Spectral Theory

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Contents

1 Adjoint Operator

Definition 1.1 (Unbounded Operator)**.** $\mathcal{H}$ is a Hilbert space.

(1) $A\colon \mathcal{H} \to \mathcal{H}$ is called a linear operator if the domain $D(A) \subset \mathcal{H}$ is a linear subspace and $A\colon D(A) \to \mathcal{H}$ is linear, denoted by $A \in \mathcal{L}(\mathcal{H})$.

(2) A is bounded if there is a c such that $\|Ax\| \le c\|x\|$ for any $x \in D(A)$. In such case, $D(A)$ can be extended to $\mathcal{H}$. Then A is viewed as a bounded operator on $\mathcal{H}$.

(3) If $D(A) \subset \mathcal{H}$ is dense, A is called densely defined.

Remark 1.2. For $A, B \in \mathcal{L}(\mathcal{H})$, $A \subset B$ means that $D(A) \subset D(B)$ and $Ax = Bx$ for $x \in D(A)$. It follows that $A \subset B$ and $B \subset A$ imply $A = B$. So $A \subset B$ and $D(B) \subset D(A)$ imply $A = B$.

Definition 1.3 (Adjoint Operator)**.** Let $A\colon \mathcal{H} \to \mathcal{H}$ be densely defined. Let

$$D(A^*) := \{y \in \mathcal{H}\colon x \mapsto \langle Ax, y\rangle \text{ is a bounded linear functional on } D(A)\}\,.$$

Because $D(A)$ is dense, it can be extended on $\mathcal{H}$ by the Hahn-Banach Theorem. Then by Riesz representation theorem, there is a $z \in \mathcal{H}$ such that

$$\langle Ax, y\rangle = \langle x, z\rangle = \langle x, A^* y\rangle\,,$$

which is defined as $A^* y$.

Remark 1.4. Note that $D(A^*)$ may be not dense.

Proposition 1.5. *For $A, B \in \mathcal{L}(\mathcal{H})$,*

(1) if A, B are densely defined and $A \subset B$, then $B^* \subset A^*$.

(2) if $D(A+B) = D(A) \cap D(B)$ is dense, then

$$A^* + B^* \subset (A+B)^*,$$

where "=" if B is bounded.

(3) if $D(BA)$ is dense, then

$$A^* B^* \subset (BA)^*,$$

where "=" if B is bounded.

Proof. (1) For $y \in D(B^*)$, $x \in D(B) \mapsto \langle Bx, y\rangle$ is bounded. So $x \in D(A) \mapsto \langle Ax, y\rangle$ is bounded, i.e. $y \in D(A^*)$. And $\langle Ax, y\rangle = \langle Bx, y\rangle$ implies that $A^* y = B^* y$.

(2) Fix $y \in D(A^*) \cap D(B^*)$. For any $x \in D(A) \cap D(B) = D(A+B)$,

$$\langle Ax, y\rangle + \langle Bx, y\rangle = \langle x, A^* y\rangle + \langle x, B^* y\rangle\,,$$

i.e., $\langle (A+B)x, y\rangle = \langle x, (A+B)^* y\rangle$. It follows that $y \in D\left((A+B)^*\right)$ and $(A+B)^* y = A^* y + B^* y$.

Let B be bounded. Fix $y \in D\left((A+B)^*\right)$. For any $x \in D(A)$,

$$\langle Ax, y\rangle = \langle Ax + Bx, y\rangle - \langle Bx, y\rangle = \langle x, (A+B)^* y\rangle - \langle x, B^* y\rangle\,.$$

So $y \in D(A^*) = D(A^* + B^*)$.

(3) Let $y \in D(A^*B^*)$. For any $x \in D(BA)$,

$$\langle BAx, y \rangle = \langle x, A^*B^*y \rangle .$$

So $y \in D((BA)^*)$ and $(BA)^*y = A^*B^*y$.

Let B be bounded. If $y \in D\left((BA)^*\right)$, for any $x \in D(A) = D(BA)$,

$$\langle Ax, B^*y \rangle = \langle BAx, y \rangle = \langle x, (BA)^*y \rangle .$$

So $B^*y \in D(A^*)$, i.e. $y \in D(A^*B^*)$.

□

2 Closable Operator

Definition 2.1 (Closed Operator)**.** For $A \in \mathcal{L}(\mathcal{H})$, if the graph of A

$$G(A) := \{(x, Ax) \colon x \in D(A)\} \subset \mathcal{H} \oplus \mathcal{H}$$

is closed, then A is called closed.

Remark 2.2. (1) For any sequence (x_n, Ax_n) in $G(A)$ such that

$$(x_n, Ax_n) \to (x, y),$$

because $(x, y) \in G(A)$, $Ax = y$.

(2) By the closed graph theorem, if A is closed and $D(A) = \mathcal{H}$, then A is bounded.

Proposition 2.3. *Let $A, B \in \mathcal{L}(\mathcal{H})$.*

(1) If A is densely defined, then A^* is closed.

(2) If A is closed and B is bounded, then $A + B$ is closed.

Proof. (1) Let $y_n \in D(A^*)$ such that $y_n \to y$ and $A^*y_n \to z$. For $x \in D(A)$,

$$\langle Ax, y_n \rangle = \langle x, A^*y_n \rangle .$$

As $n \to \infty$, $\langle Ax, y \rangle = \langle x, z \rangle$. Therefore, $y \in D(A^*)$ and $A^*y = z$. It follows that $G(A^*)$ is closed.

(2) Let x_n in $D(A + B) = D(A)$ such that $x_n \to x$ and $(A + B)x_n \to y$. Because B is bounded, $Bx_n \to Bx$ and $Ax_n \to y - Bx$. Furthermore, since A is closed, $x \in D(A)$ and $Ax = y - Bx$. Therefore, $x \in D(A + B)$ and $(A + B)x = y$.

□

Definition 2.4 (Closable Operator)**.** Let $A \in \mathcal{L}(\mathcal{H})$. A is called closable if there is a closed operator B such that $A \subset B$. In such case, B is called a closed extension of A.

Proposition 2.5. *Let $A \in \mathcal{L}(\mathcal{H})$. A is closable if and only if for any sequence x_n in $D(A)$ such that $x_n \to 0$ and $Ax_n \to y$, $y = 0$.*

Proof. If A is closable, the statements are clearly true. Conversely, it is sufficient to prove that $\overline{G(A)}$ is a graph of some linear operator. Because the linearity of $\overline{G(A)}$ is clear, it is sufficient to prove $y = y'$ for any $(x, y), (x, y') \in \overline{G(A)}$. Let x_n and x'_n be two sequences in $\overline{G(A)}$ such that

$$(x_n, Ax_n) \to (x, y), \quad (x'_n, Ax'_n) \to (x, y')$$

Because $x_n - x'_n \to 0$ and $Ax_n - Ax'_n \to y - y'$, $y = y'$ by the assumption. □

Definition 2.6 (Closure of Closable Operator)**.** By above proposition, when $A \in \mathcal{L}(\mathcal{H})$ is closable, the closed operator whose graph is $\overline{G(A)}$ is called the closure of A, denoted by $\overline{A}$.

Proposition 2.7. *Let $A, B \in \mathcal{L}(\mathcal{H})$.*

(1) $A \subset B$ implies that $\overline{A} \subset \overline{B}$.

(2) $\overline{\mathrm{Im}(\overline{A})} = \overline{\mathrm{Im}(A)}$.

(3) If A is closable and B is bounded, then $A + B$ is closable and

$$\overline{A+B} = \overline{A} + B$$

Proof. (1) It is clear.

(2) Clearly, $\overline{\mathrm{Im}(\overline{A})} \supset \overline{\mathrm{Im}(A)}$. Let $y \in \mathrm{Im}(\overline{A})$, so by definition there is a sequence x_n in $D(A)$ such that $Ax_n \to y$, i.e., $\mathrm{Im}(\overline{A}) \subset \overline{\mathrm{Im}(A)}$. Therefore, $\overline{\mathrm{Im}(\overline{A})} \subset \overline{\mathrm{Im}(A)}$.

(3) Let x_n be a sequence in $D(A+B) = D(A)$ such that $x_n \to 0$ and $(A+B)x_n \to y$. Because $Bx_n \to 0$, $Ax_n \to y$. Because A is closable, $Ax_n \to 0 = y$. So $A + B$ is closable.

First, $A \subset \overline{A}$, so $A + B \subset \overline{A} + B$ and thus

$$\overline{A+B} \subset \overline{A} + B$$

Conversely, let $x \in D(\overline{A} + B) = D(\overline{A})$. Choose a sequence x_n in $D(A)$ such that $x_n \to x$ and $Ax_n \to \overline{A}x$. Then

$$x_n \to x, \quad (A+B)x_n \to \overline{A}x + Bx.$$

Therefore, $x \in D(\overline{A+B})$ and $\overline{A+B} = \overline{A} + B$.

