Topology and Geometry

Zhiyuan Zhan
<thaleszhan@gmail.com>

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Contents

Chapter 1

Shortest Curves in Riemannian Manifolds

1.1 Riemannian Metric

Let (M, g) be a m-dimensional Riemannian manifold (always assumed being connected). And we use the Einstein summation convention.

1. ***Riemannian metric:*** For open $U \subset M$, let homeomorphism $x\colon U \to x(U) \subset \mathbb{R}^m$ be local coordinates, that is, for any $p \in U \subset M$, $x(p) = (x^1(p), \cdots, x^m(p))$. Then the local expression of g is a symmetric matrix

$$(g_{ij}(x))_{1 \le i,j \le m}$$

or in tensor form

$$g(x) = g_{ij}(x) dx^i \otimes dx^j$$

Then for any $v, w \in T_pM$ with local coordinates $v = (v^1, \cdots, v^m)$ and $w = (w^1, \cdots, w^m)$,

$$\langle v, w \rangle_p := g_p(v, w) = g_{ij}(x(p)) v^i w^j$$

Moreover, g induces a canonical measure defined on M. Let

$$\sqrt{|g|(x)} := \sqrt{\det(g_{ij}(x))}$$

Then for any smooth $F\colon U \to \mathbb{R}$,

$$\int_U F(p) d\mu(p) := \int_{x(U)} F \circ x^{-1} \sqrt{g} dx^1 \cdots dx^m$$

Remark. So $d\mu$ is basically the push-forward of measure $\sqrt{\det(g_{ij})} dx^1 \cdots dx^m$ on $x(U) \subset \mathbb{R}^m$ by $x^{-1}\colon x(U) \to U$, that is

$$d\mu = x^{-1}_{\#} \left(\sqrt{\det(g_{ij})} dx^1 \cdots dx^m \right)$$

Note that we usually do not distinguish $x(p)$ and p, so the integral is usually expressed as

$$\int_U F(x) \sqrt{|g|(x)} dx^1 \cdots dx^m$$

Moreover, for $F \in C_c^\infty(M)$, we can also define

$$\int_M F(x)\sqrt{|g|(x)}dx^1 \cdots dx^m$$

by the partition of unity.

For any $p \in M$, $g_p\colon T_pM \times T_pM \to \mathbb{R}$ with matrix expression (g_{ij}). Then let $(g^{ij}) = (g_{ij})^{-1}$, *i.e.*

$$g^{ik}g_{kj} = \delta_j^i$$

Then for any $\omega, \eta \in T_p^*M$ with coordinates

$$\omega = \omega_i dx^i, \qquad \eta = \eta_i dx^i$$

we define $g_p\colon T_p^*M \times T_p^*M \to \mathbb{R}$,

$$g_p^*(\omega, \eta) = \langle \omega, \eta \rangle_p^* := g^{ij}(p)\omega_i(p)\eta_j(p)$$

Then we can see g_p^* is independent with the choice of coordinates. Moreover, by the continuity of g^{ij},

$$g^*\colon \Gamma(T^*M) \times \Gamma(T^*M) \to C^\infty(M)$$

is a $(2,0)$-tensor field.

For $p \in M$, consider two isomorphism.

$$\flat : T_pM \to T_p^*M, \quad \flat\left(X_p\right)\left(Y_p\right) := g_p\left(X_p, Y_p\right)$$

In local coordinate, for $X = X^i \frac{\partial}{\partial x_i}$,

$$\flat(X^i \frac{\partial}{\partial x_i}) = g_{ij}X^i dx^j$$

Then second isomorphism is

$$\sharp : T_p^*M \to T_pM$$

in local coordinates, it is defined as

$$\sharp\left(w_i dx^i\right) = g^{ij} w_i \frac{\partial}{\partial x_j}$$

and thus

$$g_p(\sharp\omega, \sharp\eta) = g_{ij}g^{ki}\omega_k g^{lj}\eta_l = \delta_j^k \omega_k \eta_l g^{lj} = g^{kl}\omega_k\eta_l = g_p^*(\omega, \eta)$$

For any $f \in C^\infty(M)$, let $\nabla f = \sharp(df)$ and in local coordinates

$$\nabla f = g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}$$

Clearly, when c is regular value of f, then $f^{-1}(c)$ is a hyper-surface and ∇f is perpendicular to it.

2. ***Change of coordinates:*** Let $x = (x^1, \cdots, x^m)$ and $y = (y^1, \cdots, y^m)$ be two coordinates on U. Let $y = f(x)$ be the transition map. Then for $p \in U$ and $v, w \in T_pM$, if on (x, U),

$$v = v^i \frac{\partial}{\partial x_i}, \quad w = w^i \frac{\partial}{\partial x_i}$$

and on (y, U),

$$v = \tilde{v}^\alpha \frac{\partial}{\partial y_\alpha}, \quad w = \tilde{w}^\alpha \frac{\partial}{\partial y_\alpha}$$

then by

$$\begin{aligned} v &= v^i \frac{\partial}{\partial x_i} = v^i \frac{\partial f^\alpha}{\partial x_i} \frac{\partial}{\partial y_\alpha} \\ w &= w^i \frac{\partial}{\partial x_i} = w^i \frac{\partial f^\alpha}{\partial x_i} \frac{\partial}{\partial y_\alpha} \end{aligned}$$

we get

$$\tilde{v}^\alpha = v^i \frac{\partial f^\alpha}{\partial x_i}, \quad \tilde{w}^\alpha = w^i \frac{\partial f^\alpha}{\partial x_i}$$

If g has matrix form $(g_{ij}(x))$ on (x, U) and $(h_{\alpha\beta}(x))$ on (x, U), then

$$\begin{aligned} \langle v, w \rangle_p &= v^i g_{ij}(x) w^j \\ &= \tilde{v}^\alpha h_{\alpha\beta}(f(x)) \tilde{w}^\beta \\ &= v^i \frac{\partial f^\alpha}{\partial x_i} h_{\alpha\beta}(f(x)) \frac{\partial f^\beta}{\partial x_j} w^j \end{aligned}$$

Therefore, we get

$$g_{ij}(x) = \frac{\partial f^\alpha}{\partial x_i} h_{\alpha\beta}(f(x)) \frac{\partial f^\beta}{\partial x_j} \Rightarrow (g_{ij}(x))_{ij} = \left[\frac{\partial f^\alpha}{\partial x_i}\right]_{\alpha i}^\top (h_{\alpha\beta}(f(x)))_{\alpha\beta} \left[\frac{\partial f^\beta}{\partial x_j}\right]_{\alpha j}$$

Besides, for any $\Phi \in C_c^\infty(M)$, the integral

$$\int_M \Phi(y) \sqrt{h(y)} dy^1 \cdots dy^m = \int_M \Phi(f(x)) \sqrt{|g|(x)} dx^1 \cdots dx^m$$

Remark. Or in tensor form, we can have an easier proof. By

$$dy^\alpha = \frac{\partial f^\alpha}{\partial x_i} dx_i$$

we get

$$\begin{aligned} h_{\alpha\beta} dy^\alpha \otimes dy^\beta &= h_{\alpha\beta} \left(\frac{\partial f^\alpha}{\partial x_i} dx_i\right) \otimes \frac{\partial f^\beta}{\partial x_j} dx_j \\ &= h_{\alpha\beta} \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\beta}{\partial x_j} dx^i \otimes dx^j \end{aligned}$$

Remark. Let's review some basic knowledge about the tensor field. Considering a map

$$\theta \colon \underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{r} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{s} \to C^\infty(M)$$

such that θ is function-linear for each component. Then θ is called a (r, s)-tensor field. For a (r, s)-tensor field θ and any $p \in M$,

$$\theta_p \colon \underbrace{T_p^*M \times \cdots \times T_p^*M}_{r} \times \underbrace{T_pM \times \cdots \times T_pM}_{s} \to \mathbb{R}$$

is linear for each component and so $\theta_p \in \bigotimes^{r,s} T_pM$. When employing a chart (x, U) containing p,

$$\theta_p = \theta^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

for $1 \leq i_1, \cdots, i_r \leq m$ and $1 \leq j_1, \cdots, j_s \leq m$, where

$$\theta^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} = \theta_p \left(dx^{i_1}, \cdots, dx^{i_r}, \frac{\partial}{\partial x_{j_1}}, \cdots, \frac{\partial}{\partial x_{j_s}} \right)$$

Let (y, U) be another chart and θ_p has

$$\theta_p = \tilde{\theta}^{k_1, \cdots, k_r}_{l_1, \cdots, l_s} \frac{\partial}{\partial y_{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial y_{k_r}} \otimes dy^{l_1} \otimes \cdots \otimes dy^{l_s}$$

Then we have

$$\begin{aligned} \tilde{\theta}^{k_1, \cdots, k_r}_{l_1, \cdots, l_s} &= \theta_p \left(dy^{k_1}, \cdots, dy^{k_r}, \frac{\partial}{\partial y_{l_1}}, \cdots, \frac{\partial}{\partial y_{l_s}} \right) \\ &= \theta_p \left(\frac{\partial y^{k_1}}{\partial x^{i_1}} dx^{i_1}, \cdots, \frac{\partial y^{k_r}}{\partial x^{i_r}} dx^{i_r}, \frac{\partial x^{j_1}}{\partial y^{l_1}} \frac{\partial}{\partial x_{j_1}}, \cdots, \frac{\partial x^{j_s}}{\partial y^{l_s}} \frac{\partial}{\partial x_{j_s}} \right) \\ &= \frac{\partial y^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{k_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial y^{l_1}} \cdots \frac{\partial x^{j_s}}{\partial y^{l_s}} \theta^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} \end{aligned}$$

This formula of the coefficients $\theta^{i_1, \cdots, i_r}_{j_1, \cdots, j_s}$ is so-called the tensor transformation law. The converse is true.

Lemma 1.1.1. *θ is a smooth (r, s)-tensor field if and only on smooth manifold M if and only if for any chart (x, U),*

$$\theta = \theta^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

and so satisfy the tensor transformation law, and $\theta^{i_1, \cdots, i_r}_{j_1, \cdots, j_s}$ is smooth on U.

3. ***Induced Metric:*** Let $N \subset M$ be a submanifold with dimension $n \leq m$. Let $i \colon N \hookrightarrow M$ be an inclusion map. Then metric h on N is induced by g if

$$i^* h = g$$

which means for any $p \in N$, the map

$$i_{*,p} \colon T_p N \to T_p M$$

preserves the metric, that is

$$h(v, w) := g(i_{*,p}(v), i_{*,p}(w)), \quad v, w \in T_p N$$

More explicitly, when viewing v as same as $i_{*,p}(v)$, it means

$$h(v, w) = g(v, w), \quad v, w \in T_p N$$

Locally, let $(x = (x^1, \cdots, x^m), U)$ be coordinate on $U \subset M$ and $(\theta = (\theta^1, \cdots, \theta^n), U \cap N)$ be coordinates on N and $x = f(\theta)$, where $f \colon N \hookrightarrow M$. Then we have

$$f_{*,p} \left(\frac{\partial}{\partial \theta^\alpha} \right) = \frac{\partial f^i}{\partial \theta^\alpha} \frac{\partial}{\partial x^i}$$

If g of M has the matrix form g_{ij} and h of N has the form $h_{\alpha\beta}$, then for any $v, w \in T_p N \subset T_p M$ with

$$v = \tilde{v}^\alpha \frac{\partial}{\partial \theta^\alpha}, \quad w = \tilde{w}^\alpha \frac{\partial}{\partial \theta^\alpha}$$

we have

$$\begin{aligned}
h(v,w) &= h_{\alpha\beta}\tilde{v}^\alpha\tilde{w}^\beta \\
&= g(f_{*,p}(v), f_{*,p}(w)) \\
&= \tilde{v}^\alpha\tilde{w}^\beta g\left(f_{*,p}\left(\frac{\partial}{\partial\theta^\alpha}\right), f_{*,p}\left(\frac{\partial}{\partial\theta^\beta}\right)\right) \\
&= g_{ij}\frac{\partial f^i}{\partial\theta^\alpha}\frac{\partial f^j}{\partial\theta^\beta}\tilde{v}^\alpha\tilde{w}^\beta
\end{aligned}$$

It follows that

$$h_{\alpha\beta} = g_{ij}\frac{\partial f^i}{\partial\theta^\alpha}\frac{\partial f^j}{\partial\theta^\beta} \;\Rightarrow\; (h_{\alpha\beta})_{\alpha\beta} = \left[\frac{\partial f^i}{\partial\theta^\alpha}\right]_{i\alpha}^\top (g_{ij})_{ij}\left[\frac{\partial f^j}{\partial\theta^\beta}\right]_{j\beta}$$

Remark. Also, we can use the tensor form to obtain a easier proof. By

$$dx^i = \frac{\partial f^i}{\partial\theta^\alpha}d\theta^\alpha \;\Rightarrow\; f^*(dx^i) = \frac{\partial f^i}{\partial\theta^\alpha}d\theta^\alpha$$

because $x^i\circ f = f^i$ and $f^*g = h$,

$$\begin{aligned}
h &= h_{\alpha\beta}d\theta^\alpha\otimes d\theta^\beta \\
&= f^*g \\
&= f^*\left(g_{ij}dx^i\otimes dx^j\right) \\
&= g_{ij}f^*\left(dx^i\right)\otimes f^*\left(dx^j\right) \\
&= g_{ij}\frac{\partial f^i}{\partial\theta^\alpha}\frac{\partial f^j}{\partial\theta^\beta}d\theta^\alpha\otimes d\theta^\beta
\end{aligned}$$

Remark. If we consider two Riemannian manifolds (M,g) and (N,h) with same dimension. And $f\colon N\to M$ is a local isometry, *i.e.* $f^*g = h$. If (θ,V) is a chart of N around θ_0 and (x,U) is a chart of M around $x_0 = f(\theta_0)$, then on matrix expression of g and h also satisfy above formula,

$$h_{\alpha\beta} = g_{ij}\frac{\partial f^i}{\partial\theta^\alpha}\frac{\partial f^j}{\partial\theta^\beta}$$

Note that it is as same as the change of coordinates. So local isometric f does basically as same as change of coordinates, *i.e.* locally M and N are same.

Example 1.1.1. Let $\mathbb{S}^2\subset\mathbb{R}^2$ with the induced metric by

$$\begin{array}{cccc}
i\colon & \mathbb{S}^2 & \hookrightarrow & \mathbb{R}^3 \\
& (\theta,\varphi) & \mapsto & (x,y,z)
\end{array}$$

where

$$\begin{cases}
x = \cos\theta\cos\varphi \\
y = \sin\theta\cos\varphi \\
z = \sin\varphi
\end{cases}$$

Then the metric g on $\mathbb{S}^2$ is

$$g = d\varphi\otimes d\varphi + \cos^2 d\theta\otimes d\theta$$

4. ***More on Riemannian measure:*** We have defined the Riemannian measure $d\mu$ on (M, g). Then it gives
$$\Phi\colon C_c^\infty(M) \to \mathbb{R}, \quad \Phi(f) := \int_M f d\mu$$
Then we can equip $C_c^\infty(M)$ with L^p norm for $1 \leq p < \infty$ by
$$\|f\|_p := \left(\int_M |f|^p d\mu\right)^{1/p}$$
Then we can define
$$L^p(M, g) = \overline{C_c^\infty(M)}^{\|\cdot\|_p}$$
We have seen impact of the change of coordinates for $d\mu$. In general, if $\varphi\colon (N, h) \to (M, g)$ is a local isometry, then
$$\int_N f \circ \varphi d\mu_h = \int_M f \mu_g, \quad \forall\ f \in L^1(M, g)$$
Or more generally, for $\varphi\colon (N, h) \to (M, g)$ diffeomorphism,
$$\int_N f \circ \varphi d\mu_{\varphi^* g} = \int_M f d\mu_g, \quad \forall\ f \in L^1(M, g)$$
A direct consequence is if $\varphi\colon (N, h) \hookrightarrow (M, g)$ is an embedding and $\iota\colon \varphi(N) \hookrightarrow M$ is an inclusion, then
$$\int_N f \circ \varphi \frac{dV_{\varphi^* g}}{d\mu_h} d\mu_h = \int_{\varphi(N)} f d\mu_{\iota^* g}, \quad \forall f \in L^1(M, g),$$
Fix $u \in C^\infty(M)$. Let
$$\Omega_t := u^{-1}((-\infty, t)), \quad \Gamma_t := u^{-1}(t)$$
If t is a regular value of u, Γ_t is a hyper-surface. And by Sard's Theorem, the measure of critical point of μ is actually 0.

Theorem 1.1.1 (Co-area Formula)**.** *Let (M, g) be a Riemannian manifold with measure μ_g. For a regular value t of u, let Γ_t be equipped with the induced metric g_t and the corresponding measure μ_t. Then for any $f \in L^1(M, g)$, we have*
$$\int_M f\,|\nabla u|\,d\mu_g = \int_{\mathbb{R}} \left(\int_{\Gamma_t} f d\mu_t\right) dt$$

Proof. Let C be the set of all critical points. By Sard's Theorem, C is closed has measure 0. So $M \backslash C$ is an open submanifold. By replacing M with $M \backslash C$, we can assume u has no critical point. Consider the vector field
$$X = \frac{\nabla u}{|\nabla u|^2}$$
which is perpendicular to $T_q \Gamma_c$ at any $q \in \Gamma_c$ for any c. Let φ_t be the (local) flow generated by X. So
$$\frac{d}{dt} u\left(\varphi_t(q)\right) = du\left(X\left(\varphi_t(q)\right)\right) = \langle \nabla u, X \rangle_{\varphi_t(q)} = 1$$

It follows that if $q \in \Gamma_c$, then $\varphi_t(q) \in \Gamma_{c+t}$ for t small enough. Choose a neighborhood A of q in Γ_c such that the map

$$\psi : (-\varepsilon, \varepsilon) \times A \to M, \quad (t, y) \mapsto \varphi_t(y)$$

is a diffeomorphism onto open $U = \psi((-\varepsilon, \varepsilon) \times A)$ in M. By shrinking A such that A is contained in a chart of Γ_c, denoted by $(y^1, \cdots, y^{m-1})$. Then $(t, y^1, \cdots, y^{m-1})$ forms a chart on U. By viewing $X = \frac{\partial}{\partial t}$, because $X \top \frac{\partial}{\partial y^i}$,

$$g = \langle X, X\rangle dt \otimes dt + h_{ij} dy^i \otimes dy^j, \quad h_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$$

Since $\langle X, X\rangle = 1/|\nabla u|^2$,

$$d\mu_g = \frac{1}{|\nabla u|}\sqrt{\det (h_{ij})} dt dy^1 \cdots dy^{m-1} = \frac{1}{|\nabla u|} dt d\mu_t$$

So we conclude that for any $\rho \in C_c(U)$,

$$\int_M \rho f |\nabla u| d\mu_g = \int_U \rho f \sqrt{\det g_t} dt dy^1 \cdots dy^{m-1} = \int_{c-\varepsilon}^{c+\varepsilon} \left(\int_{\Gamma_t \cap U} \rho f dmu_t\right) dt \qquad \square$$

Corollary 1.1.1. *Suppose the critical values of u form a closed subset in $\mathbb{R}$, and $\mathrm{Vol}(\Omega_t) < \infty$, then the function $t \mapsto \mathrm{Vol}(\Omega_t)$ is smooth at regular value t, and*

$$\frac{d}{dt}\mathrm{Vol}(\Omega_t) = \int_{\Gamma_t} \frac{1}{|\nabla u|} dS_t$$

Proof. For any regular t, take $\varepsilon > 0$ so that $(t, t+\varepsilon)$ is free of critical values. By taking $f = \frac{1}{|\nabla u|}$ we get, for $h \in (0, \varepsilon)$,

$$\mathrm{Vol}(\Omega_{t+h}) - \mathrm{Vol}(\Omega_t) = \int_t^{t+h} \left(\int_{\Gamma_t} \frac{1}{|\nabla u|} d\mu_t\right) dt$$

It follows

$$\frac{d}{dt}\mathrm{Vol}(\Omega_t) = \lim_{h \to 0} \frac{1}{h}\int_t^{t+h} \left(\int_{\Gamma_t} \frac{1}{|\nabla u|} d\mu_t\right) dt = \int_{\Gamma_t} \frac{1}{|\nabla u|} d\mu_t \qquad \square$$

5. ***Length and distance:*** For any smooth curve $\gamma\colon [a, b] \to M$, the length of γ is defined as

$$L(\gamma) := \int_a^b \|\dot\gamma(t)\| \, dt$$

that is, when working in coordinates $x(t) = (x^i(t))$,

$$L(\gamma) = \int \sqrt{g_{ij}(x(t)) x^i(t) x^j(t)} dt$$

Note that the length can be also defined for piecewise smooth curves.

Remark. The length is independent with the choice of parametrization, that is

$$L(\gamma \circ \psi) = L(\gamma)$$

for any change of variable $\psi : [\alpha, \beta] \to [a, b]$, because

$$L(\gamma \circ \psi) = \int_\alpha^\beta \left(g_{ij}(x(\gamma(\psi(\tau)))) \dot x^i(\psi(\tau)) \dot x^j(\psi(\tau)) \left(\frac{d\psi}{d\tau}\right)^2\right)^{\frac{1}{2}} d\tau = L(\gamma)$$

For any $p, q \in M$, the distance of p, q is

$$d(p,q) := \inf \{L(\gamma) \colon \gamma \colon [a,b] \to M \text{ piecewise smooth with } \gamma(a) = p, \gamma(b) = q\}$$

And by the connectedness of M, $d(p,q)$ is always well-defined.

Lemma 1.1.2. *(M,d) is a metric space, that is,*

(i) $d(p,q) \geq 0$ for any $p, q \in M$,
(ii) $d(p,q) > 0$ for any $p \neq q$,
(iii) $d(p,q) = d(q,p)$ for any $p, q \in M$,
(iv) $d(p,q) \leq d(p,r) + d(q,r)$ for any $p, q, r \in M$.

Proof. It is sufficient to prove the non-degeneratity and the triangular inequality. First, for the triangular inequality, let σ_1 and σ_2 be two smooth curves connecting p with r and connecting r with q. Then there are $\varepsilon_1, \varepsilon_2 > 0$ such that

$$d(p,q) \leq L(\sigma_1) + L(\sigma_2) \leq d(p,r) + d(r,q) + \varepsilon_1 + \varepsilon_2$$

Then as $\varepsilon_1, \varepsilon_2 \to 0$, we have the triangular inequality.
Let (x, U) be a chart containing p. Then there is a $\varepsilon > 0$ with

$$D_\varepsilon(x(p)) := \left\{y \in \mathbb{R}^d : |y - x(p)| \leq \varepsilon\right\} \subset x(U)$$

and $q \notin x^{-1}(D_\varepsilon(x(p)))$. Let $(g_{ij}(x))$ be the metric matrix on (x,U) and so continuous in x on $D_\varepsilon(x(p))$. So, by the compactness of $D_\varepsilon(x(p))$ and the positivity of g, there is a $\lambda > 0$ such that

$$g_{ij}(y)\xi^i\xi^j \geq \lambda|\xi|^2$$

for all $y \in D_\varepsilon(x(p)), \xi = \left(\xi^1, \ldots, \xi^d\right) \in \mathbb{R}^d$. Therefore, for any smooth curve $\gamma \colon [a,b] \to M$ with $\gamma(a) = p, \gamma(b) = q$,

$$\begin{aligned} L(\gamma) &\geq L\left(\gamma \cap x^{-1}\left(D_\varepsilon(x(p))\right)\right) \\ &\geq \lambda\varepsilon > 0, \end{aligned}$$

where the first inequality is because γ contains $z \in \partial D_\varepsilon(x(p))$. □

Remark. Let $A(x) \in R^{n\times n}$ be symmetric for any $x \in [a,b]$ and has eigenvalues $\lambda_1(x), \cdots, \lambda_n(x)$. If $A(x)$ is continuous in x, then $\lambda_i(x)$ continuous in x for all $i = 1, 2, \cdots, n$.

Corollary 1.1.2. *The topology on M induced by d coincides with the original topology on M.*

Proof. Only need to compare the distance topology with the topology in Euclidean space, so we consider it on Euclidean space. For any x in some chart, there is a $\varepsilon > 0$ such that $D_\varepsilon(x)$ is in the same chart. By the proof of above theorem, there are $\lambda, \mu > 0$ such that

$$\lambda^2|\xi|^2 \leq g_{ij}(x)\xi^i\xi^j \leq \mu^2|\xi|^2 \quad \text{for all } y \in D_\varepsilon(x), \xi \in \mathbb{R}^d$$

Thus

$$\lambda|y - x| \leq d(y,x) \leq \mu|y - x| \quad \text{for all } y \in D_\varepsilon(x)$$

which means, if we set $B(z,\delta) := \{y \in M : d(z,y) \leq \delta\}$, then

$$\overset{\circ}{D}_{\lambda\delta}(x) \subset \overset{\circ}{B}(x,\delta) \subset \overset{\circ}{D}_{\mu\delta}(x)$$ □

Remark. The important result of this is d on M is continuous with respect to its original topology.

1.2 Geodesic

Example 1.2.1. For $\mathbb{R}^2$ with the Euclidean metric g, if we use the (x, y) coordinate, it can be expressed as

$$g = dx \otimes dx + dy \otimes dy$$

If we use the polar coordinates (r, θ), by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

then with this coordinates

$$g = dr \otimes dr + r^2 d\theta \otimes d\theta$$

So for any curve with the polar coordinates

$$\gamma(t) = (r(t), \theta(t))$$

its length

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt \\ &= \int_a^b \sqrt{r'(t)^2 + r^2(t)\theta'(t)^2} \\ &\geq |r(b) - r(a)| \end{aligned}$$

"=" if and only if $\theta' \equiv= 0$ and $r(t)$ is monotonic.

Example 1.2.2. Considering the settings in Example 1.1.1. For any curve

$$\gamma(t) = (\theta(t), \varphi(t))$$

its length

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt \\ &= \int_a^b \sqrt{\varphi'(t)^2 + \cos^2\varphi(t)\theta'(t)^2} \\ &\geq |\varphi(b) - \varphi(a)| \end{aligned}$$

"=" if and only if $\theta' \equiv= 0$ and $\varphi(t)$ is monotonic., which is the arc.

Because the globally shortest implies the locally shortest, we first consider the locally shortest property.

Definition 1.2.1 (Energy function)**.** For any $p, q \in M$, let $\gamma \in C_{p,q}$, the set of all piecewise smooth curves connecting $p, q \in M$.

$$E(\gamma) := \frac{1}{2}\int_a^b g(\dot\gamma(t), \dot\gamma(t))dt$$

Note that E dependents on the choice of parametrization.

Lemma 1.2.1. *For any $\gamma \in C_{p,q}$, $\gamma\colon [a, b] \to M$,*

$$L(\gamma)^2 \leq 2(b - a)E(\gamma)$$

and "=" if and only if $\|\dot\gamma(t)\| = const.$.

Proof. It is by Hölder's Inequality,

$$\int_a^b \|\dot{\gamma}(t)\| \, dt \leq (b-a)^{\frac{1}{2}} \left(\int_a^b \|\dot{\gamma}(t)\|^2 \, dt \right)^{\frac{1}{2}}$$

and "=" if and only if $\|\dot{\gamma}(t)\| = c$ for some c. □

Remark. Therefore, if γ is parametrized by arc length, then $\|\dot{\gamma}(t)\| = 1$ and $b - a = L(\gamma)$. So

$$L(\gamma) = 2E(\gamma)$$

Next, given a chart (x, U), assume $\gamma \colon [a, b] \to U$, let

$$x(t) := x(\gamma(t)) = (x^1(t), \cdots, x^n(t))$$

For any smooth curve $y \in C^\infty([a, b], x([a, b]))$ with $y(a) = y(b) = 0$,

$$\gamma_\varepsilon(t) = x(t) + \varepsilon y(t)$$

Lemma 1.2.2. *If γ is a smooth curve with the shortest length, then γ with parametrization $\gamma \colon [a, b] \to U$ such that $\|\dot{\gamma}(t)\| = c$ s a critical point of E,* i.e.

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(\gamma_\varepsilon) = 0$$

Proof. By assumptions,

$$\begin{aligned} L(\gamma) &= \sqrt{2(b-a)E(\gamma)} \\ &\leq L(\gamma_\varepsilon) \\ &\leq \sqrt{2(b-a)E(\gamma_\varepsilon)} \end{aligned}$$

So $E(\gamma) \leq E(\gamma_\varepsilon)$. □

Therefore, to find the shortest curve, it is sufficiently to find the critical point of E.

$$E(\gamma_\varepsilon) = E(\varepsilon) = \frac{1}{2} \int_a^b g_{ij}(x(t) + \varepsilon y(t)) \frac{d(x^i(t) + \varepsilon y^i(t))}{dt} \frac{d(x^j(t) + \varepsilon y^j(t))}{dt} dt$$

So

$$\begin{aligned} 0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(\varepsilon) &= \frac{1}{2} \int_a^b \frac{\partial g_{ij}(x(t))}{\partial x_k} y^k(t) \frac{dx^i(t)}{dt} \frac{dx^j(t)}{dt} dt \\ &+ \frac{1}{2} \int_a^b g_{ij}(x(t)) \frac{dy^i(t)}{dt} \frac{dx^j(t)}{dt} dt \\ &+ \frac{1}{2} \int_a^b g_{ij}(x(t)) \frac{dx^i(t)}{dt} \frac{dy^j(t)}{dt} dt \end{aligned}$$

Let $\frac{\partial g_{ij}}{\partial x_k} = g_{ij,k}$. Because of integral by parts,

$$\begin{aligned} 0 &= \frac{1}{2} \int_a^b \left(g_{ij,k} \frac{dx^i}{dt} \frac{dx^j}{dt} \right) y^k dt - \frac{1}{2} \int_a^b \frac{d}{dt} \left(\left(g_{ij} \frac{dx^i}{dt} \right) \right) y^j dt - \frac{1}{2} \int_a^b \frac{d}{dt} \left(\left(g_{ij} \frac{dx^j}{dt} \right) \right) y^i dt \\ &= \frac{1}{2} \int_a^b \left(g_{ij,k} \frac{dx^i}{dt} \frac{dx^j}{dt} - \frac{d}{dt} \left(g_{kj} \frac{dx^j}{dt} \right) - \frac{d}{dt} \left(g_{ik} \frac{dx^i}{dt} \right) \right) y^k dt \end{aligned}$$

So for any $k = 1, 2, \cdots, m$

$$\begin{aligned}
0 &= g_{ij,k}\frac{dx^i}{dt}\frac{dx^j}{dt} - \frac{d}{dt}\left(g_{kj}\frac{dx^j}{dt}\right) - \frac{d}{dt}\left(g_{ik}\frac{dx^i}{dt}\right) \\
&= g_{ij,k}\frac{dx^i}{dt}\frac{dx^j}{dt} - g_{kj,l}\frac{dx^j}{dt}\frac{dx^l}{dt} - g_{kj}\frac{d^2x^j}{dt^2} \\
&\quad - g_{ik,l}\frac{dx^i}{dt}\frac{dx^l}{dt} - g_{ik}\frac{d^2x^i}{dt^2} \\
&= \left(g_{ij,k} - g_{kj,i} - g_{ik,j}\right)\frac{dx^i}{dt}\frac{dx^j}{dt} - 2g_{ik}\frac{d^2x^i}{dt^2}
\end{aligned}$$

Because $(g^{ij}) = (g_{ij})^{-1}$, *i.e.* $g^{il}g_{lj} = \delta^i_j$, by multiplying with g^{lk} on both sides in above equation, we get

$$\begin{aligned}
0 &= 2\delta^l_i\frac{d^2x^i}{dt^2} - g^{lk}\left(g_{ij,k} - g_{kj,i} - g_{ik,j}\right)\frac{dx^i}{dt}\frac{dx^j}{dt} \\
&= 2\frac{d^2x^l}{dt^2} + g^{lk}\left(g_{kj,i} + g_{ik,j} - g_{ij,k}\right)\frac{dx^i}{dt}\frac{dx^j}{dt}
\end{aligned}$$

By $g_{kj,i} = g_{jk,i}$ and $g_{ik,j} = g_{ki,j}$, we get

$$\frac{d^2x^l}{dt^2} + \frac{1}{2}g^{lk}\left(g_{jk,i} + g_{ki,j} - g_{ij,k}\right)\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

Let

$$\Gamma^l_{ij} = \frac{1}{2}g^{lk}\left(g_{jk,i} + g_{ki,j} - g_{ij,k}\right) = \Gamma^l_{ji}$$

which is called the Christoffel symbol. We have the so-called geodesic equation.

Definition 1.2.2. Let (x, U) be a chart of M. A smooth (regular) curve $\gamma\colon [a, b] \to U$ is called a geodesic if it satisfies

$$\frac{d^2x^l}{dt^2} + \Gamma^l_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \quad \forall\, l = 1, 2, \cdots, m \tag{1.1}$$

Remark. In the following, we can see that the left hand side of geodesic equation is the coefficient of a $(1, 0)$-tensor field. In other words, it satisfies the tensor transformation law. So it is independent with the choice of coordinates and it can be defined globally.

Note that geodesic equation is dependent with the parametrization of curve. In above, if the parametrization of γ satisfies $\|\dot\gamma(t)\| = c$ and γ satisfies the geodesic equation, then γ is a critical point of E. The problem is if these two condition of γ can be both satisfied.

Lemma 1.2.3. *Any smooth curve γ satisfies the geodesic equation is parametrized proportionally by arc length, that is $\|\dot\gamma(t)\| = c$.*

Remark. Note that it means that the regularity condition of geodesic is not important. Besides, we have known geodesics are critical points of E.

Proof. First,

$$\frac{d}{dt}\left(g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right) = g_{ij,l}\frac{dx^i}{dt}\frac{dx^j}{dt}\frac{dx^l}{dt} + g_{ij}\frac{d^2x^i}{dt^2}\frac{dx^j}{dt} + g_{ij}\frac{dx^i}{dt}\frac{d^2x^j}{dt^2}$$

By replacing $\frac{d^2x^i}{dt^2}$, $\frac{d^2x^j}{dt^2}$ with the geodesic equation, we have

$$\frac{d}{dt}\left(g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right) = \left(g_{ij,l} - g_{kj}\Gamma^k_{il} - g_{ik}\Gamma^k_{jl}\right)\frac{dx^i}{dt}\frac{dx^j}{dt}\frac{dx^l}{dt}$$

Claim: $g_{ij,l} = g_{kj}\Gamma^k_{il} + g_{ik}\Gamma^k_{jl}$.
For the RHS,

$$\begin{aligned}\text{RHS} &= \frac{1}{2}g_{kj}g^{kp}\left(g_{pi,l} + g_{il,p} - g_{lp,i}\right) + \frac{1}{2}g_{ik}g^{kp}\left(g_{pj,l} + g_{jl,p} - g_{lp,j}\right)\\ &= \frac{1}{2}\left(g_{ji,l} + g_{il,j} - g_{lj,i}\right) + \frac{1}{2}\left(g_{ij,l} + g_{jl,i} - g_{li,j}\right)\\ &= g_{ij,l}\end{aligned}$$

$\square$

Remark. By this, if $\gamma(t)$ is a geodesic for $t \in [0,a]$ with $\|\dot\gamma(0)\| = c$, then

$$L(\gamma) = ca$$

Moreover, from the theory of ODE, geodesics exist.

Theorem 1.2.1. *Let $p \in M$ and $v \in T_pM$. Then there is a $\varepsilon > 0$ and a unique geodesic*

$$\gamma\colon (-\varepsilon,\varepsilon) \to M$$

such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Such γ is also denoted by $\gamma_{q,v}$

Remark. The geodesic equation is a ODE by considering $t \mapsto (\gamma(t), \dot\gamma(t)) \in TM$.

And also by the smoothness of ODE with respect to initial conditions, we have the following theorem.

Theorem 1.2.2. *For any $p \in M$, there are*

$$\mathcal{U}_{V,\delta} := \{(q,v)\colon p,q \in V \subset_{open} M,\ v \in T_pM,\ \|v\| \leq \delta\}$$

and $\varepsilon > 0$ and smooth

$$\gamma\colon (-\varepsilon,\varepsilon) \times \mathcal{U}_{V,\delta} \to M$$

such that for any $(q,v) \in \mathcal{U}_{V,\delta}$, $\gamma_{q,v}(t) = \gamma(t,q,v)$ is a geodesic.

Proposition 1.2.1 (Homogeneity). *If $\gamma(t,q,v)$ is a geodesic defined on $(-\varepsilon,\varepsilon)$, then the geodesic $\gamma(t,q,\lambda v)$ is defined on $(-\varepsilon/\lambda, \varepsilon/\lambda)$ and*

$$\gamma(t,q,\lambda v) = \gamma(\lambda t, q, v)$$

Proof. Let $\varphi(t) = \gamma(\lambda t, q, v) = (\varphi^1(t), \cdots, \varphi^m(t))$, where $\varphi^i(t) = x^i(\lambda t)$. Then

$$\begin{aligned}&\frac{d^2\varphi^i(t)}{dt^2} + \Gamma^i_{jk}(\varphi(t))\frac{d\varphi^j(t)}{dt}\frac{d\varphi^k(t)}{dt}\\ &= \lambda^2\left(\frac{d^2x^i(\lambda t)}{dt^2} + \Gamma^i_{jk}(x(\lambda t))\frac{dx^j(\lambda t)}{dt}\frac{dx^k(\lambda t)}{dt}\right) = 0\end{aligned}$$

So $\varphi(t)$ is a geodesic with $\varphi(0) = p$ and $\varphi'(0) = \lambda v$. By the uniqueness,

$$\varphi(t) = \gamma(t,q,\lambda v)$$

$\square$

1.3 Exponential Map

For any $p \in M$, the exponential map is defined as

$$\begin{array}{rccc} \exp_p\colon & V_p & \longrightarrow & M \\ & v & \mapsto & \gamma(1,p,v) \end{array}$$

where $V_p := \{v \in T_pM \colon \gamma(t,p,v) \text{ is defined on } [0,1]\}$. By above homogeneity of geodesics, we have the following results.

Proposition 1.3.1. *(1) For any $v \in V_p$, $\lambda v \in V_p$ for all $\lambda \in [0,1]$.*

(2) For any $p \in M$, there is $\varepsilon > 0$ such that $B(0,\varepsilon) = \{v \in T_pM \colon \|v\| < \varepsilon\} \subset V_p$.

Theorem 1.3.1. *Then exponential map $\exp_p\colon B(0,\varepsilon) \subset T_pM \to M$ is diffeomorphic for some $\varepsilon > 0$.*

Proof. We can assume that $\exp_p(B(0,\varepsilon)) \subset U$ for some chart (x,U) by choosing sufficiently small $\varepsilon' > 0$. Let $T_pM \simeq \mathbb{R}^m$. For $v = (v^1, \cdots, v^m) \in B(0,\varepsilon')$, let

$$x(\exp_p(v)) = x(\gamma(1,p,v^i\frac{\partial}{\partial x^i})) = (y^1, \cdots, y^m)$$

Then $d\exp_p\big|_O : T_O(T_pM) \simeq \mathbb{R}^m \to T_pM$ in U can be expressed as

$$d\exp_p\big|_O = \left.\begin{pmatrix} \frac{\partial y^1}{\partial v^1} & \frac{\partial y^1}{\partial v^2} & \cdots & \frac{\partial y^1}{\partial v^m} \\ \frac{\partial y^2}{\partial v^1} & \frac{\partial y^2}{\partial v^2} & \cdots & \frac{\partial y^2}{\partial v^m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y^m}{\partial v^1} & \frac{\partial y^m}{\partial v^2} & \cdots & \frac{\partial y^m}{\partial v^m} \end{pmatrix}\right|_O$$

Because

$$\begin{aligned} \frac{\partial y^k}{\partial v^i} &= \left.\frac{dy^k(0,\cdots,t,\cdots,0)}{dt}\right|_{t=0} \\ &= \left.\frac{d}{dt}\right|_{t=0} x^k(\gamma(1,p,t\frac{\partial}{\partial x^i})) \\ &= \left.\frac{d}{dt}\right|_{t=0} x^k(\gamma(t,p,\frac{\partial}{\partial x^i})) \\ &= \delta_{ki} \end{aligned}$$

we have $d\exp_p\big|_O = I$. So there is a smaller $\varepsilon > 0$ such that $\exp_p\colon B(0,\varepsilon) \subset T_pM \to M$ is a diffeomorphism. □

By this result, we have the normal coordinates.

Definition 1.3.1. For $p \in M$, there is a diffeomorphism $\exp_p^{-1}\colon \exp_p(B_p) \to B_p$ for some ball $B_p \subset \mathbb{R}^m \simeq T_pM$. Therefore, $\left\{(B_p, \exp_p^{-1})\right\}_{p\in M}$ is an atlas of M. Such coordinates system is called the normal coordinate.

Remark. Any geodesic starting from p is $\gamma(t,p,v)$ for some $v \in T_pM$. Because

$$\gamma(t,p,v) = \gamma(1,p,tv) = \exp_p(tv), \quad t \in (-\delta,\delta)$$

for small $\delta > 0$, $tv = (tv^1, \cdots, tv^m) = \exp_p^{-1}(\gamma(t,p,v))$, which means on the normal chart $(B_p, \exp_p^{-1})$, radical lines in $\mathbb{R}^m$ are corresponding to the geodesics in M.

Example 1.3.1. Consider n-dimensional sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, let $\mathbb{S}^n$ with the induced metric. Then we have shown the small arc of the greatest circle is the geodesic, *i.e.* for any given $p \in \mathbb{S}^n$ and $0 \neq v \in T_p\mathbb{S}^n$,

$$\gamma(t,p,v) = \cos(t\,\|v\|)p + \sin(t\,\|v\|)\frac{v}{\|v\|}$$

Therefore,

$$\exp_p(v) = \cos(\|v\|)p + \sin(\|v\|)\frac{v}{\|v\|},$$

and $\exp_p(0) = p$. So $\exp_p$ is well-defined on $T_p\mathbb{S}^n$. Furthermore, let p' be the antipodal point of p. Then

$$\exp_p \colon B(O,\pi) \subset T_p\mathbb{S}^n \longrightarrow \mathbb{S}^n \backslash \{p'\}$$

is a diffeomorphism.

Theorem 1.3.2. *In a normal coordinate, we have for any* $i,j,k = 1,2,\cdots,m$

(1) $g_{ij}(0) = \delta_{ij}$,

(2) $\Gamma^i_{jk}(0) = 0$,

(3) $g_{ij,k}(0) = 0$.

Proof. (1) For the isomorphism $T_pM \simeq \mathbb{R}^m$, we can choose a basis in T_pM such that it is orthonormal with respect to g_p.

(2) We know any geodesic in a normal coordinate should be $x(t) = tv$. Thus, by the geodesic equation, we have

$$\Gamma^i_{\ell n}(tv)v^\ell v^n = 0, \quad \forall\ i = 1,2,\cdots,m$$

In particular, $\Gamma^i_{\ell n}(0)v^j v^k = 0$. Then by choosing $v = \frac{1}{2}(e_j + e_k)$ for all $j,k = 1,\cdots,m$,

$$\Gamma^i_{jk}(0) = 0, \quad i,j,k = 1,2,\cdots,m$$

(3) By above claim, we have

$$g_{ij,l}(0) = g_{kj}(0)\Gamma^k_{il}(0) + g_{ik}\Gamma^k_{jl}(0)$$

So we have the desired result. □

Theorem 1.3.3. *When choosing a normal coordinate for M and applying the polar coordinate $(r,\varphi) \in (0,\infty) \times \mathbb{S}^{m-1}$ on $\mathbb{R}^m$ (called the Riemannian polar coordinates), we have*

$$g_{ij} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & g_{\varphi\varphi}(r,\varphi) & \\ 0 & & & \end{pmatrix}$$

where $g_{\varphi\varphi}(r,\varphi)\mathbb{R}^{(m-1)\times(m-1)}$ is positive.

Proof. In such case, any geodesic has the form

$$x(t) = (t,\varphi_0)$$

for some fixed φ_0. Therefore, by the geodesic equation, we have

$$\Gamma^i_{rr} = 0, \quad \forall\ i = 1,2,\cdots,m$$

and thus

$$g^{i\ell}\left(2g_{r\ell,r}-g_{rr,\ell}\right)=0,\quad \forall\ i=1,2,\cdots,m$$

By multiplying $g_{\ell i}$,

$$2g_{r\ell,r}-g_{rr,\ell}=0,\quad \forall\ \ell=1,2,\cdots,m$$

In particular, let $\ell=r$,

$$g_{rr,r}=0$$

and so $g_{rr}(t,\varphi_0)=g_{rr}(0)=1$ by continuity. Since φ_0 is general, $g_{rr}(r,\varphi)=1$ for any (r,φ). So $g_{rr,\ell}=0$ for all ℓ, and it implies

$$g_{r\ell,r}(t,\varphi_0)=0,\quad \forall\ \ell=1,2,\cdots,m$$

So $g_{r\ell}(t,\varphi_0)=g_{r\ell}(0)=0$ for all $\ell=2,\cdots,m$. Similarly, we have $g_{r\varphi}(r,\varphi)=1$ for any (r,φ). $\square$

1.4 Local Shortest Curve

First, let's see the shortest curve connecting p,q when they are closed enough. First, there is a result that can be obtained directly by using Riemannian polar coordinates.

Corollary 1.4.1 (Locally Shortest)**.** *For any $p\in M$, there is a $\rho>0$ such that $B(p,\rho):=\{q\in M\colon d(p,q)\le\rho\}$ is contained in a normal chart. Then for any $q\in\partial B(p,\rho)$, there is a unique one geodesic of shortest length ($=\rho$) connecting p,q, which is given by $x(t)=(t,\varphi_0)$. Here "shortest length" means over all curves connecting p,q in M.*

Proof. Using the Riemannian polar coordinates, let $c(t)=(r(t),\varphi(t))$ with $t\in[0,T]$ be any curve connecting p,q and note that $c(t)$ may be not fully contained in $B(p,\rho)$. So let

$$t_0:=\inf\{t\le T\colon d(c(t),p)\ge\rho\}$$

Then $c|_{[0,t_0]}$ if fully contained in $B(p,\rho)$ and clearly $L(c)\ge L(c|_{[0,t_0]})$. Then by above theorem, we get

$$\begin{aligned}
L\left(c_{[0,t_0]}\right)&=\int_0^{t_0}\left(g_{ij}(c(t))\dot{c}^i\dot{c}^j\right)^{\frac{1}{2}}dt\\
&\ge\int_0^{t_0}\left(g_{rr}(c(t))\dot{r}\dot{r}\right)^{\frac{1}{2}}dt\\
&=\left|\int_0^{t_0}|\dot{r}|dt\right|\\
&\ge\int_0^{t_0}\dot{r}dt=r(t_0)=\rho
\end{aligned}$$

where the equality is obtained by

$$g_{\varphi\varphi}\dot{\varphi}\dot{\varphi}\equiv 0\ \Leftrightarrow\ \varphi(t)\equiv const.$$

and $r(t)$ is monotonic. So $c(t)$ is the geodesic. $\square$

Remark. Therefore, for any $p\in M$, there is a $\rho>0$ such that

$$d_\rho(0):=\left\{y\in\mathbb{R}^d:|y|\le\rho\right\}\subset T_pM\simeq\mathbb{R}^m$$

is diffeomorphic to $B(p,\rho)$ by map $\exp_p$. So $(B(p,\rho),\exp_p^{-1})$ is a Riemannian polar coordinates. But we should note that such ρ is dependent on p. Clearly, if M is compact, we can get a uniform ρ_0.

Corollary 1.4.2. *If M is compact, then there is a ρ_0 such that for any $p, q \in M$ with $d(p,q) \leq \rho_0$, there is a unique shortest geodesic connecting p, q. Moreover, by the theory from ODE, such geodesic is continuous with respect to end points p and q.*

Corollary 1.4.3. *For any $p, q \in M$, if there is a piecewise smooth curve $\gamma\colon [0,l]$ such that $\gamma(0) = p$ and $\gamma(l) = q$ of shortest length, then γ should be a (smooth) geodesic.*

Proof. By the compactness of $\gamma([0,l])$, there is a ε_0 such that for any $t_1 < t_2 \in [0,l]$ with $t_2 - t_1 < \varepsilon_0$, there a geodesic γ_{t_1,t_2} connecting γ_{t_1} and $\gamma(t_2)$. Then $\gamma_{t_1,t_2} = \gamma|_{[t_1,t_2]}$ by the shortestness and uniqueness of geodesic. Therefore, γ is smooth and a geodesic. $\square$

Remark. However, in a Riemannian manifold, there may be two points p, q and no shortest curve can connect them. For example, on $\mathbb{R}^2 \setminus \{0\}$, there is no shortest curve that can connect $p = x$ with $q = -x$. Also, shortest curves are geodesics but it does not mean the shortest curve is uniquely exists. For example, in $\mathbb{S}^2$, there are infinitely many shortest curves connecting $p = x$ with $q = -x$.

To see the uniformness of ρ, we need more details.

Definition 1.4.1. (1) Let $p \in M$. If an open set U contains p and $(U, \exp_p^{-1})$ is a normal coordinates, then U is called a normal neighborhood for p.

(2) Let $p \in M$. If $p \in W$ and W is a normal neighborhood such that for any $q \in W$, W is a normal neighborhood for q, then W is called a totally normal neighborhood.

Remark. Ball $B(0,\varepsilon) \subset T_pM$ on which $\exp_p$ is well-defined is called a normal ball and a geodesic in $\exp_p(B(0,\varepsilon))$ starting from p is called a radical geodesic.

Considering the map

$$\begin{array}{rccc} \exp\colon & TM & \longrightarrow & M \times M \\ & (p,v) & \mapsto & (p, \exp_p(v)) \end{array}$$

Lemma 1.4.1. *For* $\exp$*,*

$$d\exp(p, O_p)\colon T_{(p,O_p)}TM \longrightarrow T_{(p,p)}(M \times M) \simeq T_pM \times T_pM$$

is non-singular.

Proof. First, for any $p \in M$ and $v \in T_pM$, consider the curve $c(t)$ on TM defined as

$$c(t) = (\gamma(t,p,v), O_{\gamma(t,p,v)})$$

Then we have

$$c(0) = (p, O_p), \quad \dot{c}(0) = (v, 0)$$

Therefore,

$$\begin{aligned} d\exp(p,O_p)(v,0) &= \left.\frac{d}{dt}\right|_{t=0} \exp(c(t)) = \left.\frac{d}{dt}\right|_{t=0} \exp(\gamma(t,p,v), O_{\gamma(t,p,v)}) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\gamma(t,p,v), \exp_{\gamma(t,p,v)}(O_{\gamma(t,p,v)})) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\gamma(t,p,v), \gamma(t,p,v)) \\ &= (v,v) \end{aligned}$$

Next, considering $\bar{c}(t) = (p, tv)$ with

$$\bar{c}(0) = (p, O_p), \quad \dot{\bar{c}}(0) = (0, v)$$

Therefore,

$$\begin{aligned} d\exp(p, O_p)(0, v) &= \frac{d}{dt}\bigg|_{t=0} \exp(\bar{c}(t)) = \frac{d}{dt}\bigg|_{t=0} \exp(p, tv) \\ &= \frac{d}{dt}\bigg|_{t=0} (p, \exp_p(tv)) \\ &= (0, v) \end{aligned}$$

Combining these two results, we can see

$$d\exp(p, O_p)(0, v) = \begin{pmatrix} I & O \\ I & I \end{pmatrix}$$

□

Theorem 1.4.1. *For any $p \in M$, there are a neighborhood W of p and $\delta > 0$ such that for any $q \in W$, $\exp_q$ is a diffeomorphism on $B(O_q, \delta) \subset T_qM$ and $W \subset \exp_p(B(O_q, \delta))$, that is, W is contained in a normal chart of any $q \in W$,* i.e. *W is a totally normal neighborhood.*

Proof. By above lemma, there is a neighborhood $\mathcal{U}$ of (p, O_p) in TM such that

$$\exp\colon \mathcal{U} \longrightarrow \exp(\mathcal{U}) = \Phi$$

is a diffeomorphism, where $(p, p) \in \Phi \subset M \times M$. By shrinking, we can set

$$\mathcal{U} := \{(q, v)\colon q \in V \subset M, \ v \in B(O_q, \delta) \subset T_qM\}$$

by the topology defined on TM, where $p \in V$. Since $(p, p) \in \Phi$, we can find a neighborhood W of p such that

$$W \times W \subset \Phi = \left\{(q, \tilde{q})\colon q \in V, \ \tilde{q} \in \exp_q(B(O_q, \delta))\right\}$$

For any $\tilde{q} \in W$, because $q \in W$,

$$(q, \tilde{q}) \in W \times W \subset \left\{(q, \tilde{q})\colon q \in V, \ \tilde{q} \in \exp_q(B(O_q, \delta))\right\}$$

which means $\tilde{q} \in \exp_q(B(O_q, \delta))$. So $W \subset \exp_q(B(O_q, \delta))$. □

Remark. Note that such δ is independent with choice of q. So W is also called a δ-uniform totally normal neighborhood of p.

1.5 Cut Locus

Consider the maximal geodesic $\gamma(t, p, v)$ for $t \in [0, b)$ with $\mathbb{R}_\infty$, where the right end point should be open by the theory of ODE. Let

$$A := \{t > 0\colon d(p, \gamma(t)) = t\}$$

Then if $A = (0, a]$ with $a < b$, then $\gamma(a)$ is called a cut point. Otherwise, when $A = (0, b)$, there is not cut point. Let

$$C(p) := \{\text{all cut points of geodesics starting from } p\}$$

For $v \in T_pM$ with $\|v\| = 1$, let

$$\tau(v) := \begin{cases} a, & \text{if } \exp_p(av) \text{ is a cut point} \\ b, & \text{if no cut point} \end{cases}$$

Remark. If $S_p = \{v \in T_pM\colon\ \|v\| = 1\}$, then

$$\tau\colon S_p \to \mathbb{R}_\infty$$

Moreover, let $\tilde{C}(p) := \{tv\colon v \in S_p,\ t = \tau(v)\}$. Note that $\tilde{C}(p) \cap E(p) = \varnothing$.

Let

$$E(p) := \{tv\colon v \in T_pM,\ \|v\| = 1,\ 0 \le t < \tau(v)\}$$

Proposition 1.5.1. *The map* $\exp_p\colon E(p) \to M$ *is injective.*

Proof. First, for any $p, q \in M$, if there are two shortest curves (geodesics) γ_1, γ_2 connecting p, q, then clearly at q, $v_1 = \dot{\gamma}_1 \neq \dot{\gamma}_2$, otherwise, by the uniqueness of geodesic, $\gamma_1 = \gamma_2$. Then at q, we can find a geodesic γ_3 with $\gamma_3(0) = q$ and $\dot{\gamma}_3(0) = v_1$. Let $q' = \gamma_3(\varepsilon)$. If q is not a cut point of p, then there is a small $\varepsilon > 0$ such that the geodesic $\gamma_1 \cup \gamma_3$ is a shortest curve connect p with q'. Then $\gamma_2 \cup \gamma_3$ is also a shortest curve connecting p with q'. But it is contradicted to the smoothness of shortest curves. So q is a cut point of p.
If there are $v \neq w \in E(p)$ such that

$$\exp_p(v) = \exp_p(w) = q$$

then there are two geodesics

$$t \mapsto \exp_p\left(t\frac{v}{\|v\|}\right)\ (t \in [0, \|v\|]), \quad t \mapsto \exp_p\left(t\frac{w}{\|w\|}\right)\ (t \in [0, \|w\|])$$

connecting p with q. Because $v, w \in E(p)$, they are shortest curves and so q is a cut point, which is contradicted to $v, w \in E(p)$. $\square$

Remark. In fact, $\exp_p\colon E(p) \to \exp_p(E(p))$ is a diffeomorphism when (M, g) is complete, which is followed by two facts:

(i) $\exp_p$ is a local diffeomorphism on $E(p)$ because $E(p)$ contains no conjugate points of p;

(ii) any bijective local diffeomorphism is a diffeomorphism.

Corollary 1.5.1. $\exp_p(E(p)) \cap C(p) = \varnothing$.

Proof. Assume there is a $q \in \exp_p(E(p)) \cap C(p)$. Then there are $v \in \tilde{C}(p)$ and $w \in E(p)$ such that

$$\exp_p(v) = \exp_p(w) = q$$

because

$$\gamma_1(t) \exp_p\left(t\frac{v}{\|v\|}\right)\ (t \in [0, \|v\|]), \quad \gamma_2(t) = \exp_p\left(t\frac{w}{\|w\|}\right)\ (t \in [0, \|w\|])$$

are two different shortest geodesics connecting p with q by $v \neq w$, we cannot have either $\|v\| < \|w\|$ or $\|v\| > \|w\|$. Assume $\|v\| = \|w\|$. Then because $w \in E(p)$, there is a small $\varepsilon > 0$ such that $\gamma_2(t)$ is defined on $[0, \|w\| + \varepsilon]$ and it is a shortest curve connecting p and $q' = \gamma_2(\|w\| + \varepsilon)$. But because $\|v\| = \|w\|$, *i.e.* $L(\gamma_1) = L(\gamma_2|_{[0,\|w\|]})$, $\gamma_1 \cup \gamma_2|_{[\|w\|,\|w\|+\varepsilon]}$ is also a shortest curve, which contradicts to the smoothness of shortest curve. $\square$

1.6 Existence: Hopf-Rinow Theorem

The next problem is when for any $p, q \in M$, they can be connected by a shortest curve. First, by starting from p, choose a normal ball $B(p, \rho_0)$, we can choose $p_0 \in \partial B(p, \rho_0)$ such that

$$p_0 = \min_{r_0 \in B(p,\rho_0)} d(r_0, q)$$

and the existence of p_0 is by the continuity of d and compactness of $\partial B(p, \rho_0)$. Then let γ_0 be the unique geodesic connecting p and p_0. Then by starting from p_0, we can choose normal ball $B(p_0, \rho_1)$ and similarly find $p_1 \in \partial B(p_0, \rho_)$ and the geodesic γ_1 connecting p_0 and p_1. Then we have $\{p_n\}_{n=0}^{\infty}$, $\{\rho_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$. Then let

$$\gamma = \bigcup_{n=0}^{\infty} \gamma_n$$

There are two problems.

