My Report about Operator Theory

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Chapter 1

Topological Vector Spaces and Banach Spaces

1.1 Topological Vector Spaces

Linear operations, i.e addition and scalar multiplication, provide an algebraic structure on a set, therefore constructing a vector space. In the course, linear algebra, we have learn the algebraic structure of finite dimensional vector spaces. But how to deal with infinite dimensional vector spaces? By learning the topological spaces, we know that the topological structure can give us a method to research properties of infinity. Thus, we need to equip a vector space with an additional topological structure, which should coincide with the algebraic structure. This is the reason why we define the topological vector space.

1.1.1 Topological Spaces

■ **Definition:** First, we define the topological structure on a general set.

Definition 1. A topological space $X = (X, \mathscr{T}_X)$ consists of a set X, called the underlying space of X, and a family $\mathscr{T}_X$ of subsets of X s.t.

1) $X, \varnothing \in \mathscr{T}_X$.

2) if $U_\alpha \in \mathscr{T}_X$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}_X$.

3) if $U_1, U_2 \in \mathscr{T}_X$, then $U_1 \bigcap U_2 \in \mathscr{T}_X$.

And, $\mathscr{T}_X$ is called a topology on X. The element in $\mathscr{T}_X$ is called open set.

Thus, the topological structure on a set X is totally determined by the family $\mathscr{T}_X$. In particular, from 2), we can simplify $\mathscr{T}_X$. In other words, like the basis of a vector space, there is a "basis" of $\mathscr{T}_X$.

Definition 2. If X is a set, a basis for a topology on X is a family $\mathscr{B}$ of subsets of X, s.t.

1) $\forall\ x \in X,\ \exists\ B \in \mathscr{B}$, s.t. $x \in B$.

2) if $x \in B_1 \bigcap B_2$, where $B_1,\ B_2 \in \mathscr{B}$, then there is a $B_3 \in \mathscr{B}$, s.t. $B_3 \subset B_1 \bigcap B_2$.

$\mathscr{B}$ can generate a topology $\mathscr{T}_X$ on X by doing infinite times union of elements in $\mathscr{B}$. In fact,

$$\mathscr{T}_X\ =\ \{\ U \subset X \colon U = \bigcup_{\alpha \in A} U_\alpha,\ U_\alpha \in \mathscr{B}\ \},\ \text{where A is any index set.}$$

We can do more on the basis by 3).

Definition 3. A subbasis of a topology on X is a collection $\mathscr{S}$ of subsets of X, whose union is X. The topology generated by $\mathscr{S}$ is noted by $\mathscr{T}(\mathscr{S})$.

In fact, we can generate the basis of $\mathscr{T}(\mathscr{S})$ by $\mathscr{S}$, that is,

$$\mathscr{B}\ =\ \{\ U \subset X \colon U = \bigcap_{\alpha=1}^{n} U_\alpha,\ U_\alpha \in \mathscr{S},\ n \in \mathbb{N}\ \}$$

Therefore, this basis can generate the conincided topology on X.

■ **Continuous maps:** Next, we need to endow the general maps with the topological structure.

Definition 4. Let X, Y be topological spaces and $f \colon X \to Y$ be a map. We say f is continous if

$$\forall\ V \in \mathscr{T}_Y,\ f^{-1}(V) \in \mathscr{T}_X.$$

In other words, if $f \colon X \to Y$ is a continuous map, $\mathscr{T}_X$ has "more" elements than $\mathscr{T}_Y$ has. To get more rigorous discription, we define the following concept.

Definition 5. Let X be a set and $\mathscr{T}$, $\mathscr{T}'$ be two topologies on X we say that $\mathscr{T}$ is coaser than $mathscrT'$ if $\mathscr{T} \subset \mathscr{T}'$.

Therefore, if $f \colon X \to Y$ is a continuous map, the topology $\mathscr{T}(f^{-1}(\mathscr{T}_Y))$ is coaser than $\mathscr{T}_X$.

■ **Generating topologies:** We use above methods to generate some interested topologies on a set.

1) Inital topology: Given maps $f_\alpha\colon X \to Y_\alpha$ $(\alpha \in A)$ from a set X to a family of topological spaces $\{Y_\alpha\colon \alpha \in A\}$. Let

$$\mathscr{S} \;=\; \{\, f_\alpha^{-1}(V)\colon V \in \mathscr{T}_{Y_\alpha},\; \alpha \in A \,\}$$

Then $\mathscr{T}(\mathscr{S})$ is the coarsest topology on X such that each f_α is continuous, called the initial topology induced by the family of maps $\{f_\alpha : \alpha \in A\}$.

2) Final topology: Given maps $f_\alpha\colon X_\alpha \to Y$ $(\alpha \in A)$ from a family of topological spaces $\{X_\alpha\colon \alpha \in A\}$ to a set Y. Let

$$\mathscr{S} \;=\; \{\, V\colon f_\alpha^{-1}(V) \in \mathscr{T}_{X_\alpha},\; \alpha \in A \,\}$$

Then $\mathscr{T}(\mathscr{S})$ is the finest topology on Y such that each f_α is continuous, called the final topology induced by the family of maps $\{f_\alpha\colon \alpha \in A\}$.

Here is some important examples using above way to generate topologies.

Example 6. (Initial topology)

1) Subspace topology: Let Y be a topological space and $X \subset Y$ be a subset of Y. The inclusion map $i\colon X \to Y$ can generate the initial topology $\mathscr{T}_X$ on X, then X become a subspace of Y. In fact,

$$\mathscr{T}_X \;=\; \{\, X \bigcap U\colon U \in \mathscr{T}_Y \,\}$$

2) Product topology: Let $\{Y_\alpha\colon \alpha \in A\}$ be a family of topological spaces. The product set is

$$\prod_{\alpha \in A} Y_\alpha \;=\; \{\, A \xrightarrow{f} \bigcup_{\alpha \in A} Y_\alpha\colon \forall \alpha \in A,\; f(\alpha) \in Y_\alpha \,\}$$

There is a family of maps $\{p_\beta\colon \prod_{\alpha \in A} Y_\alpha \to Y_\beta(\beta \in A)\}$. Therefore, these maps can generate the initial topology $\mathscr{T}$ on $\prod_{\alpha \in A} Y_\alpha$ and let $\prod_{\alpha \in A} Y_\alpha$ be the product topological space. In fact,

$$\mathscr{T} \;=\; \{\, \prod_{\alpha \in A} V_\alpha\colon V_\alpha \in \mathscr{T}_{Y_\alpha} \;\&\; \#\{\alpha \in A\colon V_\alpha \neq Y_\alpha\} < \infty \,\}$$

Remark. The condition, $\#\{\alpha \in A\colon V_\alpha \neq Y_\alpha\} < \infty$, is because that when the subbasis generates the basis, only finite many elements can do intersection.

Example 7. (Final topology)
Quotient topology: For a topological space X on which an equivalent relation R is fixed, $\pi\colon X \to X/R$ is the quotient map, then the quotient set can be equiped with the final topology $\mathscr{T}$ generated by the quotient map. Therefore, X/R become a quotient topological space. In fact,

$$\mathscr{T} \;=\; \{\, U \subset X/R\colon \pi^{-1}(U) \in \mathscr{T}_X \,\}$$

■ **Countability and metrizability:** When learning the analysis of real functions, we usually use the sequence to discribe the topological properties. But in some general topology, we cannot just use sequence since some properties of "uncountabibily". In this case, the concept of net can be applied to some "uncountable" topologies. Furthermore, there is a class of more special topology, metrizable topology, which has some better properties.

Definition 8. (Net)

1) Direct set: A direct set $(D, \geqslant)$ consists of a nonempty set D and a relation $\geqslant$ on D, satisfies:

   i) $\forall d \in D, d \geqslant d$
   ii) $\forall d_1, d_2, d_3 \in D, if d_3 \geqslant d_2 \& d_2 \geqslant d_1, then d_3 \geqslant d_1$
   iii) $\forall d, d' \in D, \exists d'' s.t. d'' \geqslant d \& d'' \geqslant d'$.

2) if X is a set, a net is a map $x_. : D \to X$ from a direct set D to X

Example 9. If X is a topological space and $x \in X$, then let

$$D = \{ \text{ all open neighbourhoods of x } \}, \ U \geqslant V \Leftrightarrow U \subset V$$

Then D is a direct set and $x_\alpha(\alpha \in D)$ is a net. And we say $x_\alpha \to x$ if and only if
$\forall$ open neighbourhood U of x in X, $\exists \delta \in D, \forall \alpha \in D$ with $\alpha \geqslant \delta \Rightarrow x_\alpha \in U$

Nets can be used as sequences in topological spaces. Like,

Proposition 10. *If X is a topological space and the net $x_\alpha(\alpha \in D)$ defined above and $A \subset X$, then*

1) $\overline{A} = \{ x \in X : \exists x_\alpha \text{ in } A, x_\alpha \to x \}$

2) $f : X \to Y$ is continuous between two topological spaces, $x_0 \in X$, f is continuous at x_0, if and only if
   $\forall$ net $x_\alpha(\alpha \in D)$, s.t. $x_\alpha \to x_0 \Rightarrow f(x_\alpha) \to f(x_0)$

Definition 11. (Countability)

1) First countability: For a topological space X, X is called first countable if for each point $x \in X$, x has a countable neighbourhood basis.

2) Second countability: A topological space X is second countable if it has the countable topological basis.

Remark. Clearly, the second countable topological space is first countable, but the converse is not true.

In particular, if X is first countable, sequences can be used to illuminate tpological properties rather than nets. Like,

Proposition 12. *If X is first countable, then*

    1) $U \subset X$ is closed $\Leftrightarrow \forall\, x \in U,\ \exists\, a\ sequence\{x_n\} \subset U,\ s.t.\ x_n \to x$.

    2) sequential compactness is equivalent to compactness.

And for the second countability, it is about the separability.

Definition 13. (Separability)

1) A subset A of a topological space X is called dense if $\overline{A} = X$.

2) A topological space is called separable if it has a countable dense subset.

By the definition, we can clearly know that:

Proposition 14. *If X is a second countable topological space, then it is separable and every open covering of X has a finite subcollection covering X.*

We can classify topological spaces into some classes.

Definition 15. X is a topological space, then we call X is:

(T_0) $\forall\, x,\, y \in X,\ \exists$ open $U \subset X,\ s.t.\ x \in U$ but $y \notin U$ or $y \in U$ but $x \notin U$ (Kolmogorov space)

(T_1) $\forall\, x,\, y \in X,\ \exists$ open $U,\, V \subset X,\ s.t.\ x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$
    $(\Leftrightarrow \forall\, x \in X,\ \{x\}$ is closed$)$

(T_2) $\forall\, x,\, y \in X,\ \exists$ open $U,\, V \subset X,\ s.t.\ x \in U$ & $y \in V$ and $U \bigcap V = \varnothing$ (Hausdorff space)

(T_3) T_1 holds and $\forall\, x \in X$ and closed $C \subset X$, if $x \notin C$, then $\exists$ open $U,\, V \subset X,\ s.t.\ x \in U$ & $C \subset V$ and $U \bigcap V = \varnothing$ (regular space)

(T_4) T_1 holds and $\forall$ closed $C_1,\, C_2 \subset X$, if $C_1 \bigcap C_2 = \varnothing$,
    then $\exists$ open $U,\, V \subset X,\ s.t.\ C_1 \subset U$ & $C_2 \subset V$ and $U \bigcap V = \varnothing$
    (normal space)

Then we can specify a class of more powerful topological space.

Definition 16. If X is a topological space, then X is said to be metrizable if there exists a metric d on the set X that induces the topology of X.

Remark. Clearly, if X is metrizable, X is second countable and normal.

Here is two metrization theorems provides the essence of metric spaces.

Theorem 17. *(Metrization theorems)*

Urysohn A topological space is separable and metrizable if and only if it is regular, Hausdorff and second countable.

Nagata–Smirnov A topological space is metrizable if and only if regular, Hausdorff and has a σ-locally finite basis.

■ **Complete metic space:** For a metric space, we know it is first countable, so the concept of net is unnecessary. And thus sequences are enough to determine the topological structures, like that sequential compactness is equvilent to compactness.

Proposition 18. *A compact subset of a metric space is closed, bounded and separable.*

Remark. it is clearly, since compactness is also about finity.

For any metric space, we can use the following theorem to get a completion of that and this completion is unique. Thus, we can always assume a metric space is complete.

Theorem 19. *Let $(X,\ d)$ be a metric space. Then, there exists a metric space $(\hat{X},\ \hat{d})$ with the following properties:*

1) $(\hat{X},\ \hat{d})$ is complete.

2) There is an embedding σ from X to $\hat{X}$.

3) $\sigma(X)$ is dense in $\hat{X}$.

And this $(\hat{X},\ \hat{d})$ is unique with respect to isomorphism.

Complete metric space is imporatant since it is "sufficiently large". Rigorously, we can the following definition to describe it.

Definition 20. (Baire Category) A metric space is said to be of the first category if it can be written as a countable union of sets that are nowhere dense. Otherwise, it is of the second category.

Proposition 21. *A complete metric space is a space of the second category.*

■ **Filters:** For convenience, we define some terminologies.

Definition 22. A filter on a set X is a family $\mathscr{F}$ of subsets of X satisfying the following conditions:

1) $\varnothing \notin \mathscr{F}$

2) $\mathscr{F}$ is closed under finite many intersections

3) Any subset of X containing a set in $\mathscr{F}$ belongs to $\mathscr{F}$.

Example 23. For a topological space X and $x \in X$, and let

$$\mathscr{F}(x) \;=\; \{ \text{ all neighbourhoods of x } \}$$

Then $\mathscr{F}(x)$ is a filter and $\mathscr{F}(x)$ satisfies the following properties:

1) $\forall\, U \in \mathscr{F}(x),\; x \in U$

2) $\forall\, U \in \mathscr{F}(x),\; \exists\, V \in \mathscr{F}(x),\; s.t.\; \forall\, y \in V,\; U \in \mathscr{F}(y)$

And conversely, if we can find $\mathscr{F}(x)$ for any $x \in X$ with above two properties, these can define a unique topology $\mathscr{T}$ s.t. $\mathscr{F}(x)$ is the filter of neighbourhoods of x for any $x \in X$. In fact,

$$\mathscr{T} = \{\, U \subset X \colon x \in U \Rightarrow U \in \mathscr{F}(x) \,\}$$

Also, we can define the basis of $\mathscr{F}(x)$, noted by $\mathscr{B}(x)$. That is $\mathscr{B}(x) \subset \mathscr{F}(x)$ with the following properties:

1) $\forall\, U \in \mathscr{B}(x),\; x \in U$

2) $\forall\, U_1 \;\&\; U_2 \in \mathscr{B}(x),\; \exists\, U_3 \in \mathscr{B}(x),\; s.t.\; U_3 \subset U_1 \bigcap U_2$

3) If $y \in U \in \mathscr{B}(x),\; \exists\, W \in \mathscr{B}(y),\; W \subset U$

1.1.2   Definition and Properties

■ **Definition:** Now, we need to endow the topological structure on a vector spaces. And the most important thing is that the topological structure should coincide with the algebraic structure.

Definition 1. A vector space X over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{C} \, or \, \mathbb{R}$) is called a topological vector space if X is equiped with a topology $\mathscr{T}$ s.t. the addition and the scalar multiplication, i.e.

$$(x,\; y) \mapsto x + y$$
$$(\lambda,\; x) \mapsto \lambda x$$

are continuous with respect to the topology $\mathscr{T}$.

In this definition, the most important part is that the addition and the scalar multiplication are continuous. This condition provides some additional properties for the topology and also for the linear operations. First, it can simply the topology.

Proposition 2. *Given a t.v.s. X,*

1) *For any $x_0 \in X$, the map $x \mapsto x + x_0$ is a homeomorphism.*

2) For any $\lambda \in \mathbb{K}$, then map $x \mapsto \lambda x$ is a homeomorphism.

Proof. It is clearly, since by the definition, $x \mapsto x - x_0$ and $x \mapsto \frac{1}{\lambda} x$ are continuous. $\qquad\square$

Therefore, the topology of a t.v.s is completely determined by the filter of neighbourhoods of any point. Or, more rigorously,

Corollary 3. *For a t.v.s X, the filter $\mathscr{F}(x)$ of neighbourhoods of $x \in X$ is as same as $\{\, U + x \colon U \in \mathscr{F}(e) \,\}$, where e is the unit element in X.*

Thus, to research the topology of a t.v.s. X, we just need to research the filter $\mathscr{F}(e)$ of neighbourhoods of e. First, there are two special properties of some subsets of a t.v.s. X.

Definition 4. For a subset U of a t.v.s. X,

1) U is absorbing if $\forall\, x \in X$, $\exists\, \rho > 0$ s.t. $\forall\, \lambda \in \mathbb{K}$ with $|\lambda| \leqslant \rho$, we have $\lambda x \in U$.

2) U is balanced if $\forall\, x \in U$, $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$, we have $\lambda x \in U$.

Then, the following theorem reveals the essence of $\mathscr{F}(e)$.

Theorem 5. *A filter $\mathscr{F}$ of a vector space X over $\mathbb{K}$ is the filter of neighbourhoods of the unit element e w.r.t. some topology compatible with the algebraic structure of X if and only if*

1) $\forall\, U \in \mathscr{F}$, $e \in U$

2) $\forall\, U \in \mathscr{F}$, $\exists\, V \in \mathscr{F}$ *s.t.* $V + V \subset U$

3) $\forall\, U \in \mathscr{F}$, $\forall\, \lambda \in \mathbb{K}$ *with* $\lambda \neq 0$, $\lambda U \in \mathscr{F}$

4) $\forall\, U \in \mathscr{F}$, U *is absorbing*

5) $\forall\, U \in \mathscr{F}$, $\exists\, V \in \mathscr{F}$ *s.t.* $V \subset U$ *is balanced*

Proof. If $\mathscr{F} = \mathscr{F}(e)$, these statements clearly hold.
1) is trivial.
2) is true since the addition is continuous.
3) and 4) hold since the scalar multiplication is continuous.
For 5), because the scalar multiplication is continuous, we can find a $W \in \mathscr{F}$ s.t. $\lambda W \subset U$ for any $|\lambda| \leqslant \rho$, then let $V = \bigcup_{|\lambda| \leqslant \rho} \lambda W$. Clearly, $V \in \mathscr{F}$ and V is balanced.
Conversely, We can define

$$\mathscr{F}(x) = \{\, U + x \colon U \in \mathscr{F} \,\}$$

for any $x \in X$. It can be easily checked that $\mathscr{F}(x)$ satisfies the conditions in Example 23 in last subsection. Therefore, these $\mathscr{F}(x)$ can determine a unique topology $\mathscr{T}$ on X.

Now, we just need to check the continuity of the addition and the scalar multiplication. The addition is continuous, since $\mathscr{F}$ satisfies 2). Using conditions 2) and 4) and 5) to get a balanced absorbing open neighbourhood in $\mathscr{F}$, and this neighbourhood prove the continuity of the scalar multiplication. $\qquad\square$

Here is some simple properties of a t.v.s. X. These properties are directly obtained by definition and above theorem.

Proposition 6. *For a t.v.s. X,*

1) *proper subspaces of X are never absorbing. In particular, if $M \subset X$ is a open subspace, then $M = X$.*

2) *each linear subspace of X, endowed with subspace topology, is also a t.v.s.*

3) *if H is a linear subspace of X, then $\overline{H}$ is also a linear subspace of X.*

4) *if Y is also a t.v.s. and $f \colon X \to Y$ is a linear map, then f is continuous if and only if f is continuous at the unit element e.*

■ **Hausdorff t.v.s. :** The Hausdorff Space is important since it can let the concept of limit make sense. And the topology of a t.v.s. can be simplified and has some additional properties, we can get a easier condition that make a t.v.s. become Hausdorff.

Proposition 7. *A t.v.s X is a Hausdorff space if and only if for any $x \in X$ with $x \neq e$ there exists a $U \in \mathscr{F}(e)$ s.t. $x \notin U$.*

Proof. Since the open neighbourhoods of any point in X is completely determined by the open neighbourhoods of e, this proposition is equivalent to the statement that (T_1) implies Hausdorff.

The proof can be accomplished by obtaining a contradiction to the given condition that $x \neq e$, $\exists\, U \in \mathscr{F}(e)$ s.t. $x \notin U$. For that U, there is a balanced $V \in \mathscr{F}(e)$ s.t. $V + V \subset U$ and the balance implies that $V - V \subset U$. Therefore, $(x + V) \bigcap V = \varnothing$. If not, $x + v_1 = v_2$ for $v_1,\, v_2 \in V$. This implies that $x = v_1 - v_2 \in V - V \subset U$. Thus it is a contradiction. $\qquad\square$

The following theorem is more explicit.

Theorem 8. *For t.v.s. X the following statements are equivalent.*

1) *X is Hausdorff.*

2) the intersection of all neighbourhoods of the unit element e is $\{e\}$.

3) $\{e\}$ is closed.

Proof. Before the rigorously proving, the intuition is clearly. Since in a t.v.s. (T_1) is equivalent to Hausdorff, the equivalence of 1) and 3) is clearly true.

1) $\Rightarrow$ 2) It is because that elements in $\mathscr{F}(e)$ can separate e and other points.

2) $\Rightarrow$ 3) If $x \in \overline{\{e\}}$, i.e. $\forall V_x \in \mathscr{F}(x)$, $V_x \bigcap \overline{\{e\}} \neq \varnothing \Rightarrow e \in V_x$, and $V_x = U + x$ for some $U \in \mathscr{F}(e)$, then $u + x = e$ for some $u \in U$. Thus, $x = -u \in -U$ for all $U \in \mathscr{F}(e)$. That implies $x = e$.

3) $\Rightarrow$ 1) By above mentioned, it just needs to check that if for any topology space Y, $\{y\}$ is closed $\forall y \in Y$, Y is (T_1).
Since $\{y_1\}$ is closed, $Y \backslash \{y_1\}$ is open. That means if $y_2 \neq y_1$, there exists a open neighbourhood U of y_2 s.t. $y_1 \notin U$. Similarly, we can find a open neighbourhood V of y_1 s.t. $y_2 \notin V$. Therefore, Y is (T_1). $\square$

■ **Quotient t.v.s. :** For a linear subspace M of a t.v.s. X, the quotient topology on X/M can be obtained by the quotient map $\pi \colon X \to X/M$. But because of the algebraic structure, it has more properties.

Proposition 9. *For a linear subspace M of a t.v.s. X, the quotient map $\pi \colon X \to X/M$ is open.*

Proof. Let $V \subset X$ be open, then we have

$$\pi^{-1}(\pi(V)) = V + M = \bigcup_{m \in M} (V + m)$$

Since V is open, $V + m$ is open. Thus $\pi^{-1}(\pi(V))$ is open. And by the definition of the topology on X/M, $\pi(V)$ is open. $\square$

Corollary 10. *For a linear subspace M of a t.v.s. X, the quotient space X/M endowed with the quotient topology is a t.v.s..*

Proof. We have the following commutative graph, where f and g are corresponding addition maps or scalar multiplication maps on X and X/M.

$$
\begin{array}{ccc}
X \times X & \xrightarrow{\ f\ } & X \\
\downarrow{\scriptstyle \pi \times \pi} & & \downarrow{\scriptstyle \pi} \\
X/M \times X/M & \xrightarrow{\ g\ } & X/M
\end{array}
$$

Then for an open set $V \subset X/M$, since f and π are continuous, and π is open, $(\pi \times \pi) \circ f^{-1} \circ \pi^{-1}(V)$ is open. By above commutative graph, we have $g \circ (\pi \times \pi) = \pi \circ f$. Therefore, $g^{-1}(V)$ is open, i.e. g is continuous. $\square$

Also, we can find the condition that lets the quotient topological vector space be Hausdorff.

Proposition 11. *Let X be a t.v.s..*

1) *M be a linear subspace of X. Then X/M is Hausdorff if and only if M is closed.*

2) *$X/\overline{\{e\}}$ is Hausdorff.*

Proof. 2) is true because 1). And 1) clearly holds since M is the unit element in X/M and Theorem 8 in this subsection. $\qquad\square$

Remark. By this method, for any t.v.s., we can find a Hausdorff space w.r.t it.

1.1.3 Continuous Linear Maps

The interesting maps between two topological vector spaces not only preserve the algebraic structure, but also the topological structure, thus these are continuous linear maps.

First, for a linear map $f\colon X \to Y$ between vector spaces X and Y, we have the commutative graph, where $\tilde{f}(x + \ker f) = f(x)$ is well-defined.

$$X \xrightarrow{\ f\ } \operatorname{Im} f \xrightarrow{\ i\ } Y$$
$$\downarrow{\scriptstyle \pi} \qquad {\scriptstyle \tilde{f}} \nearrow$$
$$X/\ker f$$

Proposition 1. *Let $f\colon X \to Y$ be a linear map between two t.v.s.'s X and Y.*

1) *If Y is Hausdorff and f is continuous, then $\ker f$ is closed.*

2) *By above notation, f is continuous if and only if $\tilde{f}$ is continous.*

Proof. 1) is because that $\ker f = f^{-1}(\{e\})$ and Y is Hausdorff.
For 2), if $\tilde{f}$ is continuous, it is clearly that $f = i \circ \tilde{f} \circ \pi$ is continuous. Conversely, it is because of the universal property of quotient maps. And in this case, let $U \subset \operatorname{Im} f$ be open, then $f^{-1}(U)$ is open and $\tilde{f}^{-1}(U) = \pi(f^{-1}(U))$. Since π is open, $\tilde{f}^{-1}(U)$ is open. Thus, $\tilde{f}$ is continuous. $\qquad\square$

1.1.4 Complete Topological Vector Spaces

We have just defined the completeness on a metric space by using sequence, but in metric spaces, we know the topology is so powerful that sequences can do any thing, but in general topology, or the topology in a t.v.s., we need an equivalent concept to describe the completeness.

Definition 1. (Completeness)

1) A filter $\mathscr{F}$ on a subset A of a t.v.s. X is said to be a Cauchy filter if

$$\forall\, U \in \mathscr{F}(0) \text{ in } X,\ \exists\, M \subset A\ s.t.\ M \in \mathscr{F}\ \&\ M - M \subset U$$

2) A subset A of a t.v.s. X is said to be complete if every Cauchy filter on A converges to a point $x \in A$.

Remark. Said "the filter converges to a point" means that we can define a net on this filter, and this net converge a point. And this definition is also valid without the algebraic structure.

By this definition, and using the factor that Hausdorff spaces let the limit point of a net uniquely exist, we have similar results comparing with the metric spaces.

Proposition 2. *Let X be a t.v.s..*

1) If X is Hausdorff, any complete set is closed.

2) If X is complete, any closed set is complete.

We known any metric space can be completion. Similarly, the same result can obtained in any t.v.s..

Theorem 3. *Let X be a Hausdorff t.v.s., then there exists a complete Hausdorff t.v.s. $\hat{X}$ and a map $i\colon X \to \hat{X}$ with the following properties.*

1) i is a topological monomorphism.

2) $\overline{i(X)} = \hat{X}$.

3) For any complete Hausdorff t.v.s. Y and for every continuous linear map $f\colon X \to Y$, there exists a continuous map $\hat{f}\colon \hat{X} \to Y$, s.t. the following graph is commutative

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle i}\big\downarrow & \nearrow{\scriptstyle \hat{f}} & \\
\hat{X} & &
\end{array}
$$

And $(\hat{X},\ \hat{f})$ is unique with respect to the isomorphism

Proof. The proof is similar as the proof of the completion of metric spaces, which contructs the $\hat{X}$ as a set of equivalent classes of Cauchy sequences. In a t.v.s., we just need to replace Cauchy sequences by Cauchy filters (in fact, Cauchy nets). Let

$$\begin{aligned}
\tilde{X} &= \{ \text{ all Cauchy filters in } X \} \\
R &: \quad \mathscr{F} \, R \, \mathscr{G} \Leftrightarrow \forall \, U \in \mathscr{F}(e), \ \exists \, A \in \mathscr{F} \ \& \ B \in \mathscr{G} \ s.t. \ A - B \subset U \\
\hat{X} &= \tilde{X}/R
\end{aligned}$$

We can easily define linear operations and topology, s.t. $\hat{X}$ become a complete t.v.s.. Then we just need to check the statements in above theorem hold. $\qquad\square$

1.1.5 Finite Dimensional Topological Vector Spaces

For a finite dimensional topological vector space, the topology compatible with the algebraic structure has some properties coincided with the "finity". First, continuous linear functionals on a t.v.s. have some properties.

Lemma 1. *Let X be a t.v.s. over $\mathbb{K}$. Fixed $v \in X$, then the $\phi_v \colon \mathbb{K} \to X$ by $\xi \mapsto \xi v$ is continuous,*

Proof. It is because that $\phi_v = f \circ \psi_v$ where f is the multiplication map.

$$\begin{array}{ccccc}
\mathbb{K} & \overset{\psi_v}{\to} & \mathbb{K} \times X & \overset{f}{\to} & X \\
\xi & \mapsto & (\xi, v) & \mapsto & \xi v
\end{array}$$

$\qquad\square$

Lemma 2. *For a non-zero linear functional $L \colon X \to \mathbb{K}$, where X is a t.v.s. over $\mathbb{K}$, the following statements are equivalent.*

1) *L is continuous,*

2) *$\ker L$ is closed,*

3) *$\ker L$ is not dense in X,*

4) *L is bounded in some neighbourhood of the origin in X.*

Proof. The equivalence of 1) and 2) and 4) is clearly.

1) $\Rightarrow$ 2) It is because that $\ker L = L^{-1}(\{0\})$.

2) $\Rightarrow$ 3) Since L is non-zero, it clearly holds.

3) $\Rightarrow$ 4) By the assumption, there exists a balanced set $V \in \mathscr{F}(e)$ and a point $x \notin \overline{\ker L}$ s.t. $(x + V) \bigcap \ker L = \varnothing$. $L(V)$ is balanced on $\mathbb{K}$, therefore $L(V)$ is bounded or $L(V) = \mathbb{K}$. But since $(x + V) \bigcap \ker L = \varnothing$, $L(V) \neq \mathbb{K}$.

4) $\Rightarrow$ 1) This implies that L is continuous at e. But since X is a t.v.s., L is continuous at every point. $\qquad\square$

Theorem 3. *Let X be a finite dimensional Hausdorff t.v.s. over $\mathbb{K}$ (endowed with the standard topology), and $\dim X = d$. Then we have:*

1) X is topologically isomorphic to $\mathbb{K}^d$,

2) every linear functional on X is continuous,

3) every linear map from X to any t.v.s. Y is continuous

Proof. For 1), we just need to find a homeomorphic isomorphism from $\mathbb{K}^d$ to X, like the following map, where $\{e_i\}_{i=1}^d$ is the basis of X.

$$
\begin{array}{ccc}
\mathbb{K}^d & \xrightarrow{\phi} & X \\
(\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_d) & \longmapsto & \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_d e_d
\end{array}
$$

ϕ is clearly an algebraic isomorphism. Thus we just need to check ϕ is both continuous and open.

Check: ϕ is continuous.

When $d = 1$, it is continuous by above lemma. For the general case, since d is finite, ϕ is continuous.

Check: 2) holds and ϕ is open.

When $d = 1$, it is clearly 1) and 2) are trued. And suppose 1) and 2) hold for $\dim X \leqslant d-1$, then when $\dim X = d$, let L be a non-zero linear function on X. Then since $X/\ker L \cong \operatorname{Im} L \subset \mathbb{K}$, $\dim \ker L = d - 1$. Therefore, $\ker L \cong \mathbb{K}^{d-1} \Rightarrow \ker L$ is complete $\Rightarrow \ker L$ is closed $\Rightarrow L$ is continuous by above lemma. And,

$$
\begin{array}{ccc}
X & \xrightarrow{\phi^{-1}} & \mathbb{K}^d \\
\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_d e_d & \longmapsto & (\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_d)
\end{array}
$$

is continuous since each

$$
\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_d e_d \longmapsto \lambda_i
$$

is continuous.

Then for 3), it is clealy since $\dim \operatorname{Im} L < \infty$. $\qquad \square$

Corollary 4.

1) Every finite dimensional Hausdorff t.v.s. is complete.

2) Every finite dimensional subspace of a Hausdorff t.v.s. is closed.

3) For a finite dimensional vector space, there is only one topology w.r.t. homeomorphism that can make it be a Hausdorff t.v.s..

4) Every bounded subset on a finite dimensional Hausdorff t.v.s. is compact.

Proof. These properties can be easily obtained by regarding the t.v.s. as $\mathbb{K}^d$ endowed with the standard topology. $\qquad \square$

Finally, the most important theorem in this subsection is that the converse of 4) in above corollary is also true.

Theorem 5. *A Hausforff t.v.s. is locally compact if and only if it is finite dimensional.*

Proof. Let X be a locally compact Hausdorff t.v.s. and K be a compact heighbourhood of e in X, i.e.

$$\exists\ x_1,\ \cdots,\ x_r \in X\ s.t.\ K \subset \bigcup_{i=1}^{r}(x_i + \frac{1}{2}K)$$

Let $M = \text{span}\{x_1,\ \cdots,\ x_r\}$, and M is closed. Therefore, X/M is a Hausdorff t.v.s.. Let $\pi\colon X \to X/M$ be the quotient map.
Since $K \subset M + \frac{1}{2}K$, $\pi(K) \subset \pi(\frac{1}{2}K)$. Thus, by iterating $\pi(2^n K) \subset \pi(K)$.
As K is absorbing, $X = \bigcup_{n=1}^{\infty} 2^n K$,

$$X/M = \pi(X) = \bigcup_{n=1}^{\infty} \pi(2^n K) \subset \pi(K) \subset X/M$$

And since π is continuous, $\pi(K)$ is compact, i.e. X/M is compact.
claim: $\dim X/M = 0$
Suppose $\dim X/M > 0$, then for some $\overline{x_0} \in X$ with $\overline{x_0} \neq \overline{e}$, $\mathbb{R}\overline{x_0} \subset X/M$. And since X/M is Hausdorff compact and $\mathbb{R}\overline{x_0}$ is closed, $\mathbb{R}\overline{x_0}$ is compact, which is a contradiction. $\qquad\square$

1.2 Locally Convex Topological Vector Spaces

The locally convex topological vector space is a topological vector spaces whose topology is generated by a family of seminorms, thus it can provide more properties.

1.2.1 Definition by Convex Sets

Firstly, the conception of locally convex space can be obtained by convex sets. So, we need to research some elementary traits of convex sets.

Definition 1.

1) Let S be any subset of a vector space X over $\mathbb{K}$. The convex hull of S, $conv(S)$, is the smallest convex subset containing S. In fact,

$$conv(S) = \{\ \sum_{i=1}^{n}\lambda_i x_i\colon x_i \in S,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n}\lambda_i = 1,\ n \in \mathbb{N}\ \}$$

2) A subset S of a vector space X over $\mathbb{K}$ is absolutely convex, if S is convex and balanced

3) A subset S of a vector space X over $\mathbb{K}$ is called a barrel if S is closed, absorbing and absolutely convex.

Proposition 2.

1) *Arbitrary intersections of convex sets are convex sets, and the sum of two convex sets is convex, and linear maps preserve convex.*

2) *The convex hull of a balanced set is balanced.*

3) *The closure and the interior of a convex set in a t.v.s. is convex.*

4) *Every neighbourhood of the origin of a t.v.s. is contained in a neighbourhood of the origin which is a barrel.*

Proof. 1) and 2) are easily obtained by the definition.
To prove 3), for any $\lambda \in [0,1]$, we define the map

$$\phi : \quad \begin{array}{ccc} X \times X & \longrightarrow & X \\ (x,y) & \mapsto & \lambda x + (1-\lambda)y \end{array}$$

Using the fact that $\phi(S \times S) \subset S$, $S \times S \subset \phi^{-1}(S)\phi^{-1}(\overline{S})$. And since X is a t.v.s., ϕ is continuous. Thus $\phi^{-1}(\overline{S})$ is closed, i.e. $\overline{S} \times \overline{S} \subset \phi^{-1}(\overline{S}) \Rightarrow \phi(\overline{S} \times \overline{S}) \subset \overline{S}$. $\overline{S}$ is convex.
Since $z+u = \lambda x+(1-\lambda)y+\lambda u+(1-\lambda)u$ if $z = \lambda x+(1-\lambda)y$, $x+U$ & $y+U \subset S \Rightarrow z+U \subset S$. That means $\mathring{S}$ is convex.
For 4), $\forall\, U \in \mathscr{F}(e)$, the set

$$\overline{conv\left(\bigcup_{|\lambda| \leqslant 1} \lambda U\right)}$$

is a barrel. $\qquad \square$

Corollary 3. *Every neighbourhood of the origin in a t.v.s. X is contained in a neighbourhood of the origin which is absolutely convex.*

Now, we can get the definition of a locally convex t.v.s..

Definition 4. A t.v.s. X is said to be locally convex if there is a basis of neighbourhoods of the origin in X consisting of convex sets.

By this definition, the structure of the neighbourhoods of the origin in a locally convex t.v.s. can be more explicit by using above proposition.

Proposition 5. *Let X be a locally convex t.v.s..*

1) X has a basis of neighbourhoods of origin consisting with oepn absorbing absolutely convex sets.

2) X has a basis of neighbourhoods of origin consisting with barrels.

Theorem 6. *If X is a locally convex t.v.s., then there exists a basis $\mathscr{B}$ of neighbourhoods of origin consisting of absorbing absolutely convex set s.t.*

$$\forall\ U \in \mathscr{B},\ \forall\ \rho > 0,\ \exists\ W \in \mathscr{B}\ s.t.\ W \subset \rho U$$

Conversely, if $\mathscr{B}$ is a collection of absorbing absolutely convex subsets of a vector space satisfying above condition, it can generate a unique locally convex t.v.s..

1.2.2 Definition by Seminorms

Definition 1. Let X be a vector space over $\mathbb{K}$. A map $p\colon X \to \mathbb{R}^+$ is called a seminorm if it satisfies:

1) $p(x + y) \leqslant p(x) + p(y),\ \forall\ x,\ y \in X$,

2) $p(\lambda x) = |\lambda|\, p(x),\ \forall\ x \in X,\ \forall\ \lambda \in \mathbb{K}$.

Remark. In fact, $\ker p$ is a linear sbuspace and if $\ker p = \{0\}$, p is called a norm.

By the intuition, the seminorm could construct the continuity of the addition and multiplication since it satisfies above properties. Now, we can build this rigorously.

Definition 2. Let X be a vector space and $A \subset X$ be a nonempty subset. The Minkowski functional of A is the map

$$\begin{array}{cccc} p_A: & X & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \inf\{\ \lambda > 0\colon x \in \lambda A\ \} \end{array}$$

Let X be a vector space and p is a seminorm on X, then let $\mathring{U}_p = \{\ x \in X\colon p(x) < 1\ \}$, $U_p = \{\ x \in X\colon p(x) \leqslant 1\ \}$. Thus, U may be the basis generating the topology. To see it, the following proposition is helpful.

Proposition 3. *Let $A \subset X$ be a nonempty subset of a vector space, which is absorbing and absolutely convex, then p_A is a seminorm and $\mathring{U}_{p_A} \subset A \subset U_{p_A}$. Conversely, if q is a norm on X then $\mathring{U}_q$ is an absorbing absolutely convex set and $q = p_{\mathring{U}_q}$.*

Proof. Since A is balanced, $\xi A \in \lambda A \Leftrightarrow x \in \frac{\lambda}{|\xi|} A$. Thus,

$$p_A(x) = |\xi| \inf \left\{\ \frac{\lambda}{|\xi|}\colon x \in \frac{\lambda}{|\xi|} A\ \right\} = |\xi|\, p_A(x)$$

And $p_A(x) < \infty$ ($\forall\, x \in X$) since A is absorbing.

Fixed $x,\ y \in X$, $\forall \varepsilon > 0$, $\exists\ \lambda,\ \mu > 0$, s.t. $x \in \lambda A$ & $y \in \mu A$ and

$$\lambda \leqslant p_A(x) + \varepsilon\ ,\ \ \mu \leqslant p_A(y) + \varepsilon,$$

By convexity of A, $\lambda A + \mu A \subset (\lambda + \mu)A$. Thus,

$$p_A(x) = \inf \{\ \delta > 0\colon x + y \in \delta A\ \} \leqslant \lambda + \mu \leqslant p_A(x) + p_A(y) + 2\varepsilon$$

$\Rightarrow\ p_A(x)$ is a seminorm.

$$\begin{aligned} x \in \mathring{U}_{p_A} &\ \Rightarrow\ \exists\, \lambda \in [0,1]\ s.t.\ x \in \lambda A \subset A \\ x \in A &\ \Rightarrow\ 1 \in \inf\{\ \lambda > 0\colon x \in \lambda A\ \}\ \Rightarrow\ p_A(x) \leqslant 1 \Rightarrow x \in U_{p_A} \end{aligned}$$

That means $\mathring{U}_{p_A} \subset A \subset U_{p_A}$.

Finally, the statements for q can be obtained easily by the definition.　　$\square$

Now, we can give the definition of a locally convex t.v.s. by seminorms coinciding with the definition by convex sets.

Theorem 4. *Let X be a vector space and $\mathscr{P} = \{p_i\}_{i \in I}$ be a family of seminorms. Then the initial topology $\mathscr{T}_P$ generated by $\mathscr{P}$ makes X be a locally convex t.v.s. In fact, the basis of neighbouhoods of the origin in X is like*

$$\mathscr{B} = \{\ \{x \in X\colon p_{i_1}(x) < \varepsilon, \cdots, p_{i_n}(x) < \varepsilon\}\colon i_1, \cdots, i_n \in I, n \in \mathbb{N}, \varepsilon > 0\ \}$$

Conversely, the topology of any locally convex t.v.s. can be generated by a family of seminorms.

Proof. Each element in the subbasis of $\mathscr{T}_P$ is like $\{x \in X\colon p_i(x) < \varepsilon\} = \varepsilon \mathring{U}_{p_i}$, which is clearly absorbing and absolutely convex. Therefore, every element in $\mathscr{B}$ is convex. $(X, \mathscr{T}_P)$ is a locally convex t.v.s..

Convesrly, if $(X, \mathscr{T})$ is a locally convex t.v.s., the basis of neighbourhoods of the origin in X consists of absorbing and absolutely convex stes, which can generate a family of seminorms by above proposition. Therefore, these seminorms can generate a locolly convex topology $\mathscr{T}_P$. In fact, $\mathscr{T}_P = \mathscr{T}$, since $\mathring{U}_{p_A} \subset A \subset U_{p_A}$.　　$\square$

Remark. By this theorem, in a vector space X, the seminorms on X can coincide with a locally convex topology making X be a t.v.s..

There some extra properties for the seminorms on a vector space.

Proposition 5. *Let X be a vector space and p be a seminorm on X. Then,*

1) $\forall\, r > 0$, $r\mathring{U}_p = \{x \in X\colon p(x) < r\} = \mathring{U}_{\frac{1}{r}p}$.

2) $\forall x \in X$, $x + \mathring{U}_p = \{y \in X\colon p(y - x) < 1\}$.

3) if q is a seminorm on X, $p \leqslant q \Leftrightarrow \mathring{U}_q \subset \mathring{U}_p$.

4) if $\{s_i\}_{i=1}^n$ are seminorms on X, then $s(x) = \max_{i=1,\cdots,n} s_i(x)$ is also a seminorm and $\mathring{U}_s = \bigcap_{i=1}^n \mathring{U}_{s_i}$

Theorem 6. *Let $\mathscr{P} = \{p_i\}_{i \in I}$ and $\mathscr{Q} = \{q_j\}_{j \in J}$ be two families of seminorms on a vector space X inducing $\mathscr{T}_P$ and $\mathscr{T}_Q$, then*

$$\mathscr{T}_Q \subset \mathscr{T}_P \Leftrightarrow \forall \, q \in \mathscr{Q}, \, \exists \, \{i_k\}_{k=1}^n \subset I, \, \exists \, C > 0, \, s.t. \, Cq(x) \leqslant \max_{k=1,\cdots,n} p_{i_k}(x)$$

Proof. This right side of above statement is equivalent to that

$$\forall \, q \in \mathscr{Q}, \, \exists \, \{i_k\}_{k=1}^n \subset I, \, \exists \, C > 0, \, s.t. \, C \bigcap_{k=1}^n \mathring{U}_{p_{i_k}} \subset \mathring{U}_q$$

So, it is clearly equivalent to $\mathscr{T}_Q \subset \mathscr{T}_P$. □

Remark. Because of this, we have the definition of equivalent norms. In fact, two norms p and q are said to be equivalent if and only if there exists C_1, $C_2 > 0$, s.t. $C_1 p(x) \leqslant q(x) \leqslant C_2 p(x)$, for all $x \in X$. This definition means that two equivalent norms can generate one same topology.

Corollary 7. *The family $\mathscr{P} = \{p_i\}_{i \in I}$ of seminorms and $\mathscr{Q} = \{ \max_{i \in B} p_i \colon \varnothing \neq B \subset I \text{ with } B \text{ finite} \}$ can generate one same topology on X*

1.2.3   Separability and Metrizability

For a t.v.s., (T_1) is equivalent to Hausdorff and (T_1) is associated with the ability of a topology separating points. Therefore, wheather a t.v.s. is a Hausdorff space or not is completely determined by wheather the topology of it can separate points or not. Then for a locally convex t.v.s., whose topology is induced by a family of seminorms, this separability is related to these seminorms.

Definition 1. A family of seminorms $\mathscr{P} = \{ p_i \}_{i \in I}$ on a vector space X is said to be speparating, if

$$\forall \, x \in X \backslash \{0\}, \, \exists \, i \in I \, s.t. \, p_i(x) \neq 0$$

Remark. In fact, above condition is equivalent to

$$\text{if } p_i(x) = 0, \, \forall \, i \in I \Rightarrow x = 0$$

Now, we can give the condition that makes a locally convex t.v.s. be Hausdorff.

