Stochastic Analysis on Manifolds

Zhiyuan Zhan
<thaleszhan@gmail.com>

November 2, 2025

Contents

Chapter 1

SDE and Diffusion

Fix probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of σ-fields contained in $\mathcal{F}$ with assumption that $\mathcal{F}_\infty = \mathcal{F}$. Also, $\mathbb{F}$ satisfies the usual condition, i.e., complete and right-continuous. Without specification, all semimartingales are assumed continuous. We use Einstein summation convention for convenience.

1.1 SDE on Euclidean Space

Solution of SDE: Let $Z = (Z_t)_{t\geq 0}$ be a continuous $\mathbb{R}^\ell$-valued $\mathbb{F}$-semimartingale, i.e.,

$$Z = M + A,$$

where M is a local martingale and A is an adapted process of local bounded variation such that $A_0 = 0$. Let $\sigma \colon \mathbb{R}^N \to \mathbb{R}^{N\times\ell}$ be smooth and locally Lipschitz, i.e. for all $R > 0$ there exists $R > 0$ and $C(R) > 0$ such that

$$\|\sigma(x) - \sigma(y)\|_{\mathrm{F}} \leq C(R)\,\|x - y\|\,, \quad \forall\, x, y \in B(R).$$

Let $X_0 \in \mathbb{R}^N$ be $\mathcal{F}_0$ measurable. For a stopping time τ, consider a semimartingale $X = (X_{t\wedge\tau})_{t\geq 0}$ of the form

$$X_t = X_0 + \int_0^t \sigma(X_s) dZ_s, \quad 0 \leq t \leq \tau.$$

We say X is a solution of a SDE driving by the semimartingale Z, and such equation denoted as $\mathrm{SDE}(\sigma, Z, X_0)$. Moreover, by Itô formula, for $f \in C^2(\mathbb{R}^N)$,

$$\begin{aligned}
f(X_t) &= f\left(X_0\right) + \int_0^t f_{x_i}\left(X_s\right)\sigma_\alpha^i\left(X_s\right) dZ_s^\alpha \\
&\quad + \frac{1}{2}\int_0^t f_{x_i x_j}\left(X_s\right)\sigma_\alpha^i\left(X_s\right)\sigma_\beta^j\left(X_s\right) d\left\langle Z^\alpha, Z^\beta\right\rangle_s \\
&= f\left(X_0\right) + \int_0^t f_{x_i}\left(X_s\right)\sigma_\alpha^i\left(X_s\right) dM_s^\alpha + \int_0^t f_{x_i}\left(X_s\right)\sigma_\alpha^i\left(X_s\right) dA_s^\alpha \\
&\quad + \frac{1}{2}\int_0^t f_{x_i x_j}\left(X_s\right)\sigma_\alpha^i\left(X_s\right)\sigma_\beta^j\left(X_s\right) d\left\langle M^\alpha, M^\beta\right\rangle_s
\end{aligned}$$

Remark 1.1.1. Note that if $X_0 \in L^2$ and σ is globally Lipschitz continuous, then we don't need the stopping time τ and $\mathrm{SDE}(\sigma, Z, X_0)$ has a unique solution $X = (X_t)_{t\geq 0}$.

The problem why we need to consider a stopping time is because of the local Lipschitz property of σ. The local Lipschitz would make the usual solution exploded. To consider it rigorously, we first need the definition of explosion time.

Let M be a locally compact metric space and $\widehat{M} = M \cup \{\partial_M\}$ be its one-point compactification. A continuous map $x \colon [0, \infty) \to \widehat{M}$ has an explosion time $e = e(x) > 0$ if $x_t \in M$ for $0 \leq t < e$ and $x_t = \partial M$ for $t \geq e$ when $e < \infty$, i.e.

$$e \colon W(M) \to [0, \infty]$$

where $W(M) \subset C([0, \infty), \widehat{M})$ such that e can be defined. Moreover, $W(M)$ can be equipped with the canonical filtration $\mathbb{F}_M = (\mathcal{F}_{M,t})_{t \geq 0}$, i.e., the natural filtration generated by coordinate processes. Then e is a $\mathbb{F}_M$-stopping time.

Definition 1.1.2. Let τ be a stopping time. A continuous $X = (X_{t \wedge \tau})_{t \geq 0}$ is called a semimartingale up to τ if there exists a sequence of stopping times $\tau_n \uparrow \tau$ such that $X^{\tau_n} = (X_{t \wedge \tau_n})$ is a semimartingale.

Definition 1.1.3. A semimartingale X up to a stopping time τ is a solution of $\mathrm{SDE}(\sigma, Z, X_0)$ if there exists a sequence of stopping times $\tau_n \uparrow \tau$ such that $X^{\tau_n} = (X_{t \wedge \tau_n})$ is a semimartingale and

$$X_{t \wedge \tau_n} = X_0 + \int_0^{t \wedge \tau_n} \sigma\left(X_s\right) dZ_s, \quad t \geq 0.$$

Theorem 1.1.4. *If σ is locally Lipschitz continuous, then there exists a unique $W(\mathbb{R}^N)$-valued random variable X which is a solution of $\mathrm{SDE}(\sigma, Z, X_0)$ up to its explosion time $e(X)$. Moreover, if X, Y are two solutions up to τ and η respectively, then $X_{t \wedge \tau \wedge \eta} = Y_{t \wedge \tau \wedge \eta}$. In particular, if X is a solution up to $e(X)$, then $\eta \leq e(X)$ and $X_{t \wedge \tau} = Y_{t \wedge \tau}$.*

Theorem 1.1.5. *Suppose σ and b are locally Lipschitz. Then the weak uniqueness holds for*

$$X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds,$$

where W is a Brownian motion.

Stratonovich integral. For two continuous semimartingales H, Z, the Stratonovich integral is defined as

$$\int_0^t H_s \circ dZ_s = \int_0^t H_s dZ_s + \frac{1}{2} \left\langle H, Z \right\rangle_t.$$

In such case, the Itô formula becomes

$$df(X_t) = \frac{\partial f}{\partial x^\alpha}(Z_t) \circ dZ_t.$$

Suppose that V_α, $\alpha = 1, 2, \cdots, \ell$ are smooth vector fields in $\mathbb{R}^N$, that is,

$$V_\alpha = (V_\alpha^1, \cdots, V_\alpha^N) \colon \mathbb{R}^N \to \mathbb{R}^N$$

and vector fields means that for any $f \in C^1(\mathbb{R}^N)$,

$$V_\alpha f = \sum_{i=1}^N V_\alpha^i \frac{\partial f}{\partial x^i}.$$

More generally, if

$$X_t = X_0 + \int_0^t V_\alpha\left(X_s\right) \circ dZ_s^\alpha,$$

then

$$X_t = X_0 + \int_0^t V_\alpha\left(X_s\right) dZ_s^\alpha + \frac{1}{2} \int_0^t \nabla_{V_\beta} V_\alpha\left(X_s\right) d\left\langle Z^\alpha, Z^\beta \right\rangle_s.$$

Proof. By definition,

$$dX_t = V_\alpha(X_t) \circ dZ_t^\alpha = V_\alpha(X_t) dZ_t^\alpha + \frac{1}{2} d \langle V_\alpha(X_t), Z_t^\alpha \rangle .$$

or

$$dX_t^i = \sum_{\alpha=1}^{\ell} V_\alpha^i(X_t) dZ_t^\alpha + \frac{1}{2} \sum_{\alpha=1}^{\ell} d \left\langle V_\alpha^i(X_t), Z_t^\alpha \right\rangle .$$

For $d \langle V_\alpha^i(X_t), Z_t^\alpha \rangle$, first because $\sum_\alpha V_\alpha^i(X_t) dZ_t^\alpha$ is the local martingale part of dX_t^i,

$$d \left\langle X^i, Z^\alpha \right\rangle_t = \sum_{\beta=1}^{\ell} V_\beta^i(X_t) d \left\langle Z^\beta, Z^\alpha \right\rangle_t .$$

Then by Itô formula,

$$dV_\alpha^i(X_t) = \sum_{j=1}^{n} \frac{\partial V_\alpha^i}{\partial x^j} dX_t^j + \text{ bounded variation part } .$$

So

$$\begin{aligned} d \left\langle V_\alpha^i(X_t), Z_t^\alpha \right\rangle &= \sum_{j=1}^{n} \frac{\partial V_\alpha^i}{\partial x^j} d \left\langle X^j, Z^\alpha \right\rangle_t \\ &= \sum_{\beta=1}^{\ell} \sum_{j=1}^{n} V_\beta^j \frac{\partial V_\alpha^i}{\partial x^j} d \left\langle Z^\beta, Z^\alpha \right\rangle_t \\ &= \nabla_{V_\beta} V_\alpha^i d \left\langle Z^\beta, Z^\alpha \right\rangle_t . \end{aligned}$$

Therefore,

$$dX_t = V_\alpha(X_t) dZ_t^\alpha + \frac{1}{2} \nabla_{V_\beta} V_\alpha(X_t) d \left\langle Z^\beta, Z^\alpha \right\rangle_t . \qquad \square$$

In the sense of Stratonovich integral, if $X_t = X_0 + \int_0^t V_\alpha(X_s) \circ dZ_s^\alpha$ and $f \in C^2(\mathbb{R}^N)$, then

$$f(X_t) = f(X_0) + \int_0^t (V_\alpha f)(X_s) \circ dZ_s^\alpha .$$

In differential form, if $dX_t = V_\alpha(X_t) \circ dZ_t^\alpha$

$$df(X_t) = (V_\alpha f)(X_t) dZ_t^\alpha .$$

1.2 SDE on Manifolds

Definition 1.2.1. Let M be a smooth manifold. A continuous M-valued process X is called a M-valued semi-martingale if $f(X)$ is a real-valued semi-martingale for all $f \in C^\infty(M)$.

Remark 1.2.2. We can also define it up to a stopping time τ as before. By Itô formula, it is obviously that $M = \mathbb{R}^N$ gives the usual definition of semimartingales. Moreover, the test function space can be chosen as $C_c^\infty(M)$.

Let $V_1, \cdots, V_\ell \in \Gamma(TM)$, Z be an $\mathbb{R}^\ell$-valued semimartingale, and an M-valued random variable $X_0 \in \mathcal{F}_0$. Consider an equation symbolically written as

$$dX_t = V_\alpha \circ dZ_t^\alpha ,$$

and denoted it as $\mathrm{SDE}(V_1, \cdots, V_\ell; Z, X_0)$.