I. Whether γ is shortest, *i.e.* γ is a smooth geodesic: The answer is always yes. It is sufficient to prove $\gamma_0 \cup \gamma_1$ is a shortest curve connecting p and p_1, *i.e.* $d(p, p_1) = \rho_0 + \rho_1$. Let $r = d(p, q)$.

   Claim: $d(p_0, q) = r - \rho_0$.

$$d(p_0, q) + d(p, p_0) \geq d(p, q) \;\Rightarrow\; d(p_0, q) \geq r - \rho_0$$

   The let $\tilde{\gamma}$ be any curve connecting p, q and let $\tilde{\gamma} = \tilde{\gamma}_1 \cup \tilde{\gamma}_2$, where $\tilde{\gamma}_1 = \tilde{\gamma} \cap B(p, \rho_0)$. Then

$$L(\tilde{\gamma}) = L(\tilde{\gamma}_1) + L(\tilde{\gamma}_2) \geq \rho_0 + d(p_0, q)$$

   By taking infimum on the both sides, we have

$$d(p, q) \geq \rho_0 + d(p_0, q) \;\Rightarrow\; d(p_0, q) \leq r - \rho_0$$

   Claim: $d(p, p_1) = \rho_0 + \rho_1$.

   Because $d(p_0, q) = r - \rho_0$, by above we similarly have $d(p_1, q) = r - \rho_0 - \rho_1$. So

$$\begin{aligned} d(p, p_1) &\geq d(p, q) - d(p_1, q) \\ &= r - (r - \rho_0 - \rho_1) \\ &= \rho_0 + \rho_1 \end{aligned}$$

   but $L(\gamma_0 \cup \gamma_1) = \rho_0 + \rho_1$, so $d(p, p_1) = \rho_0 + \rho_1$ and $\gamma_0 \cup \gamma_1$ is the smooth geodesic.

II. Whether γ can reach at q: The answer is not always. So we need more assumptions.

   (i) By compactness: γ may not reach at q because $\rho_n \to 0$. To solve such problem, we need to avoid this situation.

   Definition 1.6.1 (Injective Radius). For any $p \in M$,

$$i(p) = \sup\{\rho > 0\colon B(O_p, \rho) \text{ is a normal ball.}\}$$

   and for any $U \subset M$, let $i(U) = \inf_{p \in U} i(p)$.

   Proposition 1.6.1. *For $p, q \in M$ with $d(p, q) = r$, if $\overline{B(p, r)}$ is compact, then p, q can be connected by a geodesic of shortest length.*

Proof. Clearly, by the compactness of $\overline{B(p,r)}$,

$$\rho_0 = i\left(\overline{B(p,r)}\right) > 0$$

Then in above sequence, we can choose each $\rho_n = \rho_0$. Then at finitely many steps, γ can reach at q. $\square$

(ii) By completeness of geodesics:

Proposition 1.6.2. *For $p, q \in M$ with $d(p,q) = r$, if $\exp_p$ can be defined on T_pM, then p, q can be connected by a geodesic of shortest length.*

Proof. For $v \in T_pM$ with $\|v\| = 1$,

$$c(t) = \exp_p(tv)$$

can be well-define on $t \in [0, \infty)$. Consider the set

$$I := \{t \in [0,r] \colon d(c(t), q) = r - t\}$$

Check: $I = [0, r]$.

- First, $I \neq \varnothing$ by $0 \in I$.
- I is closed, because

  $$I = f^{-1}(0) \cap [0, r]$$

  where

  $$f(t) = d(c(t), q) - r - t$$

  is continuous.
- I is open: Let $t \in I$, *i.e.* $d(c(t), q) = r - t$. So

  $$d(p, c(t)) \geq r - (r - t) = t$$

  and clearly $d(p, c(t)) \leq t$. So $d(p, c(t)) = t$, which means $c|_{[0,t]}$ is the shortest geodesic connecting p and $c(t)$. Then for any $t_1 \in [0, t]$, $c|_{[0,t_1]}$ is the shortest geodesic connecting p and $c(t_1)$, so

  $$d(p, c(t_1)) = t_1$$

  Moreover, we have

  $$d(c(t_1), q) \leq d(c(t_1), c(t)) + d(c(t), q) = r - t_1$$

  and

  $$d(c(t_1), q) \geq d(p, q) - d(p, c(t_1)) = r - t_1$$

  Therefore, $d(c(t_1), q) = r - t_1$, that is $t_1 \in I$. Thus, $[0, t] \subset I$. On the other direction, there is a $\delta > 0$, such that $B(c(t), \delta)$ is a normal ball. And by above construction, we can find a $p' \in \partial B(c(t), \delta)$ such that $c(t)$ can be extended to $[0, t + \delta]$ with $c(t + \delta) = q$ and $d(c(t + \delta), q) = r - t - \delta$. Therefore, $t + \delta \in I$ and thus $[0, t + \delta) \subset I$, so I is open. $\square$

Theorem 1.6.1 (Hopf-Rinow)**.** *Let (M, g) be a Riemannian manifold, then TFAE.*

(1) (M, g) is a complete metric space.

(2) All bounded and closed subset of M is compact.

(3) There is a $p \in M$ such that $\exp_p$ *can be defined on* T_pM.

(4) For all $p \in M$, $\exp_p$ *can be defined on* T_pM.

Remark. By above propositions, we know that each one of above conditions implies that for any $p, q \in M$, there is a geodesic connects p, q of shortest length.

Proof. (4) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (2): Let $p \in M$ such that $\exp_p$ can be defined on T_pM. Choose an arbitrary $r > 0$.
Check: $\overline{B(p,r)} = \exp_p \overline{B(O_p, r)}$.
For any $v \in \overline{B(O_p, r)}$, we get

$$d(p, \exp_p(v)) = \|v\| \le r \;\Rightarrow\; \exp_p(v) \in \overline{B(p,r)}$$

Conversely, for any $q \in \overline{B(p,r)}$, then there is a unique shortest geodesic connecting p, q, *i.e.*

$$q = \exp_p(v), \quad \|v\| = d(p,q) \le r$$

So $q \in \exp_p \overline{B(O_p, r)}$. Then

$$\overline{B(p,r)} = \exp_p \overline{B(O_p, r)}$$

And by the compactness of $\exp_p \overline{B(O_p, r)}$, $\overline{B(p,r)}$ is compact. So because any bounded set is contained in some $\overline{B(p,r)}$, closedness implies that it is compact.
(2) $\Rightarrow$ (1): Let $\{p_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in M. So for any $p \in M$, by the triangular inequality, $\{d(p, p_n)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ and thus $\lim_{n\to\infty} d(p, p_n)$ exists. First, assume $\{p_n\}_{n\in\mathbb{N}}$ has no accumulative point, which means the set

$$\{p_n : n \in \mathbb{N}\}$$

is closed. And the Cauchy property implies that it is bounded. Then it is a compact set but this cannot happen. So $\{p_n\}_{n\in\mathbb{N}}$ has an accumulative point p_0. Then by choosing $p = p_0$,

$$\lim_{n\to\infty} d(p_0, p_n) = 0 \;\Rightarrow\; \lim_{n\to\infty} p_n = p_0 \in M$$

(1) $\Rightarrow$ (4): For any $p \in M$ and $v \in T_pM$,

$$\exp_p(\cdot v) \colon [0, b) \longrightarrow M, \quad b < \infty$$

Choosing $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n < b$ and $t_n \to b$. Then we know $\left\{\exp_p(t_n v)\right\}_{n\in\mathbb{N}}$ is Cauchy, so

$$\lim_{n\to\infty} \exp_p(t_n v) = p_0 \in M$$

Therefore, for any $\delta > 0$, there is an $N \in \mathbb{N}$ such that

$$\exp_p(t_n v) \in \overline{B(p_0, \delta)}, \quad \forall\, n > N$$

but it is contradicted to the assumption $b < \infty$ by the following lemma because there is a sufficiently small δ such that $\overline{B(p_0, \delta)}$ is compact. $\square$

Lemma 1.6.1. *Let $p \in M$ and $v \in T_pM$ such that* $\exp_p(tv)$ *defined on* $[0, b)$ *with* $b < \infty$. *For any compact $\Omega \subset M$ such that* $\exp_p(t_0 v) \in \Omega$. *Then there is $t_1 \in [t_0, b)$ such that* $\exp_p(t_1 v) \notin \Omega$.

Proof. By the compactness of Ω, there is a $\rho_0 > 0$ such that $\exp_q$ is diffeomorphic on $B(O_q, \rho_0)$ for any $q \in \Omega$. So if for any $t \in [t_0, b)$, $\exp_p(tv) \in \Omega$. But it means for all $t \in [t_0, b)$, $t + \rho_0/\|v\| \in [t_0, b)$, which is contradicted to $b < \infty$. $\square$

Corollary 1.6.1. *Let (M, g) be a complete Riemannian manifold. For any $p \in M$,*

$$M = \exp_p(E(p)) \sqcup C(p)$$

Theorem 1.6.2. *Let (M, g) be a complete Riemannian manifold. Let $\gamma\colon [0, \infty) \to M$ be a geodesic with $\gamma(0) = p, \dot{\gamma}(0) = v$. Let $\gamma(a)$ be a cut point. Then at least one of the following holds.*

(1) $d\exp_p$ is singular at $av \in T_pM$, that is, for any $\delta > 0$, there is $\tilde{a} \in [a, a+\delta)$ and $\tilde{v} \in E(p) \cap B(av, \delta)$ such that $\exp_p(\tilde{v}) = \exp_p(\tilde{a}v)$.

(2) There are at least two shortest curves from $\gamma(0) = p$ to $\gamma(a) = q$ and a is the minimal value such that it happens.

Proof. Choose a decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n \to a$. By completeness, there is a $v_i \in T_pM$ with $\|v_i\| = 1$ such that

$$\gamma_i(t) = \exp_p(tv_i), \quad t \in [0, b_i]$$

is a shortest curve from $\gamma(0)$ to $\gamma(a_i)$, where $b_i = d(p, \gamma(a_i))$. Note $v_i \neq v$ because of uniqueness of geodesics.

$$\lim_{i\to\infty} b_i = \lim_{i\to\infty} d(p, \gamma(a_i)) = d(p, \gamma(a)) = a$$

So the set $\{b_iv_i\colon i \in \mathbb{N}\}$ is contained in a compact subset in $E(p)$, which means it has a convergent subsequence. WLTG, assume $\{b_iv_i\}_{i\in\mathbb{N}}$ is convergent,

$$\lim_{i\to\infty} b_iv_i = aY, \quad Y \in T_pM,\ \|Y\| = 1$$

So

$$\exp_p(aY) = \lim_{i\to\infty} \exp_p(b_iv_i) = \lim_{i\to\infty} \gamma(a_i) = \gamma(a) = \exp_p(av)$$

Then $t \mapsto \exp_p(tY)$ with $t \in [0, a]$ is a geodesic.

(1) $v \neq Y$: we have two shortest curve connecting p, q.

(2) $v = Y$: then

$$\lim_{i\to\infty} b_iv_i = av = \lim_{i\to\infty} a_iv$$

and because $\exp_p(b_iv_i) = \exp_p(a_iv)$, $\exp_p$ is not injective around av. $\square$

1.7 Shortest Curve in Homotopic Class

1. ***Compact Case:*** Considering the shortest curve in a homotopic class.

   Definition 1.7.1. Two closed curves c_0, c_1 in M,

   $$c_0, c_1\colon \mathbb{S}^1 \longrightarrow M$$

   is called homotopic if there is a continuous map

   $$C\colon \mathbb{S}^1 \times [0, 1] \longrightarrow M$$

   such that $c(t, 0) = c_0(t)$ and $c(t, 1) = c_1(t)$ for all $t \in \mathbb{S}^1$.

Theorem 1.7.1. *Let (M, g) be a compact Riemannian manifold. Then every homotopic class of closed curve in M contains a curve which is shortest in the homotopic class and a geodesic.*

Lemma 1.7.1. *Let (M, g) be a compact Riemannian manifold. There is a $\rho_0 > 0$ such that for all γ_0, γ_1 closed curves with $d(\gamma_0(t), \gamma_1(t)) < \rho_0$, γ_0 is homotopic to γ_1.*

Proof. Since M is compact,

$$\rho_0 = i(M) > 0$$

Therefore, for any fixed t, $d(\gamma_0(t), \gamma_1(t)) < \rho_0$ implies there is a shortest geodesic $\Gamma(t, s)$ connecting $\gamma_0(t)$ and $\gamma_1(t)$

$$\Gamma(t, s) = \exp_{\gamma_0(t)}\left(s \exp^{-1}_{\gamma_0(t)}(\gamma_1(t))\right)$$

The continuity of t is by Corollary 1.4.2. □

Corollary 1.7.1. *Let (M, g) be a compact Riemannian manifold. A shortest curve in a homotopic class is a geodesic.*

Proof. Suppose it is not a geodesic. First, $\rho_0 = i(M) > 0$. There there are p, q in such curve with $d(p, q) < \rho_0$ and the curve from p to q is not a geodesic. Then we can find a geodesic connecting p and q in the same normal ball , whose length is shorter. Then by replacing this part with the geodesic part, the new curve is homotopic to the original one by above lemma, but it has shorter length, which is contradicted to the shortest condition. □

Proof of Theorem 1.7.1. Assume all curves in the homotopic class are parametrized by arc length. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ in the homotopic class such that $L(\gamma_n) \to \inf$. Choose a partition of $\mathbb{S}^1$,

$$0 = t_0 < t_1 < \cdots < t_{k+1} = 2\pi$$

such that $L(\gamma_n|_{[t_i,t_{i+1}]}) \leq \rho_0/2$, where $\rho_0 = i(M)$. Because $L(\gamma_n) \to \inf$, k can be chosen independent with n. For γ_n, by replacing $\gamma_n|_{[t_i,t_{i+1}]}$ with a geodesic, we obtain a curve γ_n', which is homotopic to γ_n by above lemma (by $L(\gamma_n|_{[t_i,t_{i+1}]}) \leq \rho_0/2$). For $i = 1, \cdots, k$, let $p_i = \lim_{n\to\infty} \gamma_n(i)$ by compactness of M. Let γ be curve obtained by connecting p_i and p_{i+1} with a geodesic. Then $\gamma_n' \to \gamma$ by the continuity of geodesic with respect to end points. So by above lemma, γ is in the homotopic class. Moreover,

$$\inf \leq \lim_{n\to\infty} L(\gamma_n') = L(\gamma) \leq \lim_{n\to\infty} L(\gamma_n) = \inf$$

So $L(\gamma) = \inf$. And by above corollary, γ is a geodesic. □

Remark. Above result can be easily extended to homotopic class of curves connecting $p \neq q$ in a compact Riemannian manifold.

2. ***Covering Spaces:*** The next problem is if above result can be extended to a complete Riemannian manifold. To do that, we need a new technique from the theory from covering spaces.

Definition 1.7.2 (Covering Space)**.** Let X be a topological space. A covering space of X is a topological space $\widetilde{X}$ with a surjective continuous map $\pi\colon \widetilde{X} \to X$, called covering map, such that for any $x \in X$, there is a open U containing x with the property

(1) $\pi^{-1}(U) = \bigcup_\alpha V_\alpha$, where V_α's are open and disjoint in $\widetilde{X}$,

(2) for each α, $\pi_\alpha = \pi|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Remark. (i) Smoothness: Let $\widetilde{X}, X$ be equipped with smooth structure and π be C^∞ map and π_α be diffeomorphic.

(ii) Riemannian: Let $\widetilde{X} = \widetilde{M}, X = M$ be Riemannian manifolds with metrics $\widetilde{g}$ and g. Then we require $\pi^* g = \widetilde{g}$.

Definition 1.7.3 (Lifting)**.** Let $\widetilde{X}$ be a covering space of X with covering map π and Y be a topological space and $f : Y \to X$ be a continuous map. Then a lifting of f is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $f = \pi \circ \widetilde{f}$.

Proposition 1.7.1. *Let $\pi : \widetilde{X} \to X$ be a covering map and $f : Y \to X$ be a continuous map. Let $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ be two liftings of f. Suppose Y is connected and there exists $y_0 \in Y$ such that $\widetilde{f_1}(y_0) = \widetilde{f_2}(y_0)$. Then $\widetilde{f_1} = \widetilde{f_2}$ on Y.*

Proof. Let $Y_0 = \left\{ y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \right\}$. Then by $y_0 \in Y$, $Y \neq \varnothing$. It is sufficient to prove Y_0 open and closed.

- Y is closed: Suppose $y \neq Y_0$. Let U be an open neighborhood of $f(y)$ in X such that
$$\pi^{-1}(U) = \bigcup_\beta V_\beta$$
and $\pi_\beta := \pi|_{V_\beta} : V_\beta \to U$ homeomorphism. Take V_1 and V_2 such that $\widetilde{f_1}(y) \in V_1$ and $\widetilde{f_2}(y) \in V_2$. Then $V_1 \neq V_2$. Otherwise, if $V_1 = V_2$, $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ because $\pi \circ \widetilde{f_1}(y) = \pi \circ \widetilde{f_2}(y)$ and π is homeomorphic on V_1. Thus $V_1 \cap V_2 = \varnothing$. By continuity of $\widetilde{f_1}$ and $\widetilde{f_2}$, there is an open neighborhood $W \subset Y$ of y such that
$$\widetilde{f_1}(W) \subset V_1, \quad \widetilde{f_2}(W) \subset V_2$$
So $W \cap Y_0 = \varnothing$ and thus Y_0^c is closed.
- Y is open: Suppose $y \in Y_0$. Similarly, we have V_1 and V_2. Because $V_1 \cap V_2 \neq \varnothing$, $V_1 = V_2 = V$. Then by homeomorphism of π, $\widetilde{f_1} = \widetilde{f_2}$ on V. So $V \subset Y_0$. □

Theorem 1.7.2 (Universal Covering)**.** *If X is a connected and locally simply connected topological space, there exists a simply connected topological space $\widetilde{X}$ and a covering map $\pi : \widetilde{X} \to X$, which is unique up to homeomorphism. If $\hat{\pi} : \hat{X} \to X$ is any other simply connected covering of X, there is a homeomorphism $\varphi : \widetilde{X} \to \hat{X}$ such that $\hat{\pi} \circ \varphi = \pi$.*

Corollary 1.7.2. *If X is simply connected, then every covering $\pi : \widetilde{X} \to X$ is a homeomorphism.*

Lemma 1.7.2 (Homotopy Lifting)**.** *Let $\pi : \widetilde{X} \to X$ be a covering map. Given any continuous $F : P \times [0,1] \to X$ and lifting $\widetilde{F_0} : P \times \{0\} \to \widetilde{X}$ of $F_0 = F|_{P\times\{0\}} : P \times \{0\} \to X$, there is a unique lifting $\widetilde{F} : P \times [0,1] \to \widetilde{X}$ of F s.t. $\widetilde{F}|_{P\times\{0\}} = \widetilde{F_0}$.*

Remark. By setting $P = \{*\}$, we have the so-called path-lifting lemma, for any continuous curve $\gamma : [0,1] \to X$, if there is a $\widetilde{p} \in \widetilde{X}$ such that $\pi(\widetilde{p}) = \gamma(0)$, then there is a unique lifting $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ of γ with $\tilde{\gamma}(0) = \widetilde{p}$.

To prove this lemma, we need the Lebesgue covering lemma for $[0,1]$.

Lemma 1.7.3 (Lebesgue Covering Lemma). *Let (X, d) be a compact metric space. For any open covering $\mathscr{U}$ of X, there is a $\delta > 0$ such that for any $A \subset X$ with diam$(X) < \delta$, there is $U \in \mathscr{U}$* s.t. *$A \subset U$.*

Proof. Assume for any $n \in \mathbb{N}$, there is a $C_n \subset X$ with $\operatorname{diam}(C_n) < 1/n$ such that C_n is not contained in any $U \in \mathscr{U}$. And we pick some $x_n \in C_n$. By the compactness of X, there is a subsequence such that $x_{n_k} \to x_0 \in X$. Since $\mathscr{U}$ is an open covering, there is a $U \in \mathscr{U}$ such that $x_0 \in U$. Then we choose a $\varepsilon_0 > 0$ such that $B(x_0, \varepsilon_0) \subset U$. Choose a n_k such that

$$\frac{1}{n_k} < \frac{\varepsilon_0}{2}, \qquad d\left(x_{n_k}, x_0\right) < \frac{\varepsilon_0}{2}$$

It follows that

$$C_{n_k} \subset B\left(x_{n_k}, \frac{1}{n_k}\right) \subset B\left(x_0, \varepsilon_0\right) \subset U$$

which contracts the assumption. □

Lemma 1.7.4 (Tube Lemma). *Let X, Y be two topological spaces. If $A \subset X$ is compact and $B \subset Y$ is compact, $N \subset X \times Y$ is open and $A \times B \subset N$, then there are $U \subset X$ open and $V \subset Y$ open such that*

$$A \times B \subset U \times V \subset N$$

Proof. First, let $A = \{x_0\}$. Then for any y_0 with $(x_0, y_0) \in N$, there is are open $U_{x_0}^{y_0}$ in X and V_{y_0} in Y such that

$$(x_0, y_0) \in U_{x_0}^{y_0} \times V_{y_0} \subset N$$

Since $B = \bigcup_{y \in B}\{y\} \subset \bigcup_{y \in Y} V_y$, we can find $y_1, \cdots, y_k \in Y$ *s.t.*

$$B \subset \bigcup_{i=1}^{k} V_{y_i} =: V$$

Let $U = \bigcap_{i=1}^{k} U_{x_0}^{y_i}$. Then U is an open neighborhood of x_0 and so

$$N \supset \bigcup_{y} \left(U_{x_0}^{y} \times V_y\right) \supset \bigcup_{i=1}^{k} \left(U_{x_0}^{y_i} \times V_{y_i}\right) \supset \bigcup_{i=1}^{k} (U \times V_{y_i}) = U \times \bigcup_{i=1}^{k} V_{y_i} = U \times V \supset \{x_0\} \times B$$

So for any $x_0 \in A$, there are open U_{x_0} and V_{y_0} such that

$$\{x_0\} \times B \subset U_{x_0} \times V_{x_0} \subset N$$

By the compactness of A,

$$A \subset U_{x_1} \cup \cdots \cup U_{x_m} =: U$$

Let $V = \cap_{i=1}^{m} V_{x_i}$, then $B \subset V$ and V is open. So

$$A \times B \subset U \times V \subset \bigcup_{i=1}^{m} (U_{x_i} \times V_{x_i}) \subset N$$ □

Proof of Lemma 1.7.2. Considering an open covering $\{U_\alpha\}$ of X such that for each α there is a disjoint collection $\left\{V_\alpha^\beta\right\}_\beta$ of $\widetilde{X}$ such that $\pi^{-1}(U_\alpha) = \bigcup_\beta V_\alpha^\beta$ and $\pi|_{V_\alpha^\beta} : V_\alpha^\beta \to U_\alpha$ homeomorphism. For each (α, β), let $q_{\alpha,\beta} : U_\alpha \to V_\alpha^\beta$ the inverse of this homeomorphism. Let $I = [0, 1]$.

I. Local lifting of $p_0 \in P$: For $(p_0, t) \in P \times I$, there is a $U_{\alpha(t)}$ such that

$$(p_0, t) \in F^{-1}(U_{\alpha(t)})$$

So $\{p_0\} \times \{t\} \subset F^{-1}(U_{\alpha(t)})$. Because $\{p_0\}$ in P is compact and $\{t\}$ in I is compact, by the Tube Lemma, there are open sets $V \subset P$ and $I_t \subset I$ such that

$$\{p_0\} \times \{t\} \subset V \times I_t \subset F^{-1}(U_{\alpha(t)})$$

So $F(V \times I_t) \subset U_{\alpha(t)}$. Moreover, such V can be chosen as connected. Because $\{I_t\}_{t \in I}$ is an open covering of I, by the Lebesgue Covering Lemma for I, there is a partition $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ such that $[t_i, t_{i+1}] \subset I_t$ for some t. So

$$F(V \times [t_i, t_{i+1}]) \subset U_{\alpha(t)}$$

That is, for a fixed $p_0 \in P$, there is an open neighborhood V of p_0 and a partition $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ such that for all i, $F(V \times [t_i, t_{i+1}]) \subset U_\alpha$ for some α.

Claim: There is a sequence of maps $\widetilde{F}_V^k$ such that

(i) $\widetilde{F}_V^k \colon V \times [0, t_k] \to \widetilde{X}$ is a lift of $F|_{V \times [0, t_k]}$,

(ii) $(\widetilde{F}_V^k)|_{V \times \{0\}} = \widetilde{F}_0|_V$,

(iii) $\widetilde{F}_V^{k+1}|_{V \times [0, t_k]} = \widetilde{F}_V^k$

Proof of Claim. For $k = 0$, define $\widetilde{F}_V^0 \colon V \times \{0\} \to \widetilde{X}$ as

$$\widetilde{F}_V^0 = \widetilde{F}_0|_V$$

Then we construct the sequence inductively. Assume we already have $\widetilde{F}_V^j$ for $j = k$. For $k + 1$, first, by above, there is an α such that

$$F(V \times [t_k, t_{k+1}]) \subset U_\alpha$$

Because $\pi \circ \widetilde{F}_V^k = F_{V \times [0, t_k]}$,

$$\widetilde{F}_V^k(V \times [t_k, t_{k+1}]) \subset \pi^{-1}(U_\alpha) = \bigcup_\beta V_\alpha^\beta$$

Then by the connectedness of V and continuity of $\widetilde{F}_V^k$, there is a β,

$$\widetilde{F}_V^k(V \times [t_k, t_{k+1}]) \subset V_\alpha^\beta$$

and if we define

$$\widetilde{E} := q_\alpha^\beta \circ F_{V \times [t_k, t_{k+1}]}$$

then

$$\widetilde{E}|_{V \times \{t_k\}} = \widetilde{F}_V^k|_{V \times \{t_k\}}$$

We define

$$\widetilde{F}_V^{k+1}(p, t) := \begin{cases} \widetilde{F}_V^k(p, t), & t \in [0, t_k] \\ \widetilde{E}(p, t), & t \in [t_k, t_{k+1}] \end{cases}$$

Then by Pasting Lemma, $\widetilde{F}_V^{k+1}$ is continuous on $V \times [0, t_{k+1}]$ and satisfies above three conditions. $\square$

Therefore, $\widetilde{F}_V$ is a lifting of $F|_{V \times I}$ satisfying $(\widetilde{F}_V)|_{V \times \{0\}} = \widetilde{F}_0|_V$.

II. Uniqueness of $\widetilde{F}_V$: Above three conditions determine such sequence uniquely and so we denote $\widetilde{F}_V = \widetilde{F}_V^n$. Otherwise, assume there are two constructions $\widetilde{F}_V = \widetilde{F}'_V$. We only need to check if

$$\widetilde{F}_V|_{\{p\}\times I} = \widetilde{F}'_V|_{\{p\}\times I}, \quad \forall\, p \in V$$

As before, by replacing p_0 with p, there is a partition $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ such that for all i, $F(\{p\} \times [t_i, t_{i+1}]) \subset U_\alpha$ for some α. First, it is clear that

$$\widetilde{F}_V|_{\{p\}\times\{0\}} = \widetilde{F}'_V|_{\{p\}\times\{0\}}$$

We can prove the result by induction. Assume $\widetilde{F}_V|_{\{p\}\times[0,t_k]} = \widetilde{F}'_V|_{\{p\}\times[0,t_k]}$. Similarly, by connectedness, there is a β such that $\widetilde{F}_V(\{p\} \times [t_k, t_{k+1}]) \subset V_\alpha^\beta$ and there is a β' such that $\widetilde{F}'_V(\{p\} \times [t_k, t_{k+1}]) \subset V_\alpha^{\beta'}$. But by assumption $\tilde{F}_N(p, t_k) = \tilde{F}'_N(p, t_k)$, $\beta = \beta'$. Then because$q_\alpha^\beta = \pi^{-1}\colon V_\alpha^\beta \to U_\alpha$ is a homeomorphism and

$$\pi \circ \widetilde{F}_V|_{\{p\}\times[t_k,t_{k+1}]} = \pi \circ \widetilde{F}'_V|_{\{p\}\times[t_k,t_{k+1}]}$$

we have $\widetilde{F}_V|_{\{p\}\times[t_k,t_{k+1}]} = \widetilde{F}'_V|_{\{p\}\times[t_k,t_{k+1}]}$. So we get

$$\widetilde{F}_V|_{\{p\}\times[0,t_{k+1}]} = \widetilde{F}'_V|_{\{p\}\times[0,t_{k+1}]}$$

And this also implies that, for V, W open,

$$\widetilde{F}_V|_{V\cap W\times I} = \widetilde{F}_W|_{V\cap W\times I}$$

III. Global Lifting: By above, for any $p \in P$, there is a V_p such that $\widetilde{F}_p := \widetilde{F}_{V_p}$ is a lifting of $F|_{V_p\times I}$ satisfying $(\widetilde{F}_p)|_{V_p\times\{0\}} = \widetilde{F}_0|_{V_p}$. Because $\{V_p\}_{p\in P}$ is an open covering, we define $\tilde{F}\colon P \times I \to \tilde{X}$ by

$$\widetilde{F}|_{V_p\times I} = \widetilde{F}_p$$

which is well-defined and continuous and the uniqueness is by the uniqueness of $\widetilde{F}_p$. □

3. ***Complete Case:*** For a complete Riemannian manifold (M, g), we will use its Riemannian covering $\pi\colon (\widetilde{M}, \tilde{g}) \to (M, g)$ to prove the existence of shortest curve in homotopic class in (M, g). But the first two questions are if the Riemannian covering exists and if it is also complete.

Definition 1.7.4 (Local Isometry)**.** Let $\varphi\colon (M, g) \to (N, h)$ be C^∞ between two Riemannian manifolds. If for any $p \in M$,

$$d\varphi_p\colon T_pM \longrightarrow T_{\varphi(p)}N$$

is orthogonal, *i.e.* $\varphi^*h = g$, then φ is called a local isometry.

Note that any Riemannian covering map is locally isometric.

Proposition 1.7.2. *Let $\varphi\colon (M, g) \to (N, h)$ be a local isometry.*

(1) φ map geodesics in M to geodesics in N.

(2) For any $p \in M$, the following diagram is commutative

$$\begin{array}{ccc} T_pM & \xrightarrow{d\varphi_p} & T_{\varphi(p)}N \\ \Big\downarrow{\scriptstyle \exp_p} & & \Big\downarrow{\scriptstyle \exp_{\varphi(p)}} \\ M & \xrightarrow{\quad\varphi\quad} & N \end{array}$$

(3) For any $p, q \in M$,

$$d_N(\varphi(p), \varphi(q)) \le d_M(p, q)$$

(4) If φ is bijective, then φ is an isometry.

Proof. (1) It is because φ is a local diffeomorphism and isometry. So it is as same as the change of coordinates. But the geodesic equation is independent with the change of coordinates. So φ preserves geodesics.

(2) For any $v \in T_{\varphi(p)}N$, let $\tilde{v} = (d\varphi_p)^{-1}(v) \in T_pM$. By (1), consider the geodesic

$$\gamma \colon t \mapsto \varphi(\exp_p(t\tilde{v}))$$

which satisfies $\gamma(0) = \varphi(p)$ and

$$\dot{\gamma}(0) = \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp_p(t\tilde{v})) = d\varphi_p(\tilde{v}) = v$$

By the uniqueness of geodesic,

$$\varphi(\exp_p(t\tilde{v})) = \exp_{\varphi(p)}(tv)$$

(3) It is because φ preserves the distance of curves.

$$\begin{aligned} d(p, q) &= \inf\{L(\gamma) \colon \gamma(0) = p, \gamma(1) = q\} \\ &= \inf\inf\{L(\tilde{\gamma}) \colon \tilde{\gamma} = \varphi(\gamma)\} \\ &\ge \inf\inf\{L(\tilde{\gamma}) \colon \tilde{\gamma}(0) = \varphi p, \tilde{\gamma}(1) = \varphi q\} = d(\varphi(p), \varphi(q)) \end{aligned}$$

(4) It is clear by (3). □

Proposition 1.7.3. *If $\pi \colon (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering, then (M, g) is complete if and only if $(\widetilde{M}, \widetilde{g})$ is complete.*

Proof. ($\Rightarrow$): For any $\tilde{p} \in \widetilde{M}$ and $\tilde{v} \in T_{\tilde{p}}\widetilde{M}$, let $p = \pi(\tilde{p})$ and $v = d\pi_{\tilde{p}}(\tilde{v})$. The we have the geodesic

$$\gamma(t) = \exp_p(tv) \in M, \quad t \in [0, \infty)$$

by the completeness of M. Then by the path lifting property, there is a lifting $\tilde{\gamma}(t) \colon [0, \infty) \to \tilde{M}$ of γ, *i.e.* $\tilde{\gamma}(0) = \tilde{p}$ and $\pi \circ \tilde{\gamma} = \gamma$, which implies

$$v = \left.\frac{d}{dt}\right|_{t=0} \gamma = \left.\frac{d}{dt}\right|_{t=0} \pi \circ \tilde{\gamma} = d\pi_{\tilde{p}}(\dot{\tilde{\gamma}}(0)) \;\Rightarrow\; \dot{\tilde{\gamma}}(0) = \tilde{v}$$

Besides, because π is locally isometric, $\tilde{\gamma}$ is also a geodesic. So

$$\tilde{\gamma}(t) = \exp_{\tilde{p}}(t\tilde{v}), \quad t \in [0, \infty)$$

which implies the completeness of $(\widetilde{M}, \widetilde{g})$ by Hopf-Rinow Theorem.

($\Leftarrow$): For any $p \in M$ and $v \in T_pM$, let $\tilde{p} \in \widetilde{M}$ be $\pi(\tilde{p}) = p$ and $\tilde{v} = \tilde{v} = (d\pi_p)^{-1}(v)$, then we have

$$\pi(\exp_{\tilde{p}}(t\tilde{v})) = \exp_p(tv)$$

where $t \in [0, \infty)$ by the completeness of $(\widetilde{M}, \widetilde{g})$. Therefore, (M, g) is complete by Hopf-Rinow Theorem. □

Remark. The ($\Leftarrow$) only needs the local isometry of π, which means if $\pi\colon (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a local isometry, then the completeness of $\widetilde{M}$ implies the completeness of M.

Theorem 1.7.3. *Let (M, g) be a complete Riemannian manifold and $p, q \in M$. Then every homotopic class of curves contain a shortest curve.*

Proof. Because M is connected and locally simply connected, by the Universal Covering Theorem, there is a covering $\pi\colon \widetilde{M} \to M$, where $\widetilde{M}$ is simply connected. Then we can equip $\widetilde{M}$ with a smooth structure such that $\pi \in C^\infty$ and a Riemannian metric $\widetilde{g} = \pi^* g$. So $\pi\colon (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering and $(\widetilde{M}, \widetilde{g})$ is complete.

Let $\sigma\colon [0, 1] \to M$ be a curve connecting p, q in the given homotopic class. Let $\tilde{p} \in \pi^{-1}(p)$, then there is a unique lifting $\tilde{\sigma}\colon [0, 1] \to \widetilde{M}$ of σ such that $\tilde{\sigma}(0) = \tilde{p}$. Because $(\widetilde{M}, \widetilde{g})$ is complete, there is a minimizing geodesic $\tilde{\gamma}$ connecting $\tilde{p} = \tilde{\sigma}(0)$ and $\tilde{q} := \tilde{\sigma}(1)$ and so $\gamma = \pi \circ \tilde{\gamma}$ is a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$, because π is locally isometric. Since $\widetilde{M}$ is simply connected, $\tilde{\gamma}$ is homotopic to $\tilde{\sigma}$. So γ is homotopic to σ, *i.e.* γ is in the give homotopic class. Finally, suppose σ_1 be any piecewise smooth curve connecting p, q in the given homotopic class. Let $\tilde{\sigma}_1$ be its lifting in $\widetilde{M}$ with $\tilde{\sigma}_1(0) = \tilde{p}$. Then it must end at $\tilde{\sigma}_1(1) = \tilde{q}$ by the Homotopy Lifting Lemma. Then

$$L(\gamma) = L(\tilde{\gamma}) \leq L(\tilde{\sigma}_1) = L(\sigma_1)$$

So γ is shortest. □

Theorem 1.7.4 (Ambrose)**.** *Let $\pi\colon (\widetilde{M}, \widetilde{g}) \to (M, g)$ be a local isometry. If $(\widetilde{M}, \widetilde{g})$ is complete, then (M, g) is complete and π is a Riemannian covering.*

Lemma 1.7.5. *Let $\pi\colon (\widetilde{M}, \widetilde{g}) \to (M, g)$ be a local isometry and $(\widetilde{M}, \widetilde{g})$ be complete. Given any geodesic $\gamma\colon [0, a] \to M$ with $\gamma(0) = p$ and any $\tilde{p} \in \pi^{-1}(p)$, we have a unique geodesic $\tilde{\gamma}\colon [0, a] \to \widetilde{M}$ such that $\tilde{\gamma}(0) = \tilde{p}$ and $\gamma(t) = \pi \circ \tilde{\gamma}(t)$, and such $\tilde{\gamma}$ is also called a geodesic lifting.*

Proof. By local isometry of π, let $\tilde{v} = (d\pi_{\tilde{p}})^{-1}(\dot{\gamma}(0))$. By completeness of $(\widetilde{M}, \widetilde{g})$, the geodesic

$$\tilde{\gamma}(t) := \exp_{\tilde{p}}(t\tilde{v})$$

can be defined on $t \in [0, a]$ and so

$$\pi(\exp_{\tilde{p}}(t\tilde{v})) = \exp_p(tv) = \gamma(t), \quad t \in [0, a]$$

The uniqueness is by the local isometry of π and the uniqueness of geodesic. □

Proof of Theorem 1.7.4. First, (M, g) is complete by above. Next, we need to show π is a covering.

(1) π is surjective: For any $\tilde{p} \in \widetilde{M}$, let $p = \pi(\tilde{p})$. For any $q \in M$, because M is complete, there is a shortest geodesic γ from p to $q = \gamma(t_0)$. By above lemma, γ has a lifting $\tilde{\gamma}$ in $\widetilde{M}$ such that $\tilde{\gamma}(0) = \tilde{p}$ and

$$\pi(\tilde{\gamma}(t)) = \gamma(t)$$

So let $\tilde{q} = \tilde{\gamma}(t_0)$, then $\pi(\tilde{q}) = q$.

(2) π is a evenly covering: For any $p \in M$, let $U = B(p, \varepsilon)$ be a normal ball. Consider $\pi^{-1}(p) = \{\tilde{p}_\alpha\}_{\alpha \in \Lambda}$. Let $\tilde{U}_\alpha = B(\tilde{p}_\alpha, \varepsilon)$.

I. $\pi^{-1}(U) = \bigcup_\alpha \tilde{U}_\alpha$: For any $\tilde{q} \in \tilde{U}_\alpha$ for some α, let $\tilde{\gamma}$ be the shortest geodesic from $\tilde{p}_\alpha$ to $\tilde{q}$. Then $\pi \circ \tilde{\gamma}$ is the shortest geodesic from p to $q := \pi(\tilde{q})$. Therefore,

$$d(p, q) \leq d(\tilde{p}_\alpha, \tilde{q}) < \varepsilon$$

and so $q \in U$, *i.e.* $\tilde{q} \in \pi^{-1}(U)$.
Conversely, for any $\tilde{q} \in \pi^{-1}(U)$, let $q = \pi(\tilde{q}) \in U$. Let $\gamma\colon [0, a] \to M$ be the shortest geodesic connecting $\gamma(0) = q, \gamma(1) = p$ and let $\tilde{\gamma}\colon [0, a] \to \widetilde{M}$ be the geodesic lifting of γ starting from $\tilde{\gamma}(0) = \tilde{q}$. Then $\pi(\tilde{\gamma}(1)) = \gamma(1) = p$. So $\tilde{\gamma}(1) = \tilde{p}_\alpha$ for some α. Then

$$d(\tilde{q}, \tilde{p}_\alpha) \leq L(\tilde{\gamma}) = L(\gamma) = d(p, q) < \varepsilon$$

So $\tilde{q} \in \tilde{U}_\alpha$.

II. $\pi\colon \tilde{U}_\alpha \to U$ diffeomorphic: By above, we already have $\pi(U_\alpha) \subset U$. First, for sujectivity, let $q \in U$, choose a geodesic $\gamma\colon [0, a] \to M$ in U connecting $\gamma(0) = p$ and $\gamma(a) = q$. Then there is a geodesic lifting $\tilde{\gamma}\colon [0, a] \to M$ starting from $\tilde{\gamma}(0) = \tilde{p}_\alpha$ to $\tilde{\gamma}(a) := \tilde{q}_\alpha$. Because

$$d(\tilde{p}_\alpha, \tilde{q}) \leq L(\tilde{\gamma}) = L(\gamma) < \varepsilon$$

$\tilde{q} \in U_\alpha$. And we have

$$\pi(\tilde{q}) = \pi(\tilde{\gamma}(a)) = \gamma(a) = q$$

For the injectivity, let $\tilde{q}_1 \neq \tilde{q}_2 \in \tilde{U}_\alpha$ with

$$\pi(\tilde{q}_1) = \pi(\tilde{q}_2) = q$$

Similarly, let γ be a geodesic in U from q to p and $\tilde{\gamma}_i$ be two geodesic liftings from $\tilde{q}_i$ to $\tilde{p}_\alpha$ in U_α. Let $\tilde{v}_i = \dot{\gamma}_i(a)$. Then $\tilde{v}_1 \neq \tilde{v}_2$. But because $\pi \circ \tilde{\gamma}_1 = \pi \circ \tilde{\gamma}_1$,

$$d\pi_{\tilde{p}_\alpha}(\tilde{v}_1) = d\pi_{\tilde{p}_\alpha}(\tilde{v}_2)$$

contradicted to the local isometry of π.

III. $\tilde{U}_\alpha \cap \tilde{U}_\beta = \varnothing$ for $\alpha \neq \beta$: Assume $\tilde{q} \in \tilde{U}_\alpha \cap \tilde{U}_\beta$. Let $\tilde{\gamma}_i$ the shortest geodesics connecting $\tilde{p}_i, \tilde{q}$ in $\tilde{U}_i$ for $i = \alpha, \beta$. Because by above $\pi\colon U_i \to U$ is isometric, $\pi(\tilde{\gamma}_\alpha)$ and $\pi(\tilde{\gamma}_\beta)$ are two shortest geodesics connecting $p, \pi(\tilde{q})$. So $\pi(\tilde{\gamma}_\alpha) = \pi(\tilde{\gamma}_\beta)$. Then by the uniqueness of geodesic lifting $\tilde{p}_\alpha = \tilde{p}_\beta$, which induces a contradiction. $\square$

Chapter 2

Connections

2.1 Affine Connections

The main goal is to define the derivative of tensor fields, which can be compatible with the Riemannian structure.

Definition 2.1.1. An affine connection ∇ on a smooth manifold M is a map

$$\nabla\colon \Gamma(TM)\times\Gamma(TM)\longrightarrow\Gamma(TM)$$

denoted by $(X,Y)\to\nabla_X Y$ satisfying for any $X,Y,Z\in\Gamma(TM)$

I. linearity:
$$\nabla_X(Y+Z)=\nabla_X Y+\nabla_Y Z$$

II. function linearity:
$$\nabla_{fX+gY}Z=f\nabla_X Z+g\nabla_Y Z,\quad \forall\ f,g\in C^\infty(M)$$

III. Leibniz property:
$$\nabla_X(fY)=X(f)Y+f\nabla_X Y,\quad \forall\ f\in C^\infty(M)$$

Moreover, $\nabla_X Y$ is called the covariant derivative of Y along X.

Remark. (1) Note that by Leibniz property, we have $\nabla_X f=X(f)$.

(2) We can formally define the covariant differential. Consider the map
$$\nabla\colon \Gamma(TM)\longrightarrow\Gamma(TM)\otimes\Gamma(T^*M)$$
for any $Y\in\Gamma(TM)$, $\nabla Y\in\Gamma(TM)\otimes\Gamma(T^*M)$ viewed as
$$\nabla Y\colon \Gamma(T^*M)\times\Gamma(TM)\longrightarrow C^\infty(M)$$
is defined
$$\nabla Y(\omega,X)=\omega(\nabla_X Y)$$

(3) Note that, locally covariant derivative is not the directional derivative. Considering two charts (x,U) and (y,V) with $p\in U\cap V$, let
$$Y=f^i\frac{\partial}{\partial x_i}=g^j\frac{\partial}{\partial y_j}$$

If $\nabla_X Y$ is the directional derivatives, then

$$\nabla_X Y(p) = (D_{X(p)} f^i) \frac{\partial}{\partial x_i} = (D_{X(p)} g^j) \frac{\partial}{\partial y_j}$$

However,

$$\begin{aligned}
D_{X(p)} f^i \frac{\partial}{\partial x_i} &= (D_{X(p)} g^j) \frac{\partial x^l}{\partial y_j} \frac{\partial}{\partial x_l} \\
&= \left(D_{X(p)} f^k \frac{\partial y^j}{\partial x^k} \right) \frac{\partial x^l}{\partial y_j} \frac{\partial}{\partial x_l} \\
&= (D_{X(p)} f^k) \frac{\partial y^j}{\partial x^k} \frac{\partial x^l}{\partial y_j} \frac{\partial}{\partial x_l} + f^k \left(D_{X(p)} \frac{\partial y^j}{\partial x^k} \right) \frac{\partial x^l}{\partial y_j} \frac{\partial}{\partial x_l} \\
&= (D_{X(p)} f^k) \frac{\partial}{\partial x^k} + f^k \left(D_{X(p)} \left(\frac{\partial y^j}{\partial x^k} \frac{\partial x^l}{\partial y_j} \right) - \frac{\partial y^j}{\partial x^k} D_{X(p)} \left(\frac{\partial x^l}{\partial y_j} \right) \right) \frac{\partial}{\partial x_l} \\
&= (D_{X(p)} f^k) \frac{\partial}{\partial x^k} - g^j D_{X(p)} \left(\frac{\partial x^l}{\partial y_j} \right) \frac{\partial}{\partial x_l}
\end{aligned}$$

which induces a contradiction.

Remark. In fact, affine connection exists infinitely. For any atlas $\{U_\alpha\}_{\alpha \in \Lambda}$ of M, we can find a partition of unity $(V_\beta, \varphi_\beta)_{\beta \in B}$. Then

$$\nabla_X Y(p) := \sum_{\beta \in B} \varphi_\beta(p) D_X^{V_\beta} Y(p)$$

Moreover, it is clear that the affine combination of affine connections is also an affine connection, that is, for affine connections $\nabla^{(1)}, \cdots, \nabla^{(k)}$,

$$\nabla := \sum_{i=1}^k f_i \nabla^{(i)}, \quad f_1, \cdots, f_k \in C^\infty(M) \text{ with } \sum_{i=1}^k f_i = 1$$

is an affine connection.

Similarly as derivatives, affine connections also have the local property.

Proposition 2.1.1. *For any open set $U \subset M$, if*

$$X|_U = \tilde{X}|_U, \quad Y|_U = \tilde{Y}|_U$$

then we have

$$(\nabla_X Y)|_U = (\nabla_{\tilde{X}} \tilde{Y})|_U$$

Proof. (1) First, check $(\nabla_X Y)|_U = (\nabla_{\tilde{X}} Y)|_U$. By function linearity, it is sufficient to check

$$X|_U = 0 \;\Rightarrow\; (\nabla_X Y)|_U = 0$$

For any $p \in U$, there is a compact $V \subset U$ containing p such that $f \in C^\infty(U)$ and $f \equiv 1$ on V. Then we have

$$(1 - f)X = X$$

So function linearity,

$$\nabla_X Y(p) = \nabla_{(1-f)X} Y(p) = (1 - f(p)) \nabla_X Y(p) = 0$$

(2) Next, check $(\nabla_{\tilde{X}}Y)|_U = (\nabla_{\tilde{X}}\tilde{Y})|_U$. By linearity, it is sufficient to check

$$Y|_U = 0 \;\Rightarrow\; (\nabla_X Y)|_U = 0$$

For any $p \in U$, as similar choosing, we have $(1-f)Y = Y$. Then by Leibniz property,

$$\nabla_X Y(p) = \nabla_X (1-f)Y(p) = X(1-f)(p)Y(p) + (1-f(p))\nabla_X Y(p) = 0$$

because $X(1-f)(p) = 0$ by the local property of vector fields. $\square$

Let's see the local expression of connection. On a chart (x, U), let

$$X = X^i \frac{\partial}{\partial x_i}, \quad Y = Y^j \frac{\partial}{\partial x_j}$$

Then by the properties of the affine connection,

$$\begin{aligned}
\nabla_X Y(p) &= \nabla_{X^i \frac{\partial}{\partial x_i}} Y \\
&= X^i \nabla_{\frac{\partial}{\partial x_i}} Y^j \frac{\partial}{\partial x_j} \\
&= X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial}{\partial x_j} + X^i Y^j \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right)
\end{aligned}$$

So the main part is $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$. Then we introduce new coefficients

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \begin{Bmatrix} k \\ i, j \end{Bmatrix} \frac{\partial}{\partial x_k}$$

It follows that

$$\nabla_X Y(p) = \left(X^i \frac{\partial Y^k}{\partial x_i} + X^i Y^j \begin{Bmatrix} k \\ i, j \end{Bmatrix} \right) \frac{\partial}{\partial x_k}$$

So by this calculation, we can directly obtain the following properties.

Proposition 2.1.2. *If $X(p) = \tilde{X}(p)$, then*

$$\nabla_X Y(p) = \nabla_{\tilde{X}} Y(p)$$

Proof. By function linearity, it is sufficient to prove

$$X(p) = 0 \;\Rightarrow\; \nabla_X Y(p) = 0$$

From above, $X(p) =$ implies $X^i(p) = 0$ for all i, and so by above $\nabla_X Y(p) = 0$ $\square$

Remark. First, from this, we have for any $v \in T_pM$

$$\nabla_v Y = \nabla_X Y(p)$$

for any X with $X(p) = v$.
Note that if $Y(p) = 0$, it does not implies $\nabla_X Y(p) = 0$ because it just satisfies the linearity. However, if along a curve, we can have similar result.

Proposition 2.1.3. *Let C^∞ curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ satisfy $\gamma(0) = p$ and $\dot{\gamma}(0) = v \in T_pM$. Suppose $Y(\gamma(t)) = \tilde{Y}(\gamma(t))$.*

$$\nabla_v Y(p) = \nabla_v \tilde{Y}(p)$$

Proof. Similarly, it is sufficient to prove that

$$Y(\gamma(t)) = 0 \;\Rightarrow\; \nabla_v Y(p) = 0$$

First, at p, $Y^i(p) = 0$. So the second term of $\nabla_v Y(p) = 0$. For the first term, because

$$0 = \left.\frac{d}{dt}\right|_{t=0} Y^k(\gamma(t)) = X^i(p) \frac{\partial Y^k}{\partial x_i}(p)$$ $\square$

2.2 Parallel Moving

For a C^∞ curve $c\colon [a,b] \to M$, a vector field along c is a map

$$V\colon [a,b] \longrightarrow TM$$

such that for any $t \in [a,b]$

$$V(t) \in T_{c(t)}M$$

or in local expression

$$V(t) = V^i(t)\, \frac{\partial}{\partial x^i}\bigg|_{c(t)}$$

Remark. The reason why we need the definition of vector fields along curves is that

$$c_*\left(\frac{d}{dt}\right) = \dot{c}(t)$$

in general, is not a vector field, for example, c intersects itself. But $\dot{c}(t)$ is a vector field along c.

Moreover, we denote the covariant derivative of V along c by $\frac{D}{dt}V$, which is defined as the following proposition.

Proposition 2.2.1. *Let smooth manifold M be equipped with an affine connection ∇. Then there is a unique map $V \mapsto \frac{D}{dt}V$ such that*

(1) linearity

$$\frac{D}{dt}(V+W) = \frac{D}{dt}V + \frac{D}{dt}W$$

(2) Leibniz property

$$\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{D}{dt}V, \quad \forall\ f \in C^\infty([a,b])$$

(3) if $V = Y|_{c(t)}$, where Y is a C^∞ vector field defined on a neighborhood of c, then

$$\frac{D}{dt}V = \nabla_{\dot{c}(t)}Y$$

Proof. For the uniqueness, similarly as the proof of connections, by the linearity and Leibniz property, it is local, so we can assume $c(t) \in U$ a chart (x,U). Then

$$V(t) = V^i(t)\, \frac{\partial}{\partial x^i}\bigg|_{c(t)}$$

So we have

$$\begin{aligned}
\frac{D}{dt}V(t) &= \frac{D}{dt}\left(V^i(t)\, \frac{\partial}{\partial x^i}\bigg|_{c(t)}\right) \\
&= \frac{dV^i(t)}{dt}\, \frac{\partial}{\partial x^i}\bigg|_{c(t)} + V^i(t)\frac{D}{dt}\, \frac{\partial}{\partial x^i}\bigg|_{c(t)} \\
&= \frac{dV^i(t)}{dt}\, \frac{\partial}{\partial x^i}\bigg|_{c(t)} + V^i(t)\nabla_{\dot{c}(t)}\, \frac{\partial}{\partial x^i}\bigg|_{c(t)}
\end{aligned}$$

which is uniquely determined.

For existence, locally we defined

$$\frac{D}{dt}V(t) := \frac{dV^i(t)}{dt} \left.\frac{\partial}{\partial x^i}\right|_{c(t)} + V^i(t)\nabla_{\dot{c}(t)} \left.\frac{\partial}{\partial x^i}\right|_{c(t)}$$

We need to check it is independent with the choice of coordinates. For

$$V(t) = V^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)} = \tilde{V}^j(t) \left.\frac{\partial}{\partial y^j}\right|_{c(t)}, \quad V^i(t) = \tilde{V}^j(t)\frac{\partial x^i}{\partial y^j}(c(t))$$

we get

$$\begin{aligned}
\frac{D}{dt}V(t) &= \frac{d}{dt}\left(\tilde{V}^j(t)\frac{\partial x^i}{\partial y^j}(c(t))\right) \left.\frac{\partial}{\partial x^i}\right|_{c(t)} + \tilde{V}^j(t)\frac{\partial x^i}{\partial y^j}(c(t))\nabla_{\dot{c}(t)}\frac{\partial y^j}{\partial x^i}(c(t)) \left.\frac{\partial}{\partial y^j}\right|_{c(t)} \\
&= \frac{d\tilde{V}^j(t)}{dt} \left.\frac{\partial}{\partial y^j}\right|_{c(t)} + \tilde{V}^j(t)\nabla_{\dot{c}(t)} \left.\frac{\partial}{\partial y^j}\right|_{c(t)} \\
&\quad + \tilde{V}^j(t)\left(\frac{d}{dt}\frac{\partial x^i}{\partial y^j}(c(t))\right) \left.\frac{\partial}{\partial x^i}\right|_{c(t)} + \tilde{V}^j(t)\frac{\partial x^i}{\partial y^j}(c(t))\left(\frac{d}{dt}\frac{\partial y^j}{\partial x^i}(c(t))\right) \left.\frac{\partial}{\partial y^j}\right|_{c(t)}
\end{aligned}$$

Besides, note that

$$\begin{aligned}
\tilde{V}^j(t)\frac{\partial x^i}{\partial y^j}(c(t))\left(\frac{d}{dt}\frac{\partial y^j}{\partial x^i}(c(t))\right) \left.\frac{\partial}{\partial y^j}\right|_{c(t)} &= -\tilde{V}^j(t)\frac{\partial y^j}{\partial x^i}(c(t))\left(\frac{d}{dt}\frac{\partial x^i}{\partial y^j}(c(t))\right) \left.\frac{\partial}{\partial y^j}\right|_{c(t)} \\
&= -\tilde{V}^j(t)\left(\frac{d}{dt}\frac{\partial x^i}{\partial y^j}(c(t))\right) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}
\end{aligned}$$

Therefore,

$$\frac{D}{dt}V(t) = \frac{d\tilde{V}^j(t)}{dt} \left.\frac{\partial}{\partial y^j}\right|_{c(t)} + \tilde{V}^j(t)\nabla_{\dot{c}(t)} \left.\frac{\partial}{\partial y^j}\right|_{c(t)}$$

Next, we need to prove the properties. The linearity and Leibniz property are clear. For the third one, if $c = (c^1, \cdots, c^m)$, then $\dot{c}(t) = \frac{d}{dt}c^i(t)\frac{\partial}{\partial x^i}$ and so

$$\nabla_{\dot{c}(t)}\frac{\partial}{\partial x^j} = \frac{d}{dt}c^i(t)\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \frac{d}{dt}c^i(t)\left\{\begin{matrix} k \\ i,j \end{matrix}\right\}\frac{\partial}{\partial x_k}$$

It follows that

$$\frac{D}{dt}V(t) = \left(\frac{dV^k(t)}{dt} + V^j(t)\frac{d}{dt}c^i(t)\left\{\begin{matrix} k \\ i,j \end{matrix}\right\}\right)\frac{\partial}{\partial x_k}$$

If $V(t) = Y(c(t))$ for some $Y \in \Gamma(TM)$, then by above

$$\frac{D}{dt}V(t) = \nabla_{\dot{c}(t)}Y(c(t))$$

□

Definition 2.2.1. Let M be a smooth manifold with an affine connection ∇. A vector field V along a curve $c\colon [a,b] \to M$ is called parallel if

$$\frac{D}{dt}V(t) = 0, \quad \forall\, t \in [a,b]$$

Moreover, by the uniqueness of ODE, we clearly have the following result.