□

Lemma 2.8. *Let $A \in \mathcal{L}(\mathcal{H})$ be densely defined.*

$$G(A)^{\perp} = \{(-A^*y, y) \colon y \in D(A^*)\}.$$

Proof. For any $x \in D(A)$ and $y \in D(A^*)$,

$$(-A^*y, y) \perp (x, Ax)$$

in $\mathcal{H} \oplus \mathcal{H}$. So

$$G(A)^{\perp} \supset \{(-A^*y, y) \colon y \in D(A^*)\}.$$

Conversely, let $(x', y') \perp G(A)$. For any $x \in D(A)$, $(x, Ax) \perp (x', y')$ implies that

$$\langle x, x' \rangle = - \langle Ax, y' \rangle$$

Therefore, $y' \in D(A^*)$ and $A^*y' = -x'$. It follows that

$$G(A)^{\perp} \subset \{(-A^*y, y) \colon y \in D(A^*)\}.$$

□

Theorem 2.9. *Let $A \in \mathcal{L}(\mathcal{H})$ be densely defined. A is closable if and only if $D(A^*)$ is dense. In such case, we have*

$$A^{**} = \overline{A}, \quad \overline{A}^* = A^*.$$

Proof. Assume A is closable. Let $z \in D(A^*)^\perp$. Then

$$(0, z) \in \{(-A^* y, y) \mid y \in D(A^*)\}^\perp$$

Therefore,

$$(0, z) \in G(A)^{\perp\perp} = \overline{G(A)} = G(\bar{A}),$$

which implies that $z = 0$. So $D(A^*)$ is dense.

Conversely, assume $D(A^*)$ is dense. Let $(0, z) \in \overline{G(A)}$. Then

$$(0, z) \in \{(-A^* y, y) \mid y \in D(A^*)\}^\perp$$

by above lemma. Therefore, $z \in D(A^*)^\perp = \{0\}$. It follows that A is closable.

In such case, $D(A^*)$ is dense, so it can consider A^{**}. By the similar prove of above lemma,

$$G(A)^{\perp\perp} = \{(-A^* y, y) \colon y \in D(A^*)\}^\perp = \{(z, A^{**} z) \colon z \in D(A^{**})\} = G(A^{**})$$

Therefore,

$$G(\overline{A}) = \overline{G(A)} = G(A)^{\perp\perp} = G\left(A^{**}\right),$$

i.e., $\overline{A} = A^{**}$.

Because

$$G(\overline{A})^\perp = G(A)^\perp$$

by above lemma, $G(A^*) = G(\overline{A}^*)$. So $A^* = \overline{A}^*$. □

3 Spectrum

Definition 3.1 (Resolvent Set)**.** Let $A \in \mathcal{L}(\mathcal{H})$ be densely defined. For $\lambda \in \mathbb{C}$, if

(1) $A - \lambda I$ is injective;

(2) $\operatorname{ran}(A - \lambda I)$ is dense;

(3) $(A - \lambda I)^{-1}$ is bounded on $\operatorname{ran}(A - \lambda I)$,

then $\lambda \in \rho(A)$, which is called the resolvent set, and $\sigma(A) = \mathbb{C} \backslash \rho(A)$ is called the spectrum of A. Furthermore, $R_\lambda(A) = (A - \lambda I)^{-1}$.

Definition 3.2 (Spectrum)**.** Let $A \in \mathcal{L}(\mathcal{H})$ be densely defined.

(1) If $A - \lambda I$ is not injective, then λ is called an eigenvalue of A, and the set $\sigma_p(A)$ of all eigenvalues is called the point spectrum of A.

(2) If $A - \lambda I$ is injective but $\operatorname{ran}(A - \lambda I)$ is not dense, then the set $\sigma_r(A)$ of all such λ is called the residual spectrum of A.

(3) If $A - \lambda I$ is injective and $\operatorname{ran}(A - \lambda I)$ is dense, but $(A - \lambda I)^{-1}$ is not bounded, then the set $\sigma_c(A)$ of all such λ is called the continuous spectrum of A.

Remark 3.3. Note that the spectrum of A

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

Proposition 3.4. *Let $A \in \mathcal{L}(\mathcal{H})$ be densely defined. TFAE.*

(1) $\lambda \in \sigma_c(A)$.

(2) $A-\lambda I$ *is injective,* $\overline{\mathrm{Im}(A-\lambda I)} = \mathcal{H}$ *and*

$$\forall\, c>0, \quad \exists\, x \in D(A)\backslash\{0\}, \quad \|(A-\lambda I)x\| \le c\|x\|.$$

Proof. (1) $\Rightarrow$ (2): Assume that there is a $c>0$ such that $\|(A-\lambda I)x\| \le c\|x\|$ for all $x \in D(A)\backslash\{0\}$. Then for any $y \in \mathrm{Im}(A-\lambda I)$, let $x \in D(A)$ be $y=(A-\lambda I)x$. So

$$\left\|(A-\lambda I)^{-1}y\right\| = \|x\| \le \frac{1}{c}\|y\|,$$

i.e., $(A-\lambda I)^{-1}$ is bounded, contradicting to $\lambda \in \sigma_c(A)$.
(2) $\Rightarrow$ (1): Similarly as above, if $(A-\lambda I)^{-1}$ is bounded, $A-\lambda I$ is bounded below. □

Proposition 3.5. *Let* $A \in \mathcal{L}(\mathcal{H})$ *be densely defined and closable.*

(1) When $\lambda \in \rho(A)$, $\mathrm{Im}(\overline{A}-\lambda I) = \mathcal{H}$.

(2) $\rho(A) = \rho(\overline{A})$.

(3) When $\lambda \in \rho(A)$, $\overline{R_\lambda(A)} = R_\lambda(\overline{A})$.

Proof. (1) Let $\lambda \in \rho(A)$ and $y \in \mathcal{H}$. Choose a sequence x_n in $D(A)$ such that

$$y_n = (A-\lambda I)x_n \to y.$$

and so y_n is Cauchy. Because $(A-\lambda I)^{-1}$ is bounded,

$$x_n = (A-\lambda I)^{-1}y_n$$

is also Cauchy. So $x_n \to x$ for some x. Because $A-\lambda I$ is closable, $x \in D(\overline{A-\lambda I}) = D(\overline{A}-\lambda I)$. Therefore, $(\overline{A}-\lambda I)x = y$. Therefore, $\overline{A}-\lambda I$ is bijective.

(2) First, fix $\lambda \in \rho(A)$. Let $x \in \ker(\overline{A}-\lambda I)$ and choose x_n in $D(A)$ such that $x_n \to x$ and $(A-\lambda I)x_n \to 0$. Because $(A-\lambda I)^{-1}$ is bounded, $x_n \to 0 = x$. So $\overline{A}-\lambda I$ is injective. By (1), $\overline{A}-\lambda I$ is surjective. Then the closedness $G(\overline{A}-\lambda I)$ implies that the closedness of $G\left((\overline{A}-\lambda I)^{-1}\right)$. By the closed graph theorem, $(\overline{A}-\lambda I)^{-1}$ is bounded. Therefore, $\lambda \in \rho(\overline{A})$.

Conversely, for $\lambda \in \rho(\overline{A})$, by $A-\lambda I \subset \overline{A}-\lambda I$, the injectivity of $A-\lambda I$ and the boundedness of $(A-\lambda I)^{-1}$ are clear. Furthermore, $\overline{\mathrm{Im}(A-\lambda I)} = \overline{\mathrm{Im}(\bar{A}-\lambda I)} = \mathcal{H}$. So $\lambda \in \rho(A)$.

(3) For $\lambda \in \rho(A)$, it is because $\overline{R_\lambda(A)}$ and $R_\lambda(\overline{A})$ have same graphs. □

Proposition 3.6. *Let* $A \in \mathcal{L}(\mathcal{H})$ *be densely defined and closed.* $\lambda \in \rho(A)$ *if and only if* $A-\lambda I\colon D(A) \to \mathcal{H}$ *is bijective.*

Proof. If $\lambda \in \rho(A)$, by above (1), $A-\lambda I$ is bijective.

Conversely, it is sufficient to prove $(A-\lambda I)^{-1}$ is bounded, i.e., its graph is closed by the closed graph theorem. It is because A is a closed operator. □

Proposition 3.7. *Let* $A \in \mathcal{L}(\mathcal{H})$ *be densely defined and closed. For* $\lambda, \mu \in \rho(A)$,

$$R_\lambda(A) - R_\mu(A) = (\lambda-\mu)R_\lambda(A)R_\mu(A)$$

Proof. It is because

$$R_\lambda(A) - R_\mu(A) = R_\lambda(A)((A-\mu I)-(A-\lambda I))R_\mu(A).$$ □

Proposition 3.8. *Let $A \in \mathcal{L}(\mathcal{H})$ be densely defined and closed. Let $\lambda_0 \in \rho(A)$. For $\varepsilon = \|R_{\lambda_0}(A)\|^{-1}$,*

$$B(\lambda_0, \varepsilon) \subset \rho(A),$$

i.e., $\rho(A)$ is open. Moreover, if $\lambda \in B(\lambda_0, \varepsilon)$,

$$R_\lambda(A) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}(A)^{n+1}.$$

Proof. For $\lambda \in B(\lambda_0, \varepsilon)$, let

$$K_\lambda = (\lambda - \lambda_0) R_{\lambda_0}(A),$$

whose domain is $\mathcal{H}$ by the definition. So

$$A - \lambda I = (I - K_\lambda)(A - \lambda_0 I)$$

Moreover, because $\|K_\lambda\| = |\lambda - \lambda_0| \, \|R_{\lambda_0}(A)\| < 1$,

$$(I - K_\lambda)^{-1} = \sum_{n=0}^{\infty} K_\lambda^n$$

in norm convergence. Therefore, $I - K_\lambda$ is bijective from $\mathcal{H}$ to $\mathcal{H}$. Because $A - \lambda_0 I$ is bijective from $D(A)$ to $\mathcal{H}$, $A - \lambda I = (I - K_\lambda)(A - \lambda_0 I)$ is also bijective from $D(A)$ to $\mathcal{H}$. So $\lambda \in \rho(A)$ by above proposition. Then

$$(A - \lambda I)^{-1} = (A - \lambda_0 I)^{-1} (I - K_\lambda)^{-1} = R_{\lambda_0}(A) \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}(A)^n. \qquad \square$$

Proposition 3.9. *Let $A \in \mathcal{L}(\mathcal{H})$ be bounded.*

$$\varnothing \neq \sigma(A) \subset \{\lambda \in \mathbb{C} | |\lambda| \leq \|A\|\}$$

Proof. If $|\lambda| > \|A\|$,

$$(A - \lambda I)^{-1} = -\lambda^{-1} \left(I - \lambda^{-1} A\right)^{-1} = -\lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} A^n,$$

which implies that $\lambda \notin \sigma(A)$.

