My Report about Operator Theory

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July 26, 2019

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Chapter 1

Topological Vector Spaces and Banach Spaces

1.1 Topological Vector Spaces

Linear operations, i.e addition and scalar multiplication, provide an algebraic structure on a set, therefore constructing a vector space. In the course, linear algebra, we have learn the algebraic structure of finite dimensional vector spaces. But how to deal with infinite dimensional vector spaces? By learning the topological spaces, we know that the topological structure can give us a method to research properties of infinity. Thus, we need to equip a vector space with an additional topological structure, which should coincide with the algebraic structure. This is the reason why we define the topological vector space.

1.1.1 Topological Spaces

▪ **Definition:** First, we define the topological structure on a general set.

Definition 1. A topological space $X = (X, \mathscr{T}_X)$ consists of a set X, called the underlying space of X, and a family $\mathscr{T}_X$ of subsets of X s.t.

1) $X, \varnothing \in \mathscr{T}_X$.

2) if $U_\alpha \in \mathscr{T}_X$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}_X$.

3) if $U_1, U_2 \in \mathscr{T}_X$, then $U_1 \bigcap U_2 \in \mathscr{T}_X$.

And, $\mathscr{T}_X$ is called a topology on X. The element in $\mathscr{T}_X$ is called open set.

Thus, the topological structure on a set X is totally determined by the family $\mathscr{T}_X$. In particular, from 2), we can simplify $\mathscr{T}_X$. In other words, like the basis of a vector space, there is a "basis" of $\mathscr{T}_X$.

Definition 2. If X is a set, a basis for a topology on X is a family $\mathscr{B}$ of subsets of X, s.t.

1) $\forall\ x \in X,\ \exists\ B \in \mathscr{B},$ s.t. $x \in B$.

2) if $x \in B_1 \bigcap B_2$, where $B_1,\ B_2 \in \mathscr{B}$, then there is a $B_3 \in \mathscr{B}$, s.t. $B_3 \subset B_1 \bigcap B_2$.

$\mathscr{B}$ can generate a topology $\mathscr{T}_X$ on X by doing infinite times union of elements in $\mathscr{B}$. In fact,

$$\mathscr{T}_X \ = \{\ U \subset X \colon U = \bigcup_{\alpha \in A} U_\alpha,\ U_\alpha \in \mathscr{B}\ \},\ \text{where A is any index set.}$$

We can do more on the basis by 3).

Definition 3. A subbasis of a topology on X is a collection $\mathscr{S}$ of subsets of X, whose union is X. The topology generated by $\mathscr{S}$ is noted by $\mathscr{T}(\mathscr{S})$.

In fact, we can generate the basis of $\mathscr{T}(\mathscr{S})$ by $\mathscr{S}$, that is,

$$\mathscr{B} \ = \{\ U \subset X \colon U = \bigcap_{\alpha=1}^{n} U_\alpha,\ U_\alpha \in \mathscr{S},\ n \in \mathbb{N}\ \}$$

Therefore, this basis can generate the conincided topology on X.

▪ **Continuous maps:** Next, we need to endow the general maps with the topological structure.

Definition 4. Let X, Y be topological spaces and $f \colon X \to Y$ be a map. We say f is continous if

$$\forall\ V \in \mathscr{T}_Y,\ f^{-1}(V) \in \mathscr{T}_X.$$

In other words, if $f \colon X \to Y$ is a continuous map, $\mathscr{T}_X$ has "more" elements than $\mathscr{T}_Y$ has. To get more rigorous discription, we define the following concept.

Definition 5. Let X be a set and $\mathscr{T}$, $\mathscr{T}^{'}$ be two topologies on X we say that $\mathscr{T}$ is coaser than $mathscrT^{'}$ if $\mathscr{T} \subset \mathscr{T}^{'}$.

Therefore, if $f \colon X \to Y$ is a continuous map, the topology $\mathscr{T}(f^{-1}(\mathscr{T}_Y))$ is coaser than $\mathscr{T}_X$.

▪ **Generating topologies:** We use above methods to generate some interested topologies on a set.

1) Inital topology: Given maps $f_\alpha \colon X \to Y_\alpha$ ($\alpha \in A$) from a set X to a family of topological spaces $\{Y_\alpha \colon \alpha \in A\}$. Let

$$\mathscr{S} \;=\; \{\; f_\alpha^{-1}(V) \colon V \in \mathscr{T}_{Y_\alpha},\; \alpha \in A \;\}$$

Then $\mathscr{T}(\mathscr{S})$ is the coarsest topology on X such that each f_α is continuous, called the initial topology induced by the family of maps $\{f_\alpha \,:\, \alpha \in A\}$.

