Stochastic Analysis

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Chapter 1

Symmetric Markov Operator

In this chapter, let E be a Polish space that is a separable complete metric space and let E be equipped with the Borel σ-field $\mathcal{F}$. Then the measure decomposition theorem implies that for any probability measure μ on the product σ-field $\mathcal{F} \otimes \mathcal{F}$ on $E \times E$ with $\mu_1 = \pi_1^{\#}\mu$, the first projection, then

$$\mu(dx, dy) = k(x, dy)\mu_1(dx)$$

for some probability transition kernel $k \colon E \times \mathcal{F} \to [0, 1]$. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space.

Remark 1.0.1. Moreover, because of the existence of kernels, by Ionescu–Tulcea theorem, for any probability measure μ on E^n, there are kernels k_i from E^{i-1} to E such that

$$\mu(dx_1, dx_2, \cdots, dx_n) = \mu_1(dx_1)k_2(x_1, dx_2)k_3(x_1, x_2, dx_3) \cdots k_n(x_1, \cdots, x_{n-1}, dx_n).$$

For now on any measure μ is assumed to be σ-finite.

1.1 Markov Operator

Definition 1.1.1. A Markov operator P on $(E, \mathcal{F})$ is a linear operator $P \colon \mathcal{B}_b(E) \to \mathcal{B}_b(E)$ such that

(1) (*Mass conversation*) $P\mathbb{1} = \mathbb{1}$ for constant function $\mathbb{1}(x) \equiv 1$,

(2) (Positivity preserving) for $f \geq 0$, $Pf \geq 0$.

Remark 1.1.2. For $0 \leq f \leq 1$,

$$P(\mathbb{1} - f) \geq 0 \;\Rightarrow\; 0 \leq Pf \leq P\mathbb{1} \leq \mathbb{1}.$$

Therefore, $\|Pf\|_\infty \leq \|f\|_\infty$ for all $f \in \mathcal{B}_b(E)$.

Proposition 1.1.3 (Jensen's inequality)**.** *For any convex $\phi \colon \mathbb{R} \to \mathbb{R}$ and any $f \in \mathcal{B}_b(E)$, if P is a Markov operator, then*

$$P(\phi(f)) \geq \phi(Pf)$$

Proof. Because ϕ is convex, for any $b \in \mathbb{R}$, there is $a = a(b)$ such that

$$\phi(c) \geq \phi(b) + a(b)(c - b), \quad \forall\, c \in \mathbb{R}$$

For any $x \in E$, let $c = f(x)$. We have

$$\phi(f(x)) \geq \phi(b) + a(b)(f(x) - b) \quad \Rightarrow \quad \phi(f) \geq \phi(b) + a(b)(f - b)$$

By the positivity and mass properties of P, we have

$$P(\phi(f)) \geq \phi(b) + a(b)(Pf - b)$$

So for any $x \in E$,

$$P(\phi(f))(x) \geq \phi(b) + a(b)(Pf(x) - b)$$

Then let $b = Pf(x)$, we get

$$P(\phi(f))(x) \geq \phi(Pf(x))$$

which is true for any $x \in E$. $\square$

Definition 1.1.4 (Invariant Measure)**.** A measure μ on $(E, \mathcal{F})$ is called invariant for a Markov operator P if

$$\int_E Pf d\mu = \int_E f d\mu,$$

for all $f \in \mathcal{B}_b(E)$.

Remark 1.1.5. When $f \in \mathcal{B}_b(E)$ is 0 μ-a.e., $Pf = 0$ μ-a.e.. Therefore, P can be extended on $L^\infty(\mu)$. Moreover, μ is invariant for P if

$$\int_E Pf d\mu = \int_E f d\mu, \quad \forall\ f \in L^1(\mu) \cap L^\infty(\mu).$$

Note that for $1 \leq p < \infty$, $L^1(\mu) \cap L^\infty(\mu) \subset L^\infty(\mu)$ that is because

$$\|f\|_p^p = \int |f|^p d\mu \leq \|f\|_\infty^{p-1} \int |f| d\mu = \|f\|_\infty^{p-1} \|f\|_1.$$

So by Jensen's inequality for $\phi(x) = |x|^p$ $(1 \leq p < \infty)$,

$$\int |Pf|^p d\mu \leq \int P\left(|f|^p\right) d\mu = \int |f|^p d\mu, \quad \forall\ f \in L^1(\mu) \cap L^\infty(\mu).$$

Lemma 1.1.6. *For any $1 \leq p < \infty$,*

$$L^1(\mu) \cap L^\infty(\mu) \subset L^p(\mu)$$

is dense.

