Chapter 1

Symbolic Logic

Mathematical logic is introduced to give precise meaning to mathematical statements. There are certain rules used in mathematical logic to distinguish between valid and invalid arguments. The aim of this chapter is to introduce correct procedure of studying mathematical arguments. Mathematical logic has numerous applications in computer science. The rules of the logic are used in the design of computer circuits, construction of computer programs, the verification of the correctness of the programs and in many other ways.

We begin the study of logic by designing the idea of propositions.

Definition: A *Proposition* is a statement which can be classified as true or false. The following statements are propositions.

(1) Chennai is the capital of Tamilnadu.

(2) $2 + 3 = 6$

(3) There are two solutions to the equation $x^2 - x - 6 = 0$

(4) If x is an integer then x is a positive integer.

(5) Every integer is the sum of three perfect squares.

The following are not propositions.

(1) Read this carefully

(2) Go for it.

(3) What are you doing ?

(4) $x + y = z$

(1), (2) and (3) are not propositions since they are not statements.

(4) is not a proposition since it is neither true nor false.

To describe the proposition of mathematical logic we need a way of symbolizing propositions. It has become a tradition of sorts in elementary propositional logic to use lower case letters starting from p and q, r, s... to represent propositions when we wish to study logic in symbolic form. The reason for the use of p is that p is the first letter in the word *proposition*.

The truth value of a proposition is denoted by T if it is a true proposition. The truth value of a proposition is denoted by F, if it is a false proposition.

Logical Operations

There are several ways in which we commonly combine simple propositions into compound ones. In order to produce compound propositions from simple ones we use words *and or, not, if*. We now precisely define each one of them and introduce its standard symbol. With the exception of negation, all othe. operators act on pairs of propositions. Since each proposition has two possible truth values, there are four ways that truth can be assigned to two propositions. In defining the effect that a logical operator has on two propositions, the result must be specified for all four cases. The most convenient way of defining this is with a truth table which we will illustrate in defining operators.

Definition : Let p and q be two propositions. The proposition "p and q", denoted by $p \wedge q$ is true when both p and q are true and is false otherwise. The proposition $p \wedge q$ is called the **conjunction** of p and q.

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note that there are four rows in this truth table one row for each possible combination of truth values for the propositions p and q.

Example : Suppose p is 'today is a sunday' and q is 'it is a rainy day'. The statement $p \wedge q$ is the proposition "today is a sunday and it is a rainy day". This proposition is true on rainy sundays and false on any day not a sunday and sundays when it is not rainy.

Definition 2 : Let p and q be two propositions. The Proposition "p or q", denoted by $p \vee q$ is the proposition that is false when p and q are both false and true otherwise. The proposition $p \vee q$ is called **disjunction.**

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, *i.e.*, in an inclusive way. A disjunction is true when either of the two propositions in it is true or when both are true.

Example 1: This program runs or there was an error in keying in the day. What we meant here is that atleast one of the two propositions has occurred.

Example 2 : Suppose p is 'it is a sunday' and q is 'it is a rainy day'. The proposition $p \vee q$ is true on any day which is sunday or rainy day (including rainy sundays). It is false on days that are not sundays when it does not rain.

Definition 3 : Let p be the proposition. "It is not the case that p" is another proposition, called the negation of p. The negation of p is denoted by $\sim p \left(\text{or } \neg p\right)$.

The proposition $\sim p$ is read as "not p".

Truth Table for $\sim p$

p	$\sim p$
T	F
F	T

Example

p is the proposition "today is a sunday". The negation $\sim p$ is that "today is not a sunday". The negation can also be expressed as "today is not a sunday". The negation can also be expressed as "it is not true that today is a sunday".

Conditional and Biconditional Operators

Definition : Let p and q be two propositions. The implication $p \rightarrow q$ is the proposition that is false when p is true and q is false and true otherwise. In this implication p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Truth Table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The propositions $p \rightarrow q$ arise in many mathematical reasoning and a wide variety of terminology is used to express $p \rightarrow q$. The following are some of the common usages of the implication :

(1) If p, then q
(2) P Implies q
(3) p only if q
(4) p is sufficient for q
(5) q if p
(6) q whether p
(7) q is necessary for p.

Examples

(1) If ABC is a right triangle with right angle at A then $a^2 = b^2 + c^2$
(2) If $x = y$ and $y = z$, then $x = z$.
(3) If ABC is an isosceles triangle with $a = b$ then angles A and B are equal.
(4) If I pass the final examination, then I will graduate.
(5) I will be going to movies provided only my car starts.
(6) If an integer is a multiple of 4, then it is even.
(7) $x^2 = y$ is a necessary condition for $x>y$.
(8) The fact that a polygon is a square is a sufficient condition that it is a rectangle.

Definition : Let p and q be propositions. The biconditional $p \leftrightarrow q$ is the proposition that it is true when p and q have the same truth values and is false otherwise.

Truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The bicondition $p \leftrightarrow q$ is true when both $p \to q$ and $q \to p$ are true. Because of this terminology "p if and only if q" *is* used for this bicondition. Other ways of expressing the biconditional $p \leftrightarrow q$ are : (1) "p is necessary and sufficient for q" and (2) "if p then q" and conversely.

Examples

(1) Two triangles are congruent if and only if the corresponding sides are congruent.

(2) Two lines are parallel if and only if they have the same slope.

(3) The necessary and sufficient condition for a finite monoid to be a group is that its identity element is its only idempotent.

Converse, Inverse and Contrapositive

Definition : Let p and q be propositions. $p \rightarrow q$ is a conditional proposition. The proposition $q \rightarrow p$ is called the converse of the proposition $p \rightarrow q$. The proposition $\sim p \rightarrow \sim q$ is called the inverse of the proposition. The proposition $\sim q \rightarrow \sim p$ is called the contrapositive of the proposition $p \rightarrow q$.

Truth Tables

p	q	Conditional $p \rightarrow q$	Converse $q \rightarrow p$	Inverse $\sim p \rightarrow \sim q$	Contrapositive $\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Note : Conditional proposition and its converse are not logically equivalent. But a conditional proposition $p \rightarrow q$ and contrapositive $\sim q \rightarrow \sim p$ are logically equivalent.

Example : p : A is square

q : A is a rectangle

$p \rightarrow q$: If A is a square then A is a rectangle.

$q \rightarrow p$: If A is a rectangular then A is a square

$p \rightarrow q$ is true but $q \rightarrow p$ is false.

Example : p : x^2 is odd

q : x is odd

$\sim p$: x^2 is even

$\sim q$: x is even

We show that the contrapositive $\sim q \rightarrow \sim p$ is true (*i.e.*,) if x is even. Let x be even. Then $x = 2n$ when n is an integer.

$x^2 = 4n^2 = 2(2n^2)$ which is even.

Since the contrapositive $\sim q \to \sim p$ is true, the original proposition $p \to q$ is also true.

Tautology and Contradiction

Definition : A compound proposition that is always true, no matter what the truth values of the propositions that occur is called a *tautology.*

A compound proposition that is always false is called *contradiction.* A proposition that is neither tautology nor a contradiction is called *contingency.*

Definition : The propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $\Leftrightarrow$ is used to denote p and q are logically equivalent.

One way to determine whether two propositions are logically equivalent is to use truth table. The propositions p and q are logically equivalent only if the columns giving their truth values agree.

Example 1

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

$p \vee \sim p$ is a tautology. Therefore p and $\sim p$ are logically equivalent.

Example 2

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

$p \wedge \sim p$ is a contradiction.

Example 3 : We will show that $\sim (p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the last two columns

$\sim (p \wedge q) \equiv \sim p \vee \sim q$

$\therefore \sim (p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent.

Algebra of Propositions

Proportions under the relation of logical equivalence, satisfy various laws which are given below :

Laws of Algebra of Propositions

(1) Idempotent Laws

$p \vee p \equiv p$

$p \wedge p \equiv p$

(2) Associative Laws

$p \vee (q \vee r) \equiv (p \vee q) \vee r$

$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

(3) Commutative Laws

$p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

(4) Distributive Laws

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(5) De Morgan's Laws

$\sim (p \vee q) \equiv \sim p \wedge \sim q$

$\sim (p \wedge q) \equiv \sim p \vee \sim q$

(6) Identity Laws

$p \vee f \equiv p$

$p \wedge t \equiv p$

$p \vee t \equiv t$

$p \wedge f \equiv f$

(7) Complement Laws

$p \vee \sim p \equiv t$

$p \wedge \sim p \equiv f$

$\sim \sim p \equiv p$

$\sim t \equiv f$

$\sim f \equiv t$

Note : In the above table t and f denote variables which are restricted to the truth values True and False respectively. Let us prove some of these propositions.

Distributive Law 1 : $p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r)$

p	q	r	$q\wedge r$	$p\vee(q\wedge r)$	$(p\vee q)$	$(p\vee r)$	$(p\vee q)\wedge(p\vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

From Columns (5) and (8)

$$p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r)$$

Distributive Law 2 : $p\wedge(q\vee r)\equiv(p\wedge q)\vee(p\wedge r)$

p	q	r	$q\vee r$	$p\wedge(q\vee r)$	$(p\wedge q)$	$(p\wedge r)$	$(p\wedge q)\vee(p\wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

From columns (5) and (8) $p\wedge(q\vee r)\equiv(p\wedge q)\vee(p\wedge r)$

De Morgan's Laws

(a) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(b) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(a)

p	q	$(p \vee q)$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

From columns (4) and (7) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(b)

p	q	$(p \wedge q)$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

From columns (4) and (7) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Arguments

An Argument is an assertion that a given set of propositions $p_1, p_2, \ldots p_n$ yields another proposition q. The propositions $p_1, p_2, \ldots p_n$ are called premises and the proposition q is called conclusion. The argument is denoted by $p_1, p_2, \ldots p_n \vdash q$. The symbol $\vdash$ is called turnstile.

An argument is true if q is true whenever the premises $p_1, p_2, \ldots p_n$ are true. Otherwise the argument is false. If an argument is true it is called a valid argument. If an argument is false it is called a *fallacy*.

Procedure for testing the validity of an argument

An argument is $p_1, p_2, \ldots, p_n \vdash q$. The propositions $p_1, p_2, \ldots, p_n$ are true simultaneously if and only if the proposition $p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n$ is true. Thus the argument $p_1, p_2, \ldots, p_n \vdash q$ is valid if and only if q is true whenever $p_1 \wedge p_2 \wedge p_n$ is true. In other words, the proposition $p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q$ is a tautology.

Hence, in order to prove that argument $p_1, p_2, \ldots, p_n \vdash q$ is valid we have to show that $p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q$ is a tautology. Hence, we can use the truth table to test the validity of an argument. Alternatively, in the truth table in those rows where premises are true, the conclusion is also true we say that the argument is valid.

Example: Test the validity of the argument:

I will either get an A, in this course or I will not graduate. If I don't graduate, I will go to my army. I got an A. Thus I will not go to the army.

Solution: Let p : I will get an A

q : I will graduate

r : I will go to the army

The premises are $p \vee \sim q, \sim q \rightarrow r, p$

The conclusion is $\sim r$

The argument is $p \vee \sim q, \sim q \rightarrow r, p \vdash \sim r$

In order to test the validity of the argument we have to show that $(p \vee \sim q) \wedge (\sim q \rightarrow r) \wedge p \rightarrow \sim r$ is a tautology.

p	q	r	$\sim q$	$\sim r$	$p \vee \sim q$	$\sim q \rightarrow r$	$(p \vee \sim q) \wedge$ $(\sim q \rightarrow r) \wedge p$	$(p \vee \sim q) \wedge$ $(\sim q \rightarrow r) \wedge p$ $\rightarrow \sim r$
T	T	T	F	F	T	T	T	F
T	T	F	F	T	T	T	T	T
T	F	T	T	F	T	T	T	F
T	F	F	T	T	T	F	F	T
F	T	T	F	F	F	T	F	T
F	T	F	F	T	F	T	F	T
F	F	T	T	F	T	T	F	T
F	F	F	T	T	T	F	F	T

Since all the truth values in the last column are not *T*, argument is not valid.

Aliter

p	q	r	$\sim q$	$p \vee \sim q$	$(\sim q \rightarrow r)$	$\sim r$
T	T	T	F	T	T	F
T	T	F	F	T	T	T
T	F	T	T	T	T	F
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	F	F	F	F	F	T
F	T	T	T	T	T	F
F	F	F	T	T	F	T

Premises are given in cloumns (5), (6) and (1). The conclusion is given in column (7). In order that the argument is valid, whenever the premises are true the conclusion should be true. However in the first row the premises are true, but the conclusion is false. Therefore the argument is not valid.

Example 1: Let p be the proposition "He is tall" and q be the proposition. "He is handsome". Give a simple verbal sentence which describes each of the following propositions.

(i) $p \wedge q$ (ii) $p \wedge \sim q$ (iii) $\sim(\sim p \vee q)$

(iv) $\sim p \wedge \sim q$ (v) $p \vee (\sim p \wedge q)$ (vi) $\sim(\sim p \vee \sim q)$

Solution:

(i) He is tall and handsome

(ii) He is tall and not handsome

(iii) It is not true that he is not tall and handsome

(iv) He is not tall and not handsome

(v) He is tall, or he is short and handsome

(vi) It is not true that he is not táll or not handsome.

Example 2: Classify the following as propositions or non-propositions.

(1) The popûlation of India is 100 crores

(2) April 14 is the Tamil New Year's day

(3) $2 + 3 = 6$

(4) What are you doing?

(5) Bring that book.

Solution:

(1) Proposition

(2) Proposition

(3) Proposition

(4) Not a proposition

(5) Not a proposition

Example 3: Classify the following as propositions or non-propositions

(a) a is greater than b

(b) Girls are smarter

(c) There is a greatest prime number

(d) Are yc : going to school?

(e) Bring water from the river.

(f) Delhi is the capital of India

(g) Tiruchi is the capital of Tamilnadu

(h) May is a winter month in Tamilnadu

Solution:

(a) Not a proposition

(b) Not a proposition

(c) Proposition

(d) Not a proposition

(e) Not a proposition

(f) Proposition

(g) Proposition

(h) Proposition

Example 4: Let p be the proposition "It is cold" and q be the proposition "It is raining". Write each of the following propositions in symbolic form using p and q.

(a) It is not cold

(b) It is cold and raining

(c) It is cold or it is not raining

(d) It is not cold and it is not raining

(e) It is raining or it is not cold

(f) It is not true that it is not raining.

Solution:

(a) $\sim p$

(b) $p \wedge q$

(c) $p \vee \sim q$

(d) $\sim p \wedge \sim q$

(e) $q \vee \sim p$

(f) $\sim(\sim q)$

Example 5: Write each of the following in symbolic form where p "Satish is poor" and q is "Satish is happy".

(a) Satish is poor but not happy

(b) Satish is neither rich nor happy

(c) Satish is either rich or unhappy

(d) Satish is poor or else he is rich and unhappy

(e) Satish is either happy or poor

(f) It is false that Satish is not rich and not happy.

Solution:

(a) $p \wedge \sim q$

(b) $p \wedge \sim q$

(c) $\sim p \vee \sim q$

(d) $p \vee (\sim p \wedge \sim q)$

(e) $q \vee p$

(f) $\sim (p \wedge \sim q)$

Example 6: Let p be the proposition "Mahesh reads Hindu", q be the proposition "Mahesh reads Economic Times" and r be the proposition Mahesh reads Discrete Mathematics".

Write each of the following propositions in symbolic form.

(a) Mahesh reads Hindu but not Economic Times.

(b) Mahesh reads Hindu and Economic Times but not Discrete Mathematics.

(c) It is not true that Mahesh reads Hindu but not Discrete Mathematics.

(d) It is not true that Mahesh reads Economic Times but not Discrete Mathematics and Hindu.

Solution:

(a) $p \wedge \sim q$

(b) $p \wedge q \wedge \sim r$

(c) $\sim (p \wedge \sim r)$

(d) $\sim q \wedge \sim (r \wedge p)$

Example 7: Let p be the statement Rama speaks English, q be the statement Rama speaks Tamil. Give a simple verbal statements for each of the following.

(i) $p \vee q$

(ii) $p \wedge q$

(iii) $p \wedge \sim q$

(iv) $\sim (\sim p)$

(v) $\sim p \vee \sim q$

(vi) $\sim (\sim p \wedge \sim q)$

Solution:

(i) Rama speaks English or Tamil.

(ii) Rama speaks English and Tamil.

(iii) Rama speaks English and not Tamil

(iv) It is false that Rama does not speak English.

(v) Rama does not speak English or Tamil.

(vi) It is not true that Rama does not speak English and does not speak Tamil.

Example 8: Determine the truth value of the following statements.

(i) $3 + 3 = 6$ and Chennai is the capital of Tamil Nadu.

(i) Delhi is the capital of India or Chennai is the Capital of Tamil Nadu.

(iii) It is not true that Tirupathi is in Tamil Nadu.

(iv) It is false that London is in France

(v) It is false that $3 + 3 = 6$ and London is in France.

(vi) $3 + 2 = 5$ and $1 + 4 = 6$

Solution:

(i) True

(ii) True

(iii) True

(iv) True

(v) True

(vi) False

Example 9: Find the truth table of $\sim p \wedge q$

Solution:

p	$\sim p$	q	$\sim p \wedge q$
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F

Example 10: Find the truth table for $\sim(p \vee \sim q)$

Solution:

p	q	$\sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

Example 11: Find the truth table for

(i) $p\wedge(q\vee r)$ (ii) $(p\wedge q)\vee(p\wedge r)$

Solution:

p	q	r	$q\vee r$	$p\wedge(q\vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

p	q	r	$p\wedge q$	$p\wedge r$	$(p\wedge q)\vee(p\wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Note: From the last columns of the two tables we note that

$$p\wedge(q\vee r)=(p\wedge q)\vee(p\wedge r)$$

Example 12: Find the truth tables for

(i) $p\vee(q\wedge r)$ and $(p\vee q)\wedge(p\vee r)$ and shows that $p\vee(q\wedge r)=(p\vee q)\wedge(p\vee r)$

Solution: (i)

p	q	r	$q\wedge r$	$p\vee(q\wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(ii)

p	q	r	$p\vee q$	$p\vee r$	$(p\vee q)\wedge(p\vee r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

From the last columns of the above two tables.

$p\vee(q\wedge r)=(p\vee q)\wedge(p\vee r)$

Example 13: Find the truth table for each of the following:

(i) $p\vee\sim q$

(ii) $\sim p\wedge\sim q$

(iii) $\sim(\sim p\wedge q)$

(iv) $\sim(\sim p\vee\sim q)$

(v) $(p\wedge\sim q)\vee r$

(vi) $\sim p\vee(q\wedge\sim r)$

(vii) $(p\vee\sim r)\wedge(q\vee\sim r)$

(viii) $\sim(p\vee\sim q)\wedge(\sim p\vee r)$

Solution: (i) Truth table for $p \vee \sim q$

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(ii) Truth table for $\sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(iii) Truth table for $\sim(\sim p \wedge q)$

p	q	$\sim p$	$\sim p \wedge q$	$\sim(\sim p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

(iv) Truth table for $\sim(\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(\sim p \vee \sim q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(v) Truth table for $(p \wedge \sim q) \vee r$

p	q	r	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \vee r$
T	T	T	F	F	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	T	F	F	F	F
F	F	T	T	F	T
F	F	F	T	F	F

(vi) Truth table for $\sim p \vee (q \wedge \sim r)$

p	q	r	$\sim p$	$\sim r$	$q \wedge \sim r$	$\sim p \vee (q \wedge \sim r)$
T	T	T	F	F	F	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	F	F
F	T	T	T	F	F	T
F	T	F	T	T	T	T
F	F	T	T	F	F	T
F	F	F	T	T	F	T

(vii) Truth table for $(p \vee \sim r) \wedge (q \vee \sim r)$

p	q	r	$\sim r$	$p \vee \sim r$	$q \vee \sim r$	$(p \vee \sim r) \wedge (q \vee \sim r)$
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	F	T	F
F	T	F	T	T	T	T
F	F	T	F	F	F	F
F	F	F	T	T	T	T

(viii) Truth table for $(p \vee \sim q) \wedge (\sim p \vee r)$

p	q	r	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim p \vee r$	$(p \vee \sim q) \wedge (\sim p \vee r)$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	F	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	T	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Example 14: Construct a truth table for the following

(i) $[(p \to q) \wedge q] \to q$

(ii) $(p \leftrightarrow q) \leftrightarrow r$

(iii) $p \leftrightarrow (q \to r)$

(iv) $(p \wedge \sim p) \to q$

(i) Truth table for $[(p \to q) \wedge q] \to q$

p	q	$p \to q$	$(p \to q) \wedge q$	$[(p \to q) \wedge q] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	T

(ii) Truth table for $(p \leftrightarrow q) \leftrightarrow r$

p	q	r	$p \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	T	T
F	F	F	T	F

(iii) Truth table for $p \leftrightarrow (q \rightarrow r)$

p	q	r	$q \rightarrow r$	$p \leftrightarrow (q \rightarrow r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	F
F	T	F	F	T
F	F	T	T	F
F	F	F	T	F

(iv) Truth table for $(p \wedge \sim p) \rightarrow q$

p	q	$\sim p$	$p \wedge \sim p$	$(p \wedge \sim p) \rightarrow q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Example 15: Show that $p \rightarrow q$ and $q \rightarrow p$ are not logically equivalent.

Solution:

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

From the last two columns $p \rightarrow q$ and $q \rightarrow p$ are not logically equivalent.

Example 16: Show that each of the following are equivalent to $p \rightarrow q$

(a) $\sim p \vee q$ (b) $\sim q \rightarrow \sim p$ (c) $\sim (p \wedge \sim q)$

Solution: Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth table for $\sim p \vee q$

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Truth table for $\sim q \to \sim p$

p	q	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Truth table for $\sim (p \wedge \sim q)$

p	q	$\sim q$	$p \wedge \sim q$	$\sim (p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

From the last columns of the above 4 tables

$\sim p \vee q$, $\sim q \to \sim p$ and $\sim (p \wedge \sim q)$ are equivalent to $p \to q$

Example 17: Prove that the proposition $p \vee \sim (p \wedge q)$ is a tautology.

Solution:

Truth table for $p \vee \sim (p \wedge q)$

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since all the truth values in the last column are T; $p \vee \sim (p \wedge q)$ is tautology

Example 18: Verify that $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

Solution:

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since all the truth values in the last column are F, $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction.

Example 19: Prove the De Morgan's laws

(i) $\sim(p \wedge q) \equiv \sim p \vee \sim q$ (ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Solution: (i)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

From columns (4) and (7)

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

(ii)

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

From columns (4) and (7) $\sim(p \vee q) = \sim p \wedge \sim q$

Example 20: Prove $\sim(p \vee q) \vee (\sim p \wedge q) \equiv \sim p$ by constructing the appropriate truth table.

Solution:

p	q	$p\vee q$	$\sim(p\vee q)$	$\sim p$	$(\sim p\wedge q)$	$\sim(p\vee q)\vee(\sim p\wedge q)$	$\sim p$
T	T	T	F	F	F	F	F
T	F	T	F	F	F	F	F
F	T	T	F	T	T	T	T
F	F	F	T	T	F	T	T

From the last two columns

$\sim(p\vee q)\vee(\sim p\wedge q)\equiv\sim p$

Example 21: Verify that $(p\wedge q)\to p\vee q$ is a tautology.

Solution:

p	q	$p\wedge q$	$p\vee q$	$p\wedge q\to p\vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

From the last column, $p(\wedge q)\to(p\vee q)$ is a tautology.

Example 22: Prove by truth table

$p\to(q\vee r)\equiv(p\to q)\wedge(p\to r)$

Solution:

p	q	r	$q\wedge r$	$p\to(q\wedge r)$	$p\to q$	$p\to r$	$(p\to q)\wedge(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

From columns (5) and (8) $p\to(q\wedge r)\equiv(p\to q)\wedge(p\to r)$

Example 23: Prove that $(p \to q) \wedge (q \to p) \equiv p \leftrightarrow q$

Solution:

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

From column (5) and (6)

$(p \to q) \wedge (q \to p) \equiv p \leftrightarrow q$

Example 24: Determine the truth value of $(p \leftrightarrow \sim q) \leftrightarrow (q \to p)$

Solution:

p	q	$\sim q$	$p \leftrightarrow \sim q$	$q \to p$	$(p \leftrightarrow \sim q) \leftrightarrow q \to p$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

Example 25: Verify the truth table the following

(i) $\sim (p \to q) \equiv p \wedge \sim q$ (ii) $\sim (p \leftrightarrow q) \equiv \sim p \leftrightarrow q \equiv \sim p \leftrightarrow q$

Solution: (i)

p	q	$p \to q$	$\sim (p \to q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From column (4) and (6) $\sim (p \to q) \equiv p \wedge \sim q$

(ii)

p	q	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$	$\sim p$	$\sim p \leftrightarrow q$	$\sim q$	$p \leftrightarrow \sim q$
T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	F	T	T	T	F	T
F	F	T	F	T	F	T	F

From columns (4), (6) and (8),

$\sim(p \leftrightarrow q) \equiv \sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$

Example 26: Find the truth table for

(i) $(\sim p \vee q) \to q$ (ii) $q \leftrightarrow (\sim q \wedge p)$

Solution: (i)

p	q	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \to p$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

(ii)

p	q	$\sim q$	$\sim q \wedge p$	$q \leftrightarrow (\sim q \wedge p)$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	T

Example 27: Find the truth table for each of the following

(i) $(p \leftrightarrow \sim q) \to (\sim p \wedge q)$

(ii) $(\sim q \vee p) \leftrightarrow q \sim p$

Solution: (i)

p	q	$\sim q$	$\sim p$	$p \leftrightarrow \sim q$	$\sim p \wedge q$	$(p \leftrightarrow \sim q) \rightarrow (\sim p \wedge q)$
T	T	F	F	F	F	T
T	F	T	F	T	F	F
F	T	F	T	T	T	T
F	F	T	T	F	F	T

(ii)

p	q	$\sim q$	$\sim q \vee p$	$\sim p$	$q \rightarrow \sim p$	$(\sim q \vee p) \leftrightarrow q \rightarrow \sim p$
T	T	F	T	F	F	F
T	F	T	T	F	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T

Example 28: Find the truth table for the following proposition

$$p \wedge (\sim q \rightarrow p) \wedge \sim [(p \leftrightarrow \sim q) \rightarrow (q \vee \sim p)]$$

Solution:

p	q	$\sim p$	$\sim q$	$\sim q \rightarrow p$	$p \leftrightarrow \sim q$	$q \vee \sim p$	$(p \leftrightarrow \sim q) \rightarrow (q \vee \sim p)$	$p \wedge (\sim q \rightarrow p)$	$\sim \left[\begin{matrix}(p \leftrightarrow \sim q) \\ \rightarrow (q \vee \sim p)\end{matrix}\right]$	$p \wedge (\sim q \rightarrow p) \wedge \sim \left[\begin{matrix}(p \leftrightarrow \sim q) \\ \rightarrow (q \vee \sim p)\end{matrix}\right]$
T	T	F	F	T	F	T	T	T	F	F
T	F	F	T	T	T	F	F	T	T	T
F	T	T	F	T	T	T	T	F	F	F
F	F	T	T	F	F	T	T	F	F	F

Example 29: Find the truth table of the proposition

$$[q \leftrightarrow (r \rightarrow \sim p)] \vee [(\sim q \rightarrow p) \rightarrow r]$$

Solution:

p	q	r	$\sim p$	$\sim q$	$(r \to \sim p)$	$q \leftrightarrow (r \to \sim p)$	$(\sim q \to p)$	$(\sim q \to p) \leftrightarrow r$	$q \leftrightarrow (r \to \sim p) \vee (\sim q \to p) \leftrightarrow r$
T	T	T	F	F	F	F	T	T	T
T	T	F	F	F	T	T	T	F	T
T	F	T	F	T	F	T	T	T	T
T	F	F	F	T	T	F	T	F	F
F	T	T	T	F	T	T	T	T	T
F	T	F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	F	F	F
F	F	T	T	T	T	F	F	F	F

Example 30: Prove that $(p \wedge q) \to r \equiv (p \to r) \vee (q \to r)$

Solution:

p	q	r	$p \wedge q$	$(p \wedge q \to r)$	$(p \to r)$	$(q \to r)$	$(p \to r) \vee (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

From columns (5) and (8)

$p \wedge q \to r \equiv (p \to r) \vee (q \to r)$

Example 31: Prove that $p \to (q \to r) \equiv (p \wedge \sim r) \to \sim q$

Solution:

p	q	r	$p\wedge q$	$(p\wedge q)\to r$	$p\to r$	$q\to r$	$(p\to r)\vee(q\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

From columns (5) and (8)

$(p\wedge q)\to r \equiv (p\to r)\vee(q\to r)$

Example 32: Prove that $p\to(q\to r) \equiv (p\wedge\sim r)\to\sim q$

Solution:

p	q	r	$q\to r$	$p\to(q\to r)$	$\sim r$	$p\wedge\sim r$	$\sim q$	$(p\wedge\sim r)\to\sim q$
T	T	T	T	T	F	F	F	T
T	T	F	F	F	T	T	F	F
T	F	T	T	T	F	F	T	T
T	F	F	T	T	T	T	T	T
F	T	T	T	T	F	F	F	T
F	T	F	F	T	T	F	F	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T	T

From columns (5) and (9)

$p\to(q\to r) \equiv (p\wedge\sim r)\to\sim q$

Example 33: Let p denote "It is cold" and q denote "it rains". Write down the following statements in symbolic form.

(a) It rains only if it is cold

(b) A necessary condition for *it is cold* is that it rains

(c) A sufficient condition for it to be cold is that it rains.

(d) Wherever it rains it is cold.

(e) It never rains when it is cold.

Solution:

(a) $q \rightarrow p$

(b) $p \rightarrow q$

(c) $q \rightarrow p$

(d) $q \rightarrow p$

(e) $p \rightarrow \sim q$

Example 34: Let p be the proposition "He is rich" and q is the proposition "He is happy". Write down each of the following in symbolic form.

(a) If he is rich then he is happy.

(b) He is neither rich nor happy.

(c) It is necessary to be poor in order to be happy.

(d) To be poor is to be unhappy.

(e) Being rich is a sufficient condition for being happy.

(f) Being rich is a necessary condition to be happy.

(g) One is never happy when one is rich.

(h) He is poor only if he is happy.

(i) To be rich means to be the same as to be happy.

(f) He is poor or else he is both rich and happy.

Solution:

(a) $p \rightarrow q$

(b) $\sim p \wedge \sim q$

(c) $q \rightarrow \sim p$

(d) $\sim p \leftrightarrow \sim q$

(e) $p \rightarrow q$

(f) $q \rightarrow p$

(g) $p \rightarrow \sim q$

(h) $\sim p \rightarrow q$

(i) $p \leftrightarrow q$

(j) $\sim p \vee (p \wedge q)$

Example 35: Write down the negation of each of the following propositions.

(a) If he studies he will pass the examinations.

(b) He swims if and only if the water is warm.

(c) If it snows then he does not drive his car.

(d) If the stock price falls then unemployment rises.

(e) He has blond hair if and only if he has blue eyes.

(f) If Rama is rich then Ravi and Roy are happy.

(g) Mahesh speaks English or Hindi if he speaks Tamil.

Solution:

(a) p: He studies

q: He will pass the examination.

If he studies he will pass the examination.

$p \rightarrow q$

The negation is $\sim(p \rightarrow q)$

$\sim(p \rightarrow q) = p \wedge \sim q$

$\therefore$ The negational proposition is

He studies and he will not pass the examination.

(b) p: He swims

q: Water is warm

He swims if and only if the water is warm

(*i.e.,*) $p \leftrightarrow q$

The negation is $\sim(p \leftrightarrow q)$

$\equiv p \leftrightarrow \sim q$

$\equiv -p \leftrightarrow q$

(*i.e.,*) He does not swim if and only if the water is warm.

(c) p: It snows

q: He drives car

The given proposition is $p \rightarrow \sim q$

The negation is $\sim(p \rightarrow \sim q)$

$\equiv p \wedge \sim\sim q$

$= p \wedge q$

It snows and he drives the car.

(d) p: Stock price falls

q: Unemployment rises

The given proposition $p \rightarrow q$

The negation $\sim(p \rightarrow q)$

$\equiv p \wedge \sim q$

(*i.e.*,) The stock price falls and unemployment does not rise.

(e) p: He has blond hair

q: He has blue eyes

He has blond hair if and only if he has blue eyes.

$p \leftrightarrow q$

The negation is $\sim(p \leftrightarrow q)$

$\equiv p \wedge \sim q$

He has blond hair but does not have blue eyes.

(f) p: Rama is rich

q: Ravi is happy

r: Roy is happy

The given proposition $p \rightarrow (q \wedge r)$

The negation is $\sim[p \rightarrow (q \wedge r)]$

$\equiv p \wedge \sim(q \wedge r)$

(*i.e.*,) Rama is rich and Ravi and Roy are not happy.

(g) p: Mahesh speaks English

q: Mahesh speaks Hindi

r: Mahesh speaks Tamil

The given proposition is $(p \vee q) \rightarrow r$

The negation is $\sim[(p \vee q) \leftrightarrow r]$

$\equiv (p \vee q) \wedge \sim r$

Mahesh speaks English or Hindi but not Tamil.

Example 36: Determine the contrapositive of the following propositions:

(a) If Ram is a poet then he is poor

(b) Only if Ravi studies he will pass the test

(c) If x is less than zero then x is not positive.

Solution:

For $p \to q$, the contrapositive is $\sim q \to \sim p$.

(a) p: Ram is a poet

q: Ram is poor

The given proposition is $p \to q$

The contrapositive is $\sim q \to \sim p$

(*i.e.*,) If Ram is not poor then he is not a poet.

(b) p: Ravi passes the test

q: Ravi studied

The given proposition is $p \to q$

The contrapositive is $\sim q \to \sim p$

If Ravi does not study he will not pass the test.

(c) p: x is less than zero

q: x is positive

The given proposition is $p \to \sim q$

The contrapositive is $\sim\sim q \to \sim p$

(*i.e.*,) $q \to \sim p$

(*i.e.*,) If x is positive then x is not less than zero.

Example 37: Write the inverse, converse and contrapositive for the following:

(a) If the weather is cold then it will snow.

(b) Being able to type is sufficient to learn word processing.

(c) If you do not practise you will never learn how to play your horn.

(d) I will keep this job if I get a raise in salary.

Solution:

(a) p: Weather is cold

q: It will snow

The given proposition is $p \to q$

The inverse is $\sim p \to \sim q$

(*i.e.*,) If the weather is not cold then it will not snow.

The converse is $q \to p$

If it snows then the weather is cold.

The contrapositive is $\sim q \to \sim p$

If it will not snow then the weather is not cold.

(b) p: able to type

q: learn word processing

The given statement is $p \to q$

Converse is $q \to p$

(*i.e.,*) If he learns word processing then he will be able to type.

Inverse: $\sim p \to \sim q$

If he is not able to type then he will not learn word processing.

Contrapositive: $\sim q \to \sim p$

If he will not learn word processing then he will not be able to type.

(c) p: You practise

q: You play horn

The given proposition is $\sim p \to \sim q$

Converse $\sim q \to \sim p$

If you never practise then you will never play horn.

Inverse $p \to q$

If you practise then you will play horn

Contrapositive $q \to p$

If you play horn then you will practise.

(d) p: I will keep the job

q: I will get a pay rise in salary

The given proposition $q \to p$

The *converse* $p \to q$

If I will keep the job then I will get a pay raise in salary.

The *inverse* : $\sim q \to \sim p$

If I will not get a pay rise then I will not keep the job.

The *contrapositive* $\sim p \to \sim q$

If I will not keep the job then I will not get a pay rise.

Example 38: Use the algebra of propositions to simplify the following:

(i) $\sim(\sim p \wedge \sim q)$

(ii) $\sim(\sim p \vee \sim q)$

(iii) $\sim(\sim p \vee \sim q)$

Solution:

(i) $\sim(\sim p \wedge \sim q) \equiv \sim\sim p \vee \sim\sim q$ De Morgan's law

$\equiv p \vee q$

(ii) $\sim(\sim p \vee q) \equiv \sim\sim p \wedge \sim q$ De Morgan's law

$\equiv p \wedge \sim q$

(iii) $\sim(\sim p \vee \sim q) \equiv \sim\sim p \wedge \sim\sim q$ (De Morgan's law)

$\equiv p \wedge q$

Example 39: Prove that $p \vee (p \wedge q) \equiv p$ by using the algebra of proposition.

Solution:

$p \vee (p \wedge q) \equiv (p \wedge t) \vee (p \wedge q)$ Identity law

$\equiv p \wedge (t \vee q)$ Distributive law

$\equiv p \wedge t$ Identity law

$\equiv p$ Identity law

Example 40: Prove that $\sim(p \vee q) \vee (\sim p \wedge q)$ by using the laws of algebra of proposition.

Solution:

$\sim(p \vee q) \vee (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ De Morgan's law

$\equiv \sim p \wedge (\sim q \vee q)$ Distributive law

$\equiv \sim p \wedge t$ Complement law

$\equiv \sim p$ Identity law

Example 41: Prove that $p \wedge (p \vee q) \equiv p$ by using the algebra of propositions.

Solution:

$p \wedge (p \vee q) \equiv (p \vee t) \wedge (p \vee q)$ Identity law

$\equiv p \vee (t \wedge q)$ Distributive law

$\equiv p \vee t$ Identity law

$\equiv p$ Identity law

Example 42: Prove that $(p\wedge q)\vee\sim p\equiv\sim p\vee q$ by using algebra of propositions.

Solution:

$$(p\wedge q)\vee\sim p$$

$\equiv\sim p\vee(p\wedge q)$ Commutative law

$\equiv(\sim p\vee p)\wedge(\sim p\wedge q)$ Distributive law

$\equiv t\wedge(\sim p\wedge q)$ Complement law

$\equiv\sim p\wedge q$ Identity law

Example 43: Prove that $p\wedge(\sim p\vee q)\equiv p\wedge q$

Solution:

$$p\wedge(\sim p\vee q)$$

$\equiv(p\wedge\sim p)\vee(p\wedge q)$ Distributive law

$\equiv t\vee(p\wedge q)$ Complement law

$\equiv p\wedge q$ Identity law

Example 44: Test the validity of the argument.

If a man is a bachelor he is unhappy.

If a man is unhappy he dies young.

Therefore bachelors die young.

Solution:

Let p: a man is a bachelor

q: a man is unhappy

r: a man dies young

The given argument is $p\rightarrow q, q\rightarrow r \vdash p\rightarrow r$

In order to prove that this argument is valid we have to show that $(p\rightarrow q)\wedge(q\rightarrow r)\rightarrow(p\rightarrow r)$ is a tautology.

p	q	r	$p\to q$	$q\to r$	$(p\to q)\wedge(q\to r)$	$(p\to r)$	$(p\to q)\wedge(q\to r)\to(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the last column $(p\to q)\wedge(p\to r)\to(p\to r)$ is a tautology. Hence the given argument is valid.

Example 45: Test the validity of the argument $p\leftrightarrow q, q\vdash p$

Solution:

In order to show that the argument is valid we have to show that $(p\leftrightarrow q)\wedge q\to p$ is a tautology.

p	q	$p\leftrightarrow q$	$(p\leftrightarrow q)\wedge q$	$(p\leftrightarrow q)\wedge q\to p$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

Since $(p\leftrightarrow q)\wedge q\to p$ is a tautology, the given argument is valid.

Example 46: Test the validity of the argument $p\to q, \sim q\vdash \sim p$

Solution:

In order to show that the given argument is valid we have to show that $(p\to q)\wedge(\sim q)\to\sim p$ is a tautology.

p	q	$p\to q$	$\sim q$	$(p\to q)\wedge(\sim q)$	$\sim p$	$(p\to q)\wedge(\sim q)\to\sim p$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Since $(p\to q)\wedge(\sim q)\to\sim p$ is a tautology the given argument is valid.

p	q	r	$p\to q$	$q\to r$	$(p\to q)\wedge(q\to r)$	$(p\to r)$	$(p\to q)\wedge(q\to r)\to(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the last column $(p\to q)\wedge(p\to r)\to(p\to r)$ is a tautology. Hence the given argument is valid.

Example 45: Test the validity of the argument $p\leftrightarrow q, q\vdash p$

Solution:

In order to show that the argument is valid we have to show that $(p\leftrightarrow q)\wedge q\to p$ is a tautology.

p	q	$p\leftrightarrow q$	$(p\leftrightarrow q)\wedge q$	$(p\leftrightarrow q)\wedge q\to p$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

Since $(p\leftrightarrow q)\wedge q\to p$ is a tautology, the given argument is valid.

Example 46: Test the validity of the argument $p\to q, \sim q\vdash\sim p$

Solution:

In order to show that the given argument is valid we have to show that $(p\to q)\wedge(\sim q)\to\sim p$ is a tautology.

p	q	$p\to q$	$\sim q$	$(p\to q)\wedge(\sim q)$	$\sim p$	$(p\to q)\wedge(\sim q)\to\sim p$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Since $(p\to q)\wedge(\sim q)\to\sim p$ is a tautology the given argument is valid.

Example 47: Test the valid of the argument $\sim p \to q, p \vdash q$

Solution:

In order to show that the argument is valid we have to show that

$(\sim p \to q) \wedge p \to \sim q$ is a tautology.

p	q	$\sim p$	$\sim p \to q$	$(\sim p \to q) \wedge p$	$\sim q$	$(\sim p \to q) \wedge p \to \sim q$
T	T	F	T	T	F	F
T	F	F	T	T	T	T
F	T	T	T	F	F	T
F	F	T	F	F	T	T

Since $(\sim p \to q) \wedge p \to \sim q$ is not a tautology, the given argument is not valid and hence is a fallacy.

Example 48: Test the validity of the argument $\sim p \to \sim q, q \vdash p$

Solution:

In order to show that the argument is valid we have to show that $(\sim p \to \sim q) \wedge q \to p$ is a tautology.

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$	$(\sim p \to \sim q) \wedge q$	$(\sim p \to \sim q) \wedge q \to p$
T	T	F	F	T	T	T
T	F	F	T	T	F	T
F	F	T	F	F	F	T
F	F	T	T	T	F	T

Since $(\sim p \to \sim q) \wedge q \to p$ is a tautology, the given argument is valid.

Example 49: Test the validity of the argument $p \to \sim q, \sim r \to \sim q \vdash p \to \sim r$

Solution:

p	q	r	$\sim q$	$\sim r$	$(p \to \sim q)$	$\sim r \to \sim q$	$(p \to \sim r)$	$(p \to \sim q) \wedge (\sim r \to \sim q)$	$(p \to \sim q) \wedge (\sim r \to \sim q) \to p \to \sim r$
T	T	T	F	F	F	T	F	F	T
T	T	F	F	T	F	F	T	F	T
T	F	T	T	F	T	T	F	T	F
T	F	F	T	T	T	T	T	T	T
F	T	T	F	F	T	T	T	T	T
F	T	F	F	T	T	F	T	F	T
F	F	T	T	F	T	T	T	T	T
F	F	F	T	T	T	T	T	T	T

Since $p(\to\sim q)\wedge(\sim r\to\sim q)\to(p\to\sim r)$ is not a tautology, the given argument is not valid and hence is a fallacy.

Example 50: Test the validity of the argument $p\to\sim q, r\to p, \vdash\sim r$

Solution:

p	q	r	$\sim q$	$p\to\sim q$	$r\to p$	$(p\to\sim q)\wedge(r\to p)\wedge q$	$\sim r$	$(p\to\sim q)\wedge(r\to p)\wedge q\to\sim r$
T	T	T	F	F	T	F	F	T
T	T	F	F	F	T	F	T	T
T	F	T	T	T	T	F	F	T
T	F	F	T	T	T	F	T	T
F	T	T	F	T	F	F	F	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	F	F	F	T
F	F	F	T	T	T	F	T	T

Since $(p\to\sim q)\wedge(r\to p)\wedge q\to\sim r$ is a tautology, the given argument is valid.

Example 51: Test the validity of the argument.

If I work I cannot study.

Either I work or I pass mathematics.

I passed mathematics.

Therefore I studied.

Solution:

p: I work

q: I study

r: I pass mathematics

The argument is $p\to\sim q,\ p\vee r, r\vdash q$

p	q	r	$(\sim q)$	$(p \to \sim q)$	$(p \vee r)$	$(p \to \sim q) \wedge (p \vee r) \wedge r$	$(p \to \sim q) \wedge (p \vee r) \wedge r \to q$
T	T	T	F	F	T	F	T
T	T	F	F	F	T	F	T
T	F	T	T	T	T	T	F
T	F	F	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	F	F	T

Since $(p \to \sim q) \wedge (p \vee q) \wedge r \to q$ is not a tautology, the given argument is not valid and hence is a fallacy.

Example 51: Test the validity of the argument.

If it rains, Ramesh will be sick.

It did not rain

Therefore Ramesh was not sick

Solution:

p: It rains

q: Ramesh will be sick

The argument is $p \to q, \sim p \vdash \sim q$

In order to test this argument, we have to show that $(p \to q) \wedge (\sim p) \to \sim q$ is a tautology.

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge (\sim p)$	$(p \to q) \wedge (\sim p) \to \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Since $(p \wedge q) \wedge (\sim p) \to \sim q$ is not a tautology, the given argument is a fallacy.

Example 52: If I study, then I will not fail in mathematics.

If I do not play basketball then I will study.

But I failed in mathematics.

Therefore I played basketball.

Solution:

p: I study

q: I will fail in mathematics

r: I played basket ball.

Example 53: The argument is $p \rightarrow \sim q, \sim r \rightarrow p, q \vdash r$

Solution:

In order to test the validity of the argument we have to show that $(p \rightarrow \sim q) \wedge (\sim r \rightarrow p) \wedge q \rightarrow r$ is a tautology.

p	q	r	$\sim q$	$\sim r$	$p \rightarrow \sim q$	$\sim r \rightarrow p$	$(p \rightarrow \sim q) \wedge (\sim r \rightarrow) \wedge q$	$[(p \rightarrow \sim q) \wedge (\sim r \rightarrow p) \wedge q \rightarrow r]$
T	T	T	F	F	F	T	F	T
T	T	F	F	T	F	T	F	T
T	F	T	T	F	T	T	F	T
T	F	F	T	T	T	T	F	T
F	T	T	F	F	T	T	T	T
F	T	F	F	T	T	F	F	T
F	F	T	T	F	T	T	F	T
F	F	F	T	T	T	F	F	T

Since the last column has all truth values T, $(p \rightarrow \sim q) \wedge (\sim r \rightarrow p) \wedge q \rightarrow r$ is a tautology. Hence the argument is valid.

Example 54: Prove the following argument is valid : $p, p \rightarrow q \vdash q$

Solution:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In row one p and $p \rightarrow q$ are true and so also the conclusion q is true. *i.e.*, whenever the premises are true the conclusion is also true.

$\therefore$ The argument is valid.

Example 55: Test the validity of the argument $p \to q, \sim p \vdash \sim q$

Solution:

The premises are $p \to q$ and $\sim p$

The conclusion is $\sim q$

p	q	$\sim p$	$p \to q$	$\sim q$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

In the 3rd row the premises $\sim p, p \to q$ are true but the conclusion $\sim q$ is false.

Therefore the given argument is a fallacy.

Example 57: Test the validity of the argument.

If it rains Mahesh will be sick

Mahesh was not sick

Therefore it did not rain

Solution:

p: It rains

q: Mahesh was sick

The premises are $p \to q, \sim q$

The conclusion is $\sim p$

The arrangement is $p \to q, \sim q \vdash \sim p$.

p	q	$\sim p$	$\sim q$	$p \to q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

In the last row the premises are true and the conclusion is also true.

$\therefore$ The argument is valid.

Example 57: Test the validity of the argument: On my wife's birthday, I bring her flowers. Either it is my wife's birthday or I work late.

I did not bring my wife flowers today.

Therefore, I worked late.

Solution:

p: My wife's birthday

q: I bring her flowers

r: I work late

The premises are $p \to q, \sim p \vee r, \sim q$

The conclusion is r

The argument is $p \to q, \sim p \vee r, \sim q \vdash r$

p	q	r	$\sim q$	$p \to q$	$p \vee r$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	T	F

In the last but one row the premises are true and the conclusion is also true.

$\therefore$ The argument is valid.

Example 58: Translate into symbolic form and test the validity of the argument.

If 6 is even then 2 does not divide 7

Either 5 is not prime or 2 divides 7

But 5 is a prime.

Therefore, 6 is odd.

Solution:

p: 6 is even

q: 2 does not divide 7

r: 5 is a prime.

The premises are $p \to q, \sim r \vee \sim q, r$

The conclusion is $\sim p$

The argument is $p \to q, \sim r \vee \sim q, r \vdash \sim p$.

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \to q$	$\sim r \vee \sim q$
T	T	T	F	F	F	T	F
T	T	F	F	F	T	T	T
T	F	T	F	T	F	F	T
T	F	F	F	T	T	F	T
F	T	T	T	F	F	T	F
F	T	F	T	F	T	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	T	T	T

In the 7th row the premises are true and also the conclusion is true.

Therefore, the argument is valid.

Example 59: Test the validity of the argument.

$[(p \to \sim q), r \to q, r] \vdash \sim p$

Solution: Statement — Reason

Statement	Reason
(1) $p \to \sim q$ is true	(1) Given
(2) $r \to q$ is true	(2) Given
(3) $\sim q \to \sim r$ is true	(3) Contrapositive of 2
(4) $p \to \sim r$ is true	(4) Law of syllogism, using (1) and (3)
(5) $r \to \sim p$ is true	(5) Contrapositive of (4)
(6) r is true	(6) Given
(7) $\sim p$ is true	(7) Law of detachment, using (5) and (6).

Example 60: Show that $\sim (p \wedge q) \wedge (\sim q \vee r) \wedge \sim r \to \sim p$

Solution:

Premises are $\sim (p \wedge q), \sim q \vee r, \sim r$

Statement	*Reason*
(1) $\sim (p \wedge q)$ is true	(1) given
(2) $\sim p \vee \sim q$ is true	(2) De Morgan's law
(3) $p \to q$ is true	(3) $\sim p \vee q \leftrightarrow p \to q$
(4) $\sim q \vee r$ is true	(4) given

(5) $q \to r$ is true	(5) $\sim q \vee r \Leftrightarrow q \to r$
(6) $p \to r$ is true	(6) from (3) and (5)
(7) $\sim r$ is true	(7) given
(8) $\sim p$ is true	(8) from (6) and (7)

$\therefore$ conclusion is true

Hence $\sim(p \wedge q) \wedge (\sim q \vee r) \wedge r \to \sim p$.

Exercises

1. Let p: Trichi is in Tamil Nadu
   q: Tirupathi is in Andhra
   r: Nagpur is in Delhi

   Write down the following propositions in symbolic form and determine in each case whether the proposition is true or false.
   (a) Trichi is in Tamil Nadu and Tirupathi is in Andhra.
   (b) Trichi is not in Tamil Nadu and Nagpur is not in Delhi.
   (c) Trichi is in Tamil Nadu or Tirupathi is in Andhra.
   (d) Trichi is not in Tamil Nadu and Nagpur is in Delhi.
   (e) If Trichi is in Tamil Nadu then Nagpur is not in Delhi.
   (f) Trichi is in Tamil Nadu if and only if Nagpur is not in Delhi.
2. Symbolise the following propositions.
   (a) Either I will wash my car or it will not rain.
   (b) It is humid and cloudy or it is raining, but at the sametime, it is false that it is both humid and raining.
   (c) Ramesh is either rich or he is unhappy.
   (d) It is not true that Ramesh is rich or he is unhappy.
3. Let p be the proposition: he works hard q; be the proposition he sleeps well and r; be he plays. Write down the following in verbal statement form.
   (a) $(p \wedge q)$ (b) $p \vee \sim q$ (c) $p \to q$ (d) $q \leftrightarrow r$
   (e) $p \vee (q \wedge r)$ (f) $\sim p \to \sim q$ (g) $(\sim p \wedge r) \vee q$.
4. Let p: A byte has 7 bits
   q: A word is 2 bytes
   r: A bit is a 0 or 1

   p is false and q and r are true. Write verbal statement form for the following symbolic statements. In each case determine whether the statement is true or false.

(a) $p \vee q$ (b) $p \vee r$ (c) $\sim p \wedge q$ (d) $\sim (p \wedge q)$

(e) $\sim p \vee \sim q$ (f) $[(p \wedge q) \vee r] \wedge [\sim (p \wedge r)]$

5. Write down the truth value of the compound statements
   (a) $3 + 2 = 5$ and $4 + 3 = 8$
   (b) Bangalore is in India and $3 + 4 = 7$
   (c) London is France and $3 + 2 = 3$
   (d) It is not true that London is not in USA.
   (e) It is false that Delhi is in Pakistan or London is in France.
6. Construct truth tables for
   (i) $\sim (p \wedge q)$ (ii) $p \wedge \sim q$
   (iii) $(p \wedge q) \wedge r$ (iv) $(p \wedge q) \vee (q \wedge r) \vee (r \vee p)$
   (v) $p(\wedge q) \wedge \sim r$ (vi) $(p \wedge \sim q) \vee (\sim p \vee \sim r)$
7. Rewrite the following with the minimum parenthesis as possible.
   (a) $[(\sim p) \wedge (r)] \vee (s)$
   (b) $[(p) \vee (q)] \wedge [(r) \vee (q)]$
8. Show that the following propositions are tautologies.
   (a) $\sim (p \wedge q) \leftrightarrow (\sim p) \vee (\sim q)$
   (b) $p \vee \sim p$
   (c) $(p \wedge q) \rightarrow p$
   (d) $q \rightarrow (p \vee q)$
   (e) $(p \vee q) \leftrightarrow (q \vee p)$
9. Show that the following are contradictions
   (a) $p \wedge (\sim p)$
   (b) $(p \vee q) \wedge (\sim p \vee \sim q)$
10. Test whether the following are tautologies
   (i) $[(p \vee r) \wedge (q \vee r)] \wedge [\sim p \vee \sim r]$
   (ii) $[p \wedge (q \vee r)] \wedge [q \wedge (p \vee r)]$
   (iii) $\sim (p \wedge q) \vee (\sim p \vee \sim q)$
   (iv) $(p \wedge \sim q) \vee [\sim p \wedge q]$

11. Show that $p\wedge(q\vee r)$ is not equivalent to $(p\wedge q)\vee r$.

12. Show that $p\vee(q\wedge r)$ is equivalent to $(p\vee q)\wedge(p\vee r)$.

13. Show that $p\vee\sim(q\wedge r)$ is equivalent to $(p\vee\sim q)\vee\sim r$.

14. Show that the following propositions are equivalent to $p\rightarrow q$.

    (a) $\sim p\vee q$

    (b) $\sim q\rightarrow\sim p$

    (c) $\sim(p\wedge\sim q)$

15. Show that $p\rightarrow q$ and $q\rightarrow p$ are not equivalent.

16. Whether the following pairs are equivalent?

    (a) $(p\rightarrow q)\rightarrow r,\ (p\wedge\sim q)\rightarrow r$

    (b) $p\leftrightarrow q;\ (p\wedge q)\vee(\sim p\wedge\sim q)$

    (c) $p\leftrightarrow q;\ \sim p\rightarrow\sim q$

17. Prove that the following rules of inference are valid by using truth tables.

    (i) $p, p\rightarrow q, \vdash q$

    (ii) $p\rightarrow q, q\rightarrow r, \vdash p\rightarrow r$

    (iii) $p\vdash p\vee q$

    (iv) $p\wedge q\vdash p$

    (v) $p, q\vdash p\wedge q$

    (vi) $\sim p\vdash p\rightarrow q$

    (vii) $q\vdash p\rightarrow q.$

    (viii) $p\rightarrow r, q\rightarrow r\ p\vee q\ \vdash r$

    (ix) $p\leftrightarrow q\vdash(p\rightarrow q)\wedge(q\rightarrow p)$

    (x) $(p\rightarrow q)\wedge(q\rightarrow p)\vdash p\leftrightarrow q$

18. Show that $r\rightarrow\sim q, r\vee s, s\rightarrow\sim q, p\rightarrow\Rightarrow\sim p$

19. Show that $p\rightarrow q, p\rightarrow r, q\rightarrow\sim r$ are inconsistent.

20. Show that $\sim p\vee q, \sim q\vee r, r\rightarrow s, \Leftrightarrow p\rightarrow s$.

21. Let p be the statement "the southwest monsoon is very good this year" and q be "the statement the rivers are rising". Give the verbal translations for (a) and verify the statement (b).

(a) (i) $p \vee \sim q$ (ii) $\sim(\sim p \wedge \sim q)$

(b) The statement $x > 1 \Leftrightarrow x^2 > 1$ is false.

22. Prove the following equivalence relations by use of algebra of propositions.

    (i) $p \wedge (p \vee q) \equiv p$

    (ii) $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$

    (iii) $\sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

23. Prove the following using truth tables.

    (i) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$

    (ii) $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q)$

24. Use truth tables to prove that $p \rightarrow q$ is equivalent to $\sim p \vee q$.

25. Find the contrapositive of the following propositions.

    (a) If he has courage he will win.

    (b) It is necessary to be strong in order to be a sailor.

    (c) Only if he does not tire he will win.

    (d) If is sufficient for it to be a square in order to be a rectangle.

26. State the truth value of the following statements.

    (a) Matrix multiplication is commutative

    (b) Composition of two functions is always commutative.

    (c) Every square matrix is diagonisable

    (d) Every equilateral triangle has its centroid at its circumcentre

27. (a) Write the dual of $(p \wedge q) \vee r$

    (b) Write down the negation of the following

    (i) Delhi is a small town

    (ii) Every city in India is clean

28. Write the following in symbolic form.

    (a) If John takes calculus, Ravi takes Analytical geometry then Mohan will take English.

29. Show that $q \wedge (p \wedge \sim q) \vee (\sim p \wedge \sim q)$ is a tautology.

30. Show that $s \vee r$ is a tautology implied by $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow s)$.

31. Show that $r \rightarrow s$ can be derived from the premises $p \rightarrow (q \rightarrow s), \sim r \vee p$ and q.

32. Show that $(p \to r) \wedge (q \to r) \Leftrightarrow (p \vee q) \to r$.

33. Given the following propositions generated by p, q and r which are equivalent to one another?

(i) $(p \wedge r) \vee q$

(ii) $\sim r \vee p$

(iii) $r \to p$

(iv) $r \vee \sim p$

(v) $p \to r$

34. Give the direct proof of

(i) $p \to q,\ q \to r,\ \sim(p \wedge r),\ p \vee r \Rightarrow r$

(ii) $p \to q,\ r \to s,\ s \to (p \vee r),\ s \Rightarrow q$

35. Show that $\sim p \vee (\sim p \wedge q)$ and $\sim p \wedge \sim q$ are logically equivalent.

36. Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

37. Show that the following implications are tautologies.

(a) $(p \wedge q) \to p$

(b) $p \to (p \vee q)$

(c) $\sim p \to (p \to q)$

(d) $(p \wedge q) \to p \to q$

(e) $\sim(p \to q) \to p$

(f) $\sim(p \to q) \to q$

(g) $\sim p \wedge (p \vee q) \to q$

(h) $(p \to q) \wedge (q \to r) \to (p \to r)$

(i) $p \wedge (p \to q) \to q$

38. Determine whether the following implications are tautologies.

(a) $\sim p \wedge (p \to q) \to \sim q$

(b) $\sim q \wedge (p \to q) \to \sim p$

39. Test the validity of the argument:

    If he works hard he will be successful.

    If he is successful then he will be happy.

    Therefore hard work leads to happiness.

40. Test the validity of the argument:

    A firm of Chartered Accountants makes the following declaration. An articled clerk from the firm passing the final CA examination in the first attempt will be awarded a prize of Rs. 1,000. Five clerks P, Q, R, S, T appeared for the first time from the firm and P and Q could pass. The firm awards prizes not only to them but also to R and S. Is this action totally justified? T claims the prize comparing himself with R and S but the firm refuses. Is the refusal logically justified? How should the statements be worded so that only P and Q will be entitled for the prizes?

41. Test the validity of the argument:

    All teachers are poor

    In order to be a teacher one must graduate from college.

    Some mathematicians are not graduates.

    Therefore no college graduate is poor.

Chapter 2

Sets

1.1 A set is a collection of well-defined objects. By well-defined objects we mean that given any object it is possible to determine whether the object is a member of the set or not. The objects of a set are called elements. Usually sets are denoted by capital letters A,B,C,......and the elements are denoted by small letters a,b,c,...... If a is an element of a set S then we write $a \in S$. The symbol $\in$ means 'belongs to' or a 'member of'. $a \in S$ is read as a belongs to S. If a is not an element of A then we write $a \notin A$. This is read as a is not an element of A. The symbol $\notin$ means 'does not belong to'.

Sets may be any collection of objects. The following are some of the examples of sets.

(1) The set S of all vowels in English alphabet. Here $e \in S$ and $b \notin S$.

(2) The set W of all days in a week. Here Friday $\in W$ and March $\notin W$.

(3) The set N of all natural numbers. Here $5 \in N$ and $-5 \notin N$.

(4) The set Q of all rational numbers. Here $\sqrt{2} \notin Q$, $\frac{3}{5} \in Q$, $5 \in Q$.

1.2 Finite and infinite sets :

A set which contains specific number of distinct elements is called a finite set, i.e., in counting the number of elements the counting process comes to an end. If a set is not finite then it is said to be inifinite.

Examples of finite sets :

(1) The set of all days in a week.

(2) The set of all students in a class.

(3) The set of all letters in English alphabet.

(4) The set of all books in our college library.

Examples of infinite sets :

(1) The set of all natural numbers.

(2) The set of all rational numbers.

(3) The set of all points in a line.

(4) The set of all prime numbers.

(5) The set of all integers divisible by 5.

1.3 Description of sets :

There are two methods of representing sets. They are (1) Tabulation method, (2) Set builder method. This is also called defining property method.

The tabulation method : Lists the individual elements, separates them by commas and encloses them in braces.

Example 1 :

The set of all natural numbers less than 5 is represented by, {1, 2, 3, 4}. The order of listing the elements is unimportant.

We can also write this set as {1, 3, 2, 4}, {3, 4, 1, 2}

Example 2 :

The set of all days in a week is represented by {Sun, Mon, Tues, Wed, Thurs, Fri, Sat}.

Example 3 :

The set of all natural numbers is represented by {1,2,3,......}

Some sets cannot be described by an enumeration. In such cases the second method is more compact and convenient.

Set-builder method :

The set of all natural numbers N is represented in set builder form as N = {x | x is a natural number.} This is read as the set of all elements x such that x is a natural number. The vertical line | (or :) is read as 'such that'.

Examples :

(1) The set A of all one digit prime numbers is represented by A={x | x is a one-digit prime number.}

(2) The set R={1,2,..................9} is represented by R={x | x ∈ N, x < 10}

1.4 Singleton set:

If a set contains only one element it is called a singleton set.

Example :

A={o}, B={e} and the set of all even primers are all singleton sets.

1.5 Null set :

A set with no elements is called a null set or an empty set. It is denoted by ϕ or { }. We note that {o} is not a null set but it is the set containing only one element o.

Examples of null sets :

(1) The set of all numbers which are both odd and even.

(2) The set of all even primes greater than 9.

(3) The set of all positive roots of the equation $x^2+3x+2=0$.

(4) The set of all integers whose square is negative.

1.6 Sub-set :

A set B is called a sub-set of A if every element of B is in A.

For example,

(1) The set of all vowels is a sub-set of the set of all letters in English alphabet.

(2) The set of all natural numbers is a sub-set of the set of all integers.

(3) The set of all even primes is a sub-set of the set of all primes.

Notation :

If B is a sub-set of A we write $B \subset A$. This is read as B is a sub-set of A. The symbol $\subset$ means 'subset of'. With this definition of sub-set we see that every set is a sub-set of itself.

For example, $\{1, 2, 3\} \subset \{1, 2, 3\}$

It B is a sub-set of A and A contains atleast one element which is not in B then we call B a proper sub-set of A.

Note 1 :

For B to be a proper sub-set of A we have $B \subset A$ and $A \not\subset B$.

$\{1, 2, 3\}$ is a proper subset of $\{1, 2, 3, 4, 5\}$, $B \not\subset A$ means that B is not a sub-set of A. If B is a proper sub-set of A then A is called a superset of B and is written as $A \supset B$.

Example :

If $A=\{1, 3, 4, 5\}$, $B=\{3, 5\}$ then $A \supset B$.

Note 2 :

The null-set is a sub-set of every set

1.7 Equality of sets :

Two sets A and B are said to be equal if every element of A is in B and every element of B is in A. If two sets A and B are equal we denote this by writing $A = B$. In other words the sets A and B are equal if $A \subset B$ and $B \subset A$.

Note :

Two sets are said to be *equivalent* if they have the same number of elements. If two sets are equal then they are equivalent. If two sets are equivalent they need not be equal.

Example 1 :

Let $A=\{1, 2, 3\}$, B=The set of all natural numbers less than 4; then $A=B$.

Example 2 :

Let C = The set of all vowels in English alphabet and $D = \{a, e, i, o, u\}$

Then $C = D$.

1.8 Number of sub-sets of a set :

Consider the set $S=\{a\}$. The sub-sets of S are $\{a\}$ and ϕ.

Consider the set $T = \{a, b\}$. The sub-sets of T are ϕ, $\{a\}$, $\{b\}$ and $\{a, b\}$.

Consider the set $R = \{a, b, c\}$. The sub-sets of R are ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{b,c\}$, $\{c,a\}$ and $\{a, b, c\}$. Here we note that if a set contains only one element there are two sub-sets of it, if a set contains 2 elements there are 4 sub-sets and if a set contains 3 elements there are 8 sub-sets of it. In general, if a set contains n elements, there are 2^n sub-sets. The set of all sub-sets of a given set A is called the power set of A. It is denoted by 2^A.

(i.e.) $2^A = \{X \mid X \subset A\}$

If $A = \{a, b\}$ then

$2^A = \{ \phi, \{a\}, \{b\}, \{a,b\} \}$

Here we note that the elements of 2^A are also sets. A set which contains its elements as sets is called a family of sets or class of sets.

1.9 Disjoint sets :

Two sets A and B are said to be disjoint if they have no element in common. For example, the sets {1, 2, 3} and {7, 8} are disjoint.

1.10 Universal set :

If all sets under consideration are sub-sets of a given set then the given set is called the universal set. It is usually denoted by U.

Example :

A = set of all primes

B = set of all 3 digit even natural numbers

C = {5, 10, 15, 20, ...}.

For all these sets the universal set can be taken as the set N of all natural numbers.

1.11 Set operations :

In number system we have the operations called addition, subtraction, multiplication and division. Given any two numbers we find a new number using each of these operations. Like-wise we have operations on sets to produce new sets. Given two sets

A and B we define operations called (1) union of sets (2) intersection of sets (3) difference of sets. Given A and U we define the complement of the set A.

Union of sets :

The union of sets A and B is the set of all elements which are either in A or in B or in both. The union of sets A and B is denoted by $A \cup B$.

(i.e.) $A \cup B = \{ x \mid x \in A \text{ or } x \in B\}$.

Example 1 :

Let $A = \{1, 2, 3, 4, 5\}$
$B = \{4, 5, 7, 9\}$

Then $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$

Example 2 :

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$

Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$

Example 3 :

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5\}$

Then $A \cup B = \{1, 2, 3, 4, 5\}$

Note : If $B \subset A$ then $A \cup B = A$.

Intersection of sets :

The intersection of sets A and B is the set of all elements which are in both A and B. The intersection of sets A and B is denoted by $A \cap B$.

Then $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Example 1 :

Let $A = \{1, 3, 5, 7\}$
$B = \{2, 5, 7, 10, 12\}$

$A \cap B = \{5, 7\}$

Example 2 :

Let $A = \{1, 2, 3, 4, 5\}$
$B = \{1, 2, 3\}$ then $A \cap B = \{1, 2, 3\}$

Note : If $A \subset B$ then $A \cap B = A$.

Example 3 :

Let $A = \{1, 2, 3\}$, $B = \{5, 4, 7\}$

Here $A \cap B = \phi$

Note : If A and B are disjoint then $A \cap B = \phi$.

Difference of sets :

The difference of sets A and B is defined to be the set which contains all those elements in A which are not in B. The difference of set A and B is denoted by $A - B$.

(i.e.) $A - B = \{x \mid x \in A, x \notin B\}$

Similarly $B - A = \{x \mid x \in B, x \notin A\}$

Note that $A - B \neq B - A$.

Example :

$A = \{1, 2, 3, 4, 5\}$; $B = \{3, 5, 7, 9, 10\}$

$A - B = \{1, 2, 4\}$; $B - A = \{7, 9, 10\}$

Complement of a set :

Let U be the universal set. The complement of a set A is defined to be the set of all elements which are in U and not in A. The complement of A is denoted by A' or $\overline{A}$ or A^c.

(i.e.) $A' = \{x \mid x \in U, x \notin A\}$

Example 1 :

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 3, 5, 7, 9\}$

$A' = \{2, 4, 6, 8, 10\}$

Example : 2

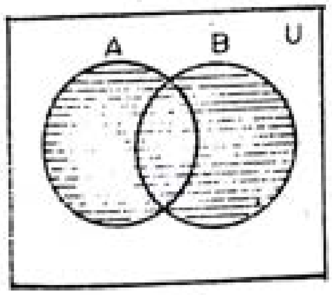
U = set of all integers.

A = set of all even integers.

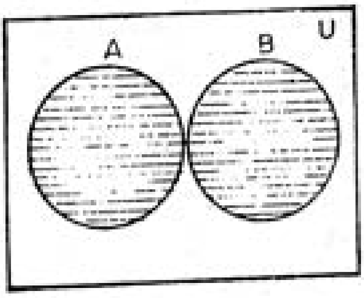
A′ = set of all odd integers.

1.12 Venn diagram :

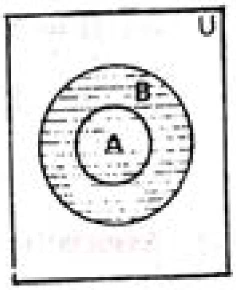
Venn diagram is a method of representing the relationship between sets. A rectangle is drawn to represent the universal set U and sets are represented by circles. Elements of a set are represented by points within these circle, representing the set. Where points are not specified it is understood that every point inside the circle represents an element of the set. Below we give some venn diagrams for union, intersection, difference and complement of sets.