Theorem 2. *A locally convex t.v.s. X is Hausdorff if and only if its topology can be induced by a separating family of seminorms $\mathscr{P} = \{ p_i \}_{i \in I}$.*

Proof. If $\mathscr{P} = \{ p_i \}_{i \in I}$ is separating, the fact that $\mathscr{T}_P$ is Hausdorff can be obtained easily by the definition.

Conversely, if X is Hausdorff, for $x \neq 0$, we can find a $U \in \mathscr{F}(0)$, s.t. U can separate x and 0. But since X is locally convex, U can be chosen as $\mathring{U}_p$ for a seminorm p. Thus, for this p, $p(x) \neq 0$. $\qquad\square$

For the metrizability of a locally convex t.v.s., the consequece is also easier than general case.

Theorem 3. *A locally convex t.v.s. X is metrizable if and only if its topology is determined by a countable separating family of seminorms.*

Proof. It can be directly obtained by the Nagata–Smirnov's Metrization Theorem. Also, there is a more explicit proof. If the topology of X is generated by a countable separating family of seminorms $\mathscr{P} = \{ p_n \}_{n=1}^{\infty}$, we can define the metric d on X by

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)}$$

Conversely, if (X, d) is the metric space, the subbasis of the topology generated by this metric is like $U_n = \{x \colon d(x, 0) < 1/n\}$. And these U_n can provide a countable separating family of seminorms. In fact, if $\mathscr{Q} = \{ q_i \}_{i \in I}$ generates the topology of X, for each U_n, there are $q_1, \cdots, q_k \in \mathscr{Q}$ and $\varepsilon_1, \cdots, \varepsilon_k > 0$, s.t. $\bigcap_{i=1}^{k} \{x \colon q_i(x) < \varepsilon_i\} \subset U_n$. Then let $p_n = \sum_{i=1}^{k} \varepsilon_i^{-1} q_i$. It can check that the family $\{p_n\}_{n=1}^{\infty}$ generate the coincided topology on X. $\qquad\square$

1.2.4 Continuous Linear Maps on LCTVS

To give the special property of continuous linear maps on LCTVS, we need firstly refine the family of seminorms.

Definition 1. A family $\mathscr{Q} = \{ q_j \}_{j \in J}$ of seminorms on a vector space X is said to be directed if

$$\forall\, j_1, \cdots, j_n \in \mathscr{Q},\ \exists\, j \in J\ \&\ C > 0,\ s.t.\ C q_j(x) \geqslant \max_{k=1,\cdots,n} q_{j_k}(x),\ \forall\, x \in X$$

Remark. By **Proposition 5** in **1.2.2**, this definition is equivalent to that

$$\forall\, \mathring{U}_{q_{j_1}}, \cdots, \mathring{U}_{q_{j_n}},\ \exists\, \mathring{U}_{q_j}\ s.t.\ \mathring{U}_{q_j} \subset \bigcap_{k=1}^{n} \mathring{U}_{j_i}$$

And thus the basis of this directed family of seminorms should be like

$$\mathscr{B}_d = \{\, r\mathring{U}_q \colon q \in \mathscr{Q}, r > 0 \,\}$$

By this special topology, we can find the condition making linear functional continuous.

Proposition 2. *Let $\mathscr{T}$ be a locally convex topology on a vector space X generated by a directed family $\mathscr{Q}$ of seminorms on X. Then*

$$L\colon \ X \longrightarrow \mathbb{K}$$

is a $\mathscr{T}$-continuous linear functional if and only if $\exists \ q \in \mathscr{Q}$ s.t. L is q-continuous, i.e. $|L(x)| \leqslant Cq(x)$ for some $C > 0$.

Proof. In fact, this property of continuous linear functional is because the element in a directed locally convex topology is like $r\mathring{U}_q$. In fact, we just need to check the origin point.
If L is continuous, there exists a $r\mathring{U}_q$ s.t. $r\mathring{U}_q \subset L^{-1}(B_1(0))$, where $B_1(0)$ is the unit ball centered at 0. This is equivalent to the q-continuity of L.
Conversely, it is clearly by the fact $\mathscr{T}_q \subset \mathscr{T}$. $\qquad\square$

We can easily see that the topology of a locally convex t.v.s. can be always induced by a directed family of seminorms by the **Corollary** 7 in **1.2.2**. Thus, we have the corollary.

Corollary 3. *$(X, \mathscr{T})$ is a locally convex t.v.s. and $\mathscr{T}$ is generated by the family $\mathscr{P} = \{p_i\}_{i \in I}$. Then $L\colon \ X \longrightarrow \mathbb{K}$ is a continuous linear functional if and only if*

$$\exists \ i_1, \cdots, i_n \in I, \ \exists \ C > 0 \ s.t. \ |L(x)| \leqslant C \max_{k=1,\cdots,n} p_{i_k}(x), \ \forall \ x \in X$$

And this corollary can be easily extended to linear maps. And the proof is similar as above statement In fact, we just need to replace $B_1(0)$ by $\mathring{U}_q$

Theorem 4. *Let X and Y be two locally convex t.v.s.'s generated by $\mathscr{P}$ and $\mathscr{Q}$. Then linear map $f\colon X \to Y$ is continuous if and only if*

$$\forall \ q \in \mathscr{Q}, \ \exists \ p_1, \cdots, p_n \in \mathscr{P}, \ \exists \ C > 0 \ s.t. \ q(f(x)) \leqslant C \max_{i=1,\cdots,n} p_i(x)$$

1.3 The Hahn-Banach Theorem

1.3.1 Two Forms of Hahn-Banach Theorem

Theorem 1 (Geometric form)**.** *Let X be a t.v.s. over $\mathbb{K}$, N be a linear subspace of X and Ω be a convex open subset of X with $N \bigcap \Omega = \varnothing$. Then there is a closed hyperplane H of X s.t.*

$$N \subset H \ \ \& \ \ H \bigcap \Omega = \varnothing$$

Proof. Assume that $\Omega \neq \varnothing$.

Let $\mathscr{F} = \{$ all linear subspace L of X s.t. $N \subset L$ and $L \bigcap \Omega = \varnothing \}$.

Since $N \in \mathscr{F}$, $\mathscr{F} \neq \varnothing$. And $\mathscr{F}$ can be ordered by "$\subset$". Clearly, for every chain $\mathscr{C} = \{C_i\}_{i \in I} \in \mathscr{F}$, it has a maximal element $C = \bigcup_{i \in I} C_i \in \mathscr{F}$. Then by Zorn's Lemma, there exists a maximal $H \in \mathscr{F}$. And this H is closed because the maximality of H.

In fact, we just need to check $\dim X/H = 1$. In the case $\mathbb{K} = \mathbb{R}$, if $\dim X/H \geqslant 2$, then we can find one dimensional subspace L in X s.t. $L \oplus H$ satisfies above condition. Then it is a contradiction to the maximality of H. And for $\mathbb{K} = \mathbb{C}$, using above process to get a real hyperplane H_0 and then $H = H_0 \bigcap i H_0$ is the hyperplane we need.

The main difficulty of this proof is to construct L. In fact, we just need to find a line $\tilde{L}$ in X/H s.t. $\tilde{L} \bigcap A = \varnothing$, where $A = \bigcup_{\lambda > 0} \lambda \pi(\Omega)$ is a cone. This $\tilde{L}$ can easily be found because we can find $\overline{x}, -\overline{x} \in X/H \backslash A$ with $\overline{x} \neq \overline{0}$ by using the fact that $\dim X/H \geqslant 2$. $\qquad \square$

Remark. In this theorem, if N is an affine linear subspace, then the H can be chosen as a affine hyperplane satisfying above condition. It is clear by using translation.

Theorem 2 (Analitic form)**.** *Let p be a seminorm on a vector space X over $\mathbb{K}$, M is a linear space of X and f is a linear functional on M s.t. $|f(x)| \leqslant p(x)$, $\forall\ x \in M$. Then there is a linear function $\tilde{f}$ on X s.t. $\tilde{f}(x) = f(x)$, $\forall\ x \in M$ and $\left|\tilde{f}(x)\right| \leqslant p(x)$, $\forall\ x \in X$.*

Proof. Let $N = \{x \in M \colon f(x) = 1\}$(affine hyperplane) and $\Omega = \{x \in X \colon p(x) < 1\}$(open convex set). Then since $N \bigcap \Omega$, we can find a closed affine hyperplane H of X, s.t. $N \subset H$ and $H \bigcap \Omega = \varnothing$. Fixed $x_0 \in N \subset H$, $H - x_0$ is a hyperplane, thus it can generate a functional $\tilde{f}$ on X. Set $\tilde{f}(x_0) = 1$, then define

$$\tilde{f} \colon \quad \begin{aligned} X = (H - x_0) \oplus \mathbb{K}x_0 \quad &\longrightarrow \quad \mathbb{K} \\ (h - x_0) + \lambda x_0 \quad &\longmapsto \quad \lambda \end{aligned}$$

Since $M = (N - x_0) \oplus \mathbb{K}x_0$ and $f(x_0) = 1$ and $H \bigcap \Omega = \varnothing$, we can easily check that $\tilde{f}(x) = f(x)$, $\forall\ x \in M$ and $\left|\tilde{f}(x)\right| \leqslant p(x)$, $\forall\ x \in X$. $\qquad \square$

1.3.2 Applications

By the geometric form of the Hahn-Banach Theorem, it says about hyperplanes can separate some non-intersecting subsets. And moreover, by the analytic form of the Hahn-Banach Theorem, since hyperplanes are linked with functionals, functionals can also separate some some non-intersecting subsets in a t.v.s. First, we can define this association more explicit.

Definition 1. Let X be a t.v.s. over $\mathbb{R}$ and H is a closed affine hyperplane. For A, $B \subset X$ and $A \bigcap B = \varnothing$, we say A and B separated by H, if

$$\exists\ a \in \mathbb{R},\ s.t.\ H = f^{-1}(\{a\}) \text{ for some } f \colon X \to \mathbb{R}\ \&\ f(A) \geqslant a,\ f(B) \leqslant a.$$

Theorem 2. *Let X be a t.v.s. over $\mathbb{R}$ and A, B be two disjoint convex non-empty subsets of X, then we have:*

1) *if A is open, then there exists a closed affine hyperplane H of X separating A and B.*

2) *if A, B are open, H can strictly separate A and B.*

3) *if A is a cone and B is open, then H can be chosen as a hyperplane.*

Proof. Let $U = A - B$. Clearly, U is open and convex and $N = \{0\} \bigcap H = \varnothing$. Therefore, there exists a continuous functional f on X, s.t. $f(U) > 0$ i.e. $f(x) > f(y)$, $\forall\ x \in A\ \&\ y \in B$. Since $B \neq \varnothing$, let $a = \inf_{x \in A} f(x) > -\infty$. And thus 1) and 2) can be obtained.

For 3), if A is a cone, then $tf(x) = f(tx) \geqslant a$, $\forall\ t > 0 \Rightarrow f(x) \geqslant \frac{a}{t}$. Thus, $f(A) \geqslant 0$. H can be chosen as a hyperplane. $\qquad\square$

For a locally convex t.v.s X, each point in X has some convex heighbourhoods. Therefore, we have following corollaries.

Corollary 3. *Let X be a locally convex t.v.s over $\mathbb{R}$.*

1) *If A and B are two disjoint closed subsets and B is compact, then A and B are strictly separated.*

2) *If A is a closed convex subset of X and $x \notin A$, then x is strictly separated from A.*

3) *If A is a subset of X, then $\overline{\text{span}\{A\}}$ is the intersection of all closed hyperplanes containing A.*

And these consequenses can be extended to $\mathbb{C}$.

Theorem 4. *Let X be a locally convex t.v.s over $\mathbb{C}$ and A, B be two disjoint closed convex subsets of X. If B is compact, then there is a continuous linear functional $f \colon X \to \mathbb{C}$, and $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ s.t.*

$$\text{Re}\, f(x) \leqslant \alpha < \alpha + \varepsilon \leqslant \text{Re}\, f(y),\ \forall\ x \in A,\ \forall\ y \in B$$

1.4 Banach Spaces

The Banach space is a very special locally convex topological space, whose topology is generated by only one seminorm. To make this space be a Hausdorff space, this seminorm is actually a norm. And more, we need it become complete. So, a Banach space is a complete normed space. Because it is definitely a locally convex Hausdorff t.v.s., all results mentioned above can be apllied on it. And we can have more interesting results of the Banach space because of its simple structure.

1.4.1 Elementary Properties

Definition 1. A normed space is a vector space X with a compatible norm $\|\cdot\|$, which makes it be a locally convex Hausdorff topological vector space. A Banach space is a complete normed space.

Remark. By this definition, we know that any result mentioned above can be also true for the Banach space $(X, \|\cdot\|)$ by replacing the family of seminorms by the $\|\cdot\|$. Moreover, two equivalent norms on X provide same topology.

The properties of finite dimensional Banach spaces is in the subsection **1.1.5**. And for the quotient space of a Banach space, we can get a more explicit expression of the induced quotient norm.

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space and M be a closed linear subspace of X and $\pi\colon X \to X/M$, then the induced quotient norm on X/M is defined as $\|x + M\| = \inf\{\|x + y\| : y \in M\}$. Thus, using the consequences of the quotient t.v.s., we have following results.*

1) π is continuous and $\|\pi(x)\| \leqslant \|x\|$.

2) X/M is a Banach space.

3) $W \subset X/M$ is open if and only if $\pi^{-1}(W)$ is open.

4) π is open.

5) if N is a finite dimensional subspace of X, then $M + N$ is closed.

Proof. The element in the subbasis of the topology generated by $\|\cdot\|$ at 0 is like $U_\varepsilon = \{x \in X\colon \|x\| < \varepsilon\}$ for a fixed $\varepsilon > 0$. Therefore, the element V_ϵ in the subbasis of the induced topology on X/M at 0 satisfies $\pi^{-1}(V_\epsilon) = \{x\colon \|x\| < \epsilon\}$ for some $\epsilon > 0$, i.e. $V_\epsilon = \{x + M\colon \|x + y_0\| < \epsilon, \exists y_0 \in M\}$ .Clearly, V_ϵ is an absorbing absolutely convex set. Then the Minkowski functional p_ϵ of V_ϵ is like

$$
\begin{aligned}
p_\epsilon(x + M) &= \inf\{\lambda > 0\colon x + M \in \lambda V_\epsilon\} \\
&= \inf\{\lambda > 0\colon \|x + y_0\| < \lambda\epsilon, \exists y_0 \in M\} \\
&= \frac{1}{\epsilon}\inf\{\|x + y\| : y \in M\}
\end{aligned}
$$

Clearly, for any $\epsilon > 0$, p_ϵ is a norm and it is equivalent to the norm $\|x + M\|$. Thus the quotient topology is definitely generated by $\|x + M\|$.

Then, 1) and 3) clearly hold by the definition. 2) is true since H is a Banach space. 4) is true for any general t.v.s.. 5) holds since $N + M$ is a finite dimensional subspace of X/M. $\qquad\square$

Similarly, we can define the norm on the product space of some Banach spaces.

Definition 3. Let $\{X_i\}_{i=1}^p$ is a family of Banach spaces with norm $\|\cdot\|$. Then,

1) if $1 \leqslant p < \infty$,

$$\oplus_{i=1}^p X_i = \left\{ x \in \prod_i^p X_i \colon \|x\| = \left[\sum_{i=1}^p \|x_i\|^p\right]^{1/p} \right\}$$

2) if $p = \infty$,

$$\oplus_{i=1}^\infty X_i = \left\{ x \in \prod_i^p X_i \colon \|x\| = \sup_i\{x(i)\} < \infty \right\}$$

1.4.2 Linear Transformations and Linear Functionals

By results in **Theorem** 4 in the subsection **1.2.4** the linear transformation $T\colon X \to Y$ between two Banach spaces is continuous if and only if $\exists\, C > 0$, s.t. $\|Tx\| \leqslant \|x\|$, $\forall\, x \in X$. Because of this property, we can define the norm of continuous linear transformation.

Definition 1. Let $T\colon X \to Y$ between two Banach spaces be a linear transformation. Then the norm of T is defined as

$$\|T\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Remark. By this definition, we know T is continuous if and only if $\|T\| < \infty$, and we call T is bounded. Also, the formula of the norm can be

$$\|T\| = \sup_{\|x\|=1} \frac{\|Tx\|}{\|x\|}$$

Let $\mathscr{B}(X, Y) = \{$all bounded linear transformations between X and $Y\}$. $\mathscr{B}(X, Y)$ is a Banach space with this norm. Then in the space $\mathscr{B}(X, Y)$, there are two topologies. The first one is generated by a family of seminorms $\{p_x\}_{x \in X}$, where $p_x(T) = \|Tx\|$. And the second one is generated by the norm $\|\cdot\|$. In fact, the convergence of a sequense in $\mathscr{B}(X, Y)$ with respect to the first one is about pointwise convergence, and with respect to the second one is about uniform convergence. Clearly, the second one is stronger. But in some case the first one can give some information of the first one.

Theorem 2 (The Principle of Uniform Boundedness)**.** *Let X be a Banach space and Y be a normed space, and $\{T_i\}_{i \in I}$ be a subset of $\mathscr{B}(X,Y)$. If for any $x \in X$ $\{\|T_i x\|\}_{i \in I}$ is bounded, then $\{\|T_i\|\}_{i \in I}$ is bounded.*

Proof. Firstly, there is a open ball $B(x_0, \varepsilon)$ such that for any $x \in B(x_0, \varepsilon)$, $\|T_i\|\, x \leqslant K$ for some constant K. Asssume it is not true. We can find a family of open balls $B(x_n, \varepsilon_n)$ s.t. $B(x_n, \varepsilon_n) \subset B(x_{n-1}, \varepsilon_{n-1})$, $\varepsilon_n < 1/n$, and a sequence $i_n \subset I$, satisfying $\|T_{i_n} x\| > n$. And since each $\overline{B(x_n, \varepsilon_n)}$ is compact, there is a $z \in \bigcap_{n=1}^{\infty} \overline{B(x_n, \varepsilon_n)}$. This contradicts to the boundedness of $\{T_i z\}_{i \in I}$.

Next, with this $B(x_0, \varepsilon)$, if $y \in X$ and $y \neq 0$, then $z = \frac{\varepsilon}{\|y\|} y + x_0 \in B(x_0, \varepsilon)$. Therefore,

$$\frac{\varepsilon}{\|y\|} \|T_i y\| - \|T_i x_0\| \leqslant \left\| \frac{\varepsilon}{\|y\|} \|T_i y\| + \|T_i x_0\| \right\| = \|T_i z\| \leqslant K$$

Then since $K' = \sup_{i \in I} \|T_i x_0\| < \infty$,

$$\|T_i y\| \leqslant \frac{K + K'}{\varepsilon} \|y\| < \infty$$

Thus $\|T_i\|_{i \in I}$ is bounded. $\qquad\square$

Remark. Let $X^* = \mathscr{B}(X, \mathbb{K})$ be the continuous linear functional space. Above theorem can easily apply to it.

Corollary 3. *For a Banach space X and $S \subset X^*$ a subset, S is norme bounded if and only if for every $x \in X$, $\sup\{|f(x)| : f \in S\}$.*

Another important property of the linear transformation betweeen two Banach spaces is that any continuous linear surjective transformation is open.

Theorem 4 (Open Mapping Theorem)**.** *Let X and Y be two Banach spaces and $T \colon X \to Y$ be a continuous linear surjection. Then T is open.*

Proof. Let X_ε and Y_ε be two open balls in X and Y with centering at 0 and and radius ε.

1) For any $\varepsilon > 0$, there is a $\delta > 0$, s.t. $Y_\delta \subset \overline{T(X_\varepsilon)}$.
   By $X = \bigcup_{n=1}^{\infty} n X_\varepsilon$, $Y = \bigcup_{n=1}^{\infty} T(n X_\varepsilon)$. Then since Y is the second Baire category, there exists n_0 and a ball $B_r(z) \subset Y$ s.t. $B_r(z) \subset \overline{T(n_0 X_\varepsilon)}$ i.e. $B_\delta(y_0) \subset \overline{T(X_\varepsilon)}$ with $\delta = r/n$ and $y_0 = z/n$. Then since for any $y \in Y_\delta$, $y = (y + y_0) - y_0$

   $$Y_\delta \subset \{y_1 - y_2 \colon y_1, y_2 \in B_\delta(y_0)\} \subset \overline{T(\{x_1 - x_2 \colon x_1, x_2 \in X_\varepsilon\})} \subset \overline{T(X_{2\varepsilon})}$$

2) For any $\varepsilon_0 > 0$, there is a $\delta_0 > 0$, s.t. $Y_{\delta_0} \subset T(X_{2\varepsilon_0})$.

Choose a positive sequence $\{\varepsilon_n\}$ with $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon_0$, then there exists a positive sequence δ_n with $\delta_n \to 0$ s.t. $Y_{\delta_n} \subset \overline{T(X_{\varepsilon_n})}$

For any $y \in Y_{\delta_0}$, there is a $x_0 \in X_{\varepsilon_0}$, s.t. $\|y - Tx_0\| < \delta_1$. since $y - Tx_0 \in Y_{\delta_0}$, there is a $x_1 \in X_{\varepsilon_1}$ s.t. $norm y - Tx_0 - Tx_1 < \delta_2$. Thus by induction, we can find a sequence $\{x_n\}$ s.t. $x_n \in X_{\varepsilon_n}$ and

$$\left\| y - T\left(\sum_{k=0}^{n} x_k \right) \right\| < \delta_{n+1}$$

And since $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon_0$, the sequence $\sum_{k=0}^{n} x_k$ absolutely converges to x in X. And

$$\|x\| \leqslant \sum_{n=1}^{\infty} \|x_n\| \leqslant \sum_{n=1}^{\infty} \varepsilon_n < 2\varepsilon_0$$

By the continuity of T and $\delta_n \to 0$, $y = Tx$. Therefore, $Y_{\delta_0} \subset T(X_{2\varepsilon_0})$.

3) For any open set $G \subset X$ and $y_0 = Tx_0$ with $x_0 \in G$, there is a open ball Y_δ s.t. $y_0 + Y_\delta \subset T(G)$.

Since G is open, there is a open ball X_ε with $x_0 + X_\varepsilon \subset G$. Then there is a $\delta > 0$ s.t. $Y_\delta \subset T(X_\varepsilon)$. Thus

$$y_0 + Y_\delta \subset Tx_0 + T(X_\varepsilon) = T(x_0 + X_\varepsilon) \subset T(G) \qquad \square$$

Then there is a directed corollary from the Open Mapping Theorem.

Corollary 5 (The Inverse Mapping Theorem)**.** *Let X and Y be two Banach spaces and $T\colon X \to Y$ be a continuous linear isomorphism, then T^{-1} is also bounded.*

Corollary 6 (The Closed Graph Theorem)**.** *Let X and Y be two Banach spaces and $T\colon X \to Y$ be a linear transformation. If the graph of T, $G = \{x \oplus Tx \colon x \in X\}$ is closed, then T is continuous.*

Proof. Since X and Y are two Banach spaces and G is closed, G is also a Banach space. Let $P_1\colon G \to X$ be $P_1(x \oplus Tx) = x$ and $P_2\colon G \to Y$ be $P_2(x \oplus Tx) = Tx$. Then P_1 is bounded and bijective. Therefore, by the Inverse Mapping Theorem, $P_1^{-1}\colon X \to G$ is continuous. Since P_2 is also continuous, $A = P_2 \circ P_1^{-1}$ is continuous. $\qquad \square$

The next important theorem is Hahn-Banach Theorem. We have known the Hahn-Banach Theorem holds in any general t.v.s.. But for the Banach space, it can induce some interesting corollaries of the bounded linear functional space.

Corollary 7. *Let X be a Banach space. Then,*

1) if $\{x_i\}_{i=1}^d$ is a linearly independent in X, and $\{\alpha_i\}_{i=1}^d$ are arbitrary scalars, then there is a $f \in X^*$ s.t. $f(x_i) = \alpha_i$ for $i = 1, \cdots, d$.

2) if Y is a linear subspace of X and $x_0 \in X$ with $\inf\{\|x_0 - y\| : y \in Y\} = d > 0$, then there is a $f \in X^*$ s.t. $f(x_0) = 1$ and $f(y) = 0 \; \forall \; y \in Y$ and $\|f\| = d^{-1}$.

3) if $x \in X$, then

$$\|x\| = \sup_{\|f\| \neq 0} \frac{|f(x)|}{\|f\|} = \sup\{|f(x)| : f \in X^*, \;\; \|f\| \leqslant 1\}$$

4) if $x \neq y$ in X, then there is a $f \in X^*$ s.t. $f(x) \neq f(y)$.

5) if Y is a linear subspace of X and Y is not dense, then there is $f \in X^*$ with $f \neq 0$ s.t. $f(y) = 0 \; \forall \; y \in Y$.

6) if Y is a linear subspace of X, then

$$\overline{Y} = \bigcap \{\ker f : f \in X^*, \; Y \subset \ker f\}$$

Proof. For 1), let $Y = \text{span}\,\{x_i\}_{i=1}^d$ and g be a linear functional on Y with $g(x_i) = \alpha_i$, then g can extend to f.

For 2), let $Y_1 = \text{span}\,\{Y, \{x_0\}\}$ and g be a linear functional on Y_1 with $g(y + \lambda x_0) = \lambda$.

Then since $\|y + \lambda x_0\| = |\lambda| \left\|\frac{1}{\lambda}y + x_0\right\| \geqslant |\lambda|\, d$, $\|g\| \leqslant d^{-1}$. Let $\{y_n\} \in Y$ s.t. $\|x_0 - y_n\| \to d$, then $1 = g(x_0 - y_n) \leqslant \|g\|\, \|x_0 - y_n\| \to \|g\|\, d$.

Thus $\|g\| = d^{-1}$. g can extend to f on X.

For 3), clearly, $\|x\| \geqslant \sup\{|f(x)| : f \in X^*, \;\; \|f\| \leqslant 1\}$. But by above proof, there is a $f(x) = \|x\|$ with $\|f\| = 1$.

4) and 5) can be easily obtained by the separability of continuous linear fuctionals and the fact that X is locally convex.

6) is a special case of **Corollary** 3 in the subsection **1.3.2**. $\qquad\square$

For linear functionals on a general vector space, there is a interesting result.

Proposition 8. *Let f, $\{f_k\}_{k=1}^n$ be linear functionals on a vector space X. If $\bigcap_{k=1}^n \ker f_k \subset \ker f$, then there are scalars $\alpha_1, \cdots, \alpha_n$ such that $f = \sum_{k=1}^n f_k$.*

Proof. Assume for $1 \leqslant k \leqslant n$, $\bigcap_{j \neq k} \ker f_j \neq \bigcap_{j=1}^n \ker f_j$.

Therefore, there is a $y_k \in \bigcap_{j \neq k} \ker f_j$ s.t. $y_k \notin \bigcap_{j=1}^n \ker f_j$, i.e. $f_j(y_k) = 0$ for $j \neq k$ and $f_k(y_k) \neq 0$.

Thus we can find $x_k{}_{k=1}^n$, s.t. $f_k(x_k) = 1$ and $f_j(x_k) = 0$ for $j \neq k$.

Let $\alpha_k = f(x_k)$. For $x \in X$, let $y = x - \sum_{k=1}^n f_k(x)x_k$.

Then $f_j(y) = f_j(x) - \sum_{k=1}^n f_k(x)f_j(x_k) = 0$. Thus $f(y) = 0$,

i.e. $f(x) = \sum_{k=1}^n \alpha_k f_k(x)$. $\qquad\square$

1.4.3 Weak Topologies

In a Banach space X, X^* denoted the set of all bounded linear functionals is also a Banach space with the induced norm. Then we can also find all bounded linear functionals on the space $(X^*, \|\cdot\|)$, denoted by X^{**}. By the Hahn-Banach Theorem, we can easily know the following proposition.

Proposition 1. *Let X be a Banach space. Then X can be isometrically embeded in X^{**}.*

Proof. Define the map ϕ, where $\hat{x}(f) = f(x)$,

$$\phi\colon \quad X \quad \longrightarrow \quad X^{**}$$
$$x \quad \longmapsto \quad \hat{x} = \phi(x)$$

Then by the corollary of Hahn-Banach Theorem, $\|\hat{x}\| = \|x\|$. $\qquad\square$

Remark. In a special case, if $X^{**} = \hat{X}$, where $\hat{X} = \pi(X)$, then X is called a reflexive space.

For a Banach space X and X^*, we can define different topologies.

Definition 2. Let X be a Banach space.

1) The weak topology on X, denoted by wk, is generated by the family of seminorms $\{p_f\colon f \in X^*\}$, where $p_f(x) = |f(x)|$.

2) The weak* topology on X^*, denoted by wk^*, is generated by the family of seminorms $\{p_x\colon x \in X\}$, where $p_x(f) = |f(x)|$.

Remark. Since the $|f(x)| \leqslant \|f\| \|x\|$, we can easily know wk on X and wk^* on X^* are weaker than the norm topology respectively.

1) The subbasis of the weak topology at x_0 is like

$$U_\varepsilon(x_0) = \{x \in X\colon |f(x - x_0)| < \varepsilon\}$$

Therefore, a net $\{x_\alpha\}$ in X converges weakly to x_0 if and only if $f(x_\alpha) \to f(x)$ for all $f \in X^*$.

2) The subbasis of the weak* topology at f_0 is like

$$V_\varepsilon(f_0) = \{f \in X^*\colon |(f - f_0)(x)| < \varepsilon\}$$

Therefore, a net $\{f_\alpha\}$ in X^* converges weakly to f_0 if and only if $f_\alpha(x) \to f(x)$ for all $x \in X$.

There are some easy properties between these two topologies and the respective norm topology.

Proposition 3. *Let X be a Banach space. Then we have*

1) $(X, wk)^* = X^*$

2) $(X^*, wk^*)^* = X$

3) *if* $A \subset X$ *is convex, then* $\overline{A} = \overline{A}^{wk}$

Proof. 1) can be obtained easily by the fact that wk is weaker than the norm topology and by the definition of continuous functionals.

For 2), because of the **Corollary** 3 in the subsction **1.2.4** and the **Proposition** 5 in the subsection **1.2.2** and the **Propostion** 8.

$$F \in (X^*, wk^*)^* \quad \Rightarrow \quad |F(f)| \leqslant \sum_{i=1}^{n} |\hat{x}_k| \text{ for some } \{\hat{x}_k\}_{k=1}^{n} \subset \hat{X}$$

$$\Rightarrow \quad \bigcap_{k=1}^{n} \ker \hat{x}_k \subset \ker F \Rightarrow F = \sum_{k=1}^{n} \alpha_k \hat{x}_k \in \hat{X} \cong X$$

And the converse is because wk^* is weaker than norm topology.

For 3), because wk is weaker than the norm topology, $\overline{A} \subset \overline{A}^{wk}$. By the separability of continuous functionals in **Theorem** 4 in the **1.3.2**, we can fine a $f \in X^*$ s.t. f separates $\overline{A}$ and any $x \in X \backslash \overline{A}$ in using a positive real number α. But $\overline{A} \subset B = \{y \in X : \operatorname{Re} f(y) \leqslant \alpha\}$ and B is clearly wk-closed. Therefore, $\overline{A}^{wk} \subset B$. $x \notin \overline{A}$ implies $x \notin B$, thus $x \notin \overline{A}^{wk}$, i.e. $\overline{A}^{wk} \subset \overline{A}$. $\square$

Also, by the Hahn-Banach Theorem and the **Theorem** 4 in the **1.2.4**, the bounded linear map can be charasterized as the following statement.

Proposition 4. *Let* $T: X \to Y$ *be a linear map between two Banach spaces. Then* T *is bounded if and only if* $T: (X, wk) \to (Y, wk)$ *is continuous.*

By the **Theorem** 3 in the **1.2.3**, we can easily see how to make X^* be wk^*-metrizable.

Theorem 5. *If* X *is a Banach space, then* X^* *is* wk^*-*metrizable if and only if* X *is separable.*

Proof. It is because that wk^* is generated by $\{p_x : x \in X\}$. $\square$

Let Y be a closed subspace a Banach space X. We can define the orthogonal complement of Y, $Y^{\perp} = \{f \in X^* : Y \subset \ker f\}$ (similar definition in X^*). Then we have the following theorem.

Theorem 6. *If* Y *is a closed subspace a Banach space* X *and* $\pi: X \to X/Y$ *is the quotient map, then*

1) the following map is an isometric isomorphism, i.e. $X^*/Y^{\perp} \cong Y^*$.

$$\rho: \quad X^*/Y^{\perp} \quad \longrightarrow \quad Y^*$$
$$f + Y^{\perp} \quad \longmapsto \quad f|_Y$$

2) the the following map is an isometric isomorphism, i.e. $(X/Y)^* \cong Y^\perp$.

$$\kappa: \quad \begin{array}{ccc} (X/Y)^* & \longrightarrow & Y^\perp \\ f & \longmapsto & f \circ \pi \end{array}$$

Proof. For 1), clearly, ρ is linear and injective. Since for any $f \in X^*$ and $g \in Y^\perp$,

$$\|f|_Y\| = \|(f+g)|_Y\| \leqslant \|(f+g)\|$$

we have $\|f|_Y\| \leqslant \|f + Y^\perp\|$. By the Hahn-Banach Theorem, for any $\phi \in Y^*$, there is a $f \in X^*$ with $f|_Y = \phi$ and $\|\phi\| = \|f\| \geqslant \|f + Y^\perp\|$.
For 2), clearly $\|\kappa(f)\| \leqslant \|f\|$. We can find a sequence $x_n + y_n$ with $x_n \in X$ and $y_n \in Y$ and $\|x_n + y_n\| < 1$ s.t.

$$\|\kappa(f)\| \geqslant |\kappa(f)(x_n + y_n)| = f(x_n + Y) \to \|f\|$$

thus κ is an isometry. And by the universal property of quotient map, in the **Propostion** 1 in the subsection **1.1.3**, we can prove that κ is surjective.

$\square$

We have an imprtant theorem to discribe the wk^*-compactness in dual space.

Theorem 7 (Alaoglu's Theorem). *Let X be a normed space, then the unit ball in X^* is wk^*-compact.*

Proof. Let $D_x = \{\alpha \in \mathbb{K} \colon |\alpha| \leqslant 1\}$ for any $x \in X$, then put $D = \prod_{x \in X} D_x$. By Tychonoff's Theorem, D is compact. Let the unit ball in X^* be denoted by B and τ be defined as

$$\tau: \quad \begin{array}{ccc} B & \longrightarrow & D \\ f & \longmapsto & \tau(f) \end{array}$$

where $\tau(f)(x) = f(x)$. Then we can prove that B is homeomorphic to $\tau(B)$ with respect to the induced topology of D and $\tau(B)$ is closed in D. Thus B is compact.

1) Injection: $\tau(f) = \tau(g) \Rightarrow f(x) = g(x) \ \forall \ x \in B \Rightarrow f = g$.

2) Continuity: $f_i \to f \ wk^* \Rightarrow \tau(f_i)(x) \to \tau(f)(x) \ \forall \ x \in B \Rightarrow \tau(f_i) \to \tau(f)$.

3) Closedness: $\tau(f_i) \to f \in D \Rightarrow f_i(x) \to f(x) \; \forall \, x \in B$, then we extend f by defining $\tilde{f}(x) = \alpha^{-1} f(\alpha x)$ for any $x \in X$ and $\alpha > 0$ with $\|\alpha x\| \leqslant 1$. It is well-defined by the linearity. Then $\tilde{f} \in B$. $\tau(B)$ is closed, and $\tau(B)$ is complete.

4) Homeorphism: $\tau \colon B \to \tau(B)$ is continuous linear map between two Banach spaces. Thus it is a homeomorphism.

$\square$

Corollary 8. *Let X be a Banach space and Y is a closed linear subspace of X.*

1) *X is reflexive if and only if the unit ball in X is weakly compact.*

2) *If X is reflexive and $x_0 \in X \backslash Y$, then there is a point $y_0 \in Y$ s.t. $\|x_0 - y_0\| = \inf y - x_0 \colon y \in Y$.*

Proof. For 1), by above theorem, the unit ball in X^{**} is wk^*-compact. Since $X = X^{**}$, the unit ball in X is wk-compact. The converse is similar as above proof.
For 2), Y is wk-compact by above corollary. Since by the Hahn-Bnanch Theorem, for each x, we can find a $f \in X^*$ with $\|f\| = 1$ and $f(x) = x$. The map $x \mapsto \|x - x_0\|$ is weakly lower semicontinuous. Therefore, there is a $y_0 \in Y$ s.t. $\|x_0 - y_0\| = \inf\{y - x_0 \colon y \in Y\}$. $\square$

1.4.4 Adjoint Operators

Definition 1. Let $T \colon X \to Y$ be a bounded linear map between two Banach spaces. Then the adjoint of operator of T is defined as

$$\begin{aligned} T^* \colon \quad Y^* &\longrightarrow X^* \\ f &\longmapsto f \circ T \end{aligned}$$

There are some easy properties of the adjoint operator, which can be obtained by the definition.

Proposition 2. *Let X, Y and Z be three Banach spaces and $T \in \mathscr{B}(X,Y), S \in \mathscr{B}(Y,Z)$. Then, we have*

1) *$T^{**}|_X = T$*

2) *the map $T \to T^*$ is an isometric isomorphism.*

3) *$(ST)^* = T^* S^*$.*

4) *$\ker T^* = (\operatorname{ran} T)^\perp$ and $\ker T = (\operatorname{ran} T^*)^\perp$.*

Then for the 4) in above theorem, we can find dual consequences. Firstly, there is a useful lemma.

Lemma 3. *Let $T\colon X \to Y$ be a bounded linear map between two Banach spaces. Then* $\operatorname{ran} T$ *is closed if and only if there exists a constant $C > 0$, for any $y \in \operatorname{ran} T$ there is a point $x \in X$ s.t. $\|y\| \geqslant C \|x\|$.*

Proof. By the Open Mapping Theorem, for the unit open ball $B \subset X$ and some $\delta > 0$, s.t.

$$\{y \in \operatorname{ran} T\colon \|y\| < \delta\} \subset T(B)$$

Thus for any nonzero $y \in \operatorname{ran} T$,

$$\exists\, z \in B \text{ s.t. } Tz = \frac{\delta}{2\|y\|}y \Rightarrow \|y\| \geqslant \frac{2}{\delta}\|x\| \text{ where } x = \frac{2\|y\|}{\delta}z \text{ with } Tx = y$$

Conversely, if $y \in \overline{\operatorname{ran} T}$, there is $\{y_n\} \subset \operatorname{ran} T$ with $y_n \to y$. By the assumption, there is a constant $C > 0$ and a sequence $\{x_n\} \subset X$ s.t. $\|x_n - x_m\| \leqslant C \|Tx_n - Tx_m\|$. Since $\{y_n\}$ is Cauchy, $\|x_n - x_m\|$ is Cauchy. Thus there exists a $x \in X$ s.t. $x_n \to x$ and $Tx = y$. $\qquad\square$

Remark. There is a more special case for this lemma. $\ker T = \{0\}$ and $\operatorname{ran} T$ is closed if and ony if there exists a constant $C > 0$ s.t. $\|Tx\| \geqslant C \|x\|$ for any $x \in X$.

Theorem 4. *Let $T\colon X \to Y$ be a bounded linear map between two Banach spaces.*

  1) $\overline{\operatorname{ran} T} = (\ker T^*)^{\perp}$.

  2) If $\operatorname{ran} T$ *is closed, then* $\operatorname{ran} T^*$ *is closed and* $\operatorname{ran} T^* = (\ker T)^{\perp}$.

Proof. For 1), let $y \in \overline{\operatorname{ran} T}$ and $y_n \subset \operatorname{ran} T$ with $y_n \to y$. If $g \in \ker T^*$, then

$$g(y_n) = g(Tx_n) = T^*g(x_n) = 0 \to g(y) \text{ i.e. } y \in (\ker T^*)^{\perp}$$

Conversely, if $y_0 \notin \overline{\operatorname{ran} T}$, then by the Hahn-Banach Theorem, there is a $g \in Y^*$ s.t.

$$\operatorname{ran} T \subset \ker g \text{ i.e. } g \in \ker T^* \ \& \ g(y_0) \neq 0 \Rightarrow y_0 \notin (\ker T^*)^{\perp}$$

For 2), if $f \in (\ker T)^{\perp}$ i.e $\ker T \subset \ker f$, then there is a $\tilde{g} \in (\operatorname{ran} T)^*$ s.t. $\tilde{g}(Tx) = f(x)$ since $\operatorname{ran} T \cong X/\ker T$. By above lemma, there exists a constant $C > 0$ s.t. for any $y \in \operatorname{ran} T$ there is a $x \in X$ with $Tx = y$ and $\|x\| \leqslant C \|y\|$. Therefore,

$$|\tilde{g}(y)| = |f(x)| \leqslant C \|f\| \|x\|$$

Thus $\tilde{g}$ can extends to g defining on Y^* s.t.

$$T^*g(x) = g(Tx) = f(x) \ \text{ i.e. } \ T^*g = f$$

Therefore, $(\ker T)^{\perp} \subset \operatorname{ran} T^*$.
Coversely, if $f \in \operatorname{ran} T^*$, i.e. there is a $g \in Y^*$ with $T^*g = f$, then for any $x \in \ker T$, $f(x) = T^*g(x) = g(Tx) = 0$. Therefore, $f \in (\ker T)^{\perp}$. $\qquad\square$

Corollary 5. *Let $T\colon X \to Y$ be a bounded linear map between two Banach spaces. Then T is invertible if and only if T^* is invertible. In this case, $(T^*)^{-1} = (T^{-1})^*$.*

1.5 Hilbert Spaces

A Hilber space is a special Banach space, which is endowed with a inner product. And the structure of inner product can provide better properties of Hilbert spaces.

1.5.1 Projection Theorem and Riesz Theorem

Definition 1. On a vector space $\mathscr{H}$ over $\mathbb{C}$, an inner product is a map $\langle \cdot, \cdot \rangle \colon \mathscr{H} \times \mathscr{H} \to \mathbb{K}$ satisfies that for any $\alpha, \beta \in \mathbb{K}$ and $x, y, z \in \mathscr{H}$,

1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

3) $\langle x, x \rangle \geqslant 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

Remark. The vector space $\mathscr{H}$ with the inner product $\langle \cdot, \cdot \rangle$ is called an inner product space. And the inner product can induce the CBS-Inequality,

$$|\langle x, y \rangle|^2 \leqslant \langle x, x \rangle \langle y, y \rangle$$

by using the positivity of the inner product, $\langle x - \alpha y, x - \alpha y \rangle \geqslant 0$.

Then on the $\mathscr{H}$, a nature norm can be induced by the inner product, defined as $\|x\|^2 = \langle x, x \rangle$ for any $x \in \mathscr{H}$. This definition is valid since the linearity, positivity and CBS-Inequality of the inner product. And there are two forms of the norm.

Proposition 2. *Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be an inner product space, and $\|\cdot\|$ be the coincided with this inner product. Then, we have the following identities.*

Polarization identity:

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x - iy\|^2 - i\|x + iy\|^2)$$

Parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Remark. The first identity can be used to construct an inner product, but not all norm can do this successfully. So, the second identity can be used to check whether a norm can construct an inner product on a normed space.

Definition 3. A Hilbert space is an inner product space, which is complete with respect to the induced norm.

Therefore, a Hilbert space is indeed a Banach space, whose norm can construct an inner product. Now, we can find the extra properties provided by the inner product. The following theorem is the most important property the Hilbert space has.

Theorem 4. *If $\mathscr{H}$ is a Hilbert space and K is a closed convex nonempty subset of $\mathscr{H}$ and $x \in \mathscr{H} \backslash K$, then there exists a unique $k_0 \in K$ s.t.*

$$\|x - k_0\| = \inf \|x - k\| : k \in K$$

Proof. Let $d = \inf \|x - k\| : k \in K$, then there is a sequence $\{k_n\} \subset K$ s.t. $\|x - k_n\| \to d$. By the parallelogram law,

$$4 \left\| x - \frac{1}{2}(k_m + k_n) \right\|^2 + \|k_m - k_n\|^2 = 2(\|x - k_m\|^2 + \|x - k_n\|^2) \to 4d^2$$

Since $\frac{1}{2}(k_m + k_n) \in K$,

$$4 \left\| x - \frac{1}{2}(k_m + k_n) \right\|^2 \geqslant 4d^2$$

Thus $\|k_m - k_n\| \to 0$. By the facts that $\mathscr{H}$ is complete and K is closed, there exists $k_0 \in K$ s.t. $k_n \to k_0$. Then $\|x - k_0\| = \lim_{n \to \infty} \|x - k_n\| = d$. If there is another $k_1 \in K$, $\|x - k_0\| = \|x - k_1\|$.

$$d \leqslant \left\| x - \frac{1}{2}(k_0 + k_1) \right\| \leqslant \frac{1}{2}(\|x - k_0\| + \|x - k_1\|) = 2d$$

Then

$$\left\| x - \frac{1}{2}(k_0 + k_1) \right\| = \frac{1}{2}(\|x - k_0\| + \|x - k_1\|)$$

Thus by the parallelogram law, $k_0 = k_1$. $\square$

Remark. The main method of above proof is the parallelogram law, which is the most eesential property of the norm induced by the inner product.

For $x, y \in \mathscr{H}$, if $\langle x, y \rangle = 0$, we say x is orthogonal to y. Similarly, if W is a subset of $\mathscr{H}$, we define $W^\perp = \{y \in \mathscr{H} : \langle x, y \rangle = 0 \ \forall \ x \in W\}$. Then we can use above theorem to obtain a important structure of a Hilbert space.

Theorem 5 (Projection Theorem). *Let $\mathscr{M}$ be a closed subspace of a Hilbert space H. Then $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$.*

Proof. Clearly, $\mathscr{M} \cap \mathscr{M}^\perp = \{0\}$. We just need to prove that for any $x \in \mathscr{H}$, there exist a unique $x_1 \in \mathscr{M}$ and $x_2 \in \mathscr{M}^\perp$, s.t. $x = x_1 + x_2$. If $x \in \mathscr{M}$, let $x_1 = 0$ and $x_2 = 0$.