2) Final topology: Given maps $f_\alpha \colon X_\alpha \to Y$ ($\alpha \in A$) from a family of topological spaces $\{X_\alpha \colon \alpha \in A\}$ to a set Y. Let

$$\mathscr{S} \;=\; \{\; V \colon f_\alpha^{-1}(V) \in \mathscr{T}_{X_\alpha},\; \alpha \in A \;\}$$

Then $\mathscr{T}(\mathscr{S})$ is the finest topology on Y such that each f_α is continuous, called the final topology induced by the family of maps $\{f_\alpha \colon \alpha \in A\}$.

Here is some important examples using above way to generate topologies.

Example 6. (Initial topology)

1) Subspace topology: Let Y be a topological space and $X \subset Y$ be a subset of Y. The inclusion map $i \colon X \to Y$ can generate the initial topology $\mathscr{T}_X$ on X, then X become a subspace of Y. In fact,

$$\mathscr{T}_X \;=\; \{\; X \bigcap U \colon U \in \mathscr{T}_Y \;\}$$

2) Product topology: Let $\{Y_\alpha \colon \alpha \in A\}$ be a family of topological spaces. The product set is

$$\prod_{\alpha \in A} Y_\alpha \;=\; \{\; A \xrightarrow{f} \bigcup_{\alpha \in A} Y_\alpha \colon \forall \alpha \in A,\; f(\alpha) \in Y_\alpha \;\}$$

There is a family of maps $\{p_\beta \colon \prod_{\alpha \in A} Y_\alpha \to Y_\beta (\beta \in A)\}$. Therefore, these maps can generate the initial topology $\mathscr{T}$ on $\prod_{\alpha \in A} Y_\alpha$ and let $\prod_{\alpha \in A} Y_\alpha$ be the product topological space. In fact,

$$\mathscr{T} \;=\; \{\; \prod_{\alpha \in A} V_\alpha \colon V_\alpha \in \mathscr{T}_{Y_\alpha} \;\&\; \#\{\alpha \in A \colon V_\alpha \neq Y_\alpha\} < \infty \;\}$$

Remark. The condition, $\#\{\alpha \in A \colon V_\alpha \neq Y_\alpha\} < \infty$, is because that when the subbasis generates the basis, only finite many elements can do intersection.

Example 7. (Final topology)
Quotient topology: For a topological space X on which an equivalent relation R is fixed, $\pi \colon X \to X/R$ is the quotient map, then the quotient set can be equiped with the final topology $\mathscr{T}$ generated by the quotient map. Therefore, X/R become a quotient topological space. In fact,

$$\mathscr{T} \;=\; \{\; U \subset X/R \colon \pi^{-1}(U) \in \mathscr{T}_X \;\}$$

▪ **Countability and metrizability:** When learning the analysis of real functions, we usually use the sequence to discribe the topological properties. But in some general topology, we cannot just use sequence since some properties of "uncountabibily". In this case, the concept of net can be applied to some "uncountable" topologies. Furthermore, there is a class of more special topology, metrizable topology, which has some better properties.

Definition 8. (Net)

1) Direct set: A direct set $(D, \ \geqslant)$ consists of a nonempty set D and a relation $\geqslant$ on D, satisfies:

   i) $\forall \ d \in D, \ d \geqslant d$

   ii) $\forall \ d_1, \ d_2, \ d_3 \in D, if \ d_3 \geqslant d_2 \ \& \ d_2 \geqslant d_1, \ then \ d_3 \geqslant d_1$

   iii) $\forall \ d, \ d^{'} \in D, \ \exists \ d^{''} \ s.t. \ d^{''} \geqslant d \ \& \ d^{''} \geqslant d^{'}.$

2) if X is a set, a net is a map $x_{\cdot}: D \to X$ from a direct set D to X

Example 9. If X is a topological space and $x \in X$, then let

$$D \ = \ \{ \text{ all open neighbourhoods of x } \}, \ U \geqslant V \Leftrightarrow U \subset V$$

Then D is a direct set and $x_\alpha(\alpha \in D)$ is a net. And we say $x_\alpha \to x$ if and only if
$\forall$ open neighbourhood U of x in X, $\exists \delta \in D$, $\forall \ \alpha \in D$ with $\alpha \geqslant \delta \Rightarrow x_\alpha \in U$

