University of Moratuwa, Sri Lanka

Department of Electronic & Telecommunication Engineering

Branched Cylinders: Dendritic Tree Approximations

BM2102 Modelling and Analysis of Physiological Systems

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Question 01

Branch Condition

At $X = 0$:

$$\left.\frac{dV_1}{dX}\right|_{X=0} = (-r_i\lambda_c)_1 I_{\text{app}}$$

$$V_1(X) = A_1 e^{-X} + B_1 e^{X}$$

For $0 \le X \le L_1$:

$$-A_1 e^{-X} + B_1 e^{X} \quad \text{at} \quad X = 0 = (-r_i\lambda_c)_1 I_{\text{app}}$$

$$\begin{aligned} -A_1 + B_1 &= (-r_i\lambda_c)_1 I_{\text{app}} \\ A_1 - B_1 &= (r_i\lambda_c)_1 I_{\text{app}} \quad (1) \end{aligned}$$

Boundary conditions:

$$V(L_{21}) = 0, \quad V(L_{22}) = 0 \quad \text{(End voltage zero)}$$

For $L_1 \le X \le L_{21}$:

$$A_{21} e^{-X} + B_{21} e^{X} = V_{21}(X)$$

For $L_1 \le X \le L_{22}$:

$$A_{22} e^{-X} + B_{22} e^{X} = V_{22}(X)$$

At $X = L_{21}$:

$$\begin{aligned} A_{21} e^{-L_{21}} + B_{21} e^{L_{21}} &= V(L_{21}) \\ A_{21} e^{-L_{21}} + B_{21} e^{L_{21}} &= 0 \quad (2) \end{aligned}$$

At $X = L_{22}$:

$$\begin{aligned} A_{22} e^{-L_{22}} + B_{22} e^{L_{22}} &= V(L_{22}) \\ A_{22} e^{-L_{22}} + B_{22} e^{L_{22}} &= 0 \quad (3) \end{aligned}$$

Nodal Condition

Continuity:

$$V_1(L_1) = V_{21}(L_1) = V_{22}(L_1)$$

Sample voltage at the common point:

$$A_1 e^{-L_1} + B_1 e^{L_1} = A_{21} e^{-L_1} + B_{21} e^{L_1} = A_{22} e^{-L_1} + B_{22} e^{L_1} \quad (*)$$

Expanding:

$$\begin{aligned} A_1 e^{-L_1} + B_1 e^{L_1} &= A_{21} e^{-L_1} + B_{21} e^{L_1} \\ A_1 e^{-L_1} + B_1 e^{L_1} - A_{21} e^{-L_1} - B_{21} e^{L_1} &= 0 \quad (4) \end{aligned}$$

$$\begin{aligned} A_{21} e^{-L_1} + B_{21} e^{L_1} &= A_{22} e^{-L_1} + B_{22} e^{L_1} \\ A_{21} e^{-L_1} + B_{21} e^{L_1} - A_{22} e^{-L_1} + B_{22} e^{L_1} &= 0 \quad (5) \end{aligned}$$

Continuation

Differentiation:

$$\left.\frac{dV_{21}}{dX}\right|_{X=L_1} = -A_{21}e^{-L_1} + B_{21}e^{L_1}$$

$$\left.\frac{dV_{22}}{dX}\right|_{X=L_1} = -A_{22}e^{-L_1} + B_{22}e^{L_1}$$

Applying KCL at $X = L_1$:

$$-\frac{1}{(r_i\lambda_c)_1}\left.\frac{dV_1}{dX}\right|_{X=L_1} = -\frac{1}{(r_i\lambda_c)_{21}}\left.\frac{dV_{21}}{dX}\right|_{X=L_1} + -\frac{1}{(r_1 i\lambda_c)_{22}}\left.\frac{dV_{22}}{dX}\right|_{X=L_1}$$

Substituting the derivatives:

$$-\frac{1}{(r_i\lambda_c)_1}\left(-A_1e^{-L_1} + B_1e^{L_1}\right) = -\frac{1}{(r_i\lambda_c)_{21}}\left(-A_{21}e^{-L_1} + B_{21}e^{L_1}\right) + -\frac{1}{(r_i\lambda_c)_{22}}\left(-A_{22}e^{-L_1} + B_{22}e^{L_1}\right)$$

Expanding:

$$\frac{-A_1e^{-L_1}}{(r_i\lambda_c)_1} + \frac{B_1e^{L_1}}{(r_i\lambda_c)_1} + \frac{A_{21}e^{-L_1}}{(r_i\lambda_c)_{21}} - \frac{B_{21}e^{L_1}}{(r_1\lambda_c)_{21}} + \frac{A_{22}e^{-L_1}}{(r_i\lambda_c)_{22}} - \frac{B_{22}e^{L_1}}{(r_i\lambda_c)_{22}} = 0 \qquad (6)$$