Proof. Let $f \in L^p$ and $\varepsilon > 0$.

- Step 1: For any $n \in \mathbb{N}$, let

  $$g_n(x) := \max\left(-n, \min(f(x), n)\right) \in [-n, n].$$

  Therefore, $g_n \in L^\infty$ and

  $$\|f - g_n\|_p^p = \int_{|f|>n} ||f| - n|^p d\mu \leq \int_{|f|>n} |f|^p d\mu \underset{n\to\infty}{\longrightarrow} 0.$$

  So let n be sufficiently large such that $\|f - g_n\|_p \leq \varepsilon/2$.

- By σ-finiteness, choose $E_k \uparrow E$ with $\mu(E_k) < \infty$ and put $h_k = g_n \mathbb{I}_{E_k}$. So $h_k \in L^1 \cap L^\infty$. Because $\mathbb{I}_{E_k^x} \to 0$ as $k \to \infty$, by DCT,

  $$\|g_n - h_k\|_p^p = \int_{E_k^c} |g_n|^p\, d\mu \leq \int_{E_k^c} |f|^p d\mu \underset{k\to\infty}{\longrightarrow} 0.$$

  Therefore,

  $$\|f - h_k\|_p \leq \|f - g_n\|_p + \|g_n - h_k\|_p < \varepsilon.$$ $\square$

Then because

$$\|Pf\|_p \le \|f\|_p, \quad \forall\, f \in L^1(\mu) \cap L^\infty(\mu),$$

and the density,

$$P\colon L^p(\mu) \to L^p(\mu)$$

for all $1 \le p \le \infty$. Note that this definition should fix an invariant μ.

Definition 1.1.7 (Reversible Measure)**.** A measure μ is called reversible for a Markov operator P if

$$\int fPgd\mu = \int gPfd\mu, \quad \forall\, f, g \in L^2(\mu).$$

Remark 1.1.8. It is obviously that if μ is reversible, then it is invariant, because it can choose $g_n \in L^2(\mu)$ such that $g_n \uparrow \mathbb{1}$ by the σ-finiteness of μ.

Definition 1.1.9. A symmetric Markov semigroup on $(E, \mathcal{F}, \mu)$ is a family of $(P_t)_{t\ge 0}$ of Markov operators such that

(i) (*Initial Condition*) $P_0 f = f$ for all $f \in L^\infty$;

(ii) (*Semigroup*) for every $t, s \ge 0$, $P_t P_s = P_{t+s}$;

(iii) (*Symmetry*) for every $t \ge 0$, μ is reversible for P_t;

(iv) (*Strong Continuity*) for all $f \in L^2(\mu)$, $P_t f \to f$ in $L^2(\mu)$ as $t \to 0$.

Remark 1.1.10. Note that strong continuity implies that $P_t \to P_{t_0}$ in the strong operator topology on $L^2(\mu)$ as $t \to t_0$ with the help of the initial condition and the semigroup property.

Theorem 1.1.11 (Kernel Representation)**.** *Let P be a Markov operator on $(E, \mathcal{F})$ that is continuous on $L^1(\nu)$. Then there exists a probability kernel p on $(E, \mathcal{F})$ such that for every $f \in L^\infty(\nu)$ and ν-a.e. $x \in E$,*

$$Pf(x) = \int_E f(y) p(x, dy).$$

1.2 Generator

For a given symmetric Markov semigroup $(P_t)_{t\ge 0}$ on $(E, \mathcal{F}, \mu)$, we can similarly define the generator but the domain is different,

$$\mathcal{D}(L) := \left\{ f \in L^2(\mu)\colon \lim_{t\to 0} \frac{Q_t f - f}{t} \text{ exists in } L^2(\mu) \right\}.$$

And we also define $\mathcal{D}_p(L)$ for considering the convergence in $L^p(\mu)$. Except for the domain, some properties are as same as the generator of a Feller semigroup, like, $\mathcal{D}(L) \subset L^2(\mu)$ dense, and $L(P_t f) = P_t(Lf)$. So

$$P_t f - f = \int_0^t P_s(Lf) ds = \int_0^t L(P_s f) ds.$$

Moreover, by the symmetry of $(P_t)_{t\ge 0}$, for any $f, g \in \mathcal{D}(L)$

$$\int_E fLgd\mu = \int_E gLfd\mu,$$

and for every $f \in \mathcal{D}_1(L)$,

$$\int Lfd\mu = 0.$$

Assume there exists an algebra $\mathcal{A} \subset \mathcal{D}(L)$, for example, $\mathcal{A} = C_c^\infty(\mathbb{R}^n)$.