Proposition 2.2.2. *Let M be a smooth manifold with an affine connection ∇ and $c\colon [a,b] \to M$ be a smooth curve. Then for any $V_0 \in T_{c(t_0)}M$, there is a unique parallel vector field V along c such that $V(t_0) = V_0$.*

Based on this, we can consider a map from T_pM to T_qM, where $p = c(0)$ and $q = c(t_0)$ for a smooth curve $c\colon [0, b] \to M$,

$$\mathcal{P}_{c,0,t_0}\colon T_pM \longrightarrow T_qM$$

For any $V_0 \in T_pM$, let V be the unique parallel vector field along c and so

$$\mathcal{P}_{c,0,t_0}(V_0) := V(t_0)$$

Moreover,

(1) by the linearity of ODE, $\mathcal{P}_{c,0,t_0}$ is linear,

(2) because we can clearly get the inverse of $\mathcal{P}_{c,0,t_0}$ by the uniqueness of parallel vector field, $\mathcal{P}_{c,0,t_0}$ is bijective.

Therefore, $\mathcal{P}_{c,0,t_0}$ is a linear isomorphism.

Proposition 2.2.3. *Let c be a smooth curve with $c(0) = p$ and $\dot{c}(0) = X(p)$. Let $Y \in \Gamma(TM)$. Then we have*

$$\nabla_X Y(p) = \lim_{h\to 0} \frac{\mathcal{P}^{-1}_{c,0,h}(Y(c(h))) - Y(c(0))}{h}$$

Proof. Let $V_1, V_2, \cdots, V_m$ be parallel vector fields along c, which are linearly independent at each point, which can be done by choosing a basis in $T_{c(0)}M$ and then generate the linear independent vector fields because $\mathcal{P}_{c,0,t}$ is isomorphic. Then we have $f^i(t)$ such that

$$Y(c(t)) = f^i(t)V_i(t)$$

So

$$\begin{aligned}
\text{RHS} &= \lim_{h\to 0} \frac{f^i(h)V_i(0) - f^i(0)V_i(0)}{h} \\
&= \left.\frac{df^i}{dt}\right|_{t=0} V_i(0) \\
&= \left.\frac{D}{dt}\right|_{t=0} (f^i(t)V^i(t)) \\
&= \left.\frac{D}{dt}\right|_{t=0} Y(c(t)) \\
&= \nabla_X Y(p)
\end{aligned}$$

□

Let $\varphi\colon N \to M$ be a smooth map between two smooth manifolds and let ∇ be an affine connection on M. A vector field V along φ is a smooth map

$$x \in N \mapsto V(x) \in T_{\varphi(x)}M$$

Let E_i be a frame vector field of M of a neighborhood containing $\varphi(x) \in N$, *i.e.*

$$V(x) = V^i(x)E_i(\varphi(x))$$

Then given any $u \in T_xN$, define

$$\widetilde{\nabla}_u V(\varphi(x)) := u(V^i(x))E_i(\varphi(x)) + V^i(x)\nabla_{d\varphi_x(u)}E_i(\varphi(x))$$

and $\widetilde{\nabla}_u$ is so-called the induced connection on N. Similarly as above, it is well-defined, *i.e.* it is independent with the choice of E_i. Note that

$$\frac{D}{dt} = \widetilde{\nabla}_{\frac{d}{dt}}$$

of $\varphi = c\colon [a,b] \to M$. Moreover, if $V(x) = W(\varphi(x))$ for some $W \in \Gamma(TM)$ (such V is called extendible), then

$$\widetilde{\nabla}_u V(x) = \nabla_{d\varphi(u)} W(\varphi(x))$$

For a smooth map $\varphi\colon N \to M$ and $X \in \Gamma(TN)$, we cannot define $d\varphi(X)$ as a vector field in M, even in $\mathrm{Im}\varphi$. In general, there are following two methods.

I. We define related vector field, that is, a vector field $X \in \Gamma(TN)$ is called φ-related to a vector field $\bar{X} \in \Gamma(TM)$ if for any $p \in N$,

$$d\varphi_p(X_p) = \bar{X}\big|_{\varphi(p)}$$

For example, if φ is a diffeomorphism, then $d\varphi(X) \in \Gamma(TM)$ is well-defined by

$$d\varphi(X)\big|_q = d\varphi_{\varphi^{-1}(q)}(X_{\varphi^{-1}(q)}), \quad \forall\ q \in M$$

Then X is φ-related to $d\varphi(X)$.

Lemma 2.2.1. *For a smooth map $\varphi\colon N \to M$, a vector field $X \in \Gamma(TN)$ is φ-related to a vector field $\bar{X} \in \Gamma(TM)$ if and only if for any $g \in C^\infty(M)$*

$$X(g \circ \varphi) = (\bar{X} g) \circ \varphi$$

Proof. For any $g \in C^\infty(M)$ and any $p \in M$, by

$$\begin{aligned}(\bar{X}g) \circ \varphi(p) &= (\bar{X}g)(\varphi(p)) = \bar{X}_{\varphi(p)}(g),\\ X(g \circ \varphi)(p) &= X_p(g \circ \varphi) = (d\varphi_p(X_p))(g)\end{aligned}$$

we have

$$X(g \circ \varphi) = (\bar{X}g) \circ \varphi \ \Leftrightarrow\ \bar{X}_{\varphi(p)} = d\varphi_p(X_p)$$

□

Proposition 2.2.4. *For a smooth map $\varphi\colon N \to M$, if $X, Y \in \Gamma(TN)$ are φ-related to $\bar{X}, \bar{Y} \in \Gamma(TM)$ respectively, then $[X, Y]$ is φ-related to $[\bar{X}, \bar{Y}]$. In particular, if φ is a diffeomorphism, then*

$$d\varphi([X, Y]) = [d\varphi(X), d\varphi(Y)]$$

Proof. For any $g \in C^\infty(M)$,

$$\begin{aligned}[X, Y](g \circ \varphi) &= XY(g \circ \varphi) - YX(g \circ \varphi)\\ &= X\left((\bar{Y}g) \circ \varphi\right) - Y\left((\bar{X}g) \circ \varphi\right)\\ &= (\bar{X}\bar{Y}g) \circ \varphi - (\bar{Y}\bar{X}g) \circ \varphi\\ &= ([\bar{X}, \bar{Y}]g) \circ \varphi\end{aligned}$$

Then it can get by above lemma. □

II. We can define $d\varphi(X)$ as a vector field along φ, because for any $p \in N$

$$d\varphi(X)(p) := d\varphi_p(X_p) \in T_{\varphi(p)}M$$

Moreover, we also have the similar result.

Lemma 2.2.2. *For a smooth map $\varphi\colon N \to M$, for any $X, Y \in \Gamma(TN)$,*

$$d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)], \quad X, Y \in \Gamma(TN)$$

Proof. It is sufficient to prove that locally. Let $(U,(x^1,\cdots,x^n))$ be a chart in N around p and $(V,(y^1,\cdots,y^m))$ be a chart in M around $\varphi(p)$, and $\varphi = (\varphi^1,\cdots,\varphi^m)$ with $y^j = \varphi^j(x^1,\cdots,x^n)$. Let

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^k \frac{\partial}{\partial x^k}$$

So by the following **Lemma** 2.2.3

$$\begin{aligned}[X,Y] &= \left[X^i \frac{\partial}{\partial x^i}, Y^k \frac{\partial}{\partial x^k}\right] \\ &= X^i \frac{\partial Y^k}{\partial x^i}\frac{\partial}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k}\frac{\partial}{\partial x^i}\end{aligned}$$

and thus

$$d\varphi([X,Y]) = \left(X^i \frac{\partial Y^k}{\partial x^i}\frac{\partial y^j}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k}\frac{\partial y^j}{\partial x^i}\right)\frac{\partial}{\partial y^j}$$

Moreover,

$$d\varphi(X) = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}, \quad d\varphi(Y) = Y^k \frac{\partial y^l}{\partial x^k}\frac{\partial}{\partial y^l}$$

By the fact,

$$\frac{\partial^2 y^j}{\partial x^i \partial x^k} = \frac{\partial^2 y^j}{\partial x^k \partial x^i}$$

we have

$$\begin{aligned}[d\varphi(X), d\varphi(Y)] &= \left[X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}, Y^k \frac{\partial y^l}{\partial x^k}\frac{\partial}{\partial y^l}\right] \\ &= X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}\left(Y^k \frac{\partial y^l}{\partial x^k}\right)\frac{\partial}{\partial y^l} - Y^k \frac{\partial y^l}{\partial x^k}\frac{\partial}{\partial y^l}\left(X^i \frac{\partial y^j}{\partial x^i}\right)\frac{\partial}{\partial y^j} \\ &= X^i \frac{\partial y^j}{\partial x^i}\left(\frac{\partial Y^k}{\partial y^j}\frac{\partial y^l}{\partial x^k} + Y^k \frac{\partial^2 y^l}{\partial y^j \partial x^k}\right)\frac{\partial}{\partial y^l} \\ &\quad - Y^k \frac{\partial y^l}{\partial x^k}\left(\frac{\partial X^i}{\partial y^l}\frac{\partial y^j}{\partial x^i} + X^i \frac{\partial^2 y^j}{\partial y^l \partial x^i}\right)\frac{\partial}{\partial y^j} \\ &= \left(X^i \frac{\partial Y^k}{\partial x^i}\frac{\partial y^l}{\partial x^k} + X^i Y^k \frac{\partial^2 y^l}{\partial x^i \partial x^k}\right)\frac{\partial}{\partial y^l} \\ &\quad - \left(Y^k \frac{\partial X^i}{\partial x^k}\frac{\partial y^j}{\partial x^i} - Y^k X^i \frac{\partial^2 y^j}{\partial x^k \partial x^i}\right)\frac{\partial}{\partial y^j} \\ &= \left(X^i \frac{\partial Y^k}{\partial x^i}\frac{\partial y^j}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k}\frac{\partial y^j}{\partial x^i}\right)\frac{\partial}{\partial y^j}\end{aligned}$$

Therefore,

$$[d\varphi(X), d\varphi(Y)] = d\varphi([X,Y])$$

□

Lemma 2.2.3. *Let M be a smooth manifold. For $X, Y \in \Gamma(TM)$ and $f, g \in C^\infty(M)$,*

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$$

2.3 Levi-Civita Connection

1. ***Affine connection on tensor fields***:

   Proposition 2.3.1. *Let M be a smooth manifold with an affine connection ∇. Then there is a unique map*

$$\nabla\colon \Gamma(TM)\times\Gamma(\bigotimes^{r,s}TM)\longrightarrow\Gamma(\bigotimes^{r,s}TM)$$

   such that

   (1) function linearity: for $A\in\Gamma(\bigotimes^{r,s}TM)$, $X,Y\in\Gamma(TM)$,

$$\nabla_{fX+gY}A=f\nabla_X A+g\nabla_Y A,\quad \forall\ f,g\in C^\infty(M)$$

   (2) linearity: for $A_1,A_2\in\Gamma(\bigotimes^{r,s}TM)$, $X\in\Gamma(TM)$,

$$\nabla_X(A_1+A_2)=\nabla_X A_1+\nabla_X A_2$$

   (3) Leibniz property: for $A\in\Gamma(\bigotimes^{r,s}TM)$, $X\in\Gamma(TM)$,

$$\nabla_X(fA)=X(f)A+f\nabla_X A,\quad \forall\ f\in C^\infty(M)$$

   (4) for $r=1,s=0$, ∇ coincides with ∇ as above

   (5) tensor property: $A_1,A_2\in\Gamma(\bigotimes^{r,s}TM)$, $X\in\Gamma(TM)$

$$\nabla_X(A_1\otimes A_2)=(\nabla_X A_1)\otimes A_2+A_1\otimes(\nabla_X A_2)$$

   (6) contraction: $c\colon\Gamma(\bigotimes^{r,s}TM)\to\Gamma(\otimes^{r-1,s-1}TM)$ contraction

$$c(\nabla_X A)=\nabla_X c(A)$$

   Proof. In local expression,

$$A=A^{i_1\cdots i_r}_{j_1\cdots j_s}Y_{i_1}\otimes\cdots\otimes Y_{i_r}\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_s}$$

   To guarantee above properties, we need

$$\begin{aligned}\nabla_X A&=\sum\nabla_X\left(A^{i_1\cdots i_r}_{j_1\cdots j_s}Y_{i_1}\otimes\cdots\otimes Y_{i_r}\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_s}\right)\\&=\sum X\left(A^{i_1\cdots i_r}_{j_1\cdots j_s}\right)Y_{i_1}\otimes\cdots\otimes Y_{i_r}\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_s}\\&\quad+\sum A^{i_1\cdots i_r}_{j_1\cdots j_s}\nabla_X\left(Y_{i_1}\otimes\cdots\otimes Y_{i_r}\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_s}\right)\end{aligned}$$

   where the second term can be divided into $\nabla_X Y$ and $\nabla_X\omega$. So the main goal is to define

$$\nabla\colon\Gamma(TM)\times\Gamma(T^*M)\longrightarrow\Gamma(T^*M)$$

   For any $X,Y\in\Gamma(TM)$ and $\omega\in\Gamma(T^*M)$, we have

$$\begin{aligned}X(\omega(Y))&=\nabla_X(\omega(Y))\\&=\nabla_X(c(Y\otimes\omega))\\&=c((\nabla_X Y)\otimes\omega+Y\otimes(\nabla_X\omega))\\&=\omega(\nabla_X Y)+(\nabla_X\omega)(Y)\end{aligned}$$

   Therefore, we define $\nabla_X\omega$ by

$$(\nabla_X\omega)(Y)=X(\omega(Y))-\omega(\nabla_X Y)$$

   □

Remark. For $A \in \Gamma(\bigotimes^{r,s} TM)$ $(r, s > 0)$, then contraction $c(A) \in \Gamma(\bigotimes^{r-1,s-1} TM)$ is defined as

$$c(A)(\omega_1, \cdots, \omega_{r-1}, X_1, \cdots, X_{s-1}) := \sum_i A(E_i^*, \omega_1, \cdots, \omega_{r-1}, E_i, X_1, \cdots, X_{s-1})$$

where $\{E_i\}$ and $\{E_i^*\}$ are a pair of dual orthonormal bases. Note that the definition is independent with the choice of orthonormal basis by linearity. Locally,

$$c(A)^{i_1 \cdots i_{r-1}}_{j_1 \cdots j_{s-1}} = A^{i i_1 \cdots i_{r-1}}_{i j_1 \cdots j_{s-1}}$$

In particular, for $A \in \Gamma(\bigotimes^{0,2} TM)$,

$$c(A) = \sum_i A(E_i, E_i), \quad E_i \text{ orthonormal basis of } TM$$

Locally,

$$A = A_{ij} dx^i \otimes dx^j \;\Rightarrow\; c(A) = g^{ik} A_{ki}$$

Similarly, for $A \in \Gamma(\bigotimes^{2,0} TM)$,

$$c(A) = \sum_i A(E_i^*, E_i^*), \quad E_i^* \text{ orthonormal basis of } T^*M$$

Locally,

$$A = A^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \;\Rightarrow\; c(A) = g_{ik} A^{ki}$$

Also, $c = \mathrm{tr}$.

Remark. Note that $(1) - (6)$ are not independent.

(i) (3) can be obtained by (5) because $fA = f \otimes A$.

(ii) (1) can be obtained by $(2) - (6)$ because

$$(fX + gY)(\omega(Z)) = (\nabla_{fX+gY}\omega)(Z) + \omega(\nabla_{fX+gY} Z)$$

where the LHS equals to

$$f\left((\nabla_X \omega)(Z) + \omega(\nabla_X Z)\right) + g\left((\nabla_Y \omega)(Z) + \omega(\nabla_Y Z)\right)$$

we have (1).

Corollary 2.3.1. *For any $A \in \Gamma(\bigotimes^{r,s} TM)$, let $\omega^1, \cdots, \omega^r \in \Gamma(T^*M)$ and $Y_1, \cdots, Y_s \in \Gamma(TM)$. Then we get*

$$\begin{aligned}(\nabla_X A)(\omega^1, \cdots, \omega^r, Y_1, \cdots, Y_s) = X(A(\omega^1, \cdots, \omega^r, Y_1, \cdots, Y_s)) &- \sum A(\cdots, \nabla_X \omega^i, \cdots) \\ &- \sum A(\cdots, \nabla_X Y_j, \cdots)\end{aligned}$$

Besides, in local expression, for the coordinates, we already have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \begin{Bmatrix} k \\ i, j \end{Bmatrix} \frac{\partial}{\partial x^k}$$

So

$$\begin{aligned}(\nabla_{\frac{\partial}{\partial x^i}} dx^j)(\frac{\partial}{\partial x^k}) &= \frac{\partial}{\partial x^k}\left(dx^j(\frac{\partial}{\partial x^k})\right) - dx^j\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right) \\ &= -dx^j\left(\begin{Bmatrix} l \\ i,k\end{Bmatrix}\frac{\partial}{\partial x^l}\right) \\ &= -\begin{Bmatrix} j \\ i,k\end{Bmatrix}\end{aligned}$$

It follows that

$$\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\begin{Bmatrix} j \\ i,k\end{Bmatrix} dx^k$$

In conclusion, we have

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \begin{Bmatrix} k \\ i,j\end{Bmatrix}\frac{\partial}{\partial x^k}, \quad \nabla_{\frac{\partial}{\partial x^i}} dx^j = -\begin{Bmatrix} j \\ i,k\end{Bmatrix} dx^k$$

Apart from the connection, the parallel moving map can also be extended. First,

$$\mathcal{P}_{c,0,t} \colon T_{c(0)}M \longrightarrow T_{c(t)}M$$

Then by canonical duality, we have

$$\mathcal{P}^*_{c,0,t} \colon T^*_{c(t)}M \longrightarrow T^*_{c(0)}M$$

Moreover, since $\mathcal{P}_{c,0,t}$ is isomorphic, we can consider $(\mathcal{P}^*_{c,0,t})^{-1} \colon T^*_{c(0)}M \longrightarrow T^*_{c(t)}M$. Therefore, the parallel moving can be extended on tensors.

$$\tilde{\mathcal{P}}_{c,0,t} \colon \bigotimes^{r,s} T_{c(0)}M \longrightarrow \bigotimes^{r,s} T_{c(t)}M$$

By choosing a basis $V_i(0)$ in $T_{c(0)}M$, we can generate $V_i(t) = \mathcal{P}_{c,0,t}(V_i(0))$ that is also a basis in $T_{c(t)}M$ because $\mathcal{P}_{c,0,t}$ is isomorphic. Besides, if we choose the canonical dual basis $\omega^j(0)$ of $V_i(0)$ in $T^*_{c(0)}M$ and let $\omega^j(t) = (\mathcal{P}^*_{c,0,t})^{-1}(\omega^j(0))$ that is also a basis in $T^*_{c(t)}M$, then by

$$\begin{aligned}\omega^j(t)\,(V_i(t)) &= \omega^j(t)\,(\mathcal{P}_{c,0,t}(V_i(0))) \\ &= \mathcal{P}^*_{c,0,t}\omega^j(t)(V_i(0)) \\ &= \omega^j(0)(V_i(0)) \\ &= \delta^j_i\end{aligned}$$

it is also the dual basis. So as similar as above, by choosing the basis $V_{i_1}(t)\otimes\cdots\otimes V_{i_r}(t)\otimes \omega^{j_1}(t)\otimes\cdots\otimes\omega^{j_s}(t)$ in $\bigotimes^{r,s} T_{c(t)}M$, we can prove

$$\nabla_X A(p) = \lim_{h\to 0}\frac{\tilde{\mathcal{P}}^{-1}_{c,0,t}(A(c(h))) - A(c(0))}{h}$$

where $X(p) = \dot{c}(0)$.

Definition 2.3.1. A tensor field A is called parallel if

$$\nabla_X A = 0, \quad \forall\ X \in \Gamma(TM)$$

2. ***Levi-Civita connection:*** Based on above calculation, we have seen an affine connection ∇ can be determined by the coefficients

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \begin{Bmatrix} k \\ i,j\end{Bmatrix}\frac{\partial}{\partial x_k}$$

Proposition 2.3.2. *Let (M, g) be a Riemannian manifold with an affine connection ∇ that provides the coefficients $\left\{ {i \atop j,k} \right\}$.*

(1) $\nabla_X g = 0$ for all $X \in \Gamma(TM)$ if and only if

$$g_{ij,l} = g_{ik}\left\{ {k \atop l, j} \right\} + g_{kj}\left\{ {k \atop l, i} \right\}$$

(2) $\left\{ {k \atop i,j} \right\} = \left\{ {k \atop j,i} \right\}$ if and only if

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

Proof. (1) On local chart (x, U),

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^l}} g &= \nabla_{\frac{\partial}{\partial x^l}} \left(g_{ij} dx^i \otimes dx^j \right) \\
&= g_{ij,l} dx^i \otimes dx^j - g_{ij}\left\{ {i \atop l, \alpha} \right\} dx^\alpha \otimes dx^j - g_{ij}\left\{ {j \atop l, \beta} \right\} dx^i \otimes dx^\beta \\
&= \left(g_{ij,l} - g_{kj}\left\{ {k \atop l, i} \right\} - g_{ik}\left\{ {k \atop l, j} \right\} \right) dx^i \otimes dx^j
\end{aligned}$$

So $\nabla_X g = 0$ for all $X \in \Gamma(TM)$ if and only if $\nabla_{\frac{\partial}{\partial x^l}} g = 0$ for all l and it is if and only if

$$g_{ij,l} - g_{kj}\left\{ {k \atop l, i} \right\} - g_{ik}\left\{ {k \atop l, j} \right\} = 0, \quad \forall\, i, j, l$$

(2) Clearly, by definition,

$$\left\{ {k \atop i, j} \right\} = \left\{ {k \atop j, i} \right\} \Leftrightarrow \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$$

In a local chart, let

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j}$$

So we have

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^i \frac{\partial}{\partial x^i}} Y = X^i \nabla_{\frac{\partial}{\partial x^i}} Y^j \frac{\partial}{\partial x^j} \\
&= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^i Y^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \\
&= Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} + X^i Y^j \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} + X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} \\
&\quad + X^i Y^j \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \right) \\
&= \nabla_Y X + [X, Y] + X^i Y^j \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \right)
\end{aligned}$$

Therefore, $T(X, Y) = 0$ for all X, Y if and only if

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$$

for all i, j. □

Remark. (i) Clearly, the Christoffel coefficients

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(g_{lk,j} + g_{kj,l} - g_{jl,k} \right)$$

satisfies above two conditions.

(ii) For a curve $c(t) = (c^1(t), \cdots, c^m(t))$ in a chart (x, U), if we choose the Christoffel coefficients, then

$$\begin{aligned}
\frac{D}{dt}(\dot{c}(t)) &= \frac{D}{dt}\left(\frac{dc^i(t)}{dt} \frac{\partial}{\partial x^i} \right) \\
&= \frac{d^2 c^i(t)}{dt^2} \frac{\partial}{\partial x^i} + \frac{dc^i(t)}{dt} \nabla_{\dot{c}(t)} \frac{\partial}{\partial x^i} \\
&= \frac{d^2 c^i(t)}{dt^2} \frac{\partial}{\partial x^i} + \frac{dc^i(t)}{dt} \frac{dc^j(t)}{dt} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \\
&= \left(\frac{d^2 c^k(t)}{dt^2} + \Gamma^k_{ij} \frac{dc^i(t)}{dt} \frac{dc^j(t)}{dt} \right) \frac{\partial}{\partial x^k} = 0
\end{aligned}$$

if and only if

$$\frac{d^2 c^k(t)}{dt^2} + \Gamma^k_{ij} \frac{dc^i(t)}{dt} \frac{dc^j(t)}{dt} = 0$$

which is the geodesic equation. So

$$c(t) \text{ is a geodesic} \Leftrightarrow \tfrac{D}{dt}(\dot{c}(t)) = 0.$$

(iii) Because

$$Z(g(X,Y)) = (\nabla_Z g)(X,Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

(1) implies that

$$Z \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$$

(iv) The (1) condition is called that the connection is compatible with the Riemannian metric and the (2) condition is called torsion-free.

Definition 2.3.2. Let (M, g) be a Riemannian manifold. An affine connection ∇ on M is called a Levi-Civita connection if it is compatible with g and it is torsion-free.

Theorem 2.3.1. *For any Riemannian manifold (M, g), there is a unique Levi-Civita connection ∇.*

Proof. First, to prove the uniqueness, let $X, Y, Z \in \Gamma(TM)$.

$$\begin{aligned}
\langle \nabla_X Y, Z \rangle &= X \langle Y, Z \rangle - \langle Y, \nabla_X Z \rangle \\
&= X \langle Y, Z \rangle - \langle Y, \nabla_Z X + [X, Z] \rangle \\
&= X \langle Y, Z \rangle - \langle Y, [X, Z] \rangle - \langle Y, \nabla_Z X \rangle \\
&= X \langle Y, Z \rangle - \langle Y, [X, Z] \rangle - Z \langle Y, X \rangle + \langle \nabla_Z Y, X \rangle \\
&= X \langle Y, Z \rangle - \langle Y, [X, Z] \rangle - Z \langle Y, X \rangle + \langle \nabla_Y Z + [Z, Y], X \rangle \\
&= X \langle Y, Z \rangle - \langle Y, [X, Z] \rangle - Z \langle Y, X \rangle + \langle [Z, Y], X \rangle + \langle \nabla_Y Z, X \rangle \\
&= X \langle Y, Z \rangle - \langle Y, [X, Z] \rangle - Z \langle Y, X \rangle + \langle [Z, Y], X \rangle \\
&\quad + Y \langle Z, X \rangle - \langle Z, \nabla_Y X \rangle \\
&= X \langle Y, Z \rangle - \langle Y, [X, Z] \rangle - Z \langle Y, X \rangle + \langle [Z, Y], X \rangle \\
&\quad + Y \langle Z, X \rangle - \langle Z, \nabla_X Y + [Y, X] \rangle \\
&= X \langle Y, Z \rangle - \langle Y, [X, Z] \rangle - Z \langle Y, X \rangle + \langle [Z, Y], X \rangle \\
&\quad + Y \langle Z, X \rangle - \langle Z, [Y, X] \rangle - \langle Z, \nabla_X Y \rangle
\end{aligned}$$

Therefore, we have

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - Z\langle Y,X\rangle + \langle [Z,Y],X\rangle + Y\langle Z,X\rangle - \langle Z,[Y,X]\rangle\right)$$

which means ∇ is uniquely determined by the Riemannian metric g.

For existence, locally, we define

$$\nabla_X Y = \left(X^i \frac{\partial Y^k}{\partial x_i} + X^i Y^j \Gamma^k_{ij}\right)\frac{\partial}{\partial x_k}$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(g_{jl,i} + g_{li,j} - g_{ij,l}\right)$$

for

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j}$$

on a chart (x,U). To check that it is well-defined, we need to prove it is independent with the choice of coordinates. So let (y,U) be another coordinates. On (y,U),

$$X^i = \tilde{X}^\alpha \frac{\partial x^i}{\partial y^\alpha}, \quad Y^j = \tilde{Y}^\beta \frac{\partial x^j}{\partial y^\beta}$$

and

$$g_{ij} = \tilde{g}_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}, \quad g^{kl} = \tilde{g}^{\mu\nu}\frac{\partial x^k}{\partial y^\mu}\frac{\partial x^l}{\partial y^\nu}$$

So

$$\begin{aligned}
g_{ij,l} &= \frac{\partial}{\partial x^l}\left(\tilde{g}_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\right)\\
&= \left(\frac{\partial}{\partial x^l}\tilde{g}_{\alpha\beta}\right)\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \tilde{g}_{\alpha\beta}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^l}\frac{\partial y^\beta}{\partial x^j} + \tilde{g}_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial^2 y^\beta}{\partial x^j \partial x^l}\\
&= \tilde{g}_{\alpha\beta,\gamma}\frac{\partial y^\gamma}{\partial x^l}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \tilde{g}_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial^2 y^\beta}{\partial x^j \partial x^l} + \tilde{g}_{\alpha\beta}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\alpha}{\partial x^l \partial x^i}
\end{aligned}$$

and thus

$$\begin{aligned}
\Gamma^k_{ij} = \frac{1}{2}\tilde{g}^{\alpha\nu}\frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\nu}\Bigg(&\tilde{g}_{ab,c}\frac{\partial y^c}{\partial x^i}\frac{\partial y^a}{\partial x^j}\frac{\partial y^b}{\partial x^l} + \tilde{g}_{ab}\frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^b}{\partial x^l \partial x^i} + \tilde{g}_{ab}\frac{\partial y^b}{\partial x^l}\frac{\partial^2 y^a}{\partial x^i \partial x^j}\\
&+ \tilde{g}_{ab,c}\frac{\partial y^c}{\partial x^j}\frac{\partial y^a}{\partial x^l}\frac{\partial y^b}{\partial x^i} + \tilde{g}_{ab}\frac{\partial y^a}{\partial x^l}\frac{\partial^2 y^b}{\partial x^i \partial x^j} + \tilde{g}_{ab}\frac{\partial y^b}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^l}\\
&- \tilde{g}_{ab,c}\frac{\partial y^c}{\partial x^l}\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j} - \tilde{g}_{ab}\frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^b}{\partial x^j \partial x^l} - \tilde{g}_{ab}\frac{\partial y^b}{\partial x^j}\frac{\partial^2 y^a}{\partial x^l \partial x^i}\Bigg)\\
= \frac{1}{2}\tilde{g}^{\alpha\nu}\frac{\partial x^k}{\partial y^\alpha}\Bigg(&\tilde{g}_{a\nu,c}\frac{\partial y^c}{\partial x^i}\frac{\partial y^a}{\partial x^j} + \tilde{g}_{a\nu}\frac{\partial^2 y^a}{\partial x^i \partial x^j} + \tilde{g}_{\nu b,c}\frac{\partial y^c}{\partial x^j}\frac{\partial y^b}{\partial x^i} + \tilde{g}_{\nu b}\frac{\partial^2 y^b}{\partial x^i \partial x^j} - \tilde{g}_{ab,\nu}\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}\Bigg)\\
= \frac{1}{2}\tilde{g}^{\alpha\nu}&\left(\tilde{g}_{\gamma\nu,\beta} + \tilde{g}_{\nu\beta,\gamma} - \tilde{g}_{\beta\gamma,\nu}\right)\frac{\partial x^k}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j} + \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial y^\alpha}\\
= \tilde{\Gamma}^\alpha_{\beta\gamma}&\frac{\partial x^k}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j} + \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial y^\alpha}
\end{aligned}$$

which means that the Christoffel symbol is not a tensor.

It follows that

$$
\begin{aligned}
\nabla_X Y &= \left(X^i \frac{\partial Y^k}{\partial x_i} + X^i Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k} \\
&= \frac{\partial y^{\alpha'}}{\partial x^k} \frac{\partial}{\partial y^{\alpha'}} \left(\tilde{X}^\beta \frac{\partial x^i}{\partial y^\beta} \frac{\partial}{\partial x_i} \left(\tilde{Y}^\alpha \frac{\partial x^k}{\partial y^\alpha} \right) \right. \\
&\quad \left. + \tilde{X}^{\beta'} \frac{\partial x^i}{\partial y^{\beta'}} \tilde{Y}^{\gamma'} \frac{\partial x^j}{\partial y^{\gamma'}} \left(\tilde{\Gamma}_{\beta\gamma}^\alpha \frac{\partial x^k}{\partial y^\alpha} \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} + \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial y^\alpha} \right) \right) \\
&= \left(\tilde{X}^\beta \frac{\partial \tilde{Y}^\alpha}{\partial y^\beta} \frac{\partial x^k}{\partial y^\alpha} + \tilde{X}^\beta \tilde{Y}^\alpha \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta} + \tilde{X}^\beta \tilde{Y}^\gamma \tilde{\Gamma}_{\beta\gamma}^\alpha \frac{\partial x^k}{\partial y^\alpha} \right) \frac{\partial y^{\alpha'}}{\partial x^k} \frac{\partial}{\partial y^{\alpha'}} \\
&= \left(\tilde{X}^\beta \frac{\partial \tilde{Y}^\alpha}{\partial y^\beta} + \tilde{X}^\beta \tilde{Y}^\gamma \tilde{\Gamma}_{\beta\gamma}^\alpha \right) \frac{\partial}{\partial y^\alpha}
\end{aligned}
$$

□

Example 2.3.1. (1) Euclidean space: For $\mathbb{R}^n$, let g be the standard Euclidean metric. It follows that all Christoffel coefficients are 0. So the Levi-Civita connection $\overline{\nabla}$

$$\overline{\nabla}_X Y = X^i \frac{\partial Y^k}{\partial x^i} \frac{\partial}{\partial x^k}$$

(2) Sphere: For $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, let g be the induced Riemannian metric. Let $\overline{\nabla}$ be the Levi-Civita connection on $\mathbb{R}^{n+1}$. For any $X, Y \in \Gamma(T\mathbb{S}^n)$, we can extend X, Y to $\bar{X}, \bar{Y} \in \Gamma(T\mathbb{R}^{n+1})$. By the local properties of connection, $\overline{\nabla}_{\bar{X}} \bar{Y}$ is independent with the choice of the extension, so it can be written as $\overline{\nabla}_X Y$. Then define

$$\nabla_X Y := \overline{\nabla}_X Y - \left\langle \overline{\nabla}_X Y, \vec{n} \right\rangle \vec{n}$$

where $\vec{n} = (x^1, \cdots, x^{n+1})$ is the outer unit normal vector. Note that

$$\overline{\nabla}_X \vec{n} = X^i \frac{\partial x^j}{\partial x^i} \frac{\partial}{\partial x^j} = X$$

Because $0 = X \langle Y, \vec{n} \rangle = \left\langle \overline{\nabla}_X Y, \vec{n} \right\rangle + \left\langle Y, \overline{\nabla}_X \vec{n} \right\rangle$,

$$\left\langle \overline{\nabla}_X Y, \vec{n} \right\rangle = -\left\langle Y, \overline{\nabla}_X \vec{n} \right\rangle = -\langle X, Y \rangle$$

Therefore

$$\nabla_X Y = \overline{\nabla}_X Y + \langle X, Y \rangle \vec{n}$$

First, the linearity, functional linearity and Leibniz property are clearly by the corresponding properties of $\overline{\nabla}$ and the linearity of $\langle \cdot, \cdot \rangle$. For the torsion-free property,

$$\nabla_X Y - \nabla_Y X = \overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y]$$

and the compatibility, because g is the induced metric,

$$X \langle Y, Z \rangle = \left\langle \overline{\nabla}_X Y, Z \right\rangle + \left\langle Y, \overline{\nabla}_X Z \right\rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

Therefore, ∇ is the Levi-Civita connection on $\mathbb{S}^n$.

Remark. In general, if (M, g) is a Riemannian manifold with the Levi-Civita connection ∇^M and N is a submanifold of M by $\iota \colon N \hookrightarrow M$ with the induce metric $\iota^* g$, then

$$\nabla_X^N Y := \left(\nabla_X^M Y \right)^T$$

the orthogonal projection of $\nabla_X^M Y$ onto TN, is the Levi-Civita connection.

(3) Hyperbolic space: Let $\mathbb{H}^n$ be the upper half-space in the $\mathbb{R}^n$,

$$\mathbb{H}^n := \left\{ \left(x^1, \cdots, x^n\right) \in \mathbb{R}^n \mid x^n > 0 \right\},$$

equipped with the hyperbolic metric

$$g = \frac{1}{(x^n)^2} \left(dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n\right) \quad \Rightarrow \quad g_{ij} = \frac{1}{(x^n)^2} \delta_{ij}$$

So $g^{ij} = (x^n)^2 \delta^{ij}$ and

$$g_{ij,k} = \begin{cases} 0, & k \neq n \\ -\frac{2\delta_{ij}}{(x^n)^3}, & k = n \end{cases} \quad \Rightarrow \quad g_{ij,k} = -\frac{2\delta_{ij}\delta_{kn}}{(x^n)^3}$$

It follows that

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2} g^{kl} \left(g_{jl,i} + g_{li,j} - g_{ij,l}\right) \\ &= -\frac{1}{x^n} \delta^{kl} \left(\delta_{jl}\delta_{in} + \delta_{li}\delta_{jn} - \delta_{ij}\delta_{ln}\right) \\ &= -\frac{1}{x^n} \left(\delta_j^k \delta_{in} + \delta_i^k \delta_{jn} - \delta_n^k \delta_{ij}\right) \end{aligned}$$

So the nonzero cases are

$$\Gamma_{ii}^n = \frac{1}{x^n} \ (i \neq n), \quad \Gamma_{nk}^k = -\frac{1}{x^n}$$

And this provides the Levi-Civita connection on $\mathbb{H}^n$.

Lemma 2.3.1. *Let (M, g) be a Riemannian manifolds with an affine connection ∇ that is compatible with g and $c\colon [a, b] \to M$ be a C^∞ curve. Let $V(t)$ and $W(t)$ be two vector fields along c. Then we have*

$$\frac{d}{dt} \langle V(t), W(t) \rangle = \left\langle \frac{D}{dt} V(t), W(t) \right\rangle + \left\langle V(t), \frac{D}{dt} W(t) \right\rangle$$

Proof. WLTG assume $c \subset U$ of a char (x, U) and

$$V(t) = V^i(t) \frac{\partial}{\partial x^i}, \quad W(t) = W^j(t) \frac{\partial}{\partial x^j}$$

Then by the compatibility of Levi-Civita connection,

$$\begin{aligned} \text{LHS} &= \frac{d}{dt} \left(V^i(t) W^j(t) \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \right) \\ &= \frac{d}{dt} \left(V^i(t)\right) W^j(t) \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle + V^i(t) \frac{d}{dt} \left(W^j(t)\right) \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \\ &+ V^i(t) W^j(t) \left(\left\langle \nabla_{\dot{c}(t)} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle + \left\langle \frac{\partial}{\partial x^i}, \nabla_{\dot{c}(t)} \frac{\partial}{\partial x^j} \right\rangle \right) \\ &= \text{RHS} \end{aligned}$$

□

Remark. If $c(t)$ is a geodesic, then $\dot{c}(t)$ is clearly a vector field along c and

$$\frac{d}{dt} \langle \dot{c}(t), \dot{c}(t) \rangle = 2 \left\langle \frac{D}{dt} \dot{c}(t), \dot{c}(t) \right\rangle = 0$$

which also provides the proof of $|c(t)| \equiv const.$

Proposition 2.3.3. *Let (M, g) be a Riemannian manifolds with an affine connection ∇ that is compatible with g. For C^∞ map $\varphi \colon N \to M$ between two manifolds, N with the induced connection $\widetilde{\nabla}$, we have*

$$u \langle V, W \rangle = \left\langle \widetilde{\nabla}_u V, W \right\rangle + \left\langle V, \widetilde{\nabla}_u W \right\rangle$$

for all $u \in T_x N$ and V, W vector fields along φ.

Proof. By the linearity, we only need to check for $u = \frac{\partial}{\partial \tilde{x}_i}$, which is clearly true by above lemma. $\square$

Proposition 2.3.4. *Let (M, g) be a Riemannian manifolds with an affine connection ∇. ∇ is compatible with g if and only if any parallel map is isometric.*

Proof. Let $c \colon [a, b] \to M$ be a C^∞ curve and

$$\mathcal{P} := \mathcal{P}_{c,0,t} \colon T_{c(0)}M \longrightarrow T_{c(t)}M$$

be a parallel map.

- "$\Rightarrow$": For any $V_0, W_0 \in T_{c(0)}M$, let

  $$V_t = \mathcal{P}(V_0), \quad W_t = \mathcal{P}(W_0)$$

  Then by parallel moving, we have

  $$\frac{d}{dt} \langle V_t, W_t \rangle = \left\langle \frac{D}{dt} V_t, W_t \right\rangle + \left\langle V_t, \frac{D}{dt} W_t \right\rangle = 0$$

- "$\Leftarrow$": For any $X, Y, Z \in \Gamma(TM)$ and any $p \in M$, choosing a smooth curve such that $c(0) = p$ and $\dot{c}(0) = X(p)$. By the parallel moving, let $\{E_i(t)\}_{i=1}^m$ be an orthonormal frame along c, which can be done because of isometry of any parallel map. let $Y(t), Z(t)$ be the restriction of Y and Z on c, so

  $$Y(t) = Y^i(t) E_i(t), \quad Z(t) = Z^j(t) E_j(t)$$

  Then we have

  $$\begin{aligned} X \langle Y, Z \rangle (p) &= \left.\frac{d}{dt}\right|_{t=0} \langle Y, Z \rangle (c(t)) \\ &= \sum_{i=1}^m \left.\frac{d}{dt}\right|_{t=0} Y^i(t) Z^i(t) \\ &= \sum_{i=1}^m \left(\left.\frac{d}{dt}\right|_{t=0} \left(Y^i(t)\right) Z^i(t) + Y^i(t) \left.\frac{d}{dt}\right|_{t=0} \left(Z^i(t)\right) \right) \\ &= \langle \nabla_X Y, Z \rangle (p) + \langle Y, \nabla_X Z \rangle (p) \end{aligned}$$

  where the final equality is can be seen in the proof of **Proposition** 2.2.3. $\square$

Proposition 2.3.5. *Let (M, g) be Riemannian with an affine connection ∇ that is torsion-free and N be a smooth manifold. For $\varphi \colon N \to M$ and X, Y vector fields in N, we have*

$$\widetilde{\nabla}_X d\varphi(Y) - \widetilde{\nabla}_Y d\varphi(X) = d\varphi([X, Y])$$

Proof. Let $x \in N$ and consider a neighborhood U around $\varphi(x)$ with coordinate y. Let

$$d\varphi(X)(x) = d\varphi_x(X_x) = X^i(x) \left.\frac{\partial}{\partial y^i}\right|_{\varphi(x)}$$
$$d\varphi(Y)(x) = d\varphi_x(Y_x) = Y^i(x) \left.\frac{\partial}{\partial y^i}\right|_{\varphi(x)}$$

Then by definition,

$$\widetilde{\nabla}_X d\varphi(Y)(x) = X_x(Y^i) \left.\frac{\partial}{\partial y^i}\right|_{\varphi(x)} + Y^i(x)\nabla_{d\varphi_x(X_x)}\frac{\partial}{\partial y^i}(\varphi(x))$$
$$\widetilde{\nabla}_Y d\varphi(X)(x) = Y_x(X^i) \left.\frac{\partial}{\partial y^i}\right|_{\varphi(x)} + X^i(x)\nabla_{d\varphi_x(Y_x)}\frac{\partial}{\partial y^i}(\varphi(x))$$

Therefore, we get

$$\begin{aligned}\widetilde{\nabla}_X d\varphi(Y)(x) - \widetilde{\nabla}_Y d\varphi(X)(x) &= d\varphi_x([X,Y]_x) \\ &\quad + \left(Y^i\nabla_{d\varphi_x(X_x)}\frac{\partial}{\partial y^i} - X^i\nabla_{d\varphi_x(Y_x)}\frac{\partial}{\partial y^i}\right)(\varphi(x))\end{aligned}$$

and for the second term,

$$Y^i\nabla_{d\varphi(X)}\frac{\partial}{\partial y^i} - X^j\nabla_{d\varphi(Y)}\frac{\partial}{\partial y^j} = Y^iX^j\nabla_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} - X^jY^i\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0$$

because ∇ is torsion-free. □

Note that $d\varphi([X,Y])$ is not a vector field on M but a vector field along φ.

Corollary 2.3.2. *Let (M,g) be Riemannian with an affine connection ∇ that is torsion-free. Let $s\colon \mathbb{R}^2 \to M$ be smooth and $V(x,y) \in T_{s(x,y)}M$ be a vector field along s. For convenience,*

$$\frac{\partial s}{\partial x} := ds\left(\frac{\partial}{\partial x}\right), \quad \frac{\partial s}{\partial y} := ds\left(\frac{\partial}{\partial y}\right)$$

By equipping $\mathbb{R}^2$ with induced connection $\widetilde{\nabla}$, we have

$$\widetilde{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial s}{\partial y} = \widetilde{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial s}{\partial x}$$

Note that $\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}$ are not vector fields in $\Gamma(TM)$ but vector fields along s.

2.4 Variation and Curvature Tensor

1. ***First variation formula:*** Let $c\colon [a,b] \to M$ be a smooth curve. A variation of c is a smooth map

$$F\colon [a,b]\times(-\varepsilon,\varepsilon) \longrightarrow M$$

such that $F(t,0) = c(t)$.

Remark. For convenience, let

$$\frac{\partial F}{\partial t} = dF\left(\frac{\partial}{\partial t}\right), \quad \frac{\partial F}{\partial s} = dF\left(\frac{\partial}{\partial s}\right)$$

The variational field is

$$V(t) = \frac{\partial F}{\partial s}(t, 0) \in T_{c(t)}M$$

Let $c_s(t) = F(t, s)$ be the curve at s. Then

$$\begin{aligned} E(s) := E(c_s) &= \frac{1}{2}\int_a^b \langle \dot{c}_s(t), \dot{c}_s(t)\rangle \, dt \\ &= \frac{1}{2}\int_a^b \left\langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle dt \end{aligned}$$

and $E\colon (-\varepsilon, \varepsilon) \to M$. By taking derivative, we have

$$\begin{aligned} E'(s) &= \frac{1}{2}\int_a^b \frac{d}{ds}\left\langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle dt \\ &= \int_a^b \left\langle \frac{D}{ds}\frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle dt \\ &= \int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle dt \\ &= \int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle dt \\ &= \int_a^b \left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle dt \\ &= \int_a^b \frac{d}{dt}\left\langle \frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle - \left\langle \frac{\partial F}{\partial s}(s,t), \frac{D}{dt}\frac{\partial F}{\partial t}(s,t)\right\rangle dt \\ &= \left\langle \frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle\Bigg|_{t=a}^{t=b} - \int_a^b \left\langle \frac{\partial F}{\partial s}(s,t), \frac{D}{dt}\frac{\partial F}{\partial t}(s,t)\right\rangle dt \end{aligned}$$

The first variation formula of energy is

$$E'(s) = \left\langle \frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle\Bigg|_{t=a}^{t=b} - \int_a^b \left\langle \frac{\partial F}{\partial s}(s,t), \frac{D}{dt}\frac{\partial F}{\partial t}(s,t)\right\rangle dt \tag{2.1}$$

In particular, at $s = 0$,

$$E'(0) = \langle V(t), \dot{c}(t)\rangle|_a^b - \int_a^b \left\langle V(t), \frac{D}{dt}\dot{c}(t)\right\rangle dt$$

Remark. (1) If end points are fixed, *i.e.* $V(a) = V(b) = 0$, then c with the minimal length implies

$$0 = E'(0) = -\int_a^b \left\langle V(t), \frac{D}{dt}\dot{c}(t)\right\rangle dt$$

Because it is true for any variation,

$$\frac{D}{dt}\dot{c}(t) = 0$$

which means c satisfies the geodesic equation.

(2) Setting $a = 0, b = 1$ and $c(0) = p$. Consider a geodesic variation, *i.e.* $c_s(t)$ is a geodesic for any s, let

$$v\colon (-\varepsilon, \varepsilon) \longrightarrow T_pM$$

with $v(0) = \dot{c}(0) = v$ and $\dot{v}(0) = w$. Moreover, WTLG, assume $|v(s)| \equiv r$. Then let

$$F(t, s) = \exp_p(tv(s)) = c_s(t)$$

So $F(t, s)$ is a geodesic variation with $F(t, 0) = c(t)$ and $F(0, s) = p$ ($V(0) = 0$). Besides,

$$V(t) = dF\left(\frac{\partial}{\partial s}\right)\bigg|_{s=0} = (d\exp_p)_v(tw)$$

Because

$$E(s) = \frac{1}{2}\int_0^1 \langle \dot{c}_s(t), \dot{c}_s(t)\rangle \, dt = \frac{1}{2}r^2$$

we have

$$\begin{aligned} 0 = E'(0) &= \langle V(t), \dot{c}(t)\rangle|_0^1 - \int_0^1 \left\langle V(t), \frac{D}{dt}\dot{c}(t)\right\rangle dt \\ &= \langle V(1), \dot{c}(1)\rangle - \langle V(0), \dot{c}(0)\rangle \\ &= \langle V(1), \dot{c}(1)\rangle \\ &= \big\langle (d\exp_p)_v(w), (d\exp_p)_v(v)\big\rangle \end{aligned}$$

by $\dot{c}(1) = dF\left(\frac{\partial}{\partial t}\right)\big|_{s=0,t=1} = (d\exp_p)_v(v)$.

Lemma 2.4.1 (Gauss)**.** *Let (M, g) be a Riemannian manifold $p \in M$. There is a $\varphi > 0$ such that for any $v, w \in B_\varepsilon(0) \subset T_pM (\simeq T_vT_pM)$,*

$$\big\langle (d\exp_p)_v(w), (d\exp_p)_v(v)\big\rangle_{\exp_p(v)} = \langle w, v\rangle_p$$

Proof. First, for $v \in B_\varepsilon(0)$, let $c(t) = \exp_p(tv)$ be the geodesic starting from p with v. So we have

$$\begin{aligned} \big\langle (d\exp_p)_v(v), (d\exp_p)_v(v)\big\rangle &= \langle \dot{c}(1), \dot{c}(1)\rangle \\ &= \langle \dot{c}(0), \dot{c}(0)\rangle \\ &= \langle v, v\rangle \end{aligned}$$

by $|\dot{c}(t)| \equiv const$. Then for v, w, let $w = w_T + w_N$ with $w_T = av$ and $w_N \top v$. By above, we get

$$\begin{aligned} \big\langle (d\exp_p)_v(w), (d\exp_p)_v(v)\big\rangle &= \big\langle (d\exp_p)_v(w_T), (d\exp_p)_v(v)\big\rangle + \big\langle (d\exp_p)_v(w_N), (d\exp_p)_v(v)\big\rangle \\ &= a\big\langle (d\exp_p)_v(v), (d\exp_p)_v(v)\big\rangle \\ &= \langle w_T, v\rangle \\ &= \langle w, v\rangle \end{aligned}$$

□

Lemma 2.4.2. *Let N_1, N_2 be two submanifolds of a complete Riemannian manifold (M, g), and let $\gamma : [0, a] \to M$ be a geodesic such that $\gamma(0) \in N_1, \gamma(a) \in N_2$ and γ is the shortest curve from N_1 to N_2. Prove that $\dot{\gamma}(0)$ is perpendicular to $T_{\gamma(0)}N_1$, and $\dot{\gamma}(a)$ is perpendicular to $T_{\gamma(t)}N_2$.*

Proof. Let $V(t)$ be a vector field along γ such that $V(0) \in T_{\gamma(0)}N_1$ and $V(a) = 0 \in T_{\gamma(a)}N_2$. Consider the variation

$$F(t,s) = \exp_{\gamma(t)} sV(t)$$

which has $\frac{\partial}{\partial s}\big|_{s=0} F(t,s) = V(t)$. Because γ is the shortest curve,

$$0 = E'(0) = \langle V(0), \dot{\gamma}(0)\rangle$$

Therefore, $\dot{\gamma}(0) \perp N_1$. For N_2, it is similar. $\square$

2. ***Second variation formula:*** Let $c\colon [a,b] \to M$ be a geodesic. Consider a two-variable variation of c that is a smooth map

$$F\colon [a,b] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta) \longrightarrow M$$

such that $F(t,v,w)|_{v=0,w=0} = c(t)$, it has two canonical variation fields,

$$V(t) = \frac{\partial F}{\partial v}(t,0,0), \quad W(t) = \frac{\partial F}{\partial w}(t,0,0)$$

Let $c_{v,w}(t) = F(t,v,w)$ be a curve at v, w. Then for its energy $E(v,w) = E(c_{s,w})$,

$$\begin{aligned}
\frac{\partial^2 E}{\partial w \partial v} &= \frac{\partial}{\partial w}\left(\frac{\partial E}{\partial v}\right) \\
&= \frac{\partial}{\partial w}\int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle dt \\
&= \int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial w}}\left(\widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}\right), \frac{\partial F}{\partial t}\right\rangle + \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial t}\right\rangle dt = \int_a^b (\mathrm{I} + \mathrm{II})\, dt
\end{aligned}$$

First, for I,

$$\begin{aligned}
\mathrm{I} &= \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}}\left(\widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}\right), \frac{\partial F}{\partial t}\right\rangle + \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial w}}\left(\widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}\right), \frac{\partial F}{\partial t}\right\rangle - \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}}\left(\widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}\right), \frac{\partial F}{\partial t}\right\rangle \\
&= \frac{d}{dt}\left\langle \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle + \left\langle \left(\widetilde{\nabla}_{\frac{\partial}{\partial w}}\widetilde{\nabla}_{\frac{\partial}{\partial t}} - \widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{\frac{\partial}{\partial w}}\right)\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle - \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}, \frac{D}{dt}\frac{\partial F}{\partial t}\right\rangle
\end{aligned}$$

and by setting $v = w = 0$, the last term is 0 because $F(t,0,0)$ is a geodesic. For II,

$$\mathrm{II} = \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w}\right\rangle$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 E}{\partial w \partial v}(0,0) = {} & \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{c}(t)\right\rangle\Big|_a^b + \int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \widetilde{\nabla}_{\frac{\partial}{\partial t}} W(t)\right\rangle dt \\
& + \int_a^b \left\langle \left(\widetilde{\nabla}_{\frac{\partial}{\partial w}}\widetilde{\nabla}_{\frac{\partial}{\partial t}} - \widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{\frac{\partial}{\partial w}}\right) V(t), \dot{c}(t)\right\rangle dt
\end{aligned}$$

In particular, when considering one-variable variation,

$$\begin{aligned}
E''(0) = {} & \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial s}} V(t), \dot{c}(t)\right\rangle\Big|_a^b + \int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \widetilde{\nabla}_{\frac{\partial}{\partial t}} V(t)\right\rangle dt \\
& + \int_a^b \left\langle \left(\widetilde{\nabla}_{\frac{\partial}{\partial s}}\widetilde{\nabla}_{\frac{\partial}{\partial t}} - \widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{\frac{\partial}{\partial s}}\right) V(t), \dot{c}(t)\right\rangle dt
\end{aligned}$$

3. ***Curvature tensor:*** For any $X, Y, Z \in \Gamma(TM)$,

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X,Y)Z$$

Proposition 2.4.1. *R is a $(1,3)$-tensor, that is a function-linear map*

$$R\colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

Proof. For any $f \in C^\infty(M)$, by **Lemma** 2.2.3,

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\ &= f \nabla_X \nabla_Y Z - \nabla_Y (f \nabla_X Z) - \nabla_{f[X,Y]-(Yf)X} Z \\ &= f \nabla_X \nabla_Y Z - Y(f) \nabla_X Z - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z + Y(f) \nabla_X Z \\ &= f R(X,Y)Z \end{aligned}$$

Similarly proof for $R(X, fY)Z$ by the following remark.

$$\begin{aligned} R(X,Y)fZ &= \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ) \\ &= \nabla_X (Y(f)Z + f\nabla_Y Z) - \nabla_Y (X(f)Z + f \nabla_X Z) \\ &\quad - [X,Y](f)Z - f\nabla_{[X,Y]} Z \\ &= XY(f)Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f \nabla_X \nabla_Y Z \\ &\quad - YX(f)Z - X(f)\nabla_Y Z - Y(f) \nabla_X Z - f \nabla_Y \nabla_X Z \\ &\quad - [X,Y](f)Z - f \nabla_{[X,Y]} Z \\ &= fR(X,Y) \end{aligned}$$

□

Proposition 2.4.2. *Let $s\colon \mathbb{R}^2 \to M$ be smooth and V be a vector field along s. Then*

$$\widetilde{\nabla}_{\frac{\partial}{\partial x}} \widetilde{\nabla}_{\frac{\partial}{\partial y}} V - \widetilde{\nabla}_{\frac{\partial}{\partial y}} \widetilde{\nabla}_{\frac{\partial}{\partial x}} V = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right) V$$

Proof. It is sufficient to prove in a local chart. Let $V = V^i(x)\frac{\partial}{\partial x^i}$ on M. Then by the linearity of R,

$$\begin{aligned} R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right) V &= V^i R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right) \frac{\partial}{\partial x^i} \\ &= V^i \left(\nabla_{\frac{\partial s}{\partial x}} \nabla_{\frac{\partial s}{\partial y}} - \nabla_{\frac{\partial s}{\partial y}} \nabla_{\frac{\partial s}{\partial x}} - \nabla_{[\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}]}\right) \frac{\partial}{\partial x^i} \\ &= V^i \left(\nabla_{\frac{\partial s}{\partial x}} \nabla_{\frac{\partial s}{\partial y}} - \nabla_{\frac{\partial s}{\partial y}} \nabla_{\frac{\partial s}{\partial x}}\right) \frac{\partial}{\partial x^i} \end{aligned}$$

where the final equality is because

$$\left[\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right] = ds\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$$

by **Lemma** 2.2.2. For the other hand, by definition

$$\widetilde{\nabla}_{\frac{\partial}{\partial y}} V = \frac{\partial V^i}{\partial y} \frac{\partial}{\partial x^i} + V^i \nabla_{\frac{\partial s}{\partial y}} \frac{\partial}{\partial x^i}$$

and so

$$\begin{aligned}\widetilde{\nabla}_{\frac{\partial}{\partial x}}\widetilde{\nabla}_{\frac{\partial}{\partial y}}V &= \widetilde{\nabla}_{\frac{\partial}{\partial x}}\left(\frac{\partial V^i}{\partial y}\frac{\partial}{\partial x^i} + V^i\nabla_{\frac{\partial s}{\partial y}}\frac{\partial}{\partial x^i}\right) \\ &= \frac{\partial^2 V^i}{\partial x\partial y}\frac{\partial}{\partial x^i} + \frac{\partial V^i}{\partial y}\nabla_{\frac{\partial s}{\partial x}}\frac{\partial}{\partial x^i} + \frac{\partial V^i}{\partial x}\nabla_{\frac{\partial s}{\partial y}}\frac{\partial}{\partial x^i} + V^i\nabla_{\frac{\partial s}{\partial x}}\nabla_{\frac{\partial s}{\partial y}}\frac{\partial}{\partial x^i}\end{aligned}$$

Similarly,

$$\widetilde{\nabla}_{\frac{\partial}{\partial y}}\widetilde{\nabla}_{\frac{\partial}{\partial x}}V = \frac{\partial^2 V^i}{\partial y\partial x}\frac{\partial}{\partial x^i} + \frac{\partial V^i}{\partial x}\nabla_{\frac{\partial s}{\partial y}}\frac{\partial}{\partial x^i} + \frac{\partial V^i}{\partial y}\nabla_{\frac{\partial s}{\partial x}}\frac{\partial}{\partial x^i} + V^i\nabla_{\frac{\partial s}{\partial y}}\nabla_{\frac{\partial s}{\partial x}}\frac{\partial}{\partial x^i}$$

Therefore,

$$\begin{aligned}\widetilde{\nabla}_{\frac{\partial}{\partial x}}\widetilde{\nabla}_{\frac{\partial}{\partial y}}V - \widetilde{\nabla}_{\frac{\partial}{\partial y}}\widetilde{\nabla}_{\frac{\partial}{\partial x}}V &= V^i\left(\nabla_{\frac{\partial s}{\partial x}}\nabla_{\frac{\partial s}{\partial y}} - \nabla_{\frac{\partial s}{\partial y}}\nabla_{\frac{\partial s}{\partial x}}\right)\frac{\partial}{\partial x^i} \\ &= R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right)V\end{aligned}$$

□

Therefore, if the variation is with fixed end points, then

$$E''(0) = \int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}}V(t), \widetilde{\nabla}_{\frac{\partial}{\partial t}}V(t)\right\rangle dt + \int_a^b \langle R\left(W(t), \dot{c}(t)\right)V(t), \dot{c}(t)\rangle\, dt$$

Furthermore, in a local chart (U, x), let

$$\begin{aligned}R^k_{lij}\frac{\partial}{\partial x^k} &= R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} \\ &= \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l} \\ &= \nabla_{\frac{\partial}{\partial x^i}}\left(\Gamma^p_{jl}\frac{\partial}{\partial x^p}\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma^p_{il}\frac{\partial}{\partial x^p}\right) \\ &= \frac{\partial\Gamma^p_{jl}}{\partial x^i}\frac{\partial}{\partial x^p} + \Gamma^p_{jl}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^p} - \frac{\partial\Gamma^p_{il}}{\partial x^j}\frac{\partial}{\partial x^p} - \Gamma^p_{il}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^p} \\ &= \left(\frac{\partial\Gamma^k_{jl}}{\partial x^i} - \frac{\partial\Gamma^k_{il}}{\partial x^j} + \Gamma^p_{jl}\Gamma^k_{ip} - \Gamma^p_{il}\Gamma^k_{jp}\right)\frac{\partial}{\partial x^k}\end{aligned}$$

Therefore,

$$R^k_{lij} = \frac{\partial\Gamma^k_{jl}}{\partial x^i} - \frac{\partial\Gamma^k_{il}}{\partial x^j} + \Gamma^p_{jl}\Gamma^k_{ip} - \Gamma^p_{il}\Gamma^k_{jp}$$

Note that by $R(X, Y)Z = -R(Y, X)Z$,

$$R^k_{lij} = -R^k_{lji}$$

Example 2.4.1. (1) Euclidean space: Clearly, because all Christoffel coefficients are 0, $R^k_{lij} = 0$ on $\mathbb{R}^n$ and it also means

$$\overline{\nabla}_X\overline{\nabla}_Y Z + \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z = 0$$

Besides, note that $\overline{\nabla}_{[X,Y]}Z = 0$.