Assume $\sigma(A) = \emptyset$. For $\lambda_0 \in \mathbb{C}$, when λ is closed to λ_0,

$$f(\lambda) := (R_\lambda(A)x, y) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n \left(R_{\lambda_0}(A)^{n+1} x, y\right),$$

i.e., f is regular on $\mathbb{C}$. Moreover, for $\lambda > \|A\|$,

$$|f(\lambda)| \leq \sum_{n=0}^{\infty} |\lambda|^{-n-1} \|A\|^n \|x\| \|y\| = |\lambda|^{-1}(1 - \|A\|/|\lambda|)$$

Therefore, $f(\lambda) \to 0$ as $\lambda \to \infty$. By Liouville's Theorem, $f(\lambda) = 0$, i.e., $R_\lambda(A) = 0$, which induces a contradiction. $\square$

4 Symmetric Operator

Definition 4.1 (Symmetric Operator)**.** Let $A \in \mathcal{L}(\mathcal{H})$ be densely defined. If for any $x, y \in D(A)$,

$$\langle Ax, y \rangle = \langle x, Ay \rangle ,$$

A is called symmetric or Hermitian.

Remark 4.2. Note that for densely defined A, it is symmetric if and only if $A \subset A^*$. So symmetric operators are closable.

Proposition 4.3. *Let $A \in \mathcal{L}(\mathcal{H})$ be densely defined.*

(1) A is symmetric if and only if for any $x \in D(A)$, $\langle Ax, x \rangle \in \mathbb{R}$.

(2) If A is symmetric, then A is closable and $\overline{A}$ is also symmetric.

(3) If A is symmetric and $D(A) = \mathcal{H}$, then A is bounded.

Proof. (1) If A is symmetric, $\langle Ax, x \rangle \in \mathbb{R}$ clearly. In converse, because $\langle Ax, x \rangle \in \mathbb{R}$,

$$\langle Ax, y \rangle = \sum_{k=0}^{3} i^k \left\langle A\left(x + i^k y\right), x + i^k y \right\rangle ,$$

and

$$\sum_{k=0}^{3} i^k \left\langle x + i^k y, A\left(x + i^k y\right) \right\rangle = (x, Ay).$$

Therefore, $\langle Ax, y \rangle = \langle x, Ay \rangle$.

(2) If A is symmetric, $A \subset A^*$. Because $D(A)$ is dense, A^* is closed. So A is closable. Moreover,

$$\overline{A} = A^{**} \subset A^* = \overline{A}^*.$$

So $\overline{A}$ is symmetric.

(3) Because $A \subset A^*$ and $D(A) = \mathcal{H}$, $A = A^*$. Therefore, A is closed. By the closed graph theorem, A is bounded. □

Proposition 4.4. *Let A be symmetric.*

(1) $\sigma_p(A) \subset \mathbb{R}$.

(2) For $\lambda, \mu \in \sigma_p(A)$ with $\lambda \neq \mu$, and $Ax = \lambda x$, $Ay = \mu y$, $\langle x, y \rangle = 0$.

Definition 4.5 (Order)**.** Let A be symmetric. For $\alpha \in \mathbb{R}$, if

$$\alpha \|A\|^2 \leq \langle Ax, x \rangle , \quad \forall\, x \in D(A),$$

then $\alpha \leq A$.

Definition 4.6 (Self-adjoint Operator)**.** Let A be densely defined. If $A = A^*$, then A is called self-adjoint.

Proposition 4.7. *Let A be self-adjoint. If B is symmetric and $A \subset B$, then $A = B$.*

Proof. It is because $A \subset B \subset B^* \subset A^* = A$. □

Proposition 4.8. *Let A be closed.*

(1) $\ker A$ *is closed.*

(2) If A is dense, then

$$\mathcal{H} = \ker A \oplus \overline{\operatorname{Im} A^*}$$

Proof. (1) Let x_n be a sequence in $\ker A$ and $x_n \to x$. Then $x - x_n \to 0$ and $A(x - x_n) = Ax - Ax_n \to Ax$. Because A is closed, $Ax = 0$, i.e., $x \in \ker A$.

(2) Let $x \in \ker A$ and $y \in D(A^*)$. Because

$$0 = \langle Ax, y\rangle = \langle x, A^* y\rangle\,,$$

$\ker A \perp \operatorname{Im} A^*$. Moreover, if $z \perp \operatorname{Im} A^*$ and $y \in D(A^*)$, i.e., $\langle z, A^* y\rangle = 0$, then $z \in D(A^{**}) = D(A)$. Furthermore, because $D(A^*)$ is dense, $Az = A^{**}z = 0$, i.e., $z \in \ker A$. □

Lemma 4.9. *Let A be a closed operator. The following statements are equivalent.*

(1) $\operatorname{Im} A$ *is closed.*

(2) There exists $C > 0$ such that for all $x \in D(A) \cap (\ker A)^\perp$,

$$\|Ax\| \geq C\|x\|.$$

Proof. Assume (1). Consider $A\colon D(A) \cap (\ker A)^\perp \to \operatorname{Im} A$. It is a bijection between Hilbert spaces $D(A) \cap (\ker A)^\perp$ and $\operatorname{Im} A$ by closedness. Then by the closed graph theorem, A^{-1} is bounded.

Conversely, assume (2). Choose $x_n \in D(A)$ such that $Ax_n \to y$. Let

$$x_n = x_n' + x_n''$$

with $x_n' \in D(A) \cap (\ker A)^\perp$ and $x_n'' \in \ker A$. So $Ax_n' \to y$. By (2), x_n' is Cauchy, so $x_n' \to x \in \mathcal{H}$. Because $G(A)$ is closed, $(x_n', Ax_n') \to (x, y) \in G(A)$. It follows that $Ax = y \in \operatorname{Im} A$. □

Proposition 4.10. *Let A be a symmetric operator.*

(1) For $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $x \in D(A)$,

$$\|(A - \lambda I)x\| \geq |\operatorname{Im} \lambda|\, \|x\|.$$

(2) If $A \geq 0$ and $\lambda > 0$, then

$$\|(A + \lambda I)x\| \geq \lambda\|x\|.$$

(3) If A is closed, for $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $\operatorname{Im}(A - \lambda I)$ *is closed.*

(4) If A is closed and $A \geq 0$, for any $\lambda > 0$, then $\operatorname{Im}(A + \lambda I)$ *is closed.*

Proof. (1) Let $\lambda = \alpha + \beta i$ for $\alpha, \beta \in \mathbb{R}$.

$$\langle (A - \lambda I)x, (A - \lambda I)x\rangle = \|(A - \alpha I)x\|^2 + \beta^2\|x\|^2 \geq \beta^2\|x\|^2$$

(2) It is because

$$\langle (A + \lambda I)x, (A + \lambda I)x\rangle = \|Ax\|^2 + 2\lambda(Ax, x) + \lambda^2\|x\|^2 \geq \lambda^2\|x\|^2.$$

(3) It is by above $\ker(A - \lambda I) = 0$, (1), and above lemma.

(4) It is by above $\ker(A + \lambda I) = 0$, (2), and above lemma.

$\square$

Theorem 4.11. *Let A be self-adjoint.*

(1) $\sigma(A) \subset \mathbb{R}$.

(2) For $\gamma \in \mathbb{R}$, if $A \geq \gamma$, $\sigma(A) \subset [\gamma, \infty)$.

(3) $\sigma_r(A) = \emptyset$.

(4) $\sigma(A) = \sigma_{ap}(A)$, where $\sigma_{ap}(A)$ is the approximated point spectrum of A, defined as

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} \colon \exists\ x_n \text{ with } \|x_n\| = 1 \text{ such that } (A - \lambda I)x_n \to 0\}$$

Proof. (1) For $\lambda \in \mathbb{C} \backslash \mathbb{R}$, by (1) in the above proposition, $A - \lambda I$ is injective and its inverse is bounded. Because $A = A^*$,

$$\mathcal{H} = \ker(A - \overline{\lambda} I) \oplus \overline{\operatorname{Im}(A - \lambda I)}$$

So $\mathcal{H} = \overline{\operatorname{Im}(A - \lambda I)}$.

(2) When $\lambda < \gamma$, by (2) in the above proposition, $A - \lambda I$ is injective and its inverse is bounded. Similarly, $\mathcal{H} = \overline{\operatorname{Im}(A - \lambda I)}$.

(3) For $\lambda \in \mathbb{R}$,

$$\mathcal{H} = \ker(A - \lambda I) \oplus \overline{\operatorname{Im}(A - \lambda I)}$$

Then if $\ker(A - \lambda I) = 0$, $\operatorname{Im}(A - \lambda I)$ is dense.