Definition 1.2.1 (Carré du Champ)**.** The carré du champ associated to L is the bilinear form Γ on $\mathcal{A} \times \mathcal{A}$ defined by

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf).$$

$\Gamma(f) = \Gamma(f,f)$.

Note that

$$\frac{d}{dt}\left(P_t f\right)^2 = 2f\frac{d}{dt}P_t f = 2fLP_t f,$$

and by Jensen's inequality,

$$L\left(f^2\right) = \lim_{t\to 0}\frac{P_t\left(f^2\right) - f^2}{t} \geq \lim_{t\to 0}\frac{\left(P_t f\right)^2 - f^2}{t} = \left.\frac{d\left(P_t f\right)^2}{dt}\right|_{t=0} \leq 2fLf,$$

which implies that $\Gamma(f) \geq 0$. Then by the Cauchy-Schwartz inequality,

$$\Gamma(f,g)^2 \leq \sqrt{\Gamma(f)\Gamma(g)}.$$

Proposition 1.2.2. *Let $(P_t)_{t\geq 0}$ on $(E, \mathcal{F}, \mu)$ be a given symmetric Markov semigroup and L be its generator. Then L is a self-adjoint operator on L^μ and so it is closed.*

Moreover, because

$$0 \leq \int_E \Gamma(f) d\mu = -\int_E fLf d\mu,$$

L is non-positive definite.

Construct semigroup from generator L. Let's assume

$$L = \sum_{i,j=1}^n \sigma_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i},$$

where b_i and σ_{ij} are continuous functions and $\sigma = (\sigma_{ij}(x)) \in \mathbb{R}^{n\times n}$ is symmetric and nonnegative. $\mathcal{D}(L) = C_c^\infty(\mathbb{R}^n)$. Moreover, of σ is invertible, L is called an elliptic diffusion operator. A Borel measure μ is called symmetric for L if for any $f, g \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} gLf d\mu = \int_{\mathbb{R}^n} fLg d\mu.$$

In the following, let's fix a measure μ symmetric for L.
Note that because $\mathcal{D}(L) = C_c^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, \mu)$ is dense, L is a non-positive symmetric operator that is densely defined on $L^2(\mathbb{R}^n, \mu)$. But it is not self-adjoint. However, it can be extended to a self-adjoint operator.

Theorem 1.2.3 (Friedrichs Extension)**.** *On the Hilbert space $L^2(\mathbb{R}^n, \mu)$, for L defined above, there exists a densely defined non-positive self-adjoint extension of L.*

In fact, if L is essentially self-adjoint, then the Friedrichs extension is the closed operator $\overline{L}$. In such case,

$$\ker(-L^* + \lambda I) = \{0\}, \ \lambda > 0.$$

It means

$$-Lf + \lambda f = 0 \ \Rightarrow \ f = 0,$$

where Lf, the differential in the sense of distribution.

Therefore, in the following, we assume L is essentially self-adjoint and replace $\overline{L}$ by L. Then L is self-adjoint on $L^2(\mathbb{R}^n, \mu)$. So we can define

$$P_t = e^{tL} = \int_{\mathbb{R}} e^{t\lambda} dE_L(\lambda) = \int_0^\infty e^{-t\lambda} dE_L(\lambda), \forall\, t \geq 0,$$

where E_L is the spectral measure associated with L. The $P_t\colon L^2(\mathbb{R}^n, \mu) \to L^2(\mathbb{R}^n, \mu)$ is a bounded operator. Note that

(i) $P_t P_s = P_{t+s}$ for all $t, s \geq 0$.

(ii) for all $f \in L^2$,
$$\|P_t f\|_2 \leq \|f\|_2 .$$

(iii) for all $f \in L^2$, $t \mapsto P_t f$ is continuous in $L^2(\mu, \mathbb{R}^n)$.

(iv) for all $f, g \in L^2$,
$$\int_{\mathbb{R}^n} f P_t g d\mu = \int_{\mathbb{R}^n} g P_t f d\mu,$$
i.e., μ is reversible for P_t.

(v) for all $f \in L^2$,
$$\lim_{t\to 0} \|P_t f - f\|_2 = 0$$

(vi) for all any $f \in \mathcal{D}(L)$,
$$\lim_{t\to 0} \left\| \frac{P_t f - f}{t} - Lf \right\|_2 = 0.$$

(vii) if $\mathbb{1} \in \mathcal{D}(L)$ and $L\mathbb{1} = 0$, then $P_t \mathbb{1} = \mathbb{1}$.