Assume $x \notin \mathscr{H}$, then by above theorem, we can find a unique $m_0 \in \mathscr{M}$ s.t. $\|x - m_0\| = \inf \{\|x - m\| : m \in \mathscr{M}\}$. Let $x_1 = m_0$. Then $x = x_1 + (x - m_0)$. Therefore, it is sufficient to prove that $x - m_0 \in \mathscr{M}^\perp$.

For any $m \in \mathscr{M}$ and any $\lambda \in \mathbb{C}$, since $m_0 + \lambda m \in \mathscr{M}$, we have

$$
\begin{aligned}
\|x - m_0\|^2 &\leqslant \|x - m_0 - \lambda m\|^2 \\
&= \|x - m_0\|^2 - 2 \operatorname{Re} \lambda \langle m, x - m_0 \rangle + |\lambda|^2 \|m\|^2
\end{aligned}
$$

Thus $-2 \operatorname{Re} \lambda \langle m, x - m_0 \rangle + |\lambda|^2 \|m\|^2 \geqslant 0$. Then taking the $\lambda = \varepsilon > 0$ and let $\varepsilon \to 0$, therefore

$$
\operatorname{Re} \langle m, x - m_0 \rangle \leqslant 0
$$

Similarly, taking $\lambda = -i\varepsilon$ and let $\varepsilon \to 0$,

$$
\operatorname{Im} \langle m, x - m_0 \rangle \leqslant 0
$$

Also, these are true for $-m$. Therefore, $\langle m, x - m_0 \rangle = 0$ for any $m \in \mathscr{M}$, i.e. $x - m_0 \in \mathscr{M}^\perp$. $\qquad \square$

By the Projection Theorem, we can directly obtained the following corrolary.

Corollary 6. *Let $\mathscr{H}$ be a Hilbert space.*

1) *If $\mathscr{M}$ is a closed subsapce of $\mathscr{H}$, then $(\mathscr{M}^\perp)^\perp = \mathscr{M}$.*

2) *If $A \subset \mathscr{H}$ is a subset, then $(A^\perp)^\perp = \overline{\operatorname{span} A}$.*

Using the Projection Theorem, the dual space of a Hilbert space can be more explicit.

Theorem 7 (Riesz Theorem). *Let $\mathscr{H}$ be a Hilbert space. Then the map σ is defined as*

$$
\begin{aligned}
\sigma: \quad \mathscr{H} &\longrightarrow \mathscr{H}^* \\
x &\longmapsto L_x
\end{aligned}
$$

where $L_x(y) = \langle y, x \rangle$. Then σ is an isometric antilinear bijection, i.e. $\mathscr{H} \cong \mathscr{H}^*$.

Proof. Clearly, L_x is antilinear. And by the CBS-Inequality, $\|L_x\| = \|x\|$, thus σ is definitely an isometry. Then we just need to prove σ is surjective. If $L \in \mathscr{H}^*$ is nonzero, then there is a $x_0 \in (\ker L)^\perp$. Thus we can assume $L(x_0) = 1$. If $y \in \mathscr{H}$, then $y - L(y)x_0 \in \ker L$. Therefore,

$$
0 = \langle y - L(y)x_0, x_0 \rangle = \langle y, x_0 \rangle - L(y) \|x_0\|^2
$$

Then let $x = \|x_0\|^{-2} x_0$, $L_x = L$. $\qquad \square$

Corollary 8. *A Hilbert space $\mathscr{H}$ is relexive. Thus, $\mathscr{H}$ is weakly complete and a subset in $\mathscr{H}$ is weakly compact if and only if it is bounded and weakly closed.*

And the Riesz Theorem can extend to bounded sesquilinear forms.

Definition 9. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces. The map

$$f \colon \mathscr{H}_1 \times \mathscr{H}_2 \to \mathbb{C}$$

is called a sesquilinear form, if it satisfies

1) $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta(y, z)$

2) $f(x, \alpha y + \beta z) = \overline{\alpha} f(x, y) + \overline{\beta}(x, z)$

Remark. If f is continuous, then we know $|f(x, y)| \leqslant C \|x\| \|y\|$ for some $C < 0$. Also, the converse is ture. Then we can define the norm of f as

$$\|f\| = \sup_{x \in \mathscr{H}_1, y \in \mathscr{H}_2} \frac{|f(x, y)|}{\|x\| \|y\|}$$

Theorem 10. *Let $f \colon \mathscr{H}_1 \times \mathscr{H}_2 \to \mathbb{C}$ be a bounded sesquilinear form of two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. Then there is a unique bounded linear map with $\|S\| = \|f\|$*

$$S \colon \mathscr{H}_1 \to \mathscr{H}_2 \ s.t. \ f(x, y) = \langle Sx, y \rangle \ \forall \ x \in \mathscr{H}_1 \ \forall \ y \in \mathscr{H}_2$$

1.5.2 Adjoint Operators

Let $T \colon \mathscr{H} \to \mathscr{H}$ be a bounded linear operator on a Hilbert space $\mathscr{H}$. Since $\mathscr{H}$ is also a Banach space, we can define the adjoint operator of T as

$$
\begin{array}{rccc}
T^* \colon & \mathscr{H}^* & \longrightarrow & \mathscr{H}^* \\
& f & \longmapsto & T^* f
\end{array}
$$

By the Riesz Theorem, each $f \in \mathscr{H}^*$ can be expressed as $f = \langle \cdot, x \rangle$ for some $x \in \mathscr{H}$. Then for any $y \in \mathscr{H}$ we have

$$T^*(\langle \cdot, x \rangle)(y) = (\langle \cdot, x \rangle)(Ty) = \langle Ty, x \rangle$$

Clearly, $f(y) = \langle Ty, x \rangle \in \mathscr{H}^*$, thus there is a unique $\tilde{x} \in \mathscr{H}$ s.t. $f(y) = \langle y, \tilde{x} \rangle$. Then, we have

$$T^*(\langle \cdot, x \rangle)(y) = \langle Ty, x \rangle = \langle y, \tilde{x} \rangle = (\langle \cdot, \tilde{x} \rangle)(y)$$

Thus, we have

$$T^*(\langle \cdot, x \rangle) = \langle \cdot, \tilde{x} \rangle$$

Since $\mathscr{H} \cong \mathscr{H}^*$, in fact $T^* \colon \mathscr{H} \to \mathscr{H}$, and by above mention, $T^*(x) = \tilde{x}$, where $\tilde{x}$ is determined by the equation

$$\langle y, \tilde{x} \rangle = \langle Ty, x \rangle, \quad \forall\, y \in \mathscr{H}$$

Therefore, we can give the definition of the adjoint operator on a Hilbert space.

Definition 1. Let $T \colon \mathscr{H} \to \mathscr{H}$ be a bounded linear operator on a Hilbert space $\mathscr{H}$. Then the adjoint operator $T^* \colon \mathscr{H} \to \mathscr{H}$ is defined as

$$\langle y, T^*x \rangle = \langle Ty, x \rangle \quad \forall\, y \in \mathscr{H}$$

Remark. Equivalently, $\langle T^*x, y \rangle = \langle x, Ty \rangle \; \forall\, y \in \mathscr{H}$. By above mention, this definition is well-defined and nature.

Since this definition is induced by the definition of a Banach space, thus all results in the subsection 1.4.4 also hold for the adjoint operators 0n a Hilbert space, except the $T \to T^*$ is linear. In fact, on a Hilbert space, this map $T \to T^*$ is antilinear, i.e. $((\alpha + \beta)T)^* = \overline{\alpha}T^*T^* + \overline{\beta}T^*$. Moreover, Since any Hilbert space is reflexive, the first result in **Proposition** 2 in the subsection **1.4.4** can be $T^{**} = T$.
And by the **Corollary** 6 of the Projection Theorem and the last result in **Proposition** 2 in the subsection **1.4.4**, we have the following proposition.

Proposition 2. *Let $T \colon \mathscr{H} \to \mathscr{H}$ be a bounded linear operator on a Hilbert space $\mathscr{H}$. Then, we have*

$$\ker T^* = (\operatorname{ran} T)^{\perp} \quad , \quad \ker T = (\operatorname{ran} T^*)^{\perp}$$
$$\overline{\operatorname{ran} T} = (\ker T^*)^{\perp} \quad , \quad \overline{\operatorname{ran} T^*} = (\ker T)^{\perp}$$

And there is a result, which provide another important information of operators on a Hilbert space and make it more interesting than the operators defined on a Banach space.

Theorem 3. *Let $T \in \mathscr{B}(\mathscr{H})$ for a Hilbert space. Then*

$$\|T\| = \|T^*\| = \|T^*T\|^{\frac{1}{2}}$$

Proof. Firstly, for any $h \in \mathscr{H}$ with $\|h\| \leqslant 1$

$$\|Ah\|^2 = \langle Ah, Ah \rangle = \langle A^*Ah, h \rangle \leqslant \|A^*Ah\|\,\|h\| \leqslant \|A^*A\| \leqslant \|A^*\|\,\|A\|$$

Therefore, $\|A\|^2 \leqslant \|A^*A\| \leqslant \|A^*\|\,\|A\|$. and $\|A\| \leqslant \|A^*\|$. But $(A^*)^* = A$, thus this identity can hold. $\qquad \square$

1.5.3 Orthonormal Sets and Schauder Basis

For a Hilbert space $\mathcal{H}$, subset $W \subset \mathcal{H}$ is called orthonormal if elements in W are pariwise orthogonal and each element has norm 1. Then we can find a maximal orthonormal set, which is called a Schauder basis. Since the argebraic basis, Hamel basis, of a Hilbert space may be uncountable, we firstly need to define the uncountable summation.

Definition 1. Let $\{h_i \colon i \in I\}$ be family of elements in $\mathcal{H}$ and $\mathscr{F}$ be a collection of all finite subsets of I. $\mathscr{F}$ can be endowded with the orter by $\subset$, then $\mathscr{F}$ is a directed set. Define the net, where $F \in \mathscr{F}$

$$h_F = \sum \{h_i \colon i \in F\}$$

Therefore, we can define sum of $\{h_i \colon i \in I\}$ as

$$\sum \{h_i \colon i \in I\} = \lim h_F$$

where the limit is with respect to the norm topology on H.

For finite orthonormal set, the results can be obtained by the algebraic structure. And if the orthonormal set is infinite, we have similar results.

Theorem 2 (Bessel's Inequality). *If $\{e_n\}_{n=1}^\infty$ is an orthonormal set of a Hilbert space and $h \in \mathcal{H}$, then*

$$\sum_{n=1}^\infty |\langle h, e_n \rangle|^2 \leqslant \|h\|^2$$

Proof. Let $h_n = h - \sum_{k=1}^n \langle h, e_k \rangle e_k$. Then h_n is clearly orthogonal to $\{e_k\}_{k=1}^n$.

$$
\begin{aligned}
\|h\|^2 &= \|h_n\|^2 + \left\| \sum_{k=1}^n \langle h, e_k \rangle e_k \right\|^2 \\
&= \|h_n\|^2 + \sum_{k=1}^n |\langle h, e_k \rangle|^2 \\
&\geqslant \sum_{k=1}^n |\langle h, e_k \rangle|^2
\end{aligned}
$$

$\square$

Corollary 3. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{E}$ be an orthonormal subset.*

1) *For any $h \in \mathcal{H}$, $|\langle h, e \rangle| \neq 0$ for at most a countable many $e \in \mathscr{E}$.*

2) *$\sum_{e \in \mathscr{E}} |\langle h, e \rangle|^2 \leqslant \|h\|^2$.*

3) $\sum\{\langle h, e\rangle e\colon e \in \mathscr{E}\}$ *is converges.*

Proof. For 1), let $\mathscr{E}_n = \{e \in \mathscr{E}\colon |\langle h, e\rangle| \geqslant \frac{1}{n}\}$, then $\mathscr{E}_n$ has to be finite. But

$$\bigcup_{n=1}^{\infty} \mathscr{E}_n = \{e \in \mathscr{E}\colon |\langle h, e\rangle| \neq 0\}$$

For 2), it is clearly by the Bessel's Inequality and above corollary.

For 3), it is because the net $h_F = \sum\{h_i \colon i \in F\}$, where $F \subset \mathscr{E}$ is finite, is Cauchy by using the fact that for any $\varepsilon > 0$ there is a $N \in \mathbb{N}$ s.t.

$$\sum_{n=N}^{\infty} |\langle h, e_n\rangle| < \varepsilon$$

$\square$

Definition 4 (Schauder basis). A orthonormal set $\mathscr{E}$ of a Hilbert space $\mathscr{H}$ is called a Schauder basis if for any $h \in \mathscr{H}$,

$$h = \sum\{\langle h, e\rangle e\colon e \in \mathscr{E}\}$$

Remark. By above corollary, this definition is equivalent to that for any $h \in \mathscr{H}$, there exists $\{e_n\}_{n=1}^{\infty} \subset \mathscr{E}$, s.t.

$$h = \sum_{n=1}^{\infty} \langle h, e_n\rangle e_n$$

Moreover, by the Zorn's Lemma, every Hilbert space has a Schauder basis.

By the definition, and above theorem and corollaries, we can find the properties of the Schauder basis.

Theorem 5. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{E}$ be an orthonormal subset. Then the following statements are equivalent.*

1) $\mathscr{E}$ is a Schauder basis.

2) $h \in \mathscr{H}$ with $h \perp \mathscr{E}$, then $h = 0$.

3) $g, h \in \mathscr{H}$, then

$$\langle g, h\rangle = \sum\{\langle g, e\rangle\langle e, h\rangle\colon e \in \mathscr{E}\}$$

4) (Parseval's Identity)$h \in \mathscr{H}$, then

$$\|h\|^2 = \sum\{|\langle h, e\rangle|^2 \colon e \in \mathscr{E}\}$$

Proposition 6. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{E}$ be an orthonormal subset.*

1) $\mathscr{H}$ is separable if and only if $\mathscr{E}$ is countable.

2) Any separable Hilbert space is isometrically isomorphic to $l^2(\mathbb{C})$.

1.5.4 Unitaries and Projections

If we can find a bijection, which is continuous and the inverse of which is also continuous, between two topological spaces, these two topological spaces are regarded as same. Also, if there is a bijection, which is linear between two linear spaces, then we say these two linear spaces are same. Now, we also want to use a special map to classify Hilbert spaces. We know the Hilbert space has two structures, linear structure and the inner product structure. So following definition is nature.

Definition 1. Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert spaces. If there is a linear isomorphism U from $\mathscr{H}$ to $\mathscr{K}$, s.t $\langle Ux, Uy \rangle = \langle x, y \rangle$, then U is called a unitary.

Remark. In fact, if there exists a unitary U between $\mathscr{H}$ and $\mathscr{K}$, U preserves not only the linear structure, but also the inner product structure of Hilbert spaces. Therefore, these two Hilbert spaces are regarded as same. Moreover, if the U is not assumed bijection, then U is called an isometry.

As above definition, the following proposition is clearly.

Proposition 2. *If $U \in \mathscr{B}(\mathscr{H})$, then the following statements are equivalent.*

1) *U is a unitary.*

2) *$U^*U = UU^* = I$.*

3) *U is a normal isometry.*

Remark. $A \in \mathscr{B}(\mathscr{H})$ is called normal, if $A^*A = AA^*$.

Now, the another important operator is projection. Like its name showing, the projection can project the elements in a Hilbert space to a corresponding subspace, and because of the Projection Theorem, we can make the definition be well-defined.

Definition 3. Let $\mathscr{M}$ be a any closed subspace of $\mathscr{H}$. Then by the Projection Theorem, $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$, i.e. for any $x \in \mathscr{H}$, there exist un $x_{//} \in \mathscr{M}$ and $x_\perp \in \mathscr{M}^\perp$ s.t.$x = x_{\shortparallel} + x_\perp$. Therefor we can define a map $P\colon \mathscr{H} \to \mathscr{M}$ as $Px = x_{\shortparallel}$. P is called a projection.

Remark. P is a linear operator can be easily proved.

There is an equivalent definition of the projection. Firstly, $A \in \mathscr{B}(\mathscr{H})$ is called self-adjoint if $A^* = A$.

Theorem 4. *Let $\mathscr{H}$ be a Hilbert space. $P \in \mathscr{B}(\mathscr{H})$ is a projection if and only if P is self-adjoint and $P^2 = P$.*

Proof. Let $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$. If P is a projection to $\mathscr{M}$, $P^2 = P$ is clearly true.

$$
\begin{aligned}
\langle P(x = x_{\shortparallel} + x_\perp), y = y_{\shortparallel} + y_\perp \rangle &= \langle x_{\shortparallel}, y_{\shortparallel} \rangle \\
\langle P^*(x = x_{\shortparallel} + x_\perp), y = y_{\shortparallel} + y_\perp \rangle &= \langle x = x_{\shortparallel} + x_\perp, P(y = y_{\shortparallel} + y_\perp) \\
&= \langle x_{\shortparallel}, y_{\shortparallel} \rangle
\end{aligned}
$$

Therefore, P is self-adjoint.

Conversely, $\mathscr{M} = P(\mathscr{H})$ is clearly a subspace. And for any $y \in \mathscr{M}$, let $x = y$, we have $Px = y$. Thus by the **Lemma 3** in the subsection **1.4.4**, $\mathscr{M}$ is closed. Since P is sefl-adjoint,

$$
\langle x - Px, Px \rangle = \langle Px - P^2 x, x \rangle = 0
$$

That means $x - Px \in \mathscr{M}^\perp$. Therefore, $x = Px + (x - Px)$ is the unique decomposition of x with respect to $\mathscr{M}$. P is a projection. $\square$

Remark. By the fact that $P^2 = P$ and the **Theorem 3** in the subsection **1.5.2**, $\|P\| = 1$. Therefore, P is bounded.

The one of the importance of projections is that any closed subspace of a Hilbert space can be regarded as a operator on the Hilbert space. And then this can provide us an effective method to research the invariant space.

Definition 5. For $T \in \mathscr{B}(\mathscr{H})$ and a closed subspace $\mathscr{M}$ of $\mathscr{H}$, if $T(\mathscr{M}) \subset \mathscr{M}$, $\mathscr{M}$ is called an invariant space for T. If $T(\mathscr{M}) \subset \mathscr{M}$ and $T(\mathscr{M}^\perp) \subset \mathscr{M}^\perp$, then $\mathscr{M}$ is called a reducing space for T.

Proposition 6. *Let $\mathscr{M}$ be a closed subspace of a Hilbert space $\mathscr{H}$ and P be the projection to $\mathscr{M}$ and $T \in \mathscr{B}(\mathscr{H})$. T acts on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$ can be expressed as*

$$
T = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}
$$

Then the following statements are equivalent.

1) $\mathscr{M}$ is invariant for T.

2) $PTP = TP$.

3) $Y = 0$.

Proof. 1) $\Rightarrow$ 2) is clearly.

2) $\Rightarrow$ 3): The P can be expressed as

$$
P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}
$$

Then by $PTP = TP$, we can find $Y = 0$.

3) $\Rightarrow$ 1) is also clearly. $\qquad\qquad$ $\square$

Proposition 7. *Let $\mathscr{M}, \mathscr{H}$ and T, P be defined as above proposition. Then the following statements are equivalent.*

1) *$\mathscr{M}$ reduces $\mathscr{H}$.*

2) *$PT = TP$.*

3) *X and Y are 0.*

4) *$\mathscr{M}$ is invariant for T and T^*.*

Proof. 1) $\Rightarrow$ 2): By above proposition, $PTP = TP$ and $(I - P)T(I - P) = T(I - P)$, thus we have $PT = TP$.

2) $\Rightarrow$ 3): Expressing P as above proposition, then by $PT = TP$, we find X and Y are 0.

3) $\Rightarrow$ 4): Since X and Y are 0, then

$$T^* = \begin{pmatrix} W^* & 0 \\ 0 & Z^* \end{pmatrix}$$

Clearly, $\mathscr{M}$ is invariant for T and T^*.

4) $\Rightarrow$ 1): If $h \in \mathscr{M}^{\perp}$ and $g \in \mathscr{M}$, $\langle g, Ah \rangle = \langle A^* g, h \rangle = 0$ since $A^* g \in \mathscr{M}$. Therefore, $T(\mathscr{M}^{\perp}) \subset \mathscr{M}^{\perp}$. $\qquad$ $\square$

Chapter 2

Banach Algebras and C^*-Algebras

2.1 Banach Algebras

Let $\mathscr{H}$ be a Hilbert space. Then $\mathscr{B}(\mathscr{H})$ is indeed a Banach space. But we have more sructure on it. Any two element $S, T \in \mathscr{B}(\mathscr{H})$ can do multiplication, defined as $ST = S \circ T$, then $ST \in \mathscr{B}(\mathscr{H})$ by the definition of norm, and moreover $\|ST\| \leqslant \|S\| \|T\|$. Therefore, $\mathscr{B}(\mathscr{H})$ is an algebra with the extra property of the multiplication, called a Banach algebra.

2.1.1 Elementary Properties

Definition 1. A Banach algebra $\mathfrak{A}$ is an algebra over $\mathbb{C}$ with a norm $\|\cdot\|$ relative to which $\mathfrak{A}$ is a Banach space and s.t. for all $a, b \in \mathfrak{A}$,

$$\|ab\| \leqslant \|a\| \|b\|.$$

Remark. The extra condition guarantees the multiplication is norm continuous. In fact, $\mathfrak{A}$ with the multiplication and the norm is a topological semigroup.

If $\mathfrak{A}$ has an identity 1, it assumes $\|1\| = 1$. But if $\mathfrak{A}$ does not have an identity, we can add an identity to it.

Proposition 2. *If $\mathfrak{A}$ is a Banach algebra without the identity. The let $\tilde{\mathfrak{A}} = \mathfrak{A} \oplus \mathbb{C}$ be an induced vector space. Then we define a norm on $\tilde{\mathfrak{A}}$ as*

$$\|(a, \lambda)\| = \|a\| + |\lambda|$$

and define a multiplication on $\tilde{\mathfrak{A}}$ as

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$$

Then $\mathfrak{A}$ is a Banach algebra with the identity $(0, 1)$.

Then $\mathfrak{A}$ can be isometrically imbedded into $\tilde{\mathfrak{A}}$. Therefore, we can always assume $\mathfrak{A}$ has an identity. For a unit algebra, the invertibility of a element is important.

Theorem 3. *Let $\mathfrak{A}$ be a Banach algebra and $a \in \mathfrak{A}$. If $\|a - 1\| < 1$, then a is invertible.*

Proof. For a nonzero real number $\lambda \in \mathbb{R}$ with $|\lambda| < 1$, we know

$$(1 - \lambda)^{-1} = \sum_{n=0}^{\infty} \lambda^n$$

Similar, for $a \in \mathfrak{A}$ with $\|a - 1\| < 1$, set

$$b = \sum_{n=0}^{\infty} (1 - a)^n$$

Firstly, since $\|(1 - a)^n\| \leqslant \|(1 - a)\|^n$, b is well-defined. Then we can prove $b = (1 - (1 - a))^{-1} = a^{-1}$. $\qquad\square$

Remark. This result is important. It says a small perturbation of an invertible element is also invertible. It is because that the multiplication is norm continuous. And by the continuity of multiplication, this result can be true at any point other than 1.

Corollary 4. *Let $\mathfrak{A}$ be a Banach algebra and*

$$
\begin{aligned}
G_l &= \{ a \in \mathfrak{A} \colon a \text{ is left invertible } \} \\
G_r &= \{ a \in \mathfrak{A} \colon a \text{ is right invertible } \} \\
G &= G_l \bigcap G_r = \{ a \in \mathfrak{A} \colon a \text{ is invertible } \}
\end{aligned}
$$

Then G_l and G_r and G are open. Moreover, the map $a \to a^{-1}$ from G to G is continuous.

Proof. G_l and G_r and G are open by above theorem.
We just need to check this map is continuous at 1 because of the continuity of multiplication. For $\{a_n\} \subset G$ with $a_n \to 1$, thus $\|1 - a_n\| < \delta < 1$. Since

$$a_n^{-1} = (1 - (1 - a_n))^{-1} = \sum_{k=0}^{\infty} (1 - a_n)^k = 1 + \sum_{k=1}^{\infty} (1 - a_n)^k$$

Therefore, we have

$$
\begin{aligned}
\|1 - a_n^{-1}\| &= \left\| \sum_{k=1}^{\infty} (1 - a_n)^k \right\| \\
&\leqslant \sum_{k=1}^{\infty} \|1 - a_n\|^k \\
&< \frac{\delta}{1 + \delta} < \delta = \|1 - a_n\|
\end{aligned}
$$

i.e. $\lim a_n = 1$. □

Corollary 5. *Let $\mathfrak{A}$ be a Banach algebra.*

   1) The closure of a proper ideal is a proper ideal.

   2) A maximal ideal is closed.

   3) every ideal contained in a maximal ideal.

If $\mathfrak{B}$ is a closed ideal of a Banach algebra $\mathfrak{A}$, then then quotient algbra $\mathfrak{A}/\mathfrak{B}$ with the induced norm is also a Banach algebra since

$$\|(a+\mathfrak{B})(b+\mathfrak{B})\| = \|ab+\mathfrak{B}\| \leqslant \|(a+b_1)(b+b_2)\| \leqslant \|(a+b_1)\|\,\|(b+b_2)\|$$

for any $b_1, b_2 \in \mathfrak{B}$.

2.1.2 Spectrum

Definition 1. Let $\mathfrak{A}$ be a Banach algebra and $a \in \mathfrak{A}$. The spectrum of a, denoted by $\sigma(a)$ defined as

$$\sigma(a) = \{\ \lambda \in \mathbb{C} \colon a - \lambda \text{ is invertible }\}$$

And the resolvents of a, $\rho(a) = \mathbb{C}\backslash\sigma(a)$.
Moreover, we can define the spectral radius of a as

$$r(a) = \sup\{\ |\lambda| : \lambda \in \sigma(a)\ \}$$

Firstly, there are some elementary properties of the spectrum.

Theorem 2. *Let $\mathfrak{A}$ be a Banach algebra and $a \in \mathfrak{A}$.*

   1) If $|\lambda| > \|a\|$, then $\lambda \notin \sigma(a)$.

   2) $\sigma(a)$ is a compact subset of $\mathbb{C}$.

   3) the map $\lambda \mapsto (a-\lambda)^{-1}$ from $\rho(a)$ to $\mathfrak{A}$ is analytic and $\sigma(a)$ is nonempty.

   4) $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$.

Proof. 1) holds by above theorem.
For 2), since $\lambda \to a - \lambda$ is continuous from $\mathbb{C}$ to $\mathfrak{A}$ and G is open, $\rho(a)$ is open i.e. $\sigma(a) = \mathbb{C}\backslash\rho(a)$ is closed. Then by 1), $\sigma(a)$ is compact.
For 3), by the identity $a^{-1} - b^{-1} = a^{-1}(b-a)b^{-1}$ and the continuity of $a \to a^{-1}$, we can compute the derivative of $F(\lambda) = (a-\lambda)^{-1}$,

$$F^{'}(\lambda) = (a-\lambda)^{-2}$$

And clearly, $F'(\lambda)$ is continuous. Thus it is analytic and it vanishes at ∞. By the Liouville's Theorem, if $\rho(a) = \mathbb{C}$, F is constant. Therefore, $\rho(a) \neq \mathbb{C}$ i.e. $\sigma(a) \neq \varnothing$.

For 4), let $U = \{\lambda \in \mathbb{C} \colon \lambda = 0 \text{ or } \lambda^{-1} \in \rho(a)\}$ and

$$f(\lambda) = \begin{cases} (\lambda^{-1} - a)^{-1} & x \neq 0, \\ 0, & x = 0 \end{cases}$$

Then f is analytic on U, i.e $f(\lambda) = \lambda \sum_{n=0}^{\infty} \lambda^n a^n$ is well-defined. Therefore, the convergent radius $R = r(a)^{-1}$

$$R^{-1} = \limsup_{n \to \infty} \|a^n\|^{\frac{1}{n}} = r(a)$$

Conversely, by the identity $(a^n - \lambda^n) = (a - \lambda)(a^{n-1} + \lambda a^{n-2} + \lambda^2 a^{n-3} + \cdots + \lambda^{n-1})$. Then, if $(a^n - \lambda^n)$ is invertible, then $(a - \lambda)$ is invertible, i.e. $\sigma(a) \subset \sigma(a^n)$. Thus $|\lambda|^n \leqslant \|a^n\|$ for any $\lambda \in \sigma a$. $r(a) = \liminf_{n \to \infty} \|a^n\|^{\frac{1}{n}}$. Therefore, $r(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}$. $\qquad \square$

If $\mathfrak{B} \subset \mathfrak{A}$ is a subalgebra with the same identity of a Banach algebra $\mathfrak{A}$, then we know for any element $b \in \mathfrak{B}$, $\sigma_{\mathfrak{A}}(b) \subset \sigma_{\mathfrak{B}}(b)$. Then we can have more results other than it. Since the spectrum is a subset of $\mathbb{C}$, we need some topological properties results of $\mathbb{C}$.

Lemma 3. *If K is any compact subset of $\mathbb{C}$, then $\mathbb{C} \backslash K$ has a countable components, only one of which is unbounded. And the boundary of each component is in K.*

Proof. Let $\tilde{K} = \mathbb{C} \backslash K$, then $\tilde{K}$ is open.

Firstly, the connected component of open set in $\mathbb{C}$ is open.
Let U be an connected component in $\tilde{K}$ and $x \in U$. For any point $x \in U$, Since any open neighbourhood of x is connected, and K is open, there is a open neighbourhood V of x s.t. $V \subset U$.

Secondly, $\mathbb{C}$ has just at most countable many open sets, which are pairwise disjoint.
This result is because any open set in $\mathbb{C}$ contains a rational point.

For any two disjoint open sets A and B in $\mathbb{C}$, $\partial A \bigcap B = \varnothing$. Thus the boundary of some compoment of $\tilde{K}$ can not be contained in any component of $\tilde{K}$, i.e. it is contained K.

Finally, since K is bounded, there is a closed ball B containing K. But the complement of B is connected, thus there is only one component of $\tilde{K}$ containing B. Thus the other components of $\tilde{K}$ are bounded. $\qquad \square$

Remark. The bounded component of $\mathbb{C} \backslash K$ is called a hole of K.

Definition 4. If $f\colon A \to \mathbb{C}$, where A is a set, then the norm of f on A is defined as

$$\|f\|_A = \sup \{|f(x)| : x \in A\}$$

For a compact set $K \in \mathbb{C}$, the polynomially convex hull of K is defined as

$$\hat{K} = \{\, z \in \mathbb{C} \colon |p(z)| \leqslant \|p\|_K \text{ for any polynomial } p \,\}$$

If $K = \hat{K}$, K is called polynomially convex.

Proposition 5. *Let K be a compact subset of $\mathbb{C}$. Then $\mathbb{C}\backslash\hat{K}$ is the unbounded component of $\mathbb{C}\backslash K$. Therefore, K is polynomially convex if and only if $\mathbb{C}\backslash K$ is connected.*

Proof. Let L be the set containing K and all bounded component of $\mathbb{C}\backslash K$. Then by the Maximal Principle, $L \subset \hat{K}$. Conversely, if $\alpha \notin L$, then $(z - \alpha)^{-1}$ is analytic in a neighbourhood of L. Therefore, there is a sequence of polynomials $\{p_n\}$ s.t. $p_n \to (z - \alpha)^{-1}$. Let $q_n = (z - \alpha)p_n$. Then $q_n \to 1$, i.e. $\|q_n - 1\| < \frac{1}{2}$ for some n. But $|q_n(\alpha) - 1| = 1$, this implies $\alpha \notin \hat{K}$, i.e. $\hat{K} \subset L$. $\qquad\square$

By above results, now we can provide the relationships betweem $\sigma_{\mathfrak{A}}(b)$ and $\sigma_{\mathfrak{B}}(b)$.

Theorem 6. *If $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras with same identity s.t. $\mathfrak{B} \subset \mathfrak{A}$ and $b \in \mathfrak{B}$, then*

1) $\sigma_{\mathfrak{A}}(b) \subset \sigma_{\mathfrak{B}}(b)$ and $\partial\sigma_{\mathfrak{B}}(b) \subset \partial\sigma_{\mathfrak{A}}(b)$

2) $\hat{\sigma_{\mathfrak{A}}}(b) = \hat{\sigma_{\mathfrak{B}}}(b)$

3) if G is a hole of $\sigma_{\mathfrak{A}}(b)$, then $G \subset \sigma_{\mathfrak{B}}(b)$ or $G \bigcap \sigma_{\mathfrak{B}}(b) = \varnothing$

Proof.

For 1), let $\lambda \in \partial\sigma_{\mathfrak{B}}(b)$. Since $\overset{\circ}{\sigma_{\mathfrak{A}}}(b) \subset \overset{\circ}{\sigma_{\mathfrak{B}}}(b)$, it is sufficient to show $\lambda \in \sigma_{\mathfrak{A}}(b)$. Suppose $\lambda \notin \sigma_{\mathfrak{A}}(b)$, i.e. $(b - \lambda)$ is invertible in $\mathfrak{A}$. But since $\lambda \in \partial\sigma_{\mathfrak{B}}(b)$, there are $\lambda_n \in \mathbb{C}\backslash\mathfrak{B}$ with $\lambda_n \to \lambda$. Thus $(b - \lambda_n)^{-1} \in \mathfrak{B}$. But $(b - \lambda_n)^{-1} \to (b - \lambda)^{-1} \in \sigma_{\mathfrak{B}}(b)$, contradicting to $\lambda \in \sigma_{\mathfrak{A}}(b)$.

2) holds because of the result of 1) and the Maxiamal Principle.

For 3), let $G_1 = G \bigcap \sigma_{\mathfrak{B}}(b)$ and $G_2 = G\backslash\sigma_{\mathfrak{B}}(b)$. Since $\partial\sigma_{\mathfrak{B}}(b) \subset \sigma_{\mathfrak{A}}(b)$ and $G \bigcap \sigma_{\mathfrak{A}}(b) = \varnothing$, $G_1 = G \bigcap \overset{\circ}{\sigma_{\mathfrak{B}}}(b)$ is open. By the facts that G_2 is clearly open and $G = G_1 \bigcup G_2$ and $G_1 \bigcap G_2 = \varnothing$, either G_1 or G_2 is empty. $\qquad\square$

Then we can have some useful corollaries.

Corollary 7. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras with same identity s.t. $\mathfrak{B} \subset \mathfrak{A}$ and $b \in \mathfrak{B}$.*

1) *If $\sigma_{\mathfrak{A}}(b)$ has no holes, then $\sigma_{\mathfrak{A}}(b) = \sigma_{\mathfrak{B}}(b)$.*

2) *If $\sigma_{\mathfrak{B}}(b) \subset \mathbb{R}$, then $\sigma_{\mathfrak{A}}(b) = \sigma_{\mathfrak{B}}(b)$.*

3) *$\sigma_{\mathfrak{A}}(b) = \sigma_{\mathfrak{B}}(b)$ if and only if $\rho_{\mathfrak{A}}(b)$ is connected.*

Proof. 1) is clearly true since ubbouded component does not intersect $\sigma_{\mathfrak{B}}(b)$. 2) is because $\mathbb{C} \backslash \sigma_{\mathfrak{A}}(b)$ has no holes. 3) is similar as 2). $\qquad\square$

2.1.3 Riesz Functional Calculus

For any polynomial p with complex coefficients,

$$p(z) = \sum_{k=0}^{n} \alpha_k z^k$$

we can define $p(a)$ for some $a \in \mathfrak{A}$, where $\mathfrak{A}$ is a Banach algebra

$$p(a) = \sum_{k=0}^{n} \alpha_k a^k$$

Clearly, $p(a)$ is well-defined. But we can do more. If f is an analytic funcion on $A \subset \mathbb{C}$, then f can be approximated by a sequence of polynomials

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

Similarly, we can define $f(a)$ for $a \in \mathfrak{A}$ as

$$f(a) = \sum_{n=0}^{\infty} \alpha_n a^n$$

If the radius of convergence of this sequence is R, then it can be well-defined for $\|a\| \leqslant R$. By the fact that $r(a) \leqslant \|a\|$, for the analytic function f, if $\sigma(a) \subset A$, $f(a)$ can be well-defined.

Let $\text{Hol}(a)$ denote all functions that are analytic in a neighbourhood of $\sigma(a)$. Then there is a map from $\text{Hol}(a)$ to $\mathfrak{A}$ defined as $f \mapsto f(a)$. Now, we can find more properties of this map. Firstly, we can give another formula of $f(a)$.

If $f \in \text{Hol}(a)$, then for any closed curve γ which encloses $\text{Hol}(a)$ and any point $z_0 \in \text{Hol}(a)$,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z - z_0)^{-1} dz$$

Therefore, replacing z_0 by $a \in \mathfrak{A}$, then we have

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z - a)^{-1} dz$$

Clearly, by the Cauchy's Integral Formula, this definition is well-defined and is coincided with above definition. But this definition can provide us a conivient method to research the map $f \mapsto f(a)$.

Theorem 1 (Riesz Functional Calculus). *Let $\mathfrak{A}$ be a Banach algebra and $a \in \mathfrak{A}$ and the map*

$$\rho: \quad \begin{aligned} \mathrm{Hol}\,(a) &\longrightarrow \quad \mathfrak{A} \\ f &\longmapsto \quad f(a) = \tfrac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1}dz \end{aligned}$$

has the following properties.

1) *ρ is an algebra homomorphism.*

2) *$\rho(1) = 1$ and $\rho(z) = a$.*

3) *If $\{f_n\} \subset \mathrm{Hol}\,(a)$ and $f \in \mathfrak{A}$ with $f_n \to f$ uniformly on a compact set of $\mathrm{Hol}\,(a)$, then $\rho(f_n) \to \rho(f)$ in norm.*

Moreover, if any map $\tau \colon \mathrm{Hol}\,(a) \to \mathfrak{A}$ satifies above conditions, then $\tau = \rho$.

Proof. For 1), ρ is clearly linear. And

$$
\begin{aligned}
f(a)g(a) &= -\frac{1}{4\pi^2} \int_{\gamma_1} f(z)(z-a)^{-1}dz \int_{\gamma_2} g(\zeta)(\zeta-a)^{-1}d\zeta \\
&= -\frac{1}{4\pi^2} \int_{\gamma_1} \int_{\gamma_2} f(z)g(\zeta) \frac{(z-a)^{-1} - (\zeta-a)^{-1}}{\zeta - z} d\zeta dz \\
&= -\frac{1}{4\pi^2} \int_{\gamma_1} f(z) \int_{\gamma_2} \frac{g(\zeta)}{\zeta - z} d\zeta (z-a)^{-1}dz \\
&\quad + \frac{1}{4\pi^2} \int_{\gamma_2} g(\zeta) \int_{\gamma_1} \frac{f(z)}{\zeta - z} dz (\zeta-a)^{-1}d\zeta
\end{aligned}
$$

We can choose γ_2 to enclose γ_1, thus

$$\int_{\gamma_1} \frac{f(z)}{\zeta - z}dz = 0, \quad \int_{\gamma_2} \frac{g(\zeta)}{\zeta - z}d\zeta] = 2\pi i g(z)$$

Therefore,

$$f(a)g(a) = 2\pi i \int_{\gamma_1} f(z)g(z)(z-a)^{-1}dz = (fg)(a)$$

For 2), let $f(z) = z^k$ and $\gamma = Re^{2\pi it}$, where $R > \|a\|$ and $t \in [0, 1]$, then

$$
\begin{aligned}
f(a) &= \frac{1}{2\pi i} \int_\gamma z^k (z-a)^{-1} dz \\
&= \frac{1}{2\pi i} \int_\gamma z^{k-1} (1 - \frac{a}{z})^{-1} dz \\
&= \frac{1}{2\pi i} \int_\gamma z^{k-1} \sum_{n=0}^{\infty} \frac{a^n}{z^n} dz \\
&= \sum_{n=0}^{\infty} (\frac{1}{2\pi i} \int_\gamma \frac{1}{z^{n-k+1}}) a^n \\
&= a^k
\end{aligned}
$$

For 3),

$$
\begin{aligned}
&\left\| \int_\gamma f_n(z)(z-a)^{-1} dz - \int_\gamma f(z)(z-a)^{-1} dz \right\| \\
&= \left\| \int_0^1 (f_n(\gamma(t)) - f(\gamma(t)))(\gamma(t) - a)^{-1} d\gamma(t) \right\| \\
&\leqslant \int_0^1 |f_n(\gamma(t)) - f(\gamma(t))| \left\| (\gamma(t) - a)^{-1} \right\| d|\gamma|(t) \\
&\leqslant M \|\gamma\| \sup\{|f_n(z) - f(z)| : z \in \gamma(t)\}
\end{aligned}
$$

where M is the bound of $\left\| (\gamma(t) - a)^{-1} \right\|$ since $t \mapsto \left\| (\gamma(t) - a)^{-1} \right\|$ is continuous on $\gamma(t)$. Therefore, by the fact that $f_n \to f$ uniformly,

$$
\|f_n(a) - f(a)\| \to 0
$$

Finally, the uniquness is because any $f \in \text{Hol}(a)$ can be approximated uniformly by a sequence of polynomials. Thus, 1) and 2) means $\tau(p) = \rho(p)$ for any polynomial p, and 3) provides the fact that $\tau(f) = \rho(f)$ for any $f \in \text{Hol}(a)$. $\qquad \square$

Remark. we have mentioned that the integral definition is coincided with the convergent difinition. In fact, by 2), this statement can be proved rigorously.

Theorem 2 (Spectral Mapping Theorem). *If $a \in \mathfrak{A}$ and $f \in \text{Hol}(a)$, then*

$$
\sigma(f(a)) = f(\sigma(a))
$$

Proof. Firstly, there is a $g \in \text{Hol}(a)$ s.t. for $\alpha \in \sigma(a)$, $f(z) - f(\alpha) = (z - \alpha)g(z)$, that means $f(\sigma(a)) \subset \sigma(f(a))$.

Conversely, if $\alpha \notin f(\sigma(a))$, $g(z) = (f(z) - \alpha)^{-1} \in \text{Hol}(a)$. Thus, $g(a)(f(a) - \alpha) = 1$. Therefore, $\alpha \notin \sigma(f(a))$. $\qquad \square$

Proposition 3. *Let $\mathfrak{A}$ be a Banach algebra and $a \in \mathfrak{A}$. $\sigma(a) = F_1 \bigcup F_2$, where F_1 and F_2 are disjoint nonempty closed sets. Then there is a nontrial idempotent e, i.e. $e^2 = e$, s.t.*

1) if $ab = ba$, then $eb = be$.

2) if $a_1 = ae$ and $a_2 = a(1 - e)$, then $a_1 a_2 = a_2 a_1 = 0$.

3) $\sigma(a_1) = F_1 \bigcup \{0\}$ and $\sigma(a_2) = F_2 \bigcup \{0\}$.

Proof. Since F_1 and F_2 are disjoint closed set, there are two disjoint open sets G_1 and G_2 separating F_1 and F_2. Let f be the characteristic function of G_1 and $e = f(a)$. Thus $e^2 = e$ by $f^2 = f$.

For 1), there is a more genera result, $f(a)b = bf(a)$ for any $f \in \text{Hol}\,(a)$. It is because by extending the fact $p(a)b = bp(a)$ for any polynomial p.

2) is clearly true.

Let $f_1(z) = zf(z)$ and $f_2(z) = z(1 - f(z))$. Then $a_j = f_j(a)$ for $j = 1, 2$. Then by the Spectral Mapping Theorem $\sigma(a_j) = f_j(\sigma(a_j)) = F_j \bigcup \{0\}$. $\quad\square$

Corollary 4. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. If $\sigma(T) = F_1 \cup F_2$, where F_1 and F_2 are disjoint nonempty closed sets, then there are closed subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ with $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, s.t.*

1) $B\mathscr{H}_j \subset \mathscr{H}_j$ for $j = 1, 2$, whenever $BT = TB$.

2) If $T_j = T|_{\mathscr{H}_j} : \mathscr{H}_j \to \mathscr{H}_j$, then $\sigma(T_j) = F_j$, for $j = 1, 2$.

Proof. From above proposition, e is actually the propjection from $\mathscr{H}$ to $\mathscr{H}_1$. Therefore by the result in above proposition, 1) clearly holds.

For 2), since e is in fact the indentity from $\mathscr{H}_1$ to $\mathscr{H}_1$, similarly, by the Spectral Mapping Theorem, $\sigma(T_j) = F_j$. $\quad\square$

2.1.4 Abelian Banach Algebras

Theorem 1 (Gelfand-Mazur Theorem)**.** *If $\mathfrak{A}$ is a Banach algebra and a division ring, then $\mathfrak{A} = \mathbb{C}$.*

Proof. It is because that for any $a \in \mathfrak{A}$, $\sigma(a) \neq \varnothing$. $\quad\square$

Next, we reach the structure of an abelian Banach algebra. The structure of abelian Banach algebras can be explicit by constructing a map from an abelian Banach algebra to a continuous function space on a compact space. Firstly, we can find this compact space. Let

$$
\begin{aligned}
\Sigma(\mathfrak{A}) &= \{\text{all algebra homomorphism } h \colon \mathfrak{A} \to \mathbb{C}\} \\
\mathscr{M} &= \{\text{all maximal ideals of } \mathfrak{A}\}
\end{aligned}
$$

for an abelian Banach algebra $\mathfrak{A}$. Then we can find the relationship between $\Sigma(\mathfrak{A})$ and $\mathscr{M}$.

Theorem 2. *Let* $\mathfrak{A}$ *be an abelian Banach algebra. Define a map*

$$\gamma\colon \quad \begin{array}{ccc} \Sigma(\mathfrak{A}) & \longrightarrow & \mathscr{M} \\ h & \longmapsto & \ker h \end{array}$$

Then γ is a bijection.

Proof. Since $\mathfrak{A}/\ker h \cong \mathbb{C}$, $\ker h \in \mathscr{M}$, i.e. γ is well-defined.