Question 02

$$AX = b$$

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-L_{21}} & e^{L_{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-L_{22}} & e^{L_{21}} \\ e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{-L_1} & e^{-L_1} & 0 \\ 0 & 0 & e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{-L_1} \\ \frac{-e^{-L_1}}{(r_i\lambda_c)_1} & \frac{e^{L_1}}{(r_i\lambda_c)_1} & \frac{e^{-L_1}}{(r_i\lambda_c)_{21}} & \frac{-e^{-L_1}}{(r_i\lambda_c)_{21}} & \frac{e^{-L_1}}{(r_i\lambda_c)_{22}} & \frac{-e^{L_1}}{(r_i\lambda_c)_{22}} \end{pmatrix}$$

$$X = \begin{pmatrix} A_1 \\ B_1 \\ A_{21} \\ B_{21} \\ A_{22} \\ B_{22} \end{pmatrix}$$

$$b = \begin{pmatrix} (r_i x_c)_1 I_{\text{opp}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A_n \times X = b_n \quad \text{(general)}$$

Where: - A_n : nth row of A - b_n : nth row of b

From the equations:

$$A_1 - B_1 = (r_i\lambda_c)_1 I_{\text{app}} \tag{1'}$$

$$e^{-L_{21}} A_{21} + e^{L_{21}} B_{21} = 0 \tag{2'}$$

$$e^{-L_{22}} A_{22} + e^{L_{21}} B_{22} = 0 \tag{3'}$$

$$e^{-L_1} A_1 + e^{L_1} B_1 - e^{-L_1} A_{21} - e^{L_1} B_{21} = 0 \tag{4'}$$

$$e^{-L_1} A_{21} + e^{L_1} B_{21} - e^{-L_1} A_{22} - e^{L_1} B_{22} = 0 \tag{5'}$$

$$-A_1 \frac{e^{-L_1}}{(r_i\lambda_c)_1} + B_1 \frac{e^{L_1}}{(r_i\lambda_c)_1} + A_{21} \frac{e^{-L_1}}{(r_i\lambda_c)_{21}} - B_{21} \frac{e^{L_4}}{(r_i\lambda_c)_{21}} + A_{22} \frac{e^{-L_1}}{(r_i\lambda_c)_{22}} - B_{22} \frac{e^{L_1}}{(r_i\lambda_c)_{22}} = 0 \tag{6'}$$

Matlab Stimulation

```
% electrical constants and derived quantities for typical
% mammalian dendrite

% Dimensions of compartments

d1 = 75e-4;             % cm
d21 = 30e-4;             % cm
d22 = 15e-4;             % cm


l1 = 1.5;            % dimensionless
l21 = 3.0;            % dimensionless
l22 = 3.0;            % dimensionless

% Electrical properties of compartments

Rm = 6e3;            % Ohms cm^2
Rc = 90;            % Ohms cm
Rs = 1e6;            % Ohms

c1 = 2*(Rc*Rm)^(1/2)/pi;

rl1 = c1*d1^(-3/2);        % Ohms
rl21 = c1*d21^(-3/2);        % Ohms
rl22 = c1*d22^(-3/2);        % Ohms


% Applied current

iapp = 1e-9;     % Amps

% Coefficient matrices

A = [1 -1 0 0 0 0;
     0 0 exp(-l21) exp(l21) 0 0;
     0 0 0 0 exp(-l22) exp(l22);
     exp(-l1) exp(l1) -exp(-l1) -exp(l1) 0 0;
     0 0 exp(-l1) exp(l1) -exp(-l1) -exp(l1);
     -exp(-l1) exp(l1) rl1*exp(-l1)/rl21 -rl1*exp(l1)/rl21 rl1*exp(-l1)/rl22
-rl1*exp(-l1)/rl22];

b = [rl1*iapp 0 0 0 0 0]';
```