Definition 1.2.3. An M-valued semimartingale X defined up to a stopping time τ is a solution of $\mathrm{SDE}(V_1, \cdots, V_\ell; Z, X_0)$ up to τ if for all $f \in C^\infty(M)$,

$$f(X_t) = f(X_0) + \int_0^t (V_\alpha f)(X_s) \circ dZ_s^\alpha.$$

Proposition 1.2.4. *Suppose $\Phi\colon M \to N$ is a diffeomorphism and X a solution of $\mathrm{SDE}(V_1, \cdots, V_\ell; Z, X_0)$. Then $\Phi(X)$ is a solution of $\mathrm{SDE}(\Phi_* V_1, \cdots, \Phi_* V_\ell; Z, \Phi(X_0))$ on N.*

Proof. Let $Y = \Phi(X)$. For any $f \in C^\infty(N)$, because $f \circ \Phi \in C^\infty(M)$ and X is a solution of $\mathrm{SDE}(V_1, \cdots, V_\ell; Z, X_0)$,

$$f \circ \Phi(X_t) = f \circ \Phi(X_0) + \int_0^t V_\alpha (f \circ \Phi)(X_s) \circ dZ_s^\alpha.$$

Note that $\Phi_*\colon \Gamma(TM) \to \Gamma(TN)$ such that for and $f \in C^\infty(N)$,

$$\Phi_* V(f)(\Phi(x)) = V(f \circ \Phi)(x).$$

It follows that

$$f(Y_t) = f(Y_0) + \int_0^t \Phi_* V_\alpha(f)(Y_s) \circ dZ_s^\alpha,$$

which means that Y is a solution of $\mathrm{SDE}(\Phi_* V_1, \cdots, \Phi_* V_\ell; Z, \Phi(X_0))$. □

Note that by Whitney's embedding theorem, any smooth manifold M can be properly embedded in a Euclidean space, i.e., $i\colon M \hookrightarrow \mathbb{R}^N$, where i is a smooth proper map. Therefore, $i(M)$ is a closed submanifold in $\mathbb{R}^N$. Note that here closed submanifold means that a submanifold that is a closed subset. Therefore, we can always view $M \subset \mathbb{R}^N$ as an embedded submanifold that is closed. Moreover, we usually assume M without boundary. When M is non-compact, we can consider its one-point compactification $\widehat{M} = M \cup \{\partial_M\}$. So $\{x_n\}_n \subset M$ such that $x_n \to \partial_M$ in $\widehat{M}$ if and only if $\|x_n\|_{\mathbb{R}^N} \to \infty$.
In the following we always assumed that $M \subset \mathbb{R}^N$ is a submanifold that is closed and without boundary. Moreover, let

$$i = (f^1, \cdots, f^N)\colon M \hookrightarrow \mathbb{R}^N$$

and f^i are coordinate functions that $f^i(x) = x^i$.

Proposition 1.2.5. *Suppose $M \subset \mathbb{R}^N$ is a submanifold that is closed. Let $f^1, \cdots, f^N$ be coordinate functions. Let X be an M-valued continuous process.*

(1) X is a semimartingale on M if and only if it is an $\mathbb{R}^N$-valued semimartingale, i.e. $f^i(X)$ is a $\mathbb{R}$-valued semimartingale for each $i = 1, 2, \cdots, N$.

(2) X is a solution of $\mathrm{SDE}(V_1, \cdots, V_\ell; Z, X_0)$ up to a stopping time σ if and only if for each $i = 1, 2, \cdots, N$,

$$f^i(X_t) = f^i(X_0) + \int_0^t (V_\alpha f^i)(X_s) \circ dZ_s^\alpha, \quad 0 \le t < \sigma.$$

Proof. (1) $\Rightarrow$: It is by the definition.

$\Leftarrow$: For any $f \in C^\infty(M)$, since M is closed $\mathbb{R}^N$, it can be extended on $\tilde{f} \in C^\infty(\mathbb{R}^N)$ such that $f(X) = \tilde{f}(X)$. Because X is $\mathbb{R}$-valued semimartingale, $f(X) = \tilde{f}(X)$ is also a semimartingale.

(2) $\Rightarrow$: It is by the definition.

$\Leftarrow$: For any $f \in C^\infty(M)$, let $\tilde{f} \in C^\infty(\mathbb{R}^N)$ be an extension. Then

$$f(X_t) = \tilde{f}\left(f^1(X_t), \cdots, f^N(X_t)\right).$$

Therefore,

$$\begin{aligned} df(X_t) &= \tilde{f}\left(f^1(X_t), \cdots, f^N(X_t)\right) \circ d(f^i(X_t)) \\ &= \tilde{f}\left(f^1(X_t), \cdots, f^N(X_t)\right) \circ V_\alpha f^i(X_t) \circ dZ_t^\alpha \\ &= \left(\tilde{f}\left(f^1(X_t), \cdots, f^N(X_t)\right) V_\alpha f^i(X_t)\right) \circ dZ_t^\alpha \\ &= V_\alpha \tilde{f}(X_t) \circ dZ_t^\alpha = V_\alpha f(X_t) \circ dZ_t^\alpha, \end{aligned}$$

where the final equality is because $V_\alpha \in \Gamma TM$ and $\tilde{f}$ is an extension of f. $\square$

Remark 1.2.6. Note that for $f \in \mathbb{R}^k \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}^k$

$$\begin{aligned} d(f \circ g(M)) &= f_{x_i}(g(M)) \circ dg^i(M) \\ &= f_{x_i}(g(M)) \circ g^i_{x_j}(M) \circ dM^j \\ &= \nabla f(g(M))^\top Dg(M) \circ dM \end{aligned}$$

under Stratonovich integral. In fact, if H, X, W are continuous semimartingales, we have

$$H \circ d\left(\int X \circ dW\right) = (HX) \circ dW,$$

because

$$d\left\langle HX, W\right\rangle = Hd\left\langle X, W\right\rangle + Xd\left\langle H, W\right\rangle,$$

which is because of Itô formula, $d(HX) = HdX + XdH + d\left\langle H, X\right\rangle$.

Remark 1.2.7. If X is a solution of $\mathrm{SDE}(V_1, \cdots, V_\ell; Z, X_0)$ up to its explosion time $e(X)$ and $X_0 \in M$, then $X_t \in M$ for $0 \le t < e(X)$. Moreover, $\mathrm{SDE}(V_1, \cdots, V_\ell; Z, X_0)$ is unique up its explosion time.

1.3 Diffusion Process

Let L be a smooth second order elliptic, but not necessarily non degenerate, differential operator on a smooth manifold M, that is,

$$L\colon C^\infty(M) \to C^\infty(M)$$

such that on every chart $(U; x^1, \cdots, x^d)$

$$Lf(x) = a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} f(x) + b^i(x)\frac{\partial}{\partial x^i} f(x) + c(x)f(x)$$

for some $a^{ij}, b^i, c \in C^\infty(U)$ and $\sum_{i,j} a^{ij}(x)\xi_i\xi_j \ge 0$.

Remark 1.3.1. If M is equipped with a torsion-free connection ∇, like a Riemannian manifold, then L can be written as a coordinate-free formulation, that is,

$$Lf := \left\langle A, \nabla^2 f\right\rangle + \left\langle b, \nabla f\right\rangle + cf,$$

for $b \in \Gamma(TM)$, $c \in C^\infty(M)$, and A, a symmetric $(2,0)$-tensor, where $\langle \cdot, \cdot \rangle$ is the natural dual operation.

For $f \in C^2(M)$ and $\omega \in W(M)$, let

$$M^f(\omega)_t = f(\omega_t) - f(\omega_0) - \int_0^t Lf(\omega_s)ds.$$

Definition 1.3.2. An $\mathbb{F}$-adapted process X that is valued in M is called a diffusion process generated by L if X is a M-valued semimartingale up to $e(X)$ and

$$M^f(X)_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds, \quad 0 \le t < e(X),$$

is a local martingale for all $f \in C^\infty(M)$.

A probability measure μ defined on $(W(M), \mathcal{F}_{M,\infty})$ is called a diffusion measure generated by L if

$$M^f(\omega)_t = f(\omega_t) - f(\omega_0) - \int_0^t Lf(\omega_s)\,ds, \quad 0 \le t < e(\omega),$$

is a local $\mathbb{F}_M$ local martingale for all $f \in C^\infty(M)$.

Remark 1.3.3. If X is an L-diffusion, then $\mu^X = X_\# \mathbb{P}$ is an L-diffusion measure. Conversely, if μ is an L-diffusion measure, then the coordinate process is an L-diffusion process.

For given L, assume locally

$$L = \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(x)\frac{\partial}{\partial x^i},$$

where $(a^{ij}(x))$ is positive semi-definite. Define

$$\Gamma(f,g) = L(fg) - fLg - gLf, \quad f,g \in C^\infty(M).$$

So locally

$$\Gamma(f,g) = a^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}.$$

And the semi-elliptic means that for any $f \in C^\infty(M)$, $\Gamma(f,f) \ge 0$. Combining this with bilinearity, it implies that if $g^1, \cdots, g^n$, the matrix $(\Gamma(g^i, g^j))$ is positive semimartingale.

Assume M is a submanifold of $\mathbb{R}^N$, which is closed. Let $z = (z^1, \cdots, z^N)$ be coordinates in $\mathbb{R}^N$ and f^α $(\alpha = 1, 2, \cdots, N)$ are coordinate maps $M \to \mathbb{R}^N$ by $f^\alpha(z) = z^\alpha$. Let

$$\tilde{a}^{\alpha\beta} = \Gamma(f^\alpha, f^\beta), \quad \tilde{b}^\alpha = Lf^\alpha.$$

Then they are in $C^\infty(M)$ and $(\tilde{a}^{\alpha\beta})$ is positive semi-definite. By the closedness of M, $\tilde{a}, \tilde{b}$ can be extended to $\mathbb{R}^N$ and let

$$\tilde{L} = \frac{1}{2}\sum_{\alpha,\beta=1}^N \tilde{a}^{\alpha\beta}\frac{\partial^2}{\partial z^\alpha \partial z^\beta} + \sum_{\alpha=1}^N \tilde{b}^\alpha \frac{\partial}{\partial z^\alpha}$$

be defined on $\mathbb{R}^N$. It is a true extension of L.

