Symplectic Structure, Geometric Invariant Theory and Scaling Problem

Zhan Zhiyuan

Contents

1 Introduction

Considering a complex reductive Lie group $G = K_{\mathbb{C}}$ acting linearly on a finite-dimensional complex vector space V equipped with a K-invariant inner product, there is a canonical symplectic structure on V such that the induced action of K on V is Hamiltonian.

Moreover, by the symplectic reduction [MW74], the projective space $\mathbb{P}(V)$ can be equipped with a canonical symplectic form ω_{FS}, called the *Fubini-Study form*. G acting on V induces G acting on $\mathbb{P}(V)$ and the action of K on $\mathbb{P}(V)$ is also a Hamiltonian action with the moment map $\mu\colon \mathbb{P}(V) \to \mathfrak{k}^*$. The image of μ has some convexity properties. If K is commutative, then by the Atiyah–Guillemin–Sternberg convexity theorem [Ati82; GS82], $\operatorname{Im}\mu$ is a convex polytope in $\mathfrak{k}^*$. For the noncommutative case, by Guillemin and Sternberg [GS84],

$$\mathcal{P}_v = \mu(\overline{G \cdot [v]}) \cap \mathfrak{t}_+^*$$

is a convex polytope, called the *moment polytope*, where $\mathfrak{t} = \mathrm{Lie}(T)$ for a maximal torus T in K and $\mathfrak{t}_+^* \subset \mathfrak{t}^*$ is a closed positive Weyl chamber.

In the invariant theory, it is important to determine if $0 \in \overline{G \cdot v}$ when giving a $v \in V$ because $0 \in \overline{G \cdot v}$ if and only if all invariant polynomials cannot separate v and 0. If $0 \in \overline{G \cdot v}$, then v is called in the null cone. By the Hilbert-Mumford criterion [MFK94], v is in the null cone if and only if there is a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$ s.t.

$$\lim_{t \to 0} \lambda(t) \cdot v = 0$$

When considering the symplectic structure on V and $\mathbb{P}(V)$ and G acting on V and $\mathbb{P}(V)$, by the Kempf-Ness theorem [KN79], there is a dual statement of Hilbert-Mumford criterion. For $v \in V \setminus \{0\}$ and let $x = [v] \in \mathbb{P}(V)$,

$$0 \notin \overline{G \cdot v} \;\Leftrightarrow\; \overline{G \cdot x} \cap \mu^{-1}(0) \neq \varnothing$$

and such x in $\mathbb{P}(V)$ is called μ-semistable i.e. there exists a $y \in \overline{G \cdot x}$ s.t. $\mu(y) = 0$. Moreover, for the Hamiltonian action $K \curvearrowright \mathbb{P}(V)$ and $x \in \mathbb{P}(V)$, there is a Kempf-Ness function $\Phi_x\colon G/K \to \mathbb{R}$ that has many nice properties. For example, when equipping G/K with a canonical Riemannian metric, Φ_x is geodesically convex. And x is μ-semistable if and only if Φ_x is bounded below. [Bür+19] applied the properties of Φ_x to get an equivalent definition of the moment map and then to obtain two algorithms to solve this problem.

The *Scaling problem* is considering $G \curvearrowright V$ and $G \curvearrowright \mathbb{P}(V)$ with the moment map μ and $v \in V$, determine if there is a $w \in \overline{G \cdot v}$ s.t. $\mu(w) = 0$ (μ can be lifted on V by its definition). For example, if $G = ST(n) \times ST(n)$ acts $V = M(n, \mathbb{C})$ by

$$(A, B) \cdot X := AXB$$

where $ST(n)$ is the set of $n \times n$ diagonal matrices with determinant 1, then the scaling problem is equivalent to the doubly stochastic scaling problem for matrices.

If $G = SL(n, \mathbb{C}) \times SL(n, \mathbb{C})$ acts $V = M(n, \mathbb{C})^{\oplus m}$ by

$$(A, B) \cdot (X_1, \cdots, X_m) := (AX_1B, \cdots, AX_mB)$$

then the scaling problem is corresponding to the operator scaling problem.

2 Lie Groups and Representations

2.1 Lie groups and Lie Algebras

Definition 1. (1) G is a Lie group if it is a group and a smooth manifold such that the following maps are smooth

$$\begin{array}{rcl} G \times G & \to & G, \\ (g,h) & \mapsto & gh \end{array} \qquad \begin{array}{rcl} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

(2) $H \subset G$ is a Lie subgroup if H is a subgroup with smooth structure s.t. the inclusion map $i \colon H \hookrightarrow G$ is an immersion.

Remark. For $H \subset G$ is a subgroup, then H is closed if and only if H is a (regular) submanifold of G. So any closed subgroup is a Lie subgroup.

Theorem 2. *Let G be a connected commutative group. Then there is a Lie group isomorphism*

$$G \simeq \mathbb{T}^m \times \mathbb{R}^n$$

where $\mathbb{T}^m = S^1 \times \cdots \times S^1$ is an m-torus.

Corollary 3. *Any connected commutative compact group $G \simeq \mathbb{T}^m$ for some $m \in \mathbb{N}$.*

Let $X \in \mathfrak{g} := T_eG$. Define the left-invariant vector field $v_X \in \mathrm{Vect}(G)$ as

$$v_X(g) := T_e l_g X \in T_gG$$

where $l_g \colon G \to G$ as $l_g(h) = gh$. Then for any X, since v_X is left-invariant, there is a unique complete integral curve $\alpha_X \colon \mathbb{R} \to G$ of v_X through e. Therefore, define the exponential map $\exp\colon \mathfrak{g} \to G$ as $\exp(X) := \alpha_X(1)$.

Proposition 4. *(1)* $\exp$ *is locally diffeomorphic at* 0.

(2) $\alpha_X(t) = \exp(tX)$ *for any* $X \in \mathfrak{g}$.

(3) $t \mapsto \exp tX$ *is one-parameter group of G for $X \in \mathfrak{g}$. Converserly, any one-parameter subgroup $\alpha(t)$ has the form* $\alpha(t) = \exp(tX)$ *for some* $X \in \mathfrak{g}$.

(4) If $\varphi\colon G \to H$ is a Lie group homomorphism, then the diagram is commutative

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "T_e\varphi"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd} \qquad \varphi \circ \exp = \exp \circ T_e\varphi$$

For $g \in G$, the conjugation $\rho_g \colon G \to G$ is defined as $\rho_g(h) = ghg^{-1}$. Then let $\mathrm{Ad}_g = T_e\rho_g$. So it can get

$$\mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{gh}, \ h\exp(X)h^{-1} = \exp \mathrm{Ad}_h X$$

Therefore, $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ is a Lie group homomorphism. Let

$$\mathrm{ad} := T_e \mathrm{Ad}\colon \mathfrak{g} \to End(\mathfrak{g})$$

Then define the Lie bracket on $\mathfrak{g}$ as

$$\begin{array}{rccc} [\cdot,\cdot]\colon & \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & (X,Y) & \longmapsto & [X,Y] := \mathrm{ad}_X(Y) \end{array}$$

$[\cdot,\cdot]$ is bilinear, anti-symmetry and satisfies the Jacobi identity. So $(\mathfrak{g},[\cdot,\cdot])$ is a Lie algebra.

Example 5. For $G = GL(n,\mathbb{C})$ a Lie group, then $\mathfrak{g} = M(n,\mathbb{C})$ and

$$\exp(X) = e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k,\ \forall\, X \in M(n,\mathbb{C})$$

And by definition $Ad_A(X) = AXA^{-1}$ and $[X,Y] = XY - YX$.

Theorem 6. *(1) Suppose H is a Lie subgroup of G. Then the Lie algebra $\mathfrak{h}$ of H can be viewed as a Lie subalgebra of $\mathfrak{g}$ and*

$$\mathfrak{h} = \{X \in \mathfrak{g}\colon\ \exp(tX) \in H,\ \forall\, t \in \mathbb{R}\}$$

(2) Let G_e be the component of G containing e. Then G_e is generated by $\exp\mathfrak{g}$ *i.e.*

$$G_e = \{\exp X_1 \exp X_2 \cdots \exp X_n\colon\ X_j \in \mathfrak{g},\ n \in \mathbb{N}\}$$

Example 7. (1) $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C})\colon\ \det A = 1\}$. Then by $\det\left(e^A\right) = e^{\operatorname{tr} A}$,

$$\mathfrak{sl}(n,\mathbb{C}) = \{X \in M(n,\mathbb{C})\colon\ \operatorname{tr} X = 0\}$$

(2) $U(n) = \left\{A \in GL(n,\mathbb{C})\colon A^\dagger A = I\right\}$. Then by $e^{A+B} = e^A e^B$ for $[A,B] = 0$,

$$\mathfrak{u}(n) = \left\{X \in M(n,\mathbb{C})\colon X^\dagger = -X\right\}$$

2.2 Lie Group Actions

Let G be a Lie group and M be a smooth manifold. Group action $G \curvearrowright M$ is a Lie group action if the map

$$\begin{array}{rccc} ev\colon & G\times M & \longrightarrow & M \\ & (g,m) & \longmapsto & g\cdot m \end{array}$$

is smooth with respect to the product smooth structure on $G \times M$. That induces a group homomorphisim

$$\begin{array}{rccc} \tau\colon & G & \longrightarrow & \mathrm{Diff}(M) \\ & g & \longmapsto & \tau(g) \end{array}$$

where $\tau(g)m := g \cdot m$. Let a Lie group act two manifolds M and N smoothly and $f\colon M \to N$ smooth. f is called G-equivariant if $f(g\cdot m) = g \cdot f(m)$.

Example 1 (Adjoint action)**.** The adjoint action of G on $\mathfrak{g}$ is as,

$$\begin{array}{rccc} \mathrm{Ad}\colon & G & \longrightarrow & \mathrm{Diff}(\mathfrak{g}) \\ & g & \longmapsto & \mathrm{Ad}_g \end{array}$$

where vector space $\mathfrak{g}$ is equipped with the canonical smooth structure. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. The coadjoint action of G on $\mathfrak{g}^*$ is as

$$\begin{array}{rccl} \mathrm{Ad}^*\colon & G & \longrightarrow & \mathrm{Diff}(\mathfrak{g}^*) \\ & g & \longmapsto & \mathrm{Ad}^*_g \end{array}$$

where for $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$,

$$\left\langle \mathrm{Ad}^*_g \xi, X \right\rangle := \langle \xi, \mathrm{Ad}_{g^{-1}} X \rangle$$

Also $\mathfrak{g}$ can act on $\mathfrak{g}$ by adjoint action

$$\begin{array}{rccl} \mathrm{ad}\colon & \mathfrak{g} & \longrightarrow & \mathrm{Diff}(\mathfrak{g}) \\ & X & \longmapsto & \mathrm{ad}_X \end{array}$$

And $\mathfrak{g}$ acts on $\mathfrak{g}^*$ by coadjoint action

$$\begin{array}{rccl} \mathrm{ad}^*\colon & \mathfrak{g} & \longrightarrow & \mathrm{Diff}(\mathfrak{g}) \\ & X & \longmapsto & \mathrm{ad}^*_X \end{array}$$

where for $\xi \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$,

$$\langle \mathrm{ad}^*_X \xi, Y \rangle := -\langle \xi, \mathrm{ad}_X Y \rangle$$

Remark. If K is a compact Lie group, then there is a Haar measure $d\alpha$ on K. Therefore, for any linear action of K on some vector space V with any inner product $\langle \cdot, \cdot \rangle$, there is an inner product $\langle \cdot, \cdot \rangle_K$ on V called K-invariant defined as

$$\langle v, w \rangle_K := \int_K \langle k \cdot v, k \cdot w \rangle \, d\alpha(k)$$

such that the action is unitary i.e $\langle k \cdot v, k \cdot w \rangle_K = \langle v, w \rangle_K$. Therefore, considering the adjoint action of K on $\mathfrak{k}$, there is a K-invariant inner product on $\mathfrak{k}$. Then $\mathfrak{k}^* \simeq \mathfrak{k}$ with this inner product. So $\mathrm{Ad}^*_k = \mathrm{Ad}_{k^{-1}}$ and by differentiation $\mathrm{ad}^*_X = -\mathrm{ad}_X$.

Let $G \curvearrowright M$ be a Lie group action. Then there is a map

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathrm{Vect}(M) \\ X & \longmapsto & X_M \end{array}$$

defined as

$$X_M(m) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot m$$

and $\gamma(t) = \exp(tX) \cdot m$ is the integral curve of X_M starting at m. Moreover, by $[X, Y]_M = -[X_M, Y_M]$, this map is an anti-homomorphism.

Proposition 2. *Let $m \in M$. The orbit of G through m is defined as*

$$G \cdot m = \{g \cdot m \colon g \in G\}$$

and the isotopic subgroup (stabilizer) of m in G is defined as

$$G_m = \{g \in G \colon g \cdot m = m\}$$

(1) $G \cdot m$ is an immersed submanifold of M with the tangent space at m is

$$T_m(G \cdot m) = \{X_M(m) \colon X \in \mathfrak{g}\}$$

(2) G_m is a Lie subgroup of G with the Lie algebra

$$\mathfrak{g}_m = \{X \in \mathfrak{g} \colon X_M(m) = 0\}$$

For $G \curvearrowright M$, define the equivalence relation on M as

$$m \sim n \ \Leftrightarrow \ n \in G \cdot m$$

Then the quotient set is

$$M/G := M/\sim = \{[m] \colon m \in M\}$$

The question is about the smooth structure on M/G.

Definition 3. Let G act M smoothly.

(1) The action is called proper if the map

$$\begin{array}{cccc} \alpha \colon & G \times M & \longrightarrow & M \times M \\ & (g, m) & \longmapsto & (g \cdot m, m) \end{array}$$

is proper i.e. the pre-image of any compact set is compact.

(2) The action is called free if $G_m = \{e\}$ for any $m \in M$.

Remark. (1) If the Lie group action $G \curvearrowright M$ is proper, then, in fact, any orbit $G \cdot m$ is a closed regular submanifold of M so that M/G with the quotient topology is Hausdorff. And if G is compact, any Lie group action of G is proper.

(2) Let G be a Lie group and $H \subset G$ be a closed normal subgroup. Then the action $H \curvearrowright G$ is defined as

$$h \cdot g = gh^{-1}, \ \forall \ g \in G, \ \forall \ h \in H$$

is free and proper. Moreover, G/H has the same sense when considering as the quotient group or as the set of orbit classes.

Theorem 4. *Suppose the Lie group action $G \curvearrowright M$ is proper and free.*

(1) There is a unique smooth structure on M/G s.t. the quotient map $\pi \colon M \to M/G$ is a submersion. In fact, $(\pi, M, M/G)$ is a G-principle fiber bundle.

(2) If $G \curvearrowright M$ is transitive, then for any $m \in M$, the map

$$\begin{array}{ccc} G/G_m & \longrightarrow & M \\ [g] & \longmapsto & g \cdot m \end{array}$$

is a diffeomorphism. And such M is called a homogeneous space.

Corollary 5. *Let G be a Lie group and H be a closed normal group. Then G/H is a Lie group with the Lie algebra*

$$\mathrm{Lie}\,(G/H) \simeq \mathfrak{g}/\mathfrak{h}$$

Example 6 (Coadjoint orbits)**.** Let G be a Lie group acting $\mathfrak{g}^*$ by coadjoint action. The for $\xi \in \mathfrak{g}^*$,

$$G_\xi = \left\{g \in G \colon \operatorname{Ad}_g^* \xi = \xi\right\}, \ \mathfrak{g}_\xi = \operatorname{Lie}(G_\xi) = \{X \in \mathfrak{g} \colon \operatorname{ad}_X^*(\xi) = 0\}$$

Since G_ξ is a closed normal subgroup, there is a homeomorphism

$$M_\xi := G/G_\xi \simeq G \cdot \xi$$

with the Lie algebra

$$\mathfrak{g}/\mathfrak{g}_\xi \simeq T_\xi M_\xi$$

More generally, since $M_\xi = M_{g\cdot\xi}$,

$$T_{g\cdot\xi} M_\xi = T_{g\cdot\xi} M_{g\cdot\xi} = \left\{X_{M_\xi}(g\cdot\xi) \colon X \in \mathfrak{g}\right\}$$

Denote $X_{M_\xi}(g\cdot\xi)$ by $X_{g\cdot\xi}$, then for any $X_{g\cdot\xi}$,

$$X_{g\cdot\xi} = \left.\frac{d}{dt}\right|_{t=0} \exp tX \cdot g \cdot \xi$$

2.3 Complex Structure

Definition 1. (1) Let V be a real vector space. A complex structure on V is a real-linear isomorphism $J \colon V \to V$ s.t. $J^2 = -id$. So if define

$$(x+iy)v := xv + yJv$$

then V is a complex vector space.

(2) Let M be a smooth manifold. An almost complex structure on M is a smooth map $J \colon TM \to TM$ s.t. for any $m \in M$, $J_m \colon T_mM \to T_mM$ is a complex structure on T_mM.

Remark. Note that if a real-vector space can be equipped with a complex structure, it should be even $\mathbb{R}$-dimensional.

(M, J) is an almost complex manifold. Let $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ and extending J on $T_{\mathbb{C}}M$ by

$$J(v \otimes z) = Jv \otimes z, \ \forall \ v \in TM$$

Since J has two eigenvalues i and $-i$, the eigenspace decomposition of $T_{\mathbb{C}}M$ is $T_{\mathbb{C}}M = T_{1,0} \oplus T_{0,1}$,

$$T_{1,0} = \{w \in T_{\mathbb{C}}M \colon Jw = iw\}, \ T_{0,1} = \{w \in T_{\mathbb{C}}M \colon Jw = -iw\}$$

Any vector in $T_{1,0}$ (or $T_{0,1}$) is called (anti-)holomorphic.