Nets can be used as sequences in topological spaces. Like,

Proposition 10. *If X is a topological space and the net $x_\alpha(\alpha \in D)$ defined above and $A \subset X$, then*

1) $\overline{A} \ = \ \{ \ x \in X : \exists \ x_\alpha \ in \ A, \ x_\alpha \to x \ \}$

2) $f : X \to Y$ *is continuous between two topological spaces,* $x_0 \in X$*,* f *is continuous at* x_0*, if and only if*
$\forall \ net \ x_\alpha(\alpha \in D), \ s.t. \ x_\alpha \to x_0 \Rightarrow f(x_\alpha) \to f(x_0)$

Definition 11. (Countability)

1) First countability: For a topological space X, X is called first countable if for each point $x \in X$, x has a countable neighbourhood basis.

2) Second countability: A topological space X is second countable if it has the countable topological basis.

Remark. Clearly, the second countable topological space is first countable, but the converse is not true.

In particular, if X is first countable, sequences can be used to illuminate tpological properties rather than nets. Like,

Proposition 12. *If X is first countable, then*

1) $U \subset X$ is closed $\Leftrightarrow \forall\ x \in U,\ \exists$ a sequence$\{x_n\} \subset U,\ s.t.\ x_n \to x$.

2) sequential compactness is equivalent to compactness.

And for the second countability, it is about the separability.

Definition 13. (Separability)

1) A subset A of a topological space X is called dense if $\overline{A} = X$.

2) A topological space is called separable if it has a countable dense subset.

By the definition, we can clearly know that:

Proposition 14. *If X is a second countable topological space, then it is separable and every open covering of X has a finite subcollection covering X.*

We can classify topological spaces into some classes.

Definition 15. X is a topological space, then we call X is:

(T_0) $\forall\ x,\ y \in X,\ \exists$ open $U \subset X,\ s.t.\ x \in U$ but $y \notin U$ or $y \in U$ but $x \notin U$ (Kolmogorov space)

(T_1) $\forall\ x,\ y \in X,\ \exists$ open $U,\ V \subset X,\ s.t.\ x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$
($\Leftrightarrow \forall\ x \in X,\ \{x\}$ is closed)

(T_2) $\forall\ x,\ y \in X,\ \exists$ open $U,\ V \subset X,\ s.t.\ x \in U\ \&\ y \in V$ and $U \bigcap V = \varnothing$ (Hausdorff space)

(T_3) T_1 holds and $\forall\ x \in X$ and closed $C \subset X$, if $x \notin C$, then $\exists$ open $U,\ V \subset X,\ s.t.\ x \in U\ \&\ C \subset V$ and $U \bigcap V = \varnothing$ (regular space)

(T_4) T_1 holds and $\forall$ closed $C_1,\ C_2 \subset X$, if $C_1 \bigcap C_2 = \varnothing$,
then $\exists$ open $U,\ V \subset X,\ s.t.\ C_1 \subset U\ \&\ C_2 \subset V$ and $U \bigcap V = \varnothing$ (normal space)

Then we can specify a class of more powerful topological space.

Definition 16. If X is a topological space, then X is said to be metrizable if there exists a metric d on the set X that induces the topology of X.

Remark. Clearly, if X is metrizable, X is second countable and normal.

Here is two metrization theorems provides the essence of metric spaces.

Theorem 17. *(Metrization theorems)*

Urysohn *A topological space is separable and metrizable if and only if it is regular, Hausdorff and second countable.*

Nagata–Smirnov *A topological space is metrizable if and only if regular, Hausdorff and has a σ-locally finite basis.*

▪ **Complete metic space:** For a metric space, we know it is first countable, so the concept of net is unnecessary. And thus sequences are enough to determine the topological structures, like that sequential compactness is equvilent to compactness.

Proposition 18. *A compact subset of a metric space is closed, bounded and separable.*

Remark. it is clearly, since compactness is also about finity.

For any metric space, we can use the following theorem to get a completion of that and this completion is unique. Thus, we can always assume a metric space is complete.

Theorem 19. *Let $(X,\ d)$ be a metric space. Then, there exists a metric space $(\hat{X},\ \hat{d})$ with the following properties:*

1) $(\hat{X},\ \hat{d})$ is complete.