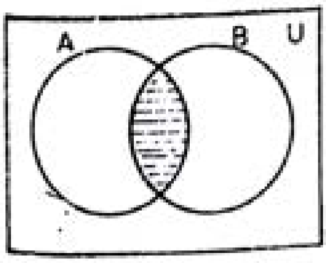
$A \cup B$



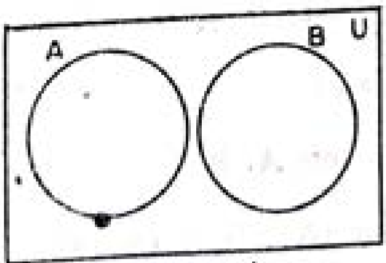
$A \cup B$



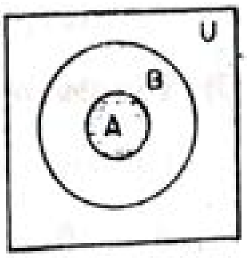
$A \cup B$



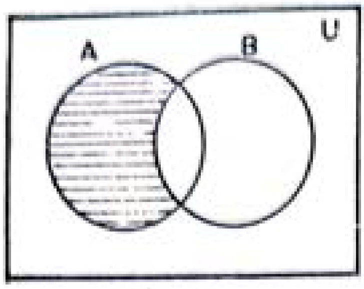
$A \cap B$



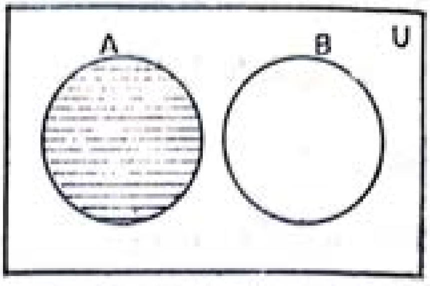
$A \cap B = \emptyset$



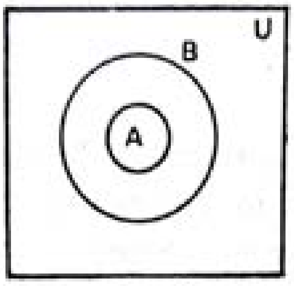
$A \cap B = A$



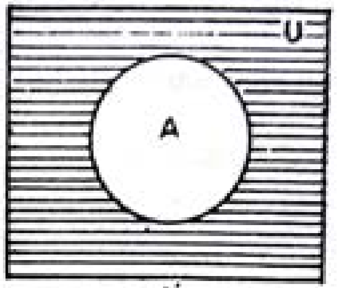
A – B



A – B = A



A-B = ∅



A′

1.13 Law of sets :

(1) Commutative laws :

For any two sets A and B,

$A \cup B = B \cup A$

$A \cap B = B \cap A$

But $A - B \neq B - A$

(2) Associative laws :

For any three sets A, B, C,

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

(3) Distributive laws :

For any three sets A, B, C,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) **De Morgan's laws :**

For any two sets **A** and **B,**

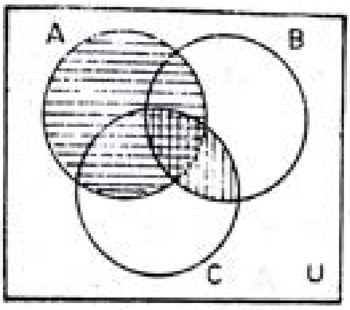
i) $(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

These can be verified using venn diagrams. We shall now verify the distributive laws and De Morgan's laws.

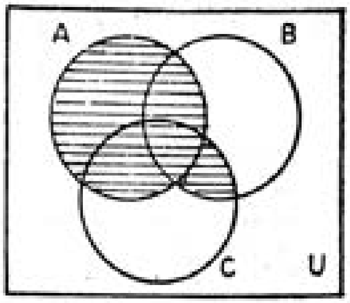
(a) Verify by Venn diagram $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

1



A and $B \cap C$

2

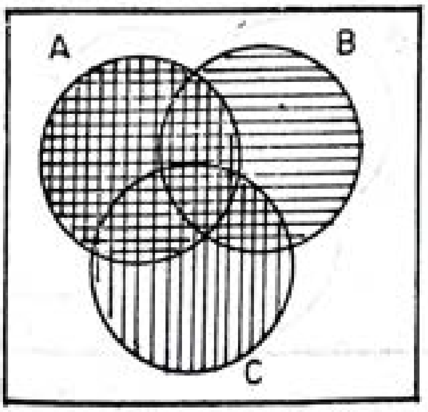


$A \cup (B \cap C)$

$B \cap C$ is shaded by vertical lines

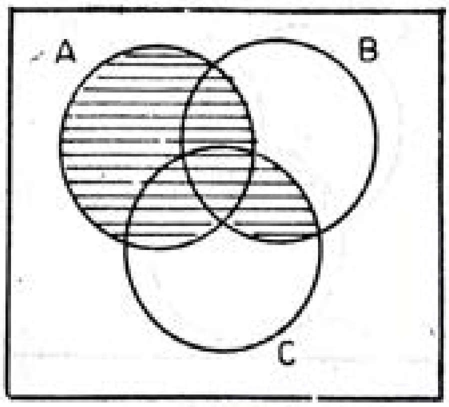
A is shaded by horizontal lines.

3



$(A \cup B)$ and $(A \cup C)$

4



$(A \cup B) \cap (A \cup C)$

$A \cup B$ is shaded by horizontal lines

$A \cup C$ is shaded by vertical lines

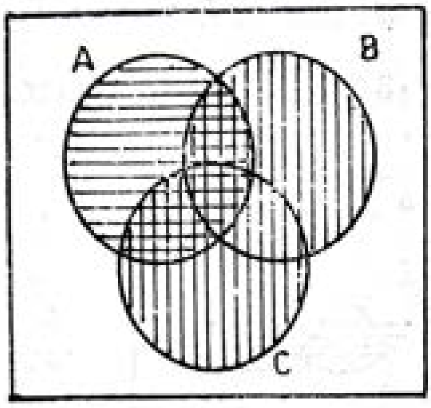
From (2) and (4),

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) verify by Venn diagram :

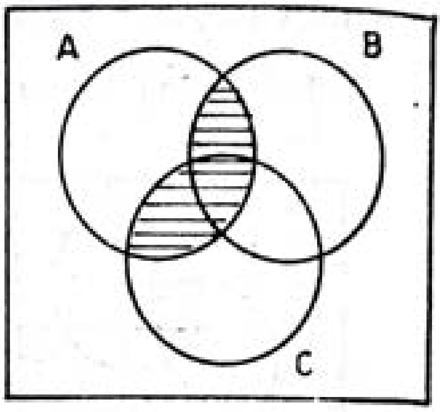
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

1



A and B ∪ C

2

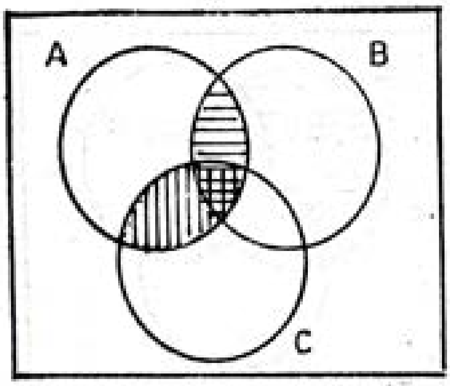


A ∩ (B ∪ C)

A is shaded by horizontal lines

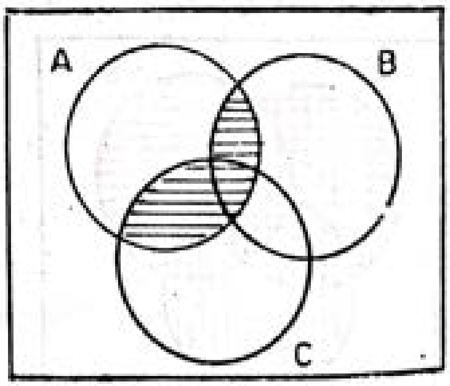
$B \cup C$ is shaded by vertical lines

3



A ∩ B and A ∩ C

4



(A ∩ B) ∪ (A ∩ C)

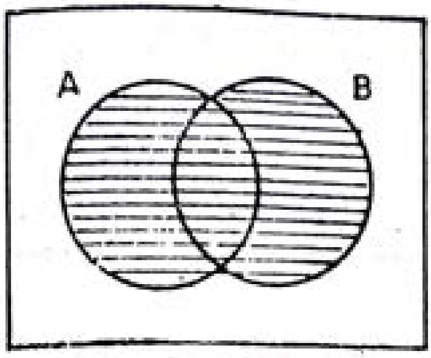
$A \cap B$ is shaded by horizontal lines

$A \cap C$ is shaded by vertical lines

From (2) and (4), $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

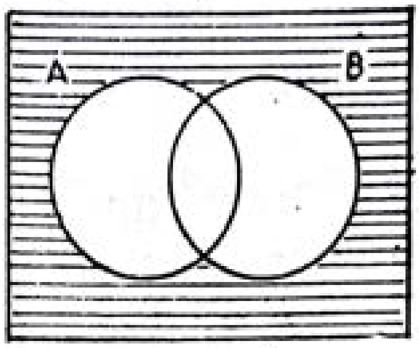
(c) Verify by Venn diagram : $(A \cup B)' = A' \cap B'$

1



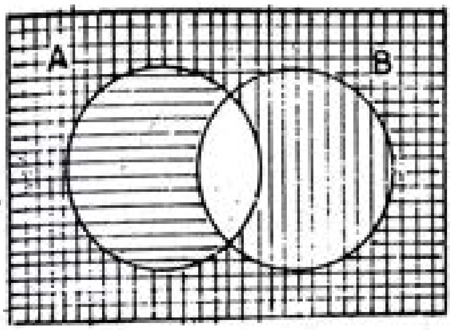
A U B

2

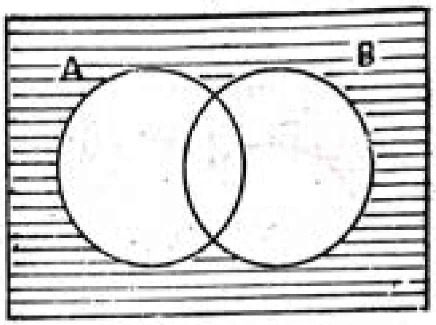


$(A \cup B)'$

3



4



$A' \cap B'$

A' and B'

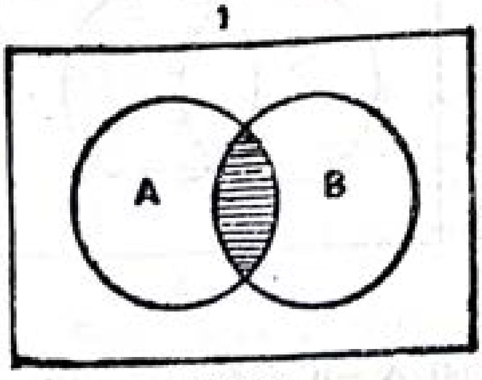
A' is shaded by vertical lines

B' is shaded by horizontal lines

From diagrams (2) and (4), $(A \cup B)' = A' \cap B'$

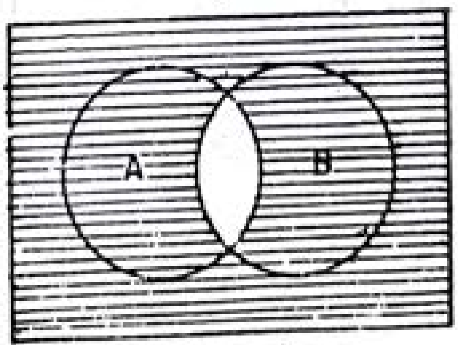
(d) Verify by venn diagram $(A \cap B)' = A' \cup B'$

1



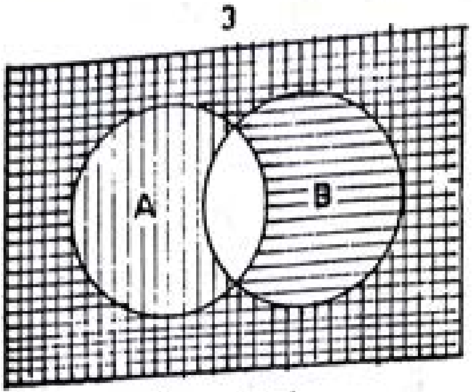
$A \cap B$

2



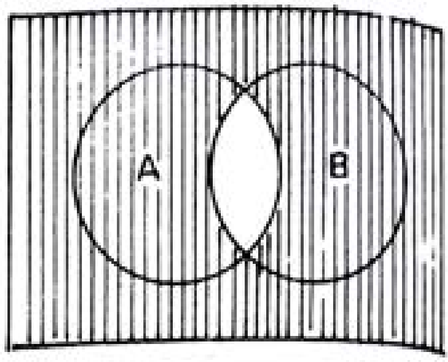
$(A \cap B)'$

3



A' and B'

4

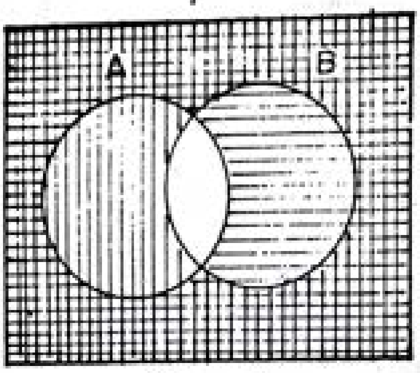


A' ∪ B'

From diagrams (2) and (4), $A' \cup B' = (A \cap B)'$

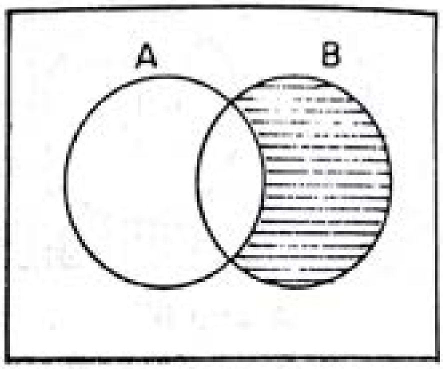
(e) Verify by venn diagram $A' - B' = B - A$

1



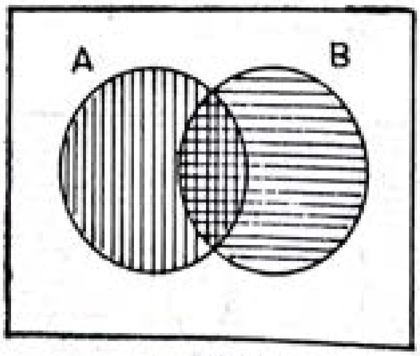
A' and B'

2



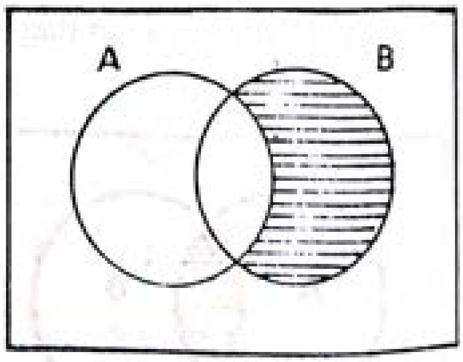
A' – B'

3



A and B

4



B – A

From diagrams (2) and (4), $A' - B' = B - A$

Some important properties :

(1) If $A \subset B$, $A \cup B = B$
$$A \cap B = A$$
$$A - B = \phi$$

(2) For any set A, $A \cup A' = U$
$$A \cap A' = \phi$$

(3)
$$U' = \phi$$
$$\phi' = U$$

(4) For any set A, $(A')' = A$

(5) If $A \subset B$ then $B' \subset A'$

(6) For any set A, $A \cup \phi = A$
$$A \cap \phi = \phi$$
$$A \cup U = U$$
$$A \cap U = A$$

Distributive laws : (Proof)

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof for (i) : In order to prove this we have to show that

(1) $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

(2) $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

Let $x \in A \cup (B \cap C)$.

$$\begin{aligned} x \in A \cup (B \cap C) &\Rightarrow x \in A \text{ or } x \in B \cap C \\ &\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\ &\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C) \\ &\Rightarrow x \in (A \cup B) \cap (A \cup C) \end{aligned}$$

Since x is an arbitrary element, we have shown that every element of $A \cup (B \cap C)$ is an element of $(A \cup B) \cap (A \cup C)$.

$$\therefore\ A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C) \qquad \ldots (1)$$

Let $y \in (A \cup B) \cap (A \cup C)$

Then, $y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in A \cup B$ and $y \in (A \cup C)$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow y \in A$ or $(y \in B$ and $y \in C)$

$\Rightarrow y \in A$ or $y \in B \cap C$

$\Rightarrow y \in A \cup (B \cap C)$

Since y is arbitrary,

$$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C) \qquad \ldots (2)$$

$\therefore$ From (1) and (2), $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof for (ii) is left as an exercise.

Demorgan laws : (Proof)

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Proof for (i)

Let $x \in (A \cup B)'$

$x \in (A \cup B)' \Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

Since x is an arbitrary element,

$$(A \cup B)' \subset A' \cap B' \qquad \ldots (1)$$

Let $y \in A' \cap B'$

$y \in A' \cap B' \Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

Since y is an arbitrary element of $A' \cap B'$ we have

$$A' \cap B' \subset (A \cup B)' \quad \text{... (2)}$$

$\therefore$ From (1) and (2), $(A \cup B)' = A' \cap B'$.

Proof for (ii):

Let $x \in (A \cap B)'$

$$\begin{aligned} x \in (A \cap B)' &\Rightarrow x \notin A \cap B \\ &\Rightarrow x \notin A \text{ or } x \notin B \\ &\Rightarrow x \in A' \text{ or } x \in B' \\ &\Rightarrow x \in A' \cup B' \end{aligned}$$

Since x is an arbitrary element of $(A \cap B)'$ it follows that

$$(A \cap B)' \subset A' \cup B' \quad \text{... (3)}$$

Let $y \in A' \cup B'$

$$\begin{aligned} y \in A' \cup B' &\Rightarrow y \in A' \text{ or } y \in B' \\ &\Rightarrow y \notin A \text{ or } y \notin B \\ &\Rightarrow y \notin (A \cap B) \\ &\Rightarrow y \in (A \cap B)' \end{aligned}$$

Since y is an arbitrary element of $A' \cup B'$ it follows that,

$$A' \cup B' \subset (A \cap B)' \quad \text{... (4)}$$

$\therefore$ From (3) and (4) we have,

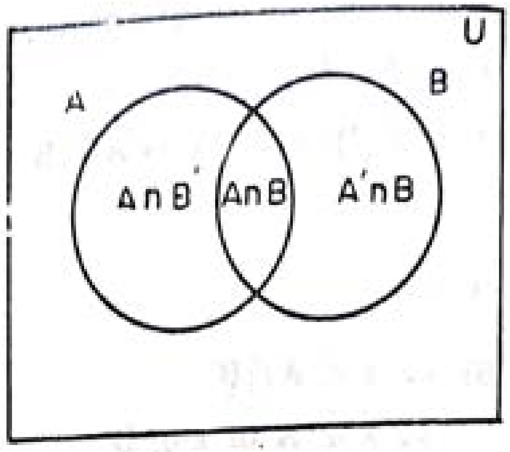
$$(A \cap B)' = A' \cup B'$$

1.14 Number of elements in a set :

If n (A) denotes the number of elements in the set A show that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Hence show that,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$



From the diagram we see that

$$A \cup B = A \cup (A' \cap B)$$

If A and B are disjoint sets,

$$n(A \cup B) = n(A) + n(B)$$

$\therefore$ $n(A \cup B) = n(A) + n(A' \cap B)$ ... (1)

since A and $A' \cap B$ are disjoint.

Also $B = (A \cap B) \cup (A' \cap B)$

Since $(A \cap B)$ and $(A' \cap B)$ are disjoint,

$$n(B) = n(A \cap B) + n(A' \cap B)$$

$$\therefore \; n(A' \cap B) = n(B) - n(A \cap B)$$

Substituting in (1),

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Let $B \cup C = D$

Then $n(A \cup B \cup C) = n(A \cup D)$

$$= n(A) + n(D) - n(A \cap D)$$
$$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$$

$$= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (A \cap C)]$$

$$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$

1.15 Cartesian Product :

A pair of objects, one of which is designated as the first component and the other as the second component, is called an ordered pair. The ordered pair with 5 as first component and 6 as second component is denoted by (5, 6). In general, (a, b) is referred to as ordered pair. It is not necessary that the components of an ordered pair to be different. For example, (a, a), (o, o), (5, 5) are all ordered pairs. The ordered pair (a, b) is different from the ordered pair (b, a) unless $a=b$. Two ordered pairs (a, b) and (c, d) are equal if and only if $a=c$ and $b=d$. Then we write $(a, b) = (c, d)$.

Let A and B be two *Sets*. The Cartesian product of sets A and B is defined to be the set of all ordered pairs with first element in A and the second element in B. The cartesian product is denoted by $A \times B$.

(i.e.) $A \times B = \{ (a, b) \mid a \in A, b \in B \}$

Example :

Let $A = \{ a, b, c \}$, $B = \{ 1, 2 \}$

Then $A \times B = \{ (a, 1) (a, 2) (b, 1) (b, 2) (c, 1), (c, 2) \}$
$B \times A = \{ (1, a) (2, a) (1, b) (2, b) (1, c) (2, c) \}$

Note that $A \times B \neq B \times A$

Let us now prove two important results on Cartesian products. (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof for (i) : We have to prove that,

(1) $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$

(2) $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$

Let $(a, b) \in A \times (B \cup C)$

$$(a, b) \in A \times (B \cup C) \Rightarrow a \in A,\ b \in B \cup C$$
$$\Rightarrow a \in A,\ (b \in B \text{ or } b \in C)$$
$$\Rightarrow (a \in A,\ b \in B) \text{ or } (a \in A,\ b \in C)$$
$$\Rightarrow (a,b) \in A \times B \text{ or } (a,b) \in A \times C$$
$$\Rightarrow (a,b) \in (A \times B) \cup (A \times C)$$

Since (a,b) is an arbitrary element of $A \times (B \cup C)$,

$$A \times (B \cup C) \subset (A \times B) \cup (A \times C) \quad \ldots \text{ I}$$

Let $(x,y) \in (A \times B) \cup (A \times C)$

$$(x,y) \in (A \times B) \cup (A \times C) \Rightarrow (x,y) \in A \times B \text{ or } (x,y) \in A \times C$$
$$\Rightarrow (x \in A,\ y \in B) \text{ or } (x \in A,\ y \in C)$$
$$\Rightarrow x \in A,\ (y \in B \text{ or } y \in C)$$
$$\Rightarrow x \in A,\ y \in B \cup C$$
$$\Rightarrow (x,y) \in A \times (B \cup C)$$

Since (x,y) is an arbitrary element,

$$(A \times B) \cup (A \times C) \subset A \times (B \cup C) \quad \ldots \text{ II}$$

From I and II we get,

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Proof for (ii) is left as an exercise.

Example 1 :

Write down the following sets in tabulation method.

(i) The set A of all odd natural numbers less than 10.

(ii) The set B of all square numbers less than 100

(iii) The set C of all roots of the equation $x^2 - 5x + 4 = 0$

(iv) The set D of all prime numbers between 10 and 20.

Solution :

(i) $A = \{1, 3, 5, 7, 9\}$

(ii) $B = \{1, 4, 9, 16, 25, 36, 49, 64, 81\}$

(iii) $x^2-5x+4=0$

$(x-1)(x-4)=0$

$\therefore$ The roots are 1 and 4.

$\therefore$ $C=\{1,4\}$

(iv) $D=\{11, 13, 17, 19\}$.

Example 2 :

Write down the following sets in set-builder form.

(i) The set A of all months in a year.

(ii) The set B of all English alphabet.

(iii) $C=\{3, 6, 9, \ldots\ldots\ldots\ldots\}$

(iv) $D=\{1, 3, 5, \ldots\ldots\ldots\ldots\}$

Solution :

$A=\{x \mid x \text{ is a month of a year}\}$

$B=\{x \mid x \text{ is an English alphabet}\}$

$C=\{x \mid x\in N \text{ and is a multiple of } 3\}$

$D=\{x \mid x\in N \text{ and } x \text{ is odd}\}$

Example 3 :

Write down the sub-sets of $A=\{a,b,c,d\}$. The sub-sets of A are ϕ, {a}, {b}, {c}, {d}, {a,b}, {b,c}, {c,d}, {a,d}, {a,c}, {b,d}, {a,b,c}, {a,b,d} {a,c,d}, {b,c,d} and {a,b,c,d}

Example 4 :

If $A=\{1, 2, 3\}$ $B=\{1, 3, 5\}$ $C=\{2, 3, 4, 6\}$ find

(i) $A\cup B$ (ii) $A\cap B$ (iii) $A-B$ (iv) $A\cup(B\cap C)$ (v) $A-(B\cap C)$

(i) $A\cup B=\{1, 2, 3, 5\}$

(ii) $A\cap B=\{1, 3\}$

(iii) $A-B=\{2\}$

(iv) $A=\{1,2,3\}$, $B\cap C=\{3\}$

$\therefore \quad A \cup (B \cap C) = \{ 1, 2, 3 \}$

(v) $A - (B \cap C) = \{ 1, 2 \}$

Example 5 :

Given that $A = \{ 0, 1, 3, 5 \}$, $B = \{ 1, 2, 4, 7 \}$
$C = \{ 1, 2, 3, 5, 8 \}$ prove that

(i) $(A \cap B) \cap C = A \cap (B \cap C)$

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$

(iii) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(iv) $(A \cap B) \cup C = (A \cup C) \cap (C \cup B)$

Solution :

(i) $A = \{ 0, 1, 3, 5 \}$

$B = \{ 1, 2, 4, 7 \}$

$C = \{ 1, 2, 3, 5, 8 \}$

$A \cap B = \{ 1 \}$

$(A \cap B) \cap C = \{1\} \cap \{1, 2, 3, 5, 8\}$
$= \{1\}$ ... I

$B \cap C = \{1, 2\}$

$A \cap (B \cap C) = \{1\} \cap \{1, 2\}$
$= \{1\}$ ... II

$\therefore$ From I and II $(A \cap B) \cap C = A \cap (B \cap C)$

(ii) $A \cup B = \{0, 1, 2, 3, 4, 5, 7\}$

$B \cup C = \{1, 2, 3, 4, 5, 7, 8\}$

$(A \cup B) \cup C = \{0, 1, 2, 3, 4, 5, 7\} \cup \{1, 2, 3, 5, 8\}$
$= \{0, 1, 2, 3, 4, 5, 7, 8\}$ ... (1)

$A \cup (B \cup C) = \{0, 1, 3, 5\} \cup \{1, 2, 3, 4, 5, 7, 8\}$
$= \{0, 1, 2, 3\ 4, 5. 7, 8\}$ ... (2)

$\therefore$ From (1) and (2), $(A \cup B) \cup C = A \cup (B \cup C)$

(iii) $(A \cup B) \cap C = \{0, 1, 2, 3, 4, 5, 7\} \cap \{1, 2, 3, 5, 8\}$
$= \{1, 2, 3, 5\}$ ... (1)

$A \cap C = \{0, 1, 3, 5\} \cap \{1, 2, 3, 5, 8\}$
$= \{1, 3, 5\}$

$(A \cap C) \cup (B \cap C) = \{1, 3, 5\} \cup \{1, 2\}$
$= \{1, 2, 3, 5\}$ ... (2)

$\therefore$ From (1) and (2), $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(iv) $(A \cap B) \cup C = \{1\} \cup \{1, 2, 3, 5, 8\}$
$= \{1, 2, 3, 5, 8\}$ ... (1)

$(A \cup C) \cap (B \cup C) = \{0, 1, 2, 3, 5, 8\} \cap \{1, 2, 3, 4, 5, 7, 8\}$
$= \{1, 2, 3, 5, 8\}$ ... (2)

$\therefore$ From (1) and (2), $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Example 6 :

Given $U = \{1, 2, 3, 4, 5, 6, 7\}$

$A = \{1, 2, 3, 4, 5\}$

$B = \{1, 3, 5, 7\}$

$C = \{2, 5, 6, 7\}$

find (i) $A \cup C$ (ii) $B \cap A$ (iii) $C - B$ (iv) $C' \cap A$

(i) $A \cup C = \{1, 2, 3, 4, 5, 6, 7\}$

(ii) $B \cap A = \{1, 3, 5\}$

(iii) $C - B = \{2, 6\}$

(iv) $C' = \{1, 3, 4\}$

$C' \cap A = \{1, 3, 4\} \cap \{1, 2, 3, 4, 5\}$
$= \{1, 3, 4\}$

Example 7 :

List the sets A, B and C given that

$A \cup B = \{p, q, r, s\}$

$A \cup C = \{q, r, s, t\}$

$A \cap B = \{q, r\}$

$A \cap C = \{q, s\}$

Solution :

$A \cap B = \{q, r\}$ and $A \cap C = \{q, s\}$

$\therefore \quad q, r, s \in A$

$q, r \in B$

$q, s \in C$

$A \cup C = \{q, r, s, t\} \Rightarrow p \notin A, p \notin C$

$A \cup B = \{p, q, r, s\} \Rightarrow t \notin A, t \notin B$

$\therefore \quad A = \{q, r, s\}$

Since $A \cup B = \{p, q, r, s\}$ and $p \notin A$, $p \in B$

Since $s \notin A \cap B$, $s \in A$ and $s \notin B$

$\therefore \quad B = \{p, q, r\}$

Since $r \notin A \cap C$, and $r \in A$, $r \notin C$

$\therefore \quad C = \{q, s, t\}$

Example 8 :

Prove that

(i) $A - (B \cap C) = (A - B) \cup (A - C)$

(ii) $A - (B \cup C) = (A - B) \cap (A - C)$

(B.Com., May 1982)

Proof for (i) ; Let $x \in A - (B \cap C)$

$$
\begin{aligned}
x \in A - (B \cap C) &\Rightarrow x \in A \text{ and } x \notin B \cap C \\
&\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\
&\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\
&\Rightarrow x \in A - B \text{ or } x \in (A - C) \\
&\Rightarrow x \in (A - B) \cup (A - B) \qquad \ldots (1)
\end{aligned}
$$

Let $y \in (A-B) \cup (A-C)$

$$
\begin{aligned}
y \in (A-B) \cup (A-C) &\Rightarrow y \in (A-B) \text{ or } y \in (A-C) \\
&\Rightarrow (y \in A \text{ and } y \notin B) \text{ or } \\
&\qquad (y \in A \text{ and } y \notin C) \\
&\Rightarrow y \in A \text{ and } (y \notin B \text{ or } y \notin C) \\
&\Rightarrow y \in A \text{ and } y \notin (B \cap C) \\
&\Rightarrow y \in A - (B \cap C)
\end{aligned}
$$

Since y is an arbitrary element,

$$(A-B) \cup (A-C) \subset A-(B \cap C) \qquad \ldots (2)$$

$\therefore$ From (1) and (2),

$$A-(B \cap C) = (A-B) \cup (A-C)$$

Proof for (ii) is left for exercise.

Example 9 :

Prove that,

(i) $[A' \cup (A \cap B')]' = A \cap B$

(ii) $(A' \cup B')' \cup (A' \cup B)' = A$

(iii) $A \cup B = (A \cap B) \cup (A \cap B') \cup (A' \cap B)$

(i) $[A' \cup (A \cap B')]' = A \cap (A \cap B')'$ (Demorgan's la

$= A \cap (A' \cup B)$ (Demorgan's law)

$= (A \cap A') \cup (A \cap B)$ (Distributive law)

$= \phi \cup (A \cap B)$

$= A \cap B.$

(ii) $(A' \cup B')' \cup (A' \cup B)' = (A \cap B) \cup (A \cap B')$

$= A \cap (B \cup B')$

$= A \cap U$

$= A$

(iii) $(A \cap B) \cup (A \cap B') \cup (A' \cap B)$

$= [(A \cap B) \cup (A \cap B')] \cup (A' \cap B)$

$= [A \cap (B \cup B')] \cup (A' \cap B)$

$= (A \cap U) \cup (A' \cap B)$

$= A \cup (A' \cap B)$

$= (A \cup A') \cap (A \cup B)$

$= U \cap (A \cup B)$

$= A \cup B$

Example 10 :

If $A = \{1,2,3\}$ $B = \{3, 4, 5\}$ find $A \times B$ and $B \times A$.

$A \times B = \{ (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)$
$(3, 3), (3, 4), (3, 5) \}$ ... (1)

$B \times A = \{ (3, 1) (3, 2), (3, 3), (4, 1), (4, 2), (4, 3),$
$(5,1), (5, 2), (5, 3) \}$ ... (2)

From (1) and (2) $A \times B \neq B \times A$.

Example 11 :

If $A = \{1, 4\}$, $B = \{4, 5\}$ $C = \{5, 7\}$ find

(a) $(A \times B) \cup (A \times C)$

(b) $(A \times B) \cap (A \times C)$ (B.Com., May '85)

$A \times B = \{ (1, 4) (4, 4) (1, 5) (, 4, 5) \}$

$A \times C = \{ (1, 5) (1, 7) (4, 5), (4, 7) \}$

$(A \times B) \cup (A \times C) = \{ (1, 4) (1, 5) (1, 7) (4, 4) (4, 5) (4, 7) \}$

$(A \times B) \cap (A \times C) = \{ (1, 5) (4, 5) \}$

Example 12 :

If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $S = \{1, 3, 4\}$

$T = \{2, 4, 5\}$ verify that,

$(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$

$A \times B = \{ (1, 2)\ (1, 3)\ (1, 4)\ (2, 2)\ (2, 3)\ (2, 4)\ (3, 2)\ (3, 3)\ (3, 4) \}$

$S \times T = \{ (1, 2)\ (1, 4)\ (1, 5)\ (3, 2)\ (3, 4)\ (3, 5)\ (4, 2)\ (4, 4)\ (4, 5) \}$

$(A \times B) \cap (S \times T) = \{ (1, 2)\ (1, 4)\ (3, 2)\ (3, 4) \}$ ... I

$A \cap S = \{1, 3\}$, $B \cap T = \{2, 4\}$

$(A \cap S) \times (B \cap T) = \{ 1, 2)\ (1, 4)\ (3, 2)\ (3, 4) \}$ ... II

$\therefore$ From I and II,

$(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$

Example 13 :

In a survey of 5000 persons, it was found that 2,800 read Indian Express and 2,300 read statesman while 400 read both papers. How many read neither Indian Express nor statesman?

(B.Com., Sep. '85)

Let E be the set of people reading Indian Express and S be the set of people reading statesman.

$$n(E) = 2{,}800$$
$$n(S) = 2{,}300$$
$$n(E \cap T) = 400$$

$\therefore$ Number of people reading either Indian Express or statesman is

$$n(E \cup T) = n(E) + n(T) - n(E \cap T)$$
$$= 2800 + 2300 - 400$$
$$= 4700.$$

$\therefore$ Number of people who read neither Indian Express nor statesman $= 5000-4700$
$= 300.$

Example 14 :

In a city three daily newspapers A,B,C are published; 42% of the people in that city read A; 51% read B;68% read C; 30% read both A and B; 28% read B and C; 36% read both A and C; 8% do not read any of the three newspapers. Find the percentage of persons who read all the three papers?

$n(A) = 42 \qquad n(A \cap B) = 30$

$n(B) = 51 \qquad n(B \cap C) = 28$

$n(C) = 68 \qquad n(A \cap C) = 36$

8% do not read any of the three newspapers $\therefore$ 92% read atleast one of the newspapers.

$\therefore \; n(A \cup B \cup C) = 92$

$$n(A \cup B \cup C) = n(A)+n(B)+n(C)-n(A \cap B)-n(A \cap C)-n(B \cap C)+n(A \cap B \cap C)$$

$\therefore \; 92 = 42+51+68-(30+28+36)+n(A \cap B \cap C)$

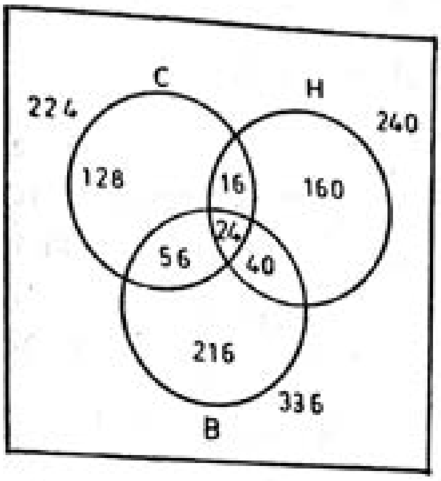
$\therefore \; n(A \cap B \cap C) = 92-161+94$
$= 186-161 = 25$

$\therefore$ 25% of the people read all the three newspapers.

Example 15 :

Out of 880 boys in a school, 224 played cricket, 240 played hockey and 336 played Basketball; Of the total 64 played both basketball and hockey; 80 played cricket and basketball and 40 played cricket and hockey. 24 played all the three games. How many did not play any of the games and how many played only one game?

Let C, H, B denote the set of players who played cricket, hockey and basketball respectively. We represent the above in the following Venn diagram.



Number of boys who played only cricket,

$= 224 - (16+24+56) = 128.$

Number of boys who played hockey only,

$= 240 - (16+24+40) = 160.$

Number of boys who played basketball only,

$= 336 - (56+24+40) = 216.$

$\therefore$ Number of boys who played atleast one game,

$= 128+160+216+16+24+56+40$

$= 640.$

$\therefore$ Number of boys who did not play any game

$= 880-640 = 240$

Number of boys who played only one game

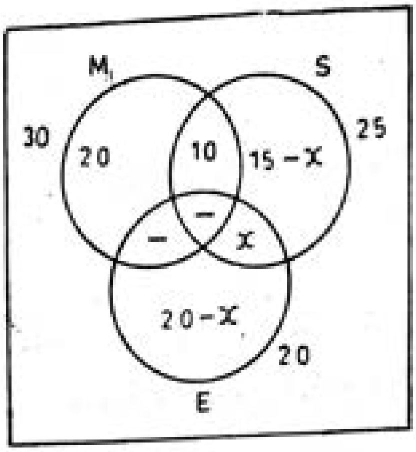
$= 128+160+216 = 504.$

Example 16 :

Out of a group of 50 teachers in a High school 30 teach Mathematics, 20 teach English and 25 teach Science. 10 teach both Mathematics and science, and none teach mathematics and English.

(1) How many teach Science and English?

(2) How many teach only English?



Total number of Teachers = 50

Let x teach Science and English

Then 15−x teach only science; 20−x teach only English ; 20 teach only Mathematics.

$\therefore \quad 50 = 20+10+15-x+x+20-x$

$x = 65-50$

$\therefore \quad x = 15$

(1) Number of teachers teaching science and English = 15,

(2) Number of teachers teaching only English

$= 20-x = 20-15 = 5$

Exercise 1 :

(1) Define (i) Subset

(ii) a proper sub-set

(iii) null set. Give examples.

(2) Express the following sets in tabulation method.

(1) The set of all days in a week.

(2) The set of all even positive integers less than 25.

(3) The set of all numbers whose square is less than 100.

(4) The set of all positive roots of the equation $x^2-7x+10 = 0$.

(5) The set of all vowels in English alphabet.

(6) The set of all primes between 30 and 50.

(7) The set of all reciprocals of integers which are integers.

(8) $\{ x \mid x$ is a prime less than $15 \}$

(9) $\{x \mid x \in N, \ x^2 < 50\}$

(10) $\{ x \mid x$ is a root of $x^2-3x+2 = 0 \}$

(11) $\{x \mid x \in N, \ 3 < x < 10 \}$

(3) Express the following sets in set builder form.

(i) The set of even positive integers less than 9.

(ii) The set of all months in a year.

(iii) $\{ 1, 2, 3, \ldots\ldots\ldots\ldots \}$

(iv) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots \}$

(v) $\{ 1, 4, 9, 16, \ldots\ldots\ldots\ldots \}$

(vi) $\{ 0, 3, 8, 15, 24, 35, 48, \ldots\ldots\ldots\ldots \}$

VII The set obtained by adding 2 to each of the odd natural numbers.

(4) List the elements in the following sets :

(I) $\{x \in N : x = 2y, \ y \in N, \ 2 < y < 5 \}$

(2) $\{x \in Z : 3x-2 = 3\}$

(3) $\{x \in Z : x^2 = 5x-4\}$

(4) $\{x : x = \text{a odd prime and } x < 15\}$

(5) $\left\{x : y \in N,\ x = \dfrac{y+1}{y}\right\}$

(5) Define finite and infinite sets. Give examples.

(6) You are given the following numbers.

$-2{\cdot}8,\ -4{\cdot}3,\ -4,\ 0{\cdot}5,\ 2{\cdot}1,\ 1{\cdot}7,\ -3{\cdot}3,\ 4$

Find $A = \{x \mid x \geqslant -4\}$
$B = \{x \mid x \leqslant 3\}$

(7) If $A = \{\frac{1}{3}, \frac{1}{2}, \frac{2}{5}, \frac{3}{8}, \frac{4}{7}, \frac{1}{10}, \frac{7}{8}\}$

Write the elements of the following sets.

$P = \{x \mid x < \frac{1}{2}\}$

$Q = \{x \mid x > \frac{1}{2}\}$

(8) State whether the following statements are true or false.

(1) The set $\{0\}$ is an empty set.

(2) The null set is a subset of every set, including itself.

(3) ϕ is a proper subset of itself.

(4) If two sets are equal then they are equivalent.

(5) If two sets are equivalent then they are equal.

(6) $5 \in \{3, 5, 7, 9\}$

(7) $\{5\} \in \{3, 5, 7, 8\}$

(8) If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{\text{whole numbers less than } 6\}$

then $A = B$.

(9) If A and B are disjoint sets then $A \cap B = \phi$

(10) The empty set has no element.

(9) State whether the following statements are true or false.

(i) $1 \in \{1, 2\}$

(ii) $\{1\} \in \{1, 2\}$

(iii) $1 \subset \{1, 2\}$

(iv) $\{1\} \subset \{1, 2\}$

(v) $\{1\ 3\} \subset \{1, 2, 1, 2, 1\}$

(vi) $a \in \{9, \{a\}\}$

(vii) $\{3\} \supset \{1, 3\}$

(viii) $\{1, 2\} \subseteq \{2, 1\}$

(10) State whether the following are true or false.

(i) The empty set has no subset.

(ii) If $x \in A'$, then $x \in (A')'$

(iii) $A \cap \phi = A$.

(vi) A finite set with n elements has 2^{n-1} proper subsets.

(v) If $A = \{2, 4, 6, 8, \ldots\ldots\ldots\ldots\}$ and $B = \{x \mid x-6x +8 = 0\}$ then $A \cap B = \{2, 4\}$

(vi) If $A \cap B = \phi$ then $A \cap \phi = B$.

(vii) If $S = \{x \mid 3 < x \leqslant 20, x \in N\}$ then $3 \in S$.

(viii) If N = set of all natural numbers and

Q = set of all rational numbers then N and Q are disjoint.

(11) Given $A = \{1, 2, 3, 4\}$

$B = \{2, 4, 6, 8\}$

$C = \{2, 4, 5, 6\}$

List the elements of the sets $(A \cup B) \cup C$ and $A \cup (B \cup C)$.

(12) If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$B = \{1, 2, 3, 4, 5\}$
$C = \{2, 4, 6, 8\}$
$D = \{4, 5, 6, 7\}$

find (1) $B \cup C$ (2) $B \cap D$ (3) $(B \cup C) \cap D$
(4) $(A \cup D) \cup B$ (5) $(B \cup C) \cap (A \cup D)$.

(13) $U = \{1, 2, 3, 4, 5, 6, 7\}$ where U is the universal set, $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{3, 4, 5, 6\}$. verify that,

(i) $(A \cap B)' = A' \cup B'$

(ii) $A - B = A \cap B' = B' - A'$

(iii) $A - B = A \cap B' \subset A \cup B$.

(14) Given that $A = \{0, 1, 3, 5\}$ $B = \{1, 2, 4, 7\}$
$C = \{1,2,3,5,8\}$ prove that

(i) $(A \cap B) \cap C = A \cap (B \cap C)$

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$

(iii) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(iv) $(A \cap C) \cup B = (A \cup B) \cap (C \cup B)$

(15) If $A = \{a, b, c, d, e\}$ $B = \{a, c, e, q\}$ and $C = \{b, e, f, q,\}$ verify that

(i) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(ii) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

(16) If $A = \{1, 4\}$, $B = \{4, 5\}$, $C = \{5, 7\}$ find

(i) $(A \times B) \cup (A \times C)$

(ii) $(A \times B) \cap (A \times C)$ (B.Com. May '85)

(17) If $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$ verify that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(18) If A and B are any two sets, verify by venn diagram the following :

(i) $A \cup B = (A - B) \cup B$

(ii) $(A \cup B) \cap A = A$

(iii) $A - (A - B) = A \cap B$

(iv) $A - B = A \cap B' = B' - A'$

(19) If A and B are any two sets, prove the results given in exercise (18).

(20) Prove that

(i) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(ii) $A \cap (B - C) = (A \cap B) - C$.

(21) The college newspaper reports that the combined membership of the Mathematics club and the Commerce club is 125 students. What is the total membership of the Commerce club if 50 students are known to be members of the Mathematics club and 30 students are members of both clubs?

(22) In a survey of 1000 consumers, it is found that 720 consumers liked product A and 450 liked product B. What is the least number that must have liked both the products?

(23) In a graduate course of 200 students of a college, records indicate 80 students have taken physics, 90 have taken biology, 35 have taken chemistry, 32 have taken both biology and physics, 23 have taken both chemistry and physics, 10 have taken both biology and chemistry, and 8 have taken all the three subjects. How many have not taken any of the three subjects?

(24) Out of 600 boys in a college, 168 play cricket, 180 play hockey, 252 play foot-ball, 48 play both foot-ball and hockey, 60 play cricket and foot-ball, 30 play cricket and

hockey, and 18 play all the three. How many do not play any game and how many play one game?

(B.Com. Nov-'84)

(25) A company studies the product preferences of 300 consumers. It was found that 226 liked product A, 51 liked product B and 54 liked product C, 21 liked products A and B, 54 liked products A and C, 39 liked products B and C and 9 liked all the three products. Prove that the study results are not correct; assume that each consumer likes atleast one of the three products.

(26) In a survey of 100 students it was found that 50 used the college library books, 45 had their own books and 30 used borrowed books; 20 used college library and their own books only; 15 used their own books and borrowed books only, whereas 10 used college library books and borrowed books only and 5 used all the three sources.

(i) How many did not use any of the three sources?

(ii) How many used their own books only?

(iii) How many used only one source of books?

(27) A salesman visits 274 housewives in a town to find out their views about three products A, B and C. He finds that 157 use A, 98 use only A, 22 use all the three, 14 use A and C but not B, 39 use B and C, 48 use only B.

(i) Which product is most popular according to his inquiry?

(ii) How many use product C only?

(iii) What fraction use atleast two products?

(iv) What per cent use only one of the products?

ANSWERS

CHAPTER - 2

Sets (page 2.30)

2. (1) { Sun, Mon, Tue, Wed, Thu., Fri., Sat. }

(2) { 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24 }

(3) { 1, 2, 3, 4, 5, 6, 7, 8, 9 } (4) { 2, 5 }

(5) { a, e, i, o, u } (6) { 31, 37, 41, 43, 47 }

(7) { 1 } (8) { 2, 5, 7, 11, 13 }

(9) { 1, 2, 3, 4, 5, 6, 7 } (10) { 1, 2 }

(11) { 4, 5, 6, 7, 8, 9 }

3. (i) $\{x/x \in 2N, x < 9\}$ (ii) $\{x/x$ is a month in a year $\}$

(iii) $\{x/x \in N\}$ (iv) $\left\{x/x = \frac{1}{y}, y \in N\right\}$

(v) $\{x/x = y^2, y \in N\}$ (vi) $\{x/x = n^{2-1}, n \in N\}$

(vii) $\{x/x = (2n-1) + 2, n \in N\}$

4. (1) { 6, 8, 10 } (2) ϕ (3) { 1, 4 }

(4) { 3, 5, 7, 11, 13 } (5) $\left\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots\ldots\right\}$

6. A = { –2.8, 4, 0.5, 2.1, 1.7, –3.3, 4 }

B = { –2.8, –4.3, –4, 0.5, 2.1, 1.7, –3.3 }

7. $P = \left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{8}, \frac{1}{10}\right\}$ $Q = \left\{\frac{4}{7}, \frac{7}{8}\right\}$

8. (1) F (2) T (3) F (4) T (5) F

(6) T (7) F (8) F (9) T (10) F

9. (i) T (ii) F (iii) F (iv) T (v) F

(vi) F (vii) F (viii) T

10. (i) T (ii) F (iii) F (iv) T (v) F

(vi) F (vii) F (viii) F

11. $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8, \}$
$A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$

12. (1) $B \cup C = \{1, 2, 3, 4, 5, 6, 8\}$ (2) $B \cap C = \{4,5\}$
(3) $\{4,5,6\}$ (4) $\{1,2,3, \ldots\ldots\ldots\ldots 9\}$
(5) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

16. (i) $\{(1,4),(1,5),(1,7),(4,4),(4,5),(4,7)\}$ (ii) $\{(1,5),(4,5)\}$

21. 105 22. 170 23. 52 24. 120, 378 26. 15, 15, 50

27. Assume that everyone uses atleast one product
(i) *Product A* (ii) 52 (iii) $\frac{38}{99}$ (iv) $\frac{198}{174} \times 150$

Relations and Functions (Page 48)

(1) Not a function (2) Range is { a, b } (4) $1 + \frac{1}{x^2}$

(5) $\{0, 3, 8\}$ (8) Not a function (9) Commutative

(10) Not a function (11) $\{0, 2, 5\}$ (12) $1 \to 2, 3 \to 4, 5 \to 6, 7 \to 8$

(14) Not a function (15) $\{(2, 2), (4, 3), (6, 4), (8, 5)\}$

(17) $m = 12$ (18) $\{6, 2, 4\}$ (20) many to one function

(21) $(x + 1)^2 + 1$ (22) range $\neq$ co-domain (24) $4(2x - 10)^2$

(25) $a = 3$ (27) (i) many to one functions (ii) Constant functio

(iii) one-one function (iv) many-one function

(28) one to one (not an onto function).

CHAPTER - 2

Ratio and Proportion (Page 58)

(1) $\frac{49}{9}$ (15) 4 (16) Rs 600 and Rs 700

(17) 3 litres from A and 8 litres from B (18) Rs 500 and Rs 300

(19) Rs 8.10, Rs 2.10 (20) 49 % (21) 49 Kg

Chapter 3

Relations

A binary relation or simply a relation R from a set A to a set B assigns to each pair (a, b) in $A \times B$ exactly one of the following

(i) "a" is related to "b", written as $a\,R\,b$

(ii) "a" is not related to "b" written as $a \not R\, b$

A relation from a set A to the same set is called a relation on A.

Example 1 : " Subset of " is a relation in a class of sub sets. For any pair of subsets A and B, $A \subset B$ or $A \not\subset B$.

Example 2 : "less than" ($<$) is a relation in the set of real numbers. For a given ordered pair a, b of real numbers either $a < b$ or $a \not< b$.

Example 3 : Perpendicularity is a relation in the set of lines for any given pair of lines. For any given pair of lines (l_1, l_2) either l_1 is perpendicular to l_2 or l_1 is not perpendicular to l_2.

Hence any relation from a set A to a set B uniquely defines a subset of $A \times B$.

Definition: A relation R from a set A to a set B is a subset of $A \times B$.

Example 1: Let $A = \{1, 2, 3\}$, $\qquad B = \{2, 5\}$

Then $R = \{(1, 2), (1, 5)\ (2, 3), (2, 5)\}$ is a relation.

Example 2: Let S be the set of integers and let the relation xRy be defined such that $x - y$ is an even integer.

Here R is defined by

$R = \{(x, y)\colon\ x, y \in S,\ x - y = 2n,\ n \in N\}$

Example 3: Let $A = \{1, 2, 3, 4\}$

$B = \{4, 5\}$

Let R be the relation "less than"

Then the ordered pairs belonging to R are

$= \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

Note that $R \subset A \times B$

Domain and Range of a relation

Let R be the relation from A to B. The domain D of the relation R is the set of all first elements of the ordered pairs which belong to R.

(*i.e.,*) $D = \{x / x \in A$ and $(x, y) \in R$ for some $y \in B\}$

The range E of the relation R is the set of all second elements of the ordered pairs, which belong to R.

(*i.e.,*) $E = \{y / y \in B$ and $(x, y) \in R$ for some $x \in A\}$

Note that $D \subset A$ and $E \subset B$

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$

Every subset of $A \times B$ is a relation from A to B.

If $R = \{(2, a), (4, b), (4, c)\}$. Then the domain of D is the set $\{2, 4\}$ and the range of R is the set $\{a, b, c\}$.

Number of distinct relations from a set A to a set B

Suppose the set A has m elements and the set B has n elements. Then the product set $A \times B$ will have mn elements. Therefore the power set of $A \times B$ will have 2^{mn} elements. Thus $A \times B$ has 2^{mn} different subsets. Every subset of $A \times B$ is a relation from A to B and hence 2^{mn} different relations from A to B.

Inverse Relation: Let R be a relation from A to B. The inverse of relation R is the relation from B to A which consists those ordered pairs which when reversed belong to R and the inverse relation is denoted by R^{-1}.

i.e., $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Example 1: Let $A = \{a, b, c\}$

$B = \{1, 2, 3\}$

and $R = (a, 1), (a, 3)\ (b, 3), (c, 3)\}$.

$R^{-1} = \{(1,a), (3,a), (3,b), (3,c)\}$

Example 2: The inverse of the relation defined "x is the husband of y" is 'x is the wife of y'.

Example 3: x is greater than y for $x, y \in Z^+$

The inverse relation; x is less than y, $x, y \in Z^+$

Theorem: If R is a relation from A to B, the domain of R is identical with the range of R^{-1} and the range of R is identical with the domain of R^{-1}.

Domain and Range of a relation

Let R be the relation from A to B. The domain D of the relation R is the set of all first elements of the ordered pairs which belong to R.

(*i.e.*,) $D = \{x / x \in A \text{ and } (x, y) \in R \text{ for some } y \in B\}$

The range E of the relation R is the set of all second elements of the ordered pairs, which belong to R.

(*i.e.*,) $E = \{y / y \in B \text{ and } (x, y) \in R \text{ for some } x \in A\}$

Note that $D \subset A$ and $E \subset B$

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$

Every subset of $A \times B$ is a relation from A to B.

If $R = \{(2, a), (4, b), (4, c)\}$. Then the domain of D is the set $\{2, 4\}$ and the range of R is the set $\{a, b, c\}$.

Number of distinct relations from a set A to a set B

Suppose the set A has m elements and the set B has n elements. Then the product set $A \times B$ will have mn elements. Therefore the power set of $A \times B$ will have 2^{mn} elements. Thus $A \times B$ has 2^{mn} different subsets. Every subset of $A \times B$ is a relation from A to B and hence 2^{mn} different relations from A to B.

Inverse Relation: Let R be a relation from A to B. The inverse of relation R is the relation from B to A which consists those ordered pairs which when reversed belong to R and the inverse relation is denoted by R^{-1}.

i.e., $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Example 1: Let $A = \{a, b, c\}$

$B = \{1, 2, 3\}$

and $R = (a, 1), (a, 3) (b, 3), (c, 3)\}$.

$R^{-1} = \{(1,a), (3,a), (3,b), (3,c)\}$

Example 2: The inverse of the relation defined "x is the husband of y" is 'x is the wife of y'.

Example 3: x is greater than y for $x, y \in Z^+$

The inverse relation; x is less than y, $x, y \in Z^+$

Theorem: If R is a relation from A to B, the domain of R is identical with the range of R^{-1} and the range of R is identical with the domain of R^{-1}.

Proof: Let $y \in R^{-1}$ domain of R^{-1}. Then $y \in B$ and there exists a $x \in A$ such that $(y, x) \in R^{-1}$

This means $(x, y) \in R$

$\therefore$ y belongs to the range of R.

$\therefore$ Domain of $R^{-1} \subset$ range of R .... (1)

Similarly we can show that range of $R^{-1} \subset$ domain of R .... (2)

Hence from (1) and (2), Domain of R^{-1} = Range of R

Similarly we can prove that domain of R = Range of R^{-1}

Identity Relation

Let A be a set. The relation I_A defined by $I_A = \{(x, y) : x \in A, y \in A, x = y\}$ is called the *identity relation* in A. Thus the identity relation in a set is the set of all ordered pairs (x, y) of $A \times A$ for which $x = y$.

If $A = \{1, 2, 3, 4, 5\}$

$I_A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

Let A be any set and R be the set $A \times A$. Then R is called the *universal relation* in A. Since the null set is a subset of $A \times A$, the null set ϕ is also a relation in A. This relation is called the *void relation.*

Equivalence relation: We now study a special type of relation called equivalence relation. To introduce equivalence relation, we need to introduce the following properties.

I Reflexive Relation

A relation R in a set A is called *reflexive* if aRa *i.e.,* $(a, a) \in R$, for every $a \in A$.

Example 1: Let R be the relation *similarity* in the set of triangles in the plane. Then R is a reflexive relation since every triangle is similar to itself.

Example 2: Let $A = \{1, 2, 3, 4\}$

Let $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

Then R is a reflexive relation

But $R_1 = \{(1, 1), (3, 3), (4, 4)\}$ is not a reflexive relation since $(2, 2) \notin R$

Example 3: Let N be the set of all natural numbers.

The relation R in N defined by "x is a divisor of y" is a reflexive relation since every natural number is a divisor of itself.

II Symmetric Relation

A relation R in a set is *a symmetric* if whenever aRb then bRa.

$$i.e., \text{ if } (a,b) \in R \text{ then } (b,a) \in R.$$

Example 1: Let A be the set of all lines in a plane. The relation R in A defined by "line x is perpendicular to line y" is a symmetric relation since if a line a is perpendicular to a line b then b is perpendicular to a.

Example 2: The relation similarity is a symmetric relation since in the set of all triangles in a plane if a triangle Δ_1 in similar to the triangle to Δ_2 then Δ_2 is similar to Δ_1.

III Antisymmetric relation

Let R be a relation in a set A. Then R is said to be an *antisymmetric* relation if $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$.

Note: R is antisymmetric if we have never both aRb and bRa except when $a = b$.

Example: Let N be the set of all natural numbers. Let R be the relation in N defined by "x is a divisor of y". Then R is *antisymmetric* since a divides b and b divides a implies $a = b$.

IV Transitive relation

A relation R in a set A is called *transitive* if whenever aRb and bRc then aRc.

$$(i.e.,) \text{ if } (a,b) \in R \text{ and } (b,c) \in R \text{ then } (a,c) \in R.$$

Example 1: The relation R of similarity of triangles is transitive since if triangle a is similar to triangle b and triangle b is similar to triangle c then triangle a is similar to triangle c.

Example 2: Let N be the set of all natural numbers and let R be the relation in N defined by 'x is less than y'.

Then $a < b$ and $b < c$ implies $a < c$ and therefore R is transitive.

Example 3: The relation perpendicularity of lines in a plane is *not transitive* for if a line a is perpendicular to a line b and line b is perpendicular to c then a is not perpendicular to c.

Equivalence relation: Let R be a relation in A. R is an equivalence relation in A if and only if

(i) R is reflexive *i.e.*, $\forall a \in A,\ aRa$

(ii) R is symmetric *i.e.*, $aRb, \Rightarrow bRa$

(iii) R is transitive *i.e.*, $aRb, bRc \Rightarrow aRc$

The equivalence relation is denoted by the symbol $\sim$.

Example 1: Let A be the set of all triangles in a plane. Let R be the relation in A defined by xRy if and only if x is congruent to y whenever $x, y \in A$. Then

(i) xRx for every $x \in A$; *i.e.*, R is reflexive

(ii) $xRy \Rightarrow yRx$ *i.e.*, R is symmetric

(iii) xRy, $yRz \Rightarrow xRz$ *i.e.*, R is transitive.

Since R is reflexive, symmetric and transitive R is an equivalence relation.

Example 2: Let R be the relation "parallel to" in the set of all lines in a plane.

(i) R is reflexive since $a \|^{l}$ to a for every a

(ii) R is symmetric since $a \|^{l}$ to b then $b \|^{l}$ to a

(iii) R is transitive since $a \|^{l}$ to b, $b \|^{l}$ to c then $a \|^{l}$ to c

R is an equivalence relation since it is reflexive, symmetric and transitive.

Equivalence class: Let A be a non empty set and R be an equivalence relation in A. Further let a be an arbitrary element of A. The element $x \in A$ satisfying xRa constitute a subset A_a of A called an equivalence class of a with respect to R. We shall denote this equivalence class by A_a or $[A]$

Thus $A_a = [a] = \{x : x \in A$ and $(x, a) \in R$ or $xRa\}$

Note: The set of all equivalence classes is denoted by X/R and this is called the quotient set of X by R.

Example 1: Let A be the set of all triangles in a plane and R be an equivalence relation in A defined by "x is congruent to y", $x \in A$, $y \in A$. When $a \in A$ the equivalence class $[a]$ is the set of all triangles of A congruent to the triangle a. The equivalence class $[b]$ is the set of all triangles congruent to the triangle b and so on.

Example 2: Let us now obtain the equivalence classes in the set I of all integers with respect to the equivalence relation "congruent to modulo 5". The integer congruent to 0 modulo 5 form an equivalence class denoted by I_a. The elements of the class are $5k$ where k is an integer. The integers congruent to 1 modulo 5 form an equivalence class denoted by I_1. The elements of this class are $5k$+1 and the equivalence class is denoted by I_1. The integers congruent to 2 modulo 5 form an equivalence class denoted by I_2. The elements of this class are denoted by I_2. The elements of this class are $5k$+2. Similarly we define

I_3 and I_4. Thus, we see that the set of all integers can be divided into the following five equivalence classes with respect to the relation congruent to modulo 5.

$$I_0 = \{........ -10, -5, 0, 5, 10, ...\}$$

$$I_1 = \{......, -9, -4, 1, 6, 11, ...\}$$

$$I_2 = \{......., -8, -3, 2, 7, 12, ...\}$$

$$I_3 = \{......., -7, -2, 3, 8, 13, ...\}$$

$$I_4 = \{......... -6, -1, 4, 9, 14, ...\}$$

Observe that each integer x which is uniquely expressible in the form $x = 5q + r$ where $0 \le r < 5$ is a member of the equivalence class I_r where r is the remainder. Note that the equivalence classes are pairwise disjoint and that

$$I = I_0 \cup I_1 \cup I_2 \cup I_3 \cup I_4$$

Properties of Equivalence Classes

Let A be a non empty set and R be an equivalence relation in A. Let a, b the two arbitrary elements in A. Then

(i) $a \in [a]$

(ii) If $b \in [a]$ then $[b] = [a]$

(iii) $[a] = [b]$ if and only if aRb.

(iv) Either $[a] = [b]$ or $[a] \cap [b] = \phi$

Proof: (i) Since R is reflexive, aRa

But $[a] = \{x \ : \ x \in A$ and $x R a]$

$\therefore a R a \Rightarrow a \in [a]$

(ii) Given $b \in [a]$

This means bRa

Let x be an arbitrary element of $[b]$

Then $x \in [b] \Rightarrow xRb$

But R is transitive.

$\therefore x R b$ and $b R a \Rightarrow x R a \Rightarrow x \in [a]$

$\therefore x \in [b] \Rightarrow x \in [a]$

$\therefore [b] \subset [a]$ .... (1)

Let y be an arbitrary element of $[a]$

Then $y \in [a] \Rightarrow y\,R\,a$

Since R is symmetric, $b\,R\,a \Rightarrow a\,R\,b$.

yRa and $a\,R\,b \Rightarrow y\,R\,b$

$\therefore y \in [b]$

i.e., $y \in [a] \Rightarrow y \in [b]$

$\therefore [a] \subset [b]$ .... (2)

From (1) and (2) $[a] = [b]$

(iii) Suppose $[a] = [b]$

Then we will prove that aRb

Since R is reflexive, we have aRa

$$aRa \Rightarrow a \in [a]$$
$$\Rightarrow a \in [b] \qquad \because [a] = [b]$$
$$\Rightarrow aRb$$

$\therefore [a] = [b] \Rightarrow aRb$

Converse: Suppose aRb

Then we will prove that $[a] = [b]$

Let x be an arbitrary element of $[a]$

Then xRa

But it is given that aRb

$\therefore xRa$ and $aRb \Rightarrow xRb$

$$\Rightarrow x \in [b]$$

i.e., $x \in [a] \Rightarrow x \in [b]$

$\therefore [a] \subset [b]$ .... (1)

Let y be an arbitrary element of $[b]$

Then $y \in [b] \Rightarrow yRb$

We are given that aRb

Since R is symmetric, bRa

yRb and $bRa \Rightarrow yRa$

This implies $y \in [a]$

Then $y \in [b] \Rightarrow y \in [a]$

$\therefore [b] \subset [a]$ .... (2)

From (1) and (2), $[a] = [b]$

Since $[a] = [b] \Rightarrow aRb$,

and $aRb \Rightarrow [a] = [b]$,

$[a] = [b]$ if any only if aRb

(iv) If $[a] \cap [b] = \phi$ then the two equivalent classes are disjoint.

If $[a] \cap [b] \neq \phi$ then we will prove that $[a] = [b]$

Since $[a] \cap [b] \neq \phi$, then there exists an element $x \in A$ such that $x \in [a] \cap [b]$

$x \in [a] \cap [b] \Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow xRa$ and xRb

$\Rightarrow aRx$ and xRb ($\because$ R is symmetric)

$\Rightarrow aRb$ ($\therefore$ R is transitive)

$\Rightarrow [a] = [b]$

$\therefore$ If $[a] \cap [b] \neq \phi$ then $[a] = [b]$

i.e., if $[a] \neq [b]$ then $[a] \cap [b] = \phi$

Partition: A partition of a set X is a subdivision of X into subsets which are disjoint and whose union is X. *i.e.*, such that $a \in X$ belongs to one and only one of the subsets. The subsets in a partition are called cells.

Thus, the collection $\{A_1, A_2, \ldots\ldots, A_n\}$ of subsets of X is a partition if and only if

(i) $X = A_1 \cup A_2 \cup \ldots\ldots\ldots \cup A_n$

(ii) For any A_i, A_j, either $A_i = A_j$ or $A_i \cap A_j = \phi$

Example 1: Consider the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ and its subsets

$B_1 = \{1, 5\}$ $B_2 = \{2, 4, 7\}$

$B_3 = \{3, 6\}$

(i) $A = B_1 \cup B_2 \cup B_3$

(ii) The intersection of any two subsets is ϕ

i.e., $B_1 \cap B_2 = \phi$, $B_2 \cap B_3 = \phi$, $B_1 \cap B_3 = \phi$

$\therefore \{B_1, B_2, B_3\}$ is a partition of A.

Example 2: Let $N = \{1, 2, 3, 4, 5, 6, 7\}$

$A_1 = \{1, 3, 7\}$

$A_2 = \{2, 4, 7\}$

$A_3 = \{5, 6\}$

$\{A_1, A_2, A_3\}$ is not a partition of N since $A_1 \cap A_2 \neq \phi$

Example 3: $N = \{1, 2, 3 .... 9\}$

$A_1 = \{1, 2, 5\}$

$A_2 = \{3, 6, 9\}$

$A_3 = \{4, 8, 3\}$

$\{A_1, A_2, A_3, A_3\}$ is not a partition of N since $N \neq A_1 \cup A_2 \cup A_3$

Example 4: Let $X = \left\{1,2,5,a,b,c,\sqrt{2},\sqrt{3},e,u,\pi,\frac{1}{2},\frac{1}{3},3,4,\frac{1}{4}\right\}$

Let $A_1 = \{1, 2, 3, 4, 5\}$

$A_2 = \left\{\frac{1}{2},\frac{1}{3},\frac{1}{4}\right\}$

$A_3 = \{a,b,c,u\}$

$A_4 = \{\sqrt{2}, \sqrt{3}, \pi, e\}$

Also $A_i \cap A_j = \phi$, $i, j = 1, 2, 3, 4, i \neq j$ and $X = A_1 \cup A_2 \cup A_3 \cup A_4$

$\therefore \{A_1, A_2, A_3\}$ is a partition of X.

Theorem: An equivalence relation defined on a set decomposes the set into disjoint equivalent classes.

Proof: Let an equivalence relation R *be* defined on S. Let $a \in S$ and T be a subset of S consisting of all those elements which are equivalent to a.

i.e., $T = \{x: \ x \in S \ \text{ and } \ xRa\}$

Then $a \in T$, for aRa (since R is reflexive)

Any two elements of T are equivalent to each other for if $x, y \in T$ then xR and yRa.

Again, $xRa, yRa \Rightarrow xRa, aRy$ $(\because R$ is symmetric)

$\Rightarrow xRy$, $(\because R$ is symmetric)

$\therefore T$ is an equivalence class.

Let T_1 be another equivalence class.

i.e., $T_1 = \{x: \ x \in S \ \text{and} \ xRb\}$

where b is not equivalent to a. Then the classes T and A must be disjoint. For if they have a common element s, sRa and sRb so that bRa which is contrary to our hypothesis.

$\therefore$ The set S can be decomposed into equivalence classes $T_1, T_2, T_3, \ldots\ldots$ such that every element of S belongs to one of these classes. Since these classes are mutually disjoint,

$\{T_1, T_2, T_3, \ldots\ldots\}$ is a partition of S.

Theorem: If R is an equivalence relation in a non-empty set X then the quotient set X/R is a partition of X.

Proof: Each $x \in X$ must belong to some equivalence class. Also the equivalence classes are pairwise disjoint for if $z \in x/R \cap y/R$ then xRz, yRz. Since $yRz \Rightarrow zRy$, xRz and $zRy \Rightarrow xRy$, it follows that x/R and y/R must be identical. Hence the two are equivalent. Hence two equivalent classes are either identical or disjoint. The set of equivalent classes is therefore a partition. Further if x and y be any two members of the same set of this partition, they stand in a relation to each other, so that the partition induces the relation.

Note: If C be a partition of X, then the induced partition is an equivalence relation whose set of equivalence classes is X/R.

Partially and Totally ordered sets

Definition: A set S is said to be partially ordered by a binary relation R if for arbitrary $a,b,c \in S$.

(i) R is reflexive *i.e.*, aRa

(ii) R is anti-symmetric *i.e.*, aRb and bRa if and only if $a = b$.

(iii) R is transitive *i.e.*, aRb and $bRc \Rightarrow aRc$

The relation R is called a partial order.

Example 1: Let R be the relation in the set of natural numbers defined by "x is a multiple of y" then R is a partial order in N.

Example 2: Let A be a subset of real numbers. Then the relation in A defined by "$x \leq y$" is a partial order in A.

Example 3: The relation R in a set defined by "x is a subset of y" is a partial order in the set of sets".

Note 1: Two elements a and b in a partially ordered set are said to be "not comparable" if either $(a,b) \in R$ or $(b,a) \in R$ or $a = b$

Note 2: If a relation R in a set is reflexive, antisymmetric and transitive, R^{-1} is also reflexive, antisymmetric and transitive.

i.e., if R defines a partial order in A then R^{-1} also defines a partial order in A which is called the inverse order.

Note 3: The word "partial" is used in defining a partial order in a set A because some elements in A need not be comparable. If on the other hand, every two elements in a partially ordered set A are comparable then the partial order in A is called a total order.