In fact, any holomorphic vector has the form $v \otimes 1 - Jv \otimes i$ for some $v \in TM$, while any anti-holomorphic vector has the form $v \otimes 1 + Jv \otimes i$ for some $v \in TM$. So for any $w \in T_{\mathbb{C}}M$ written as $w = w_{1,0} + w_{0,1}$, then

$$w_{1,0} = \frac{1}{2}(w - iJw), \ w_{0,1} = \frac{1}{2}(w + iJw)$$

Similarly, let $T^*_{\mathbb{C}}M = T^*M \otimes \mathbb{C}$. Then the corresponding decomposition of $T_{\mathbb{C}}M$ is $T^*_{\mathbb{C}}M = T^{1,0} \oplus T^{0,1}$, where

$$\begin{aligned} T^{1,0} = T^*_{1,0} &= \{\eta \in T^*_{\mathbb{C}}M \colon \eta(Jw) = i\eta(w),\ \forall\ w \in T_{\mathbb{C}}M\} \\ &= \{\xi \otimes 1 - (\xi \circ J) \otimes i \colon \xi \in T^*M\} \\ T^{0,1} = T^*_{0,1} &= \{\eta \in T^*_{\mathbb{C}}M \colon \eta(Jw) = -i\eta(w),\ \forall\ w \in T_{\mathbb{C}}M\} \\ &= \{\xi \otimes 1 + (\xi \circ J) \otimes i \colon \xi \in T^*M\} \end{aligned}$$

Then let

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{l+m=k} \Omega^{l,m}(M, \mathbb{C})$$

where $\Omega^{l,m}(M, \mathbb{C}) = \Gamma^\infty\left(\Lambda^l T^{1,0} \wedge \Lambda^k T^{1,0}\right)$. Let $\pi^{l,m} \colon \Omega^k(M, \mathbb{C}) \to \Omega^{l,m}(M, \mathbb{C})$ be the projection. For $l + m = k$,

$$\partial \colon \Omega^{l,m}(M, \mathbb{C}) \xrightarrow{d} \Omega^{k+1}(M, \mathbb{C}) \xrightarrow{\pi^{l+1,m}} \Omega^{l+1,m}(M, \mathbb{C})$$

$$\overline{\partial} \colon \Omega^{l,m}(M, \mathbb{C}) \xrightarrow{d} \Omega^{k+1}(M, \mathbb{C}) \xrightarrow{\pi^{l,m+1}} \Omega^{l,m+1}(M, \mathbb{C})$$

Remark. (1) If $f \colon M \to \mathbb{C}$, then it can see $df = \partial f + \overline{\partial} f$. But in general, $d \neq \partial + \overline{\partial}$.

(2) For smooth $f \colon M \to \mathbb{C}$, f is called holomorphic if $(df)_m \in T^{1,0}_m$ for any $m \in M$.

(3) Let (M, J_M) and (N, J_N) be two almost complex manifolds and $g \colon M \to N$ be smooth. g is called holomorphic if $Tg(T_{1,0}) \subset T_{1,0}$.

Example 2. Let $\mathbb{C}^n$ be a real manifold with the coordinate $\{x_1, y_1, \cdots, x_n, y_n\}$. Let J be an almost complex structure defined as

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j},\ J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$$

For complex coordinate $\{z_1, \cdots, z_n\}$, i.e. $z_j = x_j + iy_j$, let

$$\begin{gathered} \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \in T_{1,0},\ \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \in T_{0,1} \\ dz_j = dx_j + idy_j \in \Omega^{1,0},\ d\overline{z}_j = dx_j - idy_j \in \Omega^{0,1} \end{gathered}$$

Then for $f \colon \mathbb{C}^n \to \mathbb{C}$, it can see

$$df = \underbrace{\sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j}_{=\partial f \in \Omega^{1,0}} + \underbrace{\sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j}_{=\overline{\partial} f \in \Omega^{0,1}}$$

f is holomorphic if and only if $\overline{\partial} f = 0$ that is the Cauchy-Riemann equation.

Definition 3. An almost complex manifold (M, J) is called a complex manifold of dimension n if, for any $m \in M$, there is a neighborhood U of m s.t. $U \to V \subset \mathbb{C}^n$ is biholomorphic. Then this J is called integrable.

Remark. For a complex manifold (M, J), the tangent space of M at m is

$$T_m M := T_{1,0,m} M = \{X \in T_{\mathbb{C}} M \colon JX = iX\}$$

i.e. all holomorphic vectors, because as this definition, for any $f \colon M \to \mathbb{C}$ is holomorphic and smooth vector field $X_M \in TM = \cup T_m M$, $X_M f$ is holomorphic.

For any real vector fields u, v and a complex structure J, define the Nijenhuis tensor N_J

$$N_J(u, v) = [Ju, Jv] - J[Ju, v] - J[u, Jv] - [u, v]$$

Theorem 4 (Newlander-Nirenberg)**.** *For an almost complex manifold (M, J),*

$$(M, J) \text{ is complex} \;\Leftrightarrow\; N_J = 0 \;\Leftrightarrow\; [T_{1,0}, T_{1,0}] \subset T_{1,0} \;\Leftrightarrow\; d = \partial + \overline{\partial} \;\Leftrightarrow\; \partial^2 = \overline{\partial}^2 = 0$$

2.4 Complexification of Lie Groups

Definition 1. A complex Lie group G is a group and also a complex manifold such that

$$\begin{array}{ccccccc} G \times G & \to & G, & \quad & G & \to & G \\ (g, h) & \mapsto & gh & \quad & g & \mapsto & g^{-1} \end{array}$$

are holomorphic.

Remark. Let $\mathfrak{g} = \mathrm{Lie}(G)$ with the complex structure $\mathfrak{g} \to \mathfrak{g} \colon X \mapsto JX := iX$. $[\cdot, \cdot]$ is complex bilinear by $\mathrm{ad} \circ J = J \circ \mathrm{ad}$. Therefore, $\mathfrak{g}$ is a complex Lie algebra. Conversely, if the Lie algebra $\mathfrak{g}$ is equipped with a complex structure, then there is an integrable complex structure J on G.

Theorem 2. *Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. There is a unique complex Lie group G and a Lie group homomorphism $\iota \colon K \to G$ s.t.*

(1) (G, ι) has the universal property, that is for any complex Lie group H and a Lie group homomorphism $\rho \colon K \to H$, there is unique holomorphic Lie group homomorphism $\rho_{\mathbb{C}} \colon G \to H$ s.t. $\rho = \rho_{\mathbb{C}} \circ \iota$.

(2) ι is injective, $\iota(K)$ is a maximal compact subgroup of G, $G/\iota(K)$ is connected and $T_e\iota(\mathfrak{k})$ is a totally real subspace of $\mathfrak{g}$.

Remark. In fact, complexifying a real Lie algebra $\mathfrak{k}$ as $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$ with the extended Lie bracket defined as

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1])$$

$\mathfrak{k}_{\mathbb{C}}$ is a complex Lie algebra. And $G = K_{\mathbb{C}}$ with the Lie algebra $\mathfrak{k}_{\mathbb{C}}$.

Theorem 3 (Cartan Decomposition)**.** *Let K be a compact Lie group with $\mathfrak{k}$ and $G = K_{\mathbb{C}}$ with $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Then the map is a diffeomorphism.*

$$\begin{array}{ccc} K \times \mathfrak{k} & \longrightarrow & G \\ (k, X) & \longmapsto & \exp(iX)k \end{array}$$

Definition 4. A complex Lie group G is called reductive if $G = K_{\mathbb{C}}$ for some compact Lie group K. If G is reductive and the center of $\mathfrak{g}$ is trivial, then $\mathfrak{g}$ is called semisimple.

Example 5. $GL(n,\mathbb{C})$ and $SL(n,\mathbb{C})$ are reductive. And $\mathfrak{sl}(n,\mathbb{C})$ is semisimple but $\mathfrak{gl}(n,\mathbb{C})$ is not.

(1) $GL(n,\mathbb{C}) = U(n)_{\mathbb{C}}$ and $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n)_{\mathbb{C}}$. Moreover, the polar decomposition is the Cartan decompostion, that is for $A \in GL(n,\mathbb{C})$,

$$A = e^X U = e^{iY} U, \text{ where } U \in U(n),\ X = iY \in \mathrm{Herm}(n) \ \Rightarrow\ Y \in \mathfrak{u}(n)$$

(2) $SL(n,\mathbb{C}) = SU(n)_{\mathbb{C}}$ and $\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{su}(n)_{\mathbb{C}}$.

Let compact Lie group K act on a complex manifold (M,J) smoothly i.e.

$$\begin{array}{rccc} \tau\colon & K & \longrightarrow & \mathrm{Diff}(M) \\ & g & \longmapsto & \tau(g) \end{array}$$

is a Lie group homomorphism. Then by the above theorem, it can be uniquely extended on $G = K_{\mathbb{C}}$. Therefore, G acts on M holomorphically. And the map is defined as

$$\begin{array}{ccc} \mathfrak{g} = \mathfrak{k} + i\mathfrak{k} & \longrightarrow & \mathrm{Vect}(M) \\ X + iY & \longmapsto & X_M + JY_M \end{array}$$

where $\mathrm{Vect}(M)$ is the holomorphic vector field on M. It is well-defined since X_M and Y_M are holomorphic.

Let K be a compact Lie group. Then there is a bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on K and

$$\langle \mathrm{ad}_Z X, Y\rangle_{\mathfrak{k}} = -\langle X, \mathrm{ad}_Z Y\rangle_{\mathfrak{k}},\ \forall\ X,Y,Z \in \mathfrak{k}$$

Therefore, K is a Riemannian geometry. Moreover, the usual exponential map of the Lie group coincides with the exponential map of Riemannian manifold.

Let $G = K_{\mathbb{C}}$ be the complexified Lie group and define the inner product on $\mathfrak{g}$

$$\langle X_1 + iY_1, X_2 + iY_2\rangle_{\mathfrak{g}} := \langle X_1, X_2\rangle_{\mathfrak{k}} + \langle Y_1, Y_2\rangle_{\mathfrak{k}}$$

And this metric is K-invariant and $\mathfrak{k} \perp i\mathfrak{k}$. So instead of considering the Riemannian structure on G, defining a metric on $N := G/K$, the right cosets of G. Let $\pi\colon G \to N$ be the projection. There is a vector bundle isomorphism

$$\begin{array}{ccc} G \times \mathfrak{k} & \longrightarrow & TN \\ (g,\eta) & \longmapsto & T_e(\pi \circ L_g)(i\eta) \end{array}$$

where $L_g(h) = gh$. Then it can define a Riemannian metric $\langle\cdot,\cdot\rangle$ on N.

For $v_1, v_2 \in T_{\pi(g)}M$, there are $\eta_1, \eta_2 \in \mathfrak{k}$ s.t. $v_j = T_e(\pi \circ L_g)(i\eta_j)$.

$$\langle v_1, v_2\rangle_{\pi(g)} := \langle \eta_1, \eta_2\rangle_{\mathfrak{k}}$$

Therefore, it is a G-invariant Riemannian metric. And with this Riemannian structure, N is a complete, connected, and simply connected Riemannian metric with nonpositive sectional curvature. Moreover, any geodesic line on N has the form

$$\gamma(t) = \pi(g \exp(tiX)),\ \text{for some } g \in G,\ X \in \mathfrak{k}$$

2.5 Roots and Weyl Chambers

Definition 1. Let K be a compact Lie group with Lie algebra $\mathfrak{k}$.

(1) A maximal torus T in K is a maximal connected commutative subgroup of K.

(2) A Cartan subalgebra of $\mathfrak{k}$ is a maximal commutative subalgebra of $\mathfrak{k}$.

Remark. (1) If T is a maximal torus, then $\mathfrak{t} = \operatorname{Lie} T$ is a Cartan subalgebra. Conversely, if $\mathfrak{t}$ is a Cartan subalgebra, then $T := \exp \mathfrak{t}$ is a maximal torus.

(2) Let $G = K_{\mathbb{C}}$ with Lie algebra $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$. If $\mathfrak{t}$ is a Cartan algebra of $\mathfrak{k}$, then $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}$ is a maximal commutative subalgebra of $\mathfrak{g}$. $\mathfrak{h}$ is called a Cartan subalgebra of $\mathfrak{g}$.

Theorem 2. *Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. Then*

$$\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{z}(\mathfrak{k})$$

where $\mathfrak{z}(\mathfrak{k})$ is the center of $\mathfrak{k}$ and $\mathfrak{k}'$ is the ideal generated by $[\mathfrak{k}, \mathfrak{k}]$, the set of commutators. And $\mathfrak{k}'$ is semisimple i.e. the center of $\mathfrak{k}'$ is trivial.

Remark. (1) For complex reductive Lie group $G = K_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$, it is also true

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$$

where $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{k})_{\mathbb{C}}$, the center of $\mathfrak{g}$, and $\mathfrak{g}' = \mathfrak{k}'_{\mathbb{C}}$ that is semisimple.

(2) In fact, $\mathfrak{k}'$ is the Lie algebra of K' which is the closed normal subgroup of K generated by commutators $k_1 k_2 k_1^{-1} k_2^{-1}$ for $k_1, k_2 \in K$.

Let G act on $\mathfrak{g}$ by adjoint action. Since K is compact, there is a K-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Therefore, by differentiating of Ad,

$$\langle \operatorname{ad}_X Y, Z \rangle = - \langle Y, \operatorname{ad}_X Z \rangle, \ \forall\ X \in \mathfrak{k}$$

i.e. ad_X is skew-symmetry for any $X \in \mathfrak{k}$ and so ad_X is diagonalizable. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$. Then for any $H = H_1 + iH_2 \in \mathfrak{h}$, $\operatorname{ad}_H = \operatorname{ad}_{H_1} + i \operatorname{ad}_{H_2}$ is diagonalizable by the commutativity of ad_{H_1} and ad_{H_2}. So $\{\operatorname{ad}_H \colon H \in \mathfrak{h}\}$ is simultaneously diagonalizable. Then for $\alpha \in \mathfrak{h}^*$ i.e. $\alpha \colon \mathfrak{t} \to \mathbb{C}$ is linear, let

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \colon [H, X] = \alpha(H) X,\ \forall\ H \in \mathfrak{h}\}$$

So $\mathfrak{g}_\alpha$ is a eigenspace for $\{\operatorname{ad}_H \colon H \in \mathfrak{h}\}$ for some $\alpha \in \mathfrak{h}^*$. Clearly, $\mathfrak{g}_0 = \mathfrak{h}$.

Definition 3. The root system of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ is

$$R := \{\alpha \in \mathfrak{h}^* \colon \alpha \neq 0,\ \mathfrak{g}_\alpha \neq 0\}$$

Remark. (1) Since V is finite-dimensional, R is finite and the eigenspace decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

(2) Since ad_X is skew-symmetry for each $X \in \mathfrak{t}$, any root α is pure imaginary on $\mathfrak{t}$. Therefore, $R \subset (i\mathfrak{t})^*$, where $i\mathfrak{t}$ is viewed as a real vector space.

(3) For any $\alpha, \beta \in R \cup \{0\}$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(4) If $\mathfrak{g}$ is semisimple i.e. $\mathfrak{z}(\mathfrak{g}) = \{0\}$, then $(i\mathfrak{t})^* = \operatorname{span}_{\mathbb{R}} R$. In general, let $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{k}'$, then $(i\mathfrak{t}')^* = \operatorname{span}_{\mathbb{R}} R$.

(5) For any $\alpha \in R$, $-\alpha \in R$. Moreover, $\mathfrak{g}_{-\alpha} = \vartheta \mathfrak{g}_\alpha$ where $\vartheta(X + iY) = X - iY$ for $X + iY \in \mathfrak{g}$ and $X, Y \in \mathfrak{k}$.

Definition 4 (Killing Form)**.** Let $G = K_{\mathbb{C}}$ be a reductive Lie Group with Lie algebra $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$. The Killing form is a bilinear form

$$\begin{array}{rccc} B\colon & \mathfrak{k} \times \mathfrak{k} & \longrightarrow & \mathbb{R} \\ & (X, Y) & \longmapsto & \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y) \end{array}$$

Moreover, B can be extended to $\mathfrak{g}$.

Theorem 5. *(1) For $k \in K$ and $X, Y, Z \in \mathfrak{g}_{\mathbb{C}}$,*

$$B(\operatorname{Ad}_k X, \operatorname{Ad}_k Y) = B(X, Y),\ B(\operatorname{ad}_Z X, Y) = -B(X, \operatorname{ad}_Z Y)$$

(2) (Cartan's Criterion) $\mathfrak{g}_{\mathbb{C}}$ is semisimple if and only if B is nonsingular on $\mathfrak{g} \times \mathfrak{g}$.

(3) In general, B is nonsingular on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ for $\alpha \in R$ and is negative definite on $\mathfrak{g}' \times \mathfrak{g}'$. And thus B is an inner product on $i\mathfrak{t}'$.