2) There is an embedding σ from X to $\hat{X}$.

3) $\sigma(X)$ is dense in $\hat{X}$.

And this $(\hat{X},\ \hat{d})$ is unique with respect to isomorphism.

Complete metric space is imporatant since it is "sufficiently large". Rigorously, we can the following definition to describe it.

Definition 20. (Baire Category) A metric space is said to be of the first category if it can be written as a countable union of sets that are nowhere dense. Otherwise, it is of the second category.

Proposition 21. *A complete metric space is a space of the second category.*

▪ **Filters:** For convenience, we define some terminologies.

Definition 22. A filter on a set X is a family $\mathscr{F}$ of subsets of X satisfying the following conditions:

1) $\varnothing \notin \mathscr{F}$

2) $\mathscr{F}$ is closed under finite many intersections

3) Any subset of X containing a set in $\mathscr{F}$ belongs to $\mathscr{F}$.

Example 23. For a topological space X and $x \in X$, and let

$$\mathscr{F}(x) \;=\; \{ \text{ all neighbourhoods of x } \}$$

Then $\mathscr{F}(x)$ is a filter and $\mathscr{F}(x)$ satisfies the following properties:

1) $\forall\, U \in \mathscr{F}(x),\ x \in U$

2) $\forall\, U \in \mathscr{F}(x),\ \exists\, V \in \mathscr{F}(x),\ s.t.\ \forall\, y \in V,\ U \in \mathscr{F}(y)$

And conversely, if we can find $\mathscr{F}(x)$ for any $x \in X$ with above two properties, these can define a unique topology $\mathscr{T}$ s.t. $\mathscr{F}(x)$ is the filter of neighbourhoods of x for any $x \in X$. In fact,

$$\mathscr{T} = \{\ U \subset X : x \in U \Rightarrow U \in \mathscr{F}(x)\ \}$$

Also, we can define the basis of $\mathscr{F}(x)$, noted by $\mathscr{B}(x)$. That is $\mathscr{B}(x) \subset \mathscr{F}(x)$ with the following properties:

1) $\forall\, U \in \mathscr{B}(x),\ x \in U$

2) $\forall\, U_1\ \&\ U_2 \in \mathscr{B}(x),\ \exists\, U_3 \in \mathscr{B}(x),\ s.t.\ U_3 \subset U_1 \bigcap U_2$

3) If $y \in U \in \mathscr{B}(x),\ \exists\, W \in \mathscr{B}(y),\ W \subset U$

1.1.2 Definition and Properties

▪ **Definition:** Now, we need to endow the topological structure on a vector spaces. And the most important thing is that the topological structure should coincide with the algebraic structure.

Definition 1. A vector space X over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) is called a topological vector space if X is equiped with a topology $\mathscr{T}$ s.t. the addition and the scalar multiplication, i.e.

$$\begin{aligned}(x,\ y) &\mapsto x + y \\ (\lambda,\ x) &\mapsto \lambda x\end{aligned}$$

are continuous with respect to the topology $\mathscr{T}$.

In this definition, the most important part is that the addition and the scalar multiplication are continuous. This condition provides some additional properties for the topology and also for the linear operations. First, it can simply the topology.

Proposition 2. *Given a t.v.s. X,*

1) For any $x_0 \in X$, the map $x \mapsto x + x_0$ is a homeomorphism.

2) For any $\lambda \in \mathbb{K}$, then map $x \mapsto \lambda x$ is a homeomorphism.

Proof. It is clearly, since by the definition, $x \mapsto x - x_0$ and $x \mapsto \frac{1}{\lambda}x$ are continuous. $\square$

Therefore, the topology of a t.v.s is completely determined by the filter of neighbourhoods of any point. Or, more rigorously,

Corollary 3. *For a t.v.s X, the filter $\mathscr{F}(x)$ of neighbourhoods of $x \in X$ is as same as $\{ U + x \colon U \in \mathscr{F}(e) \}$, where e is the unit element in X.*

Thus, to research the topology of a t.v.s. X, we just need to research the filter $\mathscr{F}(e)$ of neighbourhoods of e. First, there are two special properties of some subsets of a t.v.s. X.

Definition 4. For a subset U of a t.v.s. X,

1) U is absorbing if $\forall\ x \in X,\ \exists\ \rho > 0$ *s.t.* $\forall\ \lambda \in \mathbb{K}$ with $|\lambda| \leqslant \rho$, we have $\lambda x \in U$.