(2) Sphere: We have seen the Levi-Civita connection on $\mathbb{S}^n$ is defined by

$$\nabla_X Y = \overline{\nabla}_X Y + \langle X, Y\rangle \vec{n}$$

It follows that

$$\begin{aligned}
\nabla_X \nabla_Y Z &= \overline{\nabla}_X \nabla_Y Z + \langle X, \nabla_Y Z\rangle \,\vec{n} \\
&= \overline{\nabla}_X \left(\overline{\nabla}_Y Z + \langle Y, Z\rangle \vec{n}\right) + Y\langle X, Z\rangle \vec{n} - \langle \nabla_Y X, Z\rangle \,\vec{n} \\
&= \overline{\nabla}_X \overline{\nabla}_Y Z + (X(\langle Y, Z\rangle) + \langle Y, Z\rangle X + Y\langle X, Z\rangle)\,\vec{n} - \langle \nabla_Y X, Z\rangle \,\vec{n}
\end{aligned}$$

Therefore,

$$\begin{aligned}
R(X, Y)Z &= X(\langle Y, Z\rangle)\vec{n} + \langle Y, Z\rangle X + Y(\langle X, Z\rangle)\vec{n} - \langle \nabla_Y X, Z\rangle \,\vec{n} \\
&\quad - Y(\langle X, Z\rangle)\vec{n} - \langle X, Z\rangle Y - X(\langle Y, Z\rangle)\vec{n} + \langle \nabla_X Y, Z\rangle \,\vec{n} - \langle \nabla_Y X, Z\rangle \,\vec{n} \\
&= \langle Y, Z\rangle X - \langle X, Z\rangle Y
\end{aligned}$$

(3) Hyperbolic space: We have already have

$$\Gamma_{ij}^k = -\frac{1}{x^n}\left(\delta_j^k \delta_{in} + \delta_i^k \delta_{jn} - \delta_n^k \delta_{ij}\right)$$

for the Christoffel coefficients. So by above we have

$$\begin{aligned}
R_{lij}^k &= \frac{\partial \Gamma_{jl}^k}{\partial x^i} - \frac{\partial \Gamma_{il}^k}{\partial x^j} + \Gamma_{jl}^p \Gamma_{ip}^k - \Gamma_{il}^p \Gamma_{jp}^k \\
&= \frac{1}{(x^n)^2}\left(\delta_j^k \delta_{il} - \delta_i^k \delta_{lj}\right)
\end{aligned}$$

Remark. There is another source to consider the curvature tensor by answering the problem that is if a metric g can be transformed to Euclidean metric. Let (M, g) be a Riemannian manifold. For a chart (U, y),

$$g = g_{ij} dy^i \otimes dy^j$$

with Christoffel Γ_{ij}^l and a chart (V, x),

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta$$

with Christoffel $\widetilde{\Gamma}_{\alpha\beta}^\lambda$. Then by the transformation rule, we have

$$\Gamma_{jk}^p = \widetilde{\Gamma}_{\alpha\beta}^\eta \frac{\partial y^p}{\partial x^\eta} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} + \frac{\partial^2 x^\mu}{\partial y^j \partial y^k} \frac{\partial y^p}{\partial x^\mu}$$

So

$$\frac{\partial x^\lambda}{\partial y^p} \Gamma_{jk}^p = \widetilde{\Gamma}_{\alpha\beta}^\lambda \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} + \frac{\partial^2 x^\lambda}{\partial y^j \partial y^k}$$

It follows that

$$\frac{\partial}{\partial y^k}\left(\frac{\partial x^\lambda}{\partial y^j}\right) = \frac{\partial x^\lambda}{\partial y^p} \Gamma_{jk}^p - \widetilde{\Gamma}_{\alpha\beta}^\lambda \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k}$$

So we have a ODE system for $\frac{\partial x^\lambda}{\partial y^j}$, where $\lambda, j, k = 1, 2, \cdots, m$. In particular, consider the Euclidean metric $g_{\alpha\beta} = \delta_{\alpha\beta}$, which implies $\widetilde{\Gamma}_{\alpha\beta}^\lambda = 0$, the system becomes

$$\frac{\partial}{\partial y^k}\left(\frac{\partial x^\lambda}{\partial y^j}\right) = \frac{\partial x^\lambda}{\partial y^p} \Gamma_{jk}^p$$

If such system has a solution $\frac{\partial x^\lambda}{\partial y^j}$, then we have

$$\frac{\partial}{\partial y^l}\frac{\partial}{\partial y^k}\left(\frac{\partial x^\lambda}{\partial y^j}\right) = \frac{\partial}{\partial y^k}\frac{\partial}{\partial y^l}\left(\frac{\partial x^\lambda}{\partial y^j}\right) \tag{2.2}$$

First, for the left-hand side

$$\begin{aligned}
\text{LHS} &= \frac{\partial}{\partial y^l}\left(\frac{\partial x^\lambda}{\partial y^p}\Gamma^p_{jk}\right) \\
&= \frac{\partial \Gamma^p_{jk}}{\partial y^l}\frac{\partial x^\lambda}{\partial y^p} + \Gamma^p_{jk}\frac{\partial}{\partial y^l}\left(\frac{\partial x^\lambda}{\partial y^p}\right) \\
&= \frac{\partial \Gamma^p_{jk}}{\partial y^l}\frac{\partial x^\lambda}{\partial y^p} + \Gamma^p_{jk}\Gamma^s_{pl}\frac{\partial x^\lambda}{\partial y^s} \\
&= \left(\frac{\partial \Gamma^p_{jk}}{\partial y^l} + \Gamma^i_{jk}\Gamma^p_{il}\right)\frac{\partial x^\lambda}{\partial y^p}
\end{aligned}$$

Similarly,

$$\text{RHS} = \left(\frac{\partial \Gamma^p_{jl}}{\partial y^k} + \Gamma^i_{jl}\Gamma^p_{ik}\right)\frac{\partial x^\lambda}{\partial y^p}$$

Therefore, equation (2.2) is satisfied if and only if

$$\frac{\partial \Gamma^p_{kj}}{\partial y^l} - \frac{\partial \Gamma^p_{lj}}{\partial y^k} + \Gamma^i_{kj}\Gamma^p_{li} - \Gamma^i_{lj}\Gamma^p_{ki} = R^p_{jlk} = 0$$

So we have the result that if (M,g) is locally isometric to $\mathbb{R}^m$, then the curvature tensor $R = 0$. In fact the converse is also true.

Theorem 2.4.1. *Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇ such that the curvature tensor $R = 0$. Then M is locally isometric to $\mathbb{R}^m$.*

Proof. Let (U,y) be a local chart and $g = g_{ij}dy^i \otimes dy^j$. So the goal is to find a local chart (V,x) with $V \subset U$ such that $g = \sum_i dx^i \otimes dx^i$.

- Step 1: For any $X_0 \in T_O U$, there is an $X \in \Gamma(TU)$ such that $\nabla X = 0$ and $X(0) = X_0$.
  First, considering the curve $t \mapsto (t, 0, \cdots, 0)$, we move X_0 in parallel to get a vector field $\tilde{X}$ along this curve. Second, for any fixed y^1, considering the curve $t \mapsto (y^1, t, 0, \cdots, 0)$, we move $\tilde{X}(y_1)$ in parallel. Then by construction, we get a vector field X that is along
  $$s\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^m$$
  defined by $s(x,y) = (x, y, 0, \cdots, 0)$. Then we directly have
  $$\widetilde{\nabla}_{\frac{\partial}{\partial y^2}} X = 0 \quad \text{along } s$$
  But for the other direction, we only know
  $$\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X(y^1, 0, \cdots, 0) = 0$$
  Because
  $$\widetilde{\nabla}_{\frac{\partial}{\partial y^1}}\widetilde{\nabla}_{\frac{\partial}{\partial y^2}} X - \widetilde{\nabla}_{\frac{\partial}{\partial y^2}}\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X = R\left(\frac{\partial s}{\partial y^1}, \frac{\partial s}{\partial y^2}\right) X = 0$$

we know

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^2}} \widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X = 0$$

Then by the uniqueness of ODE, we get

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X = 0 \quad \text{along } s$$

Therefore, $\widetilde{\nabla} X = 0$ along s. By induction we have such X.

- Step 2: Let $X_1^{(0)}, \cdots, X_m^{(0)} \in T_O U$ be orthonormal. Let move them in parallel to get global vector fields $X_1, \cdots, X_m$ as in Step 1. So they are also orthonormal. Moreover,

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$$

Then by Frobenius Theorem, there is a chart (V, x) with $V \subset U$ such that

$$X_i = \frac{\partial}{\partial x_i}$$

So $g = \delta_{ij} dx^i \otimes dx^j$. □

2.5 Covariant Differential

In this section, let (M, g) be a Riemannian manifold with the Levi-Civita connection ∇. Let A be a (r, s)-tensor. Its covariant differential ∇A is a $(r, s+1)$-tensor defined as

$$(\nabla A)(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s, X) := (\nabla_X A)(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s)$$

Moreover, $\nabla^2 A = \nabla(\nabla A)$ is a $(r, s+2)$-tensor and

$$\begin{aligned}
(\nabla^2 A)(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s, X, Y) &= (\nabla(\nabla A))(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s, X, Y) \\
&= (\nabla_Y(\nabla A))(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s, X) \\
&= Y((\nabla A)(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s, X)) \\
&\quad - \sum_{i=1}^r (\nabla A)(\cdots, \nabla_Y \omega_i, \cdots, X) \\
&\quad - \sum_{i=1}^s (\nabla A)(\cdots, \nabla_Y X_i, \cdots, X) - (\nabla A)(\cdots, \nabla_Y X)
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\nabla^2 A)(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s, X, Y) &= Y((\nabla_X A)(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s)) \\
&\quad - \sum_{i=1}^r (\nabla_X A)(\cdots, \nabla_Y \omega_i, \cdots) \\
&\quad - \sum_{i=1}^s (\nabla_X A)(\cdots, \nabla_Y X_i, \cdots) - (\nabla_{\nabla_Y X} A)(\cdots)
\end{aligned}$$

It follows that

$$\nabla^2 A(\cdots, X, Y) = \nabla_Y \nabla_X A - \nabla_{\nabla_Y X} A$$

and so

$$\nabla^2 A(\cdots, X, Y) - \nabla^2 A(\cdots, Y, X) = -(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})A = -R(X, Y)A$$

Example 2.5.1 (Hessian)**.** For $A = f \in C^\infty(M)$ $(0,0)$-tensor,

$$\begin{aligned}(\nabla^2 f)(X,Y) &= (\nabla_Y(\nabla f))(X) \\ &= Y((\nabla f)(X)) - (\nabla f)(\nabla_Y X) \\ &= Y(X(f)) - (\nabla_Y X)(f)\end{aligned}$$

So we get

$$(\nabla^2 f)(X,Y) - (\nabla^2 f)(Y,X) = -[X,Y]f + [X,Y]f = 0$$

i.e. $\nabla^2 f$ is a symmetric $(0,2)$-tensor, called the Hessian of f. Locally, for

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j}$$

we have

$$\begin{aligned}(\nabla^2 f)(X,Y) &= Y^j \frac{\partial}{\partial x^j}\left(X^i \frac{\partial f}{\partial x^i}\right) - Y^j \nabla_{\frac{\partial}{\partial x^j}}\left(X^i \frac{\partial}{\partial x^i}\right)(f) \\ &= Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} + X^i Y^j \frac{\partial^2 f}{\partial x^j \partial x^i} - Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} - X^i Y^j \left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}\right)(f) \\ &= X^i Y^j \frac{\partial^2 f}{\partial x^j \partial x^i} - X^i Y^j \Gamma^k_{ij} \frac{\partial f}{\partial x^k}\end{aligned}$$

Therefore, in matrix form,

$$(\nabla^2 f)_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}$$

from which we can also see that $\nabla^2 f$ is symmetric.

Considering the covariant differential locally, let (U,x) be a chart and $A \in \Gamma(\bigotimes^{r,s} TM)$ with

$$A = A^{i_1\cdots i_r}_{j_1\cdots j_s}\frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

Then $\nabla A \in \Gamma(\bigotimes^{r,s+1} TM)$. Assume

$$\nabla A = B^{i_1\cdots i_r}_{j_1\cdots j_s k}\frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes dx^k$$

where

$$\begin{aligned}B^{i_1\cdots i_r}_{j_1\cdots j_s k} &= (\nabla A)\left(dx^{i_1}, \cdots, dx^{i_r}, \frac{\partial}{\partial x_{j_1}}, \cdots, \frac{\partial}{\partial x_{j_s}}, \frac{\partial}{\partial x_k}\right) \\ &= (\nabla_{\frac{\partial}{\partial x_k}} A)\left(dx^{i_1}, \cdots, dx^{i_r}, \frac{\partial}{\partial x_{j_1}}, \cdots, \frac{\partial}{\partial x_{j_s}}\right) \\ &= \frac{\partial}{\partial x_k}\left(A(dx^{i_1}, \cdots, dx^{i_r}, \frac{\partial}{\partial x_{j_1}}, \cdots, \frac{\partial}{\partial x_{j_s}})\right) \\ &\quad - \sum_{h=1}^r A\left(\cdots, \nabla_{\frac{\partial}{\partial x_k}} dx^{i_h}, \cdots\right) - \sum_{h=1}^s A\left(\cdots, \nabla_{\frac{\partial}{\partial x_k}}\frac{\partial}{\partial x_{j_h}}, \cdots\right) \\ &= \frac{\partial}{\partial x_k} A^{i_1\cdots i_r}_{j_1\cdots j_s} + \sum_{h=1}^r A^{i_1\cdots i_{h-1} l i_{h+1}\cdots i_r}_{j_1\cdots j_s}\Gamma^{i_h}_{kl} - \sum_{h=1}^s A^{i_1\cdots i_r}_{j_1\cdots j_{h-1} l j_{h+1}\cdots j_s}\Gamma^l_{k j_h}\end{aligned}$$

If we denote

$$A^{i_1\cdots i_r}_{j_1\cdots j_s, k} = \frac{\partial}{\partial x_k} A^{i_1\cdots i_r}_{j_1\cdots j_s}, \quad A^{i_1\cdots i_r}_{j_1\cdots j_s; k} = B^{i_1\cdots i_r}_{j_1\cdots j_s k}$$

the it follows that

$$A^{i_1\cdots i_r}_{j_1\cdots j_s; k} = A^{i_1\cdots i_r}_{j_1\cdots j_s, k} + \sum_{h=1}^r A^{i_1\cdots i_{h-1} l i_{h+1}\cdots i_r}_{j_1\cdots j_s}\Gamma^{i_h}_{kl} - \sum_{h=1}^s A^{i_1\cdots i_r}_{j_1\cdots j_{h-1} l j_{h+1}\cdots j_s}\Gamma^l_{k j_h}$$

Example 2.5.2. (1) Consider vector field, *i.e.* $(1,0)$-tensor $X = X^i \frac{\partial}{\partial x^i}$, then

$$\begin{aligned}\nabla X &= X^i_{;k} \frac{\partial}{\partial x^i} \otimes dx^k \\ &= \left(\frac{\partial X^i}{\partial x^k} + X^h \Gamma^i_{kh} \right) \frac{\partial}{\partial x^i} \otimes dx^k\end{aligned}$$

Therefore, for $Y = Y^l \frac{\partial}{\partial x^l}$,

$$\nabla_Y X = \nabla X(-, Y) = \left(Y^k \frac{\partial X^i}{\partial x^k} + Y^k X^h \Gamma^i_{kh} \right) \frac{\partial}{\partial x^i}$$

(2) Consider 1-form, *i.e.* $(0,1)$-tensor $\omega = \omega_i dx^i$, then

$$\begin{aligned}\nabla \omega &= \omega_{i;k} dx^i \otimes dx^k \\ &= \left(\frac{\partial \omega_i}{\partial x^k} - \omega_h \Gamma^h_{ki} \right) dx^i \otimes dx^k\end{aligned}$$

Therefore, for $Y = Y^l \frac{\partial}{\partial x^l}$,

$$\nabla_Y \omega = \nabla \omega(-, Y) = \left(Y^k \frac{\partial \omega_i}{\partial x^k} - Y^k \omega_h \Gamma^h_{ki} \right) dx^i$$

(3) For $f \in C^\infty(M)$, $\nabla^2 f$ is a $(0,2)$-tensor with

$$\nabla^2 f = f_{;i;j} dx^i \otimes dx^j$$

Then

$$f_{;i} = \frac{\partial f}{\partial x_i}$$

and so

$$\begin{aligned}f_{;i;j} &= \frac{\partial f_{;i}}{\partial x_j} - f_{;k} \Gamma^k_{ji} \\ &= \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial f}{\partial x_k} \Gamma^k_{ij}\end{aligned}$$

which is as same as above.

Theorem 2.5.1. *Let $A^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ be a (r,s)-tensor.*

$$A^{i_1 \cdots i_r}_{j_1 \cdots j_s ;k;l} - A^{i_1 \cdots i_r}_{j_1 \cdots j_s ;l;k} = \sum_{\alpha=1}^{s} A^{i_1 \cdots i_r}_{j_1 \cdots j_{\alpha-1} h j_{\alpha+1} \cdots j_s} R^h_{j_\alpha k l} - \sum_{\beta=1}^{r} A^{i_1 \cdots i_{\beta-1} h i_{\beta+1} \cdots i_r}_{j_1 \cdots j_s} R^{i_\beta}_{hkl}$$

where $R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \frac{\partial}{\partial x_l} = R^k_{lij} \frac{\partial}{\partial x_k}$.

Proof. For any $p \in M$, let (U, x) be a normal chart and then $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$ and $g_{ij,k}(p) = 0$. Then

$$\begin{aligned}A^{i_1 \cdots i_r}_{j_1 \cdots j_s ;k;l}(p) &= \left.\frac{\partial}{\partial x_l}\right|_p A^{i_1 \cdots i_r}_{j_1 \cdots j_s ;k} \\ &= \left.\frac{\partial}{\partial x_l}\right|_p \left(\frac{\partial}{\partial x_k} A^{i_1 \cdots i_r}_{j_1 \cdots j_s} + \sum_{\beta=1}^{r} A^{\cdots i_{\beta-1} h \cdots}_{j_1 \cdots j_s} \beta^{i_\beta}_{kh} - \sum_{\alpha=1}^{s} A^{i_1 \cdots i_r}_{\cdots j_{\alpha-1} h \cdots} \beta^h_{k j_\alpha} \right) \\ &= \frac{\partial^2 A^{i_1 \cdots i_r}_{j_1 \cdots j_s}}{\partial x_l \partial x_k}(p) + \sum_{\beta=1}^{r} A^{\cdots i_{\beta-1} h \cdots}_{j_1 \cdots j_s}(p) \frac{\partial \beta^{i_\beta}_{kh}}{\partial x_l}(p) - \sum_{\alpha=1}^{s} A^{i_1 \cdots i_r}_{\cdots j_{\alpha-1} h \cdots}(p) \frac{\partial \beta^h_{k j_\alpha}}{\partial x_l}(p)\end{aligned}$$

It follows that at p,

$$\begin{aligned}A^{i_1\cdots i_r}_{j_1\cdots j_s;k;l}-A^{i_1\cdots i_r}_{j_1\cdots j_s;l;k}&=\sum_{\beta=1}^{r}A^{\cdots i_{\beta-1}h\cdots}_{j_1\cdots j_s}\left(\frac{\partial\beta^{i_\beta}_{kh}}{\partial x_l}-\frac{\partial\beta^{i_\beta}_{lh}}{\partial x_k}\right)-\sum_{\alpha=1}^{s}A^{i_1\cdots i_r}_{\cdots j_{\alpha-1}h\cdots}\left(\frac{\partial\beta^{h}_{kj_\alpha}}{\partial x_l}-\frac{\partial\beta^{h}_{lj_\alpha}}{\partial x_k}\right)\\&=\sum_{\beta=1}^{r}A^{\cdots i_{\beta-1}h\cdots}_{j_1\cdots j_s}R^{i_\beta}_{hlk}-\sum_{\alpha=1}^{s}A^{i_1\cdots i_r}_{\cdots j_{\alpha-1}h\cdots}R^{h}_{j_\alpha lk}\\&=\sum_{\alpha=1}^{s}A^{i_1\cdots i_r}_{j_1\cdots j_{\alpha-1}hj_{\alpha+1}\cdots j_s}R^{h}_{j_\alpha kl}-\sum_{\beta=1}^{r}A^{i_1\cdots i_{\beta-1}hi_{\beta+1}\cdots i_r}_{j_1\cdots j_s}R^{i_\beta}_{hkl}\end{aligned}$$

□

Next, let's consider some classical differential.

I. Divergence: For a vector field X, *i.e.* $(1,0)$-tensor, ∇X is a $(1,1)$-tensor. Locally, if $X=X^i\frac{\partial}{\partial x^i}$, then

$$\nabla X=X^i_{;k}\frac{\partial}{\partial x^i}\otimes dx^k=\left(\frac{\partial X^i}{\partial x^k}+X^h\Gamma^i_{kh}\right)\frac{\partial}{\partial x^i}\otimes dx^k$$

Then the divergence of X is

$$\operatorname{div}(X):=\operatorname{tr}(\nabla X)\left(=\sum_i\nabla X\left(dx^i,\frac{\partial}{\partial x^i}\right)=\sum_i X^i_{;i}\right)$$

Proposition 2.5.1. *Locally, we have*

$$\operatorname{div}(X)=\sum_i X^i_{;i}=\sum_i\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}X^i\right)$$

Proof. First, by above the left-hand side is

$$\text{LHS}=\sum_i\frac{\partial X^i}{\partial x^i}+X^h\Gamma^i_{ih}$$

For the right-hand side,

$$\text{RHS}=\sum_i\frac{\partial X^i}{\partial x^i}+X^i\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i}$$

Therefore, LHS=RHS if and only if we have

$$\sum_i\Gamma^i_{ih}=\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^h}=\frac{\partial}{\partial x^h}(\log\sqrt{g})=\frac{1}{2}\frac{\partial}{\partial x^h}(\log|g|)$$

where $|g|=\det(g)$.

Claim: $\sum_i\Gamma^i_{ih}=\frac{1}{2}\frac{\partial}{\partial x^h}(\log|g|)$

For the left-hand side,

$$\begin{aligned}\text{LHS}&=\frac{1}{2}\sum_i\sum_k g^{ik}\left(g_{hk,i}+g_{ki,h}-g_{ih,k}\right)\\&=\frac{1}{2}\sum_i\sum_k g^{ik}g_{ki,h}\\&=\frac{1}{2}\operatorname{tr}\left((g^{ik})(g_{ki,h})\right)\\&=\frac{1}{2}\frac{\partial}{\partial x^h}(\log|g|)\end{aligned}$$

where the final identity is by the Jacobi's formula, $\frac{d}{dt}\log\det(A(t)) = \operatorname{tr}\left(A(t)^{-1}\frac{d}{dt}A(t)\right)$. $\square$

II. Gradient: For $f \in C^\infty(M)$, f is also a $(0,0)$-tensor. So ∇f is a $(0,1)$-tensor

$$\nabla f = f_{;i}dx^i$$

However, $f_{;i} = \frac{\partial f}{\partial x^i}$. So

$$\nabla f = df$$

But in some case, we also use ∇f to denote the gradient of f

$$\operatorname{grad} f = \nabla f = \sharp(df) = g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}$$

III. Laplacian: For $f \in C^\infty(M)$, let

$$\Delta f := \operatorname{div}(\operatorname{grad} f)$$

called the Laplacian of f.

Proposition 2.5.2. *For $f \in C^\infty(M)$, $\Delta f = \operatorname{tr}(\nabla^2 f)$.*

Proof. It is sufficient to prove that in a local chart (U, x). First,

$$\nabla f = \frac{\partial f}{\partial x^j}dx^j \;\Rightarrow\; \operatorname{grad}(f) = g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}$$

Therefore, by definition

$$\begin{aligned}
\Delta f &= \sum_i \left(g^{ij}\frac{\partial f}{\partial x^j}\right)_{;i} \\
&= \sum_i \left(\frac{\partial}{\partial x^i}\left(g^{ij}\frac{\partial f}{\partial x^j}\right) + g^{hj}\frac{\partial f}{\partial x^j}\Gamma^i_{hi}\right) \\
&= g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{\partial}{\partial x^i}\left(g^{ij}\right)\frac{\partial f}{\partial x^j} + g^{hj}\frac{\partial f}{\partial x^j}\Gamma^i_{hi}
\end{aligned}$$

Then

$$\begin{aligned}
\Delta f = \operatorname{tr}(\nabla^2 f) &= g^{ij} f_{;ij} \\
&= g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial x^h}\Gamma^h_{ij}\right)
\end{aligned}$$

if and only if

$$0 = \frac{\partial}{\partial x^i}\left(g^{ij}\right) + g^{hj}\Gamma^i_{hi} + g^{ih}\Gamma^j_{ih} = g^{ij}_{;i}$$

Check: $g^{ij}_{;h} = 0$.

$$\nabla g = g_{ij;h}dx^i \otimes dx^j \otimes dx^h = 0 \;\Leftrightarrow\; g_{ij;h} = 0$$

and

$$\begin{aligned}
\left(g_{ij}g^{jk}\right)_{;p} &= (\delta^k_i)_{;p} \\
&= \frac{\partial \delta^k_i}{\partial x^p} + \delta^h_i\Gamma^k_{ph} - \delta^k_h\Gamma^h_{pi} \\
&= 0
\end{aligned}$$

Moreover, by the contraction property of ∇,

$$\begin{aligned}\left(g_{ij}g^{jk}\right)_{;p} &= \nabla_p c\left(g_{ij}\otimes g^{lk}\right)\\ &= c\left(\nabla_p(g_{ij})\otimes g^{lk}\right)+c\left(g_{ij}\otimes\nabla_p(g^{lk})\right)\\ &= g_{ij;p}g^{jk}+g_{ij}g^{jk}_{;p}\end{aligned}$$

It follows that

$$g^{jk}_{;p}=0$$

□

Remark. For any $X, Y \in \Gamma(TM)$,

$$\begin{aligned}g(\sharp\nabla^2 f(X,-),Y) &= \nabla^2 f(X,Y)\\ &= \nabla^2 f(Y,X)=\nabla_X(\nabla f)(Y)\\ &= X(\nabla f(Y))-\nabla f(\nabla_X Y)\\ &= X(g(\mathrm{grad} f,Y))-g(\mathrm{grad} f,\nabla_X Y)\\ &= g(\nabla_X \mathrm{grad} f,Y)\end{aligned}$$

Therefore,

$$\sharp\nabla^2 f(X,-)=\nabla_X \mathrm{grad} f$$

Note that

$$\Delta f=\mathrm{tr}(\nabla^2 f)=\mathrm{tr}(X\mapsto\sharp\nabla^2 f(X,-))=\mathrm{tr}(\nabla_X \mathrm{grad} f)$$

Chapter 3

Curvature and Jacobian Field

3.1 Riemannian Curvatures

In this section, let (M, g) be a Riemannian manifold.

1. ***Bianchi identities:*** Let R be the curvature tensor, *i.e.* a $(1,3)$-tensor.

   Proposition 3.1.1. *For any $X, Y, Z, W \in \Gamma(TM)$ and torsion-free connection ∇,*

   (1) $R(X,Y)Z = -R(Y,X)Z$.

   (2) (first Bianchi identity)

   $$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

   (3) (second Bianchi identity)

   $$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0$$

   Proof. Denote

   $$\mathfrak{S}T(X,Y,Z) = T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y)$$

   So we need to prove

   $$\mathfrak{S}R(X,Y)Z = 0, \quad \mathfrak{S}\nabla_X R(Y,Z)W = 0$$

   (1) has been obtained by definition.

   (2) By the definition of R,

   $$\begin{aligned}\mathfrak{S}R(X,Y)Z &= \mathfrak{S}\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\right)\\ &= \mathfrak{S}\left(\nabla_X\nabla_Y Z\right) - \mathfrak{S}\left(\nabla_Y\nabla_X Z\right) - \mathfrak{S}\left(\nabla_{[X,Y]}Z\right)\\ &= \mathfrak{S}\left(\nabla_Z\nabla_X Y\right) - \mathfrak{S}\left(\nabla_Z\nabla_Y X\right) - \mathfrak{S}\left(\nabla_{[X,Y]}Z\right)\\ &= \mathfrak{S}\left(\nabla_Z[X,Y]\right) - \mathfrak{S}\left(\nabla_{[X,Y]}Z\right)\\ &= \mathfrak{S}\left([Z,[X,Y]]\right) = 0\end{aligned}$$

   because of Jacobi's identity.

(3) First, formally

$$\begin{aligned}[\nabla_X,[\nabla_Y,\nabla_Z]] &= [\nabla_X,\nabla_Y\nabla_Z-\nabla_Z\nabla_Y] \\ &= \nabla_X(\nabla_Y\nabla_Z-\nabla_Z\nabla_Y)-(\nabla_Y\nabla_Z-\nabla_Z\nabla_Y)\nabla_X \\ &= \nabla_X\nabla_Y\nabla_Z-\nabla_X\nabla_Z\nabla_Y-\nabla_Y\nabla_Z\nabla_X+\nabla_Z\nabla_Y\nabla_X\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathfrak{S}[\nabla_X,[\nabla_Y,\nabla_Z]] &= \mathfrak{S}(\nabla_X\nabla_Y\nabla_Z)-\mathfrak{S}(\nabla_X\nabla_Z\nabla_Y)-\mathfrak{S}(\nabla_Y\nabla_Z\nabla_X)+\mathfrak{S}(\nabla_Z\nabla_Y\nabla_X) \\ &= \mathfrak{S}(\nabla_X\nabla_Y\nabla_Z)-\mathfrak{S}(\nabla_X\nabla_Z\nabla_Y)-\mathfrak{S}(\nabla_X\nabla_Y\nabla_Z)+\mathfrak{S}(\nabla_X\nabla_Z\nabla_Y) \\ &= 0\end{aligned}$$

Next, by the following remark,

$$\begin{aligned}(\nabla_X R)(Y,Z)W &= \nabla_X(R(Y,Z)W)-R(\nabla_XY,Z)W-R(Y,\nabla_XZ)W-R(Y,Z)\nabla_XW \\ &= [\nabla_X,R(Y,Z)]W-R(\nabla_XY,Z)W-R(Y,\nabla_XZ)W\end{aligned}$$

Besides, formally

$$\begin{aligned}R(X,Y)Z &= \nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z \\ &= [\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z\end{aligned}$$

So

$$\begin{aligned}\mathfrak{S}(\nabla_XR)(Y,Z)W &= \mathfrak{S}[\nabla_X,R(Y,Z)]W-\mathfrak{S}R(\nabla_XY,Z)W-\mathfrak{S}R(Y,\nabla_XZ)W \\ &= \mathfrak{S}[\nabla_X,[\nabla_Y,\nabla_Z]]W-\mathfrak{S}[\nabla_X,\nabla_{[Y,Z]}]W \\ &\quad -\mathfrak{S}R(\nabla_XY,Z)W-\mathfrak{S}R(Y,\nabla_XZ)W \\ &= -\mathfrak{S}[\nabla_X,\nabla_{[Y,Z]}]W-\mathfrak{S}R(\nabla_XY,Z)W+\mathfrak{S}R(\nabla_XZ,Y)W \\ &= -\mathfrak{S}[\nabla_X,\nabla_{[Y,Z]}]W-\mathfrak{S}R(\nabla_XY,Z)W+\mathfrak{S}R(\nabla_YX,Z)W \\ &= -\mathfrak{S}[\nabla_X,\nabla_{[Y,Z]}]W-\mathfrak{S}R([X,Y],Z)W \\ &= -\mathfrak{S}[\nabla_X,\nabla_{[Y,Z]}]W-\mathfrak{S}[\nabla_{[X,Y]},\nabla_Z]W+\mathfrak{S}\nabla_{[[X,Y],Z]}W \\ &= \mathfrak{S}[\nabla_{[Y,Z]},\nabla_X]W-\mathfrak{S}[\nabla_{[Y,Z]},\nabla_X]W \\ &= 0\end{aligned}$$

□

Remark. Locally, $R(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l}=R^k_{lij}\frac{\partial}{\partial x^k}$ and

$$R^k_{lij}=\frac{\partial\Gamma^k_{lj}}{\partial x^i}-\frac{\partial\Gamma^k_{li}}{\partial x^j}+\Gamma^h_{jl}\Gamma^k_{ih}-\Gamma^h_{il}\Gamma^k_{jh}$$

And above three properties become

(1) $R^k_{lij}=-R^k_{lji}$.

(2) $R^k_{lij}+R^k_{ijl}+R^k_{jli}=0$.

(3) $R^k_{lij;h}+R^k_{ljh;i}+R^k_{lhi;j}=0$.

Remark. Because R is a $(1,3)$-tensor, for any $\omega\in\Gamma(T^*M)$ and $X,Y,Z\in\Gamma(TM)$,

$$\begin{aligned}(\nabla_XR)(\omega,X,Y,Z) &= X(R(\omega,X,Y,Z))-R(\nabla_X\omega,X,Y,Z)-R(\omega,\nabla_XX,Y,Z) \\ &\quad -R(\omega,X,\nabla_XY,Z)-R(\omega,X,Y,\nabla_XZ)\end{aligned}$$

Besides,

$$\begin{aligned}R(\nabla_X\omega, X, Y, Z) &= \nabla_X\omega(R(X,Y)Z)\\&= X(R(\omega, X, Y, Z)) - \omega(\nabla_X(R(X,Y)Z))\end{aligned}$$

Therefore,

$$\begin{aligned}(\nabla_X R)(\omega, X, Y, Z) &= \omega((\nabla_X R)(X,Y)Z)\\&= \omega(\nabla_X(R(X,Y)Z)) - \omega(R(\nabla_X X, Y)Z)\\&\quad - \omega(R(X, \nabla_X Y)Z) - \omega(R(X,Y)\nabla_X Z)\end{aligned}$$

It follows that

$$(\nabla_X R)(X,Y)Z = \nabla_X(R(X,Y)Z) - R(\nabla_X X, Y)Z - R(X, \nabla_X Y)Z - R(X,Y)\nabla_X Z$$

2. ***Riemannian curvature:*** For any $X, Y, Z, W \in \Gamma(TM)$, let

$$R(W, Z, X, Y) := \langle R(X,Y)Z, W\rangle \in C^\infty(M)$$

So R is a $(0,4)$-tensor, called the Riemannian curvature tensor.

Proposition 3.1.2. *Let ∇ be the Levi-Civita connection.*

(1) $R(X,Y,W,Z) = -R(X,Y,Z,W)$.
(2) $R(X,Y,W,Z) = -R(Y,X,W,Z)$.
(3) $\mathfrak{S}_{YZW} R(X,Y,Z,W) = 0$.
(4) $R(X,Y,Z,W) = R(Z,W,X,Y)$.
(5) $\mathfrak{S}_{ZWV}(\nabla R)(X,Y,Z,W,V) = 0$.

Proof. (1) is clear by the anti-symmetry of curvature tensor.

(2) First, by the compatibility of g,

$$\begin{aligned}\langle \nabla_W\nabla_Z Y, X\rangle &= W\langle \nabla_Z Y, X\rangle - \langle \nabla_Z Y, \nabla_W X\rangle\\&= W\left(Z\langle Y, X\rangle - \langle Y, \nabla_Z X\rangle\right) - \langle \nabla_Z Y, \nabla_W X\rangle\\&= W(Z(\langle Y, X\rangle)) - \langle \nabla_W Y, \nabla_Z X\rangle - \langle Y, \nabla_W\nabla_Z X\rangle - \langle \nabla_Z Y, \nabla_W X\rangle\end{aligned}$$

and similarly,

$$\langle \nabla_Z\nabla_W Y, X\rangle = Z(W(\langle Y, X\rangle)) - \langle \nabla_Z Y, \nabla_W X\rangle - \langle Y, \nabla_Z\nabla_W X\rangle - \langle \nabla_W Y, \nabla_Z X\rangle$$

Also, we have

$$\begin{aligned}\left\langle \nabla_{[W,Z]} Y, X\right\rangle &= [W,Z]\langle Y, X\rangle - \left\langle Y, \nabla_{[W,Z]} X\right\rangle\\&= W(Z(\langle Y, X\rangle)) - Z(W(\langle Y, X\rangle)) - \left\langle Y, \nabla_{[W,Z]} X\right\rangle\end{aligned}$$

It follows that

$$\begin{aligned}R(X,Y,W,Z) &= \langle R(W,Z)Y, X\rangle\\&= \langle \nabla_W\nabla_Z Y, X\rangle - \langle \nabla_Z\nabla_W Y, X\rangle - \left\langle \nabla_{[W,Z]} Y, X\right\rangle\\&= -\langle Y, R(W,Z)X\rangle = -\langle R(W,Z)X, Y\rangle\\&= -R(Y,X,W,Z)\end{aligned}$$

(3) is clearly by the first Bianchi identity of curvature tensor.

(4) In fact, any $(0,4)$-tensor satisfying $(1)-(3)$ has the property (4). Therefore, it is sufficient to prove this general version. First, by (3),

$$R(W,Z,X,Y)=-R(W,X,Y,Z)-R(W,Y,Z,X)$$

and by $(1),(2),(3)$,

$$\begin{aligned}R(W,Z,X,Y)&=-R(Z,W,X,Y)\\&=R(Z,X,Y,W)+R(Z,Y,W,X)\\&=-R(X,Z,Y,W)-R(Y,Z,W,X)\end{aligned}$$

Besides,

$$\begin{aligned}-R(W,X,Y,Z)-R(X,Z,Y,W)&=R(X,W,Y,Z)+R(X,Z,W,Y)\\&=-R(X,Y,Z,W)=R(Y,X,Z,W)\end{aligned}$$

and

$$\begin{aligned}-R(W,Y,Z,X)-R(Y,Z,W,X)&=R(Y,W,Z,X)+R(Y,Z,X,W)\\&=-R(Y,X,W,Z)=R(Y,X,Z,W)\end{aligned}$$

Combining these, we have

$$R(W,Z,X,Y)=R(Y,X,Z,W)\ \Rightarrow\ R(Z,W,X,Y)=R(X,Y,Z,W)$$

(5) First,

$$\begin{aligned}(\nabla R)(W,Z,X,Y,V)&=(\nabla_V R)(W,Z,X,Y)\\&=V(R(W,Z,X,Y))-R(\nabla_V W,Z,X,Y)-R(W,\nabla_V Z,X,Y)\\&\quad-R(W,Z,\nabla_V X,Y)-R(W,Z,X,\nabla_V Y)\\&=V\langle R(X,Y)Z,W\rangle-\langle R(X,Y)Z,\nabla_V W\rangle-\langle R(X,Y)\nabla_V Z,W\rangle\\&\quad-\langle R(\nabla_V X,Y)Z,W\rangle-\langle R(X,\nabla_V Y)Z,W\rangle\end{aligned}$$

Because

$$(\nabla_V R)(X,Y)Z=\nabla_X(R(X,Y)Z)-R(\nabla_X X,Y)Z-R(X,\nabla_X Y)Z-R(X,Y)\nabla_X Z$$

It follows that

$$\begin{aligned}(\nabla R)(W,Z,X,Y,V)&=V\langle R(X,Y)Z,W\rangle-\langle R(X,Y)Z,\nabla_V W\rangle\\&\quad+\langle(\nabla_V R)(X,Y)Z,W\rangle-\langle\nabla_X(R(X,Y)Z),W\rangle\\&=\langle(\nabla_V R)(X,Y)Z,W\rangle\end{aligned}$$

Therefore, by the second Bianchi identity,

$$\mathfrak{S}_{XYV}(\nabla R)(W,Z,X,Y,V)=0$$

□

Remark. In a local chart (U,x), let $R=R_{klij}dx^k\otimes dx^l\otimes dx^i\otimes dx^j$. Then

$$\begin{aligned}R_{klij}&=R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\\&=\left\langle R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^k}\right\rangle\\&=\left\langle R^m_{lij}\frac{\partial}{\partial x^m},\frac{\partial}{\partial x^k}\right\rangle\\&=g_{km}R^m_{lij}\end{aligned}$$

So $(1),(2),(4)$ become

$$R_{klij}=-R_{lkij}=-R_{klji}=R_{ijkl}$$

and $(3),(5)$ become

$$\mathfrak{S}_{lij}R_{klij}=0,\quad \mathfrak{S}_{ijh}R_{klij;h}=0$$

3. ***Sectional curvature:*** The next problem is how many parameters R_{klij} can determine the Riemannian curvature R.

Proposition 3.1.3. *Suppose R_1 and R_2 are two $(0,4)$-tensors satisfying $(1)-(3)$ in* ***Proposition*** *3.1.2. If for any $X,Y\in\Gamma(TM)$, then*

$$R_1=R_2$$

Proof. Let $R=R_1-R_2$. Then R satisfies $(1)-(3)$ and (4) by **Proposition** 3.1.2. It is sufficient to show $R=0$ when $R(X,Y,X,Y)=0$ for any $X,Y\in\Gamma(TM)$. First, by(4),

$$\begin{aligned}0=R(X,Y+W,X,Y+W)&=R(X,Y,X,W)+R(X,W,X,Y)\\&=2R(X,Y,X,W)\end{aligned}$$

So $R(X,Y,X,W)=0$. Next,

$$\begin{aligned}0=R(X+Z,Y,X+Z,W)&=R(X,Y,Z,W)+R(Z,Y,X,W)\\&=-R(X,Z,W,Y)-R(X,W,Y,Z)+R(Z,Y,X,W)\\&=-R(X,Z,W,Y)+2R(Z,Y,X,W)\end{aligned}$$

So $2R(Z,Y,X,W)=R(X,Z,W,Y)$. And similarly by interchanging Z and W,

$$2R(W,Y,X,Z)=R(X,W,Z,Y)$$

It follows that

$$\begin{aligned}2R(Z,Y,X,W)&=2R(X,W,Z,Y)=4R(W,Y,X,Z)\\&=4R(X,Z,W,Y)=8R(Z,Y,X,W)\end{aligned}$$

So $R(Z,Y,X,W)=0$. □

Remark. In particular, if let R be the Riemannian curvature, then R is determined by

$$\{R(X,Y,X,Y)\colon X,Y\in\Gamma(TM)\}$$

Note that if X,Y are linearly dependent, then $R(X,Y,X,Y)=0$ by the anti-symmetry. Besides, if

$$\begin{cases}X'=aX+bY,\\Y'=cX+dY\end{cases}$$

then

$$R(X',Y',X',Y')=\det\begin{pmatrix}a&b\\c&d\end{pmatrix}^2R(X,Y,X,Y)$$

Consider a $(0,4)$-tensor G given by

$$G(X,Y,Z,W):=g(X,Z)g(Y,W)-g(X,W)g(Y,Z)$$

it follows that

(1) $G(X,Y,Z,W) = -G(X,Y,W,Z)$.
(2) $G(X,Y,Z,W) = -G(Y,X,W,Z)$.
(3) $\mathfrak{S}_{YZW} G(X,Y,Z,W) = 0$.
(4) $G(X,Y,Z,W) = G(Z,W,X,Y)$.

By above proposition, $G(X,Y,Z,W)$ only dependents on

$$G(X,Y,X,Y) = \|X\|^2 \|Y\|^2 - \langle X, Y\rangle^2$$

Definition 3.1.1. At $p \in M$, let $X_p, Y_p \in T_pM$ linearly independent.

$$K(X_p, Y_p) = \frac{R_p(X_p, Y_p, X_p, Y_p)}{G_p(X_p, Y_p, X_p, Y_p)}$$

Then K_p only dependents on a two dimensional subspace $\Pi_p = \mathrm{span}(X_p, Y_p) \subset T_pM$, called the sectional curvature of M at p with respect to Π_p, denoted by $K(\Pi_p)$.

Proposition 3.1.4. *For $\Pi_p = \mathrm{span}(X_p, Y_p) \subset T_pM$, let $\Theta \subset \Pi_p$ such that $\exp_p \colon \Theta \to \exp_p(\Theta) \subset M$ is a diffeomorphism. Let $i \colon \exp_p(\Theta) \hookrightarrow M$ be an inclusion and $\bar{R}$ be the curvature tensor of $\exp_p(\Theta)$ with respect to i^*g. Then*

$$\bar{R}(X_p, Y_p, X_p, Y_p) = R(X_p, Y_p, X_p, Y_p)$$

In particular, $K(\Pi_p)$ is the Gaussian curvature.

Definition 3.1.2. A Riemannian manifold (M, g) is said to have constant (sectional) curvature if $K(\Pi_p)$ is constant for all p and Π_p.

Proposition 3.1.5. *(M, g) has constant curvature k if and only if $R = kG$.*

Proof. Assume for any p and X_p, Y_p

$$R_p(X_p, Y_p, X_p, Y_p) = kG_p(X_p, Y_p, X_p, Y_p)$$

If defines a $(0,4)$-tensor $S = R - kG$, then S satisfied the conditions $(1)-(3)$ in **Proposition 3.1.2** and

$$S(X, Y, X, Y) = 0, \quad X, Y \in \Gamma(TM)$$

So $S = 0$. □

Example 3.1.1. (1) Euclidean space: For $\mathbb{R}^n$, because the curvature tensor is 0, the Riemannian curvature tensor $R \equiv 0$. So it has constant sectional curvature 0.

(2) Sphere: For $\mathbb{S}^n$ with the induced metric, we have already obtained

$$R(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

It follows that the sectional curvature satisfies

$$\begin{aligned} K(X,Y) &= \frac{\langle R(X,Y)Y, X\rangle}{G(X,Y,X,Y)} \\ &= \frac{\langle \langle Y,Y\rangle X - \langle X,Y\rangle Y, X\rangle}{\|X\|^2 \|Y\|^2 - \langle X,Y\rangle^2} \\ &= 1 \end{aligned}$$

for any X, Y. So $\mathbb{S}^n$ has the constant sectional curvature 1.

(3) Hyperbolic space: We have already obtained

$$R^k_{lij} = \frac{1}{(x^n)^2}\left(\delta^k_j \delta_{il} - \delta^k_i \delta_{lj}\right), \quad g_{ij} = \frac{1}{(x^n)^2}\delta_{ij}$$

Therefore,

$$\begin{aligned} R_{klij} &= g_{km} R^m_{lij} \\ &= \frac{1}{(x^n)^4}\delta_{km}\left(\delta^m_j \delta_{il} - \delta^m_i \delta_{lj}\right) \\ &= \frac{1}{(x^n)^4}\left(\delta_{kj}\delta_{il} - \delta_{ki}\delta_{jl}\right) \end{aligned}$$

and

$$\begin{aligned} G_{klij} &= G\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ &= g_{ki} g_{lj} - g_{kj} g_{li} \\ &= \frac{1}{(x^n)^4}\left(\delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}\right) \end{aligned}$$

It follows that

$$R = -G$$

So the hyperbolic space $\mathbb{H}^n$ has the constant sectional curvature -1.

4. ***Ricci curvature:*** For $Y, Z \in \Gamma(TM)$, $R(-, Y, -, Z) = R(Y, -, Z, -)$ is a $(0,2)$-tensor and let

$$\operatorname{Ric}(Y, Z) := \operatorname{tr} R(-, Y, -, Z) = \operatorname{tr} R(Y, -, Z, -)$$

Locally, in a chart (U, x),

$$\operatorname{Ric}_{pq} = \operatorname{Ric}\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^q}\right) = g^{ij} R_{ipjq}$$

Besides, note that $\operatorname{tr}(R(-, Y, -, Z)) = \operatorname{tr}(X \mapsto \sharp R(X, Y, -, Z))$ and

$$\begin{aligned} g(\sharp R(X, Y, -, Z), W) &= R(X, Y, W, Z) \\ &= R(W, Z, X, Y) \\ &= \langle R(X, Y)Z, W\rangle \end{aligned}$$

Therefore,

$$\sharp R(X, Y, -, Z) = R(X, Y)Z \;\Rightarrow\; \operatorname{Ric}(Y, Z) = \operatorname{tr}(X \mapsto R(X, Y)Z)$$

Locally,

$$\operatorname{Ric}_{pq} = \sum_j R^j_{pjq}$$

Moreover, by above $\{\operatorname{Ric}(X, Y) \colon X, Y \in \Gamma(TM)\}$ is determined by $\{\operatorname{Ric}(X, X) \colon X \in \Gamma(TM)\}$.

Definition 3.1.3. For any p and $X_p \in T_pM$,

$$\operatorname{Ric}(X_p) := \operatorname{Ric}_p(X_p, X_p)$$

is called the Ricci curvature, also we call $\operatorname{Ric}(X, Y)$ the Ricci curvature tensor.

Remark. Choose an $X_p \in T_pM$ with $\|X_p\| = 1$ and an orthonormal basis $\{e_i\}$ with $e_1 = X_p$, then

$$\begin{aligned}\mathrm{Ric}(X_p, X_P) &= \sum_{i=2}^{n} R(e_i, X_p, e_i, X_p) \\ &= \sum_{i=2}^{n} K(e_i, X_p)\end{aligned}$$

Remark. A Riemannian manifold (M, g) is called an Einstein manifold with Einstein constant k if

$$\mathrm{Ric}(X, X) = kg(X, X), \quad \forall\ X \in \Gamma(TM)$$

Theorem 3.1.1 (Schur)**.** *Let (M, g) be a Riemannian manifold with $m = \dim M \geq 3$.*

(1) If $K(\Pi_p) = f(p)$ for all $\Pi_p \subset T_pM$, where $f(p)$ only dependents on p, then $f(p) \equiv$ const. for all $p \in M$.

(2) If $\mathrm{Ric}(X_p, X_p) = f(p)g(X_p, X_p)$ for all $X_p \in T_pM$, then $f(p) \equiv$ const..

Proof. By above remark, (2) implies (1). So it is sufficient to prove (2). Let (U, x) be a normal chart $(g_{ij}(p) = \delta_{ij})$ and suppose

$$\mathrm{Ric} = fg \ \Rightarrow\ \mathrm{Ric}_{kl} = fg_{kl}$$

Then for any $h \in \{1, 2, \cdots, m\}$,

$$\mathrm{Ric}_{kl;h} = (fg_{kl})_{;h} = f_{,h}g_{kl} + fg_{kl;h} = f_{,h}g_{kl}$$

On the other hand, by the contraction property and $g^{ij}_{;h} = 0$

$$\mathrm{Ric}_{kl;h} = \left(g^{ij}R_{ikjl}\right)_{;h} = g^{ij}R_{ikjl;h}$$

So $g^{ij}R_{ikjl;h} = f_{,h}g_{kl}$. By the second Bianchi identity,

$$\begin{aligned}0 &= g^{ij}\left(R_{ikjl;h} + R_{iklh;j} + R_{ikhj;l}\right) \\ &= g^{ij}R_{ikjl;h} + g^{ij}R_{iklh;j} - g^{ij}R_{ikjh;l} \\ &= f_{,h}g_{kl} - f_{,l}g_{kh} + g^{ij}R_{iklh;j}\end{aligned}$$

By setting $l = k$ and then summing k, it follows that

$$\begin{aligned}0 &= \left(f_{,h}\sum_k g_{kk} - \sum_k f_{,k}g_{kh} + \sum_k g^{ij}R_{ikkh;j}\right)(p) \\ &= mf_{,h}(p) - f_{,h}(p) + \sum_k \sum_i R_{ikkh;i}(p) \\ &= (m-1)f_{,h}(p) - \sum_{k,i} R_{kikh;i}(p) \\ &= (m-1)f_{,h}(p) - \sum_i \mathrm{Ric}_{ih;i} \\ &= (m-1)f_{,h}(p) - \sum_i f_{,i}g_{ih} \\ &= (m-2)f_{,h}(p)\end{aligned}$$

So when $m \geq 3$, $f_{,h} = 0$. $\square$

Definition 3.1.4. Let $S = \operatorname{tr}\operatorname{Ric}(-,-)$. S is called the scalar curvature. Locally,

$$S = g^{pq}\operatorname{Ric}_{pq}$$

Remark. Let $\{e_i\}$ be an orthonormal basis of T_pM. Then

$$\begin{aligned} S_p = \operatorname{tr}\operatorname{Ric}(-,-) &= \sum_i \operatorname{Ric}(e_i, e_i) \\ &= \sum_{i,j} R(e_j, e_i, e_j, e_i) \\ &= \sum_{i,j} K(e_i, e_j) = 2\sum_{i<j} K(e_i, e_j) \end{aligned}$$

where the final inequality is by the symmetry of K and $K(e_i, e_i) :== 0$.

Corollary 3.1.1. *(1) If (M, g) has the constant curvature k, then*

$$\operatorname{Ric} = (m-1)kg, \quad S = m(m-1)kg$$

(2) If (M, g) is an Einstein manifold with k, then

$$S = mk$$

(3) If $\dim M \geq 3$, then M is Einstein if and only if

$$\operatorname{Ric} = \frac{S}{m} g$$

Remark. (i) For $m = 2$, K only dependents on p and

$$K(p) = \frac{\operatorname{Ric}(X, X)}{g(X, X)} = \frac{1}{2} S(p)$$

(ii) For $m = 3$, let $\{e_1, e_2, e_3\}$ be an orthonormal basis of T_pM. Then we get

$$\begin{pmatrix} \operatorname{Ric}(e_1) \\ \operatorname{Ric}(e_2) \\ \operatorname{Ric}(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} K(e_1, e_2) \\ K(e_2, e_3) \\ K(e_1, e_3) \end{pmatrix}$$

It follows that (M^3, g) is Einstein if and only if it has constant curvature.

5. ***In analysis:*** For a constant k, we call $\operatorname{Ric} \geq k$ when

$$\operatorname{Ric}(X, X) \geq kg(X, X), \quad \forall\ X \in \Gamma(TM)$$

When considering $\operatorname{Ric} = A$ as a symmetric matrix, it means $\min \lambda(A) \geq k$, where $\lambda(A)$ is the set of all eigenvalues of A. In geometric analysis, Ricci curvature is related to the measure defined on manifold.

Theorem 3.1.2. *For any C^∞ manifold M with $\dim M \geq 3$, there is a complete Riemannian metric g such that the Ricci curvature is negative and bounded below.*

Theorem 3.1.3 (Bochner's formula)**.** *For any $f \in C^\infty(M)$,*

$$\frac{1}{2}\Delta\left(\|\operatorname{grad} f\|^2\right) = \|\operatorname{Hess} f\|^2 + \langle \operatorname{grad}(\Delta f), \operatorname{grad} f\rangle + \operatorname{Ric}(\operatorname{grad} f)$$

where $\|\operatorname{Hess} f\|^2 = g^{ki}g^{jl}f_{;kl}f_{;ij}$.

