

# My Note about Operator Theory

Z. Zhan

July 17, 2019

# Contents

<b>1</b>	<b>Topological Vector Spaces and Banach Spaces</b>	<b>4</b>
1.1	Topological Vector Spaces . . . . .	5
1.1.1	Definition and Properties . . . . .	5
1.1.2	Hausdorff TVS and Quotient TVS . . . . .	5
1.1.3	Continuous Linear Maps . . . . .	5
1.1.4	Finite Dimensional TVS . . . . .	5
1.2	Locally Convex Topological Vector Spaces . . . . .	5
1.2.1	Seminorms and LCTVS . . . . .	5
1.2.2	Hausdorff LCTVS and Direct Limits . . . . .	5
1.2.3	Continuous Linear Maps . . . . .	5
1.3	Hahn-Banach Theorem . . . . .	5
1.3.1	Analytic Form and Geometric Form . . . . .	5
1.3.2	Applications . . . . .	5
1.4	Banach Spaces . . . . .	5
1.4.1	Elementary Properties . . . . .	5
1.4.2	Linear Operators and Functionals . . . . .	5
1.4.3	Weak Topology . . . . .	5
1.5	Hilbert Spaces . . . . .	5
1.5.1	Properties and Orthogonality . . . . .	5
1.5.2	The Riesz Representation Theorem . . . . .	5
<b>2</b>	<b>Banach Algebras and <math>C^*</math>-Algebras</b>	<b>6</b>
2.1	Banach Algebras . . . . .	7
2.1.1	Definition . . . . .	7
2.1.2	The Spectrum . . . . .	7
2.1.3	The Riesz Functional Calculus . . . . .	7
2.1.4	Abelian Banach Algebras . . . . .	7
2.2	$C^*$ -Algebras . . . . .	7
2.2.1	Properties . . . . .	7
2.2.2	Ideals and Quotients . . . . .	7
2.2.3	Abelian $C^*$ -Algebras and Functional Calculus . . . . .	7
2.2.4	Positive elements and Positive Maps . . . . .	7
2.2.5	Approximate Identities . . . . .	7

2.2.6	Representations and GNS Construction . . . . .	7
2.2.7	von Neumann Algebras . . . . .	7
2.3	Operators on Hilbert Spaces . . . . .	7
2.3.1	Elementary Properties . . . . .	7
2.3.2	The Adjoint of an Operator . . . . .	7
2.3.3	Projections . . . . .	7
2.3.4	The Spectrum of a Linear Operator . . . . .	7
<b>3</b>	<b>Some Important Operator Algebras</b>	<b>8</b>
3.1	Normal Operators . . . . .	9
3.1.1	Spectral Measures . . . . .	9
3.1.2	Spectral Theorem for Normal Operators . . . . .	9
3.1.3	Star-Cyclic Operators . . . . .	9
3.1.4	Functional Calculus for Normal Operators . . . . .	9
3.1.5	Multiplicity Theory . . . . .	9
3.2	Compact Operators . . . . .	9
3.2.1	Elementary Properties . . . . .	9
3.2.2	Spectral Theorem for Compact Operators . . . . .	9
3.2.3	$C^*$ -Algebras of Compact Operators . . . . .	9
3.3	Trace Class and Hilbert-Schmit Class . . . . .	9
3.3.1	Definition and Properties . . . . .	9
3.3.2	Dual Space and Weak* Topology . . . . .	9
3.3.3	Inflation and Topologies . . . . .	9
3.4	Fredholm Operators . . . . .	9
3.4.1	Definition . . . . .	9
3.4.2	The Fredholm Index . . . . .	9
3.4.3	The Essential Spectrum . . . . .	9
3.5	Isometries and Shifts . . . . .	9
3.5.1	The von Neumann-Wold Decomposition . . . . .	9
3.5.2	Properties of Shifts . . . . .	9
3.5.3	Toeplitz Operators . . . . .	9
3.5.4	Cuntz Algebra and Its Classifications . . . . .	9
<b>4</b>	<b>Compact Perturbation</b>	<b>10</b>
4.1	The Weyl-von Neumann Theorem . . . . .	10
4.2	The Weyl-von Neumann-Berg Theorem . . . . .	10
4.3	The Voiculescu's Theorem . . . . .	10
4.4	Equivalent Representations and Approximately Equivalent Representations	10
<b>5</b>	<b>Von Neumann Algebras</b>	<b>11</b>
5.1	Elementary Properties . . . . .	11
5.2	The Kaplansky Density Theorem . . . . .	11
5.3	Normal Homomorphisms . . . . .	11
5.3.1	The Up-Down Theorem . . . . .	11