2) U is balanced if $\forall\ x \in U$, $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$, we have $\lambda x \in U$.

Then, the following theorem reveals the essence of $\mathscr{F}(e)$.

Theorem 5. *A filter $\mathscr{F}$ of a vector space X over $\mathbb{K}$ is the filter of neighbourhoods of the unit element e w.r.t. some topology compatible with the algebraic structure of X if and only if*

1) $\forall\ U \in \mathscr{F},\ e \in U$

2) $\forall\ U \in \mathscr{F},\ \exists\ V \in \mathscr{F}$ s.t. $V + V \subset U$

3) $\forall\ U \in \mathscr{F},\ \forall\ \lambda \in \mathbb{K}$ with $\lambda \neq 0$, $\lambda U \in \mathscr{F}$

4) $\forall\ U \in \mathscr{F}$, U is absorbing

5) $\forall\ U \in \mathscr{F},\ \exists\ V \in \mathscr{F}$ s.t. $V \subset U$ is balanced

Proof. If $\mathscr{F} = \mathscr{F}(e)$, these statements clearly hold.
1) is trivial.
2) is true since the addition is continuous.
3) and 4) hold since the scalar multiplication is continuous.
For 5), because the scalar multiplication is continuous, we can find a $W \in \mathscr{F}$ *s.t.* $\lambda W \subset U$ for any $|\lambda| \leqslant \rho$, then let $V = \bigcup_{|\lambda| \leqslant \rho} \lambda W$. Clearly, $V \in \mathscr{F}$ and V is balanced.
Conversely, We can define

$$\mathscr{F}(x) = \{ U + x \colon U \in \mathscr{F} \}$$

for any $x \in X$. It can be easily checked that $\mathscr{F}(x)$ satisfies the conditions in Example 23 in last subsection. Therefore, these $\mathscr{F}(x)$ can determine a unique topology $\mathscr{T}$ on X.
Now, we just need to check the continuity of the addition and the scalar multiplication. The addition is continuous, since $\mathscr{F}$ satisfies 2). Using conditions 2) and 4) and 5) to get a balanced absorbing open neighbourhood in $\mathscr{F}$, and this neighbourhood prove the continuity of the scalar multiplication. $\square$

Here is some simple properties of a t.v.s. X. These properties are directly obtained by definition and above theorem.

Proposition 6. *For a t.v.s. X,*

1) *proper subspaces of X are never absorbing. In particular, if $M \subset X$ is a open subspace, then $M = X$.*

2) *each linear subspace of X, endowed with subspace topology, is also a t.v.s.*

3) *if H is a linear subspace of X, then $\overline{H}$ is also a linear subspace of X.*

4) *if Y is also a t.v.s. and $f\colon X \to Y$ is a linear map, then f is continuous if and only if f is continuous at the unit element e.*

▪ **Hausdorff t.v.s. :** The Hausdorff Space is important since it can let the concept of limit make sense. And the topology of a t.v.s. can be simplified and has some additional properties, we can get a easier condition that make a t.v.s. become Hausdorff.

Proposition 7. *A t.v.s X is a Hausdorff space if and only if for any $x \in X$ with $x \neq e$ there exists a $U \in \mathscr{F}(e)$ s.t. $x \notin U$.*

Proof. Since the open neighbourhoods of any point in X is completely determined by the open neighbourhoods of e, this proposition is equivalent to the statement that (T_1) implies Hausdorff.
The proof can be accomplished by obtaining a contradiction to the given condition that $x \neq e$, $\exists\ U \in \mathscr{F}(e)$ s.t. $x \notin U$. For that U, there is a balanced $V \in \mathscr{F}(e)$ *s.t.* $V + V \subset U$ and the balance implies that $V - V \subset U$. Therefore, $(x+V)\bigcap V = \varnothing$. If not, $x + v_1 = v_2$ for $v_1,\ v_2 \in V$. This implies that $x = v_1 - v_2 \in V - V \subset U$. Thus it is a contradiction. $\square$

The following theorem is more explicit.

Theorem 8. *For t.v.s. X the following statements are equivalent.*

1) *X is Hausdorff.*

2) the intersection of all neighbourhoods of the unit element e is $\{e\}$.

3) $\{e\}$ is closed.

Proof. Before the rigorously proving, the intuition is clearly. Since in a t.v.s. (T_1) is equivalent to Hausdorff, the equivalence of 1) and 3) is clearly true.

