My Report about Operator Theory

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July 24, 2019

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Chapter 1

Topological Vector Spaces and Banach Spaces

1.1 Topological Vector Spaces

Linear operations, i.e addition and scalar multiplication, provide an algebraic structure on a set, therefore constructing a vector space. In the course, linear algebra, we have learn the algebraic structure of finite dimensional vector spaces. But how to deal with infinite dimensional vector spaces? By learning the topological spaces, we know that the topological structure can give us a method to research properties of infinity. Thus, we need to equip a vector space with an additional topological structure, which should coincide with the algebraic structure. This is the reason why we define the topological vector space.

1.1.1 Topological Spaces

- **Definition:** First, we define the topological structure on a general set.

Definition 1. A topological space $X = (X, \mathscr{T}_X)$ consists of a set X, called the underlying space of X, and a family $\mathscr{T}_X$ of subsets of X s.t.

1) $X, \varnothing \in \mathscr{T}_X$.
2) if $U_\alpha \in \mathscr{T}_X$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}_X$.
3) if $U_1, U_2 \in \mathscr{T}_X$, then $U_1 \bigcap U_2 \in \mathscr{T}_X$.

And, $\mathscr{T}_X$ is called a topology on X. The element in $\mathscr{T}_X$ is called open set.

Thus, the topological structure on a set X is totally determined by the family $\mathscr{T}_X$. In particular, from 2), we can simplify $\mathscr{T}_X$. In other words, like the basis of a vector space, there is a "basis" of $\mathscr{T}_X$.

Definition 2. If X is a set, a basis for a topology on X is a family $\mathscr{B}$ of subsets of X, s.t.

1) $\forall\ x \in X,\ \exists\ B \in \mathscr{B},$ s.t. $x \in B$.
2) if $x \in B_1 \bigcap B_2$, where $B_1,\ B_2 \in \mathscr{B}$, then there is a $B_3 \in \mathscr{B}$, s.t. $B_3 \subset B_1 \bigcap B_2$.

$\mathscr{B}$ can generate a topology $\mathscr{T}_X$ on X by doing infinite times union of elements in $\mathscr{B}$. In fact,

$$\mathscr{T}_X \ = \ \{\ U \subset X \colon U = \bigcup_{\alpha \in A} U_\alpha,\ U_\alpha \in \mathscr{B}\ \},\ \text{where A is any index set.}$$

We can do more on the basis by 3).

Definition 3. A subbasis of a topology on X is a collection $\mathscr{S}$ of subsets of X, whose union is X. The topology generated by $\mathscr{S}$ is noted by $\mathscr{T}(\mathscr{S})$.

In fact, we can generate the basis of $\mathscr{T}(\mathscr{S})$ by $\mathscr{S}$, that is,

$$\mathscr{B} \ = \ \{\ U \subset X \colon U = \bigcap_{\alpha=1}^{n} U_\alpha,\ U_\alpha \in \mathscr{S},\ n \in \mathbb{N}\ \}$$

Therefore, this basis can generate the conincided topology on X.

- **Continuous maps:** Next, we need to endow the general maps with the topological structure.

Definition 4. Let X, Y be topological spaces and $f \colon X \to Y$ be a map. We say f is continous if

$$\forall\ V \in \mathscr{T}_Y,\ f^{-1}(V) \in \mathscr{T}_X.$$

In other words, if $f \colon X \to Y$ is a continuous map, $\mathscr{T}_X$ has "more" elements than $\mathscr{T}_Y$ has. To get more rigorous discription, we define the following concept.

Definition 5. Let X be a set and $\mathscr{T}$, $\mathscr{T}^{'}$ be two topologies on X we say that $\mathscr{T}$ is coaser than $mathscrT^{'}$ if $\mathscr{T} \subset \mathscr{T}^{'}$.

Therefore, if $f \colon X \to Y$ is a continuous map, the topology $\mathscr{T}(f^{-1}(\mathscr{T}_Y))$ is coaser than $\mathscr{T}_X$.

- **Generating topologies:** We use above methods to generate some interested topologies on a set.

1) Inital topology: Given maps $f_\alpha\colon X \to Y_\alpha$ $(\alpha \in A)$ from a set X to a family of topological spaces $\{Y_\alpha\colon \alpha \in A\}$. Let

$$\mathscr{S} \;=\; \{\; f_\alpha^{-1}(V)\colon V \in \mathscr{T}_{Y_\alpha},\; \alpha \in A \;\}$$

Then $\mathscr{T}(\mathscr{S})$ is the coarsest topology on X such that each f_α is continuous, called the initial topology induced by the family of maps $\{f_\alpha\ :\alpha \in A\}$.

2) Final topology: Given maps $f_\alpha\colon X_\alpha \to Y$ $(\alpha \in A)$ from a family of topological spaces $\{X_\alpha\colon \alpha \in A\}$ to a set Y. Let

$$\mathscr{S} \;=\; \{\; V\colon f_\alpha^{-1}(V) \in \mathscr{T}_{X_\alpha},\; \alpha \in A \;\}$$

Then $\mathscr{T}(\mathscr{S})$ is the finest topology on Y such that each f_α is continuous, called the final topology induced by the family of maps $\{f_\alpha\colon \alpha \in A\}$.