Check: $\mathfrak{A}/M \cong \mathbb{C}$ for any $M \in \mathscr{M}$
Let $\pi\colon \mathfrak{A} \to \mathfrak{A}/M$. If $\pi(a)$ is not invertible, then $\pi(a\mathfrak{A})$ is a proper ideal in $\mathfrak{A}/M$. Thus $I = \pi^{-1}(\pi(a\mathfrak{A}))$ is a proper ideal in $\mathfrak{A}$ and $M \subset I$. Then by the maximality of M, $I = M$, i.e. $\pi(a) = 0$. In fact, for any commutative ring, this result is true. Therefore, by Gelfand-Mazur Theorem, $\mathfrak{A}/M \cong \mathbb{C}$.

Check: γ is surjective. Let $M \in \mathscr{M}$. Define $\tilde{h}\colon \mathfrak{A}/M \to \mathbb{C}$ as the algebraic isomorphism. Then $h = \pi \circ \tilde{h} \in \Sigma(\mathfrak{A})$ with $\ker h = M$.

Check: γ is injective. If $\ker h = \ker h'$ for $h, h' \in \Sigma(\mathfrak{A})$, then by the **Propostion** 8 in the subsection **1.4.2**, $h = \alpha h'$. And since $h(1) = h'(1) = 1$, $h = h'$. $\qquad\square$

Then we have some properties of $h \in \Sigma(\mathfrak{A})$.

Proposition 3. *Let* $\mathfrak{A}$ *be an abelian Banach algebra and* $h \in \Sigma(\mathfrak{A})$.

   1) h is continuous.

   2) $\|h\| = 1$ for $h \neq 0$.

Proof. 1) holds since $\ker h$ is maximal, thus it is closed.

Let $\lambda = h(a)$. Suppose $|\lambda| > \|a\|$. Then $1 - \frac{1}{\lambda}$ is invertible. Set $b = (1 - \frac{1}{\lambda})^{-1}$, then

$$1 = h(b(1 - \frac{1}{\lambda})) = h(b) - \frac{h(b)h(a)}{\lambda} = 0$$

Therefore, $|h(a)| \leqslant \|a\|$ i.e $\|n\| \leqslant 1$. Since $h(1) = 1$, $\|h\| = 1$. $\qquad\square$

Definition 4. Let $\mathfrak{A}$ be an abelian Banach algebra. Then $\Sigma(\mathfrak{A}) \subset \mathfrak{A}^*$ endowed with the induced wk^*-topology, is called the maximal ideal space of $\mathfrak{A}$.

Proposition 5. *If* $\mathfrak{A}$ *is an abelian Banach algebra, then* $\Sigma(\mathfrak{A})$ *is a compact Hausdorff space. Moreover, if* $a \in \mathfrak{A}$, *then*

$$\sigma(a) = \Sigma(a) = \{\, h(a)\colon h \in \Sigma(\mathfrak{A}) \,\}$$

Proof. Since $\Sigma(\mathfrak{A}) \subset \mathfrak{A}^*$, we just need to show $\Sigma(\mathfrak{A})$ is wk^*-closed. Let $\{h_i\}$ be a net in $\Sigma(\mathfrak{A})$ s.t. $h_i \to h$ wk^* for some h in the unit closed ball of $\mathfrak{A}^*$. Then for $a, b \in \mathfrak{A}$,

$$h(ab) = \lim_i h_i(ab) = \lim_i h_i(a)h_i(b) = h(a)h(b)$$

and $h(1) = \lim_i h_i(1) = 1$, thus $h \in \Sigma(\mathfrak{A})$. $\Sigma(\mathfrak{A})$ is compact.

If $h \in \Sigma(\mathfrak{A})$ and $h - h(a) \in \ker h \in \mathscr{M}$, then $h - h(a)$ is not invertible, i.e. $\Sigma(a) \subset \sigma(a)$. Conversely, if $a - \lambda$ is not invertible, $(a - \lambda)\mathfrak{A}$ is a proper ideal, which can be contained in a maximal ideal. Then $(a - \lambda) \in \ker h$ with some $h \in \Sigma(\mathfrak{A})$, $\lambda = h(a) \in \Sigma(a)$. $\qquad \square$

Therefore, $\Sigma(\mathfrak{A})$ is the compact space we need. Then we define the map from $\mathfrak{A}$ to $C(\Sigma(\mathfrak{A}))$.

Theorem 6. *If $\mathfrak{A}$ is an abelian Banach algebra, the Gelfand transform is defined as*

$$\Gamma\colon \quad \mathfrak{A} \quad \longrightarrow \quad C(\Sigma(\mathfrak{A}))$$
$$a \quad \longmapsto \quad \hat{a} = \Gamma(a)$$

where $\hat{a}(h) = h(a)$.

1) Γ is a continuous homomorphsim.

2) $\|\Gamma\| = 1$.

3) $\ker \Gamma = \bigcap \{M \colon M \in \mathscr{M}\}$.

4) $\|\hat{a}\|_\infty = r(a)$.

Proof. Firstly, for 1) if $h_i \to h$ in wk^*, $\hat{a}(h_i) = h_i(a) \to h(a) = \hat{a}(h)$. Thus Γ is well-defined. And

$$\Gamma(ab)(h) = h(ab) = h(a)h(b) = \Gamma(a)(h)\Gamma(b)(h)$$

Thus Γ is a homormophsim.

For 2), since $|\hat{a}(h)| = |h(a)| \leqslant \|a\|$, $\|\hat{a}\|_\infty \leqslant \|a\|$. Then $\|\Gamma\| \leqslant 1$. By $\Gamma(1) = 1$, $\|\Gamma\| = 1$.

For 3), $a \in \ker \Gamma$ if and only if $h(a) = 0$ for any $h \in \Sigma(\mathfrak{A})$, i.e. $a \in \bigcap \{M \colon M \in \mathscr{M}\}$.

For 4), it holds since $\sigma(a) = \{ h(a) \colon h \in \Sigma(\mathfrak{A}) \}$. $\qquad \square$

If $a \in \mathfrak{A}$ s.t. $\overline{\{p(a) \colon p \text{ is any polynomial}\}} = \mathfrak{A}$, then a is called a generator of $\mathfrak{A}$. Clearly, this $\mathfrak{A}$ is commutative. Then we can find an extral property of this special algebra.

Proposition 7. *If $\mathfrak{A}$ is an abelian Banach algebra with a generator a, then there is a homeomorphism $\tau\colon \Sigma(\mathfrak{A}) \to \sigma(a)$ s.t. $\Gamma(p(a)) = p \circ \tau$.*

Proof. In fact, τ can be defined as $\tau(h) = h(a)$. By above mention, τ is continuous and surjective. If $\tau h_1 = \tau h_2$, then $h_1(a) = h_2(a)$. By the fact that $h_1, h_2 \in \mathfrak{A}$ and a is a generator of $\mathfrak{A}$, $h_1 = h_2$. Thus τ is a bijection. And because $\Sigma(\mathfrak{A})$ is compact, f is a closed map, i.e. f is a homeomorphism.

$$\Gamma(p(a))(h) = p(\Gamma(a))(h) = p(\Gamma(a)(h)) = p(\tau(h))$$

$\qquad \square$

Remark. If $\mathfrak{A}$ is generate by a, then $\Gamma\colon \mathfrak{A} \to C(\sigma(a))$ can be defined as $\Gamma(p(a)) = p$.

In fact, above proposition can extends to n generators. If $\{a_i\}_{i=1}^n$ are generators of $\mathfrak{A}$, i.e $\overline{\{p(a_1, \cdots, a_n)\colon p \text{ is any } n \text{ variables polynomial}\}} = \mathfrak{A}$, then we have similar results as above proposition.

2.2 C^*-Algebras

Now, we have known $\mathscr{B}(\mathscr{H})$ is a Banach algebra. But there is another algebraic operation on $\mathscr{B}(\mathscr{H})$, which let $\mathscr{B}(\mathscr{H})$ be more interesting than the general Banach algebra. This operation is a map $T \to T^*$ on $\mathscr{B}(\mathscr{H})$, called an involution, and moreover, it satisfies the condition $\|T\| = \|T^*\| = \|T^*T\|^{\frac{1}{2}}$. This identity provides a strong relation between the topological structure and the algebraic structure on $\mathscr{B}(\mathscr{H})$. In fact, the topology is completely determined by the algebraic structure on $\mathscr{B}(\mathscr{H})$. In order to research this structure, we firstly define an general algebra satisfying above condition, called a C^*-algebra. By digging its topological structures and algebraic structures, we can embed it into $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. Therefore, any C^*-algebra can be regarded as a subalgebra of $\mathscr{B}(\mathscr{H})$.

2.2.1 Elementary Properties

Definition 1. If $\mathfrak{A}$ is a Banach algebra, an involution is a map $a \to a^*$ from $\mathfrak{A}$ to $\mathfrak{A}$ satisfying for any $a, b \in \mathfrak{A}$ and any $\alpha \in \mathbb{C}$,

1) $(a^*)^* = a$,

2) $(ab)^* = b^*a^*$,

3) $(\alpha a + b)^* = \overline{\alpha}a^* + b^*$.

Definition 2. A C^*-algebra is a Banach algebra $\mathfrak{A}$ with an involution s.t. for every $a \in \mathfrak{A}$,
$$\|a^*a\| = \|a\|^2$$

Then we can get some easy properties for the norm and the involution.

Proposition 3. *Let $\mathfrak{A}$ be a C^*-algebra and $a \in \mathfrak{A}$.*

1) $\|a^*\| = \|a\|$.

2) $\|aa^*\| = \|a\|^2$.

3) $\|a\| = \sup\{\|ax\| \colon \|x\| \leqslant 1\} = \sup\{\|xa\| \colon \|x\| \leqslant 1\}$.

Proof. For 1), $\|a\|^2 = \|a^*a\| \leqslant \|a^*\|\,\|a\|$, thus $\|a\| \leqslant \|a^*\|$. Taking the involution, $\|a^*\| \leqslant \|a\|$.

2), $\|aa^*\| = \|(a^*)^*\| = \|a^*\|^2 = \|a\|^2$.

For 3), let α be the supremum, then $\alpha \leqslant \|a\|$. $a = 0$ is clearly true. For $a \neq 0$, let $x = a^*/\|a\|$. Thus $\alpha \geqslant \|a\|$. $\qquad\square$

If a C^*-algebra $\mathfrak{A}$ is without the identity, we can use same method of the Banach algebra to extend it to a unit C^*-algebra $\tilde{\mathfrak{A}}$. The only thing we need to prove is the identity. And this can be obtained the result in above proposition. Therefore, we always assume a C^*-algebra is with the identity.

Definition 4. Let $\mathfrak{A}$ be a C^*-algebra and $a \in \mathfrak{A}$.

1) a is self-adjoint if $a = a^*$.

2) a is normal if $aa^* = a^*a$.

3) a is unitary if $aa^* = a^*a = 1$.

4) a is a projection if a is self-adjoint and $a = a^2$.

Then we can see the algebraic strucuture on a C^*-algebra completely determine its norm topology.

Theorem 5. *Let $\mathfrak{A}$ be a C^*-algebra and $a \in \mathfrak{A}$. If a is self-adjoint, then*

$$r(a) = \|a\|$$

Proof. Since a is sef-sdjoint,

$$\|a\|^2 = \|a^*a\| = \|a^2\|$$

Thus by induction, we have $\|a\|^{2n} = \|a^{2n}\|$. Then

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2n}\|^{\frac{1}{2n}} = \|a\| \qquad\square$$

Remark. For any $b \in \mathfrak{A}$, we know b^*b is self-adjoint,

$$r(b^*b) = \|b^*b\| = \|b\|^2$$

Thus, the norm in a C^*-algebra is completely determined by the spectral radius, which is totally an algebraic trait.

Now, we can see how the algebraic property influences the topological structure.

Proposition 6. *Let $\rho\colon \mathfrak{A} \to \mathfrak{B}$ be a $*$-homomorphsim between two C^*-algebras.*

1) *ρ is continuous, and moreover $\|\rho(a)\| \leqslant \|a\|$.*

2) *If ρ is a $*$-isomorphism, then ρ is an isometry.*

Proof. 2) is the direct corollary from 1). For 1), clearly $\sigma(\rho(a)) \subset \sigma(a)$, thus

$$\|\rho(a)\|^2 = r(\rho(a^*a)) \leqslant r(a^*a) = \|a\|^2 \qquad \square$$

Let $\operatorname{Re}\mathfrak{A}$ denote the set of all self-adjoint elements in a C^*-algebra $\mathfrak{A}$. Then, for any $a \in \mathfrak{A}$, there are $x, y \in \operatorname{Re}\mathfrak{A}$, s.t.

$$a = x + iy, \text{ where } x = \frac{a+a^*}{2}, y = \frac{a-a^*}{2i}$$

Therefore, any element in a C^*-algebra $\mathfrak{A}$ can be combined by two self-adjoint elements. And the self-adjoint element play a important role in the algebraic structure of a C^*-algebra .

Proposition 7. *If $h \colon \mathfrak{A} \to \mathbb{C}$ is an algebraic homomorphis on a C^*-algebra $\mathfrak{A}$.*

1) *If $a \in \operatorname{Re}\mathfrak{A}$, $h(a) \in \mathbb{R}$.*

2) *For any $a \in \mathfrak{A}$, $h(a^*) = \overline{h(a)}$.*

3) *$h(a^*a) \geqslant 0 \ \forall \ a \in \mathfrak{A}$.*

4) *If $u \in \mathfrak{A}$ is a unitary, then $|u| = 1$.*

Proof. For 1), let $h(a) = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$ and $C^*(a+it)$ be the C^*-algebra generated by $a + it$ and 1, which is abelian. Therefore, $\|h\|_{C^*(a+it)} = 1$ by **Proposition 3** in the subsection **2.1.4**. Then we have

$$\begin{aligned} |h(a+it)| &\leqslant \ \|a+it\|^2 \\ &= \ \|(a+it)^*(a+it)\| \\ &= \ \|a^2+t^2\| \\ &\leqslant \ \|a\|^2 + t^2 \end{aligned}$$

i.e.

$$\begin{aligned} \|a\|^2 + t^2 &\geqslant \ |\alpha + i(t+\beta)|^2 \\ &= \ \alpha^2 + (\beta+t)^2 \\ &= \ \alpha^2 + \beta^2 + 2\beta t + t^2 \end{aligned}$$

Therefore, for any $t \in \mathbb{R}$, $\|a\|^2 \geqslant \alpha^2 + \beta^2 + 2\beta t$. Thus, $\beta = 0$.
2) and 3) and 4) is the direct results from 1). $\qquad \square$

Corollary 8. *If $a \in \operatorname{Re} A$, then $\sigma(a) \subset \mathbb{R}$.*

Proof. Let $C^*(a)$ be the C^*-algebra generated by 1 and a. Thus $C^*(a)$ is abelian. Then by **Proposition** 5 in the subsection **2.1.4**,

$$\sigma_{C^*(a)}(a) = \{\ h(a)\colon h \in \Sigma(\mathfrak{A})\ \} \subset \mathbb{R}$$

And by the *Corollary* 7 in the subsection **2.1.2**,

$$\sigma(a) = \sigma_{C^*(a)}(a) \subset \mathbb{R}$$

$\square$

And the spectrum of a element in a C^*-algebra has better property.

Theorem 9. *If $\mathfrak{B}$ is a C^*-subalgebra of a C^*-algebra $\mathfrak{A}$ and $b \in \mathfrak{B}$, then*

$$\sigma_{\mathfrak{B}}(b) = \sigma_{\mathfrak{A}}(b)$$

Proof. It suffices to show that if b is invertible in $\mathfrak{A}$ with the inverse x, then $x \in \mathfrak{B}$. Then $(a^*a)(xx^*) = 1$. Since $a^*a \in \mathfrak{B}$ and by above corollary, we know $xx^* \in \mathfrak{B}$. But $x = (xx^*)a^*$, thus $x \in \mathfrak{B}$. $\square$

2.2.2 Abelian C^*-Algebras

The abelian C^*-algebra is firstly an ablian Banach algebra, thus all results in the subsection 2.1.4 can be applied to it. But since the involution and the related norm provide more information, we can have better results than the general abelian Banach algebra has. Firsly, we strengthen the **Theorem** 6 in subsection **2.1.4**.

Theorem 1. *If $\mathfrak{A}$ is an abelian C^*-algebra , then the Gelfand transform $\Gamma\colon \mathfrak{A} \to C(\Sigma(\mathfrak{A}))$ is an isometric $*$-isomorphism.*

Proof. Firstly, Γ is a $*$-homormophism, since the result in **Theorem** 6 and

$$\Gamma(a^*)(h) = h(a^*) = \overline{h(a)} = \overline{\Gamma(a)}(h)$$

Now, we can easily see Γ is an isometry by 4) in the **Theorem** 6,

$$\|\hat{a}\|_\infty^2 = \left\|a^{\hat{*}}a\right\|_\infty = r(a^*a) = \|a\|^2$$

Finally, we need to check Γ is surjective. It is because $\Gamma(\mathfrak{A})$ is a closed subalgebra of $C(\Sigma(\mathfrak{A}))$, which is closed under the complex conjugate and separates points in $\Sigma(\mathfrak{A})$. Then by the Stone-Weierstrass Theorem $\Gamma(\mathfrak{A}) = C(\Sigma(\mathfrak{A}))$ $\square$

We know if $\mathfrak{A} = C^*(a)$ for some normal element a, then $\mathfrak{A}$ is an abelian C^*-algebra . In fact,

$$\mathfrak{A} = \overline{\{\ p(a, a^*)\colon p(z, \overline{z})\ \text{is a polynomial}\ \}}$$

Then, we can modify the result in the **Propostion** 7 in subsection **2.1.4**.

Theorem 2. *Let* $\mathfrak{A} = C^*(a)$ *for some normal element* a. *Then there is a unique isometric $*$-isomorphism* $\rho\colon \mathfrak{A} \to C(\sigma(a))$.

Proof. Firstly, we have a similar homeomorphism

$$\begin{array}{rrcl} \tau\colon & \Sigma(\mathfrak{A}) & \longrightarrow & \sigma(a) \\ & h & \longmapsto & h(a) \end{array}$$

Then $\rho(x) = \Gamma(x)\circ\tau^{-1}$ is indeed an isometric $*$-isomorphism by the property of Gelfand transform. And moreover, by the result of **Propostion** 7, for any $z \in \sigma(a)$, $z = h(a)$ for some $h \in \Sigma(\mathfrak{A})$

$$\begin{aligned} \rho(p(a,a^*))(z) &= \rho(p(a,a^*))(h(a)) = \Gamma(p(a,a^*)) \circ \tau^{-1}(h(a)) \\ &= \Gamma(p(a,a^*))(h) = h(p(a,a^*)) \\ &= p(h(a),h(a^*)) = p(h(a),\overline{h(a)}) \\ &= p(z,\overline{(z)}) \end{aligned}$$

That means that ρ maps the polynomials in $\mathfrak{A}$ to polynomails in $\sigma(a)$. Therefore, ρ is unique. $\qquad\square$

By the Riesz Functional Calculus, we have the map $f \mapsto f(a)$ from $\mathrm{Hol}\,(a)$ to $\mathfrak{A}$ for a C^*-algebra $\mathfrak{A}$ and any element $a \in \mathfrak{A}$. Now, we can extend this definition a $C(\sigma(a))$ by above ρ, but we need a is normal, then

$$\begin{array}{rrcl} \rho^{-1}\colon & C(\sigma(a)) & \longrightarrow & C^*(a) \\ & f & \longmapsto & f(a) \end{array}$$

defined above is an isometric isomorphism and ρ^{-1} maps

$$\begin{aligned} 1 &\longmapsto 1 \\ z &\longmapsto a \\ \overline{z} &\longmapsto a^* \\ z^{-1} &\longmapsto a^{-1} \\ p(z,\overline{z}) &\longmapsto p(a,a^*) \end{aligned}$$

Therefore, this map is unique and it is clearly the extension of the Riesz Functional Calculus, called Continuous Functional Calculus. Like the Riesz Functional Calculus, there is also a Spectral Theorem.

Theorem 3 (Spectral Theorem)**.** *Let* $\mathfrak{A}$ *be a C^*-algebra and* $a \in \mathfrak{A}$ *be a normal element, then for* $f \in C(\sigma(a))$,

$$\sigma(f(a)) = f(\sigma(a))$$

Proof. For some compact space X and $f \in C(X)$, then $C(X)$ with the supremum norm is a C^*-algebra and $\sigma(f) = \mathrm{ran}\,f$. Then since $f \mapsto f(a)$ is a $*$-isomorphism,

$$\sigma(f(a)) = \sigma_{C(\sigma(a))}(f) = \mathrm{ran}\,f = f(\sigma(a)) \qquad\square$$

There is an important example.

Example 4. Let μ be a compactly supported, regular Borel measure on $\mathbb{C}$ and (X, Ω, μ) be the measure space. For each $\pi \in L^\infty(\mu)$, we define the map

$$M_\phi: \quad \begin{aligned} L^2(\mu) &\longrightarrow L^2(\mu) \\ f(z) &\longmapsto \phi(z)f(z) \end{aligned}$$

Then clearly M_ϕ is in $\mathscr{B}(L^2(\mu))$.

1) $(M_\phi)^* = M_{\overline{\phi}}$ and M_ϕ is a normal element in $\mathscr{B}(L^2(\mu))$.

2) $\phi \mapsto M_\phi$ is a $*$-homomorphism from $L^\infty(\mu)$ to $L^2(\mu)$.

3) $\|M_\phi\| = \|\phi\|_\infty$.

4) $\sigma(M_\phi) = \bigcap\{\overline{\phi(U)} : U \in \Omega \ \& \ \mu(X\backslash U) = 0\}$.

5) If $f \in C(\sigma(M_\phi))$, then $f(M_\phi) = M_{f\circ\phi}$.

If $\phi(z) = z$, we set denote $N_\mu = M_\phi$ and in fact, $\sigma(N_\mu) = \operatorname{supp}\mu$.

2.2.3 Positive Elements

We have known that self-adjoint elements play a important role in a C^*-algebra $\mathfrak{A}$. The self-adjoint element a in $\mathfrak{A}$ is like the real number in $\mathbb{C}$, and the relationship between them can be revealed by the fact $a \in \operatorname{Re}\mathfrak{A}$ if and only if $\sigma(a) \subset \mathbb{R}$. The converse is obtained by the Continuous Functional Calculus. In fact, Continuous functional calculus can provide more relation between the element in $\mathfrak{A}$ and the elment in $\mathbb{C}$, like positivity.

Definition 1. Let $\mathfrak{A}$ be a C^*-algebra and $a \in \operatorname{Re}\mathfrak{A}$. Then a is called a positive element if and only if $\sigma(a) \subset \mathbb{R}^+$, denoted by $a \geqslant 0$. And let $\mathfrak{A}_+$ be the set of all positive elements.

This definition is nature by above mention, but it may not be very explicit. So we need to show more direct equivalent definitions of positive elements.

Theorem 2. *Let $\mathfrak{A}$ be a C^*-algebra . Then the following statements are equivalent.*

1) $a \geqslant 0$.

2) $a = b^2$ for some $b \geqslant 0$.

3) $a \in \operatorname{Re}\mathfrak{A}$ and $\|t - a\| \leqslant 0$ for all $t \geqslant \|a\|$.

4) $a \in \operatorname{Re}\mathfrak{A}$ and $\|t - a\| \leqslant 0$ for some $t \geqslant \|a\|$

Proof. All of these can be done by the functional calculus. And our goal is to find some vilid functions in $C(\sigma(a))$ to complete these.

1) $\Rightarrow$ 2) Let $f(x) = \sqrt{x}$ in $C(\sigma(a))$ and since $\sigma(a) \subset \mathbb{R}^+$, f is well-defined. Let $b = f(a)$. Then, we have $a = b^2$. And by Spectral Theorem, $\sigma(b) = \sigma(f(a)) \subset \mathbb{R}+$.

2) $\Rightarrow$ 3) Let $f(x) = x^2$ defined on $\sigma(b)$, then $a = f(b)$ and $\|a\| = \|f\|_\infty$. By this condition, $f(x) \geqslant 0$. Thus $\sigma(a) = f(\sigma(b)) \subset \mathbb{R}^+$.

3) $\Rightarrow$ 4) It is trivial.

4) $\Rightarrow$ 1) Let $f(x) = x$ defined on $\sigma(a) \subset \mathbb{R}$. Then this condition means

$$\|t - f\|_\infty = \|t - f(a)\| = \|t - a\| \leqslant t$$

for some $t \geqslant \|a\| = \|f\|_\infty$. Therefore, $f(x) \geqslant 0$ for all $x \in \sigma(a)$. Thus $\sigma(a) = f(\sigma(a)) \subset \mathbb{R}^+$. □

Like the fact that any element in a C^*-algebra can be combined by two self-ajoint elements, any self-adjoint element can be combined by two positive elements.

Proposition 3. *Let $\mathfrak{A}$ be a C^*-algebra . If $a \in \operatorname{Re}\mathfrak{A}$, then there are unique $u, v \in \mathbb{R}^+$, s.t.*

$$a = u - v \quad \& \quad uv = vu = 0$$

Proof. Let $f(x) = \max x, 0$ and $g(x) = -\min x, 0$. Then $f, g \in C(\sigma(a))$ and $f(x) - g(x) = x$ and $f(x)g(x) = 0$. Then $u = f(a)$ and $v = g(a)$ satisfy above conditions.

If $a = u_1 - v_1$, then we can know $C^*(a, u, v, u_1, v_1)$ is an abelian C^*-algebra , thus for some compact space X, $C^*(a, u, v, u_1, v_1) \cong C(X)$. And this uniqueness can be proved in a continuous function space. □

Corollary 4. *Let $\mathfrak{A}$ be a C^*-algebra . Then $\mathfrak{A}_+$ is a cone.*

Proof. Let $\{a_n\} \subset \mathfrak{A}_+$ be a sequence s.t. $a_n \to a$. Then by above proposition, $\|a_n - \|a_n\|\| \leqslant \|a_n\|$. Taking norm limit, $\|a - \|a\|\| \leqslant \|a\|$, thus $a \in \mathfrak{A}_+$.

Clearly, $\alpha\mathfrak{A}_+ \subset \mathfrak{A}_+$ for any $\alpha > 0$. For $a, b \in \mathfrak{A}$, we can assume that $\|a\| \leqslant 1$ and $\|b\| \leqslant 1$, then

$$\left\| 1 - \frac{1}{2}(a+b) \right\| = \frac{1}{2} \|(1-a) + (1-b)\| \leqslant 1$$

Thus $\frac{1}{2}(a+b) \in \mathfrak{A}_+$, i.e. $a + b \in \mathfrak{A}_+$. □

Then, we can build an order on $\operatorname{Re}\mathfrak{A}$ by defining $a \leqslant b \Leftrightarrow b - a \in \mathfrak{A}_+$. And moreover, let $\mathfrak{A}_- = -\mathfrak{A}_+$, then $\mathfrak{A}_- \bigcap \mathfrak{A}_+ = \{0\}$. There are other properties of positivity.

Proposition 5. *Let $\mathfrak{A}$ be a C^*-algebra .*

1) If $a \geqslant 0$, then there is a unique $b \geqslant 0$ s.t. $a = b^n$.

2) If $a \in \mathfrak{A}$, then $a^*a \in \mathfrak{A}_+$.

3) If $a \leqslant b$ in $\operatorname{Re}\mathfrak{A}$, then $c^*ac \leqslant c^*bc$ for any $c \in \mathfrak{A}$.

4) For any $a \in \operatorname{Re}\mathfrak{A}$, $-\|a\| \leqslant a \leqslant \|a\|$ and if $a \in \mathfrak{A}$, $0 \leqslant a^*a \leqslant \|a\|^2$.

5) If $0 \leqslant a \leqslant b$, then $b^{-1} \leqslant a^{-1}$.

6) For any $a \in \mathfrak{A}$, we define $|a| = \sqrt{a^*a}$, then $|a| = a_+ + a_-$.

7) If $0 \leqslant a \leqslant b \in \mathfrak{A}$, then $\|a\| \leqslant \|b\|$.

Proof. For 1), let $f(x) = \sqrt[n]{x}$ defined on $\sigma(a) \subset \mathbb{R}$, then $b = f(a)$ sastisfying $a = b^n$.

For 2), let $b = a^*a = b_+ - b_-$ and $c = \sqrt{b_+}$ and $d = ac$. Since $d^*d = b_-^2 \in \mathfrak{A}_+$, $d^*d \in \mathfrak{A}_-$. Let $d = x + iy$, then $dd^* + d^*d = 2(x^2 + y^2) \in \mathfrak{A}_+$. Thus

$$dd^* = dd^* + d^*d - d^*d \in \mathfrak{A}_+$$

By the fact that $\sigma(dd^*) \bigcup \{0\} = \sigma(d^*d) \bigcup \{0\}$, $b_-^2 = -d^*d = 0$. Therefore, $b = a^*a \in \mathfrak{A}_+$.

For 3), let $d = \sqrt{b - a}$, then

$$c^*bc - c^*ac = c^*(b - a)c = c^*d^*dc = (dc)^*dc \in \mathfrak{A}_+$$

For 4), if $x \in \sigma(a)$, $|x| \leqslant \|a\|$. Therefore, $f(x) = \|a\| - x$ and $g(x) = \|a\| + x$ are positive on $\sigma(a)$.

For 5), since $0 \leqslant a \leqslant b$,

$$1 - b^{-\frac{1}{2}}ab^{-\frac{1}{2}} = b^{-\frac{1}{2}}(b - a)b^{-\frac{1}{2}} \geqslant 0$$

i.e. $(a^{\frac{1}{2}}b^{-\frac{1}{2}})^*(a^{\frac{1}{2}}b^{-\frac{1}{2}}) \leqslant 1$, therefore $\left\|a^{\frac{1}{2}}b^{-\frac{1}{2}}\right\| \leqslant 1$ by functional calculus as similar as 4). And thus $1 \geqslant (a^{\frac{1}{2}}b^{-\frac{1}{2}})(a^{\frac{1}{2}}b^{-\frac{1}{2}})^* = a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}$. Therefore, $a^{-1} = a^{-\frac{1}{2}}1a^{-\frac{1}{2}} \geqslant b^{-1}$.

6) holds by the functional calculus and the uniqueness is by 1).

For 7), $\sigma(a) = \{h(a) \colon h \in \Sigma(\mathfrak{A})\} \subset \mathbb{R}^+$ and $\sigma(b) = \{h(b) \colon h \in \Sigma(\mathfrak{A})\} \subset \mathbb{R}^+$ and $\sigma(b - a) = \{h(b - a) \colon h \in \Sigma(\mathfrak{A})\} \subset \mathbb{R}^+$, therefore $h(b) \geqslant h(a) \geqslant 0$ for any $h \in \Sigma(\mathfrak{A})$. That means $r(b) \geqslant r(a)$, i.e. $\|b\| \geqslant \|a\|$. $\qquad\square$

By using 2) on above proposition, we can easily see that

Corollary 6. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. If T is positive, then for any $h \in \mathscr{H}$*

$$\langle Th, h \rangle \geqslant 0$$

Remark. There is a $A \in \mathscr{B}(\mathscr{H})$ s.t. $T = A^*A$, thus above statement is true. In fact, the converse is also true.

2.2.4 Approximate Identities

When we research the proper ideal of an algebra, this ideal does not contain the identity. So for the ideal of a C^*-algebra, we want to find some element has similar property as the identity has in the ideal.

Definition 1. Let $\mathfrak{A}$ be a C^*-algebra and $\{e_i\}$ be a net in $\mathfrak{A}$ s.t.

   1) $0 \leqslant e_i \leqslant 1$ for all i,

   2) $e_i \leqslant e_j$ for $i \leqslant j$,

   3) $\lim_i ae_i = \lim_i e_i a = a$ for any $a \in \mathfrak{A}$,

Then $\{e_i\}$ is called an approximate identity for $\mathfrak{A}$.

Theorem 2. *Every C^*-algebra $\mathfrak{A}$ has an approximate identity.*

Proof. Firstly, let $\Lambda = \{e \in \mathfrak{A}_+ : e < 1\}$. We can check Λ is indeed a direct set with respect to $\leqslant$. Define two functions as

$$
\begin{aligned}
f(t) &= \frac{t}{1-t}, \ \forall \, t \in [0,1), \\
g(t) &= \frac{t}{1+t} = 1 - \frac{1}{1+t}, \ \forall \, t \in [0,\infty).
\end{aligned}
$$

In fact, $g(f(t)) = t$. Then for any $a, b \in \Lambda$, let $y = f(a) + f(b)$ and $c = g(y)$. And since $\|g\|_\infty < 1$, $c \in \Lambda$. The fact that $x = f(a) \leqslant y$ implies $1+x \leqslant 1+y$. Then $(1+x)^{-1} \geqslant (1+y)^{-1}$.

$$
a = g(f(a)) = g(x) = 1 - (1+x)^{-1} \leqslant 1 - (1+y)^{-1} = c
$$

Similarly, $b \leqslant c$. Therefore, Λ is direct.

If $a \in \mathfrak{A}_+$, let $e_n = g(na) \in \Lambda$. Define

$$
h(t) = t^2(1 - g(nt)) = \frac{t^2}{1+nt} \leqslant \frac{t}{n}
$$

Thus $h(a) = a^2(1 - g(na)) = a(1 - e_n)a$, that means

$$
\|a(1-e_n)a\| = \|h\|_\infty \leqslant \frac{\|a\|}{n}
$$

For any $\varepsilon > 0$, we can choose a N, s.t. for $n > N$, $\|a(1-e_n)a\| < \varepsilon$. Moreover, since for $0 \leqslant d \leqslant b \leqslant 1 \in \mathfrak{A}$, $c^*(1-b)c \leqslant c^*(1-d)c$ for any $c \in \mathfrak{A}$. Therefore,

$$
\|c^*(1-b)c\| \leqslant \|c^*(1-d)c\|
$$

And combining above mention and the fact for $0 \leqslant d \leqslant b \leqslant 1 \in \mathfrak{A}$,

$$
\begin{aligned}
\|c - dc\|^2 &\leqslant \|c^*(1-d)c\| \\
\|c - cd\|^2 &\leqslant \|c^*(1-d)c\|
\end{aligned}
$$

implies for $e \geqslant e_N$,

$$\|a - ea\|^2 < \varepsilon$$
$$\|a - ae\|^2 < \varepsilon$$

Therefore,

$$\lim_i ae_i = \lim_i e_i a = a, \quad \forall \ a \in \mathfrak{A}$$

For arbitrary $a \in \mathfrak{A}$, a can be write as the linear combination of four positive elements. $\qquad\square$

Remark. The result that Λ is direct is also true for $\mathfrak{A}$ without the identity, since $g(0) = f(0) = 0$, which means $x, y, c \in \mathfrak{A}$. And if $\mathfrak{A}$ is separable, we can find a sequential approximate identity in its countable dense subset. Moreover, this sequential approximate identity can apply in whole $\mathfrak{A}$.

Then we can use the approximate identity to get some interesting results. First, we need a lemma.

Lemma 3. *If $\mathfrak{A}$ is a C^*-algebra and $x, y \in \mathfrak{A}$, $a \in \mathfrak{A}_+$ s.t. $x^*x \leqslant a^\alpha$ and $y^*y \leqslant a^\beta$ for some positive scalars α and β with $\alpha + \beta > 1$, then the sequence $u_n = x(n^{-1} + a)^{-\frac{1}{2}}y$ converges to a $u \in \mathfrak{A}$ with $\|u\| \leqslant \left\|a^{\frac{1}{2}(\alpha + \beta - 1)}\right\|$.*

Proof. Let $d_n m = (n^{-1} + a)^{-\frac{1}{2}} - (m^{-1} + a)^{-\frac{1}{2}}$.

$$
\begin{aligned}
\|u_n - u_m\|^2 &= \|xd_nmy\|^2 = \|y^*d_nmx^*xd_nmy\| \\
&\leqslant \left\|a^{\frac{\alpha}{2}}d_nmy\right\|^2 = \left\|a^{\frac{\alpha}{2}}d_nmy^*yd_nma^{\frac{\alpha}{2}}\right\| \\
&\leqslant \left\|a^{\frac{\alpha}{2}}d_nma^\beta d_nma^{\frac{\alpha}{2}}\right\| \\
&= \left\|d_nma^{\frac{\alpha+\beta}{2}}\right\|^2
\end{aligned}
$$

Since $f_n(t) = (n^{-1} + t)t^{\frac{\alpha+\beta}{2}}$ is an increasing positive sequence, $d_n = (n^{-1} + a)a^{\frac{\alpha+\beta}{2}}$ is an increasing positive sequence in $\mathfrak{A}_+$. By Dini's Theorem, since $\sigma(a)$ is compact, there is a continuous fuction f s.t. $f_n \to f$ uniformly. Let $d = f(a)$, thus $d_n \to d$ in norm. Thus $\{u_n\}$ is Cauchy and there exists $u = \lim_{n \to \infty} u_n$.

$$\|u_n\| = \left\|x(n^{-1} + a)^{-\frac{1}{2}}\right\| \leqslant \left\|(n^{-1} + a)a^{\frac{\alpha+\beta}{2}}\right\| \leqslant \left\|a^{\frac{1}{2}(\alpha+\beta-1)}\right\| \qquad\square$$

Proposition 4. *If $\mathfrak{A}$ is a C^*-algebra and $a \in \mathfrak{A}+$ and $x \in \mathfrak{A}$ with $x^*x \leqslant a$, and $0 < \alpha < \frac{1}{2}$, then there exists $u \in \mathfrak{A}$ with $\|u\| \leqslant \left\|a^{\frac{1}{2}-\alpha}\right\|$ and $x = ua^\alpha$.*

Proof. Let $u_n = x(n^{-1} + a)^{-\frac{1}{2}} a^{\frac{1}{2}-\alpha}$. By above lemma, $u_n \to u$ with $\|u\| \leqslant \left\| a^{\frac{1}{2}-\alpha} \right\|$. We use similar prove as above lemma,

$$
\begin{aligned}
\|x - u_n a^{\alpha}\|^2 &= \left\| x(1 - (n^{-1} + a)^{-\frac{1}{2}} a^{\frac{1}{2}}) \right\|^2 \\
&\leqslant \left\| (1 - (n^{-1} + a)^{-\frac{1}{2}} a^{\frac{1}{2}}) a (1 - (n^{-1} + a)^{-\frac{1}{2}} a^{\frac{1}{2}}) \right\| \\
&= \left\| a^{\frac{1}{2}} (1 - (n^{-1} + a)^{-\frac{1}{2}} a^{\frac{1}{2}}) \right\|^2 \\
&= \left\| a^{\frac{1}{2}} - (n^{-1} + a)^{-\frac{1}{2}} a \right\|
\end{aligned}
$$

By the Dini's Theorem, $(n^{-1} + a)^{-\frac{1}{2}} a \to a$ in norm. Therefore, $u_n a^{\alpha} \to a$ i.e. $a = ua^{\alpha}$. $\qquad\square$

Corollary 5. *If $\mathfrak{A}$ is a C^*-algebra and $x \in \mathfrak{A}$ and $0 < \beta < 1$, then there is a $u \in \mathfrak{A}$ s.t.*
$$
x = u \, |x|^{\beta}
$$

2.2.5   Ideals and Quotients

Firstly, there are two easy results of closed ideal in a C^*-algebra.

Proposition 1. *Let $\mathfrak{A}$ be a C^*-algebra.*

1) *If $\mathfrak{I}$ is a closed left or right ideal of $\mathfrak{A}$, $a \in \mathfrak{I}$ with $a = a^*$, then for $f \in C(\sigma(a))$ with $f(0) = 0$, $f(a) \in \mathfrak{A}$.*

2) *If $\mathfrak{I}$ is a closed ideal, then $a \in \mathfrak{I}$ implies $a^* \in \mathfrak{A}$.*

Proof. For 1), if $\mathfrak{I}$ is proper, $0 \in \sigma(a)$. Then $f(0) = 0 and \sigma(a) \subset \mathbb{R}$, f can be approximated by a sequence of polynomials p_n with $p_n(0) = 0$. Therefore, $p_n(a) \in \mathfrak{I}$ and by the fact that $\mathfrak{I}$ is closed, $f(a) \in \mathfrak{I}$.

For $a \in \mathfrak{I}$, by the corollary in above subsection, we know there is a $u \in \mathfrak{A}$ s.t. $a = u \, |a|^{\frac{1}{2}}$. By 1), $|a|^{\frac{1}{2}} \in \mathfrak{I}$. Therefor,

$$
a^* = |a|^{\frac{1}{2}} u \in \mathfrak{I} \qquad\square
$$

Definition 2. If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{B}$ is a $*$-subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ is called hereditary if for any $b \in \mathfrak{B}_+$ and $x \in \mathfrak{A}$ with $0 \leqslant x \leqslant b$, $x \in \mathfrak{B}$.

Now, we can give more profound properties of the closed left ideals.

Theorem 3. *Let $\mathfrak{A}$ be a C^*-algebra.*

1) *If $\mathfrak{I}$ is a closed left ideal of $\mathfrak{A}$ and $\mathfrak{B} = \mathfrak{I} \bigcap \mathfrak{I}^*$, then $\mathfrak{B}$ is a hereditary subalgebra of $\mathfrak{A}$.*

2) If $\mathfrak{B}$ is a hereditary subalgebra of $\mathfrak{A}$ and $\mathfrak{I} = \{x \in \mathfrak{A} \colon x^*x \in \mathfrak{B}\}$, then $\mathfrak{I}$ is a closed left ideal of $\mathfrak{A}$.

3) If $\mathfrak{I}$ is a closed left ideal of $\mathfrak{A}$ and $\mathfrak{B} = \mathfrak{I} \bigcap \mathfrak{I}^*$, then $\mathfrak{I} = \{x \in \mathfrak{A} \colon x^*x \in \mathfrak{B}\}$.

4) If $\mathfrak{B}$ is a hereditary subalgebra of $\mathfrak{A}$ and $\mathfrak{I} = \{x \in \mathfrak{A} \colon x^*x \in \mathfrak{B}\}$, then $\mathfrak{B} = \mathfrak{I} \bigcap \mathfrak{I}^*$.

Proof. For 1), clearly, $\mathfrak{B}$ is a C^*-algebra. Let $0 \leqslant x \leqslant b$ for some $b \in \mathfrak{B}_+$. Since $x^{\frac{1}{2}}x^{\frac{1}{2}} \leqslant b$ there is a $u \in \mathfrak{A}$, s.t. $x^{\frac{1}{2}} = ub^{\frac{1}{3}}$. But $b^{\frac{1}{3}} \in \mathfrak{B}_+ \subset \mathfrak{I}$. Therefore, $x^{\frac{1}{2}} \in \mathfrak{I}$ and thus $x \in \mathfrak{I}$. Since x is self-adjoint, $x \in \mathfrak{B}$.

For 2), if $x \in \mathfrak{I}$ and $a \in \mathfrak{A}$, then

$$(ax)^*ax = x^*a^*ax \leqslant \|a\|^2 \, x^*x \in \mathfrak{B}.$$

Therefore, $(ax)^*ax \in \mathfrak{B}$, i.e. $ax \in \mathfrak{I}$. If $x, y \in I$, then

$$(x + y)^*x + y \leqslant (x + y)^*(x + y) + (x - y)^*(x - y) = 2(x^*x + y^*y) \in \mathfrak{B}$$

Thus, $\mathfrak{I}$ is a closed left ideal.

For 3), if $x \in \mathfrak{A}$ and $x^*x \in \mathfrak{B}$, $x^*x \in \mathfrak{I}$. Thus $|x|^{\frac{1}{2}} \in \mathfrak{I}$ since $\mathfrak{I}$ is a closed left ideal. Therefore, $x = u|x|^{\frac{1}{2}} \in \mathfrak{I}$. The converse is clearly true.

For 4), if $x \in \mathfrak{I}_+$, then $x^2 \in \mathfrak{B}$. So $x = \sqrt{x^2} \in \mathfrak{B}_+$. Conversely, $x \in \mathfrak{B}_+$ means $(x^{\frac{1}{2}})^*(x^{\frac{1}{2}}) = x \in \mathfrak{B}$, $x^{\frac{1}{2}} \in \mathfrak{I}_+$. Therefore, $\mathfrak{B}_+ = \mathfrak{I}_+$. Thus, $\mathfrak{B} = \mathfrak{I} \bigcap \mathfrak{I}^*$. $\qquad\square$

Theorem 4. *If $\mathfrak{I}$ is a closed ideal of a C^*-algebra $\mathfrak{A}$, then the quotient algebra $\mathfrak{A}/\mathfrak{I}$ with the induced norm and the induced involution, i.e. $a + \mathfrak{I}^* = a^* + \mathfrak{I}$ is also a C^*-algebra and the norm can be*

$$\|a + \mathfrak{I}\| = \inf \{ \ \|a - ax\| : x \in \mathfrak{I}_+, \|x\| \leqslant 1 \ \}$$

Proof. Firstly, let $\{e_i\}$ be an approximate identity for $\mathfrak{I}$. Since $0 \leqslant e_i \leqslant 1$, if $a \in \mathfrak{A}$ and $y \in \mathfrak{I}$, $\|(a + y)(1 - e_i)\| \leqslant \|a + y\|$.

$$
\begin{aligned}
\|a + y\| &\geqslant \liminf_i \|(a + y)(1 - e_i)\| \\
&= \liminf_i \|(a - ae_i) + (y - ye_i)\| \\
&= \liminf_i \|(a - ae_i)\|
\end{aligned}
$$

The converse is clearly true.

The involution defined on the quotient algebra is well-defined and satifies the conditions since $\mathfrak{I}$ is self-adjoint. We just need to show the induced norm satisfies the C^*-identity. For $a \in \mathfrak{A}$, we have $\|a^* + \mathfrak{I}\| = \|a + \mathfrak{I}\|$, and

$$
\begin{aligned}
\|a + \mathfrak{I}\|^2 &= \|(a^* + \mathfrak{I})(a + \mathfrak{I})\| \\
&\leqslant \|a^* + \mathfrak{I}\| \|a + \mathfrak{I}\| \\
&= \|a + \mathfrak{I}\|^2
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
\|a + \mathfrak{I}\|^2 &= \inf \{ \|a - ax\|^2 : x \in \mathfrak{I}_+, \|x\| \leqslant 1 \} \\
&= \inf \{ \|(1 - x)a^*a(1 - x)\| : x \in \mathfrak{I}_+, \|x\| \leqslant 1 \} \\
&\leqslant \inf \{ \|a^*a(1 - x)\| : x \in \mathfrak{I}_+, \|x\| \leqslant 1 \} \\
&= \|a^*a + \mathfrak{I}\|
\end{aligned}
$$

Therefore, $\mathfrak{A}/\mathfrak{I}$ is indeed a C^*-algebra. $\qquad\square$

Then it can provide a C^*-algebra with some similar results as general rings have. These results can also show us how the algebraic structure in a C^*-algebra affect the topological structure. In my oppion, for researching a C^*-algebra , researching the algebraic structure may be more important.