Question 03

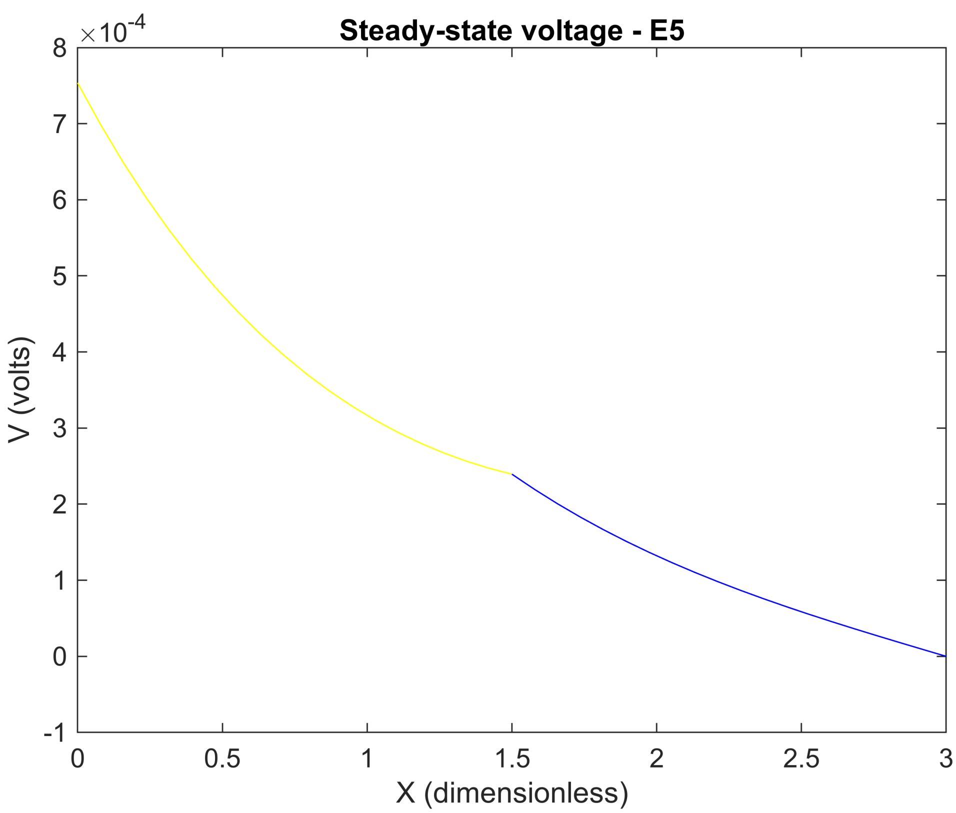
```
x=A\b;
```

```
disp(x);
```

```
    0.0007
    0.0000
    0.0011
   -0.0000
    0.0011
   -0.0000
```

Question 04

```
y1 = linspace(0, l1, 20);
y21 = linspace(l1, l21, 20);
y22 = linspace(l1, l22, 20);
v1 = x(1)*exp(-y1) + x(2)*exp(y1);
v21 = x(3)*exp(-y21) + x(4)*exp(y21);
v22 = x(5)*exp(-y22) + x(6)*exp(y22);
plot(y1, v1, 'y-', y21, v21, 'r-', y22, v22, 'b-');
xlabel('X (dimensionless)');
ylabel('V (volts)');
title('Steady-state voltage - E5');
```



The red line is not visible in the diagram, indicating it is either equal to or very close to the blue line.
Since the yellow line (the parent branch's membrane potential) does not affect the daughter branches,
it is concluded that the red and blue lines are equal.
Therefore, the steady state voltage profiles of the two daughter branches are identical.
The code also supports this conclusion by plotting the red and blue lines, with the red line being hidden by the blue line,
confirming their equality and the identical voltage profiles of the branches.
Additionally, it mentions that the plots of and the plots of seem to be identical,
which further supports the conclusion. As a result, the steady-state voltage profile through the two daughter branches is equal.

Question 05

```
A_a=A;
b_a=b;
```

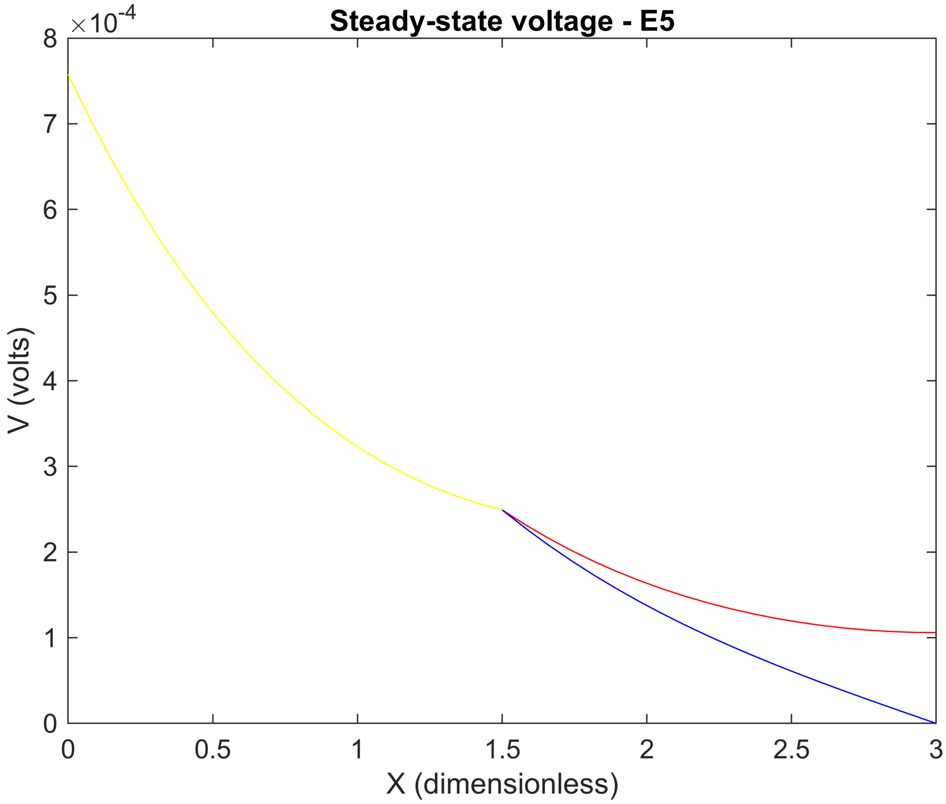
(i) $\left(\frac{d}{dX}V_{21}\right)_{X=L_{21}} = 0$

