xi e_1 - y$,

$$\begin{aligned}
&\left| \frac{\psi(x + he_1 - y) - \psi(x - y)}{h} \right| = |\partial_1 \psi(z)| \\
&= \frac{1}{(1 + |y|)^{M+d+1}} \frac{(1 + |y|)^{M+d+1}}{(1 + |z|)^{M+d+1}} (1 + |z|)^{M+d+1} |\partial_1 \psi(z)| \\
&\leq \frac{(1 + |x + \xi e_1|)^{M+d+1}}{(1 + |y|)^{M+d+1}} C_2 \|\psi\|_{M+d+2} \leq \frac{C_3}{(1 + |y|)^{M+d+1}}.
\end{aligned}$$

Therefore,

$$\left| f(y) \frac{\psi(x + he_1 - y) - \psi(x - y)}{h} \right| \leq \frac{CC_3}{(1 + |y|)^{d+1}}$$

Then let $h \to 0$, by the DCT

$$D_{e_1}(f * \psi) = f * (D_{e_1} \psi)$$

□

Proposition 1.5.13. *For any $T \in \mathcal{S}'$ and $\psi \in \mathcal{S}$,*

(1) when writing $T = \partial^\alpha f$ for some polynomial growth f, $T * \psi = f * \partial^\alpha \psi$;

(2) $T * \psi \in C^\infty$ *is polynomial growth.*

Proposition 1.5.14. *For any* $T \in \mathcal{S}'$ *and* $\psi, \varphi \in \mathcal{S}$ *and* α,

(1) $\langle T * \psi, \varphi \rangle = \langle T, \psi^\sim * \varphi \rangle$;

(2) $\partial^\alpha (T * \psi) = (\partial^\alpha T) * \psi = T * (\partial^\alpha \psi)$.

Proposition 1.5.15. *Let* $T \in \mathcal{D}'$ *with compact support* ($T \in \mathcal{S}'$).

(1) $\widehat{T}$ *is polynomial growth and* C^∞;

(2) For any $\xi \in \mathbb{R}^d$,

$$\widehat{T}(\xi) = \left\langle \bar{T}, \varphi_\xi \right\rangle$$

where $\varphi_\xi(x) = e^{-i\xi \cdot x}$.

Proof. We only need to show the case of $T = \partial^\alpha f$ for some continuous f with compact support. First,

$$\widehat{T} = \widehat{\partial^\alpha f} = i^{|\alpha|} \xi^\alpha \widehat{f}$$

Because $\widehat{f}$ is bounded, $\widehat{T}$ is polynomial growth. Besides, by

$$\widehat{f}(\xi) = \int_{\operatorname{supp} f} f(x) e^{-i\xi \cdot x} dx$$

and the DCT on compact $\operatorname{supp} f$, $\widehat{f} \in C^\infty$. Let $\eta \in \mathcal{D}$ such that $\eta = 1$ near around $\operatorname{supp} f$. Then

$$\begin{aligned}
\left\langle \bar{T}, \varphi_\xi \right\rangle &= \langle T, \eta\varphi_\xi \rangle = \langle \partial^\alpha f, \eta\varphi_\xi \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha (\eta\varphi_\xi) \rangle \\
&= (-1)^{|\alpha|} \int_{\operatorname{supp} f} f(x) \partial^\alpha (\eta\varphi_\xi)(x) dx = (-1)^{|\alpha|} \int_{\operatorname{supp} f} f(x) (\partial^\alpha \varphi_\xi)(x) dx \\
&= \int_{\operatorname{supp} f} f(x) i^{|\alpha|} \xi^\alpha e^{-i\xi \cdot x} dx = i^{|\alpha|} \xi^\alpha \widehat{f}(\xi) = \widehat{T}(\xi).
\end{aligned}$$

□

Recall, for $\varphi, \psi \in \mathcal{S}$,

$$\widehat{\widehat{\varphi}} = (2\pi)^d \varphi^\sim, \quad \widehat{\varphi}^\sim = \widehat{\varphi^\sim}, \quad (\varphi * \psi)^\sim = \varphi^\sim * \psi^\sim$$

Proposition 1.5.16. *For* $T, S \in \mathcal{S}'$ *and* $\psi \in \mathcal{S}$,

(1) $\widehat{T * \psi} = \widehat{\psi} \widehat{T}$;

(2) if S *is with compact support, then* $\widehat{T * S} \in \mathcal{S}'$ *and* $\widehat{T * S} = \widehat{S} \widehat{T}$.

Proof. (1) For $\varphi \in \mathcal{S}$,

$$\begin{aligned}
\left\langle \widehat{T * \psi}, \varphi \right\rangle &= \langle T * \psi, \widehat{\varphi} \rangle = \langle T, \psi^\sim * \widehat{\varphi} \rangle = \frac{1}{(2\pi)^d} \left\langle T, (\widehat{\widehat{\psi^\sim * \widehat{\varphi}}})^\sim \right\rangle \\
&= \frac{1}{(2\pi)^d} \left\langle \widehat{T}, \widehat{\psi * \widehat{\varphi}^\sim} \right\rangle = \langle \widehat{T}, \widehat{\psi} \varphi \rangle = \langle \widehat{\psi} \widehat{T}, \varphi \rangle.
\end{aligned}$$

(2) First, $T = \partial^\alpha f$ with f polynomial growth and continuous. It is sufficient to prove when $S = \partial^\beta g$ with continuous g and compact $\operatorname{supp} g$. First,

$$T * S = \partial^{\alpha+\beta}(f * g),$$

where $f * g$ is polynomial growth. So $T * S \in \mathcal{S}'$.
Note that $\widehat{f * g} = \widehat{g}\widehat{f}$. Because for any $\varphi \in \mathcal{S}$,

$$\begin{aligned}\langle \widehat{f * g}, \varphi \rangle &= \langle f * g, \widehat{\varphi} \rangle = \langle f, g^{\sim} * \widehat{\varphi} \rangle = \frac{1}{(2\pi)^d} \langle f, \widehat{\widehat{g^{\sim} * \widehat{\varphi}^{\sim}}} \rangle \\ &= \frac{1}{(2\pi)^d} \langle \widehat{f}, \widehat{g * \widehat{\varphi}^{\sim}} \rangle = \langle \widehat{f}, \widehat{g}\varphi \rangle = \langle \widehat{g}\widehat{f}, \varphi \rangle\end{aligned}$$

Then we have

$$\widehat{T * S} = \widehat{\partial^{\alpha+\beta}(f * g)} = i^{|\alpha+\beta|}\xi^{\alpha+\beta}\widehat{f * g} = i^{|\alpha+\beta|}\xi^{\alpha+\beta}\widehat{g}\widehat{f} = \widehat{S}\widehat{T}.$$

$\square$

Chapter 2

L^p Space

Let $(X, \mathscr{A}, \mu)$ be a measure space in the following sections.

2.1 Basic Properties

For $1 \leq p < \infty$, let $L^p(X, \mathscr{A}, \mu) = L^p(X, d\mu) = L^p(X)$ be the space of $\mathbb{C}$-measurable functions with equivalent class $u(x) = v(x)$ $\mu - a.e.$ and

$$\|u\|_{L^p(X)} = \|u\|_p := \left(\int_X |u(x)|^p d\mu(x)\right)^{1/p} < \infty$$

An $\mathscr{A}$-measurable function u on X is essentially bounded if there is an $M > 0$ such that

$$|u(x)| \leq M, \quad \mu - a.e.$$

and $\operatorname{ess\,sup}_{x\in X} |u(x)|$ denotes the infimum of such M. If no such M, $\operatorname{ess\,sup}_{x\in X} |u(x)| = \infty$ and u is called essentially unbounded. In fact,

$$\operatorname{ess\,sup}_{x\in X} |u(x)| = \inf\left\{t \geq 0 \mid \mu\left(\{x \in X \mid |u(x)| > t\}\right) = 0\right\} \tag{2.1}$$

Let $L^\infty(X, \mathscr{A}, \mu) = L^\infty(X, d\mu) = L^\infty(X)$ be the space of all essentially bounded u with the equivalent class $u(x) = v(x)$ $\mu - a.e.$ and denote

$$\|u\|_{L^\infty(X)} = \|u\|_\infty := \operatorname{ess\,sup}_{x\in X} |u(x)|$$

Remark. On $\mathbb{R}^N$, let $\mathscr{L}_N$ be the set of all Lebesgue measurable sets. And let $E \in \mathscr{L}_N$. Then define

$$\mathscr{L}_N|_E := \{B \in \mathscr{L}_N \mid B \subset E\}$$

Let m_N denote the Lebesgue measure on $\mathbb{R}^N$. Denote $L^p\left(E, \mathscr{L}_N|_E, m_N\right) = L^p(E)$. Besides, denote $dm_N(x) = dx$.

Proposition 2.1.1 (Hölder's Inequality). *For $1 \leq p \leq \infty$, let q be its conjugate ($1/p + 1/q = 1$). Then for any $u \in L^p(X)$ and $v \in L^q(X)$,*

$$\int_X |u(x)v(x)| d\mu(x) \leq \|u\|_p \|v\|_q$$

Theorem 2.1.1. *Let $\varnothing \neq \Omega \subset \mathbb{R}^N$ be open. Then for any $1 \leq p, q < \infty$, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega) \cap L^q(\Omega)$.*

2.2 Dual Space of L^p Space

For $1 \le p \le \infty$, let q be its conjugate of p. For $g \in L^q(X)$, define

$$\Phi_g(u) := \int_X u(x)g(x)d\mu(x), \quad u \in L^p(X)$$

Then by the Hölder's Inequality we have

$$|\Phi_g(u)| \le \|g\|_q \|u\|_p$$

So $\Phi_g \in L^p(X)^*$ and

$$\|\Phi_g\|_{L^p(X)^*} \le \|g\|_{L^q(X)}$$

Proposition 2.2.1. *For $1 \le p \le \infty$, let q be its conjugate of p. When $p = 1$ and $q = \infty$, we further assume $(X, \mathscr{A}, \mu)$ σ-finite. Then for $g \in L^q(X)$,*

$$\|\Phi_g\|_{L^p(X)^*} = \|g\|_q$$

In particular, define $J\colon L^q(X) \to L^p(X)^*$ by

$$J(g) := \Phi_g, \quad g \in L^q(X)$$

Then J is an isometric linear map from $L^q(X)$ to $L^p(X)^*$.

Note that for $z \in \mathbb{C}$,

$$\operatorname{sgn} z = \begin{cases} 0, & z = 0 \\ \frac{\bar{z}}{|z|}, & z \in \mathbb{C}\setminus\{0\} \end{cases}$$

and so we have

$$z \operatorname{sgn} z = |z|, \quad |\operatorname{sgn} z| = \begin{cases} 0, & z = 0 \\ 1, & z \in \mathbb{C}\setminus\{0\} \end{cases}$$

For $a \in \mathbb{R}$,

$$\operatorname{sgn} a = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0 \end{cases}$$

Proof. Only need to prove $\|\Phi_g\|_{L^p(X)^*} \ge \|g\|_q$. If $\|g\|_q = 0$, it's clear. So we assume $\|g\|_q > 0 (g \in L^q(X)\setminus\{0\})$.

(i) $1 < p \le \infty$: So $1 \le q < \infty$. Define $\mathscr{A}$-measurable function v_g as

$$v_g(x) := |g(x)|^{q-1} \operatorname{sgn}(g(x)), \quad x \in X$$

So $v_g(x)g(x) = |g(x)|^q$ for $x \in X$ and $v_g \in L^q(X)\setminus\{0\}$.
Claim: $\|v_g\|_p = \|g\|_q^{q-1}$.
First, for $1 < p < \infty$, $1 < q < \infty$ and $p = \frac{q}{q-1}$. For $x \in X$,

$$|v_g(x)| = |g(x)|^{q-1} |\operatorname{sgn}(g(x))| = |g(x)|^{q-1}$$

and so $|v_g(x)|^p = |g(x)|^{(q-1)p} = |g(x)|^q$ and thus

$$\|v_g\|_p = \left(\int_X |v_g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} = \left(\int_X |g(x)|^q d\mu(x)\right)^{\frac{q-1}{q}} = \|g\|_{L^q(X)}^{q-1} < \infty$$

If $p=\infty$, then $q=1$. So $g \in L^1(X) \setminus \{0\}$. So for $x \in X$,

$$|v_g(x)| = |\operatorname{sgn}(g(x))| = \begin{cases} 0, & g(x)=0 \\ 1, & g(x) \neq 0 \end{cases}$$

Therefore, $\|v_g\|_\infty = 1$.

Next, let's continue the proof. Note that $v_g \in L^p(X)\setminus\{0\}$ and $v_g \cdot g \in L^1(X)$. Therefore,

$$\begin{aligned} |\Phi_g(v_g)| &= \left| \int_X v_g(x) g(x) d\mu(x) \right| = \int_X |g(x)|^q d\mu(x) \\ &= \|g\|_{L^q(X)}^q = \|g\|_{L^q(X)} \|g\|_{L^q(X)}^{q-1} = \|g\|_{L^q(X)} \|v_g\|_{L^p(X)} \end{aligned}$$

Therefore,

$$\|\Phi_g\|_{L^p(X)^*} \geq \|g\|_q$$

(ii) $P=1$: Then $q=\infty$, $g \in L^\infty(X)\setminus\{0\}$ and $\Phi_g \in L^1(X)^*$.

For any ε with $0<\varepsilon<\|g\|_\infty$, let

$$A_\varepsilon := \{x \in X \,|\, |g(x)| > \|g\|_\infty - \varepsilon\}$$

Then $A_\varepsilon \in \mathscr{A}$ and $\mu(A_\varepsilon)>0$ by the definition of $\|\cdot\|_\infty$. Since $(X,\mathscr{A},\mu)$ is σ-finite, there is a $B_\varepsilon \in \mathscr{A}$ such that

$$B_\varepsilon \subset A_\varepsilon, \quad 0<\mu(B_\varepsilon)<\infty$$

Define a $\mathscr{A}$-measurable function $w_{\varepsilon,g}$ on X as

$$w_{\varepsilon,g}(x) := \chi_{B_\varepsilon}(x) \operatorname{sgn}(g(x)), \quad x \in X$$

Because for any $x \in B_\varepsilon$, $|g(x)| > \|g\|_{L^\infty(X)} - \varepsilon > 0$, $|\operatorname{sgn}(g(x))| = 1$. Thus,

$$|w_{\varepsilon,g}(x)| = \chi_{B_\varepsilon}(x) |\operatorname{sgn}(g(x))| = \chi_{B_\varepsilon}(x), \quad x \in X$$

Then we have

$$\int_X |w_{\varepsilon,g}(x)| \, d\mu(x) = \int_X \chi_{B_\varepsilon}(x) d\mu(x) = \mu(B_\varepsilon) < \infty$$

which means $w_{\varepsilon,g} \in L^1(X)\setminus\{0\}$ with $\|w_{\varepsilon,g}\|_1 = \mu(B_\varepsilon)$. Besides,

$$w_{\varepsilon,g}(x) g(x) = \chi_{B_\varepsilon}(x) \{\operatorname{sgn}(g(x))\} g(x) = \chi_{B_\varepsilon}(x) |g(x)|$$

so $w_{\varepsilon,g} \cdot g \in L^1(X)$. Then

$$\begin{aligned} |\Phi_g(w_{\varepsilon,g})| &= \left| \int_X w_{\varepsilon,g}(x) g(x) d\mu(x) \right| = \int_X \chi_{B_\varepsilon}(x) |g(x)| d\mu(x) \\ &= \int_{B_\varepsilon} |g(x)| d\mu(x) > \int_{B_\varepsilon} (\|g\|_\infty - \varepsilon) \, d\mu(x) \\ &= (\|g\|_\infty - \varepsilon) \mu(B_\varepsilon) = (\|g\|_\infty - \varepsilon) \|w_{\varepsilon,g}\|_1 \end{aligned}$$

So

$$\|\Phi_g\|_{L^1(X)^*} \geq \|g\|_\infty - \varepsilon \quad \Rightarrow \quad \|\Phi_g\|_{L^1(X)^*} \geq \|g\|_\infty$$

$\square$

Definition 2.2.1. For measure space $(X, \mathscr{A}, \mu)$, let $S_0(X, \mathscr{A}, \mu) = S_0(X)$ be the set of all $\mathscr{A}$-measurable simple functions ϕ with $\mu(\{x \in X \mid \phi(x) \neq 0\}) < \infty$.

Remark. (1) If $(X, \mathscr{A}, \mu)$ is finite, then S_0 is the set of all simple functions.

(2) For $1 \leq p \leq \infty$, $S_0(X) \subset L^p(X)$.

Proposition 2.2.2. *For $1 \leq p < \infty$, if $f \in L^p(X)$, then there is a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ in $S_0(X)$ such that*

(1) for any $n \in \mathbb{N}$ and $x \in X$,

$$|\phi_n(x)| \leq |\phi_{n+1}(x)|\,, |\phi_n(x)| \leq |f(x)|;$$

(2) for any $x \in X$,

$$\lim_{n\to\infty} \phi_n(x) = f(x)$$

(3) moreover,

$$\lim_{n\to\infty} \|\phi_n - f\|_{L^p(X)} = 0$$

In particular, for $1 \leq p < \infty$, then $S_0(X)$ is dense in $L^p(X)$.

Theorem 2.2.1. *Let $(X, \mathscr{A}, \mu)$ be σ-finite. For $1 \leq p \leq \infty$, let q be its conjugate. f is a complex-valued $\mathscr{A}$-measurable function and for any $\phi \in S_0(X)$ with*

$$\int_X |\phi(x)f(x)|d\mu(x) < \infty$$

and

$$\sup\left\{\left|\int_X \phi(x)f(x)d\mu(x)\right| \mid \phi \in S_0(X), \|\phi\|_{L^p(X)} = 1\right\} < \infty$$

Then $f \in L^q(X)$,

$$\|f\|_q = \sup\left\{\left|\int_X \phi(x)f(x)d\mu(x)\right| \mid \phi \in S_0(X), \|\phi\|_p = 1\right\}$$

Proof. Let

$$M_q(f) := \sup\left\{\left|\int_X \phi(x)f(x)d\mu(x)\right| \mid \phi \in S_0(X), \|\phi\|_{L^p(X)} = 1\right\}$$

Therefore, it is sufficient to check for any $f \in L^q(X)$

$$\|f\|_q = M_q(f)$$

If $\mu(X) = 0$, it is clear. Besides, if $\|f\|_q = 0$, it is also clear. So assume $0 < \mu(X) \leq \infty$ and $\|f\|_q \neq 0$.

(i) **Check:** If v is complex-valued, bounded, $\mathscr{A}$-measurable, and

$$\mu(\{x \in X \mid v(x) \neq 0\}) < \infty, \quad \|v\|_{L^p(X)} = 1$$

then $v \cdot f \in L^1(X)$ and

$$\left|\int_X v(x)f(x)d\mu(x)\right| \leq M_q(f)$$

First, let $E := \{x \in X \mid v(x) \neq 0\}$. For $x \in X$,

$$|v(x)f(x)| = |v(x)\chi_E(x)f(x)| \leq \|v\|_{L^\infty(X)}\chi_E(x)|f(x)|$$

Because $\mu(E) < \infty$, $\chi_E \in S_0(X)$ and so $\chi_E \cdot f \in L^1(X)$. And $v \cdot f \in L^1(X)$. There is a sequence of $\mathscr{A}$-measurable functions $\{\phi_n\}_{n\in\mathbb{N}}$ such that

- $|\phi_n(x)| \leq |\phi_{n+1}(x)|$ and $|\phi_n(x)| \leq |v(x)|$;
- $\phi_n(x) \to v(x)$ point-wisely;
- $\|\phi_1\|_p > 0$.

In particular, $\phi_n \in S_0(X)$ and

$$0 < \|\phi_n\|_{L^p(X)} \leq \|v\|_{L^p(X)} = 1 \tag{2.2}$$

Then we have

$$\left| \int_X \phi_n(x) f(x) d\mu(x) \right| \leq M_q(f) \|\phi_n\|_{L^p(X)} \leq M_q(f)$$

By DCT, as $n \to \infty$,

$$\left| \int_X v(x) f(x) d\mu(x) \right| \leq M_q(f)$$

(ii) For $1 \leq q < \infty$, $1 < p \leq \infty$. For f, there is a sequence of $\mathscr{A}$-measurable functions $\{\psi_n\}_{n\in\mathbb{N}}$ such that

- $|\psi_n(x)| \leq |\phi_{n+1}(x)|$ and $|\psi_n(x)| \leq |v(x)|$;
- $\psi_n(x) \to v(x)$ point-wisely;
- It is not that $\psi_1(x) = 0$ for $\mu - a.e.\ x \in X$.

Since $(X, \mathscr{A}, \mu)$ is σ-finite with $\mu(X) > 0$, there is a sequence $\{E_n\}_{n\in\mathbb{N}}$ such that

- $X = \bigcup_n E_n$,
- $E_n \subset E_{n+1}$ and $0 < \mu(E_n) < \infty$,
- $\|\psi_1 \chi_{E_1}\|_q > 0$

Then $\psi_n \chi_{E_n} \in L^q(X), \|\psi_n \chi_{E_n}\|_{L^q(X)} > 0$. Define

$$f_n(x) := \psi_n(x) \chi_{E_n}(x), \quad x \in X$$

So $f_n \in S_0(X)$ with $\|f_n\|_q = \|\psi_n \chi_{E_n}\|_q > 0$. Moreover, we have

- for any $x \in X$,
$$f_n(x) = \psi_n(x) \chi_{E_n}(x) \longrightarrow f(x) \cdot 1 = f(x)$$
- for any $n \in \mathbb{N}$ and $x \in X$,
$$|f_n(x)| \leq |\psi_n(x)| \leq |f(x)|.$$

By Fatou's lemma,

$$\left(\int_X |f(x)|^q d\mu(x) \right)^{\frac{1}{q}} \leq \liminf_{n\to\infty} \left(\int_X |f_n(x)|^q d\mu(x) \right)^{\frac{1}{q}}$$

Remark. In fact, it is the idea of the proof of Proposition 2.2.2.

Define

$$v_n(x) := \frac{1}{\|f_n\|_q^{q-1}} |f_n(x)|^{q-1} \chi_{E_n}(x) \operatorname{sgn}(f(x)), \quad x \in X$$

Then for any $n \in \mathbb{N}$ and $x \in X$,

$$|v_n(x)| = \begin{cases} \frac{1}{\|f_n\|_{L^q(X)}^{q-1}} |f_n(x)|^{q-1}, & 1 < q < \infty \\ \chi_{E_n}(x) |\operatorname{sgn}(f(x))|, & q = 1 \end{cases}$$

First, for any n, v_n is bounded and $\mathscr{A}$-measurable and

$$\mu(\{x \in X \mid v_n(x) \neq 0\}) < \infty$$

Check: $\|v_n\|_p = 1$.

(a) For $1 < q < \infty$, $1 < p = \frac{q}{q-1} < \infty$, by the definition

$$\begin{aligned} \|v_n\|_{L^p(X)} &= \left(\int_X |v_n(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \\ &= \frac{1}{\|f_n\|_{L^q(X)}^{q-1}} \left(\int_X |f_n(x)|^{(q-1)p} \, d\mu(x) \right)^{\frac{q-1}{q}} \\ &= \frac{1}{\|f_n\|_{L^q(X)}^{q-1}} \|f_n\|_{L^q(X)}^{q-1} = 1 \end{aligned}$$

(b) For $q = 1$, $p = \infty$. Clearly, $v_n \in L^\infty(X)$. For any $x \in X$,

$$|v_n(x)| = \chi_{E_n}(x) |\operatorname{sgn}(f(x))|$$

Therefore,

$$|\operatorname{sgn}(f_n(x))| = \chi_{E_n}(x) \, |\operatorname{sgn}(f_n(x))| \leq |v_n(x)| \leq 1$$

Because $\|f_n\|_1 > 0$, $\|\operatorname{sgn}(f_n)\|_\infty = 1$. Thus, $\|v_n\|_\infty = 1$.

Besides, by the definition of v_n, we have

$$\begin{aligned} |v_n(x) f_n(x)| = |v_n(x)| |f_n(x)| &= \begin{cases} \frac{1}{\|f_n\|_q^{q-1}} |f_n(x)|^q, & 1 < q < \infty \\ \chi_{E_n}(x) |\operatorname{sgn}(f(x))| \, |f_n(x)|, & q = 1 \end{cases} \\ &= \frac{1}{\|f_n\|_q^{q-1}} |f_n(x)|^q \end{aligned}$$

Therefore,

$$\left(\int_X |f_n(x)|^q \, d\mu(x) \right)^{\frac{1}{q}} = \int_X |v_n(x) f_n(x)| \, d\mu(x)$$

Check: $v_n(x) f(x) = |v_n(x) f(x)|$.

For $1 < q < \infty$,

$$\begin{aligned} v_n(x) f(x) &= \frac{1}{\|f_n\|_{L^q(X)}^{q-1}} |f_n(x)|^{q-1} \chi_{E_n}(x) \operatorname{sgn}(f(x)) \cdot f(x) \\ &= \frac{1}{\|f_n\|_{L^q(X)}^{q-1}} |f_n(x)|^{q-1} |f(x)| = |v_n(x)| |f(x)| \end{aligned}$$

and for $q = 1$,

$$\begin{aligned} v_n(x) f(x) &= \chi_{E_n}(x) \operatorname{sgn}(f(x)) \cdot f(x) = \chi_{E_n}(x) |f(x)| \\ &= \chi_{E_n}(x) |\operatorname{sgn}(f(x))| |f(x)| = |v_n(x)| \, |f(x)| \end{aligned}$$

Combining these results, we have

$$\left(\int_X |f_n(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} = \int_X |v_n(x) f_n(x)| \, d\mu(x)$$
$$\leq \int_X |v_n(x)| \, |f(x)| d\mu(x) = \int_X v_n(x) f(x) d\mu(x) \leq M_q(f)$$

Therefore,

$$\left(\int_X |f(x)|^q d\mu(x)\right)^{\frac{1}{q}} \leq \liminf_{n\to\infty} \left(\int_X |f_n(x)|^q d\mu(x)\right)^{\frac{1}{q}} \leq M_q(f)$$

So $f \in L^q(X)$ with $\|f\|_q \leq M_q(f)$. For the other side, let $\phi \in S_0(X)$ with $\|\phi\|_p = 1$. By the Hölder's Inequality,

$$\left|\int_X \phi(x) f(x) d\mu(x)\right| \leq \|\phi\|_p \|f\|_q = \|f\|_q$$

So $M_q(f) \leq \|f\|_q$. Therefore, for $1 \leq q < \infty$,

$$\|f\|_q = M_q(f)$$

(iii) For $q = \infty$, $p = 1$. Check $f \in L^\infty$ and $\|f\|_\infty = M_\infty(f)$. For any $\varepsilon > 0$, let $A_\varepsilon \in \mathscr{A}$

$$A_\varepsilon := \{x \in X || f(x) | \geq M_\infty(f) + \varepsilon\}$$

Check: $\mu(A_\varepsilon)$ for all $\varepsilon > 0$.

Assume there is a $\varepsilon_0 > 0$ such that $\mu(A_{\varepsilon_)}) > 0$. The by the σ-finiteness of $(X, \mathscr{A}, \mu)$, there is $B_{\varepsilon_0} \in \mathscr{A}$ such that

$$B_{\varepsilon_0} \subset A_{\varepsilon_0}, \quad 0 < \mu\left(B_{\varepsilon_0}\right) < \infty$$

Define

$$v_{\varepsilon_0}(x) := \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \chi_{B_{\varepsilon_0}}(x) \operatorname{sgn}(f(x)), \quad x \in X$$

Then v_{ε_0} is bounded and $\mathscr{A}$-measurable and

$$\mu\left(\{x \in X \mid v_{\varepsilon_0}(x) \neq 0\}\right) < \infty$$

and in particular, $v_{\varepsilon_0} \in L^1(X)$. Moreover, for $x \in B_{\varepsilon_0}$, $|f(x)| \geq M_\infty(f) + \varepsilon_0 > 0$ and so $|\operatorname{sgn}(f(x))| = 1$. Therefore,

$$\|v_{\varepsilon_0}\|_{L^1(X)} = \int_X |v_{\varepsilon_0}(x)| \, d\mu(x) = \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \int_X \chi_{B_{\varepsilon_0}}(x) |\operatorname{sgn}(f(x))| d\mu(x)$$
$$= \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \int_{B_{\varepsilon_0}} |\operatorname{sgn}(f(x))| d\mu(x) = \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \int_{B_{\varepsilon_0}} 1 d\mu(x) = 1$$

The by (i), $v_{\varepsilon_0} \cdot f \in L^1(X)$. Moreover,

$$v_{\varepsilon_0}(x) f(x) = \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \chi_{B_{\varepsilon_0}}(x) \operatorname{sgn}(f(x)) \cdot f(x) = \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \chi_{B_{\varepsilon_0}}(x) |f(x)| \geq 0$$

by (i),

$$\int_X v_{\varepsilon_0}(x) f(x) d\mu(x) \leq M_\infty(f)$$

On the other hand,

$$\begin{aligned}
&\int_X v_{\varepsilon_0}(x) f(x) d\mu(x) \\
&= \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \int_X \chi_{B_{\varepsilon_0}}(x) \operatorname{sgn}(f(x)) \cdot f(x) d\mu(x) \\
&= \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \int_{B_{\varepsilon_0}} |f(x)| d\mu(x) \\
&\geq \frac{1}{\mu\left(B_{\varepsilon_0}\right)} \int_{B_{\varepsilon_0}} \left(M_\infty(f)+\varepsilon_0\right) d\mu(x) = M_\infty(f)+\varepsilon_0
\end{aligned}$$

which induces a contradiction.