Corollary 5. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be two C^*-algebras.*

1) *If $\rho \colon \mathfrak{A} \to \mathfrak{C}$ is a $*$-homomorphism, then $\operatorname{ran} \rho$ is a C^*-algebra and the induced map $\tilde{\rho} \colon \mathfrak{A}/\ker \rho \to \operatorname{ran} \rho$ is an $*$-isomorphism.*

2) *If $\mathfrak{I}$ is a closed ideal of $\mathfrak{A}$ and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then there is a $*$-isomorphism*
$$
\mathfrak{B}/(\mathfrak{B} \cap \mathfrak{I}) \cong (\mathfrak{B} + \mathfrak{I})/\mathfrak{I}
$$

Proof. For 1), we just need to show $\operatorname{ran} \rho$ is closed.
Since $\tilde{\rho}$ is a $*$-monomorphism from $\mathfrak{A}/\ker \rho$ to $\mathbb{C}$, by the result in **Proposition 6** in the subsection **2.1.1**, ρ is an isometry. Thus $\operatorname{ran} \rho = \rho(\mathfrak{A})$ is closed.

For 2), there is a commutative graph like

$$
\begin{array}{ccc}
\mathfrak{B} & \xrightarrow{\ i\ } & \mathfrak{A} \\
{\scriptstyle Q}\downarrow & \swarrow{\scriptstyle \pi} & \\
\mathfrak{A}/\mathfrak{I} & &
\end{array}
$$

$Q = \pi \circ i \colon \mathfrak{B} \to \mathfrak{B}/\mathfrak{I}$ and thus $\pi^{-1}(Q(\mathfrak{B})) = \mathfrak{B} + \mathfrak{I}$. By restricting π on $\mathfrak{B} + \mathfrak{I}$, then we have

$$
(\mathfrak{B} + \mathfrak{I})/\mathfrak{I} \cong Q(\mathfrak{B}) \cong \mathfrak{B}/(\mathfrak{B} \cap \mathfrak{I}) \qquad\square
$$

2.2.6 Positive Functionals and GNS Construction

We have known that $\mathscr{B}(\mathscr{H})$ is a C^*-algebra. In fact, defining the general C^*-algebra is to research the operator algebra. Moreover, by the properties, we can see any C^*-algebra is a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ for some $\mathscr{H}$. To prove that, we just need to find a faithful representation of the fixed C^*-algebra. Therefore, the main idea is to construct a representation $(\pi, \mathscr{H})$. How to construct it is a question. But, fortunately, some special functionals can provide us a method.

Definition 1. Let $\mathfrak{A}$ be C^*-algebra and ϕ is a linear functional on $\mathfrak{A}$. ϕ is called positive if for any $a \in \mathfrak{A}_+$, $\phi(a) \leqslant 0$. A positive linear functional ϕ is called a state if $\phi(1) = 1$.

Remark. Let $\mathcal{S}_{\mathfrak{A}}$ denote the set of all states on $\mathfrak{A}$.

Then there are some properties of positive functionals.

Proposition 2. *Let $\mathfrak{A}$ be a C^*-algebra and ϕ be a positive functional.*

1) For any $x, y \in \mathfrak{A}$, then

$$|\phi(y^*x)| \leqslant \phi(x^*x)\phi(y^*y)$$

2) ϕ is bounded and if $\{e_i\}$ is an approximate identity of $\mathfrak{A}$, then

$$\|\phi\| = \lim_i \phi(e_i)$$

Proof. For 1), it can easily check that $< x, y >= \phi(y^*x)$ is a semi-inner product, thus by the CBS inequality, above inequality is true.

Assume $\mathfrak{A}$ without identity. If ϕ is unbounded, then there is a sequence $\{a_k\} \in \mathfrak{A}_+$ with $\|a_k\| \leqslant 1$ s.t. $\phi(a_k) > 2^k$. Let $a = \sum_{k=1}^{\infty} 2^{-k}a_k$. Then

$$\phi(a) \geqslant \phi(\sum_{k=1}^{n} 2^{-k}a_k) > n$$

which is a contradiction.

$$\alpha = \sup\left\{ \phi(a) \colon a \in \mathfrak{A}_+, \|a\| \leqslant 1 \right\} < \infty$$

Since any element in $\mathfrak{A}$ can be a linear combination of four positive elements, $\|\phi\| \leqslant 4\alpha$. Therefore, ϕ is bounded.

Let $\beta = \lim_i \phi(e_i)$. Clearly, $\beta \leqslant \|\phi\|$. And since for $a \in \mathfrak{A}$ with $\|a\| \leqslant 1$ then $0 \leqslant a^*a \leqslant 1$

$$|\phi(a)|^2 = \lim_i |\phi(e_i a)|^2 \leqslant \lim_i \phi(e_i)\phi(a^*a) \leqslant \beta \|\phi\|$$

$$\|\phi\|^2 \leqslant \beta \|\phi\|$$

i.e. $\|\phi\| \leqslant \beta$. $\qquad\qquad\square$

Remark. For 2), if $\mathfrak{A}$ has an identity, then $\|\phi\| = \phi(1)$. It is because for $a \in \mathfrak{A}$ with $\|a\| \leqslant 1$

$$|\phi(a)|^2 \leqslant \phi(a^*a)\phi(1) \leqslant \phi(1)^2$$

i.e. $|\phi(a)| \leqslant \phi(1)$.

In fact, the converse of 2) in above proposition is also true.

Proposition 3. *Let $\mathfrak{A}$ be C^*-algebra and ϕ is a bounded linear functional with $\|\phi\| = \phi(1)$, then $\phi \leqslant 0$.*

Proof. If $\mathfrak{A} = C(X)$ for some compact space X, ϕ is a measure μ with $\mu(X) = \|\mu\|$, then $\mu \geqslant 0$, i.e. ϕ is positive. Then for any $\mathfrak{A}$, if $a \in \mathfrak{A}_+$, $\mathfrak{B} = C^*(a) \cong C(\sigma(a))$. Then $\phi|_{\mathfrak{B}}(1) \leqslant \|\phi\| = \phi(1) = \phi|_{\mathfrak{B}}(1)$, thus $\phi|_{\mathfrak{B}}(a) = \phi(a) \geqslant 0$. $\qquad\square$

Using above proposition and the Hahn-Banach Theorem, we can get the corollary.

Corollary 4. *If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{B}$ is a C^*-subalgebra of $\mathfrak{A}$, then every state on $\mathfrak{B}$ can extend to $\mathfrak{A}$.*

Now, we can construct the a representation of a C^*-algebra. A representation $(\pi, \mathcal{H})$ of a C^*-algebra $\mathfrak{A}$ is called cyclic if there is a unit cyclic vector e s.t. $\overline{\pi(\mathfrak{A})e} = \mathcal{H}$.

Theorem 5 (Gelfand-Naimark-Segal Construction). *Let $\mathfrak{A}$ be a C^*-algebra and $\mathcal{S}_{\mathfrak{A}}$ be the coincided state space.*

1) *If $\phi \in \mathcal{S}_{\mathfrak{A}}$, then there is a cyclic representation $(\pi_\phi, \mathcal{H}_\phi)$ with the unit cyclic vector e_ϕ s.t.*

$$\phi(a) = \langle \pi_\phi(a)e_\phi, e_\phi \rangle, \quad \forall\, a \in \mathfrak{A}$$

2) *If $(\pi, \mathcal{H})$ is a cyclic representation with the unit cyclic vector e, then there is a $\phi \in \mathcal{S}_{\mathfrak{A}}$ defined as*

$$\phi(a) = \langle \pi(a)e, e \rangle, \quad \text{for } a \in \mathfrak{A}$$

And for the $(\pi_\phi, \mathcal{H}_\phi)$ defined as above mention, $\pi_\phi \cong \pi$.

Proof. For 1), the prove can be completed by several nature steps.

1) Constructing semi-inner product: Define a semi-inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{A}$ like

$$\langle x, y \rangle = \phi(y^*x), \quad \text{for } x, y \in \mathfrak{A}$$

2) Constructing $\mathscr{H}_\phi$: $\langle \cdot, \cdot \rangle$ is just a semi-inner product on $\mathfrak{A}$, thus we need to make it be nondegenerate. The nature method is constructing a equilaten relation or a closed ideal, and then inducing a quotient space. Naturally, we define

$$\mathfrak{B} = \{ \ x \in \mathfrak{A} \colon \phi(x) = 0 \ \}$$

Clearly, $\mathfrak{B}$ is a hereditary subalgebra. Then by the **Theorem 3** in the subsection **2.2.5**, $\mathfrak{B}$ can induced a closed left ideal

$$\mathfrak{I} = \{ \ x \in \mathfrak{A} \colon x^*x \in \mathfrak{B} \ \} = \{ \ x \in \mathfrak{A} \colon \phi(x^*x) = 0 \ \}$$

And define the inner product on $\mathfrak{A}/\mathfrak{I}$ as

$$\langle x + \mathfrak{I}, y + \mathfrak{I} \rangle = \phi(y^*x)$$

Therefore, we can easily check that $\mathfrak{A}/\mathfrak{I}$ is a inner product space. Then let $\mathscr{H}_\phi$ be the completion of $\mathfrak{A}/\mathfrak{I}$.

3) Constructing π_ϕ: Firstly, let $\pi_\phi(a)$ be defined on $\mathfrak{A}/\mathfrak{I}$ for $a \in \mathfrak{A}$.

$$\pi_\phi(a) \colon \quad \begin{aligned} \mathfrak{A}/\mathfrak{I} &\longrightarrow \mathfrak{A}/\mathfrak{I} \\ x + \mathfrak{I} &\longmapsto ax + \mathfrak{I} \end{aligned}$$

But since

$$\begin{aligned} \|ax + \mathfrak{I}\|^2 &= \langle ax + \mathfrak{I}, ax + \mathfrak{I} \rangle = \phi(ax^*ax) \\ &\leqslant \|a\|^2 \, \phi(x^*x) = \|a\|^2 \, \|x + \mathfrak{I}\|^2 \end{aligned}$$

$\|\pi_\phi(a)\| \leqslant \|a\|$. Therefore, $\pi_\phi(a)$ can extend to $\mathscr{H}_\phi$ for any $a \in \mathfrak{A}$. $\pi_\phi \colon \mathfrak{A} \to \mathscr{B}(\mathscr{H}_\phi)$ is a representation.

4) Check the conditions: Let $e_\phi = 1 + \mathfrak{I}$. Then

$$\pi_\phi(\mathfrak{A})e_\phi = \{ \ a + \mathfrak{I} \colon a \in \mathfrak{A} \ \} = \mathfrak{A}/\mathfrak{I}$$

Therefore, π_ϕ with e_ϕ is indeed a cyclic representation of $\mathfrak{A}$. And clearly,

$$\langle \pi_\phi(a)e_\phi, e_\phi \rangle = \phi(a)$$

For 2), we just need to construct a unitary from $\mathscr{H}_\phi$ to $\mathscr{H}$. By observation,

$$\langle \pi_\phi(a)e_\phi, e_\phi \rangle = \langle \pi(a)e, e \rangle$$

and the facts that $\mathscr{H}_\phi = \overline{\pi_\phi(\mathfrak{A})e_\phi}$ and $\mathscr{H} = \overline{\pi(\mathfrak{A})e}$, we can find the unitary U. Firstly, let U be defined on $\pi_\phi(\mathfrak{A})e_\phi$

$$U \colon \quad \begin{aligned} \pi_\phi(\mathfrak{A})e_\phi &\longrightarrow \pi(\mathfrak{A})e \\ \pi_\phi(a)e_\phi &\longmapsto \pi(a)e \end{aligned}$$

Since
$$\|\pi(a)e\|^2 = \langle \pi(a)e, \pi(a)e \rangle = \phi(a^*a) = \|\pi_\phi(a)e_\phi\|^2$$

U can extend to an unitary from $\mathscr{H}_\phi$ to $\mathscr{H}$. And

$$U\pi_\phi(a)(\pi_\phi(x)e_\phi) = U\pi_\phi(ax)e_\phi = \pi(ax)e = \pi(a)(\pi(x)e) = \pi(a)U(\pi_\phi(x)e_\phi)$$

Then $U\pi_\phi U^* = \pi$, $\pi_\phi \cong \pi$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. In fact, in above theorem, if ϕ is a positive functional, the results is also true.

By above theorem, our faithful representation of a C^*-algebra can not be constructed by a state, since π_ϕ is not injective. But we may choose enough many states to construct a faithful representation. Therefore, we need more properties of states.

Proposition 6. *If $\mathfrak{A}$ is C^*-algebra and $a \in \mathfrak{A}$ is self-adjoint, let $\alpha = \min \sigma(a)$ and $\alpha = \max \sigma(a)$, then*

$$[\alpha, \beta] = \{ \phi(a) \colon \phi \in \mathcal{S}_\mathfrak{A} \}$$

Proof. Let $\mathfrak{B} = C^*(a)$. Then $\mathfrak{B} = \{f(a) \colon f \in C(\sigma(a))\}$. For $\phi \in \mathcal{S}_\mathfrak{A}$ and $\phi_0 = \phi|_\mathfrak{B}$, then there is a measure μ s.t.

$$\phi(f(a)) = \phi_0(f(a)) = \int_{\sigma(a)} f d\mu, \quad \forall\, f \in C(\sigma)$$

In particular, $\phi(a) = \int_{\sigma(a)} t d\mu \in [\alpha, \beta]$.
Conversely, if $\alpha \leqslant t_0 \leqslant \beta$, define $\phi_0 \in \mathcal{S}_\mathfrak{B}$ as

$$\phi_0(f(a)) = \frac{t_0 - \alpha}{\beta - \alpha} f(\alpha) + \frac{\beta - t_0}{\beta - \alpha} f(\alpha)$$

Then ϕ_0 can extend to $\mathfrak{A}$, and $t_0 = \phi(\alpha)$. $\qquad\qquad\qquad\qquad\square$

By this proposition, we can get an important corollary.

Corollary 7. *If $\mathfrak{A}$ is a C^*-algebra and $a \in \mathfrak{A}$ and $\mathcal{S}$ is wk^*-dense subset of $\mathcal{S}_\mathfrak{A}$, then*
$$\|a\|^2 = \sup \{ \phi(a^*a) \colon \phi \in \mathcal{S} \}$$

Remark. In fact, we can easily check that $\mathcal{S}_\mathfrak{A} \subset \mathfrak{A}^*$ is a wk^*-compact convex subset. And this corollary is equivalent to saying that $\mathcal{S}$ can seperate the points in $\mathfrak{A}_+$.

Now, using this corollary, we can finally construct a faithful representation.

Theorem 8 (Gelfand-Naimark Theorem). *Every C^*-algebra $\mathfrak{A}$ has a faithful representation $(\pi, \mathscr{H})$. Moreover, $\mathscr{H}$ is separable if and only if there are countable number of states on $\mathfrak{A}$ that can separates points in $\mathfrak{A}_+$, each of which defines a separable representation. In particular, each separable C^*-algebra has a faithful, separable representation.*

Proof. Let $\mathcal{S}$ be wk^*-dense subset of $\mathcal{S}_{\mathfrak{A}}$. Define

$$\mathscr{H} = \oplus\{ \; \mathscr{H}_\phi \colon \phi \in \mathcal{S} \; \}$$

$$\pi = \oplus\{ \; \pi_\phi \colon \phi \in \mathcal{S} \; \}$$

Then we can see

$$
\begin{aligned}
\|a\|^2 &= \sup\{ \; \phi(a^*a) \colon \phi \in \mathcal{S} \; \} \\
&= \sup\{ \; \langle \pi_\phi(a)e_\phi, \pi_\phi(a)e_\phi \rangle \colon \phi \in \mathcal{S} \; \} \\
&= \sup\{ \; \|\pi_\phi(a)e_\phi\|^2 \colon \phi \in \mathcal{S} \; \} \\
&= \|\pi(a)e\|^2 \\
&\leqslant \|\pi(a)\|^2
\end{aligned}
$$

And in the GNS Construction, we have seen $\|\pi_\phi(a)\| \leqslant \|a\|$ for any $\phi \in \mathcal{S}_{\mathfrak{A}}$. Therefore,

$$\|a\| \geqslant \|\pi(a)\|$$

Thus $\|\pi(a)\| = \|a\|$ i.e. $(\pi, \mathscr{H})$ is indeed a faithful representation.

If $\mathscr{H}$ is separable, let $\{e_n\}$ be the dense subset of $\{h \in \mathscr{H} \colon \|h\| = 1\}$. Define $\mathscr{H}_n = \overline{\pi(\mathfrak{A})e_n}$ and $\pi_n = \pi|_{\mathscr{H}_n}$. Therefore, in above construction,

$$\{(\pi_\phi, \mathscr{H}_\phi, e_\phi) \colon \phi \in \mathcal{S}\}$$

can be replaced by

$$\{(\pi_n, \mathscr{H}_n, e_n) \colon n \in \mathbb{N}\}$$

Then for $a \in \mathfrak{A}_+$, there exists $b \in \mathfrak{A}$ s.t. $a = b^*b$, if for $n \in \mathbb{N}$

$$0 = \phi_n(a) = \langle \pi_n(b^*b)e_n, e_n \rangle = \|\pi(b)e_n\|^2 \tag{$*$}$$

But $\|b\| = \sup_n \|\pi(b)e_n\|$, that implies $a = 0$.
Conversely, if $\{\phi_n\}$ is these states, we can use the GNS Construction to get $\{(\pi_n, \mathscr{H}_n, e_n) \colon n \in \mathbb{N}\}$. Similarly, we can use direct sum to get the representation $(\pi, \mathscr{H})$. Then by using the $(*)$, we can know this representation is definitely faithful.

Finally, if $\mathfrak{A}$ is separable, then the closed unit ball in $\mathfrak{A}^*$ is wk^*-compactly metrizable. Therefore, there is a countable wk^*-dense subset of $\mathcal{S}_{\mathfrak{A}}$. Thus by above corollary, this subset can separate the points in $\mathfrak{A}_+$ $\qquad \square$

Remark. Therefore, any C^*-algebra can be isometrically imbedded in a operator algebra on some Hilbert space. Researching a abstract C^*-algebra is actually researching the C^*-subalgebra of an operator algebra on some Hilbert space.

Now, we can see the importance of states or positive funtionals. Thus, we want to find the relation between general linear functionals and the positive linear functionals. Firstly, we observe that for a positive functional ϕ, $\phi(a^*) = \overline{\phi(a)}$ for all a.

Definition 9. If $\mathfrak{A}$ is a C^*-algebra and L is a linear functional, L is called self-adjoint if
$$L(a^*) = \overline{L(a)}, \quad \forall\, a \in \mathfrak{A}$$
Or equivalently, $L(a) \in \mathbb{R}$ for all $a \in \operatorname{Re}\mathfrak{A}$.

The bounded self-adjoint linear functional in $\mathfrak{A}^*$ is like the self-adjoint element in $\mathfrak{A}$. So by the fact that any self-adjoint element can be a linear combination of two positive element, we want to find similar result of bounded linear functionals. There is a clearly lemma of bounded self-adjoint linear functional.

Lemma 10. *If $\mathfrak{A}$ is a C^*-algebra and L is a bounded self-adjoint linear functional, then*

$$\|L\| = \sup\{\, L(a)\colon a \in \operatorname{Re}\mathfrak{A}, \ \|a\| \leqslant 1 \,\}$$

Theorem 11 (Jordan Decomposition)**.** *If $\mathfrak{A}$ is a C^*-algebra and L is a bounded self-adjoint linear functional, then there are positive linear functionals ϕ_+ and ϕ_- s.t.*

$$L = \phi_+ - \phi_- \quad and \quad \|L\| = \|\phi_+\| + \|\phi_-\|$$

Proof. Let Ω be the set of all bounded self-adjoint linear functional in the unit ball of $\mathfrak{A}^*$ and $\tilde{\Omega}$ be the wk^*-closed convex hull of $\mathcal{S}_\mathfrak{A} \cup (-\mathcal{S}_\mathfrak{A})$.

1) Claim: $\Omega = \tilde{\Omega}$
   Clearly, $\tilde{\Omega} \subset \Omega$. If there is a $L_0 \in \Omega \backslash \tilde{\Omega}$, then there exists a $x_0 \in \mathfrak{A}$ separating L_0 and $\tilde{\Omega}$, and let $a_0 = \operatorname{Re} x_0$, we have

$$L(a_0) \leqslant \alpha < L_0(a_0), \quad \forall\, L \in \tilde{\Omega}$$

That means

$$\|a_0\| = \sup\{|\phi(a_0)| : \phi \in \mathcal{S}_\mathfrak{A}\} \leqslant \alpha < L_0(a_0)$$

but $\|L_0\| \leqslant 1$, which is a contradiction.

2) Claim: $\Omega = \{ s\phi - t\psi : \phi, \psi \in \mathcal{S}_\mathfrak{A}, \ s + t = 1 \ \& \ s, t \geqslant 0 \}$

By the fact that $(s, t, \phi, \psi) \to s\phi - t\psi$ is continuous and $\mathcal{S}_\mathfrak{A}$ is wk^*-compact, above claim is clearly true.

3) Existence: Therefore, for any self-adjoint linear functional L with $\|L\| = 1$, there are two $\phi, \psi \in \mathcal{S}_\mathfrak{A}$, and positive real numbers s, t with $s + t = 1$ s.t. $L = s\phi - t\psi$. Then let $\phi_+ = s\phi$ and $\phi_- = t\phi$, then $L = \phi_+ - \phi_-$ and

$$\|\phi_+\| + \|\phi_-\| = s + t = 1 = \|L\|$$

For general L, just let $L/\|L\|$ for $L \neq 0$, the statements are true. $\qquad \square$

In fact, for any $f \in \mathfrak{A}^*$, $f = \phi + i\psi$ for some self-adjoint linear funtional ϕ, ψ, where

$$\phi(a) = \frac{f(a) + \overline{f(a^*)}}{2} \ \& \ \psi(a) = \frac{f(a) - \overline{f(a^*)}}{2i}$$

By combining this and the Jordan Decomposition Theorem, we can have the following corollaries.

Corollary 12. *Let $\mathfrak{A}$ be a C^*-algebra and $\pi \colon \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ be the faithful representation constructed in the Gelfand-Naimark Theorem.*

1) *Any bounded linear functional can be a linear combination of four positive linear functionals.*

2) *If $L \in \mathfrak{A}^*$, then there are $g, h \in \mathscr{H}$, s.t. for any $a \in \mathfrak{A}$*

$$L(a) = \langle \pi(a)g, h \rangle$$

2.2.7 Representations

Definition 1. Let $\pi \colon \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ be a representation of a C^*-algebra $\mathfrak{A}$.

1) If there is no invariant subspace for $\pi(\mathfrak{A})$, π is called algebraically irreducible. If there is no invariant closed subspace, π is called topologically irreducible.

2) If $\overline{\pi(\mathfrak{A})e} = \mathscr{H}$ for some $e \in \mathscr{H}$, π is called cyclic and e is called a cyclic vector for π.

3) If $\overline{\pi(\mathfrak{A})\mathscr{H}} = \mathscr{H}$, π is called non-degenerate.

Remark. Since π is $*$-homomorphism and $\mathfrak{A}$ is $*$-closed, the definition of irreducibily of π is valid. In fact, if π is algebraically irreducible, then π is clearly topologically irreducible. But the converse is also true. Then we have,

$$\pi \text{ irreducible} \Rightarrow \pi \text{ cyclic} \Rightarrow \pi \text{ is non-degenerate}$$

Theorem 2. *Every representation of a C^*-algebra is equivalent to the direct sum of cyclic representations.*

Proof. Let $\pi\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ be a representation and $e \in \mathscr{H}$ be any unit element. Then $\mathscr{M} = \overline{\pi(\mathfrak{A})e}$ is clearly a invariant space for π. If $\mathscr{M} \neq \mathscr{H}$, then $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$. We have another unit vector $e^{'} \in \mathscr{M}^\perp$, then let $\mathscr{M}^{'} = \overline{\pi(\mathfrak{A})e^{'}}$ and $\mathscr{M}^{'} \perp \mathscr{M}$. If $\mathscr{M}^{'} \neq \mathscr{M}^\perp$, we can continue above process again. By induction and the Zorn's Lemma, the theorem can be obtained. $\qquad\square$

Theorem 3. *If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, then every representation $\rho\colon \mathfrak{I} \to \mathscr{B}(\mathscr{H})$ can be extended to a representation $\tilde{\rho}\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H})$. Moreover, if ρ is non-degenerate, then $\tilde{\rho}$ is unique.*

Proof. Firstly, assume ρ is non-degenerate and by above theorem we can also assume ρ is cyclic with the cyclic vector e. Then for any $a \in \mathfrak{A}$, we define

$$\tilde{\rho(a)}\colon \quad \begin{array}{ccc} \rho(\mathfrak{I})e & \longrightarrow & \mathscr{H} \\ \rho(x)e & \longmapsto & \rho(ax)e \end{array}$$

And since for an approximate identity $\{e_i\}$ for $\mathfrak{I}$, we have

$$\begin{aligned} \|\rho(ax)e\| &= \lim_i \|\rho(ae_i x)e\| \\ &\leqslant \lim_i \|\rho(ae_i)\rho(x)e\| \\ &\leqslant \sup_i \|ae_i\|\,\|\rho(x)e\| \\ &\leqslant \|a\|\,\|\rho(x)e\| \end{aligned}$$

i.e. $\|\tilde{\rho}(a)\| \leqslant \|a\|$, $\tilde{\rho}(a)$ can extend to $\mathscr{H}$. Then, we can check that $\tilde{\rho}\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ is indeed a representation.
If $T \in \mathscr{B}(\mathscr{H})$ s.t. for any $x \in \mathfrak{I}$, $T\rho(x) = \rho(ax)$, then

$$T\rho(x)h = \tilde{\rho}\rho(x)h, \quad \forall\, x \in \mathfrak{I} \,\&\, \forall\, h \in \mathscr{H}$$

Since ρ is non-degenerate, $T = \tilde{\rho}(a)$.
Now, if ρ is degenerate, let $\mathscr{H}_0 = \overline{\rho\mathfrak{I}}$, and $\rho_0(x) = \rho(x)|_{\mathscr{H}_0}$, then $\rho_0\colon \mathfrak{I} \to \mathscr{B}(\mathscr{H}_0)$ is a non-degenerate representation. Therefore, it can extend to $\tilde{\rho}_0\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H}_0)$. Let $\kappa\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H}_0^\perp)$ be any representation. Then $\tilde{\rho} = \tilde{\rho}_0 \oplus \kappa$ is the extension. $\qquad\square$

Corollary 4. *If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, and $(\rho, \mathscr{H})$ and $(\kappa, \mathscr{K})$ are two representations of $\mathfrak{A}$ s.t.*

1) $\rho|_{\mathfrak{I}}$ is a non-degenerate representation of $\mathfrak{I}$.

2) $\rho|_{\mathfrak{I}} \cong \kappa|_{\mathfrak{I}}$.

then $\rho \cong \kappa$

We know the irreducible representation play a important role in the representation theory. Therefore, we want to find the necessary and sufficient conditions make a representation be irreducible.

Definition 5. If $\mathfrak{A} \subset \mathscr{B}(\mathscr{H})$ is a C^*-subalgebra, the commutant of $\mathfrak{A}$

$$\mathfrak{A}' = \{\ T \in \mathscr{B}(\mathscr{H}) \colon TS = ST,\ \forall\ S \in \mathfrak{A}\ \}$$

Remark. Clearly, $\mathfrak{A}'$ is a C^*-algebra.

Theorem 6. *Let $\mathfrak{A}$ be a C^*-algebra and $(\rho, \mathscr{H})$ be a representation, the following statements are equivalent. Then ρ is irreducible if and only if $\rho(\mathfrak{A})' = \mathbb{C}$*

Proof. If ρ is irreducible and $\mathbb{C} \neq \rho(\mathfrak{A})'$, then there is a $T \in \rho(\mathfrak{A})' \backslash \mathbb{C}$. Therefore, there is a proper projection $P \in \rho(\mathfrak{A})' \backslash \mathbb{C}$. Then $P\mathscr{H}$ is a proper invariant space for ρ, which is a contradiction. The converse is clearly true by using similar prove as above. $\qquad \square$

Remark. The existence of P is because that $\rho(\mathfrak{A})'$ is a von Neumann algebra.

Firstly, by the GNS construction, the representations generated by positive linear functionals are important. So we need more properties of these representations.

Proposition 7. *Let $\mathfrak{A}$ be a C^*-algebra and ψ and ϕ be two positive linear functionals on $\mathfrak{A}$ and $(\pi, \mathscr{H}, e)$ be the representation generated by ϕ. Then $\psi \leqslant \phi$ if and only if there is unique a $T \in \pi(\mathfrak{A})'$ with $0 \leqslant T \leqslant 1$, s.t. $\psi(a) = \langle \pi(a)Te, e \rangle$ for all $a \in \mathfrak{A}$.*

Proof. Assume there is a $T \in \pi(\mathfrak{A})'$ with $0 \leqslant T \leqslant 1$. If $a \in \mathfrak{A}_+$, then

$$0 \leqslant T^{\frac{1}{2}} \pi(a) T^{\frac{1}{2}} = \pi(a)T = \pi(a)^{\frac{1}{2}} T \pi(a)^{\frac{1}{2}} \leqslant \pi(a)$$

Therefore, $\psi(a) \leqslant \phi(a)$.

Conversely, we define a sesquilinear form on $\pi(\mathfrak{A})e$,

$$f(\pi(a)e, \pi(b)e) = \psi(b^*a)$$

And since $\psi \leqslant \phi$, by CBS Inequality, we have

$$
\begin{aligned}
|f(\pi(a)e, \pi(b)e)| &= |\psi(b^*a)|^2 \\
&\leqslant \psi(a^*a)\psi(b^*b) \leqslant \phi(a^*a)\phi(b^*b) \\
&= \|\pi(a)e\|^2 \|\pi(b)e\|^2
\end{aligned}
$$

Therefore, f is bounded and can be extended to $\mathscr{H} = \overline{\pi(\mathfrak{A})e}$. By **Theorem 10** in the subsection **1.5.1**, we have $T \in \mathscr{B}(\mathscr{H})$, s.t.

$$\psi(b^*a) = \langle T\pi(a)e, \pi(b)e \rangle$$

In particular, $\psi(a) = \langle T\pi(a)e, e \rangle$. For $a, b, c \in \mathfrak{A}$,

$$
\begin{aligned}
\langle T\pi(a)\pi(b)e, \pi(c)e \rangle &= \langle T\pi(ab)e, \pi(c)e \rangle = \psi(c^*(ab)) \\
&= \psi((a^*c)^*b) = \langle T\pi(b)e, \pi(a^*c)e \rangle \\
&= \langle \pi(a)T\pi(b)e, \pi(c)e \rangle
\end{aligned}
$$

And by the fact that $\mathscr{H} = \overline{\pi(\mathfrak{A})e}$, $T\pi(a) = \pi(a)T$ i.e. $T \in \pi(\mathfrak{A})'$. And the uniqueness of T can be easily checked. $\qquad\square$

Proposition 8. *Let $(\pi_j, \mathscr{H}_j, e_j)$ for $j = 1, 2$ be two cyclic representations of a C^*-algebra $\mathfrak{A}$. Then $\pi_1 \cong \pi_2$ by a unitary $U \colon \mathscr{H}_1 \to \mathscr{H}_2$ satisfying $Ue_1 = e_2$ if and only if for any $a \in \mathfrak{A}$*

$$\langle \pi_1(a)e_1, e_1 \rangle = \langle \pi_2(a)e_2, e_2 \rangle$$

Proof. We define for any $a \in \mathfrak{A}$

$$
\begin{array}{cccc}
U \colon & \pi_1(\mathfrak{A})e_1 & \longrightarrow & \pi_2(\mathfrak{A})e_2 \\
& \pi_1(a)e_1 & \longmapsto & \pi_2(a)e_2
\end{array}
$$

If $\langle \pi_1(a)e_1, e_1 \rangle = \langle \pi_2(a)e_2, e_2 \rangle$,

$$\|\pi_2(a)e_2\|^2 = \langle \pi_2(a)e_2, \pi_2(a)e_2 \rangle = \langle \pi_1(a)e_1, \pi_1(a)e_1 \rangle = \|\pi_1(a)e_1\|^2$$

U is a unitary on $\pi_1(\mathfrak{A})e_1$ and U can extend to $\mathscr{H}_1$. And since

$$U(\pi_1(a)\pi_1(b)e_1) = \pi_2(ab)e_2 = \pi_2(a)\pi_2(b)e_2 = \pi_2(a)U(\pi_1(b)e_1)$$

U is the unitary make $\pi_1 \cong \pi_2$. The converse is trivial. $\qquad\square$

Remark. Using this proposition, we can strengthen the result of the **Corollary** 4 in the subsection **2.2.6**. In fact, every state on an closed ideal $\mathfrak{I}$ of a C^*-algebra $\mathfrak{A}$ can extend to a unique state on $\mathfrak{A}$. If there are two extensions, they can generate two cyclic representations. And these two representations are equivalent on the ideal, thus these two representations are equivalent. And by above proposition, these two extensions are equal.

By above two propositions, we can get the relationship between positive functionals and the generated representations.

Corollary 9. *Let $\mathfrak{A}$ be a C^*-algebra and ψ and ϕ be two positive linear functionals on $\mathfrak{A}$ with $\psi \leqslant \phi$. Then π_ψ is equivalent to a subrepresentation of π_ϕ.*

Proof. By above proposition, there is a $T \in \pi(\mathfrak{A})^{'}$ s.t.

$$\psi(a) = \langle \pi_\phi(a)Te_\phi, e_\phi \rangle$$

Put $\mathscr{M} = \overline{\pi_\phi(\mathfrak{A})T^{\frac{1}{2}}e_\phi}$. Clearly, $\mathscr{M}$ reduces $\pi_\phi(\mathfrak{A})$. Since

$$\langle \pi_\psi(a)e_\psi, e_\psi \rangle = \psi(a) = \langle \pi_\phi(a)T^{\frac{1}{2}}e_\phi, T^{\frac{1}{2}}e_\phi \rangle$$

by above proposition $\pi_\psi \cong \pi_\phi|_{\mathscr{M}}$ $\qquad\qquad\square$

Definition 10. A state $\phi \in \mathcal{S}_\mathfrak{A}$ on some C^*-algebra $\mathfrak{A}$ is called pure if for any positive linear functional ψ on $\mathfrak{A}$ with $\psi \leqslant \phi$, there is a sclalar $\lambda \in [0,1]$ s.t. $\psi = \lambda\phi$.

Theorem 11. *Let $\mathfrak{A}$ be a C^*-algebra and $(\rho, \mathscr{H})$ be a representation, the following statements are equivalent. Then ρ is irreducible if and only if ρ is equivalent to some cyclic representation generated by a pure state.*

Proof. Suppose ρ is irreducible. Then for any nonzero unit vector $h \in \mathscr{H}$, $\overline{\rho(\mathfrak{A})h} = \mathscr{H}$. Define

$$\begin{array}{rccc} \phi\colon & \mathfrak{A} & \longrightarrow & \mathbb{C} \\ & a & \longmapsto & \langle \rho(a)h, h \rangle \end{array}$$

Then ϕ is clearly a state. If $(\pi, \mathscr{K}, e)$ is the representation generated by ϕ, then

$$\langle \pi(a)e, e \rangle = \langle \rho(a)h, h \rangle$$

Thus, by above proposition, $\pi \cong \rho$.
If there is a positive linear functional $\psi \leqslant \phi$, then

$$\exists \, T \in \pi(\mathfrak{A})^{'}, \text{ s.t. } \psi(a) = \langle \pi(a)Te, e \rangle$$

But by above theorem, $\pi(\mathfrak{A})^{'} = \mathbb{C}$, therefore $T = \lambda \in \mathbb{C}$, $\psi = \lambda\phi$.

Conversely, if $(\rho, \mathscr{H}, e) \cong (\pi_\phi, \mathscr{H}_\phi, e_\phi)$ for some pure state ϕ,

$$\phi(a) = \langle \rho(a)e, e \rangle = \langle \pi_\phi e_\phi, e_\phi \rangle$$

Then if a projection $P \in \rho(\mathfrak{A})^{'}$, then

$$\psi(a) = \langle \rho(a)Pe, Pe \rangle = \langle \rho(a)Pe, e \rangle$$

is a positive linear functional and $\psi \leqslant \phi$, therefore $\psi = \lambda\phi$ for some $\lambda \in [0,1]$, i.e.

$$\langle \rho(a)Pe, e \rangle = \langle Pe, \rho(a^*)e \rangle = \lambda \langle \rho(a)e, e \rangle = \langle \lambda e, \rho(a^*)e \rangle$$

Since $\mathscr{H} = \overline{\rho(\mathfrak{A})e}$, $P = \lambda$, thus $P = 1$ or 0. Therefore ρ has no proper invariant space. $\qquad\qquad\square$

Proposition 12. *Let $\mathfrak{A}$ be a C^*-algebra and $\phi \in \mathcal{S}_{\mathfrak{A}}$. Then ϕ is pure if and only if ϕ is the extreme point of $\mathcal{S}_{\mathfrak{A}}$.*

Remark. This proposition can be obtained by definition. In fact, we usually use the extreme point to define the pure state.

We have similar results for pure states as general states. Firstly, we need a lemma.

Proposition 13. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{I}$ be a closed ideal of $\mathfrak{A}$.*

1) *If $(\rho, \mathcal{H})$ is an irreducible representation of $\mathfrak{I}$, then it can extend to a unique irreducible representation $(\tilde{\rho}, \mathcal{H})$ on $\mathfrak{A}$.*

2) *If $(\rho, \mathcal{H})$ is an irreducible representation of $\mathfrak{A}$, then $(\rho|_{\mathfrak{I}}, \mathcal{H})$ is an irreducible representation of $\mathfrak{I}$.*

Proof. For 1), since ρ is irreducible, ρ is non-degenerate. By the **Theorem 3** in this subsection, there is a unique extension $\tilde{\rho}$ of $\mathfrak{A}$. And clearly,

$$\mathbb{C} \subset \tilde{\rho}(\mathfrak{A})' \subset \rho(\mathfrak{I})' = \mathbb{C}$$

$\tilde{\rho}(\mathfrak{A})' = \mathbb{C}$, i.e. $\tilde{\rho}$ is irreducible.

For 2), ρ can be generated by a pure state ϕ on $\mathfrak{A}$. Then $\phi|_{\mathfrak{I}}$ is a pure state on $\mathfrak{I}$ since if $\phi|_{\mathfrak{I}} = s\psi_1 + (1-s)\psi_2$ for some $\psi_1, \psi_2 \in \mathcal{S}_{\mathfrak{I}}$ and $s \in (0,1)$, then ψ_1, ψ_2 have extension ψ_1', ψ_2' on $\mathfrak{A}$, $\phi = s\psi_1' + (1-s)\psi_2'$ by the uniqueness of extensions, which is a contradiction. Therefore, $\phi|_{\mathfrak{I}}$ can generate an irreducible representation κ on $\mathfrak{I}$ and $\kappa \cong \rho|_{\mathfrak{I}}$. $\square$

Proposition 14. *Let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. Then every pure state on $\mathfrak{B}$ can extend to a pure state on $\mathfrak{A}$.*

Proof. Define $T \colon \mathcal{S}_{\mathfrak{A}} \to \mathcal{S}_{\mathfrak{B}}$ by $\phi \to \phi|_{\mathfrak{B}}$. Then T is surjective, affine and wk^*-continuous. Therefore, T maps extreme points to extreme points. Thus the proposition holds. $\square$

Corollary 15. *Let $\mathfrak{A}$ be a C^*-algebra and $a \in \mathfrak{A}$. Then*

$$\|a\|^2 = \sup \{ \, \phi(a^*a) \colon \phi \text{ is a pure state on } \mathfrak{A} \, \}$$

Proof. If $\mathfrak{A} = C(X)$ for some compact space X, this corollary is true. And then by above proposition, any pure state on $C^*(a^*a)$ can extend to a pure state on $\mathfrak{A}$. Therefore, this corollary holds. $\square$

Theorem 16. *If $\mathfrak{A}$ is a C^*-algebra, then there is a family of irreducible representations $\{\pi_i\}_{i \in I}$ s.t. $\oplus_{i \in I}\pi_i$ is a faithful representation of $\mathfrak{A}$.*

Proof. Let $\{\pi_i\}_{i \in I}$ be generated by all pure state. Then by above corollary, $\oplus_{i \in I}\pi_i$ is faithful. $\square$

2.3 Operators on Hilbert Spaces

We have known that each C^*-algebra can be isometrically imbedded a $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. Therefore, we just research operator algebras other than C^*-algebras. Firstly, we need some general theories of operators on a Hilbert space.

2.3.1 Spectrums for Operators

By using the fact that $T \to T^*$ in $\mathscr{B}(\mathscr{H})$ is bijective and isometrically antilinear, we have the following proposition.

Proposition 1. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Then*

$$\sigma(T^*) = \sigma(T)^*$$

For self-adjoint operators in $\mathscr{B}(\mathscr{H})$, we have some extral properties related to the inner product.

Proposition 2. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Then T is self-adjoint if and only if $\langle Th, h \rangle \in \mathbb{R}$ for any $h \in \mathscr{H}$.*

Proof. If $\langle Th, h \rangle \in \mathbb{R}$ for any $h \in \mathscr{H}$, then for any $h, g \in \mathscr{H}$ and $\alpha \in \mathbb{C}$,

$$\langle T(h + \alpha g), h + \alpha g \rangle = \langle Th, h \rangle + \overline{\alpha}\langle Th, g \rangle + \alpha\langle Tg, h \rangle + |\alpha|^2 \langle Tg, g \rangle \in \mathbb{R}$$

Then by taking its conjugate,

$$\begin{aligned} \overline{\alpha}\langle Th, g \rangle + \alpha\langle Tg, h \rangle &= \overline{\alpha}\langle h, Tg \rangle + \alpha\langle g, Th \rangle \\ &= \overline{\alpha}\langle T^*h, g \rangle + \alpha\langle T^*g, h \rangle \end{aligned}$$

Taking $\alpha = 1$ and i, we have

$$\begin{aligned} \langle Th, g \rangle + \langle Tg, h \rangle &= \langle T^*h, g \rangle + \langle T^*g, h \rangle \\ i\langle Th, g \rangle - i\langle Tg, h \rangle &= -i\langle T^*h, g \rangle + i\langle T^*g, h \rangle \end{aligned}$$

Therfore, $\langle Th, g \rangle = \langle T^*g, h \rangle$ for any $h, g \in \mathscr{H}$.
The converse is clearly true by definition. $\qquad\square$

Proposition 3. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$ be self-adjoint. Then*

$$\sigma(T) \subset \{\ \langle Th, h \rangle \colon \|h\| = 1\ \}$$

Proof. If $\lambda \notin \{\ \langle Th, h \rangle \colon \|h\| = 1\ \}$, then there is a $\varepsilon > 0$, s.t.

$$\varepsilon < |\lambda - \langle Th, h \rangle| \leqslant \|Th\|, \quad \forall\ h \in \mathscr{H},\ \|h\| = 1$$

Therefore, by the **Lemma 3** in the subsection **1.4.4**,

$$\ker(T - \lambda) = \{0\}\ \&\ \operatorname{ran}(T - \lambda) \text{ is closed}$$

Since T is self-adjoint, $\overline{\lambda} \notin \sigma(T)$, so $\ker(\overline{\lambda} - T) = \{0\}$. Thus

$$\operatorname{ran}(T - \lambda)^{\perp} = \ker(T - \overline{\lambda}) = \{0\}$$

i.e. $\operatorname{ran}(T - \lambda) = \mathscr{H}$. Then by the Inverse Mapping Theorem, $(T - \lambda)^{-1} \in \mathscr{B}(\mathscr{H})$, i.e. $\lambda \notin \sigma(a)$. $\qquad \square$

Corollary 4. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$ be self-adjoint.*

  1) $\|T\| = \sup\{\langle Th, h\rangle \colon \|h\| = 1\}$.

  2) If $\langle Th, h\rangle = 0$ for any $h \in \mathscr{H}$, then $T = 0$.

Proof. For 1),

$$\|T\| = r(T) \leqslant \sup\{\langle Th, h\rangle \colon \|h\| = 1\} \leqslant \|T\|$$

2) is the direct result of 1). $\qquad \square$

Remark. In fact, for any $T \in \mathscr{B}(\mathscr{H})$ with $\langle Th, h\rangle = 0$ for any $h \in \mathscr{H}$, then $T = 0$ since any T can be the linear combination of two self-adjoint operators.

By above proposition, we see the spectrum of operators can be described by using another method. Firstly, we can generize the consequence in **Proposition 3**.

Proposition 5. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Then let*

$$\chi(\lambda) = \inf\{\ \|(T - \lambda)h\| \colon \|h\| = 1\ \}$$

$$\sigma_l(T) = \{\ \lambda \in \mathbb{C} \colon \chi(\lambda) = 0\ \}$$

Proof. By above proposition, we have seen if $\chi(\lambda) > 0$,

$$\ker(T - \lambda) = \{0\}\ \&\ \operatorname{ran}(T - \lambda)\text{ is closed}$$

Then let $\mathscr{K} = \operatorname{ran}(T - \lambda)$, then $T - \lambda \colon \mathscr{H} \to \mathscr{K}$ is a bijection and $(T - \lambda)^{-1} \colon \mathscr{K} \to \mathscr{H}$ is bounded. Let

$$B = (T - \lambda)^{-1}P \colon \mathscr{H} \to \mathscr{H}$$

where P is a projection to $\mathscr{K}$. Then $B \in \mathscr{B}(\mathscr{H})$ and $B(T - \lambda) = 1$. Therefore, $\sigma_l(T) \subset \{\ \lambda \in \mathbb{C} \colon \chi(\lambda) = 0\ \}$.