Example1: Let B be the partial order in

$$A = \{1, 2, 3, 4, 5, 6\} \text{ defined by " } x/y \text{ ".}$$

Then R is not a total order in A since 3 and 5 are not comparable.

Example 2: If N, the set of all natural numbers be ordered by "x is a multiple of y" then A is not totally ordered since 3 and 7 are not comparable. But the set

$\{2,4,8,.........2^n.....\}$ is totally ordered subset of N.

Illustrative Examples

Example 1: Let R be the relation from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$

Write R as a set ordered pairs and find the inverse relation R^{-1}.

Solution: R has all the ordered pairs $(a, b) \in A \times B$ for which $a < b$

$$R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$$

$$R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$$

Example 2: Let R be the relation from $A = \{2, 3, 4, 5\}$ to $B = \{3, 6, 7, 10\}$ defined by "x divides y".

Write R as a set of ordered pairs and find R^{-1}.

Solution: $R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$

$$R^{-1} = \{(6, 2), (10, 2), (3, 3), (6, 3), (10, 5)\}$$

Example 3: Let $N = \{1, 2, 3 ....\}$ and R be the relation on N defined by $x + 2y = 8$.

i.e., $R = \{(x, y) : x, y \in N, \ x + 2y = 8\}$

Write R as a set of ordered pairs and find R^{-1}.

Solution:

$$N = \{1, 2, 3, ...\}$$

$$x + 2y = 8, \ x, y \in N$$

$$R = \{(2, 3), (4, 2), (6, 1)\}$$

$$R^{-1} = \{(3, 2), (2, 4), (1, 6)\}$$

Example 4: Let I be the set of all integers. Let us define a relation R in I such that xRy holds if and only if $x - y$ divisible by 5, $x \in I, \ y \in I$

i.e., $R = \{(x, y) : x \in I, \ y \in I, \ x - y$ divisible by 5$\}$

Show that R is an equivalence relation.

Solution:

For each $x \in I, x - x = 0$ is divisible by 5.

$\therefore \forall x \in I$, we have xRx and hence R is reflexive.

Suppose xRy. Then $x - y$ is divisible by 5 and hence $y - x = -(x - y)$ is also divisible by 5. $\therefore xRy \Rightarrow yRx$

Hence R is symmetric.

(iii) Suppose xRy and yRz.

Then $x - y$ and $y - z$ are both divisible by 5.

Hence 5 is also a divisor of $(x - y) + (y - z)$

i.e., 5 is also a divisor of $x - z$.

$\therefore xRy, \ yRz \Rightarrow xRz$.

Hence R is transitive and hence is an equivalence relation.

Example 5: Let A be the set of all triangle in a plane. Let R be the relation in A defined by xRy if and only if x is congruent to y, for all $x, y \in A$. Then show that R is an equivalence relation.

Solution: I Every triangle is congruent to itself in (i.e.,) xRx and therefore R is reflexive.

II If xRy then yRx $\therefore$ R is symmetric

III If xRy, yRz, then xRz $\therefore$ R is transitive.

Hence R is reflexive, symmetric and transitive and hence R is an equivalence relation.

Example 6: Let R be the relation $\leq$ in $N = \{1, 2, 3 .....\}$ Determine whether R is an equivalence relation.

Solution: (i) R is reflexive since for $a \leq a$ for $\forall a \in N$

(ii) R is not symmetric since $a \leq b$ does not imply $b \leq a$, $\forall a, b \in R$

(iii) R is transitive since $a \leq b$, $b \leq c \Rightarrow a \leq c$.

Since R is not symmetric, R is not an equivalence relation.

Example 7: Show that for the set of all points in a plane, the relation "at the same distance from the origin as" is an equivalence relation.

Solution:

(i) Every point x is at the same distance from the origin as x.
i.e., xRx and hence R is reflexive

(ii) If the point x is at the same distance from the origin as the point y then y is at the same distance from the origin as the point x.
i.e., if xRy then yRx
Hence R is symmetric

(iii) If x is at the same distance from the origin as y and y is at the same distance from the origin as z then x is at the same distance from the origin as z.
i.e., if xRy, yRz then xRz.
Hence R is transitive
Since R is reflexive, symmetric and transitive,
R is an equivalence relation.

Example 8: Let I be the set of all integers and if the relation R be defined over the set I by xRy if $x - y$ is an even integer where $x, y \in I$, show that R is an equivalence relation.

Solution:

(i) Since $x - x = 0$ is an even integer,
xRx for $x \in I$ $\therefore$ R is reflexive.

(ii) If $x - y$ is an even integer then $y - x$ is also an even integer.
i.e., xRy, then yRx and hence R is symmetric

(iii) Let xRy and yRz
i.e., $x - y$ and $y - z$ are even integers.
Then $x - z = (x - y) + (y - z)$
$= \text{an even integer}$
$\therefore$ if xRy, yRz then xRz

Hence R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

Example 9: Let $N = \{1, 2, 3, ...\}$ and R be the relation $\simeq$ in $N \times N$ defined by

$(a, b) \simeq (c, d)$ if and only if $ad = bc$

Prove that R is an equivalence relation.

Solution:

(i) For every $(a.b) \in N \times N$,

$(a, b) \simeq (a, b)$ since $ab \simeq ba$

Hence R is reflexive

(ii) Suppose $(a, b) \simeq (c, d)$ then $ad = bc$

$ad = bc \Rightarrow cb = da$

Hence $(c, d) \simeq (a, b)$ and hence

R is symmetric.

(iii) Suppose $(a, b) \simeq (c, d)$ and $(c, d) \simeq (e, f)$

Then $ad = bc$ and $ef = de$

Then $(ad)(ef) = (bc)(de)$

i.e., $af = be$

This implies $(a, b) \approx (e, f)$

$\therefore$ R is transitive

Hence R is reflexive, symmetric and transitive and hence R is an equivalence relation.

Example 10: Let $X = \{1, 2, 3, 4\}$

Consider the relations in X

$$R_1 = \{(1, 1), (1, 2)\}$$

$$R_2 = \{(1, 1), (2, 3), (4, 1)\}$$

$$R_3 = \{(1, 3), (2, 4)\}$$

$$R_4 = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_5 = X \times X$$

$$R_6 = \phi$$

Determine whether or not each relation is reflexive

Solution:

(i) R_1 is not reflexive

(ii) R_2 is not reflexive

(iii) R_3 is not reflexive

(iv) R_4 is not reflexive

(v) R_5 is reflexive

(vi) R_6 is not reflexive

Example 11: Show that the relation of congruence modulo m (*i.e.,*) $a = b \ (mod\ m)$ in the set Z is an equivalence relation.

Solution:

$R = \{(x, y) : a - b = km$ for some fixed integer m and $x, y, k \in Z$

(i) Since $x - x = 0.m, \ \forall x \in Z, \ (x,x) \in R$

$\therefore$ R is reflexive

(ii) Let $(x,y) \in R$

Then $x - y = k.m.$

This implies $y - x = (-k)m$ and $-m \in Z$

$\therefore (y, x) \in R$

$\therefore$ R is symmetric

(iii) Let $(x, y) \in R$ and $(y, z) \in R$

Then $x - y = k_1 m$

$y - z = k_2 m$ for some integers k_1, k_2

$x - z = (x - y) + (y - z)$

$= k_1 m + k_2 m$

$= (k_1 + k_2)m$ and $k_1 + k_1 \in Z$

i.e., $(x, z) \in R$

$\therefore$ R is transitive

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

Example 12: If R and S are equivalence relations in X, prove that $R \cap S$ is an equivalence relation in X.

Solution:

(i) For all $x \in X$, $(x,x) \in R$ and $(x,x) \in S$

since R and S are equivalence relations.

$(x,x) \in R, (x, x) \in S \Rightarrow (x,x) \in R \cap S$

i.e., for all $x \in X$, $(x, x) \in R \cap S$

$\therefore R \cap S$ is reflexive

(ii) Since R and S are symmetric,

$$(x, y) \in R \Rightarrow (y,x) \in R$$

$$(x, y) \in S \Rightarrow (y,x) \in S$$

$$(x, y) \in R, (x, y) \in S \Rightarrow (x,y) \in R \cap S$$

$$(y, x) \in S, (y, x) \in S \Rightarrow (y,x) \in R \cap S$$

$$\therefore (x, y) \in R \cap S \Rightarrow (y,x) \in R \cap S$$

Hence $R \cap S$ is symmetric

(iii) Let $(x, y) \in R \cap S$ and $(y,z) \in R \cap S$

$$(x, y) \in R \cap S \Rightarrow (x,y) \in R \text{ and } (x,y) \in S$$

$$(y, z) \in R \cap S \Rightarrow (y,z) \in R \text{ and } (y,z) \in S$$

$$(x, y) \in R, (y,z) \in R \Rightarrow (x,z) \in R \quad (\because R \text{ is transitive})$$

$$(x, y) \in S, (y,z) \in S \Rightarrow (x,z) \in S \quad (\because R \text{ is transitive})$$

$$(x, z) \in R \text{ and } (x,z) \in S \Rightarrow (x,z) \in R \cap S$$

$\therefore R \cap S$ is transitive

Since $R \cap S$ is reflexive, symmetric and transitive

$R \cap S$ is an equivalence relation.

Example 13: Give an example of a relation which is

(i) symmetric and reflexive but not transitive

(ii) symmetric and transitive but not reflexive

(iii) reflexive but neither symmetric and transitive

(iv) transitive but neither symmetric nor reflexive

(v) transitive and antisymmetric but not reflexive.

Solution:

Let X be the set of all points in a plane

Define a relation R such that for $x, y \in X$, xRy if and only if the distance between x and y is less than one unit.

(i) The relation is symmetric and reflexive but not transitive.

(ii) The relation aRb in the set of integers "if a and b are both odd" is symmetric and transitive but not reflexive.

(iii) See problem number 19 in exercise

(iv) The relation "greater than" in the set of natural numbers is transitive but neither reflexive nor symmetric.

(v) In the set of all real numbers, the relation "less than" is transitive and antisymmetric but not reflexive.

Example 14: Find all the partition of $X = \{a, b, c, d\}$

Solution:

Each partition of X contains either one, two, three or four distinct sets.

The partitions of X are

(i) $\{a, b, c, d\}$

(ii) $[\{a\},\{b,c,d\}]$, $[\{b\},\{a,c,d\}]$, $[\{c\}, \{a,b,d\}]$

$[\{d\}, \{a,b,c\}]$, $[\{a,b\}, \{c,d\}]$, $[\{a,c\}, \{b,d\}]$

$[\{a,b\}, \{b,c\}]$

(iii) $[\{a\},\{b\},\{c,d\}]$, $[\{a\},\{c\}\{b,d\}]$, $[\{a\}, \{d\},\{b,c\}]$

$[\{b\}, \{c\},\{a,d\}]$, $[\{b\}, \{d\} \{a,c\}]$

$[\{c\}, \{d\}, \{a,b\}]$

(iv) $[\{a\}, \{b\}, \{c\},\{d\}]$

There are 15 different partitions of X

Example 15: Let $X = \{1, 2, 3, 4, 5, 6\}$

Determine whether each of the forthwing is a partition of X.

(i) $[\{1,3,5\}, \{2,4\}, \{3,6\}]$

(ii) $[\{1,5\}, \{2\}, \{3,6\}]$

(iii) $[\{1,5\}, \{2\}, \{4\}, \{1,5\},\{3,6\}]$

(iv) $[\{1,2,3,4,5,6\}]$

Solution:

(i) Not a partition; {1, 3, 5}, {3, 6} are not disjoint.

(ii) Not a partition of X since 4 does not belong to any subsets of X.

(iii) It is a partition.

(iv) It is a partition.

Example 16: Show that in the set of all natural numbers the set R is defined as aRb if any only if a divides b is a partial order relation.

Solution:

(i) For all $a \in N$, a is a divisor of a.

i.e., aRa $\therefore R$ is reflexive

(ii) Also if a is a divisor of b then b is a divisor of a if and only if $a = b$

i.e., aRb and $bRa \Rightarrow a = b$ $\therefore R$ is antisymmetric

If a is a divisor of b and b is a divisor of c then a is a divisor of c.

$\therefore$ R is transitive

Since R is reflexive, antisymmetric and transitive, R is a partial order relation.

Example 17: If R is defined by "x is divides y" find out whether or not each of the following subsets of N is totally ordered.

(a) {3, 6, 3, 9}

(b) {7, 77, 11}

(c) {3, 6, 24, 12}

(d) {1, 2, 3, 4.....}

(e) {5}

Solution:

(i) 3 divides 9, 9 divides 36

$\therefore$ the set is totally ordered.

(b) Since 7 and 11 are not comparable the set is not totally ordered.

(c) 3 divides 6, 6 divides 12, 12 divides 24.

$\therefore$ the set is totally ordered.

(d) 2 and 3 are not comparable.

$\therefore$ the set is not totally ordered.

(e) Any singleton set is totally ordered.

Example 18: Define a relation ~ on $N \times N$ as if $(a, b), (c, d) \in N \times N$ then $(a,b) \sim (c,d)$ if $a + d = b + c$

Prove that ~ is an equivalence relation on $N \times N$

Solution:

(i) $(a,b) \sim (a,b)$ since $a+b=b+a, \forall a,b \in N$

$\therefore$ the relation $\sim$ is reflexive

(ii) If $(a, b) \sim (c, d)$ then $a + d = b + c$

Then $d + a = c + b$

This means $(c, d) \sim (a, b)$

$\therefore$ The relation is symmetric

(iii) Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ .... (1)

$(a,b) \sim (c,d) \Rightarrow a+d=b+c$ ...... (2)

$(c,d) \sim (e,f) \Rightarrow c+f=d+e$

Adding (1) and (2) $a+f=b+e$

This implies $(a, b) \sim (e, f)$

$\therefore$ The relation $\sim$ is transitive

Since the relation is reflexive, symmetric and transitive it is an equivalence relation.

Example 19: If R is reflexive and transitive relation on a set A show that $R \cap R^{-1}$ is an equivalence relation on A.

Solution:

(i) Since R is reflexive $(a,a) \in R$ for $a \in A$

$\therefore (a,a) \in R^{-1}$ for all $a \in A$

Hence $(a,a) \in R \cap R^{-1}$ for all $a \in A$

Hence $R \cap R^{-1}$ is reflexive.

(ii) Let $(a,b) \in R \cap R^{-1}$

Then $(a,b) \in R$ and $(a,b) \in R^{-1}$

$(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

$(a,b) \in R^{-1} \Rightarrow (b,a) \in R$

$\therefore (b,a) \in R \cap R^{-1}$

$\therefore R \cap R^{-1}$ is symmetric

(iii) Let $(a,b) \in R \cap R^{-1}$ and $(b,c) \in R \cap R^{-1}$

Then $(a,b) \in R$ and $(a,b) \in R^{-1}$

$(b,c) \in R$ and $(b,c) \in R^{-1}$

Since R is transitive $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \Rightarrow R$

$(a,b) \in R^{-1}, (b,c) \in R^{-1} \Rightarrow (a,c) \Rightarrow R^{-1}$

$\therefore (a,c) \in R \cap R^{-1}$

$\therefore R \cap R^{-1}$ is transitive

Since $R \cap R^{-1}$ is reflexive, symmetric and transitive; $R \cap R^{-1}$ is an equivalence relation.

Example 20: Let R be a relation on A. Show that R is symmetric if and only if $R = R^{-1}$.

Solution:

Suppose R is symmetric.

Let $(a,b) \in R$. Then $(b,a) \in R$ since R is symmetric.

Hence $(a,b) \in R^{-1}$ and hence $R \subset R^{-1}$ .... (1)

Let $(a,b) \in R^{-1}$. Then $(b,a) \in R$ and by symmetry $(a,b) \in R$.

$\therefore R^{-1} \subset R$ .... (2)

From (1) and (2) $R = R^{-1}$

Example 21: If R and S are reflexive and symmetric then $R \cup S$ is also reflexive and symmetric.

Solution:

(i) Since R and S are reflexive,

$(a,a) \in R$ and $(a,a) \in S$ for all $a \in A$

This implies $(a,a) \in R \cup S$ for all $a \in A$

$\therefore R \cup S$ is reflexive

(ii) Let $(a,b) \in R \cup S$

$(a,b) \in R \cup S \Rightarrow (a,b) \in R$ or $(a,b) \in S$

$(b,a) \in R$ or $(b,a) \in S$ since R and S are symmetric; $\therefore$

$(b,a) \in R \cup S$

$\therefore R \cup S$ is symmetric

Note: If R and S are equivalence relations in A then $R \cap S$ is an equivalence relation in A, but $R \cup S$ is not all equivalence relation in A.

Exercises

1. Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$

   List all the elements of $A \times B$

2. Let $A = \{1, 2, 3, 4\}$. Find all the pairs of $A \times A$ that belong to R.

   (a) $R = \{(x, y) : x \geq y\}$

   (b) $R = \{(x, y) : x = y^2\}$

3. Let A and B be two sets with m and n elements. Find the number of relations in $A \times B$.

4. Let A be the set of all points in a plane. Let R be a relation such that for any two points a and b, $(a,b) \in R$ if b is one unit distance from a. Show that R is not an equivalence relation.

5. Give an example of transitive relations R and S for which $R \cup S$ is not transitive.

6. Which of the following relations in the set of real numbers are equivalence relation.

   (i) $R = \{(x, y) : |x| = |y|\}$

   (ii) $R = \{(x, y) : x \geq y\}$

7. A relation R is called circular if $(x, y) \in R$ and $(y, z) \in R$ implies $(z, x) \in R$. Show that a relation is reflexive and circular if any only if it is reflexive, symmetric and transitive.

8. Let R be a relation "perpendicular to" in the set of all lines in a plane. Determine whether R is an equivalence relation.

9. Let R be the relation in $A = \{2, 3, 4, 5\}$ defined by "x and y are relatively prime". Write down R as a set of ordered pairs and find R^{-1}.

10. Let R be a relation in A and

    $\Delta = \{(a,a) : a \in A\}$, show that R is reflexive if and only if $\Delta \subset R$.

11. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and let
   (i) $A_1 = \{1, 3, 5\}$, $A_2 = \{2\}$, $A_3 = \{4, 7\}$
   (ii) $B_1 = \{1, 5, 7\}$ $B_2 = \{3, 4\}$ $B_3 = \{2, 5, 6\}$
   (iii) $C_1 = \{1, 2, 5, 7\}$, $C_2 = \{3\}$, $C_3 = \{4, 6\}$
   (iv) $D = \{1, 2, 3, 4, 5, 6, 7\}$
   Which of the above sets are partitions of X?
12. Find all the partitions of $\{a, b, c\}$
13. Let $\{A_1, A_2, A_3, \ldots\ldots A_m\}$ and $\{B_1, B_2, \ldots\ldots B_3\}$ be partitions of a set X. Show that the collection of sets $\{A_I \cap B_J = i = 1,2,\ldots..m,\ j = 1,2,\ldots.n\}$ is also a partition of X.
14. Let A be any set and the relation "$\subset$" defined on $P(A)$, the power set of A. Show that this relation is a partial order relation.
15. Let $A = \{2, 3, 5, 6\}$ and R be "divides" show that $R = \{(x, y)\} : x$ divides $y\}$ is a partial order relation.
16. Given the relation $\{(1, 1), (1, 2)\}$ on the set $\{1, 2, 3\}$ add a minimum number of ordered pairs so that the enlarged relation is an equivalence relation in the set.
17. For any relation R on a set A, define R^{-1} by $a\ R^{-1}\ b$ if bRa. Show that a reflexive R is an equivalence relation on A if

   $R.R = R$ and $R = R^{-1}$
18. Let $A = \{n \mid n \in N$ and $n > 1\}$. If $a, b \in A$ define $a \sim b$ means that a and b have the same number of positive prime factors. Show that $\sim$ is an equivalence relation.
19. Prove that the relation R is the set of natural numbers N defined by aRb if $a^2 - 4ab + 3b^2 = 0$ $(a, b \in N)$ is reflexive but neither symmetric nor transitive.
20. Prove that the relation R in the set of integers I defined by aRb if a and b are both odd, are symmetric and transitive but not reflexive.
21. Show that the relation "greater than" in the set of natural numbers is transitive but neither reflexive not symmetric.
22. If R is an equivalence relation and zRx and zRy, prove that xRy.
23. Show that in the set of all natural numbers, the divisibility relation is reflexive, transitive but not symmetric and hence is not an equivalence relation.

24. Let Q be the set of all non-zero rational numbers and the relation R be defined over Q by xRy if $x = \frac{1}{y}$, $x, y \in Q$. Show that R is not an equivalence relation.
25. Let $R : X \to X$ where $X = \{1, 2, 3\}$
   $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (1, 3), (3, 1), (2, 3), (3, 2)\}$
   Is this relation reflexive, symmetric and transitive?
26. Let I be the set of non-zero integers and the relation R be defined over the set I by xRy if $x^y = y^x$ where $x, y \in I$, show that R is an equivalence relation.
27. Define an equivalence relation. A is the set of living people. Find out which of the two relations (a) "is mother of (b)" is brother of "is an equivalence relation" and why?
28. Let $X = \{1, 2, \ldots\ldots, 7\}$ and
   $R = \{(x, y) : x - y$ is divisible by $3\}$
   Show that R is an equivalence relation.
29. Let $A = \{0, 1, 2, 3\}$ and let
   $R = \{(0, 0), (1, 1), (2, 2), (3, 3, (1, 2), (2, 1), (3, 2), (2, 3), (3, 1), (1, 3)\}$
   (a) Show that R is an equivalence relation.
   (b) Let $a \in A$ and define $C(a) = \{b \in A / aRb\}$ $C(a)$ is called equivalence class of the element $a \in A$.
   Find $C(a)$ for each element $a \in A$.
   (c) Show that $\{C(a) : a \in A\}$ forms a partition of A for this set A.
   (d) Let R be an equivalence relation on any arbitrary set A. Prove that the set of all equivalence classes form a partition of A.
30. When is a relation in a set is an equivalence relation? Prove that any two equivalence classes in a set are either identical or they have null intersection?

Chapter 4

Functions

Definition : Suppose to each element of a set A there is assigned a unique element of a set B. Then the collection of such assignments is called a function or mapping from A into B and is denoted by

$$f: A \to B.$$

The unique element in B assigned to $a \in A$ by f is denoted by $f(a)$ and is called the image of a under f. The set A is called the domain of f and B is called the co-domain. The set of all images under f is called the range of f and is denoted by $f(A)$.

$$(i.e)\ f(A) = \{f(a) : a \in A\}$$

The range of f is a subset of the co-domain B.

Example 1: Let f assign to each country in the World its Capital. Here the domain is the set of all countries in the world. The co-domain is the set of all cities in the world. For example, the image of India is Delhi.

Example 2 : Suppose $A = \{a, b, c\}$

$$B = \{1, 2, 3, 4\}$$

The mapping of A into B is given by

$f(a) = 4, f(b) = 1, f(c) = 3.$

$\therefore$ The range of f is $\{1, 2, 3\}$.

Example 3 : Let R be the set of real numbers and let $f: R \to R$ assign to each rational number, the number 1 and to each irrational number the number -1.

The $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$

Example 4 : If f assigns to each real number its square $\forall x \in R, f(x) = x^2$. The domain and the codomain of f are equal to real numbers and so we write $f: R \to R$. Also we see that $\begin{matrix} f(-2) = 4 & f(-3) = 9 \\ f(2) = 4 & f(3) = 9 \end{matrix}$

Example 5 : Let $f = \{(0,b),(1,c),(2,c),(3,d),(0,d)\}$. f is not a function where $X = \{0,1,2,3\}$ and $Y = \{b,c,d\}$ since the image of 0 is not unique.

Example 6: Let $X = \{0,1,2,3\}$

$$y = \{a,b,c,d\}$$

Let $f = \{(0,a),(1,a),(2,a)\}$

This is not a function since the element 3 has no image

Difference between Relation and Function

Let A and B be two sets. Let f be a function from A to B. Then from the definition of function, f is a subset of $A \times B$ in which each $a \in A$, appears in one and only one ordered pair belonging to f. In other words f is a subset of $A \times B$ satisfying the following conditions. (*i.e.,*) for each $a \in A,(a,b) \in f$ for some $b \in B$.

If $(a,b) \in f$ and $(a,c) \in f$ then $a = c$.

On the other hand every subset of $A \times B$ is a relation from A to B. Therefore every function is a relation but every relation is not a function. If R is a relation from A to B then the domain of R may be a subset of R. But if R is a function from A to B then the domain of f is equal to A. Also in a relation from A to B an element of A may be related to more than one element in B. Also some element of A may not be related to an element of B. But in a function from A to B each element of A must be related to any one element of B.

Example 1 : Let $A = \{1,2,3,4\}; B = \{x,y,z\}$

Let $R = (1,a),(2,b),(3,c),(4,c)$

Here R is a relation as well as a function from A to B.

Let $R_1 = \{(1,a),(2,b),(1,c),(2,d)\}$.

Here R_1 is a relation but not a function.

Types of Functions

I **Into Function:** If the function $f: A \to B$ is such that there is atleast one element in B, which is not the image of any element in A then we say that the function from A to B is an *into* function.

In this case the range of f is a proper subset of the co-domain of f (*i.e*) $f(A) \subset B$.

An into function is called an *injective* function.

II **Onto Function :** If the function $f: A \to B$ is such that each element in B is the image of atleast one element of A then we say that the function f is an *onto* function from A to B.

An onto function is called an *surjective* function.

In this case the range of f is equal to the co-domain of f. (i.e.) $f(A) = B$

III One-One Function

A Function $f: A \to B$ is said to be one-one function if different elements in A have different images in B.

(i.e.,) If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2, x_1, x_2 \in A$.

In one-one function an element in B has only one pre image in A.

IV Many-one Function : A function $f: A \to B$ is said to be *many-one* if distinct elements of A has the same image in B.

(i.e.,) $f(x_1) = f(x_2)$ for $x \neq x_2, x_1, x_2 \in A$.

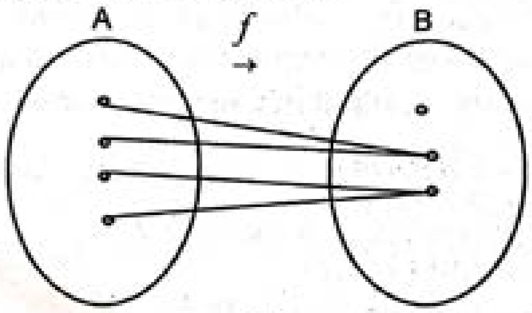
In *many-one* function some elements in B have more than one pre-image in A.

V One-one onto function : If $f: A \to B$ is *one-one* and also *onto* then f is called one-to-one correspondence between A and B.

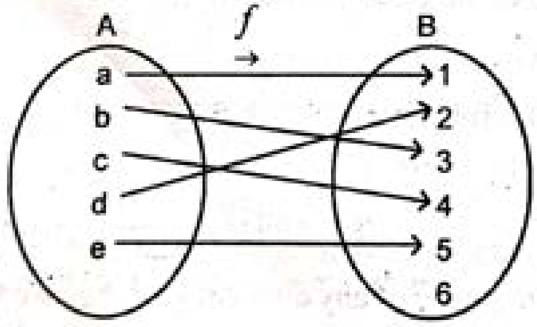
This is called a *bi-jective* function.

In this case sets A and B are said to have the same number of elements.

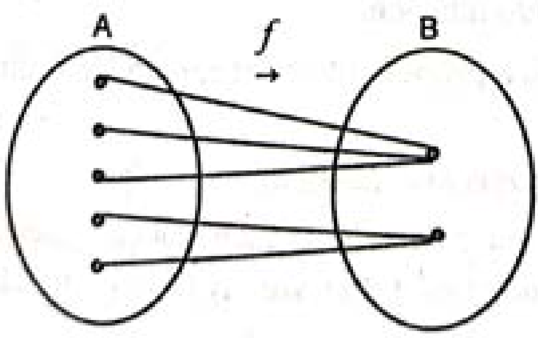
Diagrammatic Representation



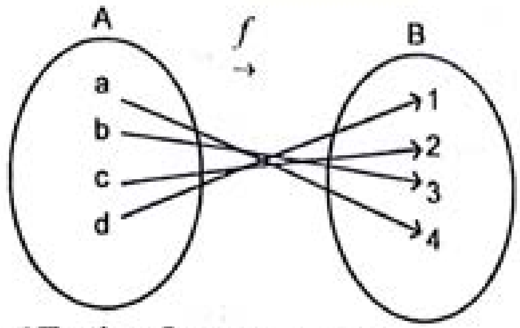
Many-one into function



One-one into function



Many-one onto function



One-one onto function

Method of Testing One-One Function

If x_1 and x_2 are two arbitrary elements of the domain whenever $x_1 \neq x_2, f(x_1) \neq f(x_2)$ implies that the function is one-one. In other words if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Method of Testing Onto Function $f: A \to B$

To prove that f is onto we show that if $y \in B$, then there exists a $x \in A$ such that $y = f(x)$. Then $y \in B \Rightarrow y \in f(A)$. Having chosen it arbitrarily every element of B is an element of $f(A)$ and hence $B \subseteq A$. But $f(A) \subset B$ therefore $B = f(A)$ and hence the function f is an onto function.

Identity Function : Let A be any set . Let the function $f: A \to A$ be defined by $f(x) = x$, for all $x \in A$. (i.e.,) each element of A is mapped onto itself. Then the function is called the identity function and is denoted by I_A.

Example

Let $f = (1,2,3,4)$

Then $f = \{(1,1),(2,2),(3,3),(4,4)\}$ is the identity function of A.

Identity function is always one-one onto.

Constant Function : A function $f: A \to B$ is called a constant function if to each element of A the same element of B is assigned. In this case the range of B contains only one element.

Example : The function $f: R \to R$ defined by $f(x) = 3$ is a constant function.

Inverse Image of an element

Let f be a function from A to B. Let B be an element of B. Then the inverse image of the element b under the function f is denoted by $f^{-1}(b)$, consisting of those elements in A which have b as their f-image.

(i.e.) $f^{-1}(b) = \{x : x \in A \text{ and } f(b) = b\}$

Note : $f^{-1}(b)$ is a sub set of A.

Example 1: If $f: R \to R$ is defined by $f(x) = x^2$ then

$f(3) = 9,\ f(-3) = 9$

$$f^{-1}(9) = \{3, -3\}$$

Also $f^{-1}(16) = \{-4, 4\}$

$f^{-1}(25) = \{-5, 5\}$ etc.

Example 2 : Let C be the set set of all complex numbers and $f: C \to C$ defined by $f(x) = -x^2$. Then $f^{-1}(-2) = \{-\sqrt{2}\,i, +\sqrt{2}\,i\}$.

Inverse Image of a Subset

Let f be a function $f: A \to B$ and let C be a subset of B (*i.e.,*) $C \subset B$. Then the inverse image of C under f, denoted by $f^{-1}(C)$ consists of all those elements in A which are mapped into some elements in C.

(*i.e.,*) $f^{-1}(C) = \{x : x \in A \text{ and } f(x) \in C\}$

$f^{-1}(C)$ is always a subset of A.

In particular $f^{-1}[f(A)] = A$

Also $f^{-1}(B) = A$.

Inverse Function

Let $f: A \to B$ be a one-one onto function. Let b be an arbitrary element of B. Since f is an onto function, there will be atleast one element in A, say a, such that $b = f(a)$, $b \in B, a \in A$. Since the functions f is one-one there will be only one element a such that $b = f(a)$. Let us denote a by $f^{-1}(b)$. Thus if $f: A \to B$ is one-one onto, we define another correspondence which associates to each element in B a unique element in A. This function is from B to A and is denoted by f^{-1}. If $f = A \to B$ then $f^{-1}: B \to A$ and f^{-1} is called the inverse function of f.

Definition : Let $f: A \to B$ be a one-one onto function. Then the function f^{-1} which associates to each element $b \in B$ the element $a \in A$ such that $f(a) = b$ is called the inverse function of f.

Note : Only one-one onto functions posses inverse function. If the function $f:A \to B$ is not onto then f^{-1} may not be defined because in this case there will be some element in B which will have no f^{-1} image in A. If $f:A \to B$ is also not one-one the function f^{-1} will not be defined. In this case, some element of B will be associated to more than one element in A and hence some element of B will not have a unique f^{-1} image in A.

Theorem 1 : Let A and B be two sets. If $f:A \to B$ is one -one onto then $f^{-1}: B \to A$ is one-one onto.

Proof: Given $f:A \to B$ is one-one. To prove $f^{-1}: B \to A$ is also one-one onto.

Let us first prove that f^{-1} is one-one onto. Let y_1 and y_2 be any two elements of B.

Let $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$ where $x_1, x_2 \in A$.

Then from the definition of f,

$$f(x_1) = y_1, f(x_2) = y_2$$

Now $f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow x_1 = x_2$

$x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ since f is one-one.

$f(x_1) = f(x_2) \Rightarrow y_1 = y_2$

$\therefore f^{-1}$ is one-one.

Then let us prove that f^{-1} is onto.

Let x be any arbitrary element of A. Since f is a function from A to B, there exists an element $y \in B$ such that $y = f(x)$ or $x = f^{-1}(y)$.

$\therefore x$ is the f^{-1} image of $y \in B$.

$\therefore$ the function f is onto. Hence f is one-one onto.

Theorem 2: If $f: A \to B$ is one-one onto then the inverse function of f is unique.

Proof : Let $f: A \to B$ be one-one and onto.

Let $g: B \to A$ and $h: B \to A$ be two inverse functions of A.

Let us then prove that $g = h$.

Let b be an arbitrary element of B.

Let $g(b) = x_1$ and $h(b) = x_2$.

Since g is the inverse function of f,

$$g(b) = x_1 \Rightarrow f(x_1) = b \qquad (1)$$

Since h is the inverse function of b,

$$g(b) = x_2 \Rightarrow f(x_2) = b \qquad (2)$$

But f is one-one function

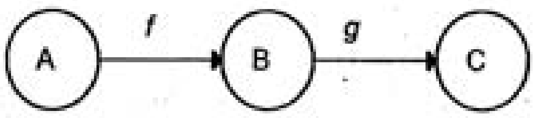
$\therefore f(x_1) = b$ and $f(x_2) = b \Rightarrow x_1 = x_2$ and $x_1 = x_2 \Rightarrow g(b) = h(b)$

$\therefore g = h$

$\therefore$ The inverse of f is unique.

Composition of Functions

Consider the function $f: A \to B$ and $g: B \to C$.



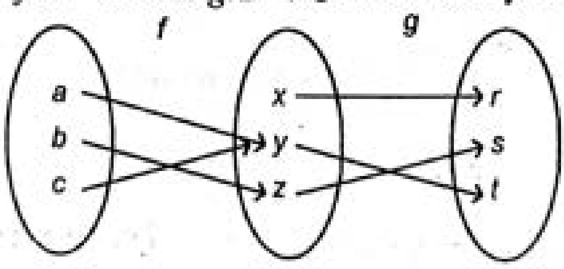
Let $a \in A$. Then its image $f(a)$ is in B, the domain of g. We can find the image of $f(a)$ under the function g *i.e.,* $g\{f(a)\}$. The function from A to C which assigns to each $a \in A$, the element $g[f(a)] \in C$ is called the composition of f and g and is denoted by gof. Formally let us define the composition of functions f and g.

Definition : Let $f: A \to B$ and $g: B \to C$. Then the composition of the functions f and g is a function from $A \to C$ given by

$(gof) : A \to C$ such that

$(gof)\, x = g[f(x)]$, for all $x \in A$.

Example 1: Let $f: A \to B$ and $g: B \to C$ be defined by the following diagram.



Now we compute $(gof) : A \to C$.

$$(gof)a = g[f(a)] = g(y) = t$$
$$(gof)b = g[f(b)] = g(z) = s$$
$$(gof)c = g[f(c)] = g(y) = t$$

Example 2: Let R be the set of real numbers. Let $f: R \to R$ and $g: R \to R$ be defined as follows :

$$f(x) = x^2 \qquad g(x) = x+5$$
$$(fog)(3) = f[g(3)] = f(8) = 64$$
$$(gof)(3) = g[f(3)] = g(9) = 14$$

Note that the composition of functions (fog) and (gof) are not the same.

In General,

$$(gof)x = g[f(x)] = g(x^2) = x^2+5$$
$$(fog)x = f[g(x)] = f(x+5) = (x+5)^2$$
$$x^2+5 \neq (x+5)^2 \quad , \quad \forall x \in R$$

Properties of Composite Functions

Theorem 1 : If $f: A \to B$ is a one-one and onto function then $fof^{-1} = I_B$ and $f^{-1}of = I_A$ where I_A and I_B are the identity functions of the sets A and B respectively.

Proof : Since $f: A \to B$ and $f^{-1} : B \to A$,

$$(f^{-1}of) : A \to A$$

Let x be an arbitrary element of A and let $f(x) = y$ where $y \in B$.

Then by the definition of the function $f^{-1}, f^{-1}(y) = x$.

Now $(f^{-1}of)(x) = f^{-1}[f(x)] = f^{-1}(y) = x$.

$$\therefore f^{-1}of = I_A$$

To prove $\therefore fof^{-1} = I_B$.

Since $f^{-1}: B \to A$ and $f: A \to B, fof^{-1}: B \to B$.

Let y be an arbitrary element of y and let $f^{-1}(y) = x$ where $x \in A$.

Then $f(x) = y$.

Now $(fof^{-1})(y) = f[f^{-1}(y)] = f(x) = y$.

$\therefore$ fof^{-1} maps every element of B onto itself.

$\therefore fof^{-1} = I_B$.

Theorem 2 : If $f: A \to B$ then $I_B of = f$ and $foI_A = f$.

(*i.e.,*) the product of any function with the identity function is the function itself.

Proof : Let x be an arbitrary element of A and let $f(x) = y$ for $x \in A, y \in B$.

Since $f: A \to B$ and $I_B : B \to B$,

$(I_B of) : A \to B$.

Then we have $(I_B of)(x) = I_B[f(x)]$

$= I_B(y) = y$. [$\because I_B$ is the identity element of B]

Also $f(x) = y$

Therefore for all $x \in A, (I_B of)(x) = f(x)$

$\therefore I_B of = f$

Again $foI_A : A \to B$

$(foI_A)(x) = f(I_A(x)] = f(x)$, since I_A is the identity element of A.

$\therefore$ for all $x \in A, (foI_A)(x) = f(x)$.

$foI_A = f$.

Theorem 3 : Let $f: A \to B$ and $g: B \to C$ be two one-one onto functions

Then

$(gof)^{-1}: C \to A = (f^{-1}og^{-1}): C \to A$

Proof : We shall prove that

(i) gof is one-one

(ii) gof is onto

(iii) $(gof)^{-1} = f^{-1}og^{-1}$

(i) Let x_1 and x_2 be any two elements in A. Then the function gof will be one-one if we show that the images of x_1 and x_2 under this function are equal if $x_1 = x_2$.

We have, $(gof)(x_1) = (gof)(x_2)$.

$$\Rightarrow g[f(x_1)] = g[f(x_2)]$$

$$\Rightarrow f(x_1) = f(x_2).$$

$(\because f(x_1), f(x_2) \in B$ and the function $g: B \to C$ is one - one).

$\Rightarrow x_1 = x_2$ since is one - one.

$\therefore$ gof is one - one.

(ii) Let z be any arbitrary element of C. Since g is an onto function of $B \to C$, there exists a $y \in B$ such that $g(y) = z$

Since f is an onto functions of $A \to B$ there exists a $x \in A$ such that $f(x) = y$

Thus for every $z \in C$, there exists a $x \in A$ such that

$$z = g(y) = g[f(x)] = (gof)(x)$$

$\therefore gof$ is a onto function of A onto C.

Thus gof is one-one onto and hence is irreversible.

(iii) Let z be an arbitrary element of C such that

$$z = g(y) \quad (y \in B_1, z \in C).$$

Then $y = g^{-1}(z)$.

Also let $y = f(x), (y \in B, x \in A)$

Then $x = f^{-1}(y)$.

$(gof)(x) = g[f(x)] = g(y) = z$

Since gof is one-one onto,

$(gof)(x) = z \Rightarrow (gof)^{-1}(z) = x$.

Also $(f^{-1}og^{-1})(z) = f^{-1}[g^{-1}(z)] = f^{-1}(y) = x$

$\therefore$ for all $z \in C, (gof)^{-1}(z) = (f^{-1}og^{-1})(z)$

Hence the function $(g^{-1}of)^{-1} = f^{-1}og^{-1}$.

Theorem 4: Associative law for composition of function

Let $f: A \to B,\ g: B \to C$ and $h: C \to D$

Then $(hog)of = ho(gof)$

Proof: $(hog)of$ is a function from A to D. Also $ho(gof)$ is a function from A to D. Therefore these two functions will be equal if they assign the same image to each element x in the domain A.

(*i.e.*) $[(hog)of](x) = [ho(gof)](x)$

Now $[(hog)of](x) = (hog)f(x)$.

$= h[g[f(x)]]$

$= h[(gof)(x)] = [ho(gof)](x)$

Hence $(hog)of = ho(gof)$

Theorem 5: Let $f: A \to B$

$g: B \to A$

Then g is the inverse function of f.

(*i.e.,*) $g = f^{-1}$ if $(gof): A \to A$ is the identify function on A and

$(fog): B \to B$ is the identify function on B their $g = f^{-1}$

Proof: In order to show that f^{-1} exists we have to show that f is one-one onto.

(i) Let $x_1, x_2 \in A$

Then $f(x_1) = f(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow (gof)(x_1) = (gof)(x_2)$

$\Rightarrow I(x_1) = I(x_2)$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one.

(ii) Let y be an arbitrary element of B.

Since g is a function of B to A, $g(y) \in A$.

Let $g(y) = x$.

Then $g(y) = x$

$\Rightarrow f[g(y)] = f(x)$

$\Rightarrow (fog)(y) = f(x)$

$\Rightarrow y = f(x)$

$\therefore$ for $y \in B$ there exists $x \in A$ such that $f(x) = y$

$\therefore f$ is onto.

(iii) Now we will show that $f^{-1} = g$.

$fog = I \Rightarrow f^{-1}(fog) = f^{-1}\, oI$

$\Rightarrow (f^{-1}of)og = f^{-1} \Rightarrow Iog = f^{-1} \Rightarrow g = f^{-1}$

$$\therefore \boxed{g = f^{-1}}$$

Similarly we can show that g^{-1} exists and $f = g^{-1}$.

Example 1: Write each of the following functions in a formula form.

(a) to each number let f assign its cube.

(b) to each number let g assign the number 7.

(c) to each positive number let h assign its square, and to each non-positive number let h assign the number 8.

Solution:

(a) To each number f assigns its cube $\therefore f(x) = x^3$

(b) To each number, g assigns 7

$\therefore f(x) = 7$

(c) To each positive h assigns its square and to each non positive number let h assigns the number 8.

$$\therefore h(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 8 & \text{if } x < 0 \end{cases}$$

Example 2: Let $X = \{1, 2, 3, 4\}$. Determine whether or not each relation is a function from X into X.

(a) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$

(b) $g = \{(3, 1), (4, 2), (1, 1)\}$

(c) $h = \{(2, 1), (3, 4), (1, 4), (2, 1) (4, 4)\}$

Solution:

(a) It is not a function since the ordered pair (2, 3) and (2, 4) in f have the same number 2 as their first element.

(b) It is not a function since $2 \in X$ and it does not appear as first element in any ordered pair of f.

(c) It is a function – Note ordered pairs (2, 1), (2, 1) are equal.

Example 3: Determine whether on not the following are functions from A to B where $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e\}$. If they are functions give the range of each.

(a) $f = \{(1, a), (2, b), (3, c), (5, e)\}$
(b) $g = \{1, e), (2, d), (3, a), (2, b), (1, d), (4, a)\}$
(c) $h = \{5, a), (1, e), (4, b), (3, c), (2, d)\}$

Solution:

(a) It is not a function since $4 \in A$, there is no image of 4 in B.

(b) It is not a function since the element $1 \in A$ has two images in B.

(c) h is a function since each element of A is associated with a unique element of B. The range is $\{a, e, b, c, d\} = B$.

Example 4: Each of the following formula defines a function from $R \to R$. Find the range of each function.

(a) $f(x) = x^3$, (b) $g(x) = \sin x$, (c) $h(x) = x^2 + 1$

Solution:

$f: R \to R$

(a) For every real number a, there is a real $\sqrt[3]{a}$. Therefore if a is any arbitrary number of R,

$$f\left(\sqrt[3]{a}\right) = \left(\sqrt[3]{a}\right)^3 = a$$

$\therefore \sqrt[3]{a} \in R$ is the pre image of $a \in R$.

$\therefore$ The range of f is R and this function is an onto function.

(b) For any real $x, -1 \le \sin x \le 1$ and all the real numbers in the interval [–1, 1] will be the sine of some real number. $\therefore$ The range f is [–1, 1] and this function is not an onto function.

(c) Let y be any real number ≥ 1.

Let $y = h(x) = x^2 + 1$

Then $x = \pm\sqrt{y-1}$ are also real numbers. $\therefore$ every real number $\geq$ is the h-image of some or other real number. Hence the range of h is $(1, \infty)$.

Example 5: Given $A = \{-1, 0, 2, 5, 6, 11\}$

$B = \{-2, -1, 0, 18, 28, 108\}$

and $f(x) = x^2 - x - 2$, Is $f(A) = B$. Find $f(A)$

Solution:

$$\begin{aligned}
f(-1) &= 1 + 1 - 2 = 0 \\
f(0) &= 0 - 0 - 2 = -2 \\
f(2) &= 4 - 2 - 2 = 0 \\
f(5) &= 25 - 5 - 2 = 18 \\
f(6) &= 36 - 6 - 2 = 28 \\
f(11) &= 121 - 11 - 2 = 108 \\
f(A) &= \{0, -2, 18, 28, 108\}
\end{aligned}$$

$-2 \in B$ and $-2 \notin f(A)$

$\therefore f(A) \neq B$.

Example 6: Let $A = \{a_1, a_2, a_3\}$

$B = \{b_1, b_2, b_3\}$

$C = \{c_1, c_2\}$

$D = \{d_1, d_2, d_3, d_4\}$

Consider the following four functions from A to B, A to D, B to C and D to B respectively.

(a) $f_1 = \{(a_1, b_2), (a_2, b_3), (a_3, b_1)\}$

(b) $f_2 = \{(a_1, d_2), (a_2, d_1), (a_3, d_4)\}$

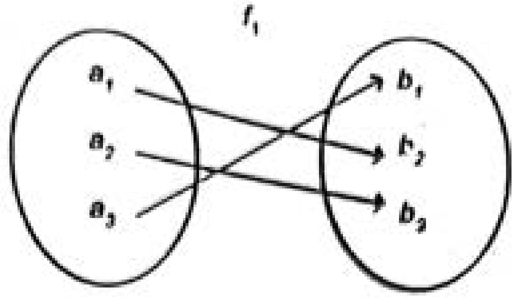
(c) $f_3 = \{(b_1, c_2), (b_2, c_2), (b_3, c_1)\}$

(d) $f_4 = \{(d_1, b_1), (d_2, b_2), (d_3, b_1)\}$

Determine whether or not each function is one-one, whether each function is onto and whether each function is every where defined.

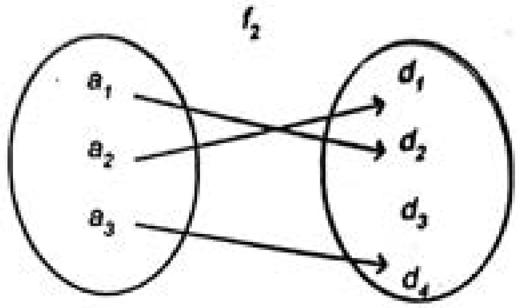
Solution:

(a)



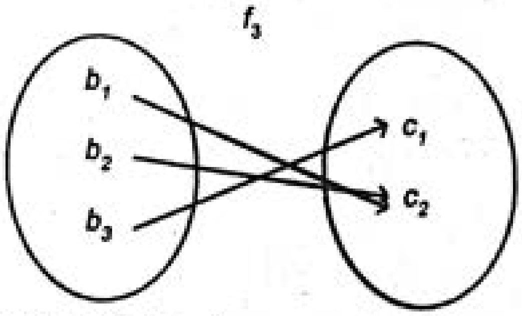
$\therefore f_1$ is every where defined, one-one and *onto.*

(b)



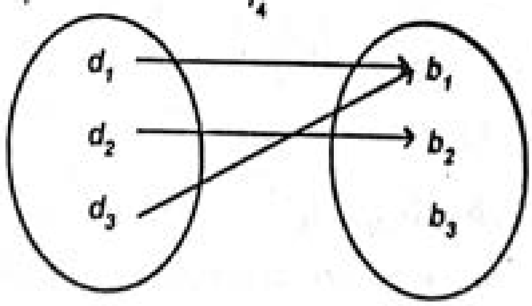
f_2 is everywhere defined one-one but not onto.

(c)



f_3 is everywhere defined not one-one, but it is onto.

(d)



f_4 is well defined; it is not one-one and it is not onto.

Example 7: Let R be a set of real numbers.

Let $f:R\rightarrow R$ be defined by $f(x)=2x+5,\ x\in R$. Show that f is one-one.

Also find a function that defines the inverse function f^{-1}.

Solution:

Let $x_1, x_2\in R$ and $x_1\neq x_2$.

$2x_1+5=2x_2+5\Rightarrow x_1=x_2$

$\therefore 2x_1+5\neq 2x_2+5$ for $x_1, x_2\in R$ and $x_1\neq x_2$.

Let y be any arbitrary element in R.

Then $y=f(x)=2x+5$

$$\text{or } x=\frac{y-5}{2}$$

$$\frac{y-5}{2}\in R$$

$\therefore \dfrac{y-5}{2}$ is the f-image of the element x

Hence f is onto.

Since f is one-one and onto, f^{-1} exists.

f^{-1} is defined by $f^{-1}:R\rightarrow R$

where $f^{-1}(y)=\dfrac{y-5}{2},\ y\rightarrow R$.

Example 8: Let $X=\left\{x : x\in R \text{ and } -\frac{\pi}{2}\leq x\leq\frac{\pi}{2}\right\}$

$Y=\{y : y\in R \text{ and } -1\leq y\leq 1\}$.

Show that the function $f : X\rightarrow Y$ defined by $f(x)=\sin x,\ \forall x\in X$ is one-one and onto. Also give the inverse function $f^{-1} : Y\rightarrow X$.

Solution:

Let p and q be two real numbers belonging to the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

$\because$ We know that any two real numbers in the interval will not have the same sine value

(a) $\sin p \neq \sin q$

(*i.e.,*) $f(p) \neq f(q)$ where $p \neq q$

$\therefore$ The function $f(x)$ is one-one.

Let y be an arbitrary number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Then there exists a x in this closed interval such that $\sin x = y$.

(i.e.,) for every y in Y, x is the f-image of $x \in X \therefore f$ is onto.

$\therefore f: X \to Y$ is one-one onto.

$\therefore f$ has an inverse function f^{-1}

(i.e.,) $f^{-1}: Y \to X$

Let y be the image of x under the function f.

Then $y = f(x) = \sin x$.

(i.e.,) x will be a image of y under the inverse function f^{-1} (i.e.,) $x = f^{-1}(y)$.

$y = \sin x \Rightarrow x = \sin^{-1} y$

The inverse function $f^{-1} = Y \to X$ defined by $f^{-1}(y) = \sin^{-1} y$.

Example 9: A function $f: R \to R$ where R is the set of real numbers is defined by.

$$f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$$

Find the intervals of values of α for which f is onto. Is this function is one-one for $\alpha = 3$? Justify your answer.

Solution:

Let $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$.

$$x^2(\alpha + 8y) + 6x(1 - y) - (8 + \alpha y) = 0$$

Since x is real $\Delta \geq 0$.

$$\therefore 36(1-y)^2+4(\alpha+8y)(8+\alpha y)\geq 0.$$

$$9(1-2y+y^2)+[8\alpha y^2+(64+\alpha^2)y+8\alpha]\geq 0$$

$$(9+8\alpha)y^2+y(46+\alpha^2)+(9+8\alpha)\geq 0$$

The conditions for this are $9+8\alpha\geq 0$ and

$$\left(46+\alpha^2\right)^2-4\left(9+8\alpha\right)^2\leq 0$$

(i.e.) $\left(46+\alpha^2+18+16\alpha\right)\left(46+\alpha^2-18-16\alpha\right)\leq 0$

$$\left(\alpha^2+16\alpha+64\right)\left(\alpha^2-16\alpha+28\right)\leq 0$$

$$\left(\alpha+8\right)^2\left(\alpha-2\right)\left(\alpha-14\right)\leq 0$$

$$\therefore 2\leq\alpha\leq 14 \text{ and } 9+8\alpha>0$$

When $\alpha=3$,

$$y=\frac{3x^2+6x-8}{3+6x-8x^2}$$

When $y=0$, we get $3x^2+6x-8=0$

$$x=\frac{-3\pm\sqrt{33}}{2}$$

$\therefore$ for $y=0$ there are two values of x.

Hence the function f is not one-one.

Example 10: Find the domain and range of the function

$$f(x)=\frac{x^2-3x-2}{x^2+x-6}$$

Solution:

$$f(x)=\frac{x^2-3x-2}{x^2+x-6}=\frac{(x-1)(x-2)}{(x+3)(x-2)}$$

$f(x)$ is not defined at $x=-3$ and $x=2$.

$\therefore$ Domain of $f=R-\{2,-3\}$

Also $y = f(x) = \frac{x-1}{x+3}$

$f(-3) = \infty$

$\underset{x \to 2}{Lt} f(x) = \underset{x \to 2}{Lt} \frac{x-1}{x+3} = \frac{1}{5}$

$$y = \frac{x-1}{x+3} \Rightarrow x = \frac{3y+1}{y-1}$$

$\therefore$ For all $y (\neq 0)$ x is real and $f(\infty) = \underset{x \to \infty}{Lt} \frac{x-1}{x+3} = 1$

$\therefore$ For real x, $f(x)$ can attain all real values except $\frac{1}{5}$ and 1.

$\therefore$ Range of $f(x) = R - \left\{\frac{1}{5}, 1\right\}$

Example 11: Is the function $f(x) = \frac{x^2 - 8x + 18}{x^2 + 4x + 30}$ an one-one function?

Solution:

Let $f(x_1) = f(x_2)$

$$\frac{x_1^2 - 8x_1 + 18}{x^2 + 4x_1 + 30} = \frac{x_2^2 - 8x_2 + 18}{x_2^2 + 4x_2 + 30}$$

(i.e.,) $12x_1^2x_2 - 12x_1 x_2^2 + 12x_1^2 - 12x_2^2 - 312x_1 + 312x_2 = 0$

(or) $12x_1 x_2(x_1 - x_2) + 12(x_1^2 - x_2^2) - 312(x_1 - x_2) = 0$

$$(x_1 - x_2)(x_1 x_2 + x_1 + x_2 - 26) = 0$$

$\therefore x_1 = x_2$ or $x_1 = \frac{26 - x_2}{1 + x_2}$

$\therefore f(x)$ is not a one-one function.

Example 12: Show that the function $f = R \to R$ defined by $f(x) = 2x + 5$

Consider the function

$f = \{(1, 3), (2, 5), (3, 3), (4, 1), (5, 2)\}$

$g = \{(1, 4), (2, 1), (3, 1), (4, 2), (5, 3)\}$

from X to X where $X = \{1, 2, 3, 4, 5\}$

(i) Determine the range of f and g.

(ii) Find gof and fog.

Solution:

Range of f $= \{1, 2, 3, 5\}$

Range of g $= \{1, 2, 3, 4\}$

$(gof)(1) = g[f(1)] = g(3) = 1$

$(gof)(2) = g[f(2)] = g(5) = 3$

$(gof)(3) = g[f(3)] = g(3) = 1$

$(gof)(4) = g[f(4)] = g(1) = 4$

$(gof)(5) = g[f(5)] = g(2) = 1$

$\therefore gof = \{(1, 1), (2, 3), (3, 1), (4, 4), (5, 1)\}$ .... (1)

$fog(1) = f[g(1)] = f(4) = 1$

$fog(2) = f[g(2)] = f(1) = 3$

$fog(3) = f[g(3)] = f(1) = 3$

$fog(4) = f[g(4)] = f(2) = 5$

$fog(5) = f[g(5)] = f(3) = 3$

$\therefore fog = \{(1, 1), (2, 3), (3, 3), (4, 5), (5, 3)\}$ .... (2)

from (1) & (2) $gof \neq fog$.

Example 13: Let f and g be functions defined by $f(x) = 3x + 4$ and $g(x) = x^2 + 2$. Find the formulate determining (1) gof and (2) fog.

Solution:

$f(x) = 3x + 1$

$g(x) = x^2 + 2$

$(gof)(x) = g[f(x)]$

$= g(3x + 1)$

$= (3x+1)^2 + 2 = 9x^2 + 6x + 3$ .... (1)

$(fog)(x) = f[g(x)]$

$= f(x^2 + 2)$

$= 3(x^2 + 2) + 1 = 3x^2 + 7$ .... (2)

From (1) and (2) $fog \neq gof$

Example 14: If the function $f : R \to R$ be given by $f(x) = 4x - 1$ and the function $g : R \to R$ be given by $g(x) = x^3 + 2$, find $(gof)x$ and $(fog)x$, R being the set of real numbers.

Solution:

$$f(x) = 4x - 1$$

$$g(x) = x^3 + 2$$

$$(gof)(x) = g[f(x)]$$

$$= g(4x-1) = (4x-1)^3 + 2$$

$$= 64x^3 - 48x^2 + 12x + 1 \quad \text{.............. (1)}$$

$$fog(x) = f[g(x)]$$

$$= f\left(x^3+2\right)$$

$$= 4\left(x^3+2\right) - 1 = 4x^3 + 7$$

Example 15: Let $A = \{1, 2, 3\}$

Define $f : A \to A$ by $f(1)=2, f(2)=1$ and $f(3)=3$. Find f^2, f^3, f^4 and f^{-1}.

Solution:

(i) $f^2 = fof$

$$f^2(1) = (fof)(1) = f[f(1)] = f(2) = 1$$

$$f^2(2) = f[f(2)] = f(1) = 2$$

$$f^2(3) = f[f(3)] = f(3) = 3$$

(ii) $f^3 = fof^2$

$$f^3(1) = f\left[f^2(1)\right] = f(1) = 2$$

$$f^3(2) = f\left[f^2(2)\right] = f(2) = 1$$

$$f^3(3) = f\left[f^2(3)\right] = f(3) = 3$$

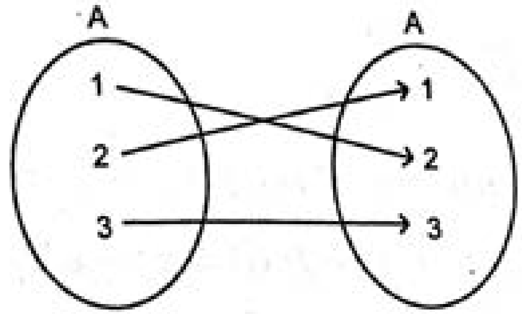
(iii) $f^4 = f^2 0 f^2$

$$f^4(1) = f^2\left[f^2(1)\right] = f^2(1) = 1$$

$$f^4(2) = f^2\left[f^2(2)\right] = f^2(2) = 2$$

$$f^4(3) = f^2\left[f^2(3)\right] = f^2(3) = 3$$

(iv)



$$f^{-1}(1) = 2$$

$$f^{-1}(2) = 1$$

$$f^{-1}(3) = 3$$

Example 16: Which of the following functions are injections, surjections or bijections on *R*, the set of real numbers?

(i) $f(x) = -2x$

(ii) $g(x) = x^2 - 1$

(iii) $h(x) = x$ if $x < 0$ and x^2 if $x \geq 0$.

(i) Let $f(x_1) = f(x_2)$ for $x_1,\ x_2 \in R$

Then $-2x_1 = -2x_2$

$x_1 = x_2$

$\therefore f$ is one-one or injective.

The range of f is the real line R.

$\therefore f$ is also onto or surjective.

Since is one-one and onto it is a bijective mapping.

(ii) $g(x) = x^2 - 1$

$g(2) = 4 - 1 = 3$

$g(-2) = 4 - 1 = 3$

$g(2) = g(-2)$

$\therefore g$ is not a injective mapping.

Also the range of g is a subset of R and not R.

$\therefore g$ is not a surjective mapping

$\therefore g$ is not a bijective mapping.

(iii) $h(x) = \begin{cases} x & x < 0 \\ x^2 & x > 0 \end{cases}$

If $x_1, x_2 \leq 0$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

If $x_1, x_2 \geq 0$ $f(x_1) = f(x_2) \Rightarrow {x_1}^2 = {x_2}^2$

$\Rightarrow x_1 = x_2$ where $x_1, x_2 \geq 0$

$\therefore R$ is a one-one function.

Clearly the range of h is R.

$\therefore f$ is in onto function.

Hence f is a one-one onto function or bijective function.

Example 17: If $f(x) = x^2 + x + 1$, $g(x) = 2x - 1$ where x is a real number find *fog*, *gof*, *fof* and *gog*.

Solution:

$f(x) = x^2 + x + 1$

$g(x) = 2x - 1$

$(fog)x = f[g(x)]$

$= f(2x - 1)$

$= (2x - 1)^2 + 2x - 1 + 1 = 4x^2 - 2x + 1$

$(gof)x = g[f(x)]$

$= g\left(x^2 + x + 1\right)$

$= 2\left(x^2 + x + 1\right) - 1$

$= 2x^2 + 2x + 1$

$$
\begin{aligned}
(fof)x &= f[f(x)] \\
&= f\left(x^2+x+1\right) \\
&= \left(x^2+x+1\right)^2+\left(x^2+x+1\right)+1 \\
&= x^4+2x^3+2x^2+2x+1+x^2+x+1+1 \\
&= x^4+2x^3+3x^2+3x+3
\end{aligned}
$$

$$
\begin{aligned}
(gog)x &= g[g(x)] \\
&= g(2x-1) \\
&= 2(2x-1)-1 \\
&= 4x-3
\end{aligned}
$$

Example 18: Determine if each of the functions is one-one.

(i) To each person, assign the number which corresponds to his age.

(ii) To each country in the world assign the latitude and longitude of its capital.

(iii) To each book written by only one author assign the author.

(iv) To each country in the world which has a president assign its president.

Solution:

(i) Not a one-one function

(ii) One-one function

(iii) Not a one-one function

(iv) One-one function.

Example 19: Given $f(x)=\dfrac{x}{1+|x|}$ in $(-\infty, \infty)$. Is f bijective? Give reason.

Solution:

$$|x| = \begin{cases} x, & x>0 \\ -x, & x\le 0 \end{cases}$$

$\therefore f(x)>0$ for $x>0$

and $f(x)<0$ for $x<0$.

$\therefore$ If two elements x_1 and x_2 are of opposites signs they cannot have the same image.

$\therefore$ for x_1, x_2 (both $+$ve),

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one-one.

But $f(x) = \dfrac{x}{1+|x|} = y$ say

Then we find for all x, $\quad y \le 1$

$\therefore f$ is not onto.

Hence f is not a bijective function.

Example 20: Find the inverse of the function $f(x) = 1 - 2^{-x}$

Solution:

$$f(x) = 1 - 2^{-x}$$

$$f(x_1) = f(x_2) \Rightarrow 1 - 2^{-x_1} = 1 - 2^{-x_2}$$

$$\Rightarrow 2^{-x_1} = 2^{-x_1}$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one-one.

Also $f(x) = 1 - 2^{-x} = y$ (say)

$$1 - y = 2^{-x}$$

or $\quad -x = \log_2(1-y)$

$$x = -\log_2(1-y)$$

$\therefore f$ is onto for $y < 1$.

$\therefore f^{-1}$ exists and is defined by

$$f^{-1} = -\log_2(1-y),\ y < 1$$

Example 21: Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false.

$f(x) = 1,\ f(y) \neq 1,\ f(z) \neq 2$

Determine $f^{-1}(1)$

Solution:

There are three possibilities

(i) $f(x) = 1$ is true but $f(y) \neq 1$ and $f(z) \neq 2$ are false.

(ii) $f(y) \neq 1$ is true but $f(x) = 1$ and $f(z) \neq 2$ are false.

(iii) $f(z) \neq 2$ is true but $f(x) = 1$ and $f(y) \neq 1$ are false

(i) is not possible since f is one-one and (i) implies $f(x) = 1, f(y) = 1$

Again (ii) contradicts that f is one-one as $f(x) = 2$ or 3

$$f(y) = 2 \text{ or } 3 \text{ and } f(z) = 2$$

In case (iii) $f(z) = 1$ or $3, f(x) = 2$ or $3, f(y) = 1$

$\therefore$ The only possibility is

$$f(z) = 3$$
$$f(x) = 2$$
$$f(y) = 1$$

$\therefore f^{-1}(1) = y$

Example 22: Let $f : R \to R$ be a function defined by

$$f(x) = \frac{e^{|x|} - e^{-x}}{e^{x} + e^{-x}}$$

Show that f is neither one-one nor onto.

Solution:

$$f(x) = \frac{e^{|x|} - e^{-x}}{e^{x} + e^{-x}}$$

$$f(0) = 0$$
$$f(-1) = 0$$
$$f(0) = f(-1)$$

$\therefore f$ is not one-one.

Suppose $f(x) = 1$ then $e^{|x|} - e^{-x} = e^{x} + e^{-x}$

Clearly $x \neq 0$

For $x > 0$, we have $e^{x} - e^{-x} = e^{x} + e^{-x}$

(i.e.,) $-e^{-x} = e^{-x}$ or $e^{-x} = 0$ which is not true.

For $x < 0$, we have $e^{-x} - e^{-x} = e^{-x} + e^{-x}$

$-e^{-x} = e^{x}$ which is not true.

$\therefore$ for $f(x) = 1$, there is no $x \in R$.

$\therefore f$ is not onto.

Hence $f(x)$ is neither one-one nor onto.

Example 23: Let f and g be functions defined by

$f(x) = \dfrac{x}{x+1}$ and $g(x) = \dfrac{x}{1-x}$. Find $(fog)^{-1}(x)$

Solution:

$$(fog)(x) = f[g(x)] = f\left(\frac{x}{1-x}\right) = \frac{\frac{x}{1-x}}{\frac{x}{1-x}+1} = x$$

Example 24: If $f(x) = \dfrac{1}{1-x}$ find $fof\,of$

Solution:

$$f(x) = \frac{1}{1-x}$$

$$(fof)(x) = f(f(x))$$

$$= f\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x}$$

$$(fofof)(x) = fo[(fof)x]$$

$$= \frac{\frac{1}{1-x}-1}{\frac{1}{1-x}} = \frac{1-1+x}{1} = x$$

Example 25: Find *fofof* if $f(x)=\frac{x}{\sqrt{1+x}}$

Solution:

Let $y=\frac{x}{\sqrt{1+x^2}}$

$(fof)(x)=f[f(x)]$
$=f(y)$

$$=\frac{y}{\sqrt{1+y^2}}$$

$$fof(x)=\left(\frac{y}{\sqrt{1+y^2}}\right)=\frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}}=\frac{x}{\sqrt{1+2x^2}}=z\ \text{(say)}$$

$fofof(x)=f(z)$

$$=\frac{z}{\sqrt{1+z^2}}=\frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}}$$

$$=\frac{x}{\sqrt{1+3x^2}}$$

Example 26: Show that the mapping $f:R\to R$ given by $f(x)=x^3+ax^2+bx+c$ is a bijection if and only if $a^2\le 3b$.

Solution:

$f(x)=x^3+ax^2+bx+c$

$f(x)\to-\infty$ as $x\Rightarrow-\infty$

$f(x)\to+\infty$ as $x\Rightarrow\infty$

$\therefore f$ is onto.

Also $f'(x) = 3x^2 + 2ax + b \geq 0$ for all $x \in R$ if and only if the discriminant of $f'(x) \leq 0$.

$$4a^2 - 12b \leq 0$$

$$a^2 - 3b \leq 0 \text{ or } a^2 \leq 3b$$

Example 27: Let $f : [0,1] \to [0,1]$ defined by

$f(x) = \dfrac{1-x}{1+x}$, $0 \leq x \leq 1$ and let

$g : [0, 1] \to [0, 1]$ defined by $g(x) = 4x(1-x)$, $0 \leq x \leq 1$

Determine *fog* and *gof*.

Solution:

$$f(x) = \frac{1-x}{1+x},\ 0 \leq x \leq 1$$

$$f[g(x)] = \frac{1-g(x)}{1-g(x)},\ 0 \leq g(x) \leq 1$$

$0 \leq g(x) \leq 1 \Rightarrow 0 \leq 4x(1-x) \leq 1$ when $0 \leq x \leq 1$.

Then $0 \leq 4x - 4x^2 \leq 1 \Rightarrow 4x^2 - 4x + 1 \geq 0$ and $x^2 - x \leq 0$

Then $0 \leq 4x - 4x^2 \leq 1 \Rightarrow 4x^2 - 4x + 1 \geq 0$ and $x^2 - x \leq 0$.

(i.e.,) $(2x-1)^2 \geq 0$ and $x(x-1) \leq 0$

But $(2x-1)^2 \geq 0$ for all real x.

$\therefore 0 \leq g(x) \leq 1 \Rightarrow 0 \leq x \leq 1$

$$f[g(x)] = \frac{1-g(x)}{1+g(x)},\ 0 \leq x \leq 1$$

$$= \frac{1-4x(1-x)}{1+4x(1-x)},\ 0 \leq x \leq 1$$

$$= \frac{4x^2 - 4x + 1}{-4x^2 + 4x + 1},\ 0 \leq x \leq 1$$

$(gof)(x) = g[f(x)]$

$$= g\left(\frac{1-x}{1+x}\right) \text{ when } 0 \le x \le 1$$

$$= 4\left(\frac{1-x}{1+x}\right)\left[1-\frac{1-x}{1+x}\right] \text{ when } 0 \le \frac{1-x}{1+x} \le 1 \text{ and } 0 \le x \le 1$$

$$= \frac{8x(1-x)}{(1+x)^2} \text{ when } 0 \le x \le 1$$

Example 28: Find the range of the function

(i) $y = \sin^2 x + \cos^4 x$

(ii) $y = 3\sin\sqrt{\frac{\pi^2}{16} - x^2}$

Solution:

$$y = \sin^2 x + \sin^4 x$$

$$= 1 - \cos^2 x + \cos^4 x$$

$$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$\therefore$ The maximum value of $y = \frac{1}{4} + \frac{3}{4} = 1$

Also y will be minimum when $\cos^2 x - \frac{1}{2} = 0$

$\therefore$ minimum $y = 0 + \frac{3}{4} = \frac{3}{4}$

$\therefore$ Range is $\left[\frac{3}{4}, 1\right]$

(i.e.) $y = 3\sin\sqrt{\frac{\pi^2}{16} - x^2}$

$\sin x$ is an increasing function.