Because $\mu(A_\varepsilon) = 0$ for all $\varepsilon > 0$, $f \in L^\infty(X)$ with

$$\|f\|_\infty \leq M_\infty(f) + \varepsilon$$

Therefore, $\|f\|_\infty \leq M_\infty(f)$.

For any $\phi \in S_0(X)$ with $\|\phi\|_1 = 1$, the Hölder's Inequality implies that

$$\left| \int_X \phi(x) f(x) d\mu(x) \right| \leq \|\phi\|_{L^1(X)} \|f\|_{L^\infty(X)} = \|f\|_{L^\infty(X)}$$

Therefore, $M_\infty(f) \leq \|f\|_\infty$.

□

Theorem 2.2.2. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and $1 \leq p < \infty$ with conjugate q. Let $\Phi \in L^p(X)^*$. Then there is a unique $g \in L^q(X)$ such that $\Phi = \Phi_g$. Moreover, $\|g\|_q = \|\Phi\|_{L^p(X)^*}$*

Proof. Note that $1 < q \leq \infty$.

- Uniqueness of $g \in L^q(X)$.

  If there are $g_1, g_2 \in L^q(X)$ such that $\Phi = \Phi_{g_1} = \Phi_{g_2}$. Then for any $u \in L^p(X)$,

  $$\begin{aligned}
  \Phi_{g_1 - g_2}(u) &= \int_X u(x) \left\{ g_1(x) - g_2(x) \right\} d\mu(x) \\
  &= \Phi_{g_1}(u) - \Phi_{g_2}(u) = \Phi(u) - \Phi(u) = 0
  \end{aligned}$$

  Therefore, $\Phi_{g_1 - g_2} = 0$ and

  $$\|g_1 - g_2\|_q = \|\Phi_{g_1 - g_2}\|_{L^p(X)^*} = 0$$

  So $g_1 = g_2$.

- Existence of $g \in L^q(X)$.

  (i) Consider $\mu(X) < \infty$. Note that $\chi_B \in L^p(X)$ for any $B \in \mathscr{A}$. Define $\nu\colon \mathscr{A} \to \mathbb{C}$ as

  $$\nu(B) := \Phi\left(\chi_B\right), \quad B \in \mathscr{A}$$

  Check: ν is a complex measure on $(X, \mathscr{A})$.

First, $\nu(\varnothing)=0$ clearly. So it only needs to prove the σ-aditivity. Let $B_k \in \mathscr{A}(k \in \mathbb{N}), B_j \cap B_k=\emptyset(j \neq k)$ and $B=\bigcup_{k=1}^{\infty} B_k$. Because for any $1 \leq p<\infty$, $\mu(X)<\infty$, by DCT

$$\left\|\sum_{k=1}^N \chi_{B_k}-\chi_B\right\|_p \longrightarrow 0$$

So by the continuity of Φ,

$$\Phi\left(\sum_{k=1}^N \chi_{B_k}\right) \longrightarrow \Phi\left(\chi_B\right.$$

and by the linearity of Φ,

$$\sum_{k=1}^N \nu\left(B_k\right)=\sum_{k=1}^N \Phi\left(\chi_{B_k}\right)=\Phi\left(\sum_{k=1}^N \chi_{B_k}\right) \longrightarrow \Phi\left(\chi_B\right)=\nu(B)$$

Check: $\nu \ll \mu$.
For any $B \in \mathscr{A}$ with $\mu(B)=0$. Then $\chi_B(x)=0$ $\mu-a.e.$, *i.e.* $\chi_B=0$ in $L^p(X)$. So

$$\nu(B)=\Phi\left(\chi_B\right)=\Phi(0)=0$$

Then by Radon-Nikodym Theorem, there is a unique $g \in L^1(X, \mathscr{A}, \mu)$ such that for any $B \in \mathscr{B}$,

$$\Phi\left(\chi_B\right)=\nu(B)=\int_B g(x) d \mu(x)=\int_X \chi_B(x) g(x) d \mu(x)$$

More generally, for any $v \in S_0(X, \mathscr{A}, \mu)$,

$$\Phi(v)=\int_X v(x) g(x) d \mu(x)$$

Note that $v \in L^p(X, \mathscr{A}, \mu)$ and $v \cdot g \in L^1 s(X, \mathscr{A}, \mu)$. Moreover,

$$\left|\int_X v(x) g(x) d \mu(x)\right|=|\Phi(v)| \leq\|\Phi\|_{L^p(X)^*}\|v\|_p$$

and so

$$\sup \left\{\left|\int_X \phi(x) g(x) d \mu(x)\right| \mid \phi \in S_0(X),\|\phi\|_{L^p(X)}=1\right\} \leq\|\Phi\|_{L^p(X)^*}<\infty$$

By above theorem, we have $g \in L^q(X)$.
Let $u \in L^p(X)$. Because $S_0(X)$ is dense in $L^p(X)$ for all $1 \leq p<\infty$, there is $\left\{v_n\right\}_{n \in \mathbb{N}} \subset S_0(X)$ such that $\left\|v_n-u\right\|_p \rightarrow 0$, which means

$$\Phi\left(v_n\right)=\int_X v_n(x) g(x) d \mu(x) \rightarrow \Phi(u)$$

On the other hand,

$$\begin{aligned}
& \left|\int_X v_n(x) g(x) d \mu(x)-\int_X u(x) g(x) d \mu(x)\right| \\
& \leq\left\|v_n-u\right\|_{L^p(X)}\|g\|_{L^q(X)} \longrightarrow 0
\end{aligned}$$

Therefore,

$$\Phi(u) = \int_X u(x)g(x)d\mu(x)$$

i.e. $\Phi = \Phi_g$. Moreover, by above proposition

$$\|g\|_{L^q(X)} = \|\Phi_g\|_{L^p(X)^*} = \|\Phi\|_{L^p(X)^*}$$

(ii) $\mu(X) = \infty$ and $(X, \mathscr{A}, \mu)$ is σ-finite. Then there is a sequence $\{E_n\}_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that $X = \bigcup_n E_n$ and $E_n \subset E_{n+1}$ with $0 < \mu(E_n) < \infty$. For any $n \in \mathbb{N}$, let

$$\mathscr{A}|_{E_n} := \{B \in \mathscr{A} \mid B \subset E_n\}$$

For $r \in [1, \infty]$ and $f \in L^r(E_n) = L^r\left(E_n, \mathscr{A}|_{E_n}, \mu\right)$, extending f on X by setting

$$f(x) = 0, \quad x \in X \backslash E_n$$

Similarly, extending it on E_{n+1}. So wwe have

$$\|f\|_{L^r(E_n)} = \|f\|_{L^r(E_{n+1})}, \quad \|f\|_{L^r(E_n)} = \|f\|_{L^r(X)}$$

Let $\Phi \in L^p(X)^*$. Clearly, by restriction, $\Phi \in L^p(E_n)^*$ with

$$\|\Phi\|_{L^p(E_n)^*} \leq \|\Phi\|_{L^p(X)^*}$$

Note that $\left(E_n, \mathscr{A}|_{E_n}, \mu\right)$ is a finite measure space. Therefore, by (i), there is a unique $g_n \in L^q(E_n)$ such that for any $u \, inL^p(X)$,

$$\begin{aligned}
\Phi\left(u\chi_{E_n}\right) &= \int_{E_n} u(x)\chi_{E_n}(x)g_n(x)d\mu(x) \\
&= \int_X u(x)\chi_{E_n}(x)g_n(x)d\mu(x)
\end{aligned}$$

and moreover

$$\|g_n\|_{L^q(X)} = \|g_n\|_{L^q(E_n)} = \|\Phi\|_{L^p(E_n)^*} \leq \|\Phi\|_{L^p(X)^*}$$

Because such $g_n \in L^q(E_n)$ uniquely exists,

$$g_n(x) = g_{n+1}(x), \quad \mu - a.e. \; x \in E_n$$

Then define g on X as

- when $x \in E_1$, $g(x) := g_1(x)$;
- when $x \in E_n \backslash E_{n-1}$, $g(x) := g_n(x)$.

So we have

$$g(x)\chi_{E_n}(x) = g_n(x), \quad \mu - a.e. \; x \in X$$

and thus $g_n \to g$ *a.e.*.

Check: $g \in L^q(X)$.

When $1 < q < \infty$, by MCT,

$$\begin{aligned}
\int_X |g(x)|^q d\mu(x) &= \lim_{n\to\infty} \int_X |g(x)|^q \chi_{E_n}(x)d\mu(x) \\
&= \lim_{n\to\infty} \int_X |g_n(x)|^q \, d\mu(x) \leq \|\Phi\|^q_{L^p(X)^*} < \infty
\end{aligned}$$

and so $g \in L^q(X)$.
When $q = \infty$ and $p = 1$, by $\|g_n\|_{L^\infty(X)} \le \|\Phi\|_{L^1(X)^*}$,

$$|g(x)|\chi_{E_n}(x) = |g_n(x)| \le \|\Phi\|_{L^1(X)^*}, \quad \mu - a.e.\ x \in X$$

So as $n \to \infty$, $|g(x)| \le \|\Phi\|_{L^1(X)^*}$ and $g \in L^\infty(X)$.
Check: $\Phi = \Phi_g$.
For any $u \in L^p(X)$,

$$\begin{aligned}\Phi\left(u\chi_{E_n}\right) &= \int_X u(x)\chi_{E_n}(x)g_n(x)d\mu(x) \\ &= \int_X u(x)\chi_{E_n}(x)g(x)d\mu(x)\end{aligned}$$

For any $1 \le p < \infty$, by DCT, $\|u\chi_{E_n} - u\|_{L^p(X)} \longrightarrow 0$. So on the LHS,

$$\Phi\left(u\chi_{E_n}\right) \longrightarrow \Phi(u)$$

On the RHS,

$$\begin{aligned}&\left|\int_X u(x)\chi_{E_n}(x)g(x)d\mu(x) - \int_X u(x)g(x)d\mu(x)\right| \\ &\le \|u\chi_{E_n} - u\|_{L^p(X)} \|g\|_{L^q(X)} \longrightarrow 0\end{aligned}$$

So

$$\Phi(u) = \int_X u(x)g(x)d\mu(x)$$

and by above proposition

$$\|g\|_{L^q(X)} = \|\Phi_g\|_{L^p(X)^*} = \|\Phi\|_{L^p(X)^*}$$

□

2.3 Riesz-Thorin Interpolation Theorem

Let $\mathbb{M}(X) = \mathbb{M}(X, \mathscr{A}, \mu)$ be the set of all complex-valued $\mathscr{A}$-measurable functions with equivalence $f(x) = g(x)$ $\mu - a.e.$.

Proposition 2.3.1. *Let $1 \le p < r < q \le \infty$.*

(1) $\theta \in (0,1)$ with

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$$

Then if $f \in L^p(X) \cap L^q(X)$, $f \in L^r(X)$ with

$$\|f\|_r \le \|f\|_p^{1-\theta} \|f\|_q^\theta$$

In particular, $L^p(X) \cap L^q(X) \subset L^r(X)$.

(2) $L^r(X) \subset L^p(X) + L^q(X)$.

So we have for any $1 \le p < r < q \le \infty$,

$$L^p(X) \cap L^q(X) \subset L^r(X) \subset L^p(X) + L^q(X)$$

Proof. (1) For $\theta \in (0,1)$ with

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q} = \frac{1}{\frac{p}{1-\theta}} + \frac{1}{\frac{q}{\theta}}$$

then

$$\frac{1}{\frac{p}{(1-\theta)r}} + \frac{1}{\frac{q}{\theta r}} = 1$$

And for $f \in L^p(X) \cap L^q(X)$, we have

$$|f|^{(1-\theta)r} \in L^{\frac{p}{(1-\theta)r}}, \quad |f|^{\theta r} \in L^{\frac{q}{\theta r}}$$

The by Hölder's Inequality,

$$\begin{aligned}\int_X |f(x)|^r \, d\mu(x) &= \int_X |f(x)|^{(1-\theta)r} \, |f(x)|^{\theta r} \, d\mu(x) \\ &\leq \left\| |f|^{(1-\theta)r} \right\|_{\frac{p}{(1-\theta)r}} \left\| |f|^{\theta r} \right\|_{\frac{q}{\theta r}} = \|f\|_p^{(1-\theta)r} \, \|f\|_q^{\theta r}\end{aligned}$$

Therefore,

$$\|f\|_r \leq \|f\|_p^{1-\theta} \|f\|_q^{\theta}$$

(2) For $g \in L^r(X)$, let $B \in \mathscr{A}$ be

$$B := \{x \in X | |g(x)| > 1\}$$

Considering $g = g \cdot \chi_B + g \cdot \chi_{X \setminus B}$.

- **Check:** $g \cdot \chi_B \in L^p(X)$.
  For $p < r$,
  $$|g(x)\chi_B(x)|^p \leq |g(x)\chi_B(x)|^r \leq |g(x)|^r \tag{2.3}$$
  Because $g \in L^p(X)$, $g \cdot \chi_B \in L^p(X)$.
- **Check:** $g \cdot \chi_{X \setminus B} \in L^q(X)$.
  When $q \neq \infty$, for any $x \in X$, by $r < q$,
  $$\left|g(x)\chi_{X \setminus B}(x)\right|^q \leq \left|g(x)\chi_{X \setminus B}(x)\right|^r \leq |g(x)|^r$$
  So $g\chi_{X \setminus B} \in L^q(X)$.
  When $q = \infty$, by $\left|g(x)\chi_{X \setminus B}(x)\right| \leq 1$, $g \cdot \chi_{X \setminus B} \in L^\infty(X)$

Therefore, $L^r(X) \subset L^p(X) + L^q(X)$.

□

Theorem 2.3.1 (Riesz-Thorin Interpolation). *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two σ-finite measure spaces. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. For $t \in (0,1)$, let $p_t, q_t \in [1, \infty]$ be*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

Let $T \colon L^{p_0}(X) + L^{p_1}(X) \to L^{q_0}(X) + L^{q_1}(X)$ be a linear map satisfying the following two conditions

- *$T(L^{p_0}(X)) \subset L^{q_0}(X)$ and there is an $M_0 \geq 0$ such that*

$$\|Tf\|_{L^{q_0}(Y)} \leq M_0 \|f\|_{L^{p_0}(X)}, \quad \forall\, f \in L^{p_0}(X)$$

- *$T(L^{p_1}(X)) \subset L^{q_1}(X)$ and there is an $M_1 \geq 0$ such that*

$$\|Tf\|_{L^{q_1}(Y)} \leq M_1 \|f\|_{L^{p_1}(X)}, \quad \forall\ f \in L^{p_1}(X)$$

Then for any $f \in L^{p_t}(X) \subset L^{p_0}(X) + L^{p_1}(X)$, $Tf \in L^{q_t}(Y)$ with

$$\|Tf\|_{L^{q_t}(Y)} \leq M_0^{1-t} M_1^t \|f\|_{L^{p_t}(X)}$$

Remark. Note that if $p_0 \neq p_1$, p_t is between p_0 and p_1. If $p_0 = p_1$, then $p_t = p_0 = p_1$. Besides, by $t \in (0,1)$, if $p_t = 1$, $p_0 = p_1 = 1$, and if $p_t = \infty$, then $p_0 = p_1 = \infty$. Similarly, q_0, q_1, q_t have the same results.

Remark. In other words, if $T \in \mathcal{B}\left(L^{p_0}(X), L^{q_0}(Y)\right)$ and $T \in \mathcal{B}\left(L^{p_1}(X), L^{q_1}(Y)\right)$, then $T \in \mathcal{B}\left(L^{p_t}(X), L^{q_t}(Y)\right)$ with

$$\|T\|_{\mathcal{B}(L^{p_t}(X), L^{q_t}(Y))} \leq \|T\|_{\mathcal{B}(L^{p_0}(X), L^{q_0}(Y))}^{1-t} \|T\|_{\mathcal{B}(L^{p_1}(X), L^{q_1}(Y))}^{t}$$

where $\mathcal{B}(V, W)$ is the set of all bounded linear map from V to W.

Proposition 2.3.2 (Hadamard Three-lines Theorem)**.** *Let $D \subset \mathbb{C}$ be a strip-shaped area*

$$D := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1\}$$

$F \colon D \to \mathbb{C}$ is continuous and bounded and regular on

$$D^\circ = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$$

Assume there are $L_0, L_1 \geq 0$ such that for any $y \in \mathbb{R}$

$$|F(iy)| \leq L_0, \quad |F(1+iy)| \leq L_1$$

Then for any $t \in (0,1)$ and any $y \in \mathbb{R}$, we have

$$|F(t+iy)| \leq L_0^{1-t} L_1^t$$

Proof. (i) $L_0, L_1 > 0$: For $a > 0$, define $F_a \colon D \to \mathbb{C}$ as

$$F_a(z) := e^{a(z^2-1)} L_0^{z-1} L_1^{-z} F(z), \quad \forall\ z \in D$$

Then F_a is continuous on D and regular on D°. Let $z \in D$ with $z = t + iy$ for $t \in [0,1]$ and $y \in \mathbb{R}$.

$$F_a(t+iy) = e^{a\left(t^2-1-y^2\right)+2iaty} L_0^{t+iy-1} L_1^{-(t+iy)} F(t+iy)$$

and so we have

$$|F_a(t+iy)| = e^{a\left(t^2-1-y^2\right)} L_0^{t-1} L_1^{-t} |F(t+iy)|$$
$$|F(t+iy)| = e^{a\left(1-t^2+y^2\right)} L_0^{1-t} L_1^{t} \, |F_a(t+iy)|$$

Check: for any $a > 0$, $t \in (0,1)$ and $y \in \mathbb{R}$,

$$|F_a(t+iy)| \leq 1$$

For $N \in \mathbb{N}$, consider

$$D_N := \{z \in \mathbb{C} | 0 \leq \operatorname{Re} z \leq 1, |\operatorname{Im} z| \leq N\}$$

we have $D_N \subset D_{N+1}$. Therefore, we only need to show that there is an N_0 such that if $N \geq N_0$, for any $z \in D_N$,

$$|F_a(z)| \leq 1$$

Since F_N is continuous on compact D_N, $|F_a|$ can take a maximum on D_N. Moreover, because F_a is regular on D_N°, $|F_a|$'s maximum is valued on ∂D_N. Note that the ∂D_N contains points as $iy, 1+iy$ for $y \in [-N, N]$ and points as $t \pm iN$ for $t \in (0,1)$. For the first case, by assumption

$$|F_a(iy)| = e^{-a\left(1+y^2\right)} L_0^{-1} |F(iy)| \leq L_0^{-1} L_0 = 1$$
$$|F_a(1+iy)| = e^{-ay^2} L_1^{-1} |F(1+iy)| \leq L_1^{-1} L_1 = 1,$$

For the second case, because F is bounded on D, $|F(z)| \leq K$ on D for some $K > 0$. So

$$|F_a(t \pm iN)| \leq e^{a\left(t^2-N^2-1\right)} L_0^{t-1} L_1^{-t} K \leq C_0 e^{-aN^2}$$

Therefore, for sufficiently large N, $C_0 e^{-aN^2} \leq 1$ and thus $|F_a(t \pm iN)| \leq 1$.

Therefore, base on this result, we have

$$|F(t+iy)| \leq e^{a\left(1-t^2+y^2\right)} L_0^{1-t} L_1^t$$

As $a \to 0^+$, we get for any $t \in (0,1)$,

$$|F(t+iy)| \leq L_0^{1-t} L_1^t$$

(ii) $L_0 = 0, L_1 > 0$ (or $L_0 > 0, L_1 = 0$): For any $\varepsilon > 0$, we have

$$|F(iy)| \leq \varepsilon, \quad |F(1+iy)| \leq L_1$$

for all $y \in \mathbb{R}$. By (i), we have for any $t \in (0,1)$ and any $y \in \mathbb{R}$,

$$|F(t+iy)| \leq \varepsilon^{1-t} L_1^t$$

So let $\varepsilon \to 0^+$,

$$|F(t+iy)| = 0, \quad \forall\, t \in (0,1),\ \forall\, y \in \mathbb{R}$$

(iii) $L_0 = L_1 = 0$: Similarly, we have

$$|F(iy)| \leq \varepsilon, \quad |F(1+iy)| \leq \varepsilon$$

for all $y \in \mathbb{R}$ and thus

$$|F(t+iy)| \leq \varepsilon^{1-t} \varepsilon^t$$

So

$$|F(t+iy)| = 0, \quad \forall\, t \in (0,1),\ \forall\, y \in \mathbb{R}$$

□

Proof of Theorem 2.3.1. Fix $t \in (0,1)$.

(I) $p_0 = p_1$: Note that it implies $p_t = p_0 = p_1$. For any $f \in L^{p_t}(X)$, that is $f \in L^{p_0}(X) = L^{p_1}(X)$. So by assumptions,

$$Tf \in L^{q_0}(Y) \cap L^{q_1}(Y) \subset L^{q_t}(X)$$

By above proposition,

$$\|Tf\|_{L^{q_t}(Y)} \leq \|Tf\|_{L^{q_0}(Y)}^{1-t} \|Tf\|_{L^{q_1}(Y)}^t$$
$$\leq \left(M_0 \|f\|_{L^{p_0}(X)}\right)^{1-t} \left(M_1 \|f\|_{L^{p_1}(X)}\right)^t = M_0^{1-t} M_1^t \|f\|_{L^{p_t}(X)}$$

(II) $p_0 \neq p_1$: Note that $1 < p_t < \infty$ and for any $\phi \in S_0(X)$, because $\phi \in L^{p_0}(X) \cap L^{p_1}(X)$,

$$T\phi \subset L^{q_0}(Y) \cap L^{q_1}(Y) \subset L^{q_t}(Y)$$

(II)-1 **Check:** For any $\phi \in S_0(X)$,

$$\|T\phi\|_{q_t} \leq M_0^{1-t} M_1^t \|\phi\|_{p_t(X)}.$$

By Theorem 2.2.1, it is sufficient to prove for any $\phi \in S_0(X)$ with $\|\phi\|_{p_t} = 1$

$$|\langle T\phi, \psi\rangle| \leq M_0^{1-t} M_1^t$$

for any $\psi \in S_0(Y)$ with $\|\psi\|_{q_t'(Y)} = 1$, where q_t' is the conjugate of q_t and $\langle u, v\rangle = \int_Y u(y)v(y)d\nu(y)$. First,

$$\phi(x) = \sum_{j=1}^{M} a_j \chi_{A_j}(x), \quad x \in X$$
$$\psi(y) = \sum_{k=1}^{N} b_k \chi_{B_k}(y), \quad y \in Y,$$

(i) $1 < q_t \leq \infty$: Then $1 \leq q_t' < \infty$. Consider the strip-shaped closed area

$$D := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1\}$$

For $z \in D$, define $\alpha(z), \beta(z)$ as

$$\alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}, \quad \beta(z) := \frac{1-z}{q_0} + \frac{z}{q_1}$$

So $\alpha(t) = \frac{1}{p_t}, \beta(t) = \frac{1}{q_t}, 1 - \beta(t) = \frac{1}{q_t'}$ and $0 < \alpha(t) < 1, 0 \leq \beta(t) < 1, 0 < 1 - \beta(t) \leq 1$. Let

$$\xi_j = \arg a_j, \eta_k = \arg b_k \quad \Rightarrow \quad a_j = |a_j| e^{i\xi_j}, b_k = |b_k| e^{i\eta_k}$$

and define $\phi_z \colon X \to \mathbb{C}$ and $\psi_z \colon Y \to \mathbb{C}$ as

$$\phi_z(x) := \sum_{j=1}^{M} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{i\xi_j} \chi_{A_j}(x) = \sum_{j=1}^{M} |a_j|^{p_t \alpha(z)} e^{i\xi_j} \chi_{A_j}(x),$$
$$\psi_z(y) := \sum_{k=1}^{N} |b_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{i\eta_k} \chi_{B_k}(y) = \sum_{k=1}^{N} |b_k|^{l_t'(1-\beta(z))} e^{i\eta_k} \chi_{B_k}(y)$$

Note that $\phi_z \in S_0(X)$, $\psi_z \in S_0(Y)$ and $\phi_t = \phi$, $\psi_t = \phi$. Also, by $\phi_z \in L^{p_0}(X) \cap L^{p_1}(X)$, $T\phi_z \in L^{q_0}(Y) \cap L^{q_1}(Y) \subset L^{q_t}(Y)$. Besides, we have $\psi_z \in L^{q_t'}(Y)$. Define $F \colon D \to \mathbb{C}$ by

$$F(z) := \langle T\phi_z, \psi_z\rangle = \int_Y (T\phi_z)(y)\psi_z(y)d\nu(y), \quad z \in D$$
$$= \sum_{j=1}^{M} \sum_{k=1}^{N} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} |b_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{i(\xi_j+\eta_k)} \left\langle T\chi_{A_j}, \chi_{B_k}\right\rangle$$

Note that F is continuous on D and regular on D°. Moreover, for $z = c+id \in D$ with $c \in [0,1], d \in \mathbb{R}$,

$$\begin{aligned}\left||a_j|^{\frac{\alpha(z)}{\alpha(t)}}\right| &= \left|\left(|a_j|^{\frac{1}{\alpha(t)}}\right)^{\frac{1-z}{p_0}+\frac{z}{p_1}}\right| = \left(|a_j|^{\frac{1}{\alpha(t)}}\right)^{\frac{1-c}{p_0}+\frac{c}{p_1}} \\ &\leq \max_{l\in[0,1]}\left\{\left(|a_j|^{\frac{1}{\alpha(t)}}\right)^{\frac{1-l}{p_0}+\frac{l}{p_1}}\right\}\end{aligned}$$

So $|a_j|^{\frac{\alpha(\cdot)}{\alpha(t)}}$ is bounded on D, and similarly for $|b_k|^{\frac{1-\beta(\cdot)}{1-\beta(t)}}$.
Therefore, the main goal is to apply Hadamard Three-lines Theorem. To do that, we need the bound of $|F(is)|$ and $|F(1+is)|$. First, note that

$$\begin{aligned}\sum_{j=1}^{M} |a_j|^{p_t}\,\mu(A_j) &= \|\phi\|_{L^{p_t}(X)}^{p_t} = 1, \\ \sum_{k=1}^{N} |b_k|^{q_t'}\,\nu(B_k) &= \|\psi\|_{L^{q_t}(Y)}^{q'} = 1\end{aligned}$$

Next, we want to show

$$\|\phi_{is}\|_{L^{p_0}(X)} = \|\phi_{1+is}\|_{L^{p_1}(X)} = 1$$

By $\alpha(is) = \frac{1-is}{p_0} + \frac{is}{p_1}, \alpha(t) = \frac{1}{p_t}$, we van get

$$\begin{aligned}\|\phi_{is}\|_{L^{p_0}(X)}^{p_0} &= \left\|\sum_{j=1}^{M} |a_j|^{\frac{\alpha(is)}{\alpha(t)}} e^{i\xi_j}\chi_{A_j}\right\|_{L^{p_0}(X)}^{p_0} = \left\|\sum_{j=1}^{M} |a_j|^{p_t\left(\frac{1-is}{p_0}+\frac{is}{p_1}\right)} e^{i\xi_j}\chi_{A_j}\right\|_{L^{p_0}(X)}^{p_0} \\ &= \left\|\sum_{j=1}^{M} |a_j|^{\frac{p_t}{p_0}}\chi_{A_j}\right\|_{L^{p_0}(X)}^{p_0} = \int_X \left(\sum_{j=1}^{M} |a_j|^{p_t}\chi_{A_j}(x)\right) d\mu(x) \\ &= \sum_{j=1}^{M} |a_j|^{p_t}\,\mu(A_j) = \|\phi\|_{L^{p_t}(X)}^{p_t} = 1.\end{aligned}$$

Similarly, by $\alpha(1+is) = \frac{-is}{p_0} + \frac{1+is}{p_1}$,

$$\begin{aligned}\|\phi_{1+is}\|_{L^{p_1}(X)}^{p_1} &= \left\|\sum_{j=1}^{M} |a_j|^{\frac{\alpha(1+is)}{\alpha(t)}} e^{i\xi_j}\chi_{A_j}\right\|_{L^{p_1}(X)}^{p_1} \\ &= \left\|\sum_{j=1}^{M} |a_j|^{p_t\left(\frac{-is}{p_0}+\frac{1+is}{p_1}\right)} e^{i\xi_j}\chi_{A_j}\right\|_{L^{p_1}(X)}^{p_1} \\ &= \left\|\sum_{j=1}^{M} |a_j|^{\frac{p_t}{p_1}}\chi_{A_j}\right\|_{L^{p_1}(X)}^{p_1} = \int_X \left(\sum_{j=1}^{M} |a_j|^{p_t}\chi_{A_j}(x)\right) d\mu(x) \\ &= \sum_{j=1}^{M} |a_j|^{p_t}\,\mu(A_j) = \|\phi\|_{L^{p_t}(X)}^{p_t} = 1\end{aligned}$$

Moreover, by similar calculation, we have

$$\|\psi_{is}\|_{L^{q_0'}(Y)} = \|\psi_{1+is}\|_{L^{q_1'}(Y)} = 1$$

Then by the Hölder's Inequality,

$$\begin{aligned}|F(is)| = |\langle T\phi_{is}, \psi_{is}\rangle| &\le \|T\phi_{is}\|_{L^{q_0}(Y)} \|\psi_{is}\|_{L^{q_0'}(Y)} \\ &\le M_0 \|\phi_{is}\|_{L^{p_0}(X)} \|\psi_{is}\|_{L^{q_0'}(Y)} = M_0\end{aligned}$$

and

$$\begin{aligned}|F(1+is)| = |\langle T\phi_{1+is}, \psi_{1+is}\rangle| &\le \|T\phi_{1+is}\|_{L^{q_1}(Y)} \|\psi_{1+is}\|_{L^{q_1'}(Y)} \\ &\le M_1 \|\phi_{1+is}\|_{L^{p_1}(X)} \|\psi_{1+is}\|_{L^{q_1'}(Y)} = M_1\end{aligned}$$

Therefore, Hadamard Three-line Theorem implies

$$|\langle T\phi, \psi\rangle| = |\langle T\phi_t, \psi_t\rangle| = |F(t)| = |F(t+i0)| \le M_0^{1-t} M_1^t$$

(ii) $q_t = 1$: Then $q_0 = q_1 = 1 = q_t$, $q_0' = q_1' = q_t' = \infty$. It has the same proof by replacing ψ_z with ψ in above proof.