1) $\Rightarrow$ 2) It is because that elements in $\mathscr{F}(e)$ can separate e and other points.

2) $\Rightarrow$ 3) If $x \in \overline{\{e\}}$, i.e. $\forall V_x \in \mathscr{F}(x),\ V_x \bigcap \overline{\{e\}} \neq \varnothing \Rightarrow e \in V_x$, and $V_x = U + x$ for some $U \in \mathscr{F}(e)$, then $u + x = e$ for some $u \in U$. Thus, $x = -u \in -U$ for all $U \in \mathscr{F}(e)$. That implies $x = e$.

3) $\Rightarrow$ 1) By above mentioned, it just needs to check that if for any topology space Y, $\{y\}$ is closed $\forall y \in Y$, Y is (T_1).
Since $\{y_1\}$ is closed, $Y\backslash\{y_1\}$ is open. That means if $y_2 \neq y_1$, there exists a open neighbourhood U of y_2 s.t. $y_1 \notin U$. Similarly, we can find a open neighbourhood V of y_1 s.t. $y_2 \notin V$. Therefore, Y is (T_1). □

- **Quotient t.v.s. :** For a linear subspace M of a t.v.s. X, the quotient topology on X/M can be obtained by the quotient map $\pi\colon X \to X/M$. But because of the algebraic structure, it has more properties.

Proposition 9. *For a linear subspace M of a t.v.s. X, the quotient map $\pi\colon X \to X/M$ is open.*

Proof. Let $V \subset X$ be open, then we have

$$\pi^{-1}(\pi(V)) = V + M = \bigcup_{m \in M} (V + m)$$

Since V is open, $V + m$ is open. Thus $\pi^{-1}(\pi(V))$ is open. And by the definition of the topology on X/M, $\pi(V)$ is open. □

Corollary 10. *For a linear subspace M of a t.v.s. X, the quotient space X/M endowed with the quotient topology is a t.v.s..*

Proof. We have the following commutative graph, where f and g are corresponding addition maps or scalar multiplication maps on X and X/M.

$$\begin{array}{ccc} X \times X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\ X/M \times X/M & \xrightarrow{g} & X/M \end{array}$$

Then for an open set $V \subset X/M$, since f and π are continuous, and π is open, $(\pi \times \pi) \circ f^{-1} \circ \pi^{-1}(V)$ is open. By above commutative graph, we have $g \circ (\pi \times \pi) = \pi \circ f$. Therefore, $g^{-1}(V)$ is open, i.e. g is continuous. □

Also, we can find the condition that lets the quotient topological vector space be Hausdorff.

Proposition 11. *Let X be a t.v.s..*

1) *M be a linear subspace of X. Then X/M is Hausdorff if and only if M is closed.*

2) *$X/\overline{\{e\}}$ is Hausdorff.*

Proof. 2) is true because 1). And 1) clearly holds since M is the unit element in X/M and Theorem 8 in this subsection. □

Remark. By this method, for any t.v.s., we can find a Hausdorff space w.r.t it.

1.1.3 Continuous Linear Maps

The interesting maps between two topological vector spaces not only preserve the algebraic structure, but also the topological structure, thus these are continuous linear maps.
First, for a linear map $f\colon X \to Y$ between vector spaces X and Y, we have the commutative graph, where $\tilde{f}(x + \ker f) = f(x)$ is well-defined.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & \operatorname{Im} f \arrow[r, "i"] & Y \\ X/\ker f \arrow[ur, "\tilde{f}"] & & \end{tikzcd}$$

Proposition 1. *Let $f\colon X \to Y$ be a linear map between two t.v.s.'s X and Y.*

1) *If Y is Hausdorff and f is continuous, then $\ker f$ is closed.*

2) *By above notation, f is continuous if and only if $\tilde{f}$ is continous.*

Proof. 1) is because that $\ker f = f^{-1}(\{e\})$ and Y is Hausdorff.
For 2), if $\tilde{f}$ is continuous, it is clearly that $f = i \circ \tilde{f} \circ \pi$ is continuous. Conversely, it is because of the universal property of quotient maps. And in this case, let $U \subset \operatorname{Im} f$ be open, then $f^{-1}(U)$ is open and $\tilde{f}^{-1}(U) = \pi(f^{-1}(U))$. Since π is open, $\tilde{f}^{-1}(U)$ is open. Thus, $\tilde{f}$ is continuous. □

1.1.4 Complete Topological Vector Spaces

We have just defined the completeness on a metric space by using sequence, but in metric spaces, we know the topology is so powerful that sequences can do any thing, but in general topology, or the topology in a t.v.s., we need an equivalent concept to describe the completeness.