$-A_{21}e^{-L_{21}} + B_{21}e^{-L_{21}} = 0 \quad \longrightarrow \quad (2'')$

Sealed End at V_{21}

Killed End at V_{22}

```
A_a(2,:)=[0 0 -exp(-l21) exp(l21) 0 0];
x_a=A_a\b;
y1 = linspace(0, l1, 20);
y21 = linspace(l1, l21, 20);
y22 = linspace(l1, l22, 20);
v1 = x_a(1)*exp(-y1) + x_a(2)*exp(y1);
v21 = x_a(3)*exp(-y21) + x_a(4)*exp(y21);
v22 = x_a(5)*exp(-y22) + x_a(6)*exp(y22);
plot(y1, v1, 'y-', y21, v21, 'r-', y22, v22, 'b-');
xlabel('X (dimensionless)');
ylabel('V (volts)');
title('Steady-state voltage - E5');
```



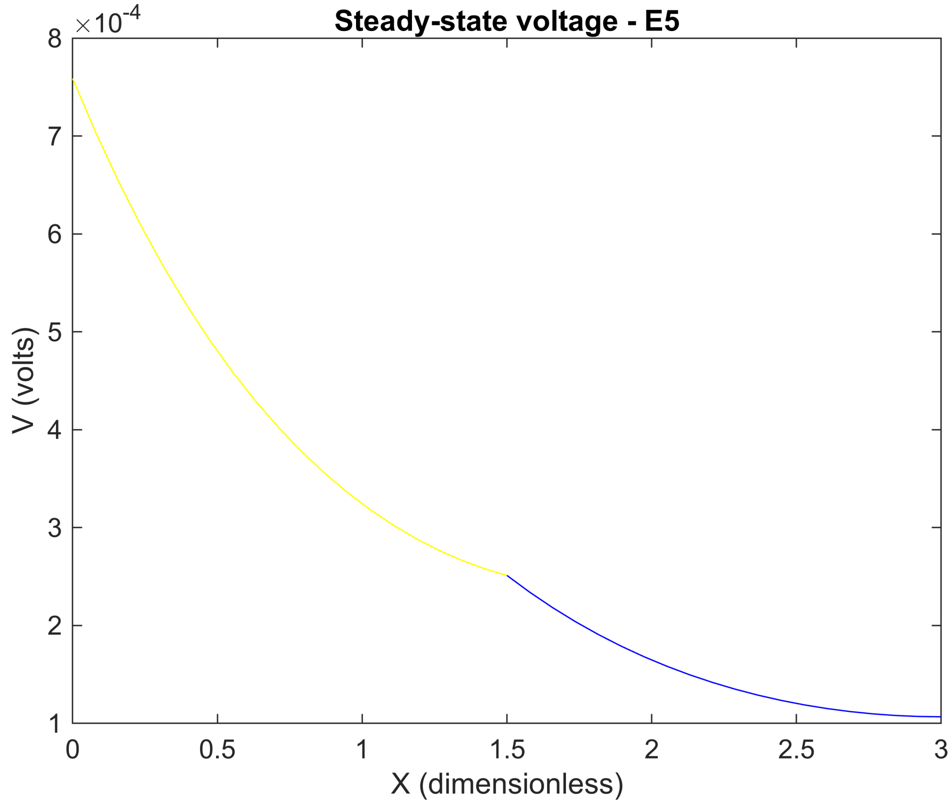
(ii) $\left(\frac{d}{dX} V_{22}\right)_{X=L_{22}} = 0$