(II)-2 **Check:** For $f \in L^{p_t}(X)$, $Tf \in L^{q_t}(Y)$ with

$$\|Tf\|_{L^{q_t}(Y)} \le M_0^{1-t} M_1^t \|f\|_{L^{p_t}(X)}$$

Because $1 < p_t < \infty$, there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $S_0(X)$ such that

- $|f_n(x)| \le |f(x)|$,
- $\lim_{n\to\infty} f_n(x) = f(x)$,
- $\lim_{n\to\infty} \|f_n - f\|_{L^{p_t}(X)} = 0$.

Let $E := \{x \in X | |f(x)| > 1\}$ and define

$$\begin{aligned} g := f\cdot\chi_E, \quad h := f - g = f\cdot\chi_{E^c}, \\ g_n := f_n\cdot\chi_E, \quad h_n := f_n - g_n = f_n\cdot\chi_{E^c}\end{aligned}$$

Assume $p_0 < p_1$. Then $p_0 < p_t < p_1$ and so

$$g \in L^{p_0}(X)\cap L^{p_t}(X), \quad h \in L^{p_1}(X)\cap L^{p_t}(X), \quad g_n, h_n \in S_0(X)$$

Therefore, by T,

$$Tg \in L^{q_0}(Y), \quad Tg_n \in L^{q_0}(Y), \quad Th \in L^{q_1}(Y), \quad Th_n \in L^{q_1}(Y)$$

By DCT,

$$\|g_n - g\|_{L^{p_0}(X)} \longrightarrow 0, \quad \|h_n - h\|_{L^{p_1}(X)} \longrightarrow 0$$

And thus

$$\begin{aligned}\|Tg_n - Tg\|_{L^{q_0}(Y)} \le M_0 \|g_n - g\|_{L^{p_0}(X)} \longrightarrow 0 \\ \|Th_n - Th\|_{L^{q_1}(Y)} \le M_1 \|h_n - h\|_{L^{p_1}(X)} \longrightarrow 0\end{aligned}$$

So there are subsequence $\{g_{n_k}\}_{k=1}^\infty$ and $\{h_{n_k}\}_{k=1}^\infty$ such that

$$\begin{aligned}(Tg_{n_k})(y) \longrightarrow (Tg)(y), \quad \nu - a.e.\ y \in Y, \\ (Th_{n_k})(y) \longrightarrow (Th)(y), \quad \nu - a.e.\ y \in Y\end{aligned}$$

And we have

$$\begin{aligned}&(Tf_{n_k})(y) = (Tg_{n_k})(y) + (Th_{n_k})(y) \\ &\longrightarrow (Tg)(y) + (Th)(y) = (Tf)(y), \quad \nu - a.e.\ y \in Y\end{aligned}$$

By above, we already have

$$\|Tf_{n_k}\|_{L^{q_t}(Y)} \leq M_0^{1-t} M_1^t \|f_{n_k}\|_{L^{p_t}(X)}$$

If $1 \leq q_t < \infty$, then by Fatou's lemma and $\lim_{k\to\infty} \|f_{n_k} - f\|_{L^{p_t}(X)} = 0$,

$$\begin{aligned}\left(\int_Y |Tf(y)|^{q_t} d\mu(x)\right)^{\frac{1}{q_t}} &\leq \varliminf_{k\to\infty} \|Tf_{n_k}\|_{L^{q_t}(Y)} \leq \varliminf_{k\to\infty} \left(M_0^{1-t} M_1^t \|f_{n_k}\|_{L^{p_t}(X)}\right) \\ &= M_0^{1-t} M_1^t \|f\|_{L^{p_t}(X)} < \infty\end{aligned}$$

Therefore, $Tf \in L^{q_t}(X)$ with

$$\|Tf\|_{L^{q_t}(Y)} \leq M_0^{1-t} M_1^t \|f\|_{L^{p_t}(X)}$$

If $q_t = \infty$,

$$|(Tf_{n_k})(y)| \leq \|Tf_{n_k}\|_{L^\infty(Y)} \leq M_0^{1-t} M_1^t \|f_{n_k}\|_{L^{p_t}(X)}, \quad \nu - a.e.\ y \in Y$$

Then by $\lim_{k\to\infty} \|f_{n_k} - f\|_{L^{p_t}(X)} = 0$,

$$|(Tf)(y)| \leq M_0^{1-t} M_1^t \|f\|_{L^{p_t}(X)}, \quad \nu - a.e.\ y \in Y$$

Thus $Tf \in L^\infty(Y)$ with

$$\|Tf\|_{L^\infty(Y)} \leq M_0^{1-t} M_1^t \|f\|_{L^{p_t}(X)}$$

□

Proposition 2.3.3 (Minkowski's Inequality). *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two σ-finite measure spaces. $F \colon X \times Y \to \mathbb{C}$ is a $\mathscr{A} \otimes \mathscr{B}$-measurable function with $F(x,y) \geq 0$ for any $(x,y) \in X \times Y$. For $1 \leq p < \infty$,*

$$\left[\int_X \left(\int_Y F(x,y) d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \leq \int_Y \left(\int_X F(x,y)^p d\mu(x)\right)^{\frac{1}{p}} d\nu(y)$$

Remark. In other words,

$$\left\|\int_Y F(\cdot, y) d\nu(y)\right\|_{L^p(X)} \leq \int_Y \|F(\cdot, y)\|_{L^p(X)} d\nu(y)$$

Proof. If $\int_Y \left(\int_X F(x,y)^p d\mu(x)\right)^{\frac{1}{p}} d\nu(y) = \infty$, then it is clear. So we assume $\int_Y \left(\int_X F(x,y)^p d\mu(x)\right)^{\frac{1}{p}} d\nu(y) < \infty$.
When $p = 1$, because F is nonnegative and $\int_Y \int_X F(x,y) d\mu(x) d\nu(y) < \infty$, by Fubini's Theorem,

$$\int_X \left(\int_Y F(x,y) d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X F(x,y) d\mu(x)\right) d\nu(y)$$

$1 < p < \infty$. Let p' be the conjugate of p. For any $g \in L^{p'}$, by Fubini's Theorem and Hölder's Inequality, we have

$$\begin{aligned}&\int_X \left(\int_Y F(x,y) d\nu(y)\right) |g(x)| d\mu(x) = \int_Y \left(\int_X F(x,y) |g(x)| d\mu(x)\right) d\nu(y) \\ &\leq \int_Y \left(\int_X F(x,y)^p d\mu(x)\right)^{\frac{1}{p}} \left(\int_X |g(x)|^{p'} d\mu(x)\right)^{\frac{1}{p'}} d\nu(y) \\ &= \int_Y \left(\int_X F(x,y)^p d\mu(x)\right)^{\frac{1}{p}} d\nu(y) \|g\|_{L^{p'}(X)} < \infty\end{aligned}$$

Therefore, the integral in the LHS is well-defined and by Theorem 2.2.1,

$$\int_Y F(\cdot, y) d\nu(y) \in L^p(X)$$

Moreover, by

$$\left|\int_X \left(\int_Y F(x,y) d\nu(y)\right) g(x) d\mu(x)\right| \leq \int_Y \left(\int_X F(x,y)^p d\mu(x)\right)^{\frac{1}{p}} d\nu(y) \|g\|_{L^{p'}(X)}$$

we have

$$\left[\int_X \left(\int_Y F(x,y) d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \leq \int_Y \left(\int_X F(x,y)^p d\mu(x)\right)^{\frac{1}{p}} d\nu(y)$$

□

Theorem 2.3.2. *Let $\Omega \subset \mathbb{R}^N$ be a Lebesgue measurable set and $1 \leq r \leq \infty$. Let $K = K(x,y) \colon \Omega \times \Omega \to \mathbb{C}$ be a Lebesgue measurable function with $K(x,\cdot) \in L^r(\Omega)$* a.e. *$x \in \Omega$ and $K(\cdot, y) \in L^r(\Omega)$* a.e. *$y \in \Omega$. And there is $M > 0$ such that*

$$\|K(x,\cdot)\|_{L^r(\Omega)} \leq M, \text{ a.e. } x \in \Omega$$
$$\|K(\cdot,y)\|_{L^r(\Omega)} \leq M, \text{ a.e. } y \in \Omega$$

where r satisfies

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$$

for $1 \leq p \leq q \leq \infty$ ($1 \leq r \leq q \leq \infty$). Then the following statements are true.

(1) For any $u \in L^p(\Omega)$,

$$\int_\Omega |K(x,y)u(y)| dy < \infty$$

(2) Let linear map $T \colon L^p(\Omega) \to \mathbb{M}(\Omega)$ defined as

$$(Tu)(x) := \int_\Omega K(x,y) u(y) dy$$

Then $Tu \in L^q(\Omega)$ and $T \in \mathcal{B}(L^p(\Omega), L^q(\Omega))$ with

$$\|Tu\|_q \leq M \|u\|_p$$

Proof. First,

$$0 \leq \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{p} - \frac{1}{r'} \leq \frac{1}{r}$$

where r' is the conjugate of r. Note that

$$1 \leq p \leq r' (\leq \infty), \quad (1 \leq) r \leq q \leq \infty$$

When $p = r'$, $q = \infty$. When $p = 1$, $q = r$.

(i) $p = r', q = \infty$: Let $u \in L^{r'}(\Omega)$. Then because $K(x,\cdot) \in L^r(\Omega)$, by Hölder's Inequality,

$$\int_\Omega |K(x,y)u(y)| dy \leq \|K(x,\cdot)\|_r \|u\|_{r'} \leq M \|u\|_{r'} < \infty$$

Therefore,

$$\left|\int_\Omega K(x,y)u(y)dy\right| \le \int_\Omega |K(x,y)u(y)|dy \le M\|u\|_{L^{r'}(\Omega)}$$

which means $\int_\Omega K(\cdot,y)u(y)dy \in L^\infty(\Omega)$ with

$$\left\|\int_\Omega K(\cdot,y)u(y)dy\right\|_\infty \le M\|u\|_{r'}$$

Note that when $r=\infty$, $r'=1$. So $p=1=r'$, $q=\infty$, which is also the above case.

(ii) $p=1, q=r<\infty$: Let $u \in L^1(\Omega)$. By Minkowski's Inequality,

$$\left\{\int_\Omega\left(\int_\Omega |K(x,y)u(y)|dy\right)^r dx\right\}^{\frac{1}{r}} \le \int_\Omega\left(\int_\Omega |K(x,y)u(y)|^r dx\right)^{\frac{1}{r}} dy$$

$$= \int_\Omega\left(\int_\Omega |K(x,y)|^r dx\right)^{\frac{1}{r}} |u(y)|dy \le M\|u\|_{L^1(\Omega)} < \infty$$

So clearly,

$$\int_\Omega |K(x,y)u(y)|dy < \infty, \quad a.e.\ x \in \Omega$$

Moreover, $\int_\Omega K(\cdot,y)u(y)dy \in L^r(\Omega)$ and

$$\left\|\int_\Omega K(\cdot,y)u(y)dy\right\|_{L^r(\Omega)} \le \left\{\int_\Omega\left(\int_\Omega |K(x,y)u(y)|dy\right)^r dx\right\}^{\frac{1}{r}} \le M\|u\|_{L^1(\Omega)}$$

Note that (i) and (ii) have already proved the theorem for $p \in \{1, r'\}$ and $q \in \{r, \infty\}$, so we only need to prove for $1 < p < r', r < q < \infty$.

(iii) $p, q \in [1,\infty]$: By $1 \le p \le r'(\le \infty)$, $(1 \le) r \le q \le \infty$,

$$L^p(\Omega) \subset L^1(\Omega) + L^{r'}(\Omega), \quad L^q(\Omega) \subset L^r(\Omega) + L^\infty(\Omega)$$

Let $u = u_1 + u_2 \in L^1(\Omega) + L^{r'}(\Omega)$ with $u_1 \in L^1(\Omega)$ and $u_2 \in L^{r'}(\Omega)$. Then by (i) and (ii), we have

$$\int_\Omega |K(x,y)u(y)|dy \le \int_\Omega |K(x,y)u_1(y)|\,dy + \int_\Omega |K(x,y)u_2(y)|\,dy < \infty$$

So $\int_\Omega K(x,y)u(y)dy$ is well-fined for any $u \in L^1(\Omega) + L^{r'}(\Omega)$, in particular, for $u \in L^p(\Omega)$. So (1) is obtained. Next, for (2), consider the linear map $T\colon L^1(\Omega) + L^{r'}(\Omega) \to \mathbb{M}(\Omega)$ defined as

$$(Tu)(x) := \int_\Omega K(x,y)u(y)dy, \quad a.e.\ x \in \Omega$$

Then by (i) and (ii),

$$T\colon L^1(\Omega) + L^{r'}(\Omega) \to L^r(\Omega) + L^\infty(\Omega)$$

with the facts for any $u \in L^{r'}(\Omega)$, $Tu \in L^\infty(\Omega)$ and

$$\|Tu\|_\infty \le M\|u\|_{r'}$$

and for any $u \in L^1(\Omega)$, $Tu \in L^r(\Omega)$ and

$$\|Tu\|_r \le M\|u\|_1$$

For $1 < p < r', r < q < \infty$, let $t := 1 - \frac{r}{q} \in (0,1)$. Then

$$\frac{1}{p} = \frac{1-t}{1} + \frac{t}{r'}, \quad \frac{1}{q} = \frac{1-t}{r} + \frac{t}{\infty}$$

Then by Riesz-Thorin Interpolation Theorem, for any $u \in L^p(\Omega)$, $Tu \in L^q(\Omega)$,

$$\|Tu\|_{L^q(\Omega)} \leq M^{1-t}M^t\|u\|_{L^p(\Omega)} = M\|u\|_{L^p(\Omega)}$$

□

Corollary 2.3.1 (Young's Inequality). *Let $p, q, r \in [1, \infty]$ satisfy*

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$$

Then for any $f \in L^r(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$,

$$\|f * g\|_q \leq \|f\|_r \|g\|_p$$

Proof. Because

$$f * g(x) = \int_{\mathbb{R}^N} f(x-y)g(y)dy = \int_{\mathbb{R}^N} K(x,y)g(y)dy$$

where $K(x,y) = f(x-y)$, it can directly obtained by above theorem. □

Recall we have already define $\mathcal{F}$ on L^1 and L^2. Moreover, for convenience, the $\mathcal{F}$ on L^1 defined as

$$\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(x)e^{-ix\cdot\xi}dx$$

and then the inverse formula becomes

$$\mathcal{F}^{-1}[u](x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(\xi)e^{ix-\xi}d\xi$$

Because under this definition, the Plancherel Theorem tells us $\mathcal{F}\colon L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is an isometry

$$\|u\|_{L^2(\mathbb{R}^N)} = \|\widehat{u}\|_{L^2(\mathbb{R}^N)}$$

But for L^p, we define $\mathcal{F}$ on it by the view of distribution because $L^p \subset \mathcal{S}'$. Now by Riesz-Thorin Interpolation Theorem, we can see $\mathcal{F}(L^p)$ more clearly when $1 \leq p \leq 2$.

Theorem 2.3.3 (Hausdorff-Young's Inequality). *Let $1 \leqslant p \leqslant 2$ and p' be the conjugate of p. Then*

$$\mathcal{F}\left(L^p\left(\mathbb{R}^N\right)\right) \subset L^{p'}, \text{ and } \mathcal{F}^{-1}\left(L^p\left(\mathbb{R}^N\right)\right) \subset L^{p'}$$

and for any $u \in L^p\left(\mathbb{R}^N\right)$,

$$\|\mathcal{F}[u]\|_{L^{p'}} \leqslant (2\pi)^{-N\left(\frac{1}{p}-\frac{1}{2}\right)}\|u\|_{L^p}$$
$$\left\|\mathcal{F}^{-1}[u]\right\|_{L^{p'}} \leqslant (2\pi)^{-N\left(\frac{1}{p}-\frac{1}{2}\right)}\|u\|_{L^p}$$

In particular, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are in $\mathcal{B}\left(L^p\left(\mathbb{R}^N\right), L^{p'}\left(\mathbb{R}^N\right)\right)$.

Proof. Firstly,

$$\begin{aligned}&\mathcal{F}\left(L^1\left(\mathbb{R}^N\right)\right) \subset L^\infty\left(\mathbb{R}^N\right) \text{ and } \|\mathcal{F}[u]\|_{L^\infty} \leqslant (2\pi)^{-\frac{N}{2}}\|u\|_{L^1}, \forall u \in L^1\left(\mathbb{R}^N\right)\\ &\mathcal{F}\left(L^2\left(\mathbb{R}^N\right)\right) \subset L^2\left(\mathbb{R}^N\right) \text{ and } \|\mathcal{F}[v]\|_{L^2} = \|v\|_{L^2}, \forall v \in L^2\left(\mathbb{R}^N\right)\end{aligned}$$

Taking $p_0 = 1, q_0 = \infty, p_1 = 2, q_1 = 2$ and $\frac{t}{2} = 1 - \frac{1}{p}$ in the Riesz-Thorin Interpolation Theorem and by

$$\frac{1}{p} = \frac{1-t}{1} + \frac{t}{2}, \frac{1}{p'} = \frac{1-t}{\infty} + \frac{t}{2}$$

it can get

$$\mathcal{F}\left(L^p\left(\mathbb{R}^N\right)\right) \subset L^{p'}\left(\mathbb{R}^N\right)$$

and

$$\|\mathcal{F}[u]\|_{L^{p'}} = \left((2\pi)^{-\frac{N}{2}}\right)^{1-t}\|u\|_{L^p} = (2\pi)^{-N\left(\frac{1}{p}-\frac{1}{2}\right)}\|u\|_{L^p}, \forall u \in L^p\left(\mathbb{R}^N\right)$$

Similarly, it holds for $\mathcal{F}^{-1}$. □

2.4 Weak L^p Space

Definition 2.4.1. Let $(X, \mathscr{A}, \mu)$ be a measure space and $f \in \mathbb{X}$. For $\alpha \in (0, \infty)$,

$$\lambda_f(\alpha) := \mu\left(\{x \in X \mid |f(x)| > \alpha\}\right)$$

Then λ_f is called the distribution function of f.

Proposition 2.4.1. *Let $(X, \mathscr{A}, \mu)$ be a measure space and $f, g \in \mathbb{X}$. Let λ_f, λ_g be the corresponding distribution functions. Then*

(1) λ_f is decreasing and right-continuous.

(2) If $|f(x)| \leq |g(x)|$ $\mu-a.e.$, then $\lambda_f(\alpha) \leq \lambda_g(\alpha)$ for any $\alpha \in (0, \infty)$.

(3) If $f \in L^\infty(X)$, then for any $\alpha \geq \|f\|_\infty$, $\lambda_f(\alpha) = 0$.

(4) For any $\alpha \in (0, \infty)$,

$$\lambda_{f+g}(\alpha) \leq \lambda_f\left(\frac{\alpha}{2}\right) + \lambda_g\left(\frac{\alpha}{2}\right)$$

Proof. (1) For $0 < \alpha < \beta$,

$$\lambda_f(\beta) = \mu(\{x \in X || f(x) |> \beta\}) \leq \mu(\{x \in X || f(x) |> \alpha\}) = \lambda_f(\alpha)$$

so it is decreasing. For any $c \in (0, \infty)$, let $\alpha_n \downarrow c$ and

$$E_n := \{x \in X || f(x) |> \alpha_n\}, \quad E := \{x \in X || f(x) |> c\}$$

Then $E_n \subset E_{n+1}$ and $\bigcup_{n=1}^\infty E_n = E$. By the monotone convergence of measure,

$$\lambda_f(\alpha_n) = \mu(E_n) \longrightarrow \mu(E) = \lambda_f(c),$$

so it is right-continuous.

(2) Because $|f(x)| \leq |g(x)|$, for any $\alpha \in (0, \infty)$, $|f(x)| > \alpha$ implies $|g(x)| > \alpha$. So

$$\lambda_f(\alpha) = \mu(\{x \in X || f(x) |> \alpha\}) \leq \mu(\{x \in X || g(x) |> \alpha\}) = \lambda_g(\alpha)$$

(3) If $\alpha \geq \|f\|_\infty$, then $|f(x)| \leq \alpha \ \mu - a.e.$, which means

$$\lambda_f(\alpha) = \mu(\{x \in X || f(x) |> \alpha\}) = 0.$$

(4) For $\alpha \in (0, \infty)$, if $|f(x) + g(x)| > \alpha$, then we have $|f(x)| > \frac{\alpha}{2}$ or $|g(x)| > \frac{\alpha}{2}$. So

$$\begin{aligned}
\lambda_{f+g}(\alpha) &= \mu(\{x \in X || f(x) + g(x) |> \alpha\}) \\
&\leq \mu\left(\left\{x \in X || f(x) \Big| > \frac{\alpha}{2}\right\} \cup \left\{x \in X || g(x) \Big| > \frac{\alpha}{2}\right\}\right) \\
&\leq \mu\left(\left\{x \in X || f(x) \Big| > \frac{\alpha}{2}\right\}\right) + \mu\left(\left\{x \in X || g(x) \Big| > \frac{\alpha}{2}\right\}\right) \\
&= \lambda_f\left(\frac{\alpha}{2}\right) + \lambda_g\left(\frac{\alpha}{2}\right)
\end{aligned}$$

$\square$

Theorem 2.4.1. *Let $(X, \mathscr{A}, \mu)$ be σ-finite and $f \in \mathbb{M}(X)$ with distribution function λ_f. If $\varphi\colon [0, \infty) \to [0, \infty)$ is increasing and C^1 with $\varphi(0) = 0$, then*

$$\int_X \varphi(|f(x)|) d\mu(x) = \int_{(0,\infty)} \lambda_f(\alpha) \varphi'(\alpha) d\alpha$$

In particular, for $1 \leq p < \infty$, let $\varphi(\alpha) = \alpha^p$ we have

$$\int_X |f(x)|^p \, d\mu(x) = p \int_{(0,\infty)} \lambda_f(\alpha) \alpha^{p-1} d\alpha$$

Proof. Note that $\varphi'(\alpha) \geq 0$. Then

$$\begin{aligned}
&\int_X \varphi(|f(x)|) d\mu(x) = \int_X \left(\int_{(0,|f(x)|)} \varphi'(\alpha) d\alpha\right) d\mu(x) \\
&= \int_X \left(\int_{(0,\infty)} \chi_{(0,|f(x)|)}(\alpha) \varphi'(\alpha) d\alpha\right) d\mu(x) \\
&= \int_{(0,\infty)} \left(\int_X \chi_{\{y \in X || f(y)| > \alpha\}}(x) d\mu(x)\right) \varphi'(\alpha) d\alpha \\
&= \int_{(0,\infty)} \mu(\{y \in X || f(y) |> \alpha\}) \varphi'(\alpha) d\alpha = \int_{(0,\infty)} \lambda_f(\alpha) \varphi'(\alpha) d\alpha
\end{aligned}$$

where we use Fubini's Theorem of nonnegative measurable function and the fact that for any $\alpha \in (0, \infty)$ and any $x \in X$,

$$\chi_{(0,|f(x)|)}(\alpha) = \chi_{\{y \in X || f(y)| > \alpha\}}(x)$$

$\square$

Definition 2.4.2. Let $(X, \mathscr{A}, \mu)$ be a measure space. For $1 \leq p < \infty$,

$$L^{p,\infty}(X, \mathscr{A}, \mu) = \left\{f \in \mathbb{M}(X) \mid \sup_{\alpha > 0} \{\alpha^p \lambda_f(\alpha)\} < \infty\right\}$$

with the equivalence $f(x) = g(x) \ \mu - a.e.$. Also, let $L^{\infty,\infty}(X, \mathscr{A}, \mu) := L^\infty(X, \mathscr{A}, \mu)$. Then for $1 \leq p \leq \infty$, $L^{p,\infty}(X) = L^{p,\infty}(X, \mathscr{A}, \mu)$ is called weak L^p space.

Proposition 2.4.2. *$L^{p,\infty}(X)$ is a $\mathbb{C}$-linear space for all $1 \leq p \leq \infty$.*

Proof. It only needs to prove for $1 \leq p < \infty$.

(1) First, $0 \in L^{p,\infty}(X)$ because $\lambda_0(\alpha) = 0$ for all $\alpha > 0$.

(2) Let $f, g \in L^{p,\infty}(X)$. For $\alpha \in (0, \infty)$, by $\lambda_{f+g}(\alpha) \leq \lambda_f\left(\frac{\alpha}{2}\right) + \lambda_g\left(\frac{\alpha}{2}\right)$,

$$\begin{aligned}\alpha^p \lambda_{f+g}(\alpha) &\leq \alpha^p \lambda_f\left(\frac{\alpha}{2}\right) + \alpha^p \lambda_g\left(\frac{\alpha}{2}\right) = 2^p\left[\left(\frac{\alpha}{2}\right)^p \lambda_f\left(\frac{\alpha}{2}\right) + \left(\frac{\alpha}{2}\right)^p \lambda_g\left(\frac{\alpha}{2}\right)\right] \\ &\leq 2^p\left[\sup_{\beta>0}\{\beta^p \lambda_f(\beta)\} + \sup_{\beta>0}\{\beta^p \lambda_g(\beta)\}\right]\end{aligned}$$

So

$$\sup_{\alpha>0}\{\alpha^p \lambda_{f+g}(\alpha)\} \leq 2^p\left[\sup_{\beta>0}\{\beta^p \lambda_f(\beta)\} + \sup_{\beta>0}\{\beta^p \lambda_g(\beta)\}\right] < \infty$$

which means $f + g \in L^{p,\infty}(X)$.

(3) Let $f \in L^{p,\infty}(X)$ and $0 \neq c \in \mathbb{C}$ (It's clearly true for $c = 0$). For $\alpha \in (0, \infty)$,

$$\lambda_{cf}(\alpha) = \mu(\{x \in X \mid |cf(x)| > \alpha\}) = \mu\left(\left\{x \in X \mid |f(x)| > \frac{\alpha}{|c|}\right\}\right) = \lambda_f\left(\frac{\alpha}{|c|}\right)$$

Therefore,

$$\alpha^p \lambda_{cf}(\alpha) = \alpha^p \lambda_f\left(\frac{\alpha}{|c|}\right) = |c|^p\left(\frac{\alpha}{|c|}\right)^p \lambda_f\left(\frac{\alpha}{|c|}\right) \leq |c|^p \sup_{\beta>0}\{\beta^p \lambda_f(\beta)\}$$

and thus

$$\sup_{\alpha>0}\{\alpha^p \lambda_{cf}(\alpha)\} \leq |c|^p \sup_{\beta>0}\{\beta^p \lambda_f(\beta)\} < \infty$$

which means $cf \in L^{p,\infty}(X)$.

$\square$

Proposition 2.4.3 (Chebyshev's Inequality)**.** *Let $(X, \mathscr{A}, \mu)$ be a measure space and $g \in \mathbb{M}(X)$. Then for any $\alpha \in (0, \infty)$,*

$$\mu(\{x \in X||g(x)| > \alpha\}) \leq \frac{1}{\alpha}\int_X |g(x)|d\mu(x)$$

Proof. For any $\alpha \in (0, \infty)$,

$$\begin{aligned}\int_X |g(x)|d\mu(x) &\geq \int_{\{y\in X||g(y)|>\alpha\}} |g(x)|d\mu(x) \geq \int_{\{y\in X||g(y)|>\alpha\}} \alpha d\mu(x) \\ &= \alpha\mu(\{y \in X \mid |g(y)| > \alpha\})\end{aligned}$$

$\square$

Proposition 2.4.4. *Let $(X, \mathscr{A}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then*

$$L^p(X) \subset L^{p,\infty}(X)$$

Proof. For $1 \leq p < \infty$, let $f \in L^p(X)$. For $\alpha \in (0, \infty)$, by Chebyshev's Inequality,

$$\begin{aligned}\lambda_f(\alpha) &= \mu(\{x \in X||f(x)| > \alpha\}) = \mu(\{x \in X||f(x)|^p > \alpha^p\}) \\ &\leq \frac{1}{\alpha^p}\int_X |f(x)|^p d\mu(x) = \frac{1}{\alpha^p}\|f\|^p_{L^p(X)}\end{aligned}$$

Therefore,

$$\sup_{\alpha>0}\{\alpha^p \lambda_f(\alpha)\} \leq \|f\|^p_{L^p(X)} < \infty$$

$\square$

Remark. Note that $L^p(\mathbb{R}^N) \neq L^{p,\infty}(\mathbb{R}^N)$. For example, $f(x) = |x|^{-\frac{N}{p}} \neq L^p$ but it is in $L^{p,\infty}$. For $1 \leq p < \infty$, $f \in L^{p,\infty}(X)$, define

$$[f]_{L^{p,\infty}(X)} := \left[\sup_{\alpha>0} \{\alpha^p \lambda_f(\alpha)\}\right]^{1/p}$$

and for $L^{\infty,\infty}(X)$, $[f]_{L^{\infty,\infty}(X)} := \|f\|_{L^\infty(X)}$. And by above proposition, for any $1 \leq p \leq \infty$,

$$[f]_{L^{p,\infty}(X)} \leq \|f\|_{L^p(X)}$$

Proposition 2.4.5. *Let $(X, \mathscr{A}, \mu)$ be a measure space and $1 \leq p < \infty$. Let $f, g \in L^{p,\infty}(X)$.*

(1) $[f]_{L^{p,\infty}(X)} \geq 0$.