Definition 6. Let $G = K_{\mathbb{C}}$ with semisimple Lie algebra $\mathfrak{g}$. Then since B is an inner product on $i\mathfrak{t}$, for any $\alpha \in (i\mathfrak{t})^*$, there is a unique $u_\alpha \in i\mathfrak{t}$ s.t.

$$\alpha(H) = B(H, u_\alpha),\ \forall\ H \in i\mathfrak{t}$$

If $\alpha \in R$, then let

$$h_\alpha := \frac{2u_\alpha}{B(u_\alpha, u_\alpha)}$$

called the coroot of α. The coroot system is

$$R^\vee := \{h_\alpha \colon \alpha \in R\} \subset i\mathfrak{t}$$

Remark. If $\mathfrak{g}$ is not semisimple, then for general $\alpha \in (i\mathfrak{t})^*$ there may no u_α because B is singular on $i\mathfrak{t}$. But if $\alpha \in R$, u_α and h_α can be also defined. Let $\mathfrak{t}' = \mathfrak{k}' \cap \mathfrak{t}$.

$$\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{z}(\mathfrak{k}) \ \Rightarrow\ i\mathfrak{t} = i\mathfrak{t}' \oplus i\,(\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{t})$$

If $H \in i\,(\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{t})$, $[H, X] = 0$ for any $X \in \mathfrak{g}$. So $\alpha(H) = 0$. Therefore, α can be viewed as a linear map on $i\mathfrak{t}'$ i.e. $\alpha \in (i\mathfrak{t}')^*$. By above, since B is nonsingular on $i\mathfrak{t}'$, there is a $u_\alpha \in i\mathfrak{t}'$ s.t.

$$\alpha(H) = B(H, u_\alpha),\ \forall\ H \in i\mathfrak{t}$$

and thus it can define the coroot h_α.

Example 7. Let $G = SL(2, \mathbb{C})$ with $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $K = SU(2)$ with $\mathfrak{k} = \mathfrak{su}(2)$. Then

$$\mathfrak{t} = \operatorname{span}_{\mathbb{R}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \mathfrak{h} = \operatorname{span}_{\mathbb{C}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then

$$[H, X] = 2X, \ [H, Y] = -2Y$$

Therefore, the two roots are $\alpha_1, \alpha_2 \colon \mathfrak{h} \to \mathbb{C}$,

$$\alpha_1(H) = 2, \ \alpha_2(H) = -2$$

and $\mathfrak{g}_{\alpha_1} = \operatorname{span}_{\mathbb{C}} X$, $\mathfrak{g}_{\alpha_2} = \operatorname{span}_{\mathbb{C}} Y$. So $\mathfrak{g} = \operatorname{span}_{\mathbb{C}} \{H, X, Y\}$ is the root decomposition.

Theorem 8 ($\mathfrak{sl}(2, \mathbb{C})$ Triple)**.** *Let $G = K_{\mathbb{C}}$ be a reductive Lie group with Lie algebra $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$. Let $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}$ and R be the corresponding root system. Let $\alpha \in R$. Then there is a $X_\alpha \in \mathfrak{g}_\alpha$ s.t. let $Y_\alpha = -\vartheta X_\alpha \in \mathfrak{g}_{-\alpha}$ and $H_\alpha = h_\alpha \in \mathfrak{h}$,*

$$\mathfrak{sl}(2, \mathbb{C}) \simeq \operatorname{span}_{\mathbb{C}} \{H_\alpha, X_\alpha, Y_\alpha\}$$

Corollary 9. *Let $G, \mathfrak{t}$ and R as above and $\alpha \in R$.*

(1) $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$.

(2) For any $\alpha, \beta \in R$, $\alpha(h_\beta) \in \mathbb{Z}$.

(3) $\dim \mathfrak{g}_\alpha = 1$.

Let $G = K_{\mathbb{C}}$ be a reductive Lie group with semisimple Lie algebra $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ be a Cartan subalgebra. Then the Killing form B is an inner product on $i\mathfrak{t}$. Transporting B to $(i\mathfrak{t})^*$ as an inner product defined as

$$B(\alpha, \beta) = B(u_\alpha, u_\beta), \ \forall \ \alpha, \beta \in (i\mathfrak{t})^*$$

With equipped B, $(i\mathfrak{t})^*$ is a Euclidean space. And by above for any $\alpha, \beta \in R$,

$$4\cos^2\theta_{\alpha\beta} = \alpha(H_\beta)\beta(H_\alpha), \ \alpha(H_\beta) = \frac{2\,\|\alpha\|\cos\theta_{\alpha\beta}}{\|\beta\|}$$

Therefore, there are only finite cases for $\theta_{\alpha\beta}$.

(1) $\theta_{\alpha\beta} = \frac{\pi}{2}$;

(2) $\|\alpha\| = \|\beta\|$, then $\theta_{\alpha\beta} = \frac{\pi}{3}$ or $\frac{2\pi}{3}$;

(3) $\|\alpha\| = \sqrt{2}\,\|\beta\|$, then $\theta_{\alpha\beta} = \frac{\pi}{4}$ or $\frac{3\pi}{4}$;

(4) $\|\alpha\| = \sqrt{3}\,\|\beta\|$, then $\theta_{\alpha\beta} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

If it is not semisimple, it can replace $i\mathfrak{t}$ and $(i\mathfrak{t})^*$ by $i\mathfrak{t}'$ and $(i\mathfrak{t}')^*$.

Definition 10. Let $G = K_{\mathbb{C}}$ be a reductive Lie group with semisimple Lie algebra $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ be a Cartan subalgebra with the corresponding root system R. (If not semisimple, replacing $\mathfrak{t}$ to $\mathfrak{t}'$.)

(1) Let $\gamma \in (i\mathfrak{t})^*$ s.t. $V = \{\alpha \in (i\mathfrak{t})^* \colon B(\alpha, \gamma) = 0\} \cap R = \varnothing$. Then define

$$R^+ = \{\alpha \in R \colon B(\alpha, \gamma) > 0\}, \ R^- = \{\alpha \in R \colon B(\alpha, \gamma) < 0\}$$

be the positive and negative root system.

(2) A system of simple roots is a subset $\Delta \subset R$ that is a basis of $(i\mathfrak{t})^*$ s.t. for any $\beta \in R$,

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$$

and either $\{k_\alpha \colon \alpha \in \Delta\} \subset \mathbb{Z}_{\geqslant 0}$ or $\{k_\alpha \colon \alpha \in \Delta\} \subset \mathbb{Z}_{\leqslant 0}$.

Remark. In fact, for any γ with $V = \{\alpha \in (i\mathfrak{t})^* \colon B(\alpha, \gamma) = 0\} \cap R = \varnothing$, there is a unique system of simple roots Δ s.t.

$$R^+ = \left\{\beta \in R \colon \beta = \sum_{\alpha \in \Delta} k_\alpha \alpha,\ k_\alpha \in \mathbb{Z}_{\geqslant 0}\right\},\ R^- = \left\{\beta \in R \colon \beta = \sum_{\alpha \in \Delta} k_\alpha \alpha,\ k_\alpha \in \mathbb{Z}_{\leqslant 0}\right\}$$

Therefore, the system of simple roots exists and there is a one-one onto relation of

$$\{\text{Positive root systems}\} \sim \{\text{Systems of simple roots}\}$$

Definition 11. Given the same settings as above, then an open Wely chamber is a connected component of $(i\mathfrak{t})^* \setminus \cup_{\alpha \in \Delta} V_\alpha$, where

$$V_\alpha = \{\beta \in (i\mathfrak{t})^* \colon B(\beta, \alpha) = 0\}$$

If Δ is a system of simple roots, then the open positive Weyl chamber is defined as

$$C(\Delta) = \{\beta \in (i\mathfrak{t})^* \colon B(\beta, \alpha) > 0,\ \forall\ \alpha \in \Delta\}$$

Remark. Conversely, choose a Weyl chamber C, then there is a unique system of simple roots Δ s.t. $C = C(\Delta)$ is the positive Wey chamber.

Let $G = K_{\mathbb{C}}$ be a reductive Lie group with Lie algebra $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ and T be a maximal torus of K with Lie algebra $\mathfrak{t}$. Let $N = \{k \in K \colon kTk^{-1} = T\}$. Then the Weyl group is defined as $W = N/T$. W acts $\mathfrak{t}$ and $\mathfrak{t}^*$. For $w := [w] \in W$ and $H \in i\mathfrak{t}$ and $\alpha \in \mathfrak{t}^*$,

$$w \cdot H := \operatorname{Ad}_w H,\ w \cdot \alpha := \operatorname{Ad}_w^* \alpha$$

The actions can be extended on $i\mathfrak{t}$ and $(i\mathfrak{t})^*$. There is a more explicit expression of W.

For $\alpha \in \Delta$, the reflection $r_\alpha \colon (i\mathfrak{t}')^* \to (i\mathfrak{t}')^*$ is defined as

$$r_\alpha(\beta) := \beta - \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)} \alpha = \beta - \beta(h_\alpha)\alpha$$

i.e. r_α is the reflection about the hyperplane orthogonal to α. Then

$$W = \text{the group generated by } \{r_\alpha \colon \alpha \in \Delta\}$$

Theorem 12. *Let W be the Weyl group acting on $(i\mathfrak{t})^*$ (or $(i\mathfrak{t}')^*$) and C be an open Weyl chamber.*

(1) For any $\alpha \in R$, $w \cdot \alpha \in R$.

(2) Let W act on the set of Weyl chambers. Then this action is transitive.

(3) Let $\alpha, \alpha' \in \overline{C}$. If $w \cdot \alpha = \alpha'$ for some $w \in W$, then $\alpha = \alpha'$.

(4) Let $\alpha \in C$. If $w \cdot \alpha = \alpha$ for some $w \in W$, then $w = id$.

(5) If Δ_1 and Δ_2 are two systems of simple roots, then there is a unique $w \in W$ s.t. $w \cdot \Delta_1 = \Delta_2$.

(6) For any $\beta \in (i\mathfrak{t})^*$, $W \cdot \beta \cap \overline{C}$ contains exact one element.

The general root system is defined as the following.

Definition 13. A root system (E, R) consists a finite-dimensional real Euclidean space E and a finite subset R containing nonzero elements s.t.

(1) $E = \operatorname{span}_{\mathbb{R}} R$.

(2) For any $\alpha \in R$, $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$.

(3) For $\alpha \in R$, $r_\alpha(R) \subset R$, where $r_\alpha \colon E \to E$ is defined as

$$r_\alpha(\beta) := \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

(4) For any $\alpha, \beta \in R$,

$$\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

Remark. For the semisimple case, $((i\mathfrak{t})^*, R)$ is a root system. For the general case, $((i\mathfrak{t}')^*, R)$ is a root system.

Definition 14. Let $G = K_{\mathbb{C}}$ be a reductive Lie group with Lie algebra $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$. Let $\mathfrak{t} = \operatorname{Lie} T$ be a Cartan subalgebra of $\mathfrak{k}$ with the corresponding root system R and choosing a system of simple roots Δ. Let $\lambda \in (i\mathfrak{t})^*$.

(1) λ is called algebraically integral (in weight lattice) if

$$\lambda(h_\alpha) \in \mathbb{Z},\ \forall\ \alpha \in R$$

(2) λ is called analytically integral if

$$\lambda(H) \in 2\pi i\mathbb{Z},\ \forall\ H \in \Gamma$$

where $\Gamma = \{H \in \mathfrak{t} \colon \exp H = I\}$

(3) λ is called dominant with respect to a system of simple roots Δ if

$$\lambda \in \overline{C(\Delta)}$$

Remark. Let $\chi \colon T \to \mathbb{C}^*$ be a group homomorphism that is called a character of T. Then there is a unique λ that is analytically integral s.t.

$$\chi(\exp H) = e^{\lambda(H)},\ \forall\ \exp H \in T = \exp \mathfrak{t}$$

Definition 15. Setting as above, for $\lambda, \mu \in (i\mathfrak{t})^*$,

$$\lambda \geqslant \mu \ \Leftrightarrow\ \mu - \lambda = \sum_{\alpha \in \Delta} k_\alpha \alpha \text{ with } k_\alpha \geqslant 0$$

Example 16. (1) Let $G = GL(n, \mathbb{C})$ and $K = U(n)$. Then let the maximal torus and the Cartan subalgebra be

$$\begin{aligned} T &= \left\{\operatorname{diag}\left(e^{i\theta_1}, \cdots, e^{i\theta_n}\right) : \theta_j \in \mathbb{R}\right\} \\ \mathfrak{t} &= \{\operatorname{diag}(i\theta_1, \cdots, i\theta_n) : \theta_j \in \mathbb{R}\} \\ \mathfrak{h} &= \{\operatorname{diag}(z_1, \cdots, z_n) : z_j \in \mathbb{C}\} \end{aligned}$$

with the root system

$$R = \{\pm(\varepsilon_k - \varepsilon_j) \colon 1 \leqslant k < j \leqslant n\}$$

where $\varepsilon_k(E_j) = \delta_{kj}$, where $E_j = \operatorname{diag}(e_j)$, with the root space

$$\mathfrak{g}_{\varepsilon_k - \varepsilon_j} = \mathbb{C}E_{k,j}$$

and the coroot $h_{\varepsilon_k - \varepsilon_j}$ for root $\varepsilon_k - \varepsilon_j$ is

$$h_{\varepsilon_k - \varepsilon_j} = E_k - E_j$$

The system of simple roots can be choosed as

$$\Delta = \{\varepsilon_k - \varepsilon_{k+1} \colon 1 \leqslant k \leqslant n-1\}$$

and the corresponding positive Weyl chamber is

$$\begin{aligned} C(\Delta) &= \{\operatorname{diag}(\theta_1, \cdots, \theta_n) : \theta_k > \theta_{k+1}, \theta_k \in \mathbb{R}\} \\ \overline{C(\Delta)} &= \{\operatorname{diag}(\theta_1, \cdots, \theta_n) : \theta_k \geqslant \theta_{k+1}\} \end{aligned}$$

And the Weyl group $W \simeq \mathcal{S}_n$, the symmetry group. Since the orbit of W intersects $\overline{C(\Delta)}$ with a single point and the action can be extended on $\mathfrak{u}(n)^*$, for any $H \in \mathfrak{u}(n)^* \simeq \operatorname{Herm}(n)$, let

$$s(H) = \operatorname{spec}(H) := \operatorname{diag}(\theta_1, \cdots, \theta_n) \in \overline{C(\Delta)}$$

where $\theta_1 \geqslant \cdots \geqslant \theta_n$ are all eigenvalues of H.

And the above mention is also true for $G = SL(n, \mathbb{C})$ and $K = SU(n)$.

(2) For a particular case $SL(3, \mathbb{C})$ with $SU(3)$, the Cartan algebra $\mathfrak{t} = \operatorname{span}_{\mathbb{R}}\{iH_1, iH_2\}$ and $\mathfrak{h} = \operatorname{span}_{\mathbb{C}}\{H_1, H_2\}$, where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

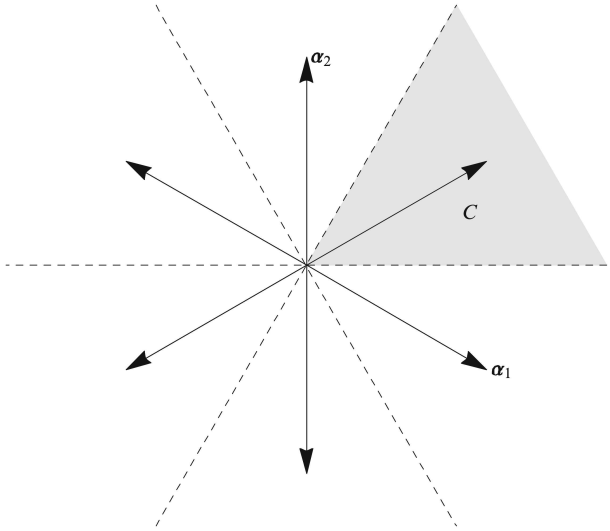
The root system is $R = \pm\{\alpha_1, \alpha_2, \alpha_3\}$. After equipping with the trace inner product and viewing the root $\alpha = (\alpha(H_1), \alpha(H_2))$, then

$$\alpha_1 = (2, -1),\ \alpha_2 = (1, -2),\ \alpha_3 = (1, 1)$$

with the root space $\mathfrak{g}_{\alpha_j} = \operatorname{span}_{\mathbb{C}} X_j$ and $\mathfrak{g}_{-\alpha_j} = \operatorname{span}_{\mathbb{C}} Y_j$ for $j = 1, 2, 3$, where

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



If let $\Delta = \{\alpha_1, \alpha_2\}$, then $C = C(\Delta)$ showed in above figure.

2.6 Representations

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. Let (Π, V) be a finite-dimensional representation of K i.e. $\Pi \colon K \to GL(V)$ is a Lie group homomorphism. Then let $\pi := T_e\Pi$ as a real linear map,

$$\begin{tikzcd} K \arrow[r, "\Pi"] & GL(V) \\ \mathfrak{k} \arrow[u, "\exp"] \arrow[r, "\pi"'] \arrow[ur, phantom, "\circlearrowleft"] & \operatorname{End} V \arrow[u, "e"'] \end{tikzcd} \qquad \Pi(\exp X) = e^{\pi(X)}$$

And by the above theorem, (Π, V) and (π, V) can be extended to representations of $G = K_{\mathbb{C}}$ and $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$. Let $\lambda \in \mathfrak{h}^*$.

$$V_\lambda = \{v \in V \colon \pi(H)v = \lambda(H)v,\ \forall\ H \in \mathfrak{h}\}$$

Similarly as the case of roots, after equipping V with a $\Pi(K)$-invariant inner product, $\pi(H)$ is skew-symmetric for any $H \in \mathfrak{t}$. Therefore, $\{\pi(H) \colon H \in \mathfrak{h}\}$ are simultaneously diagonalizable. So V_λ is an eigenspace for some λ.