Definition 1. (Completeness)

1) A filter $\mathscr{F}$ on a subset A of a t.v.s. X is said to be a Cauchy filter if

$$\forall\, U \in \mathscr{F}(0) \textbf{ in } X,\ \exists\, M \subset A \ s.t.\ M \in \mathscr{F} \ \&\ M - M \subset U$$

2) A subset A of a t.v.s. X is said to be complete if every Cauchy filter on A converges to a point $x \in A$.

Remark. Said "the filter converges to a point" means that we can define a net on this filter, and this net converge a point. And this definition is also valid without the algebraic structure.

By this definition, and using the factor that Hausdorff spaces let the limit point of a net uniquely exist, we have similar results comparing with the metric spaces.

Proposition 2. *Let X be a t.v.s..*

1) If X is Hausdorff, any complete set is closed.

2) If X is complete, any closed set is complete.

We known any metric space can be completion. Similarly, the same result can obtained in any t.v.s..

Theorem 3. *Let X be a Hausdorff t.v.s., then there exists a complete Hausdorff t.v.s. $\hat{X}$ and a map $i\colon X \to \hat{X}$ with the following properties.*

1) i is a topological monomorphism.

2) $\overline{i(X)} = \hat{X}$.

3) For any complete Hausdorff t.v.s. Y and for every continuous linear map $f\colon X \to Y$, there exists a continuous map $\hat{f}\colon \hat{X} \to Y$, s.t. the following graph is commutative

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Y \\ \hat{X} \arrow[ur, "\hat{f}"] & \end{tikzcd}$$

And $(\hat{X}, \hat{f})$ is unique with respect to the isomorphism

Proof. The proof is similar as the proof of the completion of metric spaces, which contructs the $\hat{X}$ as a set of equivalent classes of Cauchy sequences. In a t.v.s., we just need to replace Cauchy sequences by Cauchy filters (in fact, Cauchy nets). Let

$$\begin{array}{rcl} \tilde{X} & = & \{ \text{ all Cauchy filters in } X \ \} \\ R & : & \mathscr{F} \ R \, \mathscr{G} \Leftrightarrow \forall \ U \in \mathscr{F}(e), \ \exists \ A \in \mathscr{F} \ \& \ B \in \mathscr{G} \ s.t. \ A - B \subset U \\ \hat{X} & = & \tilde{X}/R \end{array}$$

We can easily define linear operations and topology, s.t. $\hat{X}$ become a complete t.v.s.. Then we just need to check the statements in above theorem hold. □

1.1.5 Finite Dimensional Topological Vector Spaces

For a finite dimensional topological vector space, the topology compatible with the algebraic structure has some properties coincided with the "finity". First, continuous linear functionals on a t.v.s. have some properties.

Lemma 1. *Let X be a t.v.s. over $\mathbb{K}$. Fixed $v \in X$, then the $\phi_v \colon \mathbb{K} \to X$ by $\xi \mapsto \xi v$ is continuous,*

Proof. It is because that $\phi_v = f \circ \psi_v$ where f is the multiplication map.

$$\begin{array}{ccccc} \mathbb{K} & \stackrel{\psi_v}{\to} & \mathbb{K} \times X & \stackrel{f}{\to} & X \\ \xi & \mapsto & (\xi, v) & \mapsto & \xi v \end{array}$$

□

Lemma 2. *For a non-zero linear functional $L \colon X \to \mathbb{K}$, where X is a t.v.s. over $\mathbb{K}$, the following statements are equivalent.*

1) L is continuous,

2) $\ker L$ is closed,

3) $\ker L$ is not dense in X,

4) L is bounded in some neighbourhood of the origin in X.

Proof. The equivalence of 1) and 2) and 4) is clearly.

1) $\Rightarrow$ 2) It is because that $\ker L = L^{-1}(\{0\})$.

2) $\Rightarrow$ 3) Since L is non-zero, it clearly holds.

3) $\Rightarrow$ 4) By the assumption, there exists a balanced set $V \in \mathscr{F}(e)$ and a point $x \notin \overline{\ker L}$ s.t. $(x + V) \bigcap \ker L = \varnothing$. $L(V)$ is balanced on $\mathbb{K}$, therefore $L(V)$ is bounded or $L(V) = \mathbb{K}$. But since $(x + V) \bigcap \ker L = \varnothing$, $L(V) \neq \mathbb{K}$.