(2) $[f]_{L^{p,\infty}(X)} = 0 \Leftrightarrow f = 0$ *in* $L^{p,\infty}(X)$.

(3) For any $c \in \mathbb{C}$, $[cf]_{L^{p,\infty}(X)} = |c|[f]_{L^{p,\infty}(X)}$.

(4) $[f+g]_{L^{p,\infty}(X)} \leq 2\left([f]_{L^{p,\infty}(X)} + [g]_{L^{p,\infty}(X)}\right)$.

Proposition 2.4.6. *Let $(X, \mathscr{A}, \mu)$ be a measure space and $f \in \mathbb{M}(X)$. For $R > 0$, let*

$$E(R) := \{x \in X || f(x) |> R\} \in \mathscr{A}$$

Define

$$\begin{aligned} h_R(x) &:= f(x)\chi_{E(R)^c}(x) + R(\overline{\operatorname{sgn} f(x)})\chi_{E(R)}(x), \\ g_R(x) &:= f(x) - h_R(x) = (\overline{\operatorname{sgn} f(x)})(|f(x)| - R)\chi_{E(R)}(x) \end{aligned}$$

Then for any $\alpha \in (0, \infty)$,

$$\lambda_{h_R}(\alpha) = \begin{cases} \lambda_f(\alpha), & 0 < \alpha < R, \\ 0, & \alpha \geq R, \end{cases} \quad \lambda_{g_R}(\alpha) = \lambda_f(\alpha + R)$$

Remark. By calculation,

$$h_R(x) = \begin{cases} R(\overline{\operatorname{sgn} f(x)}), & |f(x)| > R, \\ f(x), & 0 \leq |f(x)| \leq R, \end{cases}$$

$$|h_R(x)| = \begin{cases} R, & |f(x)| > R, \\ |f(x)|, & 0 \leq |f(x)| \leq R, \end{cases}$$

$$g_R(x) = \begin{cases} (|f(x)| - R)(\overline{\operatorname{sgn} f(x)}), & |f(x)| > R, \\ 0, & 0 \leq |f(x)| \leq R, \end{cases}$$

$$|g_R(x)| = \begin{cases} |f(x)| - R, & |f(x)| > R, \\ 0, & 0 \leq |f(x)| \leq R. \end{cases}$$

Proof. Let $R > 0$.

(i) Consider λ_{h_R}. First, let $\alpha \geq R$. Because $|h_R(x)| \leq R \leq \alpha$ for all x,

$$\lambda_{h_R}(\alpha) = \mu\left(\{x \in X || h_R(x) |> \alpha\}\right) = \mu(\emptyset) = 0$$

When $0 < \alpha < R$, for $x \in X$,

$$|h_R(x)| > \alpha \Longleftrightarrow |f(x)| > R \text{ or } \alpha < |f(x)| \leq R \Longleftrightarrow |f(x)| > \alpha$$

and so

$$\lambda_{h_R}(\alpha) = \mu\left(\{x \in X || h_R(x) |> \alpha\}\right) = \mu(\{x \in X || f(x) |> \alpha\}) = \lambda_f(\alpha)$$

(ii) Consider λ_{g_R}. For $\alpha > 0$ and $x \in X$,

$$|g_R(x)| > \alpha \iff |f(x)| > R \text{ and } |f(x)| - R > \alpha \iff |f(x)| > \alpha + R$$

and so

$$\lambda_{g_R}(\alpha) = \mu\left(\{x \in X | |g_R(x)\ | > \alpha\}\right) = \mu(\{x \in X | |f(x)\ | > \alpha + R\}) = \lambda_f(\alpha + R)$$

□

2.5 Marcinkiewicz Interpolation Theorem

Definition 2.5.1. Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two measurable spaces and subspace $\mathbb{D} \subset \mathbb{M}(X)$. Let $T\colon \mathbb{D} \to \mathbb{M}(Y)$ be a sublinear map, *i.e.*

$$|(T(f+g))(y)| \leq |(Tf)(y)| + |(Tg)(y)|, \quad |(T(cf))(y)| = |c||(Tf)(y)|.$$

Let $1 \leq p, q \leq \infty$.

(1) If $L^p(X) \subset \mathbb{D}$ and $T(L^p(X)) \subset L^q(Y)$ and there is a $C > 0$ such that for any $f \in L^p(X)$

$$\|Tf\|_{L^q(Y)} \leq C\|f\|_{L^p(X)}$$

then T is called strong (p, q)-type.

(2) If $L^p(X) \subset \mathbb{D}$ and $T(L^p(X)) \subset L^{q,\infty}(Y)$ and there is a $C > 0$ such that for any $f \in L^p(X)$

$$[Tf]_{L^{q,\infty}(Y)} \leq C\|f\|_{L^p(X)}$$

then T is called weak (p, q)-type.

Note that the strong (p, q)-type implies the weak (p, q)-type.

Theorem 2.5.1 (Marcinkiewicz Interpolation Theorem). *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ with $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$. For $0 < t < 1$, let p, q be*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

If $T\colon L^{p_0}(X) + L^{p_1}(X) \to \mathbb{M}(Y)$ be sublinear satisfying

- *T is weak (p_0, q_0)-type,* i.e. *$T(L^{p_0}(X)) \subset L^{q_0,\infty}(Y)$ and there is an M_0 such that*

$$[Tf]_{L^{q_0,\infty}(Y)} \leq M_0\|f\|_{L^{p_0}(X)}, \quad f \in L^{p_0}(X)$$

- *T is weak (p_1, q_1)-type,* i.e. *$T(L^{p_1}(X)) \subset L^{q_1,\infty}(Y)$ and there is an M_1 such that*

$$[Tf]_{L^{q_1,\infty}(Y)} \leq M_1\|f\|_{L^{p_1}(X)}, \quad f \in L^{p_1}(X)$$

The T is strong (p, q)-type, that is, $T(L^p(X)) \subset L^{q,\infty}(Y)$ and there is a $B = B(p_0, p_1, q_0, q_1, t, M_0, M_1)$ such that

$$\|Tf\|_{L^q(X)} \leq B\|f\|_{L^p(X)}, \quad f \in L^p(X)$$

Remark. By the assumptions, $1 < q < \infty$ with $q_0 < q < q_1$. And $1 \leq p < \infty$, otherwise $p = \infty$ implies $p_0 = p_1 = \infty$ and thus $q_0 = q_1 = \infty$ contradicted to $q_0 \neq q_1$. Moreover, if $p = 1$, then $p = p_0 = p_1 = 1$ by $t \in (0, 1)$. If $1 < p < \infty$, then $p_0 \neq p_1$ will imply $p_0 < p < p_1$ and $p_0 = p_1$ will imply $p = p_0 = p_1$ by $t \in (0, 1)$.

Proof. It is sufficient to prove for any $f \in L^p(X)$ with $\|f\|_{L^p(X)} = 1$ we have $Tf \in L^q(Y)$ with $\|Tf\|_{L^q(Y)} \le B$ by the sublinearity of T.

(I) $p_0 = p_1$: Then $p = p_0 = p_1 \in [1, \infty)$.

(I)-1 $p_0 = p_1$ and $q_0, q_1 < \infty$: Let $f \in L^p(X)$ with $\|f\|_{L^p(X)} = 1$. Then by above proposition

$$\int_Y |Tf(y)|^q d\nu(y) = q \int_{(0,\infty)} \beta^{q-1} \lambda_{Tf}(\beta) d\beta$$

First, by assumption,

$$\left[\sup_{\beta>0} \{\beta^{q_0} \lambda_{Tf}(\beta)\}\right]^{\frac{1}{q_0}} = [Tf]_{L^{q_0,\infty}(Y)} \le M_0 \|f\|_{L^{p_0}(X)} = M_0 \|f\|_{L^p(X)} = M_0,$$
$$\left[\sup_{\beta>0} \{\beta^{q_1} \lambda_{Tf}(\beta)\}\right]^{\frac{1}{q_1}} = [Tf]_{L^{q_1,\infty}(Y)} \le M_1 \|f\|_{L^{p_1}(X)} = M_1 \|f\|_{L^p(X)} = M_1$$

for any $\beta \in (0, \infty)$,

$$\lambda_{Tf}(\beta) \le \left(\frac{M_0}{\beta}\right)^{q_0}, \quad \lambda_{Tf}(\beta) \le \left(\frac{M_1}{\beta}\right)^{q_1}$$

Assume $q_0 < q_1$, by $q_0 < q < q_1$, we have $q - q_0 - 1 > -1, q - q_1 - 1 < -1$. So

$$\begin{aligned}
&\int_Y |Tf(y)|^q d\nu(y) = q \int_0^\infty \beta^{q-1} \lambda_{Tf}(\beta) d\beta \\
&= q \int_0^1 \beta^{q-1} \lambda_{Tf}(\beta) d\beta + q \int_1^\infty \beta^{q-1} \lambda_{Tf}(\beta) d\beta \\
&\le q \int_0^1 \beta^{q-1} \left(\frac{M_0}{\beta}\right)^{q_0} d\beta + q \int_1^\infty \beta^{q-1} \left(\frac{M_1}{\beta}\right)^{q_1} d\beta \qquad (2.4) \\
&= q M_0^{q_0} \int_0^1 \beta^{q-q_0-1} d\beta + q M_1^{q_1} \int_1^\infty \beta^{q-q_1-1} d\beta \\
&= q \left(\frac{M_0^{q_0}}{q - q_0} + \frac{M_1^{q_1}}{q_1 - q}\right) < \infty
\end{aligned}$$

Therefore, $Tf \in L^q(Y)$ and

$$B := \left\{q \left(\frac{M_0^{q_0}}{q - q_0} + \frac{M_1^{q_1}}{q_1 - q}\right)\right\}^{\frac{1}{q}} \quad \Rightarrow \quad \|Tf\|_{L^q(Y)} \le B$$

When $q_0 > q_1$, we only need to swap M_0 with M_1 in equation (2.4).

(I)-2 $p_0 = p_1$ and ($q_0 = \infty$ or $q_1 = \infty$): First, assume $q_1 = \infty$. Then $q_0 < \infty$ and $q_0 < q < \infty$. Similarly, consider the equation

$$\int_Y |Tf(y)|^q d\nu(y) = q \int_{(0,\infty)} \beta^{q-1} \lambda_{Tf}(\beta) d\beta$$

By $q_0 < \infty$, we also have for any $\beta \in (0, \infty)$,

$$\lambda_{Tf}(\beta) \le \left(\frac{M_0}{\beta}\right)^{q_0}$$

On the other hand, by $q_1 = \infty$,

$$\|Tf\|_{L^\infty(Y)} = [Tf]_{L^{\infty,\infty}}(Y) \leq M_1 \|f\|_{L^{p_1}(X)} = M_1 \|f\|_{L^p(X)} = M_1$$

So we have for any $\beta \geq M_1$, $\lambda_{Tf}(\beta) = 0$. Then combining these results with $q - q_0 - 1 > -1$,

$$\begin{aligned}
\int_Y |Tf(y)|^q d\nu(y) &= q \int_0^\infty \beta^{q-1} \lambda_{Tf}(\beta) d\beta \\
&= q \int_0^{M_1} \beta^{q-1} \lambda_{Tf}(\beta) d\beta \leq q \int_0^{M_1} \beta^{q-1} \left(\frac{M_0}{\beta}\right)^{q_0} d\beta \\
&= q M_0^{q_0} \int_0^{M_1} \beta^{q-q_0-1} d\beta = \frac{q}{q-q_0} M_0^{q_0} M_1^{q-q_0} < \infty
\end{aligned}$$

So $Tf \in L^q(Y)$ and

$$B := \left(\frac{q}{q-q_0} M_0^{q_0} M_1^{q-q_0}\right)^{\frac{1}{q}} \quad \Rightarrow \quad \|Tf\|_{L^q(Y)} \leq B$$

On the other hand, if $q_0 = \infty$ and $q_1 < \infty$, then $q_1 < q < \infty$ and $q - q_1 - 1 > -1$. So we just swap q_0 with q_1 and M_0 with M_1.

(II) $p_0 < p_1$ and $q_0, q_1 < \infty$: Then $p_0 < p < p_1 \leq q_1 < \infty$. Consider the equation

$$\begin{aligned}
\int_Y |Tf(y)|^q d\nu(y) &= q \int_0^\infty \beta^{q-1} \lambda_{Tf}(\beta) d\beta \\
&= 2^q q \int_0^\infty \beta^{q-1} \lambda_{Tf}(2\beta) d\beta
\end{aligned}$$

For $R > 0$, let $E(R)$, h_R, and g_R for f defined as in Proposition 2.4.6. By definition, we know h_R is bounded and $h_R \in L^p(X)$ by $f \in L^p(X)$. Since $p < p_1 < \infty$,

$$h_R \in L^\infty(X) \cap L^p(X) \subset L^{p_1}(X)$$

Moreover, by Proposition 2.4.6

$$\lambda_{h_R}(\alpha) = \begin{cases} \lambda_f(\alpha), & 0 < \alpha < R, \\ 0, & \alpha \geq R. \end{cases}$$

we have

$$\begin{aligned}
\|h_R\|_{L^{p_1}(X)}^{p_1} = \int_X |h_R(x)|^{p_1} \, d\mu(x) &= p_1 \int_0^\infty \alpha^{p_1-1} \lambda_{h_R}(\alpha) d\alpha \\
&= p_1 \int_0^R \alpha^{p_1-1} \lambda_f(\alpha) d\alpha.
\end{aligned}$$

Next, for g_R, by Proposition 2.4.6

$$\lambda_{g_R}(\alpha) = \lambda_f(\alpha + R)$$

and by $p_0 - p < 0$, we have

$$\begin{aligned}
&\int_X |g_R(x)|^{p_0}\, d\mu(x) = p_0 \int_0^\infty \alpha^{p_0-1}\lambda_{g_R}(\alpha)d\alpha \\
&= p_0 \int_0^\infty \alpha^{p_0-1}\lambda_f(\alpha+R)d\alpha \\
&= p_0 \int_R^\infty (\alpha-R)^{p_0-1}\lambda_f(\alpha)d\alpha \\
&\leq p_0 \int_R^\infty \alpha^{p_0-1}\lambda_f(\alpha)d\alpha = p_0 \int_R^\infty \alpha^{p_0-p}\cdot\alpha^{p-1}\lambda_f(\alpha)d\alpha \\
&\leq p_0 R^{p_0-p} \int_R^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha \\
&\leq \frac{p_0}{p} R^{p_0-p}\cdot p \int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha = \frac{p_0}{p} R^{p_0-p}\|f\|_{L^p(X)}^p = \frac{p_0}{p} R^{p_0-p} < \infty
\end{aligned}$$

So $g_R \in L^{p_0}(X)$ and

$$\|g_R\|_{L^{p_0}(X)}^{p_0} = \int_X |g_R(x)|^{p_0}\, d\mu(x) \leq p_0 \int_R^\infty \alpha^{p_0-1}\lambda_f(\alpha)d\alpha$$

Then by assumption of T, for any $R > 0$,

$$\begin{aligned}
\left[\sup_{\beta>0}\{\beta^{q_0}\lambda_{Tg_R}(\beta)\}\right]^{\frac{1}{q_0}} &= [Tg_R]_{L^{q_0,\infty}(Y)} \leq M_0 \|g_R\|_{L^{p_0}(X)}\,, \\
\left[\sup_{\beta>0}\{\beta^{q_1}\lambda_{Th_R}(\beta)\}\right]^{\frac{1}{q_1}} &= [Th_R]_{L^{q_1,\infty}(Y)} \leq M_1 \|h_R\|_{L^{p_1}(X)}
\end{aligned}$$

So for any $\beta \in (0,\infty)$,

$$\begin{aligned}
\lambda_{Tg_R}(\beta) &\leq \left(\frac{M_0}{\beta}\|g_R\|_{L^{p_0}(X)}\right)^{q_0} \\
\lambda_{Th_R}(\beta) &\leq \left(\frac{M_1}{\beta}\|h_R\|_{L^{p_1}(X)}\right)^{q_1}
\end{aligned}$$

Moreover, because $f = g_R + h_R$, by the sublinearity of T,

$$|Tf(y)| = |T(g_R+h_R)(y)| \leq |(Tg_R)(y)| + |(Th_R)(y)|\,, \quad \nu - a.e.\ y \in Y$$

Then by the properties of distribution function, for any $\beta \in (0,\infty)$,

$$\lambda_{Tf}(2\beta) \leq \lambda_{Tg_R}(\beta) + \lambda_{Th_R}(\beta)$$

Therefore,

$$
\begin{aligned}
&\int_Y |Tf(y)|^q d\nu(y) = 2^q q \int_0^\infty \beta^{q-1} \lambda_{Tf}(2\beta) d\beta \\
&\leq 2^q q \left[\int_0^\infty \beta^{q-1} \lambda_{Tg_R}(\beta) d\beta + \int_0^\infty \beta^{q-1} \lambda_{Th_R}(\beta) d\beta \right] \\
&\leq 2^q q \left[\int_0^\infty \beta^{q-1} \left(\frac{M_0}{\beta} \|g_R\|_{L^{p_0}(X)} \right)^{q_0} d\beta + \int_0^\infty \beta^{q-1} \left(\frac{M_1}{\beta} \|h_R\|_{L^{p_1}(X)} \right)^{q_1} d\beta \right] \\
&= 2^q q \left[M_0^{q_0} \int_0^\infty \beta^{q-q_0-1} \|g_R\|_{L^{p_0}(X)}^{q_0} d\beta + M_1^{q_1} \int_0^\infty \beta^{q-q_1-1} \|h_R\|_{L^{p_1}(X)}^{q_1} d\beta \right] \\
&\leq 2^q q M_0^{q_0} \int_0^\infty \beta^{q-q_0-1} \left\{ p_0 \int_R^\infty \alpha^{p_0-1} \lambda_f(\alpha) d\alpha \right\}^{\frac{q_0}{p_0}} d\beta \\
&\quad + 2^q q M_1^{q_1} \int_0^\infty \beta^{q-q_1-1} \left\{ p_1 \int_0^R \alpha^{p_1-1} \lambda_f(\alpha) d\alpha \right\}^{\frac{q_1}{p_1}} d\beta \\
&= \mathrm{I}
\end{aligned}
$$

Let

$$
\begin{aligned}
\sigma :&= \frac{p_0 (q_0 - q)}{q_0 (p_0 - p)} \\
&= \frac{p^{-1} \left(q^{-1} - q_0^{-1} \right)}{q^{-1} \left(p^{-1} - p_0^{-1} \right)} \\
&= \frac{p^{-1} \left(q^{-1} - q_1^{-1} \right)}{q^{-1} \left(p^{-1} - p_1^{-1} \right)} \\
&= \frac{p_1 (q_1 - q)}{q_1 (p_1 - p)}
\end{aligned}
$$

and $R = \beta^{\Sigma}$. Define $\Gamma = (0, \infty) \times (0, \infty)$ and

$$
D_0 := \{(\alpha, \beta) \in \Gamma \mid \alpha > \beta^\sigma\}, \quad D_1 := \{(\alpha, \beta) \in \Gamma \mid \alpha < \beta^\sigma\}
$$

Then for $j = 0, 1$, let

$$
\varphi_j(\alpha, \beta) := \chi_{D_j}(\alpha, \beta) \beta^{(q-q_j-1)\frac{p_j}{q_j}} \alpha^{p_j-1} \lambda_f(\alpha), \quad (\alpha, \beta) \in \Gamma
$$

Then

$$
\mathrm{I} = 2^q q \sum_{j=0}^{1} M_j^{q_j} p_j^{\frac{q_j}{p_j}} \int_0^\infty \left(\int_0^\infty \varphi_j(\alpha, \beta) d\alpha \right)^{\frac{q_j}{p_j}} d\beta
$$

For $j = 0, 1$, because $1 \leq \frac{q_j}{p_j} < \infty$, by the Minkowski's Inequality,

$$
\int_0^\infty \left(\int_0^\infty \varphi_j(\alpha, \beta) d\alpha \right)^{\frac{q_j}{p_j}} d\beta \leq \left[\int_0^\infty \left(\int_0^\infty \varphi_j(\alpha, \beta)^{\frac{q_j}{p_j}} d\beta \right)^{\frac{p_j}{q_j}} d\alpha \right]^{\frac{q_j}{p_j}}
$$

Therefore,

$$
\int_Y |Tf(y)|^q d\nu(y) \leq 2^q q \sum_{j=0}^{1} M_j^{q_j} p_j^{\frac{q_j}{p_j}} \left[\int_0^\infty \left(\int_0^\infty \varphi_j(\alpha, \beta)^{\frac{q_j}{p_j}} d\beta \right)^{\frac{p_j}{q_j}} d\alpha \right]^{\frac{q_j}{p_j}}
$$

First, consider φ_0. If $q_0 < q_1$, then $q - q_0 > 0$ and so $\sigma > 0$. Therefore,

$$(\alpha, \beta) \in D_0 \Longleftrightarrow \alpha > \beta^\sigma \Longleftrightarrow \alpha^{\frac{1}{\sigma}} > \beta$$

Note that $q - q_0 - 1 > -1$, so

$$\begin{aligned}
\int_0^\infty \left(\int_0^\infty \varphi_0(\alpha, \beta)^{\frac{q_0}{p_0}} d\beta \right)^{\frac{p_0}{q_0}} d\alpha &= \int_0^\infty \left(\int_0^{\alpha^{\frac{1}{\sigma}}} \varphi_0(\alpha, \beta)^{\frac{q_0}{p_0}} d\beta \right)^{\frac{p_0}{q_0}} d\alpha \\
= \int_0^\infty \left[\int_0^{\alpha^{\frac{1}{\sigma}}} \left\{ \beta^{(q-q_0-1)\frac{p_0}{q_0}} \alpha^{p_0-1} \lambda_f(\alpha) \right\}^{\frac{q_0}{p_0}} d\beta \right]^{\frac{p_0}{q_0}} d\alpha \\
= \int_0^\infty \left(\int_0^{\alpha^{\frac{1}{\sigma}}} \beta^{q-q_0-1} d\beta \right)^{\frac{p_0}{q_0}} \alpha^{p_0-1} \lambda_f(\alpha) d\alpha \\
= \frac{1}{(q-q_0)^{\frac{p_0}{q_0}}} \int_0^\infty \alpha^{\frac{p_0(q-q_0)}{q_0\sigma}} \cdot \alpha^{p_0-1} \lambda_f(\alpha) d\alpha \\
= \frac{1}{|q-q_0|^{\frac{p_0}{q_0}}} \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha &= \frac{1}{p\,|q-q_0|^{\frac{p_0}{q_0}}} p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha \\
= \frac{1}{p\,|q-q_0|^{\frac{p_0}{q_0}}} \|f\|_{L^p(X)}^p &= \frac{1}{p\,|q-q_0|^{\frac{p_0}{q_0}}}
\end{aligned}$$

If $q_0 > q_1$, the $q - q_0 < 0$ and $\sigma < 0$. So

$$(\alpha, \beta) \in D_0 \Longleftrightarrow \alpha > \beta^\sigma \Longleftrightarrow \alpha^{\frac{1}{\sigma}} < \beta$$

Note that $q - q_0 - 1 < -1$ and so

$$\begin{aligned}
\int_0^\infty \left(\int_0^\infty \varphi_0(\alpha, \beta)^{\frac{q_0}{p_0}} d\beta \right)^{\frac{p_0}{q_0}} d\alpha &= \int_0^\infty \left(\int_{\alpha^{\frac{1}{\sigma}}}^\infty \varphi_0(\alpha, \beta)^{\frac{q_0}{p_0}} d\beta \right)^{\frac{p_0}{q_0}} d\alpha \\
= \int_0^\infty \left[\int_{\alpha^{\frac{1}{\sigma}}}^\infty \left\{ \beta^{(q-q_0-1)\frac{p_0}{q_0}} \alpha^{p_0-1} \lambda_f(\alpha) \right\}^{\frac{q_0}{p_0}} d\beta \right]^{\frac{p_0}{q_0}} d\alpha \\
= \int_0^\infty \left(\int_{\alpha^{\frac{1}{\sigma}}}^\infty \beta^{q-q_0-1} d\beta \right)^{\frac{p_0}{q_0}} \alpha^{p_0-1} \lambda_f(\alpha) d\alpha \\
= \frac{1}{(q_0-q)^{\frac{p_0}{q_0}}} \int_0^\infty \alpha^{\frac{p_0(q-q_0)}{q_0\sigma}} \cdot \alpha^{p_0-1} \lambda_f(\alpha) d\alpha \\
= \frac{1}{|q-q_0|^{\frac{p_0}{q_0}}} \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha &= \frac{1}{p\,|q-q_0|^{\frac{p_0}{q_0}}} p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha \\
= \frac{1}{p\,|q-q_0|^{\frac{p_0}{q_0}}} \|f\|_{L^p(X)}^p &= \frac{1}{p\,|q-q_0|^{\frac{p_0}{q_0}}}
\end{aligned}$$

Therefore,

$$\int_0^\infty \left(\int_0^\infty \varphi_0(\alpha, \beta)^{\frac{q_0}{p_0}} d\beta \right)^{\frac{p_0}{q_0}} d\alpha = \frac{1}{p\,|q-q_0|^{\frac{p_0}{q_0}}}$$

For φ_1, it can also get

$$\int_0^\infty \left(\int_0^\infty \varphi_1(\alpha, \beta)^{\frac{q_1}{p_1}} d\beta \right)^{\frac{p_1}{q_1}} d\alpha = \frac{1}{p\,|q-q_1|^{\frac{p_1}{q_1}}}$$

And therefore,

$$\int_Y |Tf(y)|^q d\nu(y) \leq 2^q q \sum_{j=0}^{1} M_j^{q_j} \left(\frac{p_j}{p}\right)^{\frac{q_j}{p_j}} \frac{1}{|q-q_j|} < \infty$$

Let

$$B := \left\{ 2^q q \sum_{j=0}^{1} M_j^{q_j} \left(\frac{p_j}{p}\right)^{\frac{q_j}{p_j}} \frac{1}{|q-q_j|} \right\}^{\frac{1}{q}}$$

we have

$$\|Tf\|_{L^q(Y)} \leq B$$

(III) $p_0 \neq p_1$ and ($q_0 = \infty$ or $q_1 = \infty$): Omits.

□

Theorem 2.5.2. *Let $\Omega \subset \mathbb{R}^N$ be Lebesgue measurable. Let $1 < r < \infty$. $K = K(x,y)\colon \Omega \times \Omega \to \mathbb{C}$ is Lebesgue measurable such that $K(x,\cdot) \in L^r(\Omega)$ for* a.e. *$x \in \Omega$ and $K(\cdot,y) \in L^r(\Omega)$ for* a.e. *$y \in \Omega$ with $M > 0$ such that*

- $[K(x,\cdot)]_{L^{r,\infty}(\Omega)} \leq M$, a.e. $x \in \Omega$,
- $[K(\cdot,y)]_{L^{r,\infty}(\Omega)} \leq M$, a.e. $y \in \Omega$.

Let p, q satisfy

$$1 \leq p < q < \infty, \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$$

Note that $1 < r \leq q < \infty$. Then

(1) Let $f \in L^p(\Omega)$. For any $x \in X$,

$$\int_\Omega |K(x,y)f(y)|dy < \infty$$

(2) Let linear $T\colon L^p(\Omega) \to \mathbb{M}(\Omega)$ defined as

$$(Tf)(x) := \int_\Omega K(x,y)f(y)dy, \quad a.e.\ x \in \Omega$$

(i) For $p = 1$ ($q = r$), $T(L^1(\Omega)) \subset L^{r,\infty}(\Omega)$. There is a $B > 0$ such that for any $f \in L^1(\Omega)$,

$$[Tf]_{L^{r,\infty}(\Omega)} \leq BM\|f\|_{L^1(\Omega)}$$

In particular, T is weak $(1,r)$-type.

(ii) For $p \neq 1$ ($1 < p < q < \infty, 1 < r < q < \infty$), $T(L^p(\Omega)) \subset L^q(\Omega)$. There is a $C > 0$ such that for any $f \in L^p(X)$,

$$\|Tf\|_{L^q(\Omega)} \leq CM\|f\|_{L^p(\Omega)}$$

In particular, T is strong (p,q)-type.