Definition 1. For a representation (π, V), $\lambda \in \mathfrak{h}^*$ is called a weight if

$$V_\lambda \neq \{0\}$$

Let $\Delta(V)$ be the set of all weights.

Remark. Firstly, similar as roots, $\Delta(V) \subset (i\mathfrak{t})^*$ and there is a weight space decomposition of V

$$V = \bigoplus_{\lambda \in \Delta(V)} V_\lambda$$

Let R be the root system with respect to $\mathfrak{t}$. Then for any $\lambda \in \Delta(V)$, λ is algebraically integral i.e. $\lambda(h_\alpha) \in \mathbb{Z}$ for any $\alpha \in R$ by setting $\mathfrak{sl}(2, \mathbb{C})$ triples. Moreover, for any $\alpha \in R$ and $\beta \in \Delta(V)$, then

$$\pi(\mathfrak{g}_\alpha)V_\beta = V_{\alpha+\beta}$$

Let R be the root system of $\mathfrak{g}$ with a system of simple roots Δ and

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha, \ \mathfrak{n}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$$

Definition 2. For a representation (π, V), $\lambda_0 \in \Delta(V)$ is called a highest weight if

$$\mathfrak{n}^+ V_{\lambda_0} = 0, \text{ that is equivalent to } \lambda_0 + \alpha \notin \Delta(V), \ \forall\, \alpha \in R^+$$

Theorem 3. *Let $G = K_{\mathbb{C}}$ be a reductive Lie group with Lie algebra $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ with the corresponding root system R and choosing a system of simple roots Δ. Let (π, V) be an irreducible representation.*

(1) V has a unique highest weight λ_0 with $\dim V_{\lambda_0} = 1$.

(2) λ_0 is dominant.

(3) With ordering defined on $(i\mathfrak{t})^*$, $\mu \leqslant \lambda_0$ for all $\mu \in \Delta(V)$.

(4) If (π', V') is another irreducible representation with the same highest weight, then it is isomorphic to (π, V).

Remark. In fact, for any algebraically integral and dominant λ, there is an irreducible representation (π_λ, V_λ) with the highest weight λ.

Example 4 (Representations of torus)**.** Let $G = (\mathbb{C}^*)^n$. By taking logarithm of complex numbers, it can see $\mathfrak{g} = \mathbb{C}^n$. Clearly

$$\mathfrak{g} = (i\mathbb{R})^n_{\mathbb{C}} \ \Rightarrow \ \mathfrak{g} = \mathfrak{k}_{\mathbb{C}}, \text{ where } \mathfrak{k} = \{(i\theta_1, \cdots, i\theta_n) \colon \theta_j \in \mathbb{R}\}$$

By $K = (S^1)^n$ with the Lie algebra $\mathfrak{k}$, $(\mathbb{C}^*)^n = (S^1)^n_{\mathbb{C}}$. Therefore, $(\mathbb{C}^*)^n$ is a reductive Lie group with Lie algebra $\mathbb{C}^n$. Let (π, V) be a representation of $\mathbb{C}^n$ with the root system R. And let the (Π, V) be the corresponding representation of $(\mathbb{C}^*)^n$ and $\chi \colon (\mathbb{C}^*)^n \to \mathbb{C}$ defined as $\chi(\exp t) = \exp \alpha(t)$ for $\alpha \in R$, the root space decomposition is

$$V = \bigoplus_{\chi} V_\chi, \text{ where } V_\chi = \{v \in V \colon \Pi(\exp t)v = \chi(\exp t)v, \ \forall\, t \in \mathbb{C}^n\}$$

By above, for any such χ,

$$\chi(z_1, \cdots, z_n) = z_1^{k_1} \cdots z_n^{k_n}$$

for some $k_1, \cdots, k_n \in \mathbb{Z}$.

Example 5 (Representations of $\mathfrak{sl}(2, \mathbb{C})$)**.** Let (π, V) be a representation of $\mathfrak{sl}(2, \mathbb{C})$. Since $\mathfrak{h} = \operatorname{span}_{\mathbb{C}} H$, any weight $\lambda \in \mathfrak{h}^*$ can be viewed as $\lambda = \lambda(H) \in \mathbb{C}$. If λ is a weight of (π, V) with a $u \in V_\lambda$, then by above, $\pi(X)V_\lambda \subset V_{\lambda+2}$ with $\pi(X)u \in V_{\lambda+2}$. Since V is finite-dimensional, there is a N s.t. $\pi(X)^{N+1}u = 0$ and $\pi(X)^N u \neq 0$. Let

$$u_0 = \pi(X)^N u, \ \lambda_0 = \lambda + 2N$$

In fact, λ_0 is a highest weight with highest weight vetor u_0. Then let $u_k = \pi(Y)^k u_0$. Similarly, there is a m s.t. $u_m \neq 0$ and $u_{m+1} = 0$. Since $[X, Y] = H$, $\pi(H) = [\pi(X), \pi(Y)]$. So by induction,

$$\pi(X)u_k = k\left(\lambda_0 - (k-1)\right)u_{k-1}, \ k \geqslant 1$$

And

$$0 = \pi(X)u_{m+1} = (m+1)(\lambda_0 - m)u_m, \quad \Rightarrow \quad \lambda_0 = m$$

For vectors $\{u_0, u_1, \cdots, u_m\}$,

$$\begin{aligned}
\pi(H)u_k &= (m-2k)u_k \\
\pi(Y)u_k &= \begin{cases} u_{k+1}, & k < m \\ 0, & k = m \end{cases} \\
\pi(X)u_k &= \begin{cases} k\,(m-(k-1))\,u_{k-1}, & k > 0 \\ 0, & k = 0 \end{cases}
\end{aligned}$$

So $W = \operatorname{span}\{u_0, u_1, \cdots, u_m\}$ is invariant for π. Therefore, if (π, V) is an irreducible representation, then $V = W$. This shows the irreducible representation is uniquely determined by a positive integer m, i.e. the highest weight. Conversely, for any positive integer m, there is an irreducible representation (π, V) with the highest weight m by constructing $\{u_0, u_1, \cdots, u_m\}$. If $k \in \mathbb{Z}$ is a weight of π so are

$$-|k|, -|k|+2, \cdots, |k|-2, |k|$$

3 Symplectic Manifolds and Moment Map

3.1 Symplectic Manifolds

Let V be a finite-dimensional $\mathbb{R}$-vector space. A bilinear form $\omega\colon V \times V \to \mathbb{R}$ is called skew-symmetric if

$$\omega(u, v) = -\omega(v, u), \ \forall\ u, v \in V$$

If choosing an appropriate basis $\mathcal{B} = \{u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots, f_n\}$, then the matrix expression of ω is

$$\omega(u, v) = [u]_{\mathcal{B}}^T \begin{pmatrix} O & O & O \\ O & O & id \\ O & -id & O \end{pmatrix} [v]_{\mathcal{B}}$$

Let $\ker\omega = \{u \in V\colon \omega(u, v) = 0, \ \forall\ v \in V\} = \operatorname{span}\{u_1, \cdots, u_k\}$. If $\ker\omega = \{0\}$, then ω is called a symplectic form and (V, ω) is called a symplectic vector space. And then

$$\begin{array}{rccc} \omega^b\colon & V & \longrightarrow & V^* \\ & v & \longmapsto & \omega(v, \cdot) \end{array}$$

is an isomorphism.

Let (V, ω) be a symplectic vector space. A complex structure $J\colon V \to V$ is compatible with ω if

$$g(u, v) := \omega(u, Jv), \ \forall\ u, v \in V$$

is an inner product. And then

$$J^*\omega(u, v) = \omega(Ju, Jv) = g(Ju, v) = g(v, Ju) = \omega(v, -u) = \omega(u, v)$$

In fact, any two of g, ω and J can induce the left compatible structure.

Definition 1. Let M be a smooth manifold and $\omega \in \Omega^2(M)$ be a 2-form. ω is called a symplectic form if ω is closed i.e. $d\omega = 0$ and ω_m is symplectic for any $m \in M$. Then (M, ω) is called a symplectic manifold.

Example 2 (Cotangent Bundles)**.** Let N be a smooth manifold with dimension n and $M = T^*N$ be the cotangent bundle with the projection map

$$\begin{array}{rccc} \pi\colon & T^*N & \longrightarrow & N \\ & p = (x, \xi_x) & \longmapsto & x \end{array}$$

The tautological 1-form $\tau \in T^*M$ is defined as

$$\tau_p := T_p\pi^*\xi_x, \ \text{i.e.} \ \langle \tau_p, v\rangle = \langle \xi_x, T_p\pi(v)\rangle, \ \forall\ v \in T_pM$$

In fact, for any 1-form $\alpha \in T^*N$, α can be viewed as $\alpha\colon N \to T^*N$ by $x \mapsto q = (x, \alpha_x)$. Then for any $v \in T_xN$,

$$\langle (\alpha^*\tau)_x, v\rangle = \langle \tau_q, T_x\alpha(v)\rangle = \langle \alpha_x, v\rangle \ \Rightarrow \ \alpha^*\tau = \alpha$$

Conversely, the τ satisfying this property is unique. Define the canonical 2-form $\omega = -d\tau$ on $M = T^*N$. Locally, let $(q_1, \cdots, q_n)$ be a coordinate on an open neighborhood of N. Let p_k be the dual coordinate in T^*N. Then $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ is a coordinate of M. It can see

$$\tau = \sum_{k=1}^{n} p_k dq_k \ \Rightarrow \ \omega = \sum_{k=1}^{n} dq_k \wedge dp_k$$

Therefore, (M, ω) is a symplectic manifold.

Remark. (1) A special case is $\mathbb{C}^n$. For $z_k = x_k + iy_k$, $\mathbb{C}^n \simeq T^*\mathbb{R}^n$ and the canonical symplectic form

$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k$$

Moreover, by Darboux's theorem, any symplectic manifold is locally symplectomorphic to this structure.

(2) Let $\mathbb{C}^n$ equipped with the standard inner product H.

$$\begin{aligned} H &= \sum_{k=1}^{n} dz_k \otimes d\overline{z}_k \\ &= \sum_{k=1}^{n} (dx_k + idy_k) \otimes (dx_k - idy_k) \\ &= \sum_{k=1}^{n} (dx_k \otimes dx_k + dy_k \otimes dy_k) + i\,(dy_k \otimes dx_k - dx_k \otimes dy_k) \\ &= \operatorname{Re} H - i\omega \end{aligned}$$

Therefore, $\omega = -\operatorname{Im} H$. In general case, if V is a complex vector space with an inner product H, then $\omega = -\operatorname{Im} H$ is a symplectic form on V.

Example 3 (Coadjoint Orbits)**.** Considering the coadjoint action of a Lie group G on $\mathfrak{g}^*$, the orbit space of $\xi \in \mathfrak{g}^*$ is

$$M_\xi = G \cdot \xi \simeq G/G_\xi$$

Firstly, for $X, Y \in \mathfrak{g}$, let

$$\omega_\xi(X, Y) := \langle \xi, [X, Y] \rangle = \langle -\operatorname{ad}_X^* \xi, Y \rangle$$

Since

$$\begin{aligned} \ker \omega_\xi &= \{X \in \mathfrak{g} \colon \langle -\operatorname{ad}_X^* \xi, Y \rangle = 0, \ \forall\, Y \in \xi\} \\ &= \{X \in \mathfrak{g} \colon \operatorname{ad}_X^* = 0\} \\ &= \mathfrak{g}_\xi \end{aligned}$$

the induced form of ω_ξ also denoted by ω_ξ is nondegenerate on $\mathfrak{g}/\mathfrak{g}_\xi$. Therefore, ω_ξ is nondegenerate on $T_\xi M_\xi \simeq \mathfrak{g}/\mathfrak{g}_\xi$. Applying the isomorphism $G \to G$ by $h \to ghg^{-1}$, $G/G_{g\cdot\xi} \simeq G/G_\xi$. Therefore, $T_{g\cdot\xi} M_\xi \simeq \mathfrak{g}/\mathfrak{g}_\xi$. And thus, ω is well-defined on TM_ξ and nondegenerate everywhere. More explicitly, for any $X_{g\cdot\xi}, Y_{g\cdot\xi} \in T_{g\cdot\xi} M_\xi$,

$$\omega_{g\cdot\xi}\,(X_{g\cdot\xi}, Y_{g\cdot\xi}) := \langle g \cdot \xi, [X, Y] \rangle$$

The closedness of ω is by applying the global formulas for the Lie and exterior derivatives. So (M_ξ, ω) is a symplectic manifold.

On a smooth manifold M, a symplectic form ω and an almost complex structure J are compatible if

$$g_m(X, Y) = \omega_m(X, JY), \ \forall\, X, Y \in T_m M$$

is defined as a Riemannian metric on M. If (M, ω) is a symplectic manifold, then there is a compatible almost complex structure J on M because the Riemannian metric always exists.

Definition 4. A Kähler manifold (M, ω, J) is a complex manifold with a compatible symplectic form. Then ω is called a Kähler form.

If (M, ω, J) is a Kähler manifold and locally

$$\omega = \sum_{k<j} a_{kj} dz_k \wedge dz_j + b_{kj} dz_k \wedge d\overline{z}_j + c_{kj} d\overline{z}_k \wedge d\overline{z}_j$$

then by $J^*\omega = \omega$ and $J^*dz = idz$, $J^*d\overline{z} = -id\overline{z}$,

$$\omega = \frac{i}{2} \sum_{k,j=1}^{n} h_{kj} dz_k \wedge d\overline{z}_j$$

And by the properties of ω, it can see $H = (h_{kj})$ is Hermitian and positive definite. This property can be applied to define the Kähler form on a complex manifold.

Firstly, for a complex manifold M, $f \in C^\infty(M, \mathbb{R})$ is called strictly plurisubharmonic (spsh) if locally, $\left(\frac{\partial^2 f}{\partial z_k \partial \overline{z}_j}\right)$ is positive definite.

Theorem 5. *Let M be a complex manifold and $f \in C^\infty(M, \mathbb{R})$. f is spsh if and only if*

$$\omega = \frac{i}{2} \partial\overline{\partial} f$$

is a Kähler form.

Example 6. For $M = \mathbb{C}^n$ complex manifold, let $f(z) = |z|^2 = \sum_k z_k \overline{z}_k$. Then

$$\frac{\partial^2 f}{\partial z_k \partial \overline{z}_j} = \delta_{kj}$$

that is spsh. And

$$\begin{aligned} \frac{i}{2} \partial\overline{\partial} f &= \frac{i}{2} \partial \sum_k z_k d\overline{z}_k = \frac{i}{2} \sum_k dz_k \wedge d\overline{z}_k \\ &= \sum_k dx_k \wedge dy_k \\ &= \omega \end{aligned}$$

3.2 Hamiltonian Action and Moment Map

Let (M, ω) be a symplectic manifold. For a smooth function $f \colon M \to \mathbb{R}$, define $X_f \in \mathrm{Vect}(M)$ as $X_f = \left(\omega^b\right)^{-1}(df)$, and so

$$df = \omega^b(X_f) = \omega(X_f, \cdot) = \imath_{X_f}\omega$$

where $\imath_{X_f}$ is the contraction map. Then X_f is called a Hamiltonian vector field. Let

$$\mathrm{Vect}_{Ham}(M) = \{X \in \mathrm{Vect}(M) \colon \exists\ f \in C^\infty(M),\ X = X_f\}$$

For any Hamiltonian vector field X_f,

$$\mathcal{L}_{X_f} f = df(X_f) = \omega(X_f, X_f) = 0$$

which means f is perserved along X_f and by the Cartan's formula,

$$\mathcal{L}_{X_f}\omega = \imath_{X_f} d\omega + d\imath_{X_f}\omega = 0$$

which means ω is also preserved along X_f.

Example 1. Let $M = T^*\mathbb{R}^n$ with the coordinate $(q_1, \cdots .q_n, p_1, \cdots, p_n)$ and the canonical symplectic form $\omega = \sum_k dq_k \wedge dp_k$ be the configuration space. Let $q_k(t)$ be the position coordinates and $p_k(t) = \dot{q}_k(t)$ be the moment coordinate. Let $H\colon T^*\mathbb{R}^n \to \mathbb{R}$ be the energy function

$$H(q, p) = \frac{1}{2} |p|^2 + V(q)$$

where $V(q)$ is the potential function. Then the corresponding Hamiltonian vector field

$$X_H = \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial}{\partial p_k}$$

Therefore, the flow of X_H satisfies

$$\dot{q}_k(t) = \frac{\partial H}{\partial p_k} = p_k(t),\ \dot{p}_k(t) = -\frac{\partial H}{\partial q_k} = -\frac{\partial V}{\partial q_k}$$

Therefore, it is the Newton's second law

$$\ddot{q}(t) = -\nabla V$$

i.e. the energy is preserved along the flow.

Let G be a Lie group and (M, ω) be a symplectic manifold. The Lie group action $G \curvearrowright M$ is called symplectic if $\tau(g)$ is symplectomorphic, i.e. $\tau(g)^*\omega = \omega$, for any $g \in G$, where $\tau\colon G \to \mathrm{Diff}(M)$.

Definition 2. Let a Lie group action of $G \curvearrowright (M, \omega)$ be symplectic. This action is called a Hamiltonian action if there is a smooth function

$$\mu\colon M \longrightarrow \mathfrak{g}^*$$

(1) μ is G-equivariant i.e. $\mu\,(g \cdot m) = \mathrm{Ad}_g^* \,\mu(m),\ \forall\ r \in G, m \in M$

(2) for all $X \in \mathfrak{g}$,

$$d\mu_X = \imath_{X_M}\omega$$

where $\mu_X\colon M \to \mathbb{R}$ defined as $\mu_X(m) = \langle \mu(m), X \rangle$.