$\therefore$ maximum $y = 3\sin\sqrt{\frac{\pi^2}{16} - 0}$

$$= 3\sin\frac{\pi}{4} = \frac{3}{\sqrt{2}}$$

Minimum $\quad y = 3\sin\left[\frac{\pi^2}{16} - \frac{\pi^2}{16}\right] = 0$

$\therefore$ Range is $\left[0, \frac{3}{\sqrt{2}}\right]$

Example 29: If f, g, h are functions from R to R.

Such that $f(x) = x^2 - 1,\ g(x) = \sqrt{x^2 + 1}$

$$h(x) = \left.\begin{matrix} 0 \\ x \end{matrix}\right\} \begin{matrix} \text{if } x < 0 \\ \text{if } x \geq 0 \end{matrix}$$

Then find the composite function $ho(fog)$ and determine whether the function fog is invertible and h is the identify function.

Solution:

$$f(x) = x^2 - 1 \text{ for all } x$$

$$g(x) = \sqrt{x^2 + 1} \text{ for all } x$$

$$\therefore f[g(x)] = [g(x)]^2 - 1$$

$$= x^2 + 1 - 1 = x^2 \text{ for all } x$$

$$\therefore h[f(g(x)] = h(x^2) \text{ for all } x$$

$$= x^2 \text{ since } x^2 \geq 0$$

Since $x^2 > 0$, $(fog)x$ cannot be negative

$\therefore fog$ is not an onto function

$\therefore fog$ is not invertible

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$h(x) = x$ for $x \geq 0$

But given $h(x) \neq x$ for $x < 0$

Hence h is not the identity function.

Example 30: Find the inverse of the function

$$y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$$

If $y = f(x)$ then the inverse function its given by $x = f^{-1}(y)$

$$y\left(10^x + 10^{-x}\right) = 10^x - 10^{-x}$$

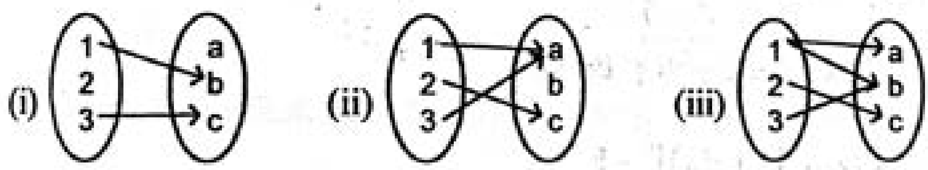
$$10^{2x} = \frac{1+y}{1-y}$$

(i.e.,) $$x = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$$

$\therefore$ the inverse function is $\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$

Exercise

1. State whether each diagram defines a function from $\{1, 2, 3\}$ into $\{a, b, c\}$

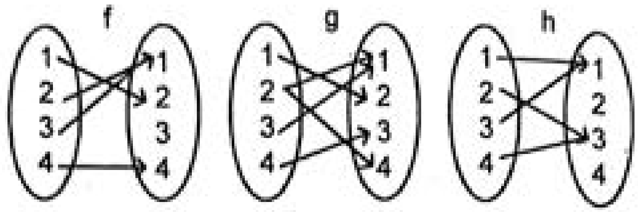


2. Define each function by a formula.
   (i) To each number f assigns its cube plus 2.
   (ii) To each number g assigns its square plus twice the number.
   (iii) To each number greater than or equal to 3, h assigns the number squared and to each number less than 3, h assign the number -2.
3. Determine the number of different functions from $\{1, 4\}$ into $\{1, 2, 3\}$
4. Let $X = \{a, b, c, d\}$. Determine whether each set of the following ordered pairs is a function from X into X.
   (i) $\{(b, a), (c, d), (d, a), (c, d), (a, d)\}$
   (ii) $\{(d, d), (c, a), (a, b), (d, b)\}$

(iii) $\{(a, b), (c, b), (c, b), (d, b)\}$

(iv) $\{(a, a), (b, a), (a, b), (c, d)\}$

5. The following diagrams defines functions f, g and h which map the set $\{1, 2, 3, 4\}$ into itself.



(i) Find the range of f, g & h.

(ii) Find the composition functions (a) fog, (b) hof, (c) g^2

6. (i) State whether each of the function is one-one.

(ii) State whether each of the function is onto.

Let $A = R - \{3\}$ and $B = R - \{1\}$ where R is the set of all real numbers. Let the function $f : A \to B$ be defined by

$f(x) = \dfrac{x-2}{x+3}$. Is this function a bijective?

7. Consider the following relations.

(i) $\{(1, 2), (3, 2), (5, 5)\}$

(ii) $\{(1, 2), (2, 3), (5, 5)\}$

(iii) $\{2, 1), (2, 3), (5, 5)\}$

Are the relations functions?

8. Which of the following sets of ordered pairs define a one-one function?

(a) $R = \{(x, y) : x^2 + y^2 = 2\}$ on R

(b) $A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 5\}$

$R = \{(x, y) : 5x + 2y \text{ is a prime}\}$ on A

(c) $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$ and

$R = \{(x, y) : y = x^2 - 3x + 3\}$ on A

9. Which of the following functions have inverse defined on their ranges?

(a) $f(x) = x^2, x \in R$

(b) $f(x) = x^3, x \in R$

(c) $f(x) = \sin x,\ 0 < x < 2\pi$

(d) $f(x) = e^x,\ x \in R$

10. Find the range of the function $\dfrac{1}{2-\cos 3x}$

11. Consider the following sets

$$R = \{(x, y) \mid x, y \in R,\ x^2 + y^2 = 25\}$$

$$R_1 = \left\{(x, y) = x, y \in R,\ y \le \frac{4x^2}{9}\right\}$$

Is $R \cap R_1$ a function? If so find the domain and range.

12. The map $f : R \to R$ is given by $f(x) = ax + b$ for constant of a and b is the function $fof = I$, I being the identity function.

13. Let $g : R \to R$ be given by $g(x) = 3 + 4x$. If $g^n(x) = gogo........g(x)$

Show that $g^n(x) = \left(4^n - 1\right) + 4^n x$. If $g^{-n}(x)$ denotes the inverse of $g^n(x)$.

prove that the above formulae holds for all negative integers.

14. Let $f : R \to R$ and $g : R \to R$ be defined by

$f(x) = x + 1$ and $g(x) = x^2 - 2$. Find $(gof)^{-1}\{[-2, -1]\}$

15. Find the inverse of the function.

$$y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}.$$

16. If the function f and g whether given by

$f = \{(1, 2), (3, 5), (4, 1)\}$

$g = \{(2, 3), (5, 1), (6, 3)\}$

Find the functions fog and gof.

17. Show that the function $f = \left\{x, \dfrac{x}{x-1}\right\};\ -1 \le x \le 1$

is one-one and find its inverse.

18. Find $g(x) = f[f(x)]$ where

$$f(x) = \begin{cases} (x+1), & x < 0 \\ (x^2+1), & x \geq 0 \end{cases}$$

19. Let R be the set of all real numbers. Using the fact that every cubic equation with real coefficients has a real root, show that $x \to x^3 - x$ defines a mapping of R onto R. Is this a one-one mapping?

20. If R is the set of real numbers, show that $f = x \to x^3 + x^2$ defines a mapping from R onto R. Is f one-one.

21. If the mapping $f: R \to R$ be given by $f(x) = 4x - 1$ and the mapping $g: R \to R$ be given by $g(x) = 3x + 2$ then find gof and fog.

22. If f and g are two functions from R to R given by $f(x) = x^3 + 3x + 1$ and $g(x) = 2x - 3$ then obtain fog and gof.

23. If $f(x) = \sqrt{(x-1)}$ and $g(x) = \sin x$, then find $(fog)x$ and $(gof)x$.

24. If $f(x) - 1 + |x - 1|, \ -3 \leq x \leq 3$
$g(x) = 2 - |x + 1|, -2 \leq x \leq 2,$
then find $(fog)x$ and $(gof)x$.

25. Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for $x \in R$, where R is the set of real numbers. Find $gof, fog, fof, gof, foh, hog, hof$, and $fohog$.

(M.C.A., April 2000)

26. A function $A(m, n)$ is defined as follows:

$$\begin{aligned} A(m,n) &= n+1 && \text{if } m = 0 \\ A(m,n) &= A(m-1, 1) && \text{if } m \neq 0, n = 0 \\ A(m,n) &= A(m-1, A(m, n-1)) && \text{if } m \neq 0, n \neq 0 \end{aligned}$$

Find the value of $A(1, 3)$

(B.C.A., October 1999)

Chapter 6

Binary Operation

In this section we discuss a general kind of mathematical operation which includes as special cases the familiar operations of arithmetic as well as many others that will be of importance in mathematics and computer science.

Definition : A binary operation in a set is a rule or operation which assigns to any pair of elements in S, taken in a definite order, another element in the same set. Such a binary operation is denoted by '*'.

Thus if a and b are two elements of the set S, then * operation assigns to the pair a, b, some element of S. We shall denote it by $a*b$. The operation * assigns to the pair (b, a) an element namely $b*a$. Observe that the definition tells nothing about the relation of $a*b$ to $b*a$.

The symbol * used to represent the operation is of no importance and presently we shall use other symbols which are more convenient. Moreover, the same set S may be equipped with several different binary operations.

Example 1: Addition of whole numbers, multiplication of whole numbers, subtraction of whole numbers are binary operations in Z.

Example 2 : Let Z be the set of integers. Operation * defined by $a*b = a+b-ab$ where $a, b \in Z$ is a binary operation. If $a, b \in Z^{+}$ then * is not a binary operation since $a+b-ab$ need not be a positive integer.

Example 3 : Let S be the set of all subsets of real numbers. For $A \in S, B \in S$, the operation *union* (*i.e.*, $\cup$) is a binary operation in S.

Example 4 : Let $G = \{1 - 1, i, -i\}$ be the set of the four fourth roots of unity. Then for $a, b \in G$, $a*b = a.b$ where . is the usual multiplication is a binary operation.

If $a*b = a+b$ where '+' is the usual addition then '+' is not a binary operation in G.

Example 5 : On the set of natural numbers, subtraction is not a binary operation.

Example 6 : The operation * on the set of real numbers defined by $a*b = |ab|$ is a binary operation.

Example 7 : Show that multiplication is a binary operation on $A = \{1, -1\}$, but not on $B = \{1, 2\}$.

Solution :

$\cdot$	1	-1
1	1	-1
-1	-1	1

Table 1

$\cdot$	1	2
1	1	2
2	2	4

Table 2

From table 1, we note that the multiplication of any two elements of A is in A. Therefore multiplication is a binary operation in A. But from table 2, we see that multiplication is not a binary operation in B since the element $4 \notin B$.

Example 8 : If $S = \{A, B, C, D\}$ where $A = \phi$, $B = \{a, b\}$, $C = \{a, c\}$, $D = \{a, b, c\}$ show that $\cup$ is a binary operation on S but $\cap$ is not.

Solution :

$A \cup B = \phi \cup \{a,b\} = \{a,b\} = B$

$A \cup C = C$

$A \cup D = D$

$B \cup C = \{a,b\} \cup \{a,b,c\} = \{a,b,c\} = D$

$C \cup D = D$

Hence $\cup$ is a binary operation in S.

Also $B \cap C = \{a,b\} \cap \{a,c\} = \{a\} \notin S$.

$\therefore \cap$ is not a binary operation in S.

Types of Binary Operations

1. Commutative Operation : A binary operation * over a set S is said to be *commutative* if for every pair of elements $a, b \in S$,

$$\boxed{a * b = b * a}$$

Example 2 : Addition and multiplication are commutative binary operations for natural numbers. But subtraction and division are not commutative binary operation for rational numbers.

2. Associative Operation : A binary operation * on a set S is called *associative* binary operation whenever $a, b, c \in S$,

$$\boxed{(a * b) * c = a * (b * c)}$$

Example : Let * be a binary operation defined on R by $a * b = a + 2b$ for all $a, b \in R$

$$(a*b)*c = (a+2b)*c \quad \text{..........} \quad (1)$$
$$= a+2b+2c$$
$$a*(b*c) = a+(b+2c)$$
$$= a+2(b+2c) \quad \text{..........} \quad (2)$$
$$= a+2b+4c$$

From (1) and (2),

$$a*(b*c) \neq (a*b)*c$$

$\therefore$ * is not an associate binary operation

3. Distributive Operation: Let S be a set on which two binary operations * and o be defined. The operation * is said to be left distributive with respect to 'o' if $a*(boc) = (a*b)o(a*c)$ and is said to be right distributive if

$$ao(b*c) = (aob)*(aoc)$$

for all $a,b,c \in S$

If the operation * is left distributive as well as right distributive we say that * is distributive with respect to 'o'.

Example: For the set of real numbers, multiplication (.) is distributive with respect to addition (+) since

$a.(b+c) = (a.b)+(a.c)$ for all $a,b,c \in R$. But addition is not distributive with respect to multiplication.

(*i.e.,*) $a+(b.c) \neq (a+b).(a+c)$ for all $a.b,c \in R$.

4. Identity: An element e in a set S is said to be an identity element with respect to the binary operation * if and only if for all $a \in S$,

$a*e = e*a = a$

Example 1: For the set R * real numbers 0 is the identity with respect to the binary operation addition and 1 is the identity with respect to the binary operation multiplication.

Example 2: In the algebra of sets, the operation $\cup$ and $\cap$ are commutative, associative and each is distributive over the other.

φ is the identity for the operation $\cup$ and U is the identity for the operation $\cap$.

5. Inverse: An element $a' \in S$ is called inverse of $a \in S$ if, $a*a' = a'*a = e$ for all $a \in S$.

6. Idempotent: The binary operation * is said to be idempotent on a set S if and only if for every $a \in S$,

$$a * a = a.$$

Example 1: How many binary operations can be defined on a set with 2 elements?

Solution: The number of binary operations on S which has 2 elements

$$= 2^4 = 16$$

Example 2: Define a binary operation on the set of positive integers by $a*b = \max(a,b)$. Show that this operation is both commutative and associative.

Solution:

$$a*(b*c) = a*\max(b,c)$$

$$= \max[a, \max(b,c)] \quad \ldots\ldots \quad (1)$$

$$(a*b)*c) = \max(a,b)*c$$

$$= \max[\max(a,b), c] \quad \ldots\ldots \quad (2)$$

But $\max[a, \max(b,c)] = \max[\max(a,b), c]$

$$\therefore a*(b*c) = (a*b)*c$$

$\therefore$ * is associative

Also $a*b = \max(a,b)$

$$= \max(b,a)$$

$$= b*a$$

$\therefore$ * is commutative

Example 3: Define a binary operation on the set of non-negative integers by $a*b = a^2 + b^2$. Does the operation have

(i) an identity element?

(ii) an inverse element?

Solution:

$$a*e = a$$

$$a^2 + e^2 = a$$

or $e = \pm\sqrt{a - a^2}$ which is not a non-negative integer.

$\therefore$ The operation * has no identity and hence no inverse for any non-negative integer.

Example 4: Give examples of binary operations on a finite set which is (i) associative but not commutative (ii) commutative but not associative.

Solution:

(i) Let $S=\{a,b\}$ with binary operation . designed by

$\cdot$	a	b
a	a	a
b	b	b

This binary operation is associative and not commutative.

(ii) Let $S=\{a,b,c\}$

$\cdot$	a	b	c
a	a	a	b
b	a	c	a
c	b	a	b

Here the binary operation is commutative but not associative.

(*i.e.,*) $(a\cdot b)\cdot c \neq a\cdot(b\cdot c)$

$$(a\cdot b)\cdot c = a\cdot c = b$$

$$a\cdot(b\cdot c) = a\cdot a = a$$

Example 5: Let S be a set and * be a binary operation on S satisfying the conditions

$$a*a=a \quad \forall a\in S$$

and $(a*a)*c=(b*c)*a, \forall a,c\in S$.

Show that * is both commutative and associative.

Solution:

$$a*b=(a*b)*(a*b)$$

$$=b*(a*b)*a$$

$$=((a*b)*a)*b$$

$$=((b*a)*a)*b$$

$$=((a*a)*b)*b$$

$=(a*b)*a$

$=b*a$

$\therefore *$ is commutative

(ii) For $a,b,c \in S$,

$$(a*b)*c=(b*c)*a$$

$$=a*(b*c)$$

$\therefore *$ is also an associative operation

Example 6: Define on Z, $a*b=-a-b,\ \forall a,b \in Z$

Show that * is not associative.

Solution:

$a*(b*c)=a*(-b-c)$

$=-a-(-b-c)$

$=-a+b+c$ .... (1)

$(a*b)*c=(-a-b)*c$

$=-(-a-b)-c$

$=a+b-c$ .... (2)

From (1) and (2)

$a*(b*c) \neq (a*b)*c$

$\therefore *$ is not associative

Exercises

1. Define a binary operation on the set of even integers which is different from addition, subtraction and multiplication.
2. Let $G=\{a, b, c, d\}$ be a set with four elements.

   Define a binary operation on G by the following table

	a	b	c	d
a	a	b	c	d
b	b	c	c	d
c	c	d	d	a
d	d	b	d	c

Is this binary operation (i) associative (ii) commutative (iii) does it have an identity element?

3. Give an example of a set with two elements and an associative binary operation on it which is not commutative.
4. Show that for the binary operations given below all the following properties are satisfied.
   (i) associativity
   (ii) existence of identity
   (iii) existence of inverse for each element

   (a) The set of real valued continuous functions on $[a, b]$ for the operation
   $$f * g = h \text{ where } h(t) = f(t) + g(t),\ 0 \le t \le 1.$$
   (b) The set of all ordered pair of real numbers for the operation
   $$(a,b) * (c,d) = (ac - bd, ad + bc)$$
   (c) The set $G = \{1, -1, i, -i\}$ for the product
5. Let $*$ be a binary operation on a set S satisfying
   $$x * (x * y) = y,\ \forall x, y \in S$$
   $$(y * x) * x = y,\ \forall x, y \in S$$
   Show that $*$ is commutative but not necessarily associative.
6. Given that
   $$S = \{A, B, C, D\} \text{ where } A = \phi,\ B = \{a\}\ C = \{a,b\},\ D = \{a,b,c\}$$
   show that S is closed under the binary operations of $\cup$ and $\cap$.
7. If the binary operation $*$ on the set of rational numbers is defined by
   $$a * b = a + b - ab,\ \text{for every } a, b \in Q$$
   (a) show that $*$ is commutative and associative
   (b) show that identity is zero
8. For each of the following determine whether $*$ is commutative and associative.
   (a) on Z, define $a * b = a - b$
   (b) on Z^+, define $a * b = \dfrac{ab}{2}$
   (c) on Z^+ define $a * b = 2^{ab}$
   (d) on Z, define $a * b = a + b + 1$
   (e) On $R - \{1\}$ define $a * b = a + b$

9. Prove that if * is an associative or commutative binary operation on a set S, then $(a*b)*(c*d)=[(d*c)*a]*b$ for all $a,b,c,d \in S$

10. The binary operations '*' and '.' on the set of real numbers are defined by

$$a*b=|a-b| \text{ and } a.b=a$$

Show that * is commutative but not associative and '.' is associative but not commutative and the * is distributive over '.'.

11. On Z^+ define $a*b=a^b$. Is * associative?

12. On the set of 2×2 non singular matrices define * as matrix multiplication. Is * commutative?

13. On Z^+ define * by $a*b=a^b$. Is * commutative?

14. Let G denote the set of all matrices of the form $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$ where $x \in R-\{O\}$

Show that matrix multiplication is a binary operation is G and show that it is also associative.

15. Let S be the set of all real numbers except -1. Define * on S by

$$a*b=a+b+ab$$

(a) Show that * is a binary operation

(b) Find the solution of the equation $2*x*3=7$ in S.

16. Let M_2 be the set of all 2×2 matrices over the rational numbers. For $A,B\in M_2$ define

$$AoB=\frac{AB+BA}{2}$$

Prove the following

(a) o is commutative

(b) $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, C=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$(AoB)oC \neq Ao(BoC)$ $(i.e.,)$ o is not associative

(a) $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity under '0'.

17. For any two matrices A, B in M_2 define

$$A * B = \frac{AB - BA}{2}$$

Prove the following

(a) For any $A \in M_2$, $A * A = 0$ where 0 is zero matrix

(b) For $A, B, C \in M_2$

$$A*(B*C) + B*(C*A) + C*(A*B) = 0$$

(c) There exists no identity in M_2 with respect to *

(d) For $A, B, C \in M_2$

$$A*(B+C) = A*B + A*C$$

$$(B+C)*A = B*A + C*A$$

(e) Show that * is neither commutative nor associative.

Symmetric Difference

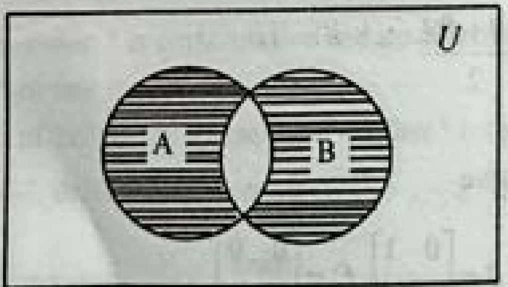
Definition: The symmetric difference of two sets A and B is defined to be $(A \cup B) - (A \cap B)$ and is denoted by $A \Delta B$

(*i.e.,*) $A \Delta B = (A \cup B) - (A \cap B)$

Also $A \Delta B = \{x\} \,|x \in A \cup B$ and $x \notin A \cap B|\}$

Note: $A \Delta B$ is the set of all elements which are either in A or in B but not in both A and B.

Venn diagram for $A \Delta B$



$A \Delta B$

Example :

$A = \{1, 2, 3, ... 9\}$

$B = (4, 5, a, b, c, d\}$

$A \Delta B = \{1, 2, 3, 6, 7, 8, 9, a, b, c, d\}$

Properties of Symmetric difference

(1) $A\Delta A=\phi$

(2) $A\Delta(B\Delta C)=(A\Delta B)\Delta C$

(3) $A\Delta\phi = A$

(4) $A\Delta B = B\Delta A$

(5) $A\Delta B=(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$

The proofs of properties (1) to (4) are simple. Let us now prove the property 5.

Proof: Let $x \in(A-B)\cup(B-A)$ Then

$$x\in(A-B)\cup(B-A)\Rightarrow x\in A-B \text{ or } x\in B-A$$
$$\Rightarrow x\in A \text{ and } x\notin B \text{ or } x\in B \text{ and } x\notin A$$
$$\Rightarrow x\in A\cup B \text{ and } x\notin A\cap B$$
$$\Rightarrow x\in(A\cup B)-(A\cap B)$$

$\therefore (A-B)\cup(B-A)\subset(A\cup B)-A\cap B$ I

Let $y\in(A\cup B)-(A\cap B)$

$$y\in(A\cup B)-(A\cap B)\Rightarrow y\in A\cup B \text{ and } y\notin A\cap B$$
$$\Rightarrow y\in A \text{ or } y\in B \text{ and } y\notin A\cap B$$
$$\Rightarrow y\in A \text{ and } y\notin A\cap B \text{ or } y\in B \text{ and } y\notin A\cap B$$
$$\Rightarrow y\in A \text{ and } y\notin B \text{ or } y\in B \text{ and } y\notin A$$
$$\Rightarrow y\in A-B \text{ or } y\in B-A$$
$$\Rightarrow y\in(A-B)\cup(B-A)$$

$\therefore (A\cup B)-(A\cap B)\subset(A-B)\cup(B-A)$ II

From I and II

$$(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$$

Hence $A\Delta B=(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$

Exercise

For the operation symmetric difference designed above (i) find the identity element and the inverse of an arbitrary set,

(ii) Show also by Venn diagram that the operation of union does not distributive over α where α is defined by

$$A\Delta B=(A\cup B)-(A\cap B)$$

Chapter 7

PERMUTATIONS AND COMBINATIONS

Permutations

Combinatorial analysis involves determining the number of possibilities of some event without enumerating all the possibilities. In order to develop the general procedure for obtaining possibilities we have to introduce the concepts called permutations and combinations.

Consider the example of finding all the arrangements of the letters a, b, c.

The possible arrangements are abc, acb, bac, bca, cab and cba.

There are 6 possibilities

Consider the following example; suppose there are ten students and two prizes. If all students are capable of getting any one of the prizes, in how many ways can the 1st and 2nd prizes be distributed among the 10 students $a_1, a_2 \ldots\ldots a_{10}$? The list of 1st and 2nd prize winners is enumerated as follows

$a_1 a_2$	$a_2 a_1$	$a_3 a_1$	....	$a_{10} a_1$
$a_1 a_3$	$a_2 a_3$	$a_3 a_2$	....	$a_{10} a_2$
$a_1 a_4$	$a_2 a_4$	$a_3 a_4$	....	$a_{10} a_3$
....	....	....		....
....	....	....		....
....	....	....		....
$a_1 a_{10}$	$a_2 a_{10}$	$a_3 a_{10}$	....	$a_{10} a_9$

Total number of arrangements $= 10 \times 9 = 90$

If there are 3 different prizes then the total number of ways of distributing the 3 prizes are $10 \times 9 \times 8 = 720$

This result is generalised as follows.

Theorem:

If some event can occur in m ways and if, following this event a second event can occur in 'n' ways then both these events can occur in $m \times n$ ways.

Note:- If some event can occur in 'm' ways, following this event a 2^{nd} event can occur in 'n' different ways and following this a 3^{rd} event can occur in 'p' different ways, then all the 3 events can occur together in $m \times n \times p$ ways. This result can be extended to any number of events.

Example (I): In how many ways can 3 scholarships of unequal value be awarded among 8 students giving not more than one to a student?

The 1st scholarship can be awarded to any one of the 8 students, the 2^{nd} to any one of the 7 students, the 3^{rd} to any one of the 6 students.

$\therefore$ All the 3 scholarships can be awarded in $8 \times 7 \times 6$ ways = 336 ways.

Example (II) : There are 5 picture nails on a wall and 7 pictures to choose from. In how many different ways can the pictures be hung on all the nails?

From the 1st nail any one of the 7 pictures can be hung. There are 7 ways for it. From the second nail we can hang any one of the other 6 pictures. Proceeding like this the total number of ways in which the pictures can be hung $= 7 \times 6 \times 5 \times 4 \times 3 = 2520$ ways.

Let us now give the definition for permutation and obtain a formula for permutation of n things taken 'r' at a time.

Any arrangement of a set of 'n' different objects in a given order is called a permutation of the objects taken all at a time. If we consider only 'r' objects ($r \leq n$) for arrangement at a time it is called the permutation of 'n' things taken 'r' at a time. It is denoted by the symbol nP_r or $P(n, r)$. Let us now obtain a formula for nP_r

I. Formula for nP_r:

$$\boxed{1} \quad \boxed{2} \qquad \boxed{3} \qquad \cdots\cdots \qquad \boxed{r}$$

$$n \quad n-1 \qquad n-2 \qquad\qquad n-(r-1)$$

Consider the arrangements of n persons and 'r' seats in a line.

The first seat can be occupied by any one of the 'n' persons and therefore there are 'n' ways for it. Having filled up the 1^{st} seat there are remaining $(n-1)$ persons and $(r-1)$ seats. The 2^{nd} seat can be filled in $(n-1)$ ways. Now the 1st two seats can be filled in $n(n-1)$ ways. The 3^{rd} seat can be filled in $(n-2)$ ways and all the three seats can be filled in $n(n-1)(n-2)$ ways.

Proceeding like this all the r seats can be filled in

$$n(n-1)(n-2)\ldots(n-r+1) \text{ ways}$$

$$\therefore nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1) \qquad \ldots (1)$$

The above formula can be expressed in the factorial form.

The product of 'n' natural numbers is called factorial n and is denoted by $\lfloor n$ (or $n!$)

Now $nP_r = n(n-1)(n-2)\ldots(n-r+1)$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2.1}{(n-r)(n-r-1)\ldots\ldots 2.1}$$

$$nP_r = \frac{\lfloor n}{\lfloor n-r} \qquad \ldots (2)$$

Note: When $r = n$ from formula (1),

$$nP_n = n(n-1)(n-2)\ldots\ldots 2.1 = \lfloor n$$

From formula (2), $nP_n = \dfrac{\lfloor n}{\lfloor 0}$

$\therefore$ we conventionally take $\lfloor 0 = 1$ even though it has no meaning.

Example:

$7P_2 = 7 \times 6 = 42$

$10P_3 = 10 \times 9 \times 8 = 720$

Formula : Prove that $nP_r = (n-1)P_r + r.(n-1)P_{r-1}$

Proof:

$$nP_r = \frac{\lfloor \underline{n}}{\lfloor \underline{n-r}}$$

$$\therefore (n-1)P_r = \frac{\lfloor \underline{n-1}}{\lfloor \underline{n-r-1}}$$

$$(n-1)P_{r-1} = \frac{\lfloor \underline{n-1}}{\lfloor \underline{n-r}}$$

Now $(n-1)P_r + r.(n-1)P_{r-1}$

$$= \frac{\lfloor \underline{n-1}}{\lfloor \underline{n-r-1}} + r.\frac{\lfloor \underline{n-1}}{\lfloor \underline{n-r}}$$

$$= \frac{\lfloor \underline{n-1}\,(n-r)}{\lfloor \underline{n-r-1}\,(n-r)} + \frac{r\,\lfloor \underline{n-1}}{\lfloor \underline{n-r}}$$

$$= \frac{\lfloor \underline{n-1}\,(n-r)}{\lfloor \underline{n-r}} + \frac{r\,\lfloor \underline{n-1}}{\lfloor \underline{n-r}}$$

$$= \frac{\lfloor \underline{(n-1)}\,[n-r+r]}{\lfloor \underline{n-r}}$$

$$= \frac{n\,\lfloor \underline{n-1}}{\lfloor \underline{n-r}} = \frac{\lfloor \underline{n}}{\lfloor \underline{n-r}} = nP_r$$

Aliter:

The result can also be obtained from the definition of nP_r

Proof:

The number of permutations of 'n' things taken 'r' at a time can be grouped as

(i) the number of arrangements in which a particular thing is absent.

(ii) the number of arrangements in which that particular thing is present.

To find the number of permutations in which a particular thing is absent, first reject that and then arrange 'r' things at a time out of the remaining $n-1$ things. This can be done in $n-1P_r$ ways.

To find the number of permutations in which a particular thing is present first select that and then arrange $(r-1)$ remaining things out of the remaining 'n' things. This can be done in $n-1P_{r-1}$ ways. Now the selected thing can be arranged in 'r' ways in every one of the permutations.

$\therefore$ The number of permutations in this case $= r.(n-1)P_{r-1}$

Combining these two results we have

$$nP_r = (n-1)P_r + r.(n-1)P_{r-1}$$

Note: By similar reasoning we can prove the following results

(i) $nP_r = n.(n-1)P_{r-1}$

(ii) $nP_r = (n-r+1)P_{r-1}$

(iii) $nP_r = {}^{n-2}P_r + 2r.{}^{n-2}P_{r-1} + r(r-1).{}^{n-2}P_{r-2}$

II. Permutations when some of the things are alike taken all at a time

Consider n letters of which the letter a occurs p times, the letter b occurs q times, the letter c occurs r times and other letters are distinct.

Assume that the total number of permutations in the above case is x.

Suppose in every permutation we replace p a's by $a_1, a_2, ...a_p$. Then every one of the permutations will result into $\lfloor p$ new permutations.

$\therefore x$ permutations with a's in p places will now be replaced by $x \lfloor p$ new permutations. Similarly if all the b's are replaced by $b_1, b_2 \ldots b_q$ then the number of distinct permutations will be $x \lfloor p \lfloor q$. Also if all the r's is are replaced by $c_1, c_2, \ldots c_r$ then the total number of distinct permutations will be $x \lfloor p \lfloor q \lfloor r$. Since all the like letters are replaced by distinct letters the number of arrangements of n things is $\lfloor n$.

$$\therefore \lfloor n = x \lfloor p \lfloor q \lfloor r$$

(ie) $$x = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}$$

III. Permutations when each thing may be repeated once, twice, ...upto r times in any arrangements.

The required number of permutations is the same as the number of ways of filling up r spaces in a row, with each blank place being filled up with any one of the n things.

The first blank place can be filled up in n different ways. Having filled up the first place the 2^{nd} place can be filled up in n ways. Now by fundamental therom the first two places can be filled up in $n \times n$ ways.

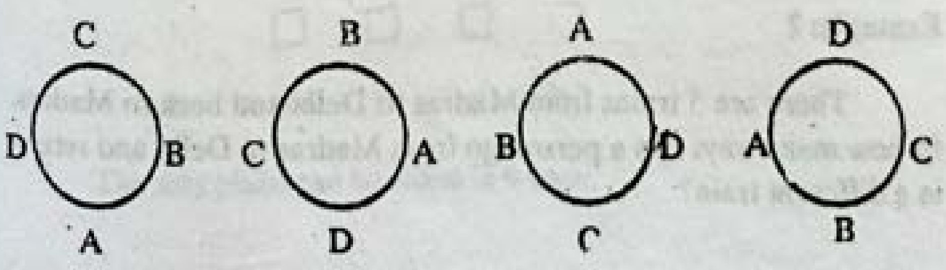
Proceeding like this all the r places can be filled in

$$n \times n \times n \times \ldots\ r \text{ times} = n^r \text{ ways.}$$

IV. Circular permutations

We have seen permutations of n things in a row. Now we consider the permutations of n things along a circle.

Consider four letters A, B, C, D arranged along a circle. The permutations ABCD, DABC, CDAB, BCDA represented along a circle are one and the same.



$\therefore$ These permutations which are distinct when represented along a line correspond to one permutation along a circle.

$\because$ Four things can be arranged along a line in $\lfloor 4$ ways. Every one permutation along a circle corresponds to 4 permutations along a line.

$\therefore$ The number of permutations of 4 things along a circle is

$$\frac{\lfloor 4}{4} = \lfloor 3$$

In general n distinct things can be arranged along a circle in

$\frac{\lfloor n}{n} = \lfloor n-1$ ways.

Example: 1

A committee of three is to be chosen out of 5 Englishmen, 4 French men and 3 Indians, the committee to contain one of each nationality. (1) In how many ways can this be done? (2) In how many arrangements will a particular Indian be included?

Solution

(i) One Englishman can be chosen from 5 Englishmen in 5 ways. One Frenchman can be chosen from 4 Frenchmen in 4 ways. One Indian can be chosen from 3 Indians in 3 ways.

$\therefore$ Total number of ways of forming the

committee $= 5 \times 4 \times 3$
$= 60$ ways.

(ii) If a specific Indian is to be included in the committee there is only one way of chosing an Indian.

$\therefore$ Total number of ways of forming the

committee $= 5 \times 4 \times 1 = 20$ ways.

Example: 2

There are 5 trains from Madras to Delhi and back to Madras. In how many ways can a person go from Madras to Delhi and return in a different train?

Solution

There are 5 ways of choosing a train from Madras to Delhi. There are 4 ways of choosing a train from Delhi to Madras since he cannot choose to return by the same train.

$\therefore$ Total number of ways of making the journey $= 5\times4$

$= 20$

Example: 3

There is a letter lock with 3 rings each marked with 5 letters and do not know the keyword. How many different useless attempts may be made to open the lock?

Solution:

For trying to open the lock the position in the first ring can be fixed in 5 ways. The position in the 2^{nd} ring can be fixed in 5 ways. The position in the 3^{rd} ring can be fixed in 5 ways.

$\therefore$ The total number of attempts to open the lock

$= 5\times5\times5 = 125$ ways.

Of these attempts only one will open the lock.

$\therefore$ The number of useless attempts $= 125 - 1 = 124$

Example: 4

How many numbers of 4 digits can be formed out of the digits 1, 2, 3.....9 if repetition of digits is (i) not allowed (ii) allowed?

Solution:

100	100	10	until place
□	□	□	□
6	7	8	9

The unit place can be filled in 9 ways.

The 10^{th} place can be filled in 8 ways since repetition is not allowed.

The 100^{th} place can be filled in 7 ways.

The 1000^{th} place can be filled in 6 ways.

$\therefore$ All the 4 places can be filled in $9\times8\times7\times6$ ways.

$\therefore$ The number of 4 digit numbers formed = 3024.

(ii) If repetition is allowed all the 4 places can be filled in $9\times9\times9\times9$ ways. = 6561 ways

Example: 5

How many numbers of 4 digits can be formed out of the digits 0, 1, 2, ... 9 if (i) repetition of digits is not allowed (ii) repetition of digits is allowed.

Solution:

1000	100	10	unit
□	□	□	□
9	9	8	7

(i) Since zero cannot occur in the 1000^{th} place, this place can be filled only in 9 ways (1, 2, ...9). Since zero can occur in other places and repetition is not allowed the 100^{th} place can be filled in 9 ways, 10^{th} place in 8 ways and unit place in 7 ways.

$\therefore$ Total number of 4 digit numbers formed $= 9\times9\times8\times7$

$= 4536$

(ii) When the repetition of digits is allowed the total number of 4 digit numbers formed $= 9\times10\times10\times10 = 9000$

Example: 6

How many odd numbers of 4 digits can be formed out of the digits 1, 2.......... 9 if repetition of digits is (i) not allowed (ii) allowed ?

Solution:

1000	100	10	unit
☐	☐	☐	☐
6	7	8	5

(i) For a number to be odd the unit place should be filled up by any one of the numbers 1, 3, 5, 7, 9.

There are 5 ways of filling the unit place.

The other places can be filled in 8, 7, 6 ways.

∴ Total number of 4 digit odd numbers

$$= 5 \times 8 \times 7 \times 6 = 1680$$

(ii) If repetition is allowed total number of 4 digit odd numbers

$$= 5 \times 9 \times 9 \times 9 = 3645$$

Example: 7

How many odd numbers of 4 digits can be formed out of the digits 0, 1, 2..........9 if repetition of digits is (i) not allowed (ii) allowed?

Solution:

1000	100	10	unit
☐	☐	☐	☐
8	8	7	5

(i) The unit place can be filled in 5 ways. Since repetition of digits is not allowed and zero cannot occur in the 1000^{th} place, the 1000^{th} place can be filled in only 8 ways. The other two places can be filled in 8 and 7 ways.

∴ Total number of 4 digit odd numbers $= 8 \times 8 \times 7 \times 5 = 2240$

(ii) If repetition is allowed the total number of 4 digit numbers = $5 \times 9 \times 10 \times 10 = 4500$

Example: 8

How many even numbers of 4 digits can be formed out of the digits 1, 2, 3..........9 if the repetition of digits is (i) not allowed (ii) allowed ?

Solution:

1000	100	10	unit
□	□	□	□
6	7	8	4

(i) **Repetition not allowed**

For a number to be even the unit place can be filled by one of the numbers 2, 4 6, 8. There are 4 ways of filling the unit place.

The other places can be filled in 8, 7, 6 ways.

$\therefore$ Total number of 4 digit even numbers = $4 \times 8 \times 7 \times 6$

= 1344

(ii) If repetition is allowed total number of 4 digit even numbers

= $4 \times 9 \times 9 \times 9$ = 2916

Example: 9

How many even numbers of 4 digits can be formed out of the digits 0, 1, 2, ...9 if repetition of digits is (i) not allowed (ii) allowed?

Solution:

(i) For a number to be even the unit place should be filled up by one of the digits 0, 2, 4, 6, 8 . Here we have to consider 2 cases. (i) Number of even numbers of 4 digits ending with zero. (2) number of even numbers of 4 digits not ending with zero. When the unit place is filled up with zero, the other places can be filled up in 9, 8, 7 ways respectingly.

$\therefore$ Number of 4 digit even numbers ending with zero

$$= 9 \times 8 \times 7 \times 1$$

$$= 504$$

$$\square \quad \square \quad \square \quad \square$$
$$8 \quad 8 \quad 7 \quad 4$$

When unit place is filled up by a non zero digit there are 4 ways for filling it. In this case we should avoid zero occuring in 1000^{th} place.

$\therefore$ This place can be filled in 8 ways. As zero can occur in the other two places, they can be filled in 8 and 7 ways.

$\therefore$ Number of 4 digit even numbers in this case $= 8 \times 8 \times 7 \times 4 = 1792$

$\therefore$ Total of 4 digit even numbers $= 504 + 1792 = 2296$

(ii) When repetition is allowed number of 4 digit even numbers $= 9 \times 10 \times 10 \times 5 = 4500$

Example: 10

The figures 4, 5, 6, 7, 8 are written in every possible order. How many of the numbers so formed will be greater than 56000?

Solution:

Case	1000	1000	100	10	unit	Number of ways
	$\square$	$\square$	$\square$	$\square$	$\square$	
1. When 5 occurs in 10000^{th} place	1	3	3	2	1	$1 \times 3 \times 3 \times 2 \times 1 = 18$
2. When 6, 7, 8 occur on 10000^{th} place	3	4	3	2	1	$3 \times 4 \times 3 \times 2 \times 1 = 72$
					Total	$= 90$

$\therefore$ Total number of numbers formed greater than 56000 = 90.

Example: 11

Find the number of significant numbers which can be formed out of the digits 0, 5, 6, 7, 8 but using each digit only once in each number.

Solution:

The significant numbers that can be formed can be of single digit, 2 digits, 3 digits, 4 digits and 5 digits. Any number other than zero is a significant number.

∴ significant number of single digit $= 4$

"	"	"	2 digits	$= 4 \times 4$	$= 16$
"	"	"	3 digits	$= 4 \times 4 \times 3$	$= 48$
"	"	"	4 digits	$= 4 \times 4 \times 3 \times 2$	$= 96$
"	"	"	5 digits	$= 4 \times 4 \times 3 \times 2 \times 1$	$= 96$
				Total	260

∴ Total number of significant numbers $= 260$

Example: 12

Find the number of ways in which 10 examination papers be arranged so that two particular papers may not come together.

Solution:

The 10 papers are classified into two groups (i) containing two particular papers (ii) containing other papers.

$$\times P \times P \times P \times P \times P \times P \times P \times P \times$$

First arrange the 8 other papers in a line. They can be arranged in $\lfloor 8$ ways. Now the two particular papers can be arranged in the places marked '×' in the diagram. There are 9 such places. The two particular papers can be arranged in these 9 places in $9P_2$ ways.

$\therefore$ Total number of arrangements of 10 papers so that the 2 particular papers do not come together $= \lfloor 8 \times 9P_2$

$$= \lfloor 8 \times 9 \times 8 = 8 \lfloor 9$$

Example: 13

Show that the number of ways in which 'n' books can be arranged on a shelf so that two particular books are not together is $(n-2) \lfloor n-1$

$$\times B_1 \times B_2 \times B_3 \text{..........} \times B_n - 2 \times$$

First arrange the $(n-2)$ other books in a line. They can be arranged among themselves is $\lfloor n-2$ ways. Now the two particular books can be arranged in the plalces marked $\times$. There are $(n-1)$ such places. The 2 books can be arranged in these $(n-1)$ places in $(n-1)P_2$ ways.

$\therefore$ Total number of arrangements

$$= \lfloor n-2 (n-1)P_2$$

$$= \lfloor n-2 \ (n-1)(n-2)$$

$$= (n-2) \lfloor n-1$$

Example: 14

Find the number of arrangements of 6 boys and 4 girls in a line so that no two girls sit together.

Solution:

$$\times B \times B \times B \times B \times B \times B \times$$

First arrange the 6 boys in a line. They can be arranged among themselves in $\lfloor 6$ ways.

Now the 4 girls can be arranged in the places marked $\times$ in the diagram. These are 7 such places. The 4 girls can be arranged in these 7 places in ${}^7P_4$ ways.

$\therefore$ Total number of arrangements $= \lfloor 6 \times {}^7P_4$

$$= 720 \times 7 \times 6 \times 5 \times 4$$

$$= 720 \times 840 = 604800$$

Example: 15

Find the number of arrangements of 5 boys and 5 girls in a row so that no two boys and no two girls sit together.

Solution:

Suppose the arrangements start with a boy; then the arrangements are of the form BG BG BG BG BG. The 5 boys can be arranged in the odd places in $\lfloor 5$ ways and the 5 girls can be arranged in the even places in $\lfloor 5$ ways.

$\therefore$ Total number of arrangements $= \lfloor 5 \times \lfloor 5$

If the arrangements start with a girl then the total number of arrangements $= \lfloor 5 \times \lfloor 5$

$\therefore$ Total number of arrangements

$$= \lfloor 5 \times \lfloor 5 + \lfloor 5 \times \lfloor 5 = 120 \times 120 + 120 \times 120$$

$$= 28800$$

Example: 16

Find the number of arrangements in which 6 boys and 4 girls can be arranged in a line so that all the girls sit together and all the boys sit together.

Solution: Consider the 6 boys as one unit and 4 girls as another unit

$$\boxed{B\,B\,B\,B\,B\,B} \quad \boxed{G\,G\,G\,G}$$

The 2 units can be arranged in $\lfloor 2$ ways. In each of these arrangements the 6 boys can be arranged among themselves in $\lfloor 6$ ways and 4 girls can be arranged among themselves in $\lfloor 4$ ways.

$\therefore$ The total number of arrangements $= 2 \times \lfloor 6 \times \lfloor 4$

$= 2 \times 720 \times 24 = 34566$

Example: 17

A family of 4 brothers and 3 sisters are to be arranged for a photograph in one now. In how many ways can they be seated if all the sisters sit together?

Solution:

Consider the 3 sisters as one unit and 4 brothers as 4 different units. There are now 5 units. They can be arranged among themselves in $\lfloor 5$ ways. In each of these arrangements the 3 sisters can be arranged in $\lfloor 3$ ways.

$\therefore$ Total number of arrangements $= \lfloor 5 \lfloor 3 = 720$

Example: 18

There are 6 books on Economics, 3 on Mathematics and 2 on Accountancy. In how many ways can they be arranged on a shelf if the books of the same subject are always to be together?

Solution:

Consider the 6 Economics books as one unit, 3 Mathematics books as one unit and 2 Accountancy books as one unit. These 3 units can be arranged in $\lfloor 3$ ways. In each of these arrangements, the 6 Economics books can be arranged in $\lfloor 6$ ways, 3 Mathematics books in $\lfloor 3$ ways and 2 Accountancy books in $\lfloor 2$ ways.

$\therefore$ Total number of arrangements $= \lfloor 3 \times \lfloor 6 \times \lfloor 3 \times \lfloor 2$

$= 51840$

Example: 19

There are 6 students of whom 2 are Indians, 2 are Americans and 2 are Russians. In how many ways can they stand on a line if students of same nationality are always to be together?

Solution:

Consider the 2 Indians as one unit, the 2 Americans as one unit and the 2 Russians as one unit. These 3 units can be arranged among themselves in $\lfloor 3$ ways. In each of these arrangements the Indians can be arranged among themselves in $\lfloor 2$ ways, the Americans in $\lfloor 2$ ways and the Russians in $\lfloor 2$ ways.

$\therefore$ Total number of arrangements $= \lfloor 3 \times \lfloor 2 \times \lfloor 2 \times \lfloor 2 = 48$

Example: 20

There are 4 bus lines between A and B and 3 bus lines between B and C (i) In how many ways can a man travel by bus from A to C by way of B? (ii) In how many ways can a man travel round trip by bus from A to C by way of B if he does not want to use a bus line more than once?

Solution:

(i) A man can travel from A to B in 4 ways and he can travel from B to C in 3 ways.

$\therefore$ Total number of ways of travelling from A to C through B is $4 \times 3 = 12$ ways.

(ii) The number of ways of moving from A to B and then from B to $C = 4 \times 3 = 12$

While returning the number of ways of moving from C to B is 2 and B to A is 3 since he does not want to use a line more than once.

$\therefore$ Number of ways of returning from C to A $= 2 \times 3 = 6$

$\therefore$ Total number of ways of making round trip $= 12 \times 6 = 72$

Example: 21

In how many ways can the letters of the word MOBILE be arranged so that the consonants always occupy the odd places.

Solution:

There are 6 places in the word MOBILE. There are 3 consonants and 3 vowels. The 3 consonants can be arranged in the 3 odd places in $3P_3$ ways.

$\therefore$ The 3 vowels can be arranged in the 3 even places in $3P_3$ ways.

$\therefore$ Total number of arrangements $= 3P_3 \times 3P_3$

$= 6 \times 6 = 36$ ways

Example: 22

In how many ways can the letters of the word STRANGE be arranged so that the vowels may appear in the odd places.

Solution :

There are 7 places in the word STRANGE. There are 4 odd places and 3 even places.

There are 2 vowels and 5 consonants. The 2 vowels can be arranged in the 4 odd places in $4P_2$ ways. The 5 consonants can be arranged in the other 5 places in $5P_5$ ways.

$\therefore$ Total number of arrangements $= 4P_2 \times 5P_5$

$= 4 \times 3 \times \lfloor 5 = 1440$

Example: 23

In how many ways can the letters of the word NAGERKOIL be arranged? How many of them begin with NA? In how many of them the 4 vowels come together? How many of them begin with the 4 vowels.

Solution:

There are 9 letters in the word NAGERKOIL.

(i) They can be arranged in $\lfloor 9$ ways.

(ii) Fixing the first two positions for N and A, the rest can be arranged the $\lfloor 7$ ways.

(iii) Consider the 4 vowels as one unit and the other 5 consonants as 5 different units. There are 6 different units. They can be arranged among themselves in $6P_6$ ways. In each of these arrangements the 4 vowels can be arranged in $4P_4$ ways.

$\therefore$ Total number of arrangements $= 6P_6 \times 4P_4$

$= \lfloor 6 \times \lfloor 4$

$=$ 17280 ways.

(iv) The first 4 places can be arranged by the 4 vowels in $4P_4$ ways. The rest 5 places can be filled in by the 5 consonanats in $5P_5$ ways.

$\therefore$ Total number of arrangements $= 4P_4 \times 5P_5$

$= \lfloor 4 \times \lfloor 5 = 2880$

Example: 24

Find the sum of all the numbers that can be formed with the digits 1, 2, 3, 4, 5 taken 4 at a time.

Solution:

Number of 4 digit numbers that can be formed out of the digits 1, 2, 3, 4, 5 is $5P_4 = 120$

Out of the 120 numbers, 1 appears in the unit place 24 times, 2 appears 24 times and so on.

$\therefore$ If we write all the 120 numbers and add, the sum of all the numbers in the unit place $= 24\,(1+2+3+4+5) = 360$

Similarly the sum of all the numbers in the 10^{th} place $= 24(1+2+3+4+5) = 360$

Sum of all the numbers in the 100^{th} place is 360.

Sum of all the numbers in the 1000^{th} place $= 360$

$\therefore$ Total of all the numbers =

$$360 \times 1000 + 360 \times 100 + 360 \times 10 + 360 = 399960$$

Example: 25

The letters of the word NATURE are permuted and the words so formed are arranged as in a dictionary. Find the rank of the word NATURE.

Solution:

The alphabetical order of the letters are A E N R T U

In this dictionary,

Number of words beginning with A is	$\lfloor 5$	=	120
" " " "	E is $\lfloor 5$	=	120
" " " "	NAE is $\lfloor 3$	=	6
" " " "	NAR is $\lfloor 3$	=	6
" " " "	NATE is $\lfloor 2$	=	2
Number of words beginning with	NATR is $\lfloor 2$	=	2
" " " "	NATUE is $\lfloor 1$	=	2
" " " "	NATURE is	=	1
			259

$\therefore$ The rank of the word NATURE is 259

Example: 26

Five persons are to address a meeting. If a specified person A is to speak before another specified person B, find the number of ways in which this could be arranged. In how many of these arrangements will B comes immediately after A?

Solution:

(i)

Cases		1 □	2 □	3 □	4 □	5 □	Number of ways
1.	Suppose A speaks first	A	4	3	2	1	24
2.	Suppose A speaks second	3	A	3	2	1	18
3.	Suppose A speaks third	3	2	A	2	1	12
4.	Suppose A speaks fourth	3	2	1	A	B	6
	A cannot be the 5^{th} speaker.		Total number of ways			=	60

(ii) Consider AB as one unit and B, C, D as 3 other units. These 4 units can be arranged in $\lfloor 4$ ways.

$\therefore$ Total number of arrangements in which B speaks immediately after A = $\lfloor 4 = 24$ ways.

Example: 27

Find the total number of numbers greater than 2000 that can be formed with the digits 1, 2, 3, 4, 5 no digit being repeated in any number.

Solution:

With the digits given we can form 5 digit numbers, all of which are all greater than 2000.

Number of 5 digit numbers that can be formed

$$= 5P_5 = \lfloor 5 = 120.$$

Now we have to find the number of 4 digit numbers which are greater than 2000.

1000	100	10	unit	
□	□	□	□	Number of ways = 96
4	4	3	2	

For a number greater than 2000, 1000^{th} place can be filled up only by the digits 2, 3, 4 5. There are 4 ways for filling up the 1000^{th} place. The other places can be filled in 4, 3, 2 ways.

$\therefore$ Number of 4 digit numbers greater than 2000

$$= 4 \times 4 \times 3 \times 2 = 96$$

$\therefore$ Number of numbers greater than 2000 $= 120 + 96$

$$= 216$$

Example: 28

If 6 times the number of permutation of 'n' things taken together is equal to seven times the number of permutations $(n - 1)$ things chosen 3 at a time find n.

Solution:

Given that $6 \, {}^{n}P_3 = 7 \, . \, {}^{(n-1)}P_3$

$6n(n-1)(n-2) = 7(n-1)(n-2)(n-3)$

$6n = 7(n-3)$ since $n - 1 \neq 0, n - 2 \neq 0$

$6n = 7n - 21$ or $n = 21$

Example: 29

If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$

find the value of r

Solution:

$$\frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = \frac{30800}{1}$$

$$\frac{\lfloor 56}{\lfloor 50 - r} \times \frac{\lfloor 51 - r}{\lfloor 54} = \frac{30800}{1}$$

$$\frac{\lfloor 56}{\lfloor 54} \cdot \frac{\lfloor 51-r}{\lfloor 50-r} = 30800$$

$$56.55 \,.\, (51-r) = 30800$$

$$51-r = \frac{30800}{56 \times 55}$$

$$= 10$$

$$\therefore \quad = 41$$

Example: 30

Find the number of permutations of the letters of the word ASSASSINATION.

Solution:

Number	of	A's	=	3
"	"	S's	=	4
"	"	N's	=	2
"	"	I's	=	2
"	"	T's	=	1
"	"	O's	=	1
		Total	=	13

$$\therefore \text{Number of arrangements} = \frac{\lfloor 3}{\lfloor 4 \lfloor 3 \lfloor 2 \lfloor 2}$$

Example: 31

How many different permutations are there if all the letters of the word ALABAMA? Of these permutations how many contain the word LAMB?

Solution:

There are 7 letters = A, A, A, A, L, E, M

$$\therefore \text{Number of arrangements} = \frac{\lfloor 7}{\lfloor 4} = 210$$

Consider LAMB as one unit

Now the units are LAMB, A, A, A.

$\therefore$ Total number of arrangements $= \frac{\lfloor 4}{\lfloor 3} = 4$

Example: 32

Suppose a licence plate contains three distinct English alphabets followed by four digits with the Ist digit not zero. How many different licence plates can be pinted?

Solution:

Leters				Digits		
☐	☐	☐	☐	☐	☐	☐
26	25	24	9	10	10	10

The 3 places for the alphabets can be filled in $26 \times 25 \times 24$ ways.

The 4 places for the digits can be filled in $9 \times 10 \times 10 \times 10$ ways.

$\therefore$ Total number of different licence plates that can be formed.

$$= 26 \times 25 \times 24 \times 9 \times 10 \times 10 \times 10$$

$$= 14\,0400{,}000$$

Example: 33

In how many ways can 5 gentlemen and 3 ladies can be arranged along a round table so that no two ladies are together?

Solution:

5 gentlemen can be arranged along a circle in $\lfloor 4$ ways. There are 5 places for the ladies and the three ladies can be arranged in these places in $5P_3$ wages.

$\therefore$ Total number of arrangements $= \lfloor 4 \times 5P_3 = 24 \times 60$

$= 1440$

Example: 34

Find the number of permutations of the letters of thw word MISSISSIPPI

Solution:

Number of letters in the word MISSISSIPPI

M	-	1
I	-	4
S	-	4
P	-	2
Total	-	11

$\therefore$ Number of arrangements $= \dfrac{\lfloor 11}{\lfloor 4 \lfloor 4 \lfloor 2}$

Example: 35

In how many different ways can the letters of the word 'POSSESSIVE' be arranged? (i) In how many of these will the S's come together? (ii) In how many of these will the relative positions of vowels and consonents remain unchanged?

Solution

Number of letters in the word POSSESSIVE:

P	-	1
O	-	1
S	-	4
E	-	2
I	-	1
V	-	1
Total	-	10

$\therefore$ Total number of arrangements of the letters $= \dfrac{\lfloor 10}{\lfloor 4 \lfloor 2}$

$$= 75600$$

(i) Consider 4 S's as one unit and other 6 letters as six units. These 7 units can be arranged in $\dfrac{\lfloor 7}{\lfloor 4 \lfloor 2}$

$$= 2520 \text{ ways}$$

(ii) Vowels have to appear in the 2^{nd}, 5^{th}, 8^{th} and 10^{th} positions.

The 4 vowels can be arranged in these positions in $\dfrac{\lfloor 4}{\lfloor 2} = 12$ ways.

The consonants can be arranged in the other positions in $\dfrac{\lfloor 6}{\lfloor 4}$

$$= 30 \text{ ways}$$

$\therefore$ Total number of arrangements $= 12 \times 30 = 360$ ways

Exercises;

1. There are 4 picuture nails on a wall and 8 pictures to choose form. In how many ways can a picture be hung on all the nails?
2. There are 3 scholarships of unequal value. In how many ways can they be distributed among 10 students giving not more than one to a student?
3. How many numbers of 3 digits can be formed out of the digits 1, 2, 3.....................9 if repetition of digits is not allowed ?
4. In how many ways can 3 students be associated with 4 Chartered accountants, assuming that each Chartered accountant can take one student?
5. Show that there are 300 numbers greater than 1000 and less than 10000 which can be formed with the six digits 0, 1, 2, 3, 4, 5 each digit appearing not more than once in each number. Show that 156 of these are even and 108 are divisible by 5.

6. Find the sum of all numbers that can be formed with the digits 1, 2, 3, 4, 5 taken 4 at a time.

7. The figures 4, 5, 6, 7, 8 are written in every possible order. How many numbers so formed is greater than 56000?

8. In how many ways can 7 boys and 4 girls can be arranged in a row so that (i) all the girls sit together (ii) no two girls sit together?

9. In how many ways can 4 Mathematics, 5 Physics and 6 Chemistry books be arranged in a row so that books of the same subject are to be together?

10. How many arrangements of the letters of the word STRANGE can be done if the vowels have to occupy only odd places?

11. The letters of the word (i) FATHER (ii) MOTHER (iii) ZENITH are permuted and the words so formed are arranged as in a dictionary. Find the rank of the word (i) FATHER (ii) MOTHER (iii) ZENITH.

12. How many four-digit numbers can be formed out of the digits 1, 2, 3..............9 if no digit is repeated in any number? How many of these will be greater than 3000?

13. Find the number of arrangements of the letters of the word FAILURE if the consonants many occupy only odd places.

14. In how many ways the letters of the word PETROL be arranged? How many of these do begin with *P*? How many begin with P but not end in *L*? Also find the number of words which can be formed if *O* and *L* are never together.

15. How many mumbers greater than 7000 can be formed with the digits 3, 5, 7, 8, 9 no digit being repeated?

16. How many numbers of six digits can be formed from the digits: 1, 2, 3, 4, 5 6 (no digit being repeated)? How many of these are not divisable by 5?

17. Four men and four women are to be seated for a dinner such that no two women sit together. Find the number of ways in which this can be arranged.

18. How many numbers each lying between 100 and 1000 can be formed with the digits 2, 0, 3, 4, 5 (no digit being repeated). How many of these are odd?

19. How many natural number not exceeding 4321 can be formed with the digits 1, 2, 3, 4? (Remember that repetitions are allowed)

20. Find the number of words that can be formed by considering all possible permutation of the letters of the word FATHER. How many of these words begin with A and end with R?

21. Six papers are set in an examination of which two are in Mathematics. In how many different orders can the papers be arranged so that (a) the two Mathematics papers are together and (b) the two Mathematics papers are not consecutive

22. How many arrangements of the letters of the word COMRADE can be made (i) if the vowels are never separated (ii) if the vowels are to occupy only odd places (iii) the relative positions of vowels and consonants are not changed.

23. How many arrangements can be formed with the letters of the word DELHI? How many of there will begin with D and how many do not? In how many words LH will be together?

24. The letters of the word ZENITHare written in all possible order. How many words are possible? If these words are written out as in a dictionary what is the rank of the word ZENITH

25. How many numbers less than 1000 and divisible by 5 can be formed using the digits 0, 1, 2, ... 9 such that each digit does not occur more than once in each number?

26. Find the number of permutations of the word CONTAMINATION.

27. In how many ways can the letters of the word CZECHOSLOVAKIA be arranged?

28. How many different words can be formed with the letters of the word BHARAT? In how many of these Band H are never together? How many of these begin with *B* and end with *T*?

29. In how many different ways can the letters of the word 'CONSTITUTION' be arranged? How many of these will have the letter '*N*' both at the beginning and at the end?

30. In how many ways can the leters of the word ARRANGE be arranged? How many of these arrangements are there in which (i) the two R's come together (ii) the two R's do not come together? (iii) the two R's and the two A's come together?

31. How many different words can be formed with the letters of the word 'CAPTAIN'? In how many of these C and T are never together?

32. There are 3 copies each of 4 different books. In how many ways can they be arranged on a shelf.

33. Find the number of arrangements that can be made out of the letters of the word. 'MISSISSIPPI'.

34. In how many ways can 7 people be arranged at a round table so that two particular persons may be together?

35. In how many ways can the letters of the word 'MATHEMATICS' be arranged so that the vowels come together?

36. If the letters of the word "WOMAN" lie permuted and the word so formed be arranged as in a dictionary, what will be the rank of word "WOMAN"?

37. In how many ways can 8 examinations papers be arranged in a row so that the best and the worst papers are never together ?

38. How many words can be formed from the letters of the word FATEHPUR when :-

(i) the three letter PUR occur together ?
(ii) vowels occur at even places ?

Combinations

In permutations of n things taken r at a time we have considered the number of different arrangements. Here we pay due regard to the order in which the different things occur. On the other hand if we do not give importance to the order but only consider the selections of the r things out of n things we call it combination.

The number of combinations of n things taken r of a time is denoted by nC_r

Consider the example of selecting 3 letters out of 4 letters. A, B, C, D.

The possible selections are ABC, ABD, ACD, BCD. There are 4 selections. $\therefore 4C_3 = 4$.

In each of the above selections we do not give importance to the order of selection. This means ABC, ACB, BCA, BAC, CAB, CBA all correspond to the same selection.

We note that for each selection there corresponds 6 different permutations.

$\therefore$ The total number of arrangements of all the selections = $4 \times 6 = 24$

We use this idea to derive the combinatorial formula for nC_r

I. Formula for nC_r

Let $a_1, a_2, a_3 \ldots\ldots\ldots a_n$ be n things. We have to select r things at a time out of these n things. The number of selections of r things at a time is denoted by nC_r.

In each of these selections the r things can be arranged among themselves in $\lfloor r$ ways.

$\therefore$ The number of arrangements of nC_r selections

$$= \lfloor r \, nC_r$$

This gives the number of arrangements of n things taken r at a time.

$$\therefore nP_r = \lfloor r . nC_r$$

$$nC_r = \frac{nP_r}{\lfloor r} \quad \text{.... (1)}$$

$$nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} \quad \text{.... (2)}$$

Note:- (i) $nC_o = \dfrac{\lfloor n}{\lfloor o \lfloor n} = 1$

(ii) $nC_n = \dfrac{\lfloor n}{\lfloor n \lfloor o} = 1$

For example $7C_2 = \dfrac{7P_2}{\lfloor 2} = \dfrac{7\times6}{1\times2} = 21$

$$10C_3 = \frac{10P_3}{\lfloor 3} = \frac{10\times9\times8}{1\times2\times3} = 120$$

Properties :

(i) $nC_r = nC_{n-r}$

Proof:

$$\text{LHS} = nC_r$$

$$= \frac{\lfloor n}{\lfloor r \lfloor n-r} \quad \text{... (1)}$$

$$\text{RHS} = nC_{n-r}$$

$$= \frac{\lfloor n}{\lfloor n-r \lfloor n-(n-r)} = \frac{\lfloor n}{\lfloor n-r \lfloor r} \quad \text{...... (2)}$$

$\therefore$ From (1) and (2) $\quad nC_r = nC_{n-r}$.

(ii) $\dfrac{nC_r}{nC_{r-1}} = \dfrac{n-r+1}{r}$

Proof:
$$nC_r = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}$$

$$nC_{r-1} = \frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r+1}}$$

$$\frac{nC_r}{nC_{r-1}} = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}\cdot\frac{\lfloor\underline{r-1}\,\lfloor\underline{n-r+1}}{\lfloor\underline{n}}$$

$$= \frac{\lfloor\underline{r-1}}{\lfloor\underline{r}}\cdot\frac{\lfloor\underline{n-r+1}}{\lfloor\underline{n-r}}$$

$$= \frac{\lfloor\underline{r-1}\,(n-r+1)\,\lfloor\underline{n-r}}{r\,\lfloor\underline{r-1}\,\lfloor\underline{n-r}}$$

$$= \frac{n-r+1}{r}$$

(iii) $nC_r + nC_{r-1} = (n+1)C_r$

Proof:
$$nC_r = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}$$

$$nC_{r-1} = \frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r+1}}$$

$$\therefore\ nC_r + nC_{r-1} = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}} + \frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r+1}}$$

$$= \frac{\lfloor\underline{n}\,(n-r+1) + \lfloor\underline{n}\,r}{\lfloor\underline{r}\,\lfloor\underline{n-r+1}}$$

$$= \frac{\lfloor\underline{n}\,[n-r+1+r]}{\lfloor\underline{r}\,\lfloor\underline{n-r+1}}$$

$$= \frac{(n+1)\,\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r+1}} = \frac{\lfloor\underline{n+1}}{\lfloor\underline{r}\,\lfloor\underline{n-r+1}} = (n+1)C_r$$

(iv) If $nC_x = nC_y$

then $x = y$ or $x = n - y$ since $nC_y = nC_{n-y}$

$\therefore x = y$ or $x + y = n$

(v) $nC_r = (n-2)C_{r-2} + 2.(n-2)C_{r-1} + (n-2)C_r$

Proof:

RHS $= (n-2)C_{r-2} + (n-2)C_{r-1} + (n-2)C_{r-1} + (n-2)C_r$

$= (n-1)C_{r-1} + (n-1)C_r$

$= nC_r$

$=$ LHS

(vi) Prove by reasoning that $(n+1)C_r = nC_r + nC_{r-1}$

Proof:

Consider $(n+1)$ things. The selections of r things out of these $n+1$ things can be grouped into

(i) the selections in which a particular thing is present.

(ii) the selections in which that particular thing is absent.

To find the number of selections in which a particular thing is present, first select that and then $(r-1)$ things from the remaining n things. This can be done in nC_{r-1} ways.

To find the number selections in which a particular thing is absent, reject that and then select r things out of the remaining n things. This can be done in nC_r ways.

$\therefore$ Combining these two $(n+1)C_r = nC_{r-1} + nC_r$

II. Combinations of n different things taken some or all at a time

The total number of ways of selections taking one or more out of n things is $2^n - 1$

Proof:

There are n things. Each thing may be dealt in two ways (selecting or rejecting it) $\therefore$ From the fundamental theorem the number of selections $= 2\times2\times2\times$..........n times

$= 2^n$

This also includes the case of rejecting all the n things. Excluding this, the number of ways of selecting one or more things out of n things $= 2^n - 1$

III. Combination of n things taken one or more at a time when n_1 of them belong to one kind, n_2 of them belong to a 2^{nd} kind n_3 of them belong to 3^{rd} kind and so on is $(n_1 + 1)(n_2 + 1)(n_3 + 1)$.....$-1$

Proof:

The n_1 things of one kind can be dealt in $(n_1 + 1)$ ways namely rejecting all, selecting one, selecting 2 and so on.

Similarly n_2 things of a second kind can be dealt in $(n_2 + 1)$ ways, n_3 things of 3^{rd} kind can be dealt in $(n_3 + 1)$ ways and so on.

$\therefore$ Total number of selections $= (n_1 + 1)(n_2 + 1)(n_3 + 1)$... This also includes the case of rejecting all.

$\therefore$ The number of ways of selecting one or more is $(n_1 + 1)(n_2 + 1)(n_3 + 1)$......$-1$

Example: 1

If $16C_r = 16C_{r+2}$ find rC_3

Solution:

$$16C_r = 16C_{r+2}$$

$$\therefore r = r+2 \text{ or } r = 16-(r+2)$$

$$0 = 2 \text{ which is not possible; } 2r = 14$$

$$\therefore r = 7$$

$$rC_3 = 7C_3 = \frac{7\times6\times5}{3\times2\times1} = 35$$

Example: 2 If $4 \,.\, nC_2 = (n+2)\,C_3$ find n

Solution: $4\,.\,nC_2 = (n+2)\,C_3$

$$4\frac{n(n-1)}{1.2} = \frac{(n+2)(n+1)n}{1.2.3}$$

$$12(n-1) = (n+2)(n+1)$$

$$12n-12 = n^2+3n+2$$

$$n^2-9n+14 = 0$$

$$(n-7)(n-2) = 0$$

$$\therefore n = 2, 7$$

Example: 3

If $(n+2)\,C_n = 45$, find n

Solution: $(n+2)\,C_n = 45$

$$(n+2)\,C_{n+2-n} = 45$$

$$(n+2)\,C_2 = 45$$

i.e $$\frac{(n+2)(n+1)}{2} = 45$$

$$n^2 + 3n + 2 = 90$$

$$n^2 + 3n - 88 = 0$$

$$(n + 11)(n - 8) = 0$$

$$\text{or } n = -11 \text{ or } 8$$

$$n = -11 \text{ is inadmisable } \therefore n = 8$$

Example: 4

If $28C_{2r} : 24C_{2r-4} = 225 : 11$ find the value of 'r'.

Solution:

$$\frac{28\,C_{2r}}{24\,C_{2r-4}} = \frac{225}{11}$$

$$\frac{\frac{28}{2r}\cdot\frac{27}{2r-1}\cdot\frac{26}{2r-2}\cdot\frac{25}{2r-3}\cdot 24\,C_{2r-4}}{24C_{2r-4}} = \frac{225}{11}$$

$$\therefore \quad 2r(2r-1)(2r-2)(2r-3) = \frac{28.27.26.25}{225}$$

$$= 28.26.11.3$$

$$= 14.13.12.11$$

$r = 7$ satisfies the equation

$\therefore r = 7$

Example: 5

If $n\,C_{10} = n\,C_6$ find $n\,C_{11}$

Solution : $nC_{10} = nC_6$

$$\therefore \quad 10 = 6 \quad \text{or} \quad 10 = n - 6$$

$$10 = 6 \text{ is inadmissable}$$

$$\therefore \quad 10 = n-6$$

$$\text{or } n = 16$$

$$\therefore nC_{11} = 16\,C_{11} = 16\,C_5 = \frac{16\times15\times14\times13\times12}{5\times4\times3\times2\times1}$$

$$= 16\times13\times21$$

$$= 4368$$

Example: 6

A company has seven Chartered Accountants, six Engineers and 3 Scientists in their managerial cadre. In how many ways can they form a committee if the committee must contain 2 members from each discipline?