Proof. WLTG, assume $M = 1$. Let r', p' be the conjugate of r, p. By

$$0 < \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{p} - \frac{1}{r'} = \frac{1}{r} - \frac{1}{p'}$$

we have

$$1 \leq p < r', \quad 1 < r < p', \quad 1 < r \leq q < \infty$$

By the assumptions of K, we know

- for any $\beta > 0$,
$$\lambda_{K(x,\cdot)}(\beta) \leq \beta^{-r}, \quad a.e.\ x \in \Omega$$
- for any $\alpha > 0$,
$$\lambda_{K(\cdot,y)}(\alpha) \leq \alpha^{-r}, \quad a.e.\ y \in \Omega$$

(I) $R > 0$ (which will be determined in the following): Let $E(R) \subset \Omega \times \Omega$ be

$$E(R) := \{(x, y) \in \Omega \times \Omega || K(x, y) |> R\}$$

and also define the H_R as

$$\begin{aligned} H_R(x,y) &:= K(x,y)\chi_{E(R)^c}(x,y) + R(\overline{\operatorname{sgn} K(x,y)})\chi_{E(R)}(x,y) \\ &= \begin{cases} R(\overline{\operatorname{sgn} K(x,y)}), & |K(x,y)| > R, \\ K(x,y), & 0 \leq |K(x,y)| \leq R, \end{cases} \end{aligned}$$

and G_R as

$$\begin{aligned} G_R(x,y) &:= K(x,y) - H_R(x,y) = (\overline{\operatorname{sgn} K(x,y)})(|K(x,y)| - R)\chi_{E(R)}(x,y) \\ &= \begin{cases} (|K(x,y)| - R)(\overline{\operatorname{sgn} K(x,y)}), & |K(x,y)| > R, \\ 0, & 0 \leq |K(x,y)| \leq R, \end{cases} \end{aligned}$$

Note they are as same as the definitions in Proposition 2.4.6 for K. Moreover, when fix $x \in \Omega$ and let $E_x(R) := \{y \in \Omega || K(x, y) |> R\}$, $\chi_{E_x(R)}(y) = \chi_{E(R)}(x, y)$. So we have

$$\lambda_{G_R(x,\cdot)}(\beta) = \lambda_{K(x,\cdot)}(\beta + R), \quad \lambda_{G_R(\cdot,y)}(\alpha) = \lambda_{K(\cdot,y)}(\alpha + R)$$

Then because $r > 1$, it can get

$$\begin{aligned} \int_\Omega |G_R(x,y)|\,dy &= \int_0^\infty \lambda_{G_R(x,\cdot)}(\beta)d\beta \\ &= \int_0^\infty \lambda_{K(x,\cdot)}(\beta + R)d\beta = \int_R^\infty \lambda_{K(x,\cdot)}(\beta)d\beta \\ &\leq \int_R^\infty \beta^{-r}d\beta = \frac{1}{r-1}R^{1-r} < \infty \end{aligned}$$

for $a.e.\ x \in \Omega$ by the weak boundedness of $K(x, \cdot)$and similarly by the weak boundedness of $K(\cdot, y)$,

$$\int_\Omega |G_R(x,y)|\,dx \leq \frac{1}{r-1}R^{1-r} < \infty$$

$a.e.\ y \in \Omega$. Then by setting $r = 1, p = q$ in Theorem 2.3.2, we have for any any $f \in L^p(\Omega)$,

$$\int_\Omega |G_R(x,y)f(y)|\,dy < \infty, \quad a.e.\ x \in \Omega$$

Moreover, if define $T_{1,R}\colon L^p(\Omega) \to \mathbb{M}(\Omega)$ by

$$(T_{1,R}f)(x) := \int_\Omega G_R(x,y)f(y)dy, \quad a.e.\ x \in \Omega$$

then

$$\|T_{1,R}f\|_{L^p(\Omega)} \le \frac{1}{r-1} R^{1-r} \|f\|_{L^p(\Omega)}, \quad \forall\ f \in L^p(\Omega)$$

Next, for H_R,

$$\lambda_{H_R(x,\cdot)}(\beta) = \begin{cases} \lambda_{K(x,\cdot)}(\beta), & 0 < \beta < R, \\ 0, & \beta \ge R, \end{cases}$$

When $p \ne 1$ ($p' \ne \infty$ and $r < p'$), for $a.e.\ x \in \Omega$,

$$\begin{aligned} \int_\Omega |H_R(x,y)|^{p'} dy &= p' \int_0^\infty \beta^{p'-1} \lambda_{H_R(x,)}(\beta) d\beta = p' \int_0^R \beta^{p'-1} \lambda_{K(x,)}(\beta) d\beta \\ &\le p' \int_0^R \beta^{p'-1} \cdot \beta^{-r} d\beta = p' \int_0^R \beta^{p'-r-1} d\beta = \frac{p'}{p'-r} R^{p'-r} = \frac{q}{r} R^{\frac{r}{q}p'} < \infty \end{aligned}$$

Therefore, for $a.e.\ x \in \Omega$, $H_R(x,\cdot) \in L^{p'}(\Omega)$ with

$$H_R(x,\cdot) \in L^{p'}(\Omega)$$

When $p = 1$ ($p' = \infty$ and $q = r$), because $(x,y) \in \Omega \times \Omega$ with

$$|H_R(x,y)| \le R$$

So $H_R(x,\cdot) \in L^\infty(\Omega)$ with

$$\|H_R(x,\cdot)\|_{L^\infty(\Omega)} \le R$$

Therefore,

$$\|H_R(x,\cdot)\|_{L^{p'}(\Omega)} \le \left(\frac{q}{r}\right)^{\frac{1}{p'}} R^{\frac{r}{q}}$$

And by Hölder's Inequality, for $f \in L^p(\Omega)$,

$$\begin{aligned} \int_\Omega |H_R(x,y)f(y)|\, dy &\le \|H_R(x,\cdot)\|_{L^{p'}(\Omega)} \|f\|_{L^p(\Omega)} \\ &\le \left(\frac{q}{r}\right)^{\frac{1}{p'}} R^{\frac{r}{q}} \|f\|_{L^p(\Omega)} < \infty \end{aligned}$$

So define $T_{2,R}\colon L^p(\Omega) \to \mathbb{M}(\Omega)$ as

$$(T_{2,R}f)(x) := \int_\Omega H_R(x,y)f(y)dy$$

and thus

$$|(T_{2,R}f)(x)| \le \left(\frac{q}{r}\right)^{\frac{1}{p'}} R^{\frac{r}{q}} \|f\|_{L^p(\Omega)}, \quad a.e.\ x \in \Omega$$

and $T_{2,R}f \in L^\infty(\Omega)$ with

$$\|T_{2,R}f\|_{L^\infty(\Omega)} \le \left(\frac{q}{r}\right)^{\frac{1}{p'}} R^{\frac{r}{q}} \|f\|_{L^p(\Omega)}$$

Then we have for any $f \in L^p(\Omega)$

$$\int_\Omega |K(x,y)f(y)| dy \le \int_\Omega |G_R(x,y)f(y)|\, dy + \int_\Omega |H_R(x,y)f(y)|\, dy < \infty$$

which proves (1). And so we have define $T\colon L^p(\Omega) \to \mathbb{M}(\Omega)$ as

$$(Tf)(x) := \int_\Omega K(x,y)f(y)dy, \quad a.e.\ x \in \Omega$$

Moreover,

$$Tf = T_{1,R}f + T_{2,R}f \quad (\in L^p(\Omega) + L^\infty(\Omega))$$

(II) For $1 < r < \infty, 1 \le p < q < \infty, \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, check T is weak (p,q)-type.

Let $f \in L^p(\Omega)$ with $\|f\|_{L^p(\Omega)} = 1$. For any $\alpha > 0$,

$$\lambda_{Tf}(\alpha) \le \lambda_{T_{1,R}f}\left(\frac{\alpha}{2}\right) + \lambda_{T_{2,R}f}\left(\frac{\alpha}{2}\right)$$

with

$$R := \left(\frac{\alpha}{2}\right)^{\frac{q}{r}} \left(\frac{q}{r}\right)^{-\frac{q}{rp'}}$$

Then by above

$$\begin{aligned}\|T_{2,R}f\|_{L^\infty(\Omega)} &\le \left(\frac{q}{r}\right)^{\frac{1}{p'}} R^{\frac{r}{q}} \|f\|_{L^p(\Omega)} = \left(\frac{q}{r}\right)^{\frac{1}{p'}} R^{\frac{r}{q}} \\ &= \left(\frac{q}{r}\right)^{\frac{1}{p'}} \left[\left(\frac{\alpha}{2}\right)^{\frac{q}{r}} \left(\frac{q}{r}\right)^{-\frac{q}{rp'}}\right]^{\frac{r}{q}} = \frac{\alpha}{2}\end{aligned}$$

Therefore, $\lambda_{T_{2,R}f}\left(\frac{\alpha}{2}\right) = 0$.

Let m_N be the Lebesgue measure. By Chebyshev's Inequality,

$$\begin{aligned}\lambda_{Tf}(\alpha) &\le \lambda_{T_{1,R}f}\left(\frac{\alpha}{2}\right) + \lambda_{T_{2,R}f}\left(\frac{\alpha}{2}\right) = \lambda_{T_{1,R}f}\left(\frac{\alpha}{2}\right) \\ &= m_N\left(\left\{x \in \Omega \left| \left| (T_{1,R}f)(x) \right| > \frac{\alpha}{2}\right.\right\}\right) \\ &= m_N\left(\left\{x \in \Omega \left| \left| (T_{1,R}f)(x)\right|^p > \left(\frac{\alpha}{2}\right)^p\right.\right\}\right) \\ &\le \left(\frac{2}{\alpha}\right)^p \|T_{1,R}f\|^p_{L^p(\Omega)} \le \left(\frac{2}{\alpha}\right)^p \left(\frac{1}{r-1}R^{1-r}\right)^p \|f\|^p_{L^p(\Omega)} \\ &= \left(\frac{2}{\alpha}\right)^p \left(\frac{1}{r-1}R^{1-r}\right)^p \\ &= \left(\frac{2}{\alpha}\right)^p \frac{1}{(r-1)^p} \left[\left(\frac{\alpha}{2}\right)^{\frac{q}{r}} \left(\frac{q}{r}\right)^{-\frac{q}{rp'}}\right]^{p(1-r)} \\ &= C_p^q \alpha^{-p+\frac{qp(1-r)}{r}} = C_p^q \alpha^{-q}\end{aligned}$$

where

$$C_p := \left[\frac{1}{(r-1)^p} 2^{p-\frac{qp(1-r)}{r}} \left(\frac{q}{r}\right)^{-\frac{qp(1-r)}{rp'}}\right]^{\frac{1}{q}} > 0$$

and the final equality is because

$$-p + \frac{qp(1-r)}{r} = -pq\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = -pq \cdot \frac{1}{p} = -q$$

Therefore,

$$\sup_{\alpha > 0} \{\alpha^q \lambda_{Tf}(\alpha)\} \le C_p^q < \infty$$

and $T_f \in L^{q,\infty}(\Omega)$ with

$$[Tf]_{L^{q,\infty}(\Omega)} = \left[\sup_{\alpha>0} \{\alpha^q \lambda_{Tf}(\alpha)\}\right]^{\frac{1}{q}} \leq C_p$$

So it proves (i)

(III) To show the strong type, we need Marcinkiewicz Interpolation Theorem. When $p \neq 1$,

$$1 < p < q < \infty, \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1, \quad 1 < p < r' < \infty$$

Choose p_0, p_1 such that

$$1 < p_0 < p < p_1 < r' < \infty$$

and define $t \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}$$

then

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 = \frac{1-t}{p_0} + \frac{t}{p_1} + \frac{1}{r} - 1$$

Therefore, define

$$\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r} - 1, \quad \frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{r} - 1$$

and thus

$$1 < p_0 < q_0 < \infty, \quad 1 < p_1 < q_1 < \infty, \quad q_0 < q_1$$

Moreover,

$$\frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

By (II), T is both weak (p_0, q_0)-type and weak (p_1, q_1)-type. So by Marcinkiewicz Interpolation Theorem, $T\left(L^p(\Omega)\right) \subset L^q(\Omega)$ with

$$\|Tf\|_{L^q(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

□

Theorem 2.5.3 (Hardy-Littlewood-Sobolev's Inequality). *Let $0 < a < N$ and p, q satisfy*

$$1 \leq p < q < \infty, \quad \frac{1}{q} = \frac{1}{p} + \frac{a}{N} - 1$$

(1) For $f \in L^p(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \frac{|f(y)|}{|x-y|^a} dy < \infty, \quad a.e.\ x \in \mathbb{R}^N$$

(2) Assume $p \neq 1$ $(1 < p < q < \infty)$. Define $T_a \colon L^p(\mathbb{R}^N) \to \mathbb{M}(\mathbb{R}^N)$ as

$$\left(T_a f\right)(x) := \left(|\cdot|^{-a} * f\right)(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^a} dy$$

Then we have $T_a\left(L^p\left(\mathbb{R}^N\right)\right) \subset L^q\left(\mathbb{R}^N\right)$ with a constant $C > 0$ such that

$$\|T_a f\|_{L^q(\mathbb{R}^N)} = \left\||\cdot|^{-a} * f\right\|_{L^q(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)}$$

Proof. For $0 < a < N$ and $1 \leq q < \infty$,

$$\frac{1}{q} = \frac{1}{p} + \frac{a}{N} - 1 = \frac{1}{p} + \frac{1}{N/a} - 1$$

Let $K(x, y) := |x - y|^{-a}$. So the idea is to apply Theorem 2.5.2 by setting $\Omega = \mathbb{R}^N$ and $r = \frac{N}{a} > 1$.
Check: There is an $M > 0$ such that

- for any $x \in \mathbb{R}^N$, $[K(x, \cdot)]_{L^{\frac{N}{a},\infty}(\mathbb{R}^N)} \leq M$, *i.e.* $K(x, \cdot) \in L^{\frac{N}{a},\infty}\left(\mathbb{R}^N\right)$,
- for any $y \in \mathbb{R}^N$, $[K(\cdot, y)]_{L^{\frac{N}{a},\infty}(\mathbb{R}^N)} \leq M$, *i.e.* $K(\cdot, y) \in L^{\frac{N}{a},\infty}\left(\mathbb{R}^N\right)$.

For any $a > 0$ and $x \in \mathbb{R}^N$,

$$\begin{aligned}
\alpha^{\frac{N}{a}} \lambda_{K(x,\cdot)}(\alpha) &= \alpha^{\frac{N}{a}} m_N \left(\left\{y \in \mathbb{R}^N || K(x, y) |> \alpha\right\}\right) \\
&= \alpha^{\frac{N}{a}} m_N \left(\left\{y \in \mathbb{R}^N ||x - y|^{-a} > \alpha\right\}\right) \\
&= \alpha^{\frac{N}{a}} m_N \left(\left\{y \in \mathbb{R}^N ||x - y\Big| < \alpha^{-\frac{1}{a}}\right\}\right) \\
&= \alpha^{\frac{N}{a}} v_N \cdot \left(\alpha^{-\frac{1}{a}}\right)^N = v_N
\end{aligned}$$

where v_N is the volume of unit ball in $\mathbb{R}^N$. Therefore, $K(x, \cdot) \in L^{\frac{N}{a},\infty}\left(\mathbb{R}^N\right)$ with

$$[K(x, \cdot)]_{L^{\frac{N}{a},\infty}(\mathbb{R}^N)} = \left[\sup_{\alpha>0} \left\{\alpha^{\frac{N}{a}} \lambda_{K(x,\cdot)}(\alpha)\right\}\right]^{\frac{a}{N}} = v_N^{\frac{a}{N}}$$

And it is similar for $K(\cdot, y)$. Therefore, by Theorem 2.5.2, we have the result. $\square$

Chapter 3

Sobolev Space

3.1 Sobolev Space $W^{m,p}(\Omega)$

Definition 3.1.1. For $1 \le p \le \infty$ and $m \in \mathbb{N}_0$,

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) \mid \partial^\alpha u \in L^p(\Omega),\ \forall\ |\alpha| \le m\}$$

is called a Sobolev space.

Remark. (1) For $1 \le p \le \infty$, $W^{m,p}(\Omega) \subset L^p(\Omega)$ subspace. And $W^{0,p}(\Omega) = L^p(\Omega)$.

(2) For $m_1 \le m_2$, $W^{m_2,p}(\Omega) \subset W^{m_1,p}(\Omega)$.

(3) $C_c^\infty(\Omega) \subset W^{m,p}(\Omega)$.

Then consider the norm on $W^{m,p}(\Omega)$.

- For $1 \le p < \infty$, $u \in W^{m,p}(\Omega)$

$$\|u\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha|\le m} \|\partial^\alpha u\|_{L^p(\Omega)}^p\right)^{1/p} = \left(\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha u(x)|^p \, dx\right)^{1/p}$$

- For $p = \infty$, $u \in W^{m,\infty}(\Omega)$

$$\|u\|_{W^{m,\infty}(\Omega)} := \sup_{|\alpha|\le m} \|\partial^\alpha u\|_{L^\infty(\Omega)}$$

Note that $\|u\|_{W^{0,p}(\Omega)} = \|u\|_{L^p(\Omega)}$.

Proposition 3.1.1. *For $1 \le p \le \infty$ and $m \in \mathbb{N}_0$, $W^{m,p}(\Omega)$ with $\|\cdot\|_{W^{m,p}(\Omega)}$ is a Banach space.*

Proof. $\|\cdot\|_{W^{m,p}(\Omega)}$ is a norm clearly, so we only need to prove the completeness. And assume $1 \le p \le \infty$ and $m \in \mathbb{N}$ ($m = 0$ is clear.)
Let $\{u_n\}_{n\in\mathbb{N}}$ in $W^{m,p}(\Omega)$ be Cauchy in $\|\cdot\|_{W^{m,p}(\Omega)}$. Then by the definition, for any α with $|\alpha| \le m$, $\{\partial u_n\}_{n\in\mathbb{N}}$ in $L^p(\Omega)$ is Cauchy. So by the completeness of L^p, there is a v_α such that

$$\|\partial^\alpha u_n - v_\alpha\|_{L^p(\Omega)} \longrightarrow 0$$

Then let $u = v_{(0,\dots,0)}$. So it is sufficient to prove $\partial^\alpha u = v_\alpha$, which is equivalent to proving that for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega u(x)\partial^\alpha \varphi(x)dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\varphi(x)dx$$

First,

$$\int_\Omega u_n(x)\partial^\alpha\varphi(x)dx = (-1)^{|\alpha|}\int_\Omega \partial^\alpha u_n(x)\varphi(x)dx$$

Then because

$$\|u_n - u\|_{L^p(\Omega)} \to 0, \quad \|\partial^\alpha u_n - v_\alpha\|_{L^p(\Omega)} \to 0$$

by Hölder's Inequality, as $n \to \infty$

$$\int_\Omega u(x)\partial^\alpha\varphi(x)dx = (-1)^{|\alpha|}\int_\Omega v_\alpha(x)\varphi(x)dx$$

So $\partial^\alpha u = v_\alpha$ by the density of $C_c^\infty(\Omega)$ in $L^p(\Omega)$. □

Remark. For $1 \le p \le \infty$ and $m_1 \le m_2$, it is clear that for any $u \in W^{m_2,p}(\Omega) \subset W^{m_1,p}(\Omega)$,

$$\|u\|_{W^{m_1,p}(\Omega)} \le \|u\|_{W^{m_2,p}(\Omega)}$$

So $W^{m_2,p}(\Omega) \hookrightarrow W^{m_1,p}(\Omega)$.

When considering $p = 2$, for any $u, v \in W^{m,2}(\Omega)$,

$$\langle u, v\rangle_{W^{m,2}(\Omega)} := \sum_{|\alpha|\le m}\langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2(\Omega)} = \sum_{|\alpha|\le m}\int_\Omega \partial^\alpha u(x)\overline{\partial^\alpha v(x)}dx$$

Theorem 3.1.1. *Let $m \in \mathbb{N}_0$. $\langle\cdot,\cdot\rangle_{W^{m,2}(\Omega)}$ is a inner product on $_{W^{m,2}(\Omega)}$ with*

$$\|u\|_{W^{m,2}(\Omega)} = \sqrt{\langle u, u\rangle_{W^{m,2}(\Omega)}}$$

and thus $\left(W^{m,2}(\Omega), \langle\cdot,\cdot\rangle_{W^{m,2}(\Omega)}\right)$ is a Hilbert space.

Sobolev spaces refine the differentiability of distributions.

Proposition 3.1.2. *Let $1 \le p \le \infty$ and $m \in \mathbb{N}_0$. Let $u \in L^p(\Omega)$. Then the following statements are equivalent.*

(a) $u \in W^{m,p}(\Omega)$.

(b) For any α with $|\alpha| \le m$, there is a $v_\alpha \in L^p(\Omega)$ such that

$$\partial^\alpha T_u = T_{v_\alpha}$$

Remark. When (a) or (b) holds, $v_\alpha = \partial^\alpha u \in L^p(\Omega)$.

Proof. $(a) \Rightarrow (b)$: Assume $u \in W^{m,p}(\Omega)$. There is a $u_\alpha \in L^p(\Omega)$ s.t.

$$\int_\Omega u(x)\partial^\alpha\varphi(x)dx = (-1)^{|\alpha|}\int_\Omega u_\alpha(x)\varphi(x)dx, \forall\varphi \in C_c^\infty(\Omega)$$

By definition of T_u,

$$\langle T_u, \varphi\rangle = \int_\Omega u(x)\varphi(x)dx, \forall\varphi \in \mathcal{D}(\Omega)$$

Then

$$\begin{aligned}\langle \partial^\alpha T_u, \varphi\rangle &= (-1)^{|\alpha|}\langle T_u, \partial^\alpha\varphi\rangle \\ &= (-1)^{|\alpha|}\int_\Omega u(x)\partial^\alpha\varphi(x)dx \\ &= \int_\Omega u_\alpha(x)\varphi(x)dx \\ &= \langle T_{u_\alpha}, \varphi\rangle, \quad \forall\varphi \in \mathcal{D}(\Omega)\end{aligned}$$

Therefore, $v_\alpha = u_\alpha \in L^p(\Omega)$ and $\partial^\alpha T_u = T_{v_\alpha}$.
$(b) \Rightarrow (a)$: That is

$$\begin{aligned}\langle \partial^\alpha T_u, \varphi\rangle &= (-1)^{|\alpha|}\int_\Omega u(x)\partial^\alpha \varphi(x)dx \\ &= \langle T_{v_\alpha}, \varphi\rangle \\ &= \int_\Omega v_\alpha(x)\varphi(x)dx\end{aligned}$$

Then

$$\int_\Omega u(x)\partial^\alpha\varphi(x)dx = (-1)^{|\alpha|}\int_\Omega v_\alpha(x)\varphi(x)dx$$

Thus, $v_\alpha = u_\alpha$. Since $v_\alpha \in L^p(\Omega), u_\alpha \in L^p(\Omega)$. So $u \in W^{m,p}(\Omega)$. □

Let $1 \le p \le \infty$ and $m \in \mathbb{N}_0$. By $C_c^\infty(\Omega) \subset W^{m,p}(\Omega)$,

$$W_c^{m,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}$$

Theorem 3.1.2. *For $1 \le p < \infty$ and $m \in \mathbb{N}_0$,*

$$W_c^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$$

i.e. *$C_c^\infty(\mathbb{R}^N)$ is dense in $W^{m,p}(\mathbb{R}^N)$.*

The idea is to choose a mollifier (summability kernel). Let $\eta \in C_c^\infty(\mathbb{R}^N)$ with

- $\eta(x) \ge 0$ for all $x \in \mathbb{R}^N$,
- $\operatorname{supp}\eta \subset \{x \in \mathbb{R}^N \mid |x| \le 1\}$,
- $\int_{\mathbb{R}^N} \eta(x)dx = 1$.

Then for $\varepsilon > 0$, define η_ε as

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^N}\eta\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N$$

Then it satisfies

- $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\eta_\varepsilon \subset \{x \in \mathbb{R}^N \mid |x| \le \varepsilon\}$,
- $\eta_\varepsilon(x) \ge 0$ for all $x \in \mathbb{R}^N$,
- $\int_{\mathbb{R}^N} \eta_\varepsilon(x)dx = 1$.

and it is called a mollifier (summability kernel), because for any $u \in L^p(\varphi)$ $(1 \le p < \infty)$, $\eta_\varepsilon * u \in C^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with

$$\partial^\alpha\left(\eta_\varepsilon * w\right)(x) = \int_{\mathbb{R}^N} \partial^\alpha(\eta_\varepsilon)(x-y)w(y)dy$$

and

$$\lim_{\varepsilon \to +0} \|\eta_\varepsilon * u - u\|_{L^p(\mathbb{R}^N)} = 0$$

where the proof is in Theorem 1.2.1.

Proof of Theorem 3.1.2. Assume $1 \le p < \infty$ and $m \in \mathbb{N}$.

(i) **Check:** $C^{\infty}\left(\mathbb{R}^N\right) \cap W^{m,p}\left(\mathbb{R}^N\right)$ is dense in $W^{m,p}\left(\mathbb{R}^N\right)$.

Choose a mollifier η_ε. Let $u \in W^{m,p}\left(\mathbb{R}^N\right) \subset L^p\left(\mathbb{R}^N\right)$. Then $\partial^\alpha u \in L^p\left(\mathbb{R}^N\right)$. By Proposition 1.4.3,

$$\partial^\alpha\left(\eta_\varepsilon * u\right) = \partial^\alpha\left(\eta_\varepsilon\right) * u = \eta_\varepsilon * \left(\partial^\alpha u\right)$$

which mean $\eta_\varepsilon * u \in C^{\infty}\left(\mathbb{R}^N\right) \cap W^{m,p}\left(\mathbb{R}^N\right)$ and

$$\partial^\alpha\left(\eta_\varepsilon * u\right) \to \partial^\alpha u$$

in L^p. So

$$\|\eta_\varepsilon * u - u\|_{W^{m,p}(\mathbb{R}^N)} \longrightarrow 0$$

(ii) **Check:** $C_c^{\infty}\left(\mathbb{R}^N\right)$ is dense in $W^{m,p}\left(\mathbb{R}^N\right)$.

Let $u \in W^{m,p}\left(\mathbb{R}^N\right)$ and fix any $\gamma > 0$. By (i), there is a $v \in C^{\infty}\left(\mathbb{R}^N\right) \cap W^{m,p}\left(\mathbb{R}^N\right)$ such that

$$\|u - v\|_{W^{m,p}(\mathbb{R}^N)} < \frac{\gamma}{2}$$

So it is sufficient to prove that there is a sequence $\{v_n\}_{n\in\mathbb{N}}$ in $C_c^{\infty}(\mathbb{R}^N)$ such that

$$\|v_n - v\|_{W^{m,p}(\mathbb{R}^N)} \to 0, \quad \text{as } n \to \infty$$

Choose $\rho \in C_c^{\infty}(\mathbb{R}^N)$ such that $0 \leq \rho(x) \leq 1$, $\rho(x) = 1$ for $|x| \leq 1$, and $\rho(x) = 0$ for $|x| \geq 2$. Then define

$$v_n(x) := \rho\left(\frac{x}{n}\right) v(x)$$

for all $n \in \mathbb{N}$. Clearly, $v_n \in C_c^{\infty}(\mathbb{R}^N)$. First, for any $x \in \mathbb{R}^N$,

$$|v_n(x) - v(x)|^p = \left|\left\{\rho\left(\frac{x}{n}\right) - 1\right\} v(x)\right|^p \longrightarrow 0, \quad \text{as } n \to \infty$$

Moreover, $|v_n(x) - v(x)|^p \leq |v(x)|^p$. So by DCT,

$$\|v_n - v\|_{L^p(\mathbb{R}^N)} \to 0, \quad \text{as } n \to \infty$$

Let α with $1 \leq |\alpha| \leq m$. Next, we need to consider $\|\partial^\alpha v_n - \partial^\alpha v\|_{L^p(\mathbb{R}^N)}$. For any β with $\beta \leq \alpha$, denote

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

Then we have

$$\begin{aligned}
\partial^\alpha v_n(x) - \partial^\alpha v(x) &= \partial^\alpha\left\{\rho\left(\frac{x}{n}\right) v(x)\right\} - \partial^\alpha v(x) \\
&= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta\left\{\rho\left(\frac{x}{n}\right)\right\} \partial^{\alpha-\beta} v(x) - \partial^\alpha v(x) \\
&= \left\{\rho\left(\frac{x}{n}\right) - 1\right\} \partial^\alpha v(x) \\
&\quad + \sum_{\beta\leq\alpha, |\beta|\neq 0} \binom{\alpha}{\beta} \frac{1}{n^{|\beta|}} \left(\partial^\beta \rho\right)\left(\frac{x}{n}\right) \partial^{\alpha-\beta} v(x)
\end{aligned}$$

For the first term, because $\partial^\alpha v \in L^p(\mathbb{R}^N)$, similarly we have

$$\left\|\left\{\rho\left(\frac{\cdot}{n}\right) - 1\right\} \partial^\alpha v\right\|_{L^p(\mathbb{R}^N)} \longrightarrow 0, \quad \text{as } n \to \infty$$

For the second term, there are $C, C' > 0$ such that

$$\begin{aligned}
&\left\| \sum_{\beta \le \alpha, |\beta| \neq 0} \binom{\alpha}{\beta} \frac{1}{n^{|\beta|}} \left(\partial^\beta \rho\right) \left(\frac{\cdot}{n}\right) \partial^{\alpha-\beta} v \right\|_{L^p(\mathbb{R}^N)} \\
&\le \frac{C}{n} \left\{ \sup_{|\beta| \le m} \left\| \left(\partial^\beta \rho\right) \left(\frac{\cdot}{n}\right) \right\|_{L^\infty(\mathbb{R}^N)} \right\} \|v\|_{W^{m,p}(\mathbb{R}^N)} \\
&\le \frac{C'}{n} \|\rho\|_{W^{m,\infty}(\mathbb{R}^N)} \|v\|_{W^{m,p}(\mathbb{R}^N)} \longrightarrow 0, \quad (n \to \infty)
\end{aligned}$$

Therefore,

$$\|\partial^\alpha v_n - \partial^\alpha v\|_{L^p(\mathbb{R}^N)} \to 0, \quad \text{as } n \to \infty$$

and thus $\|v_n - v\|_{W^{m,p}(\mathbb{R}^N)} \to 0$.