4) $\Rightarrow$ 1) This implies that L is continuous at e. But since X is a t.v.s., L is continuous at every point. □

Theorem 3. *Let X be a finite dimensional Hausdorff t.v.s. over $\mathbb{K}$ (endowed with the standard topology), and $\dim X = d$. Then we have:*

1) X is topologically isomorphic to $\mathbb{K}^d$,

2) every linear functional on X is continuous,

3) every linear map from X to any t.v.s. Y is continuous

Proof. For 1), we just need to find a homeomorphic isomorphism from $\mathbb{K}^d$ to X, like the following map, where $\{e_i\}_{i=1}^d$ is the basis of X.

$$\begin{array}{ccc} \mathbb{K}^d & \xrightarrow{\phi} & X \\ (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_d) & \longmapsto & \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_d e_d \end{array}$$

ϕ is clearly an algebraic isomorphism. Thus we just need to check ϕ is both continuous and open.
Check: ϕ is continuous.
When $d = 1$, it is continuous by above lemma. For the general case, since d is finite, ϕ is continuous.
Check: 2) holds and ϕ is open.
When $d = 1$, it is clearly 1) and 2) are trued. And suppose 1) and 2) hold for $\dim X \leqslant d-1$, then when $\dim X = d$, let L be a non-zero linear function on X. Then since $X/\ker L \cong \operatorname{Im} L \subset \mathbb{K}$, $\dim \ker L = d - 1$. Therefore, $\ker L \cong \mathbb{K}^{d-1} \Rightarrow \ker L$ is complete $\Rightarrow \ker L$ is closed $\Rightarrow L$ is continuous by above lemma. And,

$$\begin{array}{ccc} X & \xrightarrow{\phi^{-1}} & \mathbb{K}^d \\ \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_d e_d & \longmapsto & (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_d) \end{array}$$

is continuous since each

$$\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_d e_d \longmapsto \lambda_i$$

is continuous.
Then for 3), it is clealy since $\dim \operatorname{Im} L < \infty$. □

Corollary 4.

1) Every finite dimensional Hausdorff t.v.s. is complete.

2) Every finite dimensional subspace of a Hausdorff t.v.s. is closed.

3) For a finite dimensional vector space, there is only one topology w.r.t. homeomorphism that can make it be a Hausdorff t.v.s..

4) Every bounded subset on a finite dimensional Hausdorff t.v.s. is compact.

Proof. These properties can be easily obtained by regarding the t.v.s. as $\mathbb{K}^d$ endowed with the standard topology. □

Finally, the most important theorem in this subsection is that the converse of 4) in above corollary is also true.

Theorem 5. *A Hausforff t.v.s. is locally compact if and only if it is finite dimensional.*

Proof. Let X be a locally compact Hausdorff t.v.s. and K be a compact heighbourhood of e in X, i.e.

$$\exists\ x_1,\ \cdots,\ x_r \in X\ s.t.\ K \subset \bigcup_{i=1}^{r}(x_i + \frac{1}{2}K)$$

Let $M = \operatorname{span}\{x_1,\ \cdots,\ x_r\}$, and M is closed. Therefore, X/M is a Hausdorff t.v.s.. Let $\pi\colon X \to X/M$ be the quotient map.
Since $K \subset M + \frac{1}{2}K$, $\pi(K) \subset \pi(\frac{1}{2}K)$. Thus, by iterating $\pi(2^n K) \subset \pi(K)$.
As K is absorbing, $X = \bigcup_{n=1}^{\infty} 2^n K$,

$$X/M = \pi(X) = \bigcup_{n=1}^{\infty} \pi(2^n K) \subset \pi(K) \subset X/M$$

And since π is continuous, $\pi(K)$ is compact, i.e. X/M is compact.
claim: $\dim X/M = 0$
Suppose $\dim X/M > 0$, then for some $\overline{x_0} \in X$ with $\overline{x_0} \neq \overline{e}$, $\mathbb{R}\overline{x_0} \subset X/M$. And since X/M is Hausdorff compact and $\mathbb{R}\overline{x_0}$ is closed, $\mathbb{R}\overline{x_0}$ is compact, which is a contradiction. □

1.2 Locally Convex Topological Vector Spaces