Solution:

There are 7CA's, 6 Engineers and 3 Scientists. We have to select 2 from each group.

2 CA's can be selected from 7 CA's in $7\,C_2$ ways.

2 Engineers can be selected from 6 Engineers in $6\,C_2$ ways.

2 Scientists can be selected from 3 Scientists in $3\,C_2$ ways.

$\therefore$ Total number of selections $= 7\,C_2 \times 6\,C_2 \times 3\,C_2$

$= 21\times15\times3 = 945$ ways.

Example: 7

Mr. X must pass on 40 income-tax returns per day; but he has time only to audit 5 of them with care. In how many different ways can be select the five to be audited?

Solution:

Mr. X has to choose 5 returns out of 40. This can be done in $40\,C_5$ ways.

$$40\,C_5 = \frac{40\times39\times38\times37\times36}{7\times2\times3\times4\times5}$$

$$= 65008$$

Example: 8

There are six vacancies in an office. If 8 men and 5 women offer themselves, in how many ways can the posts be filled, if the conditions are such that the vacancies should go half to men and half to women?

Solution:

There are 8 men and 5 womep. There are 3 vacancies for men and 3 vacancies for women. 3 men can be selected from 8 men in $8\,C_3$ ways. 3 women can be selected from 5 women in $5\,C_3$ ways.

$\therefore$ Total number of selctions $= 8\,C_3 \times 5\,C_3$

$= 560$

Example: 9

A cricket team of 11 players is to be selected from two sets consisting of 6 and 8 players respectively. In how many ways can the selection be made on the supposition that the first set of 6 players shall contribute not fewer than 4?

Solution:

Cases	Set A 6	Set B 8	Number of ways of selection
1.	4	7	$6C_4 \times 8C_7 = 120$
2.	5	6	$6C_5 \times 8C_6 = 164$
3.	6	5	$6C_6 \times 8C_5 = 56$
			Total 340

Example: 10

In an examination paper, there are 7 questions in part A out of which any 4 are to be attempted and there are 6 questions in part B out of which 3 are to be attempted. In how many different ways can a candidate answer part A and part B in full?

Solution:

	Part A	Part B
No. of questions	7	6
No. to be attempted	4	3

Number of ways of choosing part A questions $= 7C_4$

Number of ways of choosing part B questions $= 6C_3$

$$\therefore \text{Total number of ways} = 7C_4 \times 6C_3$$

$$= 7C_3 \times 6C_3$$

$$= \frac{7\times6\times5}{3\times2\times1} \times \frac{6\times5\times4}{3\times2\times1}$$

$$= 35\times20$$

$$= 700$$

Example: 11

If $nC_2 = nC_3$ find the value of n

Solution:

$$nC_2 = nC_3$$

$$\Rightarrow nC_2 = nC_{n-3}$$

$$2 = n-3$$

$$\text{or} \quad n = 5$$

Example: 12

If $nC_r : nC_{r+1} = 1:2$ and $nC_{r+1} : nC_{r+2} = 2:3$, determine the values of n and 'r'.

Solution: $$\frac{nC_{r+1}}{nC_r} = \frac{2}{1}$$

$$\therefore \frac{n-(r+1)+1}{r+1} = 2$$

i.e $n - r = 2r + 2$ or $n - 3r = 2$ .... (1)

$$\frac{nC_{r+2}}{nC_{r+1}} = \frac{3}{2} \Rightarrow \frac{n-(r+2)+1}{r+2} = \frac{3}{2}$$

$2(n - r - 1) = 3(r + 2)$

$2n - 2r - 2 = 3r + 6$ or $2n - 5r = 8$ .... (2)

From (1) $2n - 6r = 4$

From (2) $2n - 5r = 8$

Subtracting $-r = -4$ or $r = 4$

From (1), $n = 14$

Example: 13

Find out the number of ways in which a cricket team consisting of 11 players can be selected out of 14 players. Also find out how many of these will include a particular player.? (CA Entrance May 1987)

Solution:

A team of 11 players can be selected from 14 player in $14C_{11}$ ways

$$= 14C_3$$

$$= \frac{14 \times 13 \times 12}{1 \times 2 \times 3} = 364$$

If a particular player is included then we have to select 10 players from the remaining 13.

This can be done in $13\,C_{10}$ ways $=$ $13\,C_3$ ways.

$$= \frac{13\times12\times11}{1\times2\times3} = 286$$

Example: 14

From 6 boys and 4 girls, 5 are to be selected for admission to a particular course. In how many ways can this be done if there must be exactly 2 girls? (CA Inter November 1975)

Solution:

We have to select 2 girls and 3 boys.

2 girls can be selected from 4 girls in $4C_2$ ways.

3 boys can be selected from 6 boys in $6\,C_3$ ways.

$\therefore$ Total number of ways of selection $= 4\,C_2 \times 6\,C_3 = 120$

Example: 15

From 6 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include atleast one lady? (CA Entrance December 1980)

Solution:

Cases	Gentlemen 6	Ladies 4	No. of ways
1.	4	1	$6\,C_4 \times 4\,C_1 = 60$
2.	3	2	$6\,C_3 \times 4\,C_2 = 120$
3.	2	3	$6\,C_2 \times 4\,C_3 = 60$
4.	1	4	$6\,C_1 \times 4\,C_4 = 6$
		Total number of ways	$= 246$

Example: 16

An examination paper consists of 12 questions divided into two parts A and B; Part A contains 7 questions and part B contains 5 questions. A candidate is required to answer 8 questions selcting atleast 3 questions from each part. In how many ways can he select the questions? (CA Entance November 1989)

Solution:

Number of questions

Cases	Part A 7	Part B 5	No. of ways of selection
1.	5	3	$7C_5 \times 5C_3 = 210$
2.	4	4	$7C_4 \times 5C_4 = 175$
3.	3	5	$7C_3 \times 5C_5 = 35$
		Total no. of ways	$= 420$

Example: 17

A cricket than of 11 players into be formed from 20 players including 6 bowlers and 3 wicket keepers. In how many ways can a team be formed so that the team contains exactly 2 wicket keepers and atleast 4 bowlers? (CA Entrance June 1981)

No. of Players	Bowler 6	Wicket keepers 3	others 11	No. of ways
1.	4	2	5	$6C_4 \times 3C_2 \times 11C_5 = 16170$
2.	5	2	4	$6C_5 \times 3C_2 \times 11C_4 = 5940$
3.	6	2	3	$6C_6 \times 3C_2 \times 11C_3 = 165$
		Total no. of ways		$= 22275$

Example: 18

There are 7 men and 3 ladies. Find the numbers of ways in which a committee of 6 persons can be formed if the committee is to have atleast 2 ladies.

(CA Entrance November 1987)

Solution:

No. persons	Men	Ladies	No. of ways of selection
	7	3	
Cases			
1.	4	2	$7C_4 \times 3C_2 = 105$
2.	3	3	$7C_3 \times 3C_3 = 35$
		Total no. of ways	= 140

Example: 19

In an election a voter may vote for any number of candidates not, greater than the number to be chosen. There are 7 candidates and 4 members are chosen. In how many ways can a person vote?

(CA Entrance November 1988)

Solution:

4 members are to be chosen out of 7 candidates. A person can vote to a maximum of 4 persons.

∴ A person can vote for 1 person or 2 persons or 3 persons or 4 persons.

∴ The number of possible ways.

$$= 7C_1 + 7C_2 + 7C_3 + 7C_4$$

$$= 7 + 21 + 35 + 35 = 98$$

Example: 20

Out of 4 officers and 10 clerks in an office, a committee consisting of 2 officers and 3 clerks in to be formed. In how many ways can this be done if (i) any officer and any clerk can be included (ii) one particular clerk must be on the committee (iii) one particular officer cannot be on the committee. (CA Entrance December 1981)

Solution:

There are 4 officers out of which 2 are to be chosen. This can be done in $4\,C_2$ ways.

(i) There are 10 clerks out of which 3 clerks an to be chosen. This can be done in $10\,C_3$ ways.

$\therefore$ Total no of selections $= 4\,C_2 \times 10\,C_3$

$= 6 \times 120 = 720$

(ii) If a particular clerk is to be on the committee then we have to select 2 officers from 4 officers and 2 clerks from the remaining 9 clerks. This can be done in $4\,C_2 \times 9\,C_2$ ways $= 216$

(iii) If a particular officer is to be excluded then the number of selections $= 3\,C_2 \times 10\,C_3 = 3 \times 120 = 360$

Example: 21

A reserve of 12 railway station masters is to be divided into 2 groups of 6 each, one for day duty and the other for night duty. In how many ways can this be done if two specified persons A and B should not be included in the same group?

Solution:

Divide the 12 station masters into two groups, one containing the 2 specified persons and the other containing the remaining 10 persons. For day duty select one person from the 1^{st} group and 5 persons from the 2^{nd} group. This can be done in $2\,C_1 \times 10\,C_5$ ways. The others will go for night duty.

$\therefore$ Total number of selections $= 10\,C_5 \times 2\,C_1 = 504$

Example: 22

In how many ways can a committee of 3 women and 4 men be chosen from 8 women and 7 men? What is the number of ways if Miss. A refuses to serve if Mr. B is a member?

Solution:(i) 3 women can be chosen from 8 women in $8\,C_3$ ways.

4 men can be chosen from 7 men in $7\,C_4$ ways.

$\therefore$ Total number of ways of selecting 3 women

$$\text{and 4 men} = 8\,C_3 \times 7\,C_4 = 1960$$

(ii) Since Miss A refuses to serve if Mr. B is a member consider all the selections except those in which both A and B are members. Number of selections in which Miss. A and Mr. B are members $= 7\,C_2 \times 6\,C_3 = 420$

$\therefore$ Total number of selections required $= 1960 - 420$

$= 1540$

Example: 23

There are 16 guests at a dinner party. They are to sit 8 on each side of a long table. There particular persons desire to sit on one side and two others on the other side. In how many ways can the guests be arranged?

Solution:

After permitting 3 persons to sit on one side and 2 others on the other sides there are remaining 11 persons. Now there are 5 seats on one side and 6 seats on the other side. We can choose 5 persons from the 11 persons for one side in $11\,C_5$ ways. The other six have to take seats on the other side.

$\therefore$ There are $11\,C_5$ selections in which 3 persons can choose to sit on one side and 2 others on other side. In each of these selections both sides of the table can be arranged in $\lfloor 8 \times \lfloor 8$ ways.

$\therefore$ Total no. of arrangements $= 11\,C_5 \times \lfloor 8 \times \lfloor 8$

Example:24

Find the number of diagonals in a polygon of n sides. How many triangles can be made?

Solution:

In a polygon of 'n' sides there are 'n' vertices.

(i) By joining any two vertices we get a line.

$\therefore$ There are $n\,C_2$ straight lines in a polygon.

$$\text{Number of diagonals} = n\,C_2 - n$$

$$= \frac{n\,(n-1)}{2} - n$$

$$= \frac{n\,(n-1) - 2n}{2}$$

$$= \frac{n\,(n-3)}{2}$$

(ii) In a polygon of 'n' sides no three vertices are collinear. By joining any three vertices we get a triangle. $\therefore$ Total number of triangles $= n\,C_3 = \dfrac{n\,(n-1)\,(n-2)}{6}$

Example: 25

There are 3 sections in a question peper, each containing 5 questions. A candidate has to solve only 5 questions, choosing atleast one question from each section. In how many ways can he make his choice?

(CA Inter May 1981)

Solution:

Number of ways of answering 5 questions choosing atleast one from each section. = Total number of ways of answering 5 questions out of 15 questions - number of ways of ommitting any one section + number of ways of ommitting any two sections.

$$= 15C_5 - 3.10C_5 + 3.5C_5$$

$$= \frac{15\times14\times13\times12\times11}{5\times4\times3\times2\times1} - 3\times\frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} + 3 = 2250$$

Note: In the 2[nd] term also number of ways of choosing all 5 questions from each section is counted and hence the third term.

Example: 26

A committee of 3 experts is to be selected out of a panel of 7 persons; three of them are Lawyers, three of them are Chartered Accountants and one is both a Chartered Accountant and a Lawyer. In how many ways can the committee be selected if it must have atleast a Chartered Accountant and a lawyer?

Solution:

Number of ways of selecting 3 experts such that the committee contains atleast a CA and a lawyer = Number of ways of selecting 3 persons out of the 7 persons - Number of ways of selecting all 3 lawyers - number of ways of selecting all 3 C.A's.

$$= 7C_3 - 3C_3 - 3C_3$$

$$= \frac{7\times6\times5}{1\times2\times3} - 1 - 1 = 35 - 2 = 33$$

Example: 27

A firm of Chartered Accountants in Bombay has to send 10 clerks to 5 concerns, two to each concern. Two of the concerns are in Bombay and the others are outside. Two of the clerks prefer to work in Bombay while three others prefer to work outside. In how many ways can the assignment be made if the references are to be satisfed?

Solution:

There are two concerns in Bombay and 3 concerns outside Bombay. ∴ 4 of the clerks work in Bombay and 6 of the clerks have work outside Bombay. Two prefer to work in Bombay and 3 prefer to work outside Bombay. Allowing the 5 to choose their preferences there are remaining 5 persons. We have to choose 2 out of these 5 to work in Bombay. This can be done in $5\,C_2$ ways. The other 3 will have to work outside Bombay.

$\therefore$ There are $5\,C_2$ selections each containing 4 clerks to work in Bombay and others outside Bombay.

In each of these selections the 4 clerks in Bombay can be selected for the two concerns in $4\,C_2$ ways and the 6 outside Bombay can be selected for the 3 concerns in $6\,C_2 \times 4\,C_2 \times 2\,C_2$ ways.

$= 15 \times 6 \times 1 = 90$ ways.

$\therefore$ Total number of assignments $= 5\,C_2 \times 4\,C_2 \times 6\,C_2 \times 4\,C_2$

$= 10 \times 6 \times 90 = 5400$

Example: 28

Find the number of ways in which 12 persons may be divided into 4 sets of 3 each, one to play lawn tennis, one to play cards one to play badminton and one to play table tennis.

Solution:

There are 12 players. They are to be divided into 4 sets each containing 3 players.

$\therefore$ Number of ways $= 12\,C_3 \times 9\,C_3 \times 6\,C_3 \times 3\,C_3$

$$= \frac{\lfloor 12}{\lfloor 3 \lfloor 9} \times \frac{\lfloor 9}{\lfloor 3 \lfloor 6} + \frac{\lfloor 6}{\lfloor 3 \lfloor 3} \times 1 = \frac{\lfloor 12}{(\lfloor 3)^4}$$

Exercises

1. Find the value of (i) $\dfrac{11\,C_8}{11\,C_7}$ (ii) $11\,C_7 + 11\,C_8$

2. If $36\,C_n = 36\,C_{r+4}$ find the value of r.

3. If $2n\,C_4 = 2n\,C_{r+2}$ find the value of r

4. If $m = n\,C_2$ show that $m\,C_2 = 3.(n+1)\,C_4$

5. (i) Find 'n' if $2n\,C_3 : n\,C_2 = 44:3$

   (ii) If $nC_x = 56$ and $nP_x = 336$ find 'n' and x.

6. If $(n+1)\,C_{r+1} : n\,C_r : (n-1)\,C_{r-1} = 11:6:3$ find the values of 'n' and 'r'

7. From 10 books in how many ways can a selection of 6 be made (i) when a particular book is always included (ii) when a particular book is always excluded.

8. Out of 7 consonants and 4 vowels, how many words can be formed, each containing 3 consonants and 2 vowels?

9. In how many ways can a committe of 6 men and 2 women be formed out of 10 men and 5 women?

10. From 6 boys and 4 girls 5 are to be selected for admission to a particular course. In how many ways can this be done if there must be exactly 2 girls?

11. In how many ways can be committee of 5 be formed from 4 professors and 6 students so as to include atleast a professor?

12. A committee of 7 members is to be chosen from 6 Chartered Accountants, 4 Economist and 5 Cost Accountants. In how many ways can this be done if in the committee there must be atleast one member from each group and atleast 3 Chartered Accountants?

13. A cricket team of 11 players is to be chosen from 20 players including 6 bowlers and 3 wicket keepers. In how many different ways can a team be formed so that the team contains exactly 2 wicket keepers and atleast 4 bowlers?

14. In how many ways can a committee of 5 be chosen from 10 candidates so as to exclude the youngest if it includes the oldest?

15. Out of 6 teachers and 4 boys a comittee of 8 is to be formed. In how many ways can this be done if there must be not more than two boys?

16. For a certain course of studies, a student has to select 3 subjects out of 9 subjects. The subjects are divided into 3 groups each containing 3 subjects of which one is a practical subject. A student has to choose one subject from each group in such a way that atleast one but not more than 2 practical subjects are selected. In how many ways can the selection be made?

17. A student has to answer 8 out of 10 questions in an examination (i) How many choices has he? (ii) How many if he must answer atleast four of the first five questions?

18. In a class of 16 students, there are 5 lady students. In how many ways can 10 students be selected from them so as to include atleast 4 lady students ?

19. The question paper on mathematics and statistics contains 10 questions divided into two groups of 5 questions each. In low many ways can an examinee select 6 questions taking atleast two questions from each group?

20. An examination paper consists of 12 questions divided into 2 parts A and B. Part A contains 7 questions and part B contains 5 questions. A candidate is required to answer 8 questions selecting atleast 3 questions from each part. In how many maximum ways can he select the questions?

21. A box contains 7 red, 6 white and 4 blue balls. How many sellections of three balls can be made so that (i) all three are red balls, (ii) none is a red ball (ii) there is one ball of each colour.

22. Among 20 members of a cricket club, there are two wicket keepers and five bowlers. In how many way can eleven be chosen so as to include one of the wicket keepers and atleast three bowlers?

23. From 7 gentlemen and 4 ladies, a committee of 6 is to be formed. In how many ways can this be done when the committee has to contain (i) exactly 2 ladies (ii) atleast 2 ladies.

24. An eight oared boat has 8 men of whom 3 can row on one side and 2 only on the other side. Find the number of ways is which the men can be arranged 4 on each side?

25. There are 8 dolls and 8 different fancy boxes in which they have to be packed. If 2 of the boxes are too small for 5 of the dolls in how many ways can they be packed?

26. In how many ways can 52 cards be divided among 4 players so that each may have 13 cards?

27. A man has 5 friends. In how many ways can he invite one or more of his friends for a dinner?

28. An examination paper with 10 questions consists of 6 questions in Algebra and 4 questions in Geometry. Atleast one question from each section is to be attempted in how many ways can this be done?

29. From 7 gents principals and 4 lady principals a committee of 3 is to be formed. In how many ways can this be done so as to include atleast one lady principal.

30. If 'm' parallel straight lines are intersected by 'n' parallel straight lines, show that the number of parallelograms so formed is $\frac{mn\,(m-1)\,(n-1)}{4}$

31. Show that there are 136 ways of selecting 4 letters from the word EXAMINATION.

32. Find the number of ways in which (i) a selection (ii) an arrangement of 4 letters can be made from the letters of the word 'MATHEMATICS'.

33. A committee of 7 members is to be chosen from 6 Chartered Accountants, 4 Economists and 5 Cost Accountants. In how many ways can this be done if in the committee, there must be atleast one member from each group and atleast 3 Chartered Accountants ?

CHAPTER - 7 Permutation

Exercise 7.26

(1) 1680 (2) 720 (3) 504 (4) 24 (6) 3330

(7) 90 (8) $\lfloor 8 \times \lfloor 4$; $\lfloor 7 \times {}^8P_4$ (9) $\lfloor 3 \lfloor 4 \lfloor 5 \lfloor 6$

(10) 1440 (11) (i) 261 (ii) 306 (iii) 616

(12) 3024, 2352 (13) 576 (14) 720, 600, 96, 480 (15) 192

(16) 720, 600 (17) 144 (18) 48, 18 (19) 24

(20) 720, 24 (21) 240, 480 (22) 720 ; 576 ; 144

(23) 120, 24, 96, 48 (24) 720, 616 (25) 136

(26) $\dfrac{\lfloor 13}{\lfloor 3(\lfloor 2)^4}$ (27) $\dfrac{\lfloor 14}{(\lfloor 2)^3}$ (28) 360, 240

(29) $\dfrac{\lfloor 12}{\lfloor 3(\lfloor 2)^3}$, 151200 (30) 900, 240, 660 (31) 25200, 1800

(32) $\dfrac{\lfloor 12}{(\lfloor 3)^3}$ (33) $\dfrac{\lfloor 11}{\lfloor 4 \lfloor 4 \lfloor 2}$ (34) 240 (35) $3 \lfloor 8$

(36) 117 (37) 30240 (38) (i) 720 (ii) 2880

Combinations

Exercise 7.49

1. $\frac{2}{3}$; 2. $r = 16$; 3. $r = n - 1$; 5. $n = 8, x = 3$

6. $n = 10, r = 5$; 7. $\left({}^9C_5, {}^9C_6\right)$ 8. 210 ; 9. 2100

10. 120 ; 11. 186; 12. 3570; 13. 27225; 14. 196;

15. 45; 16. 27; 17. 45, 21, 35; 18. 2772; 19. 200;

20. 420; 21. 35, 120, 168; 22. 54054; 23. 210, 371;

24. 1728; 25. 4320; 26. $\dfrac{\lfloor 52}{\lfloor (13)}$; 27. 31; 28. 945;

29. 295 32. 136, 2454. 33. 2880

Chapter 8

Matrices

Introduction

A matrix is a rectangular array of numbers written in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & & & a_{mn} \end{bmatrix}$$

The a_{ij}'s are real or complex numbers. They are called the elements of the matrix. In the above matrix there are m rows and n columns. This matrix is called a m×n matrix or a rectangular matrix of the order m×n. A matrix containing m rows and n columns is said to be of the order m×n. If m=n then the matrix is called a square matrix of the order N.

Examples

$\begin{bmatrix} 2 & 5 & -1 \\ 3 & 2 & 5 \end{bmatrix}$ is a 2×3 matrix

$\begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 7 \end{bmatrix}$ is a 3×2 matrix

$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 1 & 5 \end{bmatrix}$ is a 3×3 matrix

Equal Matrices

Two matrices A and B are said to be equal if they are of the same order and the corresponding elements are equal. In this case we write A=B.

Example

If $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ and A =B, then $a = 2$, $b = 3$, $c = 4$ and $d = 7$.

Diagonal Matrix

A square matrix in which all the elements other than the leading diagonal are zero is called a diagonal matrix.

Example

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Scalar matrix

A diagonal matrix in which all the diagonal elements are equal is called a scalar matrix.

Example

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Unit matrix

A square matrix in which all the leading diagonal elements are unity and other elements are zero is called a unit matrix. It is denoted by I.

Note: A scalar matrix in which all the leading diagonal elements are unity is a unit matrix.

Null matrix

A matrix which all the elements are zero is called a null matrix. It is denoted by 0.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ are null matrices.

A null matrix may be a square matrix or a rectangular matrix.

Row matrix

A matrix containing only one row ($1 \times n$ matrix) is called a row matrix.

$a = (a_1, a_2\, a_3 \,....\, a_n)$ is a row matrix.

Column matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot\cdot \\ \cdot\cdot \\ b_m \end{bmatrix}$$ is called a column matrix.

A matrix that consists of only one column ($m \times 1$ matrix) is called a column matrix.

Note: A row matrix is also called a row vector and a column matrix is also called a column vector.

Matrix Operation

(1) Scalar multiplication

When a matrix is multiplied by a scalar, every element in the matrix is multiplied by the scalar.

Examples

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix} = k(a_{ij})$$

$$B = \begin{bmatrix} 3 & 0 & 4 \\ 7 & -1 & 2 \end{bmatrix} \quad 3B = \begin{bmatrix} 9 & 0 & 12 \\ 21 & -3 & 6 \end{bmatrix}$$

Addition and subtraction of matrices

Matrices can be added and subtracted only if they are of the same order. The sum of the two matrices. A and B of order m×n is a matrix of order m×n whose elements are the sum of the corresponding elements of the given matrices. It is denoted by A + B. The difference of the two matrices A and B of order mxn is a matrix of order m×n whose elements are the corresponding difference of the elements of the two matrices A and B.

i.e. If $A = (a_{ij}), B = (b_{ij})$

$$\text{Then } A + B = (a_{ij} + b_{ij})$$

$$A - B = (a_{ij} - b_{ij})$$

Example

$$A = \begin{bmatrix} 4 & 5 & 3 \\ 2 & 4 & -2 \end{bmatrix}, B = \begin{bmatrix} 7 & 2 & -1 \\ 6 & 5 & 4 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4+7 & 5+2 & 3-1 \\ 2+6 & 4+5 & -2+4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 7 & 2 \\ 8 & 9 & 2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 4-7 & 5-2 & 3+1 \\ 2-6 & 4-5 & -2-4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 3 & 4 \\ -4 & -1 & -6 \end{bmatrix}$$

Multiplication of two matrices

Two matrices A and B can be multiplied if and only if the number of columns in A is equal to the number of rows in B. The product matrix is denoted by AB. In this case we say matrices A and B are conformable for multiplication and the product matrix AB has the same number of rows as A and the same number of columns as B. Thus if A is a $m \times n$ matrix and B is a $n \times p$ matrix then AB is a $m \times p$ matrix.

Note: If A and B are the square matrices of order n then AB is also a square matrix of the order n.

If $A = [a_1\, a_2\, a_3 \ldots a_n]$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ .. \\ b_n \end{bmatrix}$$

$$AB = [a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n]$$

$a_1b_1 + a_2b_2 + \ldots + a_nb_n$ is called the inner product of the two vectors (row vector and column vector)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ & \ldots & \ldots & \ldots & \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} = (a_{ij})_{m \times n}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \ldots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \ldots & b_{2p} \\ & \ldots & \ldots & & \\ b_{n1} & b_{n2} & b_{n3} & \ldots & b_{np} \end{bmatrix} = (b_{jk})_{n \times p}$$

When two matrices A and B are multiplied the element in the ith row and jth column of the product matrix is the inner product of the ith row vector of A and jth column vector of B. Thus the product of the matrices A and B can be written as AB = C where

$$C = (c_{ik})_{m \times p} = \begin{bmatrix} \sum a_{1j}b_{j1} & \cdots & \sum a_{1j}b_{jm} \\ \vdots & & \vdots \\ \sum a_{mj}b_{j1} & \cdots & \sum a_{mj}b_{jn} \end{bmatrix}$$

where $c_{ik} = \sum_j = 1\,(a_{ij}b_{jk})$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 5 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\times4+2\times2+3\times1 & 1\times5+2\times3+3\times2 \\ 3\times4+2\times2+1\times1 & 3\times5+2\times3+1\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4+3 & 5+6+6 \\ 12+4+1 & 15+6+2 \end{bmatrix} = \begin{bmatrix} 11 & 17 \\ 17 & 23 \end{bmatrix}$$

Note: The first element of the first row of the product AB is got by multiplying the elements of the first row of A with the corresponding elements of the first column of B and then adding the resulting figures. The second element of the first row of the product AB is got by multiplying the elements of the first row of A with the corresponding elements of the 2nd column and then adding them.

The first element of the 2nd row of AB is got by multiplying the elements of the second row of A with the corresponding elements of the first column of B and then adding. The second element of the 2nd row of AB is got by multiplying the elements in the 2nd row of A with the corresponding elements of the 2nd column of B and then adding. Thus the matrix AB is obtained.

In matrix multiplication, the sequence in which multiplication is performed is very important. If A is a $m \times n$ matrix and B is a $n \times p$ matrix, AB is a $n \times p$ matrix and the product of AB is derived. But the produt of BA is not defined if p is not equal to m. If A is a $m \times n$ matrix and B is a $n \times m$ matrix then AB is defined and it is a $m \times m$ matrix. Also BA is defined and it is a $n \times n$ matrix. However we cannot say that AB and BA are equal. In general $AB \neq BA$.

Transpose of a matrix

For any given matrix A, the matrix whose rows are columns of A and whose columns are rows of A is called the transpose of A and is denoted by A^T or A'. If A is a $m \times n$ matrix then A^T is $n \times m$ matrix.

Example

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 7 & 3 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & 7 \\ 2 & 3 \\ 6 & 5 \end{bmatrix}$$

Note: If a square matrix and its transpose are equal then the matrices are symmetric.

Properties of Transpose

1. If A and B are two matrices of the same order then $(A + B)^T = A^T + B^T$.
2. If A and B are conformable for multiplication then $(AB)^T = B^T \cdot A^T$

i.e. the transpose of the product of two matrices
= the product of the transposes in the reverse order

These results can be extended to n matrices.

Determinent of a matrix

Consider the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The determinent of the matrix A is the number.

$$a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

and is denoted by det. A or $|A|$.

The determinent of the matrix A is also denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Consider the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinent of the matrix A is the scalar given by

$$|A| = a_{11} \cdot (a_{22} \cdot a_{33} - a_{32} \cdot a_{23}) - (a_{21} \cdot a_{33} - a_{31} \cdot a_{23})$$
$$+ a_{13} (a_{21} \cdot a_{32} - a_{31} \cdot a_{22})$$

This can also be written as

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{11}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Let M_{ij} be the matrix obtained by deleting the ith row and jth column of A. The determinent $|M_{ij}|$ is called a minor of the matrix A.

The scalar $A_{ij} = (-1)^{i+j}\,|M_{ij}|$ is called the co-factor of the element a_{ij} of the matrix A. we now define the determinent of matrix A in terms of co factors.

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

We can also write

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$|A| = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

Note: Determinants are defined only for square matrices.

Singular and Non singular matrices

A square matrix A is said to be singular if its determinant is zero.

A square matrix A is said to be non singular if its determinant is not equal to zero.

Adjoint of a Square Matrix

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The adjoint of A is defined to be the transpose of the co-factor matrix of A and is denoted by adj.A.

$$\text{adj.}A = (A_{ij})^T$$

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\text{Adj.}A = (A_{ij})^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Reciprocal Matrix or Inverse of a matrix

If A is non-singular matrix $\frac{1}{|A|}$ adj A is defined to be the reciprocal of the matrix A or the Inverse of the matrix A. It is denoted by A^{-1}

$$A^{-1} = \frac{1}{|A|} \text{adj}.A$$

It can be shown that $AA^{-1} = A^{-1}A = I$

Note: If A and B are two matrices such that AB = BA = I then A and B are the inverses of each other.

$$\text{(i.e) } A^{-1} = B \text{ and } B^{-1} = A$$

Symmetric and skew symmetric matrices

Definition:

A square matrix $A = [a_{ij}]$ is called a symmetric matrix if the (i, j)th element of A is equal to the (j, i)th element of A.

(i.e. if $a_{ij} = a_{ij}$ for all i and j).

Definition:

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if the (i, j)th element of A is equal to the negative of the (j, i)th element of A.

(i.e. if $a_{ij} = -a_{ij}$ for all i and j)

Example (i)

$$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ are symmetric matrices.}$$

Example (ii)

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix} \text{ is a skew symmetric matrix.}$$

Note: In a skew symmetric matrix, for diagonal elements j=i so that $a_{ii} = -a_{ij}$ or $a_{ii} = 0$

$\therefore$ All diagonal elements in a skew symmetric matrix are zero.

Example 1

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, show that the products AA', $A'A$ are symmetric but

$AA' \neq A'A$

Solution

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad A' = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 4+9 & 8+15 \\ 8+15 & 16+25 \end{bmatrix} = \begin{bmatrix} 13 & 23 \\ 23 & 41 \end{bmatrix} \rightarrow (1)$$

$$A'A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4+16 & 6+20 \\ 6+20 & 9+25 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 26 \\ 26 & 34 \end{bmatrix} \quad \rightarrow (2)$$

$\therefore$ From (1) and (2),

AA' and $A'A$ are symmetric matrices. But $AA' \neq A'A$.

Note: For any square matrix A,

$(AA')' = (A')'A'$

$= AA' \quad \therefore AA'$ is symmetric.

Also $(A'A)' = (A')(A')'$

$= A'A \quad \therefore A'A$ is also symmetric.

Example 2

If A and B are both symmetric, then AB is symmetric if and only if A and B are commutative.

Proof:

Since A and B are symmetric,

$A' = A$ and $B' = B$

Hence $(AB)' = B'A' = BA$ ..... (1)

If A and B are commutative, then $AB = BA$ ..... (2)

$\therefore$ From (1) and (2) $(AB)' = AB$

$\therefore$ AB is symmetric.

Also if AB is symmetric, then

$(AB)' = AB$ $\rightarrow$ (1)

But $(AB)' = B'A' = BA$ $\rightarrow$ (2)

From (1) and (2), $AB = BA$. Hence A and B are commutative.

Example 3

If A and B are symmetric (skew symmetric), show that $A + B$ is symmetric (skew symmetric).

Proof:

Case 1

Suppose A and B are symmetric. Since A and B are symmetric,

$A' = A$ and $B' = B$.

Since A and B are symmetric matrices, they are square matrices. In order that $A + B$ to be defined, A and B must be of the same order.

Now $(A + B)' = A' + B'$

$= A + B$ since $A' = A$ and $B' = B$

Case 2

Suppose A and B are skew symmetric.

Then $A' = -A$ and $B' = -B$

$(A + B)' = A' + B'$

$= -A - B.$

$= -(A + B)$

$\therefore A + B$ is skew symmetric.

Example 4

Show that every square matrix can be uniquely expressed as the sum of a symmetric and skew-symmetric matrix.

Solution

Let A be a square matrix.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') \quad \rightarrow (1)$$

Now $(A + A')' = A' + (A')' = A' + A = A + A'$

$(A - A')' = A' - (A')' = A' - A = -(A - A')$

$\therefore A + A'$ is symmetric and $A - A'$ is skew symmetric.

$\therefore$ From (1), $A = P + Q$ ..... (2)

where $P = \frac{1}{2}(A + A')$ is symmetric.

and $Q = \frac{1}{2}(A - A')$ is skew symmetric.

$\therefore$ Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

We will now show that the representation is unique.

Suppose

$$A = R + S \quad .... (3)$$

where R is symmetric and S is skew symmetric.

Then $A' = R' + S' = R - S$ .... (4)

$\therefore$ From (3) and (4),

$A + A' = 2R$ and $A - A' = 2S$

Hence $R = \frac{1}{2}(A + A') = P$

$S = \frac{1}{2}(A - A') = Q$

$\therefore$ There is only one way of expressing a square matrix as the sum of a symmetric and skew symmetric matrices.

Example 5

Express $\begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrices.

Solution

$$\text{Let } A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 6 & 4 & 9 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

$$\frac{1}{2}(A + A') = \begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$$

$$\frac{1}{2}(A - A') = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}$$

We know that

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$= \begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}$$

Conjugate of a matrix:

The matrix obtained from any given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$.

i.e. If $A = (a_{ij})$ then $\overline{A} = (\overline{a_{ij}})$ where $\overline{a_{ij}}$ is the conjugate complex of a_{ij}.

Ex. $A = \begin{bmatrix} 4+3i & 7-3i & 5 \\ i & 4 & 3+i \end{bmatrix}$, then the conjugate of A is

$$\overline{A} = \begin{bmatrix} 4-3i & 7+3i & 5 \\ -i & 4 & 3-i \end{bmatrix}$$

Properties:

(i) $(\overline{\overline{A}}) = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$

(iii) $\overline{(KA)} = \overline{K}\,\overline{A}$ where k is any complex number.

(iv) $\overline{(AB)} = \overline{A}\,\overline{B}$, if A and B are conformable for multiplication.

If A is any matrix, the transpose of the conjugate of a matrix A is called the transposed conjugate of A and is denoted by A^*.

Note: $(\overline{A'}) = (\overline{A})' = A^*$

Example

Write down the transposed conjugate of the matrix

$$A = \begin{bmatrix} 2+i & 7+4i & 6-3i \\ 3-2i & 4-3i & 2+3i \\ 6 & 7+6i & 7 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2+i & 3-2i & 6 \\ 7+4i & 4-3i & 7+6i \\ 6-3i & 2+3i & 7 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 2-i & 3+2i & 6 \\ 7-4i & 4+3i & 7-6i \\ 6+3i & 2-3i & 7 \end{bmatrix}$$

Hermitian and skew-Hermitian matrices

Definition 1

A square matrix $A = [a_{ij}]$ is said to be Hermitian if the (i, j)th element of A is equal to the conjugate complex of the (j, i)th element of A.

i.e. $a_{ij} = \overline{a_{ij}}$ for all i and j.

Defintion 2

A square matrix $A = [a_{ij}]$ is said to be skew-Hermitian if the (i, j)th element of A is equal to the negative of the conjugate complex of the (j, i)th element of A.

i.e. if $a_{ij} = -\overline{a_{ji}}$ for all i and j.

Example (i)

$$A = \begin{bmatrix} 3 & 4+5i \\ 4-5i & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & 6+4i & 3-7i \\ 6-4i & 7 & 2+5i \\ 3+7i & 2-5i & 0 \end{bmatrix}$$

are Hermitian matrices.

Example (ii)

$$\begin{bmatrix} 0 & 3-4i \\ -3-4i & 0 \end{bmatrix}, \begin{bmatrix} -2i & 4-7i \\ -4-7i & 0 \end{bmatrix}$$ are skew-hermitian matrices

Note 1:

If A is Hermitian, then

$$a_{ii} = \overline{a_{ii}}$$

This is possible only if a_{ii} is real. Hence in a Hermitian matrix, all the diagonal elements are real.

Note 2:

If A is a skew Hermitian matrix,

$$a_{ij} = -\overline{a_{ji}}$$

$$\therefore a_{ii} = -\overline{a_{ii}}$$

$$\therefore a_{ii} + \overline{a_{ii}} = 0$$

$\therefore a_{ii}$ is either purely imaginery or zero.

$\therefore$ the diagonal elements of a skew Hermitian matrix is purely imaginary or zero.

Theorem

If A and B are Hermitian, show that AB + BA is Hermitian and AB−BA is skew hermitian.

Solution:

Let A and B be Hermitian matrices of the same order. Then

$$A^* = A \text{ and } B^* = B.$$

$$\begin{aligned} (AB + BA)^* &= (AB)^* + (BA)^* \\ &= B^*A^* + A^*B^* \\ &= BA + AB \\ &= AB + BA \end{aligned}$$

$\therefore$ AB + BA is Hermitian.

Also
$$\begin{aligned} (AB - BA)^* &= (AB)^* - (BA)^* \\ &= B^*A^* - A^*B^* \\ &= BA - AB \end{aligned}$$

$$= -(AB - BA)$$

$\therefore$ AB + BA is skew-hermitian.

Theorem

If A is a square matrix, show that $A - A^*$, A^*A are all Hermitian and $A - A^*$ is skew-Hermitian.

Solution

We know that a square matrix A is Hermitian if $A^* = A$

(i) $(A + A^*)^* = A^* + (A^*)^*$

$$= A^* + A = A + A^*$$

$\therefore$ $A + A^*$ is Hermitian.

(ii) $(AA^*)^* = (A^*)^* A^*$

$$= AA^*$$

$\therefore$ AA^* is Hermitian.

(iii) $(A^* A)^* = A^* \cdot (A^*)^*$

$$= A^* A$$

$\therefore$ $A^* A$ is Hermitian.

(iv) A square matrix A is skew Hermitian if $A^* = -A$.

$$(A - A^*)^* = A^* - (A^*)^*$$

$$= A^* - A$$

$$-(A - A^*)$$

$\therefore$ $A - A^*$ is skew Hermitian.

Theorem

Show that $B^* AB$ is Hermitian or skew Hermitian according as A is Hermitian or skew Hermitian.

Proof

Let A be a Hermitian matrix

Then $A^* = A$

Now $(B^* AB)^* = B^* A^* B$

$$= B^* AB$$

$\therefore$ $B^* AB$ is Hermitian.

Suppose A is skew Hermitian matrix

Then $A^* = -A$

$(B^* AB)^* = B^* A B^*$

$$= B^* (-A) B$$

$$= -(B^* AB)$$

$\therefore$ $B^* AB$ is skew Hermitian.

Theorem:

Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew Hermitian matrix.

Solution

Let A be a square matrix.

Then $A + A^*$ is a Hermitian matrix and $A - A^*$ is a skew Hermitian matrix.

Therefore $\frac{1}{2}(A + A^*)$ is a Hermitian matrix and $\frac{1}{2}(A - A^*)$ is a skew Hermitian matrix.

But $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = P + Q$

Here P is Hermitian and Q is skew-Hermitian. Therefore, every square matrix can be expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Suppose $A = R + S$ be another representation of A where R is a Hermitian matrix and S is a skew Hermitian matrix.

$$\begin{aligned} \text{Then } A^* &= (R + S)^* \\ &= R^* + S^* \\ &= R - S \end{aligned}$$

$$\therefore R = \frac{1}{2}(A + A^*) = P \text{ and}$$

$$S = \frac{1}{2}(A - A^*) = Q$$

$\therefore$ the representation of A is unique.

Theorem:

If A is Hermitian, show that $\overline{A}$ is Hermitian and if A is skew Hermitian show that $\overline{A}$ is a skew Hermitian.

Proof:

Case 1

Let A be a Hermitian matrix.

Then $A^* = A$

We will prove that $\overline{A}$ is a Hermitian.

$$\begin{aligned} (\overline{A})^* = [(\overline{\overline{A}})]' &= A' \quad \text{since } (\overline{\overline{A}}) = A \\ &= (A^*)' \text{ since A is Hermitian } A = A^* \\ &= [(\overline{A})']' \text{ since } A^* = (\overline{A})' \\ &= \overline{A} \quad [\because (A')' = A] \end{aligned}$$

Case 2

Suppose A is skew-Hermitian

Then $A^* = -A$

Now $(\bar{A})^* = [(\bar{A})]' = (A)' = (-A^*)'$

$= A^{\wedge} = -[(\bar{A})] = -A$

$\therefore \bar{A}$ is skew-Hermitian.

Theorem:

Show that every square matrix A can be uniquely expressed as P + iQ where P and Q are Hermitian matrices.

Solution

Let $P = \frac{1}{2}(A + A^*)$ and $Q = \frac{1}{2i}(A - A^*)$

Then $A = P + iQ$

Then $P^* = \frac{1}{2}(A + A^*)^*$

$= \frac{1}{2}(A^* + A)$

$= \frac{1}{2}(A + A^*) = P$

$\therefore$ P is Hermitian.

Also $Q^* = \left[\frac{1}{2i}(A - A^*)\right]^*$

$= \frac{1}{2i}(A - A^*)^*$

$= -\frac{1}{2i}\left[(A^* - (A^*)^*\right]$

$= -\frac{1}{2i}(A^* - A)$

$= \frac{1}{2i}(A - A^*) = Q$

$\therefore$ Q is Hermitian.

Exercise 1

1) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, show that the products AA' and $A'A$ are symmetric but not equal.

2) Show that the matrix A^2 is symmetric if either A is symmetric or A is skew-symmetric.

3) If A is symmetric (or skew symmetric) show that $B'AB$ is symmetric (or skew symmetric).

4) If A is any square matrix, show that $A + A'$ is symmetric and $A - A'$ is skew-symmetric.

5) If A and B are symmetric, show that $AB + BA$ is symmetric and $AB-BA$ is skew symmetric.

6) Express $\begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 7 \\ 6 & 2 & 1 \end{bmatrix}$ as the sum of symmetric and a skew symmetric matrix.

7) If U and V are two symmetric matrices, show that $U + V$ is also symmetric. Is UV symmetric always? Explain and illustrate by an example.

8) Show that $\begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ is hermitian.

9) Show that $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is hermitian.

10) Show that $\begin{bmatrix} 0 & -1+i \\ 1+i & 0 \end{bmatrix}$ is skew-hermitian.

11) Show that $\begin{bmatrix} i & 3+i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2 \end{bmatrix}$ is skew-hermitian.

12) If $A = \begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix}$, show that A' is hermitian.

13) Show that $\begin{bmatrix} 4 & 1-i \\ 1+i & 2 \end{bmatrix}$ is hermitian.

Show that $\begin{bmatrix} i & 1+i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$ is skew-hermitian.

Orthogonal and Unitary matrices

Definition:

A square matrix A is said to be orthogonal if $A'A = I$

Note:

$$A'A = I \Rightarrow |A'A| = |I|$$
$$\text{i.e. } |A'|\,|A| = 1$$
$$|A|^2 = 1$$
$$\text{i.e. } |A| \neq 0$$
$$\therefore A \text{ is invertible}$$
$$\therefore A'A = I \Rightarrow AA' = I$$

$\therefore$ A square matrix A is orthogonal if and only if

$$A'A = AA' = I$$

Unitary matrix

A square matrix A is said to be the unitary if $A^*A = I$.

Note:

$$A^*A = I \Rightarrow |A^*A| = 1$$
$$|A^*|\,|A| = 1$$
$$|\overline{A}|\,|A| = 1$$

Since the determinant of a unitary matrix is unit modulus,

hence $A^*A = AA^* = I$.

Theorem:

If A and B are orthogonal matrices of the same order, then AB and BA are also orthogonal.

Proof:

Since A and B are orthogonal matrices

$$AA' = A'A = I.$$
$$BB' = B'B = I.$$

Now $(AB)(AB)' = AB(B'A')$

$= A(BB')A'$ Since matrix multiplication is associative

$= A(I)A'$

$= (AI)A'$

$= AA' = I.$

Similarly we can show that $(AB)'(AB) = I$.

$\therefore$ AB is an orthogonal matrix.

$$\text{Also } (BA)(BA)' = BA(A'B')$$
$$= B(AA')B'$$
$$= (BI)B'$$
$$= BB'$$
$$= I.$$

$\therefore$ BA is also an orthogonal matrix.

Theorem:

If A and B are unitary matrices, then AB and BA are also unitary matrices.

Proof:

Since A and B are unitary matrices.

$$AA^* = A^*A = I.$$
$$BB^* = B^*B = I.$$

$$\text{Now } (AB)\ (AB)^* = AB(B^*A^*)$$
$$= A(BB^*)A^*$$
$$= (AI)A^*$$
$$= AA^* = I.$$

$\therefore$ AB is a unitary matrix.

Similarly we can show that BA is also a unitary matrix.

Theorem:

If A is orthogonal matrix, show that A' and A^{-1} are also orthogonal matrices.

Proof:

Since A is orthogonal, $A'A = I$.

Taking transpose of both sides,

$$(A'A)' = I'$$
$$A'A = I.$$

$\therefore$ A is an orthogonal matrix

Also as A is an orthogonal matrix,

$$A'A = I$$

Taking inverse on both sides,

$$(A'A)^{-1} = I^{-1}.$$

$$A^{-1}(A')^{-1} = I.$$

$$A^{-1}(A^{-1})' = I.$$

$\therefore A^{-1}$ is an orthogonal matrix.

Note: If A is an unitary, then A^{-1}, A^{*}, A^{1} are also unitary.

Theorem:

A real matrix is unitary if and only if it is orthogonal.

Proof:

If A is real, then $A^{*} = A'$.

If A is unitary, $A^{*}A = I$

If this unitary matrix is real, then

$$A'A = I.$$

$\therefore$ A is orthogonal.

Conversely if A is orthogonal, then $A'A = I$

i.e. $A^{*}A = I$ if A is real.

This implies A is unitary.

Example 1

Show that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Solution

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$A' = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore AA' = I \Rightarrow A$ is orthogonal.

Example 2

Prove that the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$\therefore$ A is orthogonal.

Example 3

Show that $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal.

Solution

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A' = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$AA' = \begin{bmatrix} \frac{1}{3}+\frac{1}{6}+\frac{1}{2} & \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{1}{6}-\frac{1}{2} \\ \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{4}{6}+0 & \frac{1}{3}-\frac{2}{6}+0 \\ \frac{1}{3}+\frac{1}{6}-\frac{1}{2} & \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{1}{6}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ A is orthogonal.

Example 4

Show that $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary.

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}, \quad A' = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$A^* = \left(\overline{A'}\right) = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-(1+i)}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$A^*A = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-(1+i)}{2} & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-i^2}{4} + \frac{1-i^2}{4} & \frac{1-i}{2} \\ \frac{-(1+i)^2}{4} + \frac{(1+i)^2}{4} & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-i^2}{4} + \frac{1-i^2}{4} & \frac{-(1-i)^2}{4} + \frac{(1-i)^2}{4} \\ \frac{-(1+i)^2}{4} + \frac{(1+i)^2}{4} & \frac{1-i^2}{4} + \frac{1-i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is unitary.

Example 5

Show that the matrix $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2 = 1$.

Solution

$$A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$$

$$A = \begin{bmatrix} a+ic & b+id \\ -b+id & a-ic \end{bmatrix}$$

$$A^* = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$AA^* = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix} \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$AA^* = \begin{bmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & b^2+d^2+a^2+c^2 \end{bmatrix}$$

$$= (a^2+b^2+c^2+d^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $AA^* = I$ if and only if $a^2+b^2+c^2+d^2 = 1$

Exercise 2

1) Show that $\begin{bmatrix} \frac{1+i}{\sqrt{7}} & \frac{2+i}{\sqrt{7}} \\ \frac{2-i}{\sqrt{7}} & \frac{-1+i}{\sqrt{7}} \end{bmatrix}$ is unitary.

2) Prove that the matrix $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal.

3) Prove that the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ is orthogonal.

4) Prove that $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is unitary.

5) If A is an orthogonal matrix and $B = AC$ where C is non-singular, prove that CB^{-1} is orthogonal.

6) Show that $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1+i \\ 1+i & 0 \end{bmatrix}$ is unitary.

7) Show that $\frac{1}{5} \begin{bmatrix} -1+2i & 2-4i \\ -4-2i & -2-i \end{bmatrix}$ is unitary.

8) If A is an orthogonal matrix, show that $A^{-1}SA$ is symmetric or skew-symmetric according as S is symmetric or skew-symmetric.

Additional Illustrative Examples and Exercise

Example 1

If $A = \begin{bmatrix} 2 & -1 & 0 & 5 \\ 3 & 2 & 6 & -4 \end{bmatrix}$

$B = \begin{bmatrix} 4 & 7 & 1 & 8 \\ -2 & 3 & 6 & 5 \end{bmatrix}$

find 2A + 3B and 3A – 2B

Solution

$$A = \begin{bmatrix} 2 & -1 & 0 & 5 \\ 3 & 2 & 6 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 7 & 1 & 8 \\ -2 & 3 & 6 & 5 \end{bmatrix}$$

$$2A = \begin{bmatrix} 4 & -2 & 0 & 10 \\ 6 & 4 & 12 & -8 \end{bmatrix}$$

$$3B = \begin{bmatrix} 12 & 21 & 3 & 24 \\ -6 & 9 & 18 & 15 \end{bmatrix}$$

$$2A + 3B = \begin{bmatrix} 16 & 19 & 3 & 34 \\ 0 & 13 & 30 & 7 \end{bmatrix}$$

$$3A = \begin{bmatrix} 6 & -3 & 0 & 15 \\ 9 & 6 & 18 & -12 \end{bmatrix}$$

$$2B = \begin{bmatrix} 8 & 14 & 2 & 16 \\ -4 & 6 & 12 & 10 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} -2 & -17 & -2 & -1 \\ 13 & 0 & 6 & -22 \end{bmatrix}$$

Example 2

Find x, y, z and w if

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$$

Solution

Since the two matrices are equal, the corresponding elements are equal.

$$\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

$$\therefore\ 3x = x + 4 \quad \text{i.e.} \quad x = 2$$

$$3y = 6 + x + y$$

$$2y = 6 + x$$

$$\text{i.e. } 2y = 6 + 2 = 8 \quad \therefore\ y = 4$$

$$3w = 2w + 3 \quad \text{i.e.} \quad w = 3$$

$$3z = -1 + z + w$$

$$3z - z = -1 + w$$

$$2z = 2 \quad \text{i.e.} \quad z = 1$$

Example 3

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix},\ B = \begin{bmatrix} 4 & 1 & 2 \\ 3 & 2 & 5 \\ 1 & 2 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 3 \\ 4 & 1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$$

Solve the equation $2(x + B) = 3(x + A) + C$

Solution

$$2(x + B) = 3(x + A) + C$$

$$2x + 2B = 3x + 3A + C$$

$$2B - 3A - C = x$$

$$x = \begin{bmatrix} 8 & 2 & 4 \\ 6 & 4 & 10 \\ 2 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 9 & 3 \\ 3 & 6 & 9 \\ 9 & 9 & 15 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 3 \\ 4 & 1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 6 - 0 & 2 - 9 - 1 & 4 - 3 - 3 \\ 6 - 3 - 4 & 4 - 6 - 1 & 10 - 9 + 2 \\ 2 - 9 - 3 & 4 - 9 - 1 & 0 - 15 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -8 & -2 \\ -1 & -3 & 3 \\ -10 & -6 & -18 \end{bmatrix}$$

Example 4

Find matrices x and y of order 2 such that

$$2x - 3y = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix};\quad 3x + 2y = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix}$$

Solution

$$2x-3y = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \quad (1)$$

$$3x + 2y = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix} \quad (2)$$

(1)×2: $$4x - 6y = \begin{bmatrix} 4 & 10 \\ 6 & 2 \end{bmatrix}$$

(2)×3: $$9x + 6y = \begin{bmatrix} 21 & 3 \\ 12 & 15 \end{bmatrix}$$

Adding $$13x = \begin{bmatrix} 25 & 13 \\ 18 & 17 \end{bmatrix}$$

$$x = \frac{1}{13}\begin{bmatrix} 25 & 13 \\ 18 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25}{13} & 1 \\ \frac{18}{13} & \frac{17}{13} \end{bmatrix}$$

From (2) $$2y = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix} - 3x$$

$$= \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \frac{75}{13} & 3 \\ \frac{54}{13} & \frac{51}{13} \end{bmatrix}$$

$$= \begin{bmatrix} 7 - \frac{75}{13} & 1 - 3 \\ 4 - \frac{54}{13} & 5 - \frac{51}{13} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{13} & -2 \\ \frac{-2}{13} & \frac{14}{13} \end{bmatrix}$$

$$y = \begin{bmatrix} \frac{8}{13} & -1 \\ \frac{-1}{13} & \frac{7}{13} \end{bmatrix}$$

Example 5

Solve for x and y if $3x + 2y = I$

$$2x - y = 0$$

where I and 0 are unit matrix and null matrix of order 3

Solution

$$3x + 2y = I \quad (1)$$

$$2x - y = 0 \quad (2)$$

$$\begin{array}{lrcl} (1)\ : & 3x + 2y & = & I \\ (2) \times 2: & \underline{4x - 2y} & = & \underline{0} \\ \text{Adding} & 7x & = & I \\ & x & = & \frac{1}{7}\ I \end{array}$$

$$= \frac{1}{7}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7} & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$$

$$y = 2x = \frac{2}{7}\ I$$

$$= \begin{bmatrix} \frac{2}{7} & 0 & 0 \\ 0 & \frac{2}{7} & 0 \\ 0 & 0 & \frac{2}{7} \end{bmatrix}$$

Example 6

When $A = \begin{vmatrix} 1 & i \\ -i & 1 \end{vmatrix}$ $B = \begin{vmatrix} i & -1 \\ -1 & -i \end{vmatrix}$

and $i = \sqrt{-1}$ determine AB. Compute also BA.

Solution

$$AB = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix}$$

$$= \begin{bmatrix} i & -i & -1 & -i^2 \\ -i^2 & -1 & +i & -i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{since } i^2 = -1$$

$$BA = \begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} i + i & i^2 - 1 \\ -1 + i^2 & -i - i \end{bmatrix}$$

$$= \begin{bmatrix} 2i & -2 \\ -2 & -2i \end{bmatrix}$$

Note: $AB \neq BA$.

Example 7

Prove that the matrix A given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the relation $A^2 - A(a + d) + (ad - bc) I = 0$ where I is the unit matrix of order 2.

Solution

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} \qquad (1)$$

$$(a+d) A = (a+d) \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} \quad \cdots (2)$$

$$(ad - bc)\,I = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \quad (3)$$

$$A^2 - (a+d)\,A + (ad-bc)\,I$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc-a^2-ad+ad-bc & ab+bd-ab-bd \\ ac+cd-ac-cd & bc+d^2-ad-d^2+ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Example 8

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

Prove that $AB = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$

Solution

$$AB = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$$

Example 9

Write the products AB and BA of two matrices A and B where

$$A = [1\ 2\ 3\ 4] \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Solution

$$AB = [1\ 2\ 3\ 4]\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

$$= [1\times1 + 2\times2 + 3\times3 + 4\times4]$$

$$= [1 + 4 + 9 + 16]$$

$$= (30)$$

$$BA = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}[1\ 2\ 3\ 4]$$

$$= \begin{bmatrix}1 & 2 & 3 & 4\\2 & 4 & 6 & 8\\3 & 6 & 9 & 12\\4 & 8 & 12 & 16\end{bmatrix}$$

Example 10

Find ABC if $A = (x\,y\,z)$

$$B = \begin{bmatrix}a & h & g\\h & b & f\\g & f & c\end{bmatrix} \quad C = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

Solution

$$ABC = (AB)C$$

$$AB = [x\ y\ z]\begin{bmatrix}a & h & g\\h & b & f\\g & f & c\end{bmatrix}$$

$$= [ax+hy+zg\ \ hx+by+fz\ \ gx+fy+cz]$$

$$(AB)C = [ax+hy+gz\ \ hx+by+fz\ \ gx+fy+cz]\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$= [\,(ax+hy+zg)\,x + (hx+by+fz)\,y + (gx+fy+cz)\,z\,]$$
$$= [\,ax^2+hxy+gxz+hxy+by^2+fzy+gxz+fyz+cz^2\,]$$
$$= [\,ax^2+by^2+cz^2+2hxy+2gxz+2fyz]$$

Example 11

If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Prove that $AB \neq BA$

Solution

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3+0 & 6+6+0 & 0+3+8 \\ 1-2+0 & 3+4+0 & 0+2+6 \\ -1-1+0 & -3+2+0 & 0+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix} \qquad (1)$$

$$BA = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+0 & 3+6+0 & 4+9+0 \\ -2+2-1 & -3+4+1 & -4+6+2 \\ 0+0-2 & 0+0+2 & 0+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix} \qquad (2)$$

From (1) and (2) we see that $AB \neq BA$

Example 12

If the matrix A is given by $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, prove that it satisfies the relation $A^2 - 4A + 3I = 0$ where I stands for the unit matrix of order 2.

Solution

$$A^2 = A.A$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & -2-2 \\ -2-2 & 1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$$A^2 - 4A + 3I = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} - 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} - \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence $A^2 - 4A + 3I = 0$

Note : We can find A^{-1} from the above relation. Premultiplying A^{-1} we get, $A^{-1}.A^2 - 4A^{-1}A + 3A^{-1}I = 0$

i.e. $A - 4I + 3A^{-1} = 0$

$(\because AA^{-1} = A^{-1}A = I \; ; \; A^{-1}I = IA^{-1} = A^{-1})$

$$3A^{-1} = 4I - A$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Example 13

Show that the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ satisfies the equation

$A^3 - 6A^2 + 9A - 4I = 0$ and hence deduce A^{-1}.

Solution

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2.A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & +5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9+0 & 21-30+9+0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9+0 \\ 21-30+9+0 & -21+30-9+0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore A^3 - 6A^2 + 9A - 4I = 0$

Multiply by A^{-1}

$A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4IA^{-1} = 0$

$A^2 - 6A + 9I - 4A^{-1} = 0$

$4A^{-1} = A^2 - 6A + 9I$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example 14

There are two families A and B .There are two men, three women and one child in family A and one man, one woman and two children in family B. The recommended daily allowance for calories is men 2400 ; women 1900 ; children 1800 and for proteins men 55 gms, women 45 gms and children 33 gms.

Represent the above information by matrices.Using matrix multiplication, calculate the total requirements of calories and proteins for each of the two families

Solution

$$\text{Let } X = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & W & C \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \end{array}$$

$$\text{Let } Y = \begin{array}{c} \\ M \\ W \\ C \end{array} \begin{array}{c} \begin{array}{cc} C & P \end{array} \\ \begin{bmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix} \end{array}$$

$$XY = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix}$$

$$= \begin{bmatrix} 4800+5700+1800 & 110+135+33 \\ 2400+1900+3600 & 55+45+66 \end{bmatrix}$$

$$= \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} P & C \end{array} \\ \begin{bmatrix} 12300 & 278 \\ 7900 & 166 \end{bmatrix} \end{array}$$

Hence required calories for family A = 12300

Required calories for family B = 7900

Required proteins for family A = 278 gms

Required proteins for family B = 166 gms

Example 15

A company is considering which of the three methods of production it should use to produce three goods A, B and C. The amount of each good produced by each method is shown in the matrix

$$\begin{array}{c} \\ \text{Method 1} \\ \text{Method 2} \\ \text{Method 3} \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 4 & 8 & 2 \\ 5 & 7 & 1 \\ 5 & 3 & 9 \end{bmatrix} \end{array}$$

The vector (or row matrix) (10 4 6) represents the profit per unit for the goods A, B and C in order. Using matrix multiplication find which method maximizes the total profit

Solution

$$\text{Let } X = \begin{array}{c} \\ \text{Method 1} \\ \text{Method 2} \\ \text{Method 3} \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 4 & 8 & 2 \\ 5 & 7 & 1 \\ 5 & 3 & 9 \end{bmatrix} \end{array}$$

$$\text{Let } Y = \begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 10 \\ 4 \\ 6 \end{bmatrix} \text{ be the matrix}$$

representing profit per unit for goods A, B and C. Total profits by methods 1, 2 and 3 are given by XY.

$$XY = \begin{bmatrix} 4 & 8 & 2 \\ 5 & 7 & 1 \\ 5 & 3 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 40 + 32 + 12 \\ 50 + 28 + 6 \\ 50 + 12 + 54 \end{bmatrix} = \begin{bmatrix} 84 \\ 84 \\ 116 \end{bmatrix}$$

Method 3 maximizes the total profit.

The maximum profit is Rs 116

Example 16

If $A = \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}$ show that A satisfies the equation $(A-10I)(A-I) = 0$. Hence find A^3.

Solution

$$A - I = \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$A - 10I = \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & -2 \\ 4 & -5 & -2 \\ -2 & -2 & -8 \end{bmatrix}$$

$$(A-10I)(A-I) = \begin{bmatrix} -5 & 4 & -2 \\ 4 & -5 & -2 \\ -2 & -2 & -8 \end{bmatrix} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -20 + 16 + 4 & -20 + 16 + 4 & 10 - 8 - 2 \\ 16 - 20 + 4 & 16 - 20 + 4 & -8 + 10 - 2 \\ -8 - 8 + 16 & -8 - 8 + 16 & 4 + 4 - 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\begin{aligned}
(A - 10I)(A - I) &= 0 \\
A^2 - 10IA - AI + 10I &= 0 \\
A^2 - 11A + 10I &= 0
\end{aligned}$$

$$A^2 = 11A - 10I$$

$$\begin{aligned}
A^3 &= 11A^2 - 10A \\
&= 11(11A - 10I) - 10A \\
&= 121A - 110I - 10A \\
&= 111A - 110I
\end{aligned}$$

$$= 111 \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} - 110 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 555 & 444 & -222 \\ 444 & 555 & -222 \\ -222 & -222 & 222 \end{bmatrix} - \begin{bmatrix} 110 & 0 & 0 \\ 0 & 110 & 0 \\ 0 & 0 & 110 \end{bmatrix}$$

$$= \begin{bmatrix} 445 & 444 & -222 \\ 444 & 445 & -222 \\ -222 & -222 & 112 \end{bmatrix}$$

Example 17

If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and

$(A+B)^2 = A^2 + B^2$, *f*ind a and b.Using the values of a and b, verify whether A B = B A.

Solution:-

$$A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$

$$(A + B)^2 = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix} \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (1+a)^2 & 0 \\ 2+b+2a+ab-4-2b & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -1 & +1 \\ 2 & -2 & -2 & +1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

Given $(A+B)^2 = A^2 + B^2$

$$\begin{bmatrix} (1+a)^2 & 0 \\ 2a-b+ab-2 & 4 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

$\therefore (1+a)^2 = a^2 + b-1; \qquad a-1 = 0$

$2a-b+ab-2 = ab-b; \qquad b = 4$

$a = 1,\ b = 4$ satisfy all the above four equations.

Hence $a = 1,\ b = 4$

When $a = 1,\ b = 4,\quad B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 1+1 \\ 2-4 & 2+1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & -1-1 \\ 4-7 & -4+1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -3 \end{bmatrix}$$

$\therefore\ AB \neq BA$

EXERCISE 1

1. If $A = \begin{bmatrix} 1 & 3 & -2 & 5 \\ 3 & 1 & 2 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 7 & 1 & 3 \\ 2 & 5 & 2 & -3 \end{bmatrix}$

Find $3A+4B$ and $4A-3B$

2. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 2 & 5 \\ 4 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \\ 4 & 1 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Evaluate the following (i) $2A - 3B + 4C$

(ii) $3A - (B - 2C)$

3. Find the values of a,b,c and d that satisfy the matrix relationship.

$$\begin{bmatrix} 2a+5 & 3-2b \\ c+7 & 4a+5 \\ d-3 & 4d+3 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 2a+8 & 4b+1 \\ c-7 & 2-3a \end{bmatrix}$$

4. Find x,y,z and w if

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$$

5. Find x, y and z if

$$\begin{bmatrix} x+3 & 3x-2y \\ -3x-z & x+y+z \end{bmatrix} = \begin{bmatrix} 2 & -7+2y \\ y+4 & 2x \end{bmatrix}$$

6. Solve the equation for x

$$\begin{bmatrix} 4 & -2 \\ 7 & 1 \end{bmatrix} + 3x = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$$

7. Solve for x and y if

$$2x + 3y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, 3x - 2y = \begin{bmatrix} 4 & -7 \\ 2 & 5 \end{bmatrix}$$

8. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$

Find AB and BA. Is AB = BA ?

9. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$ prove that

$AB \neq BA$

10. A and B are two matrices defined by

$A = \begin{bmatrix} 2 & -\frac{1}{2} & -1 & 3 \end{bmatrix}$ and

$B = \begin{bmatrix} 4 \\ 0 \\ 2 \\ -1 \end{bmatrix}$ Obtain the products AB and BA

Find, if possible, the sum of the two matrices AB and BA

11. If $A = \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix}$

verify that (i) $A(BC) = (AB)C$

(ii) $(A+B)C = AC + BC$

12. If $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & -4 \end{bmatrix}$

Show that $AB = BA = -5I$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

show that $(aA - bB)(aA + bB) = (a^2 + b^2)A$

13. If $u = [2 \ -3 \ 4]$, $x = [0 \ 2 \ 3]$

$$v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

Find (i) $uv + xy$

(ii) $4uv + 5[x(3v - y)]$

14. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 2 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

find $2(AB - BA)$

15. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -6 & 2 \\ 2 & 4 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ $C = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

show that (i) $A^2 = B^2 = C^2 = I$

(ii) $AB = BA = C$

(iii) $BC = CB = A$

(iv) $AC = CA = B$

16. If $x = \begin{bmatrix} 2 & 2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, $y = \begin{bmatrix} -1 & 2 & 4 \\ 1 & -2 & -4 \\ -1 & 2 & 4 \end{bmatrix}$

show that (i) $x + y = I$

(ii) $xy - yx = 0$

(iii) $x^2 = x$ and $y^2 = y$

17. If $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ -5 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & -1 & -7 \\ -2 & 1 & 3 & 4 \\ 3 & 2 & 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 6 & 0 & -6 \\ -1 & 2 & 4 & 5 \\ 4 & 3 & 2 & 3 \end{bmatrix}$

show that $AB = AC$ although $B \neq C$

18. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix}$

show that AB and CA are null matrices but BA and AC are not null matrices

19. If $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ verify that $A^2 - 7A + I = 0$

20. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ show that

$A^3 - 23A - 40I = 0$ where I is the unit matrix of order 3.

21. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ show that

$A^2 - 4A - 5I = 0$. Hence find the inverse of A.

21A. If $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ prove that

$A^3 - 13A + 12I = 0$

Hence find A^{-1}

22. Show that the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ satisfies the equation $A^3 - 4A^2 - 3A + 11I = 0$ where I is the unit matrix of order 3.

23. Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 2A - 5I = 0$. Hence find A^{-1}

24. If $A = \begin{bmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ show that it satisfies the equation

$$(A-2I)(A-4I) = 0$$

25. If $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ show that

$$A(A-I)(A+2I) = 0$$

26. A man buys 8 dozens of mangoes, 10 dozens of apples and 4 dozens of bananas. Mangoes cost Rs 18 per dozen, apples Rs 9 per dozen and bananas Rs 6 per dozen. Represent the quantities bought by a row matrix and the prices by a column matrix and hence obtain the total cost.

27. An engineering product requires five kinds of materials, the quantities of which are given in the form of the vector (i-e) (a row matrix) q = [17 4 36 18 6]. If p = [5 200 4 50 3] represents a vector of the corresponding prices per unit in Rs find the total expenses for the manufacture of the product.

28. A finance company has offices located in every division, every district and every taluka in a state in India. Assume that there are 5 divisions, 30 districts and 200 talukas in the state. Each office has 1 head clerk, 1 cashier, 1 clerk and 1 peon. A divisional office has, in addition, an office superintendent, 2 clerks, 1 typist and 1 peon. A district office has, in addition, 1 clerk and 1 peon. The basic monthly salaries are as follows. Office superintendent Rs 500, Head Clerk Rs 200, Cashier Rs 175, Clerks and Typists Rs 150 and peons Rs 100.

Using matrix notation, find (i) the total no. of posts of each kind in all the offices taken together, (ii) the total basic monthly salary bill of each kind of office and (iii) the total basic monthly salary of all the offices taken together.

Simultaneous Linear Equations

The concepts and operations in matrix algebra are extremely useful in solving the simultaneous linear equations. We now see some of the methods of obtaining solution to simultaneous linear equation using matrices.

Consider the linear equations:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Denote $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $E = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Now $AX = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = E$$

$$\therefore \quad AX = E$$
$$A^{-1}(AX) = A^{-1}E$$
$$(A^{-1}A)X = A^{-1}E$$
$$IX = A^{-1}E$$
$$X = A^{-1}E$$

Hence, to solve linear equations write down the coefficient matrix A and find its inverse A^{-1}. Then find $A^{-1}E$. This gives the vector X which is the solution for the given linear equations.