Then $(G \curvearrowright M, \omega, \mu)$ is called a Hamiltonian G-space.

Remark. (1) It is called Hamiltonian because for any $X \in \mathfrak{g}$, $X_M \in \mathrm{Vect}_{Ham}(M)$.

(2) If μ and ν are two moment maps for the same Hamiltonian action, then

$$d(\mu_X - \nu_X) = 0,\ \forall\ X \in \mathfrak{g}$$

So $\mu_X - \nu_X = \xi_X$ where ξ_X is a constant function on M. Moreover, $\xi\colon \mathfrak{g} \to \mathbb{R}$ is linear. So $\xi \in \mathfrak{g}^*$. Then $\mu - \nu = \xi$ and by equivariance of the moment map, ξ is fixed by the coadjoint action.

Example 3 (Operations)**.** (1) Let $(G \curvearrowright M_1, \omega_1, \mu_1)$ and $(G \curvearrowright M_2, \omega_2, \mu_2)$ be two Hamiltonian G-spaces. Then there is a canonical symplectic form $\omega_1 \times \omega_2$ on $M_1 \times M_2$ defined as

$$\omega_1 \times \omega_2 = \pi_1^* \omega_1 + \pi_2^* \omega_2$$

where $\pi_j \colon M_1 \times M_2 \to M_j$ is the projection. Moreover, after considering $G \curvearrowright M_1 \times M_2$ as

$$g \cdot (m_1, m_2) := (g \cdot m_1, g \cdot m_2)$$

This action is Hamiltonian with the moment map defined as

$$\mu_1 \times \mu_2(m_1, m_2) := \mu_1(m_1) + \mu_2(m_2)$$

(2) Let M be a symplectic manifold. Suppose the actions of G_1 and G_2 on M are Hamiltonian with moment maps μ_1 and μ_2 respectively and are commutative. Then $G_1 \times G_2$ acts M is Hamiltonian with the moment map

$$\mu_1 \oplus \mu_2 \colon M \longrightarrow \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$$

(3) Let $(G \curvearrowright M, \omega, \mu)$ be a Hamiltonian G-space. Suppose H is Lie subgroup of G with Lie algebra $\mathfrak{h}$ and the inclusion map $i \colon \mathfrak{h} \hookrightarrow \mathfrak{g}$. Then H acting on M is also Hamiltonian and the moment map is

$$\mu' \colon M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{i^*} \mathfrak{h}^*$$

where $i^* \colon \mathfrak{g}^* \to \mathfrak{h}^*$ be the dual map of i.

(4) Let $(G \curvearrowright M, \omega, \mu)$ be a Hamiltonian G-space and $N \subset M$ be a submanifold s.t. G acts N is invariant and $i^*\omega$ is a symplectic form on N, where $i \colon N \hookrightarrow M$. Then action of G on N is Hamiltonian with the moment map μ' defined as

$$\mu' \colon N \xrightarrow{i} M \xrightarrow{\mu} \mathfrak{g}^*$$

Example 4 (Cotangent Bundles)**.** Let N be a smooth manifold and $M = T^*N$ with the canonical symplectic form ω. Suppose there is a Lie group action of G on N. The induced action of G on M is defined as

$$g \cdot (n, \eta) := \left(g \cdot n, \left(T_n g^{-1}\right)^* \eta\right), \ \forall\, \eta \in T_n^* N$$

Then this action is Hamiltonian with the moment map

$$\mu_X = \imath_{X_M} \tau, \text{ i.e. } \langle \mu(p), X \rangle = \langle \eta, T_p\pi(X_M(p)) \rangle, \ p = (n, \eta) \in M$$

where τ is the tautological 1-form and $\pi \colon T^*N \to N$.

Proof. Let $p = (n, \eta) \in M$. By definiton, $\pi(g \cdot p) = g \cdot \pi(p)$. Therefore,

$$T_{g \cdot p} \circ T_p g = T_n g \circ T_p \pi, \ T_p g^* \circ T_{g \cdot p}^* = T_p \pi^* \circ T_n g^*$$

Since τ is the tautological 1-form, $\tau_p = T_p \pi^* \eta$. So

$$(g^*\tau)_p = T_p g^* \tau_{g \cdot p} = T_p g^* T_{g \cdot p} \pi^* \left(T_n g^{-1}\right)^* \eta = T_p \pi^* \eta = \tau_p$$

Therefore, τ is invariant for this action. Then by definition, $\mathcal{L}_{X_M}\tau = 0$. And by Cartan's formular,

$$\mathcal{L}_{X_M}\tau = d\imath_{X_M}\tau + \imath_{X_M}\tau \;\Rightarrow\; d\mu_X = \imath_{X_M}\omega$$

For the equivariance, since

$$\begin{aligned}\left(\operatorname{Ad}_{g^{-1}} X\right)_M(p) &= \left.\frac{d}{dt}\right|_{t=0} \exp\left(t \operatorname{Ad}_{g^{-1}} X\right)\cdot p \\ &= \left.\frac{d}{dt}\right|_{t=0} g^{-1}\exp(tX)\, g\cdot p \\ &= T_p g^{-1} X_M(g\cdot p)\end{aligned}$$

$$\begin{aligned}\langle \mu(g\cdot p), X\rangle &= \left\langle \left(T_n g^{-1}\right)^* \eta, T_{g\cdot p}\pi X_M(g\cdot p)\right\rangle \\ &= \left\langle \eta, T_n g^{-1} T_{g\cdot p}\pi X_M(g\cdot p)\right\rangle \\ &= \left\langle \eta, T_p\pi T_p g^{-1} X_M(g\cdot p)\right\rangle \\ &= \left\langle \eta, T_p\pi \left(\operatorname{Ad}_{g^{-1}} X\right)_M(p)\right\rangle \\ &= \langle \mu(p), \operatorname{Ad}_{g^{-1}} X\rangle \\ &= \left\langle \operatorname{Ad}_g^* \mu(p), X\right\rangle\end{aligned}$$

Thus, $\mu(g\cdot p) = \operatorname{Ad}_g^* \mu(p)$. □

Example 5 (Coadjoint Orbits)**.** Considering the coadjoint action of a Lie group G on $\mathfrak{g}^*$, the orbit space $G\cdot\xi$ of $\xi\in\mathfrak{g}^*$ can be equipped with a symplectic form ω. Then the coadjoint action of G on $G\cdot\xi$ is Hamiltonian with the moment map μ that is the inclusion

$$\mu\colon G\cdot\xi \hookrightarrow \mathfrak{g}^*$$

Proof. Firstly, for $g\cdot\xi\in G\cdot\xi$ and $h\in G$,

$$\mu(h\cdot(g\cdot\xi)) = h\cdot(g\cdot\xi) = h\cdot\mu(g\cdot\xi)$$

So it is clearly G-equivariant. And for $X, Y\in\mathfrak{g}$

$$\begin{aligned}\langle (d\mu_X)_{g\cdot\xi}, Y_{g\cdot\xi}\rangle &= \left.\frac{d}{dt}\right|_{t=0} \mu_X\left(\exp tY\cdot g\cdot\xi\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left\langle g\cdot\xi, \operatorname{Ad}_{\exp(-tY)} X\right\rangle \\ &= \langle g\cdot\xi, [X,Y]\rangle \\ &= \omega_{g\cdot\xi}(X_{g\cdot\xi}, Y_{g\cdot\xi})\end{aligned}$$

Therefore, $d\mu_X = \imath_{X_M}\omega$. □

Example 6 (Vector Spaces)**.** Let $M=\mathbb{C}^n$ with the standard inner product H. By above, the canonical symplectic form is given by

$$\omega = -\operatorname{Im} H$$

Let $K = U(n)$ be the Lie group acting on M naturally. This action is Hamiltonian with the moment map $\mu\colon M\to\mathfrak{k}^*$ defined as

$$\mu_X(z) = \langle \mu(z), X\rangle := \frac{iH(Xz, z)}{2}$$

where $\mathfrak{k} = \mathfrak{u}(n)$ acts on M naturally.

Proof. For any $X \in \mathfrak{k}$ and $\xi \in T_z M = \mathbb{C}^n$,

$$\begin{aligned}\langle (d\mu_X)_z, \xi\rangle &= \frac{d}{dt}\bigg|_{t=0} \mu_X\left(\exp(t\xi)\cdot z\right)\\ &= \frac{i}{2}\,\frac{d}{dt}\bigg|_{t=0} H\left(X\exp(t\xi)z, \exp(t\xi)z\right)\\ &= \frac{i}{2}\left(H(X\xi, z) + H(Xz, \xi)\right)\end{aligned}$$

Firstly, $X \in \mathfrak{k} = \mathfrak{u}(n)$ so $X^\dagger = -X$.

$$H(X\xi, z) = -H(\xi, Xz) = -\overline{H(Xz,\xi)}$$

Therefore,

$$\begin{aligned}\langle (d\mu_X)_z, \xi\rangle &= \frac{i}{2}\left(H(Xz,\xi) - \overline{H(Xz,\xi)}\right)\\ &= -\operatorname{Im} H(Xz,\xi)\\ &= \omega(Xz,\xi)\end{aligned}$$

And since K acts M linearly,

$$X_M(z) = \frac{d}{dt}\bigg|_{t=0} \exp(tX)\cdot z = \left(\frac{d}{dt}\bigg|_{t=0}\exp(tX)\right) z = Xz$$

So $\langle (d\mu_X)_z, \xi\rangle = \omega(X_M(z), \xi)$ that is $d\mu_X = \imath_{X_M}\omega$. For the equivariance, let $A \in K$,

$$\begin{aligned}\langle \mu(Az), X\rangle &= \frac{iH(XAz, Az)}{2}\\ &= \frac{iH(A^{-1}XAz, z)}{2}\\ &= \langle \mu(z), \operatorname{Ad}_{A^{-1}} X\rangle\\ &= \langle \operatorname{Ad}^*_A \mu(z), X\rangle\end{aligned}$$

□

Remark. In the general case, let $M = V$ be a complex vector space and K be a compact Lie group. And K acts on V linearly i.e. there is Lie group homomorphism

$$\Pi\colon K \to GL(V)$$

and inducing a Lie group homomorphism

$$\pi\colon \mathfrak{k} \to \operatorname{End}(V)$$

where $\pi = T_e\Pi$. And $\Pi \circ \exp = \exp \circ \pi$. Moreover, there is a K-invariant inner product H on V i.e.

$$H(\Pi(k)v, \Pi(k)w) = H(v,w),\ H\left(\pi(X)v, w\right) = -H\left(v, \pi(X)w\right)$$

So $\Pi(K) \subset U(n)$ and $\pi(\mathfrak{k}) \subset \mathfrak{u}(n)$. Therefore, when V is equipped with then canonical symplectic form $\omega = -\operatorname{Im} H$, then the action of K on V is Hamiltonian with the moment map

$$\mu_X(v) = \langle \mu(v), X\rangle := \frac{iH(\pi(X)v, v)}{2},\ \forall\ X \in \mathfrak{k}, v \in V$$

3.3 Symplectic Reduction and Projective Space

Let (M, ω) be a symplectic manifold and the action of G on M be Hamiltonian with the moment map $\mu\colon M \to \mathfrak{g}^*$. A point $m \in M$ of μ is called regular if

$$T_m\mu\colon T_mM \longrightarrow \mathfrak{g}^*$$

is surjective. And an element $\xi \in \mathfrak{g}^*$ is called a regular value if any point in $\mu^{-1}(\xi)$ is regular.

Proposition 1. *A point $m \in M$ is regular if and only if $\mathfrak{g}_m = 0$.*

Proof. Firstly, $\mathfrak{g}_m = \{X \in \mathfrak{g}\colon X_M(m) = 0\} = 0$ if and only is

$$X_M(m) = 0 \;\Rightarrow\; X = 0 \tag{$*$}$$

Since ω is nondegenerate, $X_M(m) = 0$ if and only if

$$\omega(X_M(m), v) = 0 \;\forall\; v \in T_mM \;\Leftrightarrow\; (d\mu_X)_m = 0$$

Let $i_X\colon \mathfrak{g}^* \to \mathbb{R}$ by $i_X(\xi) = \langle \xi, X\rangle$. Then

$$(d\mu_X)_m = (d(i_X \circ \mu))_m = i_X \circ T_m\mu$$

So $(d\mu_X)_m = 0$ if and only if $\operatorname{Im} T_m\mu \subset \ker i_X$. Therefore, $(*)$ is true if and only if $T_m\mu$ is surjective. □

Then it can consider the quotient symplectic space by applying the moment map.

Theorem 2 (Marsden-Weinstein-Meyer)**.** *(M, ω) is a connected symplectic manifold and G is a Lie group and the action of G on M is Hamiltonian with a moment map $\mu\colon M \to \mathfrak{g}^*$. Let $\xi \in \mathfrak{g}^*$ be a regular value of μ and G act $\mathfrak{g}^*$ by adjoint action. Suppose the restricted action of G_ξ on $\mu^{-1}(\xi)$ is proper and free so that*

$$M^\xi = \mu^{-1}(\xi)/G_\xi$$

is a smooth manifold. Then there is a unique symplectic manifold ω^ξ s.t.

$$\pi^*\omega^\xi = i^*\omega$$

where $\pi\colon \mu^{-1}(\xi) \to M^\xi$ projection and $i\colon \mu^{-1}(\xi) \hookrightarrow M$ inclustion.

Remark. (M^ξ, ω^ξ) is called the symplectic reduction of the Hamiltonian action at ξ. In particular, if $0 \in \mathfrak{g}^*$ satisfies the above conditions, then because $G_0 = G$,

$$M/\!\!/G = \mu^{-1}(0)/G$$

In fact, the general case can be induced from this by applying the shifting trick.

Moreover, on the reduction space, it can also consider the Hamiltonian action. Suppose a Lie group G acts on a symplectic manifold (M, ω) and this action is Hamiltonian with the moment map $\mu\colon M \to \mathfrak{g}^*$. Let $\tilde{H}$ be a closed subgroup of G s.t. the restricted actions of $\tilde{H}$ on M is commutative with the action of G on M. By above example, the action of $\tilde{H}$ on M is also Hamiltonian with the moment map

$$\tilde{\mu}\colon M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{\tilde{i}^*} \tilde{\mathfrak{h}}^*$$

where $\tilde{i}\colon \mathfrak{g} \hookrightarrow \tilde{\mathfrak{h}}$ is the inclusion. Let $\xi \in \tilde{\mathfrak{h}}^*$ be a regular value of $\tilde{\mu}$, which satisfies the above condition. Then considering the symplectic reduction of $\tilde{\mu}$ at ξ, let

$$\tilde{M}^\xi = \tilde{\mu}^{-1}(\xi)/\tilde{H}_\xi$$

Assume $\tilde{\mu}^{-1}(\xi)$ is G-invariant. It can induce the action of G on $\tilde{\mu}^{-1}(\xi)$. Because the action of $\tilde{H}$ on M is commutative with the action of G on M,

$$[g \cdot h \cdot m] = [g \cdot m],\ \forall\ g \in G, h \in \tilde{H}, m \in M$$

the induced action of G on $\tilde{M}^\xi$ is well-defined. Moreover, this action is also Hamiltonian with the moment map $\mu'\colon \tilde{M}^\xi \to \mathfrak{g}^*$ induced by

$$\tilde{\mu}^{-1}(\xi) \xhookrightarrow{\iota} M \xrightarrow{\mu} \mathfrak{g}^*$$

Now, considering a special case, let $M = \mathbb{C}^n$ with the standard inner product H. Thus, $(M, \omega = -\operatorname{Im} H)$ is a symplectic manifold. And let $G = U(n)$ acting on M. By above, it is a Hamiltonian action with the moment map

$$\langle \mu(z), X\rangle = \frac{iH(Xz, z)}{2}$$

Let

$$U(1) = \{z \in \mathbb{C}\colon\ |z| = 1\} \simeq \{zI_n\colon\ |z| = 1\} \subset U(n)$$

be a subgroup of $U(n)$ and the Lie algebra $\mathfrak{u}(1) \simeq i\mathbb{R}$ and an element $1^* \in \mathfrak{u}(1)^*$ defined as

$$\langle 1^*, i\rangle := 1$$

Let $U(1) = \tilde{H}$ acting on M with the induced moment map $\tilde{\mu}$. Considering $\tilde{\mu}^{-1}(-1^*)$, let $z \in \tilde{\mu}^{-1}(-1^*)$, for any $X = it \in \mathfrak{u}(1) = i\mathbb{R}$,

$$\langle \tilde{\mu}(X), z\rangle = \frac{iH(Xz, z)}{2} = \frac{-tH(z, z)}{2} = \langle -1^*, X\rangle = -t$$

So

$$\tilde{\mu}^{-1}(-1^*) = \{z \in M\colon H(z, z) = 2\}$$

Then for any $z \in \tilde{\mu}^{-1}(-1^*)$, then $z \neq 0$ and thus

$$U(1)_z = \{I\}\ \Rightarrow\ \mathfrak{u}(1)_z = 0$$

Therefore, by the above proposition, any $z \in \tilde{\mu}^{-1}(-1^*)$ is regular, i.e. -1^* is a regular value. And since

$$\langle \mathrm{Ad}_z^*(-1^*), it\rangle = \langle -1^*, \mathrm{Ad}_{z^{-1}}\, it\rangle = \langle -1^*, it\rangle$$

$U(1)_{-1^*} = U(1)$. Then it can see

$$\tilde{\mu}^{-1}(-1^*)/U(1) \simeq \mathbb{C}P^{n-1}$$

So there is a unique symplectic form ω_{FS} on $\mathbb{C}P^{n-1}$ s.t. $\pi^*\omega_{FS} = \tilde{i}^*\omega$, which is called the Fubini-Study form.