Here is some important examples using above way to generate topologies.

Example 6. (Initial topology)

1) Subspace topology: Let Y be a topological space and $X \subset Y$ be a subset of Y. The inclusion map $i\colon X \to Y$ can generate the initial topology $\mathscr{T}_X$ on X, then X become a subspace of Y. In fact,

$$\mathscr{T}_X \;=\; \{\; X \bigcap U\colon U \in \mathscr{T}_Y \;\}$$

2) Product topology: Let $\{Y_\alpha\colon \alpha \in A\}$ be a family of topological spaces. The product set is

$$\prod_{\alpha\in A} Y_\alpha \;=\; \{\; A \xrightarrow{f} \bigcup_{\alpha\in A} Y_\alpha\colon \forall\alpha \in A,\; f(\alpha) \in Y_\alpha \;\}$$

There is a family of maps $\{p_\beta\colon \prod_{\alpha\in A} Y_\alpha \to Y_\beta(\beta \in A)\}$. Therefore, these maps can generate the initial topology $\mathscr{T}$ on $\prod_{\alpha\in A} Y_\alpha$ and let $\prod_{\alpha\in A} Y_\alpha$ be the product topological space. In fact,

$$\mathscr{T} \;=\; \{\; \prod_{\alpha\in A} V_\alpha\colon V_\alpha \in \mathscr{T}_{Y_\alpha} \;\&\; \#\{\alpha \in A\colon V_\alpha \neq Y_\alpha\} < \infty \;\}$$

Remark. The condition, $\#\{\alpha \in A\colon V_\alpha \neq Y_\alpha\} < \infty$, is because that when the subbasis generates the basis, only finite many elements can do intersection.

Example 7. (Final topology)
Quotient topology: For a topological space X on which an equivalent relation R is fixed, $\pi\colon X \to X/R$ is the quotient map, then the quotient set can be equiped with the final topology $\mathscr{T}$ generated by the quotient map. Therefore, X/R become a quotient topological space. In fact,

$$\mathscr{T} \;=\; \{\; U \subset X/R\colon \pi^{-1}(U) \in \mathscr{T}_X \;\}$$

■ **Countability and metrizability:** When learning the analysis of real functions, we usually use the sequence to discribe the topological properties. But in some general topology, we cannot just use sequence since some properties of "uncountabibily". In this case, the concept of net can be applied to some "uncountable" topologies. Furthermore, there is a class of more special topology, metrizable topology, which has some better properties.

Definition 8. (Net)

1) Direct set: A direct set $(D, \ \geqslant)$ consists of a nonempty set D and a relation $\geqslant$ on D, satisfies:

    i) $\forall \ d \in D, \ d \geqslant d$
    ii) $\forall \ d_1, \ d_2, \ d_3 \in D, if \ d_3 \geqslant d_2 \ \& \ d_2 \geqslant d_1, \ then \ d_3 \geqslant d_1$
    iii) $\forall \ d, \ d^{'} \in D, \ \exists \ d^{''} \ s.t. \ d^{''} \geqslant d \ \& \ d^{''} \geqslant d^{'}.$

2) if X is a set, a net is a map $x_{.} : D \to X$ from a direct set D to X

Example 9. If X is a topological space and $x \in X$, then let

$$D \ = \ \{ \text{ all open neighbourhoods of x } \}, \ U \geqslant V \Leftrightarrow U \subset V$$

Then D is a direct set and $x_\alpha(\alpha \in D)$ is a net. And we say $x_\alpha \to x$ if and only if
$\forall$ open neighbourhood U of x in X, $\exists \delta \in D$, $\forall \ \alpha \in D$ with $\alpha \geqslant \delta \Rightarrow x_\alpha \in U$

Nets can be used as sequences in topological spaces. Like,

Proposition 10. *If X is a topological space and the net $x_\alpha(\alpha \in D)$ defined above and $A \subset X$, then*

1) $\overline{A} \ = \ \{ \ x \in X \colon \exists \ x_\alpha \ in \ A, \ x_\alpha \to x \ \}$

2) $f \colon X \to Y$ *is continuous between two topological spaces,* $x_0 \in X$*,* f *is continuous at* x_0*, if and only if*
$\forall \ net \ x_\alpha(\alpha \in D), \ s.t. \ x_\alpha \to x_0 \Rightarrow f(x_\alpha) \to f(x_0)$

Definition 11. (Countability)

1) First countability: For a topological space X, X is called first countable if for each point $x \in X$, x has a countable neighbourhood basis.

2) Second countability: A topological space X is second countable if it has the countable topological basis.

Remark. Clearly, the second countable topological space is first countable, but the converse is not true.

In particular, if X is first countable, sequences can be used to illuminate tpological properties rather than nets. Like,

Proposition 12. *If X is first countable, then*

1) $U \subset X$ is closed $\Leftrightarrow \forall\ x \in U,\ \exists$ a sequence$\{x_n\} \subset U,\ s.t.\ x_n \to x$.

2) sequential compactness is equivalent to compactness.

And for the second countability, it is about the separability.