(4) Fix $\lambda \in \sigma(A)$. Assume there exits a $C > 0$ such that

$$\|(A - \lambda I)x\| \geq C\|x\|, \quad \forall\ x \in D(A)$$

Therefore, $A - \lambda I$ is injective and its inverse is bounded. But because $\lambda \in \mathbb{R}$, by (3), $\operatorname{Im}(A - \lambda I)$ is dense. $\lambda \in \rho(A)$, which induces a contradiction. So $\lambda \in \sigma_{ap}(A)$

Conversely, it is by the following remark.

$\square$

Remark 4.12. (i) It is clear that $\sigma_p(A) \subset \sigma_{ap}(A)$.

(ii) $\sigma_c(A) \subset \sigma_{ap}(A)$: If $\lambda \in \sigma_c(A)$, for $1/n$, choose x_n with $\|x_n\| = 1$ such that

$$\|(A - \lambda)x_n\| \leq \frac{1}{n}\|x_n\|.$$

So $\lambda \in \sigma_{ap}(A)$.

(iii) $\sigma_{ap}(A) \subset \sigma(A)$: If $\lambda \in \rho(A)$, then $(A - \lambda I)^{-1}$ is bounded, which implies that

$$\exists\ C > 0, \quad \forall\ x \in D(A), \quad \|(A - \lambda I)x\| \geq C\|x\|$$

It follows that $\lambda \notin \sigma_{ap}(A)$.

(iv) For self-adjoint operator, $\sigma = \sigma_p \cup \sigma_c = \sigma_{ap}$.

Theorem 4.13. *Let A be a symmetric operator. TFAE.*

(1) A is self-adjoint.

(2) A is closed and $\ker(A^* \pm iI) = 0$.

(3) $\operatorname{Im}(A \pm iI) = \mathcal{H}$.

Proof. (1) $\Rightarrow$ (2): Clearly, A is closed. $\sigma_p(A) = \sigma_p(A^*) \subset \mathbb{R}$, so $\ker(A^* \pm iI) = 0$.

(2) $\Rightarrow$ (3): Because A is closed, A^* is densely defined. And A^* is closed. So

$$\mathcal{H} = \ker(A^* \pm iI) \oplus \overline{\operatorname{Im}(A^{**} \mp iI)}.$$

Closedness of A implies that $A = \overline{A} = A^{**}$. So $\operatorname{Im}(A \pm iI)$ is dense.

(3) $\Rightarrow$ (1): It suffices to prove $D(A^*) \subset D(A)$. For any $x \in D(A^*)$, let $y \in D(A)$ such that

$$(A^* - iI)x = (A - iI)y.$$

Since $A \subset A^*$, $(A^* - iI)(x - y) = 0$. Because $\operatorname{Im}(A + iI) = \mathcal{H}$ and $\ker(A^* - iI) \perp \operatorname{Im}(A + iI)$, $\ker(A^* - iI) = 0$. Therefore, $x = y$. □

Proposition 4.14. *Let A be a closed and symmetric operator. TFAE.*

(1) A is self-adjoint.

(2) $\sigma(A) \subset \mathbb{R}$.

Proof. (2) $\Rightarrow$ (1): Because A is closed and $\pm i \in \rho(A)$, $\operatorname{Im}(A \pm iI) = \mathcal{H}$. By above theorem, A is self-adjoint. □

Definition 4.15 (Essentially Self-adjoint)**.** Let A be a symmetric operator. If $\overline{A}$ is self-adjoint, then A is called essentially self-adjoint.

Proposition 4.16. *Let A be essentially self-adjoint. If $A \subset B$ and B is a closed symmetric operator, then $B = \overline{A}$.*

Proof. $A \subset B$ implies that $\overline{A} \subset B$. Then

$$B \subset B^* \subset \overline{A}^* = \overline{A}$$

Therefore, $B = \overline{A}$. □

Proposition 4.17. *Let A be a symmetric operator. TFAE.*

(1) A is essentially self-adjoint.

(2) $\ker(A^* \pm iI) = 0$.

(3) $\overline{\operatorname{Im}(A \pm iI)} = \mathrm{H}$.

Proof. By replacing A with $\overline{A}$ in the proof of above theorem, it can get the desired results. □

5 Resolution of Identity

Definition 5.1 (Operator Topology)**.** Consider $A_n \in \mathcal{B}(\mathrm{H})$, the space of bounded linear operators.

(1) Strong operator topology (SOT): $A_n \to A$ in SOT if

$$\forall\, x \in \mathcal{H}, \quad \|A_n x - Ax\| \to 0.$$

(2) Weak operator topology (WOT): $A_n \to A$ in WOT of

$$\forall\, x, y \in \mathcal{H}, \quad \langle A_n x, y\rangle \to \langle Ax, y\rangle\,.$$

Remark 5.2. By the Cauchy-Schwarz inequality, convergence in SOT implies convergence in WOT.

Proposition 5.3. *Let $A_n, B_n \in \mathcal{B}(\mathcal{H})$. Let $A, B \in \mathcal{B}(\mathcal{H})$.*

(1) If $A_n \to A$ and $B_n \to B$ in SOT, then $A_n B_n \to AB$ in SOT.

(2) If $A_n \to A$ in WOT, $A_n^* \to A^*$ in WOT.

Proof. (1) For any $x \in \mathcal{H}$, because $\{A_n x\}$ is convergent, it is bounded. By the principle of uniform boundedness, $\{\|A_n\|\}$ is bounded. Then

$$\begin{aligned}\|A_n B_n x - ABx\| &\le \|A_n B_n x - A_n Bx\| + \|A_n Bx - ABx\| \\ &\le \left(\sup_n \|A_n\|\right) \|B_n x - Bx\| + \|A_n Bx - ABx\| \to 0\end{aligned}$$

(2) For $x, y \in \mathrm{H}$,

$$\langle A_n^* x, y\rangle = \langle x, A_n y\rangle = \overline{\langle A_n y, x\rangle} \to \overline{\langle Ay, x\rangle} = \langle x, Ay\rangle = \langle A^* x, y\rangle\,.$$

□

Remark 5.4. Note that (1) is not true for WOT and (2) is not true for SOT.

Proposition 5.5. *Let $\{P_n\}$ be a sequence of orthogonal projections and $P \in \mathcal{B}(\mathcal{H})$.*

(1) If $P_n \to P$ in SOT, then P is also an orthogonal projection.

(2) If $P_n \to P$ in WOT and $P^2 = P$, then P is also an orthogonal projection and $P_n \to P$ in SOT.

Proof. (1) By above proposition, $P_n = P_n^* \to P^*$ in WOT, so $P = P^*$. Similarly, $P_n = P_n^2 \to P^2$ in SOT, so $P = P^2$.

(2) P is an orthogonal projection by similar reason as above. For $x \in \mathcal{H}$,

$$\begin{aligned}\|P_n x - Px\|^2 &= (P_n x, x) - 2\,\mathrm{Re}\,(P_n x, Px) + (Px, x) \\ &\to (Px, x) - 2\,\mathrm{Re}(Px, Px) + (Px, x) = 0\end{aligned}$$

□

Proposition 5.6. *Let $\{P_n\}$ be a sequence of orthogonal projections with $P_n \le P_{n+1}$. Then there is a P such that $P_n \to P$ in SOT. It is also true when $P_n \ge P_{n+1}$.*

Proof. For $x \in \mathcal{H}$, $\langle P_n x, x\rangle$ is a monotone increasing sequence with bound $\|x\|$. Therefore, it has a limit. For $m > n$,

$$\|P_m x - P_n x\| = \langle P_m x, x\rangle - \langle P_n x, x\rangle \to 0$$

Therefore, define $Px := \lim_n P_n x$, which is clearly a bounded linear operator, so it is an orthogonal projection by above proposition. The same results can be obtained for $P_n \geq P_{n+1}$. □

Definition 5.7 (Resolution of Identity)**.** Let $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be a family of orthogonal projections such that

(1) if $\lambda \leq \mu$, then $E(\lambda) \leq E(\mu)$,

(2) when $\lambda \to \infty$, $E(\lambda) \to I$ in SOT, meanwhile $E(\lambda) \to 0$ as $\lambda \to -\infty$,

(3) when $\varepsilon \to 0+$, $E(\lambda + \varepsilon) \to E(\lambda)$ in SOT.

Then it is called a resolution of the identity.

Remark 5.8. Note that if $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is a resolution of the identity, then $E(\lambda - \varepsilon)$ is also convergent to an orthogonal projection as $\varepsilon \to 0+$.

Fix $x, y \in \mathcal{H}$. Because $\lambda \mapsto \langle E(\lambda)x, x\rangle$ is monotone increasing and right-continuous, it can consider the Lebesgue-Stieltjes integral $\int_{\mathbb{R}} \cdot d\langle E(\lambda)x, x\rangle$, which can be extended to $\int_{\mathbb{R}} \cdot d\langle E(\lambda)x, y\rangle$ by the polarization

$$\langle E(\lambda)x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle E(\lambda)(x + i^k y), x + i^k y\right\rangle.$$

For a complex-valued continuous function f, consider $A(f)$, whose domain is defined as

$$D(A(f)) := \left\{x \in \mathcal{H}\colon \int_{\mathbb{R}} |f(\lambda)|\, d\langle E(\lambda)x, x\rangle < \infty\right\}.$$

Theorem 5.9. *As above settings,*

(1) $D(A(f))$ is a dense subspace of $\mathcal{H}$.