Lemma 1.3.4. *Suppose $f \in C^\infty(M)$. Then for any $\tilde{f} \in C^\infty(\mathbb{R}^N)$ which extends f from M to $\mathbb{R}^N$, $\tilde{L}\tilde{f} = Lf$ on M.*

Proof. Let $(x^1, \cdots, x^d)$ be a local coordinate on M. Note that

$$f(x) = \tilde{f}(f^1(x), \cdots, f^N(x)).$$

Then

$$\begin{aligned} Lf(x) &= L\tilde{f}(f^1(x), \cdots, f^N(x)) \\ &= \frac{1}{2}a^{ij}(x)\frac{\partial^2 \tilde{f}(f^1(x), \cdots, f^N(x))}{\partial x^i \partial x^j} + b^i(x)\frac{\partial \tilde{f}(f^1(x), \cdots, f^N(x))}{\partial x^i} \\ &= \tilde{L}\tilde{f}(z). \end{aligned}$$

□

For $\tilde{L}$ on $\mathbb{R}^n$, if $\tilde{a} = \tilde{\sigma}\tilde{\sigma}^\top$, it is the generator of diffusion process that satisfies

$$X_t = X_0 + \int_0^t \tilde{\sigma}(X_s)dW_s + \int_0^t \tilde{b}(X_s)ds.$$

We can clearly choose $\tilde{\sigma} = \tilde{a}^{\frac{1}{2}}$ and W a standard Brownian motion in $\mathbb{R}^N$ independent of X_0.

Remark 1.3.5. The existence of solution up to a stopping time needs the local Lipschitz of $\tilde{\sigma}$, which can be guaranteed by the local Lipschitz of $\tilde{a}$.

Moreover, for such diffusion process X on $\mathbb{R}^N$, $\mu^X = X_{\#}\mathbb{P}$, the distribution on $W(\mathbb{R}^N)$, is concentrated on $W(M)$, i.e., X is in fact a L-diffusion on M. In fact, we have the following theorem.

Theorem 1.3.6. *If L is a smooth second order semi-elliptic operator a C^∞-manifold M and μ_0 is a probability measure on M, then there exists a unique L-diffusion measure with initial distribution μ_0.*

Chapter 2

Stochastic Differential Geometry

2.1 Horizontal Lift

Let M be d-dimensional smooth manifold equipped with an affine connection ∇. Let Γ_{ij}^k be the corresponding Christoffel symbol, i.e,

$$\nabla_{X_i} X_j = \Gamma_{ij}^k X_k,$$

where $X_i = \frac{\partial}{\partial x^i}$. ∇ can induce concepts, such as, parallel moving ($\nabla_{\dot{c}(t)} V = 0$) and geodesic ($\nabla_{\dot{c}(t)} \dot{c}(t) = 0$).

For $x \in M$, a frame at x is a $\mathbb{R}$-linear isomorphism $u\colon \mathbb{R}^d \to T_x M$. Let $F(M)_x$ be the set of all frameworks at x. $GL(d, \mathbb{R})$ acts on $F(M)_x$ as $u \mapsto ug = u \circ g$. Let $F(M) = \bigcup_{x \in M} F(M)_x$, called the frame bundle of dimension $d + d^2$, with the canonical projection $\pi\colon F(M) \to M$. Note that each $F(M)_x \cong GL(d, \mathbb{R})$. In fact, $(F(M), M, GL(d, \mathbb{R}))$ is a principal bundle $M \cong F(M)/GL(d, \mathbb{R})$. Let

$$F(M) \times_{GL(d,\mathbb{R})} \mathbb{R}^d := F(M) \times \mathbb{R}^d / \sim,$$

where $(ug, v) \sim (u, gv)$. Then define

$$\Phi\colon F(M) \times_{GL(d,\mathbb{R})} \mathbb{R}^d \to TM$$

by

$$\Phi([u, v]) = u(v) \in T_{\pi(u)} M.$$

Φ is a linear isomorphism and so $F(M) \times_{GL(d,\mathbb{R})} \mathbb{R}^d \cong TM$.

Consider the canonical projection $\pi\colon F(M) \to M$, because it is a smooth submersion and $\pi^{-1}(x) = F(M)_x \cong GL(d, \mathbb{R})$, for

$$d\pi_u\colon T_u F(M) \to T_{\pi(u)} M,$$

the vertical space

$$V_u F(M) := \ker(d\pi_u) = T_u F(M)_{\pi(u)} \cong \mathfrak{g}(d, \mathbb{R}).$$

Let M be equipped with an affine connection ∇. Let $c(t)$ be a smooth curve on M and $u(t)$ be a lift of $c(t)$ on $F(M)$, i.e., $\pi(u(t)) = c(t)$. Because $u(t) \in F(M)_{c(t)}$, for any $e \in \mathbb{R}^d$, $u_e(t) = u(t)e \in T_{c(t)} M$, which means $u_e \in \Gamma(TM)$ is a vector field along $c(t)$. We say $u(t)$ is a horizontal lift if for any $e \in \mathbb{R}^d$, u_e is parallel along c, i.e.,

$$\nabla_{\dot{c}(t)} u_e(t) \equiv 0.$$

Lemma 2.1.1. *For any smooth curve $c\colon(-\varepsilon,\varepsilon)\to M$, let $u_0\in F(M)_{c(0)}$. Then there exists a unique $u\colon(-\varepsilon,\varepsilon)\to F(M)$ such that*

$$\pi\circ u=c,\quad u(0)=u_0,\quad \nabla_{\dot c(t)}\dot u_a(t)=0,\ \forall\ a\in\mathbb{R}^d.$$

Moreover, u dependents smoothly on (c,u_0), and if $u_1=u_0g$, the $t\mapsto u(t)g$ is the unique horizontal lift starting from u_1.

Proof. WTLG, assume $c((-\varepsilon,\varepsilon))$ is contained in a chart $(U;x^1,\cdots,x^d)$ and let Γ_{ij}^k be the corresponding Christoffel symbol for ∇. Let $e_1,\cdots,e_d$ be the canonical basis of $\mathbb{R}^d$
Then a vector field $V(t)=V^i(t)\left.\frac{\partial}{\partial x^i}\right|_{c(t)}$ along $c(t)$ is parallel to $c(t)$ if and only if $V(t)$ satisfies

$$\dot V^i(t)+A_k^i(t)V^k(t)=0,\quad\forall\ i,$$

where $A_k^i(t)=\Gamma_{jk}^i(c(t))\dot c^j(t)$. For each e_j, above equation has a unique solution

$$V_{e_j}(t)=V_j^i(t)\left.\frac{\partial}{\partial x^i}\right|_{c(t)}$$

with initial value $V_{e_j}(0)=u_0e_j=u_{0,j}^i\left.\frac{\partial}{\partial x^i}\right|_{c(0)}$. So the matrix $U(t)=(V_j^i(t))$ is the unique solution of ODE

$$\dot U(t)+A(t)U(t)=0,\quad U(0)=U_0:=(u_{0,j}^i).$$

Moreover, because

$$\frac{d}{dt}\det U(t)=\operatorname{tr}\left(U(t)^{-1}\dot U(t)\right)\det U(t)=-\operatorname{tr}(A(t))\det U(t),$$

and $u_0\in F(M)_{c(0)}$ (i.e., $\det u_0\neq 0$),

$$\det U(t)=\det U_0\cdot\exp\left(-\int_{t_0}^t\operatorname{tr}(A(s))ds\right)\neq 0,$$

it means $U(t)\in GL(d,\mathbb{R})$ for all t. Let

$$u(t)a:=V_j^i(t)a^j\left.\frac{\partial}{\partial x^i}\right|_{c(t)}$$

for any $a=(a^j)\in\mathbb{R}^d$. Then $u(t)\in F(M)_{c(t)}$ and satisfies $u(0)=u_0$ and the parallel condition. The uniqueness of $u(t)$ is directly obtained by the uniqueness of solution of the ODE, which also implies that $t\mapsto u(t)g$ is the unique lift starting from u_1. Also, the smoothness on initial values is also by the theory of ODE. □

Remark 2.1.2. Moreover, give a curve $c(t)$ on M, let $u(t),v(t)$ be its horizontal lifts starting from u_0,v_0 respectively, because $u_0=v_0(v_0^{-1}u_0)$, by above

$$u(t)=v(t)(v_0^{-1}u_0).$$

Moreover, for any t_0,t_1, it is obvious that

$$\mathcal{P}_{c,t_0,t_1}=u_{t_1}u_{t_0}^{-1}\colon T_{c(t_0)}M\to T_{c(t_1)}M$$

is the parallel moving along $c(t)$ by considering $V(t)=u_t(u_{t_0}^{-1}(X))$ and the uniqueness of parallel moving.

Remark 2.1.3. Moreover, if $\dot{c}(0) = X = X^i \frac{\partial}{\partial x^i}\big|_{c(0)}$, then we can see

$$\dot{u}(0) = -\Gamma^i_{\ell k}(c(0)) X^\ell u^k_{0,j} \left.\frac{\partial}{\partial v^i_j}\right|_{u_0} + X^i \left.\frac{\partial}{\partial x^i}\right|_{u_0}$$

which implies $c(0)$, $\dot{c}(0)$, and u_0 uniquely determined $\dot{u}(0) \in T_{u_0}F(M)$.

For a given $X \in T_xM$, let c be a smooth curve in M with $c(0) = x$ and $\dot{c}(0) = X$. Let $u \in F(M)_x$. Let $u(t)$ be the unique horizontal lift of c starting from u. Then define

$$X^H_u = \dot{u}(0) \in T_uF(M),$$

which is well-defined because it is independent of the choice of c, called a horizontal vector. Let

$$H_uF(M) := \left\{X^H_u : X \in T_xM\right\} \subset T_uF(M),$$

called the horizontal space.