□

3.2 Sobolev Space $H^s(\mathbb{R}^N)$

Note that for $1 \le p \le \infty$,

$$C_c^\infty\left(\mathbb{R}^N\right) \subset \mathcal{S}\left(\mathbb{R}^N\right) \subset W^{m,p}\left(\mathbb{R}^N\right)$$

In particular, for $1 \le p < \infty$, $\mathcal{S}$ is dense in $W^{m,p}$.
Besides, we have known for $1 \le p \le \infty$

$$\mathcal{S}\left(\mathbb{R}^N\right) \subset L^p\left(\mathbb{R}^N\right) \subset \mathcal{S}'\left(\mathbb{R}^N\right)$$

Definition 3.2.1. Let $s \in \mathbb{R}$. Considering a subspace of $\mathcal{S}'(\mathbb{R}^N)$,

$$\begin{aligned}
H^s(\mathbb{R}^N) &:= \left\{u \in \mathcal{S}'(\mathbb{R}^N) \mid \widehat{u} \in \mathbb{R}^{\mathbb{N}},\ (1+|\cdot|)^{s/2}\widehat{u} \in L^2(\mathbb{R}^N)\right\} \\
&= \left\{\mathcal{F}^{-1}(g) \mid g \in \mathbb{R}^{\mathbb{N}},\ (1+|\cdot|)^{s/2} g \in L^2(\mathbb{R}^N)\right\}
\end{aligned}$$

Remark. Recall when $u \in \mathcal{S}'$, there is a polynomial growth f ($|f(x)| \le C(1+|x|^k)$) such that $u = \partial^\alpha f$, and so $\widehat{u} = i^{|\alpha|}(\cdot)^\alpha \widehat{f} \in C^\infty$. So $u \in H^s$ when

$$i^{|\alpha|}(\cdot)^\alpha (1+|\cdot|)^{s/2} \widehat{f} \in L^2$$

Remark. (1) For $s_1, s_2 \in \mathbb{R}$ with $s_1 \le s_2$, $H^{s_2}(\mathbb{R}^N) \subset H^{s_1}(\mathbb{R}^N)$.

(2) For any $s \in \mathbb{R}$, $\mathcal{S}\left(\mathbb{R}^N\right) \subset H^s\left(\mathbb{R}^N\right)$.

Remark. By the properties of the Fourier transform on $L^2(\mathbb{R}^N)$ (Plancherel's Theorem), we have the following results.

(1) $H^0(\mathbb{R}^N) = L^2(\mathbb{R}^N)$.

(2) If $s \ge 0$, $H^s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$,

$$H^s\left(\mathbb{R}^N\right) = \left\{u \in L^2\left(\mathbb{R}^N\right) \mid \left(1+|\cdot|^2\right)^{s/2} \widehat{u} \in L^2\left(\mathbb{R}^N\right)\right\}$$

Remark. Note that for $s<0$, $H^s(\mathbb{R}^N)$ contains pure distributions, *i.e.* non-functions on $\mathbb{R}^N$. For example, consider $\delta \in \mathcal{S}'$. Then because $\widehat{\delta}=1$,

$$\left(1+|\cdot|^2\right)^{s/2}\widehat{\delta}=\left(1+|\cdot|^2\right)^{s/2}\in L^2$$

if and only if $s<-\frac{N}{2}$
Define an inner product on $H^s(\mathbb{R}^N)$ as

$$\begin{aligned}\langle u,v\rangle_{H^s(\mathbb{R}^N)} &:= \left(\left(1+|\cdot|^2\right)^{s/2}\widehat{u},\left(1+|\cdot|^2\right)^{s/2}\widehat{v}\right)_{L^2(\mathbb{R}^N)}\\ &= \int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^s\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi\end{aligned}$$

which induces a norm

$$\begin{aligned}\|u\|_{H^s(\mathbb{R}^N)} &:= \sqrt{\langle u,u\rangle_{H^s(\mathbb{R}^N)}} = \left\|\left(1+|\cdot|^2\right)^{s/2}\widehat{u}\right\|_{L^2(\mathbb{R}^N)}\\ &= \left(\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^s|\widehat{u}(\xi)|^2d\xi\right)^{1/2}\end{aligned}$$

Theorem 3.2.1. *Let $s\in\mathbb{R}$.*

(1) $\langle\cdot,\cdot\rangle_{H^s(\mathbb{R}^N)}$ is an inner product on $H^s\left(\mathbb{R}^N\right)$ and thus $\|\cdot\|_{H^s(\mathbb{R}^N)}$ is a norm.

(2) $\|\cdot\|_{H^s(\mathbb{R}^N)}$ with $\langle\cdot,\cdot\rangle_{H^s(\mathbb{R}^N)}$ is a Hilbert space.

Proof. Only need to prove the completeness. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $H^s(\mathbb{R}^N)$, that is

$$\|u_m-u_n\|_{H^s(\mathbb{R}^N)}=\left\|\left(1+|\cdot|^2\right)^{s/2}\widehat{u}_m-\left(1+|\cdot|^2\right)^{s/2}\widehat{u}_n\right\|_{L^2(\mathbb{R}^N)}\to 0,\quad m,n\to\infty$$

So $\left\{\left(1+|\cdot|^2\right)^{s/2}\widehat{u}_n\right\}_{n\in\mathbb{N}}$ is Cauchy in $L^2(\mathbb{R}^N)$. There is an $f\in L^2(\mathbb{R}^N)$ such that

$$\left\|\left(1+|\cdot|^2\right)^{s/2}\widehat{u}_n-f\right\|_{L^2(\mathbb{R}^N)}\to 0$$

Then let

$$u:=\mathcal{F}^{-1}\left[\left(1+|\cdot|^2\right)^{-s/2}f\right]$$

First, $f\in L^2(\mathbb{R}^N)$ implies there is $C>0$ and $N\in\mathbb{N}$ such that

$$\left|\left(1+|\cdot|^2\right)^{-s/2}f\right|\le C(1+|\cdot|)^N$$

and thus $\left(1+|\cdot|^2\right)^{-s/2}f\in\mathcal{S}'(\mathbb{R}^N)$. By the bijectivity of $\mathcal{F}$ on $\mathcal{S}'$, u is well-defined and in $\mathcal{S}'$. Moreover, because

$$\left(1+|\cdot|^2\right)^{s/2}\widehat{u}=f\in L^2\left(\mathbb{R}^N\right)$$

$u\in H^s(\mathbb{R}^N)$ and

$$\begin{aligned}\|u_n-u\|_{H^s(\mathbb{R}^N)} &= \left\|\left(1+|\cdot|^2\right)^{s/2}\widehat{u}_n-\left(1+|\cdot|^2\right)^{s/2}\widehat{u}\right\|_{L^2(\mathbb{R}^N)}\\ &= \left\|\left(1+|\cdot|^2\right)^{s/2}\widehat{u}_n-f\right\|_{L^2(\mathbb{R}^N)}\to 0,\end{aligned}$$

So $H^s(\mathbb{R}^N)$ is complete. $\square$

Remark. For $s_1 \leq s_2$, $u \in H^{s_2}(\mathbb{R}^N) \subset H^{s_1}(\mathbb{R}^N)$ with

$$\|u\|_{H^{s_1}(\mathbb{R}^N)} \leq \|u\|_{H^{s_2}(\mathbb{R}^N)}$$

So $H^{s_2}\left(\mathbb{R}^N\right) \hookrightarrow H^{s_1}\left(\mathbb{R}^N\right)$.

Proposition 3.2.1. *Let $s \in \mathbb{R}$. $\mathcal{S}(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$.*

Proof. For any $u \in H^s\mathbb{R}^N$, $(1+|\cdot|^2)^{s/2}\,\widehat{u} \in L^2\left(\mathbb{R}^N\right)$. Because $\mathcal{S}(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, there is a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R}^N)$ SUCH THAT

$$\left\|\left(1+|\cdot|^2\right)^{s/2}\widehat{u}-\varphi_n\right\|_{L^2(\mathbb{R}^N)} \to 0$$

Similarly as above

$$v_n(x) := \mathcal{F}^{-1}\left[\left(1+|\cdot|^2\right)^{-s/2}\varphi_n\right](x) \in H^s(\mathbb{R}^N)$$

well-defined and $\varphi_n(\xi) = (1+|\xi|^2)^{s/2}\,\widehat{v}_n(\xi)$. Therefore,

$$\begin{aligned}\|u-v_n\|_{H^s(\mathbb{R}^N)} &= \left\|\left(1+|\cdot|^2\right)^{s/2}\widehat{u}-\left(1+|\cdot|^2\right)^{s/2}\widehat{v}_n\right\|_{L^2(\mathbb{R}^N)}\\ &= \left\|\left(1+|\cdot|^2\right)^{s/2}\widehat{u}-\varphi_n\right\|_{L^2(\mathbb{R}^N)} \to 0,\end{aligned}$$

So $\mathcal{S}(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$. □

Theorem 3.2.2. *Let $m \in \mathbb{N}_0$.*

(1) $W^{m,2}(\mathbb{R}^N) = H^m(\mathbb{R}^N)$.

(2) There are $C_1, C_2 > 0$ such that for any $u \in H^m(\mathbb{R}^N) = W^{m,2}(\mathbb{R}^N)$,

$$C_1\|u\|_{H^m(\mathbb{R}^N)} \leq \|u\|_{W^{m,2}(\mathbb{R}^N)} \leq C_2\|u\|_{H^m(\mathbb{R}^N)}$$

which means these two norms are equivalent. In other words, they induce the same topology.

Proof. When $m=0$, we have $W^{0,2}\left(\mathbb{R}^N\right) = L^2\left(\mathbb{R}^N\right) = H^0\left(\mathbb{R}^N\right)$ and for any $u \in L^2(\mathbb{R}^N)$,

$$\|u\|_{W^{0,2}(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} = \|u\|_{H^0(\mathbb{R}^N)}$$

So in the following, we assume $m \in \mathbb{N}$.

(I) **Check:** There are $C_1, C_2 > 0$ such that for any $\xi \in \mathbb{R}^N$,

$$C_1^2\left(1+|\xi|^2\right)^m \leq \sum_{|\alpha|\leq m}|\xi^\alpha|^2 \leq C_2^2\left(1+|\xi|^2\right)^m$$

First, consider the inequality in the RHS. Let $|\alpha| \leq m$. If $|\alpha| = 0$, then $|\xi^\alpha|^2 = 1 \leq (1+|\xi|^2)^m$. If $0 < |\alpha| \leq m$, then for any $\xi \in \mathbb{R}^N$,

$$\begin{aligned}|\xi^\alpha|^2 &= |\xi_1|^{2\alpha_1}\cdots|\xi_N|^{2\alpha_N} \leq |\xi|^{2\alpha_1}\cdots|\xi|^{2\alpha_N} = |\xi|^{2(\alpha_1+\cdots+\alpha_N)}\\ &= |\xi|^{2|\alpha|} \leq 1+|\xi|^{2m} \leq \left(1+|\xi|^2\right)^m\end{aligned}$$

Therefore there is a $C_2 > 0$ such that

$$\sum_{|\alpha|\leq m}|\xi^\alpha|^2 \leq C_2^2\left(1+|\xi|^2\right)^m$$

Next, consider the inequality in LHS. Let $|\alpha| \le m$. There is a $K_\alpha > 0$ such that for any $\xi \in \mathbb{R}^N$,

$$\left(1+|\xi|^2\right)^m = \left(1+\xi_1^2+\cdots+\xi_N^2\right)^m \le \sum_{|\alpha|\le m} K_\alpha \left|\xi^\alpha\right|^2 \le \left(\max_{|\alpha|\le m} K_\alpha\right) \sum_{|\alpha|\le m} \left|\xi^\alpha\right|^2$$

Therefore, let $C_1 := \left(\max_{|\alpha|\le m} K_\alpha\right)^{-1/2}$ and we have

$$C_1^2 \left(1+|\xi|^2\right)^m \le \sum_{|\alpha|\le m} \left|\xi^\alpha\right|^2$$

(II) **Check:** For any $\varphi \in \mathcal{S}$ (note that $\mathcal{S} \subset H^m\left(\mathbb{R}^N\right) = W^{m,2}\left(\mathbb{R}^N\right)$),

$$C_1\|\varphi\|_{H^m(\mathbb{R}^N)} \le \|\varphi\|_{W^{m,2}(\mathbb{R}^N)} \le C_2\|\varphi\|_{H^m(\mathbb{R}^N)}$$

By (I),

$$C_1^2 \left(1+|\xi|^2\right)^m |\widehat{\varphi}(\xi)|^2 \le \sum_{|\alpha|\le m} \left|\xi^\alpha\right|^2 |\widehat{\varphi}(\xi)|^2 \le C_2^2 \left(1+|\xi|^2\right)^m |\widehat{\varphi}(\xi)|^2$$

Because $\left|\xi^\alpha\right|^2 |\widehat{\varphi}(\xi)|^2 = \left|\xi^\alpha \widehat{\varphi}(\xi)\right|^2 = \left|\mathcal{F}\left[\partial_x^\alpha \varphi\right](\xi)\right|^2$,

$$C_1^2 \left(1+|\xi|^2\right)^m |\widehat{\varphi}(\xi)|^2 \le \sum_{|\alpha|\le m} \left|\mathcal{F}\left[\partial_x^\alpha \varphi\right](\xi)\right|^2 \le C_2^2 \left(1+|\xi|^2\right)^m |\widehat{\varphi}(\xi)|^2$$

Therefore,

$$C_1^2 \left\|\left(1+|\cdot|^2\right)^{m/2} \widehat{\varphi}\right\|_{L^2(\mathbb{R}^N)}^2 \le \sum_{|\alpha|\le m} \left\|\mathcal{F}\left[\partial_x^\alpha \varphi\right]\right\|_{L^2(\mathbb{R}^N)}^2 \le C_2^2 \left\|\left(1+|\cdot|^2\right)^{m/2} \widehat{\varphi}\right\|_{L^2(\mathbb{R}^N)}^2$$

Then by Plancherel's Theorem,

$$C_1^2 \left\|\left(1+|\cdot|^2\right)^{m/2} \widehat{\varphi}\right\|_{L^2(\mathbb{R}^N)}^2 \le \sum_{|\alpha|\le m} \left\|\partial_x^\alpha \varphi\right\|_{L^2(\mathbb{R}^N)}^2 \le C_2^2 \left\|\left(1+|\cdot|^2\right)^{m/2} \widehat{\varphi}\right\|_{L^2(\mathbb{R}^N)}^2$$

and so

$$C_1\|\varphi\|_{H^m(\mathbb{R}^N)} \le \|\varphi\|_{W^{m,2}(\mathbb{R}^N)} \le C_2\|\varphi\|_{H^m(\mathbb{R}^N)}$$

(III) **Check:** For any $u \in W^{m,2}(\mathbb{R}^N)$, $u \in H^m(\mathbb{R}^N)$ with

$$C_1\|u\|_{H^m(\mathbb{R}^N)} \le \|u\|_{W^{m,2}(\mathbb{R}^N)}$$

In particular, $W^{m,2}\left(\mathbb{R}^N\right) \subset H^m\left(\mathbb{R}^N\right)$.
First, because $\mathcal{S}(\mathbb{R}^N)$ is dense in $W^{m,2}(\mathbb{R}^N)$, there is a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ such that

$$\|\varphi_n - u\|_{W^{m,2}(\mathbb{R}^N)} \to 0$$

and in particular, $\|\varphi_n - u\|_{L^2(\mathbb{R}^N)} \to 0$. Moreover, by (II),

$$C_1 \left\|\varphi_n\right\|_{H^m(\mathbb{R}^N)} \le \left\|\varphi_n\right\|_{W^{m,2}(\mathbb{R}^N)}$$

Because $\{\varphi_n\}_{n\in\mathbb{N}}$ is Cauchy in $W^{m,2}\left(\mathbb{R}^N\right)$ and

$$C_1 \left\|\varphi_k - \varphi_l\right\|_{H^m(\mathbb{R}^N)} \le \left\|\varphi_k - \varphi_l\right\|_{W^{m,2}(\mathbb{R}^N)} s \to 0, \quad \text{as } k, l \to \infty$$

$\{\varphi_n\}_{n\in\mathbb{N}}$ is Cauchy in $H^m\left(\mathbb{R}^N\right)$. So there is a $v \in H^m\left(\mathbb{R}^N\right)$ such that

$$\|\varphi_n - v\|_{H^m(\mathbb{R}^N)} \to 0$$

and in particular, $\|\varphi_n - v\|_{L^2(\mathbb{R}^N)} \to 0$ by Plancherel's Theorem. Therefore, $u = v$ in $L^2(\mathbb{R}^N)$. And by $v \in H^m\left(\mathbb{R}^N\right)$, $u \in H^m\left(\mathbb{R}^N\right)$ and $\|\varphi_n - u\|_{H^m(\mathbb{R}^N)} \to 0$. Therefore, as $n \to \infty$,

$$C_1 \|\varphi_n\|_{H^m(\mathbb{R}^N)} \le \|\varphi_n\|_{W^{m,2}(\mathbb{R}^N)} \ \Rightarrow\ C_1\|u\|_{H^m(\mathbb{R}^N)} \le \|u\|_{W^{m,2}(\mathbb{R}^N)}$$

(IV) **Check:** For any $u \in H^m(\mathbb{R}^N)$, $u \in W^{m,2}(\mathbb{R}^N)$ with

$$\|u\|_{W^{m,2}(\mathbb{R}^N)} \le C_2\|u\|_{H^m(\mathbb{R}^N)}$$

In particular, $H^m\left(\mathbb{R}^N\right) \subset W^{m,2}\left(\mathbb{R}^N\right)$.

The proof is as similar as the above by the density of $\mathcal{S}(\mathbb{R}^N)$ in $H^m(\mathbb{R}^n)$.

□

Theorem 3.2.3. *Let $s \in \mathbb{R}$. $C_c^\infty(\mathbb{R}^N)$ is dense in $H^2(\mathbb{R}^N)$.*

Proof. Let $u \in H^s(\mathbb{R}^N)$ and $\varepsilon > 0$. By the density of $\mathcal{S}(\mathbb{R}^N)$ in $H^s(\mathbb{R}^n)$, there is a $w \in \mathcal{S}(\mathbb{R}^N)$ such that

$$\|u - w\|_{H^s(\mathbb{R}^N)} < \frac{\varepsilon}{2}$$

Choosing a nonnegative integer $m \ge s$. Then we have known $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{m,2}(\mathbb{R}^N)$ and so dense in $H^m(\mathbb{R}^N)$ by above theorem. There is a $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|w - \varphi\|_{H^m(\mathbb{R}^N)} < \frac{\varepsilon}{2}$$

Therefore,

$$\begin{aligned}\|u - \varphi\|_{H^s(\mathbb{R}^N)} &\le \|u - w\|_{H^s(\mathbb{R}^N)} + \|w - \varphi\|_{H^s(\mathbb{R}^N)} \\ &\le \|u - w\|_{H^s(\mathbb{R}^N)} + \|w - \varphi\|_{H^m(\mathbb{R}^N)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

□

Note that for $s \in \mathbb{R}$ and $1 \le p \le \infty$, we can also define

$$H^{s,p}\left(\mathbb{R}^N\right) := \left\{u \in \mathcal{S}'\left(\mathbb{R}^N\right) \mid \mathcal{F}^{-1}\left[\left(1 + |\cdot|^2\right)^{s/2}\widehat{u}\right] \in L^p\left(\mathbb{R}^N\right)\right\}$$

with the norm defined as

$$\|u\|_{H^{s,p}(\mathbb{R}^N)} := \left\|\mathcal{F}^{-1}\left[\left(1 + |\cdot|^2\right)^{s/2}\widehat{u}\right]\right\|_{L^p(\mathbb{R}^N)}$$

- It is not difficult to see $\mathcal{S}(\mathbb{R}^N) \subset H^{s,p}(\mathbb{R}^N)$ for any $s \in \mathbb{R}$ and $1 \le p \le \infty$.
- When $p = 2$, by Plancherel's Theorem, $H^s(\mathbb{R}^N) = H^{s,2}(\mathbb{R}^N)$ with $\|\cdot\|_{H^s(\mathbb{R}^N)} = \|\cdot\|_{H^{s,2}(\mathbb{R}^N)}$.
- When $s = 0$, $H^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ with $\|\cdot\|_{H^{0,p}(\mathbb{R}^N)} = \|\cdot\|_{L^p(\mathbb{R}^N)}$.
- For s_1, s_2 with $s_1 \le s_2$, we have

  $$\|u\|_{H^{s_1,p}(\mathbb{R}^N)} \le \|u\|_{H^{s_2,p}(\mathbb{R}^N)}, \quad \forall\ u \in H^{s_2,p}\left(\mathbb{R}^N\right)$$

  That is $H^{s_2,p}\left(\mathbb{R}^N\right) \subset H^{s_1,p}\left(\mathbb{R}^N\right)$
- For any $s \in \mathbb{R}^N$ and $1 \le p < \infty$, $\mathcal{S}(\mathbb{R}^N)$ is dense in $H^{s,p}\left(\mathbb{R}^N\right)$.

3.3 Sobolev Embedding Theorem

In this section, let $s > 0$ and consider the relationship between $H^s(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$.

Theorem 3.3.1 (Sobolev Embedding Theorem I)**.** *Let $s > \frac{N}{2}$.*

(1) $H^s(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$. Moreover, there is a $C > 0$ such that for any $u \in H^s(\mathbb{R}^N)$,

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\|u\|_{H^s(\mathbb{R}^N)}$$

(2) For any $u \in H^s(\mathbb{R}^N)$, there is a bounded and continuous function f_u on $\mathbb{R}^N$ such that $u(x) = f_u(x)$ a.e..

Proof. Let $u \in H^s(\mathbb{R}^N)$. Then by Cauchy-Schwartz Inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} |\widehat{u}(\xi)| d\xi &= \int_{\mathbb{R}^N} \left(1+|\xi|^2\right)^{-s/2} \left\{\left(1+|\xi|^2\right)^{s/2} |\widehat{u}(\xi)|\right\} d\xi \\ &\le \left(\int_{\mathbb{R}^N} \left(1+|\xi|^2\right)^{-s} d\xi\right)^{1/2} \left(\int_{\mathbb{R}^N} \left(1+|\xi|^2\right)^{s} |\widehat{u}(\xi)|^2 d\xi\right)^{1/2}\end{aligned}$$

Because $s > \frac{N}{2}$,

$$C_0^2 := \int_{\mathbb{R}^N} \left(1+|\xi|^2\right)^{-s} d\xi < \infty$$

So we have $\widehat{u} \in L^1(\mathbb{R}^N)$ with

$$\|\widehat{u}\|_{L^1(\mathbb{R}^N)} \le C_0 \|u\|_{H^s(\mathbb{R}^N)}$$

Besides, $u \in H^s(\mathbb{R}^N)$ implies $(1+|\cdot|^2)^{s/2}\,\widehat{u} \in L^2(\mathbb{R}^N)$. Because $s > 0$, $\widehat{u} \in L^2(\mathbb{R}^N)$. So $\widehat{u} \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Then by the inverse formula

$$u(x) = \mathcal{F}^{-1}(\widehat{u})(x), \quad a.e.\ x \in \mathbb{R}^N$$

and $\widehat{u} \in L^1(\mathbb{R}^N)$ implies $\mathcal{F}^{-1}(\widehat{u})$ is continuous and bounded. So (2) is proved and $u = \mathcal{F}^{-1}(\widehat{u}) \in L^\infty(\mathbb{R}^N)$. Besides, by Hausdorff-Young's Inequality,

$$\begin{aligned}|u(x)| = \left|\mathcal{F}^{-1}[\widehat{u}](x)\right| &\le \left\|\mathcal{F}^{-1}[\widehat{u}]\right\|_{L^\infty(\mathbb{R}^N)} \\ &\le (2\pi)^{-N/2} \|\widehat{u}\|_{L^1(\mathbb{R}^N)} \le (2\pi)^{-N/2} C_0 \|u\|_{H^s(\mathbb{R}^N)}\end{aligned}$$

So let $C := (2\pi)^{-N/2} C_0$.

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\|u\|_{H^s(\mathbb{R}^N)}$$

□

Corollary 3.3.1. *Let $s > \frac{N}{2}$. Then for any $q \in [2, \infty]$, $H^s(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ with*

$$\|u\|_{L^q(\mathbb{R}^N)} \le C\|u\|_{H^s(\mathbb{R}^N)}$$

for some C.

Proof. First, for $q = 2$, by $s > 0$ and Plancherel's Theorem, we have $H^s \subset L^2$ with

$$\|u\|_{L^2(\mathbb{R}^N)} \le \|u\|_{H^s(\mathbb{R}^N)}$$

For $q = \infty$, by above theorem, $H^s \subset L^\infty$ with a $C_0 > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\|u\|_{H^s(\mathbb{R}^N)}$$

Therefore, $H^s \subset L^2 \cap L^\infty$. So for $2 < q < \infty$, $H^s \subset L^2 \cap L^\infty \subset L^q$. Besides, there is a $\theta \in (0,1)$ such that

$$\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{\infty} = \frac{1-\theta}{2}$$

Then for any $u \in H^s \subset L^2 \cap L^\infty \subset L^q$,

$$\|u\|_{L^q(\mathbb{R}^N)} \leq \|u\|_{L^2(\mathbb{R}^N)}^{1-\theta} \|u\|_{L^\infty(\mathbb{R}^N)}^{\theta} \leq \|u\|_{H^s(\mathbb{R}^N)}^{1-\theta} \left(C_0 \|u\|_{H^s(\mathbb{R}^N)}\right)^\theta = C_0^\theta \|u\|_{H^s(\mathbb{R}^N)}$$

Let $C = C_0^\theta$.

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C \|u\|_{H^s(\mathbb{R}^N)}$$

$\square$

Next, to consider $0 < s < \frac{N}{2}$, let's define the Riesz kernel. For $0 < \alpha < N$, considering $R_\alpha \colon \mathbb{R}^N \to [0, \infty]$

$$R_\alpha(x) := |x|^{-(N-\alpha)}$$

Note that $R_\alpha \in \mathcal{S}'$.