General properties of matrices

(1) If A and B are the matrices of the same order, $A+B = B+A$ (commutative law for addition)

(2) If A, B and C are matrices of the same order $A + (B+C) = (A+B) + C$ (Associate law for addition)

(3) If A and B are matrices of the same order and k is a scalar, then $k(A+B) = kA + kB$

(4) $A + 0 = A$

(5) $A + (-A) = 0$

(6) $A + B = A + C$ implies $B = C$

$B + A = C + A$ implies $B = C$

(7) If A, B, C are of order $m \times n$, $n \times p$, $p \times q$ respectively, then $A(BC) = (AB)C$ (Associative law for multiplication)

(8) If A, B, C are of order $m \times n$, $n \times p$, $p \times q$ respectively, then $A(B+C) = AB + AC$ (Distributive law)

(9) $AI = IA = 0$

(10) $A0 = 0$

(11) If n is a positive integer, then we write

$A.A.A.\ldots$ n terms $= A^n$

In particular $A.A = A^2$

(12) $A(\text{adj } A) = (\text{adj } A)A = (\det A)I$

$\text{Adj}(AB) = (\text{adj } B).(\text{adj } A)$

$\det(\text{adj } A) = (\det A)^{-1}$

(13) $(AB)^{-1} = B^{-1}.A^{-1}$

Example 18

If $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \\ 4 & 6 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 7 \\ -2 & 3 & 8 \\ 6 & -3 & 4 \end{bmatrix}$

Show that $(A+B)^T = A^T + B^T$

Solution

$$A + B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \\ 4 & 6 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ -2 & 3 & 8 \\ 6 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 7 & 11 \\ 3 & 5 & 9 \\ 10 & 3 & -1 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 3 & 3 & 10 \\ 7 & 5 & 3 \\ 11 & 9 & -1 \end{bmatrix} \qquad (1)$$

$$A^T + B^T = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -5 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 6 \\ 4 & 3 & -3 \\ 7 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 10 \\ 7 & 5 & 3 \\ 11 & 9 & -1 \end{bmatrix} \qquad (2)$$

From (1) and (2), $(A+B)^T = A^T + B^T$

Example 19

If $A = \begin{bmatrix} 2 & 2 & 5 \\ 5 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 7 \\ 2 & 5 \\ 3 & -2 \end{bmatrix}$

Verify that $(AB)^T = B^T . A^T$

Solution

$$AB = \begin{bmatrix} 2 & 2 & 5 \\ 5 & 3 & -1 \end{bmatrix} \times \begin{bmatrix} 4 & 7 \\ 2 & 5 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 + 4 + 15 & 14 + 10 - 10 \\ 20 + 6 - 3 & 35 + 15 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 14 \\ 23 & 52 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 27 & 23 \\ 14 & 52 \end{bmatrix} \qquad (1)$$

$$B^T . A^T = \begin{bmatrix} 4 & 2 & 3 \\ 7 & 5 & -2 \end{bmatrix} x \begin{bmatrix} 2 & 5 \\ 2 & 3 \\ 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 23 \\ 14 & 52 \end{bmatrix} \qquad (2)$$

From (1) and (2) $\quad (AB)^T = B^T . A^T.$

Example 20

Find the adjoint of $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 2 & 5 \\ 4 & 1 & 0 \end{bmatrix}$

Solution

$$A_{11} = \begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = 0-5 = -5$$

$$A_{12} = -\begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} = -(0-20) = 20$$

$$A_{13} = \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = 2-8 = -6$$

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{22} = \begin{vmatrix} 3 & 2 \\ 4 & 0 \end{vmatrix} = 0-8 = -8$$

$$A_{23} = -\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = -(3-4) = 1$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5-4 = 1$$

$$A_{32} = -\begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix} = -(15-4) = -11$$

$$A_{33} = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 6-2 = 4$$

$$\therefore \text{ Adjoint A } = \begin{bmatrix} -5 & 2 & 1 \\ 20 & -8 & -11 \\ -6 & 1 & 4 \end{bmatrix}$$

Example 21

For the square matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

prove that A (adj A) = |A| I

Solution

$$A_{11} = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 6 - 3 = 3$$

$$A_{12} = -\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = -(3 + 6) = -9$$

$$A_{13} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -(3 + 1) = -4$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = (3 - 2) = 1$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -(-1 - 2) = 3$$

$$A_{31} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = (-3 - 2) = -5$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -(-3 - 1) = 4$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (2 - 1) = 1$$

$$\text{Adjoint } A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 1(3) + 1(-9) + 1(-5) = -11$$

$$A(\text{adj } A) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-9-5 & -4+1+3 & -5+4+1 \\ 3-18\ 15 & -4+2-9 & -5+8-3 \\ 6+9-15 & -8-1+9 & -10-4+3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} = (-11) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{A I}$$

Example 22

Find the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

Solution

$$A_{11} = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = (-4-1) = -5$$

$$A_{12} = -\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -(-6-1) = 7$$

$$A_{13} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = (3-2) = 1$$

$$A_{21} = -\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = -(-6-4) = 10$$

$$A_{22} = \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = -4-4 = -8$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2-3) = 1$$

$$A_{31} = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = (3-8) = -5$$

$$A_{32} = -\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = -(2-12) = 10$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = (4-9) = -5$$

$$\text{adj A} = \begin{bmatrix} -5 & 10 & -5 \\ 7 & -8 & 10 \\ 1 & 1 & -5 \end{bmatrix}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 2(-5) + 3(7) + 4(1)$$

$$= 15$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A \qquad \frac{1}{15}\begin{bmatrix} -5 & 10 & -5 \\ 7 & -8 & 10 \\ 1 & 1 & -5 \end{bmatrix}$$

Example 23

If $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ and $AB = \begin{bmatrix} 7 & 3 \\ 15 & 5 \end{bmatrix}$ find B

Solution

$$AB = \begin{bmatrix} 7 & 3 \\ 15 & 5 \end{bmatrix}$$

$$A^{-1}(AB) = A^{-1}\begin{bmatrix} 7 & 3 \\ 15 & 5 \end{bmatrix}$$

$$\therefore\ B = A^{-1}\begin{bmatrix} 7 & 3 \\ 15 & 5 \end{bmatrix}$$

Now we have to find A^{-1}

$$|A| = 6-4 = 2$$

$$A_{11} = 3; \quad A_{12} = -4; \quad A_{21} = -1; \quad A_{22} = 2$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A$$

$$= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$

$$B = A^{-1}\begin{bmatrix} 7 & 3 \\ 15 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 15 & 5 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 21-15 & 9-5 \\ -28+30 & -12+10 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 4 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$$

Example 24

A square matrix A is called an orthogonal matrix, if the transpose of A is equal to A^{-1} (ie the inverse of A). Using this definition verify that the following square matrix is orthogonal.

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{-1}{3} \end{bmatrix}$$

Solution

Denote the given matrix by A

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$A_{11} = \frac{1}{9}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = \frac{1}{9}(-1+4) = \frac{1}{3}$$

$$A_{12} = -\frac{1}{9}\begin{vmatrix} 2 & -2 \\ -2 & -1 \end{vmatrix} = \frac{1}{-9}(-2-4) = \frac{2}{3}$$

$$A_{13} = \frac{1}{9}\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} = \frac{1}{9}(4+2) = \frac{2}{3}$$

$$A_{21} = -\frac{1}{9}\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} = \frac{1}{-9}(-2-4) = \frac{2}{3}$$

$$A_{22} = \frac{1}{9}\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = \frac{1}{9}(-1+4) \quad \frac{1}{3}$$

$$A_{23} = -\frac{1}{9}\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} = -\frac{1}{9}(2+4) = -\frac{2}{3}$$

$$A_{31} = -\frac{1}{9}\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} = \frac{1}{9}(-4-2) = \frac{2}{-3}$$

$$A_{32} = -\frac{1}{9}\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -\frac{1}{9}(-2-4) = \frac{2}{3}$$

$$A_{33} = \frac{1}{9}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = \frac{1}{9}(1-4) = \frac{1}{-3}$$

$$\begin{aligned} |A| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= \frac{1}{3}\left[\frac{1}{3}\right] + \frac{2}{3}\left[\frac{2}{3}\right] + \frac{2}{3}\left[\frac{2}{3}\right] \\ &= 1 \end{aligned}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} \end{bmatrix} \qquad (1)$$

$$A^{t} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix} \qquad (2)$$

From (1) and (2), $A^{-1} = A^{t}$

Example 25

Two square matrices of order 3 are given below.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Verify that one is the inverse of the other. (ICWAI Dec 1978)

Solution

In order to show that A and B are inverse of each other we have to prove that AB = I or BA = I

$$AB = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+\frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+0+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2}+0 \\ 0-\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+0+\frac{1}{2} & \frac{1}{2}-\frac{1}{2}+0 \\ 0+\frac{1}{2}-\frac{1}{2} & \frac{1}{2}+0-\frac{1}{2} & \frac{1}{2}+\frac{1}{2}-0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ A and B are inverse to each other

Example 26

If $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Show that A is an orthogonal.

Solution

For an orthogonal matrix $A \,.\, A^T = A^T.A = I$

$$A.A^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^T.A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+1 & -1+1 \\ -1+1 & 1+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since $A^T.A = A.A^T = I$, A is orthongal.

Example 27

If the matrix A is givenby

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}$$ obtain a matrix B such that

$AB = BA = I$

Solution

Since $AB = BA = I, A^{-1} = B$

$$A_{11} = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = 1-6 = -5$$

$$A_{12} = -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2-9) = 11$$

$$A_{13} = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4+3 = 7$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -(-1-2) = 3$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1-3 = -4$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -(2-3) = 1$$

$$A_{31} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 3+1 = 4$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -(3-2) = -1$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1-2 = -3$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 1(-5) + 1(11) + 1(7)$$
$$= 13$$

$$B = A^{-1} = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$$

Example 28

Find the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution

$A_{11} = d$; $A_{12} = -c$

$A_{21} = -b$; $A_{22} = a$

$A = ad - bc$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note: Inverse will exist only if $ad - bc \neq 0$

Example 29

Given two matrices A and B where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

Verify that $AB = BA = 6I$. Using this result solve the set of linear equations.

$$x - y = 3 \; ; \quad 2x + 3y + 4z = 17 \; ; \quad y + 2z = 7$$

Solution

$$A \cdot B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$\therefore$ AB $= 6$ I. Hence $A^{-1} = \frac{1}{6}B$

The solution of the given set of equations is givenby

$$x = A^{-1}E \qquad \text{where } E = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$x = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \times \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & +34 & -28 \\ -12 & +34 & -28 \\ 6 & -17 & +35 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$\therefore x = 2, \quad y = -1, \quad z = 4.$

Example 30

Using matrices solve the equations

$x + y + 2z = 4$

$2x - y + 3z = 9$

$3x - y - z = 2$

Solution

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix} \text{ and } x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} -1 & 3 \\ -1 & -1 \end{vmatrix} = 1 + 3 = 4$$

$$A_{12} = -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2-9) = 11$$

$$A_{13} = \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} = -2 + 3 = 1$$

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = -(-1 + 2) = -1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1-6 = -7$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -(-1-3) = 4$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3 + 2 = 5$$

$$A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -(3-4) = 1$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1-2 = -3$$

$$\text{Adj } A = \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

$$\begin{aligned} |A| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= 1 \times 4 + 1 \times 11 + 2 \times 1 \\ &= 4 + 11 + 2 = 17 \end{aligned}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{17}\begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

$$X = A^{-1}E$$

$$= \frac{1}{17}\begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 16 & -9 & +10 \\ 44 & -63 & +2 \\ 4 & +36 & -6 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 17 \\ -17 \\ 34 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore \quad x = 1, \; y = -1, \; z = 2$

Example 31

Solve by matrix method

$2x + 4y + z = 5 \;;\; x + y + z = 6 \;;\; 2x + 3y + z = 6$

Solution

$$\text{Let } A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \quad B = \begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix}$$

$$X = A^{-1}B$$

$$A_{11} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = 1 \;\; -3 = -2$$

$$A_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1-2) = 1$$

$$A_{13} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3-2 \quad = 1$$

$$A_{21} = \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = -(4-3) \quad = -1$$

$$A_{22} = \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 2-2 \quad = 0$$

$$A_{23} = \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = -(6-8) \quad = 2$$

$$A_{31} = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} = 4-1 \quad = 3$$

$$A_{32} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (2-1) \quad = -1$$

$$A_{33} = \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = 2-4 \quad = -2$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 2(-2) + 4(1) + 1(1)$$

$$= -4 + 4 + 1 \quad = 1$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 0 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 0 & -1 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -10 - 6 + 18 \\ 5 + 0 - 6 \\ 5 + 12 - 12 \end{bmatrix} \quad = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

$x = 2,\ y = -1,\ z = 5.$

Method of Reduction

Given a system of linear equations, certain simple operations can be performed to convert the given set of linear equations into another set of linear equations such that both these sets of equations have the same solution. A second set of equations can be obtained by performing the following operations:

1. Two equations are interchanged.

2. An equation is multiplied by a non–zero constant.

3. An equation is multiplied by a constant and added to any other equation.

In the same way elementary row operations can be performed on matrices. These elementary row operations are :–

1. Interchanging of two rows,

2. Multiplication of a row by any non–zero scalar and

3. Replacement of i^{th} row by the sums of the i^{th} row and k times the j^{th} row.

Similarly we can define column operations.

Elementary row operations or column operations can be used to obtain the inverse of a matrix. This method of finding the inverese is the method of reduction. The following result is proved in matrix algebra.

<u>Theorem:</u> If a non–singular matix A is converted into identity matrix by a series of row operations or column operations, then the same series of operations performed on the identity matrix is converted to A^{-1}.

<u>Method of converting a 3 × 3 non–singular matrix into Identity matrix:</u>

STEP I : Divide the first row of the matrix by the element in the first row and first column to make it unity. Then make the other elements in the first column zero.

STEP II : Make the element in the second row and second column unity. Then make other elements in the second column zero.

STEP III: Make the element in the third row and third column unity. Then make other elements in that column zero.

To find the inverse of a matrix X.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

consider a table of the form A / I

i.e. $$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

and convert it into the form [I / B]

i.e. $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right]$$

by row operations or column operations .

Then B is the inverse of A i.e. $B = A^{-1}$.

NOTE : The matrix A / I is called Augmented matrix.

The method of reduction is also used to solve linear equations.

To solve the equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

take $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The Augmented matrix is [A / B] . By row operations we have to convert this matrix into the form [I / C] where C is a column vector.

C vector determines the solution of the linear equations.

Example 32

Find the inverse of the matrix by the method of reduction

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -8 & -8 & -3 & 1 & 0 \\ 0 & -6 & -11 & -4 & 0 & 1 \end{array}\right]\begin{matrix} \\ R2-3R1 \\ R3-4R1 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{3}{8} & \frac{-1}{8} & 0 \\ 0 & -6 & -11 & -4 & 0 & 1 \end{array}\right]\begin{matrix} \\ -\frac{1}{8}R2 \\ \\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{3}{8} & \frac{-1}{8} & 0 \\ 0 & 0 & -5 & \frac{-7}{4} & \frac{-3}{4} & 1 \end{array}\right]\begin{matrix} R1-2R2 \\ \\ R3+6R2 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{3}{8} & \frac{-1}{8} & 0 \\ 0 & 0 & 1 & \frac{7}{20} & \frac{3}{20} & \frac{-1}{5} \end{array}\right]\begin{matrix} \\ \\ -\frac{1}{5}R3 \end{matrix}$$

$$\therefore A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{10} & \frac{1}{10} & \frac{1}{5} \\ 0 & 1 & 0 & \frac{1}{40} & \frac{-11}{40} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{7}{20} & \frac{3}{20} & \frac{-1}{5} \end{array}\right] \begin{array}{l} R1-R3 \\ R2-R3 \\ \\ \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{-1}{10} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{40} & \frac{-11}{40} & \frac{1}{5} \\ \frac{7}{20} & \frac{3}{20} & \frac{-1}{5} \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

Example 33

Solve the following equations by using matrices (augmentating the matrix of the co-efficients)and the method of reduction .

$x + y + z = 6; x - y + z = 2; 2x + y - z = 1$

Solution

The Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & -1 & -3 & -11 \end{array}\right] \begin{array}{l} \\ R2-R1 \\ R3-2R1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & -3 & -11 \end{array}\right] -\frac{1}{2} R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -9 \end{array}\right] \begin{array}{l} R1 - R2 \\ \\ R3 + R2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] -\frac{1}{3} R3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] R1 - R3$$

$\therefore$ The Solution vector is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

i.e. $x = 1, \quad y = 2, \quad z = 3.$

EXERCISE 2

1. Given that $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & -1 & 4 \end{bmatrix}$, find $A A^T$ and $A^T A$

2. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Verify that (i) $(A+B)^T = A^T + B^T$

(ii) $(AB)^T = B^T \; A^T$

3. Given $A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & 3 \end{bmatrix}$

Verify that $(AB)^T = B^T \; A^T$

4. Find the adjoints of the following matrices

(i) $\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 1 & 2 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$,

Verify that $A . (Adj.A) = (Adj\ A)\ A = |A|\ I$, where I is the unit matrix of order 3

6. Show that the adjoint of $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is A itself.

7. Let the matrix B be given by $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ Find the adjoint of B and verify that $\det(Adj.\ B) = (\det.\ B)^2$.

8. Find the Inverse of the following matrices

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

9. Find the inverse of $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and hence verify that

$$A A^{-1} = A^{-1} A = I$$

10. Find the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence obtain A^{-1}. Write down the adjoint of A^{-1} and now verify that $(adj.\ A)^{-1} = Adj.\ A^{-1}$

11. Two square matrices of order 3 are given below:

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Verify that one is the inverse of the other.

12. If $A = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$, find $(A^2)^{-1}$

13. Show that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal.

14. Show that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ is orthogonal.

15A. Solve the following matrix equations;

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$

(ii) $X \begin{bmatrix} 3 & 5 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 9 & 18 \end{bmatrix}$

15B. Let the matrix A be given by $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

Evaluate the polynomial $2A^2 + 3A + 5I$.(I is the unit matrix of order 2). Obtain a matrix B such that AB = BA = I. Using this matrix B, solve for x and y from the following equations

$2x + y = 15 \,;\, 3x + 4y = 23$

16. Solve the following equations, $2x-3y=3$ and $4x-y=11$ using matrices.

17. Using matrices, solve the equations

(i) $x+y+z=6, x+2y+3z=14, -x+y-z=-2.$

(ii) $x+y+2z=4, 2x-y+3z=9, 3x-y-z=2.$

(iii) $2x+4y+z=5, x+y+z=6, 2x+3y+z=6$

(iv) $2x+3y-z=9, x+y+z=9, 3x-y-z=-1$

(v) $3x+4y+5z=18, 2x-y+8z=13, 5x-2y+7z=20$

(vi) $x+y+z=6, x-y+z=2, 2x+y-z=1$

(vii) $3x+y+z=0, 5x+3y-z=4, (x/2)-6y+z=-5$

(viii) $2x-y+3z=9, x+3y-z=4, 3x+2y+z=10$

18. Find the inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ and hence solve

the equations $x+y+z=6, x+2y+3z=14, x+4y+9z=36$

19. Obtain the inverse of the matrix $\begin{bmatrix} 2 & 4 & -1 \\ 3 & 1 & 2 \\ 1 & 3 & -3 \end{bmatrix}$ and hence

solve the system of equations

$$2x + 4y - z = 9$$
$$3x + y + 2z = 7$$
$$x + 3y - 3z = 4$$

20. Solve the following equations by finding the inverse of the coefficient matrix.

$x + 2y - z = 2,\ 3x - 4y + 2z = 1,\quad -x + 3y - z = 4$

21. If $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ prove that $A^3 - 13A + 12I = 0$

Hence find A^{-1}

Hence or otherwise find the inverse of A

Rank of a Matrix

Submatrix of a Matrix

Let A be any matrix. Then a matrix obtained from A by leaving some rows and columns from A is called a sub matrix.

Minor of a Matrix

We know that every square matrix has a determinant. If A is $m \times n$ matrix then the determinant of every square sub matrix of A is called a *minor* of matrix A.

Consider the matrix $\begin{bmatrix} 1 & 3 & 5 & 4 \\ 6 & 1 & 2 & 7 \\ 7 & 5 & 4 & 3 \end{bmatrix}$

This is a matrix of type 3×4.

$\begin{bmatrix} 1 & 3 & 5 \\ 6 & 1 & 2 \\ 7 & 5 & 4 \end{bmatrix}$ is a square sub matrix of order 3.

$\begin{bmatrix} 2 & 7 \\ 3 & 3 \end{bmatrix}$ is a square sub matrix of order 2.

Rank of a Matrix

The number r is said to be the rank of the matrix A if

(i) A possesses atleast minor of order r which does not vanish.

(ii) Every minor of A of order $(r + 1)$ and higher orders vanish.

In other words the rank of a matrix is the order of any highest order non vanishing minor of the matrix. The rank of a matrix of A is denoted by $\rho(\text{A})$.

Note 1: If I is the unit matrix of order n, its rank is n.

Note 2: If A is any matrix of order $m \times n$ its rank is $\leq m$ and $\leq n$.

Note 3: The rank of A is the same as the rank of the transpose of A.

Note 4: The rank of a null matrix is taken as zero.

Example (i)

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix}$$

Here the highest order minor is of order 3. But $|\text{A}| = 0$

$\therefore \rho(\text{A}) \neq 3$

But the 2nd order minor $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \neq 0$. $\therefore$ The rank of A is 2.

Example (ii)

$$\text{Consider } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Here $|A| = 0$

Also every minor of order 2 vanish. But there exists non-zero minors of order 1.

$\therefore \rho(A) = 1$

Theorem 1:

The rank of the transpose of a matrix is the same as that of the original matrix.

Proof:

Let $A = [a_{ij}]_{m \times n}$, be a matrix of the type $m \times n$ and $A' = [b_{ji}]_{n \times m}$ where $b_{ji} = a_{ij}$. Suppose rank of A is r. Then there is atleast one square sub-matrix of A of order r whose determinant is not equal to zero. Let R be a square sub-matrix of order r such that $|R| \neq 0$. If R' is the transpose of the matrix R, then obviously R' is a sub-matrix of A'.

Since the value of a determinant does not change by interchanging the rows and columns, $|R'| = |R| \neq 0$.

Hence rank $A' \leq r$. (1)

On the other hand, if A' contains a square sub-matrix S of order $(r + 1)$, then corresponding to S, S' is a sub-matrix of A of order $(r + 1)$. But the rank of A is r. Therefore $|S| = |S'| = 0$.

Thus A' cannot contain an $(r + 1)$ rowed square sub-matrix with non-zero determinant. Hence rank $A' \neq r+1$ (2)

Now rank $A' \geq r$ and rank $A' \leq r$ implies rank $A' = r$.

Theorem 2

A is a non-zero column matrix and B a non-zero row matrix, show that rank $(AB) = 1$.

Proof

$$\text{Let } A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \cdots \\ a_{m1} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \end{bmatrix}$$

be two non-zero column and row matrices respectively.

$$\text{We have } AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & \cdots & a_{11}b_{1n} \\ a_{21}b_{21} & a_{21}b_{12} & a_{21}b_{13} & \cdots & a_{21}b_{1n} \\ \vdots & & & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & a_{m1}b_{13} & & a_{m1}b_{1n} \end{bmatrix}$$

Since A and B are non-zero matrices, the matrix AB will also be non-zero. The matrix AB will have atleast one non zero element obtained by multiplying corresponding non-zero elements of A and B. All the two-rowed minors of AB obviously vanish. But AB is a non-zero matrix. Hence rank A = 1.

Elementary transformation

In some cases the evaluation of determinant of matrices of higher order may not be simple. In such cases we can obtain easily the rank of a matrix using elementary transformations. An elementary transformation on a matrix is an operation of any one of the following three types.

(i) The interchange of any two rows (or columns).

(ii) The multiplication of the elements of any row (or column) by a non-zero scalar.

(iii) The addition to the elements of any row (or column) the scalar multiples of the corresponding elements of any other row (or column).

Equivalent matrices

Definition:

Two matrices A and B of the same order are said to be equivalent if one can be obtained from the other by the application of a finite chain of elementary transformations. Symbolically we write A ~ B and read it as A equivalent to B.

We will now establish a theorem on invariance of rank under elementary transformation.

Theorem:

Elementary transformations do not change the rank of a matrix.

Proof:

Let $A = [a_{ij}]$ be an $m \times n$ matrix of rank r. We shall prove the theorem in three stages.

Case 1

Interchange of a pair of rows does not change the rank.

Let B the matrix obtained from the matrix A by interchanging the p^{th} and q^{th} rows and let s be the rank of the matrix B.

Let B_1 be any (r + 1) - rowed square sub-matrix of B. The (r + 1) rows of the sub-matrix B_1 of B are also the rows of some uniquely determined sub-matrix A_1 of A. The identical rows of A_1 and B_1 may occur in the same relative positions or in different relative positions. Since the interchange of two rows of a determinant changes only the sign, we have

$$|B_1| = |A_1| \text{ or } |B_1| = -|A_1|.$$

The matrix A is of rank r. Therefore every (r + 1) rowed minor of A vanishes i.e. $|A_1| = 0$. Hence $|B_1| = 0$. Thus we see that every (r + 1) - rowed minor of B also vanishes. Therefore, s (the rank of B) cannot exceed r (the rank of A).

$$\therefore s \le r.$$

Again, as A can also be obtained from B by an interchange of rows we have $r \le s$.

Hence $r = s$.

Case 2

Multiplication of the elements of a row by a non-zero number does not change the rank.

Let B be the matrix obtained from the matrix A by the multiplying pth row by k; $k \neq 0$. Let s be the rank of the matrix B.

Now if $|B_2|$ be any (r + 1) - rowed minor of B, there exists a uniquely determined minor $|A_2|$ of A such that $|B_2| = |A_2|$ (this happens when p^{th} row is retained while obtaining B_2 from B) or $|B_2| = k\,|A_2|$ (this happens when p^{th} row is retained while obtaining B_2 from B).

Since the matrix A is of rank r, every (r + 1) - rowed minor of A vanishes i.e. , $|A_2| = 0$. Hence $|B_2| = 0$. Thus we see that (r + 1) - rowed minor of B also vanishes. Therefore, the rank of B cannot exceed r (the rank of A).

$$\therefore s \le r$$

Also, A can be obtained from B by the elementary transformation of the same type by multiplying the p^{th} row, of B by $\frac{1}{k}$. Therefore, by interchanging the roles of A and B we find that

$$r \le s$$

$$r = s$$

Case 3

Addition to the elements of a row the products by any number k of the corresponding elements of any other row does not change the rank.

Let B be the matrix obtained from the matrix A by multiplying elements in the qth row by k and adding with the corresponding elements of pth row and let s be the rank of the matrix B. Let B_3 be any (r + 1) - rowed square sub-matrix of B and A_3 be the correspondingly placed sub-matrix of A.

The above transformation has changed only the p^{th} row of the matrix A. Also the value of a determinant does not change if we add to the elements of any row the corresponding elements of any other row multiplied by some number k.

Therefore, if no row of the sub-matrix A_3 is part of the p^{th} row of A, or if two rows of A_3 are parts of the p^{th} and q^{th} rows of A, then

$$|B_3| = |A_3|$$

Since the rank of A is r, $|A_3| = 0$ and consequently $|B_3| = 0$.

Again, if a row of A_3 is a part of the p^{th} row of A, but no row is a part of the q^{th} row, then

$$|B_3| = |A_3| + k|C_3|$$

where C_3 is an (r + 1) - rowed square matrix which can be obtained from A_3 by replacing the elements of A_3 in the row which corresponds to the p^{th} row of A by the corresponding elements in the q^{th} row of A. Obviously all the (r + 1) rows of the matrix C_3 are exactly the same as the rows of some (r + 1) - rowed square sub-matrix of A, though arranged in some different order. Therefore $|C_3|$ is ± 1 times some (r + 1) - rowed minor of A. Since the rank of A is r, every (r +1) rowed minor of A is zero, so that $|A_3| = 0$, $|C_3| = 0$, and consequently $|B_3| = 0$.,

Thus we see that every (r + 1) - rowed minor of B also vanishes. Hence, s (the rank of B) cannot exceed r (the rank of A).

$$\therefore s \le r$$

Also, since A can be obtained from B by an elementary transformation of the type-multiply the q^{th} row by k and subtract from the p^{th} row. interchanging the roles of A and B, we have $r \le s$.

$$\text{Thus } r = s$$

We have thus shown that rank of a matrix remains unaltered by any elementary row transformation.

Similarly we can show that the rank of a matrix remains unaltered by elementary column transformation. Hence elementary transformations do not change the rank of a matrix.

Example 1

Find the rank of the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \\ -3 & 1 & -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 + 2R_1 \\ R_3 + R_1 \end{matrix}$$

Since all minors of order 3 and order 2 vanish $\rho(A) \neq 3$ and $\rho(A) \neq 2$. There exists non-zero minors of order 1.

$$\therefore \rho(A) = 1$$

Example 2

Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$\therefore \rho(A) = 1$$

Example 3

Show that the points (x_1, y_1) (x_2, y_2) and (x_3, y_3) are collinear if and only if the rank of the matrix is less than 3.

Solution

Condition is necessary.

Suppose the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear on the line with the equation $ax + by + c = 0$.

Then $ax_1 + by_1 + c = 0$

$$ax_2 + by_2 + c = 0$$

$$ax_3 + by_3 + c = 0$$

Eliminating a, b, c we get

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\therefore$ The rank of the given matrix is less than 3.

Case 2

Condition is sufficient

Suppose the rank of the given matrix is less than 3.

$$\text{Then } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

This means the area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is zero and hence the points are collinear.

Example 4

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix} \text{ if } a, b, c, d \text{ are all different.}$$

Solution

$$A \sim \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - R_1 \\ R_4 - R_2 \end{matrix}$$

$\therefore$ The rank of the matrix is 2.

$$\therefore \rho(A) = 2$$

Example 5

Find the rank of $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$

Solution

$$|A| = (a - b)(b - c)(c - a)$$

Case 1

Suppose a, b, c are all different.
Then $|A| \neq 0$. $\rho(A) = 3$

Case 2

Suppose two of a, b, c are equal and the 3rd is different.
i.e. say $a = b, \neq c$
Then $|A| = 0$ since 2 columns are identical.

But $\begin{vmatrix} b & c \\ b^2 & a^2 \end{vmatrix} = bc \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix}$

$bc(c - b) \neq 0$

$\therefore \rho(A) = 2$

Case 3

Suppose all a, b, c are equal
i.e. $a = b = c$.

Then $\begin{pmatrix} 1 & 1 & 1 \\ a & a & a \\ a^2 & a^2 & a^2 \end{pmatrix}$

In this case $|A| = 0$ and all minors of order 2 vanish.
$\therefore \rho(A) = 1$

Example 6

Find the rank of the matrix.

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Solution

$$A \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 6 & -1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - R_2 \\ R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - R_2 \end{matrix}$$

$\therefore \rho(A) = 2$ since all minors of order A and order 3 vanish and

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \neq 0$

Example 7

Find the rank of

$$A = \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$$

Solution

$$A \sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 0 & 33 & -11 & 22 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - 5R_1 \end{matrix}$$

$$A \sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 69 & -23 & 46 \\ 0 & 33 & -11 & 22 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - 7R_1 \\ R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 23 & 46 \\ 0 & 0 & 11 & 22 \end{bmatrix} C_1, C_2 + 3C_3, C_3 + C_4, C_4 + 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 23 & 0 \\ 0 & 0 & 11 & 0 \end{bmatrix} C_1, C_2 - 2C_1, C_3, C_4 - 2C_3$$

All 3rd order minors vanish $\therefore \rho(A) \neq 3$

But $\begin{bmatrix} 1 & 0 \\ 0 & 23 \end{bmatrix} = 23 \neq 0.$ $\qquad \therefore \rho(A) = 2$

Example 8

Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

Solution

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 - R_2 \\ R_4 - (R_2 + R_3) \end{matrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \\ R_2 \\ R_3 - R_2 \\ R_4 \end{matrix}$$

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_1 - R_3 \\ R_2 \\ R_3 \\ R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad C_1, C_2, C_3, C_4 - 2C_1$$

$$\sim \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ R_2 + R_4 \\ R_3 \\ R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ R_2 \\ R_3 \\ R_4 \end{matrix}$$

$\therefore \rho(A) \neq 4$ since $|A| = 0$

Consider the minor $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 1 \neq 0$

$\therefore \rho(A) = 3$

Example 9

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

Solution

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 7 \\ 3 & 1 & 1 & 7 \end{bmatrix} \; C_1, C_2 - C_1, C_3 - C_2, C_4 + C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 7 \\ 2 & 0 & 0 & 0 \end{bmatrix} \; \begin{matrix} R_1 \\ R_2 \\ R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; C_1, C_2 - C_3, C_3, C_4 - 7C_3.$$

All 3rd order minors vanish.

$\rho(A) \neq 3$

But $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\therefore \rho(A) = 2.$

Example 10

Find the rank of

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{bmatrix}$$ (MU March'96)

Solution:

$$A \sim \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 1 & 3 \\ 4 & 0 & 1 & 0 & 3 \\ 8 & -1 & 2 & 1 & 6 \end{bmatrix} \quad C_1 + C_2, C_2, C_3, C_4 + 2C_2\, C_5 + C_2$$

$$\sim \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & -1 & 2 & 1 & 2 \end{bmatrix} \quad \text{Divide } C_1 \text{ by 4 and } C_4 \text{ by 3}$$

$$\sim \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{bmatrix} \quad C_1 - C_5, C_2, C_3 - C_5, C_4, C_5$$

$$\sim \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_1 \\ R_2 + R_1 \\ R_3 \\ R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \\ R_2 - R_3 \\ R_3 \\ R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 - R_3 \end{matrix}$$

Rank of $A \neq 4$

Consider the minor $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq 0$

$\therefore \rho(A) = 3$

Exercise 3

Find the rank of the following matrices.

1) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ 2) $\begin{pmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{pmatrix}$ 3) $\begin{pmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \\ -3 & 1 & -2 \end{pmatrix}$

4) Show that the rank of a skew symmetric matrix cannot be 1.

5) Show that the rank of the matrix all of whose elements are 1 is 1.

Find the rank of the following matrices.

(6) $\begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{pmatrix}$ (7) $\begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix}$

(8) $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{pmatrix}$ a, b, c being real.

(9) $\begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$ (10) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{pmatrix}$

(11) $\begin{pmatrix} 1^2 & 2^3 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{pmatrix}$ (12) $\begin{pmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{pmatrix}$

(13) $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 5 & 3 & 3 & 11 \end{pmatrix}$ (14) $\begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -2 & -1 & 6 & 7 \end{pmatrix}$

(15) $\begin{pmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 4 & 2 & -1 & 1 \end{pmatrix}$

1. Test for consistency of linear equations.

Consider the system of linear equations given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This is a system of m non-homogenous equations in n unknowns $x_1, x_2, \dots x_n$.

If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}_{m \times 1}$$

where A, X, B are $m \times n$, $n \times 1$ and $m \times 1$ matrices respectively, the above equations can be written in the form of a single matrix equation $AX = B$.

Any set of values of $x_1, x_2, \dots x_n$ which simultaneously satisfy all these equations is called a solution of the system (1) - when the system of equations has one or more solutions, the equations are said to be consistent, otherwise they are said to be inconsistent.

The matrix

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n}, & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$ is called the agumented

matrix of the given system of equations.

Condition for consistency

Theorem:

The system of equations $AX = B$ is consistent i.e., possesses a solution, if and only if the coefficient matrix A and the augmented matrix [A, B] are of the same rank.

Proof:

Let $C_1, C_2, \ldots, C_n$ denote the column vectors of the matrix A. The equation $AX = B$ is then equivalent to

$$[C_1\, C_2 \ldots C_n]\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = B$$

i.e. $x_1 C_1 + x_2 C_2 + \ldots + x_n C_n = B$ ..... (2)

Let now r be the rank of the matrix A. The matrix A has then r linearly independent columns and, without loss of generality, we can suppose that the first r columns C1, C2, ... , Cr form a linearly independent set so that each of the remaining n- r columns is a linear combination of these r columns.

The combination is necessary: If the given system of equations is consistent, there must exist n scalars (numbers) k1, k2, ... kn such that

$$k_1 C_1 + k_2 C_2 + \ldots + k_n C_n = B \quad \text{..... (3)}$$

Since each of the n–r columns Cr+1, Cr+2, ... , Cn is a linear combination of first r columns $C_1, C_2, \ldots C_r$ it is obvious from (3) that B is also a linear combination of $C_1, C_2, \ldots C_r$.

Thus the maximum number of linearly independent columns of the matrix [A, B] is also r. Therefore the matrix [A, B] is also of rank r. Hence the matrices A and [A, B] are of the same rank.

The condition is sufficient

Now suppose that the matrices A and [A, B] are of the same rank r. The maximum number of linearly independent columns of the matrix [A, B] is then r. But the first r columns $C_1, C_2, \ldots C_r$ of the matrix [A, B] already form a linearly independent set. Therefore the column B should be expressed as a linear combinations of the columns $C_1, C_2, \ldots C_r$.

Thus there exist r scalars $k_1, k_2, \ldots$, kr such that

$$k_1 C_1 + k_2 C_2 + \ldots + k_r C_r = B \quad \text{..... (4)}$$

Now (4) may be written as

$$k_1 C_1 + \ldots + k_r C_r + 0.C_{r+1} + 0.C_{r+2} + \ldots + 0.C_n = B \quad \ldots (5)$$

Comparing (2) and (5), we see that

$$x_1 = k_1, \ldots x_r = k_r, x_{r+1} = 0, x_{r+2} = 0, \ldots x_n = 0,$$

constitute a solution of the equation $AX = B$

Therefore the given system of equations is consistent.

Theorem:

If A be an n-rowed non-singular matrix, X be an $n \times 1$ matrix, B be an $n \times 1$ matrix, the system of equations $AX = B$ has a unique solution;

Proof:

If A be an n-rowed non-singular matrix, the ranks of the matrices A and [A, B] are both n. Therefore the system of equations $AX = B$ is consistent i.e., possesses a solution.

Pre-multiplying both sides of $AX = B$ by $A^{-1,}$ we have

$$A^{-1}AX = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

is a solution of the equation $AX = B$.

To show that the solution is unique, let us suppose that X_1 and X_2 the two solutions of $AX = B$.

$$\text{Then} \quad AX_1 = B, AX_2 = B$$
$$\Rightarrow AX_1 = AX_2$$
$$\Rightarrow A^{-1}AX_1 = A^{-1}AX_2$$
$$\Rightarrow IX_1 = IX_2$$
$$\Rightarrow X_1 = X_2$$

Hence the solution is unique.

Working Rule for finding the solution of the equation $AX = B$.

Suppose we are given m equations and n unknowns.

Step (i) Write down the coefficient matrix A

Step (ii) Write down the augmented matrix [A, B].

Step (iii) Apply elementary transformation to find the ranks of A and [A, B] .

The following situations may arise

Case I

Rank A < Rank [A, B]

In this case equations are inconsistent

i.e. they have no solution

Case II

Rank A = Rank [A, B] = r (say)

In this case the equations are consistent i.e. they possess a solution.

If $r = n$ then the solution is unique.

If $r < n$ then there are infinite number of solutions.

Example 1

Show that the system of equations

$$3x - 4y = 2$$
$$5x + 2y = 12$$
$$-x + 3y = 1 \text{ are consistent.}$$

Solution

$$A = \begin{bmatrix} 3 & -4 \\ 5 & 2 \\ -1 & 3 \end{bmatrix}; \; B = \begin{bmatrix} 2 \\ 12 \\ 1 \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} 3 & -4 & 2 \\ 5 & 2 & 12 \\ -1 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 5 & 5 \\ 0 & 17 & 17 \\ -1 & 3 & 1 \end{bmatrix} \begin{array}{c} R_1 + 3R_3 \\ R_2 + 5R_3 \\ R_3 \end{array}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 17 & 17 \\ -1 & 3 & 1 \end{bmatrix} \begin{array}{c} R - \frac{5}{17} R_2 \\ R_2 \\ R_3 \end{array}$$

We note that $\rho(A) = 2$ and $\rho(A, B) = 2$

Hence the given system of equations is consistent and therefore possesses a solution.

Example 2

Show that the system of equations

$$x - 3y - 8z = -10$$
$$3x + 4y - 4z = 0$$
$$2x + 5y + 6z = 13 \text{ are consistent and solve then.}$$

Solution

$$A = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - \frac{11}{10} R_2 \end{matrix}$$

Here $\rho(A) = 2, \rho(A, B) = 2$

$\therefore$ The given system of equations is consistent.

Since $\rho(A) = 2 < 3$ there are infinite number of solutions.

$\therefore x - 3y - 8z = -10$

$10y + 20z = 30$

$y = 3 - 2z$ where z is a variable.

i.e. $x = -1 + 2z$

Take $z = k$, the solutions are

$x = -1 + 2k$

$y = 3 - 2k$

$z = k$ where k is any arbitrary value.

Example 3

Show that the equation

$$x + y + z = 6$$

$$x + 2y + 3z = 14$$

$x + 4y + 7z = 30$ are consistent and solve them.

Solution

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$$

The augmented matrix is

$$(A, B) = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - 3R_2 \end{matrix}$$

We note that $\rho(A) = 2$ and also

$$\rho(A, B) = 2$$

$$\therefore \rho(A) = \rho(A, B) = 2$$

Hence the given equations are consistent. But $\rho(A) = \rho(A, B) = 2 < 3$

$\therefore$ There are infinite number of solutions for the given equations.

Also $x + y + z = 6$

$$y + 2z = 8$$

$$y = 8 - 2z$$

$$x = 6 - (8 - 2z) - z$$

$$= z - 2$$

Taking $z = k$ where k is arbitrary the solutions are

$$x = k - 2$$

$$y = 8 - 2k$$

$$z = k$$

Example 4

Solve the equation

$$x + 2y - z = 3$$

$$3x - y + 2z = 1$$

$$2x - 2y + 3z = 2$$

$x - y + z = = -1$ are consistent and solve them.

Solution

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_3 \\ R_3 \\ R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - 6R_2 \\ R_4 - 3R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3/5 \\ R_4/2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 - R_3 \end{matrix}$$

$\therefore \rho = 3$ and $\rho(A, B) = 3$

$\therefore$ The given system of equations is consistent and has a unique solution.

$\therefore$ The solution is $z = 4$

$y = 4$

$x + 2y - z = 3$

$x = -1$

i.e. $x = -1, y = 4, z = 4.$

Example 5

Investigate for what values λ, μ the system of equations

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$ have

(i) no solution
(ii) a unique solution
(iii) an infinite number of solutions.

Solution

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$$

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 - R_2 \end{matrix}$$

Case 1

When $\lambda = 3$ and $\mu \neq 10$

$\rho(A) = 2$ and $\rho(A, B) = 3$

$\rho(A) \neq \rho(A, B)$

Hence the system of equations will be inconsistent if $\lambda = 3$ and $\mu \neq 10$

Case 2

If $\lambda \neq 3$ then $\rho(A) = \rho(A, B) = 3$

$\therefore$ The system of equations has a unique solution.

Case 3

If $\lambda = 3$ and $\mu = 10$ then

$\rho(A) = 2 = \rho(A, B)$

In this case the system of equations has an infinite number of solutions.

Example 6

Show that the equations

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

have no solutions unless $a + b + c = 0$ in which case they have infinitely many solutions. Find the solutions when $a = 1$, $b = 1$ and $c = -2$.

Solution

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & c \\ 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \end{bmatrix} \text{ Interchanging } R_1 \text{ and } R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & c \\ 0 & -3 & 3 & b-c \\ 0 & 3 & -3 & a+2c \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & c \\ 0 & -3 & 3 & b-c \\ 0 & 0 & 0 & a+b+c \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 + R_2 \end{matrix}$$

Case 1

If $a + b + c = 0$ then

$\rho(A) = 2$ and $\rho(A, B) = 2$

$\rho(A) = \rho(A, B) = 2 < 3$

$\therefore$ The equations are consistent and have an infinite number of solutions.

Take $a = 1, b = 1, c = 0 = -2$

then $x + y - 2z = -2$

and $-3y + 3z = 3$

$\therefore y = z - 1$

and $x = z - 1$

i.e. $x = k - 1$

$y = k - 1$

$z = k$

Where $z = k$ is any arbitrary value.

Example 7

Show that the system of equations

$\lambda x + y + z = 1$

$x + \lambda y + z = \lambda$

$x + y + \lambda z = \lambda^2$ has a unique solution provided $\lambda \neq -3$ or $\lambda \neq 1$ and find that solution. State also the nature of solution when $\lambda = 1$ and $\lambda = -2$.

Solution

$$A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & \lambda \\ 1 & 1 & \lambda & \lambda^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & \lambda & \lambda^2 \\ 1 & \lambda & 1 & \lambda \\ \lambda & 1 & 1 & 1 \end{bmatrix} \quad \text{Interchange } R_1 \text{ and } R_3$$

$$\sim \begin{bmatrix} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda - 1 & 1 - \lambda & \lambda - \lambda^2 \\ 0 & 1 - \lambda & 1 - \lambda^2 & 1 - \lambda^3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 - \lambda R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda - 1 & 1 - \lambda & \lambda(1 - \lambda) \\ 0 & 0 & 2 - \lambda - \lambda^2 & 1 + \lambda - \lambda^2 - \lambda^3 \end{bmatrix} \begin{matrix} R_{12} \\ R_2 \\ R_3 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda - 1 & 1 - \lambda & \lambda(1 - \lambda) \\ 0 & 0 & (\lambda - 1)(\lambda + 2) & (\lambda - 1)(\lambda + 1)^2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ -R_3 \end{matrix}$$

For unique solution $\rho(A) = \rho(A, B) = 3$

$(\lambda - 1)(\lambda + 2) = 0 \Rightarrow \lambda = 1, -2$

when $\lambda = 1, \rho(A) = 1, \rho(A, B) = 1 < 3$

$\therefore$ The given system of equation has an infinite number of solution with two arbitrary constants.

In this case $x + y + z = 1$.

Taking $x = k_1, y = k_2$ where k_1 and k_2 are arbitray,

the solutions are $x = k_1$

$y = k_2$

$z = 1 - k_1 - k_2$

When $\lambda = -2, \rho(A) = 2$ and $\rho(A, B) = 3$

$\therefore \rho(A) \neq \rho(A, B)$ and hence the equations are inconsistent.

Example 8

Find the value of a and b such that the equations

$$x + y + 2z = 2$$
$$2x - y + 3z = 2$$
$$5x - y + az = b \text{ have}$$

(i) no solution

(ii) a unique solution

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Solution

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & a \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & b \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & a-10 & b-10 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & a-8 & b-6 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - 2R_2 \end{matrix}$$

Case 1

If $a = 8$ and $b \neq 6$, $\rho(A) = 2$ and $\rho(A, B) = 3$

$\therefore \rho(A) \neq \rho(A, B)$

Hence the given equations has no solution.

Case 2

If $a \neq 8$ and b is arbitrary

$\rho(A) = 3 = \rho(A, B)$

Hence the given equations have a unique solution.

Exercise 4

Test for consistency and hence solve the equations

(1)

(a) $x - 2y + 3z = 2$

$2x - 3z = 3$

$x + y + z = 0$

(b) $x - y + z = 2$
$3x - y + 2z = -6$
$3x + y + z = -18$

(c) $x + 2y + 3z = 14$
$3x + y + 2z = 11$
$2x + 3y + z = 11$

(2) Apply the test of rank to examine if the following equations are consistent and if consistent find the complete solution.

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

(3) Show that the equations

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

are consistent and solve them.

(4) Using matrices solve the equations

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

(5) For what values of the parameter λ with the following equations fail to have a unique solution?

$$3x - y + \lambda z = 1$$
$$2x + y + z = 2$$
$$x + 2y - \lambda z = -1$$

Will the equations have any solution for these values of λ?

(6) Test for consistency and discuss the nature of the solution of the equations

$$x + y + 4z = 6$$
$$x + 2y - 2z = 6$$
$$\lambda x + y + z = 6$$

(7) Find the condition on a, b, c such that the equations

$$x + 2y - 3z = a$$
$$3x - y + 2z = b$$
$$x - 5y + 8z = c$$

have a solution.

(8) For what values of λ the system of equations

$$x + 2y = 1$$
$$5x + \lambda y = 5$$

has (i) no solution (ii) unique solution?

(9) For what values of λ and μ the system of equations

$$\begin{aligned} x + 5y + 3z &= 9 \\ x + \lambda y + z &= 3 \\ x + 2y + 2z &= \mu \end{aligned}$$

has a unique solution?

(10) Solve the equations

$$\begin{aligned} \lambda x + 2y - 2z - 1 &= 0 \\ 4x + 2\lambda y - z - 2 &= 0 \\ 6x + 6y + \lambda z - 3 &= 0 \end{aligned}$$

consider the case $\lambda = 2$.

Characteristic Equation of a matrix

Let A be a $n \times n$ square matrix over a field F and I be the unit matrix of the same order. Let I be an unknown. Then determinant $|A - \lambda I|$ is called the characteristic polynomial of the matrix A. The equation det $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A. The roots of this equation are called the characteristic roots of the matrix A. Characteristic roots are also called latent roots or eigen values.

The characteristic vectors of a matrix.

Let A be a $n \times n$ matrix. Let X be any non-zero column vector

$$\text{i.e. } X = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

Then any solution of the equation $AX = \lambda X$ other than $X = \acute{0}$ corresponding to some particular value of λ is called a characteristic vector or Eigen vector or latent vector.

The equation $AX = lX$ may be written as $(A - \lambda I)\, X = 0$. This equation gives a set of linear homogenous equations. If it is to possess a non-trivial solution, then $|A - \lambda I| = 0$. Then λ is a characteristic root of the matrix A. Hence to find all the characteristic vectors, we adopt the following procedure.

Step I

Solve $|A - \lambda I| = 0$ for λ.

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the characteristic roots of A

(9) For what values of λ and μ the system of equations

$$x + 5y + 3z = 9$$
$$x + \lambda y + z = 3$$
$$x + 2y + 2z = \mu$$

has a unique solution?

(10) Solve the equations

$$\lambda x + 2y - 2z - 1 = 0$$
$$4x + 2\lambda y - z - 2 = 0$$
$$6x + 6y + \lambda z - 3 = 0$$

consider the case $\lambda = 2$.

Characteristic Equation of a matrix

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Step I

Solve $|A - \lambda I| = 0$ for λ.

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the characteristic roots of A

Step II

Find the non-trivial solution of $|A - \lambda_1 I| = 0$. This solution is the characteristic vector corresponding to the characteristic root λ_1.

Step III

Similarly find the characteristic vectors corresponding to the other characteristric roots.

Step IV

Note that the number of linearly independent solutions of $(A - \lambda I)X = 0$ is equal to n – Rank of $|A - \lambda I|$.

Cayley – Hamilton's theorem

Theorem:

Every square matrix satisfies its own characteristic equation.

i.e. if the characteristic polynomial is

$$\phi(\lambda) = \lambda^n + P_1\lambda^{n-1} + P_2\lambda^{n-2} + \dots + P_{n-1}\lambda + P_n$$

Then $\phi(A) = 0 \Rightarrow$

$$A^n + P_1 A^{n-1} + P_2 A^{n-2} + \dots + P_{n-1} A + P_n I = 0$$

Proof:

$$\phi(\lambda) = |A - \lambda I|$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

$$= p_0\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} - \dots + p_n \text{ (say)}$$

Then we have to prove that

$$\phi(A) = p_0 A^n + p_1 A^{n-1} + p_2 A^{n-2} + \dots + p_n I = 0$$

Consider $(A - \lambda I)^*$ which is the adjacent of $A - \lambda I$. Let us call it B. The elements of B are minors of order $(n-1)$ of $A - \lambda I$. The elements of B are of degree at most $n-1$ in λ and hence can be written as

$$B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots + B_{n-1}$$

where $B_0, B_1, B_2, \dots B_{n-1}$ are matrices of order n whose elements are polynomials in the elements of A.

We know that

$$(A - \lambda I)\ \text{Adj.}\ (A - \lambda I) = |A - \lambda I|\,I$$
$$= \phi(\lambda)\, I.$$
$$(A - \lambda I)\left(B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \dots + B_n - 1\right) I$$
$$= (p_0 \lambda^n + p_1 \lambda^{n-1} + \dots + p_n)\, I$$

This is an identity in λ. Therefore equating the coefficient of like powers of λ, we obtain

$$-B_0 = p_0 I$$
$$AB_0 - B_1 = p_1 I$$
$$AB_1 - B_2 = p_2 I$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$AB_{r-1} - B_r = p_r I$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$AB_{n-1} - B = P_n I.$$

By pre-multiplying these equations with $A^n, A^{n-1} \dots A, I$ respectively and adding, we get

$$0 = p_0 A^n + p_1 A^{n-1} + p_2 A^{n-2} + \dots + p_n I$$

which prove Cayley-Hamilton's theorem.

Note: Cayley-Hamilton's theorem can be used to find the inverse of a matrix.

By Cayley-Hamilton's theorem,

$$0 = p_0 A^n + p_1 A^{n-1} + p_2 A^{n-2} + \dots + p_{n-1} A + p_n I$$

Multiplying both the sides by A^{-1}, we have

$$0 = p_0 A^{n-1} + p_1 A^{n-2} + \dots + p_{n-2} I + p_n A^{-1}$$
$$\therefore A^{-1} = -\frac{1}{p^n}\left[p_0 A^{n-1} + p_1 A^{n-2} + \dots + p_{n-1} I\right]$$

Note: Cayley-Hamilton's theorem can also be used in computation of arbitrary integral powers of A.

Example 1

Determine the characteristic roots of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Solution

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 1) - 1(-\lambda + 2) + 2(-1 + 2\lambda) = 0$$

$$-\lambda^3 + \lambda + \lambda - 2 - 2 + 4\lambda = 0$$

$$\lambda^3 - 6\lambda + 4 = 0$$

$$\lambda^3 - 4\lambda - 2\lambda + 4 = 0$$

$$\lambda(\lambda^2 - 4) - 2(\lambda - 2) = 0$$

$$(\lambda - 2)\left[\lambda(\lambda + 2) - 2\right] = 0$$

$$(\lambda - 2)(\lambda^2 + 2\lambda - 2) = 0$$

$$\lambda = 2, \frac{-2 \pm \sqrt{4 + 8}}{2}$$

$$= 2, \frac{-2 \pm 2\sqrt{3}}{2}$$

$$= 2, -1 \pm \sqrt{3}$$

$\therefore$ The characteristic roots are $2, -1 \pm \sqrt{3}$.

Example 2

Find the eigen values of $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Solution

$$\text{Let } A = \begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\text{i.e. } \begin{bmatrix} a-\lambda & h & g \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{bmatrix} = 0$$

i.e. $(a - \lambda)(b - \lambda)(c - \lambda) = 0$

$$\therefore \lambda = a, b, c$$

$\therefore$ The eigen values are a, b, c.

Example 3

Prove that the matrices A, B, C given below have the same characteristic values.

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & b & a \\ b & 0 & c \\ a & c & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & c & 0 \end{bmatrix}$$

Solution

The characteristic equation of A is $|A - \lambda I| = 0$.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & a & b \\ a & -\lambda & c \\ b & c & -\lambda \end{vmatrix} = 0$$

i.e. $-\lambda(\lambda^2 - c^2) - a(-a\lambda - bc) + b(ac + b\lambda) = 0$

$$-\lambda^3 + \lambda c^2 + a^2\lambda + abc + abc + b^2\lambda = 0$$

i.e. $\lambda^3 - \lambda(a^2 + b^2 + c^2) - 2abc = 0$ ... I

The characteritic equation of B is $|B - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & b & c \\ 0 & -\lambda & c \\ a & c & -\lambda \end{vmatrix} = 0$$

Expanding the determinant, we get

$$\lambda^3 - \lambda(a^2 + b^2 + c^2) - 2abc = 0$$

The characteristic equation of C is $|C - \lambda I| = 0$ ... II

$$\text{i.e.} \begin{vmatrix} -\lambda & c & b \\ c & -\lambda & a \\ d & a & -\lambda \end{vmatrix} = 0$$

Expanding this, we get

$$\lambda^2 - \lambda(a + b + c) - 2abc = 0 \quad \text{... III}$$

From I, II, III, we conclude that the matrices A, B, C have the same characteristic equation and hence the same characteristic values.

Example 4

Show that if A is a square matrix, the characteristic equations of A and A′ are identical.

Solution:

$$\text{Let } A = (a_{ij}) \; i, j = 1, 2, \ldots n$$

$$\text{Then } A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

$$\text{and } A' - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} - \lambda & \cdots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

The value of a determinant is unaffected by interchanging of rows into columns and columns into rows.

$$\therefore |A - \lambda I| = |A' - \lambda I|$$

i.e. The scalar equation $|A - \lambda I| = 0$ is the same as the scalar equation $|A' - \lambda I| = 0$.

Hence the characteristic equations of A and A′ are identical.

Example 5

Show that the two matrices A and $P^{-1}AP$ have the same charateristic roots.

Solution

$$\text{Let } B = P^{-1}AP$$

$$B - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}[A - \lambda I]P$$

$$\therefore B - \lambda I = |P^{-1}|\ |A - \lambda I|\ |P|$$
$$= |A - \lambda I|\ |P^{-1}|\ |P|$$
$$= |A - \lambda I|\ |P^{-1} P|$$
$$= |A - \lambda I|\ I$$
$$= |A - \lambda I|$$
$$\therefore |A - \lambda I| = 0 \Rightarrow |B - \lambda I| = 0$$

$\therefore$ The characteristic roots of A and $B = P^{-1}AP$ are the same.

Example 6

If A and B are two non-singular matrices, prove that AB and BA have the same characteristc roots.

Solution

$$AB = I\,(AB)$$
$$= (B^{-1}B)\,AB$$
$$= B^{-1}\,(BA)\,B.$$

$\therefore$ From example 5, it follows that AB and BA have the same characteristic roots.

Example 7

If the characteristic roots of A are $\lambda_1, \lambda_2, \dots \lambda_n$, show that the characteristic roots of A^2 are $\lambda_1^2, \lambda_2^2, \dots \lambda_n^2$.

Proof

Since $\lambda_1, \lambda_2 \dots \lambda_n$ are the characteristic roots of A, we have

$$|A - \lambda_r I| = 0, r = 1, 2, \dots n$$

Then $(A - \lambda_r I)(A + \lambda_r I) = A^2 - \lambda_r^2 I.$

$$\therefore |A^2 - \lambda_r^2 I| = |A - \lambda_r I|\ |A + \lambda_r I| = 0$$

$\therefore \lambda_r^2$ ($r = 1, 2, \dots n$) are the characteristic roots of A^2.

Example 8

If $\lambda_1, \lambda_2, \dots \lambda_n$ are the characteristic roots of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \dots \frac{1}{\lambda_n}$ are the characteristic roots of A^{-1}.

Solution

Since λ_r ($r = 1, 2, \dots n$) is a characteristic roots of A, we have

$$|A - \lambda_r I| = 0.$$

$$\Rightarrow \quad |A - \lambda_r A A^{-1}| = 0$$

$$\Rightarrow \left|-\lambda_r A\left(A^{-1}-\frac{1}{\lambda^r}I\right)\right| = 0$$

$$= (-\lambda_r)^n\,|A|\,\left|A^{-1}-\frac{1}{\lambda_r}I\right| = 0$$

Since A is a non-singular, $|A| \neq 0$ and $\lambda_r \neq 0$

$$\therefore \left|A^{-1}-\frac{1}{\lambda_r}I\right|_r = 0$$

$\therefore \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \cdots \frac{1}{\lambda_n}$ are the characteristic roots of A^{-1}.

Example 9

Find the characteristic roots of the orthogonal matrix $\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$ and verify that they are of unit modulus.

Solution

$$A = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

$$|A-\lambda I| = \begin{bmatrix}\cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda\end{bmatrix}$$

$$= (\cos\theta-\lambda)^2 + \sin^2\theta$$

The characteristic equation is $|A-\lambda I| = 0$

i.e. $(\cos\theta-\lambda)^2 + \sin^2\theta = 0$.

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\therefore \lambda = 2\cos\theta \pm \frac{\sqrt{4\cos^2\theta-4}}{2}$$

$$= \cos\theta + i\sin\theta$$

$\therefore$ the characteristic roots are $\cos\theta \pm i\sin\theta$

$$|\lambda| = |\cos\theta \pm i\sin\theta| = \sqrt{\cos^2\theta+\sin^2\theta} = 1.$$

Example 10

If A is a skew symmetric matrix of order n, show that $|A-\lambda I| = (-1)^n\,|A+\lambda I|$. Hence show that if μ is a characteristic roots of A, then $-\mu$ is also a characteristic root.

Solution

If A is a skew symmetric matrix, then $A = -A$.

$$\begin{aligned} |A - \lambda I| &= |A - \lambda I|' \\ &= |A' - (\lambda I)'| \\ &= |-A - \lambda I| \text{ since } I' = I \\ &= (-1)^n |A + \lambda I| \qquad \dots (1; \end{aligned}$$

If μ is a characteristic root of A, then $|A - \mu I| = 0$.

Then (1) gives $|A + \mu I| = 0$.

$-\mu$ is also a characteristic root of A.

Example 11

If X is a characteristic root of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is a characteristic root of adj. A.

Solution

$$\begin{aligned} [A - \lambda I] &= [AI - \lambda AA^{-1}] \\ &= A[I - \lambda A^{-1}] \\ &= \lambda A \left[\frac{I}{\lambda} - A^{-1}\right] \\ &= \lambda A \left[\frac{I}{\lambda} - \frac{\text{adj.}A}{|A|}\right] \\ &= -\frac{\lambda}{|A|} \cdot A \left[\text{adj.}A - \frac{|A|}{\lambda} I\right] \end{aligned}$$

$$\therefore\ |A - \lambda I| = \left(\frac{-\lambda}{|A|}\right) |A|^n \left|\text{Adj } A - \frac{|A|}{\lambda} I\right|$$

If λ is a characteristic root of A, then $|A - \lambda I| = 0$.

From (1),

$$|A - \lambda I| = 0 \Rightarrow \left|\text{adj:}A - \frac{|A|}{\lambda}\right|$$

Since $|A| \neq 0$ and $\lambda \neq 0$

$\therefore \frac{|A|}{\lambda}$ is a characteristic root of adj. A.

Example 12

The characteristic roots of a Hermilian matrix are real.

Solution

Let A be a Hermilian matrix.

Then $A^* = A$.

Let λ be a characteristic root and X be the corresponding characteristic vector.

Then $AX = \lambda X$ $\rightarrow (1)$

Pre-multiplying both sides by X^*,

$X^* AX = \lambda X^* X$ $\rightarrow (2)$

$(X^* AX)^* = (\lambda X^* X)^*$

$X^* A^* X = \bar{\lambda} X^* X$ since $\lambda^* = \bar{\lambda}$

i.e. $X^* AX = \bar{\lambda} X^* X$ (since A is Hermilian) $\rightarrow (3)$

$\therefore$ From (2) and (3), we have

$\lambda X^* X = \bar{\lambda} X^* X$

i.e. $(\lambda - \bar{\lambda}) X^* X = 0$

Since X is a characteristic root $X \neq 0$, and $X^* X \neq 0$.

$\therefore \lambda - \bar{\lambda} = 0$ or $\lambda = \bar{\lambda}$

$\therefore \lambda$ is real.

$\therefore$ the characteristic roots of a Hermilian matrix are real.

Example 13

Show that the characteristic roots of a real symmetric matrix are real.

Solution

Let A be a real symmetric matrix.

Then $\overline{A} = A$.

Also $A' = A$ since A is symmetric.

Accordingly, $A^* = (\overline{A})' = A$

$\therefore$ the matrix A is Hermilian.

$\therefore$ by the previous result, the characteristic roots of A are real.

Note 1 . If λ is a characteristic root of A, then $i\lambda$ is a characteristic root of iA which is Hermitian.

$\therefore$ by example 11, the root $i\lambda$ is real. $\therefore \lambda$ is zero or purely imaginery.

Note 2: It follows that the characteristic root of a real skew-symmetric matrix are either all zero or purely imaginary.

Example 14

The characteristic roots of an orthogonal matrix are of unit modulus.

Solution

Let A be an orthogonal matrix.

Then $AA' = A'A = I$

Let λ be a characteristic root of A and X be the corresponding charasteristic vector.

Then $AX = \lambda X$

Hence $(AX)'(AX) = (\lambda X)'\lambda X$

$$X'(A'A)X = \lambda^2 X'X.$$

$$X' IX = \lambda^2 X' X$$

$$X' X = \lambda^2 X' X$$

$$(1-\lambda^2)X' X = 0$$

Since $X \neq 0, X' \neq 0, X' X \neq 0$ and hence

$$1-\lambda^2 = 0 \quad \therefore \lambda = \pm 1$$

or $|\lambda| = 1$

Example 15

The characteristic roots of unitary matrix are of unit modulus.

Solution

Let A be a unitary matrix.

Then $A^*A = I.$ $\rightarrow (2)$

$$(AX)^* = (\lambda X)^*$$

$$X^*A^* = \bar{\lambda} X^* \quad \rightarrow (3)$$

From (2) and (3),

$$(X^*A^*)(AX) = \bar{\lambda} X^* (\lambda X)$$

$$X^*(A^*A)X = \lambda\bar{\lambda} X^*X$$

$$X^*X = \lambda\bar{\lambda} X^*X.$$

i.e. $(1-\lambda\bar{\lambda})X^*X = 0$

Since $X \neq 0, X^*X \neq 0,$

$$\therefore 1-\lambda\bar{\lambda} = 0$$

$$\therefore \lambda\bar{\lambda} = 1 \text{ or } |\lambda|^2 = 1 \quad \therefore |\lambda| = 1$$

Example 16

Show that 0 is a characteristic root of matrix if and only if the matrix is singular.

Solution

Let A be a square matrix and λ is a characteristic root. Then

$$|A - \lambda I| = 0$$

If $\lambda = 0$. then $|A| = 0$.

$\therefore$ 0 is a characteristic root of A. Therefore A is singular.

Conversely, A is singular, ie $|A| = 0$.

$$|A-\lambda I| = 0 \Rightarrow |-\lambda I| = 0 \therefore \lambda = 0$$

$\therefore$ 0 is a characteristic root.

Example 17

Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ and verify that it is satisfied by A. Also find A^{-1}.

Solution

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

i.e. $$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)^2 - 1] + 1[-2+\lambda+1] + [1-2+\lambda] = 0$$

$$(2-\lambda)(3-4\lambda+\lambda^2) + (\lambda-1) + (\lambda-1) = 0$$

$$6 - 8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 + 2\lambda - 2 = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

i.e. $$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$$

We have now to verify that $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = 0$$

Multiplying by A^{-1},

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\therefore 4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = 0$$

Multiplying by A^{-1},

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\therefore 4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example 18

Show that the matrix $\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley-Hermitian's theorm.

Solution

$$\text{Let } A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

The characteristic equation of $A - \lambda I$ is

$$|A - \lambda I| = \begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix}$$

$$-\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ac + b\lambda) = 0$$

$$-\lambda^3 - \lambda a^2 - c^2\lambda + abc - abc - b^2\lambda = 0$$

$$\lambda^3 + \lambda(a^2 + b^2 + c^2) = 0$$

We have to verify that $A^3 + (a^2 + b^2 + c^2)A = 0$

$$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -c^2-b^2 & ab & ac \\ ac & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ac & c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c(a^2+b^2+c^2) & b(a^2+b^2+c^2) \\ c(a^2+b^2+c^2) & 0 & -a(a^2+b^2+c^2) \\ -b(a^2+b^2+c^2) & a(a^2+b^2+c^2) & 0 \end{bmatrix}$$

$$= (a^2+b^2+c^2) \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)A$$

$$\therefore A^3 + (a^2+b^2+c^2)A = 0.$$

Example 19

Given $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, express $A^4 - 4A^3 - A^2 + 2A - 5I$ as a linear polynomial in A and hence eyluate it.

Solution

The characteristic equation of A is $|A - \lambda I| = 0$.

i.e. $\begin{bmatrix} 2-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = 0$

$$(2-\lambda)(3-\lambda) + 1 = 0$$

$$6 - 5\lambda + \lambda^2 + 1 = 0 \text{ or } \lambda^2 - 5\lambda + 7 = 0.$$

Consider the polynomial

$$f(\lambda) = \lambda^4 - 4\lambda^3 - \lambda^2 + 2\lambda - 5$$

Divide $f(\lambda)$ by $\lambda^2 - 5\lambda + 7$

$$(\lambda^2 - 5\lambda + 7) \,|\, \lambda^4 - 4\lambda^3 - \lambda^2 + 2\lambda - 5 \,|\, \lambda^2 + \lambda - 3$$

$$\frac{\lambda^4 - 5\lambda^3 + 7\lambda^2}{\lambda^3 - 8\lambda^2 + 2\lambda - 5}$$

$$\begin{array}{r} \underline{\lambda^3 - 5\lambda^2 + 7\lambda} \\ -3\lambda^2 - 5\lambda - 5 \\ \underline{-3\lambda^2 + 15\lambda - 21} \\ -20\lambda + 16 \end{array}$$

$$\therefore \lambda^4 - 4\lambda^3 - \lambda^2 + 2\lambda - 5 = (\lambda^2 - 5\lambda + 7)(\lambda^2 + \lambda - 3) - 20\lambda + 16.$$

$$= 0 - 20\lambda + 16$$

$$\therefore f(A) = A^4 - 4A^3 - A^2 + 2A - 5A.$$

$$= -20A + 16I$$

$$= -20\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + 16\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -24 & 20 \\ -20 & -44 \end{bmatrix}$$

Example 20

Find the characteristic equation of $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ and show that the matrix A satisfies the equation.

Solution

The characteristic equation is $|A - \lambda I| = 0$

$$\text{i.e.} \begin{bmatrix} 1-\lambda & 1 & 3 \\ 5 & 2-\lambda & 6 \\ -2 & -1 & -3-\lambda \end{bmatrix} = 0$$

$$\text{i.e. } (1-\lambda)[-(2-\lambda)(3+\lambda) + 6] - [-5(3+\lambda) + 12]$$
$$+ 3[-5 + 4 - 2\lambda] = 0$$

$$(1-\lambda)(\lambda^2 + \lambda) + 5\lambda + 3 - 6\lambda - 3 = 0$$

$$\lambda^2 + \lambda - \lambda^3 - \lambda^2 + 5\lambda + 3 - 6\lambda - 3 = 0$$

$$-\lambda^3 = 0 \text{ or } \lambda^3 = 0$$

We will show that the matrix satisfies the characteristic equation. We have to verify that $A^3 = O$.