Lemma 2.1.4. *For the canonical projection $\pi\colon F(M) \to M$ and any $u \in F(M)$ with $x = \pi(u)$,*

$$d\pi_u\colon H_uF(M) \to T_xM$$

is an isomorphism. Therefore,

$$T_uF(M) = V_uF(M) \oplus H_uF(M).$$

Proof. For any $X^H_u \in H_uF(M)$, $X^H_u = \dot{u}(0)$ for some $u(t)$ which is a horizontal lift of $c(t)$ with $c(0) = x$ and $\dot{c}(0) = X$ and starting from u with $\pi(u) = x$. Then

$$d\pi_u(X^H_u) = \left.\frac{d}{dt}\right|_{t=0} \pi(u(t)) = \left.\frac{d}{dt}\right|_{t=0} c(t) = X.$$

And from the construction, we have seen $X \neq 0$ implies that $X^H_u \neq 0$. $\square$

For any $e \in \mathbb{R}^d$ and $u \in F(M)$, $ue \in T_{\pi(u)}M$ and so we can consider

$$H_e(u) := (ue)^H_u = \dot{u}(0) \in H_uF(M),$$

where $u(t)$ is the horizontal lift of $c(t)$ for $c(0) = \pi(u)$ and $\dot{c}(0) = ue$ starting from u. By above, we know

$$dg_u H_e(u) = H_{g^{-1}e}(ug)$$

where $g\colon F(M) \to F(M)$ is $g(u) = u \circ g = ug$, because

$$dg_u H_e(u) = \left.\frac{d}{dt}\right|_{t=0} u(t)g.$$

Remark 2.1.5. Note that because

$$(d\pi_u)^{-1}\colon X \mapsto X^H_u$$

is $\mathbb{R}$-linear,

$$H_{\lambda e + \mu e'}(u) = \lambda H_e(u) + \mu H_{e'}(u), \quad \forall\, e, e' \in \mathbb{R}^d,\ u \in F(M),\ \lambda, \mu \in \mathbb{R}.$$

Let $e_1, \cdots, e_d$ be the canonical basis of $\mathbb{R}^d$. Then

$$H_j(u) = H_{e_j}(u) \in H_uF(M)$$

forms a basis of $H_uF(M)$ for $j = 1, \cdots, d$, because for any $v = (v^i) \in \mathbb{R}^d$, i.e., $v = v^i e_i$, then

$$H_v(u) = \sum_{i=1}^d v^i H_i(u).$$

$\{H_i\}_{i=1}^d$ are called the fundamental horizontal vector fields.

Remark 2.1.6. For any $X \in \Gamma(TM)$, its unique horizontal lift vector field $X^H \in \Gamma(TF(M))$ is defined as

$$X^H(u) := (X(\pi(u)))^H_u \in T_uF(M), \quad \forall\, u \in F(M).$$

Moreover, by definition, $d\pi_u(X^H(u)) = X(\pi(u))$, i.e,

$$\pi_* X^H = X.$$

This is the reason why X^H is called a lift.

Let $c(t)$ be a smooth curve on M and $u(t)$ be the horizontal lift of it starting from u_0. Then we can consider a curve $w(t)$ on $\mathbb{R}^d$, called the anti-development of $c(t)$ w.s.t. u_0, defined as

$$w_{u_0}(t) = \int_0^t u(s)^{-1}\dot{c}(s)ds,$$

which is because $\dot{c}(t) \in T_{c(t)}M$ and $u(t)^{-1}\colon T_{c(t)}M \to \mathbb{R}^d$. But note that $w_{u_0}(t)$ dependents on the choice of u_0. Let $v(t)$ be the horizontal lift of $c(t)$ starting from v_0. If $u_0 = v_0 g$, i.e., $v_0 = u_0 g^{-1}$, then $v(t) = u(t)g^{-1}$ and thus

$$w_{v_0}(t) = \int_0^t v(s)^{-1}\dot{c}(s)ds = g\int_0^t u(s)^{-1}\dot{c}(s)ds = g w_{u_0}(t).$$

Fix u_0 and let $w(t) = w_{u_0}(t)$. Because $\dot{w}(t) = u(t)^{-1}\dot{c}(t)$, i.e, $u(t)\dot{w}(t) = \dot{c}(t)$,

$$H_{\dot{w}(t)}(u(t)) = (u(t)\dot{w}(t))^H_{u(t)} = (\dot{c}(t))^H_{u(t)} = \dot{u}(t),$$

i.e.

$$\dot{u}(t) = \dot{w}^i(t)H_i(u(t)).$$

Note that from the definition of $H_i(u(t))$ this is linear in $u(t)$, when given $w(t)$ and u_0, this ODE has a unique solution $u(t)$ called the development of $w(t)$ in $F(M)$ and its projection $c(t) = \pi(u(t))$ is called the development of $w(t)$ in M.

2.2 Tensor Fields Scalarization

Let θ be a (r, s)-tensor field on M. For any $u \in F(M)$, let

$$X_i = ue_i \in T_{\pi(u)}M$$

for $i = 1, \cdot, d$, where $\{e_i\}$ is the canonical basis of $\mathbb{R}^d$. Then $\{X_i\}$ is a basis of $T_{\pi(u)}M$. Choose $\{X^i\}$ be the its dual basis of $T^*_{\pi(u)}M$. Then under these basis, we have

$$\theta(\pi(u)) = \theta^{i_1\cdots i_r}_{j_1\cdots j_s} X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes X^{j_1} \otimes \cdots \otimes X^{j_s}.$$

and we define the scalarization

$$\widetilde{\theta}\colon F(M) \to (\mathbb{R}^d)^{\otimes r} \otimes (\mathbb{R}^d)^{*\otimes s} =: \otimes^{(r,s)}\mathbb{R}^d,$$

as

$$\widetilde{\theta}(u) = \theta^{i_1\cdots i_r}_{j_1\cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s},$$

where $\{e^i\}$ is the dual basis of $\{e_i\}$ in $(\mathbb{R}^d)^*$. Note that $\widetilde{\theta}$ is $GL(d, \mathbb{R})$-equivariant, i.e.,

$$\widetilde{\theta}(ug) = g\tilde{\theta}(u).$$

If θ is $(0, 0)$-tensor, i.e., $\theta = f\colon M \to \mathbb{R}$,

$$\widetilde{f} = f \circ \pi\colon F(M) \to \mathbb{R}.$$

Therefore, $\widetilde{\theta}$ can be considering as a lift of θ on $F(M)$.

Remark 2.2.1. For a finite dimensional vector space V, let $\{v_i\}$ be a basis of V and $\{v_i^*\}$ be its dual basis in V^*. If

$$w_j = g_j^i v_i$$

and $\{w_j\}$ is another basis, i.e., $G = (g_j^i)_{i\times j} \in GL$, then for $w_\ell^* = \sum_k h_k^\ell (v_k)^*$, we have the matrix

$$H = (h_k^\ell)_{\ell\times k} = G^{-1}.$$

Moreover, for $u \in F(M)$, $u^{-1}\colon T_{\pi(u)}M \to \mathbb{R}^d$ and so we can consider $u^*\colon T^*_{\pi(u)}M \to (\mathbb{R}^d)^*$, which follows that we can defined $u^{-1}\colon \otimes^{(r,s)} T_{\pi(u)}M \to \otimes^{(r,s)}\mathbb{R}^d$ as

$$u^{-1} = \underbrace{u^{-1}\otimes\cdots\otimes u^{-1}}_{r} \times \underbrace{u^*\otimes\cdots\otimes u^*}_{s}.$$

Then it is obvious that for $\theta \in \Gamma(\otimes^{(r,s)}TM)$

$$\widetilde{\theta}(u) = u^{-1}\theta(\pi(u)).$$

On the other hand, for $u\colon \mathbb{R}^d \to T_xM$, consider $(u^{-1})^*\colon (\mathbb{R}^d)^* \to T^*_{\pi(u)}M$. Then it can extend u to be defined on $\otimes^{(r,s)}(\mathbb{R}^d)^* \to \otimes^{(r,s)}T^*_{\pi(u)}M$ as

$$u = \underbrace{(u^{-1})^*\otimes\cdots\otimes(u^{-1})^*}_{r} \otimes \underbrace{u\otimes\cdots\otimes u}_{s},$$

where $\otimes^{(r,s)}(\mathbb{R}^d)^* = \otimes^r(\mathbb{R}^d)^*\otimes\otimes^s\mathbb{R}^d$ and $\otimes^{(r,s)}T^*_{\pi(u)}M = \otimes^r T^*_{\pi(u)}M \otimes \otimes^s T_{\pi(u)}M$. Then for any $\theta \in \Gamma(\otimes^{(r,s)}TM)$,

$$\theta_{\pi(u)}\colon \otimes^{(r,s)} T^*_{\pi(u)}M \to \mathbb{R}, \quad \widetilde{\theta}_u\colon \otimes^{(r,s)} (\mathbb{R}^d)^* \to \mathbb{R},$$

they have

$$\theta_{\pi(u)} \circ u = \widetilde{\theta}(u),$$

because $\{(u^{-1})^* e^j\}$ is the dual basis of $\{ue_i\}$ in $T^*_{\pi(u)}M$. For $f \in C^\infty(M)$, clearly $f_{\pi(u)} = \widetilde{f}(u)$. And so

$$\nabla_X f(\pi(u)) = \widetilde{\nabla_X f}(u).$$

Proposition 2.2.2. *Let $X \in \Gamma(TM)$ and $\theta \in \Gamma(\otimes^{(r,s)}TM)$. Then*

$$\widetilde{\nabla_X\theta} = X^H\widetilde{\theta},$$

i.e., $\widetilde{\nabla_X\theta}(u) = X_u^H\widetilde{\theta}$ for all $u \in F(M)$, where $X_u^H\widetilde{\theta}$ acts on each component since $X_u^H \in T_uF(M)$.