Proof. For any $p \in M$, let (U, x) be a normal chart. At p,

$$\begin{aligned}
\text{RHS} &= g^{ki} g^{jl} f_{;kl} f_{;ij} + \left\langle g^{kl} \frac{\partial \Delta f}{\partial x^l} \frac{\partial}{\partial x^k}, g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \right\rangle \\
&\quad + \operatorname{Ric}\left(g^{kl} \frac{\partial f}{\partial x^l} \frac{\partial}{\partial x^k}, g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \right) \\
&= \sum_{k,l} (f_{:kl})^2 + \sum_j \frac{\partial \Delta f}{\partial x^j} \frac{\partial f}{\partial x^j} + \sum_{k,i} \operatorname{Ric}\left(\frac{\partial f}{\partial x^k} \frac{\partial}{\partial x^k}, \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i} \right)
\end{aligned}$$

Moreover, by $\Delta f = g^{kl} f_{;kl}$,

$$\begin{aligned}
\frac{\partial \Delta f}{\partial x^j}(p) &= \frac{\partial}{\partial x^j} \left(g^{kl} f_{;kl} \right) \\
&= g^{kl}(p) \frac{\partial}{\partial x^j} f_{;kl}(p) \\
&= \sum_k f_{;kk,j} = \sum_k f_{;kkj}(p)
\end{aligned}$$

where the final equality is because $\Gamma^k_{ij}(p) = 0$. So

$$\text{RHS} = \sum_{k,j} \left((f_{:kj})^2 + f_{;kkj} f_{,j} + f_{,k} f_{,j} \operatorname{Ric}_{kj} \right)$$

Next, for the other side,

$$\begin{aligned}
\text{LHS} &= \frac{1}{2} g^{kl} \left(g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)_{;kl} (p) = \frac{1}{2} \sum_k \left(g^{ij} \left(\frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)_{;k} \right)_{;k} (p) \\
&= \frac{1}{2} \sum_k g^{ij}(p) \left(\frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)_{;kk} (p) \\
&= \frac{1}{2} \sum_k g^{ij}(p) \left(\left(\frac{\partial f}{\partial x^i} \right)_{;k} \frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial x^i} \left(\frac{\partial f}{\partial x^j} \right)_{;k} \right)_{;k} (p) \\
&= \sum_{k,j} \left(\frac{\partial f}{\partial x^j} \left(\frac{\partial f}{\partial x^j} \right)_{;k} \right)_{;k} (p) = \sum_{k,j} \left(\frac{\partial f}{\partial x^j} \right)^2_{;k} + \frac{\partial f}{\partial x^j} \left(\frac{\partial f}{\partial x^j} \right)_{;kk} \\
&= \sum_{k,j} \left(f_{;jk}^2 + f_{;j} f_{;jkk} \right) = \sum_{k,j} \left(f_{;kj}^2 + f_{;kjk} f_{;j} \right)
\end{aligned}$$

where the forth equality is because we have

$$(A_i B_j)_{;k} = A_{i;k} B_j + A_i B_{j;k}$$

by considering the covariant differential of tensor $A_i B_j dx^i \otimes dx^j$, and the final equality is by the symmetry of $f_{;kj}$.

$$\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} \operatorname{grad} f - \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^k}} \operatorname{grad} f = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i} \right) \operatorname{grad} f$$

It follows that

$$f_{;\alpha jk} - f_{;\alpha kj} = \sum_l f_{;l} R^\alpha_{lkj}$$

and by setting $\alpha = k$, we have

$$\begin{aligned}\text{LHS} &= \sum_{k,j} f_{;kj}^2 + \sum_{k,j} f_{;kkj} f_{;j} + \sum_{k,j,l} f_{;j} f_{;l} R_{lkj}^k \\ &= \sum_{k,j} f_{;kj}^2 + \sum_{k,j} f_{;kkj} f_{;j} + \sum_{j,l} f_{;j} f_{;l} \operatorname{Ric}_{lj}\end{aligned}$$

Therefore, RHS = LHS. $\square$

Corollary 3.1.2. *For any $p \in M$, $\operatorname{Ric}_p \geq k$ if and only if*

$$\frac{1}{2}\Delta\left(\|\operatorname{grad} f\|^2\right)(p) \geq \frac{1}{m}(\Delta f)^2(p) + \langle \operatorname{grad}(\Delta f), \operatorname{grad} f\rangle (p) + k \|\operatorname{grad} f\|^2 (p)$$

Proof. Note that for a symmetric matrix $A \in \mathbb{R}^{m\times m}$,

$$m \|A\|^2 \geq \operatorname{tr}(A)^2$$

and if $A = \lambda I$, then it can take the equality. So at p, for any $f \in C^\infty(M)$

$$\|\operatorname{Hess} f\|^2 (p) \geq \frac{1}{m}(\Delta f)^2(p)$$

and we can find f such that it can take the equality.

- $\Rightarrow$: It is clearly by above theorem.
- $\Leftarrow$: By above theorem,

  $$\begin{aligned}\frac{1}{2}\Delta\left(\|\operatorname{grad} f\|^2\right)(p) - \langle \operatorname{grad}(\Delta f), \operatorname{grad} f\rangle (p) &= \langle \operatorname{grad}(\Delta f), \operatorname{grad} f\rangle (p) + \operatorname{Ric}(\operatorname{grad} f)(p) \\ &\geq \frac{1}{m}(\Delta f)^2(p) + k\|\operatorname{grad} f\|^2 (p)\end{aligned}$$

  Then for some f with $\|\operatorname{Hess} f\|^2 (p) = \frac{1}{m}(\Delta f)^2(p)$, we have

  $$\operatorname{Ric}(\operatorname{grad} f)(p) \geq k \|\operatorname{grad} f\|^2 (p)$$

  For any $X \in T_pM$, we can find a f such that $X = \operatorname{grad}(f)$ and $\|\operatorname{Hess} f\|^2 (p) = \frac{1}{m}(\Delta f)^2(p)$ by existence of solution of ODE, so

  $$\operatorname{Ric}(X) \geq k \|X\|^2, \quad \forall\, X \in T_pM$$

$\square$

3.2 Applications with Variation Formulas

Let $\gamma\colon [a,b] \to M$ be a C^∞-curve. Consider a variation of γ, $F\colon [a,b]\times(-\varepsilon,\varepsilon) \to M$ with $F(t,0) = \gamma(t)$ and let

$$V(t) = \frac{\partial F}{\partial v}(t,0), \quad T(t) = \frac{\partial F}{\partial t}(t,0) = \dot\gamma(t)$$

and energy function $E\colon (-\varepsilon,\varepsilon) \to M$

$$E(v) = \frac{1}{2}\int_a^b \left\langle \frac{\partial F}{\partial t}(t,v), \frac{\partial F}{\partial t}(t,v)\right\rangle dt$$

Then we have

- first variational formula (FVF)

$$\left.\frac{d}{dv}\right|_{v=0} E(v) = \langle V(t), T(t)\rangle|_a^b - \int_a^b \langle V(t), \nabla_T T(t)\rangle\, dt$$

- second variational formula (SVF)

$$\begin{aligned}\left.\frac{d^2}{dv^2}\right|_{v=0} E(v) = &\ \langle \nabla_V V(t), T(t)\rangle|_a^b \\ &+ \int_a^b \langle \nabla_T V(t), \nabla_T V(t)\rangle - R(V,T,V,T)dt \\ &- \int_a^b \langle \nabla_V V(t), \nabla_T T(t)\rangle\, dt\end{aligned}$$

How to determine a curve is with the minimal length. We have known that

$$E'(0) = 0 \text{ and } E''(0) \geq 0 \Rightarrow \text{local minimum}$$

For $E'(0) = 0$, it means γ is a geodesic. For $E''(0) \geq 0$,

(i) if we further assume the endpoints are fixed, then

$$F(a,v) \equiv \gamma(a),\ F(b,v) \equiv \gamma(b) \quad \Rightarrow \quad V(a) = V(b) = 0$$

So

$$E''(0) = \int_a^b \langle \nabla_T V(t), \nabla_T V(t)\rangle - R(V,T,V,T)dt$$

It follows that if the sectional curvature is negative, then $E''(0) \geq 0$. Therefore, any geodesic is with local minimal length.

(ii) if we further assume γ is closed, *i.e.*

$$\gamma(a) = \gamma(b), \quad T(a) = T(b)$$

and

$$\frac{\partial F}{\partial v}(a,0) = \frac{\partial F}{\partial v}(b,0) \ \Rightarrow\ \nabla_V V(a) = \nabla_V V(b)$$

by definition, then similarly

the sectional curvature is negative $\Rightarrow$ closed geodesic is locally minimal.

What if $E''(0) < 0$:

I. Closed curve and homotopic classes:

Theorem 3.2.1 (Synge). *(1) Any compact, orientable and even-dimensional Riemannian manifold with positive sectional curvature is simply connected.*

(2) Any compact, non-orientable, even-dimensional Riemannian manifold with positive sectional curvature has $\pi_1(M) = \mathbb{Z}_2$.

(3) Any compact, odd-dimensional Riemannian manifold with positive sectional curvature is orientable.

Remark. The compactness guarantees that in every homotopic class of closed curves, there is a geodesic with shortest length. So, for (1), the goal is to prove that any (nontrivial) closed geodesic is not with shortest length. By above, we need to construct a variation such that $\nabla_T V = 0$.

Proof. (1) Let $\gamma\colon [a,b] \to M$ be a closed geodesic with $p = \gamma(a) = \gamma(b)$. Consider the parallel map

$$\mathcal{P}_{\gamma,a,b}\colon T_pM \longrightarrow T_pM$$

we want to construct a variation such that

$$\mathcal{P}_{\gamma,a,b}(V(a)) = V(b), \quad \langle V(a), T(a)\rangle = 0$$

Note that $\mathcal{P}_{\gamma,a,b}$ is orthogonal and $\mathcal{P}_{\gamma,a,b}(T(a)) = T(a)$, *i.e.* it has eigenvalue 1. Then by the following **Lemma** 3.2.1 and **Lemma** 3.2.2, we have V_0 such that $V(t) = \mathcal{P}_{\gamma,a,t}(V_0)$ satisfies above conditions. If let

$$F(s,t) := \exp_{\gamma(t)} sV(t)$$

then we have such variation with

$$E''(0) = \int_a^b -R(V,T,V,T)dt < 0$$

which induces a contradiction.

(2) It can be obtained by above lemma and the following **Lemma** 3.2.3.

(3) Suppose M is non-orientable. Then there is a closed curve c of M such that its homotopic class is nontrivial and $\det \mathcal{P}_{\gamma,a,b} = -1$, and it can induce a contradiction by the following **Lemma** 3.2.1.

□

Lemma 3.2.1. *Let $A \in \mathbb{R}^n$ be an orthogonal matrix.*

(1) If $\det A = 1$ and 1 is its eigenvalue and n is even, then $\dim\ker(A - I) \geq 2$.

(2) If $\det A = -1$ and 1 is its eigenvalue and n is odd, then $\dim\ker(A - I) \geq 2$.

Remark. It is because all eigenvalues $|\lambda| = 1$ and A is diagonalizable in complex sense (self-adjoint).

Lemma 3.2.2. *Let (M,g) be a Riemannian manifold that is orientable. Then $\det \mathcal{P}_{\gamma,a,b} = 1$.*

Proof. Since M is orientable, there is a n-form ω that is nonzero at each point. Let $\{e_i\}$ be a basis of T_pM and $\{e_i(t)\}$ be the parallel moving of $\{e_i\}$. Then

$$t \mapsto \omega\left(e_1(t), \cdots, e_m(t)\right)$$

is a C^∞ function that does not change the sign on $[a,b]$. And

$$\omega\left(e_1(a), \cdots, e_m(a)\right) = \det \mathcal{P}_{\gamma,a,b}\omega\left(e_1(b), \cdots, e_m(b)\right)$$

So $\det \mathcal{P}_{\gamma,a,b} > 0$. □

Lemma 3.2.3. *Any non-orientable Riemannian manifold has an orientable* 2*-cover.*

II. Fixing two end-points: Then it also has

$$E''(0) = \int_a^b \langle \nabla_T V(t), \nabla_T V(t) \rangle - R(V,T,V,T)dt$$

So we need $R(V,T,V,T)$ bounded below for some appropriate constant.

Lemma 3.2.4 (Bonnet-Synge)**.** *Let (M, g) be a Riemannian manifold with the sectional curvature $\geq \kappa > 0$. Then any geodesic with length $> \frac{\pi}{\sqrt{\kappa}}$ cannot be local minimum.*

Proof. WTLG assume a arc-length parameterized geodesic $\gamma\colon [0,\ell] \to M$ with $\gamma(0) = p, \gamma(\ell) = q$. Choose $Y(0)$ be orthogonal to $\dot{\gamma}(0)$ with $\langle Y(0), Y(0) \rangle = 1$. Let $Y(t) = \mathcal{P}_{\gamma,0,t}(Y(0))$ and then $\langle Y(t), \dot{\gamma}(t) \rangle = 0$ and $\langle Y(t), Y(t) \rangle = 1$ and $\nabla_T Y(t) = 0$. So

$$R(Y,T,Y,T)(t) = K(\Pi_{\gamma(t)}(Y,T))$$

By assumption,

(i) sectional curvature $K \geq \kappa > 0$.

(ii) $\ell > \frac{\pi}{\sqrt{\kappa}} \Leftrightarrow \left(\frac{\pi}{\ell}\right)^2 < \kappa$.

Consider the variational field

$$V(t) = \sin\left(\frac{\pi}{\ell}t\right) Y(t), \quad t \in [0,\ell]$$

So we have

$$\nabla_T V(t) = \nabla_T \left(\sin\left(\frac{\pi}{\ell}t\right) Y(t) \right) = \frac{\pi}{\ell} \cos\left(\frac{\pi}{\ell} t Y(t)\right) Y(t)$$

Then $E'(0) = 0$ and

$$\begin{aligned}
E''(0) &= \int_0^\ell \langle \nabla_T V(t), \nabla_T V(t) \rangle \, dt - \int_0^\ell R(V,T,V,T)dt \\
&= \int_0^\ell \left(\frac{\pi}{\ell}\right)^2 \cos\left(\frac{\pi}{\ell}t\right)^2 dt - \int_0^\ell \sin\left(\frac{\pi}{\ell}t\right)^2 R(Y,T,Y,T)dt \\
&\leq \left(\frac{\pi}{\ell}\right)^2 \int_0^\ell \cos\left(\frac{\pi}{\ell}t\right)^2 dt - \kappa \int_0^\ell \sin\left(\frac{\pi}{\ell}t\right)^2 dt \\
&< 0
\end{aligned}$$

□

Corollary 3.2.1 (Hopf-Rinow)**.** *Let (M, g) be a complete Riemannian manifold with sectional curvature $\geq \kappa > 0$. Then*

$$\operatorname{diam}(M, g) \leq \frac{\pi}{\sqrt{\kappa}}$$

In particular, M is compact.

Lemma 3.2.5 (Myers)**.** *Let (M, g) be a Riemannian manifold with the Ricci curvature $\geq (m-1)\kappa > 0$. Then any geodesic with length $> \frac{\pi}{\sqrt{\kappa}}$ cannot be local minimum.*

Proof. By above, such $Y(0)$ has $m-1$ choice. So we can have $(m-1)$-vector fields $Y_i(t)(i=2,\cdots,m)$ and variational fields

$$V_i(t)=\sin\left(\frac{\pi}{\ell}t\right)Y_i(t),\quad i=2,\cdots,m$$

Then

$$\begin{aligned}\sum_i E''_{V_i}(0)&=\sum_i\int_0^\ell\left(\frac{\pi}{\ell}\right)^2\cos\left(\frac{\pi}{\ell}t\right)^2dt-\int_0^\ell\sin\left(\frac{\pi}{\ell}t\right)^2R(Y_i,T,Y_i,T)dt\\&\leq(m-1)\kappa\int_0^\ell\cos\left(\frac{\pi}{\ell}t\right)^2dt-\int_0^\ell\sin\left(\frac{\pi}{\ell}t\right)^2\sum_i K(\Pi_{\gamma(t)}(Y_i,T))dt\\&=(m-1)\kappa\int_0^\ell\cos\left(\frac{\pi}{\ell}t\right)^2dt-\int_0^\ell\sin\left(\frac{\pi}{\ell}t\right)^2\mathrm{Ric}(T)dt\\&<0\end{aligned}$$

Therefore, there is a variational $V=V_{i_0}$ such that $E''(0)=E''_{V_i}(0)<0$. ☐

Theorem 3.2.2 (Bonnet-Myers)**.** *Suppose (M,g) is a complete Riemannian manifold with the Ricci curvature $\geq(m-1)\kappa>0$. Then*

$$\mathrm{diam}(M,g)\leq\frac{\pi}{\sqrt{\kappa}}$$

Moreover, M is compact and (M,g) has finite fundamental group.

Proof. Consider the universal covering $\pi\colon(\widetilde{M},\widetilde{g})\to(M,g)$, which is complete and has the Ricci curvature $\geq(m-1)\kappa>0$ because π is locally isometric. Thus $(\widetilde{M},\widetilde{g})$ is also compact, which means

$$\pi^{-1}(p)=(q_1,\cdots,q_n)\,,\quad\forall p\in M$$

Because $\widetilde{M}$ is simply connected,

$$\pi^{-1}(p)\simeq\pi(M,p)/\pi_*\pi(\widetilde{M},\tilde{p})\simeq\pi(M,p)$$ ☐

Theorem 3.2.3 (Gromoll-Mayer)**.** *If $(M,g)$is a connected, complete, non-compact and with sectional curvature >0, then M is diffeomorphic to $\mathbb{R}^m$.*

Remark. Any complete non-compact Riemannian manifold M with $\dim M=2$ and sectional curvature ≥ 0, is either diffeomorphic to $\mathbb{R}^2$ or flat.

III. Fixing one end-points: Consider a variation of radical geodesics of γ, that is, consider $v\colon(-\varepsilon,\varepsilon)\to T_pM$ with $v(0)=\dot\gamma(0)$ and $|v(0)|=r$, let

$$F(t,s)=\exp_p\left(\frac{t}{r}v(s)\right),\quad t\in[0,r]$$

Then $F(0,s)=p$ for all $s\in(-\varepsilon,\varepsilon)$ and

$$\begin{aligned}V(t)&=\frac{\partial F}{\partial s}(t,0)\\&=\frac{\partial}{\partial s}\bigg|_{(t,0)}\exp_p\left(\frac{t}{r}v(s)\right)\\&=(d\exp)_{\gamma(t)}\left(\frac{t}{r}v'(0)\right)\end{aligned}$$

For any fixed s, $F(t,s) = c_s(t)$ is a geodesic, which means $\widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}(t,s) = 0$. Besides,

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial t}} V(t) &= \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s}(t,0) \\
&= \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}(t,0) \\
&= \widetilde{\nabla}_{\frac{\partial}{\partial s}} \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}(t,s)\bigg|_{s=0} + R\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right) \frac{\partial F}{\partial t}(t,0)
\end{aligned}$$

So

$$\nabla_T \nabla_T V + R(V,T)T = 0$$

Definition 3.2.1. Let $\gamma\colon [a,b] \to M$ be a geodesic with $T(t) = \dot{\gamma}(t)$. If a vector field V along γ satisfies

$$\nabla_T \nabla_T V + R(V,T)T = 0$$

we call V a Jacobian field and above equation is called the Jacobian equation.

Choosing an orthonormal frame $\{E_i(t)\}$ along γ with $E_1(t) = T(t)$, and

$$V(t) = \sum_{i=2}^{m} f^i(t) E_i(t)$$

and

$$\begin{aligned}
0 &= \langle \nabla_T \nabla_T V + R(V,T)T, E_j \rangle \\
&= \left\langle \frac{d^2 f^i}{dt^2} E_i(t) + f^i(t) R(E_i,T)T, E_j(t) \right\rangle \\
&= \frac{d^2 f^j(t)}{dt^2} + f^i(t) R(E_i,T,E_j,T)
\end{aligned}$$

So it follows that

$$\frac{d^2 f^j(t)}{dt^2} + f^i(t) R(E_i,T,E_j,T) = 0, \quad j = 2, \cdots, m$$

If (M,g) has constant sectional curvature κ, then

$$\begin{aligned}
R(E_i,T,E_j,T) = \kappa G(E_i,T,E_j,T) &= \kappa\left(\langle E_i,E_j\rangle \langle T,T\rangle - \langle E_i,T\rangle \langle E_j,T\rangle\right) \\
&= \begin{cases} 0, & i \neq j \\ 0, & i = j = 1 \\ \kappa, & i = j \neq 1 \end{cases}
\end{aligned}$$

Therefore, the equation becomes

$$\frac{d^2 f^j(t)}{dt^2} + \kappa f^j(t) = 0, \quad j = 2, \cdots, m$$

with $f^j(0) = 0$ by $V(0) = 0$. Moreover, because

$$\begin{aligned}
\nabla_T V(t)|_{t=0} &= \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s}(t,0)\bigg|_{t=0} \\
&= \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}(t,s)\bigg|_{s,t=0} \\
&= \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}(t,s)\bigg|_{s,t=0} \\
&= \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} \exp_p\left(\frac{t}{r}v(s)\right)\bigg|_{s,t=0} \\
&= \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{1}{r}v(s)\bigg|_{s=0} \\
&= \frac{1}{r}v'(s)\bigg|_{s=0}
\end{aligned}$$

where the final equality is because $v(s)$ can be a vector field along $c(s) \equiv p$,

$$\frac{d}{dt}f^i(0) = \left\langle \frac{v'(0)}{r}, E_i(t) \right\rangle$$

In particular, if we let

$$V(t) = f(t)E_2(t)$$

and $v(s)$ be the point at the circle on the subspace of T_pM spanned by E_1, E_2 ($\dot{v}(0) = rE_2$), then it follows

$$f''(t) + \kappa f'(t) = 0, \quad f(t) = 0, f'(0) = 1$$

So

$$f(t) = \begin{cases} t, & \kappa = 0 \\ \sin t, & \kappa = 1 \\ \sinh t, & \kappa = -1 \end{cases}$$

3.3 Jacobian Field and Index Form

1. ***Jacobian field:*** Let $\gamma\colon [a,b] \to M$ be a geodesic with $\gamma(a) = p, \gamma(b) = q$ and $T = \dot{\gamma}(a)$ and $|T| = 1$ (*i.e.* normal geodesic). Let $U(t)$ be a Jacobian field of γ. Let $\{E_i\}$ be a orthonormal basis of T_pM with $E_1 = T$ and $\{E_i(t)\}$ be the induced orthonormal frame. If

$$U(t) = f^i(t)E(t)$$

then as above the Jacobian equation becomes

$$\frac{d^2}{dt^2}f^j(t) + f^i(t)R(E_j, T, E_i, T) = 0, \quad j = 1, 2, \cdots, m$$

that is

$$\frac{d^2}{dt^2}\begin{pmatrix} f^1(t) \\ f^2(t) \\ \vdots \\ f^m(t) \end{pmatrix} = \begin{pmatrix} R(E_1,T,E_1,T) & R(E_1,T,E_2,T) & \dots & R(E_1,T,E_m,T) \\ R(E_2,T,E_1,T) & R(E_2,T,E_2,T) & \dots & R(E_2,T,E_m,T) \\ \vdots & \vdots & \ddots & \vdots \\ R(E_m,T,E_1,T) & R(E_m,T,E_2,T) & \dots & R(E_m,T,E_m,T) \end{pmatrix} \begin{pmatrix} f^1(t) \\ f^2(t) \\ \vdots \\ f^m(t) \end{pmatrix}$$

Because Jacobian equation is a linear ODE, we have the following proposition.

Proposition 3.3.1. *Let $\gamma\colon [0,\ell] \to M$ be a geodesic with $T = \dot\gamma(0)$.*

(1) Given $V, W \in T_{\gamma(0)}M$, there exists a unique Jacobian field U such that

$$U(0) = V, \quad \nabla_T U(0) = W$$

(2) The linear space of all Jacobian fields along γ is of dimension $2m$.

(3) The zero set of a Jacobian field U along γ is discrete if U is not 0.

Proof. (1) and (2) are clearly by linear ODE. For (3), assume the zero set has an accumulative point $\gamma(t_0)$. Then

$$U(t_0) = 0$$

Choosing an frame such that $U(t) = f^i(t)E(t)$. So

$$\nabla_T U(t_0) = \frac{d}{dt} f^i(t_0)E(t_0) = 0$$

by $\frac{d}{dt} f^i(t_0) = 0$. Then by the uniqueness of solution, $U = 0$. □

In particular, if (M, g) has the constant sectional curvature κ, by above we have

$$\frac{d^2}{dt^2}\begin{pmatrix} f^1(t) \\ f^2(t) \\ \vdots \\ f^m(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \kappa & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \kappa \end{pmatrix} \begin{pmatrix} f^1(t) \\ f^2(t) \\ \vdots \\ f^m(t) \end{pmatrix}$$

which follows that $f^1(t) = at + b$ and

$$\frac{d^2}{dt^2} f^j(t) + \kappa f^j(t) = 0, \quad j = 2, 3, \cdots, m$$

So such $U = (at + b)T + U^\perp$ with $U^\perp \perp T$ and $U^\perp$ is a Jacobian vector field.

Proposition 3.3.2. *Let $\gamma\colon [a, b] \to M$ be a normal geodesic with $T = \dot\gamma(a)$.*

(1) The vector field fT is Jacobian if and only if f is linear.

(2) Every Jacobian vector field U along γ can be uniquely decomposition

$$U = fT + U^\perp$$

where f is linear, $U^\perp$ is Jacobian and $U^\perp \perp T$, called the normal Jacobian field.

(3) If a Jacobian vector field U along γ such that $\langle U, T\rangle |_{t_0,t_1} = 0$, then

$$\langle U, T\rangle \equiv 0$$

In particular,$U(t_0) = U(t_1) = 0$ implies $\langle U, T\rangle \equiv 0$.

Proof. (1) By taking fT in the Jacobian equation, we have

$$\begin{aligned} 0 &= \nabla_T\nabla_T(fT) + R(fT, T)T \\ &= \frac{d^2}{dt^2} f(t)T \end{aligned}$$

So it is equivalent to $\frac{d^2}{dt^2} f(t) = 0$.

(2) Let
$$U = fT + U^{\perp}$$
for some f and $U^{\perp} \perp T$, by choosing an orthonormal frame $\{E_i(t)\}$. It is also unique by the uniqueness of orthogonal decomposition. Next, we need to prove they are Jacobian.
Since U is Jacobian,
$$\begin{aligned} 0 &= \nabla_T\nabla_T(fT + U^{\perp}) + R(fT + U^{\perp}, T)T \\ &= \frac{d^2}{dt^2} f(t)T + \nabla_T\nabla_T U^{\perp} + R(U^{\perp}, T)T \end{aligned}$$
By $\left\langle R(U^{\perp}, T)T, T\right\rangle = 0$,
$$\begin{aligned} 0 &= \left\langle \frac{d^2}{dt^2} f(t)T + \nabla_T\nabla_T U^{\perp} + R(U^{\perp}, T)T, T \right\rangle \\ &= \frac{d^2}{dt^2} f(t) + \left\langle \nabla_T\nabla_T U^{\perp}, T\right\rangle \end{aligned}$$
Moreover, by $\left\langle U^{\perp}, T\right\rangle = 0$,
$$\begin{aligned} 0 &= \frac{d^2}{dt^2}\left\langle U^{\perp}, T\right\rangle = \frac{d}{dt}\left(\left\langle \frac{D}{dt}U^{\perp}, T\right\rangle + \left\langle U^{\perp}, \frac{D}{dt}T\right\rangle\right) \\ &= \frac{d}{dt}\left\langle \nabla_T U^{\perp}, T\right\rangle = \left\langle \nabla_T\nabla_T U^{\perp}, T\right\rangle \end{aligned}$$
Therefore, $\frac{d^2}{dt^2} f(t) = 0$ and thus fT is Jacobian. Then by the linearity of Jacobian equation, $U^{\perp}$ is also a Jacobian field.

(3) By (2),
$$\langle U, T\rangle(t) = f(t) \ \Rightarrow \ f(t_0) = f(t_1) = 0 \ \Rightarrow \ f(t) \equiv 0$$
So $U = U^{\perp} \perp T$.

□

Proposition 3.3.3. *Let $\gamma\colon [0,1] \to M$ be a geodesic and U be a vector field along γ. Then U is Jacobian if and only if there is a variation $F(t,s)$ such that $F(t,s)$ is a geodesic for any $s \in (-\varepsilon, \varepsilon)$ and*
$$U(t) = \left.\frac{\partial F}{\partial s}\right|_{t,0}$$

Proof. We only need to prove the $\Rightarrow$ direction. For $U(0)$, considering a geodesic $\beta(s)$ with
$$\beta(0) = \gamma(0), \quad \dot\beta(0) = U(0)$$
Let $V(0) = \dot\gamma(0)$ and $V(s)$ be the vector field along $\beta(s)$ by moving $V(0)$ in parallel. Besides, let $W(0) = \nabla_T U(0)$ and $W(s)$ be the vector field along $\beta(s)$ by moving $W(0)$ in parallel. Then we construct $F\colon [0,1] \times (-\varepsilon, \varepsilon) \to M$ by
$$F(t,s) = \exp_{\beta(s)} t\left(V(s) + sW(s)\right)$$
When ε is small enough, F is well-defined (It is because we can find a totally neighborhood of $\gamma(0)$). Because for any fixed s, $F(t,s)$ is a geodesic,
$$\left.\frac{\partial F}{\partial s}\right|_{t,0}$$

is a Jacobian field.

Claim: $\frac{\partial F}{\partial s}\big|_{t,0} = U(t)$.

Because they are all Jacobian fields, they satisfy the same ODE. So it is sufficient to check their initial values.

$$\frac{\partial F}{\partial s}\bigg|_{0,0} = \frac{\partial}{\partial s}\bigg|_{s=0} \beta(s) = \dot{\beta}(0) = U(0)$$

Next,

$$\begin{aligned}\left(\widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s}\right)\bigg|_{s,t=0} &= \left(\widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}\right)\bigg|_{s,t=0} \\ &= \widetilde{\nabla}_{\frac{\partial}{\partial s}} \left(\frac{\partial F}{\partial t}\bigg|_{t=0}\right)\bigg|_{s=0} \\ &= \widetilde{\nabla}_{\frac{\partial}{\partial s}} V(s) + sW(s)\bigg|_{s=0} \\ &= W(0) = \nabla_T U(0)\end{aligned}$$

So the claim has be proved. □

Remark. In particular, if $\beta(s) \equiv \gamma(0)$, then

$$F(t,s) = \exp_{\gamma(0)} t(V + sW), \quad V = \dot{\gamma}(0),\ W = \nabla_T U(0)$$

and $U(t) = \frac{\partial F}{\partial s}(t, 0)$.

2. ***Conjugate points:*** Let's consider the zero points of Jacobian fields.

   Definition 3.3.1. Let $\gamma\colon [a,b] \to M$ be a geodesic. For $t_0, t_1 \in [a,b]$, if there exists a Jacobian field U along γ with $U \neq 0$ (nontrivial) but $U(t_0) = U(t_1) = 0$, then t_0 and t_1 are called conjugate values along γ. Moreover, the multiplicity of t_0 and t_1 is defined as

   $$\dim\left(\mathcal{J}' := \{U \text{ Jacobian} : U(t_0) = U(t_1) = 0\}\right)$$

   where the set is a vector space by linearity. We also call $\gamma(t_0)$ and $\gamma(t_1)$ conjugate points of γ.

   Remark. Note that $\dim \mathcal{J}' \leq n-1$ which is because $U(t_0) = 0$ is an initial value and the Jacobian field fT is not in the set.

   Remark. Note that if there is a Jacobian field U with $U(a) = U(b) = 0$, then

   $$0 = I(U,U) = \frac{\partial^2 E}{\partial s^2}(0)$$

   Theorem 3.3.1. *Let $\gamma\colon [0,1] \to M$ be a geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$ and $q = \gamma(1) = \exp_p(V)$. Then 0 and 1 are conjugate value of γ if and only if V is a critical point of $\exp_p$. Moreover, the multiplicity of 0 and 1 is $\dim\ker(d\exp_p)_V$.*

   Proof.
   - For "⇐", suppose $V \in T_pM$ is a critical point of $\exp_p$, that is, there is a nonzero $X \in T_V(T_pM)$ such that

     $$(d\exp_p)_V(X) = 0$$

     Then consider a (radical) variation F defined as

     $$F(t,s) := \exp_p t(V + sX)$$

then $F(t,0)=\gamma(t)$ with Jacobian field $U(t)=\frac{\partial F}{\partial s}(t,0)$ and $U(0)=0$ and

$$\begin{aligned}U(1)&=\frac{\partial F}{\partial s}\Big|_{s=0,t=1}\\&=\frac{\partial}{\partial s}\exp_p(V+sX)\Big|_{s=0}\\&=(d\exp_p)_V(X)=0\end{aligned}$$

Besides, $\nabla_T U(0)=X\neq 0$ implies that U is nontrivial. So 0 and 1 are conjugate value of γ.

- For "$\Rightarrow$", by contradiction, we assume V is not a critical point. Then there exists linearly independent $X_1,X_2,\cdots,X_m\in T_V(T_pM)$ such that

$$(d\exp_p)_V(X_1),(d\exp_p)_V(X_2),\cdots,(d\exp_p)_V(X_m)\in T_qM$$

are linearly independent. For any nontrivial Jacobian field U along γ with $U(0)=0$, *i.e.* $\nabla_T U(0)\neq 0$, consider a radical variation

$$F(t,s):=\exp_p t(V+s\nabla_T U(0))$$

whose variation field $\frac{\partial F}{\partial s}(t,0)=U(t)$. Let

$$0\neq\nabla_T U(0)=a^iX_i,\quad a^i \text{ not all } 0$$

Then we have

$$U(1)=(d\exp_p)_V(\nabla_T U(0))=\sum_i a^i(d\exp_p)_V(X_i)$$

Because a^i are not all 0 and $(d\exp_p)_V(X_i)$ are independent, $U(1)\neq 0$. □

Remark. Consider a map $A_V\colon \mathcal{J}'\to\ker(d\exp_p)_V$ defined by

$$A_V(U)=\nabla_T U(0)$$

It is well-defined because $U(0)=0$ implies that we can construct a radical variation F with $U(t)=\frac{\partial F}{\partial s}(t,0)$ and $U(1)=(d\exp_p)_V(\nabla_T U(0))=0$. In $\Leftrightarrow$, we proved that A_V is a surjection. And the injectivity is clear by the uniqueness of Jacobian field with same initial condition. So A_V is an linear isomorphism. So the $\Rightarrow$ can be directly obtained by the map A_V. When V is not a critical point, $\mathcal{J}'=0$ because A_V is an isomorphism.

3. ***Index form:*** Consider a two-parameter variation $F(t,v,w)$ with two fixed endpoints of a geodesic $\gamma\colon[a,b]\to M$ with $T=\dot\gamma(\mathrm{t})$, then

$$\frac{\partial^2E}{\partial w\partial v}(0,0)=\int_a^b\langle\nabla_T V,\nabla_T W\rangle-R(W,T,V,T)dt=:I(V,W)$$

where $V(t)=\frac{\partial F}{\partial v}(t,0,0)$ and $W(t)=\frac{\partial F}{\partial w}(t,0,0)$. Then clearly, $I(V,W)$ is symmetric. In fact,for any vector fields V,W along γ, we can also define $I(V,W)$ by above formula. Moreover, if we view $I(V,W)$ defined on the linear space of all vector fields along γ, it is a bilinear form.

Proposition 3.3.4. *Let $\gamma\colon [a,b] \to M$ be a geodesic and U be a vector field along γ. U is a Jacobian field if and only if*

$$I(U,Y) = 0$$

for all vector field Y along γ with $Y(a) = Y(b) = 0$.

Proof. By definition, for any vector field Y along γ with $Y(a) = Y(b) = 0$,

$$\begin{aligned}
I(U,Y) &= \int_a^b \langle \nabla_T U, \nabla_T Y\rangle - R(U,T,Y,T)dt \\
&= \int_a^b \frac{d}{dt}\langle \nabla_T U, Y\rangle - \langle \nabla_T\nabla_T U, Y\rangle - \langle R(U,T)T, Y\rangle\, dt \\
&= -\int_a^b \langle \nabla_T\nabla_T U + R(U,T)T, Y\rangle\, dt = 0
\end{aligned}$$

if and only if $\nabla_T\nabla_T U + R(U,T)T = 0$. □

Corollary 3.3.1. *U is Jacobian if and only if U is a critical point of variation $I(X,X)$ with respect of all vector fields X along γ when two end points are fixed.*

Proof. For any Y with $Y(a) = Y(b) = 0$,

$$\left.\frac{d}{ds}\right|_{s=0} I(X+sY, X+sY) = 2I(X,Y) = 0$$ □

For a geodesic $\gamma\colon [a,b] \to M$,

$$\begin{aligned}
\mathcal{V} &:= \text{ the set of all continuous piecewise smooth vector field along } \gamma \\
\mathcal{V}_0 &:= \{X \in \mathcal{V}\colon X(a) = X(b) = 0\}
\end{aligned}$$

Define

$$I(V,W) = \int_a^b \langle \nabla_T V, \nabla_T W\rangle - R(V,T,W,T)dt, \quad V, W \in \mathcal{V}$$

called the index form of γ, which is symmetric. Specifically, given $V, W \in \mathcal{V}$, let choose $a = t_0 < t_1 < \cdots < t_{n+1} = b$ such that V, W are smooth on $[t_i, t_{i+1}]$ for all i.

$$\begin{aligned}
I(V,W) &= \sum_i \int_{t_i}^{t_{i+1}} \langle \nabla_T V, \nabla_T W\rangle - R(V,T,W,T)dt \\
&= \sum_i \int_{t_i}^{t_{i+1}} \frac{d}{dt}\langle V, \nabla_T W\rangle - \langle V, \nabla_T\nabla_T W\rangle - \langle R(W,T)T, V\rangle\, dt \\
&= \langle V, \nabla_T W\rangle|_a^b - \sum_{i=1}^n \big\langle V(t_i), \nabla_T W(t_i^+) - \nabla_T W(t_i^-)\big\rangle \\
&\quad - \int_a^b \langle V, \nabla_T\nabla_T W\rangle - \langle R(W,T)T, V\rangle\, dt
\end{aligned}$$

Remark. (i) If W is Jacobian (so it is smooth) and $V \in \mathcal{V}_0$, then $I(V,W) = 0$.

(ii) If W is Jacobian (so it is smooth) and $V \in \mathcal{V}$, then

$$I(V,W) = \langle V, \nabla_T W\rangle|_a^b$$

(iii) If W is piecewise Jacobian and $V \in \mathcal{V}_0$, then

$$I(V,W) = -\sum_{i=1}^{n} \left\langle V(t_i), \nabla_T W(t_i^+) - \nabla_T W(t_i^-)\right\rangle$$

Consider $F\colon [a,b] \times (-\varepsilon,\varepsilon) \times (-\delta,\delta) \to M$, where F is smooth in $(-\varepsilon,\varepsilon)\times(-\delta,\delta)$ and piecewise smooth in $[a,b]$ (note that $F(t,0,0)$ is smooth), *i.e.*

$$F\colon [t_i,t_{i+1}] \times (-\varepsilon,\varepsilon) \times (-\delta,\delta) \to M \text{ smooth, } a = t_0 < t_1 < \cdots < t_n = b,$$

such that

$$V(t) = \frac{\partial F}{\partial v}(t,0,0), \quad W(t) = \frac{\partial F}{\partial w}(t,0,0)$$

then

$$\frac{\partial^2 E}{\partial w \partial v}(0,0) = I(V,W)$$

Proposition 3.3.5. *Let $\gamma\colon [a,b] \to M$ be a geodesic and $U \in \mathcal{V}$. U is a Jacobian field (then U is smooth) if and only if*

$$I(U,Y) = 0, \quad Y \in \mathcal{V}_0$$

Proof. For "$\Rightarrow$", it has been obtained by above remark. On the other side, assume U is piecewise smooth on $a = t_0 < t_1 < \cdots < t_{n+1} = b$. Let $f\colon [a,b] \to \mathbb{R}$ be smooth such that $f \geq 0$ and $f(t_i) = 0$ for all i. Constructing Y as

$$Y = f \cdot (\nabla_T\nabla_T U + R(U,T)T) \in \mathcal{V}_0$$

so we have

$$0 = I(Y,U) = -\int_a^b f(t)\, |\nabla_T\nabla_T U + R(U,T)T|^2 \, dt$$

which follows that

$$\nabla_T\nabla_T U + R(U,T)T = 0, \quad \forall\, t \in [t_i, t_{i+1}]$$

So U is piecewise Jacobian. Then

$$I(U,V) = -\sum_{i=1}^{n} \left\langle V(t_i), \nabla_T U(t_i^+) - \nabla_T U(t_i^-)\right\rangle, \quad \forall\, V \in \mathcal{V}_0$$

and also $I(U,V) = 0$. For i_0, constructing $V \in \mathcal{V}_0$ such that

$$\begin{aligned} V(t_j) &= 0, \quad j \neq i_0 \\ V(t_{i_0}) &= \nabla_T U(t_{i_0}^+) - \nabla_T U(t_{i_0}^-) \end{aligned}$$

Then $\nabla_T U(t_{i_0}^+) = \nabla_T U(t_{i_0}^-)$ for all i_0. So by the uniqueness of the solution for Jacobian equation, U is smooth. □

Theorem 3.3.2. *Let $\gamma\colon [a,b] \to M$ be a geodesic with $p = \gamma(a)$ and $q = \gamma(b)$.*

(1) p has no conjugate point along γ if and only if I is positive definite on $\mathcal{V}_0$.

(2) q is the first conjugate point along γ if and only if I is semi-positive but not positive on $\mathcal{V}_0$.

(3) *There is a* $\bar{t} \in (a,b)$ *such that* $\bar{q} = \gamma(\bar{t})$ *is conjugate if and only if there is an* $X \in \mathcal{V}_0$ *such that* $I(X,X) < 0$.

Proof. Note that we only need the "$\Rightarrow$" for (1), (2), and (3).

(1) Choosing an orthonormal basis $\{E_i\}$ of T_qM with $E_1 = \dot{\gamma}(q)$. By the following **Lemma** 3.3.1, we have m Jacobian fields $\{J_i\}$ such that $J_i(a) = 0$ and $U_i(b) = E_i$. Moreover, $\{J_i\}$ is linearly independent at each point because there is no conjugate point (otherwise, $\lambda^i J_i(t_0) = 0$ implies that p is conjugate to $\gamma(t_0)$). For any $U \in \mathcal{V}_0$,

$$U = f^i J_i, \quad f^i \text{ continuous piecewise } C^\infty,\ f^i(a) = f^i(b) = 0$$

Then it follows that

$$\begin{aligned}
I(U,U) &= \int_a^b \left\langle \nabla_T f^i J_i, \nabla_T f^j J_j \right\rangle - R(f^i J_i, T, f^j J_j, T) dt \\
&= \int_a^b \dot{f}^i \dot{f}^j \langle J_i, J_j \rangle + \dot{f}^i f_j \langle J_i, \nabla_T J_j \rangle + f^i \dot{f}^j \langle \nabla_T J_i, J_j \rangle + f^i f^j \langle \nabla_T J_i, \nabla_T J_j \rangle dt \\
&\quad - \int_a^b f^i f^j R(J_i, T, J_j, T) dt
\end{aligned}$$

First,

$$\begin{aligned}
\int_a^b f^i f^j \langle \nabla_T J_i, \nabla_T J_j \rangle dt = \int_a^b & \frac{d}{dt} \left(f^i f^j \langle \nabla_T J_i, J_j \rangle \right) - \dot{f}^i f^j \langle \nabla_T J_i, J_j \rangle \\
& - f^i \dot{f}^j \langle \nabla_T J_i, J_j \rangle - f^i f^j \langle \nabla_T \nabla_T J_i, J_j \rangle dt
\end{aligned}$$

and $\nabla_T \nabla_T J_i + R(J_i, T)T = 0$. Moreover, J_i, J_j Jacobian implies that $\langle \nabla_T J_i, J_j \rangle = \langle J_i, \nabla_T J_j \rangle$ because if let $f(t) = \langle \nabla_T J_i, J_j \rangle - \langle J_i, \nabla_T J_j \rangle$, then $f(0) = 0$ and

$$\begin{aligned}
f'(t) &= \langle \nabla_T \nabla_T J_i, J_j \rangle - \langle J_i, \nabla_T \nabla_T J_j \rangle \\
&= \langle R(J_i, T)T, J_j \rangle - \langle J_i, R(J_j, T)T \rangle = 0
\end{aligned}$$

Besides,

$$\int_a^b \frac{d}{dt} \left(f^i f^j \langle \nabla_T J_i, J_j \rangle \right) dt = \sum_i f^i f^j \langle \nabla_T J_i, J_j \rangle \Big|_{t_i}^{t_{i+1}} = 0$$

Combining these, we get

$$I(U,U) = \int_a^b \left\langle \dot{f}^i J_i, \dot{f}^j J_j \right\rangle dt \geq 0$$

Moreover, $I(U,U) = 0$ if and only if $\dot{f}^i J_i = 0$ if and only if $\dot{f}^i = 0$, *i.e.* $f^i \equiv 0$.

(2) Clearly, I is not positive. By moving in parallel, let's choose an orthonormal frame $\{E_i(t)\}$ along γ. For any $X \in \mathcal{V}_0$, $X(t) = f^i(t) E_i(t)$ with $f^i(a) = f^i(b) = 0$. For $a < c < b$, constructing

$$\widetilde{X}(t) = f^i \left(a + \frac{b-a}{c-a}(t-a) \right) E_i(t), \quad t \in [a,c]$$

So by (1), $I_c(\widetilde{X}, \widetilde{X}) > 0$. Because

$$\begin{aligned}
\lim_{c \to b} I_c(\widetilde{X}, \widetilde{X}) &= \lim_{c \to b} \int_a^c \left(\frac{b-a}{c-a} f^i \left(a + \frac{b-a}{c-a}(t-a) \right) \right)^2 \\
&\qquad - f^i \left(a + \frac{b-a}{c-a}(t-a) \right) f^j \left(a + \frac{b-a}{c-a}(t-a) \right) R(E_i, T, E_j, T) dt \\
&= I(X,X)
\end{aligned}$$

So $I(X,X) \geq 0$.

(3) Let J be Jacobian field on $\gamma|_{p\to\bar{q}}$ such that $J(p) = J(\bar{q}) = 0$. Then let $\widetilde{J}$ be $\widetilde{J}_{p\to\bar{q}}$ and $\widetilde{J}_{\bar{q}\to q} = 0$. Then

$$\nabla_T \widetilde{J}(\bar{t}^+) - \nabla_T \widetilde{J}(\bar{t}^-) = -\nabla_T \widetilde{J}(\bar{t}^-) \neq 0$$

Then let $U \in \mathcal{V}_0$ such that

$$\left\langle U(\bar{t}), \nabla_T \widetilde{J}(\bar{t}^-) \right\rangle = -1$$

Constructing

$$X = \frac{1}{c}\widetilde{J} - cU \in \mathcal{V}_0, \quad c \text{ small enough}$$

Then we have

$$\begin{aligned} I(X,X) &= \frac{1}{c^2} I(\widetilde{J}, \widetilde{J}) - 2I(\widetilde{J}, U) + c^2 I(U,U) \\ &= 0 - 2\left(-\left\langle U(\bar{t}), \nabla_T \widetilde{J}(\bar{t}^+) - \nabla_T \widetilde{J}(\bar{t}^-) \right\rangle\right) + c^2 I(U,U) \\ &= c^2 I(U,U) - 2 < 0 \end{aligned}$$

for sufficiently small c. □

Corollary 3.3.2. *Let $\gamma\colon [a,b] \to M$ be a geodesic with $p = \gamma(a)$ and $q = \gamma(b)$. If p is not conjugate to q along γ, then for any $\alpha, \beta \in \gamma$, α is not conjugate to β along γ.*

Proof. Assume $\tilde{X}$ is a Jacobian field along $\gamma|_{\alpha\to\beta}$ such that $\tilde{X}(\alpha) = \tilde{X}(\beta) = 0$. Then extending X of $\tilde{X}$ by setting $X|_{\gamma|_{p\to\alpha}} = X|_{\gamma|_{\beta\to q}} = 0$. Then $X \in \mathcal{V}_0$ with $X \neq 0$ and by above calculation

$$I(X,X) = -\sum_{t=\alpha,\beta} \left\langle X(t), \nabla_T X(t^+) - \nabla_T X(t^-) \right\rangle = 0$$

which contradicts to (1) in above theorem. □

Corollary 3.3.3. *Let (M,g) be a Riemannian manifold with sectional curvature ≤ 0. No two points are conjugate along some geodesic.*

Proof. It is clearly by

$$I(X,X) = \int_a^b \langle \nabla_T X, \nabla_T X \rangle - R(X,T,X,T)dt \geq 0$$

□

Lemma 3.3.1. *Let $\gamma\colon [a,b] \to M$ be a geodesic with $p = \gamma(a)$ and $q = \gamma(b)$. If p has no conjugate point along γ, then for any $V_a \in T_pM$ and $V_b \in T_qM$, there is a unique Jacobian field U along γ such that $U(a) = V_a$ and $U(b) = V_b$.*

Proof. Let $\mathcal{J}$ be the set of all Jacobian field along γ. Define the map $A\colon \mathcal{J} \to T_pM \times T_qM$ by $A(U) = (U(a), U(b))$. Clearly A is linear and $\dim \mathcal{J} = \dim T_pM \times T_qM = 2m$. Because p has no conjugate point along γ, $\ker A = 0$. A is injective and so it is an isomorphism. □

Definition 3.3.2. Let $\gamma\colon [a,b] \to M$ be a geodesic. The index of γ is

$$\operatorname{ind}(\gamma) := \operatorname{maxdim}\{\mathcal{A} \subset \mathcal{V}_0 \text{ subspace } : I(X,X) < 0, \quad \forall\, X \in \mathcal{A}, X \neq 0\}$$

and the nullity of γ is

$$N(\gamma) := \dim\{X \in \mathcal{V}_0\colon I(X,Y) = 0, \quad \forall\, Y \in \mathcal{V}_0\}$$

where $I\colon \mathcal{V}_0 \times \mathcal{V}_0 \to \mathbb{R}$ is considered.

Remark. By above result, we have

$$N(\gamma) := \dim \mathcal{J}' = \text{ multiplicity of conjugate points } \gamma(a), \gamma(b)$$

Theorem 3.3.3 (Morse Index Theorem)**.** *The index of a geodesic $\gamma\colon [a,b] \to M$ is the number of $\bar{t} \in (a,b)$ which are conjugate to a, with each conjugate value being counted by multiplicity, and it is always finite, that is*

$$\operatorname{ind}(\gamma) = \sum_{t \in (a,b)} N(\gamma|_{[a,t]}) < \infty$$

Lemma 3.3.2. *$\gamma\colon [a,b] \to M$ is a geodesic without conjugate points. Let U be a Jacobian field along γ and X be a piecewise smooth vector field along γ such that $X(a) = U(a)$ and $X(b) = U(b)$. Then*

$$I(U,U) \le I(X,X)$$

where "=" if and only if $X = U$.

Proof. First, $X - U$ is piecewise smooth and vanishes at ends. Then by above

$$\begin{aligned} 0 &\le I(X-U, X-U) \\ &= I(X,X) - 2I(X,U) + I(U,U) \\ &= I(X,X) - 2\left\langle \nabla_T U, X \right\rangle\big|_a^b + \left\langle \nabla_T U, U \right\rangle\big|_a^b \\ &= I(X,X) - \left\langle \nabla_T U, U \right\rangle\big|_a^b \\ &= I(X,X) - I(U,U) \end{aligned}$$

where "=" if and only if $X = U$. ☐

The first problem is how to bound the dimensions of the negative definite space and the nullity space of I. Consider the space

$$\{fT\colon f \text{ piecewise smooth on } [a,b],\ f(a) = f(b) = 0\}$$

where $T = \dot{\gamma}(0)$. Then

$$I(fT, fT) = \int_a^b \left\langle \nabla_T(fT), \nabla_T(fT) \right\rangle dt = \int_a^b \dot{f}^2(t) dt \ge 0$$

where "=" if and only if

$$f \equiv 0 \ \Rightarrow\ fT = 0$$

Besides, for all $U \in \mathcal{V}_0$ with $\langle U, T\rangle \equiv 0$,

$$\begin{aligned} I(fT, U) &= \int_a^b \langle \nabla_T(fT), \nabla_T U\rangle \, dt \\ &= \int_a^b \left\langle \dot{f}T, \nabla_T U \right\rangle dt \\ &= \int_a^b \frac{d}{dt}\left\langle \dot{f}T, U\right\rangle - \left\langle \ddot{f}T, U\right\rangle dt \\ &= \left\langle \dot{f}T, U\right\rangle \Big|_a^b = 0 \end{aligned}$$

which means that for all $W = fT + U$ with $U \perp T$, $I(W, W) = I(U, U)$.

$$\mathcal{V}_0 = \{fT\} \oplus \{U \in \mathcal{V}_0 \colon \langle U, T\rangle \equiv 0\}$$

So we only need to consider the space orthogonal to T,

$$\mathcal{V}_0^{\perp} = \{U \in \mathcal{V}_0 \colon \langle U, T\rangle \equiv 0\}$$

that is considering $I \colon \mathcal{V}_0^{\perp} \times \mathcal{V}_0^{\perp} \to \mathbb{R}$. But $\dim \mathcal{V}_0^{\perp}$ may also be infinite. So we need to reduce the dimension of $\mathcal{V}_0^{\perp}$ more. For each point at γ, we can find a totally normal neighborhood (which contains no conjugate points). So γ can be covered by totally normally neighborhoods. By compactness of γ, it can be covered by finitely many totally neighborhoods. So there is

$$a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$$

such that $\gamma|_{[t_i, t_{i+1}]}$ has no conjugate points. Let

$$T_1 := \left\{X \in \mathcal{V}_0^{\perp} : X \text{ is Jacobian along } \ \gamma|_{[t_i, t_{i+1}]} \text{ for all } i\right\}$$

Consider a linear map,

$$\varphi \colon T_1 \longrightarrow T_{\gamma(t_1)}M \times \cdots T_{\gamma(t_k)}M$$

is defined by $\varphi(J) = (J(t_1), \cdots, J(t_k))$. By **Lemma** 3.3.1, because $\gamma|_{[t_i, t_{i+1}]}$ has no conjugate points, the space of Jacobian fields along $\gamma|_{[t_i, t_{i+1}]}$ is determined by $T_{\gamma(t_i)}M \times T_{\gamma(t_{i+1})}M$, φ is an isomorphism. Therefore, $\dim T_1 = km < \infty$ by the following. Then for any $X \in \mathcal{V}_0^{\perp}$, $\{X(t_i)\}_{i=0}^{k+1}$ determine a $J_X \in T_1$ with $X - J_X$ vanishing at t_i. So by considering

$$T_2 = \left\{X \in \mathcal{V}_0^{\perp} \colon X(t_i) = 0, \ \forall \ i\right\}$$

we have

$$\mathcal{V}_0^{\perp} = T_1 \oplus T_2$$

because $T_1 \cap T_2 = \{0\}$ by the fact that $\gamma|_{[t_i, t_{i+1}]}$ has no conjugate points.

Lemma 3.3.3. *With above notations, we have*

(1) $I(X, Y) = 0$ *for any* $X \in T_1$, $Y \in T_2$.

(2) $I|_{T_2}$ *is positive definite.*

Proof. (1) For any $X \in T_1$, $Y \in T_2$, by above calculation, we have

$$I(X, Y) = -\sum_{i=1}^{n} \left\langle Y\left(t_i\right), \nabla_T X\left(t_i^{+}\right) - \nabla_T X\left(t_i^{-}\right)\right\rangle = 0$$

(2) For any $X \in T_2$, by above calculation,

$$\begin{aligned} I(X,X) &= \sum_i \int_{t_i}^{t_{i+1}} \langle \nabla_T X, \nabla_T X \rangle - R(X,T,X,T)dt \\ &= \sum_i I|_{\gamma|_{[t_i,t_{i+1}]}}(X|_{\gamma|_{[t_i,t_{i+1}]}}, X|_{\gamma|_{[t_i,t_{i+1}]}}) \geq 0 \end{aligned}$$

because $\gamma|_{[t_i,t_{i+1}]}$ has no conjugate points. □

Therefore, for negative definite and nullity space of I, we only need to consider

$$I\colon T_1 \times T_1 \longrightarrow \mathbb{R}$$

which means $\operatorname{ind}(\gamma) < \infty$ and $N(\gamma) < \infty$ because $\dim T_1 < \infty$.

Lemma 3.3.4. *Let* $\operatorname{ind}(\tau) = \operatorname{ind}(\gamma_{[a,\tau]})$.

(1) For any $\tau \in (a,b]$, there is a $\delta > 0$ such that

$$\forall\ \varepsilon \in [0,\delta], \quad \operatorname{ind}(\tau - \varepsilon) = \operatorname{ind}(\tau)$$

which mean $\operatorname{ind}(\tau)$ *is right-continuous.*

(2) For any $\tau \in [a,b)$, there is a $\delta > 0$ such that

$$\forall\ \varepsilon \in [0,\delta], \quad \operatorname{ind}(\tau + \varepsilon) = \operatorname{ind}(\tau) + N(\tau)$$

where $N(\tau) = N(\gamma|_{[a,\tau]})$, the nullity of I on $\gamma|_{[a,\tau]}$. So $\operatorname{ind}(\tau)$ *is not left-continuous.*

Remark. By the boundedness of index, this lemma directly implies the Morse Index Theorem.