(2) for $x \in D(A(f))$ and $y \in \mathcal{H}$, the operator $A(f)$ is well-defined as

$$\langle A(f)x, y\rangle = \int_{\mathbb{R}} f(\lambda) d\langle E(\lambda)x, y\rangle.$$

(3) for $x \in D(A(f))$,

$$\|A(f)x\|^2 = \int_{\mathbb{R}} |f(\lambda)|^2\, d\langle E(\lambda)x, x\rangle.$$

(4) $E(\lambda)A(f) \subset A(f)E(\lambda)$.

(5) $A(f)^* = A(\overline{f})$.

(6) $A(f)$ is closed.

(7) if f is real-valued, then $A(f) = A(f)^*$.

(8) if f is bounded, $A(f)$ is bounded and $\|A(f)\| \leq \|f\|_\infty$.

(9) *if* $|f(\lambda)| \equiv 1$, $A(f)$ *is a unitary.*

Proof. (1) For $x \in D(A(f))$, it is clear $\alpha x \in D(A(f))$. For $x, y \in D(A(f))$,

$$\langle (E(\lambda) - E(\mu))(x+y), x+y\rangle \leq 2\langle (E(\lambda) - E(\mu))(x), x\rangle + 2\langle (E(\lambda) - E(\mu))(y), y\rangle$$

Therefore, $x + y \in D(A(f))$. So $D(A(f))$ is a subspace.

For any $x \in \mathcal{H}$ and $n \in \mathbb{N}$, let $x_n = (E(n) - E(-n))x$. Then $x_n \in D(A(f))$ and $x_n \to x$. So $D(A(f))$ is dense.

(2) It is sufficient to prove that for a fixed $x \in D(A(f))$

$$y \mapsto \int_{\mathbb{R}} \overline{f(\lambda)} d\langle E(\lambda)y, x\rangle$$

is a linear functional. First, linearity is clear. For any $a, b \in \mathbb{R}$ and $y \in \mathcal{H}$, let $a = \lambda_0 < \cdots < \lambda_n = b$ be a partition of $[a, b]$.

$$\begin{aligned}
&\left|\sum_{j=1}^{n} \overline{f(\lambda_j)} \langle (E(\lambda_j) - E(\lambda_{j-1}))y, x\rangle\right| \\
\leq &\sum_{j=1}^{n} |f(\lambda_j)| \, |\langle (E(\lambda_j) - E(\lambda_{j-1}))y, x\rangle| \\
\leq &\sum_{j=1}^{n} |f(\lambda_j)| \, \|(E(\lambda_j) - E(\lambda_{j-1}))y\| \, \|(E(\lambda_j) - E(\lambda_{j-1}))x\| \\
\leq &\sqrt{\sum_{j=1}^{n} |f(\lambda_j)|^2 \|(E(\lambda_j) - E(\lambda_{j-1}))x\|^2} \sqrt{\sum_{j=1}^{n} \|(E(\lambda_j) - E(\lambda_{j-1}))y\|^2} \\
\leq &\sqrt{\sum_{j=1}^{n} |f(\lambda_j)|^2 \langle E(\lambda_j) - E(\lambda_{j-1})x, x\rangle} \|y\|
\end{aligned}$$

Therefore, we have

$$\left|\int_a^b \overline{f(\lambda)} d\langle E(\lambda)y, x\rangle\right| \leq \sqrt{\int_a^b |f(\lambda)|^2 d\langle E(\lambda)x, x\rangle} \|y\|.$$

$x \in D(A(f))$ implies that the right-hand side is bounded. So $y \mapsto \int_{\mathbb{R}} \overline{f(\lambda)} d\langle E(\lambda)y, x\rangle$ is bounded. By Riesz representation theorem, there is a unique element, defined as $A(f)x$, such that

$$\langle y, A(f)x\rangle = \int_{\mathbb{R}} \overline{f(\lambda)} d\langle E(\lambda)y, x\rangle,$$

which is clear linear in x.

(3) For $x \in D(A(f))$, let $y = A(f)x$. Then

$$\|A(f)x\|^2 = \int_{\mathbb{R}} f(\lambda) d\langle E(\lambda)x, A(f)x\rangle.$$

Furthermore,

$$\langle E(\lambda)x, A(f)x\rangle = \int_{\mathbb{R}} \overline{f(\mu)} d\langle E(\lambda)x, E(\mu)x\rangle = \int_{-\infty}^{\lambda} \overline{f(\mu)} d\langle E(\mu)x, x\rangle$$

Therefore, we have $\|A(f)x\|^2 = \int_{\mathbb{R}} |f(\mu)|^2 d\langle E(\lambda)x, x\rangle$.

(4) For $x \in D(A(f))$,

$$\int_{\mathbb{R}} |f(\mu)|^2 \, d \langle E(\mu)E(\lambda)x, E(\lambda)(x) \rangle = \int_{-\infty}^{\lambda} |f(\mu)|^2 \, d \langle E(\mu)x, x \rangle$$
$$\leq \int_{\mathbb{R}} |f(\mu)|^2 \, d \langle E(\mu)x, x \rangle < \infty.$$

So $E(\lambda)x \in D(A(f))$. Furthermore,

$$\langle E(\lambda)A(f)x, y \rangle = \int_{\mathbb{R}} f(\mu) d \langle E(\mu)x, E(\lambda)y \rangle = \int_{\mathbb{R}} f(\mu) d \langle E(\mu)E(\lambda)x, y \rangle$$
$$= \langle A(f)E(\lambda)x, y \rangle .$$

So $E(\lambda)A(f) \subset A(f)E(\lambda)$.

(5) Clearly, $D(A(f)) = D(A(\overline{f}))$. For any $x, y \in D(A(f))$,

$$\overline{\langle A(f)x, y \rangle} = \overline{\int_{\mathbb{R}} f(\lambda) d \langle E(\lambda)x, y \rangle} = \int_{\mathbb{R}} \overline{f(\lambda)} d \langle E(\lambda)y, x \rangle = \left\langle A(\overline{f})y, x \right\rangle$$

Therefore, $A(\overline{f}) \subset A(f^*)$. Conversely, for $x \in D(A(f^*))$, $y \in \mathcal{H}$, and $n \in \mathbb{N}$, let

$$x_n = (E(n) - E(-n))x, \quad y_n = (E(n) - E(-n))y.$$

Then

$$\begin{aligned}
\|A(f)^*x\| &= \lim_{n\to\infty} \|(E(n) - E(-n))A(f)^*x\| \\
&= \lim_{n\to\infty} \sup_{\|y\|\leq 1} |\langle (E(n) - E(-n))A(f)^*x, y \rangle| \\
&= \lim_{n\to\infty} \sup_{\|y\|\leq 1} |\langle A(f)^*x, y_n \rangle| \\
&= \lim_{n\to\infty} \sup_{\|y\|\leq 1} |\langle x, A(f)y_n \rangle| \\
&= \lim_{n\to\infty} \sup_{\|y\|\leq 1} |\langle x_n, A(f)y_n \rangle| \\
&= \lim_{n\to\infty} \sup_{\|y\|\leq 1} \left|\left\langle A(\bar{f})x_n, y \right\rangle\right| \\
&= \lim_{n\to\infty} \left\|A(\bar{f})x_n\right\| \\
&= \lim_{n\to\infty} \sqrt{\int_{\mathbb{R}} |f(\lambda)|^2 d \langle x_n, E(\lambda)x_n \rangle} \\
&= \lim_{n\to\infty} \sqrt{\int_{-n}^{n} |f(\lambda)|^2 d \langle x, E(\lambda)x \rangle}
\end{aligned}$$

Therefore, $x \in D(A(f)) = D(A(\overline{f}))$.

(6) By (5), $A(f) = A(\overline{f})^*$ is closed.

(7) It is also by (5).

(8) It is by (3).

(9) First, $A(f)$ is bounded and $\|A(f)x\|^2 = \|x\|^2$ by (3). So $A(f)^*A(f) = I$ and $A(\overline{f})^*A(\overline{f}) = I$, which implies that $A(f)A(f)^* = I$. □

Such $A(f)$ is denoted by

$$A(f) = \int_{\mathbb{R}} f(\lambda) dE(\lambda).$$

Remark 5.10. Note that if $A = \int_{\mathbb{R}} \lambda dE(\lambda)$, then one can see

$$A^2 = \int_{\mathbb{R}} \lambda^2 dE(\lambda)$$

So by induction, we have $A^n = \in \lambda^n dE(\lambda)$. So by linearity, for any analytic function $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$,

$$A(f) = \sum_{n}^{\infty} a_n A^n.$$

Definition 5.11. Let $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be a resolution of the identity. If there are λ_1, λ_2 such that $E(\lambda_1) = 0$ and $E(\lambda_2) = I$, then $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is called bounded.