And since for any $g \in G = U(n)$ and $z \in \tilde{\mu}^{-1}(-1^*)$

$$H(g \cdot z, g \cdot z) = H(z, z) = 2$$

$\tilde{\mu}^{-1}(-1^*)$ is G-invariant. And clearly the action of $\tilde{H} = U(1)$ and the action of $U(n)$ on M are commutative. Therefore, the action of $U(n)$ on $\mathbb{C}P^{n-1}$ is Hamiltonian with the moment map $\mu' \colon \mathbb{C}P^{n-1} \to \mathfrak{u}(n)^*$ defined as

$$\langle \mu'([z]), X \rangle = \langle \mu(z), X \rangle = \frac{iH(Xz, z)}{2}$$

for $[z] \in \mathbb{C}P^{n-1}$ i.e. $H(z, z) = 2$. Therefore,

$$\langle \mu'([z]), X \rangle = \frac{iH(Xz, z)}{H(z, z)}, \ \forall \ z \in \mathbb{C}^n \setminus \{0\}$$

Remark. Let V be an n-dimensional complex vector space and K be a compact Lie group such that K acts V linearly, i.e. there are a Lie group homomorphism and a Lie algebra homomorphism

$$\Pi \colon K \longrightarrow GL(V), \ \pi = T_e \Pi \colon \mathfrak{k} \longrightarrow End(V)$$

After equipping V with a K-invariant inner product H,

$$\Pi(K) \subset U(n), \ \pi(\mathfrak{k}) \subset \mathfrak{u}(n)$$

By above, with the symplectic form $\omega = -\operatorname{Im} H$, the action of K on V is Hamiltonian. Similarly, considering the symplectic reduction of $U(1)$ at -1^*, the projective space $\mathbb{P}(V)$ is a symplectic manifold with the Fubini-Study form ω_{FS}. And the induced action of K on $\mathbb{P}(V)$ is Hamiltonian with the moment map $\mu \colon \mathbb{P}(V) \to \mathfrak{k}^*$,

$$\langle \mu([v]), X \rangle = \frac{iH(\pi(X)v, v)}{H(v, v)}, \ \forall \ v \in V \setminus \{0\}, \ X \in \mathfrak{k}$$

In fact, μ can be also viewed as on $V \setminus \{0\}$ for some cases.

3.4 Convexity Theorems

Considering commutative Hamiltonian actions i.e. the Lie group is commutative, the image of the moment map is a convex polytope.

Theorem 1 (Atiyah-Guillemin-Sternberg)**.** *Let (M, ω) be a connected and compact symplectic manifold and T be a commutative compact Lie group. Assume the action of T on M is Hamiltonian with the moment map $\mu \colon M \to \mathfrak{t}^*$. Let fixed point set*

$$M^T = \{m \in M \colon t \cdot m = m, \ \forall \ t \in T\}$$

Then $\mu(M^T)$ is a finite set in $\mathfrak{t}^*$ and

$$\mu(M) = \operatorname{conv} \mu(M^T)$$

Remark. There is a variant version of the above theorem. Let T be a compact commutative and (M, J, ω) be a compact Kähler manifold such that T act M is Hamiltonianian and J-invariant. Let $T_{\mathbb{C}}$ be the complexification of T. Then

$$\mu\left(\overline{T_{\mathbb{C}} \cdot v}\right) = \operatorname{conv}\left\{\overline{T_{\mathbb{C}} \cdot v} \cap M^{T_{\mathbb{C}}}\right\}, \ \forall \ v \in M$$

Example 2. Let $T = (S^1)^n$ act on $\mathbb{C}P^{n-1}$ naturally i.e

$$(e^{i\theta_1}, \cdots, e^{i\theta_n}) \sim \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \in U(n)$$

This action is Hamiltonian with the moment map μ as above example. T is a compact connected Lie group and $\mathbb{C}P^{n-1}$ is connected and compact. Therefore, $\mu(\mathbb{C}P^{n-1})$ is convex. Or more explicilty, for $X = (i\theta_1, \cdots, i\theta_n) \in \mathfrak{t}$ and $z = (z_1, \cdots, z_n) \in \mathbb{C}^n \setminus \{0\}$,

$$\begin{aligned} \langle \mu([z]), X \rangle &= \frac{i \langle Xz, z \rangle}{|z|^2} \\ &= -\frac{1}{|z|^2} \sum_{j=1}^{n} \theta_j \, |z_j|^2 \end{aligned}$$

Therefore, by equipping $U(n)$ with the trace inner product,

$$\mu([z]) = \frac{i}{|z|^2} \left(|z_1|^2, \cdots, |z_n|^2 \right) \in \mathfrak{t}^*$$

If viewing $\mathfrak{t}^* \simeq \mathbb{R}^n$, then

$$\mu\left(\mathbb{C}P^{n-1}\right) = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \colon x_j \geqslant 0, \ \sum_{j=1}^{n} x_j = 1 \right\}$$

Example 3 (Horn's Theorem)**.** Let $\operatorname{Herm}(n)$ be the set of all $n \times n$ Hermitian matrices and $\operatorname{Diag}(n)$ be the set of all diagonal matrices and $HD(n) = \operatorname{Diag}(n) \cap \operatorname{Herm}(n)$

$$\begin{array}{rccc} \pi\colon & \operatorname{Herm}(n) & \longrightarrow & HD(n) \simeq \mathbb{R}^n \\ & [a_{ij}] & \longmapsto & [a_{ii}] \end{array}$$

Then for $D \in HD(n)$ and $M = \{gDg^{-1} \colon g \in U(n)\}$,

$$\pi(M) = \operatorname{conv}\left\{ \sigma D \sigma^{-1} \colon \sigma \in \mathcal{S}_n \right\}$$

Proof. Let $G = U(n)$ then

$$\operatorname{Lie} U(n) = \mathfrak{u}(n) = i \operatorname{Herm}(n) \ \Rightarrow \ \mathfrak{u}^*(n) \simeq \operatorname{Herm}(n)$$

Let $T = U(n) \cap \operatorname{Diag}(n)$ then $\mathfrak{t} = iHD(n)$ and $\mathfrak{t}^*(n) \simeq HD(n)$. Since the coadjoint action of $U(n)$ on $\mathfrak{u}(n)^*$ is

$$\operatorname{Ad}_g^* X = gXg^{-1}$$

after viewing D as an element in $\mathfrak{u}^*(n)$, $M = U(n) \cdot D$ is the coadjoint orbit. And the action of $U(n)$ on M is Hamiltonian with the moment map μ that is the inclusion $U(n) \cdot D \hookrightarrow \operatorname{Herm}(n)$. Then considering the restricted action of T on M, it is also Hamiltonian with the moment map $\tilde{\mu}$

$$\tilde{\mu} \colon M \xhookrightarrow{\ \mu\ } \operatorname{Herm}(n) \xrightarrow{\ \pi\ } HD(n)$$

Therefore, $\tilde{\mu} = \pi|_M$. And clearly

$$M^T = \left\{\sigma D \sigma^{-1} \colon \sigma \in \mathcal{S}_n\right\}$$

Thus by AGS theorem,

$$\tilde{\mu}(M) = \pi(M) = \operatorname{conv}\left\{\sigma D \sigma^{-1} \colon \sigma \in \mathcal{S}_n\right\} \qquad \square$$

For the noncommutative case, it should consider the positive Weyl Chamber. For a compact Lie group K with Lie algebra $\mathfrak{k}$, if choosing a Cartan algebra $\mathfrak{t}$ and a fundamental system of roots Δ, the closed positive Weyl chamber is in $(i\mathfrak{t})^*$ that does not interset the image of the moment map. Therefore, there are two methods to make them compatible. First, by defining the root space as

$$\mathfrak{k}_\alpha = (X \in \mathfrak{k} \colon [H, X] = i\alpha(H)X, \ \forall \ H \in \mathfrak{t})$$

It is well-defined since all eigenvalues of ad_H is pure imaginary. Then the closed positive Weyl chamber denoted by $\mathfrak{t}^*_+$ is in $\mathfrak{k}^*$ that is compatible with the image of μ. Second, by replacing μ by $i\mu$, then the image of μ is in $(i\mathfrak{k})^*$ that is compatible with the Weyl chamber. For example, considering the moment map on $\mathbb{P}(V)$

$$\langle \mu([v]), X \rangle = \frac{iH(Xv, v)}{H(v, v)} = \frac{H(v, (iX)v)}{H(v, v)} =: \langle i\mu([v]), iX \rangle \, , \ \forall \ X \in \mathfrak{k}$$

so if $\mu := i\mu$, for all $Y \in i\mathfrak{k}$,

$$\langle \mu([v]), Y \rangle = \frac{H(v, Yv)}{H(v, v)}$$

Theorem 4 (Kirwan)**.** *Let (M, ω) be a compact and connected symplectic manifold, K be a compact Lie group. The action of K on M is Hamiltonian with the moment map $\mu \colon M \to \mathfrak{k}^*$. Let $\mathfrak{t}^*_+$ be a closed Weyl chamber of a Cartan subalgebra of $\mathfrak{t}$ in $\mathfrak{k}$. Then*

$$\mu(M) \cap \mathfrak{t}^*_+$$

is a convex polytope, called the moment polytope.

For the projective space, there is another statement by Guillemin and Sternberg in [GS84].

Theorem 5 (Guillemin-Sternberg)**.** *Let V be a finite-dimensional $\mathbb{C}$-vector space with an inner product H s.t. a compact Lie group K act V unitarily, i.e. there is a Lie group homomorphism $\Pi \colon K \to U(V)$. And let $\pi = T_e\Pi$.*

Let $L \subset V \setminus \{0\}$ be a complex submanifold that is invariant for the action of K and $\mathbb{C}^*$. Let $M = L/\mathbb{C}^* \subset \mathbb{P}(V)$. Then the induced action of K on $\mathbb{P}(V)$ is Hamiltonian with the moment map $\mu \colon M \to \mathfrak{k}^*$

$$\langle \mu([v]), X \rangle = \frac{iH(\pi(X)v, v)}{H(v, v)}$$

Suppose M is connected. Then $\mu(M) \cap \mathfrak{t}^*_+$ is a convex polytope for a closed Weyl chamber $\mathfrak{t}^*_+$ of a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$.

Remark. For a special case, let $G = K_{\mathbb{C}}$. Then for any $v \in V \backslash \{0\}$, $G \cdot v$ is a complex submanifold of V. Therefore,

$$\mathcal{P}_v = \mu(\overline{G \cdot [v]}) \cap \mathfrak{t}_+^*$$

is a convex polytope. And since $\mu([v]) = \mu(v)$, $\mathcal{P}_v = \mu(\overline{G \cdot v}) \cap \mathfrak{t}_+^*$

Let $G = GL(n, \mathbb{C})$ (or $SL(n, \mathbb{C})$) and $K = U(n)$ (or $SL(n, \mathbb{C})$), so $G = K_{\mathbb{C}}$. And let $U(n)$ act on some vector space linearly (may not be standard) s.t. it is Hamiltonian with moment map μ $(= i\mu)$. Choosing the root system and positive Weyl chamber as the above example, then

$$\mathcal{P}_v = \mu(\overline{G \cdot v}) \cap \mathfrak{t}_+^* = \overline{\{\mathrm{spec}\,(\mu(w)) : w \in G \cdot v\}}$$

Example 6 (Horn's Problem)**.** Let $G = GL(n, \mathbb{C})^3$ act on $V = M(n, \mathbb{C})^{\oplus 2}$ by

$$(g_1, g_2, g_3) \cdot (X, Y) := \left(g_1 X g_3^{-1}, g_2 Y g_3^{-1}\right)$$

Then the induced action on the projective space is Hamiltonian with the moment map is

$$\mu(X, Y) = \frac{\left(XX^\dagger, YY^\dagger, -X^\dagger X - Y^\dagger Y\right)}{\|X\|^2 + \|Y\|^2} \tag{$*$}$$

with respect to the trace inner product and the moment polytope is

$$\mathcal{P} = \{(\mathrm{spec}(A), \mathrm{spec}(B), \mathrm{spec}(-A - B)) : A, B \geqslant 0,\ \ \mathrm{tr}(A) + \mathrm{tr}(B) = 1\}$$

Proof. Firstly, let $U(n)$ act $M(n, \mathbb{C})$ as $g \cdot A := gA$. Clearly, the trace inner product is $U(n)$-invariant. Therefore, by above this action is Hamiltonian with the moment map (defined as $i\mu$)

$$\langle \mu'(A), X \rangle = \mathrm{tr}\left(X^\dagger \mu'(A)\right) = \frac{\langle A, XA \rangle}{2} = \frac{\mathrm{tr}(A^\dagger X^\dagger A)}{2} \ \Rightarrow\ \mu'(A) = \frac{AA^\dagger}{2}$$

Similarly, when $U(n)$ act $M(n, \mathbb{C})$ as $g \cdot A := Ag^{-1}$, the moment map is

$$\mu''(A) = -\frac{A^\dagger A}{2}$$

Then by applying the operations of moment maps talked in above,

$$\tilde{\mu}(X, Y) = \frac{\left(XX^\dagger, YY^\dagger, -X^\dagger X - Y^\dagger Y\right)}{2}$$

So the moment map of the induced action on the projective space is $(*)$. $\square$

4 Kempf-Ness Theorem and Stability

In this section, (M, J, ω) denotes a compact Kähler manifold without boundary, such as $\mathbb{P}(V)$ with the Fubini-Study form and the nature complex structure. K is a compact Lie group acting on M such that this action preserves J and ω, like any compact Lie group acting on $\mathbb{P}(V)$. Moreover, let this action be Hamiltonian with the moment map μ.

Let $G = K_{\mathbb{C}}$ a complexified Lie group of K. So the action of K on M can be extended a holomorphic action of G on M. And for these two actions. define

$$L_m \colon \mathfrak{k} \to T_m M,\ L_m^c \colon \mathfrak{g} \to T_m M$$

as for any $X \in \mathfrak{k}$ and $Z = X + iY \in \mathfrak{g}$,

$$L_m X = X_M(m),\ L_m^c Z = Z_M(m) = L_m X + J L_m Y$$

Equipping K with a bi-invariant Riemannian metric, $\mathfrak{k} \simeq \mathfrak{k}^*$ and so the moment map μ can be viewed as $\mu \colon M \to \mathfrak{k}$. Also equipping M with the compatible Riemannian metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$, $T^*M \simeq TM$. Then it can see for any $m \in M$ and $X \in \mathfrak{k}$,

$$L_m^* = (d\mu)_m J,\ (d\mu)_m^* = J L_m,\ (d\mu)_m L_m X = -[\mu(m), X]$$

where the third equation is by $\langle \mu(m), [X, Y] \rangle = \omega(X_M(m), Y_M(m))$ that is by differentiating the equation

$$\left\langle \mu(m), \mathrm{Ad}_{-\exp(tX)} Y \right\rangle = \left\langle \mathrm{Ad}^*_{\exp(tX)} \mu(m), Y \right\rangle = \langle \mu(\exp(tX) \cdot m), Y \rangle$$

And by above, let G/K equipped with the G-invariant Riemannian s.t. it becomes a complete, connected, and simply connected Riemannian metric and let ∇ be the Levi-Civita connection.

4.1 Kempf-Ness Function

Lemma 1. *Let $x_0, x_1 \in \mu^{-1}(0)$.*

$$x_1 \in G \cdot x_0 \ \Rightarrow\ x_1 \in K \cdot x_0$$

In fact, if $x_1 = \exp(iX)k \cdot x_0$, then $k \cdot x_0 = x_1$ and $X_M(x_1) = 0$.

Proof. Assume $x_1 = \exp(iX)k \cdot x_0$. Let $x(t) = \exp(itX)k \cdot x_0$. Then

$$\begin{aligned} &x(0) = kx_0,\ x(1) = x_1 \\ &\dot{x}(t) = J X_M(x(t)) = J L_x X \end{aligned}$$

So by above identities

$$\left.\frac{d}{dt}\right| \langle \mu(x(t)), X \rangle = \langle (d\mu)_x \dot{x}, X \rangle = \omega(L_x X, \dot{x}) = \omega(L_x X, J L_x X) = \|L_x X\|^2 \geqslant 0$$

Since $\mu(x_0) = \mu(x_1) = 0$,

$$\left.\frac{d}{dt}\right| \langle \mu(x(t)), X \rangle = \left\|L_{x(t)} X\right\|^2 \equiv 0 \ \Rightarrow\ \dot{x}(t) \equiv 0$$

Therefore, $X_M(x_1) = L_{x_1} X = 0$ and $kx_0 = x_1$. □

Corollary 2. *Let $x \in \mu^{-1}(0)$ and $K_x^c = (K_x)_{\mathbb{C}}$ be the complexification of K_x. Then*

$$K_x^c = \{\exp(iX)k \colon k \in K_x,\ X \in \ker L_x\}$$

Lemma 3. *Let $x \in \mu^{-1}(0)$. The following statements are equivalent.*

(1) $(d\mu)_x \colon T_xM \to \mathfrak{k}$ is surjective.

(2) $L_x \colon \mathfrak{k} \to T_xM$ is injective.

(3) $L_x^c \colon \mathfrak{g} \to T_xM$ is injective.