Proof. Let $I^\tau = I|_{\gamma|_{[a,\tau]}}$. Let τ be fixed. WTLG, assume $\tau \in (t_j, t_{j+1})$ for some j (otherwise choosing a different cover). Let

$$T_1(\tau) = \Big\{ X \in \mathcal{V}_0^\perp(\tau) \colon X \text{ is Jacobian on } \ \gamma|_{[t_i,t_{i+1}]}\,,\ j = 0, \cdots, j-1, \text{ and on } \ \gamma|_{[t_j,\tau]} \Big\}$$

where $\mathcal{V}_0^\perp(\tau)$ is vector fields along $\gamma|_{[a,\tau]}$ and vanishing at a, τ and perpendicular to $\dot\gamma|_{[a,\tau]}$. Consider the linear isomorphism

$$\varphi^\tau \colon T_1(\tau) \longrightarrow T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_j)}M$$

Note the space of right-hand side does not change when $\tau \to \tau \pm \varepsilon$. Considering the symmetric bilinear form $I^\tau|_{T_1(\tau)}$, it can be viewed as bilinear form on $T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_j)}M$ by for any x, y in that

$$I^\tau(x,y) := I^\tau((\varphi^\tau)^{-1}(x), (\varphi^\tau)^{-1}(y))$$

Then for $\tau \to \tau \pm \varepsilon$, the base space of I^τ is fixed. For a fixed $x \in T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_j)}M$, consider the function

$$\tau \mapsto I^\tau(x,x)$$

Claim: This function is continuous *w.s.t.* τ.

Proof of Claim. Let $X = (\varphi^\tau)^{-1}(x), Y = (\varphi^\tau)^{-1}(y)$. By definition,

$$\begin{aligned} I^\tau(x,y) &= I^\tau(X,Y) \\ &= \sum_{i=0}^{j-1} \langle \nabla_T X, Y\rangle_{t_i}^{t_{i+1}} + \langle \nabla_T X, Y\rangle_{t_j}^{\tau} \end{aligned}$$

Let $X_j = X|_{[t_j,\tau]}$, $Y_j = Y|_{[t_j,\tau]}$. So when $\tau \to \tau \pm \varepsilon$, the only thing is needed to be considered is

$$\langle \nabla_T X_j, Y_j\rangle_{t_j}^{\tau} = -\left\langle \nabla_T X_j(t_j^+), Y_j(t_j)\right\rangle$$

note that $Y_j(t_j)$ is also independent with τ. So this function only dependents on $\nabla_T X_j(t_j^+)$. Besides, note that $X_j(t_j)$ is also fixed. Note the X_j is determined by $X_j(t_j)$ and $\nabla_T X_j(t_j^+)$ from a variation

$$F(t,s) = \exp_{\gamma(t_j)} t\,(V(s) + sW(s))$$

where $V(s)$ is given by $X(t_j)$ and $W(s) = \mathcal{P}_s(\nabla_T X_j(t_j^+))$. Because $X(\tau) = 0$, the variation is fixed at τ, that is $F(\tau,s) \equiv \gamma(\tau)$. So

$$V(s) + sW(s) = \exp^{-1}_{\gamma(t_j)}(\gamma(\tau)) \;\Rightarrow\; W(s) = \frac{\exp^{-1}_{\gamma(t_j)}(\gamma(\tau)) - V(s)}{W(s)}$$

and thus

$$\nabla_T X_j(t_j^+) = \mathcal{P}_{-s}\left(\frac{\exp^{-1}_{\gamma(t_j)}(\gamma(\tau)) - V(s)}{W(s)}\right)$$

which is continuity *w.s.t.* τ. $\square$

This claim directly implies

(i) $\mathrm{ind}(\tau \pm \varepsilon) \geq \mathrm{ind}(\tau)$, because $I^\tau(x,x) < 0$ implies $I^{\tau\pm\varepsilon}(x,x) < 0$.

(ii) $\mathrm{ind}_+(\tau \pm \varepsilon) \geq \mathrm{ind}_+(\tau)$, where ind_+ is the dimension of positive definite space of I^τ in $T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_j)}M$,because $I^\tau(x,x) > 0$ implies $I^{\tau\pm\varepsilon}(x,x) > 0$.

(iii) Because $\mathrm{ind}_+ = nj - \mathrm{ind} - \mathrm{null}$,

$$\mathrm{ind}(\tau \pm \varepsilon) \leq \mathrm{ind}(\tau) + N(\tau) - N(\tau \pm \varepsilon) \leq \mathrm{ind}(\tau) + N(\tau)$$

So we have

$$\mathrm{ind}(\tau) \leq \mathrm{ind}(\tau \pm \varepsilon) \leq \mathrm{ind}(\tau) + N(\tau)$$

Next, choosing a $\xi \in (t_j,\tau)$. Let $x = (x_1,\cdots,x_j) \in T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_j)}M$. By **Lemma** 3.3.2,

$$I^\xi(x,x) - I^\tau(x,x) = I^\xi(X|_{[t_j,\xi]}, X|_{[t_j,\xi]}) - I^\tau(X|_{[t_j,\tau]}, X|_{[t_j,\tau]}) \geq 0$$

because $X|_{[t_j,\tau]}$ is a Jacobian field, where "=" if and only if $X|_{[t_j,\xi]} = X|_{[t_j,\tau]} = 0$, that is $x_j = 0$.

(1) So

$$I^\xi(x,x) < 0 \;\Rightarrow\; I^\tau(x,x) < 0$$

which means $\mathrm{ind}(\xi) \leq \mathrm{ind}(\tau)$. Combining this with above result, we have

$$\mathrm{ind}(\tau - \varepsilon) = \mathrm{ind}(\tau)$$

(2) Next, we need to show

$$\operatorname{ind}(\tau+\varepsilon) \geq \operatorname{ind}(\tau) + N(\tau)$$

Because $\operatorname{Neg}(\tau) \subset \operatorname{Neg}(\tau+\varepsilon)$, where $\operatorname{Neg}(\tau)$ is the negative definite space of I^τ, it is sufficient to show $N(\tau) \subset \operatorname{Neg}(\tau+\varepsilon)$. By using notation as $\xi < \tau$, we only need to show $N(\xi) \subset \operatorname{Neg}(\tau)$. Let $x \neq 0$ in the null space of I^ξ in $T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_j)}M$.

$$I^\xi(x,y) = 0, \quad \forall\, y \in T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_j)}M$$

So $I^\xi(X,Y) = 0$ for all $\mathcal{V}_0(\xi)$ and thus X is a smooth Jacobian field on $[a,\xi]$. It follows that $x_j = X(t_j) \neq 0$ (otherwise, $x = 0$). Therefore,

$$0 = I^\xi(x,x) > I^\tau(x,x)$$

Thus, $x \in N(\tau)$. □

Theorem 3.3.4 (Cartan-Hadamard)**.** *A complete, simply connected, m-dimensional Riemannian manifold (M,g) with the sectional ≤ 0 is diffeomorphic to $\mathbb{R}^m$. More precisely, $\exp_p\colon T_pM \to M$ is a diffeomorphism.*

Proof. The aim is to prove that when (M,g) is complete,

$$\exp_p\colon T_pM \longrightarrow M$$

is a Riemannian covering map, where T_pM is equipped with metric $\exp_p^* g$, that is for any $X, Y \in T_v(T_pM)$,

$$\exp_p^* g(X,Y) = g((d\exp_p)_v(X), (d\exp_p)_v(Y))$$

which is well-defined because

$$\begin{aligned} \text{the sectional} \leq 0 \quad &\Rightarrow \quad I \text{ is always positive definite} \\ &\Rightarrow \quad \text{no conjugate points} \\ &\Rightarrow \quad (d\exp_p)_v \text{ is always injective} \end{aligned}$$

First, $\exp_p$ is clearly a local isometry by definition. To show that it is a Riemannian covering map, we only need to check the completeness, because of the **Theorem** 1.7.4. Note that $\exp_p(tX)$ is a geodesic in M for any radical line tX in T_pM. Because $\exp_p$ is locally isometric, tX is also a geodesic through 0 in T_pM. By the completeness of M, $t \in \mathbb{R}$. So tX is a geodesic that can approach infinite. It follows that $(T_pM, \exp_p^* g)$ is complete. Then when M is simply complete, $\exp_p$ is diffeomorphic. □

4. **Structure of Complete Manifold:** Let (M,g) be a complete Riemannian manifold and $p \in M$. Then

$$M = \exp_p(E(p)) \sqcup C(p)$$

Let $\gamma : [0,\infty) \to M$ be a geodesic with $\gamma(0) = p, \dot\gamma(0) = v$. Let $q = \gamma(a)$ be a cut point. By **Theorem** 1.6.2, the following statement would happen. $d\exp_p$ is singular at av, which is equivalent to q is a conjugate point of p. Moreover, between p and q, there is no conjugate point. Otherwise, there is a $X \in \mathcal{V}_0$ such that $I(X,X) < 0$, which means there is a variation given by X such that

$$E''(0) = I(X,X) < 0$$

It follows γ cannot be with the shortest length, contradicting to the definition of cut point. So q is the first conjugate point. So $d\exp_p$ is non-singular on $E(p)$. For the map,

$$\exp_p \colon E(p) \longrightarrow \exp_p(E(p))$$

is one-one and non-singular at each point. Furthermore, we can prove $E(p)$ is open. To do that, let $S_p \subset T_pM$ be all unit vectors. Recalling

$$\tau \colon S_p \longrightarrow \mathbb{R} \cup \{\infty\}$$

defined as

$$\tau(v) := \begin{cases} a, & \text{if } \exp_p(av) \text{ is a cut point} \\ \infty, & \text{if no cut point} \end{cases}$$

Theorem 3.3.5. *τ is continuous when $\mathbb{R} \cup \{\infty\}$ has equipped the canonical topology.*

Proof. For any sequence $\{X_n\} \subset S_p$ with $X = \lim_n X_n$, the goal is to prove

$$\lim_n \tau(X_n) = \tau(X)$$

I. $\tau(X) \geq \limsup_n \tau(X_n)$: First, $\{\tau(X_n)\} \subset \mathbb{R} \cup \{\infty\}$ that is a compact space, so it has a subsequence $\tau(X_{n_k})$ such that $\tau(X_{n_k}) \to \alpha$. Assume $\alpha < \infty$. Then by the continuity of γ

$$\lim_k d(p, \exp_p(\tau(X_{n_k})X_{n_K})) = d(p, \exp_p(\alpha X))$$

Because $\exp_p(\tau(X_{n_k})X_{n_K})$ is a cut point, $d(p, \exp_p(\tau(X_{n_k})X_{n_K})) = \tau(X_{n_k})$. It follows that

$$d(p, \exp_p(\alpha X)) = \lim_k \tau(X_{n_k}) = \alpha$$

So $\tau(X) \geq \alpha$. If $\alpha = \infty$, that is for any t, there is an N such that $\tau(X_{n_k}) > t$ for all $k > N$. So

$$(p, \exp_p(tX)) = \lim_k d(p, \exp_p(tX_{n_K})) = t$$

Therefore, $\tau(X) > t$ for all t, so $\tau(X) = \infty$, *i.e.* $\tau(X) \geq \alpha$.

II. $\tau(X) \leq \liminf_n \tau(X_n)$: For any convergence subsequence $\tau(X_{n_k})$ with $\tau(X_{n_k}) \to \alpha$, the aim is to prove

$$\tau(X) \leq \alpha$$

Let's assume $\tau(X) > \alpha$, *i.e.* $\exp_p(\alpha X)$ is not a cut point. By **Theorem** 1.6.2, there is a unique shortest geodesic connecting p and $\exp_p(\alpha X)$ and $d\exp_p$ is non-singular at αX. It follows that the is a neighborhood U of αX in T_pM such that $\exp_p$ is diffeomorphic on U. Besides, because $\tau(X_{n_k}) \to \alpha X$, WTLG, assume $\{\tau(X_{n_k})X_{n_k}\} \subset U$. Because $d\exp_p$ is non-singular at each $\{\tau(X_{n_k})X_{n_k}\}$. By **Theorem** 1.6.2, for each $\tau(X_{n_k})X_{n_k}$, there are two shortest geodesics connecting p with $\exp_p(\tau(X_{n_k})X_{n_k})$. So there is a $Y_{n_k} \in S_p$ such that

$$\exp_p(\tau(X_{n_k})X_{n_k}) = \exp_p(\tau(X_{n_k})Y_{n_k})$$

which means $\tau(X_{n_k})Y_{n_k} \notin U$ because $\exp_p$ is one-one on U. Since S_p is compact, there is a subsequence $Y_{n_{k_i}}$ with $Y_{n_{k_i}} \to Y \in S_p$. So

$$\lim_i \tau(X_{n_{k_i}})Y_{n_{k_i}} = \alpha Y$$

Because U is open, $\alpha Y \notin U$ and so $\alpha X \neq \alpha Y$. Furthermore,

$$\exp_p(\alpha Y) = \lim_i \exp_p(\tau(X_{n_{k_i}})Y_{n_{k_i}}) = \lim_i \exp_p(\tau(X_{n_{k_i}})X_{n_{k_i}}) = \exp_p(\alpha X) = q$$

which means there are two shortest geodesics connecting p with $\exp_p(\alpha X)$, contracting to the assumption. $\square$

Because τ is continuous,

$$\tilde{C}(p) = \{\tau(X)X\colon X \in S_p\}$$

is closed. So $E(p) = T_pM \backslash \tilde{C}(p)$ is open. Then

$$\exp_p\colon E(p) \longrightarrow \exp_p(E(p))$$

is a diffeomorphism because it is an injective local diffeomorphism by the fact that $\dim E(p) = \dim M$.

3.4 Properties with Sectional Curvature

1. ***Space form:*** A Riemannian manifold (M, g) is called a space form if it is complete and has the constant sectional curvature.

   Theorem 3.4.1. *For any $c \in \mathbb{R}$ and any $n \in \mathbb{Z}_+$, there exists a unique simply-connected n-dimensional complete Riemannian manifold with constant curvature c*

   Proof. First,

   $$\Gamma_{ij}^m = \frac{1}{2}g^{ml}\left(g_{jl,i} + g_{li,j} - g_{ij,l}\right) \quad \Rightarrow \quad \Gamma_{ij}^m(kg) = \Gamma_{ij}^m(g)$$

   for any positive constant k. It follows that the curvature tensor R is invariant with respect to the scaling of Riemannian metric g by $R_{lij}^m = \frac{\partial \Gamma_{jl}^m}{\partial x^i} - \frac{\partial \Gamma_{il}^m}{\partial x^j} + \Gamma_{jl}^p \Gamma_{ip}^m - \Gamma_{il}^p \Gamma_{jp}^m$. So the Riemannian curvature tensor $R(kg) = kR(g)$ and by definition the sectional curvature satisfies

   $$K(kg) = \frac{1}{k}K(g)$$

   for any positive constant k. Therefore, by scaling the Riemannian metric, we only need to assume $c = 0, 1, -1$. The existence is clear because we have already seen that the Euclidean space, sphere, and hyperbolic space have the constant curvature of 0, 1, and -1 respectively. So the important thing is the uniqueness, which can can be done by the following theorem. $\square$

   Theorem 3.4.2. *Assume (M, g) and $(\widetilde{M}, \widetilde{g})$ are two simply-connected n-dimensional complete Riemannian manifold with the same constant curvature $c \in \mathbb{R}$. Let $p \in M$ and $\tilde{p} \in \widetilde{M}$. Let $\{e_i\}$ be a orthonormal basis of T_pM and $\{\tilde{e}_i\}$ be a orthonormal basis of $T_{\tilde{p}}\widetilde{M}$. Then there exists a unique isometry $\varphi\colon M \to \widetilde{M}$ such that $\varphi(p) = \tilde{p}$ and $d\varphi_p(e_i) = \tilde{e}_i$.*

   Remark. Similarly, by scaling the Riemannian metric, we can assume $c = 0, 1, -1$. Besides, for $c = 0, -1$, by the above Cartan-Hadamard theorem, they are diffeomorphic to the Euclidean space.

Lemma 3.4.1. *(M,g) and $(\widetilde{M},\widetilde{g})$ are as same as the statements of above theorem. Let $p \in M$ and $\tilde{p} \in \widetilde{M}$. Suppose $\exp_p\colon \mathcal{V} \subset T_pM \to \exp_p(\mathcal{V}) = \mathcal{U} \subset M$ is a diffeomorphism. Let $\Phi\colon T_pM \to T_{\tilde{p}}\widetilde{M}$ be a linear isometry such that $\exp_{\tilde{p}}\colon \Phi(\mathcal{V}) \subset T_{\tilde{p}}\widetilde{M} \to \widetilde{M}$ is well-defined. Then there exists a local isometry*

$$\varphi\colon U \subset M \longrightarrow \exp_{\tilde{p}}\left(\Phi(\mathcal{V})\right) \subset \widetilde{M}$$

defined by

$$\varphi = \exp_{\tilde{p}} \circ \Phi \circ \exp_p^{-1}$$

Furthermore, $\varphi(p) = \tilde{p}$ and $d\varphi_p = \Phi$.

Remark. When $c = 0, -1$, by the Cartan-Hadamard theorem, the exponential map is always diffeomorphic, so φ is a diffeomorphism and this lemma would imply the existence of φ for the above theorem.

Proof. First, $\varphi(p) = \tilde{p}$ and by $(d\exp_p)_O = (d\exp_{\tilde{p}})_{\tilde{O}} = I$

$$d\varphi_p = \Phi$$

It is sufficient to prove φ is a local isometry, *i.e.* for any $q \in \mathcal{U}$ and any $X \in T_qM$,

$$\widetilde{g}(d\varphi_q(X), d\varphi_q(X)) = \widetilde{g}(\Phi(X), \Phi(X)) = g(X,X)$$

Note that if $q = p$, then it is clearly true. Assume $q \neq p$. Let $V_p = \exp_p^{-1}(q) \in T_pM$, *i.e.* the geodesic

$$\gamma(t) = \exp_p(tV_p), \quad t \in [0,1]$$

connects p with q. Then

$$\varphi(q) = \exp_{\tilde{p}}(\Phi(V_p))$$

Then the geodesic

$$\widetilde{\gamma}(t) = \exp_{\tilde{p}}(t\Phi(V_p)), \quad t \in [0,1]$$

connects $\tilde{p}$ with $\varphi(q)$. Choose a Jacobian field U along γ such that $U(0) = 0$ and $U(1) = X$, which uniquely exists because γ has no conjugate point by **Lemma** 3.3.1. Then let

$$F(t,s) = \exp_p t(V_p + sW)$$

be the corresponding variation, where

$$U(1) = (d\exp_p)_{V_p}(W) = X, \quad \Rightarrow \quad W = \left((d\exp_p)_{V_p}\right)^{-1}(X)$$

Note that $W = \nabla_{V_b} U(0)$. Other hand, let

$$\widetilde{F}(t,s) = \varphi \circ F(t,s) = \exp_{\tilde{p}}\left(t\left(\Phi(V_p) + s\Phi(W)\right)\right)$$

So $\widetilde{F}$ is a variation of $\widetilde{\gamma}$ with the Jacobian field

$$\widetilde{U}(t) = \frac{\partial}{\partial s}\widetilde{F}(t,0), \quad \widetilde{U}(0) = 0$$

Furthermore,

$$\begin{aligned}
\widetilde{U}(1) &= \left.\frac{\partial}{\partial s}\right|_{t=1,s=0} \varphi \circ F(t,s) \\
&= d\varphi_q\left(\left.\frac{\partial}{\partial s}\right|_{t=1,s=0} F(t,s)\right) \\
&= d\varphi_q(U(1)) \\
&= d\varphi_q(X)
\end{aligned}$$

and

$$\widetilde{\nabla}_{\Phi(V_p)}\widetilde{U}(0) = \Phi(W)$$

Choose an orthonormal basis $\{e_i\}$ of T_pM. Then because Φ is an isometry, $\tilde{e}_i = \Phi(e_i)$ is an orthonormal basis of $T_{\tilde{p}}\widetilde{M}$, where. Let $E_i(t)$ and $\widetilde{E}_i(t)$ be the orthonormal frames by moving e_i and $\tilde{e}_i$ in parallel along γ and $\widetilde{\gamma}$ respectively. Then

$$U(t) = f^i(t)E_i(t), \quad \widetilde{U}(t) = \widetilde{f}^i(t)\widetilde{E}_i(t)$$

So it is sufficient to prove

$$\sum_i \left(f^i(1)\right)^2 = \sum_i \left(\widetilde{f}^i(1)\right)^2$$

Note that because U is Jacobian

$$\frac{d^2}{dt^2}f^i(t) + f^i(t)R(E_j, V_b, E_i, V_b) = 0$$

Because M has the constant curvature,

$$\begin{aligned} R(E_j, V_b, E_i, V_b) &= c\left(g(E_j, E_i)g(V_b, V_b) - g(E_j, V_b)g(E_i, V_b)\right) \\ &= c\left(g(e_j, e_i)g(V_b, V_b) - g(e_j, V_b)g(e_i, V_b)\right) \end{aligned}$$

Then

$$\frac{d^2}{dt^2}f^i(t) + cf^i(t)\left(g(e_j, e_i)g(V_b, V_b) - g(e_j, V_b)g(e_i, V_b)\right) = 0$$

Besides, its initial values satisfy

$$f^i(0) = 0, \quad \left.\frac{d}{dt}\right|_{t=0} f^i(t) = g(W, e_i)$$

Similarly, for $\widetilde{U}$,

$$\frac{d^2}{dt^2}\widetilde{f}^i(t) + c\widetilde{f}^i(t)\left(\widetilde{g}(\tilde{e}_j, \tilde{e}_i)\widetilde{g}(\Phi(V_b), \Phi(V_b)) - \widetilde{g}(\tilde{e}_j, \Phi(V_b))\widetilde{g}(\tilde{e}_i, \Phi(V_b))\right) = 0$$

i.e.

$$\frac{d^2}{dt^2}\widetilde{f}^i(t) + c\widetilde{f}^i(t)\left(g(e_j, e_i)g(V_b, V_b) - g(e_j, V_b)g(e_i, V_b)\right) = 0$$

with the initial values

$$\widetilde{f}^i(0) = 0, \quad \left.\frac{d}{dt}\right|_{t=0} \widetilde{f}^i(t) = \widetilde{g}(\Phi(W), \tilde{e}_i) = g(W, e_i)$$

Therefore, $f^i(t)$ and $\widetilde{f}^i(t)$ satisfy the same ODE with same initial values. It follows that $f^i(t) = \widetilde{f}^i(t)$. □

Lemma 3.4.2. *(M, g_M) and (N, g_N) are two Riemannian manifolds. Let $\varphi_1, \varphi_2 \colon M \to N$ be two local isometry such that there is $x \in M$ with*

$$\varphi_1(x) = \varphi_2(x), \quad (d\varphi_1)_x = (d\varphi_2)_x$$

Then $\varphi_1 = \varphi_2$.

Proof. Let

$$A = \{z \in M \colon \varphi_1(z) = \varphi_2(z), \quad (d\varphi_1)_z = (d\varphi_2)_z\}$$

Clearly, $A \neq \varnothing$ and A is closed. So the main goal is to prove A is open for proving $A = M$. Suppose $z \in A$. Denote $z' = \varphi_1(z) = \varphi_2(z) \in N$. Choose $\delta > 0$ such that

$$\exp_z \colon B(O, \delta) \subset T_z M \longrightarrow B_z(\delta) \subset M$$

is a diffeomorphism and also $\exp_{z'}$ is well-defined on $B'(O, \delta) \subset T_{z'} N$. Then

$$\begin{array}{ccc} B(O,\delta) \subset T_z M & \xrightarrow{(d\varphi_i)_z} & B'(O,\delta) \subset T_{z'}N \\ \Big\downarrow \exp_z & & \Big\downarrow \exp_{z'} \\ B_z(\delta) \subset M & \xrightarrow{\quad\varphi_i\quad} & N \end{array}$$

is commutative for $i = 1, 2$, that is

$$\varphi_i \circ \exp_z = \exp_{z'} \circ (d\varphi_i)_z$$

by **Proposition** 1.7.2. So

$$\varphi_i = \exp_{z'} \circ (d\varphi_i)_z \circ \exp_z^{-1}$$

on $B_z(\delta)$. Because $(d\varphi_1)_z = (d\varphi_2)_z$ on $B(O, \delta) \subset T_z M$,

$$\varphi_1 = \varphi_2$$

on $B_z(\delta)$, which also implies $(d\varphi_1)_y = (d\varphi_2)_y$ for all $y \in B_z(\delta)$. □

Therefore, we have proved the **Theorem** 3.4.2 for $c = 0, -1$ by **Lemma** 3.4.1 and 3.4.2. So we only need to prove the case of $c = 1$.

*Proof of **Theorem** 3.4.2.* We can assume $M = \mathbb{S}^n$. For $p \in \mathbb{S}^n$, let p' be its antipodal point. Then

$$\exp_p \colon B(O, \pi) \subset T_p\mathbb{S}^n \longrightarrow \mathbb{S}^n \backslash \{p'\}$$

is a diffeomorphism. Let Φ be the linear map such that $\Phi(e_i) = \tilde{e}_i$, so it is an isometry. By **Lemma** 3.4.1, because $\exp_{\tilde{p}}$ is well-defined on $T_{\tilde{p}}\widetilde{M}$,

$$\varphi = \exp_{\tilde{p}} \circ \Phi \circ (\exp_p)^{-1} \colon \mathbb{S}^n \backslash \{p'\} \longrightarrow \widetilde{M}$$

is a local isometry with $\varphi(p) = \tilde{p}$ and $d\varphi_p = \Phi$. Find $z \in \mathbb{S}^n \backslash \{p'\}$ with $z \neq p$ and let z' be the antipodal point of z. Denote $\tilde{z} = \varphi(z) \in \widetilde{M}$. Then

$$d\varphi_z \colon T_z\mathbb{S}^n \longrightarrow T_{\tilde{z}}\widetilde{M}$$

is an isometry. Then similarly by **Lemma** 3.4.1,

$$\psi = \exp_{\tilde{z}} \circ d\varphi_z \circ (\exp_z)^{-1} \colon \mathbb{S}^n \backslash \{z'\} \longrightarrow \widetilde{M}$$

is a local isometry. Furthermore, because $\varphi(z) = \psi(z) = \tilde{z}$ and $d\psi_z = d\varphi_z$, by **Lemma** 3.4.2, $\varphi = \psi$ on $\mathbb{S}^n \backslash \{p', z'\}$. Then define

$$\theta \colon \mathbb{S}^n \longrightarrow \widetilde{M}$$

as

$$\theta(q) = \begin{cases} \varphi(q), & q \in \mathbb{S}^n \backslash \{p'\} \\ \psi(q), & q \in \mathbb{S}^n \backslash \{z'\} \end{cases}$$

Then clearly θ is C^∞ and a local isometry. Because $\mathbb{S}^n$ is complete, θ is a Riemannian covering by **Theorem** 1.7.4. Furthermore, because $\widetilde{M}$ is simply-connected, θ is a diffeomorphism. So θ is an isometry. And the uniqueness of θ is also by **Lemma** 3.4.2. □

Corollary 3.4.1. *Let (M,g) be a n-dimensional simply-connected complete Riemannian manifold. Then (M,g) has the constant sectional curvature if and only if for any $p, \tilde{p} \in M$ and any two orthonormal bases $\{e_i\}, \{\tilde{e}_i\}$ of $T_pM, T_{\tilde{p}}M$ respectively, there is an isometry $\varphi\colon M \to M$ such that $\varphi(p) = \tilde{p}$ and $d\varphi_p(e_i) = \tilde{e}_i$.*

Proof. The converse is because any section of M has the same geometry by constructing the isometry φ. □

Definition 3.4.1 (Homogeneous Riemannian Manifolds)**.** A Riemannian manifold (M,g) is called homogeneous if for any $p, q \in M$, there is an isometry $\varphi\colon M \to M$ such that $\varphi(p) = q$. (M,g) is called two-points homogeneous if for any $p_1, p_2, q_1, q_2 \in M$ with $d(p_1,p_2) = d(q_1,q_2)$, there is an isometry $\varphi\colon M \to M$ such that $\varphi(p_1) = q_1$ and $\varphi(p_2) = q_2$.

Corollary 3.4.2. *All simply-connected space forms are two-points homogeneous.*

Proof. By the completeness, there is a shortest geodesic γ connecting p_1 with p_2 and a shortest geodesic $\widetilde{\gamma}$ connecting q_1 with q_2. Let $e_1 = \dot{\gamma}(0)$ and extending it to an orthonormal basis $\{e_i\}$ of $T_{p_1}M$. Similarly, choose an orthonormal basis $\{\tilde{e}_i\}$ of $T_{q_1}M$. Then there is an isometry φ such that

$$\varphi(p_1) = q_1, \quad d\varphi_{p_1}(e_i) = \tilde{e}_i$$

Because γ is a geodesic, $\varphi \circ \gamma$ is also a geodesic. By

$$\varphi \circ \gamma(0) = q_1 = \widetilde{\gamma}(0), \qquad \left.\frac{d}{dt}\right|_{t=0} \varphi \circ \gamma(t) = d\varphi_{p_1}(e_1) = \tilde{e}_1 = \dot{\widetilde{\gamma}}(0)$$

we know $\varphi \circ \gamma = \widetilde{\gamma}$. So

$$\varphi(p_2) = \varphi \circ \gamma(1) = \widetilde{\gamma}(1) = q_2$$

□

2. ***Distance function:*** Let (M,g) be a Riemannian manifold. Given $o \in M$, define the distance function

$$\rho(\cdot) = d(o, \cdot)$$

Definition 3.4.2. We call a function $f\colon M \to \mathbb{R}$ is convex if for any geodesic $\gamma\colon [a,b] \to M$, $f \circ \gamma$ is convex.

Proposition 3.4.1. *Let $f\colon M \to \mathbb{R}$ be C^∞. Then f is convex if and only if*

$$\operatorname{Hess} f \geq 0$$

Proof. For any $p \in M$ and for any $V_p \in T_pM$, let $\gamma(t)$ be the geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = V_p$.

$$\begin{aligned}
\operatorname{Hess} f(V_p, V_p) &= \nabla^2 f(V_p, V_p) \\
&= \nabla^2 f(\dot{\gamma}(t), \dot{\gamma}(t))\big|_{t=0} \\
&= \left(\nabla_{\dot{\gamma}(t)}(\nabla f)\right)(\dot{\gamma}(t)) \\
&= \dot{\gamma}(t)\left((\nabla f)(\dot{\gamma}(t))\right) - (\nabla f)\left(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\right) \\
&= \dot{\gamma}(t)\left(\dot{\gamma}(t)(f)\right)\big|_{t=0} \\
&= \left.\frac{d}{dt}\right|_{t=0} \dot{\gamma}(t)(f)(\gamma(t)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\left.\frac{d}{dt}\right| f(\gamma(t))\right) = \left.\frac{d^2}{dt^2}\right|_{t=0} f(\gamma(t))
\end{aligned}$$

Therefore, it is clear by the property of convexity of real functions. □

Similarly, we call a C^∞ function $f\colon M \to \mathbb{R}$ is convex if $\operatorname{Hess} f > 0$.

Example 3.4.1. On Euclidean space, for $X = X^i \frac{\partial}{\partial x^i}$,

$$\begin{aligned}
\nabla^2 \rho^2(X, X) &= X^i X^j \frac{\partial^2}{\partial x^i \partial x^j} \sum_k (x^k)^2 \\
&= 2X^i X^j \delta_{ij} \\
&= 2\sum (X^i)^2 > 0
\end{aligned}$$

So ρ^2 is strictly convex.

Lemma 3.4.3. *Let (M, g) be a Riemannian manifold and $o \in M$. There is a neighborhood U of o such that ρ^2 is C^∞ and strictly convex on U.*

Proof. First, choosing (x, U) be the normal chart of $o = (0, \cdots, 0)$, geodesics are radical lines. So for any $x = (x^1, \cdots, x^n) \in U$,

$$\rho^2(x) = d^2(o, x) = \sum_i (x^i)^2$$

ρ^2 is smooth. Moreover, choosing a geodesic $\gamma(t)$ with $\dot\gamma(0) = V = (v^i)$,

$$x(t) = x(\gamma(t)) = (tv^1, \cdots, tv^i)$$

Then by the similar calculation as above,

$$\begin{aligned}
\nabla^2 \rho^2(V, V) &= \left.\frac{d^2}{dt^2}\right|_{t=0} \rho^2 \circ \gamma(t) \\
&= \left.\frac{d^2}{dt^2}\right|_{t=0} \sum_i (tv^i)^2 \\
&= 2\sum_i (v^i)^2 > 0
\end{aligned}$$

Therefore, $\nabla^2\rho^2 > 0$ at o. It follows that $\nabla^2\rho^2 > 0$ on a $V \subset U$. $\square$

Theorem 3.4.3. *Let (M, g) be a simply-connected complete Riemannian manifold with sectional curvature ≤ 0 and $o \in M$. $\rho^2\colon M \to \mathbb{R}$ is C^∞ and strictly convex*

Proof. First, because $\exp_o\colon T_oM \to M$ is a diffeomorphism, by

$$\rho^2(x) = d^2(o, x) = g\left(\exp_o^{-1}(x), \exp_o^{-1}(x)\right)$$

ρ^2 is smooth. For any $x \in M$ and any nonzero $V \in T_pM$, we claim

$$\nabla^2 \rho_x(V, V) \geq 0$$

Let ξ be a geodesic with $\xi(0) = x$ and $\dot\xi(0) = V$. By considering a family of geodesics connecting o with $\xi(s)$, the variation

$$F(t, s) = \exp_o \frac{t}{r} \exp_o^{-1}(\xi(s))$$

where $\rho(x) = r$, $t \in [0, r]$. Note that $F(t,0) = \gamma(t) = \exp_o t\frac{\exp_o^{-1}(x)}{r}$ is a normal geodesic because $|\dot\gamma(0)| = 1$. Then the corresponding Jacobian field

$$U(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} F(t,s)$$

which has

$$U(0) = 0, \quad U(r) = \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_o \exp_o^{-1}(\xi(s)) = \dot\xi(0) = V$$

Then

$$\begin{aligned}
\nabla^2 \rho_x^2(V,V) &= \left.\frac{d^2}{ds^2}\right|_{s=0} \rho^2 \circ \xi(s) \\
&= \left.\frac{d}{ds}\right|_{s=0} 2\rho(\xi(s)) \left(\frac{d}{ds}\rho(\xi(s))\right) \\
&= 2\rho(\xi(0)) \left.\frac{d^2}{ds^2}\right|_{s=0} \rho(\xi(s)) + 2\left(\left.\frac{d}{ds}\right|_{s=0} \rho(\xi(s))\right)^2 \\
&= 2r \left.\frac{d^2}{ds^2}\right|_{s=0} \rho(\xi(s)) + 2\left(\left.\frac{d}{ds}\right|_{s=0} \rho(\xi(s))\right)^2
\end{aligned}$$

Moreover, note that

$$\rho(\xi(s)) = \text{Length}\,(\gamma_s(t))\,, \quad \gamma_s(t) = F(t,s),\ t \in [0,r]$$

Let $L(s) = \rho(\xi(s))$, *i.e.*

$$L(s) = \int_0^r \sqrt{\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle} dt$$

Then by the similar calculation of the first and second variation of energy, we can get

$$\begin{aligned}
L'(0) &= \int_0^r \left.\frac{\left\langle \frac{\partial F}{\partial t}, \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}\right\rangle}{\sqrt{\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle}}\right|_{s=0} dt \\
&= \int_0^r \left.\left\langle \frac{\partial F}{\partial t}, \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}\right\rangle\right|_{s=0} dt = \int_0^r \left.\left\langle \frac{\partial F}{\partial t}, \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s}\right\rangle\right|_{s=0} dt \\
&= \int_0^r \langle T, \nabla_T U\rangle\, dt = \int_0^r T\langle T, U\rangle\, dt = \int_0^r \frac{d}{dt}\langle T, U\rangle\, dt \\
&= \langle T(r), U(r)\rangle = \langle \dot\gamma(r), V\rangle
\end{aligned}$$

and

$$\begin{aligned}
L''(0) &= \int_0^r \left.\frac{d}{ds}\right|_{s=0} \left\langle \frac{\partial F}{\partial t}, \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}\right\rangle - \langle T, \nabla_T U\rangle^2\, dt \\
&= \int_0^r \langle \nabla_T U, \nabla_T U\rangle - R(U,T,U,T) - (T\langle T, U\rangle)^2\, dt + \left.\left\langle \widetilde{\nabla}_{\frac{\partial}{\partial s}} \left.\frac{\partial F}{\partial s}\right|_{s=0}, T\right\rangle\right|_0^r \\
&= \int_0^r \langle \nabla_T U, \nabla_T U\rangle - R(U,T,U,T) - (T\langle T, U\rangle)^2\, dt
\end{aligned}$$

because at $t = 0$, $F(s,t) = p$ implies $\widetilde{\nabla}_{\frac{\partial}{\partial s}} U(0) = 0$ and at $t = r$, $F(s,r) = \xi(s)$ that is a geodesic implies that

$$\widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial s}(r,s) = \widetilde{\nabla}_{\frac{\partial}{\partial s}} \dot\xi(s) = 0$$

Besides, let $U^\perp = U - \langle U, T\rangle T$. Note that by the anti-symmetry of R,

$$R(U^\perp, T, U^\perp, T) = R(U, T, U, T)$$

Furthermore,

$$\begin{aligned}\langle \nabla_T U, \nabla_T U\rangle &= \left\langle \nabla_T U^\perp, \nabla_T U^\perp\right\rangle + 2\left\langle \nabla_T(\langle U, T\rangle T), \nabla_T U^\perp\right\rangle + \langle \nabla_T(\langle U, T\rangle T), \nabla_T(\langle U, T\rangle T)\rangle \\ &= \left\langle \nabla_T U^\perp, \nabla_T U^\perp\right\rangle + 2T\langle U, T\rangle \left\langle T, \nabla_T U^\perp\right\rangle + (T\langle U, T\rangle)^2\end{aligned}$$

because $\nabla_T T = 0$, $\langle T, T\rangle = 1$. And

$$\begin{aligned}\left\langle T, \nabla_T U^\perp\right\rangle &= \langle T, \nabla_T U\rangle - \langle T, \nabla_T(\langle U, T\rangle T)\rangle \\ &= \langle T, \nabla_T U\rangle - T\langle U, T\rangle \\ &= -\langle \nabla_T T, U\rangle = 0\end{aligned}$$

It follows that

$$\langle \nabla_T U, \nabla_T U\rangle = \left\langle \nabla_T U^\perp, \nabla_T U^\perp\right\rangle + (T\langle U, T\rangle)^2$$

Thus,

$$L''(0) = \int_0^r \left\langle \nabla_T U^\perp, \nabla_T U^\perp\right\rangle - R(U^\perp, T, U^\perp, T)dt \geq \int_0^r \left\langle \nabla_T U^\perp, \nabla_T U^\perp\right\rangle dt$$

because $R(U^\perp, T, U^\perp, T) \leq 0$ by the sectional curvature ≤ 0. Then

$$\begin{aligned}\nabla^2 \rho_x(V, V) &= 2r \left.\frac{d^2}{ds^2}\right|_{s=0} L(s) + 2\left(\left.\frac{d}{ds}\right|_{s=0} L(s)\right)^2 \\ &\geq 2r\int_0^r \left\langle \nabla_T U^\perp, \nabla_T U^\perp\right\rangle dt + 2\langle \dot\gamma(r), V\rangle^2\end{aligned}$$

First, if $\langle \dot\gamma(r), V\rangle > 0$, then $\nabla^2\rho_x(V, V) > 0$ and we have the desired result. Otherwise, if $\langle \dot\gamma(r), V\rangle = 0 = \langle \dot\gamma(r), U(r)\rangle$, then because $\langle \dot\gamma(0), U(0)\rangle = 0$, $\langle U, T\rangle \equiv 0$. It follows that $U^\perp = U$ and

$$\nabla^2\rho_x(V, V) \geq 2r\int_0^r \langle \nabla_T U, \nabla_T U\rangle dt$$

If $\nabla^2\rho_x(V, V) = 0$, then $\nabla_T U = 0$. Because $U(0) = 0$, $U \equiv 0$ and $U(r) = V = 0$, contradicting to the assumption. Therefore, $\nabla^2\rho_x(V, V) > 0$. □

Definition 3.4.3. Let (M, g) be a Riemannian manifold. A subset $\Omega \subset M$ is called convex if whereas $p, q \in \Omega$ and γ is the shortest geodesic, $\gamma \subset \Omega$. Ω is called totally convex if for any $p, q \in \Omega$ and any geodesic γ connecting p, q, $\gamma \subset \Omega$.

Remark. A simply connected complete Riemannian manifold with non-positive sectional curvature is called a Cartan-Hadamard manifold. For Cartan-Hadamard manifold, totally convexity is equivalent to convexity.

Proposition 3.4.2. *If $\tau\colon M \to \mathbb{R}$ is C^∞ and convex, the the sub-level of τ*

$$M_c := \{x \in M : \tau(x) < c\}$$

(or $\leq c$) is totally convex.

Proof. For any $p, q \in M_c$, *i.e.* $\tau(p), \tau(q) < c$. Let γ be a geodesic connecting p and q. Because τ is convex, $\tau \circ \gamma$ is convex. It follows that

$$\tau \circ \gamma(t) \leq \max \{\tau(p), \tau(q)\} < c$$

So $\gamma \subset M_c$. $\square$

Theorem 3.4.4. *Let (M, g) be a Riemannian manifold. Any $p \in M$ has a convex totally normal neighborhood.*

Proof. First, let W be a totally normal neighborhood of p. By above, there is a neighborhood U of p such that $d^2(p, \cdot)$ is convex on U. Choosing r sufficiently small such that

$$B(p, 3r) \subset W \cap U$$

Claim: $B(p, r)$ is convex.

For any $x, y \in B(p, r)$, there is a shortest geodesic γ connecting x and y with

$$d(x, y) = \text{Length}(\gamma) \leq d(x, p) + d(y, p) \leq 2r$$

Therefore, for any $z \in \gamma$, $x, y \in B(z, 2r)$. Because $B(z, 2r) \cap B(p, r) \neq \varnothing$,

$$d(z, p) \leq 3r$$

So $z \in U$. Because $d^2(p, \cdot)$ is convex on U and $d^2(p, x), d^2(p, y) \leq r^2$,

$$d^2(p, z) \leq \max \left\{d^2(p, x), d^2(p, y)\right\} \leq r^2$$

So $z \in B(p, r)$. $\square$

Chapter 4

Comparison Theorems

4.1 Rouch Comparison

1. 2-***Dim case:*** Consider ODEs on $\mathbb{R}$,

$$\phi''(t) + \kappa\phi(t) = 0$$
$$\eta''(t) + \ell\eta(t) = 0$$

with $\kappa, \ell > 0$. If $\kappa < \ell$ and $\phi(0) = \eta(0) = 0$ and $\phi'(0) = \eta'(0) = \sqrt{\kappa}$, then

$$\phi(t) = \sin\sqrt{\kappa}t, \quad \eta(t) = \sqrt{\frac{\kappa}{\ell}}\sin\sqrt{\ell}t$$

Then $\phi(t) \geq \eta(t)$ before the first zero point.

Theorem 4.1.1 (Sturm)**.** *Let f, h be two continuous functions such that $f(t) \leq h(t)$ for $t \in I$. Let ϕ, η satisfy*

$$\phi''(t) + f(t)\phi(t) = 0$$
$$\eta''(t) + h(t)\eta(t) = 0$$

Assume $\phi \neq 0$. Let $a, b \in I$ be two continuous zero point of ϕ.

(1) There is a zero point of η in (a, b). Otherwise, $f(t) = h(t)$ on $[a, b]$ and $\eta = c\phi$.

(2) Suppose $\eta(a) = 0$ and $\eta'(a) = \phi'(a) > 0$. If τ is the smallest zero point of η in $(a, b]$, then

$$\phi(t) \geq \eta(t), \quad \forall\ t \in [a, \tau]$$

and $\phi(t_0) = \eta(t_0)$ for some $t_0 \in [a, \tau]$ only if $f(t) = h(t)$ for all $t \in [a, t_0]$.

Proof. First, it has

$$\phi''\eta - \eta''\phi = (h - f)\phi\eta$$

(1) Suppose η has no zero point in (a, b). WTLG, $\phi, \eta > 0$ on (a, b). Therefore,

$$\phi''\eta - \eta''\phi = (\phi'\eta - \eta'\phi)' \geq 0$$

It follows that

$$(\phi'\eta - \eta'\phi)\big|_a^b = \phi'(b)\eta(b) - \phi'(a)\eta(a) \geq 0$$

On the other hand, by the continuity of η, $\eta(a), \eta(b) \geq 0$, and $\phi'(a) > 0, \phi'(b) < 0$ (Otherwise, by the uniqueness of the solution, $\phi = 0$),

$$\phi'(b)\eta(b) - \phi'(a)\eta(a) \leq 0$$

But if $f \neq h$, $\phi'(b)\eta(b) - \phi'(a)\eta(a) > 0$, which induces a contradiction. If $f = h$,

$$\phi'(b)\eta(b) - \phi'(a)\eta(a) = 0 \;\Rightarrow\; \phi'(b)\eta(b) = \phi'(a)\eta(a) = 0$$

So $\eta(a) = \eta(b) = 0$. By the structure of the solution of linear ODE, $\phi = c\eta$ for some c.

(2) By assumptions, we have $\phi, \eta > 0$ on (a, τ). So

$$\phi''\eta - \eta''\phi = (\phi'\eta - \eta'\phi)' = (h - f)\phi\eta \geq 0$$

Besides,

$$\phi'(a)\eta(a) - \eta'(a)\phi(a) = 0 \;\Rightarrow\; \phi'\eta - \eta'\phi \geq 0$$

It follows that

$$\left(\frac{\phi}{\eta}\right)' = \frac{\phi'\eta - \eta'\phi}{\eta^2} \geq 0$$

Besides,

$$\frac{\phi}{\eta}(a) := \lim_{t\to a} \frac{\phi(t)}{\eta(t)} = \lim_{t\to a} \frac{\phi'(t)}{\eta'(t)} = 1$$

So

$$\frac{\phi}{\eta}(t) \geq 1, \quad \forall\, t \in [a, \tau]$$

If there is a t_0 such that $\frac{\phi}{\eta}(t_0) = 1$, then

$$\phi(t) = \eta(t), \quad \forall\, t \in [a, t_0]$$

Then

$$\begin{aligned}\left(\frac{\phi}{\eta}\right)' = 0 \;&\Rightarrow\; \phi'\eta - \eta'\phi = 0\\ &\Rightarrow\; \phi''\eta - \eta''\phi = (h - f)\phi\eta = 0\\ &\Rightarrow\; h = f\end{aligned}$$

on $[a, t_0]$. □

For the first result, there is a geometric interpretation.

Theorem 4.1.2 (Bonnet). *Let M be a surface (2-dimensional Riemannian manifold) and $\gamma\colon [0, L] \to M$ be a normal geodesic. Let $\kappa > 0$.*

(1) If $K(\gamma(t)) \leq \kappa$ for all $t \in [a, b]$ and $L < \frac{\pi}{\sqrt{\kappa}}$, then γ has no conjugate point.

(2) If $K(\gamma(t)) \geq \kappa$ for all $t \in [a, b]$ and $L > \frac{\pi}{\sqrt{\kappa}}$, then there is a $\tau \in (0, L)$ such that $\gamma(\tau)$ is conjugate to $\gamma(0)$. In particular, γ is not shortest.

Proof. Let $Y(0) \perp \dot\gamma(0)$ with $|Y(0)| = 1$ and move it in parallel along γ to get $Y(t)$ ($|Y| = 1$). Then any normal Jacobian field can be written as $U(t) = \phi(t)Y(t)$. Then the Jacobian equation becomes

$$\phi''(t) + K(\gamma(t))\phi(t) = 0$$

Consider

$$\eta''(t) + \kappa\eta(t) = 0$$

It has a solution $\eta(t) = \sin\sqrt{\kappa}t$ that has two continuous zeros at $0, \frac{\pi}{\sqrt{\kappa}}$.

(1) If $K(\gamma(t)) \leq \kappa$, then by Sturm's Theorem $\phi(0) = 0$ implies $\phi(L) \neq 0$ for $L < \frac{\pi}{\sqrt{\kappa}}$. So there is no Jacobian field vanishing at endpoints of γ.

(2) If $\kappa \leq K$, then there is $t_0 \in (0, \frac{\pi}{\sqrt{\kappa}}]$ such that $\phi(t_0) = 0$. Because $L > \frac{\pi}{\sqrt{\kappa}}$, there is a Jacobian field vanishing at 0 and t_0. □

Similarly, there is a geometric interpretation for the second result.

Theorem 4.1.3. *Given two 2-dimensional Riemannian manifolds (M, g) and $(\overline{M}, \overline{g})$. Given two normal geodesics $\gamma\colon [a, b] \longrightarrow M$ and $\overline{\gamma}\colon [a, b] \longrightarrow \overline{M}$ such that $K(\gamma(t)) \leq \overline{K}(\overline{\gamma}(t))$. Let $\tau \in (a, b)$ such that $\gamma|_{[a,\tau]}$ and $\overline{\gamma}|_{[a,\tau]}$ has no conjugate points. Let U and $\overline{U}$ be normal Jacobian fields along γ and $\overline{\gamma}$ respectively with $U(a) = \overline{U}(a) = 0$ and $|\nabla_T U(a)| = |\nabla_{\overline{T}} U(a)|$. Then*

$$|U(t)| \geq \left|\overline{U}(t)\right|, \quad \forall\ t \in [a, \tau]$$

Furthermore, if there is a $t_0 \in [a, \tau]$ such that $|U(t_0)| \geq \left|\overline{U}(t_0)\right|$, then $K(\gamma(t)) = \overline{K}(\overline{\gamma}(t))$ for all $t \in [a, t_0]$.

2. ***High-dim case:*** Let γ be a geodesic with a orthonormal fame $\{Y_i(t)\}$ where $Y_1 = T$. Then any normal Jacobian field U can be

$$U(t) = \sum_{i=2}^{n} \phi^i(t) Y_i(t)$$

Then by the Jacobian equation $\nabla_T \nabla_T U + R(U, T)T = 0$,

$$\frac{d^2}{dt^2}\phi^j(t) + \sum_{i=2} \phi^i(t) R(Y_j, T, Y_i, T) = 0$$

Let M and $\overline{M}$ be two Riemannian manifolds with same dimension. Let $\gamma\colon [0, \ell] \to M$ and $\overline{\gamma}\colon [0, \ell] \to \overline{M}$ be normal geodesics. The problem is to compare the time of the first conjugate by comparing the curvature. The idea is to compare the index form $I(V, V)$.

Lemma 4.1.1. *Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds with same dimension n. Let $\gamma\colon [0, \ell] \to M$ and $\overline{\gamma}\colon [0, \ell] \to \overline{M}$ be normal geodesics. Then there is a linear map*

$$\Phi\colon \mathcal{V}(\gamma) \longrightarrow \mathcal{V}_{\overline{\gamma}},$$

where $\mathcal{V}(\gamma)$ =the set of all continuous piecewise smooth vector field along γ, such that for any $X \in \mathcal{V}(\gamma)$,

(1) if $\nabla_T X$ is continuous at t, then $\nabla_{\overline{T}} \Phi(X)$ is continuous at t;

(2) $\langle X(t), T(t)\rangle_g = \left\langle \Phi(X)(t), \overline{T}(t)\right\rangle_{\overline{g}}$;

(3) $|X(t)|_g = |\Phi(X)(t)|_{\overline{g}}$;

(4) $|\nabla_T X(t)|_g = |\nabla_{\overline{T}} \Phi(X)(t)|_{\overline{g}}$.

Proof. Let

$$\phi_{t_0}\colon T_{\gamma(t_0)} M \longrightarrow T_{\overline{\gamma}(t_0)} \overline{M}$$

be a linear isometry such that $\phi_{t_0}(T(t_0)) = \overline{T}(t_0)$. Then define

$$\Phi(X)(t) = \mathcal{P}_{\overline{\gamma}, t_0, t} \circ \phi_{t_0} \circ \mathcal{P}_{\gamma, t, t_0}(X(t))$$

Clearly, Φ is linear by the linearity of parallel moving. Let $\{Y_1 = T, Y_2, \cdots, Y_n\}$ be an orthonormal frame of γ. Then let

$$\overline{Y}_i(t_0) = \phi_{t_0}(Y_i(t_0))$$

Then $\left\{\overline{Y}_1(t_0) = T, \overline{Y}_2(t_0), \cdots, \overline{Y}_n(t_0)\right\}$ is orthonormal.

$$\overline{Y}_i(t) = \mathcal{P}_{\overline{\gamma},t_0,t}(\overline{Y}_i(t_0))$$

is an orthonormal frame of $\overline{\gamma}$ and $\overline{Y}_i(t) = \Phi(Y_i)(t)$. Then

$$X(t) = f^i(t)Y_i(t) \;\Rightarrow\; \Phi(X)(t) = f^i(t)\overline{Y}(t)$$

So above properties are clear, such as

$$\nabla_T X = \left(\frac{d}{dt}f^i\right)Y_i, \quad \nabla_T \Phi(X) = \left(\frac{d}{dt}f^i\right)\overline{Y}_i$$

□

Theorem 4.1.4. *Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds with same dimension n. Let $\gamma\colon [0,\ell] \to M$ and $\overline{\gamma}\colon [0,\ell] \to \overline{M}$ be normal geodesics. For any $t \in [0,\ell]$, suppose that for all 2-dim section $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\overline{\Pi}_{\overline{\gamma}(t)} \subset T_{\overline{\gamma}(t)}\overline{M}$,*

$$K(\Pi_{\gamma(t)}) \leq K(\overline{\Pi}_{\overline{\gamma}(t)})$$

Then we have

$$\operatorname{ind}(\gamma) \leq \operatorname{ind}(\overline{\gamma})$$

Proof. Let $W \in \mathcal{V}_0(\gamma)$.

$$\begin{aligned} I(W,W) &= \int_0^\ell \langle \nabla_T W, \nabla_T W\rangle - R(W,T,W,T)dt \\ &= \int_0^\ell \langle \nabla_T W, \nabla_T W\rangle - K(\Pi_{W,T})(\langle W,W\rangle \left|T,T\right| - \langle W,T\rangle^2)dt \end{aligned}$$

Let $V = \Phi(W)$ in above lemma. So $V \in \mathcal{V}_0(\overline{\gamma})$ and by above lemma

$$\begin{aligned} \overline{I}(V,V) &= \int_0^\ell \langle \nabla_{\overline{T}} V, \nabla_{\overline{T}} V\rangle - K(\Pi_{V,\overline{T}})(\langle V,V\rangle \left|\overline{T},\overline{T}\right| - \langle V,\overline{T}\rangle^2)dt \\ &= \int_0^\ell \langle \nabla_T W, \nabla_T W\rangle - K(\Pi_{V,\overline{T}})(\langle W,W\rangle \left|T,T\right| - \langle W,T\rangle^2)dt \end{aligned}$$

Therefore,

$$I(W,W) \geq \overline{I}(\Phi(W),\Phi(W))$$

and $I(W,W) < 0$ implies $\overline{I}(\Phi(W),\Phi(W)) < 0$, which means that $\Phi(\mathcal{A})$ is also a negative space with same dimension of $\overline{\gamma}$ if $\mathcal{A}$ is a negative space of γ. So

$$\operatorname{ind}(\gamma) \leq \operatorname{ind}(\overline{\gamma})$$

□

Corollary 4.1.1 (Morse-Schoenberg Comparison Theorem)**.** *Let (M, g) be n-dimensional Riemannian manifold with a normal geodesic $\gamma\colon [0,\ell] \to M$. Let $\kappa > 0$.*

(1) If $K(\Pi_{\gamma(t)}) \leq \kappa$ for any t and $\ell < \frac{\pi}{\sqrt{\kappa}}$, then $\operatorname{ind}(\gamma) = 0$, that is γ has no conjugate point.

(2) *If $K(\Pi_{\gamma(t)}) \geq \kappa$ for any t and $\ell > \frac{\pi}{\sqrt{\kappa}}$, then there is a $\tau \in (a, b)$ conjugate to 0 and so γ is not minimizing.*

Proof. (1) By setting $\overline{M} = \mathbb{S}^n\left(\frac{1}{\sqrt{\kappa}}\right)$, then

$$\operatorname{ind}(\gamma) \leq \operatorname{ind}(\overline{\gamma})$$

where $\overline{\gamma}\colon [0, \ell] \to \mathbb{S}^n\left(\frac{1}{\sqrt{\kappa}}\right)$. Because $\ell < \frac{\pi}{\sqrt{\kappa}}$, any geodesic with length $< \frac{\pi}{\sqrt{\kappa}}$ has no conjugate point. So $\operatorname{ind}(\overline{\gamma}) = 0$ and $\operatorname{ind}(\gamma) = 0$.

(2) By setting $\overline{M} = \mathbb{S}^n\left(\frac{1}{\sqrt{\kappa}}\right)$, then

$$\operatorname{ind}(\gamma) \geq \operatorname{ind}(\overline{\gamma})$$

But any geodesic in $\mathbb{S}^n\left(\frac{1}{\sqrt{\kappa}}\right)$ with length $> \frac{\pi}{\sqrt{\kappa}}$ always has at least one conjugate point. So $\operatorname{ind}(\overline{\gamma}) > 0$ and $\operatorname{ind}(\gamma) > 0$. $\square$

Theorem 4.1.5 (Rauch Comparison Theorem)**.** *Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds with same dimension n. Let $\gamma\colon [0, \ell] \to M$ and $\overline{\gamma}\colon [0, \ell] \to \overline{M}$ be normal geodesics. Let $U, \overline{U}$ be normal Jacobian fields along γ and $\overline{\gamma}$ respectively with $U(0) = \overline{U}(0) = 0$ and $|\nabla_T U(0)| = \left|\nabla_{\overline{T}} \overline{U}(0)\right|$. Suppose that*

(i) for any $t \in [0, \ell]$, suppose that for all 2-dim section $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\overline{\Pi}_{\overline{\gamma}(t)} \subset T_{\overline{\gamma}(t)}\overline{M}$,

$$K(\Pi_{\gamma(t)}) \leq K(\overline{\Pi}_{\overline{\gamma}(t)})$$

(ii) $\overline{\gamma}$ has no conjugate point on $[0, \ell]$. (In fact, it implies γ has no conjugate point.)