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 = 0$

Example 21

Use Cayley-Hermitian theorem to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

Solution

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$$(3-\lambda)(2-\lambda) + 1 = 0$$

$$\lambda^2 - 5\lambda + 7 = 0$$

By Cayley-Hermitian theorem,

$$A^2 - 5\lambda + 7I = 0$$

Consider the polynomial

$$f(\lambda) = 2\lambda^5 - 3\lambda^4 + \lambda^2 - 4$$

Divide f (y) by $\lambda^2 - 5\lambda + 7$

$$\lambda^2 - 5\lambda + 7 \mid 2\lambda^5 - 3\lambda^4 + \lambda^2 - 4 \mid (2\lambda^3 + 7\lambda^2 + 21\lambda + 57)$$

$$\underline{2\lambda^5 - 10\lambda^4 + 14\lambda^3}$$

$$7\lambda^4 - 14\lambda^3 + \lambda^2 - 4$$

$$\underline{7\lambda^4 - 35\lambda^3 + 49\lambda^2}$$

$$21\lambda^3 - 48\lambda^2 - 4$$

$$\underline{21\lambda^3 - 105\lambda^2 + 147\lambda}$$

$$57\lambda^2 - 147\lambda - 4$$

$$\underline{57\lambda^2 - 285\lambda + 399}$$

$$138\lambda - 403$$

$\therefore 2\lambda^5 - 3\lambda^4 + \lambda^2 - 4 = (\lambda^2 - 5\lambda + 7)(2\lambda^3 + 7\lambda^2 + 21\lambda + 57) + 138\lambda - 403$

$\therefore 2A^5 - 3A^4 + A^2 - 4I = (A^2 - 5A + 7I)$

$$(2A^3 + 7A^2 + 21A + 57I) + 138\,A - 403\,I$$
$$= 0 + 138\,A - 403\,I$$
$$= +\,138\,A - 403\,I$$

which is a linear polynomial in A.

Example 22

Find all the characteristic vectors of the matrix $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = 0.$

Solution

The characteristic equation is

$$\begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)^2 - 4 = 0 \text{ or } \lambda^2 - 6\lambda + 5 = 0$$
$$(\lambda - 1)(\lambda - 5) = 0$$
$$\therefore \lambda = 1, 5$$

$\therefore$ The characteristic roots are 1, 5. The characteristic vector of A corresponding to the characteristic root $\lambda = 1$,

$$(A - \lambda I)X = 0$$

i.e. $$(A - I)X = 0$$

i.e. $$\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = 0$$

$$2X_1 + 2X_2 = 0$$
$$2X_1 + 2X_2 = 0$$

Since $x_1 = k$, $x_2 = -k$ satisfies the equation for all values of k.

The characteristic vector corresponding to $\lambda = 1$, is $X_1 = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

When $\lambda = 5$, we have $\begin{bmatrix} -2 & +2 \\ +2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$-2x_1 + 2x_2 = 0$$
$$+2x_1 - 2X_2 = 0$$

Here $x_1 = k, x_2 = k$ satisfies the equations for all vectors of k.

The characteristic vector corresponding to $\lambda = 5$ is

$$X_2 = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 23

Find all the characteristic roots and the associated characteristic vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

i.e. $(8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8]$
$$+ 2[24 - 2(7-\lambda)] = 0$$

$$(8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(2\lambda + 10) = 0$$

$$8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 36\lambda - 60$$
$$+ 4\lambda + 20 = 0$$

$$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0 \text{ or } \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

The characteristic roots are $\lambda = 0, 3, 15$.

The characteristic vector corresponding to $\lambda = 0$ is given by $(A - \lambda I)X = 0$

i.e. $$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$8x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 7x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0$$

Solving these equations, we get

$$\frac{x_1}{24 - 14} = \frac{x_2}{-12 + 32} = \frac{x_3}{56 - 36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = \alpha \text{ (say)}$$

$\therefore$ The characteristic vector corresponding to $\lambda = 0$ is

$$x_1 = \alpha \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

The characteristic vector corresponding to $\lambda = 3$ is

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 = 0$$

Solving we get

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\text{i.e. } \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k \ (say)$$

$\therefore$ The characteristic vector corresponding to $\lambda = 3$ is

$$x_2 = k \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

The characteristic vector corresponding to $\lambda = 15$ is given by

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{i.e. } -7x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 - 8x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 - 12x_3 = 0$$

$$\therefore \frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k \ (Say)$$

$\therefore$ The characteristic vector corresponding to

$$\lambda = 15 \text{ is } x_3 = k\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Example 24

Obtain the characteristic roots and the associate characteristic vectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution

The characteristic equation is $|A - \lambda I| = 0$

$$\text{i.e.} \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[1(2-\lambda)-1] + 1[2-(3-\lambda)] = 0$$

$$(2-\lambda)[\lambda^2 - 5\lambda + 4] - 2[1-\lambda] + (\lambda - 1) = 0$$

$$2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 2 + 2\lambda + \lambda - 1 = 0$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\text{i.e. } \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$(\lambda - 1)^2(\lambda - 5) = 0$$

$\therefore$ The characteristic roots are 1, 1, 5. The characteristic vector corresponding to $\lambda = 1$ is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{i.e. } x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

The rank of the coefficient matrix, is 1 and therefore we can assign arbitrary values to $3 - 1 = 2$ variables. The third variable can be determined in terms of these two variables. In this case, there are 2 linear independent solution which can be taken as

$$x_1 = 1, x_1 = 2$$

$$x_2 = 0 \text{ and } x_2 = -1$$

$$x_3 = -1 \; x_3 = 0$$

$\therefore$ In this case, characteristic vector to $\lambda = 1$ is the linear combination of the vectors.

$$\begin{bmatrix}1\\0\\-1\end{bmatrix} \text{ and } \begin{bmatrix}2\\-1\\0\end{bmatrix}$$

$$\text{i.e. } \alpha_1, \begin{bmatrix}1\\0\\-1\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}\alpha_1 + 2\alpha_2\\-\alpha_2\\-\alpha_2\end{bmatrix}$$

The characteristic vector corresponding to $\lambda = 5$ is given by

$$\begin{bmatrix}-3 & 2 & 1\\1 & -2 & 1\\1 & 2 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\text{i.e. } -3x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 - 3x_3 = 0$$

Solving we get,

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x}{1} = k$$

The characteristic vector corresponding to $\lambda = 5$ is

$$x_3 = k_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}k_3\\k_3\\k_3\end{bmatrix}$$

Example 25

Find all the characteristic roots and the characteristic vectors of the matrix $A = \begin{bmatrix}1 & 2 & 3\\0 & 2 & 3\\0 & 0 & 2\end{bmatrix}$

Solution

The characteristic equation is $(A - \lambda I) = 0$

$$\text{i.e. } \begin{vmatrix}1-\lambda & 2 & 3\\0 & 2-\lambda & 3\\0 & 0 & 2-\lambda\end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)^2 = 0$$

$\therefore$ The characteristic roots are 1, 2, 2. The characteristic vector corresponding to $\lambda = 1$ is

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_2 + x_3 = 0$$
$$x_2 + 3x_3 = 0$$
$$x_3 = 0$$
$$\therefore x_2 = 0 \text{ and } x_3 = 0$$

The value of the coefficient matrix is 2 and therefore there are $3 - 2 = 1$ variables can be assigned arbitrarily.

$\therefore$ Take $x_1 = \text{k}$.

The characteristic vector is $x_1 = \begin{bmatrix} \text{k} \\ 0 \\ 0 \end{bmatrix}$

The characteristic vector corresponding to $\lambda = 2$ is

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 + 3x_3 = 0$$
$$3x_3 = 0$$
$$\therefore x_3 = 0 \text{ and } x_1 = 2x_2$$

The rank of the coefficient matrix is 2 and therefore $3 - 2 = 1$ variable can be determined arbitrarily.

$$\therefore x_2 = x_3 = \begin{bmatrix} 2\text{k} \\ \text{k} \\ 0 \end{bmatrix}$$

Example 26

Show that the roots of the equation $\begin{bmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{bmatrix} = 0$ are real if a, b, c, f, g, h are real.

Solution

$$\text{Let } A = \begin{bmatrix} a & h & g \\ g & b & f \\ h & f & c \end{bmatrix}$$

This is a symmetric matrix.

$\therefore$ Its characteristic roots are real.

The characteristic roots are given by

$$\begin{bmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{bmatrix} = 0 \qquad \text{... (1)}$$

Change x into $-$ x,

$$\begin{bmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{bmatrix} = 0 \qquad \text{... (2)}$$

The characteristic roots of (2) are real since the characteristic roots of (1) are real.

Exercise – 5

1) Verify Cayely–Hermitian theorem for the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

2) Find the characteristic equation of $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ and show that the matrix A satisfied this equation. Also find A^{-1}.

3) Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hermitian theorem. Find the inverse of A and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

4) Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ satisfies its characteristic equation and find A^{-1}

5) Show that the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ satisfies Cayley-Hermitian theorem.

6) Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies its own characteristic equation. Is it true of every square matrix?

7) Verify Cayely-Hermitian theorem for the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$ Hence or otherwise find A^{-1}.

8) Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ and then find A^{-1}.

9) Find the characteristic equation of $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and use Cayley Hermitian theorem to find A^{-1}.

10) Find the eigen values and eigen vectors of the following matrices

(i) $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

11) Find the characteristic roots and the characteristic vectors of the matrix.

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

12) Find the eigen values and eigen vectors of the matrices.

(i) $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 6 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

13) Verify Cayley-Hermitian theorem for the matrix.

(i) $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

Similarity of matrices

Definition:

Let A and B be two square matrices of the same order. Then matrix B is said to be similar to the matrix A if there exists a non-singular matrix P such that $B = P^{-1}AP$.

Theorem:

Similar matrices have the same determinant.

Proof:

Let A and B be two similar matrices. Then there exists a non-singular matrix P such that

$$B = P^{-1}AP$$

$$|B| = |P^{-1}|\ |A|\ |P|$$
$$= |P^{-1}|\ |P|\ |A|$$
$$= |P^{-1}P|\ |A|$$
$$= |I|\ |A|$$
$$= |A|$$

Theorem:

Similar matrices have the same eigen values. If X is an eigen vector of A corresponding to the eigen value λ then $P^{-1}X$ is an eigen vector of B corresponding to the eigen value λ where $B = P^{-1}AP$.

Proof:

Let A and B be two similar matrices. Then there exists a non-singular matrix P such that $B = P^{-1}AP$.

$$\text{Now } B - \lambda I = P^{-1}AP - \lambda I.$$
$$= P^{-1}AP - P^{-1}\lambda IP$$
$$= P^{-1}(A - \lambda I)P$$
$$\therefore\ |B - \lambda I| = |P^{-1}|\ |A - \lambda I|\ |P|$$
$$|P^{-1}|\ |P|\ |A - \lambda I| = |I|\ |A - \lambda I| = |A - \lambda I|$$

Hence matrices A and B have the same eigen values. Suppose λ is an eigen values of A and X be the corresponding eigen vector. Then $AX = \lambda X$.

$$\text{Here } B(P^{-1}X) = (P^{-1}AP)(P^{-1}X)$$
$$= P^{-1}(AX)$$
$$= P^{-1}(\lambda X)$$
$$= \lambda(P^{-1}X)$$

$\therefore P^{-1}X$ is eigen vector corresponding to the eigen value λ.

Corollary:

If A is similar to a diagonal matrix D, then the diagonal elements of D are the eigen values of A.

Proof:

Given A and D are similar matrices where D is a diagonal matrix. If two matrices are similar, they have the same eigen values.

But the eigen values of a diagonal matrix are the diagonal elements.

$\therefore$ the eigen values of A are also the diagonal elements of D.

Diagonolizable matrix

Definition:

A matrix A is said to be diagonalizable if it is similar to a diagonal matrix. Then there exists a non singular matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. Here we say that P diagonalise A.

Here we state an important theorem (without proof) for diagonalisation of a matrix.

Theorem:

A square matrix of order n is diagonalizable if and only if it possess n linearly independent eigen values.

Note: If the eigen values of square matrix are all distinct, then it is similar to a diagonal matrix. Now we state another important theorem for the existence of a similar matrix in the diagonal form.

Example

Let A and B be two square matrices of order n. Let A be non-singular. Show that $A^{-1}B$ and BA^{-1} have the eigen values.

Solutions

A is non-singular.

$$A^{-1}(BA^{-1})A = (A^{-1}B)(A^{-1}A)$$
$$= (A^{-1}B)I$$
$$\therefore A^{-1}(BA^{-1})A = A^{-1}B.$$

$\therefore BA^{-1}$ and $A^{-1}B$ are similar matrices.

Example

Show that if A is similar to a diagonal matrix, then A' is similar to A.

Solution

Let A be similar to a diagonal matrix D. Then there exists a non-singular matrix P such that

$$P^{-1}AP = D$$
$$P(P^{-1}AP)P^{-1} = P^{-1}DP$$
$$A = P^{-1}DP.$$
$$\therefore A' = (P^{-1}DP)'$$
$$= P'D(P^{-1})' \quad \text{(since } D^{1} = D\text{)}$$
$$= P'D(P')^{-1}$$

$\therefore A$ is similar to D. i.e. D is similar to A. Hence A is similar to D and D is similar to A'. This implies A is similar to A'.

Example 27

Show that the matrix

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} \text{ is diagonalizable.}$$

Also find the diagonal form and diagonalizing matrix P.

Solution

The characteristic equation of A is

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -(1+\lambda) & 4 & 4 \\ -(1+\lambda) & 3-\lambda & 4 \\ -(1+\lambda) & 8 & 7-\lambda \end{vmatrix} = 0 \quad C_1 + C_2 + C_3, C_2, C_3.$$

$$\begin{vmatrix} -(1+\lambda) & 4 & 4 \\ 0 & -(1+\lambda) & 0 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0 \quad \begin{matrix} R_1 \\ R_2 - R_1 \end{matrix}$$

$$0 + (1+\lambda)^2 (3-\lambda) = 0$$

$\therefore$ the characteristic roots are

$$\lambda = -1, -1, 3.$$

The characteristic root corresponding to $\lambda = -1$ is

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-8x_1 + 4x_2 + 4x_3 = 0$$

$$\text{(i.e)} \quad -2x_1 + x_2 + x_3 = 0$$

The rank of the coefficient matrix is 1.

$\therefore$ these equations have two linearly indpendent solutions.

clearly $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are two linearly independent solutions of the equation.

$\therefore$ x_1 and x_2 are two linearly independent characteristic vectors of A corresponding to $\lambda = -1$.

Consider the characteristic root $\lambda = 3$.

The characteristic vector is given by

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ 16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-12x_1 + 4x_2 + 4x_3 = 0$$
$$-8x_1 + 0 + 4x_3 = 0$$
$$16x_1 + 8x_2 + 4x_3 = 0$$

$$\frac{x_1}{16-0} = \frac{x_2}{-32+48} = \frac{x_3}{0+32}$$

$$\text{i.e. } \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2}$$

$\therefore$ the characteristic vector is $x_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\text{Let } P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

The columns of P are the linearly independent eigen vectors of A corresponding to the eigen values $-1, -1, 3$ respectively.

$\therefore$ the matrix P will transform A to the diagonal form D where

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

Example 28

Diagonalize the matrix $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution

The characteristic equation

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[-(1-\lambda^2)-3] + 2[-1-\lambda-1] + 3[3-1+\lambda] = 0$$

$$(2-\lambda)(\lambda^2-4) + 2(-\lambda-2) + 3(\lambda+2) = 0$$

$$2\lambda^2 - 8\lambda^3 + 4\lambda - 2\lambda - 4 + 3\lambda + 6 = 0$$

$$-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-1)(\lambda^2-\lambda-6) = 0$$

$$(\lambda-1)(\lambda-3)(\lambda+2) = 0$$

$\therefore$ the characteristic roots are $1, -2, 3$.

When $\lambda = 1$,

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + 0x_2 + x_3 = 0$$

$$x_1 + 3x_2 - 2x_3 = 0$$

$$\frac{x_1}{-2-0} = \frac{x_2}{3-1} = \frac{x_3}{0+2}$$

i.e. $\frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{2}$ or

$\therefore$ the eigen vector is $x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

The eigen vector corresponding to $\lambda = -2$ is given by

$$\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$$\frac{x_1}{-2-9} = \frac{x_2}{3-4} = \frac{x_3}{12+2}$$

$$\frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14}$$

$\therefore$ the eigen vector corresponding to $\lambda = -2$ is

$$\lambda_2 = \begin{bmatrix} -11 \\ -1 \\ 14 \end{bmatrix}$$

The eigen vector corresponding to $\lambda = 3$ is given by

$$\begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - 2x_2 + 3x_3 = 0$$
$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 + 3x_2 - 4x_3 = 0$$

$$\therefore \frac{x_1}{-2+6} = \frac{x_2}{3+1} = \frac{x_3}{2+2}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ the eigen vector corresponding to $\lambda = 3$ is given by $x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Then P is the matrix whose columns are the linearly independent eigen vectors.

$$\therefore P = \begin{bmatrix} -1 & 11 & 1 \\ 1 & -1 & 1 \\ 1 & 14 & 1 \end{bmatrix}$$

Then P will transform A to diagonal form D given by

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise 6

1) Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalisable.

Also find the transforming matrix and the diagonal matrix.

2) Show that the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is similar to a diagonal matrix. Also find the transforming matrix and diagonal matrix.

3) Diagonalise the matrix

(i) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

4) Show that the following matrices are similar to a diagonal matrix. Find the diagonal form and the diagonalizing matrix P.

(i) $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

CRAMER'S RULE

Solving Linear Equations by Cramer's Rule:

Consider the equation

$$a_1x+b_1y+c_1z = d_1 \quad \cdots\cdots\cdots\cdots \quad (1)$$

$$a_2x+b_2y+c_2z = d_2 \quad \cdots\cdots\cdots\cdots \quad (2)$$

$$a_3x+b_3y+c_3z = d_3 \quad \cdots\cdots\cdots\cdots \quad (3)$$

$$\text{Let } \Delta = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Multiplying both sides by x,

$$\Delta x = \begin{bmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix} = \Delta x \text{ (say)}$$

Then $x = \frac{\Delta_x}{\Delta}$. Illy $y = \frac{\Delta_y}{y}$. $z = \frac{\Delta_z}{z}$

where $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Test of Consistency :

If $\Delta \neq 0$, then $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, $z = \frac{\Delta_z}{\Delta}$ uniquely determine the solution.

If $\Delta = 0$ and atleast one of Δ_x, Δ_y, Δ_z is not equal to zero, then there is no solution to the given system of equations. However, if $\Delta_x = \Delta_y = \Delta_z = 0$, we cannot conclude that the equations are consistent. In this case, the equations may or may not be consistent. This is seen by the following argument.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\Delta = 0$ is the condition for the homogeneous equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0 \text{ to have a non-trivial solution.}$$

If $\Delta = 0$, then

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

This condition implies that there exists non-trivial solutions for l, m, n such that

$$a_1l + a_2m + a_3n = 0 \qquad \ldots\ldots (4)$$
$$b_1l + b_2m + b_3n = 0 \qquad \ldots\ldots (5)$$
$$c_1l + c_2m + c_3n = 0 \qquad \ldots\ldots (6)$$

Now multiply equations (1), (2), (3) by l, m, n and adding we get,

$$l\,(a_1x + b_1y + c_1z) + m\,(a_2x + b_2y + c_2z) + n\,(a_3x + b_3y + c_3z$$
$$= ld_1 + md_2 + nd_3$$

i.e., $x\,(a_1l + a_2m + a_3n) + y\,(b_1l + b_2m + b_3n) + z\,(c_1l + c_2m + c_3n$

$$= d_1l + d_2m + d_3n$$

i.e., $0 = d_1l + d_2m + d_3n$

Hence the equations are consistent only if $d_1l + d_2m + d_3n = 0$. If $d_1l + d_2m + d_3n \neq 0$ we can say that the equations are inconsistant.

Another method of testing consistency

If any one of the equations can be expressed as a linear combination of the other equations then these equations are ...

Example 1 : Solve the equations

$$\begin{aligned} x + y + z &= -1 \\ x + 2y + 3z &= -4 \\ x + 3y + 4z &= -6 \end{aligned}$$

Solution :

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 1(8-9) - 1(4-3) + 1(3-2)$$

$$= -1 - 1 + 1 = -1$$

$$\Delta_x = \begin{vmatrix} -1 & 1 & 1 \\ -4 & 2 & 3 \\ -6 & 3 & 4 \end{vmatrix}$$

$$= -1(8-9) - 1(-16+18) + 1(-12+12)$$

$$= 1 - 2 = -1$$

$$\Delta_y = \begin{vmatrix} 1 & -1 & 1 \\ 1 & -4 & 3 \\ 1 & -6 & 4 \end{vmatrix}$$

$$= 1(-16+18) + 1(4-3) + 1(-6+4)$$

$$= 2 + 1 - 2 = 1$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -4 \\ 1 & 3 & -6 \end{vmatrix}$$

$$= 1(-12+12) - 1(-6+4) - 1(3-2)$$

$$= 2 - 1 = 1$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-1}{-1} = 1$$

$$y = \frac{\Delta_y}{\Delta} = \frac{1}{-1} = -1$$

$$z = \frac{\Delta_z}{\Delta} = \frac{1}{-1} = -1$$

Example 2 : Solve the equations

$$\begin{aligned} 2y - 3z &= 0 \\ x + 3y &= -4 \\ 3x + 4y &= 3 \end{aligned}$$

Solution :

$$\Delta = \begin{vmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} = -3(4-9) = 15$$

$$\Delta_x = \begin{vmatrix} 0 & 2 & -3 \\ -4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} = -3(-16-9) = 75$$

$$\Delta_y = \begin{vmatrix} 0 & 0 & -3 \\ 1 & -4 & 0 \\ 3 & 3 & 0 \end{vmatrix} = -3(3+12) = -45$$

$$\Delta_z = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -4 \\ 3 & 4 & 3 \end{vmatrix} = -2(3+12) = -30$$

$$\therefore\ x = \frac{\Delta_x}{\Delta} = \frac{75}{15} = 5$$

$$y = \frac{\Delta_y}{\Delta} = -\frac{45}{15} = -3$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-30}{15} = -2$$

Example 3 : Solve the equations :

$$\frac{x^2z^3}{y} = e^8\ ;\ \frac{y^2z}{x} = e^4\ ;\ \frac{x^3y}{z^4} = 1$$

Solution : Taking logarithm, the given equations become

$$2\log x - \log y + 3\log z = 8$$
$$-\log x + 2\log y + \log z = 4$$
$$3\log x + \log y - 4\log z = 0$$

This is a set of linear equations in the variables $\log x$, $\log y$ & $\log z$.

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix}$$
$$= 2(-8-1) + 1(4-3) + 3(-1-6)$$
$$= -18 + 1 - 21 = -38$$

$$\Delta_{\log x} = \begin{vmatrix} 8 & -1 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -4 \end{vmatrix}$$
$$= 8(-8-1) + 1(-16-0) + 3(4-0)$$
$$= -72 - 16 + 12 = -76$$

$$\Delta_{\log y} = \begin{vmatrix} 2 & 8 & 3 \\ -1 & 4 & 1 \\ 3 & 0 & -4 \end{vmatrix}$$
$$= 2(-16-0) - 8(4-3) + 3(0-12)$$
$$= -32 - 8 - 36 = -76$$

$$\Delta_{\log z} = \begin{vmatrix} 2 & -1 & 8 \\ -1 & 2 & 4 \\ 3 & 1 & 0 \end{vmatrix}$$
$$= 2(0-4) + 1(0-12) + 8(-1-6)$$
$$= -8 - 12 - 56 = -76$$

$$\therefore \log x = \frac{\Delta_{\log x}}{\Delta} = \frac{-76}{-38} = 2\ \therefore x = e^2$$

$$\log y = \frac{\Delta_{\log y}}{\Delta} = \frac{-76}{-38} = 2\ \therefore y = e^2$$

$$\log z = \frac{\Delta_{\log z}}{\Delta} = \frac{-76}{-38} = 2\ \therefore z = e^2$$

Example 4 : Determine whether the system of equations $x - 3y + 2z = 4, 2x + y - 3z = -2, 4x - 5y + z = 5$ is inconsistent.

Solution :

$$\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -5 & 1 \end{vmatrix}$$

$$= 1(1-15) + 3(2+12) + 2(-10-4)$$

$$= -14 + 42 - 28 = 0$$

$$\Delta_x = \begin{vmatrix} 4 & -3 & 2 \\ -2 & 1 & -3 \\ 5 & -5 & 1 \end{vmatrix}$$

$$= 4(1-15) + 3(-2+15) + 2(10-5)$$

$$= -56 + 39 + 10 = -7 \neq 0$$

$\therefore \frac{\Delta_x}{\Delta}$ does not exist.

$\therefore$ The given equations are inconsistent.

Example 5 : Find the value of k if the linear, equations $3x - 2y + 4z = 1, 5x + 4y - 6z = 2, x - 8y + 14z = k$ are consistent and obtain the general solution.

Solution :

$$\Delta = \begin{vmatrix} 3 & -2 & 4 \\ 5 & 4 & -6 \\ 1 & -8 & 14 \end{vmatrix}$$

$$= 3(56-48) + 2(70+6) + 4(-40-4)$$

$$= 24 + 152 - 176 = 0$$

Since $\Delta = 0$, there exists numbers l, m, n such that

$$3l + 5m + n = 0 \qquad \text{...... (1)}$$

$$-2l + 4m - 8n = 0 \qquad \text{...... (2)}$$

$$4l - 6m + 14n = 0 \qquad \text{...... (3)}$$

Using rule of cross multiplication we get

$$\frac{l}{-40-4} = \frac{m}{-2+24} = \frac{n}{12+10}$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$$

Take $l = -2, m = 1, n = 1$

Also $d_1 = 1, d_2 = 2, d_3 = k$.

For consistency, $ld_1 + md_2 + nd_3 = 0$

$$\therefore -2 + 2 + k = 0$$

$$\therefore k = 0$$

In this case, there are infinite solutions. Then express x and y in terms of z.

$$3x - 2y = 1 - 4z \qquad \text{...... (1)}$$

$$5x + 4y = 2 + 6z \qquad \text{...... (2)}$$

$$(1) \times 2 : \quad 6x - 4y = 2 - 8z \qquad \text{...... (3)}$$

Adding (2) and (3)

$$11x = 4 - 2z$$

or

$$x = \frac{4-2z}{11}$$

$\therefore$ From (1),

$$2y = \frac{3(4-2z)}{11} - 1 + 4z$$

$$= \frac{1+38z}{11}$$

$\therefore x = \frac{4-2z}{11}$, $y = \frac{38z+1}{22}$ where z is any real number.

Example 6 : Find the value of a so that the system of equations $2x - 3y + 4z = a, x - y + 5z = 1$ and $x - 2y - z = 2$ are consistent.

Solution :

$$\Delta = \begin{vmatrix} 2 & -3 & 4 \\ 1 & -1 & 5 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= 2(1+10) + 3(-1-5) + 4(-2+1)$$

$$= 22 - 18 - 4 = 0$$

Since $\Delta = 0$, there exists numbers l, m, n such that

$$2l + m + n = 0 \quad \text{...... (1)}$$

$$-3l - m - 2n = 0 \quad \text{...... (2)}$$

$$4l + 5m - n = 0 \quad \text{...... (3)}$$

Using rule of cross multiplication, from (1) & (2)

$$\frac{l}{-1} = \frac{m}{1} = \frac{n}{1}$$

In order that the equations are consistent

$$ld_1 + md_2 + nd_3 = 0$$

Here $d_1 = a,\ d_2 = 1,\ d_3 = 2$

$l = -1,\ m = 1,\ n = 1$

$$\therefore ld_1 + md_2 + nd_3 = 0 \Rightarrow -a + 3 = 0$$

$\therefore a = 3$ (See also Example 12)

Example 7 : Find the value of k if the linear equations $3x - 2y + 4z = 1$; $5x + 4y - 6z = 2$; $x - 8y + 14z = k$ are consistent and obtain the general solution.

Solution :

$$3x - 2y + 4z = 1 \quad \text{...... (1)}$$

$$5x + 4y - 6z = 2 \quad \text{...... (2)}$$

$$x - 8y + 14z = k \quad \text{...... (3)}$$

Adding (2) and (3) we get,

$$6x - 4y + 8z = k + 2 \quad \text{...... (4)}$$

$$(1) \times 2 : 6x - 4y + 8z = 2 \quad \text{...... (5)}$$

If the equations are to be consistent, the RHS of the equations (4) and (5) should be the same since LHS are equal

$$\therefore k + 2 = 2$$

$$k = 0$$

General Solution :

$$3x - 2y = 1 - 4z \quad \text{...... (1)}$$
$$5x + 4y = 2 + 6z \quad \text{...... (2)}$$

(1) × 2 : $6x - 4y = 2 - 8z$

(2) : $5x + 4y = 2 + 6z$

$$11x = 4 - 2z$$
$$x = \frac{4 - 2z}{11}$$

from (1), $2y = 3x - (1 - 4z)$

$$= \frac{3(4 - 2z)}{11} - 1 + 4z$$
$$= \frac{12 - 6z - 11 + 44z}{11}$$
$$\therefore y = \frac{1 + 38z}{22}$$

∴ The general solution is

$$x = \frac{4 - 22}{11}, \quad y = \frac{1 + 38z}{22}$$

Example 8 : Show that the system $x + y + 2z = a$, $x + 3y - 2z = b$, $5x + 7y + 6z = c$ is consistent only if $c = 4a + b$. In the latter case, express x and y in terms of z, a and b.

Solution :

$$x + y + 2z = a \quad \text{...... (1)}$$
$$x + 3y - 2z = b \quad \text{...... (2)}$$
$$5x + 7y + 6z = c \quad \text{...... (3)}$$

4 × (1) : $4x + 4y + 8z = 4a$ ...... (4)

$$x + 3y - 2z = b \quad \text{...... (5)}$$

Adding, $5x + 7y + 6z = 4a + b$ ...... (6)

Comparing equations (3) and (6) for consistency we should have $c = 4a + b$

In that case (2) – (1) gives,

$$2y - 4z = b - a$$
$$\therefore y = 2z + \frac{b - a}{2}$$

(1) × 3 – (2) gives $2x + 8z = 3a - b$

$$\therefore x = -4z + \frac{3a - b}{2}$$

Example 9 : Obtain the values of λ and μ for which the equations $x + y + z = 3$, $x + 3y + 2z = 6$ and $x + \lambda y + 3z = \mu$ possess *(i)* a unique solution and *(ii)* no solution.

Solution :

(i) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & \lambda & 2 \end{vmatrix}$

$$= 1(9-2\lambda) - 1(3-2) + 1(\lambda-3)$$
$$= 9 - 2\lambda - 1 + \lambda - 3 = 5 - \lambda$$

For consistency, $\Delta \neq 0 \quad \therefore \; 5 - \lambda \neq 0$

i.e., $\lambda \neq 5$

When $\lambda \neq 5$, since $\Delta \neq 0$, $\frac{\Delta_x}{\Delta}$, $\frac{\Delta_y}{\Delta}$ and $\frac{\Delta_z}{\Delta}$ are finite for any finite value of μ.

$\therefore$ For consistency, $\lambda \neq 5$ and μ is any finite value.

(ii) When $\lambda = 5$, the equations are

$$x + y + z = 3 \qquad \text{...... (1)}$$
$$x + 3y + 2z = 6 \qquad \text{...... (2)}$$
$$x + 5y + 3z = \mu \qquad \text{...... (3)}$$

(1) + (3) gives $2x + 6y + 4z = 3 + \mu$

(2) $\times$ 2, gives $2x + 6y + 4z = 12$

$\therefore$ Equations are consistent if $3 + \mu = 12$ or $\mu = 9$.

$\therefore$ The equations are inconsistent if $\lambda = 5$ and $\mu \neq 9$.

Example 10 : Find k so that the equations $x + y - z = 3$, $kx - y + 2z = 5$, $x + 2y - z = 4$ have a unique solution.

Solution :

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ k & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$
$$= 1(1-4) - 1(-k-2) - 1(2k+1)$$
$$= -3 + k + 2 - 2k - 1 = -k - 2$$

For unique solution $\Delta \neq 0$

$\therefore -k - 2 \neq 0$

$\therefore k \neq -2$

Example 11 : Find the values of a and b so that the equations $x + y + z = 3$, $x + y + 2z = 6$ and $x + ay + 3z = b$ have *(i)* no solution *(ii)* only one solution and *(iii)* infinite number of solutions.

Solution :

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & a & 3 \end{vmatrix}$$
$$= 1(3-2a) - 1(3-2) + 1(a-1)$$
$$= 3 - 2a - 1 + a - 1$$
$$= 1 - a$$

(i) $\therefore$ If $a \neq 1$, then the equations have a unique solution. (b can be any number)

(ii) If $a = 1$, the equations are

$$x + y + z = 3 \qquad \text{...... (1)}$$
$$x + y + 2z = 6 \qquad \text{...... (2)}$$
$$x + y + 3z = b \qquad \text{...... (3)}$$

$$\begin{aligned} (1)+(3) \text{ gives } 2x+2y+4z &= b+3 \\ (2)\times 2 \text{ gives, } 2x+2y+4z &= 12 \\ \therefore \text{ For consistency, } b+3 &= 12 \\ \therefore b &= 9 \end{aligned}$$

When $a = 1$ and $b = 9$ the equations are consistent and there are infinite number of solutions.

(iii) If $a = 1$ and $b \neq 9$ then the equations are inconsistent and hence no solution.

Example 12 : Find the value of a so that the system of equations $2x - 3y + 4z = a$ $x - y + 5z = 1$ and $x - 2y - z = 2$ are consistent.

Solution :

$$2x - 3y + 4z = a \qquad \ldots\ldots (1)$$
$$x - y + 5z = 1 \qquad \ldots\ldots (2)$$
$$x - 2y - z = 2 \qquad \ldots\ldots (3)$$

One can easily observe that

$$\begin{aligned} (2)+(3) \text{ gives } 2x-3y+4z &= 3 \\ (1) \text{ is } 2x-3y+4z &= a \\ \therefore \text{ For consistency } a &= 3 \end{aligned}$$

Example 13 : Investigate for what values of λ, μ the equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have *(i)* unique solution *(ii)* infinite number of solutions and *(iii)* no solution.

Solution :

(i)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix}$$

$$= 1(2\lambda - 6) - 1(\lambda - 3) + 1(2 - 2)$$
$$= 2\lambda - 6 - \lambda + 3$$
$$= \lambda - 3$$

For a unique solution $\Delta \neq 0$

i.e., $\lambda - 3 \neq 0$ or $\lambda \neq 3$

(ii) If $\lambda = 3$, then $\Delta = 0$

$\therefore$ The equations may or may not be consistent. When $\lambda = 3$, the equations are

$$x + y + z \backslash= 6 \qquad \ldots\ldots (1)$$
$$x + 2y + 3z = 10 \qquad \ldots\ldots (2)$$
$$x + 2y + 3z = \mu \qquad \ldots\ldots (3)$$

Comparing equations (2) and (3) for consistency $\mu = 10$.

$\therefore$ For $\lambda = 3$ and $\mu = 10$ gives that the equations are consistent and there are an infinite number of solutions.

(iii) If $\lambda = 3$ and $\mu \neq 10$, then equations are inconsistent and hence there is no solution.

Example 14 : Find the value of a for which the equations $x - 3y + 2z = 4, 2x + y - z = 1, 5x - 2y + z = a$ possess a solution.

Solution :

$$\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & -1 \\ 5 & -2 & 1 \end{vmatrix}$$

$$= 1(1-2) + 3(2+5) + 2(-4-5)$$
$$= -1 + 21 - 18$$
$$= 2$$

Since $\Delta \neq 0$, for all finite values of a the given equations have a unique solution.

Note : If the 3rd equation is $3x - 2y + z = a$ then (1) + (2) is $3x - 2y + z = 5$.

$\therefore$ For consistency and a solution $a = 5$.

Example 15 : Determine whether the system of equations $4x + 7y - 14z = 10$, $2x + 3y - 4z = -4$, $3x + 5y - 9z = 3$ is consistent and obtain the general solution.

Solution :

$$4x + 7y - 14z = 10 \quad \text{...... (1)}$$
$$2x + 3y - 4z = -4 \quad \text{...... (2)}$$
$$3x + 5y - 9z = 3 \quad \text{...... (3)}$$

$$\Delta = \begin{vmatrix} 4 & 7 & -14 \\ 2 & 3 & -4 \\ 3 & 5 & -9 \end{vmatrix}$$

$$= 4(-27 + 20) - 7(-18 + 12) - 14(10 - 9)$$
$$= -28 + 42 - 14 = 0$$

$\therefore$ Equations may or may not be consistent.

(1) + (2) : $6x + 10y - 18z = 6$ ...... (4)

(3) × 2 $6x + 10y - 18z = 6$ ...... (5)

(4) and (5) are the same.

$\therefore$ The equations are consistent.

Now we can take any two equations to get the solutions for x, y, z.

$$4x + 7y - 14z = 10 \quad \text{...... (1)}$$
$$4x + 6y - 8z = -8 \quad \text{...... (2)} \times 2$$
$$\overline{\quad y - 6z = 18 \quad}$$

From (2),
$$2x = -3y + 4z - 4$$
$$= -3(6z + 18) + 4z - 4$$
$$= -18z - 54 + 4z - 4 = -14z - 58$$
$$\therefore x = -7z - 29$$

where z is any arbitrary real number.

$\therefore x = 6z + 18$, $y = -7z - 29$ where z is any real number.

Example 16 : Show that the equations

$x + y + z = a$, $3x + 4y + 5z = b$, $2x + 3y + 4z = c$

(*i*) have no solution if $a = b = c = 1$

(*ii*) have many solutions if $a = \frac{1}{2}b = c = 1$.

Solution :

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= 1\,(16-15) - 1\,(12-10) + 1\,(9-8)$$

$$= 1 - 2 + 1 = 0$$

Since $\Delta = 0$, the equations may or may not be consistent.

(1) + (3) gives $\quad 3x + 4y + 5z = a + c \quad$ ...... (4)

(2) is $\quad 3x + 4y + 5z = b \quad$ ...... (5)

When $a = 1, b = 1, c = 1$, RHS of (4) and (5) are not equal.

$\therefore$ The equations are inconsistent and hence no solution.

When $a = \frac{1}{2} b = c = 1$

i.e., $a = 1, c = 1, b = 2$

Equation (4) is $\quad 3x + 4y + 5z = 2$

Equation (5) is $\quad 3x + 4y + 5z = 2$

$\therefore$ Equations are consistent and hence there are infinite number of solutions.

To find the solutions : –

$$x + y + z = 1 \quad \text{...... (1)}$$

$$2x + 3y + 4z = 1 \quad \text{...... (3)}$$

(3) + (1) × 2 gives $\quad y + 2z = -1$

$$y = -2z - 1$$

From (1), $\quad x = -y - z + 1$

$$= 2z + 1 - z + 1$$

$$x = z + 2$$

$\therefore x = z + 2, y = -2z - 1$ where z is any arbitrary real number.

Example 17 : Find the values of λ for which the equations

$$\lambda x + 8y - 8z = 0$$

$$(\lambda + 1)\,x + \lambda y + z = 0$$

$(\lambda + 9)\,x + 9y + \lambda z = 0$ have a solution other than $x = y = z = 0$.

Solution :

$$\Delta = \begin{vmatrix} \lambda & 8 & -8 \\ \lambda+1 & \lambda & 1 \\ \lambda+9 & 9 & \lambda \end{vmatrix}$$

$$= \lambda\,(\lambda^2 - 9) + (\lambda + 1)\,(8\lambda + 72) + (\lambda + 9)\,(8 + 8\lambda)$$

$$= \lambda\,(\lambda^2 - 9) - (\lambda + 1)\,8\,(\lambda + 9) + 8\,(\lambda + 1)\,(\lambda + 9)$$

$$= \lambda(\lambda^2 - 9)$$

For the given set of homogeneous equations to have non-trivial solution the condition is $\Delta = 0$.

$$\lambda\,(\lambda^2 - 9) = 0$$

$$\therefore \lambda = 0, 3, -3.$$

7. Using Cramer's rule solve the following equations :

(i)
$$\begin{aligned} 2x + 3y - z &= 9 \\ x + y + z &= 9 \\ 3x - y - z &= -1 \end{aligned}$$

(ii)
$$\begin{aligned} 3x + 4y + 5z &= 18 \\ 2x - y + 8z &= 13 \\ 5x - 2y + 7z &= 20 \end{aligned}$$

(iii)
$$\begin{aligned} x + y + z &= 6 \\ x - y + z &= 2 \\ 2x + y - z &= 1. \end{aligned}$$

Example 1 *Show that the matrix* $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ *is unitary.*

Solution

$$\bar{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A(\bar{A})' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{-i}{2}+\frac{i}{2} \\ \frac{-i}{2}+\frac{i}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Since $A(\bar{A})' = I$, A is a Unitary matrix.

Exercises

1. Show that the following matrices are Hermitian.

   (a) $\begin{bmatrix} 3 & 1+i \\ 1-i & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2-i \\ 2+i & 3 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & 3+i & -2+i \\ 3-i & -3 & 3+4i \\ -2-i & 3-4i & 5 \end{bmatrix}$

2. Show that the following matrices are Skew-Hermitian.

   (a) $\begin{bmatrix} 0 & -2+i \\ 2+i & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1+i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$

If A is a square matrix show that $A + A'$ symmetric and $A - A'$ is Skew-symmetric.

Show that

(a) $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & \frac{-1+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{bmatrix}$

are unitary matrices.

Show that the following matrices are orthogonal.

(a) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

RANK OF A MATRIX

Definition : The minor of a given matrix **A** is the determinant composed of elements of the matrix left after striking out certain rows and columns.

Example *Suppose we have a matrix* $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$.

Third-order minors of this matrix are obtained by striking out one column and replacing the matrix symbol ( ) by the determinant sign | |. There are four such third order minors. Second-order minors are obtained by striking out two columns and one row. There are 18. First-order minors number 12.

Definition : The rank of a matrix **A** is the highest order of a non-zero minor of A.

Example *Consider the matrix* $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

It is easy to verify that the rank of the matrix is equal to 2.

Example *The rank of the matrix* $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix}$ *is 1.*

Note : If A is a square matrix of order n, then the rank k satisfies the relation $k \leq n$.

If $k = n$, then the matrix is **non-singular**.

If $k < n$, then the matrix is **singular**.

There is yet another way to compute the rank of a matrix which is not connected with the evaluation of determinants.

We use the term elementary transformations of a matrix **A** for the following transformations :

(a) interchange (transposition) of two rows or two columns;

(b) multiplication of a row (or a column) by an arbitrary non-zero scalar;

(c) addition of a multiple of one row (or column) to another row (column).

Clearly these elementary transformations do not change the rank of a matrix.

When computing the rank of a matrix, the given matrix may first be simplified by means of a combination of elementary transformations.

We say that an m by n matrix has diagonal form if all its elements are zero except the elements $a_{11}, a_{22}, \ldots, a_{rr}$ (where $0 \le r \le \min(m, n)$) which are equal to unity. The rank of this matrix is obviously 'r'.

Using elementary transformations, it is possible to reduce any matrix to diagonal form as follows.

Suppose we have a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

Step 1 : If the element $a_{11} = 0$ then an interchange of rows and columns will change element in the position a_{11} to a non-zero element.

Step 2 : Convert the element a_{11} to unity by multiplying the first row by $\dfrac{1}{a_{11}}$.

Step 3 : Subtract from the j^{th} column, $j > 1$, the first column multiplied by a_{ij}, then element a_{ij} $(j > 1)$ will be replaced by a zero.

Step 4 : Subtract from the i^{th} row, $i > 1$, the first row multiplied by a_{i1}, then element a_{i1} $(i > 1)$ will be replaced by a zero.

At this stage we get a matrix of the form

$$A' = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & a'_{22} & \ldots & a'_{2n} \\ \ldots & & & \\ 0 & a'_{m2} & \ldots & a'_{mn} \end{pmatrix}$$

sub-matrix that remains in the lower right corner, and so on, we finally-after a finite number of manipulations – arrive at a diagonal matrix with the same rank as the original matrix A.

Thus, to find the rank of a matrix it is necessary to convert the matrix, by means of elementary transformations, to diagonal form and count the number of units in the principal diagonal. The number of units gives the rank of the given matrix.

The method is explained in the following examples.

Example 1 *Find the rank of the matrix* $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$.

Solution

The given matrix is

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & -6 \\ 0 & -1 & -8 \end{pmatrix}$$

$$R_2 \to R_2(-1) \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & -1 & -8 \end{pmatrix}$$

$$R_3 \to R_3 + R_2 \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 5C_1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix}$$

$$C_3 \to C_3 - 6C_2 \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$R_3 \to \frac{R_3}{-2} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which is a unit matrix of order 3.

Hence the rank of the given matrix is 3.

Example 2 *Find the rank of the matrix* $\begin{pmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{pmatrix}$.

Solution

Given $$A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{pmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 15R_1 \end{matrix} \sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{pmatrix}$$

$$\begin{matrix} R_2 \to \dfrac{R_2}{-6} \\ R_3 \to \dfrac{R_3}{-18} \end{matrix} \sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{pmatrix}$$

$$R_3 \to R_3 - R_2 \sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 + C_1 \\ C_4 \to C_4 + C_1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} C_3 \to C_3 + 3C_2 \\ C_4 \to C_4 + 2C_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence $$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is a unit matrix of order 2. Hence the rank of A is 2.

Example 3 *Find the rank of the matrix* $\begin{pmatrix} 4 & -5 & 1 & 2 \\ 3 & 1 & -2 & 9 \\ 1 & 4 & 1 & 5 \end{pmatrix}$.

Solution

Let $$A = \begin{pmatrix} 4 & -5 & 1 & 2 \\ 3 & 1 & -2 & 9 \\ 1 & 4 & 1 & 5 \end{pmatrix}$$

$$A \sim \begin{pmatrix} 1 & 4 & 1 & 5 \\ 3 & 1 & -2 & 9 \\ 4 & -5 & 1 & 2 \end{pmatrix} \text{ [interchanging } R_3 \text{ \& } R_1]$$

$$\begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix} \sim \begin{pmatrix} 1 & 4 & 1 & 5 \\ 0 & -11 & -5 & -6 \\ 0 & -21 & -3 & -18 \end{pmatrix}$$

$$R_2 \to \frac{R_2}{-11}, \quad R_3 \to \frac{R_3}{-3} \quad \sim \begin{pmatrix} 1 & 4 & 1 & 5 \\ 0 & 1 & \frac{5}{11} & \frac{6}{11} \\ 0 & 7 & 1 & 6 \end{pmatrix}$$

$$R_3 \to R_3 - 7R_2 \quad \sim \begin{pmatrix} 1 & 4 & 1 & 5 \\ 0 & 1 & \frac{5}{11} & \frac{6}{11} \\ 0 & 0 & \frac{-24}{11} & \frac{24}{11} \end{pmatrix}$$

$$R_3 \to R_3 \times \frac{-11}{24} \quad \sim \begin{pmatrix} 1 & 4 & 1 & 5 \\ 0 & 1 & \frac{5}{11} & \frac{6}{11} \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$C_2 \to C_2 - 4C_1, \quad C_3 \to C_3 - C_1, \quad C_4 \to C_4 - 5C_1 \quad \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{11} & \frac{6}{11} \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$C_3 \to C_3 - \frac{5}{11}C_2, \quad C_4 \to C_4 - \frac{6}{11}C_2 \quad \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$C_4 \to C_4 + C_3 \quad \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus $$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which is a unit matrix of order 3. Hence the rank of A is 3.

Example 4

Find the rank of the matrix $\begin{pmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{pmatrix}$.

Solution

Let $$A = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{pmatrix}.$$

$$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1, \quad R_4 \to R_4 - R_1 \quad \sim \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & -9 & 0 & 9 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -3 & 0 \end{pmatrix}$$

$$R_2 \to \frac{R_2}{-9} \sim \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -3 & 0 \end{pmatrix}$$

$$\begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix} \sim \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -3 & 1 \end{pmatrix}$$

$$R_4 \to R_4 - 3R_3 \sim \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$

$$\begin{matrix} R_1 \to \frac{R_1}{2} \\ R_3 \to R_3(-1) \\ R_4 \to \frac{R_4}{13} \end{matrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} C_2 \to C_2 - \frac{1}{2}C_1 \\ C_3 \to C_3 - C_1 \\ C_4 \to C_4 - \frac{1}{2}C_1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C_4 \to C_4 + C_2 \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C_4 \to C_4 - 4C_3 \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus $$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is a unit matrix of order 4.

Hence the rank of the given matrix is 4.

Exercises

Find the rank of the following matrices :

1. $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{pmatrix}$ 2. $\begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \end{pmatrix}$

5. $\begin{pmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{pmatrix}$

6. $\begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{pmatrix}$ ✓

9. $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$

10. $\begin{pmatrix} 2 & -6 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$

11. $\begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 5 & 4 & 2 \end{pmatrix}$

12. $\begin{pmatrix} 2 & -1 & 3 & 15 \\ 3 & 2 & 0 & 21 \end{pmatrix}$ ✓

13. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 5 & 1 & 4 \\ 5 & 6 & 8 & 1 \\ 8 & 3 & 7 & 2 \end{pmatrix}$

14. $\begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -2 & -2 & 6 & -7 \end{pmatrix}$

ANSWERS

1. 3   2. 3   3. 2   4. 3   5. 3   6. 3   7. 4
8. 3   9. 3   10. 4   11. 2   12. 2   13. 4   14. 3

❑ *n* - DIMENSIONAL VECTOR SPACE

From the course of analytic geometry we know that any point in a plane is determined by its two coordinates viz x-coordinate and y-coordinate which can be denoted by an ordered set of two real numbers, (x, y). Any vector in a plane is determined by its two components, which again is an ordered set of two real numbers. Similarly, a point in space can be represented by three components.

In applied mechanics, physics and geometry we frequently encounter objects whose specification requires more than three real numbers. For example, let us consider a collection of spheres in three dimensional space. To specify a sphere completely we need the coordinates of its centre and the radius; this amounts to an ordered set of four real numbers, of which, incidentally, the radius can only assume positive values. On the other hand, let us consider various positions of a solid in space. The position of a solid will be fully defined if we indicate the coordinates of

Here we have m rows and n columns. *i.e.*, m row vectors and n column vectors. Let r be the rank of the matrix A. Since r is the rank of the matrix we have a non-zero determinant of order r which contains r rows and r columns.

In otherwords $| A_r | \neq 0$, where $| A_r |$ contains r rows and r columns. Since $| A_r | \neq 0$ we have r linearly independent row and column vectors (refer to the previous article). **Hence if the rank of a matrix A is r, then it has r linearly independent rows and columns.**

❑ SYSTEMS OF LINEAR EQUATIONS

The theory of systems of linear equations serves as the foundation for a vast and important division of Engineering subjects. In this chapter we will study systems with an arbitrary number of equations and unknowns: that is the number of equations of a system will not even be assumed to coincide with the number of unknowns. Suppose we have a system of m linear equations in n unknowns. Let us agree to use the following symbolism; the unknowns will be denoted by x and subscripts; $x_1, x_2, \ldots x_n$; we will consider the equations to be enumerated thus; first, second etc., the coefficient of x_j in the i^{th} equation will be given as a_{ij}. Finally the constant term of the i^{th} equation will be indicated as b_i.

Our systems of equations will now be written as follows :

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \ldots (1)$$

The coefficients of the unknowns form a rectangular array.

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

called **coefficient matrix** of the given system. If $m = n$, then the above matrix becomes a square matrix.

The matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{pmatrix}$$

is called **augmented matrix** of the given system of equations.

The solution of the system of linear equations (1) is a set of n numbers $k_1, k_2, \ldots, k_n$ such that each of the equations (1) become an identity upon substitution of the corresponding numbers k_i, $i = 1, 2, \ldots, n$ for the unknowns x_i.

Note 1 : A system of linear equations may not have any solutions; it is then called **inconsistent** such, for example, is the system.

$$x_1 + 5x_2 = 1$$
$$x_1 + 5x_2 = 7$$

The left members of these equations coincide, but the right members are different and so no set of values of the unknowns can satisfy both equations simultaneously.

Note 2 : If a system of linear equations has solutions, it is termed **consistent**. A consistent system is called **determinate** if it has a unique solutions and **indeterminate** if there are more solutions than one.

For instance

$$x_1 + 2x_2 = 7$$
$$x_1 + x_2 = 4$$

is determinate (has a unique solution); it has the solution $x_1=1$, $x_2=3$.

On the otherhand, the system

$$3x_1 - x_2 = 1$$
$$6x_1 - 2x_2 = 2$$

is indeterminate since it has infinitely many solutions of the form $x_1 = k$, $x_2 = 3k - 1$, where k is an arbitrary number.

The problem of the theory of systems of linear equations consists in elaborating methods to determine whether a given system of equations is consistent or not and, in the case of consistency, to establish the number of solutions and also to indicate a procedure for finding the solutions.

Procedure for finding the solutions of a system of equations

Let the given system of linear equations be

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

Step 1 : Construct the coefficient matrix which is denoted by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Step 2 : Construct the augmented matrix which is denoted by [A, B].

$$[A, B] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Step 3 : Find the ranks of both the coefficient matrix and augmented matrix which are denoted by R (A) and R (A, B).

Step 4 : Compare the ranks of R (A) and R (A, B) we have the following results.

(a) If R (A) = R (A, B) = n (number of unknowns) then the given system of equations are consistent and have unique solutions.

(b) If R (A) = R (A, B) < n (number of unknowns) then the given system of equations are consistent and have infinite number of solutions.

(c) If R (A) ≠ R (A, B) then the given system of equations are inconsistent (that is the given system of equations have no solution).

Note : While finding the ranks of coefficient matrix as well as augmented matrix we don't use column operations. Hence we use only row operations as explained below.

(a) interchanging of any two rows.

(b) multiplying any equation by a non-zero constant.

(c) addition or subtraction of a multiple of one row to another row.

The method is very clear from the worked out examples.

Example 1 ***Examine if the following equations are consistent and if so solve $x + y + z = 6$; $x + 2y - 2z = -3$; $2x + 3y + z = 11$.***

Solution

The coefficient matrix **The augmented matrix**

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 2 & 3 & 1 \end{pmatrix} \qquad [A, B] = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -2 & -3 \\ 2 & 3 & 1 & 11 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -3 & -9 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix} (R_2 \to R_2 - R_3) \sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & -2 & -8 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix} (R_2 \sim R_3) \qquad \sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \left(R_3 \to \frac{R_3}{-2}\right) \qquad \sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Now the rank of coefficient matrix is 3.

Rank of augmented matrix is also 3.

Hence the given systems of equations are consistent and have unique solution.

The reduced system is

$$z = 4$$
$$y - z = -1$$
$$x + y + z = 6$$

Solving these equations we get, $x = -1,\ y = 3,\ z = 4.$

Example 2 *Test for consistency and hence solve*

$$x - 2y + 3z = 2;\ 2x - 3z = 3;\ x + y + z = 0.$$

Solution

The coefficient matrix **The augmented matrix**

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix} \qquad [A, B] = \begin{pmatrix} 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & -9 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 4 & -9 & -1 \\ 0 & 3 & -2 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & \frac{-9}{4} \\ 0 & 3 & -2 \end{pmatrix} \left(R_2 \to \frac{R_2}{4}\right) \sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & 3 & -2 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & \frac{-9}{4} \\ 0 & 0 & \frac{19}{4} \end{pmatrix} (R_3 \to R_3 - 3R_2) \sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & 0 & \frac{19}{4} & \frac{-5}{4} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & \frac{-9}{4} \\ 0 & 0 & 1 \end{pmatrix} \left(R_3 \to R_3 \times \frac{4}{19}\right) \sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-5}{19} \end{pmatrix}$$

Here rank of coefficient matrix is 3.

Rank of augmented matrix is 3.

Hence the given system of equations are consistent and have unique solution.

The reduced system is
$$z = \frac{-5}{19}$$
$$y - \frac{9}{4}z = \frac{-1}{4}$$
$$x - 2y + 3z = 2$$

Solving these equations we get $x = \frac{21}{19}$, $y = \frac{-16}{19}$, $z = \frac{-5}{19}$

Example 3 *Test the consistency of the following system of equations and hence solve it $x + 2y + z = 3$; $2x + 3y + 2z = 5$; $3x - 5y + 5z = 2$; $3x + 9y - z = 4$.*

Solution

The coefficient matrix **The augmented matrix**

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{pmatrix} \qquad [A, B] = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & 5 & -5 & 2 \\ 3 & 9 & -1 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -11 & 2 \\ 0 & 3 & -4 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -11 & 2 \\ 0 & 3 & -4 \end{pmatrix} \quad R_2 \to R_2(-1) \quad \sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{pmatrix} \begin{matrix} R_3 \to R_3 + 11R_2 \\ R_4 \to R_4 - 3R_2 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad R_4 \to R_4 + 2R_3 \quad \sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here the rank of coefficient matrix is 3.

Rank of augmented matrix is also 3.

The number of unknowns is 3.

Hence the given system of equations are consistent and have unique solution.

The reduced system is
$$2z = 4 \text{ (or) } z = 2.$$
$$y = 1$$
$$x + 2y + z = 3$$

Solving these equations we get $x = -1$, $y = 1$, $z = 2$.

Example 4 *Test the consistency of the following system of equations and if consistent solve* $2x - y - z = 2;\ x + 2y + z = 2;\ 4x - 7y - 5z = 2.$

Solution

The coefficient matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 4 & -7 & -5 \end{pmatrix} \quad (R_2 \sim R_1)$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -3 \\ 0 & -15 & -9 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -3 \\ 0 & 5 & 3 \end{pmatrix} \quad R_3 \to \frac{R_3}{-3}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 \to R_3 + R_2$$

The augmented matrix

$$[A, B] = \begin{pmatrix} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 5 & 3 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here rank of coefficient matrix is $R(A) = 2$

Rank of augmented matrix is $R(A, B) = 2$.

i.e., $R(A) = R(A, B) < 3$ (the number of unknowns)

Hence the given system of equations are consistent but have infinite number of solutions. Here the reduced system is

$$5y + 3z = 2$$
$$x + 2y + z = 2$$

i.e.,

$$y = \frac{2 - 3z}{5}$$

$$x = 2 - z - 2\left(\frac{2 - 3z}{5}\right)$$

$$= \frac{6 + z}{5}$$

i.e.,

$$x = \frac{6 + k}{5}$$

$$y = \frac{2 - 3k}{5}, \quad z = k$$

where $z = k$ is the parameter.

Example 5 *Test the consistency of the following equations and if possible find the solutions of*
$4x - 2y + 6z = 8;\ x + y - 3z = -1;\ 15x - 3y + 9z = 21.$

Solution

The coefficient matrix   **The augmented matrix**

$$A = \begin{pmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{pmatrix} \qquad [A, B] = \begin{pmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -3 \\ 4 & -2 & 6 \\ 15 & -3 & 9 \end{pmatrix} \ R_2 \sim R_1 \qquad \sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -3 \\ 0 & -6 & 18 \\ 0 & -18 & 54 \end{pmatrix} \ \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 15R_1 \end{matrix} \qquad \sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -3 \\ 0 & -6 & 18 \\ 0 & 0 & 0 \end{pmatrix} \ R_4 \to R_4 - 3R_2 \qquad \sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \ R_2 \to \frac{R_2}{-6} \qquad \sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here $\quad R(A) = R(A, B) = 2 < 3$ (number of unknowns)

Hence the given system of equations are consistent but have infinite number of solutions. The reduced system is

$$y - 3z = -2 \quad \ldots (1)$$
$$x + y - 3z = -1 \quad \ldots (2)$$

Substituting (1) in (2) we get $x = 1$ and $y = 3z - 2$ where z is the parameter.

Example 6 *Test whether the following system of equations are consistent or not.* $x - y + z = -9;\ 2x - y + z = 4;$
$3x - y + z = 6;\ 4x - y + 2z = 7.$

Solution

The coefficient matrix   **The augmented matrix**

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \end{pmatrix} \qquad [A, B] = \begin{pmatrix} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 3 & -2 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 4R_1 \end{matrix} \quad \sim \begin{pmatrix} 0 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{matrix} \quad \sim \begin{pmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad R_4 \sim R_3 \quad \sim \begin{pmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & -11 \end{pmatrix}$$

Clearly the rank of coefficient matrix R (A) = 3.

The rank of augmented matrix R (A, B) = 4.

i.e., $$R(A) \neq R(A, B)$$

Hence the given system of equations are inconsistent.

Example 7 ***Find the values of a and b for which the equations $x + y + z = 3$; $x + 2y + 2z = 6$; $x + ay + 3z = b$ have (i) no solution (ii) a unique solution.***

Solution

The coefficient matrix **The augmented matrix**

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{pmatrix} \qquad [A, B] = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & a-1 & 2 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \quad \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & a-3 & 0 \end{pmatrix} \quad R_3 \to R_3 - 2R_2 \quad \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-3 & 0 & b-9 \end{pmatrix}$$

Case (i) : If $a - 3 = 0$ and $b - 9 \neq 0$ then R (A) = 2 and R (A, B) = 3.

Hence if $a = 3$ and $b \neq 9$ then the given system of equations are in consistent and have no solution.

Case (ii) : If $a - 3 \neq 0$ and b takes any value, then R (A) = 3, R (A, B) = 3.

Hence if $a \neq 3$ and b takes any value both the coefficient and augmented matrices have the same rank and is equal to the number of unknowns. Therefore in this case the system of equations are consistent and have unique solution.

Example 8 *Find the condition on a, b, c so that the following system of equations has a solution*
$x + 2y - 3z = a;\ 3x - y + 2z = b;\ x - 5y + 8z = c.$

Solution :

The coefficient matrix **The augmented matrix**

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 1 & -5 & 8 \end{pmatrix} \qquad [A, B] = \begin{pmatrix} 1 & 2 & -3 & a \\ 3 & -1 & 2 & b \\ 1 & -5 & 8 & c \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & \cancel{-8}\,7 & 11 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \quad \sim \begin{pmatrix} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & -7 & 11 & c-a \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 \to R_3 - R_2 \quad \sim \begin{pmatrix} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & 0 & 0 & 2a-b+c \end{pmatrix}$$

Case (i) : Here R (A) = 2; R (A, B) = 3 provided $2a - b + c \neq 0$.

In this case the given system have no solution.

Case (ii) : In the assumed matrix if $2a - b + c = 0$ then R (A, B) = 2. Also R (A) = 2.

Hence R (A) = R (A, B) = 2 < 3 (number of unknowns)

So when $2a - b + c = 0$, the system of equations are consistent and have infinite number of solutions.

Exercises

Test the consistency of the following systems of equations and solve if it is consistent.

I. 1. $x + y + z = 3$
$x + 2y + 3z = 4$
$x + 4y + 9z = 6$

Ans. $x = 2,\ y = 1,\ z = 0$

2. $x + y + z = 8$
$x - y + 2z = 6$
$3x + 5y - 7z = 14$

Ans. $x = 5,\ y = \frac{5}{3},\ z = \frac{4}{3}$

3. $x + y + z = 6$
$x - y + 2z = 5$
$3x + y + z = 8$

Ans. $x = 1,\ y = 2,\ z = 3$

4. $5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 11z = 5$

Ans. $x = \frac{7}{11},\ y = \frac{3}{11},\ z = 0$

5. $x + 2y + 3z = 1$

$2x + 3y + 2z = 2$

$3x + 3y + 4z = 1$

Ans. $x = \frac{-3}{7}$, $y = \frac{8}{7}$, $z = \frac{-2}{7}$

6. $x + y - 2z = 3$

$2x - y + z = 0$

$3x + y - z = 8$

Ans. $x = \frac{8}{5}$, $y = 5$, $z = \frac{9}{5}$

7. $x + y + z = 6$

$x - y + 2z = 5$

$3x + y + z = 8$

$2x - 2y + 3z = 7$

Ans : $x = 1, y = 2, z = 3$

8. $5x + 3y + 3z = 48$

$2x + 6y - 3z = 18$

$8x - 3y + 2z = 21$

Ans. $x = 3, y = 5, z = 6$

9. $4x + 2y - z = 9$

$x - y + 3z = -4$

$2x + z = 1$

Ans. $x = 1, y = 2, z = -1.$

10. $3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4$

Ans : $x = 1, \; y = 2, z = -1$

11. $x + y + z = 6$

$x - y + 2z = 5$

$3x + y + z = 8$

$2x - 2y + 3z = 7$

Ans. $x = 1, y = 2, z = 3$

12. $x + 2y - z - 3 = 0$

$3x - y + 2z = 1$

$2x - 2y + 3z = 2$

$x - y + z = -1$

Ans. $x = -1, y = 4, z = 4$

13. $x + 2y - z = 3$

$-3x + y - 2z = -1$

$x - 2y + 3z = 2$

$x - y + z = -1$

Ans. $x = -1, y = 4, z = 4$

14. $x + 2y + 2z = 2$

$3x - 2y - z = 5$

$2x - 5y + 3z = -4$

$x + 4y + 6z = 0$

Ans. $x = 2, y = 1, z = -1$

15. $3x + y + z = 8$

$-x + y - 2z = -5$

$x + y + z - 6 = 0$

$2x - 2y + 3z - 7 = 0$

Ans. $x = 1, y = 2, z = 3.$

Given α and β are the eigenvalues A. $\therefore$ α^3 and β^3 are the eigenvalues of A^3.

Now

$$\begin{aligned} A^3 &= A^2 \times A \\ &= \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 38 & -50 \\ -50 & 138 \end{pmatrix} \end{aligned}$$

Example 15

Form the matrix whose eigenvalues are $\alpha - 5$, $\beta - 5$, $\gamma - 5$ where α, β, γ are the eigenvalues of

$$A = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}.$$

Solution

If the matrix A has the eigenvalues λ_1, λ_2 and λ_3 then the matrix $A - kI$ has the eigenvalues $\lambda_1 - k$, $\lambda_2 - k$, $\lambda_3 - k$.

Hence the required matrix is

$$\begin{aligned} A - 5I &= \begin{pmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{bmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{bmatrix} \end{aligned}$$

TO FIND EIGENVALUES AND EIGENVECTORS OF A GIVEN MATRIX

Example 1

Find the eigenvalues of the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Solution

The characteristic equation is

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)\,[\,(2-\lambda)\,(5-\lambda)\,] = 0$$

$$\lambda = 2, 3, 5$$

$\therefore$ The eigenvalues are 2, 3, 5.

Example 2 *Find the eigenvalues of the matrix* $\begin{bmatrix} 6 & 10 \\ 14 & 25 \end{bmatrix}$.

Solution

The characteristic equation is

$$\begin{bmatrix} 6-\lambda & 10 \\ 14 & 25-\lambda \end{bmatrix} = 0$$

$$(6-\lambda)(25-\lambda) - 140 = 0$$

$$\lambda^2 - 31\lambda + 10 = 0$$

$$\lambda = \frac{31 \pm \sqrt{961-40}}{2} = \frac{31 \pm \sqrt{921}}{2}$$

Example 3 *Find the eigenvectors of* $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Solution

The characteristic equation is $|A - \lambda I| = 0$.

$$= 0$$

$$(1-\lambda)^2 = 0; \quad \lambda = 1, 1$$

The eigen vectors are given by

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$0\,x_1 + 2\,x_2 = 0; \quad x_1 = k$

$0\,x_1 + 0\,x_2 = 0; \quad x_2 = 0 \quad$ *i.e.*, $(k, 0)$ is the vector.

Example 4 *Find the eigenvalues of the matrix* $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

Solution

The characteristic equation is

$$\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - 3 = 0$$

$$8 - 6\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\lambda = 5 \text{ or } 1$$

$\therefore$ The eigenvalues are 1 and 5.

NON SYMMETRIC MATRICES WITH NON REPEATED EIGENVALUES

Example 1 *Find the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$. *[MU, 1997]*

Solution

Step : 1 ***To find characteristic equation and eigenvalues :***

The characteristic equation is $|A - \lambda I| = 0$.

i.e., $\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$

i.e., $(4-\lambda)(2-\lambda) - 3 = 0$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0$$

$\therefore$ $\boxed{\lambda = 1 \text{ or } 5}$

$\therefore$ The eigenvalues are 1 and 5.

Step : 2 ***To find eigenvectors :***

Case (i) : When $\lambda = 1$, the equation

$$(A - \lambda I) X = 0 \qquad \left[\text{where } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]$$

becomes $\begin{pmatrix} 4-1 & 1 \\ 3 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e., $3x_1 + x_2 = 0$

$3x_1 + x_2 = 0$ Take $x_1 = 1$

(or) $x_2 = -3x_1$ $\therefore x_2 = -3$

Note that whatever be the value of x_1, the value of x_2 is -3 times of it. Therefore, the eigenvector corresponding to $\lambda = 1$ is $\begin{pmatrix} k \\ -3k \end{pmatrix}$ which is the general form where k is a constant.

Here we get infinite number of eigenvectors by giving different values for k.

The simplest eigenvector is therefore $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ which is obtained by putting $k = 1$.

Case (ii) : When $\lambda = 5$, the equation $(A - \lambda I) X = 0$, where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, becomes

$$\begin{pmatrix} 4-5 & 1 \\ 3 & 2-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e., $$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Take $x_2 = 1$

$\therefore x_1 = 1$

i.e., $$-x_1 + x_2 = 0 \Rightarrow x_1 = x_2$$
$$3x_1 - 3x_2 = 0 \Rightarrow x_1 = x_2$$

By giving $x_2 = k$, we have $x_1 = k$ and therefore for different values of k we get different eigenvectors.

The general eigenvector is therefore $\begin{pmatrix} k \\ k \end{pmatrix}$ and the simplest is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Conclusion :

Characteristic Equation	Eigenvalues	Eigenvector
$\lambda^2 - 6\lambda + 5 = 0$	$\lambda_1 = 1$	$X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$
	$\lambda_2 = 5$	$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

NOTE : Since the eigenvalues are different the eigenvectors are linearly independent. $[\because |A| = \begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} \neq 0]$

Example 2 *Find the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

Solution

Step : 1 *To find characteristic equation and eigenvalues :*

Given $$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$.

i.e., $$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

i.e., $$(1-\lambda)(-1-\lambda) - 3 = 0$$

i.e., $$-1 - \lambda + \lambda + \lambda^2 - 3 = 0$$

i.e., $$\lambda^2 - 4 = 0$$
$$\lambda = \pm 2$$

$\therefore$ The eigenvalues are $\boxed{\lambda = 2, -2}$

Step : 2 ***To find eigenvectors:***

Case (i) : When $\lambda = 2$.

Let $X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = 2$.