1.3 Compact Markov Operators

Definition 1.3.1 (Diffusion Carré du Champ)**.** Let $\mathcal{A} \subset \mathbb{R}^E$ be an algebra such that for any $k \in \mathbb{N}$, any $f_1, \cdots, f_k \in \mathcal{A}$, and any $\Psi \in C^\infty(\mathbb{R}^k)$, $\Psi(f_1, \cdots, f_k) \in \mathcal{A}$. We say a bilinear form $\Gamma\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a diffusion carré du champ if for any Ψ and f_i as above,

$$\Gamma\left(\Psi\left(f_1, \ldots, f_k\right), g\right) = \sum_{i=1}^k \partial_i \Psi\left(f_1, \ldots, f_k\right) \Gamma\left(f_i, g\right).$$

Consider a symmetric Markov semigroup with generator L and the corresponding carré du champ

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf).$$

If Γ is a diffusion carré du champ, then

$$L\Psi\left(f_1, \ldots, f_k\right) = \sum_{i=1}^k \partial_i \Psi\left(f_1, \ldots, f_k\right) Lf_i + \sum_{i,j=1}^k \partial_i \partial_j \Psi\left(f_1, \ldots, f_k\right) \Gamma\left(f_i, f_j\right). \tag{1.1}$$

In particular, for $k = 1$,

$$\begin{aligned} \Gamma(\psi(f), g) &= \psi'(f)\Gamma(f, g) \\ L\psi(f) &= \psi'(f)Lf + \psi''(f)\Gamma(f). \end{aligned}$$

Definition 1.3.2 (Diffusion Semigroup)**.** An operator L satisfying (1.1) is called a diffusion generator. A symmetric Markov semigroup whose generator is a diffusion generator is called a diffusion semigroup.

Definition 1.3.3 (Dirichlet Form)**.** A bilinear form $\mathcal{E}\colon \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ is called a Dirichlet form if

(i) $\mathcal{D}(\mathcal{E}) \subset L^2(\mu)$ dense for some μ,

(ii) $\mathcal{E}(f,g) = \mathcal{E}(g,f)$ for $f,g \in \mathcal{D}(\mathcal{E})$,

(iii) $\mathcal{E}(f) = \mathcal{E}(f,f) \geq 0$ for $f \in \mathcal{D}(\mathcal{E})$,

(iv) $\mathcal{D}(\mathcal{E})$ is complete w.s.t.
$$\langle f,g\rangle_{\mathcal{E}} := \int_E fg d\mu + \mathcal{E}(f,g),$$

(v) for any $f \in \mathcal{D}(\mathcal{E})$, $0 \vee f \wedge 1 \in \mathcal{D}(\mathcal{E})$ and
$$\mathcal{E}(0 \vee f \wedge 1) \leq \mathcal{E}(f).$$

Remark 1.3.4. Note that for any symmetric, non-negative bilinear form $\mathcal{E}$ defined on some dense $D \subset L^2(\mu)$, if $\mathcal{E}$ satisfies that for any $f_n \to 0$ in $L^2(\mu)$ and f_n Cauchy w.s.t. $\langle \cdot,\cdot\rangle_{\mathcal{E}}$, $\mathcal{E}(f_n) \to 0$, then D can be extended to the closure of D w.s.t. $\|\cdot\|_2 + \mathcal{E}(\cdot)$.
If Γ is a diffusion carré du champ on an algebra $\mathcal{A} \subset L^2(\mu)$ dense and $\Gamma(f,f) \geq 0$ for all $f \in \mathcal{A}$, then let
$$\mathcal{E}(f,g) := \int_E \Gamma(f,g) d\mu$$
and taking $\mathcal{D}(\mathcal{E})$ be the closure of $\mathcal{A}$ w.s.t. $\langle \cdot,\cdot\rangle_{\mathcal{E}}$. It can prove that such $\mathcal{E}$ is a Dirichlet form. Moreover, by the symmetric and positivity of $\mathcal{E}$, Riesz representation theorem implies that we can define a a non-positive, symmetric operator L by
$$\int gLf d\mu = -\mathcal{E}(f,g)$$
on the domain
$$\mathcal{D}(L) = \{f \in \mathcal{D}(\mathcal{E}) : \exists C \text{ such that } \mathcal{E}(f,g) \leq C\|g\|_2 \text{ for all } g \in \mathcal{D}(\mathcal{E})\}.$$
Moreover, it can be extended to a self-adjoint operator L by using Friedrichs extension.