Then

$$|U(t)| \geq \left|\overline{U}(t)\right|, \quad t \in [0, \ell]$$

Remark. For the condition that $U, \overline{U}$ are normal, it can be replaced by

$$\langle \nabla_T U(0), \dot{\gamma}(0) \rangle = \left\langle \nabla_{\overline{T}} \overline{U}(0), \dot{\overline{\gamma}}(0) \right\rangle = 0$$

Decomposing $U = fT + U^{\perp}$, where $f(t) = at + b$, then

$$\nabla_T U = f'T + \nabla_T U^{\perp}$$

So

$$\begin{aligned} 0 &= \left\langle f'(0)T(0) + \nabla_T U^{\perp}, T(0) \right\rangle \\ &= f'(0) + \frac{d}{dt}\left\langle U^{\perp}, T \right\rangle \\ &= f'(0) \end{aligned}$$

which implies $f = 0$ and so U is normal.

Example 4.1.1. Consider $\mathbb{S}^n(r)$ that has the constant sectional curvature $\kappa = \frac{1}{r^2}$. Then

$$U_\kappa(t) = \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t E(t)$$

is a normal Jacobian field ($E \perp T$ and $|E| = 1$). Then

$$U(0) = 0, \quad \nabla_T U_\kappa(0) = E(0)$$

Then for different κ, $U_k(t)$ satisfies above conditions. And

$$|U_k(t)| = \left|\frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}t\right|$$

which coincides with above results, larger curvature implies smaller $|U(t)|$ when t is before the conjugate point.

Proof of Theorem 4.1.5. Assume $\overline{U} \neq 0$. Consider the map

$$t \mapsto \frac{|U(t)|^2}{\left|\overline{U(t)}\right|^2}$$

it is sufficient to show:

I. By L'Hospital's rule,

$$\begin{aligned}
\lim_{t\to 0}\frac{|U(t)|^2}{\left|\overline{U(t)}\right|^2} &= \lim_{t\to 0}\frac{\langle \nabla_T U, U\rangle}{\left\langle \nabla_{\overline{T}}\overline{U}, \overline{U}\right\rangle} \\
&= \lim_{t\to 0}\frac{\langle \nabla_T\nabla_T U, U\rangle + \langle \nabla_T U, \nabla_T U\rangle}{\left\langle \nabla_{\overline{T}}\nabla_{\overline{T}}\overline{U}, \overline{U}\right\rangle + \left\langle \nabla_{\overline{T}}\overline{U}, \nabla_{\overline{T}}\overline{U}\right\rangle} \\
&= 1
\end{aligned}$$

II. And

$$\frac{d}{dt}\frac{|U(t)|^2}{\left|\overline{U(t)}\right|^2} = 2\frac{\langle \nabla_T U, U\rangle\left\langle \overline{U}, \overline{U}\right\rangle - \langle U, U\rangle\left\langle \nabla_{\overline{T}}\overline{U}, \overline{U}\right\rangle}{\left\langle \overline{U}, \overline{U}\right\rangle^2} \geq 0$$

So it needs to check

$$\langle \nabla_T U(t), U(t)\rangle\left|\overline{U(t)}\right|^2 - \left\langle \nabla_{\overline{T}}\overline{U}(t), \overline{U}(t)\right\rangle |U(t)|^2 \geq 0$$

that is for any t_0

$$\langle \nabla_T U(t_0), U(t_0)\rangle \geq \frac{|U(t_0)|^2}{\left|\overline{U}(t_0)\right|^2}\left\langle \nabla_{\overline{T}}\overline{U}(t_0), \overline{U}(t_0)\right\rangle$$

Note that because $U, \overline{U}$ are Jacobian,

$$I_0^{t_0}(U,U) = \langle \nabla_T U(t_0), U(t_0)\rangle, \quad \overline{I}_0^{t_0}(\overline{U},\overline{U}) = \left\langle \nabla_{\overline{T}}\overline{U}(t_0), \overline{U}(t_0)\right\rangle$$

So it needs to show

$$I_0^{t_0}(U,U) \geq \frac{|U(t_0)|^2}{\left|\overline{U}(t_0)\right|^2}\overline{I}_0^{t_0}(\overline{U},\overline{U})$$

Consider a linear isometry $\phi_{t_0}\colon T_{\gamma(t_0)}M \to T_{\overline{\gamma}(t_0)}\overline{M}$ such that

$$\phi_{t_0}(U(t_0)) = c\overline{U}(t_0), \quad c = \frac{|U(t_0)|}{\left|\overline{U}(t_0)\right|}$$

Note that such ϕ_{t_0} also needs to satisfy $\phi_{t_0}(T(t_0)) = \overline{T}(t_0)$. The reason of the existence of such ϕ_{t_0} is because $\langle T, U\rangle = \left\langle \overline{T}, \overline{U}\right\rangle = 0$. For each t_0, we can choose orthonormal basis $Y_1(t_0) = T(t_0), Y_2(t_0) = \frac{U(t_0)}{|U(t_0)|}, \cdots$ and the corresponding $\overline{Y}_1(t_0) = \overline{T}(t_0), \overline{Y}_2(t_0) = \frac{\overline{U}(t_0)}{|\overline{U}(t_0)|}, \cdots$ to construct our ϕ_{t_0}. Define Φ as same as it in **Lemma** 4.1.1. Then we have

$$I_0^{t_0}(U,U) \geq \overline{I}_0^{t_0}(\Phi(U),\Phi(U))$$

Besides, note that

$$\Phi(U)(0) = 0, \quad \Phi(U)(t_0) = \phi_{t_0}(U(t_0)) = c\overline{U}(t_0)$$

Therefore, because $c\overline{U}$ is Jacobian and $\overline{\gamma}$ has no conjugate point, by **Lemma** 3.3.2,

$$\overline{I}_0^{t_0}(\Phi(U),\Phi(U)) \geq \overline{I}_0^{t_0}(c\overline{U},c\overline{U})$$

It follows that

$$I_0^{t_0}(U,U) \geq \overline{I}_0^{t_0}(c\overline{U},c\overline{U}) = c^2\overline{I}_0^{t_0}(\overline{U},\overline{U})$$ □

3. **Applications:** Consider the applications of the Rauch Comparison Theorem.

Theorem 4.1.6. *Let (M,g) and $(\overline{M},\overline{g})$ be two Riemannian manifolds with same dimension n. Let $p \in M$ and $\overline{p} \in \overline{M}$. Let $\phi\colon T_pM \to T_{\overline{p}}\overline{M}$ be an isometry, $V \in T_pM$ and $\overline{V} = \phi(V)$. Consider geodesics*

$$\gamma(t) = \exp_p tV, \quad \overline{\gamma}(t) = \exp_{\overline{p}} t\overline{V}, \quad t \in [0,1]$$

Let $X \in T_V(T_pM)$ and $\overline{X} = \phi(X) \in T_{\overline{V}}(T_{\overline{p}}\overline{M})$. Suppose

(i) for any $t \in [0,\ell]$, suppose that for all 2-dim section $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\overline{\Pi}_{\overline{\gamma}(t)} \subset T_{\overline{\gamma}(t)}\overline{M}$,

$$K(\Pi_{\gamma(t)}) \leq K(\overline{\Pi}_{\overline{\gamma}(t)})$$

(ii) $\overline{\gamma}$ has no conjugate point on $[0,\ell]$. (In fact, it implies γ has no conjugate point.)

Then

$$\left|(d\exp_p)_V(X)\right|_g \geq \left|(d\exp_{\overline{p}})_V(\overline{X})\right|_{\overline{g}}$$

Proof. Note that

$$X = X^{\perp} + \langle X,V\rangle V, \quad X^{\perp} = X - \langle X,V\rangle V$$

By **Lemma** 2.4.1 we have

$$\left|(d\exp_p)_V(V)\right|^2 = |V|^2, \quad \left\langle (d\exp_p)_V(X^{\perp}), (d\exp_p)_V(V)\right\rangle = \left\langle X^{\perp}, V\right\rangle = 0$$

Therefore,

$$\left|(d\exp_p)_V(X)\right|^2 = \left|(d\exp_p)_V(X^{\perp})\right|^2 + \langle X,V\rangle |V|^2$$

It follows that

$$\left|(d\exp_p)_V(X)\right|_g \geq \left|(d\exp_{\overline{p}})_V(\overline{X})\right|_{\overline{g}} \quad \Leftrightarrow \quad \left|(d\exp_p)_V(X^{\perp})\right|_g \geq \left|(d\exp_{\overline{p}})_V(\overline{X}^{\perp})\right|_{\overline{g}}$$

because ϕ is isometric, *i.e.* $\langle X,V\rangle_g = \left\langle \overline{X},\overline{V}\right\rangle_{\overline{g}}$ and $|V|^2 = \left|\overline{V}\right|^2$. So we only need to consider the case of $\langle X,V\rangle = 0$. Consider the variation of γ

$$F(t,s) = \exp_p t(V+sX)$$

It follows that the corresponding Jacobian field satisfies

$$U(0)=0,\quad \nabla_T U(0)=X,\quad U(1)=(d\exp_p)_V(X)$$

Similarly, consider the variation of $\overline{\gamma}$

$$\overline{F}(t,s)=\exp_{\overline{p}} t(\overline{V}+s\overline{X})$$

It follows that the corresponding Jacobian field satisfies

$$\overline{U}(0)=0,\quad \overline{\nabla}_{\overline{T}}\overline{U}(0)=\overline{X},\quad \overline{U}(1)=(d\exp_{\overline{p}})_V(\overline{X})$$

Because

$$\langle \nabla_T U(0),\dot{\gamma}(0)\rangle=\left\langle \nabla_{\overline{T}}\overline{U}(0),\dot{\overline{\gamma}}(0)\right\rangle=0$$

they are normal Jacobian fields. Then the result can be obtained by the Rauch Comparison Theorem. □

Corollary 4.1.2. *Let (M,g) be a complete Riemannian manifold with sectional curvature ≤ 0. Then for any $p\in M$,*

$$\left|(d\exp_p)_V(X)\right|\geq |X|\,,\quad \forall\ V\in T_pM,\ \forall\ X\in T_V(T_pM)$$

Remark. $\left|(d\exp_p)_V(X)\right|^2=g_{\exp_p(V)}\left((d\exp_p)_V(X),(d\exp_p)_V(X)\right)$ and $|X|^2=g_p(X,X)$.

Corollary 4.1.3. *Let (M,g) be a Cartan-Hadamard manifold. Consider a geodesic triangular with sides a,b,c and angle A,B,C,*

(1) $a^2+b^2-2ab\cos C\leq c^2$;

(2) $A+B+C\leq\pi$.

Proof. For (1), it can get by comparing the geodesic triangular with the triangular in T_pM with a,b being radical geodesic. For (2), let a,b,c with A',B',C' be a triangular in Euclidean space, then

$$\begin{aligned} a^2+b^2-2ab\cos c\leq c^2=a^2+b^2-2ab\cos C' &\Rightarrow\ \cos C'\leq\cos C\\ &\Rightarrow\ A+B+C\leq A'+B'+C'=\pi \end{aligned}$$ □

4.2 Hessian and Laplacian Comparison

1 ***Hessian:*** For $p\in M$, consider the distance function $\rho(\cdot)=d(p,\cdot)$. When (M,g) is complete, $\exp_p\colon E(p)\to\exp_p(E(p))$ is a diffeomorphism and so

$$\rho(x)=\sqrt{g_p\left(\exp_p^{-1}(x),\exp_p^{-1}(x)\right)}$$

is C^∞ defined on $\exp_p(E(p))\setminus\{p\}$.

Lemma 4.2.1. *Let (M,g) be a Riemannian manifold and $\gamma\colon[0,\ell]\to M$ be a normal geodesic with $p=\gamma(0)$ and $v=\dot{\gamma}(0)$. Suppose γ is minimizing and contain no cut point. For $\rho(\cdot)=d(p,\cdot)$, we have*

$$(\operatorname{grad}\rho)_t=\dot{\gamma}(t)$$

Proof. **Claim:** For any t and $E\in T_{\gamma(t)}M$ with $|E|=1$ and $\langle E,\dot{\gamma}(t)\rangle=0$, it can find a geodesic $\widetilde{\gamma}(s)$ starting from $\gamma(t)$ such that $\dot{\widetilde{\gamma}}(0)=E$ and $\rho(\widetilde{\gamma}(s))\equiv t$.

proof of the claim. Because γ is minimizing and contain no cut point, $\gamma \in \exp_p(E(p))$. Because $\exp_p$ is non-singular on $E(p)$, it can let

$$\widetilde{E} := (d\exp_p)^{-1}_{tv}(E) \in T_pM$$

Note that

$$(d\exp_p)_{tv}(tv) = \left.\frac{d}{ds}\right|_{s=0} \exp_p((t+s)v) = \dot{\gamma}(t)$$

Because $\langle E, \dot{\gamma}(t)\rangle = 0$, by Gauss's Lemma

$$\left\langle \widetilde{E}, tv \right\rangle = \left\langle (d\exp_p)_{tv}(\widetilde{E}), (d\exp_p)_{tv}(tv) \right\rangle = \langle E, \dot{\gamma}(t)\rangle = 0$$

Therefore, we can find a unit circle $v(s)$ in T_pM with $v(0) = v$ and $\dot{v}(0) = \widetilde{E}$, *i.e.* $|v(s)| = 1$. Then let

$$\widetilde{\gamma}(s) = \exp_p(tv(s))$$

First, clearly $d(p, \widetilde{\gamma}(s)) \equiv t$. Moreover,

$$\dot{\widetilde{\gamma}}(0) = \left.\frac{d}{ds}\right|_{s=0} \exp_p(tv(s)) = (d\exp_p)_{tv}(\widetilde{E}) = E \qquad \square$$

Then

$$0 = \left.\frac{d}{ds}\right|_{s=0} \rho(\widetilde{\gamma}(s)) = \langle \operatorname{grad}\rho, E\rangle = 0$$

Because E is arbitrary, $\operatorname{grad}\rho = c\dot{\gamma}(t)$. Moreover,

$$\langle \operatorname{grad}\rho, \dot{\gamma}(t)\rangle = \dot{\gamma}(t)(\rho) = \frac{d}{dt}\rho(\gamma(t)) = \frac{d}{dt}t = 1$$

So we have $(\operatorname{grad}\rho)_t = \dot{\gamma}(t)$. $\square$

Theorem 4.2.1 (Hessian Comparison Theorem)**.** *Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds with same dimension n. Let $\gamma\colon [0, \ell] \to M$ and $\overline{\gamma}\colon [0, \ell] \to \overline{M}$ be normal geodesics. Let $p = \gamma(0)$, $\overline{p} = \overline{\gamma}(0)$, $\rho(\cdot) = d(p, \cdot)$, and $\overline{\rho}(\cdot) = \overline{d}(\overline{p}, \cdot)$. Suppose that*

(i) for any $t \in [0, \ell]$, suppose that for all 2-dim section $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\overline{\Pi}_{\overline{\gamma}(t)} \subset T_{\overline{\gamma}(t)}\overline{M}$,

$$K(\Pi_{\gamma(t)}) \leq K(\overline{\Pi}_{\overline{\gamma}(t)})$$

(ii) $\overline{\gamma}, \gamma$ are minimizing and contain no cut point.

Then we have

$$\operatorname{Hess}\rho \geq \overline{\operatorname{Hess}}\overline{\rho}$$

along $\gamma, \overline{\gamma}$, that is for any $t \in [0, \ell]$, for any $X \in T_{\gamma(t)}M$, $\overline{X} \in T_{\overline{\gamma}(t)}\overline{M}$ with $|X|_g = \left|\overline{X}\right|_{\overline{g}}$ and $\langle X, \dot{\gamma}(t)\rangle = \left\langle \overline{X}, \dot{\overline{\gamma}}(t)\right\rangle$,

$$\operatorname{Hess}\rho(X, X) \geq \overline{\operatorname{Hess}}\overline{\rho}(\overline{X}, \overline{X})$$

Remark. For any $X \in T_{\gamma(t)}M$ for some t,

$$X = X^{\perp} + \langle X, \dot{\gamma}(t)\rangle\, \dot{\gamma}(t)$$

Then we have

$$\operatorname{Hess}\rho(X,X) = \operatorname{Hess}\rho(X^\perp, X^\perp) + 2\langle X, \dot\gamma(t)\rangle \operatorname{Hess}\rho(X^\perp, \dot\gamma(t)) + \langle X, \dot\gamma(t)\rangle^2 \operatorname{Hess}\rho(\dot\gamma(t), \dot\gamma(t))$$

However,

$$\operatorname{Hess}\rho(\dot\gamma(t), \dot\gamma(t)) = \frac{d^2}{dt^2}\rho(\gamma(t)) = \frac{d^2}{dt^2}t = 0$$

Besides, by $\operatorname{grad}\rho = \dot\gamma(t)$ ($|\operatorname{grad}\rho| = 1$)

$$\begin{aligned}
\operatorname{Hess}\rho(\dot\gamma(t), X^\perp) &= \nabla^2\rho(\dot\gamma(t), X^\perp) \\
&= \nabla(\nabla\rho)(\dot\gamma(t), X^\perp) \\
&= \nabla_{X^\perp}(\nabla\rho)(\dot\gamma(t)) \\
&= \nabla_{X^\perp}(\nabla_{\dot\gamma(t)}\rho) - \nabla\rho(\nabla_{X^\perp}\dot\gamma(t)) \\
&= \nabla_{X^\perp}(\langle\dot\gamma(t), \operatorname{grad}\rho\rangle) - \langle\nabla_{X^\perp}\dot\gamma(t), \operatorname{grad}\rho\rangle \\
&= \nabla_{X^\perp}(\langle\operatorname{grad}\rho, \operatorname{grad}\rho\rangle) - \langle\nabla_{X^\perp}\operatorname{grad}\rho, \operatorname{grad}\rho\rangle \\
&= \frac{1}{2}\nabla_{X^\perp}(\langle\operatorname{grad}\rho, \operatorname{grad}\rho\rangle) \\
&= 0
\end{aligned}$$

Therefore, we have

$$\operatorname{Hess}\rho(X,X) = \operatorname{Hess}\rho(X^\perp, X^\perp)$$

and in above theorem $X, \overline{X}$ can be chosen such that they are perpendicular to tangent direction.

Proof. WLTG, let $t = \ell$. Let $X \in T_{\gamma(\ell)}M$, $\overline{X} \in T_{\overline{\gamma}(\ell)}\overline{M}$ with $|X|_g = \left|\overline{X}\right|_{\overline{g}}$ and $\langle X, \dot\gamma(\ell)\rangle = \left\langle \overline{X}, \dot{\overline{\gamma}}(\ell)\right\rangle = 0$. Because $\gamma, \overline{\gamma}$ are contained in the area that the exponential map is diffeomorphic, we can apply the calculation as **Theorem** 3.4.3. Let ξ be a geodesic with $\xi(0) = \gamma(t)$ and $\dot\xi(0) = X$. By considering a family of geodesics connecting p with $\xi(s)$, the variation

$$F(t,s) = \exp_p \frac{t}{\ell}\exp_p^{-1}(\xi(s)), \quad t \in [0,\ell]$$

with the Jacobian field U that satisfies $U(0) = 0, U(\ell) = \dot\xi(0) = X$. Then

$$\begin{aligned}
\operatorname{Hess}\rho(X,X) &= \left.\frac{d^2}{ds^2}\right|_{s=0}\rho(\xi(s)) \\
&= \left.\frac{d^2}{ds^2}\right|_{s=0} L(s) \\
&= \int_0^\ell \langle\nabla_T U, \nabla_T U\rangle - R(U,T,U,T) - (T\langle T,U\rangle)^2 dt \\
&= \int_0^\ell \langle\nabla_T U, \nabla_T U\rangle - R(U,T,U,T) dt \\
&= I(U,U)
\end{aligned}$$

because $\langle T, U\rangle = 0$ by $U(0), U(\ell) \perp T$. Similarly,

$$\overline{\operatorname{Hess}}\overline{\rho}(\overline{X}, \overline{X}) = \overline{I}(\overline{U}, \overline{U})$$

Similarly as the proof of the Rauch Comparison Theorem, we need to compare the index form. Choose a linear isometry

$$\phi_t \colon T_{\gamma(t)}M \longrightarrow T_{\overline{\gamma}(t)}\overline{M}$$

such that $\phi_\ell(X) = \overline{X}$, which can be done because $|X| = |\overline{X}|$. Then by the proof of **Theorem** 4.1.4,

$$I(U,U) \geq \overline{I}(\Phi(U), \Phi(U))$$

Besides, clearly $\Phi(U)(0) = 0$ and

$$\Phi(U)(\ell) = \phi_\ell(X) = \overline{X}$$

Then because $\overline{U}$ is Jacobian with $\overline{U}(0) = 0$ and $\overline{U}(\ell) = \overline{X}$, by **Lemma** 4.1.1,

$$I(U,U) \geq \overline{I}(\Phi(U), \Phi(U)) \geq \overline{I}(\overline{U}, \overline{U})$$ □

Remark. In above calculation, because U is Jacobian,

$$\operatorname{Hess} \rho(X,X) = I(U,U) = \langle \nabla_T U, U \rangle|_0^\ell = \langle \nabla_T U(\ell), X \rangle$$

Remark. Moreover, if $f \in C^\infty(\mathbb{R}_+)$ with $f' \geq 0$, then under the same condition

$$\operatorname{Hess} f \circ \rho(X,X) \geq \overline{\operatorname{Hess}} f \circ \overline{\rho}(\overline{X}, \overline{X})$$

Because:

$$\begin{aligned}
\operatorname{Hess} f \circ \rho(X,X) &= \left.\frac{d^2}{ds^2}\right|_{s=0} f(\rho(\xi(s))) \\
&= \left.\frac{d}{ds}\right|_{s=0} \left(f'(\rho(\xi(s))) \frac{d}{ds} \rho(\xi(s)) \right) \\
&= f''(t) \langle X, X \rangle^2 + f'(t) \operatorname{Hess} \rho(X,X)
\end{aligned}$$

Because $f''(t) \langle X, X \rangle^2$ is as same as $\overline{X}$, $f'(t) \geq 0$ implies the desired result.

Corollary 4.2.1. *Let (M,g) be a Cartan-Hadamard manifold. Then*

$$\operatorname{Hess} \rho^2(X,X) \geq 2g(X,X)$$

and so $\Delta \rho^2 \geq 2n$.

Proof. It is because the Euclidean space (T_pM, g_p) has the constant sectional curvature 0 and the Hessian is $2g$. □

Example 4.2.1 (Hessian on Constant Curvature Space)**.** Let (M,g) be a Riemannian manifold with constant sectional curvature $\kappa \in \mathbb{R}$. Let $p \in M$ and $\rho(\cdot) = d(p, \cdot)$. Let $\gamma \colon [0, \ell] \to M$ with $q = \gamma(t_0)$ and $X \in T_qM$ perpendicular to $\dot{\gamma}(t_0)$. Then by above

$$\operatorname{Hess} \rho(X,X) = \langle \nabla_T U(t_0), X \rangle$$

There is an $E(t)$ be parallel along γ with $E \perp T$ such that $U(t) = f(t)E(t)$.

Proof. Let $E_1 = T, E_2, \cdots, E_n$ be an orthonormal frame along γ. Then

$$U(t) = \sum_{i=2}^n f^i(t) E_i(t)$$

and the constant sectional curvature implies that for all $i = 2, 3, \cdots, n$

$$\begin{cases} \dfrac{d^2}{dt^2} f^i(t) + \kappa f^i(t) = 0 \\ f^i(t) = 0 \end{cases}$$

Because $U \neq 0$, there is i_0 such that $f^{i_0} \neq 0$. Let $f(t) = f^{i_0}(t)$. Because all f^i satisfy the same equation with the same 0-order condition, $f^i(t) = c_i f(t)$ with a constant c_i for all i. So

$$U(t) = \sum_{i=2}^{n} f^i(t) E_i(t) = \sum_{i=2}^{n} c_i f(t) E_i(t) = f(t) \sum_{i=2}^{n} c_i E_i(t) = f(t) E(t)$$

with $E(t) = \sum_{i=2}^{n} c_i E_i(t) \perp T$. Moreover, WTLG, let $f'(0) = 1$. Then

$$f(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t, & \kappa > 0 \\ t, & \kappa = 0 \\ \frac{1}{\sqrt{\kappa}} \sinh \sqrt{\kappa} t, & \kappa < 0 \end{cases} \qquad \square$$

So we have

$$\begin{aligned} \nabla_T U(t_0) &= \nabla_T \left(f(t) E(t) \right)\big|_{t=t_0} \\ &= f'(t_0) E(t_0) \end{aligned}$$

Besides, we know $U(t_0) = X = f(t_0) E(t_0)$, that is

$$|X|^2 = f^2(t_0) \left| E(t_0) \right|^2$$

Therefore,

$$\operatorname{Hess} \rho(X, X) = \langle \nabla_T U(t_0), U(t_0) \rangle = \frac{f'(t_0)}{f(t_0)} |X|^2$$

and

$$\Delta \rho(\gamma(t_0)) = \sum_{i=2}^{n} \operatorname{Hess} \rho(E_i, E_i) = \frac{f'(t_0)}{f(t_0)} (n-1)$$

In particular, if $\kappa = 0$, then

$$\operatorname{Hess} \rho(X, X) = \frac{1}{t_0} |X|^2 = \frac{1}{r} |X|^2, \quad r = d(p, q)$$

So by Hessian Comparison theorem, if (M, g) is a Cartan-Hadamard manifold,

$$\operatorname{Hess} \rho \geq \frac{1}{r} g$$

Furthermore, in such case

$$\Delta \rho = \sum_{i=2}^{n} \operatorname{Hess} \rho(E_i, E_i) \geq \sum_{i=2}^{n} \frac{1}{r} g(E_i, E_i) = \frac{n-1}{r}$$

2 ***Laplacian:*** We want to relax the conditions of the Hessian Comparison Theorem to obtain the Laplacian Comparison Theorem.

Theorem 4.2.2. *Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds with same dimension n. Let $\gamma \colon [0, \ell] \to M$ and $\overline{\gamma} \colon [0, \ell] \to \overline{M}$ be normal geodesics. Let $p = \gamma(0)$, $\overline{p} = \overline{\gamma}(0)$, $\rho(\cdot) = d(p, \cdot)$, and $\overline{\rho}(\cdot) = \overline{d}(\overline{p}, \cdot)$. Suppose that*

(1) $\operatorname{Ric}(\dot{\gamma}, \dot{\gamma})(t) \leq \overline{\operatorname{Ric}}(\dot{\overline{\gamma}}, \dot{\overline{\gamma}})(t)$ *for any $t \in [0, \ell]$;*

(2) $\overline{\gamma}, \gamma$ are minimizing and contain no cut point.

(3) M is a space form.

Then we have

$$\Delta\rho(\gamma(t)) \geq \overline{\Delta}\overline{\rho}(\overline{\gamma}(t)), \quad t \in [0, \ell]$$

Proof. First, choosing $t = \ell$. Let $e_1 = \dot{\gamma}(\ell), \cdots, e_n$ be an orthonormal basis of $T_{\gamma(\ell)}M$ and moving them in parallel to get a frame $\{e_i(t)\}$. Then

$$\Delta\rho(\gamma(\ell)) = \sum_{i=2}^{n} \operatorname{Hess}\rho(e_i, e_i) = \sum_{i=2}^{n} I(U_i, U_i)$$

where U_i is a normal Jacobian field with $U_i(0) = 0, U_i(\ell) = e_i$. Similarly for $\overline{M}$,

$$\overline{\Delta}\overline{\rho}(\overline{\gamma}(\ell)) = \sum_{i=2}^{n} \overline{\operatorname{Hess}}\overline{\rho}(\overline{e}_i, \overline{e}_i) = \sum_{i=2}^{n} \overline{I}(\overline{U}_i, \overline{U}_i)$$

where $\overline{U}_i$ is a normal Jacobian field with $\overline{U}_i(0) = 0, \overline{U}_i(\ell) = \overline{e}_i$. Let $\phi_\ell \colon T_{\gamma(\ell)}M \to\colon T_{\overline{\gamma}(\ell)}\overline{M}$ such that $\phi_\ell(e_i) = \overline{e}_i$. Then it induces a Φ, which satisfies

$$\Phi(U_i)(0) = 0, \quad \Phi(U_i)(\ell) = \overline{U}_i(\ell)$$

So by **Lemma** 3.3.2,

$$\overline{I}(\Phi(U_i), \Phi(U_i)) \geq \overline{I}(\overline{U}_i, \overline{U}_i), \quad \forall\ i$$

Therefore, it is sufficient to show

$$\sum_{i=2}^{n} I(U_i, U_i) \geq \sum_{i=2}^{n} \overline{I}(\Phi(U_i), \Phi(U_i))$$

Because

$$\begin{aligned}
\text{LHS} &= \sum_{i=2}^{n} \int_0^\ell \langle \nabla_T U_i, \nabla_T U_i \rangle - R(U_i, T, U_i, T) dt \\
\text{RHS} &= \sum_{i=2}^{n} \int_0^\ell \left\langle \overline{\nabla}_{\overline{T}} \Phi(U_i), \overline{\nabla}_{\overline{T}} \Phi(U_i) \right\rangle - \overline{R}(\Phi(U_i), \overline{T}, \Phi(U_i), \overline{T}) dt \\
&= \sum_{i=2}^{n} \int_0^\ell \langle \nabla_T U_i, \nabla_T U_i \rangle - \overline{R}(\Phi(U_i), \overline{T}, \Phi(U_i), \overline{T}) dt
\end{aligned}$$

we need to show

$$\int_0^\ell \sum_{i=2}^{n} R(U_i, T, U_i, T) dt \geq \int_0^\ell \sum_{i=2}^{n} \overline{R}(\Phi(U_i), \overline{T}, \Phi(U_i), \overline{T}) dt$$

However,

$$R(U_i, T, U_i, T) = K(\Pi_{U_i, T}) \left(\langle U_i, U_i \rangle \langle T, T \rangle - \langle U_i, T \rangle^2 \right)$$

When $t = \ell$, because $\{e_i\}$ is orthonormal,

$$\operatorname{Ric}(\dot{\gamma(\ell)}) = \sum_{i=2}^{n} R(U_i, T, U_i, T)(\ell)$$

But it is not true for other point. So we need the further condition. Because M has constant sectional curvature and $U_i(0) = 0, U_i(\ell) = e_i$

$$U_i(t) = f(t) e_i(t), \quad \forall\ i = 2, 3, \cdots, n$$

where $f'' + \kappa f = 0$ with $f(0) = 0, f(\ell) = 1$ (Note that f is independent with i by the uniqueness of the solution.) Then for any $t \in [0, \ell]$,

$$\sum_{i=2}^{n} R(U_i, T, U_i, T)(t) = \sum_{i=2}^{n} f^2(t) R(e_i, T, e_i, T) = f^2(t) \operatorname{Ric}(\dot\gamma(t))$$

For the right hand side, because $\Phi(U_i)(t) = f(t)\overline{e}_i(t)$,

$$\sum_{i=2}^{n} \overline{R}(\Phi(U_i), \overline{T}, \Phi(U_i), \overline{T}) = f^2(t)\overline{\operatorname{Ric}}(\dot{\overline{\gamma}}(t))$$

Therefore, the inequality we need is

$$\int_0^\ell f^2(t) \operatorname{Ric}(\dot\gamma(t))dt \geq \int_0^\ell f^2(t)\overline{\operatorname{Ric}}(\dot{\overline{\gamma}}(t))dt$$

which is clearly true. $\square$

Remark. Note that if $\Delta\rho(\gamma(t)) \leq \overline{\Delta}\overline{\rho}(\overline{\gamma}(t))$, by

$$\overline{I}_0^t(\Phi(U_i), \Phi(U_i)) = \overline{I}_0^t(\overline{U}_i, \overline{U}_i)$$

$\Phi(U_i) = f(t)\overline{e}_i(t) = \overline{U}_i$, *i.e.* it is Jacobian,

$$\nabla_{\overline{T}}\nabla_{\overline{T}}\Phi(U_i) + \overline{R}(\Phi(U_i), \overline{T})\overline{T} = 0 \;\Rightarrow\; \frac{d^2}{dt^2} f(t) + f\overline{R}(\overline{e}_i, \overline{T}, \overline{e}_j, \overline{T}) = 0$$

But $f'' + \kappa f = 0$. Therefore, $\overline{R}(\overline{e}_i, \overline{T}, \overline{e}_j, \overline{T}) = \kappa$, which means $\overline{M}$ has constant sectional curvature.

Corollary 4.2.2. *Under the same conditions as above theorem. Let $f \in C^\infty(\mathbb{R}_+)$ with $f' \geq 0$. Then*

$$\Delta f(\rho)(\gamma(t)) \geq \overline{\Delta} f(\overline{\rho})(\overline{\gamma}(t))$$

Proof. By

$$\operatorname{Hess} f \circ \rho(X, X) = f''(t) \langle X, X\rangle^2 + f'(t) \operatorname{Hess} \rho(X^\perp, X^\perp)$$

we have

$$\Delta f(\rho)(t) = f''(t) + f'(t)\Delta\rho(t)$$

$\square$

Theorem 4.2.3 (Laplacian Comparison Theorem)**.** *Let (M, g) be a Riemannian manifold with dimension n and $\gamma\colon [0, \ell] \to M$ be a geodesic with $\gamma(0) = p$. Let $\rho(\cdot) = d(p, \cdot)$. Suppose that* $\operatorname{Ric} \geq (n-1)\kappa$ *and γ is minimizing and has no cut point.*

$$\Delta\rho(\gamma(t)) \leq \frac{f_\kappa'(r)}{f_\kappa(r)}(n-1)$$

where f_κ the solution of $f'' + \kappa f = 0$ with $f(0) = 0, f(\ell) = 1$.

Proof. Let $(\overline{M}, \overline{g})$ be a space form with curvature κ and $\overline{\gamma}\colon [0, \ell] \to \overline{M}$ be a geodesic. We only need to show γ is minimizing and has no cut point. It is clear when $\kappa \leq 0$. For $\kappa > 0$, because γ is minimizing and has no cut point, $\ell < \frac{\pi}{\sqrt{\kappa}}$ by Myers' Lemma. Then because $\overline{M}$ is isometric to the sphere $\mathbb{S}^n(1/\sqrt{\kappa})$, $\overline{\gamma}$ is minimizing and has no cut point. $\square$

3 ***Volume Comparison:*** Let (M, g) be a Riemannian manifold. Let (x, U) be a chart. Then define the integral by

$$\int_{x(U)} \sqrt{\det g_{ij}(x)}dx$$

Note this definition is independent with the choice of coordinate by the fact in section 1.1 and the change of variable of Euclidean integral. Moreover, by the partition of unity, we can choose (x_α, U_α) with ϕ_α and define

$$\mathrm{Vol}(M) = \sum_\alpha \int_{x_\alpha(U_\alpha)} \phi_\alpha \cdot \sqrt{\det g_{ij}(x)}dx_\alpha$$

Moreover, this definition is independent with the choice of (x_α, U_α) and ϕ_α,

$$\begin{aligned}
\mathrm{Vol}(M) &= \sum_\beta \int_{y_\beta(V_\beta)} \psi_\beta \cdot \sqrt{\det g_{ij}(y)}dy_\beta \\
&= \sum_\beta \sum_\alpha \int_{y_\beta(V_\beta)\cap x_\alpha(U_\alpha)} \phi_\alpha\psi_\beta \cdot \sqrt{\det g_{ij}(y)}dy_\beta \\
&= \sum_\beta \sum_\alpha \int_{y_\beta(V_\beta)\cap x_\alpha(U_\alpha)} \phi_\alpha\psi_\beta \cdot \sqrt{\det g_{ij}(x)}dx_\alpha \\
&= \sum_\alpha \int_{x_\alpha(U_\alpha)} \phi_\alpha \cdot \sqrt{\det g_{ij}(x)}dx_\alpha
\end{aligned}$$

Note that the final equality is because we interchange $\sum_\beta \sum_\alpha$ by $\sum_\alpha \sum_\beta$, which is valid because each term is nonnegative (by the series theory). Then we construct a linear functional

$$\mathcal{F} \colon C_c(M) \longrightarrow \mathbb{R}$$

by

$$\mathcal{F}(f) := \sum_\alpha \int_{x_\alpha(U_\alpha)} f(x)\phi_\alpha \cdot \sqrt{\det g_{ij}(x)}dx_\alpha$$

which is independent with the choice of the partition of unity (where the final inequality does not require the nonnegativity because of the compactness). Then $f \geq 0$ implies $\mathcal{F}(f) \geq 0$. Therefore, by the Riesz-Markov-Kakutani Representation Theorem (because M is a locally compact Hausdorff space), then there is a σ-algebra $\mathfrak{M}$ containing all Borel sets and a unique positive measure μ such that

$$\mathcal{F}(f) = \int_M f(x)d\mu(x), \quad \forall\ f \in C_c(M)$$

Moreover,

(1) $\mu(K) \leq \infty$ for all compact K;

(2) For any $E \in \mathfrak{M}$,

$$\mu(E) = \inf\{\mu(V) \colon E \subset V,\ V \text{ is open}\}$$

which is called outer-regular;

(3) because M is σ-compact (countable union of compact sets), for any any $E \in \mathfrak{M}$,

$$\mu(E) = \inf\{\mu(K) \colon K \subset E,\ K \text{ is compact}\}$$

which is called inner-regular.

So μ is a regular Borel measure on M.

Remark. (i) Consider the volume form

$$\Omega_0 = \sqrt{\det g_{ij}(x)} dx^1 \wedge \cdots dx^n$$

when M is orientable, above integral coincides with the integral of form.

(ii) If M is complete, then $M = \exp_p(E(p)) \sqcup C(p)$. Because any radical geodesic intersects $\tilde{C}(p)$ only once on T_pM, $\tilde{C}(p)$ has the Lebesgue measure 0. Because $\exp_p$ is smooth, $C(p)$ has measure 0.

Let (M, g) be a complete Riemannian manifold. In coordinate $\exp_p \colon E(p) \subset T_pM \to \exp_p(E(p))$, let $\left\{\frac{\partial}{\partial x^i}\right\}$ be the coordinate basis. Then g has the matrix expression as

$$g_{ij}(v) = \left(g^* \exp_p\right)_v \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{\exp_p(v)} \left((d\exp_p)_v \left(\frac{\partial}{\partial x^i}\right), (d\exp_p)_v \left(\frac{\partial}{\partial x^j}\right)\right), \quad \forall\, v \in E(p)$$

Because $C(p)$ has measure 0,

$$\operatorname{Vol}(M) = \int_{E(p)} \sqrt{\det(g_{ij}(x))} dx$$

Let $V \in E(p)$ with $|V| = 1$ and $t_0 < \tau(V)$. Let

$$F_i(t, s) = \exp_p \frac{t}{t_0} \left(t_0 V + s\frac{\partial}{\partial x^i}\right), \quad t \in [0, t_0]$$

with the corresponding Jacobian field $U_i(t)$,

$$U_i(0) = 0, \quad U_i(t_0) = (d\exp_p)_{t_0 V} \left(\frac{\partial}{\partial x^i}\right)$$

But this method is not valid because we cannot calculate it for each t_0.

Therefore, let's consider another way. Let $\varphi(x) = \sqrt{\det(g_{ij}(x))}$ and consider the polar coordinate $\varphi(t, \theta)$. Denote $\theta \sim V \in S_p$.

$$\operatorname{Vol}(M) = \int_0^{\tau(\theta)} \int_{\mathbb{S}^{n-1}} \varphi(t, \theta) t^{n-1} d\theta dt$$

where

$$\varphi(t, \theta) = \sqrt{\det g_{\exp_p(t\theta)} \left((d\exp_p)_{t\theta} \left(\frac{\partial}{\partial x^i}\right), (d\exp_p)_{t\theta} \left(\frac{\partial}{\partial x^j}\right)\right)}$$

Moreover, let's fix θ (*i.e.* $V \in S_p$) to calculate $\varphi(t) = \varphi(t, \theta)$.

$$(d\exp_p)_{t_0 V} \colon T_{t_0 V}(T_pM) \longrightarrow T_{\exp_p(t_0 V)}M$$

where $\left\{\frac{\partial}{\partial x^i}\right\}$ is an orthonormal basis of $(T_{t_0 V}(T_pM), g_p)$. If choosing an orthonormal basis $\{e_j\}$ in $T_{\exp_p(t_0 V)}M$ such that $d\exp_p\left(\frac{\partial}{\partial x^i}\right) = \alpha_i^j e_j$. Then

$$\varphi(t_0) = \det(\alpha_i^j)$$

Lemma 4.2.2. *Let $\gamma\colon [0,\ell] \to M$ be a normal geodesic containing no conjugate point. Let $J_1, \cdots, J_{n-1}$ be linearly independent normal Jacobian fields along γ such that $J_i(0) = 0$. Then we have*

$$\varphi(t) = \frac{\sqrt{\det\left(\langle J_i(t), J_j(t)\rangle\right)}}{\sqrt{\det\left(\langle t\nabla_T J_i(0), t\nabla_T J_j(0)\rangle\right)}} = \frac{\sqrt{\det\left(\langle J_i(t), J_j(t)\rangle\right)}}{t^{n-1}\sqrt{\det\left(\langle \nabla_T J_i(0), \nabla_T J_j(0)\rangle\right)}}$$

Remark. Because γ has no conjugate point, $J_1, \cdots, J_{n-1}$ are linearly independent if and only if $J_1(t_0), \cdots, J_{n-1}(t_0)$ are linearly independent for some $t_0 \neq 0$. Also, it is if and only if $\nabla_T J_1(0), \cdots, \nabla_T J_{n-1}(0)$ are linearly independent.

Proof. First, consider

$$(d\exp_p)_{t\dot\gamma(0)}\colon T_{t\dot\gamma(0)}(T_{\gamma(0)}M) \longrightarrow T_{\gamma(t)}M$$

And $J_1(t), \cdots, J_{n-1}(t), \dot\gamma(t)$ is a basis of $T_{\gamma(t)}M$ and by $J_i(t) \perp \dot\gamma(t)$ and $|\dot\gamma(t)| = 1$,

$$|J_1(t) \wedge \cdots \wedge J_{n-1}(t)| = \sqrt{\det\left(\langle J_i(t), J_j(t)\rangle\right)} = |\dot\gamma(t) \wedge J_1(t) \wedge \cdots \wedge J_{n-1}(t)|$$

Besides, $\nabla_T J_1(0), \cdots, \nabla_T J_{n-1}(0)$ are linear independent and

$$0 = \langle J_i(t), \dot\gamma(t)\rangle \;\Rightarrow\; 0 = \frac{d}{dt}\langle J_i(t), \dot\gamma(t)\rangle = \langle \nabla_T J_i(t), \dot\gamma(t)\rangle$$

that is $\nabla_T J_i(0) \perp \dot\gamma(0)$. It follows $t\nabla_T J_1(0), \cdots, t\nabla_T J_{n-1}(0), \dot\gamma(0)$ is a basis of $T_{\gamma(0)}M$. Therefore,

$$t^{n-1}\sqrt{\det\left(\langle \nabla_T J_i(0), \nabla_T J_j(0)\rangle\right)} = |\dot\gamma(0) \wedge t\nabla_T J_1(0) \wedge \cdots \wedge t\nabla_T J_{n-1}(0)|$$

For any t_0, we have

$$(d\exp_p)_{t_0\dot\gamma(0)}(t_0\nabla_T J_i(0)) = J_i(t_0), \quad (d\exp_p)_{t_0\dot\gamma(0)}(\dot\gamma(0)) = \dot\gamma(t_0)$$

by choosing the variation $F(t,s) = \exp_p t(\dot\gamma(0) + \nabla_T J_i(0))$. Then by linear algebra, we have the result. $\square$

Remark. For linear map $\Phi\colon V \to \widetilde{V}$ between two same dimensional inner product spaces. Let $\{e_i\}$ be an orthonormal basis of V and define

$$\det\Phi := \det\left(\langle \Phi(e_i), \Phi(e_j)\rangle\right)_{n\times n}$$

First, this definition is independent with the choice of the orthonormal basis $\{e_i\}$. Because if $\{e_i'\}$ is another orthonormal basis of V with $e_i' = \varepsilon_i^k e_k$, then

$$\left\langle \Phi(e_i'), \Phi(e_j')\right\rangle = \varepsilon_i^k \varepsilon_j^\ell \left\langle \Phi(e_k), \Phi(e_\ell)\right\rangle \;\Rightarrow\; \left(\left\langle \Phi(e_i'), \Phi(e_j')\right\rangle\right) = \left(\varepsilon_i^k\right)\left(\langle \Phi(e_k), \Phi(e_\ell)\rangle\right)\left(\varepsilon_j^\ell\right)^\top$$

and by $\left(\varepsilon_i^k\right)$ is orthonormal, we have $\det\left(\left\langle \Phi(e_i'), \Phi(e_j')\right\rangle\right) = \det\left(\langle \Phi(e_i), \Phi(e_j)\rangle\right)$. Moreover, if $\{v_i\}$ be another basis of V but it may be not orthonormal and $v_i = a_i^j e_j$, then similarly we have

$$\left(\langle \Phi(v_i), \Phi(v_j)\rangle\right) = \left(a_i^k\right)\left(\langle \Phi(e_k), \Phi(e_\ell)\rangle\right)\left(a_j^\ell\right)^\top$$

It follows that

$$\det\Phi = \frac{\det\left(\langle \Phi(v_i), \Phi(v_j)\rangle\right)}{\left(\det\left(a_i^k\right)\right)^2} = \frac{\det\left(\langle \Phi(v_i), \Phi(v_j)\rangle\right)}{|v_1 \wedge \cdots \wedge v_n|^2}$$

Furthermore, if let $\{\widetilde{e}_i\}$ be an orthonormal basis of $\widetilde{V}$ and $\Phi(e_i) = \alpha_i^j \widetilde{e}_j$, then we have

$$(\langle \Phi(e_i), \Phi(e_j) \rangle)_{n\times n} = \left(\alpha_i^k\right)\left(\alpha_i^k\right)^\top \Rightarrow \det \Phi = \left(\det\left(\alpha_i^k\right)\right)^2$$

Therefore, $\sqrt{\det \Phi} = |\Phi(e_1) \wedge \cdots \wedge \Phi(e_n)|$. And so

$$\sqrt{\det \Phi} = \frac{|\Phi(v_1) \wedge \cdots \wedge \Phi(v_n)|}{|v_1 \wedge \cdots \wedge v_n|}$$

Example 4.2.2 (Constant Sectional Curvature)**.** Let (M_κ, g) be a simply connected space form of sectional curvature κ. Let $e_1, e_2, \cdots, e_n = \dot{\gamma}(0)$ be a orthonormal basis of $T_{\gamma(0)}M$ and move them in parallel to get a frame $e_i(t)$. Let

$$J_i(t) = c_i f_\kappa(t) E_i(t)$$

such that $f(t)$ is the solution of

$$\begin{cases} f'' + \kappa f = 0 \\ f(0) = 0, \ f'(0) = 1 \end{cases}$$

Then

$$\varphi_\kappa(t, \theta) = \left(\frac{f_\kappa(t)}{t}\right)^{n-1}$$

Therefore,

$$\begin{aligned} \mathrm{Vol}(M_\kappa) &= \int_0^{\frac{\pi}{\sqrt{\kappa}}} \int_{\mathbb{S}^{n-1}} f_\kappa(t)^{n-1} dt d\theta \\ &= \mathrm{Vol}(\mathbb{S}^{n-1}) \int_0^{\frac{\pi}{\sqrt{\kappa}}} f_\kappa(t)^{n-1} dt \end{aligned}$$

and denote $\frac{\pi}{\sqrt{\kappa}} = \infty$ if $\kappa \le 0$.

Lemma 4.2.3. *Let $\gamma\colon [0, \ell] \to M$ be a normal geodesic containing no conjugate point. Let $\rho(\cdot) = d(\gamma(0), \cdot)$ and $\varphi(t) = \varphi(t, \dot{\gamma}(0))$. Then*

$$(\log \varphi(t))' = \frac{\varphi'(t)}{\varphi(t)} = \left(\Delta\rho - \frac{n-1}{\rho}\right)(\gamma(t))$$

Proof. WTLG, let $t = \ell$. Let $e_1, e_2, \cdot, e_n = \dot{\gamma}(\ell)$ be a orthonormal basis of $T_{\gamma(\ell)}M$. Let $J_1, \cdots, J_{n-1}$ be normal Jacobian fields along γ such that $J_i(0) = 0$, $J_i(\ell) = e_i$, and so they are independent and independent at each point. Note that

$$\varphi(t) = \frac{|J_1(t) \wedge \cdots \wedge J_{n-1}(t)|}{t^{n-1} |\nabla_T J_1(0) \wedge \cdots \wedge \nabla_T J_{n-1}(0)|} \ \Rightarrow \ \varphi(\ell) = \frac{1}{\ell^{n-1} |\nabla_T J_1(0) \wedge \cdots \wedge \nabla_T J_{n-1}(0)|}$$

Then

$$\left.\frac{\varphi'}{\varphi}\right|_{t=\ell} = \left.\frac{\frac{d}{dt}\varphi^2}{2\varphi^2}\right|_{t=\ell}$$

First,

$$\begin{aligned} \frac{d}{dt}\varphi^2 &= \frac{1}{|\nabla_T J_1(0) \wedge \cdots \wedge \nabla_T J_{n-1}(0)|^2} \frac{d}{dt}\left(\frac{1}{t^{2(n-1)}} \det\left(\langle J_i(t), J_j(t)\rangle\right)\right) \\ &= \ell^{2(n-1)} \varphi(\ell)^2 \left(-2(n-1)\frac{1}{t^{2n-1}} \det\left(\langle J_i(t), J_j(t)\rangle\right) + \frac{1}{t^{2(n-1)}} \frac{d}{dt} \det\left(\langle J_i(t), J_j(t)\rangle\right)\right) \end{aligned}$$

Besides,

$$\left.\frac{d}{dt}\right|_{t=\ell} \det\left(\langle J_i(t), J_j(t)\rangle\right) = \left.\frac{d}{dt}\right|_{t=\ell} \sum_\sigma \operatorname{sgn}(\sigma) \left\langle J_1(t), J_{\sigma(1)}(t)\right\rangle \cdots \left\langle J_{n-1}(t), J_{\sigma(n-1)}(t)\right\rangle$$

$$= \sum_\sigma \operatorname{sgn}(\sigma) \sum_{i=1}^{n-1} \left\langle J_1(l), J_{\sigma(1)}(l)\right\rangle \cdots \left.\frac{d}{dt}\right|_{t=\ell} \left\langle J_i(t), J_{\sigma(i)}(t)\right\rangle \cdots \left\langle J_{n-1}(l), J_{\sigma(n-1)}(l)\right\rangle$$

$$= \sum_{i=1}^{n-1} \left.\frac{d}{dt}\right|_{t=\ell} \langle J_i(t), J_i(t)\rangle$$

$$= 2\sum_{i=1}^{n-1} \langle \nabla_T J_i(\ell), J_i(\ell)\rangle$$

Therefore,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=\ell} \varphi^2 &= \ell^{2(n-1)}\varphi(\ell)^2 \left(-2(n-1)\frac{1}{\ell^{2n-1}} + \frac{2}{\ell^{2(n-1)}}\sum_{i=1}^{n-1} \langle \nabla_T J_i(\ell), J_i(\ell)\rangle\right) \\
&= 2\varphi(\ell)^2 \left(-(n-1)\frac{1}{\ell} + \sum_{i=1}^{n-1} \langle \nabla_T J_i(\ell), J_i(\ell)\rangle\right) \\
&= 2\varphi(\ell)^2 \left(-(n-1)\frac{1}{\ell} + \sum_{i=1}^{n-1} I(J_i, J_i)\right) \\
&= 2\varphi(\ell)^2 \left(-(n-1)\frac{1}{\ell} + \sum_{i=1}^{n-1} \operatorname{Hess}\rho(e_i, e_i)\right) \\
&= 2\varphi(\ell)^2 \left(-(n-1)\frac{1}{\rho} + \Delta\rho(\gamma(\ell))\right)
\end{aligned}$$

It follows the desired result. □

Remark. In particular, if M_κ is a space form of κ, then $\varphi(t) = \left(\frac{f_\kappa(t)}{t}\right)^{n-1}$. So

$$(\log\varphi(t))' = (n-1)\frac{t}{f_\kappa(t)}\frac{tf'_\kappa(t) - f_\kappa(t)}{t^2} = \Delta\rho - (n-1)\frac{1}{\rho}$$

Theorem 4.2.4 (Bishop)**.** *Let (M, g) be a Riemannian manifold with* Ric $\geq (n-1)\kappa$. *Let $\gamma\colon [0,\ell] \to M$ be a normal geodesic containing no cut point. Then*

$$t \mapsto \frac{\varphi(t)}{\varphi_\kappa(t)}, \quad t \in [0,\ell]$$

is non-increasing.

Remark. As $t \to 0$, by above example, $\varphi_\kappa \to 1$. In fact, by

$$\varphi(t,\theta) = \sqrt{\det g_{\exp_p(t\theta)}\left((d\exp_p)_{t\theta}\left(\frac{\partial}{\partial x^i}\right), (d\exp_p)_{t\theta}\left(\frac{\partial}{\partial x^j}\right)\right)}$$

as $t \to 0$, $\varphi \to 1$. Therefore, for any $0 < t_1 < t \leq \ell$,

$$\frac{\varphi(t)}{\varphi_\kappa(t)} \leq \frac{\varphi(t_1)}{\varphi_\kappa(t_1)} \to 0, \quad t_1 \to 0$$

It follows that $\varphi(t) \leq \varphi_\kappa(t)$.

Proof. Let $(\overline{M}, \overline{g})$ be a simply-connected space form of κ. Note that $\overline{\mathrm{Ric}} = (n-1)\kappa g$. Then by $\mathrm{Ric} \geq \overline{\mathrm{Ric}}$, We have

$$\Delta\rho \leq \overline{\Delta}\overline{\rho}$$

Therefore, by above lemma

$$\begin{aligned}
(\log\varphi(t))' &= \left(\Delta\rho - (n-1)\frac{1}{\rho}\right)(\gamma(t)) \\
(\log\varphi_\kappa(t))' &= \left(\overline{\Delta}\overline{\rho} - (n-1)\frac{1}{\overline{\rho}}\right)(\overline{\gamma}(t))
\end{aligned}$$

for $0 < t_1 < t_2 \leq \ell$, we have

$$(\log\varphi(t))' \leq (\log\varphi_\kappa(t))' \;\Rightarrow\; \log\frac{\varphi(t_2)}{\varphi_\kappa(t_2)} \leq \log\frac{\varphi(t_1)}{\varphi_\kappa(t_1)} \;\Rightarrow\; \frac{\varphi(t_2)}{\varphi_\kappa(t_2)} \leq \frac{\varphi(t_1)}{\varphi_\kappa(t_1)}$$ □

Corollary 4.2.3 (Bishop)**.** *Let (M, g) be a complete Riemannian manifold with* Ric $\geq$ *$(n-1)\kappa$ for $\kappa > 0$. Then*

$$\mathrm{Vol}(M) \leq \mathrm{Vol}\left(\mathbb{S}^n(1/\sqrt{\kappa})\right)$$

where "=" if and only if M is isometric to $\mathbb{S}^n(1/\sqrt{\kappa})$.

Proof. First,

$$\begin{aligned}
\mathrm{Vol}(M) &= \int_0^{\tau(\theta)}\int_{\mathbb{S}^{n-1}} \varphi(t,\theta)t^{n-1}d\theta dt \\
&\leq \int_0^{\frac{\pi}{\sqrt{\kappa}}}\int_{\mathbb{S}^{n-1}} \varphi_\kappa(t,\theta)t^{n-1}d\theta dt \\
&= \mathrm{Vol}\left(\mathbb{S}^n(1/\sqrt{\kappa})\right)
\end{aligned}$$

Furthermore, "=" if and only if

$$\Delta\rho = \overline{\Delta}\overline{\rho}$$

which is equivalent to M has constant sectional curvature κ. Moreover, it should be simply connected. Otherwise, because its universal covering is isometric to $\mathbb{S}^n(1/\sqrt{\kappa})$, the volume of the universal covering is $\mathrm{Vol}\left(\mathbb{S}^n(1/\sqrt{\kappa})\right)$, it follows that $\mathrm{Vol}(M) < \mathrm{Vol}\left(\mathbb{S}^n(1/\sqrt{\kappa})\right)$, contradicting to the assumption. □

Remark. If (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)\kappa$ for $\kappa \leq 0$. Then for $p \in M$

$$\mathrm{Vol}(B_p(r)) \leq \mathrm{Vol}\left(B^\kappa(r)\right), \quad \forall\, r > 0$$

and "=" if and only if M is isometric to M_κ.

Theorem 4.2.5 (Bishop-Gromov)**.** *Let (M, g) be a Riemannian manifold with* Ric $\geq$ *$(n-1)\kappa$. Let $p \in M$ be any point. Then the function*

$$r \mapsto \frac{\mathrm{Vol}\, B_p(r)}{\mathrm{Vol}\, B^\kappa(r)}$$

is non-increasing.