Conversely, if there is a $B \in \mathscr{B}(\mathscr{H})$ s.t. $B(T - \lambda) = 1$, then for $h \in \mathscr{H}$ with $\|h\| = 1$

$$1 = \|h\| = \|B(T - \lambda)h\| \leqslant \|B\|\|(T - \lambda)h\|$$

Therefore, $\|(T - \lambda)h\| \geqslant \|B\|^{-1}$, i.e. $\chi(\lambda) > 0$. $\qquad \square$

Definition 6. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$.

1) The point spectrum of T is defined as

$$\sigma_p(T) = \{\ \lambda \in \mathbb{C} \colon \ker(T - \lambda) \neq \{0\}\ \}$$

2) The approximate point spectrum of T is defined as

$$\sigma_{ap}(T) = \{\ \lambda \in \mathbb{C} \colon \exists \text{ units } x_n \subset \mathscr{B}(\mathscr{H}) \text{ s.t. } \|(T - \lambda)x_n\| \to 0\ \}$$

Proposition 7. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$.*

1. *$\lambda \notin \sigma_{ap}(T)$ if and only if $\ker(T - \lambda) = \{0\}$ and $\operatorname{ran}(T - \lambda)$ is closed.*

2. *$\partial\sigma(T) \subset \sigma_{ap}(T)$.*

3. *$\partial\sigma(T) \subset \sigma_l(T) \cap \sigma_r(T) = \sigma_{ap}(T) \cap \sigma_{ap}(T^*)$.*

Proof. For 1), if $\lambda \notin \sigma_{ap}(T)$, then there is a constant $C > 0$ s.t.

$$\|(T - \lambda)x\| \geqslant C\|x\|, \quad \forall\ x \in \mathscr{H}$$

Thus by above proposition, $\ker(T - \lambda) = \{0\}$ and $\operatorname{ran}(T - \lambda)$ is closed. Conversely, the condition that means $(T - \lambda)$ is left invertible, thus

$$\|(T - \lambda)x\| \geqslant \|B\|^{-1}\|x\|, \quad \forall\ x \in \mathscr{H}, \ \|x\| = 1$$

where B is the left inverse of T. Therefore, $\lambda \notin \sigma_{ap}(T)$.

For 2), let $\lambda \in \partial\sigma(T)$ and $\{\lambda_n\} \subset \rho(T)$ s.t. $\lambda_n \to \lambda$. Then we know $\left\|(T - \lambda_n)^{-1}\right\| \to \infty$. Otherwise, if $\left\|(T - \lambda_n)^{-1}\right\| < M$ and $|\lambda_n - \lambda| < M^{-1}$ for sufficiently large n,

$$\|(T - \lambda_n) - (T - \lambda)\| < \left\|(T - \lambda_n)^{-1}\right\|^{-1}$$

by the perturbation of inverse, $T - \lambda$ is also invertible. Let $\{x_n\}$ by any sequence of unit vectors in $\mathscr{H}$. Define

$$y_n = \frac{(T - \lambda_n)^{-1}x_n}{\|(T - \lambda_n)^{-1}x_n\|} \tag{2.1}$$

Then $\|(T - \lambda)y_n\| \to 0$, i.e. $\lambda \in \sigma_{ap}(T)$.

For 3), by 1) we have known $\sigma_l(T) = \sigma_{ap}(T)$ and by taking the complex conjugate, $\sigma_r(T^*) = \sigma_{ap}(T^*)$. Therefore, 3) holds. $\qquad\square$

Now, we can prove the inverse of the **Corollary 6** in the subsection **2.2.3**.

Theorem 8. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. If for any $h \in \mathscr{H}$,*

$$\langle Th, h \rangle \geqslant 0$$

then T is positive.

Proof. Firstly, T is self-adjoint by above proposition. And for $\lambda < 0$ and $\|h\| = 1$

$$
\begin{aligned}
\|(T - \lambda)h\| &= \|Th\|^2 - 2\lambda\langle Th, h\rangle + \lambda^2\|h\|^2 \\
&\geqslant -2\lambda\langle Th, h\rangle + \lambda^2\|h\|^2 \\
&\geqslant \lambda^2\|h\|^2
\end{aligned}
$$

Therefore, $\lambda \notin \sigma_l(T)$. Since T is self-adjoint, $\lambda \notin \sigma_r(T)$. Thus $\lambda \notin \sigma(T)$, i.e. $\sigma(T) \subset [0, \infty)$. $\qquad\square$

2.3.2 Polar Decomposition

For a $z \in \mathbb{C}$, it can be decomposed as $z = |z|\, e^{i\theta}$. Similarly, a operator in $\mathscr{B}(\mathscr{H})$ can also be decomposed as a complex number does. Firstly, we need to find the element playing same role as $e^{i\theta}$.

Definition 1. Let $\mathscr{H}$ be a Hilbert space and $U \in \mathscr{B}(\mathscr{H})$ and $\mathscr{M}$ be a closed subspace of $\mathscr{H}$. If $U|_{\mathscr{M}}$ is an isometry and $U|_{\mathscr{M}^\perp} = 0$, then U is called a partial isometry. $\mathscr{M}$ is called the initial space and $\mathscr{N} = U(\mathscr{M})$ is called the final space.

Proposition 2. *If U is a partial isometry, then U^*U is the projection on the initial space $\mathscr{M}$ and UU^* is the projection on the final space $\mathscr{N}$. Moreover, U^* is the partial isometry with the initial space $\mathscr{N}$ and final space $\mathscr{M}$.*

Proof. For any $g, h \in \mathscr{H}$, then $g = g_{\shortparallel} + g_\perp$ and $h = h_{\shortparallel} + h_\perp$, where $g_{\shortparallel}, h_{\shortparallel} \in \mathscr{M}$ and $g_\perp, h_\perp \in \mathscr{M}^\perp$

$$
\begin{aligned}
\langle U^*U(g_{\shortparallel} + g_\perp), h_{\shortparallel} + h_\perp \rangle &= \langle Ug_{\shortparallel}, Uh_{\shortparallel} \rangle = \langle g_{\shortparallel}, h_{\shortparallel} \rangle \\
&= \langle P_{\mathscr{M}}(g_{\shortparallel} + g_\perp), h_{\shortparallel} + h_\perp \rangle
\end{aligned}
$$

Then $U^*U = P_{\mathscr{M}}$. $P_{\mathscr{M}^\perp} = 1 - U^*U$ implies $U(1 - U^*U) = 0$, i.e. $U = UU^*U$ and $\ker U^* = (\operatorname{ran} U)^\perp = \mathscr{N}^\perp$, therefore, UU^* is the projection on $\mathscr{N}$. For $x \in \mathscr{N}$.

$$\langle U^*x, U^*x \rangle = \langle UU^*x, x \rangle = \langle x, x \rangle \tag{2.2}$$

And $\operatorname{ran} U^* = (\ker U)^\perp = \mathscr{M}$, thus U^* is a partial isometry with the initial space $\mathscr{N}$ and the final space $\mathscr{M}$. $\qquad\square$

Corollary 3. *If $U \in \mathscr{B}(\mathscr{H})$, then the following statements are equivalent.*

    1) U is a partial isometry.

2) U^*U is a projection.

3) $U = UU^*U$.

4) U^* is a partial isometry.

5) UU^* is a projection.

6) $U^* = U^*UU^*$.

Theorem 4 (Polar Decomposition). *Let $\mathscr{H}$ be a Hile]bert space and $T \in \mathscr{B}(\mathscr{H})$. Then there is a partial isometry $U \in \mathscr{B}(\mathscr{H})$ with the initial space $(\ker T)^\perp$ and the final space $\overline{\operatorname{ran} T}$, s.t. $T = U\,|T|$. Moreover, if $T = WS$, where $S \geqslant 0$ and W is a partial isometry with $\ker W = \ker S$, then $S = |T|$ and $W = U$.*

Proof. Define

$$\begin{array}{rccc} U: & \operatorname{ran} |T| & \longrightarrow & \operatorname{ran} T \\ & |T|\,h & \longmapsto & Th \end{array}$$

Since

$$\|Th\|^2 = \langle Th, Th \rangle = \langle T^*Th, h \rangle = \langle |T|\,h, |T|\,h \rangle = \||T|\,h\|^2$$

U is a isometry. And thus U can extend to $\overline{\operatorname{ran} |T|}$ and onto $\overline{\operatorname{ran} T}$. Then extending U to $\mathscr{H}$ by setting $U|_{\overline{\operatorname{ran} |T|}^\perp} = 0$. U is a partial isometry with the initial space $\ker |T|^\perp$ and the final space $\overline{\operatorname{ran} T}$. Clearly, $\ker |T| = \ker T$ and $T = U\,|T|$.

If $T = WS$, then $T^*T = SW^*WS = SP_{\ker S^\perp}S = S^2$. Therefore, $S = |T|$. And $W\,|T| = U\,|T|$, thus W ans U has same initial space, then $W = U$. $\square$

2.3.3 Topologies on $\mathscr{B}(\mathscr{H})$

In $\mathscr{B}(\mathscr{H})$, the norm topology may be too stronger, which means norm closed subalgebras in $\mathscr{B}(\mathscr{H})$ do not contain "enough" elements. Therefore, we need some weaker topologies

Definition 1. Let $\mathscr{H}$ be a Hilbert space.

1) The strong operator topology on $\mathscr{B}(\mathscr{H})$, denoted by SOT, is generated by a family of seminorms $\{p_x\}_{x \in \mathscr{H}}$, where $p_x(T) = \|Tx\|$.

2) The weak operator topology on $\mathscr{B}(\mathscr{H})$, denoted by WOT, is denerated by a fanmily of seminorms $\{p_{gh}\}_{g,h \in \mathscr{H}}$, where $p_{gh}(T) = |\langle Tg, h \rangle|$.

Remark. Firstly, by definition and the **Corollary** 4 in the subsection **2.3.1**, we can see

$$\bigcap_{x \in \mathscr{H}} \ker p_x = \{0\} \ \& \ \bigcap_{g,h \in \mathscr{H}} \ker p_{gh} = \{0\}$$

Therefore, both SOT and WOT can let $\mathscr{B}(\mathscr{H})$ be a Hausdorff locally convex t.v.s.. And by the CBS Inequatlity, SOT is stronger than WOT. Moreover,

1) the subbasis of the WOT at T_0 is like

$$U_{gh\varepsilon}(T_0) = \{T \in \mathscr{B}(\mathscr{H}) \colon |\langle (T - T_0)g, h \rangle| < \varepsilon\}$$

Therefore, a net $\{T_\alpha\}$ in $\mathscr{B}(\mathscr{H})$ converges in WOT to T_0 if and only if $\langle T_\alpha g, h \rangle \to \langle T_0 g, h \rangle$ for all $g, h \in \mathscr{H}$.

2) the subbasis of the SOT at T_0 is like

$$V_{x\varepsilon}(T_0) = \{T \in \mathscr{B}(\mathscr{H}) \colon \|(T - T_0)x\| < \varepsilon\}$$

Therefore, a net $\{T_\alpha\}$ in $\mathscr{B}(\mathscr{H})$ converges in TOT to T_0 if and only if $\|T_\alpha x\| \to \|T_0 x\|$ for all $x \in \mathscr{H}$.

Comparing with the norm topology, the SOT is related to the pointwise convergence, where the norm topology is related to the uniform convergence.

Proposition 2. *Let $\mathscr{H}$ be a Hilbert space and L is a linear functional on $\mathscr{B}(\mathscr{H})$. Then the following statements are equivalent.*

1) *L is SOT-continuous.*

2) *L is WOT-continuous.*

3) *There are vectors $f_1, \cdots, f_n, g_1, \cdots, g_n \in \mathscr{H}$ s.t.*

$$L(T) = \sum_{i=1}^{n} \langle Tg_i, f_i \rangle, \quad \forall \, T \in \mathscr{B}(\mathscr{H})$$

Proof. Clearly, 3) $\Rightarrow$ 2) $\Rightarrow$ 1), thus assume 1), then by the **Theorem** 4 in the subsection **1.2.4**, there are $g_1, \cdots, g_n \in \mathscr{H}$ and $\alpha_1, \cdots, \alpha_n > 0$ s.t.

$$\|L(T)\| \leqslant \sum_{i=1}^{n} \alpha_i \|Tg_i\|$$

Replacing g_i by $\alpha_i g_i$ and g_i by $\sqrt{n}g_i$ then

$$\|L(T)\| \leqslant \sum_{i=1}^{n} \|Tg_i\| \leqslant \sqrt{n}(\sum_{i=1}^{n} \|Tg_i\|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} \|Tg_i\|^2)^{\frac{1}{2}}$$

Let $\mathscr{K} = \overline{\{\oplus_{i=1}^n Tg_i \colon T \in \mathscr{B}(\mathscr{H})\}}$ be a closed subspace of $\mathscr{H}^{(n)}$. Define a linear functional F on $\mathscr{K}$ by $F(\oplus_{i=1}^n Tg_i) = L(T)$, since

$$|F(\oplus_{i=1}^n Tg_i)| = \|L(T)\| \leqslant (\sum_{i=1}^n \|Tg_i\|^2)^{\frac{1}{2}}$$

Therefore, F can extends to $\mathscr{H}^{(n)}$ with the conincided inner product and norm. By the Riesz Theorem, there is a $f_1, \cdots, f_n \in \mathscr{H}$ s.t.

$$F(\oplus_{i=1}^n h_i) = \langle \oplus_{i=1}^n h_i, \oplus_{i=1}^n f_i \rangle, \quad \forall \ \oplus_{i=1}^n h_i \in \mathscr{H}^{(n)}$$

In particular, restricting F to $\mathscr{K}$

$$L(T) = F(\oplus_{i=1}^n Tg_i) = \langle \oplus_{i=1}^n Tg_i, \oplus_{i=1}^n f_i \rangle = \sum_{i=1}^n \langle Tg_i, f_i \rangle \qquad \square$$

Proposition 3. *If S is a convex subset of $\mathscr{B}(\mathscr{H})$, then*

$$\overline{S}^{WOT} = \overline{S}^{SOT}$$

Proof. Clearly,
$$\mathscr{T}_{WOT} \subset \mathscr{T}_{SOT}$$
Therefore,
$$\overline{S}^{SOT} \subset \overline{S}^{WOT}$$

Since for linear functionals, SOT-continuity implies WOT-continuity,

$$\overline{(S, WOT)}^{wk} \subset \overline{(S, SOT)}^{wk}$$

And we have known $\overline{(S, WOT)}^{wk} = \overline{(S, WOT)} = \overline{S}^{WOT}$ and $\overline{(S, SOT)}^{wk} = \overline{(S, SOT)} = \overline{S}^{SOT}$, therefore

$$\overline{S}^{SOT} \subset \overline{S}^{WOT} \subset \overline{S}^{SOT} \qquad \square$$

WOT also have a similar property to wk-topology, the Alaoglu's Theorem.

Theorem 4. *The closed unit ball in $\mathscr{B}(\mathscr{H})$ is WOT-compact.*

Proof. Let X_h be closed unit ball in $\mathscr{H}$ with the wk-topology and indexed with $h \in \mathscr{H}$ and B be the closed unit ball in $\mathscr{H}$ with the WOT. Define

$$\tau \colon \quad B \quad \longrightarrow \quad \prod\{X_h \colon \|h\| \leqslant 1\}$$
$$T \quad \longmapsto \quad \tau(T)$$

where $\tau(T)_h = Th$. Then using similarl method, we can prove B is compact.
$\square$

Chapter 3

Normal Operators and Abelian Von Neumann Algebras

For a normal operator N, it can generate an abelian norm closed C^*-subalgebra of $\mathscr{B}(\mathscr{H})$, which is isometrically $*$-isomorphic to $C(\sigma(N))$. But the norm closed C^*-subalgebra may be too "small" to get more interesting results. Therefore, we want to see the WOT-closed C^*-subalgebra $W^*(N)$ generated by a normal operator N. This extension can be naturally done by following processes. Firstly, we will extend $C(X)$ to $B(X, \Omega)$, where $B(X, \Omega)$ denotes all Ω-measurable functions on X, and find the map from $B(X, \Omega)$ to $\mathscr{B}(\mathscr{H})$. Then using these method, we can extend the Continuous Functional Calculus to the Borel Functional Calculus. In order to illuminate the structure of $W^*(N)$, we need more properties of WOT-closed C^*-subalgebras, i.e. von Neumann algebras, and abelian von Neumann algebras. After that, we can build the isomorphism from $W^*(N)$ to $L^\infty(\sigma(N))$. Finally, by above methods, we can get the multiplicity theory of normal operators.

3.1 Spectral Theorem

3.1.1 Spectral Measures

Firstly, we want to construct the map from $B(X, \Omega)$ to $\mathscr{B}(\mathscr{H})$. Thus we need to define the operator-valued measure.

Definition 1. If X is a set and Ω is a σ-algebra of subsets of X, and $\mathscr{H}$ is a Hilbert space, then a spectral measure for $(X, \Omega, \mathscr{H})$ is a function $E \colon \Omega \to \mathscr{B}(\mathscr{H})$ satisfying the following properties.

1) $E(\Delta)$ is a projection for any $\Delta \in \Omega$.

2) $E(\varnothing) = 0$ and $E(X) = 1$.

3) $E(\Delta_1 \cap \Delta_1) = E(\Delta_1)E(\Delta_2)$ for any $\Delta_1, \Delta_2 \in \Omega$.

4) If $\{\Delta_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets in Ω,

$$E(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} E(\Delta_n)$$

Remark. In 4), this convergence is about SOT, and since $E(\Delta)$ is a projection, it is also about WOT.

Then we can see the relation between the spectral measure and the complex measure.

Proposition 2. *If E is a spectral measure for $(X, \Omega, \mathscr{H})$ and $g, h \in \mathscr{H}$, then*

$$E_{gh}(\Delta) = \langle E(\Delta)g, h \rangle$$

is a complex measure of (X, Ω). Moreover, $\|E_{gh}\| \leqslant \|g\| \, \|h\|$.

Proof. $\mu = E_{gh}(\Delta)$ is a complex measure by definition.

$$|\mu|(\Delta) = |\langle E(\Delta)g, h \rangle| \leqslant \|E(\Delta)g\| \, \|h\| \leqslant \|g\| \, \|h\| \qquad \square$$

Using the spectral measure, we can define the operator-valued integral.

Theorem 3. *If E is a spectral measure for $(X, \Omega, \mathscr{H})$ and $\phi \colon X \to \mathbb{C}$ is a bounded Ω-measurable function, then there is a unique operator $T \in \mathscr{B}(\mathscr{H})$ s.t. for any $\varepsilon > 0$, there is a Ω-partition $\{\Delta_i\}_{i=1}^{n}$ of X with*

$$\sup \{|\phi(x) - \phi(y)| : x, y \in \Delta_k\}$$

for $1 \leqslant k \leqslant n$ s.t. for any $x_k \in \Delta_k$

$$\left\| T - \sum_{k=1}^{n} \phi(k)E(\Delta_k) \right\| < \varepsilon$$

Proof. Define

$$
\begin{array}{rccc}
B \colon & \mathscr{H} \times \mathscr{H} & \longrightarrow & \mathbb{C} \\
& (g, h) & \longmapsto & \int \phi \, dE_{gh}
\end{array}
$$

Since $\|B(g, h)\| \leqslant \|\phi\|_{\infty} \|g\| \, \|h\|$, B is a bounded sesquilinear form. Then by the Riesz Theorem, there is a $B \in \mathscr{B}(\mathscr{H})$ s.t. $B(g, h) = \langle Tg, h \rangle$ for all $g, h \in \mathscr{H}$ and $\|T\| \leqslant \|\phi\|_{\infty}$. Then for and $g, h \in \mathscr{H}$ and $x_k \in \Delta_k$ with the giving partition

$$
\begin{aligned}
\left| \langle Tg, h \rangle - \sum_{k=1}^{n} \phi(x_k)\langle E(\Delta_k)g, h \rangle \right| &= \left| \sum_{k=1}^{n} \int_{\Delta_k} (\phi(x) - \phi(x_k)) dE_{gh}(x) \right| \\
&\leqslant \sum_{k=1}^{n} \int_{\Delta_k} |\phi(x) - \phi(x_k)| \, d\,|E_{gh}|\,(x) \\
&\leqslant \varepsilon \int d\,|E_{gh}|\,(x) \leqslant \varepsilon \|g\| \, \|h\|
\end{aligned}
$$

Therefore, by the Riezs Theorem,

$$\left\| T - \sum_{k=1}^{n} \phi(k) E(\Delta_k) \right\| < \varepsilon \qquad \qquad \Box$$

Remark. We define $T = \int \phi dE$, and $\langle Tg, h \rangle = \int \phi dE_{gh}$.

We have one more property on $B(X)$ for a compact space X. Let $B(X)$ denote all bounded Borel measurable functions on X and $M(X)$ be all Borel measures.

Lemma 4. *If X is a compact space and $\phi \in B(X)$, then there is a net $\{u_i\} \subset C(X)$ with $\|u_i\|_\infty \leqslant \|\phi\|$ s.t. $\int u_i d\mu \to \int \phi d\mu$ for any μ in $M(X)$.*

Proof. $C(X)^* = M(X)$ and $C(X) \subset C(X)^{**} = M(X)^*$. Therefore, the unit ball in $C(X)$ is wk-dense in the unit ball in $M(X)^*$ (by using Hahn-Banach Theorem). Identifying $B(X)$ be the subspace of $M(X)^*$, we have this lemma. $\qquad \Box$

Remark. In fact, $C(X)$ is norm closed in $C(X)^{**} = M(X)^*$ like that the C^*-algebra is norm closed in $\mathscr{B}(\mathscr{H})$. But by above lemma, we see in $M(X)^*$

$$\overline{C(X)}^{wk} = B(X)$$

Thus we may apply this to extending C^*-algebras.

$B(X, \Omega)$ with the supremum norm and complex conjugate can become a C^*-algebra. Then by above theorem, we can provide the map from $B(X, \Omega)$ to operator algebras.

Proposition 5. *If E is a spectral measure for $(X, \Omega, \mathscr{H})$, then*

$$\begin{array}{cccc} \rho: & B(X, \Omega) & \longrightarrow & \mathscr{B}(\mathscr{H}) \\ & \phi & \longmapsto & \int \phi dE \end{array}$$

is a representation of $B(X, \Omega)$.

Proof. Firstly, we need check that ρ preserves the involution. For $g, h \in \mathscr{H}$,

$$E_{gh}(\Delta) = \langle E(\Delta)g, h \rangle = \langle g, E(\Delta)h \rangle = \overline{\langle E(\Delta)h, g \rangle} = \overline{E_{hg}(\Delta)}$$

Therefore, for any $g, h \in \mathscr{H}$

$$\begin{aligned} \langle (\int \phi dE)^* g, h \rangle &= \langle g, (\int \phi dE)h \rangle \\ &= \overline{\langle (\int \phi dE)h, g \rangle} = \overline{\int \phi dE_{hg}} \\ &= \int \phi^* d\overline{E_{hg}} = \int \phi^* dE_{gh} \\ &= \langle (\int \phi^* dE)g, h \rangle \end{aligned}$$

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Also, ρ must be multiplicative. Fix $\phi, \psi \in B(X)$ and $\varepsilon > 0$, let $\{\Delta_i\}_{i=1}^n$ be the Ω-partition of X s.t. the oscillations of $\phi, \psi, \phi\psi$ on each Δ_i are less than ε. By the fact $E(\Delta_i)E(\Delta_j) = 0$ for $i \neq j$,

$$\left\| \int \phi\psi dE - (\int \phi dE)(\int \psi dE) \right\|$$

$$\leqslant \quad \varepsilon + \left\| \sum_{k=1}^n \phi(x_k)\psi(x_k)E(\Delta_k) - (\sum_{i=1}^n \phi(x_i)E(\Delta_i))(\sum_{j=1}^n \psi(x_j)E(\Delta_j)) \right\|$$

$$+ \left\| (\sum_{i=1}^n \phi(x_i)E(\Delta_i))(\sum_{j=1}^n \psi(x_j)E(\Delta_j)) - (\int \phi dE)(\int \psi dE) \right\|$$

$$\leqslant \quad \varepsilon + \left\| (\sum_{i=1}^n \phi(x_i)E(\Delta_i))(\sum_{j=1}^n \psi(x_j)E(\Delta_j) - \int \psi dE) \right\|$$

$$+ \left\| (\sum_{i=1}^n \phi(x_i)E(\Delta_i) - \int \phi dE)(\sum_{j=1}^n \psi(x_j)E(\Delta_j)) \right\|$$

$$\leqslant \quad \varepsilon(1 + \|\phi\|_\infty + \|\psi\|_\infty)$$

Then ρ is indeed multiplicative. $\qquad \square$

We have already there is a map from $C(X)$ to $\mathscr{B}(\mathscr{H})$, then by combining this proposition and above lemma, if the map is wk-continuous on $C(X)$, we can extend it from $M(X)$ to $\mathscr{B}(\mathscr{H})$.

Theorem 6. *If X is a compact space and $\rho\colon C(X) \to \mathscr{B}(\mathscr{H})$ is a representation, then there is a unique spectral measure E defined on the Borel sets of X s.t. ρ can be expressed as*

$$\rho(\phi) = \int \phi dE \quad \forall \; \phi \in C(X)$$

Moreover, $T \in \mathscr{B}(\mathscr{H})$ commutes with $\rho(\phi)$ for any $\phi \in C(X)$ if and only if T commutes with $E(\Delta)$ for any Borel set Δ.

Proof. Constructing $\tilde{\rho}$ from ρ:
For any $g, h \in \mathscr{H}$, the map $u \to \langle \rho(u)g, h \rangle$ from $C(X)$ to $\mathscr{B}(\mathscr{H})$ is bounded with $\|g\| \|h\|$. Therefore, by Riesz Theorem, there exists a unique $\mu_{gh} \in M(X)$ with $\|\mu_{gh}\| \leqslant \|g\| \|h\|$

$$\langle \rho(u)g, h \rangle = \int u d\mu_{gh}, \quad \forall \; u \in C(X)$$

For any $\phi \in B(X)$, $(g, h) \to \int \phi d\mu_{gh}$ is a bounded sesquilinear on $\mathscr{H} \times \mathscr{H}$. Therefore, there exsits a unique $T \in \mathscr{B}(\mathscr{H})$, s.t.

$$\int \phi d\mu_{gh} = \langle Tg, h \rangle$$

Then we define $\tilde{\rho}(\phi) = T$ and it is well-defined. Clearly, $\tilde{\rho}(u) = \rho(u)$ for any u in $C(X)$.

Check: $\tilde{\rho} \colon B(X) \to \mathscr{B}(\mathscr{H})$ is a representation

By the definiation, $\tilde{\rho}$ is clearly linear and preserves the involution, thus we just need to prove the multiplication. Let ϕ and ψ in $B(X)$. There is a sequence $\{u_i\} \subset C(X)$ s.t. $u_i \to \phi$ in wk-topology. Then since $\int u_i \psi d\mu \to \int \phi\psi$ for any $\mu \in M(X)$, $\rho(\tilde{u_i \psi}) \to \rho(\phi\psi)$ in WOT. For any $\psi \in C(X)$,

$$\tilde{\rho}(\phi\psi) = \lim \tilde{\rho}(u_i\psi) = \lim \rho(u_i)\rho(\psi) = \tilde{\rho}(\phi)\rho(\psi)$$

Therefore, if $\phi, \psi \in B(X)$, $\tilde{\rho}(\phi\psi) = \tilde{\rho}(\phi)\tilde{\rho}(\psi)$.

Contructing the spectral measure:

Let $E(\Delta) = \rho(\chi_\Delta(x))$ for all Borel set Δ. Clearly, $E(\Delta)$ is a projection for any Δ. For disjoint Borel sets Δ_1, Δ_2,

$$E(\Delta_1 \cap \Delta_2) = \tilde{\rho}(\chi_{\Delta_1 \cap \Delta_2}) = \tilde{\rho}(\chi(\Delta_1)\chi(\Delta_2)) = \tilde{\rho}(\chi(\Delta_1)) = E(\Delta_1)E(\Delta_2)$$

Let $\{\Delta_n\}_{n=1}^\infty$ be disjoint Borel sets and $\Delta = \cup_{n=1}^\infty \Delta_n$ and $\Lambda_n = \cup_{k=n+1}^\infty \Delta_k$. By the fact that E is finitely additive, for any $h \in \mathscr{H}$

$$\begin{aligned} \left\| E(\Delta)h - \sum_{k=1}^n E(\Delta_k)h \right\| &= \langle E(\Lambda_n)h, E(\Lambda_n)h \rangle \\ &= \langle E(\Lambda_n)h, h \rangle = \langle \tilde{\rho}(\Lambda_n)h, h \rangle \\ &= \int \chi_{\Lambda_n} d\mu_{hh} \\ &\leqslant \mu_{hh}(\Lambda_n) \to 0 \end{aligned}$$

Then by above theorem, with respect to this spectral measure E, for any $\phi \in B(X)$,

$$\tilde{\rho}(\phi) = \int \phi dE$$

In particular, for $u \in C(X)$, it is also true.

Check: The uniqueness of E. If there is another E' satisfying above conditions, then for any $g, h \in \mathscr{H}$

$$\int \phi dE'_{gh} = \int \phi dE_{gh}$$

Therefore, $E' = E$.

Finally, if T commutes with $\tilde{\rho}(\phi)$ for any $\phi \in C(X)$, we can use the WOT-convergence to get that T commutes with $\tilde{\rho}(\phi)$ for any $\phi \in B(X)$. Therefore, clearly T commutes with $E(\Delta)$ for any Borel set Δ. Conversely, if T commutes with $E(\Delta)$ for any Borel set Δ,

$$\langle E(\Delta)Tg, h \rangle = \langle E(\Delta)g, T^*h \rangle$$

i.e. $\mu_{Tg,h} = \mu_{g,T^*h}$. Thus for any $\phi \in B(X)$,

$$\int \phi d\mu_{Tg,h} = \int \phi d\mu_{g,T^*h}, \text{ i.e. } \langle \phi Tg, h \rangle = \langle \phi g, T^*h \rangle$$

Then $\langle \tilde{\rho}(\phi)Tg, h \rangle = \langle T\tilde{\rho}(\phi)g, h \rangle$ for any $g, h \in \mathscr{H}$. $\qquad \square$

Remark. Thus, we extend $\rho \colon C(X) \to \mathscr{B}(\mathscr{H})$ to $\tilde{\rho}$ defined on $B(X)$ More-over, by above proof we know the $*$-homomorphism

$$\begin{array}{cccc}
\tilde{\rho}\colon & (B(X), wk) & \longrightarrow & (\mathscr{B}(\mathscr{H}), WOT) \\
& \phi & \longmapsto & \int \phi dE
\end{array}$$

is continuous and therefore $\tilde{\rho}(B(X))$ is WOT-closed in $\mathscr{B}(\mathscr{H})$.

3.1.2 Spectral Theorem for Normal Operators

Now we can apply above theorem to a normal operator and extend the Continuous Functional Calculus to the Borel Functional Calculus. Moreover, it will promote us to consider the structure of the WOT-closed operator algebra generated by a normal operator.

Theorem 1. *If N is a normal operator, then there is a unique spectral measure E s.t.*

$$\begin{array}{cccc}
\rho\colon & B(\sigma(N)) & \longrightarrow & \mathscr{B}(\mathscr{H}) \\
& \phi & \longmapsto & \phi(N) = \int \phi dE
\end{array}$$

and $\rho|_{C(\sigma(N))}$ is the Continuous Functional Calculus. Moreover,

1) *$\tilde{\rho}\colon (B(X), wk) \to (\mathscr{B}(\mathscr{H}), WOT)$ is continuous.*

2) *if G is nonempty and relatively open subset in $\sigma(N)$, $E(G) \neq 0$.*

3) *if $T \in \mathscr{B}(\mathscr{H})$, then $TN = NT$ and $TN^* = N^*T$ if and only if T commutes with $E(\Delta)$ for any Borel set Δ.*

Proof. By above theorem, this ρ is in fact the extension of the Continuous Functional Calculus. And ρ is $wk - WOT$ continuous.

For 2), it is because that $\rho|_{C(\sigma(N))}$ is an isometry. If $\chi_G > 0$, then there is a continuous function s.t. $0 < f < \chi_G$. Therefore, $0 < \rho(f) \leqslant \rho(\chi_G)$.

Since the condition that T commutes with N and N^* is equivalent to that T commutes with $\phi(N)$ for any $\phi \in C(\sigma(N))$, by above theorem, 3) holds. $\qquad \square$

Remark. Now, for a normal operator N, we have

$$N = \int_{\sigma(N)} z dE = z(N) = \int_\Gamma (z - N)^{-1} dz$$

There are some properties of the spectral measure of normal operator N.

Proposition 2. *Let N be a normal operator with the spectral measure E.*

1) *$\lambda \in \sigma_p(N) \Leftrightarrow E(\{\lambda\}) \neq 0$.*

2) *If $\lambda \in \sigma_p(N)$, then $\operatorname{ran} E(\{\lambda\}) = \ker(N - \lambda)$.*

3) *If $\lambda \in \sigma_p(N)$ and $\phi \in B(\sigma(N))$, then for any $h \in \mathscr{H}$,*

$$\phi(N)h = \phi(\lambda)h$$

Proof. For 1), if $\lambda \in \sigma(N)$, then there is a nonzero vector $h \in \mathscr{H}$ s.t. $(N - \lambda)h = 0$.

$$0 = \|(N - \lambda)h\|^2 = \int_{\sigma(N)} |z - \lambda|^2 \, dE_{hh} = \int_{\sigma(N)\setminus\{\lambda\}} |z - \lambda|^2 \, dE_{hh}$$

Then since E_{hh} is a positive measure,

$$\int_{\sigma(N)\setminus\{\lambda\}} dE_{hh} = \|E(\sigma(N)\setminus\{\lambda\})h\|^2 = 0$$

And since $E(\sigma(N)) = 1$,

$$E(\{\lambda\})h = h, \text{ i.e } E(\{\lambda\}) \neq 0$$

Conversely, if $E(\{\lambda\}) \neq 0$, then for nonzero $h \in \operatorname{ran} E(\{\lambda\})$, $E(\{\lambda\})h = h$,

$$
\begin{aligned}
\int_{\sigma(N)} |z - \lambda|^2 \, d\langle E(z)h, h\rangle &= \int_{\sigma(N)} |z - \lambda|^2 \, d\langle E(z)E(\{\lambda\})h, h\rangle \\
&= \int_{\{\lambda\}} |z - \lambda|^2 \, d\langle E(z)h, h\rangle = 0
\end{aligned}
$$

That means

$$\|(N - \lambda)h\|^2 = \int_{\sigma(N)} |z - \lambda|^2 \, d\langle E(z)h, h\rangle = 0$$

Then $h \in \ker(T - \lambda)$, i.e. $\lambda \in \sigma_p(N)$.

2) and 3) can be obtained directly by above proof. $\qquad\square$

Remark. This means that if λ is an isolated point in $\sigma(N)$, then $\lambda \in \sigma_P(N)$.

Now, we can extend the **Example** 4 in the subsection **2.2.2**.

Example 3. Let μ be a compactly supported, regular Borel measure on $\mathbb{C}$ and (X, Ω, μ) be the measure space. For each $\phi \in L^\infty(\mu)$, we define the map

$$
\begin{aligned}
M_\phi: \quad L^2(\mu) &\longrightarrow L^2(\mu) \\
f(z) &\longmapsto \phi(z)f(z)
\end{aligned}
$$

1) the spectral measure for M_ϕ is $M_{\chi_{\phi^{-1}(\Delta)}}$.

2) If $\psi \in B(\sigma(M_\phi))$, then $\psi(M_\phi) = M_{\psi \circ \phi}$.

3.2 Abelian Von Neumann Algebras

In above section, we built the representation ρ from $B(X)$ to $\mathscr{B}(\mathscr{H})$, and more over, ρ is $wk - WOT$ continuous. Thus we want to prove $\rho(B(X))$ is indeed a WOT-closed C^*-subalgebra of $\mathscr{B}(\mathscr{H})$. If $X = \sigma(N)$, $\rho(B(\sigma(N)))$ is the WOT-closed C^*-subalgebra generated by N, denoted by $W^*(N)$. We will provide more rigorous proof of this statement. After that, we will make the structure of $W^*(N)$ more explicit. But before doing these, we some properties of WOT-closed subalgebra.

3.2.1 Double Commutant Theorem

Definition 1. A von Neumann algebra is a weakly (or strongly) closed C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ that contains the identity.

Definition 2. If $\mathcal{S} \subset \mathscr{B}(\mathscr{H})$ is a subset, then the commutant of $\mathcal{S}$, denoted by $\mathcal{S}'$,

$$\mathcal{S}' = \{\, T \in \mathscr{B}(\mathscr{H}) \colon TS = ST \ \forall\, S \in \mathcal{S} \,\}$$

There are some easy properties if the commutant.

Proposition 3. *Let $\mathcal{S}$ be a subset of $\mathscr{B}(\mathscr{H})$.*

1) *$\mathcal{S}'$ is a von Neumann algebra.*

2) *If $A = \oplus_{i=1}^{n} A_i$ acts on $\mathscr{H}^{(n)}$ and $T = [T_{ij}] \in \mathscr{B}(\mathscr{H}^{(n)})$, then $T \in \{A\}'$ if and only if $T_{ij}A_j = A_i T_{ij}$ for all $0 \leqslant i, j \leqslant n$.*

3) *$[\mathcal{S}^{(n)}]'' = [\mathcal{S}'']^{(n)}$.*

Definition 4. Let $\mathcal{S}$ be a linear manifold of $\mathscr{B}(\mathscr{H})$.

$$\operatorname{Lat} \mathcal{S} = \{\, \mathscr{M} \colon \mathscr{M} \text{ is a invariant closed subspace for } S, \ \forall S \in \mathcal{S} \,\}$$

Proposition 5. *If $\mathfrak{A}$ is a C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ containing the identity, then*

$$\overline{A}^{SOT} = \{\, T \in \mathscr{B}(\mathscr{H}) \colon \operatorname{Lat} \mathfrak{A}^n \subset \operatorname{Lat} T^n \ \forall\, n \in \mathbb{N} \,\}$$

Proof. Assume $T \in \overline{A}^{SOT}$, there exist $T_k \in \mathfrak{A}$ s.t. $T_k h \to Th$ for all $h \in \mathscr{H}$. If $\mathscr{M} \in \operatorname{Lat} \mathfrak{A}^{(n)}$, $T_k^{(n)}(\mathscr{M}) \subset \mathscr{M}$. Since $\mathscr{M}$ is closed, $T^{(n)}(\mathscr{M}) \subset \mathscr{M}$.
Conversely, let T satisfy the condition. Fix $h_1, \cdots, h_n \in \mathscr{H}$ and $\varepsilon > 0$. It suffices to show that there exists $A \in \mathfrak{A}$ s.t. $\|(T - A)h_k\| < \varepsilon$ for $1 \leqslant k \leqslant n$. Let $\mathscr{M} = \overline{\operatorname{span}\{(Ah_1, \cdots, Ah_n) \colon A \in \mathfrak{A}\}}$. Since $1 \in \mathfrak{A}$, $(h_1, \cdots, h_n) \in \mathscr{M}$. Clearly, $\mathscr{M} \in \operatorname{Lat} \mathfrak{A}^{(n)}$. Therefore $\mathscr{M} \in \operatorname{Lat} T^{(n)}$, i.e. $T^{(n)}(\mathscr{M}) \subset \mathscr{M}$

$$T^{(n)}(h_1, \cdots, h_n) = (Th_1, \cdots, Th_n) \in \mathscr{M}$$

Thus, clearly for any given $\varepsilon > 0$, there exists $A \in \mathfrak{A}$ s.t.

$$\|(Ah_1, \cdots, Ah_n) - (Th_1, \cdots, Th_n)\| < \varepsilon$$

That means,

$$\sum_{k=1}^{n} \|(A - T)h_k\|^2 < \varepsilon^2$$

Therefore, $\|(T - A)h_k\| < \varepsilon$ for $1 \leqslant k \leqslant n$. $\hfill\square$

Theorem 6 (Double Commutant Theorem). *If $\mathfrak{A}$ is a C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ containing the identity, then*

$$\overline{\mathfrak{A}}^{SOT} = \overline{\mathfrak{A}}^{WOT} = \mathfrak{A}''$$

Proof. Firslty, by above proposition, $\mathfrak{A}''$ is SOT-closed and clearly $\mathfrak{A} \subset \mathfrak{A}''$. Therefore, it is sufficient to show that $\mathfrak{A}'' \subset \overline{\mathfrak{A}}^{SOT}$. Let $T \in \mathfrak{A}''$. Then, by above proposition, we just to check that $\operatorname{Lat} \mathfrak{A}^{(n)} \subset \operatorname{Lat} T^{(n)}$.
Let $\mathscr{M} \in \operatorname{Lat} \mathfrak{A}^{(n)}$. Thus, $\mathscr{M}$ reduces $\mathfrak{A}^{(n)}$. That means the projection P from $\mathscr{H}^{(n)}$ to $\mathscr{M}$ satisfies $P\mathfrak{A}^{(n)} = \mathfrak{A}^{(n)}P$, i.e. $P \in \{\mathfrak{A}^{(n)}\}'$. Since $T \in \{\mathfrak{A}\}''$, and by above proposition $T^{(n)} \in \{\mathfrak{A}^{(n)}\}''$, $PT^{(n)} = T^{(n)}P$. That means $\mathscr{M} \in \operatorname{Lat} T^{(n)}$ $\hfill\square$

Remark. Therefore, if $\mathfrak{A}$ is a von Neumann algebra, $\mathfrak{A} = \mathfrak{A}''$. It supports the statement that the algebraic structure in a C^*-algebra can completely determine the topological structure.

3.2.2 Abelian Von Neumann Algebras

Proposition 1. *1) If $\{\mathfrak{A}_i\}_{i \in I}$ is a collection of von Neumann algebras, then $\cap_{i \in I} \mathfrak{A}_i$ is a von Neumann algebra.*

2) If $\mathfrak{A}$ is von Neumann algebra, then the center of $\mathfrak{A}$, $Z_{\mathfrak{A}} = \mathfrak{A} \cap \mathfrak{A}'$ is a von Neumann algebra.

We said that the norm closed C^*-subalgebras of $\mathscr{B}(\mathscr{H})$ not containing enough interesting elements. Now, we can see what are these "interesting elements" the WOT-closed C^*-subalgebras have.

Theorem 2. *Let $\mathfrak{A}$ be a von Neumann algebra, then $\mathfrak{A}$ is the norm closed linear span of the projections in $\mathfrak{A}$.*

Proof. Since $\mathfrak{A}$ is the linear span of $\operatorname{Re}\mathfrak{A}$, we just need to show $\operatorname{Re}\mathfrak{A}$ is the norm closed linear span of the projections in $\mathfrak{A}$.
Let $A \in \operatorname{Re}\mathfrak{A}$ with the spectral measure E. E is defined on $\sigma(A)$. If $[a, b] \subset \sigma(A)$, then there exists a sequence $\{u_n(t)\}$ in $C(\sigma(A))$ s.t.

$$u_n(t) = 1 \text{ for } a \leqslant t \leqslant b - n^{-1} \text{ and } u_n(t) = 0 \text{ for other} t$$

Then $u_n(t) \to \chi_{[a,b)}(t)$. If $h \in \mathscr{H}$,

$$\|(u_n(A) - E[a,b))h\|^2 = \int_{\sigma(A)} \left| u_n(t) - \chi_{[a,b)}(t) \right|^2 dE_{hh}(t)$$

Therefore, $u_n(A) \to E[a,b)$ in SOT. And since $u_n(A) \in \mathfrak{A}$ and $\mathfrak{A}$ is SOT-closed, $E[a,b)$ is in $\mathfrak{A}$. Thus, for any Borel subset Ω of $\sigma(A)$, $E(\Omega) \in \mathfrak{A}$. Then $A = \int_{\sigma(A)} t\,dE$ is the norm closed of linear span of all $E(\Omega)$ $\qquad \square$

Remark. In fact, for any normal operator N in a von Neumann algebra $\mathfrak{A}$ with the spectral measure E and for any $\phi \in B(\sigma(N))$, $\phi(N)$ is in $\mathfrak{A}$. It is because that $\overline{C(\sigma(N))}^{WOT} = B(\sigma(N))$, and the map from $B(\sigma(N))$ to $\mathscr{B}(\mathscr{H})$ is $wk-WOT$ continuous. That means $\phi(N)$ can be approximated by a sequence in $\mathfrak{A}$ with respect to the WOT, thus $\phi(N)$ is in $\mathfrak{A}$. In particular, for any Borel subset Ω of $\sigma(N)$, the projection $E(\Omega) \in \mathfrak{A}$.

So, these interesting elements are projections. We say the von Neumann algebra have enough projections wich can generate the von Neumann algebra. Projections play an important role in a von Neumann algebra. We will classify von Neumann algebras by classifying the containing projections.

Proposition 3. *Let $\mathfrak{A}$ be a von Neumann algebra and $A \in \mathfrak{A}$.*

   1) If E and F are the projections onto $\overline{\operatorname{ran} A}$ and $\ker A$, then $E, F \in \mathfrak{A}$.

   2) If $A = W|A|$ is teh polar decomposition, then $|A|, W \in \mathfrak{A}$.

Proof. For 1), if $h \in \ker A$ and $B \in \mathfrak{A}'$, then $BAh = ABh = 0$. That means F is invariant for $\{\mathfrak{A}\}'$ and thus $F \in \{\mathfrak{A}\}''$. And $E = \ker A^* \in \{\mathfrak{A}\}''$.