Proof. Fix $u \in F(M)$. Let $c(t)$ be a curve in M with $c(0) = \pi(u)$ and $\dot{c}(0) = X_{\pi(u)}$. Let $u(t)$ be the unique horizontal lift of $c(t)$ starting from $u(0) = u$. Then as we know, $\tau_t = u(t)u^{-1}$ is the parallel moving along $c(t)$. So we have

$$\left.\frac{d}{dt}\right|_{t=0} \tau_t^{-1}\theta(c(t)) = \nabla_X\theta(\pi(u)).$$

On the other hand, because $\pi(u(t)) = c(t)$, we have

$$\widetilde{\theta}(u(t)) = u(t)^{-1}\theta(c(t)).$$

Moreover, because $X_u^H = \dot{u}(0)$,

$$X_u^H\widetilde{\theta} = \left.\frac{d}{dt}\right|_{t=0} \widetilde{\theta}(u(t))$$

$$
\begin{aligned}
&= \left.\frac{d}{dt}\right|_{t=0} u(t)^{-1}\theta(c(t)) \\
&= u^{-1} \left.\frac{d}{dt}\right|_{t=0} \tau_t^{-1}\theta(c(t)) \\
&= u^{-1}(\nabla_X\theta)(\pi(u)) = \widetilde{\nabla_X\theta}(u),
\end{aligned}
$$

where the first equality is because $\widetilde{\theta}$ is a tensor field on $\mathbb{R}^d$, i.e. the coordinate does not change along time. □

Remark 2.2.3. By the definition of $X_u^H\widetilde{\theta}$, because it is a tensor field on $\mathbb{R}^d$,

$$
(X_u^H\widetilde{\theta})(v^1,\cdots,v^r,w_1,\cdots,w_s) = X_u^H\left(\widetilde{\theta}(u)(v^1,\cdots,v^r,w_1,\cdots,w_s)\right)
$$

for any $v^i \in (\mathbb{R}^d)^*$ and $w_j \in \mathbb{R}^d$.

Remark 2.2.4. Above is also true for $f \in C^\infty(M)$.

$$
\begin{aligned}
X_u^H\widetilde{f} &= \left.\frac{d}{dt}\right|_{t=0} \widetilde{f}(u(t)) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(\pi(u(t))) = \left.\frac{d}{dt}\right|_{t=0} f(c(t)) \\
&= \nabla_X f(c(0)) = \nabla_X f(\pi(u)) = \widetilde{\nabla_X f}(u).
\end{aligned}
$$

Example 2.2.5 (Hessian)**.** For any $f \in C^\infty(M)$, consider the Hessian $\nabla^2 f \in \Gamma(\otimes^{0,2}TM)$,

$$
\begin{aligned}
\nabla^2 f_{\pi(u)}(ue_i, ue_j) &= (\nabla_{ue_j}\nabla f)_{\pi(u)}(ue_i) = (\widetilde{\nabla_{ue_j}\nabla f})(u)(e_i) \\
&= \left((ue_j)_u^H\widetilde{\nabla f}\right)(e_i) = H_j\left(\widetilde{\nabla f}(u)(e_i)\right) \\
&= H_j\left(\nabla f_{\pi(u)}(ue_i)\right) = H_j\left((\nabla_{ue_i} f)_{\pi(u)}\right) \\
&= H_j\left(\widetilde{\nabla_{ue_i} f}(u)\right) = H_j H_i \widetilde{f}
\end{aligned}
$$

where $\widetilde{f} = f \circ \pi$.

2.3 Stochastic Horizontal Lift

Given a SDE on $F(M)$:

$$
dU_t = H(U_t) \circ dW_t = \sum_{i=1}^d H_i(U_t) \circ dW_t^i, \tag{2.1}
$$

where W is an $\mathbb{R}^d$-valued semimartingale and $\{H_i\}_{i=1}^d$ are the fundamental horizontal vector fields on $F(M)$.

Definition 2.3.1. (1) An $F(M)$-valued semimartingale is said to be horizontal if there exists an $\mathbb{R}^d$-valued semimartingale W with $W_0 = 0$ such that SDE (2.1) holds. Then W is called the anti-development of U (or of its projection $X = \pi(U)$).

(2) Let W be an $\mathbb{R}^d$-valued semimartingale with $W_0 = 0$ and U_0 is an $F(M)$-valued, $\mathcal{F}_0$-measurable random variable. The solution of SDE (2.1) with initial condition U_0 is called a stochastic development of W in $F(M)$, so is its projection $X = \pi(U)$.

(3) Let X be an M-valued semimartingale. An $F(M)$-valued horizontal semimartingale U such that $\pi(U) = X$ is called a stochastic horizontal lift of X.

Note that $W \mapsto U$ is by considering the solution of SDE (2.1), and $U \mapsto X$ is just by projection, but we need $X \mapsto U$ and $U \mapsto W$.
Assume $M \subset \mathbb{R}^N$ is a submanifold that is closed. Let $f = (f^\alpha)\colon M \to \mathbb{R}^N$ be the coordinate function and let lift it to

$$\widetilde{f}\colon F(M) \to M \subset \mathbb{R}^N$$

as $\widetilde{f} = f \circ \pi$, i.e. $\widetilde{f}(u) = f(\pi(u)) = \pi(u)$ as in $\mathbb{R}^N$.
For any M-valued semimartingale, we regard $X = (X^\alpha)_{\alpha=1}^N = (f^\alpha(X))_{\alpha=1}^N \in \mathbb{R}^N$ as an $\mathbb{R}^N$-valued semimartingale.
For any $x \in M$, let $P(x)\colon \mathbb{R}^N \to T_xM$ be the orthogonal projection by viewing $T_xM \subset \mathbb{R}^N$. Let $P_\alpha(x) = P(x)e_\alpha \in T_xM$ and so $P_\alpha \in \Gamma(TM)$. Define $P_\alpha^H \in \Gamma(TF(M))$ as

$$P_\alpha^H(u) = (P_\alpha(\pi(u)))_u^H \in T_uF(M), \quad \forall\, u \in F(M).$$

By using the fundamental horizontal lifts,

$$P_\alpha^H(u) = \big(u(u^{-1}P_\alpha(\pi(u)))\big)_u^H = (u^{-1}P_\alpha(\pi(u)))^i H_i(u).$$

Lemma 2.3.2. *For notations as above, when viewing $T_xM \subset \mathbb{R}^N$, we have*

$$P_\alpha^H(u)\widetilde{f} = P_\alpha(\pi(u)), \quad P_\alpha(\pi(u))H_i(u)\widetilde{f^\alpha} = ue_i.$$

Proof. Let $u(t)$ be the horizontal lift from $u(0) = u$ of a curve $c(t)$ on M with $c(0) = \pi(u)$ and $\dot{c}(0) = \widetilde{f}(u) = P_\alpha(\pi(u))$. It follows that $P_\alpha^H(u) = \dot{u}(0)$. Because $\widetilde{f}\colon F(M) \to \mathbb{R}^N$ and $P_\alpha^H(u) \in T_uF(M)$,

$$P_\alpha^H(u)\widetilde{f} = \frac{d}{dt}\bigg|_{t=0} \widetilde{f}(u(t)) = \frac{d}{dt}\bigg|_{t=0} \pi(u(t)) = \frac{d}{dt}\bigg|_{t=0} c(t) = P_\alpha(\pi(u)).$$

Next, let $v(t)$ be the horizontal lift start from $v(0) = u$ of curve $x(t)$ with $x(0) = \pi(u)$ and $\dot{x}(0) = ue_i$. So

$$H_i(u)\widetilde{f} = \frac{d}{dt}\bigg|_{t=0} \widetilde{f}(v(t)) = \frac{d}{dt}\bigg|_{t=0} \pi(v(t)) = \frac{d}{dt}\bigg|_{t=0} x(t) = ue_i,$$

which implies that $H_i(u)\widetilde{f} \in T_{\pi(u)}M$ and so

$$P(\pi(u))H_i(u)\widetilde{f} = H_i(u)\widetilde{f} \;\Rightarrow\; P_\alpha(\pi(u))H_i(u)\widetilde{f^\alpha} = H_i(u)\widetilde{f} = ue_i. \qquad \square$$

Remark 2.3.3. For a vector field $H \in \Gamma(TF(M))$, we usually denote $Hg(u) := H(u)g$ for any $u \in F(M)$ and $g\colon F(M) \to \mathbb{R}^n$. Under this notation,

$$P_\alpha^H\widetilde{f}(u) = P_\alpha(\pi(u)), \quad P_\alpha(\pi(u))H_i\widetilde{f}^\alpha(u) = ue_i.$$

Remark 2.3.4. Note that above two identities view $P_\alpha(\pi(u))$ as a vector in $\mathbb{R}^N$. More precisely, for any $g \in C^\infty(M)$, $g \circ \pi \in C^\infty(F(M))$. We similarly have

$$H_i(u)(g \circ \pi) = \frac{d}{dt}\bigg|_{t=0} g(c(t)) = (ue_i)g.$$

and

$$P_\alpha^H(u)(g \circ \pi) = P_\alpha(\pi(u))g$$

or in other words,

$$d\pi_u(P_\alpha^H(u)) = P_\alpha(\pi(u)),$$

which is the definition of horizontal lift.

Lemma 2.3.5. *Assume $M \subset \mathbb{R}^N$ is a submanifold that is closed. If X is an M-valued semimartingale, then*

$$X_t = X_0 + \int_0^t P(X_s) \circ dX_s = X_0 + \int_0^t P_\alpha(X_s) \circ dX_s^\alpha.$$

Proof. Let $Q_\alpha(x) = e_\alpha - P_\alpha(x)$, i.e., $Q_\alpha(x)$ normal to $T_x M$ and $Q_\alpha \in \Gamma(NM)$. Let

$$Y_t = X_0 + \int_0^t P_\alpha(X_s) \circ dX_t^\alpha$$

By the closedness of M, there exists $f \in C^\infty(\mathbb{R}^n)$ such that $f(x) \geq 0$ and $f^{-1}(0) = M$. Then by Itô formula,

$$f(Y_t) = f(X_0) + \int_0^t P_\alpha f(X_t) \circ dX_t^\alpha.$$

Because $P_\alpha \in \Gamma(TM)$ and $f|_M = 0$, $P_\alpha f(x) = 0$ for all $x \in M$. Since X is M-valued, $f(Y_t) = 0$, i.e., $Y \in f^{-1}(0) = M$, and Y is an M-valued semimartingale.
Next, consider a tubular neighborhood U of M and defined the natural projection $\tilde{h}\colon U \to M$, which can be extended a smooth $h\colon \mathbb{R}^N \to M$. Because $Q_\alpha \in \Gamma(NM)$, by choosing Fermi coordinates, we have

$$Q_\alpha h(x) = 0 \;\Rightarrow\; P_\alpha h(x) = \frac{\partial h}{\partial x^\alpha}.$$

So

$$\begin{aligned} Y_t = h(Y_t) &= h(X_0) + \int_0^t P_\alpha h(X_s) \circ dX_s^\alpha \\ &= X_0 + \int_0^t \frac{\partial h}{\partial x^\alpha} \circ dX_s^\alpha \\ &= h(X_t) = X_t. \end{aligned}$$

□

Theorem 2.3.6. *A horizontal semimartingale U on $F(M)$ has a unique anti-development W. In fact,*

$$W_t = \int_0^t U_s^{-1} P_\alpha(X_s) \circ dX_s^\alpha,$$

for $X = \pi(U)$.