Proof. First, let r is small such that $B_p(r)$ contains no cut point. Then

$$\frac{\mathrm{Vol}\, B_p(r)}{\mathrm{Vol}\, B^\kappa(r)} = \frac{\int_0^r\int_{\mathbb{S}^{n-1}} \varphi(t,\theta)t^{n-1}d\theta dt}{\int_0^r\int_{\mathbb{S}^{n-1}} \varphi_\kappa(t)t^{n-1}d\theta dt}$$

Consider the function

$$t \mapsto \frac{\int_{\mathbb{S}^{n-1}} \varphi(t,\theta)t^{n-1}d\theta}{\int_{\mathbb{S}^{n-1}} \varphi_\kappa(t)t^{n-1}d\theta} = \int_{\mathbb{S}^{n-1}} \frac{\varphi(t,\theta)}{\varphi_\kappa(t)}d\theta$$

which is non-increasing as above. Then by the following lemma, we have the desired result. Next, if $B_p(r)$ contains cut points, then let χ be the characteristic function of $E(p) \subset T_pM$. For χ_κ on M^κ, when $\kappa \leq 0$, $\chi_\kappa \equiv 1$. When $\kappa > 0$,

$$\chi_k(r,\theta) := \begin{cases} 1, & r < \dfrac{\pi}{\sqrt{\kappa}} \\ 0, & r \geq \dfrac{\pi}{\sqrt{\kappa}} \end{cases}$$

Note that because $\mathrm{Ric} \geq (n-1)\kappa$, $r \mapsto \frac{\chi(r,\theta)}{\chi_\kappa(r,\theta)}$ is non-increasing, so

$$t \mapsto= \int_{\mathbb{S}^{n-1}} \frac{\varphi(t,\theta)\chi(r,\theta)}{\varphi_\kappa(t)\chi_\kappa(r,\theta)}d\theta$$

is non-increasing. Then by

$$\begin{aligned} \frac{\operatorname{Vol} B_p(r)}{\operatorname{Vol} B^\kappa(r)} &= \frac{\operatorname{Vol} B_p(r)\backslash C(p)}{\operatorname{Vol} B^\kappa(r)\backslash C^\kappa(p)} \\ &= \frac{\int_0^r \int_{\mathbb{S}^{n-1}} \varphi(t,\theta)\chi(r,\theta)t^{n-1}d\theta dt}{\int_0^r \int_{\mathbb{S}^{n-1}} \varphi_\kappa(t)\chi_\kappa(r)t^{n-1}d\theta dt} \end{aligned}$$

and the following lemma, we have the desired result. $\square$

Lemma 4.2.4. *Let $f, g\colon (0,\infty) \to \mathbb{R}$ be two functions with $f, g \geq 0$. Suppose $t \mapsto \frac{f(t)}{g(t)}$ is non-increasing. Then*

$$t \mapsto \frac{\int_0^t f(s)ds}{\int_0^t g(s)ds}$$

is non-increasing.

Proof. For any $t_1 \leq t_2$, we want to show

$$\begin{aligned} \frac{\int_0^{t_1} f(s)ds}{\int_0^{t_1} g(s)ds} \geq \frac{\int_0^{t_2} f(s)ds}{\int_0^{t_2} g(s)ds} &\Leftrightarrow \int_0^{t_1} f(s)ds \int_0^{t_2} g(s)ds \geq \int_0^{t_2} f(s)ds \int_0^{t_1} g(s)ds \\ &\Leftrightarrow \int_0^{t_1} f(s)ds \int_{t_1}^{t_2} g(s)ds \geq \int_{t_1}^{t_2} f(s)ds \int_0^{t_1} g(s)ds \end{aligned}$$

Let $h = \frac{f}{g}$. We have

$$\begin{aligned} \int_0^{t_1} f(s)ds \int_{t_1}^{t_2} g(s)ds &= \int_0^{t_1} h(s)g(s)ds \int_{t_1}^{t_2} g(s)ds \\ &\geq \int_0^{t_1} h(t_1)g(s)ds \int_{t_1}^{t_2} g(s)ds \\ &\geq \int_0^{t_1} g(s)ds \int_{t_1}^{t_2} h(t_1)g(s)ds \\ &\geq \int_0^{t_1} g(s)ds \int_{t_1}^{t_2} h(s)g(s)ds \\ &= \int_{t_1}^{t_2} f(s)ds \int_0^{t_1} g(s)ds \end{aligned}$$

by the fact that h is non-increasing. $\square$

Theorem 4.2.6 (Maximal Diameter Theorem)**.** *Let* (M, g) *be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)\kappa$ *for* $\kappa > 0$. *Then* $\mathrm{diam}(M) = \frac{\pi}{\sqrt{\kappa}}$ *if and only is* M *is isometric to* $\mathbb{S}^n(1/\sqrt{\kappa})$.

Proof. Note that

$$\mathrm{Vol}\left(\mathbb{S}^n\left(\frac{1}{\sqrt{\kappa}}\right)\right) = \mathrm{Vol}\left(B^1(r)\right) + \mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right)$$

and

$$\mathrm{Vol}(M) \geq \mathrm{Vol}(B_p(r)) + \mathrm{Vol}\left(B_q\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right)$$

where $p, q \in M$ with $d(p,q) = \frac{\pi}{\sqrt{\kappa}}$, because $B_p(r) \cap B_q\left(\frac{\pi}{\sqrt{\kappa}} - r\right) = \varnothing$. Moreover, by above theorem

$$\begin{aligned}
\mathrm{Vol}(B_p(r)) &= \frac{\mathrm{Vol}(B_p(r))}{\mathrm{Vol}(B^1(r))}\mathrm{Vol}\left(B^1(r)\right) \\
&\geq \frac{\mathrm{Vol}\left(B_p\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)}{\mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)}\mathrm{Vol}\left(B^1(r)\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right) &= \frac{\mathrm{Vol}\left(B_p\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right)}{\mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right)}\mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right) \\
&\geq \frac{\mathrm{Vol}\left(B_p\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)}{\mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)}\mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right)
\end{aligned}$$

Therefore,

$$\mathrm{Vol}(M) \geq \frac{\mathrm{Vol}\left(B_p\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)}{\mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)}\mathrm{Vol}\left(\mathbb{S}^n\left(\frac{1}{\sqrt{\kappa}}\right)\right) = \mathrm{Vol}\left(B_p\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)$$

But $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{\kappa}}$, so

$$\mathrm{Vol}(M) = \mathrm{Vol}\left(B_p\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)$$

and

$$\mathrm{Vol}(M) = \mathrm{Vol}(B_p(r)) + \mathrm{Vol}\left(B_q\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right)$$

and

$$\frac{\mathrm{Vol}(B_p(r))}{\mathrm{Vol}(B^1(r))} = \frac{\mathrm{Vol}\left(B_p\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)}{\mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}}\right)\right)}, \quad \forall\, 0 < r \leq \frac{\pi}{\sqrt{\kappa}}$$

As $r \to 0$, the LHS $\to 1$. Therefore,

$$\mathrm{Vol}(M) = \mathrm{Vol}\left(B_p\left(\frac{\pi}{\sqrt{\kappa}}\right)\right) = \mathrm{Vol}\left(B^1\left(\frac{\pi}{\sqrt{\kappa}}\right)\right) = \mathrm{Vol}\left(\mathbb{S}^n\left(\frac{1}{\sqrt{\kappa}}\right)\right)$$

Then by above, it is if and only is M is isometric to $\mathbb{S}^n(1/\sqrt{\kappa})$. $\square$

Remark. Note that on $\mathbb{S}^n(1/\sqrt{\kappa})$, let p, q be $d(p,q) = \frac{\pi}{\sqrt{\kappa}}$, then for any $z \in \mathbb{S}^n(1/\sqrt{\kappa})$,

$$d(p,z) + d(z,q) = \frac{\pi}{\sqrt{\kappa}}$$

i.e. any z lies in some shortest geodesic connecting p, q. In fact, for any M with

$$\mathrm{Vol}\,(M) = \mathrm{Vol}\,(B_p\,(r)) + \mathrm{Vol}\left(B_q\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right)$$

it has the similar property. For any z with $d(p,z) = r$

$$d(q,z) \geq d(p,q) - d(p,z) = \frac{\pi}{\sqrt{\kappa}} - r$$

If it takes $>$, then there is a $\varepsilon > 0$ such that

$$B_z(\varepsilon) \cap B_q\left(\frac{\pi}{\sqrt{\kappa}} - r\right) = \varnothing$$

By completeness, $\mathrm{Vol}\,(B_z(\epsilon) \backslash B_p(r)) \neq 0$. Therefore,

$$\mathrm{Vol}\,(B_p\,(r)) + \mathrm{Vol}\left(B_q\left(\frac{\pi}{\sqrt{\kappa}} - r\right)\right) < \mathrm{Vol}\,(M)$$

contradicting to the assumption.

Theorem 4.2.7. *Let (M, g) be a complete non-compact Riemannian manifold of dimension n with* $\mathrm{Ric} \geq 0$. *Let $p \in M$. Then there is a constant $C = C(p,n)$ such that*

$$\mathrm{Vol}(B_p(r)) \geq cr, \quad \forall\; r > 2$$

Proof. There is a geodesic $\gamma \colon [0, \infty) \to M$ with $\gamma(0) = p$ such that $d(p, \gamma(t)) = t$ by the following lemma. For any t, consider two balls $B_{\gamma(t)}(t-1), B_{\gamma(t)}(t+1)$, then we have

$$\frac{\mathrm{Vol}(B_{\gamma(t)}(t+1))}{\mathrm{Vol}(B_{\gamma(t)}(t-1))} \leq \frac{\mathrm{Vol}(B^0(t+1))}{\mathrm{Vol}(B^0(t-1))} = \frac{(t+1)^n}{(t-1)^n}$$

Because $B_p(1) \subset B_{\gamma(t)}(t+1) \backslash B_{\gamma(t)}(t-1)$,

$$\mathrm{Vol}(B_p(1)) \leq \mathrm{Vol}(B_{\gamma(t)}(t+1)) - \mathrm{Vol}(B_{\gamma(t)}(t-1))$$

Then

$$\frac{\mathrm{Vol}(B_p(1))}{\mathrm{Vol}(B_{\gamma(t)}(t-1))} \leq \frac{\mathrm{Vol}(B_{\gamma(t)}(t+1))}{\mathrm{Vol}(B_{\gamma(t)}(t-1))} - 1 \leq \frac{(t+1)^n}{(t-1)^n} - 1$$

It follows that

$$\mathrm{Vol}(B_p(2t-1) \geq \mathrm{Vol}(B_{\gamma(t)}(t-1)) \geq \mathrm{Vol}(B_p(1))\frac{(t-1)^n}{(t+1)^n - (t-1)^n}$$

by $B_{\gamma(t)}(t-1) \subset B_p(2t-1)$. Taking $r = 2t-1$,

$$\mathrm{Vol}(B_p(r) \geq \mathrm{Vol}(B_p(1))\frac{(\frac{r+1}{2} - 1)^n}{(\frac{r+1}{2} + 1)^n - (\frac{r+1}{2} - 1)^n}$$

□

Lemma 4.2.5. *If Riemannian manifold M is non-compact and complete, then for any $p \in M$ there is a geodesic $\gamma\colon [0,\infty) \to M$ with $\gamma(0) = p$ such that $d(p, \gamma(t)) = t$.*

Proof. Because M is non-compact and complete, there is sequence $q_i \in M$ such that

$$d(p, q_i) \to \infty$$

Then let $t \mapsto \exp_p(tv_i)$ with $v_i \in S_p$ be geodesic connecting p and q_i and so $t \in [0, d(p, q_i)]$. By the compactness of S_p, we can assume $v_i \to v_0 \in S_p$. Consider

$$t \mapsto \exp_p(tv_0)$$

Because $\exp_p$ and d are continuous,

$$d(p, \exp_p(tv_0)) = \lim_i d(p, \exp_p(tv_i)) = t, \quad \forall\ t$$ □

Theorem 4.2.8. *Let (M, g) be a complete Riemannian manifold with the sectional curvature $\leq \kappa$. When $B_p(r)$ contains no cut point,*

$$\mathrm{Vol}(B_p(r)) \geq \mathrm{Vol}(B^\kappa(r))$$

4.3 Splitting Theorem

Theorem 4.3.1. *Let (M, g) be a complete non-compact Riemannian manifold of dimension n with* $\mathrm{Ric} \geq 0$. *If it contains a line,* i.e. *a geodesic $\gamma\colon (-\infty, \infty) \to M$ such that $d(\gamma(s), \gamma(t)) = |t - s|$. Then M is isometric to $M' \times \mathbb{R}$, where (M', g') is a complete Riemannian manifold with with* $\mathrm{Ric} \geq 0$.

1. ***Busemann function:*** Let $O \in M$ and a normal geodesic ray $\gamma\colon [0,\infty) \to M$ with $\gamma(0) = O$. For any $t \geq 0$,
$$b_t^\gamma(x) = t - d(x, \gamma(t))$$

   Proposition 4.3.1. *For above $b_t^\gamma(x)$,*

   (1) $b_t^\gamma(x) \leq d(O, x)$, and so $\{b_t^\gamma : t \geq 0\}$ is uniformly bounded on a compact subset.

   (2) For a given $x \in M$, $b_t^\gamma(x)$ is nondecreasing in t.

   (3) $\{b_t^\gamma : t \geq 0\}$ is equicontinuous.

   Proof. (1) It is clear by the triangular inequality.

   (2) For $0 \leq s < t$,

$$\begin{aligned} b_t^\gamma(x) - b_s^\gamma(x) &= (t - s) + d(x, \gamma(s)) - d(x, \gamma(t)) \\ &\geq d(\gamma(t), \gamma(s)) + d(x, \gamma(s)) - d(x, \gamma(t)) \\ &\geq 0 \end{aligned}$$

   (3) For any $x, y \in M$,

$$\begin{aligned} |b_t^\gamma(x) - b_t^\gamma(y)| &= |d(y, \gamma(t)) - d(x, \gamma(t))| \\ &\leq d(x, y) \end{aligned}$$

   So they are equicontinuous.

□

Then by the Arzelà–Ascoli theorem,

$$b^\gamma(x) := \lim_{t\to\infty} t - d(x, \gamma(t))$$

is well-defined. Consider the level set of b_t^γ,

$$\{x \in M : b_t^\gamma(x) = c\} \;\Leftrightarrow\; \{x \in M : d(x, \gamma(t)) = t - c\}$$

So $b^\gamma(x)$ "measure" the difference of $d(O, \gamma(\infty))$ and $d(x, \gamma(\infty))$, which is basically the "(signed)distance" between O and the projection of x onto γ.

Proposition 4.3.2. *We have $b^\gamma(\gamma(t)) = t$. If M contains a line γ, then let $\gamma^+ = \gamma|_{[0,\infty)}$ we have*

$$b^{\gamma^+}(\gamma(t)) = t, \quad \forall\, t \in (-\infty, \infty)$$

Proof. By definition,

$$b^{\gamma^+}(\gamma(t)) = \lim_{s\to\infty} s - d(\gamma(t), \gamma^+(s)) = \lim_{s\to\infty} s - |t - s| = t \qquad \square$$

2. ***Smoothness:*** Given a normal ray γ, for any $p \in M$ and any t_k with $\gamma(t_k) \to \infty$, let

$$\{\delta_k(\cdot) : k = 1, 2, \cdots\}$$

be a family of normal geodesics connecting p with $\gamma(t_k)$. Then by compactness of S_p, there is a subsequence

$$\lim_k \dot{\delta}_k(0) = v \;\Rightarrow\; \delta(t) = \exp_p(tv)$$

where the normal ray $\delta(t)$ is called an asymptote of γ. Note that δ is not unique and $\lim_k \delta_k(t) = \delta(t)$.

Proposition 4.3.3. *Given a ray γ. For $p \in M$, let δ be an asymptote of γ through p. Then we have*

(1) $b^\gamma(x) \geq b^\gamma(p) + b_r^\delta(x)$ for any $x \in M$, $r \geq 0$. In particular, as $r \to \infty$,

$$b^\gamma(x) \geq b^\gamma(p) + b^\delta(x)$$

(2) $b^\gamma(\delta(r)) = b^\gamma(p) + b^\delta(\delta(r))$ for any $r \geq 0$.

Proof. Let t_k be chosen as above such that $\lim_k \delta_k(t) = \delta(t)$.

(1) First, because δ_k connects p with $\gamma(t_k)$

$$\begin{aligned} b^\gamma(x) - b^\gamma(p) &= \lim_{k\to\infty} -d(x, \gamma(t_k)) + d(p, \gamma(t_k)) \\ &= \lim_{k\to\infty} -d(x, \gamma(t_k)) + d(p, \delta_k(r)) + d(\delta_k(r), \gamma(t_k)) \\ &= r + \lim_{k\to\infty} d(\delta_k(r), \gamma(t_k)) - d(x, \gamma(t_k)) \\ &\geq r + \lim_{k\to\infty} -d(\delta_k(r), x) \\ &= r - d(\delta(r), x) = b_r^\delta(x) \end{aligned}$$

(2) First,

$$\begin{aligned}b^\gamma(p) &= \lim_k (t_k - d(p,\gamma(t_k)))\\ &= \lim_k (t_k - d(p,\delta_k(r)) - d(\delta_k(r),\gamma(t_k)))\\ &= \lim_k (t_k - d(\delta_k(r),\gamma(t_k))) - r\end{aligned}$$

Because

$$d(\delta(r),\gamma(t_k)) - d(\delta_k(r),\delta(r)) \le d(\delta_k(r),\gamma(t_k)) \le d(\delta(r),\gamma(t_k)) + d(\delta_k(r),\delta(r))$$

and $d(\delta_k(r),\delta(r)) \to 0$ as $r \to \infty$, we have

$$b^\gamma(p) = \lim_k (t_k - d(\delta(r),\gamma(t_k))) - r = b^\gamma(\delta(r)) - r \qquad \square$$

Denote

$$\begin{aligned}b^\gamma_{p,r}(x) &= b^\gamma(p) + b^\delta_r(x)\\ &= b^\gamma(p) + r - d(x,\delta(r))\end{aligned}$$

Because δ is a ray with $d(p,\delta(t)) = t$, it is always the shortest geodesic. So it contains no cut point, which means if x is around a sufficiently small neighborhood of p, $d(x,\delta(r))$ is smooth so is $b^\gamma_{p,r}(x)$. Note that $b^\gamma_{p,r}(x) \le b^\gamma(x)$ and $b^\gamma_{p,r}(p) = b^\gamma(p)$. Moreover, by assuming $\mathrm{Ric} \ge 0$,

$$\begin{aligned}\Delta b^\gamma_{p,r}(x) &= -\Delta d(x,\delta(r))\\ &\ge -(n-1)\frac{f_\kappa'(\rho)}{f_\kappa(\rho)} = -(n-1)\frac{1}{d(x,\delta(r))}\end{aligned}$$

Therefore, for any $\varepsilon > 0$, we can choose sufficiently large r such that $\Delta b^\gamma_{p,r}(x) \ge -\varepsilon$ on a small neighborhood of p.

Definition 4.3.1. (1) A lower barrier function (or support function) for a continuous function f at x_0 is a C^2 function g is defined on a neighborhood of x such that $g(x) \le f(x)$ and $g(x_0) = f(x_0)$.

(2) A function in $f \in C^0(M)$ is said that $\Delta f \ge a$ at x_0 in the barrier sense if for any $\varepsilon > 0$, there is a lower barrier function $f_{x_0,\varepsilon}$ of f at x_0 such that $\Delta f_{x_0,\varepsilon} \ge a - \varepsilon$.

Remark. Therefore, Busemann function b^γ is subharmonic *i.e.* $\Delta b^\gamma \ge 0$ in the barrier sense.

Theorem 4.3.2 (Hopf-Calabi Maximal Principle)**.** *Let M be a connected Riemannian manifold and $f \in C^0(M)$. If f is subharmonic in the barrier sense, then f attaches no local maximum unless it is constant.*

Proof. Let p be a weak local maximum, *i.e.* for a neighborhood V of p,

$$f(p) \ge f(x), \quad \forall x \in V$$

Let $B_p(\delta)$ for a small δ. Assume there is a $z \in \partial B_p(\delta)$ such that $f(p) > f(z)$, which implies $f(p) > f(z')$ for $z' \in \partial B_p(\delta)$ closed to z. Choosing a normal chart (x,U) such that $z = (\delta, 0, \cdots, 0)$. Set

$$\phi(x) = x_1 - \alpha(x_2^2 + \cdots + x_n^2)$$

for sufficiently large α such that if $y \in \partial B_p(\delta)$ and $f(y) = f(p)$ then $\phi(y) < 0$. Besides,

$$\frac{\partial \phi}{\partial x_1} \neq 0 \;\Rightarrow\; \operatorname{grad} \phi = g^{ij} \frac{\partial \phi}{\partial x^j} \frac{\partial}{\partial x^i} \neq 0$$

Let $\psi = e^{a\phi} - 1$. Then

$$\Delta \psi = e^{a\phi} \left(a^2 \left| \operatorname{grad} \phi \right|^2 + a \Delta \phi \right) > 0$$

(Note that $\Delta(f \circ \phi) = f'' \left| \operatorname{grad} \phi \right|^2 + f' \Delta \phi$) by choosing sufficiently large a. Moreover, $\psi(p) = 0$. Then we can choose small η such that

$$(f + \eta\psi)|_{\partial B_p(\delta)} < f(p), \quad (f + \eta\psi)(p) = f(p)$$

It follows that $f + \eta\psi$ has an inner maximum $q \in B_p(\delta)$. Let $f_{q,\varepsilon}$ be a barrier of f at q. Then

$$\Delta f_{q,\varepsilon} \geq -\varepsilon$$

Consider $f_{q,\varepsilon} + \eta\psi$ that is a barrier function of $f + \eta\psi$ at q. We have

$$\Delta(f_{q,\varepsilon} + \eta\psi) \geq -\varepsilon + \eta \Delta \psi > 0$$

when ε is chosen sufficiently small. However, around q,

$$(f_{q,\varepsilon} + \eta\psi)(x) \leq (f + \eta\psi)(x) \leq (f + \eta\psi)(q) = (f_{q,\varepsilon} + \eta\psi)(q)$$

which contradicts to $\Delta > 0$. $\square$

Theorem 4.3.3 (Regularity)**.** *If f is harmonic ($\Delta f = 0$) in barrier sense, then f is smooth.*

Proposition 4.3.4. *Assume* Ric ≥ 0 *and γ is a line. Let $\gamma^+ = \gamma|_{[0,+\infty)}$ and $\gamma^- = -\gamma|_{(-\infty,0]}$. Then*

$$b^{\gamma^+} + b^{\gamma^-} = 0$$

and they are smooth. So they are harmonic.

Proof. In fact, because Busemann functions are submanifold in barrier sense, $b^{\gamma^+} + b^{\gamma^-} = 0$ implies that

$$\Delta b^{\gamma^+} = \Delta(-b^{\gamma^-})$$

so that $-b^{\gamma^-}$ is subharmonic, which means b^{γ^-} is harmonic in barrier sense, so is $b^{\gamma+}$. Then by above theorem, they are smooth. So it is sufficient to check the identity. First, we know

$$\Delta \left(b^{\gamma^+} + b^{\gamma^-} \right) \geq 0$$

in barrier sense. For any $x \in M$,

$$d(x, \gamma^+(t)) + d(x, \gamma^-(s)) \geq t + s \;\Rightarrow\; t - d(x, \gamma^+(t)) + s - d(x, \gamma^-(s)) \leq 0$$

As $t, s \to \infty$,

$$b^{\gamma^+}(x) + b^{\gamma^-}(x) \leq 0$$

At $x = \gamma(0)$, $b^{\gamma^+}(x) + b^{\gamma^-}(x) = 0$. Then by Hopf-Calabi Maximal Principle, we have the identity. $\square$

3. ***Submanifold:*** Consider the Riemannian submanifold.

Definition 4.3.2. (1) For two Riemannian manifolds (M, g) and $(\overline{M}, \overline{g})$, let $f\colon M \to \overline{M}$ be an immersion. If $f^*\overline{g} = g$, then (M, g) is called an (immersed) Riemannian submanifold.

(2) Let (M, g) be a Riemannian submanifold of $(\overline{M}, \overline{g})$. Let $p \in M$ and $p = f(p) \in \overline{M}$, *i.e.* we do not distinguish M and $f(M) \in \overline{M}$. Then

$$T_p\overline{M} = T_pM \oplus (T_pM)^{\perp}$$

$(T_pM)^{\perp}$ is called the orthogonal complement of T_pM. Let $\Pi\colon \bigsqcup_{q\in M} T_q\overline{M} \to TM$ by $\Pi(X_q) = X_q - (X_q)^{\perp}$.

(3) (M, g) is called totally geodesic if for any geodesic $\gamma \in \overline{M}$ with $\gamma(0) \in M$ and $\dot{\gamma}(0) \in T_pM$, $\gamma \in M$.

For any $X, Y \in \Gamma^{\infty}(TM)$, which can be extended to $\Gamma^{\infty}(T\overline{M})$,

$$\nabla_X Y = \Pi(\overline{\nabla}_X Y)$$

is the Levi-Civita connection of M. Therefore, for a curve $\gamma \in M$,

$$\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0 \;\Rightarrow\; \nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

Therefore, if γ is a geodesic in $\overline{M}$, then also a geodesic in M. The second fundamental form is defined as

$$B(X, Y) := \overline{\nabla}_X Y - \nabla_X Y$$

(1) For any $f \in C^{\infty}(M)$, $B(fX, Y) = B(X, fY) = fB(X, Y)$.

(2) $B(X, Y) = B(Y, X)$.

Theorem 4.3.4. *(M, g) is a totally geodesic submanifold of $(\overline{M}, \overline{g})$ if and only if $B \equiv 0$.*

Proof. "$\Rightarrow$": By the symmetry, it is sufficient to prove $B(V, V) = 0$ for any $V \in T_pM$. Let γ be a geodesic in $\overline{M}$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$, *i.e.* $\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$. Because M is totally geodesic, $\gamma \subset M$ and $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Therefore,

$$B(V, V) = \overline{\nabla}_{\dot{\gamma}}\dot{\gamma}(0) - \nabla_{\dot{\gamma}}\dot{\gamma}(0) = 0$$

"$\Leftarrow$": For any $p \in M$ and $V \in T_pM$, let γ be a geodesic in $\overline{M}$ with $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = V \in T_pM$. Let ξ be a geodesic in M with $\xi(0) = p \in M$ and $\dot{\xi}(0) = V \in T_pM$, that is $\nabla_{\dot{\xi}}\dot{\xi} = 0$. But

$$B(\dot{\xi}, \dot{\xi}) = 0 \;\Rightarrow\; \overline{\nabla}_{\dot{\xi}}\dot{\xi} = 0$$

which means ξ is also a geodesic in $\overline{M}$. By the uniqueness of geodesic, $\gamma = \xi \subset M$. $\square$

Corollary 4.3.1. *(M, g) is a totally geodesic submanifold of $(\overline{M}, \overline{g})$. For any 2-section $\Pi_p \subset T_pM \subset T_p\overline{M}$,*

$$K(\Pi_p) = \overline{K}(\Pi_p)$$

Example 4.3.1. Let (M, g) be a Riemannian manifold and $o \in M$ and $\rho(\cdot) = d(o, \cdot)$. ρ is smooth on $M \backslash \{o, C(o)\}$ and $\operatorname{grad}\rho = \dot{\gamma}$. Let $S_t = \{x \in M : \rho(x) = t\}$. For any $X, Y \in \Gamma^\infty(TS_t)$,

$$\begin{aligned}
\operatorname{Hess}(\rho)(X, X) &= \nabla_Y(\nabla_X \rho) - \nabla\rho(\nabla_Y X) \\
&= Y(X(\rho)) - \nabla_Y X(\rho) \\
&= Y \langle X, \operatorname{grad}\rho\rangle - \langle \nabla_Y X, \operatorname{grad}\rho\rangle \\
&= -\langle \nabla_Y X, \operatorname{grad}\rho\rangle \\
&= -\langle \nabla_X Y, \operatorname{grad}\rho\rangle - \langle [X, Y], \operatorname{grad}\rho\rangle \\
&= -\langle \nabla_X Y, \operatorname{grad}\rho\rangle
\end{aligned}$$

Therefore, $\operatorname{Hess}\rho(X, X) \cdot (-\operatorname{grad}\rho) = B(X, X)$

Next, let's consider the level set of b^{γ^+}.

Proposition 4.3.5. *Under same condition as above,* $\left|\operatorname{grad} b^{\gamma^+}\right| = 1$.

Proof. Because the Busemann function is 1-Lipschitz,

$$\left|\operatorname{grad} b^{\gamma^+}\right| \leq 1$$

For any $p \in M$, let δ^γ be an (normal)asymptote of γ^+ starting from p. At p,

$$\begin{aligned}
\left|\operatorname{grad} b^{\gamma^+}\right| &= \left|\operatorname{grad} b^{\gamma^+}\right| \left|\dot{\delta^\gamma}(0)\right| \\
&\geq \left|\left\langle \operatorname{grad} b^{\gamma^+}, \dot{\delta^\gamma}(0)\right\rangle\right| \\
&= \left.\frac{d}{dt}\right|_{t=0} b^{\gamma^+}(\delta(t)) \\
&= \lim_{t \to 0} \frac{b^{\gamma^+}(\delta(t)) - b^{\gamma^+}(p)}{t} = 1
\end{aligned}$$

Therefore, $\left|\operatorname{grad} b^{\gamma^+}\right| = 1$. $\square$

Corollary 4.3.2. *Under the same conditions, all level sets of* b^{γ^+} *are hyper-submanifolds.*

By the Bochner's formula(**Theorem** 3.1.3) and $\operatorname{Ric} \geq 0$,

$$\left|\operatorname{Hess}(b^{\gamma^+})\right|^2 \leq 0 \;\Rightarrow\; \operatorname{Hess}(b^{\gamma^+}) = 0$$

Besides, for any X, Y

$$\begin{aligned}
\operatorname{Hess} b^{\gamma^+}(X, Y) &= \nabla^2 \operatorname{Hess}(b^{\gamma^+}(X, Y) \\
&= \nabla\left(\nabla b^{\gamma^+}\right)(X, Y) \\
&= \nabla_Y\left(\nabla b^{\gamma^+}\right)(X) = 0
\end{aligned}$$

Therefore, $\nabla_Y \nabla b^{\gamma^+} = 0$ for any Y, that is ∇b^{γ^+} is parallel along each curve. It can induce the totally geodesic property (geodesic connecting points in the submanifold and with the initial vector lying the submanifold is still in the submanifold) of level sets of b^{γ^+}. It is because that for any $X, Y \in \Gamma^\infty(TN)$, where $p \in N \subset M$ a level set of b^{γ^+},

$$\begin{aligned}\left\langle \nabla_X Y, \operatorname{grad} b^{\gamma^+} \right\rangle &= X \left\langle Y, \operatorname{grad} b^{\gamma^+} \right\rangle + \left\langle Y, \nabla_X \operatorname{grad} b^{\gamma^+} \right\rangle \\ &= X \left\langle Y, \operatorname{grad} b^{\gamma^+} \right\rangle \\ &= 0\end{aligned}$$

By **Theorem** 4.3.4, $\left\langle \nabla_X Y, \operatorname{grad} b^{\gamma^+} \right\rangle = 0$ implies $B(X, Y) = 0$, so N is totally geodesic.

4. ***Splitting:*** Let's consider the proof of the Splitting Theorem.

Proof of Theorem 4.3.1. Let $V_0 = \left\{ x \in M : b^{\gamma^+}(x) = 0 \right\}$ that is a complete, $\operatorname{Ric} \geq 0$, totally geodesic, $(n-1)$-dim Riemannian manifold. Define

$$\phi \colon V_0 \times \mathbb{R} \longrightarrow M$$

as

$$\phi(a, t) = \exp_a t \operatorname{grad} b^{\gamma^+}$$

We need to check:

I. $t \to \exp_a t \operatorname{grad} b^{\gamma^+}$ is a line: Let δ^+ be an asymptote of γ^+ start from a and δ^- be an asymptote of γ^- start from a. Note that δ^+, δ^- are two rays. To prove $\delta = \delta^+ \cup \delta^-$ smooth, it needs to prove

$$d(\delta^+(t), \delta^-(s)) = t + s$$

First, by triangular inequality,

$$t + s \geq d(\delta^+(t), \delta^-(s))$$

Besides, because Busemann function is 1-Lipschitz and $b^{\gamma^+} + b^{\gamma^-} = 0$

$$\begin{aligned}d(\delta^+(t), \delta^-(s)) &\geq \left| b^{\gamma^+}(\delta^+(t)) - b^{\gamma^+}(\delta^-(s)) \right| \\ &= \left| b^{\gamma^+}(a) + t + b^{\gamma^-}(a) + s \right| = t + s\end{aligned}$$

Therefore, δ is a normal line for all δ^+, δ^-, which means δ^+, δ^- are unique so is δ. We want to show $\delta(t) = \exp_a t \operatorname{grad} b^{\gamma^+}$. By uniqueness, we only need to show $\dot{\delta}(0) \perp V_0$. For any $y \in V_0$,

$$d(\delta^+(t), y) \geq \left| b^{\gamma^+}(\delta^+(t)) - b^{\gamma^+}(y) \right| = t$$

and $d(\delta^+(t), a) = t$. Therefore, $\dot{\delta}(0) \perp V_0$ by **Lemma** 2.4.2. Then, because $\delta(t)$ contains no cut point, ϕ is a local diffeomorphism.

II. ϕ is injective: because $\dot{\delta}(0) = \operatorname{grad}(b^{\gamma^+})$ for any $a = \delta(0)$, which implies $\dot{\delta}(t) = \operatorname{grad}(b^{\gamma^+})$ because $\operatorname{grad}(b^{\gamma^+}$ is parallel along every geodesic. $\delta_1(t_0) = \delta_2(t_0)$ implies $a_1 = a_2$ otherwise $\dot{\delta}_1(t_0) \neq \dot{\delta}_2(t_0)$.

III. ϕ is surjective: For any $x \in M$, let $t = b^{\gamma^+}(x)$. Similarly, we apply above assertions to $V_t = \left\{ x \in M : b^{\gamma^+}(x) = t \right\}$ so that we get there is a line $\tilde{\delta}$ through x and $\dot{\tilde{\delta}}(0) = \operatorname{grad}(b^{\gamma^+}$. Moreover, because

$$b^{\gamma^+}(\tilde{\delta}(r)) = b^{\gamma^+}(x) + r = t + r$$

there is a point $a = \tilde{\delta}(-t) \in \tilde{\delta}$ with $b^{\gamma^+}(a) = 0$, that is $a \in V_0$. It implies it is the δ starting from such a.

IV. ϕ is isometric: We need to check $\phi^* g = g_{V_0} \oplus g_{\mathrm{Euc}}$. For $X = (0, a\, \frac{\partial}{\partial t}\big|_{t=0})$, because

$$d\phi(X) = a\, \frac{\partial}{\partial t}\bigg|_{t=0} \exp_a\left(t\, \mathrm{grad}(b^{\gamma^+})\right) = a\, \mathrm{grad}(b^{\gamma^+})$$

and $\left|\mathrm{grad}(b^{\gamma^+})\right| = 1$

$$\phi^* g(X, X) = g(d\phi(X), d\phi(X)) = a^2 = g_{\mathrm{Euc}}\left(a\, \frac{\partial}{\partial t}\bigg|_{t=0}, a\, \frac{\partial}{\partial t}\bigg|_{t=0}\right)$$

For $Y = (V, 0)$ with $V \in T_a V_0$, let $a(s)$ be a geodesic in V_0 with $a(0) = a$ and $\dot{a}(0) = V$. Then

$$d\phi(Y) = \frac{d}{ds}\bigg|_{s=0} \phi(a(s), 0) = \frac{d}{ds}\bigg|_{s=0} a(s) = V$$

So $\phi^* g(Y, Y) = g(V, V) = g_{V_0}(V, V)$. Besides

$$\phi^* g(X, Y) = g\left(a\, \mathrm{grad}(b^{\gamma^+}), V\right) = 0$$

because $\mathrm{grad}(b^{\gamma^+}) \perp V_0$. $\square$

Chapter 5

Submanifolds

5.1 More for Distance Function

1. ***Local Distance Function:*** Suppose (M, g) be a Riemannian manifold and $S \subset M$ be any subset, for each $\gamma(t) \in M$,

$$d(x, S) := \inf \{d(x, p) \colon p \in S\}$$

Then we can see

(1) $d(x, S) \leq d(x, y) + d(y, S)$ by the triangular inequality;

(2) $x \mapsto d(x, S)$ is a continuous function on M by definition and the continuity of distance function.

Remark. Let S be a closed set. First, clearly $d(\cdot, S)$ can be smooth around a neighborhood of $M \backslash S$. Besides, (1) implies that $d(\cdot, S)$ is 1-Lipschitz continuous, which means $d(\cdot, S)$ is continuously differentiable *a.e.* on $M \backslash S$. Moreover. if S is further a smooth submanifold, $d(\cdot, S)$ is smooth *a.e.* on $M \backslash S$.

Theorem 5.1.1. *Let (M, g) be a Riemannian manifold and $S \subset M$ be any subset and $f \colon M \to [0, \infty)$ be $f(x) = d(x, S)$. If f is C^1 on some open set $U \subset M \backslash S$, then $|\operatorname{grad} f| \equiv 1$ on U.*

Proof. Let $x \in U$. First, we want to prove $|\operatorname{grad} f(x)| \leq 1$. Assume $|\operatorname{grad} f| \neq 0$. Let $v \in T_x M$ with $|v| = 1$ and γ be the shortest geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. By above,

$$f(\gamma(t)) \leq t + f(\gamma(0)) \;\Rightarrow\; \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \leq 1$$

On the other hand, if we let $v = (\operatorname{grad} f(x)) / |\operatorname{grad} f(x)|$, then

$$\left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \langle \operatorname{grad} f(x), v \rangle = |\operatorname{grad} f(x)| \leq 1$$

Next, assume $|\operatorname{grad} f(x)| < 1$. By continuity of $\operatorname{grad} f$, there exist $\delta, \varepsilon > 0$ such that $|\operatorname{grad} f| \leq 1 - \delta$ on a closed geodesic ball $\overline{B}_\varepsilon(x) \subset U$. Let $0 < c < \varepsilon\delta$. There is arc length parametrized curve $\alpha \colon [0, b] \to M$ such that

$$\alpha(0) = x,\; \alpha(b) \in S,\; b = \operatorname{Leng}(\alpha) < d(x, S) + c$$

Furthermore, if $b < \varepsilon$, $\alpha(b) \in \overline{B}_\varepsilon(x) \subset M\backslash S$, contradicting to $\alpha(b) \in S$, so $b > \varepsilon$. It follows that

$$d(\alpha(\varepsilon), S) \leq \text{Leng}(\alpha|_{[\varepsilon,b]}) = b - \varepsilon < d(x,S) + c - \varepsilon$$

However, from another view, for $0 \leq t \leq \varepsilon$, $\alpha(t) \in \overline{B}_\varepsilon(x)$ implies that

$$\left|\frac{d}{t}f(\alpha(t))\right| = |\langle \text{grad}\, f, \dot{\alpha}(t)\rangle| \leq |\text{grad}\, f|\, |\dot{\alpha}(t)| \leq 1 - \delta \;\Rightarrow\; \frac{d}{dt}f(\alpha(t)) \geq -(1-\delta)$$

which follows that

$$f(\alpha(t)) \geq f(x) - (1-\delta)t$$

By setting $t = \varepsilon$,

$$d(\alpha(\varepsilon), S) \geq d(x,S) - (1-\delta)\varepsilon > d(x,S) + c - \varepsilon$$

which induces a contradiction. $\square$

Definition 5.1.1. Let (M,g) be a Riemannian manifold and $U \subset M$ be an open set. A local distance function on U is a C^1 function $f\colon U \to \mathbb{R}$ such that $|\text{grad}\, f| \equiv 1$ on U.

Theorem 5.1.2. *Suppose f is a smooth local distance function on a open subset $U \subset M$. Then*

$$\nabla_{\text{grad}\, f}(\text{grad}\, f) \equiv 0$$

and so each integral curve of grad f *is a normal geodesic.*

Proof. Let $F = \text{grad}\, f$, *i.e.* $Wf = df(W) = \langle F, W\rangle$ and so $Ff = \langle F, F\rangle = |\,\text{grad}\, f|^2 = 1$. Then we get

$$\begin{aligned}
\langle W, \nabla_F F\rangle &= F\langle W, F\rangle - \langle \nabla_F W, F\rangle \\
&= FWf - \langle [F,W], F\rangle - \langle \nabla_W F, F\rangle \\
&= FWf - [F,W]f - \frac{1}{2}W|F|^2 \\
&= WFf - \frac{1}{2}W|F|^2 \\
&= 0
\end{aligned}$$

Therefore, $\nabla_F F = 0$. $\square$

Lemma 5.1.1. *Let $K \subset M$ and $f\colon K \to \mathbb{R}$ is a continuous function and $f|_W$ is a smooth distance function on a open set $W \subset K$. Then for any curve $[a,b] \to K$ with $\sigma \subset W$, we have*

$$\text{Leng}(\sigma) \geq |f(\sigma(b)) - f(\sigma(a))|$$

Proof. By $|\text{grad}\, f| \equiv 1$,

$$\begin{aligned}
|f(\sigma(b)) - f(\sigma(a))| &\leq \int_a^b \left|\frac{d}{dt}f(\sigma(t))\right| dt \\
&\leq \int_a^b \left|\left\langle \text{grad}\, f, \dot{\sigma(t)}\right\rangle\right| dt \\
&\leq \int_a^b |\text{grad}\, f| \left|\dot{\sigma(t)}\right| dt \\
&= \int_a^b \left|\dot{\sigma(t)}\right| dt = \text{Leng}(\sigma)
\end{aligned}$$

so we have the desired result. $\square$

Theorem 5.1.3. *Let (M,g) be a Riemannian manifold and $U \subset M$ be an open set. Let $S \subset U$ and $f\colon U \to [0,\infty)$ be a continuous function such that $S = f^{-1}(0)$. Assume f is a smooth local distance function on $U\backslash S$. Then there is a neighborhood $U_0 \subset U$ of S in which $f(x) = d(x,S)$.*

Proof. Let

$$U_0 = \bigcup_{p\in S} B_{\varepsilon_p}(p)$$

where $B_{\varepsilon_p}(p)$ is a δ_p-uniform totally normal ball of p and ε_p is chosen such that $B_{w\varepsilon_p}(p) \subset U$. Let $x \in U_0$ arbitrary and $c = f(x)$. We need to check $d(x,S) = c$. First, if $x \in S$, it is clearly true. Assume $x \notin S$. There is some $p \in S$ such that $x \in B_{\varepsilon_p}(p)$, which means that $d(x,S) < \varepsilon_p$. Let $\alpha\colon [0,b] \to B_{\varepsilon_p}(p)$ be the radical geodesic connecting p to x. Then by above lemma,

$$\operatorname{Leng}(\alpha) \geq |f(x) - f(p)| = c \;\Rightarrow\; c \leq \operatorname{Leng}(\alpha) < \varepsilon_p$$

Let $\gamma\colon (-\varepsilon_p, \varepsilon_p) \to U$ be the normal geodesic starting from x with $\dot\gamma(0) = -\operatorname{grad} f(x)$. So

$$\frac{d}{dt} f(\gamma(t)) = \langle \operatorname{grad} f, \dot\gamma(t)\rangle = -\left|\operatorname{grad} f\right|^2 = -1$$

and

$$f(\gamma(t)) = c - t$$

for $t < c < \varepsilon$, so $f(\gamma(c)) = 0$ by continuity, *i.e.* $\gamma(c) \in S$. It follows that

$$d(x,S) \leq c$$

To prove the reverse inequality, suppose $\alpha : [a,b] \to M$ is any admissible curve starting at x and ending at a point of S. Assume first that $\alpha(t) \in U$ for all $t \in [a,b]$, and let $b_0 \in [a,b]$ be the first time that $\alpha(b_0) \in S$. Then

$$\operatorname{Leng}(\alpha) \geq \operatorname{Leng}\left(\alpha|_{[a,b_0]}\right) \geq |f(\alpha(b_0)) - f(\alpha(a))| = c$$

On the other hand, suppose $\alpha(t) \in M\backslash U$ for some t. The triangle inequality implies,

$$B_{\varepsilon_p}(x) \subset B_{2\varepsilon_p}(p) \subset U$$

so there is a first time $b_0 \in [a,b]$ such that $d(x, \alpha(b_0)) \geq \varepsilon_p$. Then

$$\operatorname{Leng}(\alpha) \geq \operatorname{Leng}\left(\alpha|_{[a,b_0]}\right) \geq \varepsilon_p > c$$

Taken together, these two inequalities show that $\operatorname{Leng}(\alpha) \geq c$ for every such α, which implies $d(x,S) \geq c$. $\square$

Corollary 5.1.1. *Let (M,g) be a Riemannian manifold, and let f be a smooth local distance function on an open subset $U \subseteq M$. If c is a real number such that $S = f^{-1}(c)$ is nonempty, then there is a neighborhood U_0 of S in U on which $|f(x) - c|$ is equal to the distance in M from x to S.*

2. ***Fermi Coordinates:*** Let (M,g) be a Riemannian manifold and $P \subset M$ be an embedded submanifold. Let $\pi\colon NP \to P$ be the normal bundle of P in m.

$$\mathcal{E} := \left\{(p,v) \in TM\colon\ \exp_p(tv) \text{ defined on } [0,1]\right\}$$

called the domain of exponential map, that is $\exp\colon \mathcal{E} \to M$ by $(p,v) \mapsto \exp_p(v)$. Then let $\mathcal{E}_P := \mathcal{E} \cap NP$ and $E = \exp|_{\mathcal{E}}\colon \mathcal{E}_P \to M$ be called normal exponential map of P in M.

Definition 5.1.2. (1) A normal neighborhood of P in M is an open subset $U \subset M$ such that $E\colon V = E^{-1}(U) \to U$ is diffeomorphic and $V \subset \mathcal{E}_P$ open, whose intersection with each fiber N_xP is star-shaped.

(2) A normal neighborhood of P in M is called a tubular neighborhood if

$$V = \{(x, v) \in NP\colon\ |v| < \delta(x)\}$$

for some continuous function $\delta\colon P \to (0, \infty)$. If $\delta(x) \equiv \varepsilon$, it is called a ε-uniform tubular neighborhood, or ε-tubular neighborhood.

Theorem 5.1.4 (Tubular Neighborhood Theorem)**.** *Let (M, g) be a Riemannian manifold. Every embedded submanifold M has a tubular neighborhood in M and every compact submanifold has a uniform tubular neighborhood.*

Proof. Let $P \subset M$ be an embedded submanifold and $P_0 = \{(x, 0)\colon x \in P\} \subset NP$.

(i) First, we want to show there is a neighborhood of P_0 such that E is a local diffeomorphism on it. It is sufficient to show $dE_{(x,0)}$ is bijective on each $(x, 0) \in P_0$. Note that $E|_{P_0}\colon P_0 \to P$ is a diffeomorphism by $P \hookrightarrow M$, so

$$dE_{(x,0)}\colon T_{(x,0)}P_0 \subset T_{(x,0)}NP \longrightarrow T_xP$$

is isomorphic. On the other hand, $E|_{N_xP} = \exp_x$, which is a local diffeomorphism. $dE_{(x,0)}\colon T_{(x,0)}N_xP \to N_xP$ is isomorphic. So

$$dE_{(x,0)}\colon T_{(x,0)}NP \longrightarrow T_xM = T_xP \oplus N_xP$$

is surjective, so it is bijective by dimension equality. So E is a local diffeomorphism. For any $(x, 0) \in P_0$, there is a sufficiently small $\delta > 0$ such that E is diffeomorphic on

$$V_\delta(x) = \{(x', v') \in NP : d\,(x, x') < \delta, |v'| < \delta\}$$

(ii) For any $x \in P$, define

$$\Delta(x) := \sup\{\delta \le 1\colon E \text{ diffeomorphic on } V_\delta(x)\}$$

By above, $\Delta(x) > 0$. Moreover, E is injective on $V_{\Delta(x)}(x)$ because any $(x_1, v_1), (x_2, v_2)$ in this set are in $V_\delta(x)$ for some $\delta < \Delta(x)$. So E is diffeomorphic on $V_{\Delta(x)}(x)$. Moreover, for any $x, x' \in P$, if $d(x, x') < \Delta(x)$, by triangular inequality

$$V_\delta(x') \subset V_{\Delta(x)}(x), \quad \delta = \Delta(x) - d(x, x')$$

This implies that $\Delta(x') \ge \Delta(x) - d(x, x')$, *i.e.*

$$\Delta(x) - \Delta\,(x') \le d\,(x, x')$$

and it is also true when $d(x, x') > \Delta(x)$. Thus

$$|\Delta(x) - \Delta\,(x')| \le d\,(x, x')$$

which means Δ is continuous.

(iii) Consider

$$V=\left\{(x,v)\in NP:|v|_g<\frac{1}{2}\Delta(x)\right\}$$

Let $(x,v),(x',v')\in V$ with $E(x,v)=E(x',v')$. Assume $\Delta(x')\leq\Delta(x)$. Because $\exp_x(v)=\exp_{x'}(v')$, there is a curve from x to x' with length $|v|+|v'|$. So

$$d\left(x,x'\right)\leq|v|+|v'|<\frac{1}{2}\Delta(x)+\frac{1}{2}\Delta\left(x'\right)\leq\Delta(x)$$

Therefore, $(x,v),(x',v')\in V_{\Delta(x)}(x)$, so $(x,v)=(x',v')$. Because E is a local diffeomorphism, $E\colon V\to E(V)$ is a diffeomorphism. □

Assume $\dim P=p$. Let $(W_0,\psi=(x^1,\cdots,x^p))$ be a coordinate chart of P and $E_1,\cdots,E_{n-p}$ be a local orthogonal frame of NP. Let

$$V_0=V\cap NP|_{W_0}\subset NP,\quad U_0=E(V_0)\subset M$$

Then define the coordinate map $\varphi\colon U_0\to\mathbb{R}^n$ by

$$E(q,v^1E_1|_q+\cdots+v^{n-p}E_{n-p}|_q)\mapsto(x^1(q),\cdots,x^p(q),v^1,\cdots,v^{n-p})$$

called Fermi coordinates.

Proposition 5.1.1. *Using notations as above and let $x^{p+j}=v^j$ for convenience.*

(i) $q\in P\cap U_0$ has the coordinate $x^{p+1}=\cdots=x^n=0$.

(ii) At each $q\in P\cap U_0$,

$$g_{ij}=g_{ji}=\begin{cases}0, & 1\leq i\leq p \text{ and } p+1\leq j\leq n,\\ \delta_{ij}, & p+1\leq i,j\leq n.\end{cases}$$

(iii) For any $q\in P\cap U_0$ and $v=v^1E_1|_q+\cdots+v^{n-p}E_{n-p}|_q\in N_qP$, the geodesic γ starting from $\gamma(0)=q$ with initial velocity v has coordinate expression

$$\gamma(t)=\left(x^1(q),\ldots,x^p(q),tv^1,\ldots,tv^{n-p}\right)$$

(iv) At each $q\in P\cap U_0$, $\Gamma_{ij}^k(q)=0$ for $p+1\leq i,j\leq n$.

(v) At each $q\in P\cap U_0$,

$$\frac{\partial}{\partial x^i}g_{jk}(q)=0,\quad p+1\leq i,j,k\leq n$$

3. ***Distance to Submanifolds:*** Let (M,g) be a n-dimensional Riemannian manifold and $S\subset M$ be a k-dimensional submanifold. Let U be any normal neighborhood of S in M.

Proposition 5.1.2. *There exist a unique continuous function $r\colon U\to[0,\infty)$ and smooth vector field V_r on $U\backslash S$ that has the coordinate representation in terms of any Fermi coordinates $(x^1,\cdots,x^k,v^1,\cdots,v^{n-k})$ for S on a subset $U_0\subset U$*

$$r\left(x^1,\ldots,x^k,v^1,\ldots,v^{n-k}\right)=\sqrt{(v^1)^2+\cdots+(v^{n-k})^2},$$

$$\partial_r=\frac{v^1}{r(x,v)}\frac{\partial}{\partial v^1}+\cdots+\frac{v^{n-k}}{r(x,v)}\frac{\partial}{\partial v^{n-k}}.$$

Moreover, r is smooth on $U\backslash S$, and r^2 is smooth on all of S.

Proof. It only needs to prove r and V_r can be defined on U and $U\backslash S$ respectively. Let $E\colon V \subset NS \to U$ be diffeomorphic.

(i) Define a function $\rho\colon V \to [0,\infty)$ by $\rho(p,v) = |v|$ and $r\colon U \to [0,\infty)$ by $r = \rho \circ E^{-1}$.

(ii) Let $q \in U\backslash S$. Then $q = \exp_p(v)$ for a unique $(p,v) \in V$, and the geodesic $\gamma\colon [0,1] \to U$ given by $\gamma(t) = \exp_p(tv)$ connecting p to q, which has the coordinate expression

$$\gamma(t) = (x^1, \cdots, x^k, tv^1, \cdots, tv^{n-k}), \quad x(q) = (x^i),\ v = v^i E_i|_p$$

Define

$$V_r(q) := \frac{1}{r(q)}\dot\gamma(1)$$

□

Theorem 5.1.5 (Gauss Lemma). *With the same notations, on $U\backslash S$, V_r is a unit vector field orthogonal to the level sets of r.*

Proof. (i) For $q \in U\backslash S$ with coordinate $(x^1, \cdots, x^k, v^1, \cdots, v^{n-k})$, consider the corresponding geodesic

$$\gamma(t) = \exp_p(tv) = (x^1, \cdots, x^k, tv^1, \cdots, tv^{n-k})$$

Then

$$|\gamma'(0)| = |v| = \sqrt{(v^1)^2 + \cdots + (v^{n-k})^2} = r(q)$$

and thus $|\dot\gamma(1)| = r(q)$, and $|V_r(q)| = 1$.

(ii) For orthogonality, let $q \in U\backslash S$ and $q = \exp_{p_0}(v_0)$ for $p_0 \in S$ and $v_0 \in N_{p_0}S$ with $v_0 \neq 0$. Let $b = r(q) = |v_0|$. So $q \in r^{-1}(b)$, which is an embedded submanifold. Let $\sigma\colon (-\varepsilon,\varepsilon) \to r^{-1}(q)$ be starting from q and denote $w = \dot\sigma(0)$. Then

$$\sigma(s) = \exp_{x(s)}(v(s))$$

where $x(s) \in S$ with $x(0) = p_0$, $v(s) \in N_{x(s)}S$ with $|v(s)| \equiv b$ and $v(0) = v_0$. Consider a variation

$$F(s,t) = \exp_{x(s)}\left(\frac{t}{b}v(s)\right) : (-\varepsilon,\varepsilon) \times [0,b] \to M$$

with $T(t) = \frac{\partial}{\partial t}F(0,t)$ and Jacobian field $J(t) = \frac{\partial}{\partial s}F(0,t)$.

$$T(0) = v_0, \quad T(b) = V_r(q)$$

and

$$J(0) = \dot x(0), \quad J(b) = w$$

Consider the energy

$$E(s) = \frac{1}{2}\int_0^b \left\langle \frac{\partial}{\partial t}F(s,t), \frac{\partial}{\partial t}F(s,t)\right\rangle dt = \frac{1}{2}\int_0^b |v(s)|^2\, dt = \frac{1}{2}r^2 b$$

we have

$$\begin{aligned} 0 = \dot E(0) &= \langle J(b), T(b)\rangle - \langle J(0), T(0)\rangle \\ &= \langle w, V_r(q)\rangle \end{aligned}$$

because $\langle \dot x(0), v_0\rangle = 0$. □

Corollary 5.1.2. *Using same notations and assumptions as above,*

(1) $V_r = \operatorname{grad} r$ on $U\backslash S$.

(2) r is a local distance function

(3) each unit-speed geodesic $\gamma : [a,b) \to U$ with $\dot\gamma(a)$ normal to S coincides with an integral curve of V_r on (a,b).

(4) S has a tubular neighborhood in which the distance in M to S is equal to r.

5.2 Riemannian Submanifold

Chapter 6

tmp

1. ***Eigenvalue estimating:*** For $u \in C^\infty(M)$, consider

$$\lambda_1(M,g) := \inf_{\int u=0} \frac{\int |\nabla u|^2}{\int u^2}$$

which is related to

$$\Delta u = -\lambda_1 u$$

for the lowest nonzero λ_1. The question is how to get the lower bound of λ_1.

Theorem 6.0.1. *Let (M,g) be a Riemannian manifold of dimension n with $\mathrm{Ric} \geq (n-1)$. Then $\lambda_1 \geq n$.*

Example 6.0.1 (Sphere)**.** Consider $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, let $x_k = (0, \cdots, x_k, \cdots, 0)$ be the coordinate function. Then we have

$$\begin{aligned}
\Delta x_k &= \sum_i \langle \nabla_{e_i} \operatorname{grad} x_k, e_i \rangle \\
&= \sum_i \langle D_{e_i} \operatorname{grad} x_k, e_i \rangle - \left\langle (D_{e_i} \operatorname{grad} x_k)^\perp, e_i \right\rangle \\
&= \sum_i \langle D_{e_i} \operatorname{grad} x_k, e_i \rangle \\
&= -\sum_i \left\langle D_{e_i} (Dx_k)^\perp, e_i \right\rangle \\
&= \sum_i \left\langle (Dx_k)^\perp, D_{e_i} e_i \right\rangle \\
&= \left\langle (Dx_k)^\perp, \sum_i (D_{e_i} e_i)^\perp \right\rangle
\end{aligned}$$

We know

$$\nabla_{e_i} e_i = D_{e_i} e_i + \langle e_i, e_i \rangle \vec{x} \;\Rightarrow\; \sum_i (D_{e_i} e_i)^\perp = \sum_i -\langle e_i, e_i \rangle \vec{x} = -n\vec{x}$$

Therefore,

$$\Delta x_k = \left\langle (Dx_k)^\perp, \sum_i (D_{e_i} e_i)^\perp \right\rangle = -n \langle Dx_k, \vec{x} \rangle = -n x_k$$

Proof. □