Definition 1.3.5 (Compact Markov Diffusion Triple)**.** Let $(E, \mathcal{F}, \mu)$ be a polished measure space and μ be a probability measure. For $\mathcal{A} \subset L^2(\mu)$, let
$$\Gamma\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$
be a symmetric bilinear form. We say (E, μ, Γ) a compact Markov diffusion triple the followings are satisfied.

(a) $\mathcal{A}$ is dense in $L^2(\mu)$,

(b) $\mathcal{A}$ is an algebra closed under composition with smooth functions,

(c) $\Gamma(f) = \Gamma(f,f) \geq 0$ for all $f \in \mathcal{A}$,

(d) Γ is a diffusion carré du champ,

(e) $\Gamma(f) = 0$ implies that f is a constant,

and let $\mathcal{E}(f,g) = \int_E \Gamma(f,g)d\mu$ for all $f,g \in \mathcal{A}$, which satisfies

(f) for every $f \in \mathcal{A}$, there exist a $C > 0$ such that $\mathcal{E}(f,g) \leq C \|g\|_2$ for all $g \in \mathcal{A}$.

It follows that $\mathcal{E}$ can be extended to a Dirichlet form. Let L be the self-adjoint operator defined on $\mathcal{D}(L)$ such that

$$\int gLf d\mu = -\mathcal{E}(f,g).$$

Note that $\mathcal{A} \subset \mathcal{D}(L)$. Let $P_t = e^{tL}$ called the semigroup be assumed that

(g) $L\mathcal{A} \subset \mathcal{A}$,

(h) $P_t\mathcal{A} \subset \mathcal{A}$.

Proposition 1.3.6. *Let $(E, \mathcal{F}, \mu)$ be a compact Markov diffusion triple and P_t be its semigroup.*

(1) P_t is a symmetric Markov semigroup for μ.

(2) For any $f \in L^2(\mu)$,

$$\lim_{t\to\infty} P_t f = \int_E f d\mu$$

in L^2, which is called the ergodic property.

Curvature.

Definition 1.3.7. Given a compact Markov diffusion triple $(E, \mathcal{F}, \mu)$. For any $f, g \in \mathcal{A}$,

$$\Gamma_2(f,g) = \frac{1}{2}\left(L\Gamma(f,g) - \Gamma(f, Lg) - \Gamma(g, Lf)\right),$$

and $\Gamma_2(f) = \Gamma_2(f,f)$.

Definition 1.3.8 (Curvature Dimension)**.** For $\rho \in \mathbb{R}$ and $n \in [1,\infty]$, a compact Markov diffusion triple $(E, \mathcal{F}, \mu)$ is to said to satisfy the curvature-dimension condition $\mathrm{CD}(\rho, n)$ if

$$\Gamma_2(f) \geq \rho\Gamma(f) + \frac{1}{n}(Lf)^2.$$

1.4 Poincaré Inequality

Proposition 1.4.1. *Let P_t be the semigroup of a compact Markov triple. TFAE.*

(1) $\mathrm{CD}(\rho,\infty)$ *holds for some $\rho \in \mathbb{R}$.*

(2) For every $f \in \mathcal{A}$ and $t \geq 0$,

$$\Gamma(P_t f) \leq e^{-2\rho t} P_t\Gamma(f).$$

(3) For every $f \in \mathcal{A}$ and $t \geq 0$,

$$P_t(f^2) - (P_t f)^2 \leq \frac{1 - e^{-2\rho t}}{\rho} P_t\Gamma(f).$$

(4) For every $f \in \mathcal{A}$ and $t \geq 0$,

$$P_t(f^2) - (P_t f)^2 \geq \frac{e^{2\rho t} - 1}{\rho} \Gamma(P_t f).$$

For the last two conditions, if $\rho = 0$, then the coefficients in RHS can be taken as $2t$.

Proof. (1) $\Rightarrow$ (2): for $f \in \mathcal{A}$, let

$$\Lambda(s) = e^{-2\rho} P_s \Gamma(P_{t-s} f).$$

Then by chain rule,

$$\Lambda'(s) = 2e^{-2\rho s} P_s \left(\Gamma_2 \left(P_{t-s} f\right) - \rho \Gamma \left(P_{t-s} f\right)\right) \geq 0,$$

because of $\mathrm{CD}(\rho, \infty)$. Therefore, $\Lambda(t) \geq \Lambda(0)$.
(2) $\Rightarrow$ (3): Let

$$\Lambda(s) = P_s \left(P_{t-s} f\right)^2.$$

So $\Lambda'(s) = 2 P_s \Gamma \left(P_{t-s} f\right)$ and

$$\begin{aligned}
\Lambda(t) - \Lambda(0) &= 2 \int_0^t P_s \Gamma \left(P_{t-s} f\right) ds \\
&\leq 2 \int_0^t e^{-2\rho(t-s)} P_t \Gamma(f) ds \\
&= \frac{1 - e^{-2\rho t}}{\rho} P_t \Gamma(f).
\end{aligned}$$