Proof. By definition, W should be

$$dU_t = H_i(U_t) \circ dW_t^i.$$

Let $\widetilde{f}$ be defined as above. Then

$$\widetilde{f}(U_t) = f(\pi(U_t)) = f(X_t) = X_t \in M \subset \mathbb{R}^N.$$

Then X_t should satisfy

$$dX_t^\alpha = H_i \widetilde{f^\alpha}(U_t) \circ dW_t^i.$$

Therefore, by above formulas, we have

$$U_t^{-1} P_\alpha(X_t) \circ dX_t^\alpha = U_t^{-1} P_\alpha(X_t) H_i \widetilde{f^\alpha}(U_t) \circ dW_t^i = e_i dW_t^i = dW_t$$

By the uniqueness of SDE, W_t uniquely exists and

$$W_t = \int_0^t U_s^{-1} P_\alpha(X_s) \circ dX_s^\alpha.$$

□

Remark 2.3.7. Note that if U_t^1 and U_t^2 are two horizontal lifts of X starting from U_0^1 and $U_0^2 = U_0^1 g$, then by above, we obviously have

$$W_t^1 = g^{-1} W_t^2.$$

Remark 2.3.8. On the other hand,

$$\begin{aligned} P_\alpha(X_t) \circ dX_t^\alpha &= P_\alpha(X_t) H_i \widetilde{f^\alpha}(U_t) \circ dW_t^i \\ &= U_t e_i \circ dW_t^i = U_t \circ (e_i \circ dW_t^i) \\ &= U_t \circ dW_t. \end{aligned}$$

Then by above,

$$X_t = X_0 + \int_0^s U_s \circ dW_s.$$

and clearly this formula is independent of the choice of U.

Theorem 2.3.9. *Suppose X is an M-valued semimartingale (up to a stopping time τ), and U_0 is an $F(M)$-valued $\mathcal{F}_0$-random variable such that $\pi(U_0) = X_0$. Then there exists a unique horizontal lift U (up to τ) of X starting from U_0.*

Proof. Consider a SDE defined as

$$dU_t = P_\alpha^H(U_t) \circ dX_t^\alpha.$$

Let U_t be its unique solution, so U_t is clearly a horizontal lift since $P_\alpha^H \in \Gamma HM$. It suffices to show $\pi(U_t) = X_t$. Let

$$Y_t = \widetilde{f}(U_t) = f(\pi(U_t)) = \pi(U_t) \in M \subset \mathbb{R}^N.$$

By above, we have

$$dY_t = P_\alpha^H \widetilde{f}(U_t) \circ dX_t^\alpha = P_\alpha(Y_t) \circ dX_t^\alpha.$$

By above lemma, we know $dX_t = P_\alpha(X_t) \circ dX_t^\alpha$. By the uniqueness of SDE, $X = Y$.
Next, assume there exists another horizontal lift Π_t of X_t. Then there exists a semimartingale W such that

$$d\Pi_t = H_i(\Pi_t) \circ dW_t^i.$$

and by above theorem

$$dW_t = \Pi_t^{-1} P_\alpha(X_t) \circ dX_t^\alpha.$$

So

$$\begin{aligned} d\Pi_t &= \left(\Pi_t^{-1} P_\alpha(X_t)\right)^i H_i(\Pi_t) \circ dX_t^\alpha \\ &= P_\alpha^H(\Pi_t) \circ dX_t^\alpha. \end{aligned}$$

So $\Pi = U$ by the uniqueness of solution of SDE. □

Remark 2.3.10. When considering explosion time, there is a fact, if X on M is a semimartingale up to τ, then its horizontal lift H on $F(M)$ is also defined up to τ.

Similarly as the deterministic case, given a semimartingale X on M, the horizontal lift U_t provide the stochastic parallel moving along X_t,

$$\tau_{t_1 t_2}^X = U_{t_2} U_{t_1}^{-1} : T_{X_{t_1}} M \to T_{X_{t_2}} M.$$

Proposition 2.3.11. *Let a semimartingale X on a manifold M be the solution of* $\mathrm{SDE}(V_1, \cdots, V_N; Z, X_0)$ *and V_α^H be the horizontal lift of V_α. Then the horizontal lift U of X starting from U_0 satisfy* $\mathrm{SDE}(V_1^H, \cdots, V_N^H; Z, U_0)$, *and the anti-development of X is given by*

$$W_t = \int_0^t U_s^{-1} V_\alpha(X_s) \circ dZ_s^\alpha.$$

Proof. Assume $M \subset \mathbb{R}^N$ that is closed and use above notations. First, we similar have

$$V_\alpha^H \widetilde{f}(u) = V_\alpha(\pi(u)).$$

Let Π be the solution of $\mathrm{SDE}(V_1^H, \cdots, V_N^H; Z, U_0)$ and $Y_t = \widetilde{f}(\Pi_t) = \pi(\Pi_t)$.

$$dY_t = V_\alpha^H \widetilde{f}(\Pi_t) \circ dZ_t^\alpha = V_\alpha(Y_t) \circ dZ_t^\alpha,$$

which implies that $Y = X$ by the uniqueness of solution of $\mathrm{SDE}(V_1, \cdots, V_N; Z, X_0)$. And by the uniqueness of horizontal lift, $U = \Pi$.
Next, the anti-development of X is given by

$$dW_t = U_t^{-1} P_\beta(X_t) \circ dX_t^\beta.$$

Because $dX_t = V_\alpha(X_t) \circ dZ_t^\alpha$, i.e., $dX_t^\beta = V_\alpha^\beta(X_t) \circ dZ_t^\alpha$,

$$dW_t = U_t^{-1} P_\beta(X_t) V_\alpha^\beta(X_t) \circ dZ_t^\alpha.$$

Because $V_\alpha(X_t) \in T_{X_t}M$,

$$P_\beta(X_t) V_\alpha^\beta(X_t) = P(X_t) V_\alpha(X_t) = V_\alpha(X_t).$$

So

$$dW_t = U_t^{-1} V_\alpha(X_t) \circ dZ_t^\alpha.$$

□

2.4 Stochastic Line Integral

Line Integral. For any 1-form $\omega = f(t)dt$ on $[a, b]$ of $\mathbb{R}$, define

$$\int_{[a,b]} \omega = \int_a^b f(t)dt.$$

Let M be a smooth manifold and $\omega \in \Gamma(T^*M)$. Let $\gamma \colon [a, b] \to M$ smooth. Define the line integral of ω along γ as

$$\int_\gamma \omega = \int_{[a,b]} \gamma^* \omega.$$

Clearly, it satisfies the linearity in ω and additivity w.s.t. γ. Moreover, for any smooth $F \colon M \to N$ and $\eta \in \Gamma(T^*N)$ and γ in M,

$$\int_\gamma F^* \eta = \int_{F \circ \gamma} \eta.$$

Proposition 2.4.1. *Let $\omega \in \Gamma(T^*M)$ and $\gamma \colon [a, b] \to M$ smooth.*

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\dot{\gamma}(t))dt$$

In particular, if $\omega = df$ for some $f \in C^\infty(M)$,

$$\int_\gamma df = \int_a^b (f \circ \gamma)'(t)dt = f(\gamma(b)) - f(\gamma(a)).$$

Proof. Let $\gamma(t) = (\gamma^1(t), \cdots, \gamma^d(t))$ and $\omega = \omega_i dx^i$ be the coordinate expression. Then

$$(\gamma^*\omega)_t = \omega_i(\gamma(t))dx^i(\gamma(t)) = \omega_i(\gamma(t))d\gamma^i(t) = \omega_i(\gamma(t))(\gamma^i)'(t)dt$$

On the other hand,

$$\omega_{\gamma(t)}(\dot\gamma(t)) = \omega_i(\gamma(t))dx^i\left((\gamma^j)'(t)\frac{\partial}{\partial x^j}\right) = \omega_i(\gamma(t))(\gamma^i)'(t).$$

Therefore, we get the desired result. □

Stochastic Line Integral. Let $x\colon [0,t] \to M$ and $u(t)$ be its horizontal lift on $F(M)$ starting from u_0. Let $w(t)$ be the anti-development of $x(t)$, i.e., $\dot w_t = u_t^{-1}\dot x_t$ and $w_0 = 0$. Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^d$ and so $\{u_t e_i\}$ is a basis of $T_{x_t}M$. Let

$$w_t = w_t^i e_i \;\Rightarrow\; \dot x_t = \dot w_t^i\,(u_t e_i)\,.$$

Let $\theta \in \Gamma(t^*M)$. Then

$$\int_x \theta = \int_0^t \theta_{x_s}(\dot x_s)ds = \int_0^t \dot w_s^i \theta_{x_s}(u_s e_i)ds.$$

Definition 2.4.2. Let $\theta \in \Gamma(t^*M)$ and X be an M-valued semimartingale. Let U be a horizontal lift of X and W be its anti-development. The stochastic line integral of θ along X is defined by

$$\int_{X_{[0,t]}} \theta = \int_0^t \theta_{X_s}(U_s e_i) \circ dW_s^i.$$

Remark 2.4.3. For such θ, let $\widetilde\theta$ be the scalarization of 1-form θ. Then by the definition, $\widetilde\theta\colon F(M) \to \mathbb{R}^d$ by viewing $(\mathbb{R}^d)^* = \mathbb{R}^d$ and

$$\widetilde\theta(u) = \left(\theta_{\pi(u)}(ue_1), \cdots, \theta_{\pi(u)}(ue_d)\right)^\top \in \mathbb{R}^d.$$

So

$$\int_{X_{[0,t]}} \theta = \int_0^t \widetilde\theta(U_s)_i \circ dW_s^i = \int_0^t \widetilde\theta(U_s)^\top \circ dW_s.$$

Note that $\int_{X_{[0,t]}} \theta$ seems depend on the choice of horizontal lift, and so the connection on M. Actually, it is independent of the above choices by the following proposition and the fact that any semimartingale on M is a solution of some SDE shown in above section.