Proposition 3.3.1. *Let $0 < \alpha < N$. Then*

$$\mathcal{F}\left[R_\alpha\right](\xi) = C_{N,\alpha} R_{N-\alpha}(\xi) = C_{N,\alpha} |\xi|^{-\alpha},$$
$$\mathcal{F}^{-1}\left[R_{N-\alpha}\right](x) = C_{N,\alpha}^{-1} R_\alpha(x) = C_{N,\alpha}^{-1} |x|^{-(N-\alpha)},$$

where

$$C_{N,\alpha} = 2^{\alpha - \frac{N}{2}} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{N-\alpha}{2}\right)}$$

Proof. For $t > 0$, define $g_t(\xi) := e^{-\frac{|\xi|^2}{2t}}$, $\xi \in \mathbb{R}^N$. Then $g_t \in \mathcal{S}(\mathbb{R}^N)$ and so $\widehat{g_t} \in \mathcal{S}(\mathbb{R}^N)$ with

$$\widehat{g_t}(x) = t^{\frac{N}{2}} e^{-\frac{t}{2}|x|^2}$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^N)$, by Plancherel's Theorem ($\mathcal{S} \subset L^2(\mathbb{R}^N)$),

$$\int_{\mathbb{R}^N} \widehat{g_t}(x) \widehat{\varphi}(x) dx = \int_{\mathbb{R}^N} g_t(\xi) \varphi(\xi) d\xi$$

and therefore, by multiplying $t^{-\frac{\alpha}{2}-1}$ in the both sides

$$t^{\frac{N-\alpha}{2}-1} \int_{\mathbb{R}^N} e^{-\frac{t}{2}|x|^2} \widehat{\varphi}(x) dx = t^{-\frac{\alpha}{2}-1} \int_{\mathbb{R}^N} e^{-\frac{|\xi|^2}{2t}} \varphi(\xi) d\xi$$

Next, consider the integrals on the both sides. In general, for $p, b > 0$,

$$\int_0^\infty t^{p-1} e^{-bt} dt = b^{-p} \int_0^\infty t^{p-1} e^{-t} dt = b^{-p} \Gamma(p)$$

So for the LHS, by integrating on $(0, \infty)$,

$$\begin{aligned}
&\int_0^\infty t^{\frac{N-\alpha}{2}-1} \left(\int_{\mathbb{R}^N} e^{-\frac{t}{2}|x|^2} |\widehat{\varphi}(x)| dx \right) dt \\
&= \int_{\mathbb{R}^N} \left(\int_0^\infty t^{\frac{N-\alpha}{2}-1} e^{-\frac{t}{2}|x|^2} dt \right) |\widehat{\varphi}(x)| dx
\end{aligned}$$

and because for any $x \in \mathbb{R}^N \setminus \{0\}$,

$$\int_0^\infty t^{\frac{N-\alpha}{2}-1} e^{-\frac{t}{2}|x|^2} dt = \left(\frac{|x|^2}{2}\right)^{-\frac{N-\alpha}{2}} \Gamma\left(\frac{N-\alpha}{2}\right)$$
$$= 2^{\frac{N-\alpha}{2}} \Gamma\left(\frac{N-\alpha}{2}\right) |x|^{-(N-\alpha)} \quad (< \infty)$$

Thus

$$\text{LHS} = \int_{\mathbb{R}^N} \left(\int_0^\infty t^{\frac{N-\alpha}{2}-1} e^{-\frac{t}{2}|x|^2} dt\right) |\widehat{\varphi}(x)| dx$$
$$= 2^{\frac{N-\alpha}{2}} \Gamma\left(\frac{N-\alpha}{2}\right) \int_{\mathbb{R}^N} |x|^{-(N-\alpha)} |\widehat{\varphi}(x)| dx < \infty$$

Similarly, for the RHS,

$$\text{RHS} = \int_0^\infty t^{-\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^N} e^{-\frac{|\xi|^2}{2t}} \varphi(\xi) d\xi\right) dt$$
$$= \int_{\mathbb{R}^N} \left(\int_0^\infty t^{-\frac{\alpha}{2}-1} e^{-\frac{|\xi|^2}{2t}} dt\right) \varphi(\xi) d\xi$$
$$= 2^{\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right) \int_{\mathbb{R}^N} |\xi|^{-\alpha} \varphi(\xi) d\xi$$

Therefore, we have

$$2^{\frac{N-\alpha}{2}} \Gamma\left(\frac{N-\alpha}{2}\right) \int_{\mathbb{R}^N} |x|^{-(N-\alpha)} \widehat{\varphi}(x) dx = 2^{\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right) \int_{\mathbb{R}^N} |\xi|^{-\alpha} \varphi(\xi) d\xi$$

Therefore,

$$\int_{\mathbb{R}^N} \left(C_{N,\alpha} |\xi|^{-\alpha}\right) \varphi(\xi) d\xi = \int_{\mathbb{R}^N} |x|^{-(N-\alpha)} \widehat{\varphi}(x) dx$$

The by the Plancherel's Theorem

$$\mathcal{F}\left[R_\alpha\right](\xi) = \mathcal{F}\left[|\cdot|^{-(N-\alpha)}\right](\xi) = C_{N,\alpha} |\xi|^{-\alpha}$$

and

$$\mathcal{F}^{-1}\left[R_{N-\alpha}\right](x) = \mathcal{F}^{-1}\left[|\cdot|^{-\alpha}\right](x) = C_{N,\alpha}^{-1} |x|^{-(N-\alpha)}$$

□

Theorem 3.3.2 (Sobolev Embedding Theorem II)**.** *Let* $0 < s < \frac{N}{2}$ *and*

$$\frac{1}{q} = \frac{1}{2} - \frac{s}{N} \ \Rightarrow \ q = \frac{2N}{N-2s}$$

Then $H^s(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$, *and there is a* $C > 0$ *such that for any* $u \in H^s(\mathbb{R}^N)$,

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C \|u\|_{H^s(\mathbb{R}^N)}$$

Proof. Let q' be the conjugate index of q,

$$\frac{1}{q'} = 1 - \frac{1}{q} = \frac{1}{2} + \frac{s}{N}$$

Note that $1 < q' < 2$. In the following, for $f, g \in \mathbb{M}\left(\mathbb{R}^N\right)$,

$$\langle f, g \rangle := \int_{\mathbb{R}^N} f(x) g(x) dx$$

Let $u \in H^s(\mathbb{R}^N)$ and

$$v := \mathcal{F}^{-1}\left[\left(1+|\cdot|^2\right)^{s/2}\mathcal{F}[u]\right]$$

Then $v \in L^2(\mathbb{R}^N)$ and

$$u = \mathcal{F}^{-1}\left[\left(1+|\cdot|^2\right)^{-s/2}\mathcal{F}[v]\right]$$

So by the Plancherel's Theorem

$$\begin{aligned}
\|v\|_{L^2(\mathbb{R}^N)} &= \left\|\mathcal{F}^{-1}\left[\left(1+|\cdot|^2\right)^{s/2}\mathcal{F}[u]\right]\right\|_{L^2(\mathbb{R}^N)} \\
&= \left\|\left(1+|\cdot|^2\right)^{s/2}\mathcal{F}[u]\right\|_{L^2(\mathbb{R}^N)} = \|u\|_{H^s(\mathbb{R}^N)}
\end{aligned}$$

For $\varphi \in \mathcal{S}(\mathbb{R}^N)$, note that $u\varphi \in L^1$ $(u, \varphi \in L^2)$. Then by $2s < N$, $|\cdot|^{-s}\mathcal{F}[\bar{\varphi}] \in L^2\left(\mathbb{R}^N\right)$. The by above proposition,

$$\begin{aligned}
|\langle u, \varphi\rangle| &= \left|\left(\mathcal{F}^{-1}\left[\left(1+|\cdot|^2\right)^{-s/2}\mathcal{F}[v]\right], \bar{\varphi}\right)_{L^2(\mathbb{R}^N)}\right| \\
&= \left|\left(v, \mathcal{F}^{-1}\left[\left(1+|\cdot|^2\right)^{-s/2}\mathcal{F}[\bar{\varphi}]\right]\right)_{L^2(\mathbb{R}^N)}\right| \\
&\leq \|v\|_{L^2(\mathbb{R}^N)}\left\|\mathcal{F}^{-1}\left[\left(1+|\cdot|^2\right)^{-s/2}\mathcal{F}[\bar{\varphi}]\right]\right\|_{L^2(\mathbb{R}^N)} \\
&= \|u\|_{H^s(\mathbb{R}^N)}\left\|\left(1+|\cdot|^2\right)^{-s/2}\mathcal{F}[\bar{\varphi}]\right\|_{L^2(\mathbb{R}^N)} \\
&\leq \|u\|_{H^s(\mathbb{R}^N)}\left\||\cdot|^{-s}\mathcal{F}[\bar{\varphi}]\right\|_{L^2(\mathbb{R}^N)} \\
&= \|u\|_{H^s(\mathbb{R}^N)}\left\|\mathcal{F}^{-1}\left[|\cdot|^{-s}\cdot\mathcal{F}[\bar{\varphi}]\right]\right\|_{L^2(\mathbb{R}^N)} \\
&= (2\pi)^{-N/2}C_{N,s}^{-1}\|u\|_{H^s(\mathbb{R}^N)}\left\||\cdot|^{-(N-s)} * \bar{\varphi}\right\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

To consider $|\cdot|^{-(N-s)} * \bar{\varphi}$, note that

$$0 < N-s < N, \quad 1 < q' < 2, \quad \frac{1}{2} = \frac{1}{q'} - \frac{s}{N} = \frac{1}{q'} + \frac{N-s}{N} - 1$$

Then by the Hardy-Littlewood-Sobolev's Inequality, $|\cdot|^{-(N-s)} * \bar{\varphi} \in L^2(\mathbb{R}^N)$ and there is a $C' > 0$ such that

$$\left\||\cdot|^{-(N-s)} * \bar{\varphi}\right\|_{L^2(\mathbb{R}^N)} \leq C'\|\bar{\varphi}\|_{L^{q'}(\mathbb{R}^N)} = C'\|\varphi\|_{L^{q'}(\mathbb{R}^N)}$$

Therefore,

$$\begin{aligned}
|\langle u, \varphi\rangle| &\leq (2\pi)^{-N/2}C_{N,s}^{-1}\|u\|_{H^s(\mathbb{R}^N)}\left\||\cdot|^{-(N-s)} * \bar{\varphi}\right\|_{L^2(\mathbb{R}^N)} \\
&\leq C\|u\|_{H^s(\mathbb{R}^N)}\|\varphi\|_{L^{q'}(\mathbb{R}^N)}
\end{aligned}$$

for some $C > 0$.
Let $g \in S_0(\mathbb{R}^N)$ with $\|g\|_{L_{q'}(\mathbb{R}^N)} = 1$. Then clearly $u \cdot g \in L^1(\mathbb{R}^N)$ and $g \in L^2\left(\mathbb{R}^N\right) \cap L^{q'}\left(\mathbb{R}^N\right)$ with $1 < q' < 2$. Because $\mathcal{S}(\mathbb{R}^N)$ is dense in $L^2\left(\mathbb{R}^N\right) \cap L^{q'}$, there is a sequence in $\mathcal{S}(\mathbb{R}^N)$ such that

$$\|\varphi_n - g\|_{L^2(\mathbb{R}^N)} \to 0, \quad \|\varphi_n - g\|_{L^{q'}(\mathbb{R}^N)} \to 0$$

And by above there is a $C > 0$ such that

$$|\langle u, \varphi_n\rangle| \leq C\|u\|_{H^s(\mathbb{R}^N)}\|\varphi_n\|_{L^{q'}(\mathbb{R}^N)}$$

Then by taking $n \to \infty$ on the both sides,

$$|\langle u, g\rangle| \leq C\|u\|_{H^s(\mathbb{R}^N)}\|g\|_{L^{q'}(\mathbb{R}^N)} = C\|u\|_{H^s(\mathbb{R}^N)}$$

So by Theorem 2.2.1,

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{H^s(\mathbb{R}^N)}.$$

□

Corollary 3.3.2. *Let $0 < s < \frac{N}{2}$ and $q \in \mathbb{R}$ satisfy*

$$\frac{1}{2} - \frac{2}{N} \leqslant \frac{1}{q} \leqslant \frac{1}{2}$$

Then it can get $H^s\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$.

Proof. Firstly, let $q_0 = \frac{2N}{N-2s}$. By the Sobolev Embedding Theorem and the fact $H^s\left(\mathbb{R}^N\right) \hookrightarrow L^2\left(\mathbb{R}^N\right)$ for any $s > 0$,

$$\begin{aligned} &H^s\left(\mathbb{R}^N\right) \hookrightarrow L^2\left(\mathbb{R}^N\right) \text{ and } \|u\|_{L^2} \leqslant \|u\|_{H^s}, \forall u \in H^s\left(\mathbb{R}^N\right) \\ &H^s\left(\mathbb{R}^N\right) \hookrightarrow L^{q_0}\left(\mathbb{R}^N\right) \text{ and } \|u\|_{L^{q_0}} \leqslant C\|u\|_{H^s}, \forall u \in H^s\left(\mathbb{R}^N\right) \end{aligned}$$

Then $H^s\left(\mathbb{R}^N\right) \subset L^2\left(\mathbb{R}^N\right) \cap L^{q_0}\left(\mathbb{R}^N\right) \subset L^q$, for any $2 \leqslant q \leqslant q_0$. Besides, for this q, let

$$t = \frac{\frac{1}{q} - \frac{1}{2}}{\frac{1}{q_0} - \frac{1}{2}} \in (0,1) \text{ i.e. } \frac{1}{q} = \frac{1-t}{2} + \frac{t}{q_0}$$

in the Reisz-Thorin Interpolation Theorem. So $H^s\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ and

$$\|u\|_{L^q} \leqslant C^t\|u\|_{H^s}$$

□

Chapter 4

Sobolev Inequalities in $\mathbb{R}^n$

Let $\mathbb{R}^n$ with the Lebesgue measure μ_n.

4.1 Sobolev Inequalities

For $f \in C_c^\infty$, we have

$$|f(x)| \leq \frac{1}{2}\int_{-\infty}^{+\infty} |f'(x)|\, dx$$

So the main idea is to control $f \in C_c^\infty(\mathbb{R}^n)$ by its gradient ∇f, that is we want

$$\|f\|_q \leq C\,\|\nabla f\|_p$$

for some $C > 0$. But if replacing f by $f(t\cdot)$, this will imply

$$t^{-n/q}\|f\|_q \leq Ct^{1-n/p}\|\nabla f\|_p$$

As $t \to 0$ and $t \to \infty$, only if

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n} \;\Rightarrow\; q = \frac{np}{n-p}$$

Theorem 4.1.1. *Fix an integer $n \geq 2$ and $1 \leq p < n$ and set $q = \frac{np}{n-p}$. Then there is a constant $C = C(n,p)$ such that*

$$\forall\ f \in C_c^\infty(\mathbb{R}^n), \quad \|f\|_q \leq C\|\nabla f\|_p$$

Proof I. Set

$$F_i(x) = \int_{-\infty}^{+\infty} |\partial_i f\,(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|\, dt,$$

which only dependents on $n-1$ variables. And

$$F_{i,m}(x) = \begin{cases} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\partial_i f(x)|\, dx_1 \ldots dx_m & \text{if } i \leq m \\ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F_i(x) dx_1 \ldots dx_m & \text{if } i > m \end{cases},$$

which only dependents on $n-m$ variables or $n-m-1$ variables, and in particular, for $m = n$,

$$F_{i,n}(x) = \int_{\mathbb{R}^n} |\partial_i f(y)|\, dy$$

By the $\mathbb{R}$ case, we have

$$|f(x)| \leq \frac{1}{2}\int_{-\infty}^{+\infty} |\partial_i f\,(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|\, dt$$

for any $i = 1, 2, \cdots, n$. Then we have

$$|f| \leq \frac{1}{2}(F_1 \dots F_n)^{\frac{1}{n}} \Rightarrow |f|^{\frac{n}{n-1}} \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} (F_1 \dots F_n)^{\frac{1}{n-1}}$$

By Hölder's Inequality,

$$\left|\int f_1 f_2 \dots f_k d\mu\right| \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \dots \|f_k\|_{p_k}$$

for $f_i \in L^{p_i}$, $1 \leq p_i \leq \infty$ with $1/p_1 + \cdots + 1/p_k = 1$. Then by setting $k = n-1, p_1 = \cdots = p_k = n-1$, we have

$$\int \cdots \int |f(x)|^{\frac{n}{n-1}} dx_1 \dots dx_m \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} (F_{1,m}(x) \dots F_{n,m}(x))^{\frac{1}{n-1}}$$

For $m = n$,

$$\|f\|_{\frac{n}{n-1}} \leq \frac{1}{2}\left(\prod_{i=1}^{n} \|\partial_i f\|_1\right)^{\frac{1}{n}}$$

As $\left(\prod_1^n a_i\right)^{1/n} \leq \frac{1}{n}\sum_1^n a_i$,

$$\|f\|_{\frac{n}{n-1}} \leq \frac{1}{2n}\sum_1^n \|\partial_i f(x)\|_1 \, dx \leq \frac{1}{2\sqrt{n}}\|\nabla f\|_1$$

Then we have proved the case of $p = 1$ and $q = \frac{n}{n-1}$.
Next, fix $p > 1$. For any $\alpha > 1$ and $f \in C_c^\infty(\mathbb{R}^n)$, $|f|^\alpha$ is C^1 with compact support and satisfies

$$|\nabla |f|^\alpha| = \alpha |f|^{\alpha-1} |\nabla f|$$

Because C_c^∞ is dense in C^1, there is a sequence $\{f_k\}_{k\in\mathbb{N}}$ in C_c^∞ such that $f_k \to |f|^\alpha$ and $\nabla f_k \to \nabla |f|^\alpha$. Therefore, it can replace f in $\|f\|_{n/(n-1)} \leq C\|\nabla f\|_1$ by $\|f\|^\alpha$, we have

$$\begin{aligned}
\|f\|_{\frac{\alpha n}{n-1}}^\alpha &\leq C\alpha \int |f(x)|^{\alpha-1}|\nabla f(x)| dx \\
&\leq C\alpha \left(\int |f(x)|^{(\alpha-1)p'} dx\right)^{1/p'} \left(\int |\nabla f(x)|^p dx\right)^{1/p}
\end{aligned}$$

where p' is the conjugate of p. If we pick

$$\alpha = \frac{(n-1)p}{n-p} \Rightarrow (\alpha - 1)q = \frac{np}{n-p}$$

then

$$\|f\|_{np/(n-p)}^{(n-1)p/(n-p)} \leq C\frac{(n-1)p}{n-p}\|f\|_{np/(n-p)}^{n(p-1)/(n-p)}\|\nabla f\|_p.$$

and by $(n-1)p/(n-p) - n(p-1)/(n-p) = 1$,

$$\|f\|_{np/(n-p)} \leq C\frac{(n-1)p}{n-p}\|\nabla f\|_p$$

□

4.2 Riesz Potential

Proposition 4.2.1. *For any $f \in C_c^\infty(\mathbb{R}^n)$,*

$$|f(x)| \leq \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|y-x|^{n-1}} dy$$

where ω_{n-1} is the $(n-1)$-dimensional volume of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ ($\omega_{n-1} = n\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$).

Proof. Using the polar coordinate (r, θ) with $r > 0$ and $\theta \in \mathbb{S}^{n-1}$, we have

$$f(x) = -\int_0^\infty \partial_r f(x + r\theta) dr$$

Then integrating along $\mathbb{S}^{n-1}$,

$$\begin{aligned} f(x) &= -\frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_0^\infty \partial_r f(x + r\theta) dr d\theta \\ &= -\frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{\partial_r f(x + r\theta)}{r^{n-1}} r^{n-1} dr d\theta \end{aligned}$$

By setting $y = x + r\theta$, we have $r = |y - x|$ and

$$dy = r^{n-1} dr d\theta, \quad \partial_r f(x + r\theta) = |y - x|^{-1} \sum_{i=1}^n (y_i - x_i) \partial_i f(y)$$

Therefore

$$f(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\langle x - y, \nabla f(y) \rangle}{|y - x|^n} dy$$

In particular,

$$|f(x)| \leq \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|y-x|^{n-1}} dy$$

□

For $0 < \alpha < N$, define the Riesz potential operator (Riesz kernel)

$$(I_\alpha f)(x) = \frac{1}{c_\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|y - x|^\alpha} dy$$

where

$$c_\alpha = 2^{\alpha - \frac{n}{2}} \Gamma(\frac{\alpha}{2}) \Gamma(\frac{n - \alpha}{2})$$

Remark. For the Laplace operator $\Delta = -\sum_i \partial_i f$ and $\beta \in \mathbb{R}$,

$$\Delta^{\frac{\beta}{2}} f := \mathcal{F}^{-1} \left(|\cdot|^\beta \widehat{f} \right)$$

Then by Proposition 3.3.1, we have known

$$I_\alpha f = \Delta^{-\frac{\alpha}{2}} f$$

For $0 < \alpha < n$, by setting $a = n - \alpha$ in Hardy-Littlewood-Sobolev's Inequality (Theorem 2.5.3), then for any $1 < p < \infty$ and $q = \frac{np}{n-\alpha p}$,

$$\|I_\alpha f\|_q \leq C\|f\|_p$$

Therefore, when $p > 1$ and $q = \frac{np}{n-p}$, by

$$(I_1 \nabla f)(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\nabla f(y)}{|y-x|^{n-1}} dy$$

we have

$$\|f\|_q \leq C_0 \|I_1 \nabla f\|_q \leq C \|\nabla f\|_p$$

for some $C > 0$.

Remark. $a > 0$ implies $n \geq \alpha$. In Sobolev Inequality, $\alpha = 1$ implies $n \geq 2$.

4.3 Different Cases in p

I. The case $p = 1$: Isoperimetry.

Let $\mathbb{B}_n(r)$ and $\mathbb{S}^{n-1}(r)$ be the ball and the sphere of radius r centered at 0 in $\mathbb{R}^n$. Let $\Omega_n = \mu_n(\mathbb{B}_n(1))$ and $\omega_{n-1} = \mathbb{S}^{n-1}(1) = n\Omega_n$. Let Ω be a bounded domain in $\mathbb{R}^n$ having a smooth boundary of given finite $(n-1)$-dimensional measure, that is

$$\mu_{n-1}(\partial\Omega) = \mu_{n-1}\left(\mathbb{S}^{n-1}(r)\right) = \omega_{n-1} r^{n-1}$$

where r is chosen such that above equality is satisfied. Then

$$\mu_n(\Omega) \leq \mu_n\left(\mathbb{B}^n(r)\right) = \Omega_n r^n$$

which is called the isoperimetric inequality, that is

$$[\mu_n(\Omega)]^{(n-1)/n} \leq C_n \mu_{n-1}(\partial\Omega)$$

for

$$C_n = \frac{\Omega_n^{1-1/n}}{\omega_{n-1}} = \frac{[\Gamma((n-1)/2)]^{1/n}}{\sqrt{\pi} n}$$

The isoperimetric inequality is equivalent to Sobolev's Inequality of the case $p = 1$, that is

$$\|f\|_{n/(n-1)} \leq C_n \|\nabla f\|_1$$

Theorem 4.3.1 (Co-area Formula)**.** *For any $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in \mathbb{R}^\ltimes$,*

$$\int g|\nabla f| d\mu_n = \int_{-\infty}^{+\infty} \left(\int_{f(x)=t} g(x) d\mu_{n-1}(x) \right) dt$$

First, Sobolev's Inequality implies the isoperimetric inequality is by choosing a sequence of $f_n \in C_c^\infty$ such that $f_n \to \chi_\Omega$. Then clearly, we have

$$\|f_n\|_{n/(n-1)} \to \mu_n(\Omega)^{(n-1)/n}$$

Moreover, in the co-area formula, by setting $g(x) = 1$ and $f(x) = f_n(x)$,

$$\int_{\mathbb{R}^n} |\nabla f_n| \, dx = \int_{-\infty}^{\infty} \mu_{n-1}\left(\{x \in \mathbb{R}^n : f_n(x) = t\}\right) dt$$

Then $\|\nabla f_n\|_1 \to \mu_{n-1}(\partial\Omega)$. Therefore, by taking limits in Sobolev's Inequality of f_n, we have

$$[\mu_n(\Omega)]^{(n-1)/n} \le C_n \mu_{n-1}(\partial\Omega)$$

Conversely, for $f \in C_c^\infty(\Omega)$ with $f \ge 0$,

$$\begin{aligned}
\int |f(x)|^{n/(n-1)} dx &\le \int_0^\infty \mu_n(\{f > t\})^{(n-1)/n} dt \\
&\le C_n \int_0^\infty \mu_{n-1}(\{f = t\}) dt \\
&= C_n \int |\nabla f| d\mu_n = \|\nabla f\|_1
\end{aligned}$$

where the first inequality is because

$$\begin{aligned}
\left\| \int_0^\infty 1_{\{f(\cdot)>t\}}(t) dt \right\|_{n/(n-1)} &\le \int_0^\infty \left\| 1_{\{f(\cdot)>t\}} \right\|_{n/(n-1)} dt \\
&= \int_0^\infty \mu_n(\{z : f(z) > t\})^{(n-1)/n} dt.
\end{aligned}$$

and by $f(x) = \int_0^\infty \chi_{\{f(x)>t\}}(t) dt$.

II. The case $1 \le p < n$:

Theorem 4.3.2. *For $1 \le p < n$, the Sobolev Inequality*

$$\forall\ f \in C_c^\infty(\mathbb{R}^n), \quad \|f\|_{n/(n-p)} \le C \|\nabla f\|_p$$

holds with $C = C(n,p)$, where

$$C(n,p) = \frac{p-1}{n-p} \left(\frac{n-p}{n(p-1)} \right)^{1/q} \left(\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}} \right)^{1/n}$$

for $1 < p < \infty$ and

$$C(n,1) = \frac{1}{n} \left(\frac{n}{\omega_{n-1}} \right)^{1/n}$$

III. The case $p > n$:

Theorem 4.3.3. *For $p > n$, there is a $C = C(n,p)$ such that for any Ω of $\mu_n(\Omega) < \infty$, we have*

$$\forall\ f \in C_c^\infty(\Omega), \quad \|f\|_\infty \le C \mu_n(\Omega)^{1/n-1/p} \|\nabla f\|_p$$

Proof. By above, we already have

$$|f(x)| \le \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy$$

Let p' be the conjugate of p. Note that $(n-1)(p'-1) = (n-1)/(p-1) < 1$. Let $R = (\mu_n(\Omega)/\Omega_n)^{1/n}$ and so $\mu_n(\Omega) = \mu_n(\mathbb{B}(R))$. Then

$$\begin{aligned}
\int_\Omega \frac{1}{|x-y|p^{p'(n-1)}} dy &\leq \int_{\mathrm{B}(R)} \frac{1}{|x-y|^{p'(n-1)}} dy \\
&\leq \omega_{n-1} \int_0^R r^{(1-n)p'+n-1} dr \\
&= \omega_{n-1} \left(1-(n-1)(p'-1)\right)^{-1} R^{1-(n-1)(p'-1)} \\
&= \omega_{n-1} \left(1-(n-1)(p'-1)\right)^{-1} R^{(p-n)/(p-1)} \\
&= \frac{\omega_{n-1}\mu_n(\Omega)^{(p-n)/n(p-1)}}{\Omega_n^{(p-n)/n(p-1)} \left(1-(n-1)(p'-1)\right)} \\
&= B\mu_n(\Omega)^{(p-n)/n(p-1)}
\end{aligned}$$

So by Hölder's Inequality,

$$\begin{aligned}
\|f\|_\infty &\leq \left(\frac{1}{\omega_{n-1}} \int_\Omega \frac{1}{|x-y|^{p'(n-1)}} dy\right)^{1/p'} \|\nabla f\|_p \\
&\leq C\mu_n(\Omega)^{1/n-1/p} \|\nabla f\|_p.
\end{aligned}$$

□

Theorem 4.3.4. *For $p > n$, there is a $C = C(n,p)$ such that for any $f \in C^\infty(\mathbb{R}^n)$ with $\|f\|_p < \infty$,*

$$\sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right\} \leq C\|\nabla f\|_p$$

with $\alpha = 1 - n/p$

Lemma 4.3.1. *Let B be a ball of radius $r > 0$. Then*

$$\forall f \in C^\infty(B), \quad \forall x \in B, \quad |f(x) - f_B| \leq \frac{2^n}{\omega_{n-1}} \int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy$$

where

$$f_B = \frac{1}{\mu_n(B)} \int_B f(x) dx$$

Proof. For any $x, y \in B$,

$$f(x) - f(y) = -\int_0^{|x-y|} \partial_\rho f\left(x + \rho\frac{y-x}{|y-x|}\right) d\rho$$

So we have

$$|f(x) - f(y)| \leq \int_0^\infty F\left(x + \rho\frac{y-x}{|y-x|}\right) d\rho$$

where

$$F(z) = \begin{cases} |\nabla f(z)| & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$
\begin{aligned}
|f(x) - f_B| &= \left| f(x) - \frac{1}{\mu_n(B)} \int_B f(y)dy \right| \\
&\le \frac{1}{\mu_n(B)} \int_B |f(x) - f(y)|dy \\
&\le \frac{1}{\Omega_n r^n} \int_B dy \left\{ \int_0^\infty F\left(x + \rho \frac{y-x}{|y-x|}\right) d\rho \right\} \\
&\le \frac{1}{\Omega_n r^n} \int_{\{y:|x-y|\le 2r\}} dy \left\{ \int_0^\infty F\left(x + \rho \frac{y-x}{|y-x|}\right) d\rho \right\} \\
&= \frac{1}{\Omega_n r^n} \int_0^\infty \int_{\mathbb{S}^{n-1}} \int_0^{2r} F(x+\rho\theta) s^{n-1} ds d\theta d\rho \\
&= \frac{2^n}{n\Omega_n} \int_0^\infty \int_{S^{n-1}} F(x + r\theta) d\theta dr \\
&= \frac{2^n}{\omega_{n-1}} \int_B \frac{|\nabla f(y)|}{|y-x|^{n-1}} dy.
\end{aligned}
$$

□

Proof of Theorem 4.3.4. By above lemma and Hölder's Inequality,

$$
|f(x) - f_B| \le C\mu_n(B)^{1/n-1/p} \left(\int_B |\nabla f|^p d\mu_n \right)^{1/p} \le C\mu_n(B)^{1/n-1/p} \|\nabla f\|_p
$$

for any ball $B \subset \mathbb{R}^n$ and $x \in B$. So for any x, y with $|x-y| \le r$, by choosing B with radius r containing x, y

$$
\begin{aligned}
|f(x) - f(y)| &\le |f(x) - f_B| + |f_B - f(y)| \\
&\le 2C\Omega_n r^{1-n/p} \|\nabla f\|_p \\
&\le 2C\Omega_n |x-y|^{1-n/p} \|\nabla f\|_p.
\end{aligned}
$$

□

IV. The case $p = n$:

First, consider

$$
\int_\Omega \frac{1}{|x-y|^{r(n-1)}} dy
$$

where $r < \frac{n}{n-1}$. By above estimating, we

$$
\int_\Omega \frac{1}{|x-y|^{r(n-1)}} dy \le \frac{\omega_{n-1}}{1-(r-1)(n-1)} \left[\mu_n(\Omega)/\Omega_n\right]^{-(n+r-nr)/n}
$$

For any $n < q < \infty$, set $1/n - 1/q = \delta$ and $1/r = 1 + 1/q - 1/n = 1 - \delta$. Then we have

$$
\begin{aligned}
|f(x)| &\le \frac{1}{\omega_{n-1}} \int \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy \\
&= \frac{1}{\omega_{n-1}} \int \frac{|\nabla f(y)|^{n/q}}{|x-y|^{r(n-1)/q}} \times |\nabla f(y)|^{n\delta} \times \frac{1}{|x-y|^{r(n-1)(1-1/n)}} dy
\end{aligned}
$$

and by setting $p_1 = q, p_2 = 1/\delta$, and $p_3 = n/(n-1)$ in Hölder's Inequality,

$$|f(x)| \leq \frac{1}{\omega_{n-1}} \left(\int \frac{|\nabla f(y)|^n}{|x-y|^{\gamma(n-1)}} dy \right)^{1/q} \times$$
$$\left(\int |\nabla f(y)|^n dy \right)^{\delta} \left(\int_{\mathrm{supp}(f)} \frac{1}{|x-y|^{r(n-1)}} dy \right)^{1-1/n}$$

So if f is supported in Ω, by Hardy-Littlewood-Sobolev's Inequality

$$\|f\|_q \leq \frac{1}{\omega_{n-1}} \|\nabla f\|_n^{n/q+n\delta} \left(\int_\Omega \frac{1}{|x-y|^{r(n-1)}} dy \right)^{1/q+1-1/n}$$
$$\leq \frac{1}{\omega_{n-1}} \|\nabla f\|_n \left(\int_\Omega \frac{1}{|x-y|^{r(n-1)}} dy \right)^{1/r}$$

Then by the estimating of $\int_\Omega 1/|x-y|^{r(n-1)} dy$,

$$\|f\|_q \leq \frac{\omega_{n-1}^{-1+1/r}}{[1-(r-1)(n-1)]^{1/r} \Omega_n^{(n+r-nr)/nr}} \mu_n(\Omega)^{(n+r-nr)/nr} \|\nabla f\|_r$$

As $1/r = 1 + 1/q - 1/n$,

$$\|f\|_q \leq \frac{\omega_{n-1}^{1/q-1/n}}{[1-(r-1)(n-1)]^{1/r} \Omega_n^{1/q}} \mu_n(\Omega)^{1/q} \|\nabla f\|_n$$
$$= \frac{n^{1/q}}{[1-(r-1)(n-1)]^{1/r} \omega_{n-1}^{1/n}} \mu_n(\Omega)^{1/q} \|\nabla f\|_n$$

Note that $1-(r-1)(n-1) = n(n+1)/(nq+n-q) \geq (n+1)/q$ by $q > n$. Hence, we get

$$\|f\|_q^q \leq q^{1+q(n-1)/n} \omega_{n-1}^{-q/n} \mu_n(\Omega) \|\nabla f\|_n^q$$

It follows that for any $k = n, n+1, \cdots$,

$$\int_\Omega \left(\frac{|f(x)|}{\|\nabla f\|_n} \right)^{kn/(n-1)} dx \leq [kn/(n-1)]^{1+k} \omega_{n-1}^{-k/(n-1)} \mu_n(\Omega)$$

Besides, by the following Poincaré's Inequality, we have

$$\|f\|_n \leq C_0 \|\nabla f\|_n$$

So for any $1 < q < n$,

$$\int_\Omega |f(x)|^q \, dx \leq \|f\|_n^q \mu_n(\Omega)^{1-\frac{q}{n}} \leq C_0^q \|\nabla f\|_n^q \mu_n(\Omega)^{-\frac{q}{n}} \mu_n(\Omega) = C\mu_n(\Omega)$$

and if $q = 0$, the above inequality is clearly true. So for $k = 0, 1, \cdots, n-1$,

$$\int_\Omega \left(\frac{|f(x)|}{\|\nabla f\|_n} \right)^{kn/(n-1)} dx \leq C\mu_n(\Omega)$$

Then because for small $\alpha > 0$ ($\alpha < (n-1)\omega_{n-1}^{1/(n-1)}/en$),

$$\sum_0^\infty \frac{\alpha^k k^k}{(k-1)!} \left(\frac{n}{(n-1)\omega_{n-1}^{1/(n-1)}} \right)^k$$

is convergent,

$$\int_\Omega \exp\left(\alpha \left(\frac{|f(x)|}{\|\nabla f\|_n} \right)^{n/(n-1)} \right) dx \leq \sum_0^\infty \frac{\alpha^k}{k!} \int_\Omega \left(\frac{|f(x)|}{\|\nabla f\|_n} \right)^{kn/(n-1)} dx$$
$$\leq C\mu_n(\Omega)$$

4.4 Sobolev-Poincaré Inequalities

Theorem 4.4.1 (Poincaré Inequality)**.** *Let $B = B(z, r)$ be a Euclidean ball with radius r centered at z in $\mathbb{R}^n$. For any $1 \le p < \infty$, we have*

$$\forall\ f \in C_c^\infty(B), \quad \left(\int_B |f|^p d\mu_n\right)^{1/p} \le r\left(\int_B |\nabla f|^p d\mu_n\right)^{1/p}$$

and for

$$\forall\ f \in C^\infty(B), \quad \left(\int_B |f - f_B|^p\, d\mu_n\right)^{1/p} \le 2^n r\left(\int_B |\nabla f|^p d\mu_n\right)^{1/p}$$

where $f_B = \mu(B)^{-1}\int_B f d\mu_n$.