Define the moment squared function as

$$\begin{array}{cccc} f\colon & M & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \frac{1}{2}\left\|\mu(x)\right\|^2 \end{array}$$

For any $x \in M$ and $X \in T_xM$,

$$\langle \nabla f, X\rangle = \langle (d\mu)_x X, \mu(x)\rangle = \omega\left(L_x\mu(x), X\right) = \langle JL_x\mu(x), X\rangle$$

Therefore, the gradient of f is $\nabla f = JL_x\mu(x)$.

Then considering the negative gradient flow $x \colon \mathbb{R} \to M$ of f,

$$\dot{x}(t) = -JL_x\mu(x),\ x(0) = x_0 \tag{$\diamondsuit$}$$

Moreover, if the $g \colon \mathbb{R} \to G$ satifies the ODE

$$g(t)^{-1}\dot{g}(t) = i\mu(x(t)),\ g(0) = e$$

then $x(t) = g(t)^{-1} \cdot x_0$ is a solution of $\diamondsuit$. And thus $x(t) \in G \cdot x_0$.

Theorem 4. *Let $x_0 \in M$ and $x \colon \mathbb{R} \to M$ be a solution of $\diamondsuit$. Then*

$$x_\infty = \lim_{t\to\infty} x(t)$$

exists and $L_{x_\infty}\mu(x_\infty) = 0$ i.e. x_∞ is a critical point of f. Moreover, there are positive C, c, T, ε and $\frac{1}{2} < \alpha < 1$ s.t. for all $t > T$,

$$\begin{aligned} d\left(x(t), x_\infty\right) &\leqslant \int_t^\infty |\dot{x}(s)|\, ds \\ &\leqslant \frac{C}{1-\alpha}\left(f(x(t)) - f(x_\infty)\right)^{1-\alpha} \\ &\leqslant \frac{c}{(t-T)^\varepsilon} \end{aligned}$$

Theorem 5 (Kempf-Ness Function)**.** *Fix $x \in M$, then there is a unique smooth function $\Phi_x \colon G \to \mathbb{R}$ s.t. for any $g \in G$ and $v \in T_gG$ and $k \in K$,*

$$(d\Phi_x)_g(v) = -\left\langle \mu(g^{-1}\cdot x), \mathrm{Im}(T_eL_{g^{-1}}v)\right\rangle,\ \Phi_x(k) = 0$$

and Φ_x is K-invariant.

Remark. This Φ_x is called the lifted Kempf-Ness function. And since it is K-invariant, the induced function

$$\Phi_x \colon G/K \to \mathbb{R}$$

is called the Kempf-Ness function.

Skech of Proof. Define $v_x \in \mathrm{Vect}(G)$ and 1-form α_x on G,

$$v_x(g) = -T_e L_g i\mu(g^{-1} \cdot x), \ \ \alpha_x(g)(v) = -\left\langle \mu(g^{-1} \cdot x), \mathrm{Im}(T_e L_{g^{-1}} v) \right\rangle$$

for any $g \in G$ and $v \in T_g G$. And it can see

$$\alpha_x(g)(v) = \langle v_x, v \rangle$$

Let $\psi_x \colon G \to G \cdot x$ by $\psi_x(g) = g^{-1} \cdot x$. Then

$$T_g \psi_x(v_x(g)) = \nabla f\left(\psi_x(g)\right)$$

There exists a $\Phi_x \colon G \to \mathbb{R}$ s.t.

$$d\Phi_x = \alpha_x, \ \ \Phi_x|_K = 0$$

and thus $\nabla \Phi_x = v_x$ and $T_g \psi_x(\nabla \Phi_x) = \nabla f\left(\psi_x(g)\right)$. Moreover, for a fixed $g \in G$, let $\Phi_x|_{Kg} : Kg \to \mathbb{R}$. Since for any $v = T_e L_g X$ where $X \in \mathfrak{k}$,

$$\alpha_x(g)(v) = -\left\langle \mu(g^{-1} \cdot x), \mathrm{Im}(X) \right\rangle = 0 \ \Rightarrow \ d\, \Phi_x|_{Kg} = 0$$

Φ_x is invariant on Kg so Φ_x is K-invariant. □

Theorem 6 (Properties)**.** *Let $N = G/K$ and the Kempf-Ness function $\Phi_x \colon N \to \mathbb{R}$.*

(1) With the Riemannian metric defined as above on N, Φ_x is geodesically convex.

(2) The critical sets of Φ_x is a closed and connected submanifold of N, given by

$$\mathrm{Crit}\, \Phi_x = \left\{[g] \in N \colon \mu\left(g^{-1} \cdot x\right) = 0\right\}$$

(3) If $\mathrm{Crit}\, \Phi_x \neq \varnothing$, then $|\Phi_x|$ attaches its minima and every negative gradient flow of Φ_x converges exponentially to a critical point.

(4) Let $g \colon \mathbb{R} \to G$ be a smooth curve and $\gamma = \pi \circ g \colon \mathbb{R} \to N$. Then γ is a negative gradient flow of Φ_x if and only if $g(t)$ satisfies

$$\mathrm{Im}\left(g^{-1}\dot{g}\right) = \mu\left(g^{-1} \cdot x\right)$$

(5) If $\mathrm{Crit}\, \Phi_x \neq \varnothing$ and $\{g_n\} \subset G$ s.t. $\sup_n \Phi_x([g_n]) < \infty$, then there is a sequence (h_n) in the identity component of G_x s.t. $(h_n g_n)$ has a convergent subsequence.

Example 7. Let V be an n-dimensional complex vector space with an inner product s.t. the action of K on V is unitary and let the action of $G = K_{\mathbb{C}}$ on V be the extension. Therefore, assume $K \subset U(n)$ and $G \subset GL(n, \mathbb{C})$. As above, the induced action of K on $\mathbb{P}(V)$ is Hamiltonian with the moment map

$$\langle \mu([v]), X \rangle = \frac{i \langle X \cdot v, v \rangle}{\|v\|^2}, \ \forall\, v \in V \setminus \{0\}\,, \ X \in \mathfrak{k}$$

also extending the action of G on $\mathbb{P}(V)$. Then the lifted Kempf-Ness function $\Phi_{[v]} \colon G \to \mathbb{R}$ is

$$\Phi_{[v]}(g) = \frac{1}{2}\left(\log \left\|g^{-1} \cdot v\right\|^2 - \log \|v\|^2\right)$$

Proof. Clearly, $\Phi_{[v]}(k) = 0$ for all $k \in K$ and $\Phi_{[v]}$ is K-invariant Let $g \in G$ and $X \in T_g G$. Let $\varphi(t)$ be the integral curve of v starting at g then

$$\left.\frac{d}{dt}\right|_{t=0} \varphi(t)^{-1}\varphi(t) = 0 \;\Rightarrow\; \left.\frac{d}{dt}\right|_{t=0} \varphi(t)^{-1} = -g^{-1}Xg^{-1}$$

Therefore,

$$\begin{aligned}
\left(d\Phi_{[v]}\right)_g X &= \frac{\langle -g^{-1}Xg^{-1}\cdot v, g^{-1}\cdot v\rangle}{\|g^{-1}\cdot v\|^2} \\
&= -\frac{\langle i\,\mathrm{Im}\,(g^{-1}X)\,g^{-1}\cdot v, g^{-1}\cdot v\rangle}{\|g^{-1}\cdot v\|^2} - \frac{\langle \mathrm{Re}\,(g^{-1}X)\,g^{-1}\cdot v, g^{-1}\cdot v\rangle}{\|g^{-1}\cdot v\|^2} \\
&= -\frac{\langle i\,\mathrm{Im}\,(g^{-1}X)\,g^{-1}\cdot v, g^{-1}\cdot v\rangle}{\|g^{-1}\cdot v\|^2} \\
&= -\left\langle \mu(g^{-1}\cdot[v]), \mathrm{Im}\left(g^{-1}X\right)\right\rangle
\end{aligned}$$

where the third equality is by $\left(d\Phi_{[v]}\right)_g Y = 0$ whenever $Y = gZ$ for some $Z \in \mathfrak{k}$. $\square$

Remark. For convenience, let the lifted Kemf-Ness function $F_{[v]}$ be defined as

$$F_{[v]}(g) = \frac{1}{2}\log\|g\cdot v\|^2$$

In fact, $F_{[v]}(g) = \Phi_{[v]}(g^{-1}) + c$. The properties of $\Phi_{[v]}$ are also valid for $F_{[v]}$. Moreover, it can see for $X \in i\mathfrak{k}$,

$$(dF_{[v]})_e X = \langle \mu([v]), X\rangle$$

And if let $F_{[v]} \colon G/K \to \mathbb{R}$, then the geodesic of X starting at $[e]$ is $\pi(\exp(itX))$.

$$\left.\frac{d}{dt}\right|_{t=0} F_{[v]}\left(\pi(\exp(itX))\right) = \left.\frac{d}{dt}\right|_{t=0} F_{[v]}\left(\exp(itX)\right) = \langle \mu([v]), X\rangle$$

If let $\mu = i\mu$ and $H = iX \in i\mathfrak{k}$ and lifting μ from on $\mathbb{P}(V)$ to on V, for $F_v(g) = \frac{1}{2}\log\|g\cdot v\|^2$

$$\left.\frac{d}{dt}\right|_{t=0} F_v\left(\exp(tH)\right) = \langle \mu(v), H\rangle$$

It is the definition of moment map used in [Bür+19].

4.2 More Properties of Moment Map

In the next subsections, let $K \subset U(n)$ and $G = K_{\mathbb{C}} \subset GL(n, \mathbb{C})$ be matrix groups. Let

$$\mathscr{T}^c := \left\{\zeta \in \mathfrak{g}\setminus\{0\} : \;\exists\, g \in G,\; g\zeta g^{-1} \in \mathfrak{k}\right\}$$

Then the *μ-weight* of $(x,\zeta) \in M \times \mathscr{T}^c$ is

$$w_\mu(x,\zeta) := \lim_{t\to\infty} \langle \mu(\exp(it\zeta)\cdot x), \mathrm{Re}(\zeta)\rangle$$

Remark. The existence of w is because for any $x \in M$ and $\zeta \in \mathscr{T}^c$,

$$x^{\pm} = \lim_{t \to \pm\infty} \exp(it\zeta) \cdot x$$

exist by the Morse theory. Moreover, $\zeta_M(x^{\pm}) = 0$. In particular, if $\zeta = X \in \mathfrak{k}$, then

$$w_\mu(x, X) = \lim_{t \to \infty} \frac{\Phi_x(\exp(-itX))}{t}$$

where Φ_x is the lifted Kempf-Ness function.

By applying the properties of μ-weight, it can prove many important results related to the moment map.

Theorem 1. *Let $x \in M$ and $\zeta = X + iY \in \mathscr{T}^c$ s.t. $\zeta_M(x) = 0$. Then $\langle \mu(x), Y \rangle = 0$, $\|X\| > \|Y\|$ and*

$$\frac{\langle \mu(x), X \rangle^2}{\|X\|^2 - \|Y\|^2} \leqslant \|\mu(g \cdot x)\|^2, \ \forall \ g \in G$$

Remark. More generally, for any $x \in M$ and $X \in \mathfrak{g} \backslash \{0\}$ and $g \in G$,

$$\frac{-w_\mu(x, X)}{\|X\|} \leqslant \|\mu(g \cdot x)\|$$

Corollary 2. *Let $x_0 \in M$ be a critical point of the moment squared funcion f i.e. $L_{x_0}\mu(x_0) = 0$. Then*

$$\|\mu(x_0)\| \leqslant \|\mu(g \cdot x_0)\|, \ \forall \ g \in G$$

In the above mention, if $x_0, x_1 \in \mu^{-1}(0)$, then $x_1 \in G \cdot x_0$ implies $x_1 \in K \cdot x_0$. This statement can be more general.

Theorem 3. *Let x_0 and x_1 be critical points of the moment squared function f. Then*

$$x_1 \in G \cdot x_0 \ \Rightarrow \ x_1 \in K \cdot x_0$$

Theorem 4. *Let $x_0 \in M$ and $x \colon \mathbb{R} \to M$ be the solution of $\diamondsuit$ with $x_\infty = \lim_{t\to\infty} x(t)$. Then*

$$\|\mu(x_\infty)\| = \inf_{g \in G} \|\mu(g \cdot x_0)\|$$

Remark. Note that x_∞ is a critical point of the moment squared function and $x_\infty \in \overline{G \cdot x_0}$. And the G-orbit of x_0 determines the G-orbit of x_∞ that is, in fact, the K-orbit of x_∞.

More generally, the infimum points are in the same K-orbit.

Theorem 5. *Let $x_0 \in M$ and $m = \inf_{g\in G} \|\mu(g \cdot x_0\|$. Then*

$$x, y \in \overline{G \cdot x_0} \ \text{s.t.} \ \mu(x) = \mu(y) = m \ \Rightarrow \ y = K \cdot x$$

Remark. It shows that for any $x \in G \cdot x_0$ s.t. $\|\mu x\| = m$, x is a critical point of the moment squared function f and is a limit point of the negative gradient flow of f starting at some point in $G \cdot x_0$.

The infimum of the norm of moment map along the G-orbits can be characterized. For $x \in \mathrm{Crit}(f)$, let

$$W^s(K \cdot x) = \left\{ y_0 \in M \colon y(t) \text{ of } \diamondsuit \text{ starting at } y_0 \text{ s.t. } \lim_{t \to \infty} y(t) \in K \cdot x \right\}$$

Corollary 6. *The following statements holds.*

(1) $M = \bigcup_{x\in \mathrm{Crit}(f)} W^s(K\cdot x)$.

(2) For $x \in \mathrm{Crit}(f)$ *and* $y_0 \in M$,

$$y_0 \in W^s(K\cdot x) \;\Leftrightarrow\; x \in \overline{G\cdot y_0},\;\; \|\mu(x)\| = \inf_{g\in G} \|\mu(g\cdot y_0)\|$$

(3) For any $x \in \mathrm{Crit}(f)$, $W^s(K\cdot x)$ *is the union of* G*-orbits.*

4.3 Stability

Definition 1. An element $x \in M$ is called

(1) μ-unstable if $\overline{G\cdot x} \cap \mu^{-1}(0) = \varnothing$.

(2) μ-semistable if $\overline{G\cdot x} \cap \mu^{-1}(0) \neq \varnothing$.

(3) μ-polystable if $G\cdot x \cap \mu^{-1}(0) \neq \varnothing$.

(4) μ-stable if $G\cdot x \cap \mu^{-1}(0) \neq \varnothing$ and G_x is discrete.

And let M^{us}, M^{ss}, M^{ps} and M^s be the corresponding sets.

Firstly, these points can be characterized by using the negative gradient flow of moment squared map.

Theorem 2. *Let* $x_0 \in M$ *and* $x(t)$ *be the solution of* $\diamondsuit$ *and* $x_\infty = \lim_{t\to\infty} x(t)$.

(1) $x_0 \in M^{ss}$ *if and only if* $\mu(x_\infty) = 0$.

(2) $x_0 \in M^{ps}$ *if and only if* $\mu(x_\infty) = 0$ *and* $x_\infty \in G\cdot x_0$.

(3) $x_0 \in M^s$ *if and only if* G_{x_∞} *is discrete.*

And M^{ss} *and* M^s *are open in* M.

Also, the Kempf-Ness function can be applied to characterize the stability.

Theorem 3. *Let* $x \in M$ *and* Φ_x *be the Kempf-Ness function.*

(1) $x \in M^{ss}$ *if and only if* Φ_x *is bounded below.*

(2) $x \in M^{ps}$ *if and only if* Φ_x *has a critical point.*

(3) $x \in M^s$ *if and only if* Φ_x *is bounded below and proper.*

Now considering a special case related to the geometric invariant theory, let $G = K_{\mathbb{C}} \subset GL(n,\mathbb{C})$ act a complex vector space V with a K-invariant inner product. Also, let the induced action of K and G act on $\mathbb{P}(V)$ with the moment map μ as in above mention.

Definition 4. Let $v \in V\setminus\{0\}$.

(1) v is called unstable if $0 \in \overline{G\cdot v}$.

(2) v is called semistable if $0 \notin \overline{G \cdot v}$.

(3) v is called polystable if $G \cdot v = \overline{G \cdot v}$.

(4) v is called stable if $G \cdot v = \overline{G \cdot v}$ and G_v is discrete.

Theorem 5 (Kempf-Ness)**.** *Let $x = [v] \in \mathbb{P}(V)$. Considering G acting on V and the Hamiltonian action of K on $\mathbb{P}(V)$ and the induced action of G on $\mathbb{P}(V)$, then*

(1) v is unstable if and only if $x = [v]$ is μ-unstable.

(2) v is semistable if and only if $x = [v]$ is μ-semistable.

(3) v is polystable if and only if $x = [v]$ is μ-polystable.

(4) v is stable if and only if $x = [v]$ is μ-stable.

Remark. Therefore, by applying the above results

$$0 \notin \overline{G \cdot v} \;\Leftrightarrow\; 0 \notin \overline{G \cdot x} \;\Leftrightarrow\; \mu(x_\infty) = 0 \;\Leftrightarrow\; \Phi_x \text{ bounded below}$$

5 Scaling Problem and Examples

5.1 Invariant Theory

Let $G = K_{\mathbb{C}}$ be a reductive Lie group acting on an n-dimensional complex space V with a K-invariant inner product, and thus assume $K \subset U(n)$ and $G \subset GL(n, \mathbb{C})$.