Proposition 5.12. *$A = \int_{\mathbb{R}} \lambda dE(\lambda)$ is bounded if and only if $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is bounded.*

Proof. If A is bounded, for any $x \in \mathcal{H}$,

$$\int_{\mathbb{R}} \lambda^2 d \langle E(\lambda)x, x\rangle = \|Ax\|^2 \leq \|A\|^2 \|x\|^2.$$

Assume there is a sequence $\lambda_n < 0$ with $\lambda_n \to -\infty$ such that $E(\lambda_n) \neq 0$. Choose a sequence $x_n \in E(\lambda_n)\mathcal{H}$ with $\|x_n\| = 1$.

$$\begin{aligned}
\int_{\mathbb{R}} \lambda^2 d \langle E(\lambda)x_n, x_n\rangle &= \int_{\mathbb{R}} \lambda^2 d \langle E(\lambda)E(\lambda_n)x_n, x_n\rangle \\
&= \int_{-\infty}^{\lambda_n} \lambda^2 d \langle E(\lambda)x_n, x_n\rangle \\
&\geq \lambda_n^2 \langle E(\lambda_n)x_n, x_n\rangle = \lambda_n^2,
\end{aligned}$$

which induces a contradiction. And the similar reasoning for $\lambda \to \infty$.

Conversely, if $E(\lambda_1) = 0$ and $E(\lambda_2) = I$, then

$$\int_{\mathbb{R}} \lambda^2 d \langle E(\lambda)x, x\rangle \leq \int_{\lambda_1}^{\lambda_2} \lambda^2 d \langle E(\lambda)x, x\rangle \leq \max\left\{\lambda_1^2, \lambda_2^2\right\} \|x\|^2.$$

Therefore, $D(A) = \mathcal{H}$ and $\|A\| \leq \max\{|\lambda_1|, |\lambda_2|\}$. □

Definition 5.13. Let $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be a resolution of the identity.

$$E(\{\lambda\}) = E(\lambda) - E(\lambda - 0).$$

If $E(\{\lambda\}) \neq 0$, then $E(\lambda)$ is called uncontinuous at λ.

Proposition 5.14. *Let $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be a resolution of the identity and $A = \int_{\mathbb{R}} \lambda dE(\lambda)$.*

(1) $x \in \ker(A - \mu I)$ if and only if the function $\lambda \mapsto \|E(\lambda)x\|^2$ is constant except for $\lambda = \mu$.

(2) $E(\{\mu\})$ is the orthogonal projection onto $\ker(A - \mu I)$.

Proof. (1) For $x \in D(A)$,

$$\|(A - \mu I)x\|^2 = \int_{\mathbb{R}} (\lambda - \mu)^2 d \langle E(\lambda)x, x\rangle.$$

(2) Let $x \in \ker(A - \mu I)$. By (1), when $\lambda < \mu$,

$$\|E(\lambda)x\| = \lim_{\lambda \to -\infty} \|E(\lambda)x\| = 0,$$

and when $\lambda > \mu$,

$$\|E(\lambda)x\| = \lim_{\lambda \to \infty} \|E(\lambda)x\| = \|x\|^2.$$

Therefore, $E(\mu - 0)x = 0$ and $E(\mu)x = x$.

Conversely, for any $x \in \operatorname{Im} E(\{\mu\})$,

$$E(\lambda)x = \begin{cases} x, & \lambda > \mu \\ 0, & \lambda < \mu. \end{cases}$$

Therefore, by (1), $x \in \ker(A - \mu I)$.

□

Lemma 5.15. *For a $A \in \mathcal{B}(\mathcal{H})$, if $0 \leq A \leq 1$, then $\|A\| \leq 1$.*

Proof. Because A is self-adjoint and $A \geq 0$, the Cauchy-Schwarz inequality implies that

$$\langle Ax, y \rangle \leq \langle Ax, x \rangle^{\frac{1}{2}} \langle Ay, y \rangle^{\frac{1}{2}}$$

Choose $y = Ax$ and $A \leq 1$,

$$\langle Ax, Ax \rangle = |\langle Ax, x \rangle|^{1/2} \left|\left\langle A^2x, Ax \right\rangle\right|^{1/2} \leq |\langle x, x \rangle|^{1/2} |\langle Ax, Ax \rangle|^{1/2}$$

Therefore, $\|Ax\| \leq \|x\|$. □

Lemma 5.16. *Let $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$ such that*

$$0 \leq A_1 \leq \cdots \leq A_n \leq \cdots \leq A.$$

Then A_n is convergent in SOT.

Proof. It can assume $A = \alpha I$. Furthermore, by replacing A_n by $\frac{1}{\alpha} A_n$, we can assume $\alpha = 1$.

For $x \in \mathcal{H}$, $\{\langle A_n x, x \rangle\}_{n \in \mathbb{N}}$ is a bounded and monotone increasing sequence, so it converges. Furthermore, if $m > n$, by Cauchy-Schwarz inequality,

$$\begin{aligned} \langle (A_m - A_n)x, (A_m - A_n)x \rangle &\leq \langle (A_m - A_n)x, x \rangle^{\frac{1}{2}} \left\langle (A_m - A_n)^2 x, (A_m - A_n)x \right\rangle^{\frac{1}{2}} \\ &\quad \langle (A_m - A_n)x, x \rangle^{\frac{1}{2}} \langle (A_m - A_n)x, (A_m - A_n)x \rangle^{\frac{1}{2}}, \end{aligned}$$

where the last inequality is because $A_m - A_n \leq 1$ and above lemma. Therefore,

$$\|(A_m - A_n)x\| \leq \langle (A_m - A_n)x, x \rangle^{\frac{1}{2}} \to 0,$$

and it is Cauchy. Let $Bx := \lim_{n \to \infty} A_n x$. Moreover, $\|A_n x\| \leq \|x\|$ by above lemma implies that $\|Bx\| \leq \|x\|$. So $B \in \mathcal{B}(\mathcal{H})$ and $A_n \to B$ in SOT. □

Theorem 5.17. *Let $A \in \mathcal{B}(\mathcal{H})$ with $A \geq 0$. Then there exists a unique $X \in \mathcal{B}(\mathcal{H})$ with $X \geq 0$ such that $X^2 = A$. Moreover, if $AT = TA$ for $T \in \mathcal{B}(\mathcal{H})$, then $XT = TX$.*

Proof. Constructing a sequence of polynomials $\{P_n(t)\}_{n\in\mathbb{N}}$ as $P_0(t) = 0$ and $P_{n+1}(t) = (t + P_n(t)^2)/2$. Therefore,

$$P_{n+1}(t) - P_n(t) = (P_n(t) - P_{n-1}(t))\,(P_n(t) + P_{n-1}(t))\,/2.$$

WLTG, assume $0 \le A \le 1$. Let $B = I - A$. Then $0 \le B \le 1$ and $B^{2k}, B^{2k+1} \ge 0$. Let $B_n = P_n(B) \ge 0$ and so $B_{n+1} - B_n = P_{n+1}(B) - P_n(B) \ge 0$. By

$$\langle B_{n+1}x, x\rangle = \frac{1}{2}\langle Bx, x\rangle + \frac{1}{2}\langle B_n x, B_n x\rangle \le \langle x, x\rangle\,,$$

$B_n \le 1$ and $\|B_n\| \le 1$. Therefore,

$$0 \le B_1 \le B_2 \le \cdots \le \cdots \le I$$

By above lemma, $B_n \to B_\infty$ in SOT and $0 \le B_\infty \le 1$. Let $X = I - B_\infty$. Because $B_\infty = (B_\infty + B_\infty^2)/2$,

$$X^2 = (I - B_\infty)^2 = I - 2B_\infty + B_\infty^2 = I - B_\infty = A.$$

Moreover, for any $T \in \mathcal{B}(\mathcal{H})$ with $TA = AT$, $TB_n = B_nT$ and so $TB_\infty = B_\infty T$ and $XT = TX$.

If there is another $Y \ge 0$ such that $Y^2 = A$. Because $YA = AY$, $XY = YX$. Therefore, $(X + Y)(X - Y) = X^2 - Y^2 = 0$. Let $X = X_1^2$ and $Y = Y_1^2$ with $X_1, Y_1 \ge 0$. For $x \in \mathcal{H}$, let $y = (X - Y)x$.

$$\|X_1 y\|^2 + \|Y_1 y\|^2 = \left\langle (X_1^2 + Y_1^2)y, y\right\rangle = \langle (X+Y)(X-Y)x, y\rangle = 0.$$

So $X_1 y = Y_1 y = 0$ and $Xy = Yy = 0$. Then

$$\|Xx - Yx\|^2 = \langle y, (X - Y)x\rangle = \langle (X - Y)y, x\rangle = 0.$$

Therefore, $X = Y$. □

Definition 5.18 (Root and Absolute)**.** For $A \ge 0$, let $X = A^{\frac{1}{2}}$. For any $A \in \mathcal{B}(\mathcal{H})$, let $|A| = (A^*A)^{\frac{1}{2}}$.

Definition 5.19 (Partial Isometry)**.** Let $U \in \mathcal{B}(\mathcal{H})$. If $U\colon (\ker U)^\perp \to \operatorname{Im} U$ is an isometry, then U is called a partial isometry.

Theorem 5.20 (Polar Decomposition)**.** *Let $A \in \mathcal{B}(\mathcal{H})$. Then there exists a unique partial isometry U such that* $\ker U = \ker A = \ker |A|$, $\operatorname{Im} U = \overline{\operatorname{Im} A}$, *and* $A = U|A|$.