$-A_{22}e^{-L_{22}} + B_{22}e^{-L_{22}} = 0 \quad \longrightarrow \quad (3'')$

Sealed End at V_{21}

Sealed End at V_{22}

```
A_a(3,:) = [0 0 0 0 -exp(-l22) exp(l22)];
x_a=A_a\b;
y1 = linspace(0, l1, 20);
y21 = linspace(l1, l21, 20);
y22 = linspace(l1, l22, 20);
v1 = x_a(1)*exp(-y1) + x_a(2)*exp(y1);
v21 = x_a(3)*exp(-y21) + x_a(4)*exp(y21);
v22 = x_a(5)*exp(-y22) + x_a(6)*exp(y22);
plot(y1, v1, 'y-', y21, v21, 'r-', y22, v22, 'b-');
xlabel('X (dimensionless)');
ylabel('V (volts)');
title('Steady-state voltage - E5');
```



(iii) $\left(\frac{d}{dX}V_1\right)_{X=0} = 0$

$-A_1 + B_1 \quad \longrightarrow \quad (1'') = 0$

$$\left(\frac{d}{dX}V_1\right)_{X=L_1} = rl21 * Iapp$$

$-A_{21}e^{-L_{21}} + B_{21}e^{-L_{21}} = 0 \quad \longrightarrow \quad (2'')$

Sealed End at V_{21}

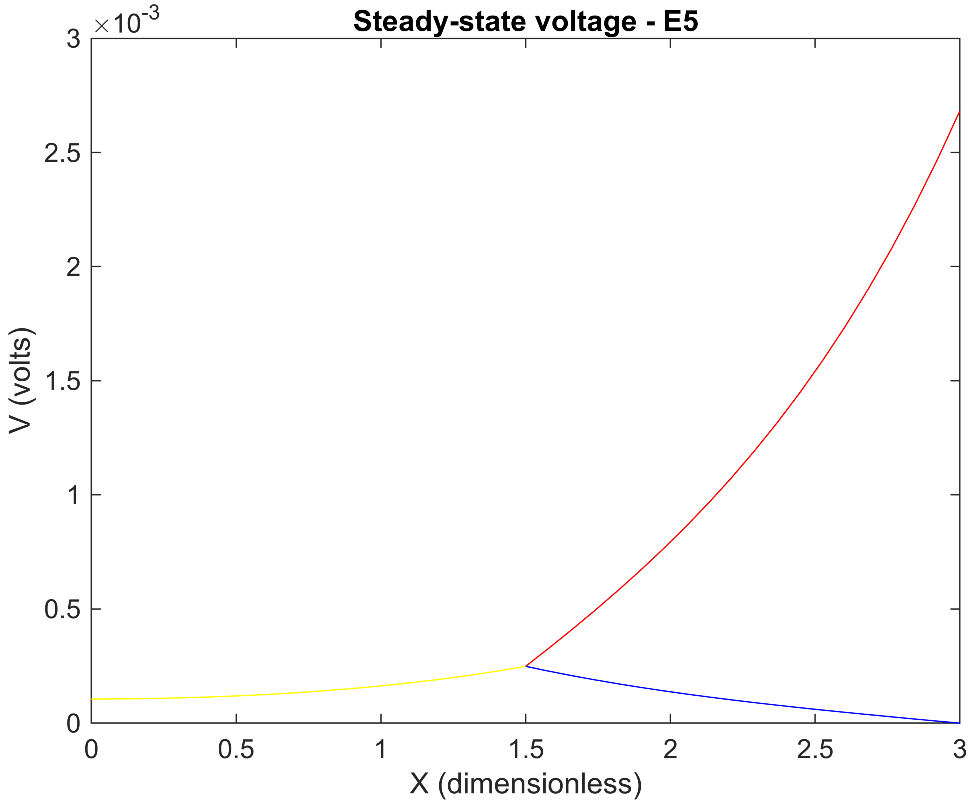
Killed End at V_{22}

Current applied at node passes through the 21 branch only

Positive Voltage gradient at end 21

```
b(1) = 0; b(2) = rl21*iapp;
A_a(3,:) = A(3,:);
x_a=A_a\b;
y1 = linspace(0, l1, 20);
y21 = linspace(l1, l21, 20);
y22 = linspace(l1, l22, 20);
v1 = x_a(1)*exp(-y1) + x_a(2)*exp(y1);
v21 = x_a(3)*exp(-y21) + x_a(4)*exp(y21);
v22 = x_a(5)*exp(-y22) + x_a(6)*exp(y22);
plot(y1, v1, 'y-', y21, v21, 'r-', y22, v22, 'b-');
xlabel('X (dimensionless)');
ylabel('V (volts)');
```

```
title('Steady-state voltage - E5');
```