The invariant theory is to consider the induced action of G on the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$. Let $(x_1, \cdots, x_n)$ be the coordinate of V and denote the polynomial ring by $\mathbb{C}[V]$. The action of G on $\mathbb{C}[V]$ is defined as

$$g \cdot f(x_1, \cdots, x_n) \mapsto h(x_1, \cdots, x_n) := f\left(g^{-1} \cdot (x_1, \cdots, x_n)\right)$$

Then the invariant polynomial ring is

$$\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] \colon g \cdot f = f, \ \forall\, g \in G\}$$

that is a subalgebra of $\mathbb{C}[V]$. Moreover, by Hilbert, $\mathbb{C}[V]^G$ is a finitely generated subalgebra i.e. there are polynomials $f_1, \cdots, f_l \in \mathbb{C}[V]$ s.t.

$$\mathbb{C}[V]^G = \mathbb{C}[f_1, \cdots, f_l] := \{p\,(f_1, \cdots, f_l) : p \in \mathbb{C}[y_1, \cdots, y_l]\}$$

The invariant polynomials can be used to separate the orbit closure of action fo G on V. In fact, for any $v, w \in V$,

$$\overline{G \cdot v} \cap \overline{G \cdot w} \neq \varnothing \ \Leftrightarrow\ f(v) = f(w)\ \forall\, f \in \mathbb{C}[V]^G$$

Let the set, called null cone, be defined as

$$\mathcal{N} := \left\{v \in V \colon 0 \in \overline{G \cdot v}\right\}$$

i.e. the set of all unstable points. By above,

$$\mathcal{N} = \left\{v \in V \colon f(v) = 0,\ \forall\, f \in \mathbb{C}[V]^G\right\} = \{v \in V \colon f_1(v) = \cdots = f_l(v) = 0\}$$

Theorem 1 (Hilbert-Mumford Criterion)**.** *For any $v \in V$,*

$$v \in \mathcal{N} \ \Leftrightarrow\ \exists\, \lambda \colon \mathbb{C}^* \to G \textit{ algebriac homomorphism s.t. } \lim_{t \to \infty} \lambda(t) \cdot v = 0$$

Remark. This λ is called a one-parameter subgroup (PSG). For some special cases, the 1-PSG is explicit.

(1) If $G = (\mathbb{C}^*)^n$, any 1-PSG has the form

$$\lambda(t) = (t^{\alpha_1}, \cdots, t^{\alpha_n})$$

for some $\alpha_1, \cdots, \alpha_n \in \mathbb{Z}$.

(2) If $G = GL(n, \mathbb{C})$ or some matrix Lie group, any 1-PSG has the form

$$\lambda(t) = S\, \mathrm{diag}\,(t^{\alpha_1}, \cdots, t^{\alpha_n})\, S^{-1}$$

for some $\alpha_1, \cdots, \alpha_n \in \mathbb{Z}$ and $S \in GL(n, \mathbb{C})$.

Example 2. Let $G = \mathcal{S}_n$ be the symmetry group acting on V. Then

$$\mathbb{C}[V]^{\mathcal{S}_n} = \mathbb{C}[e_1, \cdots, e_n]$$

where e_k is called k-th elementary symmetric polynomials given by

$$e_k(x_1, \cdots, x_n) = \sum_{1 \leqslant i_1 < \cdots i_k \leqslant n} x_{i_1} \cdots x_{i_k}$$

the $\mathcal{N} = \{0\}$.

Example 3. Let $G = GL(n, \mathbb{C})$ act on $V = M(n, \mathbb{C})$ by conjugtion, $g \cdot A = gAg^{-1}$. Considering the 1-PSG

$$\lambda(t) = \begin{pmatrix} t^{\alpha_1} & & \\ & \ddots & \\ & & t^{\alpha_n} \end{pmatrix}$$

with integers $\alpha_1 \geqslant \cdots \geqslant \alpha_n$. In fact, any 1-PSG is conjugate to this form. Then for any $A = [a_{kj}] \in V$,

$$\lambda(t) \cdot A = \lambda(t) A \lambda(t)^{-1} = \left[t^{\alpha_k - \alpha_j} a_{kj}\right]$$

Therefore,

$$\lim_{t \to \infty} \lambda(t) \cdot A = 0 \;\Leftrightarrow\; A \text{ is strictly upper triangular}$$

Then by Hilbert-Mumford criterion,

$$A \in \mathcal{N} \Leftrightarrow A \text{ is conjugate to a strictly upper triangular.} \Leftrightarrow A \text{ is nilpotent.}$$

Let $X = [x_{ij}]$. Then X is nilpotent if and only if

$$\det(tI - X) = t^n - f_1(\mathbf{x}) t^{n-1} + \cdots + (-1)^n f_n(\mathbf{x}) = t^n$$

where $\mathbf{x} = (x_{11}, \cdots, x_{nn})$. This equivalent to $f_1(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0$. Therefore,

$$\mathbb{C}[V]^G = \mathbb{C}[f_1, \cdots, f_n]$$

Example 4. Let $G = SL(n, \mathbb{C}) \times SL(n, \mathbb{C})$ act on $V = M(n, \mathbb{C})$ by

$$(A, B) \cdot H := AHB$$

Firstly, if $H \in \mathcal{N}$, then there are $\{A_k\}$ and $\{B_k\}$ in $SL(n, \mathbb{C})$ s.t. $A_k H B_k \to 0$. By the continuity of det,

$$\det(A_k H B_k) = \det(A_k) \det(H) \det(B_k) = \det H \to 0$$

So $\det H = 0$ and H is singular. Conversely, if H is singular, then there is a $S \in GL(n, \mathbb{C})$ s.t. the last row of $S^{-1}H$ is 0. So if let the 1-PSG

$$\lambda(t) = \left(S \operatorname{diag}\left(t, \cdots, t, t^{-n-1}\right) S^{-1}, I\right)$$

then $\lambda(t) \cdot H \to 0$. By Hilbert-Mumford criterion, $H \in \mathcal{N}$. Therefore, $H \in \mathcal{N}$ if and only if H is singular. And the invariant polynomial ring is

$$\mathbb{C}[V]^G = \langle \det X \rangle$$

where $X = [x_{ij}]$ are variables.

The *null cone problem* is given $v \in V$, determine if $0 \in \mathcal{N}$ or if v is unstable for the action of G on V. The dual problem is that given $v \in V$, determine if v is semistable. Then by Kempf-Ness theorem, when considering the Hamiltonian action of K on $\mathbb{P}(V)$ with the moment map μ and the induced action of G on $\mathbb{P}(V)$, it is equivalent to determine if v is μ-semistable, i.e if

$$\exists\ [w] \in \overline{G \cdot [v]} \text{ s.t. } \mu([w]) = 0$$

or simply, if there is a $w \in \overline{G \cdot v}$ s.t. $\mu(w) = 0$. It is called the *scaling problem*.

5.2 Matrix Scaling

Let $G = ST(n) \times ST(n)$ act on $V = M(n, \mathbb{C})$ by the left-right action i.e.

$$(A, B) \cdot H = AHB$$

where

$$ST(n) = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} : z_1, \cdots, z_n \in \mathbb{C}^*,\ \prod_{j=1}^{n} z_j = 1 \right\}$$

(1) **Invariant Theory:** Any 1-PSG $\lambda\colon \mathbb{C}^* \to G$ has the form

$$\lambda(t) = \left(\operatorname{diag}\left(t^{\alpha_1}, \cdots, t^{\alpha_n}\right), \operatorname{diag}\left(t^{\beta_1}, \cdots, t^{\beta_n}\right)\right)$$

where $\alpha_j, \beta_j \in \mathbb{Z}$ for any j and

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = 0$$

For any $H = [h_{kj}]$,

$$\lambda(t) \cdot H = \left[t^{\alpha_k + \beta_j} h_{kj}\right]$$

Therefore, by Hilbert-Mumford criterion,

$$\begin{aligned} H \in \mathcal{N} \ \Leftrightarrow\ & \exists\ \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n \in \mathbb{Z} \\ & \sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = 0 \\ & \text{s.t. } \alpha_k + \beta_j > 0,\ \forall\ (k, j) \in \operatorname{supp} H \\ \Leftrightarrow\ & \operatorname{supp} H \text{ has no perfect matching} \end{aligned}$$

where the bipartite graph $\operatorname{supp} H = \{(k, j) \in [n] \times [n] \colon h_{kj} \neq 0\}$. And thus

$$\mathbb{C}[V]^G = \left\langle x_{1\sigma(1)} \cdots x_{n\sigma(n)} \colon \sigma \in \mathcal{S}_n \right\rangle$$

where $X = [x_{ij}]$ are variables.

(2) **Geometric Invariant Theory:** $ST(n)$ is a commutative Lie group with Lie algebra $\mathfrak{st}(n)$, where

$$\mathfrak{st}(n) = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} : z_1, \cdots, z_n \in \mathbb{C},\ \sum_{j=1}^n z_j = 0 \right\}$$

and let

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} : \theta_1, \cdots, \theta_n \in \mathbb{R},\ \sum_{j=1}^n \theta_j = 0 \right\}$$

It can see $\mathfrak{st}(n) = \mathfrak{t}_{\mathbb{C}}$. Moreover, if let

$$T(n) = \left\{ \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} : \theta_1, \cdots, \theta_n \in \mathbb{R},\ \sum_{j=1}^n \theta_j = 0 \right\}$$

then $\operatorname{Lie} T(n) = \mathfrak{t}$ and $T(n)$ is a compact Lie group. Therefore, $ST(n) = T(n)_{\mathbb{C}}$ is a reductive Lie group. And if let $K = T(n) \times T(n)$ be the compact Lie group with the Lie algebra $\mathfrak{k} = \mathfrak{t} \times \mathfrak{t}$, then $G = K_{\mathbb{C}}$ is a reductive Lie group with Lie algebra $\mathfrak{st}(n) \times \mathfrak{st}(n)$. Moreover, the trace inner product on $\mathfrak{t}$ is a bi-invariant Riemannian metric so it is also valid on $\mathfrak{t} \times \mathfrak{t}$.

Equipping $V = M(n, \mathbb{C})$ with the trace inner product, it is clear that the action of K on V is invriant for this inner product. And thus the action of K on V is Hamiltonian with the moment map $\hat{\mu} \colon V \to (\mathfrak{t} \times \mathfrak{t})^*$. Firstly, considering the action of $T(n)$ on $M(n, \mathbb{C})$ by $A \cdot Z = AZ$ with the moment map $\tilde{\mu}_r$, since $T(n) \subset U(n)$ and the moment map of $U(n)$ on $M(n, \mathbb{C})$ is

$$\tilde{\mu}_r'(A) = \frac{iAA^\dagger}{2} \;\Rightarrow\; \tilde{\mu}_r(A) = \tilde{\mu}_r'(A)\big|_{\mathfrak{t}} = \frac{i}{2} \operatorname{diag}(r_1(A), \cdots, r_n(A))$$

where $r_k(A) = \sum_{j=1}^n |a_{kj}|^2$. Similarly, for $T(n)$ acting on $M(n, \mathbb{C})$ by $A \cdot Z = ZA$ with the moment map

$$\tilde{\mu}_l(A) = \frac{i}{2} \operatorname{diag}(l_1(A), \cdots, l_n(A))$$

where $l_j(A) = \sum_{k=1}^n |a_{kj}|^2$. Therefore,

$$\hat{\mu}(A) = \frac{i}{2} (\mathbf{r}_A, \mathbf{l}_A)$$

where $\mathbf{r}_A = \operatorname{diag}(r_1(A), \cdots, r_n(A))$ and $\mathbf{l}_A = \operatorname{diag}(l_1(A), \cdots, l_n(A))$. Then the moment map μ of the induced action of K on $\mathbb{P}(V)$ is

$$\mu(A) = \frac{i}{\|A\|^2} (\mathbf{r}_A, \mathbf{l}_A)$$

Clearly $\|A\|^2 = \sum_k r_k(A) = \sum_j l_j(A)$, so the image of μ is a convex polytope in $\mathfrak{k}^*$. If viewing $\mu \in (i\mathfrak{k})^* \simeq i\mathfrak{k}$, then

$$\mu(A) = \frac{1}{\|A\|^2} \left(\mathbf{r}_A - \frac{\|A\|^2}{n} I_n, \mathbf{l}_A - \frac{\|A\|^2}{n} I_n \right)$$

So if $\mu(A) = 0$, then $|A|^2 = \left[|a_{ij}|^2 \right]$ is doubly stochastic-scalable.

Remark. As above mention, μ also can be calculated by differentiating the Kempf-Ness function along the geodesic (in G/K) i.e.

$$\frac{d}{dt}\bigg|_{t=0} F_A(e^{tH}) = \frac{d}{dt}\bigg|_{t=0} \log \left\| e^{tH} \cdot A \right\| = \langle \mu(A), H \rangle$$

5.3 Operator Scaling

Let $G = SL(n, \mathbb{C}) \times SL(n, \mathbb{C})$ acts $V = M(n, \mathbb{C})^{\oplus m}$ by

$$(A, B) \cdot (H_1, \cdots, H_m) := (AH_1B, \cdots, AH_mB)$$

(1) **Invarian Theory:** Any 1-PSG $\lambda \colon \mathbb{C}^* \to G$ has the form

$$\lambda(t) = \left(S^{-1} \operatorname{diag}(t^{\alpha_1}, \cdots, t^{\alpha_n}) S, T^{-1} \operatorname{diag}\left(t^{\beta_1}, \cdots, t^{\beta_n}\right) T\right)$$

where $S, T \in SL(n, \mathbb{C})$ and $\alpha_j, \beta_j \in \mathbb{Z}$ for any j s.t.

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = 0$$

Similar as matrix scaling, $(H_1, \cdots, H_m) \in \mathcal{N}$ if and only if there are $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n \in \mathbb{Z}$ with

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = 0$$

s.t.

$$\alpha_k + \beta_j > 0, \ \forall \ (k, j) \in \operatorname{supp}(SH_1T^{-1}, \cdots, SH_mT^{-1})$$

where

$$\operatorname{supp}(SH_1T^{-1}, \cdots, SH_mT^{-1}) = \bigcap_{l=1}^{m} \operatorname{supp}(SH_lT^{-1})$$

for some basis change S, T. So it means there is a common Hall's block in H_l's i.e. there is a subspace U in $\mathbb{C}^n$ s.t.

$$\dim H_l(U) < \dim U, \ \forall \ l = 1, 2, \cdots, m$$

Therefore,

$$H = (H_1, \cdots, H_m) \in \mathcal{N} \ \Leftrightarrow \ H \text{ is rank-decreasing.}$$

And thus the invariant polynomial ring is

$$\mathbb{C}[V]^G = \left\langle \sum_{l=1}^{m} X_l \otimes D_l \colon \ \forall \ d \in \mathbb{N}, \ D_l \in M(d, \mathbb{C}) \right\rangle$$

where $X_l = [x_{l,k,j}]$ are variables.

(2) **Geometric Invariant Theory:** By above example, $SL(n, \mathbb{C}) = SU(n)_{\mathbb{C}}$ with a maximal torus $T \subset SU(n)$ with Lie algbebra $\mathfrak{t}$ and let the Cartan subalgebra $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$, then

$$T = \left\{ \operatorname{diag}\left(e^{i\theta_1}, \cdots, e^{i\theta_n}\right) : \theta_j \in \mathbb{R}, \ \sum_{j=1}^{n} \theta_j = 0 \right\}$$

$$\mathfrak{t} = \left\{ \operatorname{diag}\left(i\theta_1, \cdots, i\theta_n\right) : \theta_j \in \mathbb{R}, \ \sum_{j=1}^{n} \theta_j = 0 \right\}$$

$$\mathfrak{h} = \left\{ \operatorname{diag}\left(z_1, \cdots, z_n\right) : z_j \in \mathbb{C}, \ \sum_{j=1}^{n} z_j = 0 \right\}$$

with the root system as $R = \{\pm(\varepsilon_k - \varepsilon_j) \colon 1 \leqslant k < j \leqslant n\}$. Then the closed Weyl chamber is

$$\overline{C(\Delta)} = \{\operatorname{diag}(\theta_1, \cdots, \theta_n) : \theta_k \geqslant \theta_{k+1}\}$$

if choosing the fundamental system as $\Delta = \{\varepsilon_k - \varepsilon_{k+1} \colon 1 \leqslant k \leqslant n-1\}$.

Let $K = SU(n) \times SU(n)$. Then $G = K_{\mathbb{C}}$. The corresponding Lie structures are just obtained by the Cartesian product.

Clearly, the trace inner product on V is K-invariant. So similar to the last example talked in section **3.4**, the moment map of K acting on $\mathbb{P}(V)$

$$\mu \colon V \to (i\mathfrak{k})^*$$

is, for $H = (H_1, \cdots, H_m) \in V$,

$$\mu(H) = \frac{1}{\|H\|^2} \left(\sum_l H_l H_l^\dagger, \sum_l H_l^\dagger H_l \right)$$

and the moment polytope is

$$\mathcal{P} = \mu(V) \cap \overline{C(\Delta)} = \{(\operatorname{spec}(A), \operatorname{spec}(B)) : A, B \geqslant 0, \operatorname{tr}(A) = \operatorname{tr}(B) = 1\}$$

Also after viewing $(i\mathfrak{k})^* \simeq i\mathfrak{k}$,

$$\mu(H) = \frac{1}{\|H\|^2} \left(\sum_l H_l H_l^\dagger - \frac{\|H\|^2}{n} I_n, \sum_l H_l^\dagger H_l - \frac{\|H\|^2}{n} I_n \right)$$

Therefore, $\mu(H) = 0$ means H is scalable.

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