Proposition 2.4.4. *Let $\theta \in \Gamma(T^*M)$ and X be the solution of $dX_t = V_\alpha(X_t) \circ dZ_t^\alpha$. Then*

$$\int_{X_{[0,t]}} \theta = \int_0^t \theta(V_\alpha)(X_s) \circ dZ_s^\alpha.$$

Proof. Choose a horizontal lift U and corresponding anti-development W of X. Then

$$dW_t = U_t^{-1}V_\alpha(X_t) \circ dZ_t^\alpha.$$

Therefore,

$$\begin{aligned}\widetilde\theta(U_s)_i \circ dW_s^i &= \widetilde\theta(U_s)_i(U_t^{-1}V_\alpha(X_s))^i \circ dZ_s^\alpha\\ &= \theta_{X_s}(U_s e_i)(U_t^{-1}V_\alpha(X_s))^i \circ dZ_s^\alpha\\ &= \theta_{X_s}\left(U_s\left(e_i(U_t^{-1}V_\alpha(X_s))^i\right)\right) \circ dZ_s^\alpha\\ &= \theta_{X_s}(V_\alpha(X_s)) \circ dZ_s^\alpha.\end{aligned}$$

Therefore, we have

$$\int_{X_{[0,t]}} \theta = \int_0^t \theta(V_\alpha)(X_s) \circ dZ_s^\alpha.$$

□

Remark 2.4.5. Note that the stochastic line integral of 1-form θ along a given M-valued semimartingale X is independent of the choice of horizontal lift and also the connection. Symbolically,

$$\int_{X_{[0,t]}} \theta = \int_0^t \theta \circ dX_t$$

Example 2.4.6. (1) For all $f \in C^\infty(M)$, assume $dX_t = V_\alpha(X_t) \circ dZ_t^\alpha$.

$$\int_{X_{[0,t]}} df = \int_0^t V_\alpha f(X_s) \circ dZ_s^\alpha = f(X_t) - f(X_0)$$

(2) Let $(U, x^1, \cdots, x^d)$ be a local chart of M. Let $\theta = \theta_i dx^i$. For any X a semimartingale on M, let $X^i = x^i(X)$ that is a semimartingale on $\mathbb{R}$.

$$dX_t = V_i(X_t) \circ dX_t^i, \quad V_i = \frac{\partial}{\partial x^i},$$

because $df(X_t) = \frac{\partial f}{\partial x^i}(X_t) \circ dX_t^i$. Then

$$\int_{X_{[0,t]}} \theta = \int_0^t \theta(V_i(X_s)) \circ dX_s^i = \int_0^t \theta_i(X_s) \circ dX_s^i,$$

which is the local expression of stochastic line integral.

Definition 2.4.7. Let $\theta\colon \Gamma(TF(M)) \to C^\infty(F(M), \mathbb{R}^d)$, i.e., a $\mathbb{R}^d$-valued 1-form on $F(M)$. Define

$$\theta(Z)(u) = \theta_u(Z_u) := u^{-1}(d\pi_u(Z_u)),$$

for any $Z \in \Gamma(TF(M))$ and $u \in F(M)$, where $\pi\colon F(M) \to M$ is the natural projection. Then θ is called the solder form.

Proposition 2.4.8. *Let U be a horizontal semimartingale on $F(M)$. Then its the corresponding anti-development is given by*

$$W_t = \int_{U_{[0,t]}} \theta,$$

where θ is the solder form.

Proof. Let $X_t = \pi(U_t)$. Then we have

$$dU_t = P_\alpha^H(U_t) \circ dX_t^\alpha.$$

Note that $\pi_* P_\alpha^H = P_\alpha$. So we get

$$\begin{aligned}
\int_{U_{[0,t]}} \theta &= \int_0^t \theta(P_\alpha^H)(U_s) \circ dX_s^\alpha \\
&= \int_0^t U_s^{-1} d\pi_{U_s}(P_\alpha^H(U_s)) \circ dX_s^\alpha \\
&= \int_0^t U_s^{-1} P_\alpha(\pi(U_s)) \circ dX_s^\alpha = W_t.
\end{aligned}$$

□

Quadratic Variation. Let $h \in \Gamma(\otimes^{(0,2)}TM)$ and $\widetilde{h}$ be its scalarization. So

$$\widetilde{h}(u)(e, e') = h_{\pi(u)}(ue, ue'), \quad \forall\ e, e' \in \mathbb{R}^d,\ u \in F(M).$$

Let

$$h^{\text{sym}}(e, e') = \frac{h(e, e') + h(e', e)}{2}.$$

Definition 2.4.9. Let $h \in \Gamma(\otimes^{(0,2)}TM)$ and X be an M-valued semimartingale. Let U be a horizontal lift of X and W be its anti-development. Note that

$$dX_t = U_t e_i \circ dW_t^i,$$

Then the h-quadratic variation of X is defined as

$$\begin{aligned}\int_0^t h(dX_s, dX_s) &:= \int_0^t h_{X_s}(U_s e_i, U_s e_j) d\left\langle W^i, W^j \right\rangle_s \\ &= \int_0^t \widetilde{h}(U_s)(e_i, e_j) d\left\langle W^i, W^j \right\rangle_s.\end{aligned}$$

By viewing $dW_s^i dW_s^j = d\left\langle W^i, W^j \right\rangle_s$, then

$$\int_0^t h(dX_s, dX_s) = \int_0^t \widetilde{h}(U_s)(dW_s, dW_s).$$

Obviously,

$$\int_0^t h(dX_s, dX_s) = \int_0^t h^{\text{sym}}(dX_s, dX_s)$$

and if h is anti-symmetric, $\int_0^t h(dX_s, dX_s) = 0$.

Remark 2.4.10. Note that this definition is independent the choice of U, i.e., the choice of U_0, because the anti-development of $U_t g$ is $g^{-1}W_t$ and

$$\begin{aligned}h(U_t g e_i, U_t g e_j) d\left\langle (g^{-1}W)^i, (g^{-1}W)^j \right\rangle_t &= \widetilde{h}(U_t)(g e_i, g e_j) d\left\langle (g^{-1}W)^i, (g^{-1}W)^j \right\rangle_t \\ &= g_i^\alpha g_j^\beta \widetilde{h}(U_t)(e_\alpha, e_\beta) \tilde{g}_k^i \tilde{g}_\ell^j d\left\langle W^k, W^\ell \right\rangle_t \\ &= \widetilde{h}(U_t)(e_k, e_\ell) d\left\langle W^k, W^\ell \right\rangle_t,\end{aligned}$$

where $g e_i = g_i^j$ and $(\tilde{g}_i^j) = (g_i^j)^{-1}$. So h-quadratic variation is well-defined.

Proposition 2.4.11. *Let $h \in \Gamma(\otimes^{(0,2)}TM)$ and X be the solution of* $\text{SDE}(V_1, \cdots, V_N; Z, X_0)$. *Then*

$$\int_0^t h(dX_s, dX_s) = \int_0^t h(V_\alpha, V_\beta)(X_s) d\left\langle Z^\alpha, Z^\beta \right\rangle_s.$$

Proof. Note that

$$dX_t = V_\alpha(X_t) \circ dZ_t^\alpha$$

Let U be a horizontal lift of X and W be its anti-development. Then

$$dW_t = U_t^{-1} V_\alpha(X_t) \circ dZ_t^\alpha.$$

and so

$$d\left\langle W^i, W^j \right\rangle_t = \left(U_t^{-1} V_\alpha(X_t)\right)^i \left(U_t^{-1} V_\beta(X_t)\right)^j d\left\langle Z^\alpha, Z^\beta \right\rangle_t.$$

It implies that

$$\begin{aligned}h(U_t e_i, U_t e_j) d\left\langle W^i, W^j \right\rangle_t &= h(U_t e_i, U_t e_j) \left(U_t^{-1} V_\alpha(X_t)\right)^i \left(U_t^{-1} V_\beta(X_t)\right)^j d\left\langle Z^\alpha, Z^\beta \right\rangle_t \\ &= h(V_\alpha, V_\beta)(X_t) d\left\langle Z^\alpha, Z^\beta \right\rangle_t.\end{aligned}$$

□

Example 2.4.12. (1) Let $(U, x^1, \cdots, x^d)$ be a coordinate and $h \in \Gamma(\otimes^{(0,2)}TM)$ locally written as

$$h = h_{ij} dx^i \otimes dx^j.$$

For $X = (X^i)$ on M,

$$\int_0^t h(dX_s, dX_s) = \int_0^t h_{ij}(X_s) d\left\langle X^i, X^j \right\rangle_s,$$

by $dX_t = \frac{\partial}{\partial x^i}(X_t) \circ dX_t^i$.

(2) For $f \in C^\infty(M)$, $\nabla^2 f \in \Gamma(\otimes^{(0,2)}TM)$ and so

$$\begin{aligned}\int_0^t \nabla^2 f(dX_s, dX_s) &= \int_0^t \nabla^2 f_{X_s}(U_s e_i, U_s e_j) d\left\langle W^i, W^j \right\rangle_s \\ &= \int_0^t H_j H_i \widetilde{f}(U_s) d\left\langle W^i, W^j \right\rangle_s.\end{aligned}$$

(3) If $f, g \in C^\infty(M)$,

$$\int_0^t (df \otimes dg)(dX_s, dX_s) = \langle f(X), g(X) \rangle_t.$$

Proof. For X an M-valued semimartingale, WTLG assume $dX_t = V_\alpha(X_t) \circ dZ_t^\alpha$. Then

$$\int_0^t (df \otimes dg)(dX_s, dX_s) = \int_0^t V_\alpha f(X_s) V_\beta f(X_s) d\left\langle Z^\alpha, Z^\beta \right\rangle_s.$$

On the other hand, because

$$df(X_t) = V_\alpha f(X_t) \circ dZ_t^\alpha, \quad dg(X_t) = V_\beta g(X_t) \circ dZ_t^\beta,$$

we have

$$\langle f(X), g(X) \rangle_t = \int_0^t V_\alpha f(X_s) V_\beta f(X_s) d\left\langle Z^\alpha, Z^\beta \right\rangle_s.$$ □

Proposition 2.4.13. *If $\theta \in \Gamma(\otimes^{(0,2)}TM)$ is positive semi-definite, then the θ-quadratic variation of a semimartingale X is nondecreasing.*

Proof. First,

$$\int_0^t h\left(dX_s, dX_s\right) = \int_0^t h\left(U_s e_i, U_s e_j\right) d\left\langle W^i, W^j \right\rangle_s.$$

Because h is positive semi-positive, the matrix $\left(h\left(U_s e_i, U_s e_j\right)\right)$ is positive semi-definite and let $(m_i^k(s))$ be its squared root matrix. Let

$$J_t^k = \int_0^t m_i^k(s) dW_s^i.$$

Therefore,

$$\int_0^t h\left(dX_s, dX_s\right) = \sum_k \int_0^t m_i^k(s) m_j^k(s) d\left\langle W^i, W^j \right\rangle_s = \sum_k \left\langle J^k, J^k \right\rangle_t.$$ □

2.5 Martingale on Manifold

Definition 2.5.1. Suppose M is a C^∞ manifold with a connection ∇. An M-valued semimartingale X is called a ∇-martingale if its anti-development W w.s.t ∇ is an $\mathbb{R}^d$-valued local martingale.