Proof. It is sufficient to assume $B = \mathbb{B}$, the unit ball. First, we have

$$|f(x)| \le \frac{1}{\omega_{n-1}} \int \frac{|\nabla f(y)|}{|y-x|^{n-1}} dy$$

This yields

$$\int_{\mathbb{B}} |f| d\mu \le \frac{1}{\omega_{n-1}} \int_{\mathbb{B}} |\nabla f(y)| \left(\int_{\mathbb{B}} \frac{dx}{|x-y|^{n-1}}\right) dy$$

As

$$\int_{\mathbb{B}} \frac{dx}{|x-y|^{n-1}} \le \int_{\mathbb{B}} \frac{dx}{|x|^{n-1}} = \omega_{n-1}$$

we get

$$\int_{\mathbb{B}} |f| d\mu \le \int_{\mathbb{B}} |\nabla f| d\mu$$

which is the case of $p = 1$.
For $p > 1$, let $c(x) = \int_{\mathbb{B}} |x-y|^{1-n} dy$. Then consider the measure $d\rho = c(x)^{-1} |x-y|^{-n+1} \chi_{\mathbb{B}}(y) dy$, which is a normalized measure on $\mathbb{R}^n$. Then by Jensen's Inequality of $d\rho$

$$\begin{aligned}
\left(\int \frac{|\nabla f(y)|}{|y-x|^{n-1}} dy\right)^p &= \left(\int \frac{|\nabla f(y)|}{|y-x|^{n-1}} \chi_{\mathbb{B}}(y) dy\right)^p \\
&= c(x)^p \left(\int |\nabla f(y)| d\rho\right)^p \\
&\le c(x)^p \int |\nabla f(y)|^p d\rho \\
&= c(x)^{p-1} \int_{\mathbb{B}} \frac{|\nabla f(y)|^p}{|y-x|^{n-1}} dy
\end{aligned}$$

Therefore, by $c(x) \le \omega_{n-1}$,

$$|f(x)|^p \le \frac{c(x)^{p-1}}{\omega_{n-1}^p} \int_{\mathbb{B}} \frac{|\nabla f(y)|^p}{|y-x|^{n-1}} dy \le \frac{1}{\omega_{n-1}} \int_{\mathbb{B}} \frac{|\nabla f(y)|^p}{|y-x|^{n-1}} dy.$$

Then, as similar as above we have

$$\int_{\mathbb{B}} |f|^p d\mu \le \int_{\mathbb{B}} |\nabla f|^p d\mu$$

For the $C^\infty(\mathbb{R}^n)$, case, we have the similar prove by using the inequality,

$$|f(x) - f_B| \le \frac{2^n}{\omega_{n-1}} \int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy$$

□

For any open set Ω and $1 \leq p \leq \infty$, we set

$$\|f\|_{p,\Omega} = \left(\int_\Omega |f|^p d\mu \right)^{1/p}$$

Then Poincaré Inequality says

$$\forall f \in C_c^\infty(B), \quad \|f\|_{p,B} \leq r \|\nabla f\|_{p,B},$$
$$\forall f \in C^\infty(B), \quad \|f - f_B\|_{p,B} \leq 2^n r \|\nabla f\|_{p,B}$$

Above Poincaré Inequality can be obtained from Sobolev Inequality in a more general way.

Theorem 4.4.2. *Fix $1 \leq p < n$ and set $q = np/(n-p)$. Then there is a constant $C = C(n,p)$ such that for any $f \in C_c^\infty(B)$, where $B \subset \mathbb{R}^n$ is a ball with radius r, we have*

$$\|f\|_{s,B} \leq C r^{1+n(1/s-1/p)} \|\nabla f\|_{p,B}$$

for all $1 \leq s \leq q$. Moreover, for any $f \in C^\infty(B)$,

$$\|f - f_B\|_{s,B} \leq C r^{1+n(1/s-1/p)} \|\nabla f\|_{p,B}$$

Proof. Let

$$K(x,y) = \chi_B(x)\chi_B(y)\frac{1}{|x-y|^{n-1}}$$

Then by above and $1 \leq s \leq q = np/(n-p)$ and Sobolev Inequality,

$$\|f\|_{s,B} \leq C_0 \|Kf\|_{s,B} \leq C_0 \mu(B)^{1/s-1/p+1/n} \|Kf\|_{np/(n-p),B} \leq C r^{1+n(1/s-1/p)} \|f\|_{p,B}$$

And the next inequality can also be obtained by similar kernel with the inequality $|f(x) - f_B| \leq \frac{2^n}{\omega_{n-1}} \int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy$. □

Note that $1 \leq p < q$, so above is true for $s = p$, which is Poincaré Inequality.

Remark. Clearly, we have similar result for any bounded domain $\Omega \subset \mathbb{R}^n$,

$$\forall f \in C_c^\infty(\Omega), \quad \|f\|_{s,\Omega} \leq C \mu(\Omega)^{1/s-1/q} \|\nabla f\|_{p,\Omega}$$

for $1 \leq p < n$, $1 \leq s < q = np/(n-p)$. Moreover, if Ω has smooth boundary, then

$$\forall f \in C^\infty(\Omega), \quad \|f - f_\Omega\|_{p,\Omega} \leq C(p,\Omega) \|\nabla f\|_{p,\Omega}$$

4.5 Elliptic Operator

Consider a second order differential operator

$$L = -\sum_{i,j} a_{ij}(x)\partial_i\partial_j + \sum_i c_i(x) + c(x)$$

which can also be expressed as

$$L = -\sum_{i,j} \partial_i \left(a_{i,j}(x)\partial_j\right) + \sum_i b_i(x)\partial_i + c(x), \quad b_i(x) = c_i(x) + \sum_\ell \partial_\ell a_{\ell,i}$$

Then denote $A(x) = (a_{ij}(x))_{1\leq i,j\leq n}$ and $b(x) = (b_i(x), \cdots, b_n(x))$, for any smooth $X\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$Lf = -\operatorname{div}(A\nabla f) + \langle b, \nabla f\rangle + c$$

called the divergence form of L.
Consider certain properties of (weak) solutions of equation

$$Lu = 0$$

on an Euclidean ball B, where

$$Lf = -\sum_{i,j} \partial_i \left(a_{i,j} \partial_j f\right) \tag{4.1}$$

and $A = (a_{ij})$ is uniformly elliptic, that is, there is $0 < \lambda \leq 1$ such that

$$\forall x \in \mathbb{R}^n, \quad \forall \xi, \zeta \in \mathbb{R}^n, \begin{cases} \lambda |\xi|^2 & \leq \sum_{i,j} a_{i,j}(x) \xi_i \xi_j, \\ \lambda^{-1} |\xi| |\zeta| & \geq \left| \sum_{i,j} a_{i,j}(x) \xi_i \zeta_j \right| \end{cases}$$

Note that if A is symmetric, it means eigenvalues of $A(x)$ lying in $[\lambda, \lambda^{-1}]$.
Consider $W^{1,2}(B)$ with its norm $\|\cdot\|_{W^{1,2}}$, which is

$$\|f\|_{W^{1,2}} = \sqrt{\|f\|_2^2 + \|\nabla f\|_2^2}$$

and consider $W_c^{1,2}(B) = \overline{C_c^\infty(B)}^{\|\cdot\|_{W^{1,2}}}$.

Definition 4.5.1. (1) A (weak) solution of (4.1) in the ball B is a $u \in W^{1,2}(B)$ such that

$$\forall \phi \in W_0^{1,2}(B), \quad \int_{\mathbb{R}^n} \sum_{i,j} a_{i,j}(x) \partial_i u(x) \partial_j \phi(x) dx = 0$$

(2) A (weak) subsolution is a $u \in W_c^{1,2}(B)$ such that

$$\int \sum_{i,j} a_{i,j}(x) \partial_i u(x) \partial_j \phi(x) dx \leq 0$$

for all $\phi \in W_0^{1,2}(B), \phi \geq 0$. And u is called a supersolution if $-u$ is a subsolution.

Theorem 4.5.1. *Consider the settings in equation (4.1). For any $\delta > 0$, there exists $C = C(n, \lambda, \delta) > 0$ such that any positive solution u of (4.1) in a ball B satisfies Harnack Inequality*

$$\sup_{\delta B} \{u\} \leq C \inf_{\delta B} \{u\}$$

Moreover, for any $\delta \in (0, 1)$, there exist $C' = C'(n, \lambda, \delta) > 0$ and $\alpha = \alpha(n, \lambda, \delta) > 0$ such that any solution u of (4.1) in a ball B satisfies Harnack continuity estimate

$$\sup_{x,y \in \delta B} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\} \leq C' r^{-\alpha} \|u\|_{\infty, B}$$

where r is the radius of B.

Lemma 4.5.1. *If u is a subsolution of (4.1) in B and $\varepsilon \leq u \leq c$ for some $0 < \varepsilon \leq c < \infty$, then u^α is also a subsolution for all $\alpha \geq 1$.*

Proof. For any $\phi \in C_c^\infty(B)$ with $\phi \geq 0$,

$$\begin{aligned}\sum_{i,j} a_{i,j}\partial_i u^\alpha \partial_j \phi &= \alpha \sum_{i,j} a_{i,j} u^{\alpha-1}\partial_i u \partial_j \phi \\ &= \alpha \sum_{i,j} a_{i,j}\partial_i u \partial_j \left(u^{\alpha-1}\phi\right) - \alpha(\alpha-1)\left(\sum_{i,j} a_{i,j}\partial_i u \partial_j u\right) u^{\alpha-2}\phi \\ &\leq \alpha \sum_{i,j} a_{i,j}\partial_i u \partial_j \left(u^{\alpha-1}\phi\right)\end{aligned}$$

Moreover, $u^{\alpha-1}\phi \in L^2(B)$ and

$$\nabla\left(u^{\alpha-1}\phi\right) = (\alpha-1)u^{\alpha-2}\phi\nabla u + u^{\alpha-1}\nabla\phi$$

Then because $\varepsilon \leq u \leq c$, by the boundedness of $u^{\alpha-2}$ and $u^{\alpha-1}$, $\nabla\left(u^{\alpha-1}\phi\right) \in L^2(B)$, so $u^{\alpha-1}\phi \in W^{1,2}(B)$. And since $\phi \in C_c^\infty(B)$ with $\phi \geq 0$, $u^{\alpha-1}\phi \in W_c^{1,2}(B)$ with $u^{\alpha-1}\phi \geq 0$. Then because u is a subsolution,

$$\int_B \sum_{i,j} a_{i,j}\partial_i u^\alpha \partial_j \phi d\mu \leq \alpha \int_B \sum_{i,j} a_{i,j}\partial_i u \partial_j \left(u^{\alpha-1}\phi\right) d\mu \leq 0$$

□

Before proving Theorem 4.5.1, let's consider the following properties.

I. Sobolev-type Inequality for Moser's Iteration: Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$. For $n > 2$, Theorem 4.4.2 implies

$$\forall f \in C_c^\infty(\mathbb{B}), \quad \|f\|_{q,\mathbb{B}} \leq C_n \|\nabla f\|_{2,\mathbb{B}}, \; q := \frac{2n}{n-2}$$

Moreover, for any $1 \leq p \leq 1$, set $\gamma \in [0,1]$ such that

$$\frac{1}{p} = \frac{\gamma}{q} + \frac{1-\gamma}{2}.$$

by Proposition 2.3.1,

$$\|f\|_{p,\mathbb{B}} \leq \|f\|_{q,\mathbb{B}}^\gamma \|f\|_{2,\mathbb{B}}^{1-\gamma}$$

In particular, for $p = 2(1+2/n)$ *i.e.* $\gamma = n/(n+2)$,

$$\int |f|^{2(1+2/n)} d\mu \leq \|f\|_q^2 \|f\|_2^{4/n}$$

Then combining with the Sobolev Inequality, we get

$$\forall f \in C_c^\infty(\mathbb{B}), \quad \int_{\mathbb{B}} |f|^{2(1+2/n)} d\mu \leq C_n^2 \left(\int_{\mathbb{B}} |\nabla f|^2 d\mu\right)\left(\int_{\mathbb{B}} |f|^2 d\mu\right)^{2/n} \tag{4.2}$$

Note it is also true for any $f \in W_c^{1,2}(\mathbb{B})$ by taking limits.

In fact, $n \geq 2$ can guarantee the validity of Theorem 4.4.2. But in order to apply Proposition 2.3.1 for $1/p = \gamma/q + 1 - \gamma/2$, since $q > 2$, $p \geq 2$. But $p < n$, so $n > 2$. However, for $n = 1, 2$, above inequality is also true by replacing n with $\nu \geq 2$, like $\nu = 3$

$$\int_{\mathbb{B}} |f|^{2(1+2/3)} d\mu \leq C^2 \left(\int_{\mathbb{B}} |\nabla f|^2 d\mu\right)\left(\int_{\mathbb{B}} |f|^2 d\mu\right)^{2/3} \tag{4.3}$$

for all $f \in C_c^\infty(\mathbb{B})$.

II. Subsolution: Let B be a ball in $\mathbb{R}^n$ with $V = \mu(B)$. WLTG, assume B is the unit ball.

Lemma 4.5.2. *Let u be a positive subsolution in B. There is a constant $C_1 = C_1(n, \lambda)$ such that for any $0 < \rho' < \rho \leq 1$ and $p \geq 2$,*

$$\int_{\rho' B} u^{p\theta} d\mu \leq C_1(\rho - \rho')^{-2} V^{1-\theta} \left(p^2 \int_{\rho B} u^p d\mu\right)^\theta$$

with $\theta = 1 + \frac{2}{n}$ if $n > 2$ and $\theta = 1 + \frac{2}{3}$ for $n = 1, 2$.

Proof. By replacing u with $u + \varepsilon$, we can assume u is bounded below away from 0. First, for any $\phi \in W_0^{1,2}(B)$ with $\phi \geq 0$, we have

$$\int_{\mathbb{R}^n} \sum_{i,j} a_{i,j}(x)\partial_i u(x)\partial_j \phi(x) d\mu \leq 0$$

Define a function $G\colon (0, \infty) \to (0, \infty)$ such that it is piecewise C^1, non-decreasing, has non-negative G' with $G(s) \leq sG'(s)$, and $G(s) = as$ for large s. Finally, define $H(s) \geq 0$by $H'(s) = \sqrt{G'(s)}$, $H(0) = 0$. Also, we have $H(s) \leq sH'(s)$.

Let $\psi \in C_c^\infty(B)$ be non-negative. Set $\phi = \psi^w G(u)$. Then $\phi \geq 0$ and $\phi \in W_c^{1,2}(B)$ because G is non-decreasing and $G(s) = as$ for large s. Because u is a subsolution,

$$\sum_{i,j} a_{i,j}\partial_i u \partial_j \phi = \psi^2 G'(u) \sum_{i,j} a_{i,j}\partial_i u \partial_j u + 2\psi G(u) \sum_{i,j} a_{i,j}\partial_i u \partial_j \psi \leq 0$$

So by $G(u) \leq uG'(u)$,

$$\begin{aligned}\int_B \psi^2 G'(u) \sum_{i,j} a_{i,j}\partial_i u \partial_j u d\mu &\leq 2 \left| \int_B \psi G(u) \sum_{i,j} a_{i,j}\partial_i u \partial_j \psi d\mu \right| \\ &\leq 2 \int_B \psi u G'(u) \left| \sum_{i,j} a_{i,j}\partial_i u \partial_j \psi \right| d\mu\end{aligned}$$

Moreover, by the uniform ellipticity,

$$\begin{aligned}\int_B \psi^2 G'(u) |\nabla u|^2 d\mu &\leq \int_B \psi^2 G'(u) \sum_{i,j} a_{i,j}\partial_i u \partial_j u d\mu \\ &\leq 2\lambda^{-1} \int_B \psi u G'(u) |\nabla u| |\nabla \psi| d\mu\end{aligned}$$

Then by Cauchy-Schwartz Inequality,

$$\begin{aligned}&\int_B \psi^2 G'(u) |\nabla u|^2 d\mu \\ &\quad \leq 2\lambda^{-1} \left(\int_B \psi^2 G'(u) |\nabla u|^2 d\mu\right)^{1/2} \left(\int_B u^2 G'(u) |\nabla \psi|^2 d\mu\right)^{1/2}\end{aligned}$$

Thus

$$\int_B \psi^2 G'(u) |\nabla u|^2 d\mu \leq 4\lambda^{-2} \int_B u^2 G'(u) |\nabla \psi|^2 d\mu$$

As $\nabla(\psi H(u)) = \psi H'(u)\nabla u + H(u)\nabla\psi$ and

$$\begin{aligned}|\nabla\psi H(u)|^2 &\le 2\left(\psi^2\left|H'(u)\right|^2|\nabla u|^2 + H(u)^2|\nabla\psi|^2\right)\\ &\le 2\left(\psi^2 G'(u)|\nabla u|^2 + u^2G'(u)|\nabla\psi|^2\right)\end{aligned}$$

we obtain

$$\int_B |\nabla\psi H(u)|^2 d\mu \le 2\left(1+4\lambda^{-2}\right)\int_B u^2G'(u)|\nabla\psi|^2 d\mu$$

Therefore, $\psi H(u) \in W_c^{1,2}(B)$. Then by inequality (4.2),

$$\begin{aligned}&\int_B |\psi H(u)|^{2(1+2/n)}d\mu\\ &\quad\le C_n^2\left(\int_B|\nabla\psi H(u)|^2d\mu\right)\left(\int_B|\psi H(u)|^2d\mu\right)^{2/n}\\ &\le 2C_n^2\left(1+4\lambda^{-2}\right)\left(\int_B|\nabla\psi|^2|u|^2G'(u)d\mu\right)\left(\int_B|\psi|^2u^2G'(u)d\mu\right)^{2/n}\\ &\le 2C_n^2\left(1+4\lambda^{-2}\right)\|\nabla\psi\|_\infty^2\|\psi\|_\infty^{4/n}\left(\int_{\mathrm{supp}(\psi)}u^2G'(u)d\mu\right)^{1+2/n}\end{aligned}$$

Given $0<\rho'<\rho<1$, we pick $\psi\in C_c^\infty(B)$ such that $0\le\psi\le 1$, $\psi=1$ on $\rho' B$, $\psi=0$ on $\rho B\backslash\rho' B$, and $|\nabla\psi|\le 2/\left(\rho-\rho'\right)$. Then

$$\int_{\rho' B}|H(u)|^{2\theta}d\mu \le 8C_n^2\left(1+4\lambda^{-2}\right)\left(\rho-\rho'\right)^{-2}\left(\int_{\rho B}u^2G'(u)d\mu\right)^\theta$$

with $\theta = 1+2/n$. Fix $p\ge 1$ and some large N,

$$H_N(s) = \begin{cases} s^{p/2} & \text{if } s\le N\\ N^{(p/2)-1}s & \text{if } s>N\end{cases}$$

and let

$$\begin{aligned}G_N(s) &= \textstyle\int_0^s H'(t)^2dt\\ &= \frac{p^2}{4(p-1)}\begin{cases} s^{p-1} & \text{if } s\le N\\ \frac{4(p-1)}{p^2}N^{p-2}(s-N)+N^{p-1} & \text{if } s>N.\end{cases}\end{aligned}$$

Then G_N and H_N have our required properties for any $p>2$. And

$$H_N(s)\to s^{p/2},\quad G_N'(s)\to (p/2)^2 s^{p-2}$$

Therefore,

$$\int_{\rho' B}u^{p\theta}d\mu \le 2C_n^2\left(1+4\lambda^{-2}\right)\left(\rho-\rho'\right)^{-2}\left(p^2\int_{\rho B}u^pd\mu\right)^\theta$$

□

Theorem 4.5.2. *There is $C_2 = C_2(n,\lambda)$ such that for any $0<\delta<1$, any $p\ge 2$, and any positive subsolution u in a ball B of volume V,*

$$\sup_{\delta B}\{u^p\} \le C_2(1-\delta)^{-n}\left(V^{-1}\int_B u^pd\mu\right)$$

Proof. Fix $p \geq 2$ and $0 < \delta < 1$. For each $i \in \mathbb{N}_0$, set $p_i = p\theta^i$, $\rho_0 = 1$, and

$$\rho_i = 1 - (1-\delta) \sum_{j=1}^{j=i} 2^{-j}, \quad i \geq$$

Then $\rho_{i+1} - \rho_i = (1-\delta)2^{-i-1}, p_{i+1} = p_i\theta$, and by above lemma

$$\int_{\rho_{i+1}B} u^{p_{i+1}} d\mu \leq C(1-\delta)^{-2} 2^{2(i+1)} \left(p_i^2 \int_{\rho_i B} u^{p_i} d\mu \right)^{\theta}$$

or

$$\left(\int_{\rho_{i+1}B} u^{p_{i+1}} d\mu \right)^{1/p_{i+1}} \leq \left[C(1-\delta)^{-2} \right]^{1/p_{i+1}} 2^{2(i+1)/p_{i+1}} p_i^{2/p_i} \left(\int_{\rho_i B} u^{p_i} d\mu \right)^{1/p_i}$$

for $i = 0, 1, \cdots$ with $C = 2C_n^2 \left(1 + 4\lambda^{-2}\right)$. This yields

$$\left(\int_{\rho_{i+1}B} u^{p_{i+1}} d\mu \right)^{1/p_{i+1}} \leq \left[C(n)C(p) \left[C(1-\delta)^{-2} \right]^{\left(\sum_1^{i+1} \theta^{-j} \right)} \int_B u^p d\mu \right]^{1/p}$$

where

$$C(n) = 2^{2\left(\sum_1^{\infty} j\theta^{-j} \right)}, \quad C(p) = e^{2\sum_0^{\infty} \theta^{-i} \log(p\theta_i)}$$

Observe that $\rho_i \to \delta$,

$$\sum_1^{\infty} \theta^{-j} = \theta^{-1} \left(1 - \theta^{-1}\right)^{-1} = n/2$$

and

$$\lim_{p \to \infty} \|f\|_{p,B} = \|f\|_{\infty}$$

Hence

$$\sup_{\delta B} \{u\} \leq \left(C(n)C(p)C^n(1-\delta)^n \right)^{1/p} \|u\|_{p,B}$$

Moreover, by setting $G(t) = t^{p-1}$ in above lemma, instead of $G(s) \leq sG'(s)$, we can find a sharper version

$$\int_{\rho' B} u^{p\theta} d\mu \leq C_1 \left(\rho - \rho'\right)^{-2} V^{1-\theta} \left(\int_{\rho B} u^p d\mu \right)^{\theta}$$

Then the C_2 in above is independent with $p \geq 2$. □

Theorem 4.5.3. *Fix $0 < p \leq 2$. There is $C_2 = C_2(n, \lambda)$ such that for any $0 < \delta < 1$, and any positive subsolution u in a ball B of volume V,*

$$\sup_{\delta B} \{u\} \leq C_3(1-\delta)^{n/p} \left(V^{-1} \int_B u^p d\mu \right)^{1/p}$$

III. Supersolution: Let B be a fixed ball with volume V and u be a positive supersolution.

Theorem 4.5.4. *There is a constant $C_4 = C_4(n, \lambda)$ such that for any $0 < \delta < 1$ and any $p \in (0, \infty)$, we have*

$$\sup_{\delta B} \left\{ u^{-p} \right\} \leq C_4(1-\delta)^{-n} \frac{1}{V} \int_B u^{-p} d\mu$$

Theorem 4.5.5. *Fix $0 < p_0 < \theta = 1 + 2/n$ ($\theta = 1 + 2/3$ if $n = 1, 2$). There is a constant $C_5 = C_5(n, \lambda, p_0)$ such that for any $\delta \in (0, 1)$ and any $p \in (0, p_0/\theta)$,*

$$\left(\frac{1}{V} \int_{\delta B} u^{p_0} d\mu \right)^{1/p_0} \leq \left[C_5(1-\delta)^{-2n+2} \right]^{1/p - 1/p_0} \left(\frac{1}{V} \int_B u^p d\mu \right)^{1/p}$$

Consider a collection of measurable subsets $U_\sigma, 0 < \sigma \leq 1$, of a fixed measure space endowed with a measure μ, such that $U_{\sigma'} \subset U_\sigma$ if $\sigma' \leq \sigma$. In our application, U_σ will be σB for some fixed ball $B \subset \mathbb{R}^n$.

Lemma 4.5.3. *Fix $\delta \in (0,1)$. Let $\gamma, C > 0$ and $\alpha_0 \in (0, \infty]$. Let f be positive and measurable on $U_1 = U$ which satisfies*

$$\|f\|_{\alpha_0, U_{\sigma'}} \leq \left[C\left(\sigma - \sigma'\right)^{-\gamma} \mu(U)^{-1}\right]^{1/\alpha - 1/\alpha_0} \|f\|_{\alpha, U_\sigma}$$

for all $0 < \delta \leq \sigma' < \sigma \leq 1$ and $0 < \alpha \leq \min\{1, \alpha_0/2\}$. Assume further that

$$\mu(\log f > \lambda) \leq C\mu(U)\lambda^{-1}$$

for all $\lambda > 0$. Then

$$\|f\|_{\alpha_0, U_\delta} \leq A\mu(U)^{1/\alpha_0}$$

where $A = A(\delta, \gamma, C, \alpha_0)$.

1. Harnack Inequality:

   Theorem 4.5.6. *Fix $0 < p_0 < \theta = 1 + 2/n$ ($\theta = 1 + 2/3$ if $n = 1, 2$). There is a constant $C = C(n, \lambda, \delta, p)$ such that for any ball B and any positive supersolution u in B we have*

   $$\frac{1}{\mu(\delta B)} \int_{\delta B} u^p d\mu \leq C \inf_{\delta B} \{u^p\}$$

2. Hölder Continuity: Let u be a positive solution. Then

   $$\sup_{x,y \in \delta B} \left\{\frac{|u(x) - u(y)|}{|x - y|^\alpha}\right\} \leq Cr^{-\alpha} \sup_B \{u\}$$

   The following lemma provides a more rigorous speaking.

   Lemma 4.5.4. *There is a $\alpha = \alpha(n, \lambda)$ such that for any solution u in a ball B,*

   $$\forall \rho \in (0,1), \sup_{x,y \in \rho B} \{|u(x) - u(y)|\} \leq 2^{1+\alpha} \rho^\alpha \sup_B \{u\}.$$