(2) $\Rightarrow$ (4): Similarly, as above, by using $P_s \Gamma \left(P_{t-s} f\right) \geq e^{2\rho s} \Gamma \left(P_t f\right)$,

$$\Lambda(t) - \Lambda(0) \geq 2 \int_0^t e^{2\rho s} \Gamma \left(P_t f\right) ds = \frac{e^{2\rho t} - 1}{\rho} \Gamma \left(P_t f\right).$$

(3) $\Rightarrow$ (1): Note that for any $h \in \mathcal{A}$,

$$P_t h = h + tLh + \frac{t^2}{2} L^2 h + o\left(t^2\right), \quad t \to 0.$$

Therefore choosing $h = f$ and $h = f^2$, we have

$$\begin{aligned}
P_t\left(f^2\right) - \left(P_t f\right)^2 &= tL\left(f^2\right) + \frac{t^2}{2} L^2\left(f^2\right) - 2tfLf - t^2(Lf)^2 - t^2 f L^2 f + o\left(t^2\right) \\
&= 2t\Gamma(f) + \frac{t^2}{2} L^2\left(f^2\right) - t^2(Lf)^2 - t^2 f L^2 f + o\left(t^2\right).
\end{aligned}$$

On the other hand,

$$\frac{1 - e^{-2\rho t}}{\rho} P_t \Gamma(f) = 2t\Gamma(f) - 2\rho t^2 \Gamma(f) + 2t^2 L\Gamma(f) + o\left(t^2\right).$$

Therefore, by (3),

$$\frac{1}{2} L^2\left(f^2\right) - (Lf)^2 - f L^2 f + o(1) \leq -2\rho\Gamma(f) + 2L\Gamma(f) + o(1).$$

As $t \to 0$, we have

$$\frac{1}{2} L^2\left(f^2\right) - (Lf)^2 - f L^2 f \leq -2\rho\Gamma(f) + 2L\Gamma(f).$$

Then by arranging,

$$L\Gamma(f) - 2\Gamma(f, Lf) \geq 2\rho\Gamma(f).$$

(4) $\Rightarrow$ (1): It is similarly as above. $\square$

For (3) in above proposition, by the ergodic property, as $t \to \infty$,

$$\int f^2 d\mu - \left(\int f d\mu\right)^2 \leq \frac{1}{\rho} \int \Gamma(f) d\mu,$$

which is called a Poincaré inequality.

Definition 1.4.2 (Poincaré inequality)**.** Let μ be a probability measure and $\mathcal{E}$ be a Dirichlet form on $L^2(\mu)$. We say that μ and $\mathcal{E}$ satisfy a Poincaré inequality with constant C if

$$\int_E f^2 d\mu - \left(\int_E f d\mu\right)^2 \leq C\mathcal{E}(f),$$

for any $f \in \mathcal{D}(\mathcal{E})$. The Poincaré constant of μ and $\mathcal{E}$ is the smallest C such that above inequality holds for all $f \in \mathcal{D}(\mathcal{E})$.

Remark 1.4.3. When considering a compact Markov triple, because $\mathcal{A} \subset \mathcal{D}(\mathcal{E})$ is dense, it suffices to check PI on $\mathcal{A}$. Moreover, if a compact Markov triple satisfies $\mathrm{CD}(\rho, \infty)$, it satisfies $\mathrm{PI}(1/\rho)$.

Corollary 1.4.4. *The compact Markov triple (E, μ, Γ) satisfies $\mathrm{CD}(\rho, \infty)$ if and only if for any $t \geq 0$ and $x \in E$ μ-a.e., the measure $p_t(x, \cdot)$ satisfies* PI *with constant $(1 - e^{-2\rho t})/\rho$.*