(iv) $\left(\frac{d}{dX}V_1\right)_{X=L_1} = rl22 * Iapp$

$-A_{22}e^{-L_{22}} + B_{22}e^{-L_{22}} = 0 \quad \longrightarrow \quad (3'')$

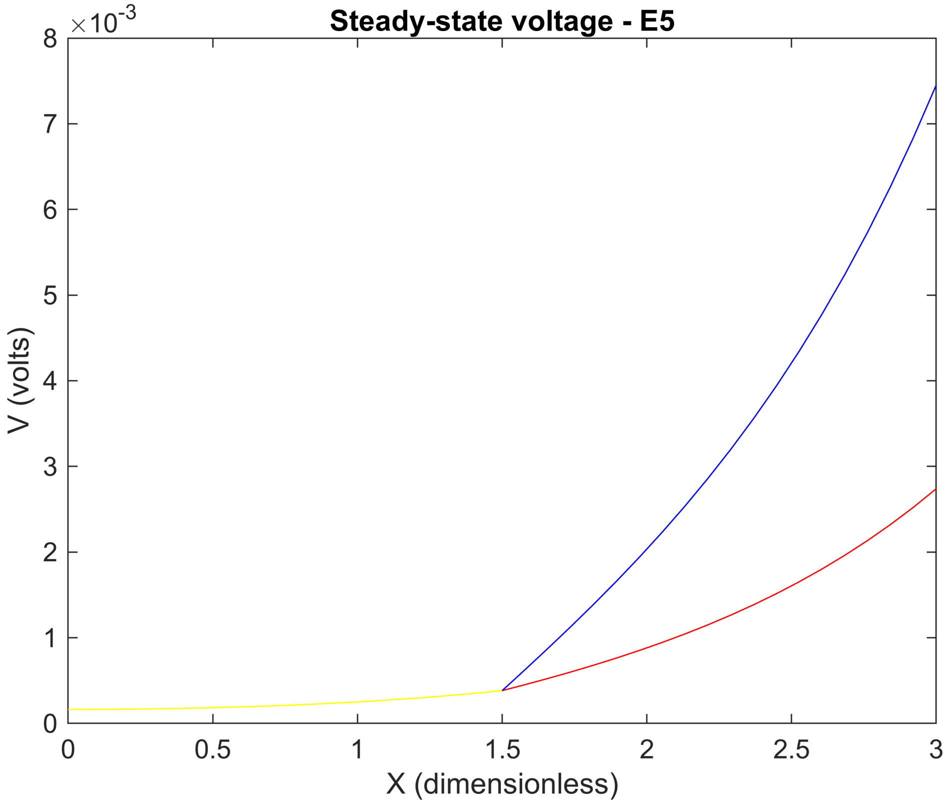
Sealed End at V_{21}

Sealed End at V_{22}

Current applied at node passes through the 21 and 22 branch, but the diameters of the branches are different

Positive Voltage gradient at both ends

```
b(3) = rl22*iapp;
A_a(3,:) = [0 0 0 0 -exp(-l22) exp(l22)];
x_a=A_a\b;
y1 = linspace(0, l1, 20);
y21 = linspace(l1, l21, 20);
y22 = linspace(l1, l22, 20);
v1 = x_a(1)*exp(-y1) + x_a(2)*exp(y1);
v21 = x_a(3)*exp(-y21) + x_a(4)*exp(y21);
v22 = x_a(5)*exp(-y22) + x_a(6)*exp(y22);
plot(y1, v1, 'y-', y21, v21, 'r-', y22, v22, 'b-');
xlabel('X (dimensionless)');
ylabel('V (volts)');
title('Steady-state voltage - E5');
```



The positive right-hand sides of $\left(\frac{d}{dX}V_{21}\right)_{X=L_{21}}$ and $\left(\frac{d}{dX}V_{22}\right)_{X=L_{22}}$

in 2(c) and 2(d) indicate that the membrane voltage gradient is positive at the rightmost nodes of the daughter branches.
This means the daughter branches transmit an electrical impulse to another neuron or branch.
The increase in membrane voltage at these rightmost nodes causes the voltage gradient to be positive, ensuring efficient electrical impulse transmission from the parent branch to the daughter branches.