Proof. For $x, y \in \mathcal{H}$, by

$$\langle Ax, Ay\rangle = \langle |A|\,x, |A|\,y\rangle\,,$$

$\ker A = \ker |A|$. Define $U_0\colon \operatorname{Im}|A| \to \operatorname{Im} A$ by

$$U_0\,|A|\,x = Ax.$$

So it is well-defined and inner product-preserving. Then it can be extended to $U\colon \overline{\operatorname{Im}|A|} \to \overline{\operatorname{Im} A}$. Moreover, let $U|_{\overline{\operatorname{Im}|A|}^\perp} = 0$. So U is a partial isometry and $\operatorname{Im} U = \overline{\operatorname{Im} A}$. Because

$$\overline{\operatorname{Im}|A|}^\perp = \ker|A| = \ker A,$$

$\ker U = \ker A$. So $A = U\,|A|$. The uniqueness is obvious. □

Definition 5.21. Let $A \in \mathcal{B}(\mathcal{H})$. Let $K \subset \mathcal{H}$ be a closed subspace and P is the orthogonal projection onto K. If $PD(A) \subset D(A)$ and $PA \subset AP$, then P (or K) is called reducing A.

Remark 5.22. In such case, $(I-P)A \subset A(I-P)$, i.e., K reducing A is equivalent to $K^{\perp}$ reducing A. Then consider $A_K\colon D(A_K) \to K$ for $D(A_K) = D(A) \cap K$, which is a linear operator on K.

Proposition 5.23. *Let $A \in \mathcal{B}(\mathcal{H})$. Let $K \subset \mathcal{H}$ be a closed subspace and P is the orthogonal projection onto K. If K reduces A, the the following statements are true.*

(1) If A is closed, then A_K is closed.

(2) If $D(A)$ is dense in $\mathcal{H}$, then $D(A_K)$ is dense in K. In such case, $(A^*)_K = (A_K)^*$.

(3) If A is unitary, then A_K is unitary.

(4) If A is self-adjoint, then A_K is self-adjoint.

(5) $\sigma(A) = \sigma(A_K) \cup \sigma(A_{K^{\perp}})$.

Proof. (1) Let $x_n \in K \cap D(A)$ with $x_n \to x$ and $Ax_n \to y$. Because K is closed, $x, y \in K$. Furthermore, because A is closed, $Ax = y$.

(2) Let $x \in K$. There exists $x_n \in D(A)$ such that $x_n \to x$ and so $Px_n \to x$. Because $Px_n \in D(A_K)$, $D(A_K)$ is dense.

$PA \subset AP$ implies that $PA^* \subset A^*P$. For $x \in D((A_K)^*) \subset K$ and $y \in D(A)$,

$$\langle (A_K)^*x, y\rangle = \langle (A_K)^*x, Py\rangle = \langle x, A_K Py\rangle = \langle x, APy\rangle = \langle x, PAy\rangle = \langle x, Ay\rangle$$

Therefore, $x \in D(A^*) \cap K$ and $(A_K)^* \subset (A^*)_K$.

Conversely, if $x \in D(A^*) \cap K$, for any $y \in D(A_K)$,

$$\langle (A^*)_K x, y\rangle = \langle A^*x, y\rangle = \langle x, Ay\rangle = \langle x, A_K y\rangle\,,$$

so $x \in D((A_K)^*)$.

(3) It is directly by (2).

(4) It is by (2).

(5) Let $\lambda \in \rho(A)$. Then $A_K - \lambda I_K$ and $A_{K^{\perp}} - \lambda I_{K^{\perp}}$ are injective. For any $y \in \mathcal{H}$, let $x_n \in D(A)$ with $(A-\lambda I)x_n \to y$. If $y \in K$, because $PA \subset AP$, $(A-\lambda I)Px_n \to Py = y$. So $\overline{\operatorname{Im}(A_K - \lambda I_K)} = K$. Similarly, $\overline{\operatorname{Im}(A_{K^{\perp}} - \lambda I_{K^{\perp}})} = K^{\perp}$. Moreover, let $\|(A-\lambda)^{-1}\| = C$. Therefore, it is clearly

$$\|(A_K - \lambda I_K)\,x\| \geq C\|x\|, \quad \forall\, x \in D(A_K),$$
$$\|(A_{K^{\perp}} - \lambda I_{K^{\perp}})\,x\| \geq C\|x\|, \quad \forall\, x \in D(A_{K^{\perp}}),$$

It follows that $\rho(A) \subset \rho(A_K) \cap \rho(A_{K^{\perp}})$.

Conversely, let $\lambda \in \rho(A_K) \cap \rho(A_{K^{\perp}})$. Then $A_K - \lambda I_K$ and $A_{K^{\perp}} - \lambda I_{K^{\perp}}$ are injective. So $A - \lambda I$ is also injective. For any $y \in \mathcal{H}$, there exist $x_n' \in D(A) \cap K$ such that $(A-\lambda I)x_n' \to Py$, and $x_n'' \in D(A) \cap K^{\perp}$ such that $(A-\lambda I)x_n'' \to (I-P)y$. Therefore, $x_n = x_n' + x_n'' \in D(A)$ and $(A-\lambda I)x_n \to y$. It follows that $\overline{\operatorname{Im}(A-\lambda I)} = \mathcal{H}$. Furthermore, let $C_1 = \|(A_K - \lambda I_K)^{-1}\|$ and $C_2 = \|(A_{K^{\perp}} - \lambda I_{K^{\perp}})^{-1}\|$. Then

$$\|(A-\lambda I)x\|^2 \geq C_1^2\|Px\|^2 + C_2^2\|(I-P)x\|^2,$$

which implies that $(A-\lambda I)^{-1}$ is bounded. □

6 Spectral Decomposition for Bounded Self-adjoint Operator

Theorem 6.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $A = U|A|$ be its polar decomposition.*

(1) $\operatorname{Im} A = \operatorname{Im}|A|$ *and* $U = U^*$.

(2) If A is commutative with all bounded operators, so is U.

(3) Therefore are reducing spaces $\mathcal{H}_\pm$ such that

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \ker A$$

and $A_+ \geq 0$ and $A_- \leq 0$.

(4) If K reduces A and $A_K \geq 0$, then $K \subset \mathcal{H}_+ \oplus \ker A$. So is $\mathcal{H}_-$.

Proof. (1) $A = A^* = |A|\, U^*$. So $\operatorname{Im} A \subset \operatorname{Im}|A|$. Conversely,

$$A^2 = AA^* = U\,|A|^2\, U^* = (U\,|A|\,U^*)^2$$

By the uniqueness of root, $|A| = U\,|A|\,U^* = AU^*$. So $\operatorname{Im}|A| \subset \operatorname{Im} A$.

Furthermore, we have

$$U^*|A| = U^*U|A|U^* = |A|U^*$$

and

$$\ker U^* = (\operatorname{Im} U)^\perp = (\operatorname{Im} A)^\perp = \ker A.$$

So by the uniqueness of the polar decomposition, $U = U^*$.

(2) If $AB = BA$, then $|A|\,B = B\,|A|$ because A is self-adjoint. So

$$BU\,|A| = U\,|A|\,B.$$

Therefore, $BU = UB$ on $\operatorname{Im}|A|$. On the other hand, for any $x \in \ker|A| = (\operatorname{Im}|A|)^\perp$, because $\ker|A| = \ker U$, $BUx = 0$. Moreover, $ABx = BAx = 0$, i.e., $Bx \in \ker A = \ker U$. It follows that $UBx = 0$. Therefore, $UB = BU$.

(3) First, $\mathcal{H} = \ker A \oplus \overline{\operatorname{Im} A}$. $U^2 = U^*U$ is the identity on $\overline{\operatorname{Im} A}$. Let $\mathcal{H}_+$ be the image of $(I+U)/2$ on $\overline{\operatorname{Im} A}$ and $\mathcal{H}_-$ be the image of $(I-U)/2$ on $\overline{\operatorname{Im} A}$. It follows that $\mathcal{H}_- \perp \mathcal{H}_+$, and $\overline{\operatorname{Im} A} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Because $UA = AU$, $\mathcal{H}_\pm$ reduces A. For $x \in \mathcal{H}_+$, $Ux = x$, so

$$Ax = U\,|A|\,x = |A|\,Ux = |A|\,x$$

and for $x \in \mathcal{H}_-$, $Ux = -x$, so

$$Ax = U\,|A|\,x = |A|\,Ux = -\,|A|\,x.$$

(4) Let Q and $P_\pm$ be the corresponding orthogonal projections onto K and $\mathcal{H}_\pm$ respectively. Because $AQ = QA$, $UQ = QU$, i.e., $P_\pm$ also commutative with Q. So QP_- is also an orthogonal projection and it suffices to show $QP_- = 0$. For any $x \in \operatorname{Im} QP_- = \operatorname{Im} P_-Q$,

$$\langle Ax, x\rangle \geq 0, \quad \langle Ax, x\rangle \leq 0,$$

so $\langle Ax, x\rangle = 0$. On K, because $A_K \geq 0$, Cauchy-Schwarz inequality implies that

$$\langle Ax, Ax\rangle \leq \langle Ax, x\rangle^{\frac{1}{2}} \left\langle A^2x, Ax\right\rangle^{\frac{1}{2}} = 0$$

It follows that $x \in \ker A \cap \mathcal{H}_{=}0$. ∎

Theorem 6.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. There is a unique resolution of the identity $\{E(\lambda)\}$ such that*

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda).$$

Furthermore, when $\lambda < -\|A\|$, $E(\lambda) = 0$, and $E(\lambda) = I$ for $\lambda > \|A\|$.