For 2), $|A|$ is clearly in $\mathfrak{A}$ by the Continuous Functional Calculus. For W, we just need to prove that $WT = TW$ for any $T \in \mathfrak{A}'$. Let $E = \ker|A|$. By 1), $E \in \mathfrak{A} = \mathfrak{A}''$. For $x \in \ker|A| = E$, $ET = TE$ implies $Tx \in \ker|A|$. Since $\ker|A| = \ker A$, $WTx = 0 = TWx$. On the other hand, if $x \in \operatorname{ran}|A|$, $x = |A|y$ for some $y \in \mathscr{H}$.

$$WTx = WT|A|y = W|A|Ty = ATy = TAy = TW|A|y = TWx$$

And this equation can be easily extended to $\overline{\operatorname{ran}|A|}$. $\mathscr{H} = \ker|A| \oplus \overline{\operatorname{ran}|A|}$, thus $TW = WT$. $\qquad \square$

Before researching the structure of abelian von Neumann algebras, there is a typical example.

Example 4. If (X, Ω, μ) is a finite measure space, and $\mathfrak{A}_\mu = \{M_\phi \colon \phi \in L^\infty(\mu)\}$, then

$$\mathfrak{A}_\mu = \mathfrak{A}'_\mu = \mathfrak{A}''_\mu$$

Proof. It suffices to show $\mathfrak{A}_\mu = \mathfrak{A}'_\mu$. And since $\mathfrak{A}_\mu$ is abelian, $\mathfrak{A}_\mu \subset \mathfrak{A}'_\mu$. Consider $T \in \mathfrak{A}'_\mu$, then $\mu(X) < \infty$ implies that $1 \in L^2(\mu)$. Let $\phi = T(1) \in L^2(\mu)$.

Check: $T = M_\phi$
Firstly, if $\psi \in L^\infty(\mu)$,

$$T(\psi) = T(M_\psi 1) = M_\psi T(1) = \phi\psi$$

And also,

$$\|T(\psi)\| = \|\phi\psi\| \leqslant \|T\| \, \|\psi\|_2$$

Using this we can show $\phi \in L^\infty(\mu)$. Let

$$\Delta_n = \{ \, x \in X \colon \, |\phi(x)| \leqslant n \, \}, \ \psi = \chi_{\Delta_n}$$

Then we have

$$\|T\|^2 \, \mu(\Delta_n) = \|T\|^2 \, \|\psi\|^2 \geqslant \|\phi\|^2 \, \|\psi\|^2 = \int_{\Delta_n} |\phi|^2 \, d\mu = n^2 \mu(\Delta_n)$$

Thus $\mu(\Delta_n) = 0$. $\phi \in L^\infty(\mu)$. And since $L^\infty(\mu)$ is dense in $L^2(\infty)$, $T = M_\phi$ can be defined on $L^2(\infty)$. $\qquad \square$

Remark. In fact, this result can be also true for (X, Ω, μ) σ-finite.

Definition 5. Let $\mathcal{S} \subset \mathscr{B}(\mathscr{H})$ and $e_0 \in \mathscr{H}$.

1) e_0 is said to be star-cyclic for $\mathcal{S}$ if $\overline{C^*(\mathcal{S})e_0} = \mathscr{H}$.

2) e_0 is said to be separating for $\mathcal{S}$ there is no nonzero $S \in \mathcal{S}$ s.t. $Se_0 = 0$.

Proposition 6. *If $\mathfrak{A}$ is a unital C^*-algebra of operators, a vector is cyclic if and only if it is separating for $\mathfrak{A}'$*

Proof. For $T \in \mathfrak{A}'$, s.t. $Te_0 = 0$. Then $T\mathfrak{A}e_0 = \mathfrak{A}Te_0 = 0$. If e_0 is the cyclic vector for $\mathfrak{A}$, then $\mathfrak{A}e_0$ is dense in $\mathscr{H}$. Therefore, $T = 0$.
Conversely, if e_0 separates $\mathfrak{A}'$, let P be the projection onto $\overline{(\mathfrak{A}e_0)}^\perp$. Then $P \in \mathfrak{A}'$ and $Pe_0 = 0$. Therefore, $P = 0$. e_0 is cyclic for $\mathfrak{A}$. $\qquad \square$

Corollary 7. *If $\mathfrak{A}$ is an abelian operator algebra, then each cyclic vector for $\mathfrak{A}$ is also a separating vector for $\mathfrak{A}$.*

Definition 8. An abelian subalgebra $\mathfrak{A} \subset \mathscr{B}(\mathscr{H})$ is said to be a maximal abelian subalgebra if there is no abelian subalgebra in $\mathscr{B}(\mathscr{H})$ containing $\mathfrak{A}$. In paticular, a maximal abelian von Neumann algebra is called a masa.

Theorem 9. *Let $\mathfrak{A}$ be a subalgebra in $\mathscr{B}(\mathscr{H})$. Then the following statement are equivalent.*

1) $\mathfrak{A}$ is a masa.

2) $\mathfrak{A} = \mathfrak{A}'$

Proof. Assume $\mathfrak{A}$ is a masa. Then $\mathfrak{A} \subset \mathfrak{A}'$. If there is a nonzero $A \in \mathfrak{A}' \backslash \mathfrak{A}$, then $\mathbb{C} = W^*(\mathfrak{A}, A)$ is an abelian von Neumann algebra containing $\mathfrak{A}$, which is a contradiction. Thus $\mathfrak{A} = \mathfrak{A}'$.

Conversely, $\mathfrak{A}$ is an abelian von Neumann algebra by above proposition. By Zorn's Lemma there is a maximal abelian von Neumann algebra $\mathfrak{B}$ s.t. $\mathfrak{A} \subset \mathfrak{B}$. Then $\mathfrak{B}' \subset \mathfrak{A}'$. By above statement, $\mathfrak{B} = \mathfrak{B}'$. Thus $\mathfrak{B} \subset \mathfrak{A}$. $\qquad\square$

Theorem 10. *If $\mathscr{H}$ is separable and $\mathfrak{A}$ is an abelian C^*-algebra containing the identity, the following statements are equivalent.*

1) $\mathfrak{A} = \mathfrak{A}'$.

2) $\mathfrak{A}$ is strongly closed and has a cyclic vector.

3) There is a compact metric space X, a regular Borel measure μ with support X, and a unitary $U \colon L^2(\mu) \to \mathscr{H}$ s.t. $U\mathfrak{A}_\mu U^{-1} = \mathfrak{A}$.

Proof. 1) $\Rightarrow$ 2) Let $\{e_n\}_{n=1}^\infty$ be a sequence of unit vectors s.t.

$$\overline{\mathfrak{A}e_n} \perp \overline{\mathfrak{A}e_m}, \text{ for } n \neq m \text{ and } \mathscr{H} = \oplus_{n=1}^\infty \overline{\mathfrak{A}e_n}$$

Set $e_0 = \sum_{n=1}^\infty 2^{-n} e_n$ and $P_n = \overline{\mathfrak{A}e_n}$. Then $P_n \in \mathfrak{A}' = \mathfrak{A}$.

$$e_n = 2^n P_n e_0 \in \overline{\mathfrak{A}e_0} \Rightarrow \overline{\mathfrak{A}e_n} \subset \overline{\mathfrak{A}e_0}$$

Therefore, $\mathscr{H} = \overline{\mathfrak{A}e_0}$.

2) $\Rightarrow$ 3) Since $\mathscr{H}$ is separable, the unit ball in $\mathfrak{A}$ is a WOT-closed compact metric space, therefore there is a countable weakly dense set. Let $\mathfrak{A}_1$ be the C^*-algebra generated by this set and X be the maximal ideal space of $\mathfrak{A}_1$. Clearly, X is metrizable. Then the inverse of Gelfand transform

$$\rho \colon C(X) \longrightarrow \mathfrak{A}_1 \subset \mathfrak{A}$$

And there is a spectral measure E, s.t. $\rho(f) = \int_X f dE$. Let e_0 be the cyclic vector of $\mathfrak{A}$. Then we can define the U,

$$
\begin{array}{cccc}
U \colon & C(X) & \longrightarrow & \mathfrak{A}_1 e_0 \\
 & \phi & \longmapsto & \rho(\phi)e_0
\end{array}
$$

Let $\mu(\Delta) = \langle E(\Delta)e_0, e_0 \rangle$, then $C(X) \subset L^2(\mu)$ by the fact that μ is positive finite measure. For $\phi \in C(X)$,

$$\|\rho(\phi)e_0\|^2 = \langle \rho(\phi^*)\rho(\phi)e_0, e_0 \rangle = \int |\phi|^2 \, d\mu = \|\phi\|^2$$

Therefore, U is an isometry. Since $C(X)$ is norm densed in $L^2(\mu)$ and $\overline{\mathfrak{A}_1 e_0} = \overline{\mathfrak{A} e_0} = \mathscr{H}$, $U \colon L^2(\mu) \to \mathscr{H}$ is a unitary.

Let $\psi \in C(X)$ and $\phi \in C(X)$

$$UM_\psi \phi = U(\psi\phi) = \rho(\psi\phi)e_0 = \rho(\psi)\rho(\phi)e_0 = \rho(\psi)U\phi$$

Since $\rho(C(X)) = \mathfrak{A}_1$, $U^*\mathfrak{A}_1 U = \{M_\phi \colon \phi \in C(X)\}$. Define the map $\Gamma \colon \mathfrak{A} \to \mathfrak{A}_\mu$ by $\Gamma(A) = U^*AU$. We have known that

$$\Gamma(\mathfrak{A}_1) \subset \{M_\phi \colon \phi \in C(X)\} \subset \mathfrak{A}_\mu,$$

$$\overline{\{M_\phi \colon \phi \in C(X)\}}^{WOT} \subset \mathfrak{A}_\mu$$

And Γ is clearly WOT-closed, thus

$$\Gamma(\mathfrak{A}) \subset \overline{\Gamma(\mathfrak{A}_1)}^{WOT} \subset \mathfrak{A}_\mu$$

Γ is well-defined. In fact, by similar proof of $\mathfrak{A}_\mu = \mathfrak{A}_\mu'$, $\{M_\phi \colon \phi \in C(X)\}' = \mathfrak{A}_\mu$, i.e.

$$\mathfrak{A}_\mu = \overline{\{M_\phi \colon \phi \in C(X)\}}^{WOT}$$

Finally, it can easily check $\overline{\Gamma(\mathfrak{A}_1)}^{WOT} \subset \Gamma(\mathfrak{A})$. Therefore, Γ is an isomorphism.

$3) \Rightarrow 1)$ It clearly holds by above example. $\qquad \square$

This theorem gives an explicit structure of the maximal abelian von Neumann algebra, which is equivalent to $L^\infty(\mu)$. And for a general abelian von Neumann algebra, by Zorn's Lemma, it can embedded into a masa.

Corollary 11. *If $\mathscr{H}$ is separable, every abelian C^*-algebra contained in $\mathscr{B}(\mathscr{H})$ has a separating vector.*

Proof. By Zorn's Lemma, there is a masa containing the abelian C^*-algebra and thus by above theorem it has a cyclic vector, which can separate the masa, in particular, this C^*-algebra. $\qquad \square$

3.2.3 Star-Cyclic Normal Operators

In above subsection, we can see $\mathfrak{A}_\mu = \{M_\phi \colon \phi \in L^\infty(\mu)\} \cong L^\infty(\mu)$ play an important role in the abelian von Neumann algebra. So we will see more properties of $\mathfrak{A}_\mu$.

Theorem 1 (Fuglede-Putnam Theorem). *Let N and M are normal operators and $A \in \mathscr{B}(\mathscr{H})$. If $NA = AM$, then $N^*A = AM^*$.*

Proof. Since for any polynomial, $p(N)A = Ap(M)$, then for a fixed $\omega \in \mathbb{C}$, by taking limits

$$e^{i\omega N}A = Ae^{i\omega M}, \text{ or } A = e^{-i\omega N}Ae^{i\omega M}$$

By the fact that $e^X e^Y = e^{X+Y}$ for commuting operators X, Y., therefore

$$
\begin{aligned}
f(\omega) &= e^{-i\omega N^*}Ae^{i\omega M^*} \\
&= e^{-i\omega N^*}e^{-i\overline{\omega}N}Ae^{i\overline{\omega}M}e^{i\omega M^*} \\
&= e^{-i(\omega N^* + \overline{\omega}N)}Ae^{i(\omega M^* + \overline{\omega}M)}
\end{aligned}
$$

Note that f is a entire operator-valued function. Since $(\omega N^* + \overline{\omega}N)$ and $(\omega M^* + \overline{\omega}M)$ are self-adjoint, $e^{-i(\omega N^* + \overline{\omega}N)}$ and $e^{i(\omega M^* + \overline{\omega}M)}$ are unitaries. Then f is bounded, thus f is constant, i.e.

$$0 = f'(\omega) = -iN^*e^{-i\omega N^*}Ae^{i\omega M^*} + ie^{-i\omega N^*}AM^*e^{i\omega M^*}$$

Let $\omega = 0$, then $0 = -iN^*A + iAM^*$. $\qquad\square$

Corollary 2. *If N is a normal operator with $N = \int z dE$, then $NA = AN$ if and only if $E(\Delta)N = NE(\Delta)$ for all Borel set Δ.*

Then using this theorem we can find the generator of $\mathfrak{A}_\mu$.

Theorem 3. *If μ is any compactly supported measure on $\mathbb{C}$, then*

$$\{N_\mu\}' = \mathfrak{A}'_\mu$$

Proof. Clearly, $\mathfrak{A}'_\mu \subset N'_\mu$. If $T \in N'_\mu$, then $TN_\mu^* = N_\mu^*T$. Therefore, T can commutes with any polynomials in z and $\overline{z}$, and taking limits, $T \in \mathfrak{A}'_\mu$. $\qquad\square$

Remark. By this, we can also see the statement in above theorem

$$\{M_\phi \colon \phi \in C(X)\}' = \mathfrak{A}_\mu$$

is true. Moreover, $\{N_\mu\}' = \mathfrak{A}'_\mu$ implies that

$$W^*(N_\mu) = \mathfrak{A}_\mu \cong L^\infty(\mu)$$

Therefore, in order to research $\mathfrak{A}_\mu$, we just research N_μ.

We have seen that if μ is finite then N_μ is indeed a cyclic normal operator on $L^2(\mu)$ as $\overline{C^*(\mu)1} = L^2(\mu)$ by the Stone-Weierstrass Theorem. Now, we can prove any cyclic normal operator is equivalent to a N_μ for some finite μ.

Theorem 4. *A normal operator N is cyclic if and only if it is equivalent to N_μ for some compactly supported measure μ on $\mathbb{C}$. Moreover, if e_0 is the cyclic vector, then the unitary $U \colon \mathscr{H} \to L^2(\mu)$ with $N_\mu = UNU^*$ can be chosen s.t. $Ue_0 = 1$.*

Proof. If E is the spectral measure s.t. $N = \int_{\sigma(N)} z \, dE$, for any Borel set Δ in $\sigma(N)$, then define

$$\mu(\Delta) = \langle E(\Delta)e_0, e_0 \rangle = \|E(\Delta)e_0\|^2$$

μ is finite, positive, and compactly supported on $\mathbb{C}$. Then $C(\sigma(N)) \subset L^2(\mu)$. We know that $C^*(N) \cong C(\sigma(N))$. Define

$$U: \quad \begin{aligned} C^*(N)e_0 &\longrightarrow C(\sigma(N)) \\ \phi(N)e_0 &\longmapsto \phi \end{aligned}$$

Then we have,

$$\|\phi(N)e_0\|^2 = \langle \phi(N^*)\phi(N)e_0, e_0 \rangle = \int_{\sigma(N)} |\phi|^2 \, d\mu = \|\phi\|_2$$

U is a unitary. By taking norm limits, $U: \mathscr{H} \to L^2(\mu)$ is a unitary and $Ue_0 = 1$ and $UNe_0 = z$. If $\phi \in C(\sigma(N))$, then

$$UNU^*(\phi) = UN\phi(N)e_0 = U(z\phi)(N)e_0 = z\phi = N_\mu(\phi)$$

Also, taking the norm limits, $UNU^* = N_\mu$. $\qquad\square$

Theorem 5. *Two normal operator N_μ and N_ν are equivalent if and only if μ and ν are mutually absolutely continuous.*

Proof. If μ and ν are mutually absolutely continuous and $\phi = \frac{d\mu}{d\nu}$, then define

$$U: \quad \begin{aligned} L^2(\mu) &\longrightarrow L^2(\nu) \\ f &\longmapsto f\sqrt{\phi} \end{aligned}$$

Then we have

$$\|Uf\|^2 = \int |f|^2 \phi \, d\nu = \int |f|^2 \, d\mu = \|f\|^2$$

Thus U is an isometry. Since $\frac{1}{\phi} = \frac{d\nu}{d\mu}$, the map $g \to \sqrt{\frac{1}{\phi}}g$ is the inverse of U. Therefore, U is unitary, and

$$UN_\mu U^* g = UN_\mu(\sqrt{\tfrac{1}{\phi}}g) = U(z\sqrt{\tfrac{1}{\phi}}g) = zg = N_\nu g$$

Conversely, $N_\mu \cong N_\nu$ means $K = \sup\mu = \sigma(N_\mu) = \sigma(N_\nu) = \sup N_\nu$. Let $V: L^2(\mu) \to L^2(\nu)$ be the unitary s.t. $VN_\mu V^* = N_\nu$. Then also $VN_\mu^* V^* = N_\nu^*$. Therefore, for any polynomial in z and $\overline{z}$, let $V(1) = \psi$

$$Vp(N_\mu, N_\mu^*)(1) = p(N_\nu, N_\nu^*)V(1)$$

That means

$$\int_K |p|^2 \, d\mu = \int_K |p|^2 \, |\psi|^2 \, d\nu$$

Taking limits, for any $f \in C(K)$, above identity is also true. Thus, $d\mu = |\psi|^2 \, d\nu$, i.e. $\mu \ll \mu$. Similarly, by V^{-1}, we can get the converse. $\qquad\square$

3.3   The structure of $W^*(N)$

For a norm operator N, by above subsection, clearly

$$\{ \, M_\phi \colon \phi \in B \, \} \subset W^*(N) = \{N\}'$$

And we have seen, $\{N_\mu\}' = \mathfrak{A}_\mu \cong L^\infty(\mu)$. Firsty, we want to prove $\{M_\phi \colon \phi \in B\} = W^*(N)$. More, if we also can find some μ s.t. $W^*(N) = W^*(N_\mu)$, then the structure of $W^*(N)$ is explicit,

$$W^*(N) = \{ \, M_\phi \colon \phi \in B \, \} \cong L^\infty(\mu)$$

Let N be a normal operator acting on $\mathscr{H}$ with the spectral measure E. If $h \in \mathscr{H}$ is nonzero, then let

$$\mu_h(\Delta) = E_{hh}(\Delta), \ \ \mathscr{H}_h = \overline{W^*(N)h}, \ \ N_h = N|_{\mathscr{H}_h}$$

In fact, $\mathscr{H}_h$ is the smallest reducing space for N containing h. And N_h is a star-cyclic normal operator with the cyclic vector h. If E_h is the spectral measure of N_h, then by the uniqueness of spectral measure

$$E_h(\Delta) = \chi_\Delta(N_h) = \chi_\Delta(N)|_{\mathscr{H}_h} = E(\Delta)|_{\mathscr{H}_h}$$

Therefore, there is a unique unitary $U_h \colon \mathscr{H}_h \to L^2(\mu_h)$ s.t. $U_h h = 1$ and $U_h N_h U_h^* = N_{\mu_h}$.

Proposition 1. *Let N be a normal operator acting on $\mathscr{H}$ and a nonzero $h \in \mathscr{H}$.*

$$
\begin{array}{cccc}
\rho_h \colon & W^*(N) & \longrightarrow & W^*(N_h) \\
& A & \longmapsto & A|_{\mathscr{H}_h}
\end{array}
$$

1) ρ_h is a $*$-homomorphism and WOT-continuous.

2) If $\phi \in B(\sigma(N))$, then $\rho_h(\phi(N)) = \phi(N_h)$.

3) If h is separating for $W^*(N)$, then ρ_h is injective.

4) If $A \in W^*(N)$, then there exists a $\phi \in B(\sigma(N))$ s.t. $\rho_h(A) = \phi(N_h)$, and thus ρ_h is surjective.

Proof. For 1), if $p(z, \overline{z})$ is any polynomial, then $\rho(p(N, N^*)) = p(N_h, N_h^*)$. For any $A \in W^*(N)$, there exists $p_n(N, N^*)$ s.t. $p_n(N, N^*) \to A$ in WOT. Then for any $g, h \in \mathscr{H}_h$

$$\langle p_n(N, N^*)g, h \rangle \to \langle Ag, h \rangle$$

Therefore, $p_n(N_h, N_h^*) \to A|_{\mathscr{H}_h}$ in WOT. And thus, ρ_h is WOT-continuous and $\rho_h(A) \in W^*(N_h)$, well-defined. And clearly, ρ_h is a $*$-homomorphism.

For 2), if $\phi \in B(\sigma(N))$, then there exist polynomials $p_n(z, \bar{z})$ s.t. $p_n \to \phi$ weakly in $M(\sigma(N))$ and $M(\sigma(N_h))$ as $\sigma(N_h) \subset \sigma(N)$. That means $p_n(N, N^*) \to \phi(N)$ in WOT and $p_n(N_h, N_h^*) \to \phi(N_h)$. But $\rho_h(p_n(N, N^*) = p_n(N_h, N_h^*)$ and ρ is WOT-continuous, $\rho_h(\phi(N)) = \phi(N_h)$.

For 3), if h is separating for $W^*(N)$, $A_h = 0$ i.e $AW^*(N)h = 0$. That means $AW^*(N) = 0$, then $A = 0$.

For 4), if $A \in W^*(N)$ and $A_h = \rho_h(A)$, then $A_h \in \{N_h\}' = U_h^*\{N_{\mu_h}\}'U_h$. There is a $\psi \in L^\infty(\mu_h)$ s.t. $A_h = U_h^* M_\psi U_h$. Then there exist $\psi_n \in B(\sigma(N))$ s.t. $M_{\psi_n} \to M_\psi$ in norm. Thus $\psi_n(N_h) \to A_h$ in norm, there is a $\phi \in M(\sigma(N))$, s.t. $\phi(N_h) = A_h$. Moreover, this also implies ρ_h is surjective. $\square$

Remark. The most important part in this proof is that for the μ_h,

$$W^*(N_{\mu_h}) = \mathfrak{A}_{\mu_h} = \{\ M_\phi \colon \phi \in L^\infty(\mu_h)\ \} = \{\ M_\phi \colon \phi \in B(\sigma(N_{\mu_h}))\ \}$$

Using above proposition we get one of the main result of this subsection.

Theorem 2. *If N is a normal operator, then*

$$W^*(N) = \{\ \phi(N) \colon \phi \in B(\sigma(N))\ \}$$

Proof. It is sufficient to show that $\mathfrak{A} = \{\phi(N) \colon \phi \in B(\sigma(N))\}$ is WOT-closed. Let $A \in \overline{\mathfrak{A}}^{WOT} = W^*(N)$. Then there exists $\{\phi)n\} \subset B(\sigma(N))$ s.t. $\phi_n(N) \to A$ in WOT. And for any $h \in \mathscr{H}$, there exists $\phi_h \in B(\sigma(N_h))$, s.t. $A_h = \phi_h(N_h)$, also $\phi_n(N_h) \to A_h$ in WOT. That means

$$Ah = \phi_h(N_h)h = \phi_h(N)h, \quad \forall\, h \in \mathscr{H}$$

But ϕ_h is dependent with the chose of h. So we want to find a $e \in \mathscr{H}$ s.t. $\phi_h(N_h) = \phi_e(N_h)$ for any $h \in \mathscr{H}$. Fix a $e \in \mathscr{H}$
Check: $\phi_h(N_h)h = \phi_e(N_h)h$
By the definition, for any Borel subset Δ of $\sigma(N)$, since

$$\begin{aligned} \mu_e &= \langle E(\Delta)e, e \rangle = \|E(\Delta)e\|^2 \\ \mu_h &= \langle E(\Delta)h, h \rangle = \|E(\Delta)h\|^2 \end{aligned}$$

mu_e and μ_h are mutually absolutely continuous. In fact, for any $g \in \mathscr{H}$, μ_h and μ_g are mutually absolutely continuous. Since $\phi_n(N_h) \to A_h$ in WOT, $\phi_n \to \phi_h$ weak* in $L^\infty(\mu_h)$. Also, $\phi_n \to \phi_e$ weak* in $L^\infty(\mu_e)$. Then for any Borel subset Δ

$$\int_\Delta \phi_n d\mu_h = \int_\Delta \phi_n \frac{d\mu_h}{d\mu_e} d\mu_e \to \int_\Delta \phi_e d\mu_h$$

and also $\int_\Delta \phi_n d\mu_h \to \int_\Delta \phi d\mu_h$, therefore, $\phi_h = \phi_e$ a.e. with repect to μ_h, in fact, it can be with respect to any μ_g. Then for $g \in \mathscr{H}_h$,

$$\langle \phi_h(N_h)g, g \rangle = \langle \phi_h(N)g, g \rangle = \int \phi_h d\mu_g = \int \phi_e d\mu_g = \langle \phi_e(N_h)g, g \rangle$$

Therefore, $\phi_h(N_h)h = \phi_e(N_h)h$. Then $A = \phi_e(N)$. $\square$

And also, by above proposition, the second result can also be proved.

Theorem 3. *If N is a normal operator on a separable Hilbert space, then there is a compactly supported, positive and finite measure μ, s.t. there is a map $\rho\colon L^\infty \to W^*(N)$, s.t.*

1) ρ is an isometrical $*$-isomorphism.

2) $\rho\colon (L^\infty(\mu), weak^*) \to (W^*(N), WOT)$ is a homeomorphism.

Proof. Since $W^*(N)$ is an abelian von Neumann algebra, it has a separating vector e by the **Corollary** 11 in the subsection **3.2.2**. Then let $\rho' = \rho_e$ and $\mu = \mu_e$. By above proposition, ρ' is a $*$-isomorphism from $W^*(N_e)$ to $W^*(N)$. And since $W^*(N_e)$ is also $*$-isomorphic to $L^\infty(\mu)$. Thus there is a $*$-isomorphism ρ from $L^\infty(\mu)$ to $W^*(N)$ s.t. for any $\phi \in B(\sigma(N))$, $\rho(\phi) = \phi(N)$. And thus ρ is isometrical.

For 2), since $W^*(N) = \{\phi(N)\colon \phi \in B(\sigma(N))\}$, if $\phi_n(N) \to \phi(N)$ in WOT, then we can easily know $\phi_n \to \phi$ in weak*. And the converse is trivial. ρ is weak*-WOT homeomorphism. $\qquad\square$

Now, we can see the structure of any abelian von Neumann algebra on a separable Hilbert space.

Theorem 4. *If $\mathfrak{A}$ is an abelian von Neumann algebra on a separable Hilbert space $\mathscr{H}$, then there is a compactly supported, positive and finite measure μ s.t. $\mathfrak{A} \cong L^\infty(\mu)$, $*$-isomorphism.*

Proof. Since $\mathscr{H}$ is separable, the closed unit ball in $\mathfrak{A}$ is WOT-compactly metrizable, thus there is a countable WOT-dense set of $\mathfrak{A}$. Since $\mathfrak{A}$ can be generated by projections in $\mathfrak{A}$, thus this WOT-dense set can be chosen as a set of projections $\{P_n\}$. Let $\mathbb{C} = C^*(\{P_n\})$, then $\mathbb{C}$ is ablelian. Thus there is a compact space X, which is totally disconnected, s.t. $\mathbb{C} \cong C(X)$. And let $\rho\colon C(X) \to \mathbb{C}$ be the inverse of Gelfand Transform. Then $p_n = \rho^{-1}(P_n)$ is the characteristic function and continuous. Define

$$f = \sum_{n=1}^{\infty} 3^{-n}(2p_n - 1)$$

Claim: $C^*(f) = C(X)$

By the Stone-Weierstrass Theorem, we just need to show that f can separate the points of X. Since $\{p_n\}$ generates $C(X)$, $\{p_n\}$ can separate the points in X. Thus, for $x \neq y \in X$, there is a p_k, s.t. $p_k(x) \neq p_k(y)$ and chosing the

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k be the smallest one satifying this condition, then

$$|f(x) - f(y)| = 2\left|\sum_{n=k}^{\infty} 3^{-n}(p_n(x) - p_n(y))\right|$$

$$\geqslant 23^{-k} - 2\sum_{n=k+1}^{\infty} 3^{-n}$$

$$= 3^{-k}$$

Therefore, let $A = \rho(f) = \sum_{n=1}^{\infty} 3^{-n}(2P_n - 1)$, $C^*(A) = \mathbb{C}$. Moreover, $W^*(A) = \overline{C^*(A)}^{WOT} = \mathfrak{A}$. And since A is self-adjoint, thus by above theorem, there exists μ on $\sigma(A)$, s.t.

$$\mathfrak{A} \cong L^\infty(\mu) \qquad \qquad \square$$

3.4 Multiplicity Theory

Finally, we want find the necessary and sufficient condition making two normal operators be equivalent. We should assume the Hilbert space is always separable.

Proposition 1. *If N is a normal operator on $\mathscr{H}$ and $e \in \mathscr{H}$, then there is a separating vector e_0 for $W^*(N)$ s.t. $e \in \overline{W^*(N)}$.*

Proof. Choosing any separating operator f_0 for $\overline{W^*(N)}$, let $\mathscr{M} = \overline{W^*(N)}f_0$, then $e = m_0 + g_0$ for $m_0 \in \mathscr{M}$ and $g_0 \in \mathscr{M}^\perp$. Now, let $\mathscr{G} = \overline{W^*(N)g_0}$ and $e_0 = f_0 + g_0$. Then $e \in \overline{W^*(N)e_0}$.
Check: e_0 is separating for $W^*(N)$
For any $A \in W^*(N)$, since both $\mathscr{M}$ and $\mathscr{G}$ reduce $W^*(N)$, if

$$Ae_0 = Af_0 + Ag_0 = 0$$

then $Af_0 = 0$. Because f_0 is separating for $W^*(N)$, $A = 0$. $\qquad \square$

Proposition 2. *Let μ be a compactly supported measure on $\mathbb{C}$ and Δ be a Borel subset of $\sup \mu$ and $\nu = \mu|_\Delta$. If $N = N_\mu \oplus N_\nu$ on $L^2(\mu) \oplus L^2(\nu)$ and $g \oplus h \in L^2(\mu) \oplus L^2(\nu)$ with $h \neq 0$ a.e. ν, then there is a $f \in L^2(\mu)$ s.t. $f \oplus h$ is a separating vector for $W^*(N)$ and $g \oplus h \in \overline{W^*(N)(f \oplus h)}$.*

Proof. Let $f(z) = g(z)$ for $z \in \Delta$ and $f(z) = 1$ for $x \in \Delta'$ where $\Delta' = \sup \mu \backslash \Delta$. In fact,

$$\mathscr{H} = \overline{W^*(N)(f \oplus h)} = \overline{\{\ \phi f \oplus \phi h \colon \phi \in L^\infty(\mu)\ \}}$$

Since $\phi\chi_{\Delta'} \oplus 0 = \phi\chi_{\Delta'}(f \oplus h)$,

$$g \oplus h = f \oplus h - (1-g)\chi_{\Delta'} \oplus 0 \in \mathscr{H}$$

If $\phi \in L^\infty(\mu)$ with $\phi f \oplus \phi h = 0$, then $\phi f = \phi h = 0$ a.e. μ and ν respectively. By definition, $\phi = 0$ a.e. μ. $\qquad \square$

Then we can decompose normal operators as a "decreasing" sequence of cyclic operators.

Theorem 3. *If N is a normal operator, then there is a sequence of measures $\{\mu_n\}$ on $\mathbb{C}$ s.t. $\mu_{n+1} \ll \mu_n$ for all n and*

$$N \cong N_{\mu_1} \oplus N_{\mu_2} \oplus N_{\mu_3} \oplus \cdots$$

Proof. Let e_1 be a separating vector for $W^*(N)$ and $\{f_n\}$ be the orthonormal basis of $\mathscr{H}$ s.t. $f_1 = e_1$. Put

$$\mathscr{H}_1 = \overline{W^*(N)e_1}, \quad \mu_1(\Delta) = \|E(\Delta)\|^2, \quad N_2 = N|_{\mathscr{H}_1}$$

Then for $f_2' = $ the projection of f_2 to $\mathscr{H}_1^\perp$, there is a separating vector e_2 for $W^*(N_2)$ s.t.

$$f_2' \in \mathscr{H}_2 = \overline{W^*(N_2)e_2}$$

And clearly, if $\mu_2\Delta = \|E(\Delta)|_{\mathscr{H}_2}e_2\|$, $\mu_2 \ll \mu_1$. By induction and the Zorn's Lemma,

$$N \cong N_{\mu_1} \oplus N_{\mu_2} \oplus N_{\mu_3} \oplus \cdots \qquad \square$$

Therefore, if there is another normal operator $M \cong N_{\nu_1} \oplus N_{\nu_2} \oplus \cdots$, we want to find the relation between the equivalence of N and M and the equivalence of the coincided μ_n and ν_n. But before that we need some lemmas.

Proposition 4. *If N_1 and N_2 are normal operators on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively and $X \colon \mathscr{H}_1 \to \mathscr{H}_2$ s.t. $XN_1 = N_2X$, then*

1) $\overline{\operatorname{ran} X}$ reduces N_2.

2) $\ker X$ reduces N_1.

3) If $M_1 = N_1|_{(\ker X)^\perp}$ and $M_2 = N_2|_{\overline{\operatorname{ran} X}}$, then $M_1 \cong M_2$.

Proof. For 1), if $f_1 \in \mathscr{H}_1$, $XN_1f_1 = N_2Xf_1 \in \operatorname{ran} X$. Therefore, $\overline{\operatorname{ran} X}$ is invariant N_2. And also, by the Fuglede-Putnam Theorem, $XN_1^* = N_2^*X$, thus $\overline{\operatorname{ran} X}$ is invariant N_2^*.
2) is similar as 1). For 3), it is sufficent to show this when $\ker X = \{0\}$ and $\operatorname{ran} X$ is dense. By the Polar Decomposition, $X = UA$ for some unitary U. Since $X^*N_2^* = N_1^*X^*$, $X^*N_2 = N_1X^*$. Therefore $A^2 \in \{N_1\}'$, so $A \in \{N_1\}'$ by the functional calculus.

$$N_2X = N_2UA = XN_1 = UAN_1 = UN_1A$$

That means $N_2U = UN_1$ on $\overline{\operatorname{ran} A} = \overline{\operatorname{ran} X} = \mathscr{H}_1$, therefore $N_1 \cong N_2$. $\square$

Proposition 5. *If N, A and B are normal operators on $\mathscr{K}$, $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively and N is $*$-cyclic and*

$$N \oplus A \cong N \oplus B$$

then $A \cong B$.

Proof. Let $U \colon \mathscr{K} \oplus \mathscr{H}_A \to \mathscr{K} \oplus \mathscr{H}_B$, and U can be

$$U = \left(\begin{array}{cc} U_{11} & U_{12} \\ U_{21} & U_{22} \end{array} \right)$$

Then we have

$$\left(\begin{array}{cc} U_{11}N & U_{12}A \\ U_{21}N & U_{22}A \end{array} \right) = \left(\begin{array}{cc} NU_{11} & NU_{12} \\ BU_{21} & BU_{22} \end{array} \right)$$

$$\left(\begin{array}{cc} U_{11}N^* & U_{12}A^* \\ U_{21}N^* & U_{22}N^* \end{array} \right) = \left(\begin{array}{cc} N^*U_{11} & N^*U_{12} \\ B^*U_{21} & B^*U_{22} \end{array} \right)$$

Then by above proposition,

$$A|_{(\ker U_{22})^\perp} = B|_{(\ker U_{22}^*)^\perp}$$

For $\ker U_{22}$ and $\ker U_{22}^*$, put $\mathscr{M}_1 = U_{12}(\ker U_{22})$, $\mathscr{M}_2 = U_{21}^*(\ker U_{22}^*)$, also by the identity, $U^*U = UU^* = 1$,

$$A|_{\ker U_{22}} \cong N|_{\mathscr{M}_1}, \quad B|_{\ker U_{22}^*} \cong N|_{\mathscr{M}_2}$$

And then it easily check that $\mathscr{M}_1 = \mathscr{M}_2$. $\qquad\square$

Theorem 6. *For two normal operators N and M and the two sequences of measures $\{\mu_n\}$ and $\{\nu_n\}$ on $\mathbb{C}$ s.t. $\mu_{n+1} \ll \mu_n$ and $\nu_{n+1} \ll \nu_n$ for all n s.t.*

$$\begin{array}{rcl} N & \cong & N_{\mu_1} \oplus N_{\mu_2} \oplus \cdots \\ M & \cong & N_{\nu_1} \oplus N_{\nu_2} \oplus \cdots \end{array}$$

then $N \cong M$ if and only if μ_n and ν_n are mutually absolutely continuous for all n.

Proof. Let $N \cong M$ with the unitary U. If e_1 is the separating vector for $W^*(N)$, then $f_1 = Ue_1$ is the separating vector for $W^*(M)$. If the μ_1 and ν_1 are the coincided measures, then μ_1 and ν_1 are mutually absolutely continuous, thus $N_{\mu_1} \cong N_{\nu_1}$. Let $\mathscr{H}_1 = \overline{W^*(N)e_1}$ and $\mathscr{H}_1' = \overline{W^*(M)e_1}$. Then by above theorem,

$$N = N_{\mu_1} \oplus N|_{\mathscr{H}_1^\perp} \cong M = N_{\nu_1} \oplus M|_{\mathscr{H}_1'^\perp}$$

we know that $N|_{\mathscr{H}_1^\perp} \cong M|_{\mathscr{H}_1'^\perp}$. Therefore, by induction, we can get the result. $\qquad\square$

Then we can get another formula of above theorem to help us build the Multiplicity Theorem.

Corollary 7. *If N is a normal operator with the coincided measure μ, then there is a decreasing sequence $\{\Delta_n\}$ of Borel subsets of $\sigma(N)$ s.t. $\Delta_1 = \sigma(N)$ and*

$$N \cong N_\mu \oplus N_{\mu|_{\Delta_2}} \oplus N_{\mu|_{\Delta_3}} \oplus \cdots$$

Also, if M is another normal operator with similar formula

$$M \cong N_\nu \oplus N_{\nu|_{\Sigma_2}} \oplus N_{\nu|_{\Sigma_3}} \oplus \cdots$$

then $N \cong M$ if and only if μ and ν are mutually absolutely continuous and $\mu(\Delta_n \backslash \Sigma_n) = 0 = \mu(\Sigma_n \backslash \Delta_n)$

Theorem 8. *If N is a normal operator, then there are mutually singular measures $\mu_\infty, \mu_1, \mu_2, \cdots$ s.t.*

$$N \cong N_\infty^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus N_{\mu_3}^{(3)} \oplus \cdots$$

If M is another normal operator with the corresponding measures $\nu_\infty, \nu_1, \cdots$, then $N \cong M$ if and only if μ_n and ν_n are mutually absolutely continuous for $1 \leqslant n \leqslant \infty$.

Proof. Let the μ and $\{\Delta_n\}$ be the sequence in above corollary. Put

$$\Sigma_\infty = \bigcap_{n=1}^\infty \Delta_n, \ \Sigma_n = \Delta_n \backslash \Delta_{n+1} \text{ and } \mu_n = \mu|_{\Sigma_n}$$

for $1 \leqslant n \leqslant \infty$ and $\nu_n = \mu|_{\Delta_n}$ for $1 \leqslant n < \infty$. Now,

$$\Delta_n = \Sigma_\infty \bigcup (\Delta_n \backslash \Delta_{n+1}) \bigcup (\Delta_{n+1} \backslash \Delta_{n+2}) \bigcup (\Delta_{n+2} \backslash \Delta_{n+3}) \bigcup \cdots$$

Therefore, $\nu_n = \mu_\infty + \mu_n + \mu_{n+1} + \mu_{n+2} + \cdots$ and $\{\mu_n\}$ are pairwise singular. Thus,

$$N_{\nu_n} = N_{\mu_\infty} \oplus N_{\mu_n} \oplus N_{\mu_{n+1}} \oplus \cdots$$

Then we have

$$
\begin{aligned}
N &= N_{\nu_1} \oplus N_{\nu_2} \oplus \cdots \\
&= N_\infty^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus N_{\mu_3}^{(3)} \oplus \cdots
\end{aligned}
$$

$\square$

Now, we can define the multiplicity function m_N for a normal operator N with the corresponding measure μ. As above preceding theorem,

$$N \cong N_\infty^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus N_{\mu_3}^{(3)} \oplus \cdots$$

If there are disjoint Borel sets $\Delta_\infty, \Delta_1, \Delta_2, \cdots$ of $\mathbb{C}$ s.t. μ_n and $mu|_{\Delta_n}$ are mutually absolutely continuous for all n, then define $m_N \colon \mathbb{C} \to \mathbb{N} \cup \{\infty\}$ as

$$m_N = \infty \chi_{\Delta_\infty} + \chi_{\Delta_1} + 2\chi_{\Delta_2} + 3\chi_{\Delta_3} + \cdots$$

Clearly, m_N is dependent with the choice of $\{\Delta_n\}$, but by the definition, m_N can be identified as an equivalent class with respect to μ. And moreover,

$$\Delta_n = \{\, z \colon m_N(z) = n \,\}$$

Therefore, we can recover the $\{\Delta_n\}$ by the m_N. By combining with above theorem, we can classify the normal operators.

Theorem 9. *Two normal operators are equivalent if and only if they have same multiplicity function.*

Chapter 4

Compact Operators, Fredholm Theory and Perturbations

The compact operator is as similar as the finite dimensional operator with respect to the spectrums, the representations and so on. In fact, all compact operators can form the unique closed ideal in $\mathscr{B}(\mathscr{H})$. Moreover, it contains some important classes of operators, like the trace class, which provides an extra topology on $\mathscr{B}(\mathscr{H})$ and this topology plays an important role in the von Neumann algebra. In fact, the topology is coincided with the WOT in some cases, but it can give a new method to obtain some properties of general von Neumann algebras. For non-compact operators, there is a "weak version" of the compact operator, the Fredholm operator. By researching the Fredholm operators, it provides us an approach to find some "invariant variable" under the compact perturbations. Finally, we have known that finite dimensional normal operator can always be diagnolizable, but for infinite dimensional case, this statement is not true. But by some small compact perturbation, the normal operator can be diagnalizable, and this result has some applications of representation theory.

4.1 Spectrums

4.1.1 Elementary Properties

Definition 1. Let $\mathscr{H}$ be a Hilbert space and $B_{\mathscr{H}}$ be the closed unit ball in $\mathscr{H}$ and $T \in \mathscr{B}(\mathscr{H})$. T is compact if and only if the closure of $T(B_{\mathscr{H}})$ is compact. Then let $\mathscr{B}_0(\mathscr{H})$ denote the set of all compact operators.

Remark. Clearly, if T is finite rank, $T \in \mathscr{B}_0(\mathscr{H})$. Denote $\mathscr{B}_{00}(\mathscr{H})$ as the

set of all finite rank operators. Therefore, $\mathscr{B}_{00}(\mathscr{H}) \subset \mathscr{B}_0(\mathscr{H})$.

$$\mathscr{B}_{00}(\mathscr{H}) = \{\, T \in \mathscr{B}(\mathscr{H}) \colon \operatorname{ran} T \text{ is finite dimension }\}$$

Since $\mathscr{H}$ is a complete metric space, the compactness of $\overline{T(B_{\mathscr{H}})}$ is equivalent to that any sequence in $\overline{T(B_{\mathscr{H}})}$ has a convergent subsequence with the convergent point in $\overline{T(B_{\mathscr{H}})}$. By this fact, we can get the following theorem.

Theorem 2. *Let $\mathscr{H}$ be a Hilbert space.*

1) *$\mathscr{B}_0(\mathscr{H})$ is a linear space.*

2) *$\mathscr{B}_0(\mathscr{H})$ is closed in norm.*

3) *$\mathscr{B}_0(\mathscr{H})$ is an ideal in $\mathscr{B}(\mathscr{H})$.*

4) *If $T \in \mathscr{B}_0(\mathscr{H})$, then $T^* \in \mathscr{B}_0(\mathscr{H})$.*

Remark. Therefore, $\mathscr{B}_0(\mathscr{H})$ is a C^*-subalgebra, and moreover, $\mathscr{B}_0(\mathscr{H})$ is a closed ideal of $\mathscr{B}(\mathscr{H})$.

Definition 3. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. T is called completely continuous if for any sequence $\{x_n\} \in \mathscr{H}$ with $x_n \to x$ weakly, then $Tx_n \to Tx$ in norm.

Proposition 4. *$T \in \mathscr{B}(\mathscr{H})$ is compact if and only if T is completely continuous.*

Proof. Let T be a compact operator and $\{x_n\} \in \mathscr{H}$ with $x_n \to 0$ weakly. By the Principle of Uniform Boundedness, $M = \sup_n \|x_n\| < \infty$. Assuming $M \leqslant 1$,
$$\{Tx_n\} \subset \overline{T(B_{\mathscr{H}})}$$
By the fact that $\overline{T(B_{\mathscr{H}})}$ is compact, there is a subsequence $\{x_{n_k}\}$ s.t. $Tx_{n_k} \to y$ in norm. Since T is also weakly continuous, $Tx_n \to 0$ weakly. Thus $y = 0$. Then T is completely continuous.
Conversely, assume T is completely continuous.
Firstly, if $\mathscr{H}$ is separable and since $\mathscr{H}$ is reflexive, $B_{\mathscr{H}}$ is a weak-compact metric space. Therefore, for any $\{x_n\} \subset B_{\mathscr{H}}$, there is a subsequence $\{x_{n_k}\}$ and x s.t. $x_{n_k} \to x$ weakly. Therefore, $Tx_{n_k} \to x$ in norm, i.e. T is compact. For general case, let $\{x_n\} \subset B_{\mathscr{H}}$, then $\mathscr{H}_1 = \overline{\operatorname{span}\{x_n\}}$ is separable. Therefore, since
$$T|_{\mathscr{H}_1}(\{x_n\}) = T(\{x_n\}) \subset T|_{\mathscr{H}_1}(B_{\mathscr{H}_1})$$
T is compact. $\qquad\qquad\square$

By above theorem, we can see the power of compact operators. Intuitively, compact operators can "strengthen" the topology. They map the weakly convergent sequences be norm convergent sequence. How can they

do that? We have known the weak topology is agree with the norm topology on the finite dimensional space. So, compact operators may be the "extension" of finite rank operators. In fact, we can describe this property more rigorously.