Remark 2.5.2. It is well defined because if W and W' are two anti-development of X, then there exists a $g \in GL(d, \mathbb{R})$ such that $W' = gW$.

Remark 2.5.3. Note that when $M = \mathbb{R}^N$ is equipped with the canonical connection. Then for any $c(t)$ on M, its horizontal lift starting from u_0 is

$$u(t)a = (u_0)^i_j a^j \left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

When viewing $T_xM \subset \mathbb{R}^N$, $u(t)a \equiv u_0 a$. Define $f\colon F(M) \to \mathbb{R}^N$ as $f(u) = ue_i$. Then

$$P^H_\alpha(u)f = \left.\frac{d}{dt}\right|_{t=0} f(u_t) = \left.\frac{d}{dt}\right|_{t=0} u_t e_i = 0.$$

Let X be an M-valued semimartingale. Because $dU_t = P^H_\alpha(U_t) \circ dX^\alpha_t$,

$$df(U_t) = P^H_\alpha f(U_t) \circ dX^\alpha_t = 0 \;\Rightarrow\; f(U_t) = U_t e_i = U_0 e_i.$$

Therefore,

$$\begin{aligned} X_t &= X_0 + \int_0^t U_s e_i \circ dW^i_s \\ &= X_0 + \int_0^t U_0 e_i dW^i_s \\ &= X_0 + U_0 W_t. \end{aligned}$$

It follows that when $M = \mathbb{R}^N$, if X is an M-martingale, then it is a local martingale in the usual sense.

Proposition 2.5.4. *An M-valued semimartingale X is a ∇-martingale if an only if*

$$N^f(X)_t := f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \nabla^2 f(dX_s, dX_s),$$

is an $\mathbb{R}$-valued local martingale for every $f \in C^\infty(M)$.

Remark 2.5.5. For Euclidean case, a continuous semimartingale X is a local martingale if and only if $N^f(X)$ is a local martingale for all $f \in C^\infty$ (or $f \in C^\infty_c$) by Itô formula.

Proof. $\Rightarrow$: Let U be a horizontal lift of X and W be its anti-development. Then

$$dU_t = H_i(U_t) \circ dW^i_t.$$

For $f \in C^\infty(M)$, let $\widetilde{f} = f \circ \pi \in C^\infty(F(M))$. So

$$\begin{aligned} f(X_t) - f(X_0) &= \widetilde{f}(U_t) - \widetilde{f}(U_0) \\ &= \int_0^t H_i \widetilde{f}(U_s) \circ dW^i_s \end{aligned}$$

$$= \int_0^t H_i\widetilde{f}(U_s)dW_s^i + \frac{1}{2}\left\langle H_i\widetilde{f}(U), W^i\right\rangle_t.$$

Note that

$$dH_i\widetilde{f}(U_t) = H_jH_i\widetilde{f}(U_t) \circ dW_t^j,$$

which implies that

$$\begin{aligned}\left\langle H_i\widetilde{f}(U), W^i\right\rangle_t &= \int_0^t H_jH_i\widetilde{f}(U_s)d\left\langle W^i, W^j\right\rangle_s \\ &= \int_0^t \nabla^2 f(dX_s, dX_s).\end{aligned}$$

Therefore,

$$N^f(X)_t = \int_0^t H_i\widetilde{f}(U_t)dW_t^i.$$

When X is an M-local semimartingale, i.e., W is an $\mathbb{R}^d$-local martingale, then $N^f(X)$ is obviously a local martingale.

$\Leftarrow$: Assume $M \subset \mathbb{R}^N$ properly embedded. Let $f\colon M \to \mathbb{R}^N$ be the coordinate function $f(x) = x$ and $\widetilde{f} = f \circ \pi$ be its lift on $F(M)$. When viewing $T_xM \subset \mathbb{R}^N$, we already have $H_i\widetilde{f}(u) = ue_i$. So similarly as above calculation,

$$\begin{aligned}N^f(X)_t &:= N^{f^\alpha}(X)_t e_\alpha \\ &= e_\alpha \int_0^t H_i\widetilde{f^\alpha}(U_s)dW_s^i \\ &= e_\alpha \int_0^t (U_se_i)^\alpha dW_t^i \\ &= \int_0^t U_se_idW_t^i = \int_0^t U_sdW_s,\end{aligned}$$

by viewing $U_s\colon \mathbb{R}^d \to \mathbb{R}^N$ where $U_s\colon \mathbb{R}^d \to T_xM$ is an isomorphism. Define $V_s\colon \mathbb{R}^N = T_xM \oplus N_xM \to \mathbb{R}^d$ as

$$V_s\xi = \begin{cases} U_s^{-1}\xi, & \xi \in T_{X_s}M, \\ 0, & \xi \perp T_{X_s}M \end{cases}$$

Then $V_sU_se = e$ for all $\mathbb{R}^d$. Therefore,

$$\int_0^t V_sdN^f(X)_s = \int_0^t V_sU_sdW_s = W_t.$$

Because $N^f(X)$ is a local martingale, W_t is a local martingale. It follows that X is an M-martingale. □

Remark 2.5.6. From the proof, we can see that when viewing $M \subset \mathbb{R}^N$ and let $f = (f^1, \cdots, f^\alpha)\colon M \to \mathbb{R}^N$ be the coordinate function, if N^{f^α} for $\alpha = 1, \cdots, N$ are local martingales, then X is an M-local martingale.

Proposition 2.5.7. *Suppose $(U, x^1, \cdots, x^d)$ is a local chart on M and $X = (X^i)$, i.e. $X^i = x^i(X)$, a semimartingale on M. Then X is an M-martingale if and only if*

$$N_t^i = X_t^i - X_0^i + \frac{1}{2}\int_0^t \Gamma_{jk}^i(X_s)d\left\langle X^j, X^k\right\rangle_s$$

is a local martingale.

Proof. $\Rightarrow$: Locally, we know

$$\nabla^2 x^i = -\Gamma^i_{jk} dx^j \otimes dx^k,$$

which implies that

$$\int_0^t \nabla^2 x^i(dX_s, dX_s) = -\int_0^t \Gamma^i_{jk}(X_s) d\left\langle X^j, X^k \right\rangle_s.$$

Therefore, this direction can be directly obtained by applying above theorem to each $f = x^i$.
$\Leftarrow$: Choose a local coordinate for $F(M)$ as $(\tilde{x}^i, e^i_j)$, where

$$\tilde{x}^i(u) = x^i(\pi(u)), \quad e^i_j(u) \text{ for } ue_j = e^i_j(u) \left.\frac{\partial}{\partial x^j}\right|_{\pi(u)}.$$

Let U be a horizontal lift with the anti-development W of X. Then

$$dU_t = H_j(U_t) \circ dW^j_t.$$

Because $X_t = \pi(U_t)$ and $U_t e_j = e^i_j(t) \left.\frac{\partial}{\partial x^j}\right|_{X_t}$, where $e^i_j(t) = e^i_j(U_t)$, we have

$$\begin{aligned} dX^i_t = d\tilde{x}^i(U_t) &= H_j \tilde{x}^i(U_t) \circ dW^j_t \\ &= (U_t e_j)(x^i) \circ dW^j_t \\ &= e^i_j(t) \circ dW^j_t. \end{aligned}$$

Moreover, we also know

$$dU_t = P^H_k(U_t) \circ dX^k_t.$$

Note that in the local chart, M can be viewed as $\mathbb{R}^d$ and so $P(x)\colon \mathbb{R}^d \to T_x M$ the orthogonal projection is the identity, which implies that

$$P_k(X_t) = P(X_t) e_k = e_k = \delta^\ell_k \left.\frac{\partial}{\partial x^\ell}\right|_{X_t}.$$

Let $v(s)$ be the horizontal lift starting from U_t of a curve $c(s)$ with $c(0) = U_t$ and $\dot{c}(0) = P_k(X_t) = \delta^m_k \left.\frac{\partial}{\partial x^m}\right|_{X_t}$. Then by using the coordinates, we have

$$\dot{v}(0) = P^H_k(U_t) = -\Gamma^i_{m\ell}(X_t)\delta^m_k e^\ell_j(t) \left.\frac{\partial}{\partial e^i_j}\right|_{U_t} + \delta^m_k \left.\frac{\partial}{\partial \tilde{x}^m}\right|_{U_t}.$$

So

$$P^H_k e^i_j(U_t) = -\Gamma^i_{k\ell}(X_t) e^\ell_j(t).$$

It follows that

$$de^i_j(t) = -\Gamma^i_{k\ell}(X_t) e^\ell_j(t) \circ dX^k_t.$$

Let $(f^i_j) = (e^j_j)^{-1}$. Then we get

$$dW^j_t = f^j_k(t) \circ dX^k_t.$$

So

$$\begin{aligned} dX^i_t &= e^i_j(t) dW^i_t + \frac{1}{2} d\left\langle e^i_j, W^j \right\rangle_t \\ &= e^i_j(t) dW^i_t - \frac{1}{2}\Gamma^i_{k\ell}\left(X_t\right) e^\ell_j(t) f^j_m(t) d\left\langle X^k, X^m \right\rangle_t \\ &= e^i_j(t) dW^i_t - \frac{1}{2}\Gamma^i_{k\ell}\left(X_t\right) d\left\langle X^k, X^\ell \right\rangle, \end{aligned}$$

which implies that

$$dN^i_t = e^i_j(t) dW^i_t \;\Rightarrow\; dW^i_t = f^i_j(t) dN^i_t.$$

Therefore, if N^i_t are all local martingales, so is W_t and X is an M-valued martingale. $\square$