Proposition 1.4.5 (Spectral Gap)**.** *If the compact Markov triple (E, μ, Γ) satisfies $\mathrm{PI}(C)$ for some constant C, then the spectrum of L*

$$\sigma(L) \subset (-\infty, -\frac{1}{C}] \cup \{0\}\,.$$

Proof. Let $\lambda \in \sigma(L)$ such that $\lambda \neq 0$. Because L is self-adjoint, i.e., $\sigma(L) = \sigma_{ap}(L)$, there exists $f_n \in \mathcal{D}(L)$ such that $\|f\|_2 = 1$ and

$$\|Lf_n - \lambda f_n\|_2 \to 0, \quad n \to \infty.$$

Note that because μ is a probability measure, $\|Lf_n - \lambda f_n\|_1 \to 0$. It follows that $\int f_n d\mu \to 0$ by $\int Lf_n d\mu = 0$ for all n. Then PI implies that

$$\int_E f_n^2 d\mu - \left(\int_E f_n d\mu\right)^2 \leq C \int_E \Gamma\left(f_n\right) d\mu = -C \int_E f_n L f_n d\mu.$$

As $n \to \infty$,

$$\lambda = \int_E f_n L f_n d\mu \leq -\frac{1}{C}.$$ □

PI **under** $\mathrm{CD}(\rho, n)$**.**

Lemma 1.4.6. *Suppose the compact Markov triple (E, μ, Γ) satisfies $\mathrm{CD}(\rho, \infty)$ with some $\rho > 0$. It satisfies $\mathrm{PI}(C)$ for some $C > 0$ if and only if*

$$\int_E \Gamma(f) d\mu \leq C \int_E (Lf)^2 d\mu, \quad \forall\ f \in \mathcal{D}(L).$$

Proof. $\Rightarrow$: Let

$$\Lambda(t) = \int_E (P_t f)^2 \, d\mu.$$

Then

$$\Lambda'(t) = -2 \int_E \Gamma(P_t f) \, d\mu, \quad \Lambda''(t) = 4 \int_E (LP_t f)^2 \, d\mu.$$

Because it satisfies $\mathrm{CD}(\rho, \infty)$ with some $\rho > 0$, by

$$\Gamma(P_t f) \leq e^{-2\rho t} P_t \Gamma(f) \leq P_t \Gamma(f) \leq \Gamma(f)$$

Then by DCT, $\lim_{t\to\infty} \Lambda'(t)$ exists. And by ergodicity,

$$\lim_{t\to\infty} \Lambda(t) = \int_E f d\mu,$$

$\lim_{t\to\infty} \Lambda'(t) = 0$. By assumption,

$$\Lambda''(t) \geq -\frac{2}{C} \Lambda'(t).$$

Therefore,

$$\begin{aligned}
\int f^2 d\mu - \left(\int f d\mu \right)^2 &= -\int_0^\infty \Lambda'(t) dt \\
&\leq \frac{C}{2} \int_0^\infty \Lambda''(t) dt \\
&= -\frac{C}{2} \Lambda'(0) \\
&= C \int_E \Gamma(f) d\mu.
\end{aligned}$$

$\Leftarrow$: Choosing $f \in \mathcal{D}(L)$ with mean 0 (otherwise, let $f - \int f d\mu$ and note that $\Gamma(c, g) = 0$ for any constant c and function g). By Cauchy-Schwartz inequality,

$$\begin{aligned}
\int_E \Gamma(f) d\mu &= \int_E f(-Lf) d\mu \\
&\leq \sqrt{\int_E f^2 d\mu \int_E (Lf)^2 d\mu} \\
&\leq \sqrt{C \int_E \Gamma(f) d\mu \int_E (Lf)^2 d\mu}.
\end{aligned}$$

□

Theorem 1.4.7. *Let (E, μ, Γ) be a compact Markov triple. If it satisfies $\mathrm{CD}(\rho, n)$ for some $\rho > 0$ and $n > 1$, then μ satisfies $\mathrm{PI}(C)$ with $C = \frac{n-1}{\rho n}$.*

Proof. Because of $\mathrm{CD}(\rho, n)$,

$$\int_E \Gamma_2(f) d\mu \geq \rho \int_E \Gamma(f) d\mu + \frac{1}{n} \int_E (Lf)^2 d\mu.$$

Because $\int Lh d\mu = 0$,

$$\int_E \Gamma_2(f) = \frac{1}{2} \left(\int_E L\Gamma(f) d\mu - \int_E \Gamma(f, Lf) d\mu \right)$$

$$
\begin{aligned}
&= \frac{1}{2}\int_E L\Gamma(f)d\mu - \frac{1}{2}\int_E L(fLf)d\mu + \frac{1}{2}\int_E (Lf)^2 + fL^2 f d\mu \\
&= \int_E (Lf)^2 d\mu.
\end{aligned}
$$