Definition 13. (Separability)

1) A subset A of a topological space X is called dense if $\overline{A} = X$.

2) A topological space is called separable if it has a countable dense subset.

By the definition, we can clearly know that:

Proposition 14. *If X is a second countable topological space, then it is separable and every open covering of X has a finite subcollection covering X.*

We can classify topological spaces into some classes.

Definition 15. X is a topological space, then we call X is:

(T_0) $\forall\ x,\ y \in X,\ \exists$ open $U \subset X,\ s.t.\ x \in U$ but $y \notin U$ or $y \in U$ but $x \notin U$ (Kolmogorov space)

(T_1) $\forall\ x,\ y \in X,\ \exists$ open $U,\ V \subset X,\ s.t.\ x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$
($\Leftrightarrow \forall\ x \in X,\ \{x\}$ is closed)

(T_2) $\forall\ x,\ y \in X,\ \exists$ open $U,\ V \subset X,\ s.t.\ x \in U\ \&\ y \in V$ and $U \bigcap V = \varnothing$ (Hausdorff space)

(T_3) T_1 holds and $\forall\ x \in X$ and closed $C \subset X$, if $x \notin C$, then $\exists$ open $U,\ V \subset X,\ s.t.\ x \in U\ \&\ C \subset V$ and $U \bigcap V = \varnothing$ (regular space)

(T_4) T_1 holds and $\forall$ closed $C_1,\ C_2 \subset X$, if $C_1 \bigcap C_2 = \varnothing$,
then $\exists$ open $U,\ V \subset X,\ s.t.\ C_1 \subset U\ \&\ C_2 \subset V$ and $U \bigcap V = \varnothing$ (normal space)

Then we can specify a class of more powerful topological space.

Definition 16. If X is a topological space, then X is said to be metrizable if there exists a metric d on the set X that induces the topology of X.

Remark. Clearly, if X is metrizable, X is second countable and normal.

Here is two metrization theorems provides the essence of metric spaces.

Theorem 17. *(Metrization theorems)*

Urysohn *A topological space is separable and metrizable if and only if it is regular, Hausdorff and second countable.*

Nagata–Smirnov *A topological space is metrizable if and only if regular, Hausdorff and has a σ-locally finite basis.*

▪ **Complete metic space:** For a metric space, we know it is first countable, so the concept of net is unnecessary. And thus sequences are enough to determine the topological structures, like that sequential compactness is equvilent to compactness.

Proposition 18. *A compact subset of a metric space is closed, bounded and separable.*

Remark. it is clearly, since compactness is also about finity.

For any metric space, we can use the following theorem to get a completion of that and this completion is unique. Thus, we can always assume a metric space is complete.

Theorem 19. *Let $(X,\ d)$ be a metric space. Then, there exists a metric space $(\hat{X},\ \hat{d})$ with the following properties:*

1) $(\hat{X},\ \hat{d})$ is complete.

2) There is an embedding σ from X to $\hat{X}$.

3) $\sigma(X)$ is dense in $\hat{X}$.

And this $(\hat{X},\ \hat{d})$ is unique with respect to isomorphism.

Complete metric space is imporatant since it is "sufficiently large". Rigorously, we can the following definition to describe it.

Definition 20. (Baire Category) A metric space is said to be of the first category if it can be written as a countable union of sets that are nowhere dense. Otherwise, it is of the second category.

Proposition 21. *A complete metric space is a space of the second category.*

▪ **Filters:** For convenience, we define some terminologies.

Definition 22. A filter on a set X is a family $\mathscr{F}$ of subsets of X satisfying the following conditions:

1) $\varnothing \notin \mathscr{F}$

2) $\mathscr{F}$ is closed under finite many intersections

3) Any subset of X containing a set in $\mathscr{F}$ belongs to $\mathscr{F}$.

Example 23. For a topological space X and $x \in X$, and let

$$\mathscr{F}(x) \ = \ \{ \text{ all neighbourhoods of x } \}$$

Then $\mathscr{F}(x)$ is a filter. And we can define the basis of $\mathscr{F}(x)$, noted by $\mathscr{B}(x)$. That is $\mathscr{B}(x) \subset \mathscr{F}(x)$ with the following properties:

1) $\forall U \in \mathscr{B}(x),\ x \in U$

2) $\forall U_1$ & $U_2 \in \mathscr{B}(x),\ \exists\ U_3 \in \mathscr{B}(x),\ s.t.\ U_3 \subset U_1 \bigcap U_2$

3) If $y \in U \in \mathscr{B}(x),\ \exists\ W \in \mathscr{B}(y),\ W \subset U$

1.1.2 Definition and Properties

▪ **Definition:** Now, we need to endow the topological structure on a vector spaces. And the most important thing is that the topological structure should coincide with the algebraic structure.

Definition 1. A vector space X over a field $\mathbb{K}$ is called a topological vector space if X is equiped with a topology $\mathscr{T}$ s.t. the addition and the scalar multiplication, i.e.

$$(x,\ y) \mapsto x + y(\lambda,\ x) \mapsto \lambda x$$

are continuous with respect to the topology $\mathscr{T}$.