Then, we have $(A - \lambda I) X_1 = 0$

i.e., $\begin{pmatrix} 1-2 & 1 \\ 3 & -1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e., $\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e., $-x_1 + x_2 = 0$ ... (1)

$3x_1 - 3x_2 = 0$ ... (2)

From equations (1) and (2), we get, $x_1 = x_2$

i.e., The eigenvector corresponding to $\lambda = 2$ is $X_1 = \begin{pmatrix} k \\ k \end{pmatrix}$ (or) $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Case (ii): When $\lambda = -2$, the eigenvector $X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is given by

$$(A - \lambda I) X_2 = 0$$

i.e., $\begin{pmatrix} 1+2 & 1 \\ 3 & -1+2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e., $\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$3x_1 + x_2 = 0$$

$$3x_1 + x_2 = 0$$

$\Rightarrow$ $x_2 = -3x_1$

Putting $x_1 = k$, we get $x_2 = -3k$

$\therefore$ $X_2 = \begin{pmatrix} k \\ -3k \end{pmatrix}$

The simplest eigenvector $X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
$\lambda^2 - 4 = 0$	$\lambda_1 = 2$	$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
	$\lambda_2 = -2$	$X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Example 3

Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Solution

Step : 1 ***To find characteristic equation and eigenvalues:***

The characteristic equation is $|A - \lambda I| = 0$.

$$\text{i.e.,} \quad \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\text{i.e.,} \quad (1-\lambda)^2 + 1 = 0$$

$$\text{i.e.,} \quad \lambda^2 - 2\lambda + 1 + 1 = 0$$

$$\text{i.e.,} \quad \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

$\therefore$ The eigenvalues are $1 + i$ and $1 - i$.

Step : 2 ***To find eigenvectors:***

<u>*Case (i) :* **When $\lambda = 1 + i$,**</u>

Let $X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $1 + i$.

Then, we have

$$(A - \lambda I) X_1 = 0$$

$$\Rightarrow \quad \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{i.e.,} \quad \left.\begin{aligned} -ix_1 + x_2 &= 0 \\ -x_1 - ix_2 &= 0 \end{aligned}\right\} \quad \dots (A)$$

$$(A) \Rightarrow \quad x_1 + ix_2 = 0$$

$$\Rightarrow \quad x_1 = -i\, x_2$$

Putting $x_2 = 1$, we get $x_1 = -i$

$\therefore$ The eigenvector $X_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

<u>*Case (ii) :* **When $\lambda = 1 - i$,**</u>

The eigenvector $X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ corresponding to the eigenvalue $1 - i$ is given by

i.e.,
$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.,
$$\left.\begin{aligned} ix_1 + x_2 &= 0 \\ -x_1 + ix_2 &= 0 \end{aligned}\right\} \quad \dots \text{(B)}$$

(B) $\Rightarrow$
$$ix_1 + x_2 = 0$$

i.e.,
$$x_2 = -ix_1$$

Putting $x_1 = 1$, we get $x_2 = -i$.

$\therefore$
$$X_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
$\lambda^2 - 2\lambda + 2 = 0$	$\lambda_1 = 1 + i$	$X_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$
	$\lambda_2 = 1 - i$	$X_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Example 4

Find the eigenvalues and eigenvectors of
$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}.$$
[A.U. 1998]

Solution

Step : 1 ***To find characteristic equation:***

The given matrix be
$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$$

The characteristic equation is
$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \quad \dots (1)$$

where
$$\begin{aligned} a_1 &= \begin{cases}\text{Sum of leading} \\ \text{diagonal elements}\end{cases} \\ &= 2 + 1 - 3 \\ &= 0 \\ a_2 &= \begin{cases}\text{Sum of the minors of the} \\ \text{leading diagonal elements}\end{cases} \\ &= \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -7 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} \\ &= -5 - 6 - 2 \\ &= -13 \end{aligned}$$

$$a_3 = |A| = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{vmatrix}$$
$$= 2(-3-2) - 2(-6+7) + 0(4+7)$$
$$= -10-2$$
$$= -12$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 0\lambda^2 - 13\lambda + 12 = 0$$
$$\lambda^3 - 13\lambda + 12 = 0$$

Step : 2 ***To find eigenvalues :***

$$\lambda^3 - 13\lambda + 12 = 0$$

When $\lambda = 1$,

$$1 - 13 + 12 = 0$$

$\therefore$ 1 is a root.

$$\begin{array}{c|cccc} 1 & 1 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

$$\lambda^2 + \lambda - 12 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1+48}}{2} = \frac{-1 \pm 7}{2}$$
$$= 3 \text{ or } -4$$

$\therefore$ Eigenvalues are $\lambda = 1, 3, -4$

Step : 3 ***To find eigenvectors :***

<u>*Case (i)* : **When $\lambda = 1$**</u>

Let $X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda = 1$.

Then from the equation

$$(A - \lambda I) X_1 = 0, \text{ we have}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -7 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e., $x_1 + 2x_2 + 0x_3 = 0$

$$2x_1 + 0x_2 + x_3 = 0$$
$$-7x_1 + 2x_2 - 4x_3 = 0$$

Considering first two equations and using cross rule method, we have

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ \hline 2 & & 0 & & 1 & & 2 \\ 0 & & 1 & & 2 & & 0 \end{array}$$

$$\frac{x_1}{2-0} = \frac{x_2}{0-1} = \frac{x_3}{0-4} = k$$

i.e.,
$$x_1 = 2k$$
$$x_2 = -k$$
$$x_3 = -4k$$

Hence the general eigenvector is $X = \begin{pmatrix} 2k \\ -k \\ -4k \end{pmatrix}$.

Putting $k = 1$, we get the simplest eigenvector, $X_1 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$.

Case (ii) : **When $\lambda = 3$**

Let $X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = 3$.

Then the equation $(A - \lambda I)\, X_2 = 0$ becomes

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

i.e.,
$$-x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 - 2x_2 + x_3 = 0$$
$$-7x_1 + 2x_2 - 6x_3 = 0$$

Considering first two equations and applying rule of cross multiplication we have

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ \hline 2 & & 0 & & -1 & & 2 \\ -2 & & 1 & & 2 & & -2 \end{array}$$

$$\frac{x_1}{2-0} = \frac{x_2}{0+1} = \frac{x_3}{2-4}$$

i.e.,
$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k$$

i.e., $$x_1 = 2k,\ x_2 = k,\ x_3 = -2k.$$

Hence the general eigenvector corresponding to $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 2k \\ k \\ -2k \end{pmatrix}$$

By putting $k = 1$, we get the simplest eigenvector, $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

<u>*Case (iii) :*</u> **<u>When $\lambda = -4$.</u>**

Let $X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = -4$.

Then the equation $(A - \lambda I) X_3 = 0$ becomes

$$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -7 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.,
$$\begin{aligned} 6x_1 + 2x_2 + 0x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \\ -7x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

Considering the first two equations

$$\begin{aligned} 6x_1 + 2x_2 + 0x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \end{aligned}$$

By applying cross rule method, we have

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \hline 2 & 0 & 6 & 2 \\ 5 & 1 & 2 & 5 \end{array}$$

$$\frac{x_1}{2-0} = \frac{x_2}{0-6} = \frac{x_3}{30-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-6} = \frac{x_3}{26} = k$$

i.e., $$x_1 = 2k,\ x_2 = -6k,\ x_3 = 26k$$

Hence the general eigenvector corresponding to $\lambda = -4$ is

$$X_3 = \begin{pmatrix} 2k \\ -6k \\ 26k \end{pmatrix}$$

The simplest eigenvector can be obtained by taking $k = \frac{1}{2}$.

i.e.,
$$X_3 = \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$$

<u>*Conclusion :*</u>

Characteristic equation	Eigenvalues	Eigenvector
	$\lambda_1 = 1$	$X_1 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$
$\lambda^3 - 13\lambda + 12 = 0$	$\lambda_2 = 3$	$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$
	$\lambda_3 = 4$	$X_3 = \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$

Note : The eigenvectors are linearly independent since the eigenvalues are different.

$$[\because |A| = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & -3 \\ -4 & -2 & 13 \end{bmatrix} \neq 0]$$

Example 5

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

Solution

Step : 1 ***To find eigenvalues :***

The given matrix is a ***triangular matrix.***

For a triangular matrix, the eigenvalues are its leading diagonal element (property).

$\therefore$ The eigenvalues of the given matrix is $\lambda = 2, 3, 5$

Step : 2 ***To find eigenvectors :***

The eigenvectors are given by $|A - \lambda I| = 0$, where λ is the eigenvalue and X is the eigenvector.

(6)

i.e., $$\begin{pmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e., $$\left.\begin{aligned} (3-\lambda)x_1 + x_2 + 4x_3 &= 0 \\ 0x_1 + (2-\lambda)x_2 + 6x_3 &= 0 \\ 0x_1 + 0x_2 + (5-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots (A)$$

From the above three equations we can determine the components x_1, x_2, x_3 of the eigenvector X for different eigenvalues.

__*Case (i) :* When $\lambda = 2$__

To find the eigenvector $X_1 = (x_1, x_2, x_3)$ corresponding to $\lambda = 2$.

When $\lambda = 2$, the above set of equations (A) becomes

$$x_1 + x_2 + 4x_3 = 0 \quad \ldots (1)$$

$$0x_1 + 0x_2 + 6x_3 = 0 \quad \ldots (2)$$

$$0x_1 + 0x_2 + 3x_3 = 0 \quad \ldots (3)$$

(2) & (3) $\Rightarrow$ $x_3 = 0$

(1) $\Rightarrow$ $x_1 + x_2 = 0$ $[\because x_3 = 0]$

i.e., $x_1 = -x_2$

$\therefore$ Putting $x_2 = k$, we get $x_1 = -k$.

$\therefore$ The eigenvector $X_1 = \begin{pmatrix} -k \\ k \\ 0 \end{pmatrix}$

The simplest eigenvector is $X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

__*Case (ii) :* When $\lambda = 3$__, equations (A) becomes

$$0x_1 + x_2 + 4x_3 = 0 \quad \ldots (4)$$

$$0x_1 - x_2 + 6x_3 = 0 \quad \ldots (5)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \quad \ldots (6)$$

(6) $\Rightarrow$ $x_3 = 0$

Substituting $x_3 = 0$ in (5), we get

$$x_2 = 0$$

Since $x_2 = 0$ & $x_3 = 0$,

(4) $\Rightarrow$ $0x_1 = 0$

$\Rightarrow$ x_1 takes any value

$$\Rightarrow \qquad x_1 = k$$

$\therefore$ The eigenvector corresponding to $\lambda = 3$ is

$$X_2 = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$$

i.e.,

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Case (iii) : When $\lambda = 5$, equations (A) becomes

$$-2x_1 + x_2 + 4x_3 = 0 \qquad \ldots (7)$$

$$0x_1 - 3x_2 + 6x_3 = 0 \qquad \ldots (8)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \qquad \ldots (9)$$

$$(8) \Rightarrow \qquad x_2 = 2x_3$$

Taking $x_3 = k$, we get $x_2 = 2k$

Substituting $x_2 = 2k, x_3 = k$ in (7), we get

$$-2x_1 + 2k + 4k = 0$$

$$\Rightarrow \qquad -2x_1 = -6k$$

$$\Rightarrow \qquad x_1 = 3k$$

$\therefore$ The eigenvector is

$$X_3 = \begin{pmatrix} 3k \\ 2k \\ k \end{pmatrix}$$

The simplest eigenvector is

$$X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Conclusion :

Eigenvalues	Eigenvector
$\lambda_1 = 2$	$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
$\lambda_2 = 2$	$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$\lambda_3 = 5$	$X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

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Example 6

Find the eigenvalues and eigenvectors of $\begin{pmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{pmatrix}$. *[A.U., 1999]*

Solution

Step : 1 *To find characteristic equation :*

Let the given matrix be $A = \begin{pmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{pmatrix}$.

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where $a_1 = \left\{ \begin{array}{l} \text{Sum of leading} \\ \text{diagonal elements} \end{array} \right. = 4 + 10 - 13$

$= 1$

$a_2 = \left\{ \begin{array}{l} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{array} \right.$

$= \begin{vmatrix} 10 & 4 \\ -30 & -13 \end{vmatrix} + \begin{vmatrix} 4 & -10 \\ 6 & -13 \end{vmatrix} + \begin{vmatrix} 4 & -20 \\ -2 & 10 \end{vmatrix}$

$= -130 + 120 - 52 + 60 + 40 - 40$

$= -2$

$a_3 = |A| = \begin{vmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{vmatrix}$

$= 4[-130 + 120] + 20[26 - 24] - 10[60 - 60]$

$= -40 + 40 + 0$

$= 0$

$\therefore$ The characteristic equation is

$$\lambda^3 - \lambda^2 - 2\lambda = 0$$

Step : 2 *To find eigenvalues :*

$$\lambda^3 - \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda^2 - \lambda - 2) = 0$$

When $\lambda = 0$, $\lambda^2 - \lambda - 2 = 0$

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$= 2$ or -1

$\therefore$ Eigenvalues are $\lambda = 0, -1, 2$

Step : 3 *To find eigenvectors :*

The eigenvectors are given by the equation $(A - \lambda I)\,X = 0$ where λ is the eigenvalue and X is the eigenvector.

i.e.,
$$\begin{pmatrix} 4-\lambda & -20 & -10 \\ -2 & 10-\lambda & 4 \\ 6 & -30 & -13-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.,
$$\left.\begin{aligned} (4-\lambda)\,x_1 - 20x_2 - 10x_3 &= 0 \\ -2x_1 + (10-\lambda)\,x_2 + 4x_3 &= 0 \\ 6x_1 - 30x_2 - (13+\lambda)\,x_3 &= 0 \end{aligned}\right\} \quad \ldots \text{(A)}$$

From the above three equations we can determine the components x_1, x_2, x_3 of the eigenvector X for different values of λ.

<u>Case (i) : When $\lambda = 0$</u>

To find the eigenvector $X_1 = (x_1, x_2, x_3)$ corresponding to $\lambda = 0$.

When $\lambda = 0$ the above set of equations (A) becomes

$$\begin{aligned} 4x_1 - 20x_2 - 10x_3 &= 0 \\ -2x_1 + 10x_2 + 4x_3 &= 0 \\ 6x_1 - 30x_2 - 13x_3 &= 0 \end{aligned}$$

To find x_1, x_2, x_3 we can take any two of the above equations, say first two equations. By cross rule method we have

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ \hline -20 & & -10 & & 4 & & -20 \\ 10 & \times & 4 & \times & -2 & \times & 10 \end{array}$$

i.e.,
$$\frac{x_1}{-80+100} = \frac{x_2}{20-16} = \frac{x_3}{40-40}$$

i.e.,
$$\frac{x_1}{20} = \frac{x_2}{4} = \frac{x_3}{0} = k$$

i.e.,
$$x_1 = 20k,\ x_2 = 4k,\ x_3 = 0$$

Hence the general eigenvector corresponding to $\lambda = 0$ is $\begin{pmatrix} 20k \\ 4k \\ 0 \end{pmatrix}$

and by taking $k = \frac{1}{4}$ we get the simplest eigenvector viz $X_1 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$.

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<u>*Case (ii)* : **When $\lambda = -1$**</u>

To find the eigenvector $X_2 = (x_1, x_2, x_3)$.

When $\lambda = -1$, the set of equations (A) becomes

$$\begin{aligned} 5x_1 - 20x_2 - 10x_3 &= 0 \\ -2x_1 + 11x_2 + 4x_3 &= 0 \\ 6x_1 - 30x_2 - 12x_3 &= 0 \end{aligned}$$

To find x_1, x_2, x_3 we consider any two of the above equations, say first two. By Cross rule method we have,

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ \hline -20 & -10 & 5 & -20 \\ 11 & 4 & -2 & 11 \end{array}$$

i.e.,
$$\frac{x_1}{-80+110} = \frac{x_2}{20-20} = \frac{x_3}{55-40}$$

i.e.,
$$\frac{x_1}{30} = \frac{x_2}{0} = \frac{x_3}{15} = k$$

i.e.,
$$x_1 = 30k, \quad x_2 = 0k, \quad x_3 = 15k$$

Therefore the general eigenvector corresponding to the eigenvalue $\lambda = 1$ is $\begin{pmatrix} 30k \\ 0k \\ 15k \end{pmatrix}$.

When $k = \frac{1}{15}$ we get the simplest eigenvector $X_2 = (2, 0, 1)$.

<u>*Case (iii)* : **When $\lambda = 2$**</u>

To find the eigenvector X_3 when $\lambda = 2$, the set of equations (A) becomes

$$\begin{aligned} 2x_1 - 20x_2 - 10x_3 &= 0 \\ -2x_1 + 8x_2 + 4x_3 &= 0 \\ 6x_1 - 30x_2 - 15x_3 &= 0 \end{aligned}$$

Taking first two equations and applying cross rule method we have

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ \hline -20 & -10 & 2 & -20 \\ 8 & 4 & -2 & 8 \end{array}$$

i.e.,
$$\frac{x_1}{-80+80} = \frac{x_2}{20-8} = \frac{x_3}{16-40} = k$$

i.e., $$\frac{x_1}{0} = \frac{x_2}{12} = \frac{x_3}{-24} = k$$

i.e., $$x_1 = 0k,\ x_2 = 12k,\ x_3 = -24k$$

Hence the simplest eigenvector can be obtained by putting $k = \frac{1}{12}$

Therefore the eigenvector corresponding to $\lambda = 2$ is $X_3 = (0, 1, -2)$

Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
$\lambda^3 - \lambda^2 - 2\lambda = 0$	$\lambda_1 = 0$	$X_1 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$
	$\lambda_2 = -1$	$X_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$
	$\lambda_3 = 2$	$X_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

These three eigenvectors are linearly independent.

❒ PROBLEMS BASED ON NON-SYMMETRIC MATRICES WITH REPEATED EIGENVALUES ❒

Example 7 *Find the eigenvalues and eigenvectors of*

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution

Step : I *To find characteristic equation and eigenvalues :*

Let $$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic equation is $|A - \lambda I| = 0$.

i.e., $$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

i.e., $(2-\lambda)(2-\lambda)(2-\lambda) = 0$

i.e., $\boxed{\lambda = 2, 2, 2}$

$\therefore$ The eigenvalues are 2, 2, 2.

Step : 2 ***To find eigenvectors :***

The eigenvector $X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is given by the equation

$$(A - \lambda I) X = 0$$

i.e., $$\begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e., $$\left.\begin{aligned} (2-\lambda)x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots (1)$$

Case (i) : <u>When $\lambda = 2$</u>, the system of equations (1) become

$$0x_1 + x_2 + 0x_3 = 0 \quad \ldots (2)$$

$$0x_1 + 0x_2 + x_3 = 0 \quad \ldots (3)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \quad \ldots (4)$$

Taking equation (2) and (3) and applying cross rule method, we get

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ \hline 1 & & 0 & & 0 & & 1 \\ 0 & & 1 & & 0 & & 0 \end{array}$$

i.e., $$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} = k$$

i.e., $$x_1 = k, \; x_2 = 0, \; x_3 = 0$$

$\therefore$ The eigenvector corresponding to $\lambda = 2$ is $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

[Taking $k = 1$]

The second and third eigenvector is also same as X_1.

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Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
	$\lambda_1 = 2$	$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$(2-\lambda)^3 = 0$	$\lambda_2 = 2$	$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\lambda_3 = 2$	$X_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ These three eigenvectors are linearly dependent.

Example 8 ***Find the eigenvalues and eigenvectors of***

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution

Step : 1 ***To find characteristic equation:***

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \quad \text{... (1)}$$

where

$$a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases} = 2+2+1$$

$$= 5$$

$$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$$

$$= \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 2-0+2-0+4-1$$

$$= 7$$

$$a_3 = |A|$$

$$= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(2-0) - 1(1-0) + 1(0-0)$$

$$= 4-1 = 3$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

Step : 2 ***To find eigenvalues :***

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

When $\lambda = 1$,

$$1 - 5 + 7 - 3 = 0$$

$\therefore$ 1 is a root.

$$\begin{array}{c|cccc} 1 & 1 & -5 & 7 & -3 \\ & 0 & 1 & -4 & 3 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2}$$

$$= 3 \text{ or } 1$$

$\therefore$ Eigenvalues are $\boxed{\lambda = 1, 1, 3}$

Step : 3 ***To find eigenvectors :***

The eigenvectors $\mathbf{X} = (x_1, x_2, x_3)$ is given by the equation $(A - \lambda I)\,X = 0$

i.e., $$\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e., $$\left.\begin{aligned} (2-\lambda)\,x_1 + x_2 + x_3 &= 0 \\ x_1 + (2-\lambda)\,x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + (1-\lambda)\,x_3 &= 0 \end{aligned}\right\} \quad \ldots (A)$$

Case (i) : When $\lambda = 1$, we get from (A)

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 0$$
$$0x_1 + 0x_2 + 0x_3 = 0$$

The above equations reduces to

$$x_1 + x_2 + x_3 = 0$$

i.e., $$x_1 = -x_2 - x_3$$

Taking $x_2 = k_1, x_3 = k_2$, we get $x_1 = -k_1 - k_2$.

$\therefore$ The eigenvector corresponding to $\lambda = 1$ is $X_1 = \begin{pmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{pmatrix}$.

The simplest eigenvector is obtained by taking $k_1 = 1, k_2 = 0$ *i.e.,*

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

By taking $k_1 = -1$, $k_2 = 0$, we get one more eigenvector

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\therefore$ The two eigenvectors corresponding to $\lambda = 1$ are
$X_1 = (-1, 1, 0)$ and $X_2 = (1, -1, 0)$.

Case (ii) : When $\lambda = 3$. We get from (A),

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 - 2x_3 &= 0 \end{aligned}$$

Considering first two equations and applying rule of cross multiplication we get

$$\begin{array}{cccccccc} & x_1 & & x_2 & & & x_3 & \\ \hline 1 & & 1 & & -1 & & 1 & \\ -1 & & 1 & & 1 & & -1 & \end{array}$$

i.e., $$\frac{x_1}{1+1} = \frac{x_2}{1+1} = \frac{x_3}{1-1}$$

i.e., $$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0} = k$$

i.e., $$x_1 = 2k,\ x_2 = 2k,\ x_3 = 0k$$

The simplest eigenvector is obtained by taking $k = \frac{1}{2}$

$\therefore$ The third eigenvector is $X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$	$\lambda_1 = 1$	$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
	$\lambda_2 = 1$	$X_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$
	$\lambda_3 = 3$	$X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Example 9 *Find the eigenvalues and eigenvectors of the matrix* $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Solution

Step : 1 ***To find characteristic equation:***

Let $$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \qquad \ldots (1)$$

where
$$a_1 = \begin{cases}\text{Sum of leading} \\ \text{diagonal elements}\end{cases}$$
$$= -2+1+0$$
$$= -1$$
$$a_2 = \begin{cases}\text{Sum of the minors of the} \\ \text{leading diagonal elements}\end{cases}$$
$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$
$$= 0 - 12 + 0 - 3 - 2 - 4$$
$$= -21$$
$$a_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$
$$= -2(0-12) - 2(0-6) - 3(-4+1)$$
$$= 24 + 12 + 9$$
$$= 45$$

$\therefore$ The characteristic equation is

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

Step : 2 ***To find eigenvalues :***

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

When $\lambda = -3$, $(-3)^3 + (-3)^2 - 21(-3) - 45$
$$= -27 + 9 + 63 - 45$$
$$= 0$$

$\therefore \lambda = -3$ is a root.

$$\begin{array}{r|cccc} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$$\lambda^2 - 2\lambda - 15 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4+60}}{2} = \frac{2 \pm 8}{2}$$

$$= 5 \text{ or } -3$$

$\therefore$ Eigenvalues are $\quad \lambda = -3, -3, 5$

Step : 3 ***To find eigenvectors :***

The eigenvectors $X = (x_1, x_2, x_3)$ is given by the equation

$$(A - \lambda I) X = 0 \quad (\lambda \text{ - eigenvalue})$$

i.e.,
$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.,
$$\left.\begin{aligned} (-2-\lambda) x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + (1-\lambda) x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - \lambda x_3 &= 0 \end{aligned}\right\} \quad \text{... (A)}$$

<u>Case (i) : When $\lambda = -3$</u> (two values), the set of equations (A) become

$$x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 + 4x_2 - 6x_3 = 0$$
$$-x_1 - 2x_2 + 3x_3 = 0$$

The above equations is equivalent to one independent equation viz., $\quad x_1 + 2x_2 - 3x_3 = 0$

Hence by giving arbitrary values to any two variables we get the value of the third variable in terms of the other two. Suppose if we give $x_2 = k_1, x_3 = k_2$ we get $x_1 = 3k_2 - 2k_1$.

Therefore the general eigenvector corresponding to $\lambda = -3$ is

$$X = \begin{pmatrix} 3k_2 - 2k_1 \\ k_1 \\ k_2 \end{pmatrix}$$

Independent eigenvectors can be obtained by giving arbitrary values to two of the quantities x_1, x_2, x_3. Giving $k_1 = 0,\ k_2 = 1$ we get the eigenvector $X_1 = (3, 0, 1)$.

Giving $k_1 = 1, k_2 = 0$ we get the eigenvector $X_2 = (-2, 1, 0)$.

Therefore the two eigenvectors corresponding to $\lambda = -3$ are (3, 0, 1) and (−2, 1, 0).

Case (ii) : When $\lambda = 5$, the set of equations (A) become

$$\begin{aligned} -7x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - 4x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - 5x_3 &= 0 \end{aligned}$$

To find x_1, x_2, x_3 we consider any two of the above equations say first two equations

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ \hline 2 & & -3 & & -7 & & 2 \\ -4 & & -6 & & 2 & & -4 \end{array}$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4} = k$$

i.e., $$x_1 = -24k,\ x_2 = -48k,\ x_3 = 24k$$

Hence the general eigenvector is $\begin{pmatrix} -24k \\ -48k \\ 24k \end{pmatrix}$.

The simplest is $X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ by taking $k = \frac{-1}{24}$.

Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$	$\lambda_1 = -3$	$X_1 = (3, 0, 1)$
	$\lambda_2 = -3$	$X_2 = (-2, 1, 0)$
	$\lambda_3 = 5$	$X_3 = (1, 2, -1)$

NOTE : The three eigenvectors (3, 0, 1); (−2, 1, 0); (1, 2, −1) are linearly independent even though the two eigenvalues are equal.

$$\left[\because |A| = \begin{vmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & 2 & -1 \end{vmatrix} \neq 0\right]$$

Example 10 *Find the eigen values and eigenvectors of* $\begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$. *[MKU, 1998]*

Solution

Step : 1 *To find characteristic equation:*

Let $$A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \qquad \ldots (1)$$

where $a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases}$

$$= 6 - 13 + 4$$
$$= -3$$

$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$

$$= \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$$
$$= -52 + 60 + 24 - 35 - 78 + 84$$
$$= -165 + 168$$
$$= 3$$

$$a_3 = |A| = \begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix}$$
$$= 6(-52 + 60) + 6(56 - 70) + 5(-84 + 91)$$
$$= 48 - 84 + 35$$
$$= -1$$

$\therefore$ The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

Step : 2 *To find eigenvalues :*

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$
$$(\lambda + 1)^3 = 0$$
$$\therefore \quad \lambda = -1, -1, -1$$

$\therefore$ Eigenvalues are $\lambda = -1, -1, -1$

Step : 3 *To find eigenvectors :*

The eigenvectors $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is given by the equation

$$(A - \lambda I)\, X = 0$$

i.e.,
$$\begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

i.e.,
$$\left.\begin{aligned} (6-\lambda)\, x_1 - 6x_2 + 5x_3 &= 0 \\ 14x_1 - (13+\lambda)\, x_2 + 10x_3 &= 0 \\ 7x_1 - 6x_2 + (4-\lambda)\, x_3 &= 0 \end{aligned}\right\} \quad \ldots (1)$$

Case (i) : When $\lambda = -1$, the system of equations (1) become

$$\begin{aligned} 7x_1 - 6x_2 + 5x_3 &= 0 \\ 14x_1 - 12x_2 + 10x_3 &= 0 \\ 7x_1 - 6x_2 + 5x_3 &= 0 \end{aligned}$$

The above equations represent the same equation $7x_1 - 6x_2 + 5x_3 = 0$.

Therefore a number of eigenvectors can be obtained by giving arbitrary values to any two of the quantities x_1, x_2, x_3. The above equations can be written as

$$7x_1 = 6x_2 - 5x_3$$

(or)
$$x_1 = \frac{6}{7}\, k_1 - \frac{5}{7}\, k_2$$

where $x_2 = k_1, x_3 = k_2$.

Therefore the general eigenvector is $\begin{pmatrix} \frac{6}{7}k_1 - \frac{5}{7}k_2 \\ k_1 \\ k_2 \end{pmatrix}$.

Putting $k_1 = 0, k_2 = 7$, we get, $x_1 = -5$

i.e., one eigenvector is $X_1 = (-5, 0, 7)$.

Putting $k_2 = 0, k_1 = 7$, we get, $x_1 = 6$

$\therefore$ Another eigenvector is $X_2 = (6, 7, 0)$.

Putting $k_1 = 5, k_2 = 6$, we get, $x_1 = 0$

One more eigenvector is $X_3 = (0, 5, 6)$

Clearly these three vectors are linearly independent.

Since $\begin{vmatrix} -5 & 0 & 7 \\ 6 & 7 & 0 \\ 0 & 5 & 6 \end{vmatrix} \neq 0$

Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
$(\lambda + 1)^3 = 0$	$\lambda_1 = -1$	$X_1 = (-5, 0, 7)$
	$\lambda_2 = -1$	$X_2 = (6, 7, 0)$
	$\lambda_3 = -1$	$X_3 = (0, 5, 6)$

NOTE : Here all the eigenvalues are equal.

Example 11

Find the eigenvalues and eigenvectors of the matrix

$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$. *[A.U., 2000]*

Solution

Step : 1 ***To find characteristic equation:***

Let $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \qquad \ldots (1)$$

where $a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases}$

$= 3 - 3 + 7$

$= 7$

$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$

$= \begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix}$

$= -21 + 20 + 21 - 15 - 9 + 20$

$= 16$

$$a_3 = |A| = \begin{vmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= 3(-21+20) - 10(-14+12) + 5(-10+9)$$

$$= -3 + 20 - 5$$

$$= 12$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

Step : 2 ***To find eigenvalues :***

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

When $\lambda = 2$, $(2)^3 - 7(2)^2 + 16(2) - 12$

$$= 8 - 28 + 32 - 12$$

$$= 0$$

$\therefore$ $\lambda = 2$ is a root.

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2}$$

$$= 3 \text{ or } 2$$

$\therefore$ Eigenvalues are $\lambda = 2, 2, 3$

Step : 3 ***To find eigenvectors :***

The eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is given by

$$\begin{pmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case (i) : When $\lambda = 3$, we get

$$0x_1 + 10x_2 + 5x_3 = 0$$

$$-2x_1 - 6x_2 - 4x_3 = 0$$

$$3x_1 + 5x_2 + 4x_3 = 0$$

Considering last two equations, we get,

$$\frac{x_1}{-24+20} = \frac{x_2}{-12+8} = \frac{x_3}{-10+18}$$

i.e., $$\frac{x_1}{-4} = \frac{x_2}{-4} = \frac{x_3}{8}$$

i.e., $$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2}$$

∴ The eigenvector when $\lambda = 3$ is

$$X_1 = (1, 1, -2)$$

Case (ii) : When $\lambda = 2$ (equal root), we get,

$$x_1 + 10x_2 + 5x_3 = 0$$
$$-2x_1 - 5x_2 - 4x_3 = 0$$
$$3x_1 + 5x_2 + 5x_3 = 0$$

From the last two equations, we get,

$$\frac{x_1}{-25+20} = \frac{x_2}{-12+10} = \frac{x_3}{-10+15}$$

i.e., $$\frac{x_1}{5} = \frac{x_2}{2} = \frac{x_3}{-5}$$

The eigenvector when $\lambda = 2$ is

$$X_2 = (5, 2, -5).$$

The third eigenvector is also same as X_2.

Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$	$\lambda_1 = 3$	$X_1 = (1, 1, -2)$
	$\lambda_2 = \lambda_3 = 2$	$X_2 = X_3 = (5, 2, -5)$

NOTE : Two eigenvalues are equal but the eigenvectors are linearly dependent.

∴ These three eigenvectors are linearly dependent

$$\left[\because \begin{vmatrix} 1 & 5 & 5 \\ 1 & 2 & 2 \\ -2 & -5 & -5 \end{vmatrix} = 0\right]$$

Example 12

Find eigenvalues and eigenvectors of the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Solution

Step : 1 ***To find characteristic equation:***

Let
$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Characteristic equation is

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = 0$$
$$(a-\lambda)^2 + b^2 = 0$$
$$\lambda^2 - 2a\lambda + a^2 + b^2 = 0$$

Step : 2 ***To find eigenvalues :***

$$\lambda^2 - 2a\lambda + a^2 + b^2 = 0 \quad \text{(Quadratic in } \lambda)$$
$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4a^2 - 4b^2}}{2}$$
$$= a \pm ib$$

The two eigenvalues are $\boxed{\begin{array}{l}\lambda = a + ib \\ \lambda = a - ib\end{array}}$

Step : 3 ***To find eigenvectors :***

The eigenvectors X are given by

$$(a - \lambda I)\,X = 0$$
$$\begin{pmatrix} a-\lambda & b \\ -b & a-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
$$(a-\lambda)\,x_1 + bx_2 = 0$$
$$-bx_1 + (a-\lambda)\,x_2 = 0$$

Case (i) : When $\lambda = a + ib$,

$$-ibx_1 + bx_2 = 0$$
$$-bx_1 - ibx_2 = 0$$

i.e.,
$$-ix_1 = -x_2$$
$\Rightarrow$
$$ix_1 = x_2$$

i.e., $$\frac{x_1}{1} = \frac{x_2}{i}$$

Hence the eigenvector is $X_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Case (ii) : When $\lambda = a - ib$, we have

$$ibx_1 + bx_2 = 0$$

$$-bx_1 + ibx_2 = 0$$

i.e., $$ix_1 = -x_2$$

i.e., $$\frac{x_1}{1} = \frac{x_2}{-i}$$

$\therefore$ The eigenvector is $X_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Conclusion :

Characteristic equation	Eigenvalues	Eigenvector
$\lambda^2 - 2a\lambda + a^2 + b^2 = 0$	$\lambda_1 = a + ib$	$X_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$
	$\lambda_2 = a - ib$	$X_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

❒ PROBLEMS UNDER SYMMETRIC MATRICES WITH DIFFERENT EIGENVALUES ❒

Example 13 *Find the eigenvalues and eigenvectors of*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$

Solution

Step : 1 *To find characteristic equation and eigenvalues :*

The characteristic equation is $|A - \lambda I| = 0$.

i.e., $$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

i.e., $$(1-\lambda)\,[(3-\lambda)^2 - 1] = 0$$

$$1 - \lambda = 0 \text{ or } \quad 9 + \lambda^2 - 6\lambda - 1 = 0$$

$$\lambda = 1 \text{ or } \quad \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda = 1 \text{ or } \qquad \lambda = \frac{6 \pm \sqrt{36-32}}{2}$$

$$\lambda = 1 \text{ or } \qquad \lambda = \frac{6 \pm 2}{2}$$

$$\Rightarrow \qquad \lambda = 4 \text{ or } 2$$

$\therefore$ The eigenvalues are $\lambda = 1, 2$ and 4.

Step : 2 ***To find eigenvectors :***

The eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is given by

$$(A - \lambda I)\, X = 0.$$

i.e.,
$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.,
$$\left.\begin{aligned} (1-\lambda)\, x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + (3-\lambda)\, x_2 - x_3 &= 0 \\ 0x_1 - x_2 + (3-\lambda)\, x_3 &= 0 \end{aligned}\right\} \qquad \ldots (A)$$

<u>*Case (i)* : When $\lambda = 1$</u>, we get from (A),

$$\begin{aligned} 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 2x_2 - x_3 &= 0 \\ 0x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

Taking 2nd and 3rd equations and applying cross rule method, we get

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ \hline 2 & & -1 & & 0 & & 2 \\ -1 & & 2 & & 0 & & -1 \end{array}$$

i.e.,
$$\frac{x_1}{4-1} = \frac{x_2}{0-0} = \frac{x_3}{0-0} = k$$

i.e.,
$$x_1 = 3k,\ x_2 = 0,\ x_3 = 0$$

$\therefore$ The simplest eigenvector corresponding to the eigenvalue $\lambda = 1$ is

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \qquad [\text{Taking } k = \frac{1}{3}]$$

<u>*Case (ii)* : When $\lambda = 2$</u>, we get from (A) ,

$$\begin{aligned} -x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + x_2 - x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= 0 \end{aligned}$$

Taking first two equations and applying cross rule method, we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ \hline 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{array}$$

i.e., $$\frac{x_1}{0-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

i.e., $$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1} = k$$

i.e., $$x_1 = 0,\ x_2 = -k,\ x_3 = -k$$

$\therefore$ The simplest eigenvector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. [By taking $k = -1$]

Case (iii) : **When $\lambda = 4$**, we get from (A)

$$-3x_1 + 0x_2 + 0x_3 = 0$$
$$0x_1 - x_2 - x_3 = 0$$
$$0x_1 - x_2 - x_3 = 0$$

Taking first two equations and applying cross rule method, we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ \hline 0 & 0 & -3 & 0 \\ -1 & -1 & 0 & -1 \end{array}$$

i.e., $$\frac{x_1}{0-0} = \frac{x_2}{0-3} = \frac{x_3}{3-0} = k$$

i.e., $$\frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3} = k$$

i.e., $$x_1 = 0k,\ x_2 = -3k,\ x_3 = 3k$$

$\therefore$ The simplest eigenvector is $X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. [Taking $k = \frac{1}{3}$]

Conclusion :

Characteristic Equation	Eigenvalues	Eigenvector
$(1-\lambda)(\lambda^2 - 6\lambda + 8) = 0$	$\lambda_1 = 1$	$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\lambda_2 = 2$	$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
	$\lambda_3 = 4$	$X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Example 14

Solution

Step : 1 ***To find characteristic equation and eigenvalues:***

The characteristic equation is $|A - \lambda I| = 0$.

i.e., $\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

i.e., $(2-\lambda)^3 - (2-\lambda) = 0$

$\Rightarrow$ $(2-\lambda)[(2-\lambda)^2 - 1] = 0$

$\Rightarrow$ $(2-\lambda)(4+\lambda^2-4\lambda-1) = 0$

$(2-\lambda)(\lambda^2-4\lambda+3) = 0$

$\lambda = 2,$ $\lambda^2 - 4\lambda + 3 = 0$

$$\lambda = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2}$$

$$= 3 \text{ or } 1$$

$\therefore$ The eigenvalues are $\lambda = 1, 2, 3$.

Step : 2 ***To find eigenvectors :***

The eigenvectors $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the solution of the equation

$$(A - \lambda I)X = 0$$

i.e., $\begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e., $\left.\begin{aligned} (2-\lambda)x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 + (2-\lambda)x_2 + 0x_3 &= 0 \\ x_1 + 0x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\}$ ... (A)

From the above three equations we can determine the components x_1, x_2, x_3 of the eigenvector X for different eigenvalues λ.

Case (i) : When $\lambda = 1$, the equations (A) becomes

$$x_1 + 0x_2 + x_3 = 0 \quad \dots (1)$$
$$0x_1 + x_2 + 0x_3 = 0 \quad \dots (2)$$
$$x_1 + 0x_2 + x_3 = 0 \quad \dots (3)$$

(2) $\Rightarrow$ $x_2 = 0$

(1) & (3) $\Rightarrow$ $x_1 + x_3 = 0$

i.e., $x_1 = -x_3$

$\therefore$ Putting $x_3 = k$, we get $x_1 = -k$.

$\therefore$ The eigenvector corresponding to $\lambda = 1$

$$X = \begin{pmatrix} -k \\ 0 \\ k \end{pmatrix}$$

The simplest eigenvector is $X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ [Taking $k = 1$]

Case (ii) : When $\lambda = 2$, the equations (A) becomes

$$0x_1 + 0x_2 + x_3 = 0 \quad \dots (4)$$
$$0x_1 + 0x_2 + 0x_3 = 0 \quad \dots (5)$$
$$x_1 + 0x_2 + 0x_3 = 0 \quad \dots (6)$$

From (4) $\Rightarrow$ $x_3 = 0$

From (6) $\Rightarrow$ $x_1 = 0$

and clearly x_2 takes any value (From (5)).

$\therefore$ The eigenvector corresponding to $\lambda = 2$ is

$$X = \begin{pmatrix} 0 \\ k_2 \\ 0 \end{pmatrix}$$

i.e., $X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Case (iii) : When $\lambda = 3$, the equations (A) becomes

$$-x_1 + 0x_2 + x_3 = 0 \quad \dots (7)$$
$$0x_1 - x_2 + 0x_3 = 0 \quad \dots (8)$$
$$x_1 + 0x_2 - x_3 = 0 \quad \dots (9)$$

From (7) & (9) $\Rightarrow$ $x_1 = x_3$

From (8) $\Rightarrow$ $x_2 = 0$

Putting $x_3 = k$, we get $x_1 = k$

$\therefore$ The eigenvector corresponding to $\lambda = 3$ is

$$X = \begin{pmatrix} k \\ 0 \\ k \end{pmatrix}$$

The simplest eigenvector is $\quad X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad$ (Taking $k = 1$)

Conclusion :

Characteristic Equation	Eigenvalues	Eigenvector
$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$	$\lambda_1 = 2$	$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
	$\lambda_2 = 2$	$X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
	$\lambda_3 = 2$	$X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Example 15 ***Find the eigenvalues and eigenvectors of***

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Solution

Step : I ***To find characteristic equation:***

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \qquad \ldots (1)$$

where
$$a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases}$$
$$= 3 + 5 + 3$$
$$= 11$$
$$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$$
$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$
$$= 15 - 1 + 9 - 1 + 15 - 1$$
$$= 36$$

$$a_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 42 - 2 - 4$$

$$= 36$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

Step : 2 *To find eigenvalues :*

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

When $\lambda = 2$, $(2)^3 - 11(2)^2 + 36(2) - 36$

$$= 8 - 44 + 72 - 36$$

$$= 0$$

$\therefore$ $\lambda = 2$ is a root.

$$\begin{array}{c|cccc} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda = \frac{9 \pm \sqrt{81-72}}{2} = \frac{9 \pm 3}{2}$$

$$= 6 \text{ or } 3$$

$\therefore$ Eigenvalues are $\lambda = 2, 3, 6$

Step : 3 *To find eigenvectors :*

The eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is given by

$$(A - \lambda I)X = 0.$$

i.e.,

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.,

$$\left.\begin{aligned} (3-\lambda)x_1 - x_2 + x_3 &= 0 \\ -x_1 + (5-\lambda)x_2 - x_3 &= 0 \\ x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots (A)$$

<u>*Case (i)* : **When $\lambda = 2$**</u>, we get from (A),

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + 3x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}$$

Taking first two equations and applying cross rule method, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ \hline -1 & & 1 & & 1 & & -1 \\ 3 & \times & -1 & \times & -1 & \times & 3 \end{array}$$

i.e., $$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$$

i.e., $$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} = k$$

i.e., $$x_1 = -2k,\ x_2 = 0,\ x_3 = 2k$$

$\therefore$ The simplest eigenvector is obtained by putting $k = \frac{1}{2}$

i.e., $$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

<u>*Case (ii)* : **When $\lambda = 3$**</u>, we get from (A),

$$\begin{aligned} 0x_1 - x_2 + x_3 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + 0x_3 &= 0 \end{aligned}$$

Taking first two equations and applying cross rule method, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ \hline -1 & & 1 & & 0 & & -1 \\ 2 & \times & -1 & \times & -1 & \times & 2 \end{array}$$

i.e., $$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1}$$

i.e., $$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} = k$$

i.e., $$x_1 = -k,\ x_2 = -k,\ x_3 = -k$$

$\therefore$ The simplest eigenvector is obtained by putting $k = -1$

i.e., $$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Case (iii) : **When $\lambda = 6$, we get from (A)**

$$-3x_1 - x_2 + x_3 = 0$$
$$-x_1 - x_2 - x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$

Taking first two equations and applying cross rule method, we get

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline \begin{array}{cc} -1 & 1 \\ -1 & -1 \end{array} & \begin{array}{cc} & -3 \\ & -1 \end{array} & \begin{array}{c} -1 \\ -1 \end{array} \end{array}$$

i.e.,
$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

i.e.,
$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} = k$$

i.e.,
$$x_1 = 2k,\ x_2 = -4k,\ x_3 = 2k$$

$\therefore$ The simplest eigenvector is obtained by putting $k = \frac{1}{2}$

i.e.,
$$X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Conclusion :

Characteristic Equation	Eigenvalues	Eigenvector
$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$	$\lambda_1 = 2$	$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
	$\lambda_2 = 3$	$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
	$\lambda_3 = 6$	$X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Example 16 ***Find the eigenvalues and eigenvectors of***

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}.$$

Solution

Step : I ***To find characteristic equation:***

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where
$$a_1 = \left\{\begin{array}{l}\text{Sum of leading}\\ \text{diagonal elements}\end{array}\right. = 5-2+5$$
$$= 8$$
$$a_2 = \left\{\begin{array}{l}\text{Sum of the minors of the}\\ \text{leading diagonal elements}\end{array}\right.$$
$$= \begin{vmatrix}-2 & 0\\ 0 & 5\end{vmatrix} + \begin{vmatrix}5 & 1\\ 1 & 5\end{vmatrix} + \begin{vmatrix}5 & 0\\ 0 & -2\end{vmatrix}$$
$$= -10+0+25-1-10-0$$
$$= 4$$
$$a_3 = |A| = \begin{vmatrix}5 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 5\end{vmatrix}$$
$$= 5(-10-0)-0+1(0+2)$$
$$= -50+2$$
$$= -48$$

$\therefore$ The characteristic equation is
$$\lambda^3 - 8\lambda^2 + 4\lambda + 48 = 0$$

Step : 2 ***To find eigenvalues :***
$$\lambda^3 - 8\lambda^2 + 4\lambda + 48 = 0$$
When $\lambda = -2$, $(-2)^3 - 8(-2)^2 + 4(-2) + 48$
$$= -8-32-8+48$$
$$= 0$$

$\therefore$ $\lambda = -2$ is a root.

-2	1	-8	4	48
	0	-2	20	-48
	1	-10	24	0

$$\lambda^2 - 10\lambda + 24 = 0$$
$$\lambda = \frac{10 \pm \sqrt{100-96}}{2} = \frac{10\pm 2}{2}$$
$$= 6 \text{ or } 4$$

$\therefore$ Eigenvalues are $\lambda = -2, 4, 6$

Step : 3 ***To find eigenvectors :***

The eigenvector $X = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$ is given by
$$(A-\lambda I)X = 0$$

i.e., $$\begin{pmatrix} 5-\lambda & 0 & 1 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e., $$\left.\begin{aligned} (5-\lambda)x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 + (-2-\lambda)x_2 + 0x_3 &= 0 \\ x_1 + 0x_2 + (5-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \text{(A)}$$

<u>***Case (i) :*** **When $\lambda = -2$**</u>, we get from (A),

$$\begin{aligned} 7x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ x_1 + 0x_2 + 7x_3 &= 0 \end{aligned}$$

Taking 1^{st} and 3^{rd} equations and applying cross rule method, we get

$$\begin{array}{cccccccc} & x_1 & & & x_2 & & x_3 & \\ \hline 0 & & 1 & & 7 & & 0 \\ 0 & & 7 & & 1 & & 0 \end{array}$$

i.e., $$\frac{x_1}{0-0} = \frac{x_2}{1-49} = \frac{x_3}{0-0}$$

i.e., $$\frac{x_1}{0} = \frac{x_2}{-48} = \frac{x_3}{0} = k$$

i.e., $$x_1 = 0k,\ x_2 = -48k,\ x_3 = 0k$$

$\therefore$ The simplest eigenvector is obtained by putting $k = -\dfrac{1}{48}$

i.e., $$X_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

<u>***Case (ii) :*** **When $\lambda = 4$**</u>, we get

$$\begin{aligned} x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 - 6x_2 + 0x_3 &= 0 \\ x_1 + 0x_2 + x_3 &= 0 \end{aligned}$$

Taking 1^{st} and 2^{nd} equations and applying cross rule method, we get

$$\begin{array}{cccccccc} & x_1 & & & x_2 & & x_3 & \\ \hline 0 & & 1 & & 1 & & 0 \\ -6 & & 0 & & 0 & & -6 \end{array}$$

i.e., $$\frac{x_1}{0+6} = \frac{x_2}{0-0} = \frac{x_3}{-6-0}$$

i.e., $$\frac{x_1}{6} = \frac{x_2}{0} = \frac{x_3}{-6} = k$$

i.e., $$x_1 = 6k,\ x_2 = 0k,\ x_3 = -6k$$

$\therefore$ The simplest eigenvector is obtained by putting $k = \frac{1}{6}$.

i.e., $$X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Case (iii) : **When $\lambda = 6$**, we get from (A)

$$\begin{aligned} -x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 - 8x_2 + 0x_3 &= 0 \\ x_1 + 0x_2 - x_3 &= 0 \end{aligned}$$

Taking 1st and 2nd equations and applying cross rule method, we get

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ \hline 0 & & 1 & & -1 & & 0 \\ & \times & & \times & & \times & \\ -8 & & 0 & & 0 & & -8 \end{array}$$

i.e., $$\frac{x_1}{0+8} = \frac{x_2}{0-0} = \frac{x_3}{8-0}$$

i.e., $$\frac{x_1}{8} = \frac{x_2}{0} = \frac{x_3}{8} = k$$

i.e., $$x_1 = 8k,\ x_2 = 0k,\ x_3 = 8k$$

$\therefore$ The simplest eigenvector is obtained by putting $k = \frac{1}{8}$

i.e., $$X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Conclusion :

Characteristic Equation	Eigenvalues	Eigenvector
	$\lambda_1 = -2$	$X_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
$\lambda^3 - 8\lambda^2 + 4\lambda + 48 = 0$	$\lambda_2 = 4$	$X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
	$\lambda_3 = 6$	$X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Example 17 *Find the eigenvalues and eigenvectors of* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. *[AU, Part-time 1996]*

Solution

Step : 1 *To find characteristic equation and eigenvalues :*

Let the given matrix be $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

The characteristic equation is $|A - \lambda I| = 0$.

$$\text{i.e.,} \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\text{i.e.,} \quad (1-\lambda)\,[(1-\lambda)(1-\lambda)-1] = 0$$

$$\text{i.e.,} \quad (1-\lambda)\,[\lambda^2 - 2\lambda] = 0$$

$$\text{i.e.,} \quad (1-\lambda)\,\lambda\,(\lambda-2) = 0$$

$$\text{i.e.,} \quad \lambda = 0, 1, 2$$

Therefore the eigenvalues are $\lambda = 0, 1, 2$.

Step : 2 *To find eigenvectors :*

The eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the solution of the equation

$(A - \lambda I)\,X = 0$.

$$\text{i.e.,} \quad \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{i.e.,} \quad \left.\begin{aligned} (1-\lambda)x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + (1-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots (A)$$

By using (A) we can find eigenvectors corresponding to each eigenvalue.

From the above three equations we can determine the components x_1, x_2, x_3 of the eigenvector X for different values of λ.

<u>Case (i) : When $\lambda = 0$</u>, the set of equations (A) become

$$x_1 + 0x_2 + 0x_3 = 0$$
$$0x_1 + x_2 + x_3 = 0$$
$$0x_1 + x_2 + x_3 = 0$$

i.e., $x_1 = 0,\ x_2 = -x_3$

Putting $x_3 = k$, we get $x_1 = 0$, $x_2 = -k$, $x_3 = k$. Therefore the general eigenvector corresponding to the eigenvalue $\lambda = 0$ is $X = \begin{pmatrix} 0 \\ -k \\ k \end{pmatrix}$ and by taking $k = 1$ we get the simplest eigenvector $X = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

<u>*Case (ii)* : **When $\lambda = 1$**</u>, the set of equations (A) become

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 = 0$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 = 0$$

i.e., $1 \cdot x_3 = 0 \Rightarrow x_3 = 0 \qquad [\because 1 \neq 0]$

$1 \cdot x_2 = 0 \Rightarrow x_2 = 0 \qquad [\because 1 \neq 0]$

Since $0 \cdot x_1 = 0$, x_1 takes any value. Let it be equal to k.

Hence the general eigenvector corresponding to the eigenvalue $\lambda = 1$ is $X = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$ and simplest eigenvector $X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ can be obtained by putting $k = 1$.

<u>*Case (iii)* : **When $\lambda = 2$**</u>, the eigenvector $X = (x_1, x_2, x_3)$ is given by the set of equations

$$-1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$$
$$0 \cdot x_1 - 1 \cdot x_2 + 1 \cdot x_3 = 0$$
$$0 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3 = 0$$

i.e., $-1 \cdot x_1 = 0 \Rightarrow x_1 = 0 \qquad [\because 1 \neq 0]$

$-x_2 + x_3 = 0 \Rightarrow x_2 = x_3$

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

i.e., $x_1 = 0$

$x_2 = k$

$x_3 = k$

$\therefore$ The eigenvector corresponding to $\lambda = 2$ is $X = \begin{pmatrix} 0 \\ k \\ k \end{pmatrix}$ and the simplest eigenvector is $X = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Conclusion :

Characteristic Equation	Eigenvalues	Eigenvector
$(1-\lambda)\lambda(\lambda-2) = 0$	$\lambda_1 = -2$	$X_1 = (0, -1, 1)$
	$\lambda_2 = 4$	$X_2 = (1, 0, 0)$
	$\lambda_3 = 6$	$X_3 = (0, 1, 1)$

❒ PROBLEMS UNDER SYMMETRIC MATRICES WITH REPEATED EIGENVALUES ❒

Example 18

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution

Step : 1 ***To find characteristic equation:***

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where

$$\begin{aligned} a_1 &= \left\{ \begin{array}{l} \text{Sum of leading} \\ \text{diagonal elements} \end{array} \right. \\ &= 2 + 2 + 2 \\ &= 6 \\ a_2 &= \left\{ \begin{array}{l} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{array} \right. \\ &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\ &= 4 - 1 + 4 - 1 + 4 - 1 \\ &= 9 \\ a_3 &= |A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} \\ &= 2(4-1) + 1(-2+1) + 1(1-2) \end{aligned}$$

$$= 6 - 1 - 1$$
$$= 4$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 5 = 0$$

Step : 2 ***To find eigenvalues :***

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

When $\lambda = 1$, $(1)^3 - 6(1)^2 + 9(1) - 4$
$$= 1 - 6 + 9 - 4$$
$$= 0$$

$\therefore$ $\lambda = 1$ is a root.

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 9 & -4 \\ & 0 & 1 & -5 & 4 \\ \hline & 1 & -5 & 4 & 0 \end{array}$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2}$$
$$= 4 \text{ or } 1$$

$\therefore$ Eigenvalues are $\lambda = 1, 1, 4$

Step : 3 ***To find eigenvectors :***

The eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is given by the equation

$$(A - \lambda I)X = 0$$

i.e.,
$$\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.,
$$\left.\begin{aligned} (2-\lambda)x_1 - x_2 + x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 - x_3 &= 0 \\ x_1 - x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots (A)$$

Case (i) : When $\lambda = 1$, we get from (A)

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$

The above equations are the same.

9. $\begin{pmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ *Ans.* $\lambda = 0, 0; (3, -1, 0)$
$\lambda = 1; (12, -4, -1)$

10. $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *Ans.* $\lambda = 0; (1, -1, 0)$
$\lambda = 1; (0, 0, 1)$
$\lambda = 4; (1, 1, 0)$

III. Symmetric matrices with different eigenvalues.

11. $\begin{pmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{pmatrix}$ *Ans.* $\lambda = -3; (1, -1, 1)$
$\lambda = -6; (1, 0, -1)$
$\lambda = 12; (1, 2, 1)$

12. $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ *Ans.* $\lambda = 0; (1, 2, 2)$
$\lambda = 3; (2, 1, -2)$
$\lambda = 15; (2, -2, 1)$

13. $\begin{pmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{pmatrix}$ *Ans.* $\lambda = -2; (1, -1, 0)$
$\lambda = 9; (2, 2, -1)$
$\lambda = -18; (1, 1, 4)$

IV. Symmetric Matrices with repeated eigenvalues.

14. $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ *Ans.* $\lambda = 1; (1, 0, -1)$
$\lambda = 3; (1, 1, 1)$
$\lambda = 3; (1, -2, 1)$

CAYLEY-HAMILTON THEOREM
(Statement Only)

Every square matrix satisfies its own characteristic equation.

METHOD OF FINDING INVERSE OF A MATRIX

Let A be a non-singular matrix. *i.e.*, $|A| \neq 0$.

From the Cayley-Hamilton Theorem we have,

$$a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0 \quad \ldots(1)$$

Premultiplying equation (1) by A^{-1}, we get,

$$a_0 A^{n-1} + a_1 A^{n-2} + \ldots + a_n A^{-1} = 0 \qquad [\because A^{-1} I = A^{-1}]$$

i.e.,
$$a_n A^{-1} = -(a_0 A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I)$$

$$A^{-1} = -\frac{1}{a_n}(a_0 A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I)$$

Example 1 *Verify that* $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ *satisfies its own characteristic equation and hence find* A^4.

Solution

Step : 1 ***To find characteristic equation :***

Characteristic equation is $|A - \lambda I| = 0$

$$\text{i.e.,} \quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 4 = 0$$

$$-1 - \lambda + \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 5 = 0$$

Step : 2 ***To find*** A^4 **:**

By Cayley-Hamilton theorem, we have

$$A^2 - 5I = 0 \qquad \ldots (A)$$

$$\text{i.e.,} \quad \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence A satisfies its own characteristic equation.

Multiplying (A) by A^2, we get

$$A^4 - 5A^2 = 0$$

$$\Rightarrow \quad A^4 = 5A^2$$

$$= 5\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

Example 2 *Use Cayley-Hamilton theorem to find the inverse of* $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$.

Solution

Step : 1 ***To find characteristic equation :***

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & 3 \\ 2 & 6-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(6-\lambda) - 6 = 0$$
$$42 + \lambda^2 - 13\lambda - 6 = 0$$
$$\lambda^2 - 13\lambda + 36 = 0$$

Step : 2 *To find A^{-1} :*

By Cayley-Hamilton Theorem, we get

$$A^2 - 13A + 36I = 0$$

i.e., $$A^{-1}A^2 - 13A^{-1}A + 36A^{-1} = 0$$
$$A - 13I + 36A^{-1} = 0$$
$$A^{-1} = \frac{1}{36}[13I - A]$$
$$= \frac{1}{36}\left[\begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} - \begin{pmatrix} 7 & 3 \\ 2 & 6 \end{pmatrix}\right]$$
$$= \frac{1}{36}\begin{bmatrix} 6 & -3 \\ -2 & 7 \end{bmatrix}$$

Example 3 *State Cayley–Hamilton theorem and use it to find inverse of* $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

Solution

Step : 1 *To find characteristic equation :*

Cayley – Hamilton Theorem : Every square matrix satisfies its own characteristic equation.

$$\text{Given } A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

The characteristic equation is

i.e., $$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

i.e., $$(1-\lambda)(3-\lambda) - 8 = 0$$
$$3 - 4\lambda + \lambda^2 - 8 = 0$$
$$\lambda^2 - 4\lambda - 5I = 0$$

i.e., $$A^2 - 4A - 5I = 0$$

Step : 2 *To find A^{-1} :*

$$A^{-1}A^2 - 4AA^{-1} - 5A^{-1} = 0 \quad \text{(Multiply by } A^{-1}\text{)}$$
$$A - 4I - 5A^{-1} = 0$$

$$A^{-1} = \frac{1}{5}[A - 4I] = \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$$

Example 4 *If $A = \begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}$, express A^3 interms of A and I using Cayley - Hamilton theorem.*

Solution

Step : 1 ***To find characteristic equation :***

The characteristic equation is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 0 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(5-\lambda) = 0$$

$$5 - \lambda - 5\lambda + \lambda^2 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

ie. $A^2 - 6A + 5I = 0$ [By C – H Theorem]. ... (1)

Step : 2 ***To find A^3 :***

i.e. $A^3 - 6A^2 + 5A = 0$ [premultiplying by A]

i.e. $A^3 = 6A^2 - 5A$

$= 6[6A - 5I] - 5A$ [From (1)]

$= 31A - 30I$

$$= 31\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix} - 30\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 0 \\ 124 & 155 \end{bmatrix} + \begin{bmatrix} -30 & 0 \\ 0 & -30 \end{bmatrix}$$

$$\therefore \quad A^3 = \begin{bmatrix} 1 & 0 \\ 124 & 125 \end{bmatrix}$$

Example 5 *Using Cayley - Hamilton theorem find the inverse of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.*

Solution

Step : 1 ***To find characteristic equation :***

Let $$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

By Cayley-Hamilton theorem we have

$$A^2 - 4A - 5I = 0$$

Step : 2 *To find A^{-1} :*

ie. $$A^{-1}A^2 - 4A^{-1}A - 5A^{-1} = 0$$

$$A - 4I - 5A^{-1} = 0$$

$$5A^{-1} = A - 4I$$

$$A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Example 6

Verify Cayley-Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

Solution

Step : 1 *To find characteristic equation :*

Let $$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where $$a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases}$$

$$= 8 - 3 + 1$$

$$= 6$$

$$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$$

$$= \begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix}$$

$$= -3 - 8 + 8 - 6 - 24 + 32$$

$$= -1$$

$$a_3 = |A| = \begin{vmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix}$$

$$= 8(-3-8) + 8(4+6) + 2(-16+9)$$

$$= -88 + 80 - 14$$

$$= -22$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 6\lambda^2 - \lambda + 22 = 0$$

Step : 2 ***Verification :***

To verify Cayley-Hamilton theorem, we have to prove that

$$A^3 - 6A^2 - A + 22I = 0$$

Now, $$A^2 = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 64-32+6 & -64+24-8 & 16+16+2 \\ 32-12-6 & -32+9+8 & 8+6-2 \\ 24-16+3 & -24+12-4 & 6+8+1 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 304-112+32 & -384+120-32 & 272-96+30 \\ 152-42-22 & -192+45+32 & 136-36-30 \\ 114-56+11 & -144+60-16 & 102-48+15 \end{bmatrix}$$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$$

$\therefore \; A^3 - 6A^2 - A + 22\,I$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix} - 6\begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + 22\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, Cayley-Hamilton theorem is verified.

Example 7 *Verify Cayley-Hamilton theorem for* $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$. ***Hence find its inverse.*** *[AU 1998]*

Solution

Step : 1 ***To find characteristic equation :***

Given $$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where $a_1 = \left\{\text{Sum of leading diagonal elements}\right.$

$= 1 + 1 + 1$

$= 3$

$a_2 = \left\{\text{Sum of the minors of the leading diagonal elements}\right.$

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$= 1 - 1 + 1 - 3 + 1 - 0$

$= -1$

$$a_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$= 1\,(1 - 1) - 0\,(2 + 1) + 3\,(-2 - 1)$

$= -9$

$\therefore$ The characteristic equation is

$$\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

Step : 2 ***Verification :***

To verify Cayley-Hamilton Theorem, we have to prove that

$$A^3 - 3A^2 - A + 9I = 0 \qquad \dots (2)$$

Now

$$A^2 = A \times A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0+0-3 & 3-0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix}$$

$$= \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}$$

$$A^3 = AA^2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}$$

$$= \begin{bmatrix} 4+0+0 & -3+0-6 & 6+0+15 \\ 8+3-0 & -6+2+2 & 12+4-5 \\ 4-3+0 & -3-2-2 & 6-4+5 \end{bmatrix}$$

$$= \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix}$$

$\therefore A^3 - 3A^2 - A + 9I \Rightarrow$

$$\begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix} + \begin{pmatrix} -12 & 9 & -18 \\ -9 & -6 & -12 \\ 0 & 6 & -15 \end{pmatrix} + \begin{pmatrix} -1 & 0 & -3 \\ -2 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$+ \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence Cayley-Hamilton theorem is verified.

Step : 3 ***To find A^{-1}:***

We know that $A^3 - 3A^2 - A + 9I = 0$

Premultiplying by A^{-1},

$$A^{-1}A^3 - 3A^{-1}A^2 - A^{-1}A + 9I = 0$$

$$A^2 - 3A - I + 9A^{-1} = 0$$

$$\therefore A^{-1} = \frac{-1}{9}(A^2 - 3A - I)$$

$$A^2 - 3A - I = \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{-1}{9}\begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{pmatrix}$$

Example 6

Verify Cayley-Hamilton theorem and hence find inverse for $\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$. *[AU, Part-time 2000]*

Solution

Step : 1 *To find characteristic equation :*

Let
$$A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where
$$a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases}$$
$$= 1 + 2 + 1 = 4$$
$$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$$
$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$
$$= 2 - 6 + 1 - 7 + 2 - 12$$
$$= -20$$
$$a_3 = |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$
$$= 1(2 - 6) - 3(4 - 3) + 7(8 - 2)$$
$$= -4 - 3 + 42 = 35$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

Step : 2 ***Verification :***

To verify Cayley-Hamilton theorem we have to prove that

$$A^3 - 4A^2 - 20A - 35I = 0$$

$$\text{Now } A^2 = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}$$

$$= \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$= \begin{bmatrix} 20+45+70 & 23+66+63 & 23+111+98 \\ 80+30+30 & 92+44+27 & 92+74+42 \\ 20+30+10 & 23+44+9 & 23+74+14 \end{bmatrix}$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix}$$

$$\therefore A^3 - 4A^2 - 20A - 35I \Rightarrow$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} + \begin{pmatrix} -80 & -92 & -92 \\ -60 & -88 & -148 \\ -40 & -36 & -56 \end{pmatrix}$$

$$+ \begin{pmatrix} -20 & -60 & -140 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{pmatrix} + \begin{pmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence Cayley-Hamilton theorem is verified.

Step : 3 ***To find A^{-1} :***

We know that $A^3 - 4A^2 - 20A - 35 = 0$

Premultiply by A^{-1},

$$A^{-1}A^3 - 4A^{-1}A^2 - 20A^{-1}A - 35A^{-1} = 0$$

$$A^2 - 4A - 20 - 35A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{35}(A^2 - 4A - 20)$$

Now $A^2 - 4A - 20$

$$= \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} + \begin{pmatrix} -4 & -12 & -28 \\ -16 & -8 & -12 \\ -4 & -8 & -4 \end{pmatrix} + \begin{pmatrix} -20 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{35}\begin{pmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{pmatrix}$$

Example 9

Find A^4 using Cayley-Hamilton Theorem for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$ *[MU, 1998]*

Solution

Step : 1 ***To find characteristic equation:***

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where $a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases}$

$$= 1 - 1 - 1 = -1$$

$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$

$$= \begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= 1 - 4 - 1 - 9 - 1 - 4$$

$$= -18$$

$$a_3 = |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= 1(1 - 4) - 2(-2 - 12) + 3(2 + 3)$$

$$= -3 + 28 + 15 = 40$$

$\therefore$ The characteristic equation is

$$\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$

Step : 2 *To find A^4 :*

By Cayley-Hamilton theorem we have

$$A^3 + A^2 - 18A - 40 = 0$$

Multiplying by 'A', we get

i.e., $$A(A^3 + A^2 - 18A - 40) = 0$$

i.e., $$A^4 + A^3 - 18A^2 - 40A = 0$$

i.e., $$A^4 = -A^3 + 18A^2 + 40A \qquad \ldots (1)$$

Now $$A^2 = A \times A$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 2-2+3 & 3+8-3 \\ 2-2+12 & 4+1+4 & 6-4-4 \\ 3+2-3 & 6-1-1 & 9+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} \qquad \ldots (2)$$

$$A^3 = A^2 \times A = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 14+6+24 & 28-3+8 & 42+12-8 \\ 12+18-6 & 24-9-2 & 36+36+2 \\ 2+8+42 & 4-4+14 & 6+16+14 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} \qquad \ldots (3)$$

Substituting (2) and (3) in (1), we get

$$A^4 = -\begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + 18\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + 40\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

Example 10 *Verify Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ *and hence evaluate the matrix equation,* $A^8-5A^7+7A^6-3A^5+A^4-5A^3-8A^2+2A-I$.

Solution

Step : 1 *To find characteristic equation:*

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where

$$a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases}$$

$$= 2+1+2 = 5$$

$$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2-0+4-1+2-0$$

$$= 7$$

$$a_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0)-1(0-0)+1(0-1)$$

$$= 4-0-0-1 = 3$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

Step : 2 *Verification :*

Verification of Cayley-Hamilton theorem.

Now

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10+0+4 & 8+1+4 & 8+0+5 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 5+0+8 & 4+1+8 & 4+0+10 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$+ 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley-Hamilton theorem is verified.

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I)$$

$$- 15A^2 + 5A - I$$

$$= A^5(0) + A(0) - 15A^2 + 5A - I$$

$$= -15A^2 + 5A - I$$

$$= -15\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= -11\begin{bmatrix} 6 & 5 & 5 \\ 0 & 1 & 0 \\ 5 & 5 & 6 \end{bmatrix}$$

Example 11

Verify Cayley-Hamilton Theorem and hence find A^{-1} and A^4, $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Solution

Step : 1 *To find characteristic equation:*

Given $$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where $$a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal elements} \end{cases}$$

$$= 3 + 5 + 3 = 11$$

$$a_2 = \begin{cases} \text{Sum of the minors of the} \\ \text{leading diagonal elements} \end{cases}$$

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ -1 & 5 \end{vmatrix}$$

$$= 15 - 1 + 9 - 1 + 15 + 1$$

$$= 38$$

$$a_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) - 1(-3+1) + 1(1-5)$$

$$= 42 + 2 - 4 = 40$$

$\therefore$ The characteristic equation is

$$\lambda^3 - 11\lambda^2 + 38\lambda - 40 = 0$$

Step : 2 *Verification :*

To verify Cayley-Hamilton theorem we have to prove that

$$A^3 - 11A^2 + 38A - 40I = 0$$

Now, $$A^2 = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1+1 & 3+5-1 & 3-1+3 \\ -3-5-1 & -1+25+1 & -1-5-3 \\ 3+1+3 & 1-5-3 & 1+1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 27-7+5 & 9+35-5 & 9-7+15 \\ -27-25-9 & -9+125+9 & -9-25-27 \\ 21+7+11 & 7-35-11 & 7+7+33 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{bmatrix}$$

$\therefore \quad A^3 - 11A^2 + 38A - 40\,I$

$$= \begin{bmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{bmatrix} - 11 \begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix} - 38 \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} + 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{bmatrix} - \begin{bmatrix} 99 & 77 & 55 \\ -99 & 275 & -99 \\ 77 & -77 & 121 \end{bmatrix} + \begin{bmatrix} 114 & 38 & 38 \\ -38 & 190 & -38 \\ 38 & -38 & 114 \end{bmatrix} + \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, Cayley Hamilton theorem is verified.

Step : 3 *To find A^{-1} :*

$\therefore$ We know that $A^3 - 11A^2 + 38A - 40\,I = 0$

Premultiplying by A^{-1} $\quad \cdots (1)$