Therefore,

$$
\frac{n-1}{\rho n}\int_E (Lf)^2 d\mu \geq \int_E \Gamma(f)d\mu.
$$

Then by above lemma, it has the result. □

1.5 Applications with PI

Decay of Variance. For a probability measure μ and $f \in L^2(\mu)$, let

$$
\mathrm{Var}_\mu(f) = \int_E f^2 d\mu - \left(\int_E f d\mu\right)^2.
$$

Proposition 1.5.1. *The compact Markov triple* (E, μ, Γ) *satisfies* $\mathrm{PI}(C)$ *if and only if*

$$
\mathrm{Var}_\mu(P_t f) \leq e^{-\frac{2t}{C}} \mathrm{Var}_\mu(f), \quad f \in L^2(\mu).
$$

Proof. $\Rightarrow$: For $f \in \mathcal{A}$,

$$
\frac{d}{dt}\int_E (P_t f)^2 \, d\mu = 2\int_E P_t f L P_t f d\mu = -2\mathcal{E}\left(P_t f\right).
$$

Define

$$
\Lambda(t) = e^{2t/C}\,\mathrm{Var}_\mu\left(P_t f\right),
$$

so

$$
\Lambda'(t) = \frac{2}{C}\mathrm{Var}_\mu\left(P_t f\right) - 2\mathcal{E}\left(P_t f\right) \leq 0
$$

by $\mathrm{PI}(C)$. It follows that $\Lambda(t) \leq \Lambda(0)$. For general $f \in L^2(\mu)$, it can get by density.
$\Leftarrow$: It suffices to prove that for $f \in \mathcal{A}$ with $\int f d\mu = 0$. Note that

$$
P_t f = f + tLf + o(t),
$$

and so

$$
\mathrm{Var}\left(P_t f\right) = \int_E f^2 d\mu + 2t\int_E fLf d\mu + o(t).
$$

On the other hand,

$$
e^{-2t/C}\,\mathrm{Var}_\mu(f) = \left(1 - \frac{2t}{C} + o(t)\right)\mathrm{Var}_\mu(f).
$$

Therefore,

$$
2t\int_E fLf d\mu + o(t) \leq \left(-\frac{2t}{C} + o(t)\right)\mathrm{Var}_\mu(f).
$$

Then dividing t on the both sides and taking $t \to 0$,

$$
2\int_E fLf d\mu \leq -\frac{2}{C}\mathrm{Var}_\mu(f).
$$

□

Log-concave measures.

Definition 1.5.2. The probability measure μ on $\mathbb{R}^n$ defined by

$$d\mu(x) = e^{-W(x)}dx$$

is called log-concave if $W\colon \mathbb{R}^n \to \mathbb{R}$ is convex. For $\rho > 0$, μ is called ρ-strongly log-concave if $W(x) - \rho\,|x|^2$ is convex.

Assume $W \in C^\infty(\mathbb{R}^n)$. And on $\mathbb{R}^n$,

$$\Gamma(f, g) = \langle \nabla f, \nabla g \rangle$$

is a carré du champ. Then by divergence theorem,

$$-\int_{\mathbb{R}^n} \Gamma(f, g) d\mu = -\int_{\mathbb{R}^n} \left\langle e^{-W} \nabla f, \nabla g \right\rangle dx = \int_{\mathbb{R}^n} f(\Delta g - \langle \nabla W, \nabla g \rangle) d\mu.$$

Therefore,

$$Lg = \Delta g - \langle \nabla W, \nabla g \rangle.$$

If all derivatives of $W(x)$ grow at most polynomially fast as $|x| \to \infty$, then $(\mathbb{R}^n, \mu, \Gamma)$ is a compact Markov triple with $\mathcal{A}$ being the class of smooth, bounded functions whose derivatives all vanish super-polynomially fast.
Moreover,

$$\Gamma_2(f, g) = \left\langle \nabla^2 f, \nabla^2 g \right\rangle + (\nabla f)^\top \left(\nabla^2 W\right) \nabla g,$$

By the strongly convexity of W,

$$\Gamma_2(f, f) \geq \rho\,\|f\|_2 = \rho \Gamma(f),$$

i.e., $(\mathbb{R}^n, \mu, \Gamma)$ is $\mathrm{CD}(\rho, \infty)$.

Corollary 1.5.3. *Every ρ-strongly log-concave probability measure satisfies* $\mathrm{PI}(1/\rho)$*, i.e.,*

$$\mathrm{Var}_\mu(f) \leq \frac{1}{\rho} \mathbb{E}_\mu[\|\nabla f\|^2].$$

1.6 Strong Gradient Bound