Proof. For $\lambda \in \mathbb{R}$, let

$$A - \lambda I = U(\lambda)\,|A - \lambda I|$$

be the polar decomposition and let

$$\mathcal{H} = \mathcal{H}_+(\lambda) \oplus \mathcal{H}_-(\lambda) \oplus \ker(A - \lambda I).$$

Denote $A_\pm(\lambda)$ be $A|_{\mathcal{H}_\pm}$, and thus $A_+(\lambda) \geq \lambda$ and $A_-(\lambda) \leq \lambda$. Denote $P_0(\lambda)$ and $P_\pm(\lambda)$ be orthogonal projections onto $\ker(A - \lambda I)$ and $\mathcal{H}_\pm(\lambda)$. Let $E(\lambda) := P_0(\lambda) + P_-(\lambda)$.

$$H(\lambda) := E(\lambda)H = H_-(\lambda) \oplus \ker(A - \lambda I).$$

and so $H(\lambda)$ reduces A.

(1) For $x \in \mathcal{H}(\lambda)$, if $\lambda < \mu$, by $\langle (A - \lambda I)x, x\rangle \leq 0$, $\langle (A - \mu I)x, x\rangle \leq 0$. Therefore, $E(\lambda) \leq E(\mu)$ by (4) in above theorem.

(2) If $\lambda < -\|A\|$, for $x \in \mathcal{H}$ with $x \neq 0$,

$$\langle (A - \lambda I)x, x\rangle > \langle Ax, x\rangle + \|A\|\|x\|^2 \geq \langle Ax, x\rangle + |\langle Ax, x\rangle| \geq 0,$$

which implies that $E(\lambda) = 0$. Similarly, $\lambda > \|A\|$ implies that $E(\lambda) = I$.

(3) When $\lambda < \mu$, by $(A - \mu I)E(\mu) \leq 0$ and $(A - \lambda I)(I - E(\lambda)) \geq 0$,

$$\begin{aligned} &(A - \mu I)(E(\mu) - E(\lambda)) \leq 0, \\ &(A - \lambda I)(E(\mu) - E(\lambda)) \geq 0. \end{aligned}$$

It follows that

$$\lambda(E(\mu) - E(\lambda)) \leq A(E(\mu) - E(\lambda)) \leq \mu(E(\mu) - E(\lambda)).$$

As $\mu \to \lambda + 0$, $(A - \lambda I)(E(\lambda + 0) - E(\lambda)) = 0$. So

$$(E(\lambda + 0) - E(\lambda))\mathcal{H} \subset \ker(A - \lambda I) \subset E(\lambda)\mathcal{H},$$

which implies that $E(\lambda + 0) - E(\lambda) = 0$.

(4) choose $\lambda_0 < -\|A\|$, $\lambda_n > \|A\|$, and $\lambda_0 < \lambda_1 < \cdots < \lambda_n$, by above

$$\begin{aligned} \sum_{k=1}^{n} \lambda_{k-1}\left(E\left(\lambda_k\right) - E\left(\lambda_{k-1}\right)\right) &\leq \sum_{k=1}^{n} A\left(E\left(\lambda_k\right) - E\left(\lambda_{k-1}\right)\right) = A \\ &\leq \sum_{k=1}^{n} \lambda_k\left(E\left(\lambda_k\right) - E\left(\lambda_{k-1}\right)\right). \end{aligned}$$

Then, for $\lambda_k' \in (\lambda_{k-1}, \lambda_k]$ and $x \in \mathcal{H}$,

$$\begin{aligned} -\sum_{k=1}^{n}\left(\lambda_k' - \lambda_{k-1}\right)\left\langle\left(E\left(\lambda_k\right) - E\left(\lambda_{k-1}\right)\right)x, x\right\rangle &\leq \left\langle\left(A - \sum_{k=1}^{n} \lambda_k'\left(E\left(\lambda_k\right) - E\left(\lambda_{k-1}\right)\right)\right)x, x\right\rangle \\ &\leq \sum_{k=1}^{n}\left(\lambda_k - \lambda_k'\right)\left\langle\left(E\left(\lambda_k\right) - E\left(\lambda_{k-1}\right)\right)x, x\right\rangle. \end{aligned}$$

Let $\Delta = \max_{1\le k\le n} |\lambda_k - \lambda_{k-1}|$. So

$$-\Delta\|x\|^2 \le \left\langle A - \sum_{k=1}^{n} \lambda_k' \left(E\left(\lambda_k\right) - E\left(\lambda_{k-1}\right)\right) x, x \right\rangle \le \Delta\|x\|^2.$$

It follows that

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda).$$

(5) If $A = \int_{-\infty}^{\infty} \lambda dF(\lambda)$ for another $F(\lambda)$. Let

$$A_\lambda = \int_{-\infty}^{\infty} |\mu - \lambda|\, dF(\mu).$$

Then we have

$$\begin{aligned}\left\langle A_\lambda^2 x, x\right\rangle = \langle A_\lambda x, A_\lambda x\rangle &= \int_{-\infty}^{\infty} |\mu-\lambda|^2\, d\left\langle F(\mu)x, x\right\rangle \\ &= \langle (A-\lambda I)x, (A-\lambda I)x\rangle = \left\langle (A-\lambda I)^2 x, x\right\rangle\end{aligned}$$

Therefore, $A_\lambda^2 = (A-\lambda I)^2$. Because $A_\lambda \ge 0$, by the uniqueness of root, $A_\lambda = |A - \lambda I|$. Let $V(\lambda) = I - F(\lambda) - F(\lambda - 0)$. For $x, y \in \mathcal{H}$,

$$\begin{aligned}\langle V(\lambda)A_\lambda x, y\rangle =& \int_{\mu<\lambda} |\mu-\lambda| d\left\langle F(\mu)x, (I - F(\lambda) - F(\lambda-0))y\right\rangle \\ &+ \int_{\mu>\lambda} |\mu-\lambda| d\left\langle F(\mu)x, (I - F(\lambda) - F(\lambda-0))y\right\rangle \\ =& -\int_{\mu<\lambda} (\lambda-\mu) d\left\langle F(\mu)x, y\right\rangle + \int_{\mu>\lambda} (\mu-\lambda) d\left\langle F(\mu)x, y\right\rangle \\ =& \left\langle (A-\lambda I)x, y\right\rangle.\end{aligned}$$

So $A - \lambda I = V(\lambda)A_\lambda$. Furthermore,

$$V(\lambda)^2 = I - F(\lambda) + F(\lambda - 0) = I - F(\{\lambda\})$$

is a projection, $V(\lambda)$ is a partial isometry. Because $\ker(A - \lambda I) = F(\{\lambda\}) = \ker V(\lambda)$, $A - \lambda I = V(\lambda)\,|A - \lambda I|$ is the polar decomposition. So $U(\lambda) = V(\lambda)$ and thus $E(\lambda) = F(\lambda)$. □

7 Spectral Measure

Definition 7.1 (Spectral Measure)**.** Let $\mathcal{H}$ be a Hilbert space and $(\mathbb{R}^d, \mathcal{B}^d)$ be the Borel measure space. If a family of orthogonal projections $\{E(B)\}_{B\in\mathcal{B}^d}$ satisfies

(1) $E(\emptyset) = 0$ and $E(\mathbb{R}^d) = I$,

(2) for $B_1, B_2 \in \mathcal{B}^d$, $E(B_1)E(B_2) = E(B_1 \cap B_2)$,

(3) for disjoint $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{B}^d$,

$$\sum_{n=1}^{N} E(B_n) \to E(\cup_n B_n)$$

in SOT as $N \to \infty$,

then E is called a d-dimensional spectral measure.

Remark 7.2. By (2), $\sum_{n=1}^{N} E(B_n)$ is still an orthogonal projection. By (3), for $B_1 \subset B_2$, $E(B_2) = E(B_1) + E(B_2 \backslash B_1)$, so $E(B_1) \leq E(B_2)$.

Proposition 7.3. *Let E be a 1-dimensional spectral measure. Then denote $E(\lambda) = E((-\infty, \lambda])$, which is a resolution of the identity. Furthermore, the integral with respect to $\langle E(\lambda)x, x\rangle$ is as same as $\langle E(B)x, x\rangle$.*

Proposition 7.4. *For $F \colon \mathcal{H} \to \mathbb{R}$, if it satisfies*

(1) there exists C such that $|F(x)| \leq C\|x\|^2$,

(2) $F(x+y) + F(x-y) = 2F(x) + 2F(y)$,

(3) for $\alpha \in \mathbb{C}$ and $x \in \mathcal{H}$, $F(\alpha x) = |\alpha|^2 F(x)$,

then there exists $A \in \mathcal{B}(\mathcal{H})$ that is self-adjoint such that $F(x) = \langle Ax, x\rangle$.

Proof. For $x, y \in \mathcal{H}$,

$$B(x,y) = \frac{1}{4}\sum_{k=0}^{3} i^k F\left(x + i^k y\right),$$

which implies that $B(y,x) = \overline{B(x,y)}$ and $B(x,0) = 0$. By (2) and (3),

$$F(x+\alpha y) + F(x+\alpha z) = \frac{1}{2}(F(2x + \alpha(y+z)) + F(\alpha(y-z)))$$

Choose $\alpha = \pm 1, \pm i$ and add them all, we get

$$4B(x,y) + 4B(x,z) = 2(B(2x, y+z) + B(0, y-z)) = 2B(2x, y+z).$$

Let $z = 0$, so $2B(x,y) = B(2x,y)$ and

$$B(x,y) + B(x,z) = B(x, y+z).$$

Furthermore, it can be extended to

□