Question 06

```
% Dimensions of compartments

d21 = 47.2470e-4;          % E9 cm
d22 = d21;                 % E9 cm

l1 = 1.5;             % dimensionless
l21 = 3.0;             % dimensionless
l22 = 3.0;             % dimensionless

% Electrical properties of compartments

Rm = 6e3;             % Ohms cm^2
Rc = 90;             % Ohms cm
Rs = 1e6;             % Ohms
```

```
c1 = 2*(Rc*Rm)^(1/2)/pi;

rl1 = c1*d1^(-3/2);        % Ohms
rl21 = c1*d21^(-3/2);        % Ohms
rl22 = c1*d22^(-3/2);        % Ohms


% Applied current

iapp = 1e-9;     % Amps

% Coefficient matrices

A = [1 -1 0 0 0 0;
     0 0 exp(-l21) exp(l21) 0 0;
     0 0 0 0 exp(-l22) exp(l22);
     exp(-l1) exp(l1) -exp(-l1) -exp(l1) 0 0;
     0 0 exp(-l1) exp(l1) -exp(-l1) -exp(l1);
     -exp(-l1) exp(l1) rl1*exp(-l1)/rl21 -rl1*exp(l1)/rl21 rl1*exp(-l1)/rl22
-rl1*exp(-l1)/rl22];

b = [rl1*iapp 0 0 0 0 0]';
```

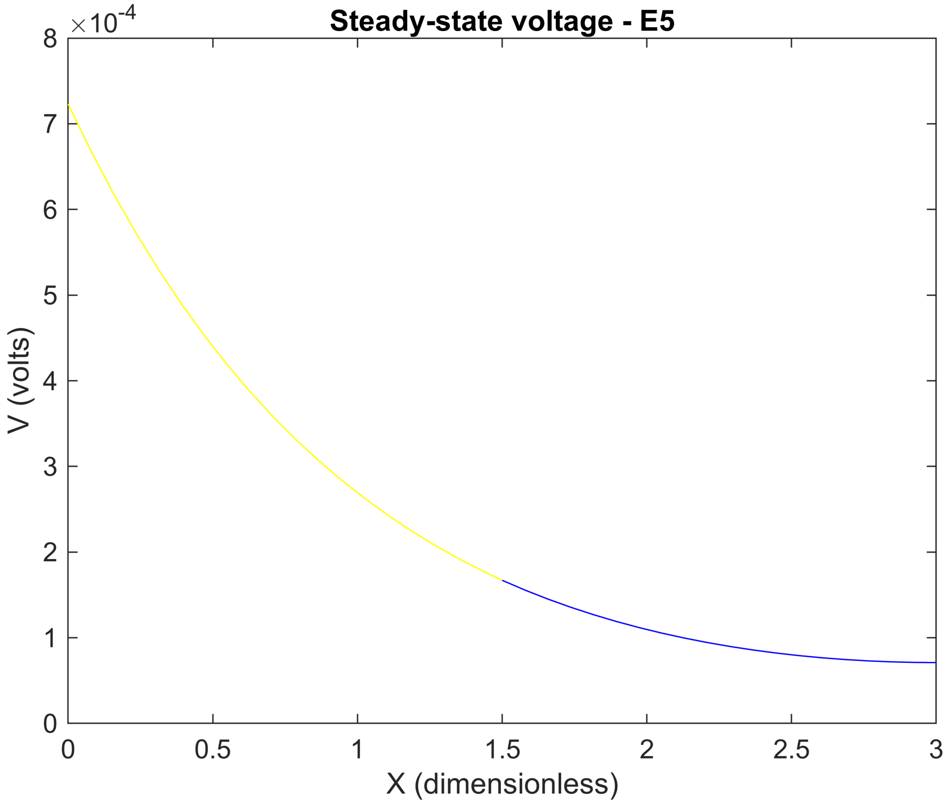
Sealed End at V_{21}

Sealed End at V_{22}

Current applied at terminal X=0

Diameter of both branch are equal

```
A_1=A;
A_1(2,:)=[0 0 -exp(-l21) exp(l21) 0 0];
A_1(3,:) = [0 0 0 0 -exp(-l22) exp(l22)];
x_a=A_1\b;
y1 = linspace(0, l1, 20);
y21 = linspace(l1, l21, 20);
y22 = linspace(l1, l22, 20);
v1 = x_a(1)*exp(-y1) + x_a(2)*exp(y1);
v21 = x_a(3)*exp(-y21) + x_a(4)*exp(y21);
v22 = x_a(5)*exp(-y22) + x_a(6)*exp(y22);
plot(y1, v1, 'y-', y21, v21, 'r-', y22, v22, 'b-');
xlabel('X (dimensionless)');
ylabel('V (volts)');
title('Steady-state voltage - E5');
```