Theorem 5. $T \in \mathscr{B}(\mathscr{H})$ *is compact if and only if there is a sequence* $\{T_n\} \subset \mathscr{B}_{00}(\mathscr{H})$ *s.t.* $T_n \to T$ *in norm.*

Proof. Assume T is compact. Therefore, $\overline{T(B_{\mathscr{H}})}$ is separable. Let $\{e_n\}_{n=1}^{\infty}$ be the basis of the dense subspace of $\overline{T(B_{\mathscr{H}})}$. Then define projections

$$P_n \colon \mathscr{H} \longrightarrow \mathrm{span}\,\{e_1, e_2, \cdots, e_n\}$$

and let $T_n = P_n T$. Clearly, $T_n \in \mathscr{B}_{00}(\mathscr{H})$. Now, clearly, $\|Th - T_n h\| \to 0$ for any $h \in \mathscr{H}$. Since $\overline{T(B_{\mathscr{H}})}$ is compact, for any given $\varepsilon > 0$, there are $h_1, \cdots, h_m \in \mathscr{H}$ s.t.

$$T(B_{\mathscr{H}}) \subset \bigcup_{i=1}^{m} B_{\varepsilon}(Th_i)$$

where $B_{\varepsilon}(h_i)$ is the open ball centred at Th_i with radius ε. Therefore, for any $h \in B_{\mathscr{H}}$, choose h_j s.t. $\|Th - Th_j\| < \varepsilon$.

$$
\begin{aligned}
\|Th - T_n h\| &\leqslant \|Th - Th_j\| + \|Th_j - T_n h_j\| + \|P_n(Th_j - Th)\| \\
&\leqslant 2\|Th - Th_j\| + \|Th_j - T_n h_j\| \\
&< 2\varepsilon + \|Th_j - T_n h_j\|
\end{aligned}
$$

Since $\|Th_j - T_n h_j\| < \varepsilon$ for any h_j and $n > n_0$ for some $n_)$,

$$\|Th - T_n h\| < 3\varepsilon \text{ for } n > n_0$$

Then $T_n \to T$ uniformly, i.e. $T_n \to T$ in norm.
The converse is clearly since $\mathscr{B}_{00}(\mathscr{H}) \subset \mathscr{B}_0(\mathscr{H})$ and $\mathscr{B}_0(\mathscr{H})$ is norm closed. $\qquad\square$

Corollary 6. *All projections* $\{P_i\}_{i \in I}$ *in* $\mathscr{B}(\mathscr{H})$ *form an approximate identity for the ideal* $\mathscr{B}_0(\mathscr{H})$.

4.1.2   Spectrums of Compact Operators

The spectrum of a compact operator also has similar properties as a finite rank operator.

Theorem 1. *If* $T \in \mathscr{B}_0(\mathscr{H})$ *and* $\lambda \neq 0$ *satisfying*

$$\inf \{ \; \|(T - \lambda)h\| : \; \|h\| = 1 \; \} = 0$$

then $\lambda \in \sigma_p(T)$.

Proof. There is a sequence $\{h_n\}$ with $\|h_n\| = 1$ s.t. $\|(T - \lambda)h_n\| \to 0$. Since T is compact, there is a subsequence $\{h_{n_k}\}$ and a h_0 s.t. $\|Th_{n_k} - h_0\| \to 0$, therefore

$$h_{n_k} = \frac{1}{\lambda}((\lambda - T)h_{n_k} + Th_{n_k}) \to \frac{1}{\lambda}h_0$$

Then $\left\|\frac{1}{\lambda}h_0\right\| = 1 = \left|\frac{1}{\lambda}\right| \|h_0\|$, thus $\|h_0\| \neq 0$. Since $Th_{n_k} \to \frac{1}{\lambda}Th_0$ and $Th_{n_k} \to h_0$,

$$h_0 = \frac{1}{\lambda}Th_0, \text{ i.e. } Th_0 = \lambda h_0$$

$\square$

In fact, we have known that

$$\sigma_{ap}(T) = \sigma_l(T) = \{ \lambda \in \mathbb{C} \colon \inf \{\|(T - \lambda)h\| : \|h\| = 1\} = 0 \}$$

and $\lambda \notin \sigma_{ap}(T)$ is equivalent to that $\ker(T - \lambda) = \{0\}$ and $\operatorname{ran}(T - \lambda)$ is closed. Combining with the Riesz Theorem, any closed and bounded subset in a normed space is compact if and only if the normed space is finite dimensional, we have the following corollary.

Corollary 2. *If $T \in \mathscr{B}_0(\mathscr{H})$ and $\lambda \neq 0$ and $\ker(T - \lambda) = \{0\}$, then $\operatorname{ran}(T - \lambda)$ is closed.*

Here is another important corollary, which says the point spectrum play an important role.

Corollary 3. *If $T \in \mathscr{B}_0(\mathscr{H})$, $\lambda \notin \sigma_p(T)$ with $\lambda \neq 0$ and $\overline{\lambda} \notin \sigma_p(T^*)$, then $\lambda \notin \sigma(T)$.*

Proof. By above theorem, $\lambda \notin \sigma_p(T)$ with $\lambda \neq 0$ means $\lambda \notin \sigma_l(T)$, thus $\ker(T - \lambda) = \{0\}$ and $\operatorname{ran}(T - \lambda)$ is closed.
Similarly, for T^*, $\ker(T^* - \overline{\lambda}) = \{0\}$ and $\operatorname{ran}(T^* - \lambda)$ is closed. Therefore,

$$\operatorname{ran}(T - \lambda) = \overline{\operatorname{ran}(T - \lambda)} = (\ker(T^* - \overline{\lambda}))^\perp = \mathscr{H}$$

Thus $T - \lambda$ is a bijection, and by the Inverse Mapping Theorem, $(T - \lambda)^{-1} \in \mathscr{B}(\mathscr{H})$, i.e. $\lambda \notin \sigma(T)$.
$\square$

Remark. In other words, for a compact operator T, if $\lambda \in \sigma(T)$ with $\lambda \neq 0$, then $\lambda \in \sigma_p(T)$ or $\lambda \in \sigma_p(T^*)$.

In fact, any nonzero point in $\sigma(T)$ for a compact operator T is isolated and moreover, it is an eigenvalue. To prove this, we need some lemmas.

Lemma 4. *If $\mathscr{M}$ and $\mathscr{N}$ are two closed linear subspaces of $\mathscr{H}$ with $\mathscr{M} \subset \mathscr{N}$, then for any $\varepsilon > 0$, there exists a $y \in \mathscr{N}$ with $\|y\| = 1$ s.t. $\operatorname{dist}(y, \mathscr{M}) \leqslant 1 - \varepsilon$.*

Proof. For $y \in \mathcal{N}$, define $\delta(y) = \text{dist}\,(y, \mathcal{M})$. Choosing $y_1 \in \mathcal{N} \backslash \mathcal{M}$, there exists a $x_0 \in \mathcal{M}$ s.t.

$$\delta(y_1) \leqslant \|x_0 - y_1\| \leqslant (1 + \varepsilon)\delta(y_1)$$

Let $y_2 = y_1 - x_0$, then

$$(1 + \varepsilon)\delta(y_2) = (1 + \varepsilon)\inf\,\{\|y_1 - x_0 - x\| : x \in \mathcal{M}\} = (1 + \varepsilon)\delta(y_1)$$

Thus $(1 + \varepsilon)\delta(y_2) > \|x_0 - y_1\| = \|y_2\|$. Let $y = \|y_2\|^{-1}\,y_2$, then $y_2 \in \mathcal{N}$.

$$
\begin{aligned}
\|y - x\| &= \left\|\|y_2\|^{-1}\,y_2 - x\right\| = \|y_2\|^{-1}\,\|y_2 - \|y_2\|\,x\| \\
&> ((1 + \varepsilon)\delta(y_2))^{-1}\,\|y_2 - \|y_2\|\,x\| \\
&\geqslant (1 + \varepsilon)^{-1} > 1 - \varepsilon
\end{aligned}
$$

$\square$

Proposition 5. *If $T \in \mathcal{B}_0(\mathcal{H})$ and $\{\lambda_n\}$ is a sequence of distinct elements in $\sigma_p(T)$, then $\lim_{n \to \infty} \lambda_n = 0$.*

Proof. For each n, choosing a nonzero $x_n \in \ker\,(T - \lambda_n)$,

$$\mathcal{M}_n = \text{span}\,\{\,x_1, x_2, \cdots, x_n\,\}$$

By preceding lemma, there is a $y_n \in \mathcal{M}_n$ with $\|y_n\| = 1$ s.t.

$$\text{dist}\,(y_n, \mathcal{M}_{n-1}) > \frac{1}{2}$$

Let $y_n = \sum_{i=1}^{n} \alpha_i x_n$, thus

$$(T - \lambda_n)y_n = \sum_{i=1}^{n-1} \alpha_i(\lambda_i - \lambda_n)x_i \in \mathcal{M}_{n-1}$$

Therefore, for $n > m$,

$$
\begin{aligned}
T(\lambda_n^{-1} y_n) - T(\lambda_m^{-1} y_m) &= \lambda_n^{-1}(T - \lambda_n)y_n - \lambda_m^{-1}(T - \lambda_m)y_m + (y_n - y_m) \\
&= y_n - (y_m + \lambda_n^{-1}(T - \lambda_n)y_n + \lambda_m^{-1}(T - \lambda_m)y_m)
\end{aligned}
$$

Since the part in the bracketed is in $\mathcal{M}_{n-1}$,

$$\left\|T(\lambda_n^{-1} y_n) - T(\lambda_m^{-1} y_m)\right\| \leqslant \text{dist}\,(y_n, \mathcal{M}_{n-1}) > \frac{1}{2}$$

Thus $\{\lambda_n^{-1} y_n\}$ has no bounded subset by the fact that T is compact.

$$\left\|\lambda_n^{-1} y_n\right\| = |\lambda_n|^{-1} \to \infty$$

That means $\lim_{n \to \infty} \lambda_n = 0$. $\square$

Proposition 6. *If $T \in \mathcal{B}_0(\mathcal{H})$ and a nonzero $\lambda \in \sigma(T)$, then λ is a isolated point in $\sigma(T)$.*

Proof. If there is a sequence $\{\lambda_n\} \subset \sigma(T)$ s.t. $\lambda_n \to \lambda$, then each λ_n is in either $\sigma_p(T)$ or $\sigma_p(T^*)$. Then if there exists a subsequence $\{\lambda_{n_k}\} \subset \sigma_p(T)$. But by above proposition, $\lim_{k\to\infty} \lambda_{n_k} = 0$, which is a contradiction. Similarly, if $\{\lambda_{n_k}\} \subset \sigma_p(T^*)$, $\lim_{k\to\infty} \lambda_{n_k} = 0$, which is also a contradiction. $\square$

Now, we can see any nonzero point in the spectrum of a compact operator is a eigenvalue.

Lemma 7. *If $T \in \mathcal{B}_0(\mathcal{H})$ and a nonzero $\lambda \in \sigma(T)$, then $\ker(T - \lambda)$ is finite dimentional.*

Proof. If there is an infinite orthonormal sequence $\{e_n\}$ in $\ker(T - \lambda)$, then $\{Te_n\}$ has a convergent subsequence $\{Te_{n_k}\}$, thus it is Cauchy.

$$\left\| Te_{n_k} - Te_{n_j} \right\|^2 = \left\| \lambda e_{n_k} - \lambda e_{n_j} \right\|^2 = 2|\lambda| > 0$$

which is a contradiction. $\square$

Theorem 8. *If $T \in \mathcal{B}_0(\mathcal{H})$ and a nonzero $\lambda \in \sigma(T)$, then $\lambda \in \sigma_p(T)$ and $\dim(T - \lambda) < \infty$.*

Proof. By the **Proposition** 3 in the subsection **2.1.3**, since each $\lambda \in \sigma(T)$ is isolated, we can define

$$E(\lambda) = \int_{\Gamma(\{\lambda\})} (z - T)^{-1} dz$$

where $\Gamma(\{\lambda\})$ is the closed curve enclosed $\{\lambda\}$ and disjoint with other λ in $\sigma(T)$. Then, each $E(\lambda)$ is a projection.

$$\mathcal{H}_\lambda = E(\lambda)\mathcal{H}, \ T_\lambda = T|_{\mathcal{H}_\lambda} : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$$

Since $\sigma(T_\lambda) = \{\lambda\}$ and $\lambda \neq 0$, T_λ is invertible. But clearly, T_λ is compact. That means any bounded and closed subset in $\mathcal{H}_\lambda$ is compact, thus $\dim \mathcal{H}_\lambda < \infty$ by the Riesz Theorem. Therefore, by the result of finite linear algebra, λ is the eigenvalue of T_λ, i.e. $\lambda \in \sigma_p(T)$. By above lemma, $\dim(T - \lambda) < \infty$. $\square$

Combining all of above results, we can get an explicit structure of the spectrum of compact operator.

Theorem 9 (Riesz)**.** *If $\mathcal{H}$ is a infinite dimensional Hilbert space and $T \in \mathcal{B}_0(\mathcal{H})$, then one and only one of the following possibilities occurs.*

   1) $\sigma(T) = \{0\}$.

2) $\sigma(T) = \{0, \lambda_1, \cdots, \lambda_n\}$, where each $\lambda_k \neq 0$, and each λ_k is the eigenvalue of T with $\dim \ker(T - \lambda_k) < \infty$.

3) $\sigma(T) = \{0, \lambda_1, \lambda_2, \cdots\}$, where each $\lambda_k \neq 0$, and each λ_k is the eigenvalue of T with $\dim \ker(T - \lambda_k) < \infty$, and moreover, $\lim_{n\to\infty} \lambda_n = 0$.

Because of its discrete spectrum, we can using the spectral measure to decompose the compact normal operator.

Theorem 10 (Spectral Theorem). *If T is a compact normal operator on a Hilbert space $\mathscr{H}$, then T has at most countable eigenvalues $\{\lambda_n\}$, and there are corresponding projections $P_n \colon \mathscr{H} \to \ker(T - \lambda_n)$ with $P_n P_m = P_m P_n = 0$ s.t.*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$

where $\{\lambda_n\}$ are all distinctive eigenvalues and the convergence is with respect to the norm topology.

Proof. Let E be the spectral measure of T, then we see

$$N = \int_{\sigma(N)} z \, dE$$

If $\dim \mathscr{H} = \infty$, since $\sigma(N)$ is consisted of at most countable eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and 0, then we can set $P_n = E(\{\lambda_n\})$ for $n = 1, 2, \cdots$, which is well-defined by the **Proposition** 2 in the subsection **3.1.2**, above integral can be

$$
\begin{aligned}
N &= \lim_{n\to\infty} \left(\int_{B_{\frac{1}{n}}(0)} z \, dz + \sum_{k=1}^{n} \lambda_k P_k \right) \\
&= \sum_{n=1}^{\infty} \lambda_n P_n
\end{aligned}
$$

And moreover, P_n is the projection from $\mathscr{H}$ onto $\ker(T - \lambda_n)$.
If $\dim \mathscr{H} < \infty$, above theorem is clearly true. $\square$

Also, we can see the Functional Calculus of the compact normal operators. Firstly, Taking same notation as above theorem and let $P_0 = 1 - \sum_{n=1}^{\infty}$, in fact, P_0 is the projection from $\mathscr{H}$ to $\ker T$. By the Functional Calculus of normal operators, we know

$$W^*(T) = \{\, \phi(T) \colon \phi \in L^{\infty}(\sigma(T)) \,\}$$

Since $\sigma(T)$ is discrete, $L^{\infty}(\sigma(T)) = l^{\infty}(\mathbb{C})$, and for any $(a_n)_{n=0}^{\infty} \in l^{\infty}(\mathbb{C})$,

$$(a_n)(T) = a_0 P_0 + \sum_{n=1}^{\infty} a_n P_n$$

Thus we can see the separating vector for $W^*(T)$, if e is a separating vector and decomposing e to $\{P_n\}_{n=0}^{\infty}$

$$e = \sum_{n=0}^{\infty} e_n, \ e_n \in P_n \text{ for each n}$$

Then for $(a_n)_{n=0}^{\infty} \in l^{\infty}(\mathbb{C})$

$$(a_n)(T)e = (a_0 P_0 + \sum_{n=1}^{\infty} a_n P_n)(\sum_{n=0}^{\infty} e_n) = \sum_{n=0}^{\infty} a_n e_n$$

Since e is separating, $\sum_{n=0}^{\infty} a_n e_n = 0$ for any $(a_n) \in l^{\infty}(\mathbb{C})$ implies $a_n = 0$ for all n. That means that $e_n \neq 0$ for any n. Then the corresponding measure μ is defined as

$$\mu(\{\lambda_n, \lambda_m\}) = \|E(\{\lambda_n, \lambda_m\})e\|^2 = \|e_n + e_m\|^2$$

Now, we can see the multiplicity function of compact normal operator. By similar construction in the **Theorem 3** and **Theorem 8** in the section **3.4**, the multiplicity function is like

$$m_T(\lambda) = \dim \ker (T - \lambda)$$

That means two compact normal operators are equivalent if and only if they have same dimension of all eigenspaces.

Finally,, we can prove $\mathscr{B}_0(\mathscr{H})$ is the unique closed ideal in $\mathscr{B}(\mathscr{H})$ if $\mathscr{H}$ is separable.

Proposition 11. *If N is a normal operator in $\mathscr{B}(\mathscr{H})$ with the specture measure E, then N is compact if and only if for any $\varepsilon > 0$,*

$$\dim E(\{z \in \mathbb{C} \colon |z| > \varepsilon\}) < \infty$$

Proof. Let $\Delta_\varepsilon = \{z \in \mathbb{C} \colon |z| > \varepsilon\}$ and $E_\varepsilon = E(\Delta_\varepsilon)$.
Assume that for any given $\varepsilon > 0$, $\dim E_\varepsilon < \infty$, then

$$
\begin{aligned}
N - N E_\varepsilon &= \int z \, dE - \int z \chi_{\Delta_\varepsilon} \, dE \\
&= \int z \chi_{\mathbb{C} \backslash \Delta_\varepsilon} \, dE = \phi(N)
\end{aligned}
$$

where $\phi(z) = z \chi_{\mathbb{C} \backslash \Delta_\varepsilon}(z)$. Therefore,

$$\|N - N E_\varepsilon\| = \sup\{|z| : \mathbb{C} \backslash \Delta_\varepsilon\} < \varepsilon$$

Thus $N \in \mathscr{B}_0(\mathscr{H})$.
Conversely, if N is compact, then for any $\varepsilon > 0$, put $\phi(z) = z^{-1} \chi_{\Delta_\varepsilon}$, then

$$N\phi(N) = \int \chi_{\Delta_\varepsilon} \, dE = E_\varepsilon$$

Since E_ε is a compact projection, i.e. $\operatorname{ran} E_\varepsilon$ is closed, by the Riesz Theorem, $\dim \operatorname{ran} E_\varepsilon < \infty$. $\qquad \square$

Theorem 12. *If $\mathscr{H}$ is a separable Hilbert space and $\mathfrak{I}$ is an ideal of $\mathscr{B}(\mathscr{H})$ that contains a non-compact operator, then $\mathfrak{I} = \mathscr{B}(\mathscr{H})$.*

Proof. Let $T \in \mathfrak{I} \backslash \mathscr{B}_0(\mathscr{H})$ then

$$T^*T = \int_{\sigma(T^*T)} t dE(t)$$

By above theorem, there is an $\varepsilon > 0$ s.t. $P = E(\varepsilon, \infty)$ has infinite rank.

$$P = (\int t^{-1} \chi_{(\varepsilon, \infty)}(t) dE(t)) T^*T \in \mathfrak{I}$$

Since $\mathscr{H}$ is separable, $\dim P\mathscr{H} = \dim \mathscr{H} = \aleph_0$, there is a unitary U from $\mathscr{H}$ to $P\mathscr{H}$. Therefore, $1 = U^*PU \in \mathfrak{I}$. $\mathfrak{I} = \mathscr{B}(\mathscr{H})$. $\qquad \square$

Proposition 13. *If $\mathfrak{I}$ is a closed ideal of $\mathscr{B}(\mathscr{H})$, the $\mathscr{B}_0(\mathscr{H}) \subset \mathfrak{I}$ or $\mathfrak{I} = \{0\}$.*

Proof. Since $\mathfrak{I}$ is closed, $\mathfrak{I}$ is self-adjoint by the **Proposition** 1 in the subsection **2.2.5**. Then by the **Theorem** 3 in the subsection **2.2.5**, we know $\mathfrak{I} = \mathfrak{I} \cap \mathfrak{I}^*$ is a hereditary subalgebra.
Therefore, if $\mathfrak{I}$ is nonzero and $T \in \mathfrak{I}$ is nonzero, then there is a finite rank projection P s.t.
$$0 \leqslant P \leqslant T^*T$$

Thus, $P \in \mathfrak{I}$. Morover, any finite rank projection is in $\mathfrak{I}$. Since $\mathfrak{I}$ is norm closed, $\mathscr{B}_0(\mathscr{H}) \subset \mathfrak{I}$. $\qquad \square$

Combining above propositions and theorems, we can get the final result.

Corollary 14. *If $\mathscr{H}$ is a separable Hilbert space, then the only nontrivial closed ideal of $\mathscr{B}(\mathscr{H})$ is $\mathscr{B}_0(\mathscr{H})$.*

4.2 Compact Operator Algebras

For a normal operator, it can be decomposed as the direct sum of $*$-cyclic normal operators. Therefore, we want to reseach the similar property of compact operators. Like researching the normal operator, we firstly begin with the representation of compact operators. Let $\mathfrak{A}$ denote a C^*-subalgebra in $\mathscr{B}_0(\mathscr{H})$.

4.2.1 Minimal Projections

Definition 1. If $\mathfrak{A}$ is a C^*-subalgebra in $\mathscr{B}_0(\mathscr{H})$

1) A projection E in $\mathfrak{A}$ is minimal if $E \neq 0$ and there are no nonzero projection P in $\mathfrak{A}$ s.t. $P < E$.

2) $\mathfrak{A}$ is called irreducible if $\mathfrak{A}$ has no proper reducing subspaces.

Proposition 2. *Let E be a projection in $\mathfrak{A}$.*

1) *E is minimal if and only if*

$$E\mathfrak{A}E = \{ \ \lambda E \colon \lambda \in \mathbb{C} \ \}$$

2) *Every projection in $\mathfrak{A}$ is the direct sum of a finite number of pairwise orthogonal minimal projections in $\mathfrak{A}$.*

Proof. For 1), clearly, $E\mathfrak{A}E$ is a C^*-subalgebra of $\mathscr{B}_0(\mathscr{H})$. Thus for any self-adjoint operator $A \in E\mathfrak{A}E$, each nonzero spectrum point λ of A is isolated. That means $\chi_{\{\lambda\}}$ is continuous, thus $E(\{\lambda\})$ is a projection in $E\mathfrak{A}E$. However, the minimality of E implies that $E(\{\lambda\}) = E$. Therefore, by the fact that $E\mathfrak{A}E$ can be generated by all self-adjoint operators in $E\mathfrak{A}E$, $E\mathfrak{A}E = \{\lambda E \colon \lambda \in \mathbb{C}\}$. The converse is trivial.

For 2), since each projection in $P\,\mathfrak{A}$ is compact, P has finite rank. Thus P can he direct sum of a finite number of pairwise orthogonal minimal projections. $\qquad\square$

Using the minimal projection, we can get an important trait the irreducible compact subalgebra has.

Theorem 3. *If $\mathfrak{A}$ is an irreducible C^*-subalgebra of $\mathscr{B}_0(\mathscr{H})$, then $\mathfrak{A} = \mathscr{B}_0(\mathscr{H})$.*

Proof. Let E be a minimal projection in $\mathfrak{A}$.
Claim: The rank of E is one.
If $\operatorname{rank} E > 1$, then there are nonzero $g, h \in E$ with $g \perp h$. Let $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$ s.t. $EAE = \lambda A$. Then

$$\langle Ag, h \rangle = \langle EAEg, h \rangle = \lambda \langle g, h \rangle = 0$$

Thus, $h \perp \overline{\mathfrak{A}g}$. But since $\mathfrak{A}$ is irreducible, $\overline{\mathfrak{A}g} = \mathscr{H}$. So $h = 0$ contradicted to the assumption.

Claim: If P is any rank one projection, then $P \in \mathfrak{A}$.
Let $e \in E$ and $p \in P$. By the irreducibility of $\mathfrak{A}$, $\mathscr{H} = \overline{\mathfrak{A}e}$. Then for any given $\varepsilon > 0$,

$$\|Ae - p\| < \varepsilon, \quad \text{for some } A \in \mathfrak{A}$$

Let $P' = AE$, then $P' \in \mathfrak{A}$. And moreover, for any $h \in \mathscr{H}$

$$
\begin{aligned}
\left\| (P - P')h \right\| &= \ \| \langle h, p \rangle p - \langle h, Ae \rangle Ae \| \\
&= \ \| \langle h, p - Ae \rangle Ae + \langle h, p \rangle (p - Ae) \| \\
&\leqslant \ \|h\| \, \|Ae\| \, \varepsilon + \|h\| \, \|p\| \, \varepsilon
\end{aligned}
$$

Thus by the fact that $\mathfrak{A}$ is norm closed, $P \in \mathfrak{A}$.
Therefore, any finite rank projection in $\mathfrak{A}$, i.e. $\mathfrak{A} = \mathscr{B}_0(\mathscr{H})$. $\qquad\square$

Corollary 4. *For any Hilbert space $\mathscr{H}$, $\mathscr{B}_0(\mathscr{H})$ is simple.*

Proof. Let $\mathfrak{I}$ be an closed ideal in $\mathscr{B}_0(\mathscr{H})$. Then $\overline{\mathfrak{I}e}$ for any nonzero e is a reducing space for $\mathscr{B}_0(\mathscr{H})$. Thus $\overline{\mathfrak{I}e} = \mathscr{H}$ since $\mathscr{B}_0(\mathscr{H})$ is irreducible. That means $\mathfrak{I}$ is irreducible. Thus by above theorem, $\mathfrak{I} = \mathscr{B}_0(\mathscr{H})$. $\qquad\square$

Remark. In fact, the converse of the result in above proof is also true. $\mathfrak{B}$ is a irreducible subalgebra contained in $\mathscr{B}(\mathscr{H})$ if and only if $\overline{\mathfrak{B}e} = \mathscr{H}$ for any $e \in \mathscr{H}$. And because of this, we can get the genera version of the corollary that if $\mathfrak{B}$ is a irreducible subalgebra contained in $\mathscr{B}(\mathscr{H})$, then any ideal of it is irreducible.

Corollary 5. *Let $\mathscr{H}$ be a Hilbert space. If $\mathfrak{B}$ is a irreducible C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ and $\mathfrak{B} \cap \mathscr{B}_0(\mathscr{H}) \neq 0$, then $\mathscr{B}_0(\mathscr{H}) \subset \mathfrak{B}$.*

Proof. Let $\mathfrak{I} = \mathfrak{B} \cap \mathscr{B}_0(\mathscr{H})$. Then $\mathfrak{I}$ is a closed ideal in $\mathfrak{B}$. By above corollary, $\mathfrak{I} = \mathscr{B}_0(\mathscr{H})$, i.e. $\mathscr{B}_0(\mathscr{H}) \subset \mathfrak{B}$. $\qquad\square$

Now, the unit element of a compact operator subalgebra can be provided.

Proposition 6. *If E is a minimal projection in $\mathfrak{A}$, e is any nonzero vector in $\mathscr{H}$ and $\mathscr{H}_e = \overline{\mathfrak{A}e}$, then $\mathfrak{A}|_{\mathscr{H}_e} = \mathscr{B}_0(\mathscr{H}_e)$.*

Proof. Let $\mathfrak{A}_e = \mathfrak{A}|_{\mathscr{H}_e}$. We just need to show $\mathfrak{A}_e$ is irreducible in $\mathscr{B}(\mathscr{H}_e)$. Let P be a projection in $\mathscr{B}(\mathscr{H}_e)$ commuting with $\mathfrak{A}_e$. Put

$$T = P - \langle Pe, e\rangle 1$$

Then $T \in \mathfrak{A}'_e$, for any $A, B \in \mathfrak{A}$,

$$\begin{aligned} \langle TAe, Be\rangle &= \langle TB^*AEe, Ee\rangle \\ &= \langle TEB^*AEe, e\rangle \end{aligned}$$

Since E is minimal, $EB^*AE = \lambda E$ for some $\lambda \in \mathbb{C}$. Therefore,

$$\langle TAe, Be\rangle = \lambda\langle Te, e\rangle = \lambda(\langle Pe, e\rangle - \langle Pe, e\rangle) = 0$$

Since A, B are arbitrary and $\mathscr{H}_e = \overline{\mathfrak{A}e}$, $T = 0$, that means $P = \langle Pe, e\rangle 1$. Thus, $P = 0$ or 1. $\qquad\square$

4.2.2 Representations of Compact Operator Algebras

For the representation of a compact operator subalgebra, it can be decomposed as the direct sum of irreducible parts.

Theorem 1. *If $(\rho, \mathscr{K})$ is a non-degenerate representation of $\mathfrak{A}$, then there are irreducible representations $\{\rho_i\}_{i \in I}$ s.t. each ρ_i is equivalent to a subrepresentation of the identity representation $i\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H})$.*

Proof. Claim: There is a minimal projection $E \in \mathfrak{A}$ s.t. $\rho(E) \neq 0$.

In fact, there is a self-adjoint operator $A \in \mathfrak{A}$ s.t. $\rho(A) \neq 0$. Otherwise, ρ is the zero representation. Then there is a spectral projection F for A with $\rho(F) \neq 0$. Then since F is the direct sum of finite numbers minimal projections, that means ρ must not vanish on at least one of them.

For this E, choosing a unit vector $e_0 \in \rho(E)$ and define $\mathscr{K}_0 = \overline{\rho(\mathfrak{A})e_0}$ and $\rho_0(A) = \rho(A)|_{\mathscr{K}_0}$. Then

Claim: $(\rho_0, \mathscr{K}_0)$ is equivalent to a subrepresentation of the identity representation of $\mathfrak{A}$.

Define the unitary U from $\mathscr{K}_0$ to $\mathscr{H}_0 = \overline{\mathfrak{A}e}$, where e is the unit vector in E.

$$U: \quad \begin{matrix} \mathscr{K}_0 & \longrightarrow & \mathscr{H}_0 \\ \rho(A)e_0 & \longmapsto & Ae \end{matrix}$$

Since E is minimal, there exists a unique λ s.t. $EA^*AE = \lambda E$ for $A \in \mathfrak{A}$.

$$
\begin{aligned}
\|\rho(A)e_0\|^2 &= \|\rho(AE)e_0\|^2 = \langle \rho(EA^*AE)e_0, e_0 \rangle \\
&= \lambda \langle e_0, e_0 \rangle = \lambda \langle e, e \rangle \\
&= \langle EA^*AEe_0, e_0 \rangle \\
&= \|Ae\|^2
\end{aligned}
$$

Therefore, U is surjective isometry, and can extend to $\mathscr{K}_0$. And moreover,

$$U\rho(A)\rho(B)e_0 = ABe \implies U\rho(A)U^*Be = A|_{\mathscr{H}_0}Be$$

Thus, $U\rho_0(A)U^* = A|_{\mathscr{H}_0}$. By above proposition, ρ_0 is irreducible.

Then by Zorn's Lemma, we can get a maximal family of $(\rho_i, \mathscr{K}_i)_{i \in I}$ as above construction. And by the maximality and the non-degenerality,

$$\rho = \oplus_{i \in I}\rho_i, \quad \mathscr{K} = \oplus_{i \in I}\mathscr{K}_i \qquad \qquad \square$$

Remark. If ρ is degenerate, then let $\mathscr{H}_0 = \overline{\rho(\mathfrak{A})\mathscr{H}}$. The representation $(\rho_0(A) = \rho(A)|_{\mathscr{H}_0}, \mathscr{H}_0)$ is a non-degenerate representation, thus above theorem can be applied to it. Moreover, $\rho = \rho_0 \oplus 0$.

Now, we can use above theorem to get the structure of any finite dimensional C^*-algebra.

Corollary 2. *For any finite dimensional C^*-algebra $\mathfrak{A}$ and let $M(n)$ be the set of $n \times n$ matrices with complex entries acting on the inner product space $\mathbb{C}^n$, there are $n_1, \cdots, n_p \in \mathbb{N}$, s.t.*

$$\mathfrak{A} \cong M(n_1) \oplus M(n_2) \oplus \cdots \oplus M(n_p)$$

Also, there are some other corollaries of above theorem.

Corollary 3. *1) If $(\rho, \mathscr{K})$ is an irreducible representation of $\mathfrak{A}$, then $\rho(\mathfrak{A}) = \mathscr{B}_0(\mathscr{K})$.*

2) If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces and $\rho\colon \mathscr{B}_0(\mathscr{H}) \to \mathscr{B}_0(K)$ is an $*$-isomorphism, then there is a unitary $U\colon \mathscr{H} \to \mathscr{K}$, s.t.

$$\rho(T) = UTU^*, \text{ for any } T \in \mathscr{B}_0(\mathscr{H})$$

3) If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces and $\rho\colon \mathscr{B}(\mathscr{H}) \to \mathscr{B}(K)$ is an $*$-isomorphism, then there is a unitary $U\colon \mathscr{H} \to \mathscr{K}$, s.t.

$$\rho(A) = UAU^*, \text{ for any } A \in \mathscr{B}(\mathscr{H})$$

Proof. 1) is the direct result from above theorem. And 2) can be obtained by 1).

For 3), since $\rho(\mathscr{B}_0(\mathscr{H}))$ is an ideal of $\mathscr{K}$ and $\mathscr{K}$ is irreducible, $\rho(\mathscr{B}_0(\mathscr{H}))$ is irreducible by above corollary. Thus $\rho(\mathscr{B}_0(\mathscr{H})) = \mathscr{B}_0(\mathscr{K})$. Then by 2), there is a unitary $U\colon \mathscr{H} \to \mathscr{K}$, s.t.

$$\rho(A) = UAU^*, \text{ for any } A \in \mathscr{B}_0(\mathscr{H})$$

Let $\{E_i\}$ be the approximatel identity for $\mathscr{B}_0(\mathscr{H})$ consisting with all finite projections. For any $A \leqslant 0$ in $\mathscr{H}$, then it can see

$$A_i = A^{\frac{1}{2}} E_i A^{\frac{1}{2}} \to A \text{ in } SOT$$

Therefore, $\rho(A_i) = UA_iU^* \to UAU^* = T$ in SOT. Since $A_i \leqslant A$, $\rho(A_i) \leqslant \rho(A)$ and thus $T \leqslant \rho(A)$ by the fact that $\{A_i\}$ is increasing. Conversely, $\rho(A_i) \leqslant T$ implies that $A_i \leqslant \rho^{-1}(T)$. Thus $A \leqslant \rho^{-1}(T)$, thus $\rho(A) \leqslant T$. $\square$

Here is an important example, which can be used to construct the *AF* algebra.

Example 4. Let $M(n)$ be the set of $n \times n$ matrices with complex entries acting on the inner product space $\mathbb{C}^n$. If

$$\rho\colon M(m) \longrightarrow M(n)$$

is a $*$-homomorphism, then there is an integer k s.t. $km \leqslant n$ and a unitary U in $M(n)$ and

$$\rho(x) = U(\underbrace{x \oplus x \oplus \cdots \oplus x}_{k} \oplus 0)U^*$$

In general, define $\mathbf{m} = (m_1, m_2, \cdots, m_p)$, where $m_i \in \mathbb{N}$, and

$$M(\mathbf{m}) = M(m_1) \oplus M(m_2) \oplus \cdots M(m_p)$$

Then for $\mathbf{m} = (m_1, m_2, \cdots, m_p)$ and $\mathbf{n} = (n_1, n_2, \cdots, n_q)$, if $\rho\colon M(\mathbf{m}) \to M(\mathbf{n})$ is a $*$-homomorphism, then there this a $q \times p$ matirx $[k_{ij}]$ with integer entries s.t.

$$\rho(x_1, \cdots, x_p) = U_1(\underbrace{x_1 \oplus \cdots \oplus x_1}_{k_{11}} \oplus 0)U_1^* \oplus \cdots \oplus U_q(\underbrace{x_p \oplus \cdots \oplus x_p}_{k_{qp}} \oplus 0)U_q^*$$

By above corollary, we see for any finite dimensional C^*-algebras $\mathfrak{A}$, there exists a $\mathbf{n} = (n_1, n_2, \cdots, n_q)$ s.t. $\mathfrak{A} \cong M(\mathbf{n})$. Then we can get the form of the $*$-homomorphism between any two finite dimensional C^*-algebras.

If ρ is a $*$-homomorphism between two finite dimensional C^*-algebras $\mathfrak{A} \cong M(\mathbf{m})$ and $\mathfrak{B} \cong M(\mathbf{n})$, where $\mathbf{m} = (m_1, m_2, \cdots, m_p)$ and $\mathbf{n} = (n_1, n_2, \cdots, n_q)$, then ρ is determined up to a $q \times p$ matirx $[k_{ij}]$ with integer entries. In fact, we can use it to construct the AF algebras.

4.2.3 Decompositions of Compact Operator Algebras

By above theorem and the Zorn's Lemma, for the non-degenerate $\mathfrak{A}$, there is a maximal family $\{E_i\}_{i \in I}$ of pairwise orthogonal minimal projections in $\mathfrak{A}$ and let $\mathscr{H}_i = \overline{\mathfrak{A} E_i \mathscr{H}}$ s.t.

$$\mathscr{H} = \oplus_{i \in I} \mathscr{H}_i \ , \ \mathfrak{A}|_{\mathscr{H}_i} = \mathscr{B}_0(\mathscr{H}_i)$$

But we cannot just using this as the decomposition of the compact operator algebra $\mathfrak{A}$, since there are some "equivalent" relationships of these $\mathscr{H}_i$. Firstly, we say that $\mathscr{H}_i$ does not dependent on $\mathscr{H}_j$, if there is a $A \in \mathfrak{A}$ s.t. $\mathfrak{A}|_{\mathscr{H}_i} = 0$ and $\mathfrak{A}|_{\mathscr{H}_j} \neq 0$.

Proposition 1. *$\mathscr{H}_i$ does not dependent on $\mathscr{H}_j$ if and only if*

$$A|_{\mathscr{H}_i \oplus \mathscr{H}_j} = \mathscr{B}_0(\mathscr{H}_i) \oplus \mathscr{B}_0(\mathscr{H}_j)$$

Proof. Assume that $\mathscr{H}_i$ does not dependent on $\mathscr{H}_j$. Let

$$\mathfrak{I} = \{ \ A \in \mathfrak{A} \colon A|_{\mathscr{H}_j} = 0 \ \}$$

Since $\mathscr{H}_j$ is reducing, $\mathfrak{I}$ is a closed ideal. Moreover, $\mathfrak{I}|_{\mathscr{H}_i}$ is a closed ideal of $\mathfrak{A}|_{\mathscr{H}_i} = \mathscr{B}_0(\mathscr{H}_i)$. By the assumption, $\mathfrak{I}|_{\mathscr{H}_i}$ is not zero, thus $\mathfrak{I}|_{\mathscr{H}_i} = \mathscr{B}_0(\mathscr{H}_i)$. Let K_i and K_j be arbitrary compact operators on $\mathscr{H}_i$ and $\mathscr{H}_j$. And $A \in \mathfrak{A}$ s.t. $A|_{\mathscr{H}_j} = K_j$. Since $\mathfrak{I}|_{\mathscr{H}_i} = \mathscr{B}_0(\mathscr{H}_i)$, there exists $B \in \mathfrak{I}$ s.t. $B|_{\mathscr{H}_i} = K_i - A|_{\mathscr{H}_i}$. Thus,

$$(A + B)|_{\mathscr{H}_i} = K_i, \ \& \ (A + B)|_{\mathscr{H}_j} = K_j$$

The converse is trivial. $\qquad\square$

Remark. By this proposition, if $\mathscr{H}_i$ does not dependent with $\mathscr{H}_j$, $\mathscr{H}_j$ does not $\mathscr{H}_i$. $\mathscr{H}_i$ and $\mathscr{H}_j$ are independent if they are not dependent.

Now we want to research the same properties of the dependent subspaces have, and then we can devide all $\mathscr{H}_{i i \in I}$ into dependency class.

Proposition 2. *If $\mathscr{H}_i$ and $\mathscr{H}_j$ are dependent, then*

$$\|A|_{\mathscr{H}_i}\| = \|A|_{\mathscr{H}_j}\| \ , \ for \ any \ A \in \mathfrak{A}$$

Moreover, two dependent spaces are isomorphic, and each dependency class is finite.

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Proof. By the hypothesis, if $A \in \mathfrak{A}$ s.t. $A|_{\mathscr{H}_i} = 0$, then $A|_{\mathscr{H}_j} = 0$. Then we can define the map
$$\rho \colon \mathscr{B}_0(\mathscr{H}_j) \longrightarrow \mathscr{B}_0(\mathscr{H}_i)$$
For any $K \in \mathscr{B}_0(\mathscr{H}_j)$, there is a $A \in \mathfrak{A}$ s.t. $A|_{\mathscr{H}_j} = K$, then we define that $\rho(K) = A|_{\mathscr{H}_i}$.

Claim: ρ is well-defined.

If $B \in \mathfrak{A}$ s.t. $B|_{\mathscr{H}_j} = K$, then $(A - B)|_{\mathscr{H}_j} = 0$. By the dependency of $\mathscr{H}_j$ and $\mathscr{H}_i$, $(A - B)|_{\mathscr{H}_i} = 0$, i.e. $A|_{\mathscr{H}_i} = B|_{\mathscr{H}_i}$.

ρ is a $*$-isomorphism.

If $K \in \mathscr{B}_0(\mathscr{H}_j)$ s.t. $\rho(K) = 0$, then there is a $A \in \mathfrak{A}$ s.t. $A|_{\mathscr{H}_j} = K$ and $A|_{\mathscr{H}_i} = 0$. Also, by the dependency, $K = A|_{\mathscr{H}_j} = 0$. Clearly, ρ is a $*$-homomorphism, then ρ is a $*$-monomorphism, i.e. ρ is an isometry. And the surjectivity of ρ is similar by the dependency of $\mathscr{H}_j$ and $\mathscr{H}_i$.

Then by the **Corollary** 3 in above subsection, we can find a unitary from $\mathscr{H}_j$ to $\mathscr{H}_i$. Thus $\dim \mathscr{H}_j = \dim \mathscr{H}_i$. And if there are infinite element in a dependency class, by the compactness of A, $\|A|_{\mathscr{H}_i}\| \to 0$, which is a contradiction. $\qquad \square$

Theorem 3. *If $\mathfrak{A}$ is a C^*-subalgebra of compact operators, then*
$$\mathscr{H} \cong \mathscr{H}_0 \oplus \oplus_{d \in D} \mathscr{H}_d^{(k_d)}$$
and
$$\mathfrak{A} \cong \{ \ 0 \oplus \oplus_{d \in D} K_d^{(k_d)} \colon K_d \in \mathscr{B}_0(\mathscr{H}_d) \ \}$$

Proof. Firstly, by above mention, if $\mathfrak{A}$ is non-degenerate, there are $\{\mathscr{H}_i\}_{i \in I}$ s.t.
$$\mathscr{H} = \oplus_{i \in I} \mathscr{H}_i \ , \ \mathfrak{A}|_{\mathscr{H}_i} = \mathscr{B}_0(\mathscr{H}_i)$$
But by above theorem, we can divide $\{\mathscr{H}_i\}$ into the dependency classes, and re-labeled as
$$\{ \ \underbrace{\mathscr{H}_1, \cdots, \mathscr{H}_1}_{k_1}, \underbrace{\mathscr{H}_2, \cdots, \mathscr{H}_2}_{k_2}, \cdots \ \}$$
Thus, for the non-degenerate $\mathfrak{A}$,
$$\mathscr{H} \cong \oplus_{d \in D} \mathscr{H}_d^{(k_d)}, \mathfrak{A} \cong \{ \ \oplus_{d \in D} K_d^{(k_d)} \colon K_d \in \mathscr{B}_0(\mathscr{H}_d) \ \}$$
If $\mathfrak{A}$ is degenerate, let $\mathscr{H}_0 = (\overline{\mathfrak{A}\mathscr{H}})^{\perp}$, then this theorem holds. $\qquad \square$

For a non-degenerate $\mathfrak{A}$, let $\hat{\mathfrak{A}}$ denote the set of all equivalent classes of irreducible representations of $\mathfrak{A}$. For any $\zeta \in \hat{\mathfrak{A}}$, let $\rho_\zeta \in \zeta$. By the theorem, for any representation ρ of $\mathfrak{A}$, there are irreducible representations $\{\rho_i\}$ s.t. $\rho = \oplus_i \rho_i$, then we define the mutiplicity function of ρ,
$$m_\rho(\zeta) = \#\{i \colon \rho_i \in \zeta\}$$
Now, we can easily apply above theorem to the representation.

Theorem 4. *If ρ is a representation of $\mathfrak{A}$ and m_ρ is the multiplicity function, then*

$$\rho \cong \oplus\{ \ \rho_\zeta^{(m_\rho(\zeta))} : \zeta \in \hat{\mathfrak{A}} \ \}$$

Moreover, any two representations of $\mathfrak{A}$ are equivalent if and only if they have same multiplicity functions.