5.3.2	Ideals . . . . .	11
5.4	Projections . . . . .	11
5.5	Classifications of von Neumann Algebras . . . . .	11
5.6	The Structure of Type I Algebras . . . . .	11



# Chapter 1

## Topological Vector Spaces and Banach Spaces

### 1.1 Topological Vector Spaces

#### 1.1.1 Definition and Properties

#### 1.1.2 Hausdorff TVS and Quotient TVS

#### 1.1.3 Continuous Linear Maps

#### 1.1.4 Finite Dimensional TVS

### 1.2 Locally Convex Topological Vector Spaces

#### 1.2.1 Seminorms and LCTVS

#### 1.2.2 Hausdorff LCTVS and Direct Limits

#### 1.2.3 Continuous Linear Maps

### 1.3 Hahn-Banach Theorem

#### 1.3.1 Analytic Form and Geometric Form

#### 1.3.2 Applications

### 1.4 Banach Spaces

#### 1.4.1 Elementary Properties

#### 1.4.2 Linear Operators and Functionals

#### 1.4.3 Weak Topology

### 1.5 Hilbert Spaces

#### 1.5.1 Properties and Orthogonality

#### 1.5.2 The Riesz Representation Theorem



## Chapter 2

# Banach Algebras and $C^*$ -Algebras

### 2.1 Banach Algebras

#### 2.1.1 Definition

#### 2.1.2 The Spectrum

#### 2.1.3 The Riesz Functional Calculus

#### 2.1.4 Abelian Banach Algebras

### 2.2 $C^*$ -Algebras

#### 2.2.1 Properties

#### 2.2.2 Ideals and Quotients

#### 2.2.3 Abelian $C^*$ -Algebras and Functional Calculus

#### 2.2.4 Positive elements and Positive Maps

#### 2.2.5 Approximate Identities

#### 2.2.6 Representations and GNS Construction

#### 2.2.7 von Neumann Algebras

### 2.3 Operators on Hilbert Spaces

#### 2.3.1 Elementary Properties

#### 2.3.2 The Adjoint of an Operator

#### 2.3.3 Projections

#### 2.3.4 The Spectrum of a Linear Operator





## Chapter 3

# Some Important Operator Algebras

### 3.1 Normal Operators

#### 3.1.1 Spectral Measures

#### 3.1.2 Spectral Theorem for Normal Operators

#### 3.1.3 Star-Cyclic Operators

#### 3.1.4 Functional Calculus for Normal Operators

#### 3.1.5 Multiplicity Theory

### 3.2 Compact Operators

#### 3.2.1 Elementary Properties

#### 3.2.2 Spectral Theorem for Compact Operators

#### 3.2.3 $C^*$ -Algebras of Compact Operators

### 3.3 Trace Class and Hilbert-Schmidt Class

#### 3.3.1 Definition and Properties

#### 3.3.2 Dual Space and Weak\* Topology

#### 3.3.3 Inflation and Topologies

### 3.4 Fredholm Operators

#### 3.4.1 Definition

#### 3.4.2 The Fredholm Index

#### 3.4.3 The Essential Spectrum

### 3.5 Isometries and Shifts<sup>9</sup>

#### 3.5.1 The von Neumann-Wold Decomposition

#### 3.5.2 Properties of Shifts

#### 3.5.3 Toeplitz Operators

#### 3.5.4 $C^*$ -Algebra and Its Classifications

## Chapter 4

# Compact Perturbation

4.1 The Weyl-von Neumann Theorem

4.2 The Weyl-von Neumann-Berg Theorem

4.3 The Voiculescu's Theorem

4.4 Equivalent Representations and Approximately  
Equivalent Representations

## Chapter 5

# Von Neumann Algebras

### 5.1 Elementary Properties

### 5.2 The Kaplansky Density Theorem

### 5.3 Normal Homomorphisms

#### 5.3.1 The Up-Down Theorem

#### 5.3.2 Ideals

### 5.4 Projections

### 5.5 Classifications of von Neumann Algebras

### 5.6 The Structure of Type I Algebras