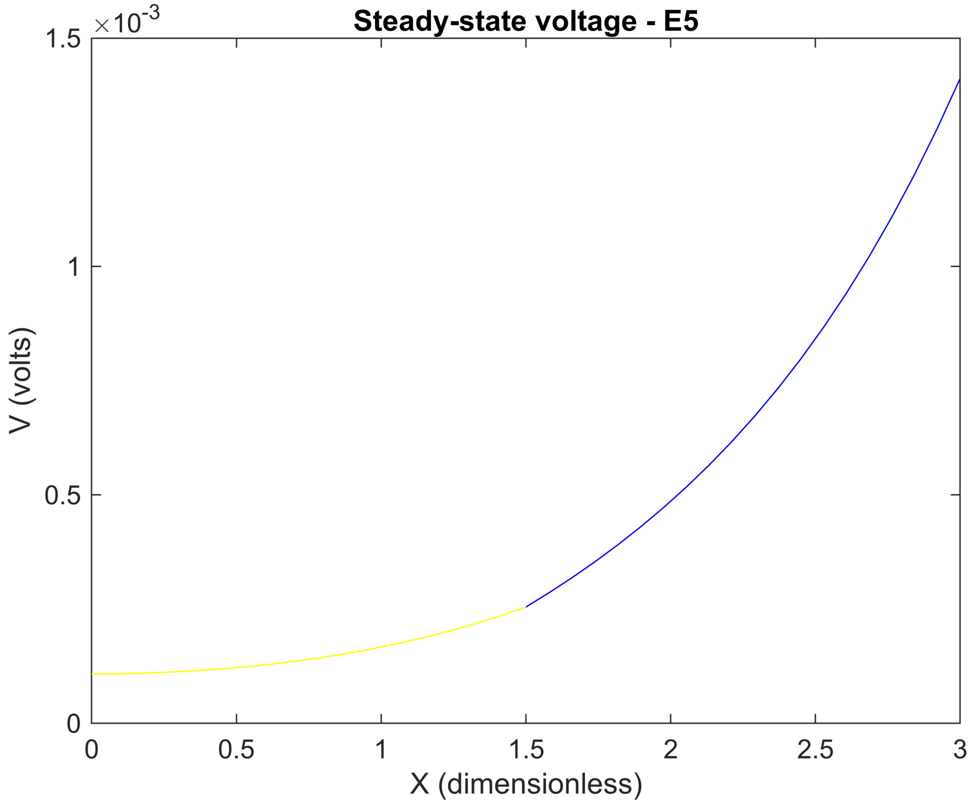
Sealed End at V_{21}

Sealed End at V_{22}

Current applied at node passes through the 21 and 22 branch, but the diameters of the branches are equal

Positive voltage gradient at both ends

```
A_2=A;
b_2=b;
A_2(2,:)=[0 0 -exp(-l21) exp(l21) 0 0];
A_2(3,:) = [0 0 0 0 -exp(-l22) exp(l22)];
b_2(1) = 0; b_2(2) = rl21*iapp;
b_2(3) = rl22*iapp;
x_a=A_2\b_2;
y1 = linspace(0, l1, 20);
y21 = linspace(l1, l21, 20);
y22 = linspace(l1, l22, 20);
v1 = x_a(1)*exp(-y1) + x_a(2)*exp(y1);
v21 = x_a(3)*exp(-y21) + x_a(4)*exp(y21);
v22 = x_a(5)*exp(-y22) + x_a(6)*exp(y22);
plot(y1, v1, 'y-', y21, v21, 'r-', y22, v22, 'b-');
xlabel('X (dimensionless)');
ylabel('V (volts)');
title('Steady-state voltage - E5');
```



The given image explains the significance of certain properties observed in graphs labeled 2(c) and 2(d).
Both graphs exhibit a continuously differentiable voltage profile, which allows smooth transmission of electrical impulses from the parent branch to the daughter branches.
Smooth transitions due to identical properties of both ends

For graph 2(c):
Both daughter branches have **sealed ends**, meaning the (zero axial current) at their terminals. Since the structural and electrical parameters (diameter, length, resistivity) of both branches are identical, they experience identical voltage drops, causing the voltage curves of both branches to **overlap**completely.In sealed end with zero axial no matter with structural parameter potential difference same

For graph 2(d):
Both daughter branches are instead **sealed at their ends**(Positive Voltage gradient), but they still have identical electrical properties (same diameter, same resistance, same length).
Due to this perfect symmetry, the resulting potential difference along both branches is the same, leading again to **overlapping voltage profiles**despite the change in boundary condition.
In conclusion, the overlap of voltage curves in both 2(c) and 2(d) is due to the identical properties of the daughter branches, and in 2(c), also due to both ends being sealed with zero axial current. These observations are complementary and reinforce that symmetry in structure and boundary conditions leads to equal voltage behavior across branches.